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Categorical representations and KLR algebras

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We prove that the KLR algebra associated with the cyclic quiver of length e is a subquotient of the KLR algebra associated with the cyclic quiver of length $e+1$. We also give a geometric interpretation of this fact. This result has an important application in the theory of categorical representations. We prove that a category with an action of $\widetilde{\mathfrak{sl}}_{e+1}$ contains a subcategory with an action of $\widetilde{\mathfrak{sl}}_e$. We also give generalizations of these results to more general quivers and Lie types.

1. Introduction

Consider the complex affine Lie algebra $\widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$. In this paper, we study categorical representations of $\widetilde{\mathfrak{sl}}_e$. Our goal is to relate the notion of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with the notion of a categorical representation of $\widetilde{\mathfrak{sl}}_{e+1}$.

The Lie algebra $\widetilde{\mathfrak{sl}}_e$ has generators e_i, f_i for $i \in [0, e-1]$. Let $\alpha_0, \dots, \alpha_{e-1}$ be the simple roots of $\widetilde{\mathfrak{sl}}_e$. Fix $k \in [0, e-1]$. Consider the following inclusion of Lie algebras $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$:

$$e_r \mapsto \begin{cases} e_r & \text{if } r \in [0, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases} \qquad f_r \mapsto \begin{cases} f_r & \text{if } r \in [0, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1]. \end{cases} \tag{1}$$

It is clear that each $\widetilde{\mathfrak{sl}}_{e+1}$-module can be restricted to the subalgebra $\widetilde{\mathfrak{sl}}_e$ of $\widetilde{\mathfrak{sl}}_{e+1}$. So it is natural to ask if we can do the same with categorical representations.

First, we recall the notion of a categorical representation. Let $\boldsymbol{k}$ be a field. Let $\mathcal{C}$ be an abelian Hom-finite $\boldsymbol{k}$-linear category that admits a direct sum decomposition $\mathcal{C} = \bigoplus_{\mu \in \mathbb{Z}^e} \mathcal{C}_\mu$. A categorical representation of $\widetilde{\mathfrak{sl}}_e$ in $\mathcal{C}$ is a pair of biadjoint functors $E_i, F_i \colon \mathcal{C} \to \mathcal{C}$ for $i \in [0, e-1]$ satisfying a list of axioms. The main axiom is that for each positive integer d there is an algebra homomorphism

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$R_d(A^{(1)}_{e-1}) \to \mathrm{End}(F^d)^{\mathrm{op}}$, where $F = \bigoplus_{i=0}^{e-1} F_i$ and $R_d(A^{(1)}_{e-1})$ is the KLR algebra of rank d associated with the quiver $A^{(1)}_{e-1}$ (i.e., with the cyclic quiver of length e).

Let $\bar{\mathcal{C}}$ be an abelian Hom-finite $\boldsymbol{k}$-linear category. Assume that $\bar{\mathcal{C}} = \bigoplus_{\mu\in\mathbb{Z}^{e+1}} \bar{\mathcal{C}}_\mu$ has a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_{e+1}$ with respect to functors $\bar{E}_i$, $\bar{F}_i$ for $i \in [0, e]$. We want to restrict the action of $\widetilde{\mathfrak{sl}}_{e+1}$ on $\bar{\mathcal{C}}$ to $\widetilde{\mathfrak{sl}}_e$. The most obvious way to do this is to define new functors $E_i, F_i : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$, for $i \in [0, e-1]$, from the functors $\bar{E}_i, \bar{F}_i : \bar{\mathcal{C}} \to \bar{\mathcal{C}}$, for $i \in [0, e]$, by the same formulas as in (1). Of course, this makes no sense because the notion of a commutator of two functors does not exist. However, we are able to get a structure of a categorical representation on a subcategory $\mathcal{C} \subset \bar{\mathcal{C}}$ (and not on the category $\bar{\mathcal{C}}$ itself). We do this in the following way.

Assume additionally that the category $\bar{\mathcal{C}}_\mu$ is zero whenever μ has a negative entry. For each e-tuple $\mu = (\mu_1, \dots, \mu_e) \in \mathbb{Z}^e$ we consider the $(e+1)$-tuple $\bar{\mu} = (\mu_1, \dots, \mu_k, 0, \mu_{k+1}, \dots, \mu_e)$ and we set $\mathcal{C}_\mu = \bar{\mathcal{C}}_{\bar{\mu}}$,

$$\mathcal{C} = \bigoplus_{\mu\in\mathbb{Z}^e} \mathcal{C}_\mu.$$

Next, consider the endofunctors of $\mathcal{C}$ given by

$$E_i = \begin{cases} \bar{E}_i|_{\mathcal{C}} & \text{if } 0 \leqslant i < k, \\ \bar{E}_k\bar{E}_{k+1}|_{\mathcal{C}} & \text{if } i = k, \\ \bar{E}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e, \end{cases} \qquad F_i = \begin{cases} \bar{F}_i|_{\mathcal{C}} & \text{if } 0 \leqslant i < k, \\ \bar{F}_{k+1}\bar{F}_k|_{\mathcal{C}} & \text{if } i = k, \\ \bar{F}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e. \end{cases}$$

The following theorem holds.

Theorem 1.1. *The category $\mathcal{C}$ has the structure of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with respect to the functors $E_0, \dots, E_{e-1}, F_0, \dots, F_{e-1}$.* □

Let us explain our motivation for proving Theorem 1.1 (see [Maksimau 2015b] for more details). Let O^ν_{-e} be the parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N = \mathfrak{gl}_N[t, t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$ with parabolic type ν at level $-e-N$. By [Rouquier et al. 2016], there is a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in O^ν_{-e}. Now we apply Theorem 1.1 to $\bar{\mathcal{C}} = O^\nu_{-(e+1)}$. It happens that in this case the subcategory $\mathcal{C} \subset \bar{\mathcal{C}}$ defined as above is equivalent to O^ν_{-e}. This allows us to compare the categorical representations in the category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N$ for two different (negative) levels.

A result similar to Theorem 1.1 has recently appeared in [Riche and Williamson 2018], where it is applied in the following way. It is known from [Chuang and Rouquier 2008] that there is a categorical representation of $\widetilde{\mathfrak{sl}}_p$ in the category $\mathrm{Rep}(\mathrm{GL}_n(\bar{\mathbb{F}}_p))$ of finite dimensional algebraic representations of $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$. Riche and Williamson used this fact to construct a categorical representation of the Hecke category on the principal block $\mathrm{Rep}_0(\mathrm{GL}_n(\bar{\mathbb{F}}_p))$ of $\mathrm{Rep}(\mathrm{GL}_n(\bar{\mathbb{F}}_p))$ for $p > n$. Their proof is in two steps. First they show that the action of $\widetilde{\mathfrak{sl}}_p$ on $\mathrm{Rep}(\mathrm{GL}_n(\bar{\mathbb{F}}_p))$ induces an action of $\widetilde{\mathfrak{sl}}_n$ on some full subcategory of $\mathrm{Rep}(\mathrm{GL}_n(\bar{\mathbb{F}}_p))$. The second step is to show that the action of $\widetilde{\mathfrak{sl}}_n$ constructed on the first step induces an action of the Hecke category on $\mathrm{Rep}_0(\mathrm{GL}_n(\bar{\mathbb{F}}_p))$. The first step of their proof is essentially $p-n$ consecutive applications of Theorem 1.1.

The main difficulty in proving Theorem 1.1 is showing that the action of the KLR algebra $R_d(A^{(1)}_e)$ on

$\overline{F}^d$, where $\overline{F}=\bigoplus_{i=0}^{e}\overline{F}_i$, yields an action of the KLR algebra $R_d(A_{e-1}^{(1)})$ on F^d. So, to prove the theorem, we need to compare the KLR algebra $R_d(A_e^{(1)})$ with the KLR algebra $R_d(A_{e-1}^{(1)})$. This is done in Section 2.

We introduce the abbreviations $\Gamma=A_{e-1}^{(1)}$ and $\overline{\Gamma}=A_e^{(1)}$. Let $\alpha=\sum_{i=0}^{e-1}d_i\alpha_i$ be a dimension vector of the quiver Γ. We consider the dimension vector $\bar{\alpha}$ of $\overline{\Gamma}$ defined by

$$\bar{\alpha}=\sum_{i=0}^{k}d_i\alpha_i+\sum_{i=k+1}^{e}d_{i-1}\alpha_i.$$

Let $R_\alpha(\Gamma)$ and $R_{\bar{\alpha}}(\overline{\Gamma})$ be the KLR algebras associated with the quivers Γ and $\overline{\Gamma}$ and the dimension vectors α and $\bar{\alpha}$. The algebra $R_{\bar{\alpha}}(\overline{\Gamma})$ contains idempotents $e(\boldsymbol{i})$ parametrized by certain sequences $\boldsymbol{i}$ of vertices of $\overline{\Gamma}$. In Section 2D we consider some sets of such sequences $\bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$ and $\bar{I}^{\bar{\alpha}}_{\mathrm{un}}$. Set $\boldsymbol{e}=\sum_{\boldsymbol{i}\in\bar{I}^{\bar{\alpha}}_{\mathrm{ord}}}e(\boldsymbol{i})\in R_{\bar{\alpha}}(\overline{\Gamma})$ and

$$S_{\bar{\alpha}}(\overline{\Gamma})=\boldsymbol{e}R_{\bar{\alpha}}(\overline{\Gamma})\boldsymbol{e}\Big/\sum_{\boldsymbol{i}\in\bar{I}^{\bar{\alpha}}_{\mathrm{un}}}\boldsymbol{e}R_{\bar{\alpha}}(\overline{\Gamma})e(\boldsymbol{i})R_{\bar{\alpha}}(\overline{\Gamma})\boldsymbol{e}.$$

The main result of Section 2 is the following theorem.

Theorem 1.2. *There is an algebra isomorphism $R_\alpha(\Gamma)\simeq S_{\bar{\alpha}}(\overline{\Gamma})$.* □

The paper has the following structure. In Section 2 we study KLR algebras. In particular, we prove Theorem 1.2. In Section 3 we study categorical representations. We prove our main result about categorical representations (Theorem 1.1). We also generalize this theorem to arbitrary symmetric Kac–Moody Lie algebras. In Appendix A we give a geometric construction of the isomorphism in Theorem 1.2. In Appendix B, we give some versions of Theorems 1.1 and 1.2 in type A over a local ring.

It is important to emphasize the relation between the present paper and [Maksimau 2015b]. That preprint contains (an earlier version of) the results of the present paper and an application of these results to the category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N$. The preprint is expected to be published as two different papers. The present paper is the first of them. It contains the results of the preprint about KLR algebras and categorical representations. The second paper will give an application of the results of the first paper to the affine category $\mathcal{O}$.

2. KLR algebras

For a noetherian ring A we denote by $\mathrm{mod}(A)$ the abelian category of left finitely generated A-modules. We denote by $\mathbb{N}$ the set of nonnegative integers.

2A. *Kac–Moody algebras associated with a quiver.* Let $\Gamma=(I,H)$ be a quiver without 1-loops with the set of vertices I and the set of arrows H. For $i,j\in I$ let $h_{i,j}$ be the number of arrows from i to j and set also $a_{i,j}=2\delta_{i,j}-h_{i,j}-h_{j,i}$. Let $\mathfrak{g}_I$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the matrix $(a_{i,j})$. Denote by e_i, f_i for $i\in I$ the Serre generators of $\mathfrak{g}_I$.

Remark 2.1. By the Kac–Moody Lie algebra associated with the Cartan matrix $(a_{i,j})$ we understand the Lie algebra with the set of generators e_i, f_i, h_i, $i\in I$, modulo the defining relations

$$
\begin{aligned}
[h_i, h_j] &= 0,\\
[h_i, e_j] &= a_{i,j} e_j,\\
[h_i, f_j] &= -a_{i,j} e_j,\\
[e_i, f_j] &= \delta_{i,j} h_i,\\
(\mathrm{ad}(e_i))^{1-a_{i,j}}(e_j) &= 0 \quad i \neq j,\\
(\mathrm{ad}(f_i))^{1-a_{i,j}}(f_j) &= 0 \quad i \neq j.
\end{aligned}
$$

In particular, if $(a_{i,j})$ is the affine Cartan matrix of type $A^{(1)}_{e-1}$, then we get the Lie algebra $\widetilde{\mathfrak{sl}}_e(\mathbb{C}) = \mathfrak{sl}_e(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ (not $\mathfrak{sl}_e(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1} \oplus \mathbb{C}\partial$).

For each $i \in I$, let α_i be the simple root corresponding to e_i. Set

$$Q_I = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \quad \text{and} \quad Q_I^+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i.$$

For $\alpha = \sum_{i\in I} d_i\alpha_i \in Q_I^+$ denote by $|\alpha|$ its height, i.e., we have $|\alpha| = \sum_{i\in I} d_i$. Set $I^\alpha = \bigl\{\boldsymbol{i} = (i_1, \dots, i_{|\alpha|}) \in I^{|\alpha|} : \sum_{r=1}^{|\alpha|} \alpha_{i_r} = \alpha\bigr\}$.

2B. *Doubled quiver.* Let $\Gamma = (I, H)$ be a quiver without 1-loops. Fix a decomposition $I = I_0 \sqcup I_1$ such that there are no arrows between the vertices in I_1. In this section we define a *doubled quiver* $\overline{\Gamma} = (\bar{I}, \overline{H})$ associated with (Γ, I_0, I_1). The idea is to "double" each vertex in the set I_1 (we do not touch the vertices from I_0). We replace each vertex $i \in I_1$ by a couple of vertices i^1 and i^2 with an arrow $i^1 \to i^2$. Each arrow entering i should be replaced by an arrow entering to i^1, each arrow coming from i should be replaced by an arrow coming from i^2.

Now we describe the construction of $\overline{\Gamma} = (\bar{I}, \overline{H})$ formally. Let $\bar{I}_0$ be a set that is in bijection with I_0. Let i^0 be the element of $\bar{I}_0$ associated with an element $i \in I_0$. Similarly, let $\bar{I}_1$ and $\bar{I}_2$ be sets that are in bijection with I_1. Denote by i^1 and i^2 the elements of $\bar{I}_1$ and $\bar{I}_2$ respectively that correspond to an element $i \in I_1$. Put $\bar{I} = \bar{I}_0 \sqcup \bar{I}_1 \sqcup \bar{I}_2$. We define $\overline{H}$ in the following way. The set $\overline{H}$ contains 4 types of arrows:

- An arrow $i^0 \to j^0$ for each arrow $i \to j$ in H with $i, j \in I_0$.
- An arrow $i^0 \to j^1$ for each arrow $i \to j$ in H with $i \in I_0$, $j \in I_1$.
- An arrow $i^2 \to j^0$ for each arrow $i \to j$ in H with $i \in I_1$, $j \in I_0$.
- An arrow $i^1 \to i^2$ for each vertex $i \in I_1$.

Set $I^\infty = \coprod_{d\in\mathbb{N}} I^d$ and $\bar{I}^\infty = \coprod_{d\in\mathbb{N}} \bar{I}^d$, where I^d and $\bar{I}^d$ are the cartesian products. The concatenation yields a monoid structure on I^∞ and $\bar{I}^\infty$. Let $\phi\colon I^\infty \to \bar{I}^\infty$ be the unique morphism of monoids such that for $i \in I \subset I^\infty$ we have

$$\phi(i) = \begin{cases} i^0 & \text{if } i \in I_0,\\ (i^1, i^2) & \text{if } i \in I_1.\end{cases}$$

There is a unique $\mathbb{Z}$-linear map $\phi\colon Q_I \to Q_{\bar{I}}$ such that $\phi(I^\alpha) \subset \bar{I}^{\phi(\alpha)}$ for each $\alpha \in Q_I^+$. It is given by

$$\phi(\alpha_i) = \begin{cases} \alpha_{i^0} & \text{if } i \in I_0, \\ \alpha_{i^1} + \alpha_{i^2} & \text{if } i \in I_1. \end{cases}$$

2C. ***KLR algebras.*** Let $\boldsymbol{k}$ be a field. Let $\Gamma = (I, H)$ be a quiver without 1-loops. For $r \in [1, d-1]$ let s_r be the transposition $(r, r+1) \in \mathfrak{S}_d$. For $\boldsymbol{i} = (i_1, \dots, i_d) \in I^d$ set $s_r(\boldsymbol{i}) = (i_1, \dots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \dots, i_d)$. For $i, j \in I$ we set

$$Q_{i,j}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ (v-u)^{h_{i,j}}(u-v)^{h_{j,i}} & \text{else.} \end{cases}$$

Definition 2.2. Assume that the quiver Γ is finite. The *KLR-algebra* $R_{d,\boldsymbol{k}}(\Gamma)$ is the $\boldsymbol{k}$-algebra with the set of generators $\tau_1, \dots, \tau_{d-1}, x_1, \dots, x_d, e(\boldsymbol{i})$ where $\boldsymbol{i} \in I^d$, modulo the following defining relations:

$$\begin{aligned}
e(\boldsymbol{i})e(\boldsymbol{j}) &= \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}), \\
\sum_{\boldsymbol{i} \in I^d} e(\boldsymbol{i}) &= 1, \\
x_r e(\boldsymbol{i}) &= e(\boldsymbol{i}) x_r, \\
\tau_r e(\boldsymbol{i}) &= e(s_r(\boldsymbol{i}))\tau_r, \\
x_r x_s &= x_s x_r, \\
\tau_r x_{r+1} e(\boldsymbol{i}) &= (x_r \tau_r + \delta_{i_r, i_{r+1}}) e(\boldsymbol{i}), \\
x_{r+1} \tau_r e(\boldsymbol{i}) &= (\tau_r x_r + \delta_{i_r, i_{r+1}}) e(\boldsymbol{i}), \\
\tau_r x_s &= x_s \tau_r \quad \text{if } s \neq r, r+1, \\
\tau_r \tau_s &= \tau_s \tau_r \quad \text{if } |r-s| > 1, \\
\tau_r^2 e(\boldsymbol{i}) &= \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ Q_{i_r, i_{r+1}}(x_r, x_{r+1}) e(\boldsymbol{i}) & \text{else,} \end{cases} \\
(\tau_r \tau_{r+1} \tau_r - \tau_{r+1} \tau_r \tau_{r+1}) e(\boldsymbol{i}) &= \begin{cases} (x_{r+2} - x_r)^{-1}(Q_{i_r, i_{r+1}}(x_{r+2}, x_{r+1}) - Q_{i_r, i_{r+1}}(x_r, x_{r+1})) e(\boldsymbol{i}) & \text{if } i_r = i_{r+2}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

for each $\boldsymbol{i}$, $\boldsymbol{j}$, r and s. We may write $R_{d,\boldsymbol{k}} = R_{d,\boldsymbol{k}}(\Gamma)$. The algebra $R_{d,\boldsymbol{k}}$ admits a $\mathbb{Z}$-grading such that $\deg e(\boldsymbol{i}) = 0$, $\deg x_r = 2$ and $\deg \tau_s e(\boldsymbol{i}) = -a_{i_s, i_{s+1}}$, for each $1 \leqslant r \leqslant d$, $1 \leqslant s < d$ and $\boldsymbol{i} \in I^d$.

For each $\alpha \in Q_I^+$ such that $|\alpha| = d$ set $e(\alpha) = \sum_{\boldsymbol{i} \in I^\alpha} e(\boldsymbol{i}) \in R_{d,\boldsymbol{k}}$. It is a homogeneous central idempotent of degree zero. We have the following decomposition into a sum of unitary $\boldsymbol{k}$-algebras $R_{d,\boldsymbol{k}} = \bigoplus_{|\alpha|=d} R_{\alpha,\boldsymbol{k}}$, where $R_{\alpha,\boldsymbol{k}} = e(\alpha) R_{d,\boldsymbol{k}}$.

Let $\boldsymbol{k}_d^{(I)}$ be the direct sum of copies of the ring $\boldsymbol{k}_d[x] := \boldsymbol{k}[x_1, \dots, x_d]$ labeled by I^d. We write

$$\boldsymbol{k}_d^{(I)} = \bigoplus_{\boldsymbol{i} \in I^d} \boldsymbol{k}_d[x] e(\boldsymbol{i}), \tag{2}$$

where $e(\boldsymbol{i})$ is the idempotent of the ring $\boldsymbol{k}_d^{(I)}$ projecting to the component $\boldsymbol{i}$. A polynomial in $\boldsymbol{k}_d[x]$ can be considered as an element of $\boldsymbol{k}_d^{(I)}$ via the diagonal inclusion. For each $i, j \in I$ fix a polynomial $P_{i,j}(u, v)$ such that we have $Q_{i,j}(u, v) = P_{i,j}(u, v)P_{j,i}(v, u)$.

Denote by ∂_r the Demazure operator on $\boldsymbol{k}_d[x]$, i.e., we have

$$\partial_r(f) = (x_r - x_{r+1})^{-1}(s_r(f) - f).$$

The following is proved in [Rouquier 2008, §3.2].

Proposition 2.3. *The algebra $R_{d,\boldsymbol{k}}$ has a faithful representation in the vector space $\boldsymbol{k}_d^{(I)}$ such that the element $e(\boldsymbol{i})$ acts by projection to $\boldsymbol{k}_d^{(I)}e(\boldsymbol{i})$, the element x_r acts by multiplication by x_r and such that for $f \in \boldsymbol{k}_d[x]$ we have*

$$\tau_r \cdot fe(\boldsymbol{i}) = \begin{cases} \partial_r(f)e(\boldsymbol{i}) & \text{if } i_r = i_{r+1}, \\ P_{i_r,i_{r+1}}(x_{r+1}, x_r)s_r(f)e(s_r(\boldsymbol{i})) & \text{otherwise.} \end{cases} \tag{3}$$

We will always choose $P_{i,j}$ in the following way:

$$P_{i,j}(u, v) = (u - v)^{h_{j,i}}.$$

Remark 2.4. There is an explicit construction of a basis of a KLR algebra (see [Khovanov and Lauda 2009, Theorem 2.5]). Assume $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$. Set $\mathfrak{S}_{\boldsymbol{i},\boldsymbol{j}} = \{w \in \mathfrak{S}_d : w(\boldsymbol{i}) = \boldsymbol{j}\}$. For each permutation $w \in \mathfrak{S}_{\boldsymbol{i},\boldsymbol{j}}$ fix a reduced expression $w = s_{p_1} \cdots s_{p_r}$ and set $\tau_w = \tau_{p_1} \cdots \tau_{p_r}$. Then the vector space $e(\boldsymbol{j})R_{\alpha,\boldsymbol{k}}e(\boldsymbol{i})$ has a basis $\{\tau_w x_1^{a_1} \cdots x_d^{a_d} e(\boldsymbol{i}) : w \in \mathfrak{S}_{\boldsymbol{i},\boldsymbol{j}}, a_1, \dots, a_d \in \mathbb{N}\}$. Note that the element τ_w depends on the reduced expression of w. Moreover, if we change the reduced expression of w, then the element $\tau_w e(\boldsymbol{i})$ is changed only by a linear combination of monomials of the form $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\boldsymbol{i})$ with $t < \ell(w)$. Note also that if $s_{p_1} \cdots s_{p_r}$ is not a reduced expression, then the element $\tau_{p_1} \cdots \tau_{p_r} e(\boldsymbol{i})$ may be written as a linear combination of monomials of the form $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\boldsymbol{i})$ with $t < r$. Moreover, in both situations above, the linear combination can be chosen in such a way that for each monomial $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\boldsymbol{i})$ in the linear combination, the expression $s_{q_1} \cdots s_{q_t}$ is reduced.

Remark 2.5. The algebra $R_{d,\boldsymbol{k}}$ in Definition 2.2 is well defined only for a finite quiver because of the second relation. However, the algebra $R_{\alpha,\boldsymbol{k}}$ is well defined even if the quiver is infinite because each α uses a finite set of vertices. Thus, for an infinite quiver we can define $R_{d,\boldsymbol{k}}$ as $R_{d,\boldsymbol{k}} = \bigoplus_{|\alpha|=d} R_{\alpha,\boldsymbol{k}}$. However, in this case the algebra $R_{d,\boldsymbol{k}}$ is not unitary.

2D. ***Balanced KLR algebras.*** From now on the quiver Γ is assumed to be finite. Fix a decomposition $I = I_0 \sqcup I_1$ as in Section 2B and consider the quiver $\overline{\Gamma} = (\bar{I}, \overline{H})$ as in Section 2B. Recall the decomposition $\bar{I} = \bar{I}_0 \sqcup \bar{I}_1 \sqcup \bar{I}_2$. In this section we work with the KLR algebra associated with the quiver $\overline{\Gamma}$.

We say that a sequence $\boldsymbol{i} = (i_1, i_2, \dots, i_d) \in \bar{I}^d$ is *unordered* if there is an index $r \in [1, d]$ such that the number of elements from $\bar{I}_2$ in the sequence $(i_1, i_2, \dots, i_r)$ is strictly greater than the number of elements from $\bar{I}_1$. We say that it is *well-ordered* if for each index a such that $i_a = i^1$ for some $i \in I_1$, we have $a < d$ and $i_{a+1} = i^2$. We denote by $\bar{I}^\alpha_{\mathrm{ord}}$ and $\bar{I}^\alpha_{\mathrm{un}}$ the subsets of well-ordered and unordered sequences in $\bar{I}^\alpha$.

The map ϕ from Section 2B yields a bijection

$$\phi\colon Q_I^+ \to \Big\{\alpha = \sum_{i\in\bar{I}} d_i\alpha_i \in Q_{\bar{I}}^+ : d_{i^1} = d_{i^2}, \forall i \in I_1\Big\}, \quad \alpha \mapsto \bar{\alpha}.$$

Fix $\alpha \in Q_I^+$. Set $\boldsymbol{e} = \sum_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} e(\boldsymbol{i}) \in R_{\bar{\alpha},\boldsymbol{k}}(\overline{\Gamma})$.

Definition 2.6. For $\alpha \in Q_I^+$, the *balanced KLR algebra* is the algebra

$$S_{\bar{\alpha},\boldsymbol{k}}(\overline{\Gamma}) = \boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}(\overline{\Gamma})\boldsymbol{e} \Big/ \sum_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{un}}} \boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}(\overline{\Gamma})e(\boldsymbol{i})R_{\bar{\alpha},\boldsymbol{k}}(\overline{\Gamma})\boldsymbol{e}.$$

We may write $S_{\bar{\alpha},\boldsymbol{k}}(\overline{\Gamma}) = S_{\bar{\alpha},\boldsymbol{k}}$.

Remark 2.7. Assume that $\boldsymbol{i} = (i_1, \dots, i_d) \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$. Let a be an index such that $i_a \in \bar{I}_1$. We have the relation $\tau_a^2 e(\boldsymbol{i}) = (x_{a+1} - x_a)e(\boldsymbol{i})$ in $R_{\bar{\alpha},\boldsymbol{k}}$. Moreover, we have $\tau_a^2 e(\boldsymbol{i}) = \tau_a e(s_a(\boldsymbol{i}))\tau_a e(\boldsymbol{i})$ and $s_a(\boldsymbol{i})$ is unordered. Thus we have $x_a e(\boldsymbol{i}) = x_{a+1}e(\boldsymbol{i})$ in $S_{\bar{\alpha},\boldsymbol{k}}$.

2E. *The polynomial representation of $S_{\bar{\alpha},\boldsymbol{k}}$.* We assume $\alpha = \sum_{i\in I} d_i\alpha_i \in Q_I^+$. Let $\boldsymbol{i} = (i_1, \dots, i_d) \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$. Denote by $J(\boldsymbol{i})$ the ideal of the polynomial ring $\boldsymbol{k}_d[x]e(\boldsymbol{i}) \subset \boldsymbol{k}_d^{(\bar{I})}$ generated by the set

$$\{(x_r - x_{r+1})e(\boldsymbol{i}) : i_r \in \bar{I}_1\}.$$

Lemma 2.8. *Assume that $\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$ and $\boldsymbol{j} \in \bar{I}^{\bar{\alpha}}_{\mathrm{un}}$. Then each element of $e(\boldsymbol{i})R_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{j})$ maps $\boldsymbol{k}_d[x]e(\boldsymbol{j})$ to $J(\boldsymbol{i})$.*

Proof. We will prove by induction on k that for all $\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$ and $\boldsymbol{j} \in \bar{I}^{\bar{\alpha}}_{\mathrm{un}}$ and all $p_1, \dots, p_k$ such that the permutation $w = s_{p_1} \cdots s_{p_k} \in \mathfrak{S}_d$ satisfies $w(\boldsymbol{j}) = \boldsymbol{i}$, the monomial $\tau_{p_1} \cdots \tau_{p_k}$ maps $\boldsymbol{k}_d[x]e(\boldsymbol{j})$ to $J(\boldsymbol{i})$.

Assume $k = 1$. Write $p = p_1$. Let us write $\boldsymbol{i} = (i_1, \dots, i_d)$ and $\boldsymbol{j} = (j_1, \dots, j_d)$. Then we have $\boldsymbol{i} = s_p(\boldsymbol{j})$. By assumptions on $\boldsymbol{i}$ and $\boldsymbol{j}$ we know that there exists $i \in I_1$ such that $i_p = j_{p+1} = i^1$ and $i_{p+1} = j_p = i^2$. In this case the statement is obvious because τ_p maps $fe(\boldsymbol{j}) \in \boldsymbol{k}_d[x]e(\boldsymbol{j})$ to $(x_{p+1} - x_p)s_p(f)e(\boldsymbol{i})$ by (3).

Now consider a monomial $\tau_{p_1} \cdots \tau_{p_k}$ such that the permutation $w = s_{p_1} \cdots s_{p_k}$ satisfies $w(\boldsymbol{j}) = \boldsymbol{i}$ and assume that the statement is true for all such monomials of smaller length. By assumptions on $\boldsymbol{i}$ and $\boldsymbol{j}$ there is an index $r \in [1, d]$ such that $i_r = i^1$ for some $i \in I_1$ and $w^{-1}(r+1) < w^{-1}(r)$. Thus w has a reduced expression of the form $w = s_r s_{r_1} \cdots s_{r_h}$. This implies that $\tau_{p_1} \cdots \tau_{p_k} e(\boldsymbol{j})$ is equal to a monomial of the form $\tau_r \tau_{r_1} \cdots \tau_{r_h} e(\boldsymbol{j})$ modulo monomials of the form $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\boldsymbol{j})$ with $t < k$, see Remark 2.4. As the sequence $s_r(\boldsymbol{i})$ is unordered, the case $k = 1$ and the induction hypothesis imply the statement. $\square$

Lemma 2.9. *Assume that $\boldsymbol{i}, \boldsymbol{j} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$. Then each element of $e(\boldsymbol{i})R_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{j})$ maps $J(\boldsymbol{j})$ into $J(\boldsymbol{i})$.*

Proof. Take $y \in e(\boldsymbol{i})R_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{j})$. We must prove that for each $r \in [1, d]$ such that $j_r = i^1$ for some $i \in I_1$ and each $f \in \boldsymbol{k}_d[x]$ we have $y((x_r - x_{r+1})fe(\boldsymbol{j})) \in J(\boldsymbol{i})$. We have $(x_r - x_{r+1})fe(\boldsymbol{j}) = -\tau_r^2(fe(\boldsymbol{i}))$ (see Remark 2.7). This implies

$$y((x_r - x_{r+1})fe(\boldsymbol{j})) = -y\tau_r^2(fe(\boldsymbol{j})) = -y\tau_r e(s_r(\boldsymbol{j}))(\tau_r(fe(\boldsymbol{j}))).$$

Thus Lemma 2.8 implies the statement because the sequence $s_r(\boldsymbol{j})$ is unordered. $\square$

The representation of $R_{\bar{\alpha},\boldsymbol{k}}$ on

$$\boldsymbol{k}_{\bar{\alpha}}^{(\bar{I})} := \bigoplus_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}} \boldsymbol{k}_{|\bar{\alpha}|}[x]e(\boldsymbol{i})$$

yields a representation of $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e}$ on

$$\boldsymbol{k}_{\bar{\alpha},\mathrm{ord}}^{(\bar{I})} := \bigoplus_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} \boldsymbol{k}_{|\bar{\alpha}|}[x]e(\boldsymbol{i}).$$

Set $J_{\bar{\alpha},\mathrm{ord}} = \bigoplus_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} J(\boldsymbol{i})$. From Lemmas 2.8 and 2.9 we deduce the following.

Lemma 2.10. *The representation of $R_{\bar{\alpha},\boldsymbol{k}}$ on $\boldsymbol{k}_{\bar{\alpha}}^{(\bar{I})}$ factors through a representation of $S_{\bar{\alpha},\boldsymbol{k}}$ on $\boldsymbol{k}_{\bar{\alpha},\mathrm{ord}}^{(\bar{I})}/J_{\bar{\alpha},\mathrm{ord}}$. This representation is faithful.*

Proof. The faithfulness is proved in the proof of Theorem 2.12. □

2F. ***The comparison of the polynomial representations.*** Fix $\alpha\in Q_I^+$. Set $d=|\alpha|$ and $\bar{d}=|\bar{\alpha}|$. For each sequence $\boldsymbol{i}=(i_1,\dots,i_d)\in I^\alpha$ and $r\in[1,d]$ we denote by r' or $r'_{\boldsymbol{i}}$ the positive integer such that $r'-1$ is the length of the sequence $\phi(i_1,\dots,i_{r-1})\in \bar{I}^\infty$.

For $r\in[1,d]$ and $r\in[1,d-1]$ consider the element $x_r^*\in S_{\bar{\alpha},\boldsymbol{k}}$ and $\tau_r^*\in S_{\bar{\alpha},\boldsymbol{k}}$, respectively, such that for each $\boldsymbol{i}\in I^\alpha$ we have

$$x_r^*e(\phi(\boldsymbol{i})) = x_{r'}e(\phi(\boldsymbol{i})),\quad \tau_r^*e(\phi(\boldsymbol{i})) = \begin{cases} \tau_{r'}e(\phi(\boldsymbol{i})), & \text{if } i_r, i_{r+1}\in I_0,\\ \tau_{r'}\tau_{r'+1}e(\phi(\boldsymbol{i})) & \text{if } i_r\in I_1, i_{r+1}\in I_0,\\ \tau_{r'+1}\tau_{r'}e(\phi(\boldsymbol{i})) & \text{if } i_r\in I_0, i_{r+1}\in I_1,\\ \tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1}e(\phi(\boldsymbol{i})) & \text{if } i_r, i_{r+1}\in I_1, i_r\neq i_{r+1},\\ -\tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1}e(\phi(\boldsymbol{i})) & \text{if } i_r=i_{r+1}\in I_1.\end{cases}$$

For each $\boldsymbol{i}\in I^\alpha$ we have the algebra isomorphism

$$\boldsymbol{k}_d[x]e(\boldsymbol{i})\simeq \boldsymbol{k}_{\bar{d}}[x]e(\phi(\boldsymbol{i}))/J(\phi(\boldsymbol{i})),\quad x_re(\boldsymbol{i})\mapsto x_{r'}e(\phi(\boldsymbol{i})).$$

We will always identify $\boldsymbol{k}_\alpha^{(I)}$ with $\boldsymbol{k}_{\bar{\alpha},\mathrm{ord}}^{(\bar{I})}/J_{\bar{\alpha},\mathrm{ord}}$ via this isomorphism.

Lemma 2.11. *The action of the elements $e(\boldsymbol{i})$, $x_re(\boldsymbol{i})$ and $\tau_re(\boldsymbol{i})$ of $R_{\alpha,\boldsymbol{k}}$ on $\boldsymbol{k}_\alpha^{(I)}$ is the same as the action of the elements $e(\phi(\boldsymbol{i}))$, $x_r^*e(\phi(\boldsymbol{i}))$, $\tau_r^*e(\phi(\boldsymbol{i}))$ of $S_{\bar{\alpha},\boldsymbol{k}}$ on $\boldsymbol{k}_{\bar{\alpha},\mathrm{ord}}^{(\bar{I})}/J_{\bar{\alpha},\mathrm{ord}}$.*

Proof. The proof is based on the observation that by construction for each $i\in I_1$ and $j\in I_0$ we have

$$P_{i^1,j^0}(u,v)P_{i^2,j^0}(u,v)=P_{i,j}(u,v),\quad P_{j^0,i^1}(u,v)P_{j^0,i^2}(u,v)=P_{j,i}(u,v). \tag{4}$$

For each $\boldsymbol{i}\in I^\alpha$, we write $\phi(\boldsymbol{i})=(i'_1,i'_2,\dots,i'_{\bar{d}})$. The only difficult part concerns the operator $\tau_re(\boldsymbol{i})$ when at least one of the elements i_r or i_{r+1} is in I_1. Assume that $i_r\in I_1$ and $i_{r+1}\in I_0$. In this case we have

$$i'_{r'}=(i_r)^1\in\bar{I}_1,\quad i'_{r'+1}=(i_r)^2\in\bar{I}_2,\quad i'_{r'+2}=(i_{r+1})^0\in\bar{I}_0.$$

In particular, the element $i'_{r'+2}$ is different from $i'_{r'}$ and $i'_{r'+1}$. Then, by (3), for each $f\in\boldsymbol{k}_{\bar{d}}[x]$ the element $\tau_r^*e(\phi(\boldsymbol{i}))=\tau_{r'}\tau_{r'+1}e(\phi(\boldsymbol{i}))$ maps $fe(\phi(\boldsymbol{i}))\in\boldsymbol{k}_{\bar{\alpha},\mathrm{ord}}^{(\bar{I})}/J_{\bar{\alpha},\mathrm{ord}}$ to

$$
\begin{aligned}
P_{i'_{r'},i'_{r'+2}}(x_{r'+1},x_{r'})s_{r'}(P_{i'_{r'+1},i'_{r'+2}}&(x_{r'+2},x_{r'+1})s_{r'+1}(f))e(s_{r'}s_{r'+1}(\phi(\boldsymbol{i})))\\
&= P_{i'_{r'},i'_{r'+2}}(x_{r'+1},x_{r'})P_{i'_{r'+1},i'_{r'+2}}(x_{r'+2},x_{r'})s_{r'}s_{r'+1}(f)e(\phi(s_r(\boldsymbol{i})))\\
&= P_{i_r,i_{r+1}}(x_{r'+1},x_{r'})s_{r'}s_{r'+1}(f)e(\phi(s_r(\boldsymbol{i}))),
\end{aligned}
$$

where the last equality holds by (4). Thus we see that the action of $\tau_r^* e(\phi(\boldsymbol{i}))$ on the polynomial representation is the same as the action of $\tau_r e(\boldsymbol{i})$. The case when $i_r \in I_0$ and $i_{r+1} \in I_1$ can be done similarly.

Assume now that $i_r \neq i_{r+1}$ are both in I_1. By the assumption on the quiver Γ (see Section 2B), there are no arrows in Γ between i_r and i_{r+1}. Thus there are no arrows in $\overline{\Gamma}$ between any of the vertices $(i_r)^1 = i'_{r'}$ or $(i_r)^2 = i'_{r'+1}$ and any of the vertices $(i_{r+1})^1 = i'_{r'+2}$ or $(i_{r+1})^2 = i'_{r'+3}$. Then, by (3), for each $f \in \boldsymbol{k}_{\bar{d}}[x]$ the element $\tau_r^* e(\boldsymbol{i}) = \tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1}e(\phi(\boldsymbol{i}))$ maps $fe(\phi(\boldsymbol{i}))$ to

$$
s_{r'+1}s_{r'+2}s_{r'}s_{r'+1}(f)e(\phi(s_r(\boldsymbol{i}))).
$$

Thus we see that the action of $\tau_r^* e(\phi(\boldsymbol{i}))$ on the polynomial representation is the same as that of $\tau_r e(\boldsymbol{i})$.

Finally, assume that $i_r = i_{r+1} \in I_1$. In this case we have

$$
(i_r)^1 = i'_{r'} = (i_{r+1})^1 = i'_{r'+2} \quad \text{and} \quad (i_r)^2 = i'_{r'+1} = (i_{r+1})^2 = i'_{r'+3}.
$$

Then, by (3), for each $f \in \boldsymbol{k}_{\bar{d}}[x]$ the element $\tau_r^* e(\phi(\boldsymbol{i})) = -\tau_{r'+1}\tau_{r'+2}\tau_{r'}\tau_{r'+1}e(\phi(\boldsymbol{i}))$ maps $fe(\phi(\boldsymbol{i}))$ to

$$
s_{r'+1}\partial_{r'+2}\partial_{r'}(x_{r'+1}-x_{r'+2})s_{r'+1}(f)e(\phi(s_r(\boldsymbol{i}))),
$$

where ∂_r is the Demazure operator (see the definition before Proposition 2.3). To prove that this gives the same result as for $\tau_r e(\boldsymbol{i})$, it is enough to check this on monomials $x_r^n x_{r+1}^m e(\boldsymbol{i})$. Assume for simplicity that $n \geqslant m$. The situation $n \leqslant m$ can be treated similarly. The element $\tau_r e(\boldsymbol{i})$ maps this monomial to

$$
\partial_r(x_r^n x_{r+1}^m)e(\boldsymbol{i}) = -\sum_{a=m}^{n-1} x_r^a x_{r+1}^{n+m-1-a} e(\boldsymbol{i}).
$$

Here the symbol $\sum_{a=x}^{y}$ means 0 when $y = x-1$. The element $\tau_r^* e(\phi(\boldsymbol{i}))$ maps $x_{r'+1}^n x_{r'+2}^m e(\phi(\boldsymbol{i}))$ to $s_{r'+1}\partial_{r'+2}\partial_{r'}[x_{r'+1}^{m+1}x_{r'+2}^n - x_{r'+1}^m x_{r'+2}^{n+1}]e(\phi(\boldsymbol{i}))$, which equals

$$
\begin{aligned}
&s_{r'+1}\left[-\left(\sum_{a=0}^{m} x_{r'}^a x_{r'+1}^{m-a}\right)\left(\sum_{b=0}^{n-1} x_{r'+2}^b x_{r'+3}^{n-1-b}\right) + \left(\sum_{a=0}^{m-1} x_{r'}^a x_{r'+1}^{m-1-a}\right)\left(\sum_{b=0}^{n} x_{r'+2}^b x_{r'+3}^{n-b}\right)\right]e(\phi(\boldsymbol{i}))\\
&= \left[-\left(\sum_{a=0}^{m} x_{r'}^a x_{r'+2}^{m-a}\right)\left(\sum_{b=0}^{n-1} x_{r'+1}^b x_{r'+3}^{n-1-b}\right) + \left(\sum_{a=0}^{m-1} x_{r'}^a x_{r'+2}^{m-1-a}\right)\left(\sum_{b=0}^{n} x_{r'+1}^b x_{r'+3}^{n-b}\right)\right]e(\phi(\boldsymbol{i}))\\
&= \left[-x_{r'}^m\left(\sum_{b=0}^{n-1} x_{r'+1}^b x_{r'+3}^{n-1-b}\right) + x_{r'+1}^n\left(\sum_{a=0}^{m-1} x_{r'}^a x_{r'+2}^{m-1-a}\right)\right]e(\phi(\boldsymbol{i}))\\
&= \left[-x_{r'+1}^m\left(\sum_{b=0}^{n-1} x_{r'+1}^b x_{r'+2}^{n-1-b}\right) + x_{r'+1}^n\left(\sum_{a=0}^{m-1} x_{r'+1}^a x_{r'+2}^{m-1-a}\right)\right]e(\phi(\boldsymbol{i})) = -\left(\sum_{a=m}^{n-1} x_{r'+1}^a x_{r'+2}^{m+n-1-a}\right)e(\phi(\boldsymbol{i})).
\end{aligned}
$$

Here the first equality follows from the following property of the Demazure operator

$$\partial_r(x_{r+1}^n) = -\partial_r(x_r^n) = \sum_{a=0}^{n-1} x_r^a x_{r+1}^{n-1-a},$$

the fourth equality follows from Remark 2.7. Other equalities are obtained by elementary manipulations with sums. □

2G. ***The isomorphism Φ.***

Theorem 2.12. *For each $\alpha \in Q_I^+$, there is an algebra isomorphism $\Phi_{\alpha,\boldsymbol{k}}\colon R_{\alpha,\boldsymbol{k}} \to S_{\bar{\alpha},\boldsymbol{k}}$ such that*

$$e(\boldsymbol{i}) \mapsto e(\phi(\boldsymbol{i})),$$
$$x_r e(\boldsymbol{i}) \mapsto x_r^* e(\phi(\boldsymbol{i})),$$
$$\tau_r e(\boldsymbol{i}) \mapsto \tau_r^* e(\phi(\boldsymbol{i})).$$

Proof. By Proposition 2.3, the representation $\boldsymbol{k}_\alpha^{(I)}$ of $R_{\alpha,\boldsymbol{k}}$ is faithful. Now, in view of Lemma 2.11, it is enough to prove the following two facts:

- The elements $e(\phi(\boldsymbol{i}))$, x_r^*, τ_r^* generate $S_{\bar{\alpha},\boldsymbol{k}}$.
- The representation $\boldsymbol{k}_{\bar{\alpha},\mathrm{ord}}^{(\bar{I})}/J_{\bar{\alpha},\mathrm{ord}}$ of $S_{\bar{\alpha},\boldsymbol{k}}$ is faithful.

Fix $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$. Set $\boldsymbol{i}' = (i'_1, \dots, i'_{\bar{d}}) = \phi(\boldsymbol{i})$ and $\boldsymbol{j}' = \phi(\boldsymbol{j})$. Let $\boldsymbol{B}$ and $\boldsymbol{B}'$ be the bases of $e(\boldsymbol{j})R_{\alpha,\boldsymbol{k}}e(\boldsymbol{i})$ and $e(\boldsymbol{j}')R_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i}')$, respectively, as in Remark 2.4. These bases depend on some choices of reduced expressions. We will make some special choices later. For each element $b = \tau_w x_1^{a_1} \cdots x_d^{a_d} e(\boldsymbol{i}) \in \boldsymbol{B}$ we construct an element $b^* \in e(\boldsymbol{j}')S_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i}')$ that acts by the same operator on the polynomial representation. We set

$$b^* = \tau_{p_1}^* \cdots \tau_{p_k}^* (x_1^*)^{a_1} \cdots (x_d^*)^{a_d} e(\boldsymbol{i}') \in e(\boldsymbol{j}')S_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i}'),$$

where $w = s_{p_1} \cdots s_{p_k}$ is a reduced expression (as we said above, some special choice of reduced expressions will be fixed later).

Let us call the permutation $w \in \mathfrak{S}_{\boldsymbol{i}',\boldsymbol{j}'}$ *balanced* if we have $w(a+1) = w(a)+1$ for each a such that $i'_a = i^1$ for some $i \in I$ (and thus $i'_{a+1} = i^2$). Otherwise we say that w is *unbalanced*. There exists a unique map $u\colon \mathfrak{S}_{\boldsymbol{i},\boldsymbol{j}} \to \mathfrak{S}_{\boldsymbol{i}',\boldsymbol{j}'}$ such that for each $w \in \mathfrak{S}_{\boldsymbol{i},\boldsymbol{j}}$ the permutation $u(w)$ is balanced and $w(r) < w(t)$ if and only if $u(w)(r') < u(w)(t')$ for each $r, t \in [1, d]$, where $r' = r'_{\boldsymbol{i}}$ and $t' = t'_{\boldsymbol{i}}$ are as in Section 2F. The image of u is exactly the set of all balanced permutations in $\mathfrak{S}_{\boldsymbol{i}',\boldsymbol{j}'}$.

Assume that $w \in \mathfrak{S}_{\boldsymbol{i}',\boldsymbol{j}'}$ is unbalanced. We claim that there exists an index a such that $i'_a \in \bar{I}_1$ and $w(a) > w(a+1)$. Indeed, let J be the set of indices $a \in [1, \bar{d}]$ such that $i'_a \in \bar{I}_1$. As $\boldsymbol{j}'$ is well-ordered, we have $\sum_{a \in J}(w(a+1) - w(a)) = \#J$. As w is unbalanced, not all summands in this sum are equal to 1. Then one of the summands must be negative. Let $a \in J$ be an index such that $w(a) > w(a+1)$. We can assume that the reduced expression of w is of the form $w = s_{p_1} \cdots s_{p_k} s_a$. In this case the element $\tau_w e(\boldsymbol{i}')$ is zero in $S_{\bar{\alpha},\boldsymbol{k}}$ because the sequence $s_a(\boldsymbol{i}')$ is unordered.

Assume that $w \in \mathfrak{S}_{\boldsymbol{i}',\boldsymbol{j}'}$ is balanced. Thus, there exists some $\tilde{w} \in \mathfrak{S}_{\boldsymbol{i},\boldsymbol{j}}$ such that $u(\tilde{w}) = w$. We choose an arbitrary reduced expression $\tilde{w} = s_{p_1} \cdots s_{p_k}$ and we choose the reduced expression $w = s_{q_1} \cdots s_{q_r}$ of w obtained from the reduced expression of $\tilde{w}$ in the following way. For $t \in \{1, \dots, k\}$ set $\boldsymbol{i}^t = s_{p_{t+1}} \cdots s_{p_k}(\boldsymbol{i})$ (in particular, we have $\boldsymbol{i}^k = \boldsymbol{i}$). We write $\boldsymbol{i}^t = (i_1^t, \dots, i_d^t)$. We construct the reduced expression of w as $w = \hat{s}_{p_1} \cdots \hat{s}_{p_k}$, where for $a = p_t$ we have

$$\hat{s}_a = \begin{cases} s_{a'} & \text{if } i_a^t, i_{a+1}^t \in I_0, \\ s_{a'+1}s_{a'} & \text{if } i_a^t \in I_0 \text{ and } i_{a+1}^t \in I_1, \\ s_{a'}s_{a'+1} & \text{if } i_a^t \in I_1 \text{ and } i_{a+1}^t \in I_0, \\ s_{a'+1}s_{a'}s_{a'+2}s_{a'+1} & \text{if } i_a^t, i_{a+1}^t \in I_1, \end{cases}$$

where $a' = a'_{\boldsymbol{i}^r}$ is as in Section 2F. Let us explain why the obtained expression of w is reduced. The fact that the expression $\tilde{w} = s_{p_1} \cdots s_{p_k}$ is reduced means the following. When we apply the transpositions $s_{p_k}, s_{p_{k-1}}, \dots, s_{p_1}$ consecutively to the d-tuple $(1, 2, \dots, d)$, if two elements of the set $\{1, 2, \dots, d\}$ are exchanged once by some s, then these two elements are never exchanged again by another s later. It is clear that the expression $w = s_{q_1} \cdots s_{q_r} = \hat{s}_{p_1} \cdots \hat{s}_{p_k}$ inherits the same property from $\tilde{w} = s_{p_1} \cdots s_{p_k}$ because for each $a, b \in \{1, 2, \dots, d\}$, $a \neq b$ we have the following (we set $a' = a'_{\boldsymbol{i}}$ and $b' = b'_{\boldsymbol{i}}$):

- If $i_a, i_b \in I_0$, then if the reduced expression of $\tilde{w}$ exchanges a and b exactly once or never exchanges them then the expression of w exchanges a' and b' exactly once or never exchanges them, respectively.
- If $i_a \in I_0$ and $i_b \in I_1$, then if the reduced expression of $\tilde{w}$ exchanges a and b exactly once or never exchanges them then the expression of w exchanges a' and b' exactly once or never exchanges them, respectively, and it also exchanges a' with $b'+1$ exactly once or, respectively, never exchanges them.
- If $i_a \in I_1$ and $i_b \in I_0$, then if the reduced expression of $\tilde{w}$ exchanges a and b exactly once or never exchanges them then the expression of w exchanges a' and b' exactly once or never exchanges them, respectively, and it also exchanges $a'+1$ with b' exactly once or, respectively, never exchanges them.
- If $i_a, i_b \in I_1$, then if the reduced expression of $\tilde{w}$ exchanges a and b exactly once or never exchanges them then the expression of w exchanges a' and b' exactly once or never exchanges them, respectively, and the same thing for a' and $b'+1$, for $a'+1$ and b', and for $a'+1$ and $b'+1$.

If the reduced expressions are chosen as above, then the element $\tau_w e(\boldsymbol{i}') = \tau_{q_1} \cdots \tau_{q_r} e(\boldsymbol{i}') \in S_{\alpha,\boldsymbol{k}}$ is equal to $\pm(\tau_{p_1} \cdots \tau_{p_k} e(\boldsymbol{i}))^*$.

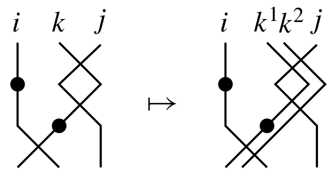
The discussion above shows that the image of an element $b' \in \boldsymbol{B}'$ in $e(\boldsymbol{j}')S_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i}')$ is either zero or of the form $\pm b^*$ for some $b \in \boldsymbol{B}$. Moreover, each b^* for $b \in \boldsymbol{B}$ can be obtained in such a way. Now we get the following:

- The elements $e(\phi(\boldsymbol{i}))$, x_r^*, and τ_r^* generate $S_{\bar{\alpha},\boldsymbol{k}}$ because the image of each element of $\boldsymbol{B}'$ in $e(\boldsymbol{j}')S_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i}')$ is either zero or a monomial in $e(\phi(\boldsymbol{i}))$, x_r^*, and τ_r^*.

- The representation $\boldsymbol{k}^{(\bar{I})}_{\bar{\alpha},\mathrm{ord}}/J_{\bar{\alpha},\mathrm{ord}}$ of $S_{\bar{\alpha},\boldsymbol{k}}$ is faithful because the spanning set $\{b^*\colon b\in\boldsymbol{B}\}$ of $e(\boldsymbol{j}')S_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i}')$ acts on the polynomial representation by linearly independent operators (because the polynomial representation of $R_{\alpha,\boldsymbol{k}}$ in Proposition 2.3 is faithful). □

Remark 2.13. (a) Note that Theorem 2.12 also remains true for an infinite quiver Γ because α is supported on a finite number of vertices (see also Remark 2.5).

(b) The formulas that define the isomorphism $\Phi_{\alpha,\boldsymbol{k}}$ become more natural if we look at them from the point of view of Khovanov–Lauda diagrams (see [Khovanov and Lauda 2009]). Diagrammatically, the isomorphism $\Phi_{\alpha,\boldsymbol{k}}$ looks in the following way. It sends a diagram representing an element of $R_{\alpha,\boldsymbol{k}}$ to the diagram (sometimes with a sign) obtained by replacing each strand with label $k\in I_1$ by two parallel strands with labels k^1 and k^2 (if there is a dot on the strand with label k, it should be moved to the strand with label k^1). For example, if $i,\ j\in I_0$ and $k\in I_1$, we have:



3. Categorical representations

3A. ***The standard representation of*** $\widetilde{\mathfrak{sl}}_e$**.** Consider the affine Lie algebra $\widetilde{\mathfrak{sl}}_e=\mathfrak{sl}_e\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\mathbf{1}$, defined over $\mathbb{C}$. Let e_i, f_i and h_i for $i=0,1,\dots,e-1$, be the standard generators of $\widetilde{\mathfrak{sl}}_e$ (see Remark 2.1). Let V_e be a $\mathbb{C}$-vector space with canonical basis $\{v_1,\dots,v_e\}$ and set $U_e=V_e\otimes\mathbb{C}[z,z^{-1}]$. The vector space U_e has a basis $\{u_r:r\in\mathbb{Z}\}$ where $u_{a+eb}=v_a\otimes z^{-b}$ for $a\in[1,e]$, $b\in\mathbb{Z}$. It has a structure of an $\widetilde{\mathfrak{sl}}_e$-module such that

$$f_i(u_r)=\delta_{i\equiv r}u_{r+1}\quad\text{and}\quad e_i(u_r)=\delta_{i\equiv r-1}u_{r-1}.$$

Let $\{v'_1,\dots,v'_{e+1}\}$ and $\{u'_r:r\in\mathbb{Z}\}$ denote the bases of V_{e+1} and U_{e+1}.

Fix an integer $0\leqslant k<e$. Consider the following inclusion of vector spaces

$$V_e\subset V_{e+1},\ v_r\mapsto\begin{cases}v'_r & \text{if } r\leqslant k,\\ v'_{r+1} & \text{if } r>k.\end{cases}$$

It yields an inclusion $\mathfrak{sl}_e \subset \mathfrak{sl}_{e+1}$ such that

$$e_r \mapsto \begin{cases} e_r & \text{if } r \in [1, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases}$$

$$f_r \mapsto \begin{cases} f_r & \text{if } r \in [1, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1], \end{cases}$$

$$h_r \mapsto \begin{cases} h_r & \text{if } r \in [1, k-1], \\ h_k + h_{k+1} & \text{if } r = k, \\ h_{r+1} & \text{if } r \in [k+1, e-1]. \end{cases}$$

This inclusion lifts uniquely to an inclusion $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$ such that

$$e_0 \mapsto \begin{cases} e_0 & \text{if } k \neq 0, \\ [e_0, e_1] & \text{else}, \end{cases}$$

$$f_0 \mapsto \begin{cases} f_0 & \text{if } k \neq 0, \\ [f_1, f_0] & \text{else}, \end{cases}$$

$$h_0 \mapsto \begin{cases} h_0 & \text{if } k \neq 0, \\ h_0 + h_1 & \text{else}. \end{cases}$$

Consider the inclusion $U_e \subset U_{e+1}$ such that $u_r \mapsto u'_{\Upsilon(r)}$, where Υ is defined in (8).

Lemma 3.1. *The embeddings $V_e \subset V_{e+1}$ and $U_e \subset U_{e+1}$ are compatible with the actions of $\mathfrak{sl}_e \subset \mathfrak{sl}_{e+1}$ and $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$, respectively.* □

3B. *Type A quivers.* Let $\Gamma_\infty = (I_\infty, H_\infty)$ be the quiver with the set of vertices $I_\infty = \mathbb{Z}$ and the set of arrows $H_\infty = \{i \to i+1 : i \in I_\infty\}$. Assume that $e > 1$ is an integer. Let $\Gamma_e = (I_e, H_e)$ be the quiver with the set of vertices $I_e = \mathbb{Z}/e\mathbb{Z}$ and the set of arrows $H_e = \{i \to i+1 : i \in I_e\}$. Then $\mathfrak{g}_{I_e}$ is the Lie algebra $\widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ (see Remark 2.1).

Assume that $\Gamma = (I, H)$ is a quiver whose connected components are of the form Γ_e, with $e \in \mathbb{N}$, $e > 1$ or $e = \infty$. For $i \in I$ denote by $i+1$ and $i-1$ the (unique) vertices in I such that there are arrows $i \to i+1$ and $i-1 \to i$.

Let X_I be the free abelian group with basis $\{\varepsilon_i : i \in I\}$. Set also

$$X_I^+ = \bigoplus_{i \in I} \mathbb{N}\varepsilon_i. \tag{5}$$

Let us also consider the following additive map

$$\iota\colon Q_I \to X_I, \quad \alpha_i \mapsto \varepsilon_i - \varepsilon_{i+1}.$$

We may omit the symbol ι and write α instead of $\iota(\alpha)$. Let ϕ denote also the unique additive embedding

$$\phi\colon X_I \to X_{\bar{I}}, \quad \varepsilon_i \mapsto \varepsilon_{i'}, \tag{6}$$

where

$$i' = \begin{cases} i^0 & \text{if } i \in I_0, \\ i^1 & \text{if } i \in I_1. \end{cases}$$

3C. *Categorical representations.* Let $\Gamma = (I, H)$ be a quiver as in Section 3B. Let $\boldsymbol{k}$ be a field. Assume that $\mathcal{C}$ is a Hom-finite $\boldsymbol{k}$-linear abelian category.

Definition 3.2. A $\mathfrak{g}_I$-categorical representation (E, F, x, τ) in $\mathcal{C}$ is the following data:

(1) a decomposition $\mathcal{C} = \bigoplus_{\mu \in X_I} \mathcal{C}_\mu$,

(2) a pair of biadjoint exact endofunctors (E, F) of $\mathcal{C}$,

(3) morphisms of functors $x\colon F \to F$ and $\tau\colon F^2 \to F^2$,

(4) decompositions $E = \bigoplus_{i \in I} E_i$ and $F = \bigoplus_{i \in I} F_i$,

satisfying the following conditions:

(a) We have $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$, $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$.

(b) For each $d \in \mathbb{N}$ there is an algebra homomorphism $\psi_d\colon R_{d,\boldsymbol{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$ such that $\psi_d(e(\boldsymbol{i}))$ is the projector to $F_{i_d} \cdots F_{i_1}$, where $\boldsymbol{i} = (i_1, \dots, i_d)$ and

$$\psi_d(x_r) = F^{d-r} x F^{r-1} \quad \text{and} \quad \psi_d(\tau_r) = F^{d-r-1} \tau F^{r-1}.$$

(c) For each $M \in \mathcal{C}$ the endomorphism of $F(M)$ induced by x is nilpotent.

Remark 3.3. (a) For a pair of adjoint functors (E, F) we have an isomorphism $\mathrm{End}(E^d) \simeq \mathrm{End}(F^d)^{\mathrm{op}}$. In particular, the algebra homomorphism $R_{d,\boldsymbol{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$ in Definition 3.2 yields an algebra homomorphism $R_{d,\boldsymbol{k}} \to \mathrm{End}(E^d)$.

(b) If the quiver Γ is infinite, the direct sums in (4) should be understood in the following way. For each object $M \in \mathcal{C}$, there is only a finite number of $i \in I$ such that $E_i(M)$ and $F_i(M)$ are nonzero.

3D. *From $\widetilde{\mathfrak{sl}}_{e+1}$-categorical representations to $\widetilde{\mathfrak{sl}}_e$-categorical representations.* As in Section 3A, we fix $0 \leqslant k < e$. Only in Section 3D, we assume that $\Gamma = (I, H)$ and $\overline{\Gamma} = (\bar{I}, \overline{H})$ are fixed as in as in Section B2 (i.e., we have $\Gamma = \Gamma_e$, $I_1 = \{k\}$ and we identity $\overline{\Gamma}$ with Γ_{e+1}).

Let $\overline{\mathcal{C}}$ be a Hom-finite abelian $\boldsymbol{k}$-linear category. Let

$$\overline{E} = \overline{E}_0 \oplus \overline{E}_1 \oplus \cdots \oplus \overline{E}_e \quad \text{and} \quad \overline{F} = \overline{F}_0 \oplus \overline{F}_1 \oplus \cdots \oplus \overline{F}_e$$

be endofunctors defining a $\widetilde{\mathfrak{sl}}_{e+1}$-categorical representation in $\overline{\mathcal{C}}$. Let $\overline{\psi}_d\colon R_{d,\boldsymbol{k}} \to \mathrm{End}(\overline{F}^d)^{\mathrm{op}}$ be the corresponding algebra homomorphism. We set $\overline{F}_{\boldsymbol{i}} = \overline{F}_{i_d} \cdots \overline{F}_{i_1}$ for any tuple $\boldsymbol{i} = (i_1, \dots, i_d) \in \bar{I}^d$ and $\overline{F}_{\bar{\alpha}} = \bigoplus_{\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}} \overline{F}_{\boldsymbol{i}}$ for any element $\bar{\alpha} \in Q_{\bar{I}}^+$. If $|\bar{\alpha}| = d$ let $\overline{\psi}_{\bar{\alpha}}\colon R_{\bar{\alpha},\boldsymbol{k}} \to \mathrm{End}(\overline{F}_{\bar{\alpha}})^{\mathrm{op}}$ be the $\bar{\alpha}$-component of $\overline{\psi}_d$.

Now, recall the notation $X_{\bar{I}}^+$ from (5). Assume that we have

$$\overline{\mathcal{C}}_\mu = 0, \quad \forall \mu \in X_{\bar{I}} \backslash X_{\bar{I}}^+. \tag{7}$$

For $\mu \in X_I^+$ set $\mathcal{C}_\mu = \overline{\mathcal{C}}_{\phi(\mu)}$, where the map ϕ is as in (6). Let $\mathcal{C} = \bigoplus_{\mu \in X_I^+} \mathcal{C}_\mu$.

Remark 3.4. (a) $\mathcal{C}$ is stable by $\overline{F}_i$, $\overline{E}_i$ for each $i \neq k, k+1$,

(b) $\mathcal{C}$ is stable by $\overline{F}_{k+1}\overline{F}_k$, $\overline{E}_k\overline{E}_{k+1}$,

(c) $\overline{F}_{i_d}\overline{F}_{i_{d-1}}\cdots\overline{F}_{i_1}(M) = 0$ for each $M \in \mathcal{C}$ whenever the sequence $(i_1, \dots, i_d)$ is unordered (see Section 2D).

Consider the following endofunctors of $\mathcal{C}$:

$$E_i = \begin{cases} \overline{E}_i|_{\mathcal{C}} & \text{if } 0 \leqslant i < k, \\ \overline{E}_k\overline{E}_{k+1}|_{\mathcal{C}} & \text{if } i = k, \\ \overline{E}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e, \end{cases} \quad \text{and} \quad F_i = \begin{cases} \overline{F}_i|_{\mathcal{C}} & \text{if } 0 \leqslant i < k, \\ \overline{F}_{k+1}\overline{F}_k|_{\mathcal{C}} & \text{if } i = k, \\ \overline{F}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e. \end{cases}$$

Similarly to the notations above we set $F_{\boldsymbol{i}} = F_{i_d}\cdots F_{i_1}$ for any tuple $\boldsymbol{i} = (i_1, \dots, i_d) \in I^d$ and $F_\alpha = \bigoplus_{\boldsymbol{i}\in I^\alpha} F_{\boldsymbol{i}}$ for any element $\alpha \in Q_I^+$. Note that we have $F_{\boldsymbol{i}} = \overline{F}_{\phi(\boldsymbol{i})}|_{\mathcal{C}}$ for each $\boldsymbol{i} \in I^\alpha$.

Let $\alpha \in Q_I^+$ and $\bar{\alpha} = \phi(\alpha)$. Note that we have

$$F_\alpha = \bigoplus_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} \overline{F}_{\boldsymbol{i}}|_{\mathcal{C}}.$$

The homomorphism $\overline{\psi}_{\bar{\alpha}}$ yields a homomorphism $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$, where $\boldsymbol{e} = \sum_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} e(\boldsymbol{i})$. By (c), the homomorphism $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$ factors through a homomorphism $S_{\bar{\alpha},\boldsymbol{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$. Let us call it $\overline{\psi}'_{\bar{\alpha}}$. Then we can define an algebra homomorphism $\psi_\alpha\colon R_{\alpha,\boldsymbol{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$ by setting $\psi_\alpha = \overline{\psi}'_{\bar{\alpha}} \circ \Phi_{\alpha,\boldsymbol{k}}$.

Now, Theorem 2.12 implies the following result.

Theorem 3.5. *For each category $\overline{\mathcal{C}}$, defined as above, that satisfies* (7), *we have a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in the subcategory $\mathcal{C}$ of $\overline{\mathcal{C}}$ given by functors F_i and E_i and the algebra homomorphisms $\psi_\alpha\colon R_{\alpha,\boldsymbol{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$.* □

Now, we describe the example that motivated us to prove Theorem 3.5. See [Maksimau 2015b] for details.

Example 3.6. Let U_e and V_e be as in Section 3A. Fix $\nu = (\nu_1, \dots, \nu_l) \in \mathbb{N}^l$ and put $N = \sum_{r=1}^{l} \nu_r$. Set $\wedge^\nu U_e = \wedge^{\nu_1} U_e \otimes \cdots \otimes \wedge^{\nu_l} U_e$.

Let O^ν_{-e} be the parabolic category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N$ with parabolic type ν at level $-e-N$. The categorical representation of $\widetilde{\mathfrak{sl}}_e$ in O^ν_{-e} (constructed in [Rouquier et al. 2016]) yields an $\widetilde{\mathfrak{sl}}_e$-module structure on the (complexified) Grothendieck group $[O^\nu_{-e}]$ of O^ν_{-e}. This module is isomorphic to $\wedge^\nu U_e$.

Let us apply Theorem 1.1 to $\overline{\mathcal{C}} = O^\nu_{-(e+1)}$. It happens that in this case the subcategory $\mathcal{C} \subset \overline{\mathcal{C}}$ defined as above is equivalent to O^ν_{-e}. The embedding of categories $O^\nu_{-e} \subset O^\nu_{-(e+1)}$ categorifies the embedding $\wedge^\nu U_e \subset \wedge^\nu U_{e+1}$ (see also Lemma 3.1).

3E. *Reduction of the number of idempotents.* In this section we show that it is possible to reduce the number of idempotents in the quotient in Definition 2.6. This is necessary to generalize Theorem 3.5. Here we assume the quivers $\Gamma = (I, H)$ and $\overline{\Gamma} = (\bar{I}, \overline{H})$ are as in Section 2B.

We fix $\alpha \in Q_I^+$ and put $\bar{\alpha} = \phi(\alpha)$. We say that the sequence $\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}$ is *almost ordered* if there exists a well-ordered sequence $\boldsymbol{j} \in \bar{I}^{\bar{\alpha}}$ such that there exists an index r such that $j_r \in \bar{I}_1$ and $\boldsymbol{i} = s_r(\boldsymbol{j})$. It is clear from the definition that each almost ordered sequence is unordered because the subsequence $(i_1, i_2, \dots, i_r)$ of $\boldsymbol{i}$ contains more elements from $\bar{I}_2$ than from $\bar{I}_1$. The following lemma reduces the number of generators of the kernel of $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to S_{\bar{\alpha},\boldsymbol{k}}$ (see Definition 2.6).

Lemma 3.7. *The kernel of the homomorphism $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to S_{\bar{\alpha},\boldsymbol{k}}$ is equal to $\sum_{\boldsymbol{i}} \boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i})R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e}$, where $\boldsymbol{i}$ runs over the set of all almost ordered sequences in $\bar{I}^{\bar{\alpha}}$.*

Proof. Denote by J the ideal $\sum_{\boldsymbol{i}} \boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}e(\boldsymbol{i})R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e}$ of $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e}$, where $\boldsymbol{i}$ runs over the set of all almost ordered sequences in $\bar{I}^{\bar{\alpha}}$.

By definition, each element of the kernel of $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to S_{\bar{\alpha},\boldsymbol{k}}$ is a linear combination of elements of the form $\boldsymbol{e}ae(\boldsymbol{j})b\boldsymbol{e}$, where a and b are in $R_{\bar{\alpha},\boldsymbol{k}}$ and the sequence $\boldsymbol{j}$ is unordered. By Remark 2.4, it is enough to prove that for each $\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}$, $\boldsymbol{j} \in \bar{I}^{\bar{\alpha}}_{\mathrm{un}}$, $b \in R_{\bar{\alpha},\boldsymbol{k}}$ and indices $p_1, \dots, p_k$ the element $e(\boldsymbol{i})\tau_{p_1}\cdots\tau_{p_k}e(\boldsymbol{j})b\boldsymbol{e}$ is in J. We will prove this statement by induction on k.

Assume that $k = 1$. Write $p = p_1$. The element $e(\boldsymbol{i})\tau_p e(\boldsymbol{j})b\boldsymbol{e}$ may be nonzero only if $\boldsymbol{i} = s_p(\boldsymbol{j})$. This is possible only if the sequence $\boldsymbol{j}$ is almost ordered. Thus the element $e(\boldsymbol{i})\tau_p e(\boldsymbol{j})b\boldsymbol{e}$ is in J.

Now, assume that $k > 1$ and that the statement is true for each value $< k$. Set $w = s_{p_1}\cdots s_{p_k}$. We may assume that $\boldsymbol{i} = w(\boldsymbol{j})$, otherwise the element $e(\boldsymbol{i})\tau_{p_1}\cdots\tau_{p_k}e(\boldsymbol{j})b\boldsymbol{e}$ is zero. By assumptions on $\boldsymbol{i}$ and $\boldsymbol{j}$ there is an index $r \in [1, d]$ such that $i_r \in \bar{I}_1$ and $w^{-1}(r+1) < w^{-1}(r)$. Thus w has a reduced expression of the form $w = s_r s_{r_1}\cdots s_{r_h}$. This implies that $\tau_{p_1}\cdots\tau_{p_k}e(\boldsymbol{j})$ is equal to a monomial of the form $\tau_r\tau_{r_1}\cdots\tau_{r_h}e(\boldsymbol{j})$ modulo monomials of the form $\tau_{q_1}\cdots\tau_{q_t}x_1^{b_1}\cdots x_d^{b_d}e(\boldsymbol{j})$ with $t < k$, see Remark 2.4. Thus the element $e(\boldsymbol{i})\tau_1\cdots\tau_k e(\boldsymbol{j})b\boldsymbol{e}$ is equal to $e(\boldsymbol{i})\tau_r\tau_{r_1}\cdots\tau_{r_h}e(\boldsymbol{j})b\boldsymbol{e}$ modulo the elements of the same form $e(\boldsymbol{i})\tau_{p_1}\cdots\tau_{p_k}e(\boldsymbol{j})b\boldsymbol{e}$ with smaller k. The element $e(\boldsymbol{i})\tau_r\tau_{r_1}\cdots\tau_{r_h}e(\boldsymbol{j})b\boldsymbol{e}$ is in J because the sequence $s_r(\boldsymbol{i})$ is almost ordered and the additional terms are in J by the induction assumption. □

3F. ***Generalization of Theorem 3.5.*** In this section we modify slightly the definition of a categorical representation given in Definition 3.2. The only difference is that we use the lattice Q_I instead of X_I. This new definition is not equivalent to Definition 3.2. In this section we work with an arbitrary quiver $\Gamma = (I, H)$ without 1-loops.

Let $\boldsymbol{k}$ be a field. Let $\mathcal{C}$ be a $\boldsymbol{k}$-linear Hom-finite category.

Definition 3.8. A $\mathfrak{g}_I$-quasicategorical representation (E, F, x, τ) in $\mathcal{C}$ is the following data

(1) a decomposition $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \mathcal{C}_\alpha$,

(2) a pair of biadjoint exact endofunctors (E, F) of $\mathcal{C}$,

(3) morphisms of functors $x\colon F \to F$, $\tau\colon F^2 \to F^2$,

(4) decompositions $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$,

satisfying the following conditions.

(a) We have $E_i(\mathcal{C}_\alpha) \subset \mathcal{C}_{\alpha - \alpha_i}$, $F_i(\mathcal{C}_\alpha) \subset \mathcal{C}_{\alpha + \alpha_i}$.

(b) For each $d \in \mathbb{N}$ there is an algebra homomorphism $\psi_d \colon R_{d,\boldsymbol{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$ such that $\psi_d(e(\boldsymbol{i}))$ is the projector to $F_{i_d} \cdots F_{i_1}$, where $\boldsymbol{i} = (i_1, \dots, i_d)$ and

$$\psi_d(x_r) = F^{d-r} x F^{r-1} \quad \text{and} \quad \psi_d(\tau_r) = F^{d-r-1} \tau F^{r-1}.$$

(c) For each $M \in \mathcal{C}$ the endomorphism of $F(M)$ induced by x is nilpotent.

If the quiver Γ is infinite, condition (4) should be understood in the same way as in Remark 3.3(b).

Now, fix a decomposition $I = I_0 \sqcup I_1$ as in Section 2B. We consider the quiver $\overline{\Gamma} = (\bar{I}, \overline{H})$ and the map ϕ as in Section 2B. To distinguish the elements of Q_I and $Q_{\bar{I}}$, we write $Q_{\bar{I}} = \bigoplus_{i \in \bar{I}} \mathbb{Z}\bar{\alpha}_i$. For each $\alpha \in Q_I$ we set $\bar{\alpha} = \phi(\alpha) \in Q_{\bar{I}}$. (See Section 2B for the notation.) However we can sometimes use the symbol $\bar{\alpha}$ for an arbitrary element of $Q_{\bar{I}}$ that is not associated with some α in Q_I. Let $\overline{\mathcal{C}}$ be a Hom-finite abelian $\boldsymbol{k}$-linear category. Let $\overline{E} = \bigoplus_{i \in \bar{I}} \overline{E}_i$ and $\overline{F} = \bigoplus_{i \in \bar{I}} \overline{F}_i$ be endofunctors defining a $\mathfrak{g}_{\bar{I}}$-quasicategorical representation in $\overline{\mathcal{C}}$. Let $\overline{\psi}_d \colon R_{d,\boldsymbol{k}}(\overline{\Gamma}) \to \mathrm{End}(\overline{F}^d)^{\mathrm{op}}$ be the corresponding algebra homomorphism. We set $\overline{F}_{\boldsymbol{i}} = \overline{F}_{i_d} \cdots \overline{F}_{i_1}$ for any tuple $\boldsymbol{i} = (i_1, \dots, i_d) \in \bar{I}^d$ and $\overline{F}_{\bar{\alpha}} = \bigoplus_{\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}} \overline{F}_{\boldsymbol{i}}$ for any element $\bar{\alpha} \in Q_{\bar{I}}^+$. If $|\bar{\alpha}| = d$, let $\overline{\psi}_{\bar{\alpha}} \colon R_{\bar{\alpha},\boldsymbol{k}} \to \mathrm{End}(\overline{F}_{\bar{\alpha}})^{\mathrm{op}}$ be the $\bar{\alpha}$-component of $\overline{\psi}_d$.

Assume that $\mathcal{C}$ is an abelian subcategory of $\overline{\mathcal{C}}$ satisfying the following conditions:

(a) $\mathcal{C}$ is stable by $\overline{F}_i$ and $\overline{E}_i$ for each $i \in I_0$.

(b) $\mathcal{C}$ is stable by $\overline{F}_{i^2}\overline{F}_{i^1}$ and $\overline{E}_{i^1}\overline{E}_{i^2}$ for each $i \in I_1$.

(c) We have $\overline{F}_{i^2}(\mathcal{C}) = 0$ for each $i \in I_1$.

(d) We have $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \mathcal{C} \cap \overline{\mathcal{C}}_{\bar{\alpha}}$.

By (d), we get a decomposition $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \mathcal{C}_\alpha$, where $\mathcal{C}_\alpha = \mathcal{C} \cap \overline{\mathcal{C}}_{\bar{\alpha}}$. For each $i \in I$ we consider the following endofunctors E_i and F_i of $\mathcal{C}$:

$$F_i = \begin{cases} \overline{F}_i|_{\mathcal{C}} & \text{if } i \in I_0, \\ \overline{F}_{i^2}\overline{F}_{i^1}|_{\mathcal{C}} & \text{if } i \in I_1, \end{cases} \quad \text{and} \quad E_i = \begin{cases} \overline{E}_i|_{\mathcal{C}} & \text{if } i \in I_0, \\ \overline{E}_{i^1}\overline{E}_{i^2}|_{\mathcal{C}} & \text{if } i \in I_1. \end{cases}$$

As in the notations above we set $F_{\boldsymbol{i}} = F_{i_d} \cdots F_{i_1}$ for any tuple $\boldsymbol{i} = (i_1, \dots, i_d) \in I^d$ and $F_\alpha = \bigoplus_{\boldsymbol{i} \in I^\alpha} F_{\boldsymbol{i}}$ for any element $\alpha \in Q_I^+$. Note that we have $F_{\boldsymbol{i}} = \overline{F}_{\phi(\boldsymbol{i})}|_{\mathcal{C}}$ for each $\boldsymbol{i} \in I^\alpha$.

Let $\alpha \in Q_I^+$. We have

$$F_\alpha = \bigoplus_{\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} \overline{F}_{\boldsymbol{i}}|_{\mathcal{C}}.$$

The homomorphism $\overline{\psi}_{\bar{\alpha}}$ yields a homomorphism $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$, where $\boldsymbol{e} = \sum_{\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} e(\boldsymbol{i})$.

Since the category $\mathcal{C}$ satisfies (a), (b) and (c), for each almost ordered sequence $\boldsymbol{i} = (i_1, \dots, i_d) \in I^\alpha$ we have $\overline{F}_{i_d} \cdots \overline{F}_{i_1}(\mathcal{C}) = 0$. By Lemma 3.7, this implies that the homomorphism $\boldsymbol{e}R_{\bar{\alpha},\boldsymbol{k}}\boldsymbol{e} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$ factors through a homomorphism $S_{\bar{\alpha},\boldsymbol{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$. Let us call it $\overline{\psi}'_{\bar{\alpha}}$. Then we can define an algebra homomorphism $\psi_\alpha \colon R_{\alpha,\boldsymbol{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$ by setting $\psi_\alpha = \overline{\psi}'_{\bar{\alpha}} \circ \Phi_{\alpha,\boldsymbol{k}}$.

Now, Theorem 2.12 implies the following result.

Theorem 3.9. *For each abelian subcategory $\mathcal{C} \subset \bar{\mathcal{C}}$ as above, that satisfies (a)–(d), we have a $\mathfrak{g}_I$-quasicategorical representation in $\mathcal{C}$ given by functors F_i and E_i and the algebra homomorphisms $\psi_\alpha : R_{\alpha,\boldsymbol{k}} \to \mathrm{End}(F_\alpha)^{\mathrm{op}}$.* □

Remark 3.10. Assume that the category $\bar{\mathcal{C}}$ is such that we have $\bar{\mathcal{C}}_{\bar{\alpha}} = 0$ whenever $\bar{\alpha} = \sum_{i \in \bar{I}} d_i \bar{\alpha}_i \in Q_{\bar{I}}$ is such that $d_{i^1} < d_{i^2}$ for some $i \in I_1$. In this case the subcategory $\mathcal{C} \subset \bar{\mathcal{C}}$ defined by $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \bar{\mathcal{C}}_{\bar{\alpha}}$ satisfies conditions (a)–(d).

Appendix A: The geometric construction of the isomorphism Φ

The goal of this section is to give a geometric construction of the isomorphism Φ in Theorem 2.12.

A1. ***The geometric construction of the KLR algebra.*** Let $\boldsymbol{k}$ be a field. Let $\Gamma = (I, H)$ be a quiver without 1-loops. See Section 2A for the notations related to quivers. For an arrow $h \in H$ we will write h' and h'' for its source and target respectively. Fix $\alpha = \sum_{i \in I} d_i \alpha_i \in Q_I^+$ and set $d = |\alpha|$. Set also

$$E_\alpha = \bigoplus_{h \in H} \mathrm{Hom}(V_{h'}, V_{h''}), \qquad V_i = \mathbb{C}^{d_i}, \qquad V = \bigoplus_{i \in I} V_i.$$

The group $G_\alpha = \prod_{i \in I} \mathrm{GL}(V_i)$ acts on E_α by base changes.

Set

$$I^\alpha = \left\{ \boldsymbol{i} = (i_1, \ldots, i_d) \in I^d : \sum_{r=1}^{d} \alpha_{i_r} = \alpha \right\}.$$

We denote by $F_{\boldsymbol{i}}$ the variety of all flags

$$\phi = (V = V^0 \supset V^1 \supset \cdots \supset V^d = \{0\})$$

in V that are homogeneous with respect to the decomposition $V = \bigoplus_{i \in I} V_i$ and such that the I-graded vector space V^{r-1}/V^r has graded dimension i_r for $r \in [1, d]$. We denote by $\tilde{F}_{\boldsymbol{i}}$ the variety of pairs $(x, \phi) \in E_\alpha \times F_{\boldsymbol{i}}$ such that x preserves ϕ, i.e., we have $x(V^r) \subset V^r$ for $r \in \{0, 1, \ldots, m\}$. Let $\pi_{\boldsymbol{i}}$ be the natural projection from $\tilde{F}_{\boldsymbol{i}}$ to E_α, i.e., $\pi_{\boldsymbol{i}} : \tilde{F}_{\boldsymbol{i}} \to E_\alpha$, $(x, \phi) \mapsto x$. For $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$ we denote by $Z_{\boldsymbol{i},\boldsymbol{j}}$ the variety of triples $(x, \phi_1, \phi_2) \in E_\alpha \times F_{\boldsymbol{i}} \times F_{\boldsymbol{j}}$ such that x preserves ϕ_1 and ϕ_2 (i.e., we have $Z_{\boldsymbol{i},\boldsymbol{j}} = \tilde{F}_{\boldsymbol{i}} \times_{E_\alpha} \tilde{F}_{\boldsymbol{j}}$). Set

$$Z_\alpha = \coprod_{\boldsymbol{i},\boldsymbol{j} \in I^\alpha} Z_{\boldsymbol{i},\boldsymbol{j}} \quad \text{and} \quad \tilde{F}_\alpha = \coprod_{\boldsymbol{i} \in I^\alpha} \tilde{F}_{\boldsymbol{i}}.$$

We have an algebra structure on $H_*^{G_\alpha}(Z_\alpha, \boldsymbol{k})$ such that the multiplication is the convolution product with respect to the inclusion $Z_\alpha \subset \tilde{F}_\alpha \times \tilde{F}_\alpha$. Here $H_*^{G_\alpha}(\bullet, \boldsymbol{k})$ denotes the G_α-equivariant Borel–Moore homology with coefficients in $\boldsymbol{k}$. See [Chriss and Ginzburg 1997, §2.7] for the definition of the convolution product.

The following result is proved by Rouquier [2008] and by Varagnolo and Vasserot [2011] in the situation $\mathrm{char}\, \boldsymbol{k} = 0$. See [Maksimau 2015a] for the proof over an arbitrary field.

Proposition A.1. *There is an algebra isomorphism $R_{\alpha,\boldsymbol{k}}(\Gamma)\simeq H_*^{G_\alpha}(Z_\alpha,\boldsymbol{k})$. Moreover, for each $\boldsymbol{i},\boldsymbol{j}\in I^\alpha$, the vector subspace $e(\boldsymbol{i})R_{\alpha,\boldsymbol{k}}(\Gamma)e(\boldsymbol{j})\subset R_{\alpha,\boldsymbol{k}}(\Gamma)$ corresponds to the vector subspace $H_*^{G_\alpha}(Z_{\boldsymbol{i},\boldsymbol{j}},\boldsymbol{k})\subset H_*^{G_\alpha}(Z_\alpha,\boldsymbol{k})$.* □

A2. ***The geometric construction of the isomorphism Φ.*** As in Section 2B, fix a decomposition $I=I_0\sqcup I_1$ and consider the quiver $\overline{\Gamma}=(\bar{I},\overline{H})$; also fix $\alpha\in Q_I^+$ and consider $\bar\alpha=\phi(\alpha)\in Q_{\bar I}^+$.

We start from the variety $Z_{\bar\alpha}$ defined with respect to the quiver $\overline{\Gamma}$. By Proposition A.1, we have an algebra isomorphism $R_{\bar\alpha,\boldsymbol{k}}(\overline{\Gamma})\simeq H_*^{G_{\bar\alpha}}(Z_{\bar\alpha},\boldsymbol{k})$. We have an obvious projection $p\colon Z_{\bar\alpha}\to E_{\bar\alpha}$ defined by $(x,\phi_1,\phi_2)\mapsto x$. For each $i\in I_1$ denote by h_i the unique arrow in $\overline{\Gamma}$ that goes from i^1 to i^2. Consider the following open subset of $E_{\bar\alpha}$: $E^0_{\bar\alpha}=\{x\in E_{\bar\alpha} : x_{h_i}\text{ is invertible }\forall i\in I_1\}$. Set $Z^0_{\bar\alpha}=p^{-1}(E^0_{\bar\alpha})$. The pullback with respect to the inclusion $Z^0_{\bar\alpha}\subset Z_{\bar\alpha}$ yields an algebra homomorphism $H_*^{G_{\bar\alpha}}(Z_{\bar\alpha},\boldsymbol{k})\to H_*^{G_{\bar\alpha}}(Z^0_{\bar\alpha},\boldsymbol{k})$ (see [Chriss and Ginzburg 1997, Lemma 2.7.46]).

Remark A.2. If the sequence $\boldsymbol{i}\in\bar{I}^{\bar\alpha}$ is unordered, then a flag from $F_{\boldsymbol{i}}$ is never preserved by an element from $E^0_{\bar\alpha}$. This implies that $Z_{\boldsymbol{i},\boldsymbol{j}}\cap Z^0_{\bar\alpha}=\varnothing$ if $\boldsymbol{i}$ or $\boldsymbol{j}$ is unordered. Thus for each $\boldsymbol{i}\in\bar{I}^{\bar\alpha}_{\mathrm{un}}$, the idempotent $e(\boldsymbol{i})$ is in the kernel of the homomorphism $H_*^{G_{\bar\alpha}}(Z_{\bar\alpha},\boldsymbol{k})\to H_*^{G_{\bar\alpha}}(Z^0_{\bar\alpha},\boldsymbol{k})$.

Let $\boldsymbol{e}$ be the idempotent as in Definition 2.6. Consider the following subset of $Z_{\bar\alpha}$:

$$Z'_{\bar\alpha}=\coprod_{\boldsymbol{i},\boldsymbol{j}\in\bar{I}^{\bar\alpha}_{\mathrm{ord}}}Z_{\boldsymbol{i},\boldsymbol{j}}.$$

The algebra isomorphism $R_{\bar\alpha,\boldsymbol{k}}(\overline{\Gamma})\simeq H_*^{G_{\bar\alpha}}(Z_{\bar\alpha},\boldsymbol{k})$ above restricts to an algebra isomorphism $\boldsymbol{e}R_{\bar\alpha}(\overline{\Gamma})\boldsymbol{e}\simeq H_*^{G_{\bar\alpha}}(Z'_{\bar\alpha},\boldsymbol{k})$.

Now, set $Z'^0_\alpha=Z'_\alpha\cap Z^0_\alpha$. Similarly to the construction above, we have an algebra homomorphism $H_*^{G_{\bar\alpha}}(Z'_{\bar\alpha},\boldsymbol{k})\to H_*^{G_{\bar\alpha}}(Z'^0_{\bar\alpha},\boldsymbol{k})$. By Remark A.2, the kernel of this homomorphism contains the kernel of $\boldsymbol{e}R_{\bar\alpha,\boldsymbol{k}}(\overline{\Gamma})\boldsymbol{e}\to R_{\alpha,\boldsymbol{k}}(\Gamma)$ (see Theorem 2.12). The following result implies that these kernels are the same.

Lemma A.3. *We have the following algebra isomorphism $R_{\alpha,\boldsymbol{k}}(\Gamma)\simeq H_*^{G_{\bar\alpha}}(Z'^0_{\bar\alpha},\boldsymbol{k})$.*

Proof. For each $i\in I_0$ we identify $V_i\simeq V_{i^0}$. For each $i\in I_1$ we identify $V_i\simeq V_{i^1}\simeq V_{i^2}$. We have a diagonal inclusion $G_\alpha\subset G_{\bar\alpha}$, i.e., the component $\mathrm{GL}(V_i)$ of G_α with $i\in I_0$ goes to $\mathrm{GL}(V_{i^0})$ and the component $\mathrm{GL}(V_i)$ with $i\in I_1$ goes diagonally to $\mathrm{GL}(V_{i^1})\times\mathrm{GL}(V_{i^2})$.

Set $G^{\mathrm{bis}}_\alpha=\prod_{i\in I_1}\mathrm{GL}(V_{i^2})\subset G_{\bar\alpha}$. We have an obvious group isomorphism $G_{\bar\alpha}/G^{\mathrm{bis}}_\alpha\simeq G_\alpha$.

Let us denote by X the choice of isomorphisms $V_{i^1}\simeq V_{i^2}$ mentioned above. Let $E^X_{\bar\alpha}$ be the subset of $E_{\bar\alpha}$ that contains only $x\in E_{\bar\alpha}$ such that for each $i\in I_1$ the component x_{h_i} is the isomorphism chosen in X.

The group G^{bis}_α acts freely on $E^0_{\bar\alpha}$ such that each orbit intersects $E^X_{\bar\alpha}$ once. This implies that we have an isomorphism of algebraic varieties $E^0_{\bar\alpha}/G^{\mathrm{bis}}_\alpha\simeq E^X_{\bar\alpha}$. Now, set $Z'^X_{\bar\alpha}=p^{-1}(E^X_{\bar\alpha})$. The same argument as above yields $Z'^0_{\bar\alpha}/G^{\mathrm{bis}}_\alpha\simeq Z'^X_{\bar\alpha}$. We get the following chain of algebra isomorphisms

$$H_*^{G_{\bar\alpha}}(Z'^0_{\bar\alpha},\boldsymbol{k})\simeq H_*^{G_{\bar\alpha}/G^{\mathrm{bis}}_\alpha}(Z'^0_{\bar\alpha}/G^{\mathrm{bis}}_\alpha,\boldsymbol{k})\simeq H_*^{G_\alpha}(Z'^X_{\bar\alpha},\boldsymbol{k}).$$

To complete the proof we have to show that the G_α-variety $Z'^X_{\bar\alpha}$ is isomorphic to Z_α. Each element of $I^{\bar\alpha}_{\mathrm{ord}}$ is of the form $\phi(\boldsymbol{i})$ for a unique $\boldsymbol{i} \in I^\alpha$, where ϕ is as in Section 2B. Let us abbreviate $\boldsymbol{i}' = \phi(\boldsymbol{i})$. By definition we have

$$Z'_{\bar\alpha} = \coprod_{\boldsymbol{i},\boldsymbol{j}\in I^\alpha} Z_{\boldsymbol{i}',\boldsymbol{j}'}.$$

Set $Z^X_{\boldsymbol{i}',\boldsymbol{j}'} = Z_{\boldsymbol{i}',\boldsymbol{j}'} \cap Z'^X_{\bar\alpha}$. We have an obvious isomorphism of G_α-varieties $Z^X_{\boldsymbol{i}',\boldsymbol{j}'} \simeq Z_{\boldsymbol{i},\boldsymbol{j}}$. (Beware, the variety $Z_{\boldsymbol{i},\boldsymbol{j}}$ is defined with respect to the quiver Γ and the variety $Z_{\boldsymbol{i}',\boldsymbol{j}'}$ is defined with respect to the quiver $\overline{\Gamma}$.) Taking the union for all $\boldsymbol{i}, \boldsymbol{j} \in I^\alpha$ yields an isomorphism of G_α-varieties $Z'^X_{\bar\alpha} \simeq Z_\alpha$. $\square$

Corollary A.4. *We have the following commutative diagram.*

$$\begin{array}{ccc}
\boldsymbol{e}R_{\bar\alpha,\boldsymbol{k}}(\overline{\Gamma})\boldsymbol{e} & \longrightarrow & R_{\alpha,\boldsymbol{k}}(\Gamma) \\
\downarrow & & \downarrow \\
H^{G_{\bar\alpha}}_*(Z'_{\bar\alpha}, \boldsymbol{k}) & \longrightarrow & H^{G_{\bar\alpha}}_*(Z'^0_{\bar\alpha}, \boldsymbol{k}).
\end{array}$$

Here the left vertical map is the isomorphism from Proposition A.1, the right vertical map is the isomorphism from Lemma A.3, the top horizontal map is obtained from Theorem 2.12 and the bottom horizontal map is the pullback with respect to the inclusion $Z'^0_{\bar\alpha} \subset Z'_{\bar\alpha}$.

Proof. The result follows directly from Lemma A.3. The commutativity of the diagram is easy to see on the generators of $R_{\bar\alpha,\boldsymbol{k}}(\overline{\Gamma})$.

Indeed, the isomorphism $R_{\alpha,\boldsymbol{k}} \simeq H^{G_\alpha}_*(Z_\alpha, \boldsymbol{k})$ is defined in the following way (see [Maksimau 2015a, §2.9, Theorem 2.4] for more details). The element $e(\boldsymbol{i})$ corresponds to the fundamental class $[Z_{\boldsymbol{i},\boldsymbol{i}}]$. The element $x_r e(\boldsymbol{i})$ corresponds to the first Chern class of some line bundle on $Z_{\boldsymbol{i},\boldsymbol{i}}$. The element $\psi_r e(\boldsymbol{i})$ corresponds to the fundamental class of some correspondence in $Z_{s_r(\boldsymbol{i}),\boldsymbol{i}}$. The commutativity of the diagram in the statement follows from standard properties of Chern classes and fundamental classes. $\square$

Appendix B: A local ring version in type A

In this appendix we give some versions of the main results of the paper (Theorems 2.12 and 3.5) over a local ring. These ring versions are interesting because the study of the category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N$ in [Maksimau 2015b] uses a deformation argument. For this we need a version of Theorem 1.2 over a local ring.

It is known that the affine Hecke algebra over a field is related with the KLR algebra (see Propositions B.5, B.6). This allows to reformulate the definition of a categorical representation (see Definition 3.2) that is given in term of KLR algebras in an equivalent way in terms of Hecke algebras (see Definition B.14). The main difficulty is that there is no known relation between Hecke and KLR algebras over a ring. Over a local ring, we can give a definition of a categorical representation using the Hecke algebra (see Definition B.17). But we have no equivalent definition in terms of KLR algebras. That is why, Proposition B.12, that is a ring analogue of Theorem 2.12, is formulated in terms of Hecke algebras and not in terms of KLR algebras.

B1. *Intertwining operators.* The center of the algebra $R_{\alpha,\boldsymbol{k}}$ is the ring of symmetric polynomials $\boldsymbol{k}_d[x]^{\mathfrak{S}_d}$, see [Rouquier 2008, Proposition 3.9]. Thus $S_{\bar{\alpha},\boldsymbol{k}}$ is a $\boldsymbol{k}_d[x]^{\mathfrak{S}_d}$-algebra under the isomorphism $\Phi_{\alpha,\boldsymbol{k}}$ in Section 2G. Let Σ be the polynomial $\prod_{a<b}(x_a-x_b)^2 \in \boldsymbol{k}_d[x]^{\mathfrak{S}_d}$. Let $R_{\alpha,\boldsymbol{k}}[\Sigma^{-1}]$ and $S_{\bar{\alpha},\boldsymbol{k}}[\Sigma^{-1}]$ be the rings of quotients of $R_{\alpha,\boldsymbol{k}}$ and $S_{\bar{\alpha},\boldsymbol{k}}$ obtained by inverting Σ. We can extend the isomorphism $\Phi_{\alpha,\boldsymbol{k}}$ from Theorem 2.12 to an algebra isomorphism

$$\Phi_{\alpha,\boldsymbol{k}}\colon R_{\alpha,\boldsymbol{k}}[\Sigma^{-1}] \to S_{\bar{\alpha},\boldsymbol{k}}[\Sigma^{-1}].$$

Assume that the connected components of the quiver Γ are of the form Γ_a for $a \in \mathbb{N}$, $a>1$ or $a=\infty$. (The quiver Γ_a is defined in Section 3B.)

Note that there is an action of the symmetric group $\mathfrak{S}_d$ on $\boldsymbol{k}_d^{(I)}$ permuting the variables and the components of $\boldsymbol{i}$. Consider the following element in $R_{\alpha,\boldsymbol{k}}[\Sigma^{-1}]$:

$$\Psi_r e(\boldsymbol{i}) = \begin{cases} ((x_r-x_{r+1})\tau_r+1)e(\boldsymbol{i}) & \text{if } i_{r+1}=i_r, \\ -(x_r-x_{r+1})^{-1}\tau_r e(\boldsymbol{i}) & \text{if } i_{r+1}=i_r-1, \\ \tau_r e(\boldsymbol{i}) & \text{else.} \end{cases}$$

The element $\Psi_r e(\boldsymbol{i})$ is called *intertwining operator*. Using the formulas (3) we can check that $\Psi_r e(\boldsymbol{i})$ still acts on the polynomial representation and the corresponding operator is equal to $s_r e(\boldsymbol{i})$. Note also that $\tilde{\Psi}_r=(x_r-x_{r+1})\Psi_r$ is an element of $R_{\alpha,\boldsymbol{k}}$.

Lemma B.1. *The images of intertwining operators by $\Phi_{\alpha,\boldsymbol{k}}\colon R_{\alpha,\boldsymbol{k}} \to S_{\bar{\alpha},\boldsymbol{k}}$ can be described in the following way. For $\boldsymbol{i} \in I^\alpha$ such that $i_r-1 \neq i_{r+1}$ we have*

$$\Phi_{\alpha,\boldsymbol{k}}(\Psi_r e(\boldsymbol{i})) = \begin{cases} \Psi_{r'} e(\phi(\boldsymbol{i})) & \text{if } i_r, i_{r+1} \in I_0, \\ \Psi_{r'}\Psi_{r'+1} e(\phi(\boldsymbol{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \Psi_{r'+1}\Psi_{r'} e(\phi(\boldsymbol{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1, \\ \Psi_{r'+1}\Psi_{r'+2}\Psi_{r'}\Psi_{r'+1} e(\phi(\boldsymbol{i})) & \text{if } i_r, i_{r+1} \in I_1. \end{cases}$$

For $\boldsymbol{i} \in I^\alpha$ such that $i_r-1 = i_{r+1}$ we have

$$\Phi_{\alpha,\boldsymbol{k}}(\tilde{\Psi}_r e(\boldsymbol{i})) = \begin{cases} \tilde{\Psi}_{r'} e(\phi(\boldsymbol{i})) & \text{if } i_r, i_{r+1} \in I_0, \\ \tilde{\Psi}_{r'}\Psi_{r'+1} e(\phi(\boldsymbol{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \Psi_{r'+1}\tilde{\Psi}_{r'} e(\phi(\boldsymbol{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1. \end{cases}$$

Here $r'=r'_{\boldsymbol{i}}$ is as in Section 2F.

Proof. By construction of $\Phi_{\alpha,\boldsymbol{k}}$, the elements $\Phi_{\alpha,\boldsymbol{k}}(\Psi_r e(\boldsymbol{i}))$ and $\Phi_{\alpha,\boldsymbol{k}}(\tilde{\Psi}_r e(\boldsymbol{i}))$ are the unique elements of $S_{\bar{\alpha},\boldsymbol{k}}$ that acts on the polynomial representation by the same operator as $\Psi_r e(\boldsymbol{i})$ and $\tilde{\Psi}_r e(\boldsymbol{i})$, respectively.

The right hand side in the formulas for $\Phi_{\alpha,\boldsymbol{k}}(\Psi_r e(\boldsymbol{i}))$ or $\Phi_{\alpha,\boldsymbol{k}}(\tilde{\Psi}_r e(\boldsymbol{i}))$ in the statement is an element X in $S_{\bar{\alpha},\boldsymbol{k}}[\Sigma^{-1}]$. To complete the proof we have to show that:

(1) X acts by the same operator as $\Psi_r e(\boldsymbol{i})$ or $\tilde{\Psi}_r e(\boldsymbol{i})$, respectively, on the polynomial representation.

(2) X is in $S_{\bar{\alpha},\boldsymbol{k}}$.

Part (1) is obvious. Part (2) follows from part (1) and from the faithfulness of the polynomial representation of $S_{\bar{\alpha},k}[\Sigma^{-1}]$ (see Lemma 2.10). (In fact, part (2) is not obvious only in the case $i_r = i_{r+1} \in I_1$.) $\square$

B2. ***Special quivers.*** From now on we will be interested only in some special types of quivers.

First, consider the quiver $\Gamma = \Gamma_e$, where e is an integer > 1. In particular, from now on we fix $I = \mathbb{Z}/e\mathbb{Z}$. Fix $k \in [0, e-1]$ and set $I_1 = \{k\}$ and $I_0 = I \backslash \{k\}$. In this case the quiver $\overline{\Gamma}$ is isomorphic to Γ_{e+1}. More precisely, the decomposition $\bar{I} = \bar{I}_0 \sqcup \bar{I}_1 \sqcup \bar{I}_2$ is such that $\bar{I}_1 = \{k\}$ and $\bar{I}_2 = \{k+1\}$. To avoid confusion, for $i \in \bar{I}$ we will write $\bar{\alpha}_i$ and $\bar{\varepsilon}_i$ for α_i and ε_i respectively.

Remark B.2. If Γ is as above, a sequence $\boldsymbol{i} = (i_1, \dots, i_d) \in \bar{I}^d$ is well ordered if for each index a such that $i_a = k$ we have $a < d$ and $i_{a+1} = k+1$. The sequence $\boldsymbol{i}$ is unordered if there is $r \leqslant d$ such that the subsequence $(i_1, \dots, i_r)$ contains more elements equal to $k+1$ than elements equal to k.

Let $\Upsilon\colon \mathbb{Z} \to \mathbb{Z}$ be the map given for $a \in \mathbb{Z}$ and $b \in [0, e-1]$ by

$$\Upsilon(ae+b) = \begin{cases} a(e+1)+b & \text{if } b \in [0,k], \\ a(e+1)+b+1 & \text{if } b \in [k+1, e-1]. \end{cases} \tag{8}$$

Now, consider the quiver $\tilde{\Gamma} = (\Gamma_\infty)^{\sqcup l}$ (i.e., $\tilde{\Gamma}$ is a disjoint union of l copies of Γ_∞). Set $\tilde{\Gamma} = (\tilde{I}, \tilde{H})$ and write $\tilde{\alpha}_i$ and $\tilde{\varepsilon}_i$ and for α_i and ε_i respectively for each $i \in \tilde{I}$. We identify an element of $\tilde{I}$ with an element $(a, b) \in \mathbb{Z} \times [1, l]$ in the obvious way. Consider the decomposition $\tilde{I} = \tilde{I}_0 \sqcup \tilde{I}_1$ such that $(a, b) \in \tilde{I}_1$ if and only if $a \equiv k \mod e$. In this case the quiver $\overline{\tilde{\Gamma}}$ is isomorphic to $\tilde{\Gamma}$. We will often write $\tilde{\Gamma}$ instead of $\overline{\tilde{\Gamma}}$ (but sometimes, if confusion is possible, we will use the notation $\overline{\tilde{\Gamma}}$ to stress that we work with the doubled quiver). More precisely, in this case we have

$$\begin{aligned} (a,b)^0 &= (\Upsilon(a), b), \\ (a,b)^1 &= (\Upsilon(a), b), \\ (a,b)^2 &= (\Upsilon(a)+1, b). \end{aligned}$$

To distinguish notations, we will always write $\tilde{\phi}$ for any of the maps $\tilde{\phi}\colon \tilde{I}^\infty \to \tilde{I}^\infty$, $Q_{\tilde{I}} \to Q_{\tilde{I}}$, $X_{\tilde{I}} \to X_{\tilde{I}}$ in Section 2B.

From now on we write $\Gamma = \Gamma_e$, $\overline{\Gamma} = \Gamma_{e+1}$ and $\tilde{\Gamma} = (\Gamma_\infty)^{\sqcup l}$. Recall that

$$I = I_e = \mathbb{Z}/e\mathbb{Z}, \quad \bar{I} = I_{e+1} = \mathbb{Z}/(e+1)\mathbb{Z}, \quad \tilde{I} = (I_\infty)^{\sqcup l} = \mathbb{Z} \times [1, l].$$

Consider the quiver homomorphism $\pi_e\colon \tilde{\Gamma} \to \Gamma$ such that

$$\pi_e\colon \tilde{I} \to I, \quad (a, b) \mapsto a \mod e.$$

Then π_{e+1} is a quiver homomorphism $\pi_{e+1}\colon \tilde{\Gamma} \to \overline{\Gamma}$. They yield $\mathbb{Z}$-linear maps

$$\pi_e\colon Q_{\tilde{I}} \to Q_I, \quad \pi_e\colon X_{\tilde{I}} \to X_I, \quad \pi_{e+1}\colon Q_{\tilde{I}} \to Q_{\bar{I}}, \quad \pi_{e+1}\colon X_{\tilde{I}} \to X_{\bar{I}}.$$

The following diagrams are commutative for $\alpha \in Q_I^+$ and $\tilde{\alpha} \in Q_{\tilde{I}}^+$ such that $\pi_e(\tilde{\alpha}) = \alpha$,

$$\begin{tikzcd} Q_{\tilde{I}} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi_e"'] & Q_{\tilde{I}} \arrow[d, "\pi_{e+1}"] \\ Q_I \arrow[r, "\phi"] & Q_{\bar{I}} \end{tikzcd} \qquad \begin{tikzcd} X_{\tilde{I}} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi_e"'] & X_{\tilde{I}} \arrow[d, "\pi_{e+1}"] \\ X_I \arrow[r, "\phi"] & X_{\bar{I}} \end{tikzcd} \qquad \begin{tikzcd} \tilde{I}^{\tilde{\alpha}} \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi_e"'] & \tilde{I}^{\tilde{\phi}(\tilde{\alpha})} \arrow[d, "\pi_{e+1}"] \\ I^{\alpha} \arrow[r, "\phi"] & \bar{I}^{\phi(\alpha)} \end{tikzcd}$$

The quiver $\tilde{\Gamma}$ is infinite. We will sometimes use its truncated version. Fix a positive integer N. Denote by $\tilde{\Gamma}^{\leqslant N}$ the full subquiver (i.e., a quiver with a smaller set of vertices and the same arrows between these vertices) of $\tilde{\Gamma}$ that contains only vertices (a, b) such that $|a| \leqslant eN$. Let $\bar{\tilde{\Gamma}}^{\leqslant N}$ be the doubled quiver associated with $\tilde{\Gamma}^{\leqslant N}$. We can see the quiver $\bar{\tilde{\Gamma}}^{\leqslant N}$ as a full subquiver of $\bar{\tilde{\Gamma}}$ that contains only vertices (a, b) such that we have

$$\begin{cases} -(e+1)N \leqslant a \leqslant (e+1)N & \text{if } k \neq 0, \\ -(e+1)N \leqslant a \leqslant (e+1)N + 1 & \text{else.} \end{cases}$$

(Attention, it is not true that the isomorphism of quivers $\tilde{\Gamma} \simeq \bar{\tilde{\Gamma}}$ takes $\tilde{\Gamma}^{\leqslant N}$ to $\bar{\tilde{\Gamma}}^{\leqslant N}$.)

B3. *Hecke algebras.* Let R be a commutative ring with 1. Fix an element $q \in R$.

Definition B.3. The *affine Hecke algebra* $H_{R,d}(q)$ is the R-algebra generated by $T_1, \dots, T_{d-1}$ and the invertible elements $X_1, \dots, X_d$ modulo the following defining relations

$$\begin{aligned} X_r X_s &= X_s X_r, \\ T_r X_r &= X_r T_r && \text{if } |r-s| > 1, \\ T_r T_s &= T_s T_r && \text{if } |r-s| > 1, \\ T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1}, \\ T_r X_{r+1} &= X_r T_r + (q-1) X_{r+1}, \\ T_r X_r &= X_{r+1} T_r - (q-1) X_{r+1}, \\ 0 &= (T_r - q)(T_r + 1). \end{aligned}$$

Assume that $R = \boldsymbol{k}$ is a field and $q \neq 0, 1$. The algebra $H_{d,\boldsymbol{k}}(q)$ has a faithful representation (see [Miemietz and Stroppel 2016, Proposition 3.11]) in the vector space $\boldsymbol{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ such that $X_r^{\pm 1}$ acts by multiplication by $X_r^{\pm 1}$ and T_r by

$$T_r(P) = q s_r(P) + (q-1) X_{r+1} (X_r - X_{r+1})^{-1} (s_r(P) - P).$$

The following operator acts on $\boldsymbol{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ as the reflection s_r

$$\Psi_r = \frac{X_r - X_{r+1}}{q X_r - X_{r+1}} (T_r - q) + 1 = (T_r + 1) \frac{X_r - X_{r+1}}{X_r - q X_{r+1}} - 1.$$

For a future use, consider the element $\tilde{\Psi}_r \in H_{d,\boldsymbol{k}}$ given by

$$\tilde{\Psi}_r = (q X_r - X_{r+1}) \Psi_r = (X_r - X_{r+1}) T_r + (q-1) X_{r+1}.$$

B4. ***The isomorphism between Hecke and KLR algebras.*** First, we define some localized versions of Hecke algebras and KLR algebras. Let $\mathscr{F}$ be a finite subset of $\boldsymbol{k}^\times$. We view $\mathscr{F}$ as the vertex set of a quiver with an arrow $i \to j$ if and only if $j = qi$. Consider the algebra

$$A_1 = \bigoplus_{\boldsymbol{i} \in \mathscr{F}^d} \boldsymbol{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}][(X_r - X_t)^{-1}, (qX_r - X_t)^{-1} : r \neq t]e(\boldsymbol{i}),$$

where $e(\boldsymbol{i})$ are orthogonal idempotents and X_r commutes with $e(\boldsymbol{i})$. Let $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ be the A_1-module given by the extension of scalars from the $\boldsymbol{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$-module $H_{d,\boldsymbol{k}}(q)$. It has a $\boldsymbol{k}$-algebra structure such that

$$T_r e(\boldsymbol{i}) - e(s_r(\boldsymbol{i}))T_r = (1-q)X_{r+1}(X_r - X_{r+1})^{-1}(e(\boldsymbol{i}) - e(s_r(\boldsymbol{i})))$$

and

$$Z^{-1}T_r = T_r Z^{-1}, \qquad \text{where } Z = \prod_{r<t}(X_r - X_t)^2 \prod_{r \neq t}(qX_r - X_t)^2.$$

In this section the KLR algebras are always defined with respect to the quiver $\mathscr{F}$. We consider the algebra

$$A_2 = \bigoplus_{\boldsymbol{i} \in \mathscr{F}^d} \boldsymbol{k}[x_1, \dots, x_d][S_{\boldsymbol{i}}^{-1}]e(\boldsymbol{i}),$$

where

$$S_{\boldsymbol{i}} = \{(x_r + 1), (i_r(x_r + 1) - i_t(x_t + 1)), (qi_r(x_r + 1) - i_t(x_t + 1) : r \neq t)\}.$$

Consider the following central element in $R_{d,\boldsymbol{k}}$

$$z = \prod_r (x_r + 1) \prod_{i,j \in \mathscr{F}, r \neq t} (i(x_r + 1) - j(x_t + 1)).$$

The A_2-module $R^{\mathrm{loc}}_{d,\boldsymbol{k}} = A_2 \otimes_{\boldsymbol{k}_d^{(\mathscr{F})}} R_{d,\boldsymbol{k}}$ has a $\boldsymbol{k}$-algebra structure because it is a subalgebra in $R_{d,\boldsymbol{k}}[z^{-1}]$, where $\boldsymbol{k}_d^{(\mathscr{F})}$ is as in (2).

Remark B.4. We assumed above that the set $\mathcal{F}$ is finite. This assumption is important because it implies that A_1 contains $\boldsymbol{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ and A_2 contains $\boldsymbol{k}[x_1, \dots, x_d]$. However, it is possible to define the algebras above (A_1, A_2, $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ and $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$) for arbitrary $\mathcal{F} \subset \boldsymbol{k}^\times$. Indeed, if $\mathcal{F}_1 \subset \mathcal{F}_2$ are finite, then the algebra defined with respect to $\mathcal{F}_1$ is obviously a nonunitary subalgebra of the algebra defined with respect to $\mathcal{F}_2$. Then we can define the algebras A_1, A_2, $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ and $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ with respect to any arbitrary $\mathcal{F}$. For example, we define the algebra $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ associated with $\mathcal{F}$ as

$$R^{\mathrm{loc}}_{d,\boldsymbol{k}}(\mathcal{F}) = \varinjlim_{\mathcal{F}_0 \subset \mathcal{F}} R^{\mathrm{loc}}_{d,\boldsymbol{k}}(\mathcal{F}_0),$$

where the direct limit is taken over all finite subsets $\mathcal{F}_0$ of $\mathcal{F}$. Note that if the set $\mathcal{F}$ is infinite, then the algebras A_1, A_2, $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ and $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ are not unitary.

From now on we assume that $\mathcal{F}$ is an arbitrary subset of $\boldsymbol{k}^\times$.

Proposition B.5. *There is an isomorphism of $\boldsymbol{k}$-algebras $R^{\mathrm{loc}}_{d,\boldsymbol{k}} \simeq H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ such that*

$$\begin{aligned} e(\boldsymbol{i}) &\mapsto e(\boldsymbol{i}),\\ x_r e(\boldsymbol{i}) &\mapsto (i_r^{-1}X_r - 1)e(\boldsymbol{i}),\\ \Psi_r e(\boldsymbol{i}) &\mapsto \Psi_r e(\boldsymbol{i}). \end{aligned}$$

Proof. The polynomial representations of $H_{d,\boldsymbol{k}}(q)$ and $R_{d,\boldsymbol{k}}$ yield faithful representations of $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ and $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ on A_1 and A_2 respectively. Moreover, there is an isomorphism of $\boldsymbol{k}$-algebras $A_2 \simeq A_1$ given by $x_r e(\boldsymbol{i}) \mapsto (i_r^{-1}X_r - 1)e(\boldsymbol{i})$.

This implies the statement. Indeed, the elements $e(\boldsymbol{i}) \in R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ and $e(\boldsymbol{i}) \in H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ act on $A_2 \simeq A_1$ by the same operators. The elements $x_r e(\boldsymbol{i}) \in R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ and $(i_r^{-1}X_r - 1)e(\boldsymbol{i}) \in H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ act on $A_2 \simeq A_1$ by the same operators. Finally, the elements $\Psi_r e(\boldsymbol{i}) \in R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ and $\Psi_r e(\boldsymbol{i}) \in H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ also act on $A_2 \simeq A_1$ by the same operators. The elements above generate the algebras $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ and $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$. $\square$

Now, we consider the subalgebra $\hat{R}_{d,\boldsymbol{k}}$ of $R^{\mathrm{loc}}_{d,\boldsymbol{k}}$ generated by

- the elements of $R_{d,\boldsymbol{k}}$,
- the elements $(x_r + 1)^{-1}$,
- the elements of the form $(i_r(x_r + 1) - i_t(x_t + 1))^{-1}e(\boldsymbol{i})$ such that $r \neq t$ and $i_r \neq i_t$,
- the elements of the form $(qi_r(x_r + 1) - i_t(x_t + 1))^{-1}e(\boldsymbol{i})$ such that $r \neq t$ and $qi_r \neq i_t$.

Similarly, consider the subalgebra $\hat{H}_{d,\boldsymbol{k}}(q)$ of $H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ generated by

- the elements of $H_{d,\boldsymbol{k}}(q)$,
- the elements of the form $(X_r - X_t)^{-1}e(\boldsymbol{i})$ such that $r \neq t$ and $i_r \neq i_t$,
- the elements of the form $(qX_r - X_t)^{-1}e(\boldsymbol{i})$ such that $r \neq t$ and $qi_r \neq i_t$.

Note that the element $\Psi_r e(\boldsymbol{i}) \in H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ belongs to $\hat{H}_{d,\boldsymbol{k}}(q)$ if $i_r \neq qi_{r+1}$. We have the following proposition, see also [Rouquier 2008, §3.2].

Proposition B.6. *The isomorphism $R^{\mathrm{loc}}_{d,\boldsymbol{k}} \simeq H^{\mathrm{loc}}_{d,\boldsymbol{k}}(q)$ from Proposition B.5 restricts to an isomorphism $\hat{R}_{d,\boldsymbol{k}} \simeq \hat{H}_{d,\boldsymbol{k}}(q)$.* $\square$

B5. ***Deformation rings.*** In this section we introduce some general definitions from [Rouquier et al. 2016] for a later use.

We call the *deformation ring* $(R, \kappa, \kappa_1, \dots, \kappa_l)$ a regular commutative noetherian $\mathbb{C}$-algebra R with 1 equipped with a homomorphism $\mathbb{C}[\kappa^{\pm 1}, \kappa_1, \dots, \kappa_l] \to R$. Let $\kappa, \kappa_1, \dots, \kappa_l$ also denote the images of $\kappa, \kappa_1, \dots, \kappa_l$ in R. A deformation ring is *in general position* if any two elements of the set

$$\{\kappa_u - \kappa_v + a\kappa + b,\ \kappa - c : a, b \in \mathbb{Z},\ c \in \mathbb{Q},\ u \neq v\}$$

have no common nontrivial divisors. A *local deformation ring* is a deformation ring which is a local ring such that $\kappa_1, \dots, \kappa_l, \kappa - e$ belong to the maximal ideal of R. Note that each $\mathbb{C}$-algebra that is a field has a *trivial* local deformation ring structure, i.e., such that $\kappa_1 = \cdots = \kappa_l = 0$ and $\kappa = e$. We always consider $\mathbb{C}$ as a local deformation ring with a trivial deformation ring structure.

We will write $\bar{\kappa} = \kappa(e+1)/e$ and $\bar{\kappa}_r = \kappa_r(e+1)/e$. We will abbreviate R for $(R, \kappa, \kappa_1, \dots, \kappa_l)$ and $\bar{R}$ for $(R, \bar{\kappa}, \bar{\kappa}_1, \dots, \bar{\kappa}_l)$.

Let R be a complete local deformation ring with residue field $\boldsymbol{k}$. Consider the elements $q_e = \exp(2\pi\sqrt{-1}/\kappa)$ and $q_{e+1} = \exp(2\pi\sqrt{-1}/\bar{\kappa})$ in R. These elements specialize to $\zeta_e = \exp(2\pi\sqrt{-1}/e)$ and $\zeta_{e+1} = \exp(2\pi\sqrt{-1}/(e+1))$ in $\boldsymbol{k}$.

B6. ***The choice of $\mathcal{F}$.*** From now on we assume that R is a complete local deformation ring in general position with residue field $\boldsymbol{k}$ and field of fractions K. In this section we define some special choice of the set $\mathcal{F}$. This choice of parameters is particularly interesting because it is related with the categorical action on the category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N$, see [Rouquier et al. 2016].

Fix a tuple $\nu = (\nu_1, \dots, \nu_l) \in \mathbb{Z}^l$. Put $Q_r = \exp(2\pi\sqrt{-1}(\nu_r + \kappa_r)/\kappa)$ for $r \in [1, l]$. The canonical homomorphism $R \to \boldsymbol{k}$ maps q_e to ζ_e and Q_r to $\zeta_e^{\nu_r}$.

Now, consider the subset $\mathscr{F}$ of R given by

$$\mathscr{F} = \bigcup_{r\in\mathbb{Z}, t\in[1,l]} \{q_e^r Q_t\}.$$

Denote by $\mathcal{F}_{\boldsymbol{k}}$ the image of $\mathcal{F}$ in $\boldsymbol{k}$ with respect to the surjection $R \to \boldsymbol{k}$. Recall from Section B4 that we consider $\mathcal{F}$ (and $\mathcal{F}_{\boldsymbol{k}}$) as a vertex set of a quiver. The set $\mathcal{F}$ is a vertex set of a quiver that is a disjoint union if l infinite linear quivers. The set $\mathcal{F}_{\boldsymbol{k}}$ is a vertex set of a cyclic quiver of length e.

Fix $k \in [0, e-1]$. To this k we associate a map $\Upsilon : \mathbb{Z} \to \mathbb{Z}$ as in (8). Now, consider the tuple

$$\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_l) \in \mathbb{Z}^l, \quad \bar{\nu}_r = \Upsilon(\nu_r)\ \forall r \in [1, l].$$

Let $\bar{R}$ be as in the previous section. Let $\bar{\boldsymbol{k}}$ and $\bar{K}$ be the residue field and the field of fractions of $\bar{R}$ respectively. Now, consider $\bar{Q} = (\bar{Q}_1, \dots, \bar{Q}_l)$, where $\bar{Q}_r = \exp(2\pi\sqrt{-1}(\bar{\nu}_r + \bar{\kappa}_r)/\bar{\kappa})$ and $\bar{\kappa}$ and $\bar{\kappa}_r$ are defined in Section B5. Consider the subset $\bar{\mathscr{F}}$ of $\bar{R}$ given by

$$\bar{\mathscr{F}} = \bigcup_{r\in\mathbb{Z}, t\in[1,l]} \{q_{e+1}^r \bar{Q}_t\}.$$

Denote by $\bar{\mathcal{F}}_{\bar{\boldsymbol{k}}}$ the image of $\bar{\mathcal{F}}$ in $\bar{\boldsymbol{k}}$ with respect to the surjection $\bar{R} \to \bar{\boldsymbol{k}}$. The set $\bar{\mathcal{F}}$ is a vertex set of a quiver that is a disjoint union of l infinite linear quivers. The set $\bar{\mathcal{F}}_{\bar{\boldsymbol{k}}}$ is a vertex set of a cyclic quiver of length $e+1$.

B7. ***Algebras $\hat{H}$, $\widehat{SH}$, $\hat{R}$ and $\hat{S}$.*** Let $\Gamma = (I, H)$, $\bar{\Gamma} = (\bar{I}, \bar{H})$ and $\tilde{\Gamma} = (\tilde{I}, \tilde{H})$ be as in Section B2.

We will use the notation $\mathcal{F}$, $\mathcal{F}_{\boldsymbol{k}}$, $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}_{\bar{\boldsymbol{k}}}$ as in previous section. (In particular, we fix some $\nu = (\nu_1, \dots, \nu_l)$.)

We have the following isomorphisms of quivers

$$
\begin{aligned}
\tilde{I} &\simeq \mathscr{F}, & i=(a,b) &\mapsto p_i := \exp(2\pi\sqrt{-1}(a+\kappa_b)/\kappa),\\
\tilde{I} &\simeq \overline{\mathscr{F}}, & i=(a,b) &\mapsto \bar{p}_i := \exp(2\pi\sqrt{-1}(a+\bar{\kappa}_b)/\bar{\kappa}),\\
I &\simeq \mathscr{F}_{\boldsymbol{k}}, & i &\mapsto p_i := \zeta_e^i,\\
\bar{I} &\simeq \overline{\mathscr{F}}_{\bar{\boldsymbol{k}}}, & i &\mapsto \bar{p}_i := \zeta_{e+1}^i.
\end{aligned}
$$

These isomorphisms yield the following commutative diagrams

$$
\begin{array}{ccc}
\tilde{I} & \xrightarrow{\sim} & \mathscr{F}\\
{\scriptstyle \pi_e}\downarrow & & \downarrow\\
I & \xrightarrow{\sim} & \mathscr{F}_{\boldsymbol{k}},
\end{array}
\qquad\qquad
\begin{array}{ccc}
\tilde{I} & \xrightarrow{\sim} & \overline{\mathscr{F}}\\
{\scriptstyle \pi_{e+1}}\downarrow & & \downarrow\\
\bar{I} & \xrightarrow{\sim} & \overline{\mathscr{F}}_{\bar{\boldsymbol{k}}}.
\end{array}
$$

We will identify

$$
I \simeq \mathscr{F}_{\boldsymbol{k}}, \quad \bar{I} \simeq \overline{\mathscr{F}}_{\bar{\boldsymbol{k}}}, \quad \tilde{I} \simeq \mathscr{F}, \quad \tilde{I} \simeq \overline{\mathscr{F}}
$$

as above.

Our goal is to obtain an analogue of Theorem 2.12 over the ring R. First, consider the algebras $\hat{H}_{d,\boldsymbol{k}}(\zeta_e)$ and $\hat{H}_{d,K}(q_e)$ defined in the same way as in Section B4 with respect to the sets $\mathscr{F}_{\boldsymbol{k}} \subset \boldsymbol{k}$ and $\mathscr{F} \subset K$. We can consider the R-algebra $\hat{H}_{d,R}(q_e)$ defined in a similar way with respect to the same set of idempotents as $\hat{H}_{d,\boldsymbol{k}}(\zeta_e)$ (i.e., with respect to the set $\mathcal{F}_{\boldsymbol{k}}$, not $\mathcal{F}$).

The algebra $\hat{H}_{d,K}(q_e)$ is not unitary because the quiver $\tilde{\Gamma}$ is infinite. To avoid this problem we consider the truncated version of this algebra. Let $\hat{H}^{\leqslant N}_{d,K}(q_e)$ be the quotient of $\hat{H}_{d,K}(q_e)$ by the two-sided ideal generated by the idempotents $e(\boldsymbol{j}) \in \tilde{I}^d$ such that $\boldsymbol{j}$ contains a component that is not a vertex of the truncated quiver $\tilde{\Gamma}^{\leqslant N}$ (see Section B2). (In fact, the algebra $\hat{H}^{\leqslant N}_{d,K}(q_e)$ is isomorphic to a direct summand of $\hat{H}_{d,K}(q_e)$).

Similarly, we define the algebras $\hat{H}_{d,\bar{\boldsymbol{k}}}(\zeta_{e+1})$, $\hat{H}_{d,\overline{K}}(q_{e+1})$ and $\hat{H}_{d,\overline{R}}(q_{e+1})$ using the sets $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}_{\bar{\boldsymbol{k}}}$ instead of $\mathcal{F}$ and $\mathcal{F}_{\boldsymbol{k}}$. We define a truncation $\hat{H}^{\leqslant N}_{d,\overline{K}}(q_{e+1})$ of $\hat{H}_{d,\overline{K}}(q_{e+1})$ using the quiver $\overline{\tilde{\Gamma}}^{\leqslant N}$.

For each $\boldsymbol{i} \in I^d$ we consider the following idempotent in $\hat{H}^{\leqslant N}_{d,K}(q_e)$:

$$
e(\boldsymbol{i}) = \sum_{\boldsymbol{j}\in\tilde{I}^d,\, \pi_e(\boldsymbol{j})=\boldsymbol{i}} e(\boldsymbol{j}).
$$

Here we mean that $e(\boldsymbol{j})$ is zero if $\boldsymbol{j}$ contains a vertex that is not in the truncated quiver $\tilde{\Gamma}^{\leqslant N}$. The idempotent $e(\boldsymbol{i})$ is well defined because only a finite number of terms in the sum are nonzero. For each $\boldsymbol{i} \in \bar{I}^d$ we can define an idempotent $e(\boldsymbol{i}) \in \hat{H}^{\leqslant N}_{d,\overline{K}}(q_{e+1})$ in a similar way.

Lemma B.7. *There is an injective algebra homomorphism $\hat{H}_{d,R}(q_e) \to \hat{H}^{\leqslant N}_{d,K}(q_e)$ such that $e(\boldsymbol{i}) \mapsto e(\boldsymbol{i})$, $X_r e(\boldsymbol{i}) \mapsto X_r e(\boldsymbol{i})$ and $T_r e(\boldsymbol{i}) \mapsto T_r e(\boldsymbol{i})$.*

Proof. It is clear that we have an algebra homomorphism $\hat{H}_{d,R}(q_e) \to \hat{H}^{\leqslant N}_{d,K}(q_e)$ as in the statement. We only have to check the injectivity.

For each $w \in \mathfrak{S}_d$ we have an element $T_w \in H_{d,R}(q)$ defined in the following way. We have $T_w = T_{i_1} \cdots T_{i_r}$, where $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression. It is well-known that T_w is independent of the choice of the reduced expression. Moreover, the algebra $H_{d,R}(q)$ is free over $R[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ with a basis $\{T_w : w \in \mathfrak{S}_d\}$.

Set

$$B = \bigoplus_{\boldsymbol{i} \in \mathcal{F}_{\boldsymbol{k}}^d} R[X_1^{\pm 1}, \dots, X_d^{\pm 1}][(X_r - X_t)^{-1}, (q_e X_r - X_t)^{-1} : r \neq t] e(\boldsymbol{i}),$$

where we invert $(X_r - X_t)$ only if $i_r \neq i_t$ and we invert $(q_e X_r - X_t)$ only if $\zeta_e i_r \neq i_t$. We have $\hat{H}_{d,R}(q_e) = B \otimes_{R[X_1^{\pm 1}, \dots, X_d^{\pm 1}]} H_{d,R}(q_e)$. This implies that the B-module $\hat{H}_{d,R}(q_e)$ is free with a basis $\{T_w : w \in \mathfrak{S}_d\}$.

Similarly, we can show that the algebra $\hat{H}_{d,K}^{\leqslant N}(q_e)$ is free (with a basis $\{T_w : w \in \mathfrak{S}_d\}$) over

$$B' = \bigoplus_{\boldsymbol{j} \in \mathcal{F}^d} K[X_1^{\pm 1}, \dots, X_d^{\pm 1}][(X_r - X_t)^{-1}, (q_e X_r - X_t)^{-1} : r \neq t] e(\boldsymbol{j}),$$

where we invert $(X_r - X_t)$ only if $j_r \neq j_t$ and we invert $(q_e X_r - X_t)$ only if $q_e j_r \neq j_t$, and we take only $\boldsymbol{j}$ that are supported on the vertices of the truncated quiver $\Gamma^{\leqslant N}$.

Now, the injectivity of the homomorphism follows from the fact that it takes a B-basis of $\hat{H}_{d,R}(q_e)$ to a B'-linearly independent set in $\hat{H}_{d,K}^{\leqslant N}(q_e)$. □

Now we define the algebra $\widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$ that is a Hecke analogue of a localization of the balanced KLR algebra $S_{\bar{\alpha},\boldsymbol{k}}$. To do so, consider the idempotent $\boldsymbol{e} = \sum_{\boldsymbol{i} \in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} e(\boldsymbol{i})$ in $\hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$. We set

$$\widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1}) = \boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})\boldsymbol{e} / \sum_{\boldsymbol{j} \in \bar{I}^{\bar{\alpha}}_{\mathrm{un}}} \boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1}) e(\boldsymbol{j}) \hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})\boldsymbol{e}.$$

Now, we define a similar algebra over K. To do this, we need to introduce some additional notation. Denote by $Q^+_{\tilde{I},\mathrm{eq}}$ the subset of $Q^+_{\tilde{I}}$ that contains only $\tilde{\alpha}$ such that for each $k \in \tilde{I}_1$, the dimension vector $\tilde{\alpha}$ has the same dimensions at vertices k^1 and k^2.

Set

$$\hat{H}^{\leqslant N}_{\bar{\alpha},\overline{K}}(q_{e+1}) = \bigoplus_{\pi_{e+1}(\tilde{\alpha}) = \bar{\alpha}} \hat{H}_{\tilde{\alpha},\overline{K}}(q_{e+1}), \quad \text{and} \quad \widehat{SH}^{\leqslant N}_{\bar{\alpha},\overline{K}}(q_{e+1}) = \bigoplus_{\pi_{e+1}(\tilde{\alpha}) = \bar{\alpha}} \widehat{SH}_{\tilde{\alpha},\overline{K}}(q_{e+1}),$$

where in the sums we take only $\tilde{\alpha} \in Q^+_{\tilde{I},\mathrm{eq}}$ that are supported on the vertices of the truncated quiver $\bar{\tilde{\Gamma}}^{\leqslant N}$ and $\widehat{SH}_{\tilde{\alpha},\overline{K}}(q_{e+1})$ is defined similarly to $\widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$. More precisely, we have

$$\widehat{SH}_{\tilde{\alpha},\overline{K}}(q_{e+1}) = \tilde{\boldsymbol{e}}_{\tilde{\alpha}} H_{\tilde{\alpha},\overline{K}}(q_{e+1}) \tilde{\boldsymbol{e}}_{\tilde{\alpha}} / \sum_{\boldsymbol{j} \in \tilde{I}^{\tilde{\alpha}}_{\mathrm{un}}} \tilde{\boldsymbol{e}}_{\tilde{\alpha}} H_{\tilde{\alpha},\overline{K}}(q_{e+1}) e(\boldsymbol{j}) H_{\tilde{\alpha},\overline{K}}(q_{e+1}) \tilde{\boldsymbol{e}}_{\tilde{\alpha}},$$

where $\tilde{\boldsymbol{e}}_{\tilde{\alpha}} = \sum_{\boldsymbol{j} \in \tilde{I}^{\tilde{\alpha}}_{\mathrm{ord}}} e(\boldsymbol{j})$.

Remark B.8. Consider the following idempotents in $\hat{H}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$:

$$\tilde{\boldsymbol{e}} = \sum_{\pi_{e+1}(\tilde{\alpha})=\bar{\alpha}} \tilde{\boldsymbol{e}}_{\tilde{\alpha}} \quad \text{and} \quad \boldsymbol{e} = \sum_{\boldsymbol{i}\in \bar{I}^{\bar{\alpha}}_{\mathrm{ord}}} e(\boldsymbol{i}),$$

where the first sum is taken only by $\tilde{\alpha} \in Q^+_{\tilde{I},\mathrm{eq}}$. (Note that $\hat{H}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ was defined as a quotient of $\hat{H}_{\bar{\alpha},\bar{K}}(q_{e+1})$. So, if $\tilde{\alpha}$ is not supported on $\bar{\tilde{\Gamma}}^{\leqslant N}$, then the idempotent $\tilde{\boldsymbol{e}}_{\tilde{\alpha}}$ is zero by definition. In particular, the sum has a finite number of nonzero terms.) Set also $\tilde{I}^{\bar{\alpha}} = \coprod_{\pi_{e+1}(\tilde{\alpha})=\bar{\alpha}} \tilde{I}^{\tilde{\alpha}}$, where the sum is taken only by $\tilde{\alpha} \in Q^+_{\tilde{I},\mathrm{eq}}$. By definition, the algebra $\widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ is a quotient of $\tilde{\boldsymbol{e}}\hat{H}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})\tilde{\boldsymbol{e}}$. But we can see this algebra as the same quotient of $\boldsymbol{e}\hat{H}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})\boldsymbol{e}$ (we do the quotient with respect to the same idempotents). Indeed, the idempotent $\boldsymbol{e}$ is a sum of a bigger number of standard idempotents $e(\boldsymbol{j})$, $\boldsymbol{j} \in \tilde{I}^{\bar{\alpha}}$ than the idempotent $\tilde{\boldsymbol{e}}$. More precisely, the idempotent $\tilde{\boldsymbol{e}}$ is the sum all $e(\boldsymbol{j})$ such that $\boldsymbol{j}$ is well-ordered while $\boldsymbol{e}$ is the sum of all $e(\boldsymbol{j})$ such that $\pi_{e+1}(\boldsymbol{j})$ is well-ordered. But each $\boldsymbol{j} \in \tilde{I}^{\bar{\alpha}}$ such that $\pi_{e+1}(\boldsymbol{j})$ is well-ordered and $\boldsymbol{j}$ is not well-ordered must be unordered. Then such $e(\boldsymbol{j})$ becomes zero after taking the quotient.

Finally, we define the R-algebra $\widehat{SH}^{N}_{\bar{\alpha},\bar{R}}(q_{e+1})$ as the image in $\widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ of the following composition of homomorphisms

$$\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e} \to \boldsymbol{e}\hat{H}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})\boldsymbol{e} \to \widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1}).$$

The lemma below shows that the algebra $\widehat{SH}^{N}_{\bar{\alpha},\bar{R}}(q_{e+1})$ is independent of N for N large enough. So, we can write simply $\widehat{SH}_{\bar{\alpha},\bar{R}}(q_{e+1})$ instead of $\widehat{SH}^{N}_{\bar{\alpha},\bar{R}}(q_{e+1})$ for N large enough.

Lemma B.9. *Assume $N \geqslant 2d$. Then the algebra $\widehat{SH}^{N}_{\bar{\alpha},\bar{R}}(q_{e+1})$ is independent of N.*

Proof. Denote by J_N the kernel of $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e} \to \widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$. Take $M > N$. It is clear that we have $J_M \subset J_N$.

Let us show that we also have an opposite inclusion if $N \geqslant 2d$. We want to show that each element $x \in J_N$ is also in J_M. It is enough to show this for x of the form $x = Xe(\boldsymbol{i})$, where $\boldsymbol{i} \in I^{\bar{\alpha}}_{\mathrm{ord}}$ and X is composed of the elements of the form T_r and X_r. Then $Xe(\boldsymbol{i}) \in J_N$ means that the element $Xe(\boldsymbol{j}) \in \widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ is zero for each $\boldsymbol{j} \in \tilde{I}^{\bar{\alpha}}$ supported on $\bar{\tilde{\Gamma}}^{\leqslant N}$ such that $\pi_{e+1}(\boldsymbol{j}) = \boldsymbol{i}$. To show that we have $Xe(\boldsymbol{i}) \in J_M$ we must check that the element $Xe(\boldsymbol{j}) \in \widehat{SH}^{\leqslant M}_{\bar{\alpha},\bar{K}}(q_{e+1})$ is zero for each $\boldsymbol{j} \in \tilde{I}^{\bar{\alpha}}$ supported on $\bar{\tilde{\Gamma}}^{\leqslant M}$ such that $\pi_{e+1}(\boldsymbol{j}) = \boldsymbol{i}$.

Let $\tilde{\alpha} \in Q^+_{\tilde{I},\mathrm{eq}}$ be such that $\boldsymbol{j} \in \tilde{I}^{\tilde{\alpha}}$. It is clear that we can find an $\tilde{\alpha}' \in Q^+_{\tilde{I},\mathrm{eq}}$ supported on $\bar{\tilde{\Gamma}}^{\leqslant 2d}$ such that we have an isomorphism $\hat{H}_{\tilde{\alpha},\bar{K}}(q_{e+1}) \simeq \hat{H}_{\tilde{\alpha},\bar{K}}(q_{e+1})$ that induces an isomorphism $\widehat{SH}_{\tilde{\alpha},\bar{K}}(q_{e+1}) \simeq \widehat{SH}_{\tilde{\alpha},\bar{K}}(q_{e+1})$ and such that this isomorphism preserves the generators X_r and T_r and sends the idempotent $e(\boldsymbol{j})$ to some idempotent $e(\boldsymbol{j}')$ such that $\boldsymbol{j}'$ is supported on $\bar{\tilde{\Gamma}}^{\leqslant 2d}$ and $\pi_{e+1}(\boldsymbol{j}) = \pi_{e+1}(\boldsymbol{j}')$. Then the element $Xe(\boldsymbol{j}) \in \widehat{SH}^{\leqslant M}_{\bar{\alpha},\bar{K}}(q_{e+1})$ is zero because $Xe(\boldsymbol{j}') \in \widehat{SH}^{\leqslant M}_{\bar{\alpha},\bar{K}}(q_{e+1})$ is zero. This implies $x \in J_M$. $\square$

Now we define the KLR versions of the algebras $\widehat{SH}_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1})$ and $\widehat{SH}^{\leqslant N}_{\bar\alpha,\bar K}(q_{e+1})$. As for the Hecke version, we denote by $\boldsymbol{e}$ the idempotent $\sum_{\boldsymbol{i}\in\bar I^{\bar\alpha}_{\mathrm{ord}}} e(\boldsymbol{i})$ in $\hat R_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)$. Set

$$\hat S_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)=\boldsymbol{e}\hat R_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)\boldsymbol{e}/\sum_{\boldsymbol{i}\in\bar I^{\bar\alpha}_{\mathrm{un}}}\boldsymbol{e}\hat R_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)e(\boldsymbol{i})R_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)\boldsymbol{e}.$$

For each $\tilde\alpha\in Q^+_{\tilde I,\mathrm{eq}}$ we consider the idempotent $\tilde{\boldsymbol{e}}_{\tilde\alpha}=\sum_{\boldsymbol{j}\in\tilde I^{\tilde\alpha}_{\mathrm{ord}}} e(\boldsymbol{j})$ in $\hat R_{\tilde\alpha,K}(\overline{\tilde\Gamma})$. Set

$$\hat S_{\bar\alpha,K}(\overline{\tilde\Gamma}^{\leqslant N})=\bigoplus_{\pi_{e+1}(\tilde\alpha)=\bar\alpha}\hat S_{\tilde\alpha,K}(\overline{\tilde\Gamma}),$$

where we take only $\tilde\alpha\in Q^+_{\tilde I,\mathrm{eq}}$ that are supported on the vertices of the truncated quiver $\overline{\tilde\Gamma}^{\leqslant N}$ and

$$\hat S_{\tilde\alpha,K}(\overline{\tilde\Gamma})=\tilde{\boldsymbol{e}}_{\tilde\alpha}\hat R_{\tilde\alpha,K}(\overline{\tilde\Gamma})\tilde{\boldsymbol{e}}_{\tilde\alpha}/\sum_{\boldsymbol{j}\in\tilde I^{\tilde\alpha}_{\mathrm{un}}}\tilde{\boldsymbol{e}}_{\tilde\alpha}\hat R_{\tilde\alpha,K}(\overline{\tilde\Gamma})e(\boldsymbol{j})R_{\tilde\alpha,K}(\overline{\tilde\Gamma})\tilde{\boldsymbol{e}}_{\tilde\alpha}.$$

Remark B.10. By Proposition B.6 we have algebra isomorphisms

$$\begin{aligned}\hat R_{\alpha,\boldsymbol{k}}(\Gamma)&\simeq\hat H_{\alpha,\boldsymbol{k}}(\zeta_e), & \hat R_{\alpha,K}(\tilde\Gamma^{\leqslant N})&\simeq\hat H^{\leqslant N}_{\alpha,K}(q_e),\\ \hat R_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)&\simeq\hat H_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1}), & \hat R_{\bar\alpha,K}(\overline{\tilde\Gamma}^{\leqslant N})&\simeq\hat H^{\leqslant N}_{\bar\alpha,\bar K}(q_{e+1}),\end{aligned}$$

from which we deduce the isomorphisms

$$\hat S_{\bar\alpha,\boldsymbol{k}}(\overline\Gamma)\simeq\widehat{SH}_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1})\quad\text{and}\quad\hat S_{\bar\alpha,K}(\overline{\tilde\Gamma}^{\leqslant N})\simeq\widehat{SH}^{\leqslant N}_{\bar\alpha,\bar K}(q_{e+1}).$$

We may use these isomorphisms without mentioning them explicitly. Using the identifications above between KLR algebras and Hecke algebras, a localization of the isomorphism in Theorem 2.12 yields an isomorphism

$$\Phi_{\alpha,\boldsymbol{k}}\colon\hat H_{\alpha,\boldsymbol{k}}(\zeta_e)\to\widehat{SH}_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1}).$$

In the same way we also obtain an algebra isomorphism

$$\Phi_{\tilde\alpha,K}\colon\hat H_{\tilde\alpha,K}(q_e)\to\widehat{SH}_{\tilde\phi(\tilde\alpha),\bar K}(q_{e+1})$$

for each $\tilde\alpha\in Q^+_{\tilde I}$. Taking the sum over all $\tilde\alpha\in Q^+_{\tilde I}$ such that $\pi_e(\tilde\alpha)=\alpha$ and such that $\tilde\alpha$ is supported on the vertices of the truncated quiver $\tilde\Gamma^{\leqslant N}$ yields an isomorphism

$$\Phi_{\alpha,K}\colon\hat H^{\leqslant N}_{\alpha,K}(q_e)\to\widehat{SH}^{\leqslant N}_{\bar\alpha,\bar K}(q_{e+1}).$$

Lemma B.11. *The homomorphism $\boldsymbol{e}\hat H_{\bar\alpha,\bar R}(q_{e+1})\boldsymbol{e}\to\boldsymbol{e}\hat H_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1})\boldsymbol{e}$ factors through a homomorphism $\widehat{SH}_{\bar\alpha,\bar R}(q_{e+1})\to\widehat{SH}_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1})$.*

Proof. In Section 2E we constructed a faithful polynomial representation of $S_{\bar\alpha,\boldsymbol{k}}$. Let us call it $\mathcal{P}ol_{\boldsymbol{k}}$. It is constructed as a quotient of the standard polynomial representation of $\boldsymbol{e}R_{\bar\alpha,\boldsymbol{k}}\boldsymbol{e}$. After localization we get a faithful representation $\widehat{\mathcal{P}ol}_{\boldsymbol{k}}$ of $\hat S_{\bar\alpha,\boldsymbol{k}}$. Thus the kernel of the algebra homomorphism $\boldsymbol{e}\hat R_{\bar\alpha,\boldsymbol{k}}\boldsymbol{e}\to\hat S_{\bar\alpha,\boldsymbol{k}}$ is the annihilator of the representation $\widehat{\mathcal{P}ol}_{\boldsymbol{k}}$. We can transfer this to the Hecke side (because the isomorphism in Proposition B.6 comes from the identification of the polynomial representations) and we obtain that the kernel of the algebra homomorphism $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})\boldsymbol{e} \to \widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$ is the annihilator of the representation $\widehat{\mathcal{P}ol}_{\boldsymbol{k}}$. Similarly, we can characterize the kernel of the algebra homomorphism $\boldsymbol{e}\hat{H}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})\boldsymbol{e} \to \widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ as the annihilator of a similar representation $\widehat{\mathcal{P}ol}^{\leqslant N}_{K}$.

The K-vector space $\widehat{\mathcal{P}ol}^{\leqslant N}_{K}$ has an R-submodule $\widehat{\mathcal{P}ol}_{R}$ stable by the action of $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e}$ such that $\boldsymbol{k}\otimes_R \widehat{\mathcal{P}ol}_{R} = \widehat{\mathcal{P}ol}_{\boldsymbol{k}}$ and it is compatible with the algebra homomorphism $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e} \to \boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})\boldsymbol{e}$. By definition of $\widehat{SH}_{\bar{\alpha},\bar{R}}(q_{e+1})$ and the discussion above, the kernel of the algebra homomorphism $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e} \to \widehat{SH}_{\bar{\alpha},\bar{R}}(q_{e+1})$ is formed by the elements that act by zero on $\widehat{\mathcal{P}ol}^{\leqslant N}_{K}$ (we assume that N is big enough). Thus each element of this kernel acts by zero on $\widehat{\mathcal{P}ol}_{R}$. This implies, that an element of the kernel of $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e} \to \widehat{SH}_{\bar{\alpha},\bar{R}}(q_{e+1})$ specializes to an element of the kernel of $\boldsymbol{e}\hat{H}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})\boldsymbol{e} \to \widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$. This proves the statement. ◻

B8. ***The deformation of the isomorphism* Φ.**

Proposition B.12. *There is a unique algebra homomorphism* $\Phi_{\alpha,R}\colon \hat{H}_{\alpha,R}(q_e) \to \widehat{SH}_{\bar{\alpha},R}(q_{e+1})$ *such that the following diagram is commutative*:

$$\begin{array}{ccc}
\hat{H}_{\alpha,\boldsymbol{k}}(\zeta_e) & \xrightarrow{\Phi_{\alpha,\boldsymbol{k}}} & \widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1}) \\
\uparrow & & \uparrow \\
\hat{H}_{\alpha,R}(q_e) & \xrightarrow{\Phi_{\alpha,R}} & \widehat{SH}_{\bar{\alpha},\bar{R}}(q_{e+1}) \\
\downarrow & & \downarrow \\
\hat{H}^{\leqslant N}_{\alpha,K}(q_e) & \xrightarrow{\Phi_{\alpha,K}} & \widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1}).
\end{array}$$

Proof. First we consider the algebras $H^{\mathrm{loc}}_{\alpha,\boldsymbol{k}}(\zeta_e)$, $H^{\mathrm{loc}}_{\alpha,R}(q_e)$ and $H^{\mathrm{loc},\leqslant N}_{\alpha,K}(q_e)$ obtained from $\hat{H}_{\alpha,\boldsymbol{k}}(\zeta_e)$, $\hat{H}_{\alpha,R}(q_e)$ and $\hat{H}^{\leqslant N}_{\alpha,K}(q_e)$ by inverting

- $(X_r - X_t)$ and $(\zeta_e X_r - X_t)$ with $r \neq t$,
- $(X_r - X_t)$ and $(q_e X_r - X_t)$ with $r \neq t$,
- $(X_r - X_t)$ and $(q_e X_r - X_t)$ with $r \neq t$

respectively. Let $SH^{\mathrm{loc}}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$ and $SH^{\mathrm{loc},\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ be the localizations of $\widehat{SH}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1})$ and $\widehat{SH}^{\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ such that the isomorphisms $\Phi_{\alpha,\boldsymbol{k}}$ and $\Phi_{\alpha,K}$ above induce isomorphisms

$$\Phi_{\alpha,\boldsymbol{k}}\colon H^{\mathrm{loc}}_{\alpha,\boldsymbol{k}}(\zeta_e) \to SH^{\mathrm{loc}}_{\bar{\alpha},\bar{\boldsymbol{k}}}(\zeta_{e+1}) \quad \text{and} \quad \Phi_{\alpha,K}\colon H^{\mathrm{loc},\leqslant N}_{\alpha,K}(q_e) \to SH^{\mathrm{loc},\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1}).$$

Let $SH^{\mathrm{loc}}_{\bar{\alpha},\bar{R}}(q_{e+1})$ be the image in $SH^{\mathrm{loc},\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})$ of the following composition of homomorphisms

$$\boldsymbol{e}H^{\mathrm{loc}}_{\bar{\alpha},\bar{R}}(q_{e+1})\boldsymbol{e} \to \boldsymbol{e}H^{\mathrm{loc},\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1})\boldsymbol{e} \to SH^{\mathrm{loc},\leqslant N}_{\bar{\alpha},\bar{K}}(q_{e+1}).$$

(We assume $N \geqslant 2d$. Then, similarly to Lemma B.9, the algebra $SH^{\mathrm{loc}}_{\bar{\alpha},\bar{R}}$ is independent of N under this assumption.)

Next, we want to prove that there exists an algebra homomorphism $\Phi_{\alpha,R}\colon H^{\mathrm{loc}}_{\alpha,R}(q_e) \to SH^{\mathrm{loc}}_{\bar\alpha,\bar R}(q_{e+1})$ such that the following diagram is commutative:

$$\begin{array}{ccc}
H^{\mathrm{loc}}_{\alpha,\boldsymbol{k}}(\zeta_e) & \xrightarrow{\Phi_{\alpha,\boldsymbol{k}}} & SH^{\mathrm{loc}}_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1}) \\
\uparrow & & \uparrow \\
H^{\mathrm{loc}}_{\alpha,R}(q_e) & \xrightarrow{\Phi_{\alpha,R}} & SH^{\mathrm{loc}}_{\bar\alpha,\bar R}(q_{e+1}) \\
\downarrow & & \downarrow \\
H^{\mathrm{loc},\leqslant N}_{\alpha,K}(q_e) & \xrightarrow{\Phi_{\alpha,K}} & SH^{\mathrm{loc},\leqslant N}_{\bar\alpha,\bar K}(q_{e+1}).
\end{array} \tag{9}$$

We just need to check that the map $\Phi_{\alpha,K}$ takes an element of $H^{\mathrm{loc}}_{\alpha,R}(q_e)$ to an element of $SH^{\mathrm{loc}}_{\bar\alpha,\bar R}(q_{e+1})$ and that it specializes to the map $\Phi_{\alpha,\boldsymbol{k}}\colon H^{\mathrm{loc}}_{\alpha,\boldsymbol{k}}(\zeta_e) \to SH^{\mathrm{loc}}_{\bar\alpha,\bar{\boldsymbol{k}}}(\zeta_{e+1})$. We will check this on the generators $e(\boldsymbol{i})$, $X_r e(\boldsymbol{i})$ and $\Psi_r e(\boldsymbol{i})$ of $H^{\mathrm{loc}}_{\alpha,R}(q_e)$.

This is obvious for the idempotents $e(\boldsymbol{i})$.

Let us check this for $X_r e(\boldsymbol{i})$. Assume that $\boldsymbol{i} \in I^\alpha$ and $\boldsymbol{j} \in \tilde I^{|\alpha|}$ are such that we have $\pi_e(\boldsymbol{j}) = \boldsymbol{i}$. Write $\boldsymbol{i}' = \phi(\boldsymbol{i})$ and $\boldsymbol{j}' = \tilde\phi(\boldsymbol{j})$. Set $r' = r'_{\boldsymbol{j}} = r'_{\boldsymbol{i}}$, see the notation in Section 2F. By Theorem 2.12 and Proposition B.5 we have

$$\Phi_{\alpha,K}(X_r e(\boldsymbol{j})) = \bar p^{-1}_{j'_{r'}} p_{j_r} X_{r'} e(\boldsymbol{j}').$$

Since, $\bar p^{-1}_{j'_{r'}} p_{j_r}$ depends only on $\boldsymbol{i}$ and r and $e(\boldsymbol{i}) = \sum_{\pi_e(\boldsymbol{j})=\boldsymbol{i}} e(\boldsymbol{j})$, we deduce that

$$\Phi_{\alpha,K}(X_r e(\boldsymbol{i})) = \bar p^{-1}_{j'_{r'}} p_{j_r} X_{r'} e(\boldsymbol{i}').$$

Thus the element $\Phi_{\alpha,K}(X_r e(\boldsymbol{i}))$ is in $SH^{\mathrm{loc}}_{\bar\alpha,\bar R}$ and its image in $SH^{\mathrm{loc}}_{\bar\alpha,\bar{\boldsymbol{k}}}$ is $\bar p^{-1}_{i'_{r'}} p_{i_r} X_{r'} e(\boldsymbol{i}') = \Phi_{\alpha,\boldsymbol{k}}(X_r e(\boldsymbol{i}))$.

Next, we consider the generators $\Psi_r e(\boldsymbol{i})$. We must prove that for each $\boldsymbol{j}$ such that $\pi_e(\boldsymbol{j}) = \boldsymbol{i}$ and for each r we have

- $\Phi_{\alpha,K}(\Psi_r e(\boldsymbol{j})) = \Xi e(\boldsymbol{j}')$, for some element $\Xi \in H^{\mathrm{loc}}_{\alpha,R}(q_e)$ that depends only on r and $\boldsymbol{i}$,
- the image of $\Xi e(\boldsymbol{i}')$ in $SH^{\mathrm{loc}}_{\bar\alpha,\bar{\boldsymbol{k}}}(q_{e+1})$ under the specialization $R \to \boldsymbol{k}$ is $\Phi_{\alpha,\boldsymbol{k}}(\Psi_r e(\boldsymbol{i}))$.

This follows from Lemma B.1.

Now we obtain the diagram from the claim of Proposition B.12 as the restriction of the diagram (9). □

B9. ***Alternative definition of a categorical representation.*** There is an alternative definition of a categorical representation, where the KLR algebra is replaced by the affine Hecke algebra.

Let R be a $\mathbb{C}$-algebra. Fix an invertible element $q \in R$, $q \neq 1$. Let $\mathcal{C}$ be an R-linear exact category.

Definition B.13. A *representation datum* in $\mathcal{C}$ is a tuple (E, F, X, T) where (E, F) is a pair of exact biadjoint functors $\mathcal{C} \to \mathcal{C}$ and $X \in \mathrm{End}(F)^{\mathrm{op}}$ and $T \in \mathrm{End}(F^2)^{\mathrm{op}}$ are endomorphisms of functors such that for each $d \in \mathbb{N}$, there is an R-algebra homomorphism $\psi_d\colon H_{d,R}(q) \to \mathrm{End}(F^d)^{\mathrm{op}}$ given by

$$\begin{aligned} X_r &\mapsto F^{d-r} X F^{r-1} && \forall r \in [1, d], \\ T_r &\mapsto F^{d-r-1} T F^{r-1} && \forall r \in [1, d-1]. \end{aligned}$$

Now, assume that $R = \boldsymbol{k}$ is a field. Assume that $\mathcal{C}$ is a Hom-finite $\boldsymbol{k}$-linear abelian category. Let $\mathscr{F}$ be a subset of $\boldsymbol{k}^\times$ (possibly infinite). As in Section B4, we view $\mathscr{F}$ as the vertex set of a quiver with an arrow $i \to j$ if and only if $j = qi$.

Definition B.14. A $\mathfrak{g}_{\mathscr{F}}$-categorical representation in $\mathcal{C}$ is the datum of a representation datum (E, F, X, T) and a decomposition $\mathcal{C} = \bigoplus_{\mu \in X_{\mathscr{F}}} \mathcal{C}_\mu$ satisfying the conditions (a) and (b) below. For $i \in \mathscr{F}$, let E_i and F_i be endofunctors of $\mathcal{C}$ such that for each $M \in \mathcal{C}$ the objects $E_i(M)$ and $F_i(M)$ are the generalized i-eigenspaces of X acting on $E(M)$ and $F(M)$ respectively, see also Remark 3.3 (a). We assume

(a) $F = \bigoplus_{i \in \mathscr{F}} F_i$ and $E = \bigoplus_{i \in \mathscr{F}} E_i$,

(b) $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$ and $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$.

If the set $\mathscr{F}$ is infinite, condition (a) should be understood in the same way as in Remark 3.3 (b).

Remark B.15. (a) By definition, for each object $M \in \mathcal{C}$ and each $d \in \mathbb{Z}_{\geqslant 0}$, we have $F_{i_d} \cdots F_{i_1}(M) \neq 0$ only for a finite number of sequences $(i_1, \dots, i_d) \in \mathcal{F}^d$. (Else, the endomorphism algebra of $F^d(M)$ is infinite-dimensional.) Then the homomorphism $H_{d,\boldsymbol{k}}(q) \to \mathrm{End}(F^d(M))^{\mathrm{op}}$ extends to a homomorphism $\hat{H}_{d,\boldsymbol{k}}(q) \to \mathrm{End}(F^d(M))^{\mathrm{op}}$ such that only a finite number of idempotents $e(\boldsymbol{j})$ has a nonzero image. (We define the action of $e(\boldsymbol{i})$ as the projection from F^d to $F_{i_d} \cdots F_{i_1}$. Note that the action of $(X_r - X_t)^{-1} e(\boldsymbol{i})$ such that $i_r \neq i_t$ is well defined because X_r and X_t have different eigenvalues. Similarly, the action of $(qX_r - X_t)^{-1} e(\boldsymbol{i})$ such that $r \neq t$ and $qi_r \neq i_t$ is well defined.) In particular, we obtain a homomorphism $\hat{H}_{d,\boldsymbol{k}}(q) \to \mathrm{End}(F^d)^{\mathrm{op}}$.

(b) As in part (a), if we have a categorical representation in the sense of Definition 3.2, then the homomorphism $R_{d,\boldsymbol{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$ extends to a homomorphism $\hat{R}_{d,\boldsymbol{k}} \to \mathrm{End}(F^d)^{\mathrm{op}}$. Then Proposition B.6 implies that the two definitions of a categorical representation of $\mathfrak{g}_{\mathcal{F}}$ (Definitions 3.2 and B.14) are equivalent.

B10. ***Categorical representations over R.*** We assume that the ring R is as in Section B6. We are going to obtain an analogue of Theorem 3.5 over R.

Let $\mathcal{C}_R$, $\mathcal{C}_{\boldsymbol{k}}$ and $\mathcal{C}_K$ be R-, $\boldsymbol{k}$- and K-linear categories, respectively. Assume that $\mathcal{C}_{\boldsymbol{k}}$ and $\mathcal{C}_K$ are Hom-finite $\boldsymbol{k}$-linear and K-linear abelian categories, respectively. Assume that the category $\mathcal{C}_R$ is exact. Fix R-linear functors $\Omega_{\boldsymbol{k}}\colon \mathcal{C}_R \to \mathcal{C}_{\boldsymbol{k}}$ and $\Omega_K\colon \mathcal{C}_R \to \mathcal{C}_K$.

Remark B.16. The first example of a situation as above that we should imagine is the following. Let A be an R-algebra that is finitely generated as an R-module. We set $\mathcal{C}_R = \mathrm{mod}\,(A)$, $\mathcal{C}_{\boldsymbol{k}} = \mathrm{mod}\,(\boldsymbol{k} \otimes_R A)$, $\mathcal{C}_K = \mathrm{mod}\,(K \otimes_R A)$, $\Omega_{\boldsymbol{k}} = \boldsymbol{k} \otimes \bullet$ and $\Omega_K = K \otimes \bullet$.

Another interesting situation (that in fact motivated the result of this section) is when $\mathcal{C}_B$, for $B \in \{R, \boldsymbol{k}, H\}$, is the category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_N$ over B at a negative level. We do not want to assume in this section that the category $\mathcal{C}_R$ is abelian because [Rouquier et al. 2016] constructs a categorical representation only in the Δ-filtered category $\mathcal{O}$ over R (and not in the whole abelian category $\mathcal{O}$ over R).

Definition B.17. A categorical representation of $(\widetilde{\mathfrak{sl}}_e, \mathfrak{sl}_\infty^{\oplus l})$ in $(\mathcal{C}_R, \mathcal{C}_{\boldsymbol{k}}, \mathcal{C}_K)$ is the following data

(1) a categorical representation of $\mathfrak{g}_I = \widetilde{\mathfrak{sl}}_e$ in $\mathcal{C}_{\boldsymbol{k}}$,

(2) a categorical representation of $\mathfrak{g}_{\tilde{I}} = \mathfrak{sl}_\infty^{\oplus l}$ in $\mathcal{C}_K$,

(3) a representation datum (E, F) in $\mathcal{C}_R$ (with respect to the Hecke algebra $H_{d,R}(q_e)$) such that the functors E and F commute with $\Omega_{\boldsymbol{k}}$ and Ω_K,

(4) lifts (with respect to $\Omega_{\boldsymbol{k}}$) of decompositions $E = \bigoplus_{i\in I} E_i$, $F = \bigoplus_{i\in I} F_i$ and $\mathcal{C}_{\boldsymbol{k}} = \bigoplus_{X_I} \mathcal{C}_{\boldsymbol{k},\mu}$ from $\mathcal{C}_{\boldsymbol{k}}$ to $\mathcal{C}_R$

such that the following compatibility conditions are satisfied:

- The decomposition $\mathcal{C}_R = \bigoplus_{\mu\in X_e} \mathcal{C}_{R,\mu}$ is compatible with the decomposition $\mathcal{C}_K = \bigoplus_{\tilde{\mu}\in X_{\tilde{I}}} \mathcal{C}_{K,\tilde{\mu}}$ (i.e., we have $\Omega_K(\mathcal{C}_{R,\mu}) \subset \bigoplus_{\pi_e(\tilde{\mu})=\mu} \mathcal{C}_{K,\tilde{\mu}}$).
- The decompositions $E = \bigoplus_{i\in I} E_i$ and $F = \bigoplus_{i\in I} F_i$ in $\mathcal{C}_R$ are compatible with the decompositions $E = \bigoplus_{j\in\tilde{I}} E_j$ and $F = \bigoplus_{j\in\tilde{I}} F_j$ in $\mathcal{C}_K$ with respect to Ω_K (i.e., the functors $E_i = \bigoplus_{j\in\tilde{I},\pi_e(j)=i} E_j$ and $F_i = \bigoplus_{j\in\tilde{I},\pi_e(j)=i} F_j$ for $\mathcal{C}_K$ correspond to the functors E_i, F_i for $\mathcal{C}_R$).
- The actions of the Hecke algebras $H_{d,R}(q_e)$, $H_{d,\boldsymbol{k}}(\zeta_e)$ and $H_{d,K}(q_e)$ on $\mathrm{End}(F^d)^{\mathrm{op}}$ for $\mathcal{C}_R$, $\mathcal{C}_{\boldsymbol{k}}$ and $\mathcal{C}_K$ are compatible with $\Omega_{\boldsymbol{k}}$ and Ω_K.

Proposition B.12 yields the following version of Theorem 3.5 over R.

Let $(\overline{\mathcal{C}}_R, \overline{\mathcal{C}}_{\boldsymbol{k}}, \overline{\mathcal{C}}_K)$ be a categorical representation of $(\widetilde{\mathfrak{sl}}_{e+1}, \mathfrak{sl}_\infty^{\oplus l})$. Assume that for each $\mu \in X_{\bar{I}} \backslash X_{\bar{I}}^+$ we have $\overline{\mathcal{C}}_{\boldsymbol{k},\mu} = \overline{\mathcal{C}}_{R,\mu} = 0$ and the for each $\tilde{\mu} \in X_{\tilde{I}} \backslash X_{\tilde{I}}^+$ we have $\overline{\mathcal{C}}_{K,\tilde{\mu}} = 0$. Let $\mathcal{C}_R$, $\mathcal{C}_{\boldsymbol{k}}$ and $\mathcal{C}_K$ be the subcategory of $\overline{\mathcal{C}}_R$, $\overline{\mathcal{C}}_{\boldsymbol{k}}$ and $\overline{\mathcal{C}}_K$, respectively, defined in the same way as in Section 3D. Then we have the following.

Theorem B.18. *There is a categorical representation of* $(\widetilde{\mathfrak{sl}}_e, \mathfrak{sl}_\infty^{\oplus l})$ *in* $(\mathcal{C}_R, \mathcal{C}_{\boldsymbol{k}}, \mathcal{C}_K)$.

Proof. We obtain a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in $\mathcal{C}_{\boldsymbol{k}}$ by Theorem 3.5. A similar argument as in the proof of Theorem 3.5 yields a categorical representation of $\mathfrak{sl}_\infty^{\oplus l}$ in $\mathcal{C}_K$ (we just have to replace the isomorphism Φ from Section 2G associated with the quiver Γ_e by a similar isomorphism associated with the quiver $\tilde{\Gamma}$). To construct a representation datum in $\mathcal{C}_R$, we use the homomorphism $\Phi_{\alpha,R}$ from Proposition B.12. All axioms of a $(\widetilde{\mathfrak{sl}}_e, \mathfrak{sl}_\infty^{\oplus l})$-categorical representation in $(\mathcal{C}_R, \mathcal{C}_{\boldsymbol{k}}, \mathcal{C}_K)$ follow automatically from the axioms of a categorical representation of $(\widetilde{\mathfrak{sl}}_{e+1}, \mathfrak{sl}_\infty^{\oplus l})$ in $(\overline{\mathcal{C}}_R, \overline{\mathcal{C}}_{\boldsymbol{k}}, \overline{\mathcal{C}}_K)$. □

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