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Grothendieck rings for Lie superalgebras and the Duflo–Serganova functor

Crystal Hoyt and Shifra Reif

We show that the Duflo–Serganova functor on the category of finite-dimensional modules over a finite-dimensional contragredient Lie superalgebra induces a ring homomorphism on a natural quotient of the Grothendieck ring, which is isomorphic to the ring of supercharacters. We realize this homomorphism as a certain evaluation of functions related to the supersymmetry property. We use this realization to describe the kernel and image of the homomorphism induced by the Duflo–Serganova functor.

1. Introduction

The Duflo–Serganova functor was originally introduced in [Duflo and Serganova 2005] together with associated varieties of modules over Lie superalgebras. On the category of finite-dimensional modules, the Duflo–Serganova functor is a tensor functor which preserves the superdimension. This functor was used by Serganova [2011] to prove the conjecture of Kac and Wakimoto that the superdimension of a finite-dimensional module is zero if and only if the atypicality of the module is maximal. The Duflo–Serganova functor was also used to give an additional proof for the superdimension formula of $GL(m \mid n)$-modules in [Heidersdorf and Weissauer 2014], and has been applied to study Deligne categories in [Comes and Heidersdorf 2017; Entova-Aizenbud et al. 2015; Heidersdorf 2015; Heidersdorf and Weissauer 2015].

Given an odd element x in a Lie superalgebra $\mathfrak{g}$ satisfying $[x, x] = 0$, we have that $x^2 = 0$ in the universal enveloping algebra of $\mathfrak{g}$, and so for every $\mathfrak{g}$-module M, we can define the cohomology

$$M_x := \operatorname{Ker}_M x / xM.$$

In fact, M_x is a module for the Lie superalgebra

$$\mathfrak{g}_x := \operatorname{Ker} \operatorname{ad}_x / \operatorname{Im} \operatorname{ad}_x,$$

which is a Lie superalgebra of smaller rank than $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{gl}(m \mid n)$ and x is a root vector, then $\mathfrak{g}_x = \mathfrak{gl}(m-1 \mid n-1)$. Duflo and Serganova [2005] defined the functor $DS_x : M \mapsto M_x$ from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules, which we refer to as the Duflo–Serganova functor.

One of the difficulties that arises in using the Duflo–Serganova functor is that it is not exact. It is therefore surprising that it induces a ring homomorphism ds_x on a natural quotient of the Grothendieck ring

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of the category of finite-dimensional $\mathfrak{g}$-modules. This quotient is defined by identifying the equivalence class of a module $[M]$ with $-[\Pi(M)]$, where Π is the shift of parity functor. We refer to this quotient as the *supercharacter ring* of $\mathfrak{g}$ and show that the homomorphism ds_x is indeed well defined.

Sergeev and Veselov [2011] described the supercharacter ring as a ring of functions admitting a certain supersymmetry condition. In this paper, we realize the homomorphism ds_x in terms of evaluation of functions related to the supersymmetry condition. For example, the supercharacter ring of the Lie supergroup $GL(m \mid n)$ corresponding to the Lie superalgebra $\mathfrak{gl}(m \mid n)$ is isomorphic to the ring of doubly symmetric Laurent polynomials in $x_1, \dots, x_m, y_1, \dots, y_n$ for which the evaluation $x_1 = y_1 = t$ is independent of t. If x is a root vector for the root $\varepsilon_i - \delta_j$ of $\mathfrak{gl}(m \mid n)$, then the homomorphism ds_x is given by the evaluation $x_i = y_j = t$, which is independent of the variable t after evaluation, by the supersymmetry property.

We use this realization to describe the kernel of the homomorphism ds_x when x is a root vector. In particular, we show that if $\mathfrak{g}$ is a Lie superalgebra of type I, the supercharacters of Kac modules form a basis for the kernel. When $\mathfrak{g}$ is a Lie superalgebra of type II, there are no Kac modules; however, we show that the kernel has a basis consisting of expressions similar to the supercharacters of Kac modules. These are the same expressions that were used by Gruson and Serganova [2010] to define Kazhdan–Lusztig polynomials for the orthosymplectic Lie superalgebras.

We also describe the image of ds_x. In particular, for $\mathfrak{g} = \mathfrak{sl}(m \mid n)$, $m \neq n$, and $\mathfrak{osp}(m \mid 2n)$, we show that the image is the supercharacter ring of G_x, where G_x is the Lie supergroup corresponding to the Lie superalgebra $\mathfrak{g}_x$. Moreover, we prove that the homomorphism induced by the Duflo–Serganova functor from the category of finite-dimensional G-modules to the category of finite-dimensional G_x-modules is surjective. For the exceptional Lie superalgebras, we explicitly describe the image using a set of generators.

2. Preliminaries

2A. *Lie superalgebras.* Lie superalgebras are a natural generalization of Lie algebras which first appeared in mathematical physics. In this paper, we study the finite-dimensional contragredient Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with indecomposable Cartan matrix. These are the Lie superalgebras $\mathfrak{sl}(m \mid n)$, $m \neq n$, $\mathfrak{gl}(n \mid n)$, $\mathfrak{osp}(m \mid 2n)$, $D(2, 1, \alpha)$, $F(4)$, or $G(3)$. We also consider the case when $\mathfrak{g} = \mathfrak{gl}(m \mid n)$ is the general linear Lie superalgebra. These Lie superalgebras resemble reductive Lie algebras in their structure theory; in particular, they are defined by a Cartan matrix and they possess an even supersymmetric invariant bilinear form $(\cdot\,,\cdot)$ which has kernel equal to the center of $\mathfrak{g}$.

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\bar{0}} \subset \mathfrak{g}$, and consider the corresponding root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

Then the set of roots $\Delta \subset \mathfrak{h}^*$ splits $\Delta = \Delta_{\bar{0}} \sqcup \Delta_{\bar{1}}$ into even roots $\Delta_{\bar{0}}$ and odd roots $\Delta_{\bar{1}}$. A choice of positive roots $\Delta^+ = \Delta^+_{\bar{0}} \sqcup \Delta^+_{\bar{1}}$ determines a triangular decomposition of $\mathfrak{g}$ given by $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^{\pm} = \bigoplus_{\alpha\in\Delta^{\pm}} \mathfrak{g}_{\alpha}$. Let $\rho_{\bar 0} = \frac{1}{2}\sum_{\alpha\in\Delta_{\bar 0}^{+}} \alpha$, $\rho_{\bar 1} = \frac{1}{2}\sum_{\alpha\in\Delta_{\bar 1}^{+}} \alpha$, and $\rho = \rho_{\bar 0} - \rho_{\bar 1}$. The Weyl group W of $\mathfrak{g}$ is by definition the Weyl group of $\mathfrak{g}_{\bar 0}$. The sign map $\operatorname{sgn} : W \to \{\pm 1\}$ is defined by $w \mapsto (-1)^{l(w)}$, where $l(w)$ denotes the length of w as a product of simple reflections with respect to a set of simple roots for $\mathfrak{g}_{\bar 0}$.

The space $\mathfrak{h}^*$ inherits an even supersymmetric bilinear form $(\cdot\,,\cdot)$. A root $\beta \in \Delta_{\bar 1}$ is called isotropic if $(\beta, \beta) = 0$. Two roots $\alpha, \beta \in \Delta$ are called orthogonal if $(\alpha, \beta) = 0$. The maximal number of linearly independent mutually orthogonal isotropic roots is called the defect of $\mathfrak{g}$. We denote by $\Delta_{\mathrm{iso}} := \{\beta \in \Delta_{\bar 1} \mid (\beta, \beta) = 0\}$ the set of all isotropic roots and by $\Delta_{\mathrm{iso}}^{+} = \Delta_{\mathrm{iso}} \cap \Delta^{+}$ the set of positive isotropic roots, and we let $\rho_{\mathrm{iso}} := \frac{1}{2}\sum_{\alpha\in\Delta_{\mathrm{iso}}^{+}} \alpha$. We define

$$\mathscr{S}_{\mathfrak{g}} = \{B \subset \Delta_{\mathrm{iso}} \mid B = \{\beta_1, \dots, \beta_k \mid (\beta_i, \beta_j) = 0,\ \beta_i \neq \pm\beta_j\}\} \tag{2-1}$$

to be the set of subsets of linearly independent mutually orthogonal isotropic roots.

The space $\mathfrak{h}^*$ has a natural basis $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n$, which for $\mathfrak{gl}(m \mid n)$ and $\mathfrak{osp}(m \mid 2n)$ satisfies $(\varepsilon_i, \varepsilon_j) = \delta_{ij} = -(\delta_i, \delta_j)$ and $(\varepsilon_i, \delta_j) = 0$. The roots of $\mathfrak{g}$ have a nice presentation in this basis (see [Cheng and Wang 2012; Musson 2012] for more details). Let $Q_{\mathfrak{g}} = \operatorname{span}_{\mathbb{Z}} \Delta$ be the root lattice of $\mathfrak{g}$, and let $Q_{\mathfrak{g}}^{+} = \operatorname{span}_{\mathbb{Z}} \Delta^{+}$. The parity function $p : \Delta \to \mathbb{Z}_2$ extends uniquely to a linear function $p : Q_{\mathfrak{g}} \to \mathbb{Z}_2$. The root lattice $Q_{\mathfrak{g}}$ is contained in the integral weight lattice $P_{\bar 0}$ for $\mathfrak{g}_{\bar 0}$, where

$$P_{\bar 0} = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha \in \Delta_{\bar 0} \right\}.$$

The set of dominant integral weights

$$P_{\bar 0}^{+} = \left\{ \lambda \in P_{\bar 0} \;\middle|\; \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \geq 0 \text{ for all } \alpha \in \Delta_{\bar 0} \right\}$$

is the set of highest weights of finite-dimensional simple $\mathfrak{g}_{\bar 0}$-modules.

The category of finite-dimensional modules $\mathscr{F}_{\mathfrak{g}}$ over a Lie superalgebra $\mathfrak{g}$ is not semisimple; that is, there exist indecomposable modules which are not irreducible. For example, a Lie superalgebra $\mathfrak{g}$ of type I has a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{+1}$, so one can define the Kac module of highest weight $\lambda \in P_{\bar 0}$ as

$$K(\lambda) = \operatorname{Ind}_{\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{+1}}^{\mathfrak{g}} L_{\bar 0}(\lambda),$$

where $L_{\bar 0}(\lambda)$ is the finite-dimensional simple $\mathfrak{g}_{\bar 0}$-module of highest weight λ and $\mathfrak{g}_{+1}$ acts trivially on $L_{\bar 0}(\lambda)$. Then $K(\lambda)$ is a finite-dimensional, indecomposable $\mathfrak{g}$-module with a unique simple quotient $L(\lambda)$, where λ is the highest weight with respect to the distinguished choice of simple roots, and $K(\lambda)$ is simple (i.e., $K(\lambda) = L(\lambda)$) if and only if λ is a typical weight: $(\lambda + \rho, \beta) \neq 0$ for all $\beta \in \Delta_{\mathrm{iso}}$ (see, for example, [Cheng and Wang 2012, Chapter 2] for more details).

If $G_{\bar 0}$ is a simply connected and connected Lie group corresponding to the Lie algebra $\mathfrak{g}_{\bar 0}$ [Serganova 2014], and $\mathscr{F}_G$ is the full subcategory of $\mathscr{F}_{\mathfrak{g}}$ consisting of all finite-dimensional $G_{\bar 0}$-integrable modules, then $\mathscr{F}_G$ is equivalent to the category of finite-dimensional modules over the corresponding algebraic supergroup G [Serganova 2014].

2B. ***Supercharacter rings of Lie superalgebras.*** The character theory of Lie superalgebras is a rich area of research which has led to interesting applications in number theory [Kac and Wakimoto 1994; 2014]. For a finite-dimensional $\mathfrak{g}$-module M, with weight decomposition $M = \bigoplus_{\mu\in\mathfrak{h}^*} M^\mu$ and weight spaces $M^\mu = M^\mu_{\bar{0}} \oplus M^\mu_{\bar{1}}$, the supercharacter of M is defined to be

$$\operatorname{sch} M = \sum_{\mu\in\mathfrak{h}^*} (\dim M^\mu_{\bar{0}} - \dim M^\mu_{\bar{1}}) e^\mu,$$

while the character of M is given by $\operatorname{ch} M = \sum(\dim\ M^\mu_{\bar{0}} + \dim\ M^\mu_{\bar{1}})e^\mu$. A finite-dimensional simple $\mathfrak{g}$-module is determined by its supercharacter, as well as by its character [Sergeev and Veselov 2011, Proposition 4.2].

The supercharacter ring $\mathcal{J}_\mathfrak{g}$ of a Lie superalgebra $\mathfrak{g}$ is defined to be the image of the map

$$\operatorname{sch} : \mathcal{F}_\mathfrak{g} \to \mathbb{Z}[P_{\bar{0}}]^W,$$

where $\mathbb{Z}[P_{\bar{0}}] := \mathbb{Z}\{e^\mu \mid \mu \in P_{\bar{0}}\}$. For an element $f \in \mathcal{J}_\mathfrak{g}$, with $f = \sum_{\mu\in P_{\bar{0}}} c_\mu e^\mu$, we call the set $\operatorname{Supp} f = \{\mu \in P_{\bar{0}} \mid c_\mu \neq 0\}$ the support of f.

For a fixed choice of positive roots $\Delta^+ = \Delta^+_{\bar{0}} \sqcup \Delta^+_{\bar{1}}$, we denote the super Weyl denominator by $R = R_{\bar{0}}/R_{\bar{1}}$ where $R_{\bar{0}} = \prod_{\alpha\in\Delta^+_{\bar{0}}}(1-e^{-\alpha})$ and $R_{\bar{1}} = \prod_{\alpha\in\Delta^+_{\bar{1}}}(1-e^{-\alpha})$. Note that the supercharacter of the Kac module equals

$$\operatorname{sch} K(\lambda) = e^{-\rho} R^{-1} \cdot \operatorname{ch} L_{\bar{0}}(\lambda),$$

where $\Delta^+ = \Delta^+_{\bar{0}} \sqcup \Delta^+_{\bar{1}}$ is the distinguished choice of simple roots.

The Grothendieck group of the category $\mathcal{F}_\mathfrak{g}$ is defined by taking the free abelian group generated by the elements $[M]$ which represent each isomorphism class of finite-dimensional $\mathfrak{g}$-modules, and modding out by the relations $[M_1] - [M_2] + [M_3]$ for all exact sequences $0 \to M_1 \to M_2 \to M_3 \to$. Since $\mathcal{F}_\mathfrak{g}$ is closed under tensor products, the Grothendieck group inherits a natural ring structure.

The Grothendieck ring of $\mathcal{F}_\mathfrak{g}$ has a natural quotient described as follows. Let Π denote the parity reversing functor from $\mathcal{F}_\mathfrak{g}$ to itself, and let $\mathcal{K}_\mathfrak{g}$ denote the quotient of the Grothendieck ring of $\mathcal{F}_\mathfrak{g}$ by the ideal $\langle [\Pi(M)] + [M] \mid M \text{ is a } \mathfrak{g}\text{-module}\rangle$. The map $\operatorname{sch} : \mathcal{K}_\mathfrak{g} \to \mathbb{Z}[P_{\bar{0}}]^{W_{\bar{0}}}$ given on generators by $[M] \mapsto \operatorname{sch} M$ is injective [Sergeev and Veselov 2011, Proposition 4.4], and its image is the supercharacter ring $\mathcal{J}_\mathfrak{g}$ of $\mathfrak{g}$.

Remark 1. In this paper, we identify the rings $\mathcal{K}_\mathfrak{g}$ and $\mathcal{J}_\mathfrak{g}$ under this isomorphism, and use the notation $\mathcal{J}_\mathfrak{g}$ to denote this ring. Given a module $M \in \mathcal{F}_\mathfrak{g}$, we write $[M]$ for its image in $\mathcal{J}_\mathfrak{g}$.

Sergeev and Veselov [2011] gave an explicit description of supercharacter rings for basic classical Lie superalgebras as follows. The supercharacter ring of $\mathfrak{g}$ is isomorphic to the space of supersymmetric exponential functions

$$\mathcal{J}_\mathfrak{g} = \{f \in \mathbb{Z}[P_{\bar{0}}]^W \mid D_\beta f \text{ is in the ideal generated by } (e^\beta - 1) \text{ for any } \beta \in \Delta_{\mathrm{iso}}\} \tag{2-2}$$

where $D_\beta(e^\lambda) = (\lambda, \beta)e^\beta$. Sergeev and Veselov [2011, §7] also described the supercharacter ring $\mathcal{J}_G \subset \mathcal{J}_\mathfrak{g}$ for the Lie supergroup G corresponding to the Lie superalgebra $\mathfrak{g}$ as a ring of Laurent polynomials subject to some additional conditions. Recall the basis $\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n$ of $\mathfrak{h}^*$, and define $x_i := e^{\varepsilon_i}$, $y_j := e^{\delta_j}$, $u_i = x_i + x_i^{-1}$, and $v_j = y_j + y_j^{-1}$.

$GL(m \mid n)$: *The supercharacter ring of* $GL(m \mid n)$ *is*

$$\mathcal{J}_G = \left\{ f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_m \times S_n} \;\middle|\; y_j \frac{\partial f}{\partial y_j} + x_i \frac{\partial f}{\partial x_i} \in \langle y_j - x_i \rangle \right\}. \tag{2-3}$$

$SL(m \mid n)$, $m \neq n$: *The supercharacter ring of* $SL(m \mid n)$ *for* $m \neq n$ *is the quotient of* (2-3) *by the ideal* $\langle x_1 \cdots x_m - y_1 \cdots y_n \rangle$.

$B(m \mid n)$: *The supercharacter ring of* $OSP(2m+1 \mid 2n)$ *is*

$$\mathcal{J}_G = \left\{ f \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \;\middle|\; u_i \frac{\partial f}{\partial u_i} + v_j \frac{\partial f}{\partial v_j} \in \langle u_i - v_j \rangle \right\}.$$

$C(n+1)$: *The supercharacter ring of* $OSP(2 \mid 2n)$ *is*

$$\mathcal{J}_G = \left\{ f \in \mathbb{Z}[u_1, v_1, \dots, v_n]^{S_m} \;\middle|\; u_1 \frac{\partial f}{\partial u_1} + v_j \frac{\partial f}{\partial v_j} \in \langle u_1 - v_j \rangle \right\}.$$

$D(m \mid n)$, $m \geq 2$: *The supercharacter ring of* $OSP(2m \mid 2n)$ *for* $m \geq 2$ *is*

$$\mathcal{J}_G = \left\{ f \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \;\middle|\; u_i \frac{\partial f}{\partial u_i} + v_j \frac{\partial f}{\partial v_j} \in \langle u_i - v_j \rangle \right\}.$$

Remark 2. Note that $f \in \mathcal{J}_{GL(m|n)}$ if and only if it is supersymmetric in $x_1, \dots, x_m, y_1, \dots y_n$, that is, if it is invariant under permutation of $x_1, \dots, x_m$ and of $y_1, \dots, y_n$, and if the substitution $x_1 = y_1 = t$ made in f is independent of t (see for example [Musson 2012, §12]).

2C. ***The Duflo–Serganova functor.*** The idea behind the Duflo–Serganova functor is simple and natural. For any odd element $x \in \mathfrak{g}_{\bar{1}}$ of a finite-dimensional contragredient Lie superalgebra $\mathfrak{g}$ which satisfies $[x, x] = 0$, we have that $x^2 = 0$ in the universal enveloping algebra of $\mathfrak{g}$, and so for any finite-dimensional $\mathfrak{g}$-module M we can define the cohomology

$$M_x := \mathrm{Ker}_M\, x / xM. \tag{2-4}$$

Then M_x is in fact a module over the Lie superalgebra

$$\mathfrak{g}_x := \mathfrak{g}^x / [x, \mathfrak{g}],$$

where $\mathfrak{g}^x = \{a \in \mathfrak{g} \mid [x, a] = 0\}$ is the centralizer of x in $\mathfrak{g}$ [Duflo and Serganova 2005, Lemma 6.2]. The Duflo–Serganova functor $DS_x : \mathcal{F}_\mathfrak{g} \to \mathcal{F}_{\mathfrak{g}_x}$ is defined from the category of finite-dimensional $\mathfrak{g}$-modules to the category of finite-dimensional $\mathfrak{g}_x$-modules by sending $M \mapsto M_x$.

The Duflo–Serganova functor is a cohomology functor and hence is a symmetric monoidal tensor functor; that is, for $\mathfrak{g}$-modules M, N one has a natural isomorphism $M_x \otimes N_x \to (M \otimes N)_x$ [Serganova 2011]. Moreover, the Duflo–Serganova functor commutes with direct sums; however, it is not exact.

Let $X_{\mathfrak{g}} = \{x \in \mathfrak{g}_{\bar{1}} : [x, x] = 0\}$, and let $\mathcal{S}_{\mathfrak{g}}$ be the set of subsets of mutually orthogonal isotropic roots (see (2-1)). Then the $G_{\bar{0}}$-orbits of $X_{\mathfrak{g}}$ are in one-to-one correspondence with the W-orbits of $\mathcal{S}_{\mathfrak{g}}$ via the correspondence

$$B = \{\beta_1, \dots, \beta_k\} \mapsto x = x_{\beta_1} + \cdots + x_{\beta_k} \in X_{\mathfrak{g}}, \tag{2-5}$$

where each $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$ is chosen to be nonzero [Duflo and Serganova 2005, Theorem 4.2].

The Lie superalgebra $\mathfrak{g}_x$ can be naturally embedded into $\mathfrak{g}^x \subset \mathfrak{g}$, in such a way that $\mathfrak{h}_x = \mathfrak{h} \cap \mathfrak{g}_x$ is a Cartan subalgebra of $\mathfrak{g}_x$ and the root spaces of $\mathfrak{g}_x$ are root spaces of $\mathfrak{g}$ [Duflo and Serganova 2005, Lemma 6.3]. More explicitly, Duflo and Serganova proved the following:

If $B = \{\beta_1, \dots, \beta_k\} \in \mathcal{S}$ and $x = x_{\beta_1} + \cdots + x_{\beta_k}$ for some nonzero $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, then $\mathfrak{g}^x \subset \mathfrak{g}$ can be decomposed into a semidirect sum $\mathfrak{g}^x = [x, \mathfrak{g}] \subsetplus \mathfrak{g}_x$, where $\mathfrak{g}_x = \mathfrak{h}_x \oplus \big(\bigoplus_{\alpha \in \Delta_x} \mathfrak{g}_\alpha\big)$, the subspace $\mathfrak{h}_x = \mathfrak{h} \cap \mathfrak{g}_x$ is a Cartan subalgebra of $\mathfrak{g}_x$, and

$$\Delta_x = \{\alpha \in \Delta \mid (\alpha, \beta) = 0 \text{ for all } \beta \in B \text{ and } \pm\alpha \notin B\} \tag{2-6}$$

is the root system of $\mathfrak{g}_x$.

For each finite-dimensional contragredient Lie superalgebra $\mathfrak{g}$ with irreducible Cartan matrix, we can explicitly describe the isomorphism type of $\mathfrak{g}_x$. If $\mathcal{B} = \{\beta_1, \dots, \beta_k\} \in \mathcal{S}$ and $x = x_{\beta_1} + \cdots + x_{\beta_k}$ for some nonzero $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, then by [Duflo and Serganova 2005, Remark 6.4] we have the following description. In particular, the defect of $\mathfrak{g}_x$ equals the defect of $\mathfrak{g}$ minus k. Note that in the last three columns the defect of $\mathfrak{g}$ is 1 and $k = 1$:

$\mathfrak{g}$	$\mathfrak{gl}(m \mid n)$	$\mathfrak{sl}(m \mid n)$, $m \neq n$	$\mathfrak{osp}(m \mid 2n)$	$D(2, 1, \alpha)$	F_4	G_3
$\mathfrak{g}_x$	$\mathfrak{gl}(m-k \mid n-k)$	$\mathfrak{sl}(m-k \mid n-k)$	$\mathfrak{osp}(m-2k \mid 2n-2k)$	$\mathbb{C}$	$\mathfrak{sl}(3)$	$\mathfrak{sl}(2)$

Remark 3. Note that when $\mathfrak{g}_x$ is simple, the embedding $\mathfrak{g}^x \subset \mathfrak{g}$ is determined by the condition that the root spaces of $\mathfrak{g}_x$ are mapped into the respective root spaces of $\mathfrak{g}$, since in this case $\mathfrak{h}_x \subset [\mathfrak{n}_x^+, \mathfrak{n}_x^-]$. For $\mathfrak{g} = \mathfrak{gl}(m, n)$, we take the matrix embedding of $\mathfrak{g}_x = \mathfrak{gl}(m-k \mid n-k)$ into $\mathfrak{gl}(m \mid n)$ which has $2k$ zero rows and $2k$ zero columns at the locations $r_i, n+s_i$, for $i = 1, \dots, k$, when $B = \{\beta_i = \varepsilon_{r_i} - \delta_{s_i}\}_{i=1,\dots,k}$ is the set of maximal isotropic roots defining x.

3. The Duflo–Serganova functor and the supercharacter ring

In this section, we prove that the Duflo–Serganova functor $DS_x : \mathcal{F}_{\mathfrak{g}} \to \mathcal{F}_{\mathfrak{g}_x}$ induces a ring homomorphism $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$, and we realize it as a certain evaluation of the functions $f \in \mathcal{J}_{\mathfrak{g}}$ related to the supersymmetry property defining $\mathcal{J}_{\mathfrak{g}}$.

3A. ***The ring homomorphism induced by the Duflo–Serganova functor.*** Let $\mathfrak{g}$ be a finite-dimensional contragredient Lie superalgebra with indecomposable Cartan matrix, or let $\mathfrak{g} = \mathfrak{gl}(m, n)$, and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $B = \{\beta_1, \dots, \beta_k\} \in \mathcal{S}_{\mathfrak{g}}$, $x \in X_{\mathfrak{g}}$, and $x = x_{\beta_1} + \cdots + x_{\beta_k}$ for nonzero $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$. Fix an embedding $\mathfrak{g}_x \subset \mathfrak{g}^x \subset \mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_x = \mathfrak{h} \cap \mathfrak{g}_x$ (see Section 2C).

Lemma 4. *For $\mathfrak{g}$-modules M and N we have*

(1) $\operatorname{sch} M_x(h) = \operatorname{sch} M(h)$ *for all* $h \in \mathfrak{h}_x$ *and*

(2) *if* $\operatorname{sch} M = \operatorname{sch} N$, *then* $\operatorname{sch} M_x = \operatorname{sch} N_x$.

Proof. We have an exact sequence $0 \to \ker_M x \to M \to xM \to 0$ of $\mathfrak{h}^x$-invariant spaces. Thus, $M/\ker_M x \cong \Pi(xM)$ as $\mathfrak{h}^x$-modules, where Π switches the parity of a superspace, and so $\operatorname{sch}(M/\ker_M x)(h) = \operatorname{sch} \Pi(xM)(h)$ for all $h \in \mathfrak{h}^x$. Hence, for all $h \in \mathfrak{h}_x \subset \mathfrak{h}^x$ we have that

$$\begin{aligned}\operatorname{sch} M(h) &= \operatorname{sch} \ker x(h) + \operatorname{sch} \Pi(xM)(h) = \operatorname{sch} \ker x(h) - \operatorname{sch} M(h) = \operatorname{sch}(\ker_M x/xM)(h) \\ &= \operatorname{sch}(M_x)(h). \qquad \square\end{aligned}$$

Remark 5. The following example shows that Lemma 4 does not hold if we replace supercharacter by character. It also shows that the Duflo–Serganova functor is not exact.

Example 6. Let $\mathfrak{g} = \mathfrak{gl}(2 \mid 1)$ with the standard choice of simple roots $\{\alpha = \varepsilon_1 - \varepsilon_2,\ \beta = \varepsilon_2 - \delta_1\}$. Let $K(0)$ be the Kac module with highest weight zero, and denote the highest weight vector by v_0. Then $K(0) = \operatorname{span}\{v_0, f_\beta v_0, f_{\alpha+\beta} v_0, f_\beta f_{\alpha+\beta} v_0\}$, where $f_\beta \in \mathfrak{g}_{-\beta}$ and $f_{\alpha+\beta} \in \mathfrak{g}_{-\alpha-\beta}$ are nonzero. The maximal submodule of $K(0)$ is $\overline{K}(0) := \operatorname{span}\{f_\beta v_0, f_{\alpha+\beta} v_0, f_\beta f_{\alpha+\beta} v_0\}$, and the simple quotient of $K(0)$ is isomorphic to the trivial $\mathfrak{g}$-module $L(0)$. Clearly, the $\mathfrak{g}$-modules $K(0)$ and $L(0) \oplus \overline{K}(0)$ have the same character and supercharacter.

Let us show that for $x = f_\beta$, the $\mathfrak{g}_x$-modules $K(0)_x$ and $(L(0) \oplus \overline{K}(0))_x$ have the same supercharacter but not the same character. In this case, $\mathfrak{g}_x = \mathfrak{gl}(1 \mid 0)$. By a direct computation using (2-4) and the basis given above, one can check that $K(0)_x = \{0\}$, $L(0)_x \cong \mathbb{C}_{1|0}$, and $\overline{K}(0)_x \cong \mathbb{C}_{0|1}$, where $\mathbb{C}_{1|0}$ and $\mathbb{C}_{0|1}$ are the even and odd trivial $\mathfrak{g}_x$-modules, respectively. Thus, $\operatorname{ch} K(0)_x = \operatorname{sch} K(0)_x = 0$ and $\operatorname{sch}(L(0) \oplus \overline{K}(0))_x = 0$, while $\operatorname{ch}(L(0) \oplus \overline{K}(0))_x = 2$.

Definition 7. We define $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$ on the generators $[M] \in \mathcal{J}_{\mathfrak{g}}$, where $M \in \mathcal{F}_{\mathfrak{g}}$, by

$$ds_x([M]) = [DS_x(M)],$$

and we extend linearly to $\mathcal{J}_{\mathfrak{g}}$.

It is not difficult to show that ds_x is a well defined linear map using Lemma 4. The fact that ds_x is a ring homomorphism then follows from the fact that DS_x is a tensor functor. Hence, we have:

Proposition 8. *Let $\mathfrak{g}$ be a finite-dimensional contragredient Lie superalgebra, and let $x \in \mathfrak{g}_{\bar{1}}$ nonzero such that $[x, x] = 0$. The functor $DS_x : \mathcal{F}_{\mathfrak{g}} \to \mathcal{F}_{\mathfrak{g}_x}$ induces a ring homomorphism on the corresponding supercharacter rings $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$.*

Remark 9. The proofs in Section 3A also work for modules in the BGG category $\mathbb{O}$, and so the Duflo–Serganova functor induces a group homomorphism on the quotient of the Grothendieck group by the parity. However, it is not a ring homomorphism since category $\mathbb{O}$ is not closed under tensor products.

3B. ***Realization of the ring homomorphism.*** Given $f \in \mathcal{J}_{\mathfrak{g}}$ we can realize $f : \mathfrak{h} \to \mathbb{C}$ as a supersymmetric function in the variables $x_1, \dots, x_m, y_1, \dots, y_n$ with $x_i = e^{\varepsilon_i}$ and $y_j = e^{\delta_j}$, using the supercharacter ring description of Sergeev and Veselov (see (2-2)). (Note that for $F(4)$ we take $x_i = e^{(1/2)\varepsilon_i}$ and $y_j = e^{(1/2)\delta_j}$.)

Theorem 10. *Suppose $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$ is defined by $x = x_{\beta_1} + \cdots + x_{\beta_k}$ for nonzero $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, where $B = \{\beta_1, \dots, \beta_k\} \in \mathcal{S}_{\mathfrak{g}}$.*

(1) *Then for any $f \in \mathcal{J}_{\mathfrak{g}}$,*
$$ds_x(f) = f|_{\mathfrak{h}_x}.$$

(2) *If $B = \{\varepsilon_1 - \delta_1\}$, then $ds_x(f)$ is given by substituting $x_1 = y_1$ into f, that is,*
$$ds_x f = f|_{x_1 = y_1}.$$

If $\mathfrak{g} = F(4)$ or $D(2, 1, \alpha)$ and $B = \{\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_1)\}$ or $B = \{\varepsilon_1 - \varepsilon_2 - \varepsilon_3\}$, then $ds_x f$ is given by substituting $y_1 = x_1 x_2 x_3$ or $x_1 = x_2 x_3$ into f, respectively.

(3) *If $B = \{\beta_i = a_i \varepsilon_{r_i} - b_i \delta_{s_i}\}_{i=1,\dots,k}$ for some $a_i, b_i \in \{\pm 1\}$, then $ds_x f$ is given by substituting $x_{r_i}^{a_i} = y_{s_i}^{b_i}$ into f, that is,*
$$ds_x f = f|_{x_{r_i}^{a_i} = y_{s_i}^{b_i},\ i=1,\dots,k}.$$

(4) *For any $f \in \mathcal{J}_{\mathfrak{g}}$,*
$$ds_x(f) = f|_{\beta_1 = \cdots = \beta_k = 0}.$$

Proof. It suffices to prove (1) for a spanning set of $\mathcal{J}_{\mathfrak{g}}$. Suppose $[M] \in \mathcal{K}_{\mathfrak{g}}$ corresponds to a module $M \in \mathcal{F}_{\mathfrak{g}}$. By Lemma 4, we have
$$ds_x([M]) = [DS_x(M)] = \operatorname{sch} M_x = (\operatorname{sch} M)|_{\mathfrak{h}_x} = [M]|_{\mathfrak{h}_x} \in \mathcal{J}_{\mathfrak{g}_x}.$$
Hence, $ds_x(f) = f|_{\mathfrak{h}_x}$ for any $f \in \mathcal{J}_{\mathfrak{g}}$.

To prove (2), fix $f \in \mathcal{J}_{\mathfrak{g}}$, and suppose that $x \in \mathfrak{g}_\beta$. If $\beta = \varepsilon_1 - \delta_1$, then the evaluation $f_{x_1 = y_1 = t}$ is well defined and independent of t due to the supersymmetry property of $f \in \mathcal{J}_{\mathfrak{g}}$. Thus,
$$f|_{x_1 = y_1 = t} := f(t, x_2, \dots, x_m \mid t, y_2, \dots, y_n)$$
is equal to the restriction of f to the hyperplane $x_1 - y_1 = 0$. Since $\mathfrak{h}_x \subset \mathfrak{h}^x = \{h \in \mathfrak{h} \mid \beta(h) = 0\}$ belongs to the hyperplane $x_1 - y_1 = 0$, we have proven that $ds_x f = f|_{\mathfrak{h}_x} = f|_{x_1 = y_1}$. The cases $\beta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_1$ and $\beta = \varepsilon_1 - \varepsilon_2 - \varepsilon_3$ are similar.

Now (3) can be proven using arguments similar to that of (2) and the fact that
$$\mathfrak{h}^x = \{h \in \mathfrak{h} \mid \beta(h) = 0 \text{ for all } \beta \in B\}.$$

Finally, (4) follows from (3) since if $\beta_i = a_i\varepsilon_{r_i} - b_i\delta_{r_i}$, then $\beta_i = 0$ if and only if $x_{r_i}^{a_i} y_{r_i}^{-b_i} = e^{\beta_i} = 1$ if and only if $x_{r_i}^{a_i} = y_{r_i}^{b_i}$. □

Corollary 11. *If $x = x_{\beta_1} + \cdots + x_{\beta_k}$ where $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$ and $B = \{\beta_1, \ldots, \beta_k\} \in \mathcal{S}$, then for all $f \in \mathcal{J}_\mathfrak{g}$*

$$ds_x(f) = ds_{x_{\beta_1}} \circ \cdots \circ ds_{x_{\beta_k}}(f).$$

4. The kernel of the ring homomorphism

In this section, we give a $\mathbb{Z}$-basis for the kernel of ds_x when $x \in \mathfrak{g}_\beta$ is a root vector of an isotropic root β for the Lie superalgebra $\mathfrak{g}$. Our basis is given by elements of the following form.

Definition 12. For each $\lambda \in P_{\bar{0}}$, we define

$$k(\lambda) := R^{-1} \cdot \sum_{w \in W} (-1)^{l(w)+p(w(\rho)-\rho)} e^{w(\lambda+\rho)-\rho}.$$

Here $p(w(\rho) - \rho)$ denotes the parity of $w(\rho) - \rho$, which is well defined since $w(\rho) - \rho \in Q$. Note that the element $w(\rho) - \rho$ may be odd, e.g., in $\mathfrak{osp}(1 \mid 2)$.

For each $\lambda \in P_{\bar{0}}^+$, the expression $k(\lambda)$ is in $\mathbb{Z}[P_{\bar{0}}]^W$ since it is the product of the W-invariant polynomial $e^{\rho_{\bar{1}}} R_{\bar{1}}$ and the character of a finite-dimensional $\mathfrak{g}_{\bar{0}}$-module given by the Weyl character formula. Moreover, since the evaluation $k(\lambda)|_{\beta=0}$ equals zero for any $\beta \in \Delta_{\mathrm{iso}}$, we have that $k(\lambda) \in \mathcal{J}_\mathfrak{g}$. It is clear that $k(\lambda)$ is in the kernel of ds_x for any $x \in \mathfrak{g}_\beta$, since $ds_x(R_{\bar{1}}) = 0$.

For Lie superalgebras of type I with the distinguished choice of simple roots, $k(\lambda)$ is the supercharacter of a Kac module when $\lambda \in P_{\bar{0}}^+$, whereas in type II, $k(\lambda)$ is a virtual supercharacter. Similar virtual characters were used by Gruson and Serganova [2010] to study the character formula of simple modules over orthosymplectic Lie superalgebras.

We need the following definition to prove the main result in this section for Lie superalgebras of type II.

Definition 13. Given a finite-dimensional Lie superalgebra $\mathfrak{g}$ with root system Δ, we define a Lie algebra $\tilde{\mathfrak{g}}$ as follows. We let $\tilde{\Delta}$ be the root system with positive even roots given by

$$\tilde{\Delta}^+ := \left\{ \alpha \in \Delta_{\bar{0}}^+ \;\middle|\; \frac{\alpha}{2} \notin \Delta_{\bar{1}} \right\} \cup \{\alpha \in \Delta_{\bar{1}}^+ \mid \alpha \notin \Delta_{\mathrm{iso}}\},$$

and we let $\tilde{\mathfrak{g}}$ be the semisimple Lie algebra with root system $\tilde{\Delta}$. If $\Delta_{\bar{1}} = \Delta_{\mathrm{iso}}$, then $\tilde{\mathfrak{g}} = \mathfrak{g}_{\bar{0}}$. If $\mathfrak{g} = B(m \mid n)$, $G(3)$, then $\tilde{\mathfrak{g}} \cong B_m \times B_n$, $G_2 \times A_1$, respectively. We set $\tilde{\rho} := \frac{1}{2}\sum_{\alpha \in \tilde{\Delta}^+} \alpha$. Note that $\rho = \tilde{\rho} - \rho_{\mathrm{iso}}$, since $\beta \in \Delta_{\mathrm{iso}}$ if and only if $\beta \in \Delta_{\bar{1}}$ but $2\beta \notin \Delta_{\bar{0}}$. Let $P_{\tilde{\mathfrak{g}}}^+$ denote the set of dominant integral weights of $\tilde{\mathfrak{g}}$. Then $P_{\bar{0}}^+ \subset P_{\tilde{\mathfrak{g}}}^+$ and the Weyl group of $\tilde{\mathfrak{g}}$ is isomorphic to the Weyl group of $\mathfrak{g}_{\bar{0}}$. We extend the definition of $k(\lambda)$ to $\lambda \in P_{\tilde{\mathfrak{g}}}$ by letting

$$k(\lambda) := R^{-1} \cdot \sum_{w \in W} (-1)^{l(w)+p(w(\lambda+\rho)-\rho)} e^{w(\lambda+\rho)-\rho}.$$

We have the following lemma.

Lemma 14. *The set* $\{k(\mu) \mid \mu \in P^+_{\tilde{\mathfrak{g}}} + \rho_{\rm iso}\}$ *is linearly independent.*

Proof. To prove linear independence we consider a completion of $\mathbb{Z}[P_{\tilde{\mathfrak{g}}}]$, where we allow expansions in the domain $|e^{-\alpha}| < 1$ for $\alpha \in \tilde{\Delta}^+$. Note that in this completion, $R^{-1} = \sum_{\nu \in -Q^+_{\tilde{\mathfrak{g}}}} b_\nu e^\nu$ for some $b_\nu \in \mathbb{Z}$. For each $\mu \in P^+_{\tilde{\mathfrak{g}}} + \rho_{\rm iso}$, we will show that $\mu+\rho$ is a strictly dominant element of $P_{\tilde{\mathfrak{g}}}$, that is, $w(\mu+\rho) < \mu+\rho$ for $w \in W$, $w \neq 1$. Indeed, if $\mu \in P^+_{\tilde{\mathfrak{g}}} + \rho_{\rm iso}$, then $\mu+\rho = \lambda+\tilde{\rho}$ for some $\lambda \in P^+_{\tilde{\mathfrak{g}}}$. Since $\lambda+\tilde{\rho}$ is strictly dominant with respect to $\tilde{\mathfrak{g}}$, it is also strictly dominant with respect to $\mathfrak{g}_{\bar{0}}$ and the claim follows. Thus,

$$k(\mu) = e^\mu + \sum_{\nu \in \mu - Q^+_{\tilde{\mathfrak{g}}}} a_\nu e^\nu$$

and linear independence follows. □

Remark 15. Note that if one takes the distinguished choice of simple roots for $\mathfrak{gl}(m,n)$, then $P^+ = P^+ + \rho_{\rm iso}$, since in this case $(\rho_{\rm iso}, \alpha) = 0$ for every even root α.

The following lemma is used in the proof of the main theorem of this section.

Lemma 16. *For each* $\mu \in P^+_{\tilde{\mathfrak{g}}}$, *we have* $k(\mu+\rho_{\rm iso}) = e^{\rho_{\rm iso}} \prod_{\alpha \in \Delta^+_{\rm iso}} (1-e^{-\alpha}) \cdot \operatorname{ch} L_{\tilde{\mathfrak{g}}}(\mu)$.

Proof. For any element $g \in \mathbb{Z}[P_{\tilde{\mathfrak{g}}}]$ with $\operatorname{Supp} g \subset \mu + Q_{\tilde{\mathfrak{g}}}$, we write $g = \sum_{\lambda \in Q_{\tilde{\mathfrak{g}}}} c_{\mu+\lambda} e^{\mu+\lambda}$, and we define $\bar{g} = \sum_{\lambda \in Q_{\tilde{\mathfrak{g}}}} (-1)^{p(\lambda)} c_{\mu+\lambda} e^{\mu+\lambda}$, where $p : Q_{\tilde{\mathfrak{g}}} \to \mathbb{Z}_2$ is the parity function. Clearly, this operation is an involution. So we have that

$$\begin{aligned}
\overline{e^{\rho_{\rm iso}} \prod_{\alpha \in \Delta^+_{\rm iso}} (1-e^{-\alpha}) \cdot \operatorname{ch} L_{\tilde{\mathfrak{g}}}(\mu)} &= (-1)^{p(\rho_{\rm iso})} e^{\rho_{\rm iso}} \prod_{\alpha \in \Delta^+_{\rm iso}} (1+e^{-\alpha}) \cdot \operatorname{sch} L_{\tilde{\mathfrak{g}}}(\mu) \\
&= (-1)^{p(\rho_{\rm iso})} e^{\rho_{\rm iso}} \prod_{\alpha \in \Delta^+_{\rm iso}} (1+e^{-\alpha}) \frac{\sum_{w \in W} (-1)^{l(w)+p(w(\mu+\tilde{\rho})-\tilde{\rho})} e^{w(\mu+\tilde{\rho})-\tilde{\rho}}}{\prod_{\alpha \in \tilde{\Delta}^+_{\bar{0}}} (1-e^{-\alpha})} \\
&= \prod_{\alpha \in \Delta^+_{\bar{1}}} (1+e^{-\alpha}) \cdot \frac{\sum_{w \in W} (-1)^{l(w)+p(w(\mu+\rho_{\rm iso}+\rho)-\rho)} e^{w((\mu+\rho_{\rm iso})+\tilde{\rho}-\rho_{\rm iso})-\tilde{\rho}+\rho_{\rm iso}}}{\prod_{\alpha \in \tilde{\Delta}^+_{\bar{0}}} (1-e^{-\alpha}) \prod_{\alpha \in \Delta^+_{\bar{1}} \setminus \Delta^+_{\rm iso}} (1+e^{-\alpha})} \\
&= \prod_{\alpha \in \Delta^+_{\bar{1}}} (1+e^{-\alpha}) \frac{\sum_{w \in W} (-1)^{l(w)+p(w((\mu+\rho_{\rm iso})+\rho)-\rho)} e^{w((\mu+\rho_{\rm iso})+\rho)-\rho}}{\prod_{\alpha \in \Delta^+_{\bar{0}}} (1-e^{-\alpha})} \\
&= \overline{k(\mu+\rho_{\rm iso})},
\end{aligned}$$

and hence, the claim follows. □

We have the following theorem.

Theorem 17. *If* β *is an odd isotropic root and* $x \in \mathfrak{g}_\beta$, *then the set*

$$\{k(\lambda) \mid \lambda \in P^+_{\bar{0}} + \rho_{\rm iso}\} \tag{4-1}$$

is a $\mathbb{Z}$*-basis for the kernel of* $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$.

Proof. Linear independence of the set (4-1) follows from Lemma 14 since $P_{\bar{0}}^+ \subset P_{\tilde{\mathfrak{g}}}^+$. So it only remains to show that the set (4-1) spans the kernel of $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$.

Let $f \in \mathcal{J}_{\mathfrak{g}}$ such that $ds_x(f) = 0$. According to Theorem 10, this means that the restriction of f to the hyperplane $\beta = 0$ is zero, or equivalently, substituting $e^{-\beta} = 1$ yields zero. Hence, f is divisible by $(1-e^{-\beta})$. Since f is W-invariant and $W\beta = \Delta_{\mathrm{iso}}$, it follows that f is divisible by $e^{\rho_{\mathrm{iso}}} \prod_{\alpha\in\Delta_{\mathrm{iso}}^+}(1-e^{-\alpha})$.

Write

$$f = e^{\rho_{\mathrm{iso}}} \prod_{\alpha\in\Delta_{\mathrm{iso}}^+} (1-e^{-\alpha}) \cdot g.$$

Then g is a W-invariant element of $\mathbb{Z}[P_{\bar{0}}]$, since both f and $e^{\rho_{\mathrm{iso}}} \prod_{\alpha\in\Delta_{\mathrm{iso}}^+}(1-e^{-\alpha})$ are W-invariant.

Case 1. First suppose that $\mathfrak{g}$ does not have nonisotropic roots; then $\Delta_{\mathrm{iso}}^+ = \Delta_{\bar{1}}^+$ and $\rho_{\mathrm{iso}} = \rho_{\bar{1}}$. By the theory of symmetric functions,

$$g = \sum_{\mu\in P_{\bar{0}}^+}^{\text{finite}} a_\mu \operatorname{ch} L_{\mathfrak{g}_{\bar{0}}}(\mu),$$

for some $a_\mu \in \mathbb{Z}$, where $P_{\bar{0}}^+$ is the set of highest weights of finite-dimensional $\mathfrak{g}_{\bar{0}}$-modules (see for example [Macdonald 1995]).

By the Weyl character formula for semisimple Lie algebras, we have that

$$\begin{aligned}
f &= e^{\rho_{\bar{1}}} R_{\bar{1}} \cdot g \\
&= e^{\rho_{\bar{1}}} R_{\bar{1}} \sum_{\mu\in P_{\bar{0}}^+} a_\mu \operatorname{ch} L_{\mathfrak{g}_{\bar{0}}}(\mu) \\
&= e^{\rho_{\bar{1}}} R_{\bar{1}} \sum_{\mu\in P_{\bar{0}}^+} a_\mu \frac{\sum_{w\in W} (-1)^{l(w)} e^{w(\mu+\rho_0)}}{e^{\rho_{\bar{0}}} R_{\bar{0}}} \\
&= e^{\rho_{\bar{1}}} R_{\bar{1}} \sum_{\lambda\in P_{\bar{0}}^+ + \rho_{\bar{1}}} b_\lambda \frac{\sum_{w\in W} (-1)^{l(w)} e^{w(\lambda+\rho_0-\rho_1)}}{e^{\rho_{\bar{0}}} R_{\bar{0}}} \\
&= \sum_{\lambda\in P_{\bar{0}}^+ + \rho_{\bar{1}}} b_\lambda k(\lambda),
\end{aligned}$$

where $b_\lambda := a_{\lambda-\rho_{\bar{1}}}$. For each $w \in W$, the parity of $w(\rho)$ equals the parity of ρ, since $\rho \in P_{\bar{0}}$. Hence, the last equality follows.

Case 2. Suppose that $\mathfrak{g}$ has nonisotropic roots. Since $P_{\bar{0}} \subset P_{\tilde{\mathfrak{g}},}$, by the theory of characters of Lie algebras

$$g = \sum_{\mu\in P_{\tilde{\mathfrak{g}}}^+}^{\text{finite}} a_\mu \operatorname{ch} L_{\tilde{\mathfrak{g}}}(\mu)$$

for some $a_\mu \in \mathbb{Z}$. By Lemma 16, we have that

$$\begin{aligned} f &= e^{\rho_{\text{iso}}} \prod_{\alpha\in\Delta^+_{\text{iso}}} (1-e^{-\alpha}) \cdot g \\ &= e^{\rho_{\text{iso}}} \prod_{\alpha\in\Delta^+_{\text{iso}}} (1-e^{-\alpha}) \sum_{\mu\in P^+_{\tilde{\mathfrak{g}}}} a_\mu \operatorname{ch} L_{\tilde{\mathfrak{g}}}(\mu) \\ &= \sum_{\mu\in P^+_{\tilde{\mathfrak{g}}}} a_\mu \cdot e^{\rho_{\text{iso}}} \prod_{\alpha\in\Delta^+_{\text{iso}}} (1-e^{-\alpha}) \cdot \operatorname{ch} L_{\tilde{\mathfrak{g}}}(\mu) \\ &= \sum_{\mu\in P^+_{\tilde{\mathfrak{g}}}} a_\mu \cdot k(\mu+\rho_{\text{iso}}) \\ &= \sum_{\lambda\in P^+_{\tilde{\mathfrak{g}}}+\rho_{\text{iso}}} b_\lambda k(\lambda) \end{aligned} \tag{4-2}$$

where $b_\lambda := a_{\lambda-\rho_{\text{iso}}}$. We are left to show that $b_\lambda = 0$ for $\lambda \notin P^+_{\bar{0}} + \rho_{\text{iso}}$. Since $\operatorname{Supp} f \subset P_{\bar{0}}$, $\operatorname{Supp} k(\lambda) \subset P_{\bar{0}}$, the elements $k(\lambda)$ for $\mu \in P^+_{\tilde{\mathfrak{g}}} + \rho_{\text{iso}}$ are linearly independent, and the sum in (4-2) is finite, we conclude that

$$f = \sum_{\lambda\in P^+_{\bar{0}}+\rho_{\text{iso}}} b_\lambda k(\lambda). \qquad \square$$

Corollary 18. *Let G be one of the Lie supergroups $SL(m\,|\,n)$, $m \neq n$, $GL(m\,|\,n)$, or $SOSP(m\,|\,2n)$, and let $\mathfrak{g}$ be the corresponding Lie superalgebra. Let β be an odd isotropic root and $x \in \mathfrak{g}_\beta$, and let $DS_x : \mathcal{F}_G \to \mathcal{F}_{G_x}$ be the Duflo–Serganova functor from the category $\mathcal{F}_G$ of finite-dimensional G-modules to the category $\mathcal{F}_{G_x}$ of finite-dimensional G_x-modules, where G_x denotes the Lie supergroup corresponding to the Lie superalgebra $\mathfrak{g}_x$. Then the kernel of the induced ring homomorphism $ds_x : \mathcal{J}_G \to \mathcal{J}_{G_x}$ has a $\mathbb{Z}$-basis*

$$\{k(\lambda) \mid \lambda \in P^+_G + \rho_{\text{iso}}\},$$

where P^+_G is the set of highest weights for finite-dimensional G-modules.

Proof. Let $P_G \subset P_{\bar{0}}$ be the sublattice of integral weights of finite-dimensional $G_{\bar{0}}$-modules. Then for $G = GL(m \mid n)$ or $SOSP(m \mid 2n)$

$$P_G = \left\{ \sum_{i=1}^m \lambda_i \varepsilon_i + \sum_{j=1}^n \mu_j \delta_j \;\middle|\; \lambda_i, \mu_j \in \mathbb{Z} \right\},$$

and the supercharacter ring for the category of finite-dimensional G-modules $\mathcal{F}_G$ is

$$\mathcal{J}_G = \left\{ f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^W \;\middle|\; y_j \frac{\partial f}{\partial y_j} + x_i \frac{\partial f}{\partial x_i} \in \langle y_j - x_i \rangle \right\}$$

as shown in [Sergeev and Veselov 2011, §7] (note that this ring is therein denoted by $J(\mathfrak{g})_0$). If $G = SL(m \mid n)$, $m \neq n$, then

$$P_G = \left\{ \sum_{i=1}^{m} \lambda_i \varepsilon_i + \sum_{j=1}^{n} \mu_j \delta_j \,\middle|\, \lambda_i, \mu_j \in \mathbb{Z},\ \sum_{i=1}^{m} \lambda_i - \sum_{j=1}^{n} \mu_j = 0 \right\},$$

and the supercharacter ring for the category of finite-dimensional G-modules $\mathcal{F}_G$ is

$$\mathcal{J}_G = \left\{ f \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^W / \langle x_1 \cdots x_m - y_1 \cdots y_n \rangle \,\middle|\, y_j \frac{\partial f}{\partial y_j} + x_i \frac{\partial f}{\partial x_i} \in \langle y_j - x_i \rangle \right\}$$

as shown in [Sergeev and Veselov 2011, §7]. Since in both cases $\mathcal{J}_G = \mathcal{J}_{\mathfrak{g}} \cap \mathbb{Z}[P_G]$, the kernel of the homomorphism $ds_x : \mathcal{J}_G \to \mathcal{J}_{G_x}$ equals $\mathrm{Ker}_G\, ds_x = \mathrm{Ker}_{\mathfrak{g}}\, ds_x \cap \mathbb{Z}[P_G]$, where $\mathrm{Ker}_{\mathfrak{g}}\, ds_x$ is the kernel of the corresponding homomorphism $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$. It follows from the linear independence of the elements $k(\lambda)$ and the fact that $\lambda \in P_G$ if and only if $\mathrm{Supp}\, k(\lambda) \in P_G$ that $\mathrm{Ker}_G\, ds_x = \mathrm{span}\{k(\lambda) \mid \lambda \in P_G + \rho_{\mathrm{iso}}\}$. Since $P_G^+ = P_{\mathfrak{g}}^+ \cap P_G$, the claim follows. □

Remark 19. On the level of categories, it was shown in [Boe et al. 2012] that a module M over a type-I finite-dimensional contragredient Lie superalgebra has a filtration of Kac modules or dual Kac modules if and only if $DS_x(M) = 0$ for all $x \in X_{\mathfrak{g}}^-$ or $x \in X_{\mathfrak{g}}^+$, respectively, where $X_{\mathfrak{g}}^{\pm} = X_{\mathfrak{g}} \cap \mathfrak{n}^{\pm}$ and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is the triangular decomposition with respect to the distinguished choice of simple roots.

5. The image of the ring homomorphism

5A. ***Image of ds_x for classical Lie superalgebras.*** Let $\mathfrak{g}$ be one of the Lie superalgebras: $\mathfrak{sl}(m \mid n)$, $m \neq n$, $\mathfrak{gl}(m \mid n)$, and $\mathfrak{osp}(m \mid 2n)$. In this section, we describe the image of ds_x for every $x \in X_{\mathfrak{g}}$. We use the realization of ds_x given in Theorem 10 and the explicit description of the supercharacter rings given by Sergeev and Veselov [2011, §7].

Theorem 20. *Let G be one of the Lie supergroups $SL(m \mid n)$, $m \neq n$, $GL(m \mid n)$, or $OSP(m, 2n)$ and $\mathfrak{g}$ be the corresponding Lie superalgebra. For any $x \in X_{\mathfrak{g}}$, the Duflo–Serganova functor $DS_x : \mathcal{F}_G \to \mathcal{F}_{G_x}$ from the category $\mathcal{F}_G$ of finite-dimensional G-modules to the category $\mathcal{F}_{G_x}$ of finite-dimensional G_x-modules induces a surjective ring homomorphism on the corresponding supercharacter rings $ds_x : \mathcal{J}_G \to \mathcal{J}_{G_x}$.*

Proof. We will use Corollary 11 to reduce to the case that $x \in \mathfrak{g}_\beta$ for some isotropic root β. Using the realization of ds_x given in Theorem 10, we will show that ds_x transfers a certain set of generators of the supercharacter ring $\mathcal{K}_G$ to a set of generators of the supercharacter ring $\mathcal{J}_{G_x}$. We use the same set of generators of $\mathcal{J}_G$ that Sergeev and Veselov [2011, §7] used to give explicit descriptions of supercharacter rings over basic Lie superalgebras and their corresponding Lie supergroups.

$GL(m, n)$: The supercharacter ring of $GL(m, n)$ is generated by $(x_1 \cdots x_m)/(y_1 \cdots y_n)$, $(y_1 \cdots y_n)/(x_1 \cdots x_m)$, $h_k(x_1, \dots, x_m, y_1, \dots, y_n)$, and $h_k(x_1^{-1}, \dots, x_m^{-1}, y_1^{-1}, \dots, y_n^{-1})$, $k \in \mathbb{Z}_{>0}$, where

$$\chi_G(t) = \frac{\prod_{i=1}^{m}(1 - x_i t)}{\prod_{j=1}^{n}(1 - y_j t)} = \sum_{k=0}^{\infty} h_k(x_1, \dots, x_m, y_1, \dots, y_n) t^k. \tag{5-1}$$

$SL(m,n), m \neq n$: The supercharacter ring of $SL(m,n), m \neq n$, is generated by $h_k(x_1,\dots,x_m,y_1,\dots,y_n)$ and $h_k(x_1^{-1},\dots,x_m^{-1},y_1^{-1},\dots,y_n^{-1})$, $k \in \mathbb{Z}_{>0}$, where h_k is given by (5-1).

$OSP(2m+1,2n)$: The supercharacter ring of $OSP(2m+1,2n)$ is generated by $h_k(x_1,\dots,x_m,y_1,\dots,y_n)$, $k \in \mathbb{Z}_{>0}$, where

$$\chi_G(t) = \frac{\prod_{j=1}^{n}(1-y_jt)(1-y_j^{-1}t)}{(1-t)\prod_{i=1}^{m}(1-x_it)(1-x_i^{-1}t)} = \sum_{k=0}^{\infty} h_k(x_1,\dots,x_m,y_1,\dots,y_n)t^k.$$

$OSP(2,2n)$: The supercharacter ring of $OSP(2,2n)$ is generated by $h_k(x_1,y_1,\dots,y_n)$, $k \in \mathbb{Z}_{>0}$, where

$$\chi_G(t) = \frac{\prod_{i=1}^{m}(1-y_jt)(1-y_j^{-1}t)}{(1-x_1t)(1-x_1^{-1}t)} = \sum_{k=0}^{\infty} h_k(x_1,y_1,\dots,y_n)t^k.$$

$OSP(2m,2n), m \geq 2$: The supercharacter ring of $OSP(2m,2n)$ is generated by $h_k(x_1,\dots,x_m,y_1,\dots,y_n)$, $k \in \mathbb{Z}_{>0}$, where

$$\chi_G(t) = \frac{\prod_{p=1}^{n}(1-y_jt)(1-y_j^{-1}t)}{\prod_{i=1}^{m}(1-x_it)(1-x_i^{-1}t)} = \sum_{k=0}^{\infty} h_k(x_1,\dots,x_m,y_1,\dots,y_n)t^k.$$

By Theorem 10, $ds_x(h_k^{\mathfrak{g}}) = (h_k^{\mathfrak{g}})|_{\beta=0}$. Since χ_G is W-invariant and $W\beta = \Delta_{\text{iso}}$ for any $\beta \in \Delta_{\text{iso}}$, it suffices to consider the case that $\beta = \varepsilon_1 - \delta_1$. In this case, $\beta = 0$ if and only if $x_1 = y_1$. It is not difficult to check that $\chi_G(t)|_{x_1=y_1} = \chi_{G_x}$, and hence, $ds_x(h_k^{\mathfrak{g}}) = h_k^{\mathfrak{g}_x}$. Thus, all the generators of $\mathcal{J}_{G_x}$ are in the image of ds_x.

The general case for arbitrary $x \in X_{\mathfrak{g}}$ now follows from Corollary 11, since the composition of surjective maps is surjective. □

Proposition 21. *Let $\mathfrak{g} = \mathfrak{sl}(m \mid n)$, $m \neq n$, or $\mathfrak{g} = \mathfrak{osp}(m \mid 2n)$. Then for any $x \in X_{\mathfrak{g}}$, the image of $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$ is the supercharacter ring $\mathcal{J}_{G_x}$ of the Lie supergroup G_x.*

Proof. We use Theorems 20 and 10, together with the description of the rings $\mathcal{J}_{\mathfrak{g}}$ given by Sergeev and Veselov [2011] to prove that the image of the map $ds_x : \mathcal{J}_{\mathfrak{g}} \to \mathcal{J}_{\mathfrak{g}_x}$ equals $\mathcal{J}_{G_x}$ in the case that $x \in \mathfrak{g}_\beta$ is an isotropic root β. The claim for any element $x \in X_{\mathfrak{g}}$ then follows Corollary 11.

The supercharacter ring of $\mathfrak{g} = \mathfrak{sl}(m \mid n)$, $m \neq n$, is $\mathcal{J}_{\mathfrak{g}} = \mathcal{J}_G \oplus \bigoplus_{a \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_a$, where

$$J(\mathfrak{g})_a = (x_1 \cdots x_n)^a \prod_{i,j}(1 - x_i y_j^{-1})\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{S_m \times S_n},$$

and $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]_0^{S_m \times S_n}$ is the quotient of the ring $\mathbb{Z}[x_1^{\pm 1},\dots,x_m^{\pm 1},y_1^{\pm 1},\dots,y_n^{\pm 1}]^{S_m \times S_n}$ by ideal $\langle x_1 \cdots x_m - y_1 \cdots y_n \rangle$. Clearly, $f|_{\beta=0} = f|_{x_i=y_j} = 0$ for any $f \in J(\mathfrak{g})_a$. Hence, $ds_x(f) = 0$ for any $x \in X_{\mathfrak{g}}$ and $f \in J(\mathfrak{g})_a$.

If $\mathfrak{g} = B(m \mid n)$, $C(n+1)$, or $D(m \mid n)$, then $\mathcal{J}_{\mathfrak{g}} = \mathcal{J}_G \oplus \tilde{\mathcal{J}}$ and $ds_x(f) = 0$ for all $f \in \tilde{\mathcal{J}}$. Indeed, for $\beta = \pm\varepsilon_i \pm \delta_j$ it is not difficult to check that $f|_{\beta=0} = f|_{x_i^{\pm 1} = y_j^{\pm 1}} = f|_{u_i=v_j} = 0$.

The supercharacter ring of $\mathfrak{g} = B(m \mid n)$ is $\mathcal{J}_\mathfrak{g} = \mathcal{J}_G \oplus J_{\mathfrak{g},1/2}$, where

$$J_{\mathfrak{g},1/2} = \left\{ \prod_{i=1}^{m} (x_i^{1/2} + x_i^{-1/2}) \prod_{i,j} (u_i - v_j) g \;\middle|\; g \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \right\}.$$

The supercharacter ring of $\mathfrak{g} = C(n+1)$ is $\mathcal{J}_\mathfrak{g} = \mathcal{J}_G \oplus \big(J(\mathfrak{g})_0^- \oplus \bigoplus_{a \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_a\big)$, where

$$J(\mathfrak{g})_0^- = \left\{ x_1 \prod_{j=1}^{n} (u_1 - v_j) g \;\middle|\; g \in \mathbb{Z}[u_1, v_1, \dots, v_n]^{S_n} \right\},$$

$$J(\mathfrak{g})_a = x_1^a \prod_{j=1}^{n} (1 - x_1 y_j)(1 - x_1 y_j^{-1}) \mathbb{Z}[x_1^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^W.$$

The supercharacter ring of $\mathfrak{g} = D(m \mid n)$ is $\mathcal{J}_\mathfrak{g} = \mathcal{J}_G \oplus \big(J(\mathfrak{g})_0^- \oplus J_{\mathfrak{g},1/2}\big)$, where

$$J(\mathfrak{g})_0^- = \left\{ \omega \prod_{i,j} (u_i - v_j) g \;\middle|\; g \in \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n]^{S_m \times S_n} \right\},$$

$$J_{\mathfrak{g},1/2} = \prod_{i,j} (u_i - v_j) ((x_1 \dots x_m)^{1/2} \mathbb{Z}[u_1, \dots, u_m, v_1, \dots, v_n])^W. \qquad \square$$

Proposition 22. *Let $\mathfrak{g} = \mathfrak{gl}(m \mid n)$ and $x \in X_\mathfrak{g}$. The image of $ds_x : \mathcal{J}_\mathfrak{g} \to \mathcal{J}_{\mathfrak{g}_x}$ is*

$$\bigoplus_{a \in \mathbb{C}/\mathbb{Z}} (x_1 \cdots x_{m-k})^a (y_1 \cdots y_{n-k})^{-a} \mathcal{J}_{G_x},$$

where k is the size of $\psi(x) \in S_\mathfrak{g}$ under the bijection $\psi : X_\mathfrak{g}/G_{\bar{0}} \to S_\mathfrak{g}/W$, and $\mathcal{J}_{G_x}$ is the supercharacter ring of the Lie supergroup G_x.

Proof. By Sergeev and Veselov [2011], the supercharacter ring of $\mathfrak{gl}(m \mid n)$ is $\mathcal{J}_\mathfrak{g} = \bigoplus_{a,b \in \mathbb{C}/\mathbb{Z}} J(\mathfrak{g})_{a,b}$ where $J(\mathfrak{g})_{0,0} = \mathcal{J}_G$,

$$J(\mathfrak{g})_{a,b} = (x_1 \cdots x_m)^a (y_1 \cdots y_n)^{-a} J(\mathfrak{g})_{0,0}$$

when $a + b \in \mathbb{Z}$, but $a \notin \mathbb{Z}$, and

$$J(\mathfrak{g})_{a,b} = (x_1 \cdots x_m)^a (y_1 \cdots y_n)^b \prod_{i,j} (1 - x_i y_j^{-1}) \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_m \times S_n}$$

when $a + b \notin \mathbb{Z}$.

Then we have that $f|_{x_i = y_j} = 0$ for any $f \in J(\mathfrak{g})_{a,b}$ with $a + b \notin \mathbb{Z}$. By Theorem 20, $ds_x(J(\mathfrak{g})_{0,0}) = J(\mathfrak{g}_x)_{0,0}$. Since $ds_x(f) = f|_{x_{r_i} = y_{s_i},\, i=1,\dots,k}$ by Theorem 10, we have that

$$ds_x(J(\mathfrak{g})_{a,b}) = (x_1 \cdots x_{m-k})^a (y_1 \cdots y_{n-k})^{-a} J(\mathfrak{g}_x)_{0,0} = J(\mathfrak{g}_x)_{a,b}$$

when $a + b \in \mathbb{Z}$, but $a \notin \mathbb{Z}$. $\square$

5B. ***The image of ds_x for the exceptional Lie superalgebras.*** In this section, we describe the image of ds_x for the Lie superalgebras $G(3)$, $F(4)$, and $D(2,1,\alpha)$, using the explicit description of the supercharacter rings given by Sergeev and Veselov [2011, §7].

Since $G(3)$, $F(4)$, and $D(2,1,\alpha)$ have defect 1, we may assume that $x \in \mathfrak{g}_\beta$ for some isotropic root β. Moreover, since $W\beta = \Delta_{\bar{1}}$, it suffices to describe the image for a fixed choice of β.

5B1. $G(3)$. Let $\beta = \varepsilon_3 + \delta_1$. Then $\mathfrak{g}_x \cong \mathfrak{sl}(2)$ with $\Delta_x = \{\pm(\varepsilon_1 - \varepsilon_2)\}$. The supercharacter ring of $G(3)$ is

$$\mathcal{J}_{\mathfrak{g}} = \{g(w) + (v_1 - u_1)(v_1 - u_2)(v_1 - u_3)h \mid h \in \mathbb{Z}[u_1, u_2, u_3, v_1]^{S_3},\ g \in \mathbb{Z}[w]\},$$

where $y_1 = e^{\delta_1}$, $v_1 = y_1 + y_1^{-1}$, $x_i = e^{\varepsilon_i}$, $u_i = x_i + x_i^{-1}$ for $i = 1, 2, 3$, and

$$w = v_1^2 - v_1(u_1 + u_2 + u_3 + 1) + u_1u_2 + u_1u_3 + u_2u_3.$$

Note that $x_1x_2x_3 = 1$, so $u_3 = x_1x_2 + x_1^{-1}x_2^{-1}$.

Theorem 10 implies that $ds_x(f) = f|_{y_1 = x_3^{-1} = x_1x_2}$ for every $f \in \mathcal{J}_{\mathfrak{g}}$. Hence, $ds_x(f) = ds_x(g(w))$ since $(v_1 - u_3)|_{y_1 = x_3^{-1} = x_1x_2} = 0$. Thus, the image of ds_x is the polynomial ring $\mathbb{Z}[w_x]$ generated by the element

$$w_x := w|_{y_1 = x_3^{-1} = x_1x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} \in \mathcal{J}_{\mathfrak{g}_x}.$$

Note that $w_x + 1$ is the supercharacter of the adjoint representation of $\mathfrak{sl}(2)$, and that $x_1/x_2 + x_2/x_1$ equals $x_1^2 + x_2^2$ in $\mathcal{J}_{\mathfrak{g}_x}$ due to the relation $x_1x_2 = 1$. Finally, we obtain that

$$\operatorname{Im} ds_x = \mathbb{Z}[x_1^2 + x_2^{-2}] \subsetneq \mathcal{J}_{G_x} = \mathcal{J}_{SL(2)} = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]^{S_2}/\langle x_1x_2 - 1\rangle \cong \mathbb{Z}[x_1 + x_1^{-1}].$$

5B2. $F(4)$. Let $\beta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \delta_1)$. Then $\mathfrak{g}_x \cong \mathfrak{sl}(3)$ with $\Delta_x = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le 3\}$. The supercharacter ring of $F(4)$ is

$$\mathcal{J}_{\mathfrak{g}} = \{g(w_1, w_2) + Qh \mid h \in \mathbb{Z}[x_1^{\pm 2}, x_2^{\pm 2}, x_3^{\pm 2}, (x_1x_2x_3)^{\pm 1}, y_1^{\pm 1}]^{W_0},\ g \in \mathbb{Z}[w_1, w_2]\},$$

where $y_1 = e^{(1/2)\delta_1}$, $x_i = e^{(1/2)\varepsilon_i}$ for $i = 1, 2, 3$, and

$$Q = (y_1 + y_1^{-1} - x_1x_2x_3 - x_1^{-1}x_2^{-1}x_3^{-1}) \prod_{i=1}^{3}\left(y_1 + y_1^{-1} - \frac{x_1x_2x_3}{x_i^2} - \frac{x_i^2}{x_1x_2x_3}\right),$$

$$w_k = \sum_{i \ne j} \frac{x_i^{2k}}{x_j^{2k}} + \sum_{i=1}^{3}(x_i^{2k} + x_i^{-2k}) + y_1^{2k} + y_1^{-2k} - (y_1^k + y_1^{-k})\prod_{i=1}^{3}(x_i^k + x_i^{-k}), \quad k = 1, 2.$$

Theorem 10 implies that $ds_x(f) = f|_{x_1x_2x_3 = y_1}$ for every $f \in \mathcal{J}_{\mathfrak{g}}$. Hence, $ds_x(f) = ds_x(g(w_1, w_2))$ since $Q|_{x_1x_2x_3 = y_1} = 0$. Thus, the image of ds_x is generated by the elements

$$w_x^1 := w_1|_{x_1x_2x_3 = y_1} = \sum_{i \ne j} \frac{x_i^2}{x_j^2},$$

$$w_x^2 := w_2|_{x_1x_2x_3 = y_1} = \sum_{i \ne j} \frac{x_i^4}{x_j^4},$$

and is a proper subring of $\mathcal{J}_{G_x} = \mathcal{J}_{SL(3)} = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm}]^{S_3}/\langle x_1x_2x_3 - 1\rangle$.

5B3. $D(2,1,\alpha)$. Let $\beta=\varepsilon_1-\varepsilon_2-\varepsilon_3$. Then $\mathfrak{g}_x\cong\mathbb{C}$.

If $\alpha\notin\mathbb{Q}$, then the supercharacter ring of $D(2,1,\alpha)$ is

$$\mathcal{J}_{\mathfrak{g}}=\{c+Qh\mid c\in\mathbb{Z},\ h\in\mathbb{Z}[u_1,u_2,u_3]\},$$

where $x_i:=e^{\varepsilon_i}$, $u_i=x_i+x_i^{-1}$ for $i=1,2,3$, and

$$\begin{aligned}Q&=(x_1-x_2x_3)(x_2-x_1x_3)(x_3-x_1x_2)(1-x_1x_2x_3)x_1^{-2}x_2^{-2}x_3^{-2}\\&=u_1^2+u_2^2+u_3^2-u_1u_2u_3-4.\end{aligned}$$

If $\alpha=p/q\in\mathbb{Q}$, then the supercharacter ring of $D(2,1,\alpha)$ is

$$\mathcal{J}_{\mathfrak{g}}=\{g(w_\alpha)+Qh\mid g\in\mathbb{Z}[w_\alpha],\ h\in\mathbb{Z}[u_1,u_2,u_3]\},$$

where

$$w_\alpha=(x_1+x_1^{-1}-x_2x_3-x_2^{-1}x_3^{-1})\frac{(x_2^p-x_2^{-p})(x_3^q-x_3^{-q})}{(x_2-x_2^{-1})(x_3-x_3^{-1})}+x_2^px_3^{-q}+x_2^{-p}x_3^q.$$

By Theorem 10, $ds_x(f)=f|_{x_1=x_2x_3}$ for every $f\in\mathcal{J}_{\mathfrak{g}}$. Since $Q|_{x_1=x_2x_3}=0$, $ds_x(f)=c$ for some $c\in\mathbb{Z}$ when $\alpha\notin\mathbb{Q}$, while $ds_x(f)=ds_x(g(w_\alpha))$ when $\alpha\in\mathbb{Q}$. Thus, the image of ds_x is $\mathbb{Z}\subset\mathcal{J}_{\mathbb{C}}$ when $\alpha\notin\mathbb{Q}$ and the image is $\mathbb{Z}[w_\alpha]\subset\mathcal{J}_{\mathbb{C}}$ when $\alpha\in\mathbb{Q}$.

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