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# Ordinary algebraic curves with many automorphisms in positive characteristic

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Let  $\mathcal{X}$  be an ordinary (projective, geometrically irreducible, nonsingular) algebraic curve of genus  $g(\mathcal{X}) \geq 2$  defined over an algebraically closed field  $\mathbb{K}$  of odd characteristic  $p$ . Let  $\text{Aut}(\mathcal{X})$  be the group of all automorphisms of  $\mathcal{X}$  which fix  $\mathbb{K}$  elementwise. For any solvable subgroup  $G$  of  $\text{Aut}(\mathcal{X})$  we prove that  $|G| \leq 34(g(\mathcal{X}) + 1)^{3/2}$ . There are known curves attaining this bound up to the constant 34. For  $p$  odd, our result improves the classical Nakajima bound  $|G| \leq 84(g(\mathcal{X}) - 1)g(\mathcal{X})$  and, for solvable groups  $G$ , the Gunby–Smith–Yuan bound  $|G| \leq 6(g(\mathcal{X})^2 + 12\sqrt{21}g(\mathcal{X})^{3/2})$  where  $g(\mathcal{X}) > cp^2$  for some positive constant  $c$ .

## 1. Introduction

In this paper,  $\mathcal{X}$  stands for a (projective, geometrically irreducible, nonsingular) algebraic curve of genus  $g(\mathcal{X}) \geq 2$  defined over an algebraically closed field  $\mathbb{K}$  of odd characteristic  $p$ . Let  $\text{Aut}(\mathcal{X})$  be the group of all automorphisms of  $\mathcal{X}$  which fix  $\mathbb{K}$  elementwise. The assumption  $g(\mathcal{X}) \geq 2$  ensures that  $\text{Aut}(\mathcal{X})$  is finite. However, the classical Hurwitz bound  $|\text{Aut}(\mathcal{X})| \leq 84(g(\mathcal{X}) - 1)$  for complex curves fails in positive characteristic, and there exist four families of curves satisfying  $|\text{Aut}(\mathcal{X})| \geq 8g^3(\mathcal{X})$  [Stichtenoth 1973; Henn 1978; Hirschfeld et al. 2008, §11.12]. Each of them has  $p$ -rank  $\gamma(\mathcal{X})$  (equivalently, its Hasse–Witt invariant) equal to zero; see for instance [Giulietti and Korchmáros 2014]. On the other hand, if  $\mathcal{X}$  is ordinary, i.e.,  $g(\mathcal{X}) = \gamma(\mathcal{X})$ , Guralnick and Zieve announced in 2004, as reported in [Gunby et al. 2015; Kontogeorgis and Rotger 2008], that for odd  $p$  there exists a sharper bound, namely  $|\text{Aut}(\mathcal{X})| \leq c_p g(\mathcal{X})^{8/5}$  with some constant  $c_p$  depending on  $p$ . It should be noticed that no proof of this sharper bound is available in the literature. In this paper, we concern ourselves with solvable automorphism groups  $G$  of an ordinary curve  $\mathcal{X}$ , and for odd  $p$  we prove the even sharper bound:

**Theorem 1.1.** *Let  $\mathcal{X}$  be an algebraic curve of genus  $g(\mathcal{X}) \geq 2$  defined over an algebraically closed field  $\mathbb{K}$  of odd characteristic  $p$ . If  $\mathcal{X}$  is ordinary and  $G$  is a solvable subgroup of  $\text{Aut}(\mathcal{X})$ , then*

$$|G| \leq 34(g(\mathcal{X}) + 1)^{3/2}. \quad (1)$$

For odd  $p$ , our result provides an improvement on the classical Nakajima bound  $|G| \leq 84(g(\mathcal{X}) - 1)g(\mathcal{X})$  [1987] and, for solvable groups, on the recent Gunby–Smith–Yuan bound  $|G| \leq 6(g(\mathcal{X})^2 + 12\sqrt{21}g(\mathcal{X})^{3/2})$  proven in [Gunby et al. 2015] under the hypothesis that  $g(\mathcal{X}) > cp^2$  for some positive constant  $c$ .

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The following example is due to Stichtenoth, and it shows that (1) is the best possible bound apart from the constant  $c$  [Korchmáros et al. 2018]. Let  $\mathbb{F}_q$  be a finite field of order  $q = p^h$ , and let  $\mathbb{K} = \overline{\mathbb{F}_q}$  denote its algebraic closure. For a positive integer  $m$  prime to  $p$ , let  $\mathcal{Y}$  be the irreducible curve with affine equation

$$y^q + y = x^m + \frac{1}{x^m} \quad (2)$$

and  $F = \mathbb{K}(\mathcal{Y})$  its function field. Let  $t = x^{m(q-1)}$ . The extension  $F|\mathbb{K}(t)$  is a non-Galois extension as the Galois closure of  $F$  with respect to  $H$  is the function field  $\mathbb{K}(x, y, z)$  where  $x, y, z$  are linked by (2) and  $z^q + z = x^m$ . Furthermore,  $g(\mathcal{Y}) = (q-1)(qm-1)$ ,  $\gamma(\mathcal{Y}) = (q-1)^2$ , and  $\text{Aut}(\mathcal{Y})$  contains a subgroup  $Q \rtimes U$  of index 2 where  $Q$  is an elementary abelian normal subgroup of order  $q^2$  and the complement  $U$  is a cyclic group of order  $m(q-1)$ . If  $m = 1$ , then  $\mathcal{Y}$  is an ordinary curve, and in this case  $2g(\mathcal{Y})^{3/2} = 2(q-1)^3 < 2q^2(q-1) = |\text{Aut}(\mathcal{Y})|$ , which shows indeed that (1) is sharp up to the constant  $c$ .

Lower bounds on the order of solvable automorphism groups of algebraic curves depending on their genera are due to Neftin and Zieve. Their [2015, Theorem 4.1] states that for every integer  $\ell > 0$  there exists a curve  $\mathcal{X}$  together with a solvable subgroup of  $\text{Aut}(\mathcal{X})$  of order  $d$  and derived length  $\ell$  such that

$$g(\mathcal{X}) \leq c_\ell d \log^{o_\ell}(d),$$

where  $c_\ell$  is a constant and  $\log^{o_\ell}$  denotes log iterated  $\ell$  times. The curve  $\mathcal{X}$  is constructed as a solvable cover of a curve with at least one rational point, in which a given set  $S$  of rational points splits completely.

## 2. Background and preliminary results

For a subgroup  $G$  of  $\text{Aut}(\mathcal{X})$ , let  $\overline{\mathcal{X}}$  denote a nonsingular model of  $\mathbb{K}(\mathcal{X})^G$ , that is, a (projective, nonsingular, geometrically irreducible) algebraic curve with function field  $\mathbb{K}(\mathcal{X})^G$ , where  $\mathbb{K}(\mathcal{X})^G$  consists of all elements of  $\mathbb{K}(\mathcal{X})$  fixed by every element in  $G$ . Usually,  $\overline{\mathcal{X}}$  is called the quotient curve of  $\mathcal{X}$  by  $G$  and denoted by  $\mathcal{X}/G$ . The field extension  $\mathbb{K}(\mathcal{X})|\mathbb{K}(\mathcal{X})^G$  is Galois of degree  $|G|$ .

Since our approach is mostly group theoretical, we prefer to use notation and terminology from group theory rather than from function field theory.

Let  $\Phi$  be the cover of  $\mathcal{X}|\overline{\mathcal{X}}$  where  $\overline{\mathcal{X}} = \mathcal{X}/G$ . A point  $P \in \mathcal{X}$  is a ramification point of  $G$  if the stabilizer  $G_P$  of  $P$  in  $G$  is nontrivial; the ramification index  $e_P$  is  $|G_P|$ ; a point  $\overline{Q} \in \overline{\mathcal{X}}$  is a branch point of  $G$  if there is a ramification point  $P \in \mathcal{X}$  such that  $\Phi(P) = \overline{Q}$ ; the ramification (branch) locus of  $G$  is the set of all ramification (branch) points. The  $G$ -orbit of  $P \in \mathcal{X}$  is the subset  $o = \{R \mid R = g(P), g \in G\}$  of  $\mathcal{X}$ , and it is *long* if  $|o| = |G|$ ; otherwise  $o$  is *short*. For a point  $\overline{Q}$ , the  $G$ -orbit  $o$  lying over  $\overline{Q}$  consists of all points  $P \in \mathcal{X}$  such that  $\Phi(P) = \overline{Q}$ . If  $P \in o$ , then  $|o| = |G|/|G_P|$  and hence  $\overline{Q}$  is a branch point if and only if  $o$  is a short  $G$ -orbit. It may be that  $G$  has no short orbits. This is the case if and only if every nontrivial element in  $G$  is fixed-point-free on  $\mathcal{X}$ , that is, the cover  $\Phi$  is unramified. On the other hand,  $G$  has a finite number of short orbits. For a nonnegative integer  $i$ , the  $i$ -th ramification group of  $\mathcal{X}$  at  $P$  is denoted by  $G_P^{(i)}$  (or  $G_i(P)$  as in [Serre 1979, Chapter IV]) and defined to be

$$G_P^{(i)} = \{g \mid \text{ord}_P(g(t) - t) \geq i + 1, g \in G_P\},$$

where  $t$  is a uniformizing element (local parameter) at  $P$ . Here  $G_P^{(0)} = G_P$ .

Let  $\bar{g}$  be the genus of the quotient curve  $\bar{\mathcal{X}} = \mathcal{X}/G$ . The Riemann–Hurwitz genus formula gives

$$2g - 2 = |G|(2\bar{g} - 2) + \sum_{P \in \mathcal{X}} d_P, \quad (3)$$

where the different  $d_P$  at  $P$  is given by

$$d_P = \sum_{i \geq 0} (|G_P^{(i)}| - 1). \quad (4)$$

If  $|G_P|$  is prime to  $p$ , then  $d_P = |G_P| - 1$ .

Let  $\gamma$  be the  $p$ -rank of  $\mathcal{X}$ , and let  $\bar{\gamma}$  be the  $p$ -rank of the quotient curve  $\bar{\mathcal{X}} = \mathcal{X}/G$ . The Deuring–Shafarevich formula (see [Sullivan 1975] or [Hirschfeld et al. 2008, Theorem 11.62]) states that if  $G$  is a  $p$ -group then

$$\gamma - 1 = |G|(\bar{\gamma} - 1) + \sum_{i=1}^k (|G| - \ell_i) \quad (5)$$

where  $\ell_1, \dots, \ell_k$  are the sizes of the short orbits of  $G$ . If  $\mathcal{X}$  is ordinary (and hence  $G_P^{(2)}$  is trivial for every  $P \in \mathcal{X}$ ; see Result 2.5(i)), then  $d_P = |G_P^{(0)}| - 1 + |G_P^{(1)}| - 1 = 2(|G_P^{(0)}| - 1) = 2(|G_P| - 1)$  and hence (5) follows from (3) and vice versa.

The Nakajima bound (see [1987, Theorem 1] or [Hirschfeld et al. 2008, Theorem 11.84]) states that the existence of large  $p$ -groups of automorphisms implies that  $\gamma = 0$ .

**Result 2.1.** *If  $\mathcal{X}$  has positive  $p$ -rank  $\gamma$ , then every  $p$ -subgroup of  $\text{Aut}(\mathcal{X})$  has order  $\leq p(\gamma - 1)/(p - 2)$ .*

A subgroup of  $\text{Aut}(\mathcal{X})$  is a prime to  $p$  group (or a  $p'$ -subgroup) if its order is prime to  $p$ . A subgroup  $G$  of  $\text{Aut}(\mathcal{X})$  is *tame* if the 1-point stabilizer of any point in  $G$  is  $p'$ -group. Otherwise,  $G$  is *nontame* (or *wild*). Obviously, every  $p'$ -subgroup of  $\text{Aut}(\mathcal{X})$  is tame, but the converse is not always true.

**Result 2.2.** *The following claims hold.*

- (i) *If  $|G| > 84(g(\mathcal{X}) - 1)$ , then  $G$  is nontame.*
- (ii) *If  $G$  is abelian, then  $|G| \leq 4g + 4$ .*
- (iii) *If  $G$  has prime order other than  $p$ , then  $|G| \leq 2g + 1$ .*

The first two claims are due to Stichtenoth [1973]; see also [Hirschfeld et al. 2008, Theorems 11.56 and 11.79]. For a proof of claim (iii), see [Homma 1980] or [Hirschfeld et al. 2008, Theorem 11.108].

Henn’s bound [1978] (see also [Hirschfeld et al. 2008, Theorem 11.127]) has the following corollary.

**Result 2.3.** *If  $|G| > 8g^3$ , then  $\mathcal{X}$  has zero  $p$ -rank, and  $G$  is not solvable.*

An orbit  $o$  of  $G$  is *tame* if  $G_P$  is a  $p'$ -group for  $P \in o$ . The structure of  $G_P$  is well known; see for instance [Serre 1979, Chapter IV, Corollary 4] or [Hirschfeld et al. 2008, Theorem 11.49].

**Result 2.4.** *The stabilizer  $G_P$  of a point  $P \in \mathcal{X}$  in  $G$  is a semidirect product  $G_P = Q_P \rtimes U$  where the normal subgroup  $Q_P$  is a  $p$ -group while the complement  $U$  is a cyclic prime to  $p$  group.*

If  $\mathcal{X}$  is ordinary, some more results are available; those used in this paper are collected below.

**Result 2.5.** *If  $\mathcal{X}$  is an ordinary curve, then*

- (i)  $Q_P^{(2)}$  is trivial,
- (ii)  $Q_P$  is elementary abelian,
- (iii) no nontrivial element of  $U$  commutes with a nontrivial element of  $Q_P$ ,
- (iv)  $|U|$  divides  $|Q_P| - 1$ , and
- (v) the quotient curve  $\mathcal{X}/G$  for a  $p$ -group  $G$  of automorphisms is also ordinary.

Claim (i) is due to Nakajima [1987, Theorem 2.1]. Claim (ii) follows from claim (i) by Serre's result [1979, Corollary 3, p. 67] stating that the factor groups  $Q_P^{(i)}/Q_P^{(i+1)}$  for  $i \geq 1$  are elementary abelian; see also [Hirschfeld et al. 2008, Theorem 11.74]. Claim (iii) follows from claim (ii) by Serre's result [1979, Corollary 1, p. 69]; see also [Hirschfeld et al. 2008, Theorem 11.75(ii)]. Claim (iv) is a consequence of claim (iii) since the latter claim together with Result 2.4 imply that  $U$  induces an automorphism group of  $Q_P$ . Claim (v) follows from comparison of (3) to (5) taking into account claim (i).

For a nontrivial  $p$ -subgroup  $G$  of  $\text{Aut}(\mathcal{X})$ , divide both sides in (3) by 2 and then subtract the result from (5). If  $G_P^{(2)}$  is trivial for every  $P \in \mathcal{X}$ , then this computation gives

$$g(\mathcal{X}) - \gamma(\mathcal{X}) = |G|(g(\bar{\mathcal{X}}) - \gamma(\bar{\mathcal{X}})) \quad (6)$$

where  $\bar{\mathcal{X}} = \mathcal{X}/Q$  [Nakajima 1987]. This shows the first two claims of the following result hold. The third one is due to Stichtenoth [1973]; see also [Hirschfeld et al. 2008, Theorem 11.79].

**Result 2.6.** *Let  $Q$  be nontrivial  $p$ -subgroup of  $\text{Aut}(\mathcal{X})$ . Assume that  $Q_P^{(2)}$  is trivial for every  $P \in \mathcal{X}$ . Then*

- (i) (6) holds,
- (ii)  $\mathcal{X}$  and its quotient curve  $\mathcal{X}/Q$  are simultaneously ordinary or not, and
- (iii)  $|Q_P| \leq pg(\mathcal{X})/(p-1)$ .

The first two claims below on low-genus curves are well known; see for instance [Hirschfeld et al. 2008, Theorems 11.94 and 11.99]. The third one is a corollary of Henn's bound.

**Result 2.7.** *If  $G$  is an automorphism group of an elliptic curve  $\mathcal{E}$  over  $\mathbb{K}$ , then for every point  $P \in \mathcal{E}$  the order of the stabilizer  $G_P$  of  $P$  in  $G$  divides 6 when  $p > 3$  and 12 when  $p = 3$ . The solvable automorphism groups of a genus-2 curve over  $\mathbb{K}$  have order at most 48. For genus-3 curves the latter bound is 216.*

We also need a technical result.

**Result 2.8.** *Assume that  $\text{Aut}(\mathcal{X})$  has a solvable subgroup  $G$  of order larger than  $34(g(\mathcal{X}) + 1)^{3/2}$ . If  $N$  is a normal subgroup of  $G$  and the quotient curve  $\bar{\mathcal{X}} = \mathcal{X}/N$  is neither rational nor elliptic, then the automorphism group  $\bar{G} = G/N$  of  $\bar{\mathcal{X}}$  has order larger than  $34(g(\bar{\mathcal{X}}) + 1)^{3/2}$ , as well.*

Since  $|N| = |G|/|\bar{G}|$ , the claim is a straightforward consequence of (3) except for the cases where  $\mathfrak{g}(\bar{\mathcal{X}}) = 2$ , or  $\mathfrak{g}(\bar{\mathcal{X}}) = 3$ ,  $\mathfrak{g}(\mathcal{X}) = 5$ ,  $|N| = 2$ , and the cover  $\mathcal{X}/\bar{\mathcal{X}}$  is unramified. Actually, the exceptional cases do not occur. In fact,  $|\bar{G}| \geq |G|(\mathfrak{g}(\bar{\mathcal{X}}) - 1)/(\mathfrak{g}(\mathcal{X}) - 1) > 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}(\mathfrak{g}(\bar{\mathcal{X}}) - 1)/(\mathfrak{g}(\mathcal{X}) - 1)$  is bigger than 48 and  $8 \cdot 27 = 216$  for  $\mathfrak{g}(\bar{\mathcal{X}}) = 2$  and  $\mathfrak{g}(\bar{\mathcal{X}}) = 3$ , contradicting Results 2.7 and 2.3, respectively.

From group theory we use Dickson's classification of finite subgroups of the projective linear group  $PGL(2, \mathbb{K})$ ; see [Valentini and Madan 1980] or [Hirschfeld et al. 2008, Theorem A.8].

**Result 2.9.** *The following is a complete list of finite solvable subgroups of  $PGL(2, \mathbb{K})$  up to conjugacy:*

- (i) *cyclic groups of order prime to  $p$ ,*
- (ii) *elementary abelian  $p$ -groups,*
- (iii) *dihedral groups with an index-2 cyclic subgroup of order prime to  $p$ ,*
- (iv) *the alternating group  $A_4$ ,*
- (v) *the symmetric group  $S_4$ ,*
- (vi) *semidirect products of an elementary abelian  $p$ -group of order  $p^h$  by a cyclic group of order  $n$  with  $n \mid (p^h - 1)$ .*

If  $PGL(2, \mathbb{K})$  is viewed as the automorphism group of the line over  $\mathbb{K}$ , any cyclic subgroup of order prime to  $p$  has exactly two points, while any  $p$ -subgroup has a unique fixed point [Valentini and Madan 1980].

We also use the Schur–Zassenhaus theorem; see for instance [Machì 2012, Corollary 7.5].

**Result 2.10.** *Let  $G$  be a finite group with a normal subgroup  $N$ . If  $|N|$  is prime to the index  $[G : N]$  of  $N$ , then  $N$  has a complement in  $G$ , that is,  $G = N \rtimes M$  for a subgroup  $M$  of  $G$ . Such complements are pairwise conjugate in  $G$ .*

From representation theory, we need the Maschke theorem; see for instance [Machì 2012, Theorem 6.1].

**Result 2.11.** *Any representation of a finite group over a field whose characteristic is prime to the order of the group is completely reducible.*

The following two lemmas of independent interest play a role in our proof of Theorem 1.1.

**Lemma 2.12.** *Let  $\mathcal{X}$  be an ordinary algebraic curve of genus  $\mathfrak{g}(\mathcal{X}) \geq 2$  defined over an algebraically closed field  $\mathbb{K}$  of odd characteristic  $p$ . Let  $H$  be a solvable automorphism group of  $\text{Aut}(\mathcal{X})$  containing a normal  $p$ -subgroup  $Q$  such that  $|Q|$  and  $[H : Q]$  are coprime. Suppose that a complement  $U$  of  $Q$  in  $H$  is abelian and that*

$$|H| > \begin{cases} 18(\mathfrak{g} - 1) & \text{for } |U| = 3, \\ 12(\mathfrak{g} - 1) & \text{otherwise.} \end{cases} \quad (7)$$

*Then  $U$  is cyclic, and the quotient curve  $\bar{\mathcal{X}} = \mathcal{X}/Q$  is rational. Furthermore,  $Q$  has exactly two (nontame) short orbits, say  $\Omega_1, \Omega_2$ . They are also the only short orbits of  $H$ , and  $\mathfrak{g}(\mathcal{X}) = |Q| - (|\Omega_1| + |\Omega_2|) + 1$ .*



*Proof.* From [Result 2.10](#),  $H = Q \rtimes U$ . Set  $|Q| = p^k$  and  $|U| = u$ . Then  $p \nmid u$ . Furthermore, if  $u = 2$ , then  $|H| = 2|Q| > 9g(\mathcal{X})$  whence  $|Q| > 4.5g(\mathcal{X})$ . From [Result 2.1](#),  $\mathcal{X}$  has zero  $p$ -rank, which is not possible as  $\mathcal{X}$  is assumed to be ordinary of genus at least 2. Therefore,  $u \geq 3$ .

Three cases are treated separately according as the quotient curve  $\bar{\mathcal{X}} = \mathcal{X}/Q$  has genus  $\bar{g}$  at least 2, or  $\bar{\mathcal{X}}$  is elliptic, or rational.

If  $g(\bar{\mathcal{X}}) \geq 2$ , then  $\text{Aut}(\bar{\mathcal{X}})$  has a subgroup isomorphic to  $U$ , and [Result 2.2\(ii\)](#) yields  $4g(\bar{\mathcal{X}}) + 4 \geq |U|$ . Furthermore, from [\(3\)](#) applied to  $Q$ ,  $g - 1 \geq |Q|(g(\bar{\mathcal{X}}) - 1)$ . Let  $c = 12$  or  $c = 18$ , according as  $|U| > 3$  or  $|U| = 3$ , so that  $|H| > c(g - 1)$  from [\(7\)](#). Then

$$(4g(\bar{\mathcal{X}}) + 4)|Q| \geq |U||Q| = |H| \geq c(g - 1) \geq c|Q|(g(\bar{\mathcal{X}}) - 1),$$

whence

$$c \leq 4 \frac{g(\bar{\mathcal{X}}) + 1}{g(\bar{\mathcal{X}}) - 1}.$$

As the right-hand side is smaller than 12, a contradiction to the choice of the constant  $c$  is obtained.

If  $\bar{\mathcal{X}}$  is elliptic, then the cover  $\mathcal{X}|\bar{\mathcal{X}}$  ramifies; otherwise  $\mathcal{X}$  itself would be elliptic. Thus,  $Q$  has some short orbits. The group  $H$  acts on the set of short orbits of  $Q$ . In this action, an orbit of a given short orbit  $o$  of  $Q$  with respect to  $H$  is a set of short orbits of  $Q$  having the same length of  $o$ . We will refer to these short orbits as images of  $o$ . Take a short orbit of  $Q$  together with its images  $o_1, \dots, o_{u_1}$  under the action of  $H$ . Since  $Q$  is a normal subgroup of  $H$ ,  $o = o_1 \cup \dots \cup o_{u_1}$  is an  $H$ -orbit of size  $u_1 p^v$  where  $p^v = |o_1| = \dots = |o_{u_1}|$ . Equivalently, the stabilizer of a point  $P \in o$  has order  $p^{k-v} u / u_1$ , and by [Result 2.4](#), it is the semidirect product  $Q_1 \rtimes U_1$  where  $|Q_1| = p^{k-v}$  and  $|U_1| = u / u_1$  for subgroups  $Q_1$  of  $Q$  and  $U_1$  of  $U$ , respectively. The point  $\bar{P}$  lying under  $P$  in the cover  $\mathcal{X}|\bar{\mathcal{X}}$  is fixed by the factor group  $\bar{U}_1 = U_1 Q / Q$ . Since  $\bar{\mathcal{X}}$  is elliptic, and  $p$  is prime to  $|\bar{U}_1|$ , [Result 2.7](#) yields  $|\bar{U}_1| \leq 4$  for  $p = 3$  and  $|\bar{U}_1| \leq 6$  for  $p > 3$ . As  $\bar{U}_1 \cong U_1$ , this yields the same bound for  $|U_1|$ , that is,  $u \leq 4u_1$  for  $p = 3$  and  $u \leq 6u_1$  for  $p > 3$ . Furthermore, since the  $p$ -group  $Q_1$  fixes  $P$ , and  $Q_1^{(0)} = Q_1^{(1)} = Q_1$ , we have  $d_P = \sum_{i \geq 0} (|Q_1^{(i)}| - 1) \geq 2(|Q_1| - 1) = 2(p^{k-v} - 1) \geq \frac{4}{3}p^{k-v}$ . From [\(3\)](#) applied to  $Q$ , since  $P \in o$  and  $|o| = p^v u_1$ , if  $p = 3$ , then

$$2g - 2 \geq 3^v u_1 d_P \geq 3^v u_1 \left( \frac{4}{3} 3^{k-v} \right) = \frac{4}{3} 3^k u_1 \geq \frac{1}{3} 3^k u = \frac{1}{3} |Q||U| = \frac{1}{3} |H|,$$

while for  $p > 3$ ,

$$2g - 2 \geq p^v u_1 d_P \geq p^v u_1 \left( \frac{4}{3} p^{k-v} \right) = \frac{4}{3} p^k u_1 \geq \frac{2}{9} p^k u = \frac{2}{9} |Q||U| = \frac{2}{9} |H|,$$

but this contradicts [\(7\)](#).

If  $\bar{\mathcal{X}}$  is rational, then  $Q$  has at least one short orbit. Furthermore,  $\bar{U} = UQ/Q$  is isomorphic to a subgroup of  $PGL(2, \mathbb{K}) \cong \text{Aut}(\bar{\mathcal{X}})$ . Since  $U \cong \bar{U}$  and  $U$  is abelian, from [Result 2.9](#),  $\bar{U}$  is cyclic,  $\bar{U}$  fixes two points  $\bar{P}_0$  and  $\bar{P}_\infty$ , but no nontrivial element in  $\bar{U}$  fixes a point other than  $\bar{P}_0$  or  $\bar{P}_\infty$ . Let  $o_\infty$  and  $o_0$  be the  $Q$ -orbits lying over  $\bar{P}_0$  and  $\bar{P}_\infty$ , respectively. Obviously,  $o_\infty$  and  $o_0$  are short orbits of  $H$ . We show that  $Q$  has at most two short orbits, the candidates being  $o_\infty$  and  $o_0$ . By absurd, there is a  $Q$ -orbit  $o$

of size  $p^m$  with  $m < k$  which lies over a point  $\bar{P} \in \bar{\mathcal{X}}$  different from both  $\bar{P}_0$  and  $\bar{P}_\infty$ . Since the orbit of  $\bar{P}$  in  $\bar{U}$  has length  $u$ , then the  $H$ -orbit of a point  $P \in o$  has length  $up^m$ . If  $u > 3$ , (3) applied to  $Q$  gives

$$2g - 2 \geq -2p^k + up^m(p^{k-m} - 1) \geq -2p^k + up^m \frac{2}{3}p^{k-m} = -2p^k + \frac{2}{3}up^k = \frac{2}{3}(u - 3)p^k > \frac{1}{6}up^k = \frac{1}{6}|H|,$$

a contradiction with  $|H| > 12(g - 1)$ . If  $u = 3$ , then  $p > 3$ , and hence,

$$2g - 2 \geq -2p^k + 3p^m(p^{k-m} - 1) = p^k - 3p^m > \frac{1}{3}p^k,$$

whence  $|H| = 3p^k < 18(g - 1)$ , a contradiction with (7). This proves that  $H$  has exactly two short orbits. Since, as we have showed,  $Q$  has either one or two short orbits, and they are contained in  $o_\infty \cup o_0$ , two cases arise correspondingly. Assume first that  $Q$  has two short orbits. They are  $o_\infty$  and  $o_0$ . If their lengths are  $p^a$  and  $p^b$  with  $a, b < k$ , then (5) (or (3)) applied to  $Q$  gives

$$g(\mathcal{X}) - 1 = \gamma(\mathcal{X}) - 1 = -p^k + (p^k - p^a) + (p^k - p^b)$$

whence  $g(\mathcal{X}) = p^k - (p^a + p^b) + 1 > 0$ . The same argument shows that if  $Q$  has just one short orbit, then  $\gamma(\mathcal{X}) = 0$ , a contradiction.  $\square$

**Lemma 2.13.** *Let  $N$  be an automorphism group of an algebraic curve of even genus such that  $|N|$  is even. Then any 2-subgroup of  $N$  has a cyclic subgroup of index 2.*

*Proof.* Let  $U$  be a subgroup of  $N$  of order  $d = 2^u \geq 2$ , and  $\bar{\mathcal{X}} = \mathcal{X}/U$  the arising quotient curve. From (3) applied to  $U$ ,

$$2g(\mathcal{X}) - 2 = 2^u(2g(\bar{\mathcal{X}}) - 2) + \sum_{i=1}^m (2^u - \ell_i)$$

where  $\ell_1, \dots, \ell_m$  are the short orbits of  $U$  on  $\mathcal{X}$ . Since  $g(\mathcal{X})$  is even,  $2g(\mathcal{X}) - 2 \equiv 2 \pmod{4}$ . On the other hand,  $2^u(2g(\bar{\mathcal{X}}) - 2) \equiv 0 \pmod{4}$ . Therefore, some  $\ell_i$  ( $1 \leq i \leq m$ ) must be either 1 or 2. Therefore,  $U$  or a subgroup of  $U$  of index 2 fixes a point of  $\mathcal{X}$  and hence is cyclic.  $\square$

### 3. The proof of Theorem 1.1

Our proof is by induction on the genus. Theorem 1.1 holds for  $g(\mathcal{X}) = 2$ , as  $|G| \leq 48$  for any solvable automorphism group  $G$  of a genus-2 curve; see Result 2.7. For  $g(\mathcal{X}) > 2$ ,  $\mathcal{X}$  is taken by absurd for a minimal counterexample with respect the genera so that for any solvable subgroup of  $\text{Aut}(\bar{\mathcal{X}})$  of an ordinary curve  $\bar{\mathcal{X}}$  of genus  $g(\bar{\mathcal{X}}) \geq 2$  we have  $|\bar{G}| \leq 34(g + 1)^{3/2}$ . Two cases are treated separately.

**Case I.**  *$G$  contains a minimal normal  $p$ -subgroup.*

**Proposition 3.1.** *Let  $\mathcal{X}$  be an ordinary algebraic curve of genus  $g$  defined over an algebraically closed field  $\mathbb{K}$  of odd characteristic  $p > 0$ . If  $G$  is a solvable subgroup of  $\text{Aut}(\mathcal{X})$  containing a minimal normal  $p$ -subgroup  $N$ , then  $|G| \leq 34(g + 1)^{3/2}$ .*



*Proof.* Before going through the proof we describe the main steps in it.

Take the largest normal  $p$ -subgroup  $Q$  of  $G$ . Let  $\bar{\mathcal{X}}$  be the quotient curve of  $\mathcal{X}$  with respect to  $Q$ , and let  $\bar{G} = G/Q$ . The first step is to show that  $\bar{\mathcal{X}}$  is rational. Then we derive from the classification in [Result 2.9](#) that  $G$  is a semidirect product of  $Q$  by cyclic group  $U$  of order prime to  $p$ . Therefore, [Lemma 2.12](#) applies to  $G$ . This gives us enough information on the action of  $Q$  on  $\mathcal{X}$ :  $Q$  has exactly two (nontame) orbits, say  $\Omega_1$  and  $\Omega_2$ , and they are also the only short orbits of  $G$ . Then a subgroup  $H$  of  $G$  of index  $\leq 2$  preserves both  $\Omega_1$  and  $\Omega_2$ , inducing a permutation group on each of them. If both  $\Omega_1$  and  $\Omega_2$  are nontrivial, that is,  $|\Omega_1| > 1$  and  $|\Omega_2| > 1$ , then two cases are possible, according as  $Q_P$  with  $P \in \Omega_1$  is sharply transitive and faithful on  $\Omega_2$  or some nontrivial element in  $Q_P$  fixes  $\Omega_2$  pointwise. So the next step is to rule out both these possibilities using elementary permutation group theory together with [Results 2.2](#) and [2.4](#). If  $\Omega_1 = \{P\}$  and  $|\Omega_2| > 1$ , then  $G$  fixes  $P$ , and the structure of  $G$  is given by [Result 2.4](#) where  $Q$  is an elementary abelian group, that is, a vector space over the prime field of  $\mathbb{K}$  and  $G$  is a linear group so that some appropriate result from representation theory can be used. In fact, combining [Result 2.11](#) with (5) allows us to rule out this possibility. If  $\Omega_1 = \{P\}$  and  $\Omega_2 = \{Q\}$ , we are able to prove a much stronger bound, namely  $|G| \leq 2(g(\mathcal{X}) + 1)$ . In this final step, our approach is function field theory rather than group theory as it uses some ideas from Nakajima's paper [\[1987\]](#) and the Riemann–Roch theorem together with some results on linearized polynomials over finite fields.

The quotient group  $\bar{G}$  is a subgroup of  $\text{Aut}(\bar{\mathcal{X}})$ , and it has no normal  $p$ -subgroup; otherwise  $G$  would have a normal  $p$ -subgroup properly containing  $Q$ . For  $\bar{g} = g(\bar{\mathcal{X}})$  three cases may occur, namely  $\bar{g} \geq 2$ ,  $\bar{g} = 1$ , or  $\bar{g} = 0$ . If  $\bar{g} \geq 2$ , then [Result 2.8](#) shows that  $|\bar{G}| > 34(\bar{g} + 1)^{3/2}$ . Since  $\bar{\mathcal{X}}$  is still ordinary by [Result 2.5\(v\)](#), this contradicts our choice of  $\mathcal{X}$  to be a minimal counterexample. If  $\bar{g} = 1$ , then the cover  $\mathbb{K}(\mathcal{X})|\mathbb{K}(\bar{\mathcal{X}})$  ramifies. Take a short orbit  $\Delta$  of  $Q$ . Let  $\Gamma$  be the nontame short orbit of  $G$  that contains  $\Delta$ . Since  $Q$  is normal in  $G$ , the orbit  $\Gamma$  partitions into short orbits of  $Q$  whose components have the same length, which is equal to  $|\Delta|$ . Let  $k$  be the number of the  $Q$ -orbits contained in  $\Gamma$ . Then

$$|G_P| = \frac{|G|}{k|\Delta|}$$

holds for every  $P \in \Gamma$ . Moreover, the quotient group  $G_P Q/Q$  fixes a place on  $\bar{\mathcal{X}}$ . Now, from [Result 2.7](#),

$$\frac{|G_P Q|}{|Q|} = \frac{|G_P|}{|G_P \cap Q|} = \frac{|G_P|}{|Q_P|} \leq 12.$$

From this together with (3) and [Result 2.5\(i\)](#),

$$2g - 2 \geq 2k|\Delta|(|Q_P| - 1) \geq 2k|\Delta| \frac{|Q_P|}{2} \geq \frac{k|\Delta||G_P|}{12} = \frac{|G|}{12},$$

which contradicts our hypothesis  $|G| > 34(g + 1)^{3/2}$ .

It turns out that  $\bar{\mathcal{X}}$  is rational. Therefore,  $\bar{G}$  is isomorphic to a subgroup of  $PGL(2, \mathbb{K})$  which contains no normal  $p$ -subgroup. From [Result 2.9](#),  $\bar{G}$  is a prime to  $p$  subgroup which is either cyclic, or dihedral, or isomorphic to one of the groups  $\text{Alt}_4$ ,  $\text{Sym}_4$ . In all cases,  $\bar{G}$  has a cyclic subgroup  $U$  of index  $\leq 6$  and of order distinct from 3. We may dismiss all cases but the cyclic one up to replacing  $\bar{G}$  with  $U$ , that is, up

to assuming that  $G = Q \rtimes U$  with  $|G| \geq \frac{34}{6}(g(\mathcal{X}) + 1)^{3/2}$ . Then  $|G| > 12(g - 1)$ . Therefore, [Lemma 2.12](#) applies to  $G$ . Thus,  $Q$  has exactly two (nontame) orbits, say  $\Omega_1$  and  $\Omega_2$ , and they are also the only short orbits of  $G$ . More precisely,

$$\gamma - 1 = |Q| - (|\Omega_1| + |\Omega_2|). \quad (8)$$

We may also observe that  $G_P$  with  $P \in \Omega_1$  contains a subgroup  $V$  isomorphic to  $U$ . In fact,  $|Q||U| = |G| = |G_P||\Omega_1| = |Q_P \rtimes V||\Omega_1| = |V||Q_P||\Omega_1|$  with a prime to  $p$  subgroup  $V$  fixing  $P$ , whence  $|U| = |V|$ . Since  $V$  is cyclic the claim follows.

We proceed with the case where both  $\Omega_1$  and  $\Omega_2$  are nontrivial, that is, their lengths are at least 2.

Assume that  $Q$  is nonabelian, and look at the action of its center  $Z(Q)$  on  $\mathcal{X}$ . Since  $Z(Q)$  is a nontrivial normal subgroup of  $G$ , we can argue as before to show that quotient curve  $\mathcal{X}/Z(Q)$  is rational, and hence that the Galois cover  $\mathcal{X}/(\mathcal{X}/Z(Q))$  ramifies at some points. Indeed, observe that in the previous arguments normality of  $Q$  was only used to dismiss all cases but the rational one, and hence we may simply replace  $Q$  with  $Z(Q)$ . In other words, there is a point  $P \in \Omega_1$  (or  $R \in \Omega_2$ ) such that some nontrivial subgroup  $T$  of  $Z(Q)$  fixes  $P$  (or  $R$ ). Suppose that the former case occurs. Since  $\Omega_1$  is a  $Q$ -orbit,  $T$  fixes  $\Omega_1$  pointwise.

The group  $G$  has an index  $\leq 2$  subgroup  $H$  that induces a permutation group on  $\Omega_1$ . Let  $M_1$  be the kernel of this permutation representation. Obviously,  $T$  is a nontrivial  $p$ -subgroup of  $M_1$ . Therefore,  $M$  contains some but not all elements from  $Q$ . Since both  $M_1$  and  $Q$  are normal subgroups of  $G$ ,  $N = M_1 \cap Q$  is a nontrivial normal  $p$ -subgroup of  $G$ . As we have proven before, the quotient curve  $\tilde{\mathcal{X}} = \mathcal{X}/N$  is rational, and hence the factor group  $\tilde{G} = G/N$  is isomorphic to a subgroup of  $PGL(2, \mathbb{K})$ . Since  $1 \not\leq N \leq Q$ , the order of  $\tilde{G}$  is divisible by  $p$ . From [Result 2.9](#),  $\tilde{G} = \tilde{Q} \rtimes \tilde{U}$  where  $\tilde{Q}$  is an elementary abelian  $p$ -group of order  $q$  and  $\tilde{U} \cong UN/N \cong U$  with  $|\tilde{U}| = |U|$  is a divisor of  $q - 1$ .

This shows that  $Q$  acts on  $\Omega_1$  as an abelian transitive permutation group. Obviously this holds true when  $Q$  is abelian. Therefore, the action of  $Q$  on  $\Omega_1$  is sharply transitive. In terms of 1-point stabilizers of  $Q$  on  $\Omega_1$ , we have  $Q_P = Q_{P'}$  for any  $P, P' \in \Omega_1$ . Moreover,  $Q_P = N$ , and hence  $Q_P$  is a normal subgroup of  $G$ .

Furthermore, since  $\mathcal{X}$  is an ordinary curve,  $Q_P$  is an elementary abelian group by [Result 2.5\(ii\)](#).

The quotient curve  $\mathcal{X}/Q_P$  is rational, and its automorphism group contains the factor group  $Q/Q_P$ . Hence, exactly one of the  $Q_P$ -orbits is preserved by  $Q$ . Since  $\Omega_1$  is a  $Q$ -orbit consisting of fixed points of  $Q_P$ ,  $\Omega_2$  must be a  $Q_P$ -orbit. Similarly, if  $Z(Q) \neq Q_P$ , the factor group  $Z(Q)Q_P/Q_P$  is an automorphism group of  $\mathcal{X}/Q_P$  and hence exactly one of the  $Q_P$ -orbits is preserved by  $Z(Q)$ . Either  $Z(Q)$  fixes a point in  $\Omega_1$  but then  $Z(Q) = Q_P$ , or  $\Omega_2$  is a  $Z(Q)$ -orbit. This shows that either  $Z(Q) = Q_P$ , or  $Z(G)$  acts transitively on  $\Omega_2$ .

Two cases arise according as  $Q_P$  is sharply transitive and faithful on  $\Omega_2$  or some nontrivial element in  $Q_P$  fixes  $\Omega_2$  pointwise.

If some nontrivial element in  $Q_P$  fixes  $\Omega_2$  pointwise, then the kernel  $M_2$  of the permutation representation of  $H$  on  $\Omega_2$  contains a nontrivial  $p$ -subgroup. Hence, the above results extend from  $\Omega_1$  to  $\Omega_2$ , and  $Q_R$  is a normal subgroup of  $Q$ .

If  $Q_P$  is (sharply) transitive on  $\Omega_2$ , then the abelian group  $Z(Q)Q_P$  acts on  $\Omega_2$  as a sharply transitive permutation group, as well. Hence, either  $Z(Q) = Q_P$ , or as before  $M_2$  contains a nontrivial  $p$ -subgroup, and  $Q_R$  is a normal subgroup of  $Q$ . In the former case,  $Q = Q_P Q_R$  with  $Q_R \cap Q_P = \{1\}$ , and  $Z(Q) = Q_P$  yields that

$$Q = Q_P \times Q_R. \quad (9)$$

This shows that  $Q$  is abelian, and hence  $|Q| \leq 4g+4$  by [Result 2.2\(ii\)](#). Also, either  $|Q_P|$  or  $|Q_R|$  is at most  $\sqrt{4g+4}$ . From [Result 2.5\(i\)](#),  $G_P^{(2)}$  at  $P \in \Omega_1$  is trivial. Furthermore, for  $G_P = Q_P \rtimes V$ , [Result 2.5\(iv\)](#) gives  $|U| = |V| \leq |Q_P| - 1$ . Hence,  $|U| < |Q_P| \leq \sqrt{|Q|} \leq \sqrt{4g+4}$  whence

$$|G| = |U||Q| \leq 8(g+1)^{3/2}. \quad (10)$$

If  $Q_R$  is a normal subgroup, take a point  $R$  from  $\Omega_2$ , and look at the subgroup  $Q_{P,R}$  of  $Q_P$  fixing  $R$ . Actually, we prove that either  $Q_{P,R} = Q_P$  or  $Q_{P,R}$  is trivial. Suppose that  $Q_{P,R} \neq \{1\}$ . Since  $Q_{P,R} = Q_P \cap Q_R$  and both  $Q_P$  and  $Q_R$  are normal subgroups of  $G$ ; the same holds for  $Q_{P,R}$ . By (ii), the quotient curve  $\mathcal{X}/Q_{P,R}$  is rational and hence its automorphism group  $Q/Q_{P,R}$  fixes exactly one point. Furthermore, each point in  $\Omega_2$  is totally ramified. Therefore,  $Q_R = Q_{P,R}$ ; otherwise  $Q_R/Q_{P,R}$  would fix any point lying under a point in  $\Omega_1$  in the cover  $\mathcal{X}/(Q/Q_{P,R})$ .

It turns out that either  $Q_P = Q_R$  or  $Q_P \cap Q_R = \{1\}$ , whenever  $P \in \Omega_1$  and  $R \in \Omega_2$ .

In the former case, from (5) applied to  $Q_P$ ,

$$\gamma - 1 = -|Q_P| + |\Omega_1|(|Q_P| - 1) + |\Omega_2|(|Q_P| - 1) = -|Q_P| + |Q| - |\Omega_1| + |Q| - |\Omega_2|.$$

This together with (8) give  $Q = Q_P$ , a contradiction.

Therefore, the latter case must hold. Thus,  $Q = Q_P \times Q_R$  and  $Q_P$  (and also  $Q_R$ ) is an elementary abelian group since it is isomorphic to a  $p$ -subgroup of  $PGL(2, \mathbb{K})$ . Also,  $|Q_P| = |Q_R| = \sqrt{|Q|}$ . Since  $Q$  is abelian, this yields  $|Q_P| \leq \sqrt{4g+4}$  by [Result 2.2\(ii\)](#). Now, the argument used after (9) can be employed to prove (10). This ends the proof in the case where both  $\Omega_1$  and  $\Omega_2$  are nontrivial.

Suppose next  $\Omega_1 = \{P\}$  and  $|\Omega_2| \geq 2$ . Then  $G$  fixes  $P$ , and hence  $G = Q \rtimes U$  with an elementary abelian  $p$ -group  $Q$ . Furthermore,  $G$  has a permutation representation on  $\Omega_2$  with kernel  $K$ . As  $\Omega_2$  is a short orbit of  $Q$ , the stabilizer  $Q_R$  of  $R \in \Omega_2$  in  $Q$  is nontrivial. Since  $Q$  is abelian, this yields that  $K$  is nontrivial, and hence it is a nontrivial elementary abelian normal subgroup of  $G$ . In other words,  $Q$  is an  $r$ -dimensional vector space  $V(r, p)$  over a finite field  $\mathbb{F}_p$  with  $|Q| = p^r$ , the action of each nontrivial element of  $U$  by conjugacy is a nontrivial automorphism of  $V(r, p)$ , and  $K$  is a  $U$ -invariant subspace. By [Result 2.11](#),  $K$  has a complementary  $U$ -invariant subspace. Therefore,  $Q$  has a subgroup  $M$  such that  $Q = K \times M$ , and  $M$  is a normal subgroup of  $G$ . Since  $K \cap M = \{1\}$ , and  $\Omega_2$  is an orbit of  $Q$ , this yields  $|M| = |\Omega_2|$ . The factor group  $G/M$  is an automorphism group of the quotient curve  $\mathcal{X}/M$ , and  $Q/M$  is a nontrivial  $p$ -subgroup of  $G/M$  whereas  $G/M$  fixes two points on  $\mathcal{X}/M$ . Therefore the quotient curve  $\mathcal{X}/M$  is not rational since the 2-point stabilizer in the representation of  $PGL(2, \mathbb{K})$  as an automorphism group of the rational function field is a prime to  $p$  (cyclic) group. We show that  $\mathcal{X}/M$  is not elliptic either.

From (5),  $g(\mathcal{X}) - 1 = \gamma(\mathcal{X}) - 1 = -|Q| + 1 + |\Omega_2|$ , and so  $g(\mathcal{X})$  is even. Since  $M$  is a normal subgroup of odd order,  $g(\mathcal{X}) \equiv 0 \pmod{2}$  yields that  $g(\mathcal{X}/M) \equiv 0 \pmod{2}$ . In particular,  $g(\mathcal{X}/M) \neq 1$ . Therefore,  $g(\mathcal{X}/M) \geq 2$ . At this point we may repeat our previous argument and prove  $|G/M| > 34(g(\mathcal{X}/M) + 1)^{3/2}$ . Again, we get a contradiction with our choice of  $\mathcal{X}$  to be a minimal counterexample, which ends the proof in the case where just one of  $\Omega_1$  and  $\Omega_2$  is trivial.

We are left with the case where both short orbits of  $Q$  are trivial. Our goal is to prove a much stronger bound for this case, namely  $|U| \leq 2$  whence

$$|G| \leq 2(g(\mathcal{X}) + 1). \quad (11)$$

We also show that if equality holds then  $\mathcal{X}$  is a hyperelliptic curve with equation

$$f(U) = aT + b + cT^{-1}, \quad a, b, c \in \mathbb{K}^*, \quad (12)$$

where  $f(U) \in \mathbb{K}[U]$  is an additive polynomial of degree  $|Q|$ .

Let  $\Omega_1 = \{P_1\}$  and  $\Omega_2 = \{P_2\}$ . Then  $Q$  has two fixed points  $P_1$  and  $P_2$ , but no nontrivial element in  $Q$  fixes a point of  $\mathcal{X}$  other than  $P_1$  and  $P_2$ . From (5),

$$g(\mathcal{X}) + 1 = \gamma(\mathcal{X}) + 1 = |Q|. \quad (13)$$

Therefore,  $|U| \leq g(\mathcal{X})$ . Actually, for our purpose, we need a stronger estimate, namely  $|U| \leq 2$ . To prove the latter bound, we use some ideas from Nakajima's paper [1987] regarding the Riemann–Roch spaces  $\mathcal{L}(D)$  of certain divisors  $D$  of  $\mathbb{K}(\mathcal{X})$ . Our first step is to show

- (i)  $\dim_{\mathbb{K}} \mathcal{L}((|Q| - 1)P_1) = 1$  and
- (ii)  $\dim_{\mathbb{K}} \mathcal{L}((|Q| - 1)P_1 + P_2) \geq 2$ .

Let  $\ell \geq 1$  be the smallest integer such that  $\dim_{\mathbb{K}} \mathcal{L}(\ell P_1) = 2$ , and take  $x \in \mathcal{L}(\ell P_1)$  with  $v_{P_1}(x) = -\ell$ . As  $Q = Q_{P_1}$ , the Riemann–Roch space  $\mathcal{L}(\ell P_1)$  contains all  $c_\sigma = \sigma(x) - x$  with  $\sigma \in Q$ . This yields  $c_\sigma \in \mathbb{K}$  by  $v_{P_1}(c_\sigma) \geq -\ell + 1$  and our choice of  $\ell$  to be minimal. Also,  $Q = Q_{P_2}$  together with  $v_{P_2}(x) \geq 0$  show  $v_{P_2}(c_\sigma) \geq 1$ . Therefore,  $c_\sigma = 0$  for all  $\sigma \in Q$ , that is,  $x$  is fixed by  $Q$ . From  $\ell = [\mathbb{K}(\mathcal{X}) : \mathbb{K}(x)] = [\mathbb{K} : \mathbb{K}(\mathcal{X})^Q][\mathbb{K}(\mathcal{X})^Q : \mathbb{K}(x)]$  and  $|Q| = [\mathbb{K} : \mathbb{K}(\mathcal{X})^Q]$ , it turns out that  $\ell$  is a multiple of  $|Q|$ . Thus  $\ell > |Q| - 1$  whence (i) follows. From the Riemann–Roch theorem,  $\dim_{\mathbb{K}} \mathcal{L}((|Q| - 1)P_1 + P_2) \geq |Q| - g + 1 = 2$ , which proves (ii).

Let  $d \geq 1$  be the smallest integer such that  $\dim_{\mathbb{K}} \mathcal{L}(dP_1 + P_2) = 2$ . From (ii)

$$d \leq |Q| - 1. \quad (14)$$

Let  $\alpha$  be a generator of the cyclic group  $U$ . Since  $\alpha$  fixes both points  $P_1$  and  $P_2$ , it acts on  $\mathcal{L}(dP_1 + P_2)$  as a  $\mathbb{K}$ -vector space automorphism  $\bar{\alpha}$ . If  $\bar{\alpha}$  is trivial, then  $\alpha(u) = u$  for all  $u \in \mathcal{L}(dP_1 + P_2)$ . Suppose that  $\bar{\alpha}$  is nontrivial. Since  $U$  is a prime to  $p$  cyclic group,  $\bar{\alpha}$  has two distinct eigenspaces, so that  $\mathcal{L}(dP_1 + P_2) = \mathbb{K} \oplus \mathbb{K}u$  where  $u \in \mathcal{L}(dP_1 + P_2)$  is an eigenvector of  $\bar{\alpha}$  with eigenvalue  $\xi \in \mathbb{K}^*$  so that

$\bar{\alpha}(u) = \xi u$  with  $\xi^{|U|} = 1$ . Therefore, there is  $u \in \mathcal{L}(dP_1 + P_2)$  with  $u \neq 0$  such that  $\alpha(u) = \xi u$  with  $\xi^{|U|} = 1$ . The pole divisor of  $u$  is

$$\operatorname{div}(u)_\infty = dP_1 + P_2. \quad (15)$$

Since  $Q = Q_{P_1} = Q_{P_2}$ , the Riemann–Roch space  $\mathcal{L}(dP_1 + P_2)$  contains  $\sigma(u)$  and hence contains all

$$\theta_\sigma = \sigma(u) - u, \quad \sigma \in Q.$$

By our choice of  $d$  to be minimal, this yields  $\theta_\sigma \in \mathbb{K}$ , and then defines the map  $\theta$  from  $Q$  into  $\mathbb{K}$  that takes  $\sigma$  to  $\theta_\sigma$ . More precisely,  $\theta$  is a homomorphism from  $Q$  into the additive group  $(\mathbb{K}, +)$  of  $\mathbb{K}$  as the following computation shows:

$$\theta_{\sigma_1 \circ \sigma_2} = (\sigma_1 \circ \sigma_2)(u) - u = \sigma_1(\sigma_2(u) - u + u) - u = \sigma_1(\theta_{\sigma_2}) + \sigma_1(u) - u = \theta_{\sigma_2} + \theta_{\sigma_1} = \theta_{\sigma_1} + \theta_{\sigma_2}.$$

Also,  $\theta$  is injective. In fact, if  $\theta_{\sigma_0} = 0$  for some  $\sigma_0 \in Q \setminus \{1\}$ , then  $u$  is in the fixed field of  $\sigma_0$ , which is impossible since  $v_{P_2}(u) = -1$  whereas  $P_2$  is totally ramified in the cover  $\mathcal{X}|\mathcal{X}/\langle \sigma_p \rangle$ . The image  $\theta(Q)$  of  $\theta$  is an additive subgroup of  $\mathbb{K}$  of order  $|Q|$ . The smallest subfield of  $\mathbb{K}$  containing  $\theta(Q)$  is a finite field  $\mathbb{F}_{p^m}$  and hence  $\theta(Q)$  can be viewed as a linear subspace of  $\mathbb{F}_{p^m}$  considered as a vector space over  $\mathbb{F}_p$ . Therefore, the polynomial

$$f(U) = \prod_{\sigma \in Q} (U - \theta_\sigma) \quad (16)$$

is a linearized polynomial over  $\mathbb{F}_p$  [Lidl and Niederreiter 1983, §4, Theorem 3.52]. In particular,  $f(U)$  is an additive polynomial of degree  $|Q|$ ; see also [Serre 1962, Chapter V, §5]. Also,  $f(U)$  is separable as  $\theta$  is injective. From (16), the pole divisor of  $f(u) \in \mathbb{K}(\mathcal{X})$  is

$$\operatorname{div}(f(u))_\infty = |Q|(dP_1 + P_2). \quad (17)$$

For every  $\sigma_0 \in Q$ ,

$$\sigma_0(f(u)) = \prod_{\sigma \in Q} (\sigma_0(u) - \theta_\sigma) = \prod_{\sigma \in Q} (u + \theta_{\sigma_0} - \theta_\sigma) = \prod_{\sigma \in Q} (u - \theta_{\sigma\sigma_0^{-1}}) = \prod_{\sigma \in Q} (u - \theta_\sigma) = f(u).$$

Thus,  $f(u) \in \mathbb{K}(\mathcal{X})^Q$ . Furthermore, from  $\alpha \in N_G(Q)$ , for every  $\sigma \in Q$  there is  $\sigma' \in Q$  such that  $\alpha\sigma = \sigma'\alpha$ . Therefore,

$$\alpha(f(u)) = \prod_{\sigma \in Q} (\alpha(\sigma(u) - u)) = \prod_{\sigma \in Q} (\alpha(\sigma(u)) - \xi u) = \prod_{\sigma \in Q} (\sigma'(\alpha(u)) - \xi u) = \prod_{\sigma \in Q} (\sigma'(\xi u) - \xi u) = \xi f(u).$$

This shows that if  $R \in \mathcal{X}$  is a zero of  $f(u)$  then  $\operatorname{Supp}(\operatorname{div}(f(u)_0))$  contains the  $U$ -orbit of  $R$  of length  $|U|$ . Actually, since  $\sigma(f(u)) = f(u)$  for  $\sigma \in Q$ ,  $\operatorname{Supp}(\operatorname{div}(f(u)_0))$  contains the  $G$ -orbit of  $R$  of length  $|G| = |Q||U|$ . This together with (17) give

$$|U|(d+1). \quad (18)$$

On the other hand,  $\mathbb{K}(\mathcal{X})^Q$  is rational. Let  $\bar{P}_1$  and  $\bar{P}_2$  be the points lying under  $P_1$  and  $P_2$ , respectively, and let  $\bar{R}_1, \bar{R}_2, \dots, \bar{R}_k$  with  $k = (d+1)/|U|$  be the points lying under the zeros of  $f(u)$  in the cover  $\mathcal{X}|\mathcal{X}/Q$ . We may represent  $\mathbb{K}(\mathcal{X})^Q$  as the projective line  $\mathbb{K} \cup \{\infty\}$  over  $\mathbb{K}$  so that  $\bar{P}_1 = \infty$ ,  $\bar{P}_1 = 0$ , and  $\bar{R}_i = t_i$  for  $1 \leq i \leq k$ . Let  $g(t) = t^d + t^{-1} + h(t)$  where  $h(t) \in \mathbb{K}[t]$  is a polynomial of degree  $k = (d+1)/|U|$  whose roots are  $r_1, \dots, r_k$ . It turns out that  $f(u), g(t) \in \mathbb{K}(\mathcal{X})$  have the same pole and zero divisors, and hence

$$cf(u) = t^d + t^{-1} + h(t), \quad c \in \mathbb{K}^*. \quad (19)$$

We prove that  $\mathbb{K}(\mathcal{X}) = \mathbb{K}(u, t)$ . From [Sullivan 1975] (see also [Hirschfeld et al. 2008, Remark 12.12]), the polynomial  $cTf(X) - T^{d+1} - 1 - h(T)T$  is irreducible, and the plane curve  $\mathcal{C}$  has genus  $g(\mathcal{C}) = \frac{1}{2}(q-1)(d+1)$ . Comparison with (13) shows  $\mathbb{K}(\mathcal{X}) = \mathbb{K}$  and  $d = 1$  whence  $|U| \leq 2$ . If equality holds, then  $\deg h(T) = 1$  and  $\mathcal{X}$  is a hyperelliptic curve with Equation (12).  $\square$

### Case II. $G$ contains no minimal normal $p$ -subgroup.

**Proposition 3.2.** *Let  $\mathcal{X}$  be an ordinary algebraic curve of genus  $g$  defined over a field  $\mathbb{K}$  of odd characteristic  $p > 0$ . If  $G$  is a solvable subgroup of  $\text{Aut}(\mathcal{X})$  with a minimal normal subgroup  $N$ , then  $|G| \leq 34(g(\mathcal{X}) + 1)^{3/2}$ .*

*Proof.* We begin with an outline of the proof.

Since  $\mathcal{X}$  is chosen to be a (minimal) counterexample, Proposition 3.1 yields that  $G$  contains no nontrivial normal  $p$ -subgroup. The factor group  $\bar{G} = G/N$  is a subgroup of  $\text{Aut}(\bar{\mathcal{X}})$  where  $\bar{\mathcal{X}} = \mathcal{X}/N$ . As in the proof of Proposition 3.1, we begin by showing that  $\bar{\mathcal{X}}$  must be rational. This time Result 2.6(ii) does not apply and some more effort is needed to rule out the possibility of  $g(\bar{\mathcal{X}}) \geq 2$  while the elliptic case does not require a different approach. If  $\bar{\mathcal{X}}$  is rational, the classification in Result 2.9 gives the possibility of the structure of  $\bar{G}$  and its action on  $\bar{\mathcal{X}}$ . A careful analysis shows that  $\bar{G}$  must be of type (vi) in Result 2.9. From this we obtain the possibilities for the action of  $G$  on  $\mathcal{X}$ . After that, (3) and (5) together with straightforward computation are sufficient to end the proof although the case where  $N$  is an elementary abelian 2-group requires some additional facts from group theory.

We prove that  $g(\bar{\mathcal{X}}) \geq 2$ . By Result 2.2(ii),  $|N| \leq 4g(\mathcal{X}) + 4$  as  $N$  is abelian. If  $\bar{\mathcal{X}}$  is also ordinary, then the choice of  $\mathcal{X}$  to have minimal genus implies that  $|\bar{G}| \leq 34(g(\bar{\mathcal{X}}) + 1)^{3/2}$ . Comparing this with Result 2.8 shows a contradiction. Therefore, the possibility for  $\bar{\mathcal{X}}$  to be nonordinary is investigated.

From Result 2.5(i), any  $p$ -subgroup  $S$  of  $G$  has trivial second ramification group at any point  $\mathcal{X}$ . The latter property remains true when  $\mathcal{X}$  and  $S$  are replaced by  $\bar{\mathcal{X}}$  and the factor group  $\bar{S} = SN/N$ , respectively. To show this claim, take  $\bar{P} \in \bar{\mathcal{X}}$  and let  $\bar{S}_{\bar{P}}$  be the subgroup of  $\bar{S}$  fixing  $\bar{P}$ . Since  $p \nmid |N|$  there is a point  $P \in \mathcal{X}$  lying over  $\bar{P}$  which is fixed by  $S$ . Hence, the stabilizer  $S_P$  of  $P$  in  $S$  is a nontrivial normal subgroup of  $G_P$ . Since  $N$  is a normal subgroup in  $G$ , so is  $N_P$  in  $G_P$ . This yields that the product  $N_P S_P$  is actually a direct product. Therefore,  $N_P$  is trivial by Result 2.5(iii), that is, the cover  $\mathcal{X}|\bar{\mathcal{X}}$  is unramified at  $\bar{P}$ . From this, the claim follows.

Actually,  $N$  may be taken to be the largest normal subgroup  $N_1$  of  $G$  whose order is prime to  $p$ . Also, by our hypothesis, the quotient curve  $\mathcal{X}_1 = \mathcal{X}/N_1$  is neither rational, nor elliptic. From Result 2.8, its



$\mathbb{K}$ -automorphism group  $G_1 = G/N_1$  has order bigger than  $34(\mathfrak{g}(\mathcal{X}_1) + 1)^{3/2}$ . Since  $G$  and hence  $G_1$  are solvable,  $G_1$  has a minimal normal  $d$ -subgroup where  $d$  must be equal to  $p$  by the choice of  $N_1$  to be the largest normal, prime to  $p$  subgroup of  $G$ . Take the largest normal  $p$ -subgroup  $N_2$  of  $G_1$ . Observe that  $N_2 \neq G_1$ . In fact, if  $N_2 = G_1$ , then  $G_1$  is  $p$ -group of order bigger than  $34(\mathfrak{g}(\mathcal{X}_1) + 1)^{3/2} > p\mathfrak{g}(\mathcal{X}_1)/(p-2)$ . From [Result 2.1](#),  $\mathcal{X}_1$  has zero  $p$ -rank, and hence  $G_1$  fixes a point  $P_1 \in \mathcal{X}_1$ . On the other hand, since  $G_1^{(2)}$  is trivial, [Result 2.6\(iii\)](#) shows  $|G_1| \leq p\mathfrak{g}(\mathcal{X}_1)/(p-1)$ , a contradiction. Now, define  $\mathcal{X}_2$  to be the quotient curve  $\mathcal{X}_1/N_2$ . Since the second ramification group of  $N_1$  at any point of  $\mathcal{X}_1$  is trivial, [Result 2.6\(i\)](#) gives  $\mathfrak{g}(\mathcal{X}_1) - \gamma(\mathcal{X}_1) = |N_2|(\mathfrak{g}(\mathcal{X}_2) - \gamma(\mathcal{X}_2))$ . In particular, if  $\mathcal{X}_2$  is ordinary or rational, then  $\mathcal{X}_1$  is an ordinary curve. From the proof of [Proposition 3.1](#), the case  $\mathfrak{g}(\mathcal{X}_2) = 1$  cannot occur as  $|G_1| > 34(\mathfrak{g}(\mathcal{X}_1) + 1)^{3/2}$ . Therefore,  $\mathfrak{g}(\mathcal{X}_2) \geq 2$  with  $\mathfrak{g}(\mathcal{X}_2) > \gamma(\mathcal{X}_2)$  may be assumed. The factor group  $G_2 = G_1/N_2$  is a  $\mathbb{K}$ -automorphism group of the quotient curve  $\mathcal{X}_2 = \mathcal{X}_1/N_2$ , and it has a minimal normal  $d$ -subgroup with  $d \neq p$ , by the choice of  $N_2$ . Define  $N_3$  to be the largest normal, prime to  $p$  subgroup of  $G_2$ . Observe that  $N_3$  must be a proper subgroup of  $G_2$ ; otherwise  $G_2$  itself would be a prime to  $p$  subgroup of  $\text{Aut}(\mathcal{X}_2)$  of order bigger than  $34(\mathfrak{g}(\mathcal{X}_2) + 1)^{3/2}$ , contradicting [Result 2.2\(i\)](#). Therefore, there exists a (maximal) nontrivial normal  $p$ -subgroup  $N_4$  in the factor group  $G_3 = G_2/N_3$ . Now, the above argument remains valid whenever  $G, N_1, G_1, N_2, \mathcal{X}_1, \mathcal{X}_2$  are replaced by  $G_2, N_3, G_3, N_4, \mathcal{X}_3, \mathcal{X}_4$  where the quotient curves are  $\mathcal{X}_3 = G_2/N_3$  and  $\mathcal{X}_4 = G_3/N_4$ . In particular,  $\mathfrak{g}(\mathcal{X}_4) \neq 1$  and  $\mathfrak{g}(\mathcal{X}_3) - \gamma(\mathcal{X}_3) = |N_4|(\mathfrak{g}(\mathcal{X}_4) - \gamma(\mathcal{X}_4))$ . Repeating the above argument, a finite sharply decreasing sequence  $\mathfrak{g}(\mathcal{X}_1) > \mathfrak{g}(\mathcal{X}_2) > \mathfrak{g}(\mathcal{X}_3) > \mathfrak{g}(\mathcal{X}_4) > \dots$  arises. If this sequence has  $n+1$  members, then  $\mathfrak{g}(\mathcal{X}_n) - \gamma(\mathcal{X}_n) = |N_{n+1}|(\mathfrak{g}(\mathcal{X}_{n+1}) - \gamma(\mathcal{X}_{n+1}))$  with  $\mathfrak{g}(\mathcal{X}_{n+1}) = \gamma(\mathcal{X}_{n+1}) = 0$ . Therefore, for some (odd) index  $m \leq n$ , the curve  $\mathcal{X}_m$  would not be ordinary, but the successive member  $\mathcal{X}_{m+1}$  would be an ordinary curve. Since  $\mathcal{X}_{m+1}$  is a quotient curve of  $\mathcal{X}_m$  with respect to a  $p$ -subgroup, this is impossible by [Result 2.6\(ii\)](#).

We continue with the elliptic case. Since  $\mathfrak{g}(\mathcal{X}) \geq 2$ , [\(3\)](#) applied to  $\bar{X}$  ensures that  $N$  has a short orbit. Let  $\Gamma$  be a short orbit of  $G$  containing a short orbit of  $N$ . Since  $N$  is a normal subgroup of  $G$ ,  $\Gamma$  is partitioned into short orbits  $\Sigma_1, \dots, \Sigma_k$  of  $N$  each of length  $|\Sigma_1|$ . Take a point  $R_i$  from  $\Sigma_i$  for  $i = 1, 2, \dots, k$ , and set  $\Sigma = \Sigma_1$  and  $S = S_1$ . With this notation,  $|G| = |G_S||\Gamma| = |G_S|k|\Sigma|$ , and [\(3\)](#) gives

$$2\mathfrak{g}(\mathcal{X}) - 2 \geq \sum_{i=1}^k |\Sigma_i|(|N_{S_i}| - 1) = k|\Sigma|(|N_S| - 1) \geq +\frac{1}{2}k|\Sigma||N_S| = \frac{1}{2}|G|\frac{|N_S|}{|G_S|}. \quad (20)$$

Also, the factor group  $G_S N/N$  is a subgroup of  $\text{Aut}(\bar{\mathcal{X}})$  fixing the point of  $\bar{\mathcal{X}}$  lying under  $S$  in the cover  $\mathcal{X}|\bar{\mathcal{X}}$ . From [Result 2.7](#),

$$\frac{|G_S N|}{|N|} = \frac{|G_S|}{|G_S \cap N|} = \frac{|G_S|}{|N_S|} \leq 12.$$

This and [\(20\)](#) yield  $|G| \leq 48(\mathfrak{g}(\mathcal{X}) - 1)$ , a contradiction with our hypothesis  $34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$ .

Therefore,  $\bar{\mathcal{X}}$  is rational. Thus,  $\bar{G}$  is isomorphic to a subgroup of  $PGL(2, \mathbb{K})$ . Since  $p$  divides  $|G|$  but not  $|N|$ ,  $\bar{G}$  contains a nontrivial  $p$ -subgroup. From [Result 2.9](#), either  $p = 3$  and  $\bar{G} \cong \text{Alt}_4, \text{Sym}_4$ , or  $\bar{G} = \bar{Q} \rtimes \bar{C}$  where  $\bar{Q}$  is a normal  $p$ -subgroup and its complement  $\bar{C}$  is a cyclic prime to  $p$  subgroup and  $|\bar{C}|$  divides  $|\bar{Q}| - 1$ .

If  $\bar{G} \cong \text{Alt}_4, \text{Sym}_4$ , then  $|\bar{G}| \leq 24$  whence  $|G| \leq 24|N| \leq 96(\mathfrak{g}(\mathcal{X}) + 1)$  as  $N$  is abelian. Comparison with our hypothesis  $|G| \geq 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$  shows that  $\mathfrak{g}(\mathcal{X}) \leq 6$ . For small genera we need a little more. If  $|N|$  is prime, then  $|N| \leq 2\mathfrak{g}(\mathcal{X}) + 1$  by [Result 2.2\(iii\)](#), and hence  $|G| \leq 48(\mathfrak{g}(\mathcal{X}) + 1)$ , which is inconsistent with  $|G| \geq 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$ . Otherwise, since  $p = 3$  and  $|N|$  has order a power of prime distinct from  $p$ , the bound  $|N| \leq 4(\mathfrak{g}(\mathcal{X}) + 1)$  with  $\mathfrak{g}(\mathcal{X}) \leq 6$  is only possible for  $(\mathfrak{g}(\mathcal{X}), |N|) \in \{(3, 16), (4, 16), (5, 16), (6, 16), (6, 25)\}$ . Comparison of  $|G| \leq 24|N|$  with  $|G| \geq 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$  rule out the latter three cases. Furthermore, since  $N$  is an elementary abelian group of order 16,  $\mathfrak{g}(\mathcal{X})$  must be odd by [Lemma 2.13](#). Finally,  $\mathfrak{g}(\mathcal{X}) = 3$ ,  $|N| = 16$ , and  $G/N \cong \text{Sym}_4$  is impossible as [Result 2.3](#) would imply that  $\mathcal{X}$  has zero  $p$ -rank.

Therefore, the case  $\bar{G} = \bar{Q} \rtimes \bar{C}$  occurs. Also,  $\bar{G}$  fixes a unique place  $\bar{P} \in \bar{\mathcal{X}}$ . Let  $\Delta$  be the  $N$ -orbits in  $\mathcal{X}$  that lie over  $\bar{P}$  in the cover  $\mathcal{X}|\bar{\mathcal{X}}$ . We prove that  $\Delta$  is a long orbit of  $N$ . By absurd, the permutation representation of  $G$  on  $\Delta$  has a nontrivial 1-point stabilizer containing a nontrivial subgroup  $M$  of  $N$ . Since  $N$  is abelian,  $M$  is in the kernel. In particular,  $M$  is a normal subgroup of  $G$  contradicting our choice of  $N$  to be minimal.

Take a Sylow  $p$ -subgroup  $Q$  of  $G$  of order  $|Q| = p^h$  with  $h \geq 1$ , and look at the action of  $Q$  on  $\Delta$ . Since  $|\Delta| = |N|$  is prime to  $p$ ,  $Q$  fixes a point  $P \in \Delta$ , that is,  $Q = Q_P$ . Since  $\mathcal{X}$  is an ordinary curve, [Result 2.5\(ii\)](#) shows that  $Q_P$  and hence  $Q$  are elementary abelian. Therefore,  $G_P = Q \rtimes U$  where  $U$  is a prime to  $p$  cyclic group. Thus,

$$|\bar{Q}||\bar{C}||N| = |\bar{G}||N| = |G| = |G_P||\Delta| = |Q||U||\Delta| = |Q||U||N|, \quad (21)$$

whence  $|Q| = |\bar{Q}|$  and  $|U| = |\bar{C}|$ . Consider the subgroup  $H$  of  $G$  generated by  $G_P$  and  $N$ . Since  $\Delta$  is a long  $N$ -orbit,  $G_P \cap N = \{1\}$ . As  $N$  is normal in  $H$  this implies that  $H = N \rtimes G_P = N \rtimes (Q \rtimes U)$  and hence  $|H| = |N||Q||U|$ , which proves  $G = H = N \rtimes (Q \rtimes U)$ .

Since  $\bar{\mathcal{X}}$  is rational and  $\bar{P}$  is the unique fixed point of nontrivial elements of  $\bar{Q}$ , each  $\bar{Q}$ -orbit other than  $\{\bar{P}\}$  is long. Furthermore,  $\bar{C}$  fixes a point  $\bar{R}$  other than  $\bar{P}$  and no nontrivial element of  $\bar{C}$  fixes a point distinct from  $\bar{P}$  and  $\bar{R}$ . This shows that the  $\bar{G}$ -orbit  $\bar{\Omega}_1$  of  $\bar{R}$  has length  $|Q|$ . In terms of the action of  $G$  on  $\mathcal{X}$ , there exist as many as  $|Q|$  orbits of  $N$ , say  $\Delta_1, \dots, \Delta_{|Q|}$ , whose union  $\Lambda$  is a short  $G$ -orbit lying over  $\bar{\Omega}_1$  in the cover  $\mathcal{X}|\bar{\mathcal{X}}$ . Obviously, if at least one of  $\Delta_i$  is a short  $N$ -orbit, then so are all.

We show that this actually occurs. Since the cover  $\mathcal{X}|\bar{\mathcal{X}}$  ramifies,  $N$  has some short orbits, and by absurd there exists a short  $N$ -orbit  $\Sigma$  not contained in  $\Lambda$ . Then  $\Sigma$  and  $\Lambda$  are disjoint. Let  $\Gamma$  denote the (short)  $G$ -orbit containing  $\Sigma$ . Since  $N$  is a normal subgroup of  $G$ ,  $\Gamma$  is partitioned into  $N$ -orbits, say  $\Sigma = \Sigma_1, \dots, \Sigma_k$ , each of them of the same length  $|\Sigma|$ . Here  $k = |Q||U|$  since the set of points of  $\bar{\mathcal{X}}$  lying under these  $k$  short  $N$ -orbits is a long  $\bar{G}$ -orbit. Also,  $|N| = |\Sigma_i||N_{R_i}|$  for  $1 \leq i \leq k$  and  $R_i \in \Sigma_i$ . In particular,  $|\Sigma_1| = |\Sigma_i|$  and  $|N_{R_1}| = |N_{R_i}|$ . From [\(3\)](#),

$$2\mathfrak{g}(\mathcal{X}) - 2 \geq -2|N| + \sum_{i=1}^k |\Sigma_i|(|N_{R_i}| - 1) = -2|N| + |Q||U||\Sigma_1|(|N_{R_1}| - 1).$$

Since  $N_{R_1}$  is nontrivial,  $|N_{R_1}| - 1 \geq \frac{1}{2}|N_{R_1}|$ . Therefore,

$$2\mathfrak{g}(\mathcal{X}) - 2 \geq -2|N| + \frac{1}{2}|Q||U||\Sigma_1||N_{R_1}| = -2|N| + \frac{1}{2}|Q||U||N| = |N|(\frac{1}{2}(|Q||U| - 2)) = \frac{1}{2}|N|(|Q||U| - 4).$$

As  $|Q||U| - 4 \geq \frac{1}{2}|Q||U|$  by  $|Q||U| \geq 4$ , this gives

$$2\mathfrak{g}(\mathcal{X}) - 2 \geq \frac{1}{4}|N||U||Q| = \frac{1}{4}|G|.$$

But this contradicts our hypothesis  $|G| > 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$ .

Therefore, the short orbits of  $N$  are exactly  $\Delta_1, \dots, \Delta_{|Q|}$ . Take a point  $S_i$  from  $\Delta_i$  for  $i = 1, \dots, |Q|$ . Then  $N_{S_1}$  and  $N_{S_i}$  are conjugate in  $G$ , and hence  $|N_{S_1}| = |N_{S_i}|$ . From (3) applied to  $N$ ,

$$2\mathfrak{g}(\mathcal{X}) - 2 = -2|N| + \sum_{i=1}^{|Q|} |\Delta_i|(|N_{S_i}| - 1) = -2|N| + |Q||\Delta_1|(|N_{S_1}| - 1) \geq -2|N| + \frac{1}{2}|Q||\Delta_1||N_{S_1}|.$$

Since  $|N| = |\Delta_1||N_{S_1}|$ , this gives  $2\mathfrak{g}(\mathcal{X}) - 2 \geq \frac{1}{2}|N|(|Q| - 4)$  whence  $2\mathfrak{g}(\mathcal{X}) - 2 \geq \frac{1}{4}|N||Q|$  provided that  $|Q| \geq 5$ . The missing case,  $|Q| = 3$ , cannot actually occur since in this case  $|\bar{C}| = |U| \leq |Q| - 1 = 2$ , whence  $|G| = |Q||U||N| \leq 6|N| \leq 24(\mathfrak{g}(\mathcal{X}) + 1)$ , a contradiction with  $|G| > 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$ . Thus,

$$|N||Q| \leq 8(\mathfrak{g}(\mathcal{X}) - 1). \quad (22)$$

Since  $|N||U| < |N||Q|$ , this also shows

$$|N||U| < 8(\mathfrak{g}(\mathcal{X}) - 1). \quad (23)$$

Therefore,

$$|G||N| = |N|^2|U||Q| < 64(\mathfrak{g}(\mathcal{X}) - 1)^2.$$

Equations (22) and (23) together with our hypothesis  $|G| \geq 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$  yield

$$|N| < \frac{64}{34}\sqrt{\mathfrak{g}(\mathcal{X}) - 1}. \quad (24)$$

From (24) and  $|G| = |N||Q||U| \geq 34(\mathfrak{g}(\mathcal{X}) + 1)^{3/2}$  we obtain

$$|Q||U| > \frac{34^2}{64}(\mathfrak{g}(\mathcal{X}) - 1) > 18(\mathfrak{g}(\mathcal{X}) - 1),$$

which shows that Lemma 2.12 applies to the subgroup  $Q \rtimes U$  of  $\text{Aut}(\mathcal{X})$ . With the notation in Lemma 2.12, this gives that  $Q \rtimes U$  and  $Q$  have the same two short orbits,  $\Omega_1 = \{P\}$  and  $\Omega_2$ . In the cover  $\mathcal{X}|\bar{\mathcal{X}}$ , the point  $\bar{P} \in \bar{\mathcal{X}}$  lying under  $P$  is fixed by  $Q$ . We prove that  $\Omega_2$  is a subset of the  $N$ -orbit  $\Delta$  containing  $P$ . For this purpose, it suffices to show that for any point  $R \in \Omega_2$ , the point  $\bar{R} \in \bar{\mathcal{X}}$  lying under  $R$  in the cover  $\mathcal{X}|\bar{\mathcal{X}}$  coincides with  $\bar{P}$ . Since  $\Omega_2$  is a  $Q$ -short orbit, the stabilizer  $Q_R$  is nontrivial, and hence  $\bar{Q}$  fixes  $\bar{R}$ . Since  $\bar{\mathcal{X}}$  is rational, this yields  $\bar{P} = \bar{R}$ . Therefore,  $\Omega_2 \cup \{P\}$  is contained in  $\Delta$ , and either  $\Delta = \Omega_2 \cup \{P\}$  or  $\Delta$  contains a long  $Q$ -orbit. In the latter case,  $|U| < |Q| < |N|$ , and hence

$$|G|^2 = |N||Q||N||U||Q||U| < |N||Q||N||U||N|^2 \leq \frac{64^2}{34}(\mathfrak{g}(\mathcal{X}) - 1)^3$$

whence  $|G| < 34(g(\mathcal{X}) + 1)^{3/2}$ , a contradiction with our hypothesis. Otherwise  $|N| = |\Delta| = 1 + |\Omega_2|$ . In particular,  $|N|$  is even, and hence it is a power of 2. Also, by (5),  $g(\mathcal{X}) - 1 = \gamma(\mathcal{X}) - 1 = -|Q| + 1 + |\Omega_2|$  where  $|\Omega_2| \geq 1$  is a power of  $p$ . This implies that  $g(\mathcal{X})$  is also even. Since  $N$  is an elementary abelian 2-group, Lemma 2.13 yields that either  $|N| = 2$  or  $|N| = 4$ .

If  $|N| = 2$ , then  $\Omega_2$  consists of a unique point  $R$  and  $Q \rtimes U$  fixes both points  $P$  and  $R$ . Since  $\Delta = \{P, R\}$ , and  $\Delta$  is a  $G$ -orbit, the stabilizer  $G_{P,R}$  is an index-2 (normal) subgroup of  $G$ . On the other hand,  $G_{P,R} = Q \rtimes U$  and hence  $Q$  is the unique Sylow  $p$ -subgroup of  $Q \rtimes U$ . Thus,  $Q$  is a characteristic subgroup of the normal subgroup  $G_{P,R}$  of  $G$ . But then  $Q$  is a normal subgroup of  $G$ , a contradiction with our hypothesis.

If  $|N| = 4$ , then  $|\Delta| = 4$  and  $p = 3$ . The permutation representation of  $G$  of degree 4 on  $\Delta$  contains a 4-cycle induced by  $N$  but also a 3-cycle induced by  $Q$ . Hence, if  $K = \ker$ , then  $G/K \cong \text{Sym}_4$ . On the other hand, since both  $N$  and  $\text{Ker}$  are normal subgroups of  $G$ , their product  $NK$  is normal, as well. Hence,  $NK/K$  is a normal subgroup of  $G/K$ , but this contradicts  $G/K \cong \text{Sym}_4$ .  $\square$

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[gabor.korchmaros@unibas.it](mailto:gabor.korchmaros@unibas.it)

*Dipartimento di Matematica, Informatica ed Economia,  
Università degli Studi della Basilicata, Potenza, Italy*

[mariamontanucci@gmail.com](mailto:mariamontanucci@gmail.com)

*Dipartimento di Tecnica e Gestione dei Sistemi Industriali,  
Università degli Studi di Padova, Vicenza, Italy*

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