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A tubular variant of Runge's method in all dimensions, with applications to integral points on Siegel modular varieties

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Runge's method is a tool to figure out integral points on algebraic curves effectively in terms of height. This method has been generalized to varieties of any dimension, but unfortunately the conditions needed to apply it are often too restrictive. We provide a further generalization intended to be more flexible while still effective, and exemplify its applicability by giving finiteness results for integral points on some Siegel modular varieties. As a special case, we obtain an explicit finiteness result for integral points on the Siegel modular variety $A_2(2)$.

Introduction

One of the major motivations of number theory is the description of rational or integral solutions of diophantine equations, which from a geometric perspective amounts to understanding the behavior of rational or integral points on algebraic varieties. In dimension one, several fundamental results provide a good overview of the situation, including the famous Faltings' theorem (for genus ≥ 2 and algebraic points) or Siegel's theorem (for integral points and a function with at least three poles). Nevertheless, the quest for general *effectivity* (meaning a bound on the height on these points, or hopefully complete

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determination of the points) is still ongoing, and effective methods are quite different from these two powerful theoretical theorems.

On the other hand, there is major interest in the study of algebraic torsion points of elliptic curves, or more generally abelian varieties, defined over a given number field. In many situations, it amounts to understanding the algebraic points of so-called *modular spaces*, parametrizing isomorphism classes of abelian varieties with additional datum. For modular curves (i.e., modular spaces of elliptic curves), the existing techniques are numerous and far-reaching (for example, with Merel's uniform boundedness theorem [1996] or Mazur's isogeny theorem [1977]), but the world of higher-dimensional abelian varieties is far less known.

We thus focus in this paper on a method for integral points on curves called *Runge's method*, and its generalizations to algebraic varieties and applications for Siegel modular varieties. This introduction is twofold: first, we give the guiding principles behind our approach and second, we flesh out the precise structure of the article and indicate where to find the details for each claim made.

Runge's method for algebraic varieties. On a smooth algebraic projective curve C over a number field K, Runge's method proceeds as follows. Let $\phi \in K(C)$ be a nonconstant rational function on C. For any finite extension L/K, we denote by M_L the set of places of L (and by M_L^∞ the archimedean ones). For S_L, a finite set of places of L containing M_L^∞, we denote the ring of S_L-integers of L by

$$\mathcal{O}_{L,S_L} = \{x \in L : |x|_v \le 1 \text{ for all } v \in M_L \backslash S_L\}.$$

Now, let r_L be the number of orbits of poles of ϕ under the action of $\mathrm{Gal}(\bar{L}/L)$. The *Runge condition* on a pair (L, S_L) is the inequality

$$|S_L| < r_L. \tag{1}$$

Then, Bombieri's generalization [Bombieri and Gubler 2006, paragraph 9.6.5 and Theorem 9.6.6] of Runge's old theorem [1887] states that given such C and ϕ, there is an *absolute* bound B such that for every pair (L, S_L) satisfying the Runge condition and every point $P \in C(L)$ such that $\phi(P) \in \mathcal{O}_{L,S_L}$,

$$h(\phi(P)) \le B,$$

where h is the Weil height. In short, as long as the point $\phi(P)$ has few nonintegrality places (the exact condition being (1)), there is an absolute bound on the height of $\phi(P)$. When applicable, this method has two important assets: it gives good bounds and is uniform in the pairs (L, S_L), which for example is not true for Baker's method [Bilu 1995].

Our first goal was to transpose the ideas for Runge's method on curves to higher-dimensional varieties. First, let us recall a previous generalization of Bombieri's theorem in higher dimensions obtained by Levin [2008, Theorem 4] under a simplified form. On a projective smooth variety X, the analogues of poles of ϕ are effective divisors $D_1, \dots, D_r$. We have to fix a smooth integral model $\mathcal{X}$ of X on $\mathcal{O}_K$, and denote by $\mathcal{D}_1, \dots, \mathcal{D}_r$ the Zariski closures of the divisors in this model, of union $\mathcal{D}$, so our integral points here are the points of $(\mathcal{X} \backslash \mathcal{D})(\mathcal{O}_{L,S_L})$. There are two major changes in higher dimension. Firstly, the divisors have to be ample (or at least big) to obtain finiteness results (this was automatic for dimension 1). Secondly, instead of the condition $|S_L| < r$ as for curves, the *higher-dimensional Runge condition* is

$$m|S_L| < r, \tag{2}$$

where m is the smallest number such that any $(m+1)$ divisors amongst $D_1, \dots, D_r$ have empty common intersection. Levin's theorem states in particular that when the divisors are ample,

$$\left(\bigcup_{\substack{(L,S_L) \\ m|S_L|<r}} (\mathcal{X} \backslash \mathcal{D})(\mathcal{O}_{L,S_L}) \right) \text{ is (effectively) finite.}$$

The issue with (2) is that the maximal number $|S_L|$ satisfying this condition is lowered by m, since the ample (or big) hypothesis tends to give a lower bound on m, condition (2) is impossible to satisfy (remember that S_L contains archimedean places, so $|S_L| \geq [L:\mathbb{Q}]/2$). This was the initial motivation for a generalization of this theorem, called "tubular Runge's theorem", designed to be more flexible in terms of the Runge condition. Let us explain its principle below.

In addition to X and $D_1, \dots, D_r$, we fix a closed subvariety Y of X which is meant to be "where the divisors $D_1, \dots, D_r$ intersect a lot". More precisely, let m_Y be the smallest number such that for any $(m_Y + 1)$ distinct divisors amongst $D_1, \dots, D_r$, their common intersection is included in Y. In particular, $m_Y \leq m$, and the goal is to have m_Y as small as possible without asking Y to be too large. Now, we fix a "tubular neighborhood" of Y, which is the datum of a family $\mathcal{V} = (V_v)_v$ where v goes through the places v of $\overline{K}$, every V_v is a neighborhood of Y in the v-adic topology, and this family is uniformly not too small in some sense. As the main example, if $\mathcal{Y}$ is the Zariski closure of Y in $\mathcal{X}$, we can define at a finite place v the neighborhood V_v to be the set of points of $\mathcal{X}(\overline{K_v})$ reducing in $\mathcal{Y}$ modulo v. A point $P \in X(\overline{K})$ does *not* belong to $\mathcal{V}$ if $P \notin V_v$ for every place v of $\overline{K}$, and intuitively, this means that P is v-adically far away from Y for *every* place v of $\overline{K}$. Now, assume our integral points are *not* in $\mathcal{V}$. It implies that at most m_Y divisors amongst $D_1, \dots, D_r$ can be v-adically close to them, hence using the same principles of proof as Levin, this gives the *tubular Runge condition*

$$m_Y|S_L| < r. \tag{3}$$

With this additional data, one can now sketch our tubular Runge's theorem.

Theorem (simplified version of the "tubular Runge's theorem" (Theorem 5.1)). *For X, $\mathcal{X}$, $D_1, \dots, D_r$, Y, m_Y and a tubular neighborhood $\mathcal{V}$ of Y as in the paragraph above, let $(\mathcal{X} \backslash \mathcal{D})(\mathcal{O}_{L,S_L}) \backslash \mathcal{V}$ be the set of points of $(\mathcal{X} \backslash \mathcal{D})(\mathcal{O}_{L,S_L})$ which do not belong to $\mathcal{V}$. Then, if $D_1, \dots, D_r$ are ample, for every such tubular neighborhood, the set*

$$\left(\bigcup_{\substack{(L,S_L) \\ m_Y|S_L|<r}} (\mathcal{X} \backslash \mathcal{D})(\mathcal{O}_{L,S_L}) \backslash \mathcal{V} \right) \text{ is finite,}$$

and bounded in terms of some auxiliary height.

As the implicit bound on the height is parametrized by the tubular neighborhood $\mathcal{V}$, this theorem can be seen as a *concentration result* rather than a finiteness one; essentially, it states that the points of $(\mathcal{X}\backslash\mathcal{D})(\mathcal{O}_{L,S_L})$ concentrate near the closed subset Y. As such, we have compared it to theorems of [Corvaja et al. 2009], notably Autissier's theorem and the CLZ theorem, in Section 5 (in particular, our version is made to be effective, whereas these results are based on Schmidt's subspace theorem, of which no effective proof is known yet). On the other hand, there is an interesting (and genuine finiteness result) variant only using the tubular neighborhood at finite places: under all above assumptions, we also have finiteness of the union of all the $(\mathcal{X}\backslash\mathcal{D})(\mathcal{O}_{L,S_L})$ minus all the points reducing in Y at some finite place, where the pairs (L, S_L) satisfy the *mixed tubular Runge condition*

$$m|M_L^\infty| + m_Y|S_L\backslash M_L^\infty| < r, \tag{4}$$

and this will be straightforward given the proof of the theorem.

In the second part of our paper, we apply the method to Siegel modular varieties, both as a proof of principle and because integral points on these varieties are not very well understood, apart from the Shafarevich conjecture proved by Faltings. As we will see below, this is also a case where a candidate for Y presents itself, thus giving tubular neighborhoods a natural interpretation.

For $n \geq 2$, the variety denoted by $A_2(n)$ is the variety over $\mathbb{Q}(\zeta_n)$ parametrizing triples (A, λ, α_n) where (A, λ) is a principally polarized abelian variety of dimension 2 and α_n is a symplectic level n structure on (A, λ). It is a quasiprojective algebraic variety of dimension 3, and its Satake compactification (which is a projective algebraic variety) is denoted by $A_2(n)^S$, the boundary being $\partial A_2(n) = A_2(n)^S \backslash A_2(n)$. The extension of scalars $A_2(n)_\mathbb{C}$ is the quotient of the half-superior Siegel space $\mathcal{H}_2$ by the natural action of the symplectic congruence subgroup $\Gamma_2(n)$ of $\mathrm{Sp}_4(\mathbb{Z})$ made up with the matrices congruent to the identity modulo n. Now, we consider some divisors ($n^4/2+2$ of them) defined by the vanishing of some modular forms, specifically *theta functions*. One finds that they intersect a lot on the boundary $\partial A_2(n)$ (m comparable to n^4), but when we fix $Y = \partial A_2(n)$, we get $m_Y \leq (n^2-3)$ hence giving the *tubular Runge condition*

$$(n^2-3)|S_L| < \tfrac{1}{2}n^4 + 2.$$

The application of our tubular Runge's theorem gives for every even $n \geq 2$ a finiteness result for the integral points for these divisors and some tubular neighborhoods associated to potentially bad reduction for the finite places; this is Theorem 7.12. In the special case $n = 2$, we made this result completely explicit in Theorem 8.2. A simplified case of this theorem (using (4)) is the following result.

Theorem (Theorem 8.2, simplified case). *Let K be either $\mathbb{Q}$ or a quadratic imaginary field.*

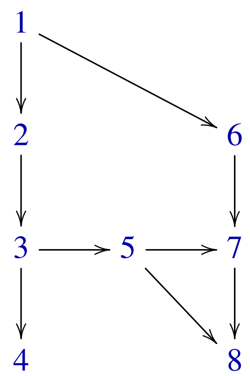
Let A be a principally polarized abelian surface defined over K, whose full 2-torsion is also defined over K and having potentially good reduction at all finite places of K.

Then, if the semistable reduction of A is a product of elliptic curves at most at 3 finite places of K, we have the explicit bound

$$h_\mathcal{F}(A) \leq 828,$$

where $h_\mathcal{F}$ is the stable Faltings height. In particular, there are only finitely many such abelian surfaces.

Let us finally explain the structure of the paper.



Section 1 is devoted to the notations used throughout the paper, including heights, M_K-constants and bounded sets. We advise the reader to pay particular attention to its reading as it introduces notations which are ubiquitous in the rest of the paper. Section 2 is where the exact definition and basic properties of tubular neighborhoods are given. In Section 3, we prove the key result for the tubular Runge's theorem (Proposition 3.1), essentially relying on a well-applied Nullstellensatz. In Section 4, we reprove Bombieri's theorem for curves with Bilu's idea, as it is not yet published to our knowledge (although this is exactly the principle behind Runge's method in [Bilu and Parent 2011] for example). Finally, we prove and discuss our tubular Runge's theorem (Theorem 5.1) in Section 5.

For the applications to Siegel modular varieties, Section 6 gathers the necessary notations and reminders on these varieties (Section 6A), their integral models and their properties (Section 6B) and the key notion of theta divisors on abelian varieties and their link with classical theta functions (Section 6C). The theta functions are essential because they define the divisors we use in our applications of the tubular Runge's theorem.

In Section 7, we focus on the case of abelian surfaces (the one we are interested in), especially regarding the behavior of theta divisors (Section 7A) and state in Section 7B the applications of our tubular Runge's theorem for the varieties $A_2(n)^S$ and the divisors mentioned above (Theorems 7.11 and 7.12).

Finally, in Section 8, we make explicit Theorem 7.11 by computations on the ten fourth powers of even characteristic theta constants. To do this, the places need to be split into three categories. The finite places not above 2 are treated by the theory of algebraic theta functions in Section 8A, the archimedean places by estimates of Fourier expansions in Section 8B and the finite places above 2 (the hardest case) by the theory of Igusa invariants and with polynomials built from our ten theta constants in Section 8C. The final estimates are given as Theorem 8.2 in Section 8D, both in terms of a given embedding of $A_2(2)$ and in terms of Faltings height.

The main results of this paper have been announced in the recently published note [Le Fourn 2017], and apart from Section 8 and some improvements can be found in the author's thesis manuscript [Le Fourn 2015] (both in French).

1. Notations and preliminary notions

The following notations are classical and given below for clarity. They will be used throughout the paper.

• K is a number field, M_K and M_K^∞ are the set of places and archimedean places of K, respectively. We also denote by $M_{\overline{K}}$ the set of places of $\overline{K}$.

• $|\cdot|_\infty$ is the usual absolute value on $\mathbb{Q}$, and $|\cdot|_p$ is the place associated to p prime, whose absolute value is normalized by

$$|x|_p = p^{-\operatorname{ord}_p(x)},$$

where $\operatorname{ord}_p(x)$ is the unique integer such that $x = p^{\operatorname{ord}_p(x)} \frac{a}{b}$ with $p \nmid ab$ (by convention, $|0|_p = 0$). Similarly, $|\cdot|_v$ is the absolute value on K associated to $v \in M_K$, normalized to extend $|\cdot|_{v_0}$ when v is above $v_0 \in M_\mathbb{Q}$, and the local degree is $n_v = [K_v : \mathbb{Q}_{v_0}]$. For every $x \in K^*$, one has the classical product formula

$$\prod_{v \in M_K} |x|_v^{n_v} = 1.$$

When v comes from a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, we indifferently write $|\cdot|_v$ and $|\cdot|_\mathfrak{p}$.

• For any place v of K, one defines the sup norm on K^{n+1} by

$$\|(x_0, \ldots, x_n)\|_v = \max_{0 \le i \le n} |x_i|_v.$$

• Every set of places $S \subset M_K$ considered is finite and contains M_K^∞. The ring of S-integers is

$$\mathcal{O}_{K,S} = \{x \in K : |x|_v \le 1 \text{ for every } v \in M_K \backslash S\}.$$

• For every $P \in \mathbb{P}^n(K)$, we denote by $x_P = (x_{P,0}, \ldots, x_{P,n}) \in K^{n+1}$ any possible choice of projective coordinates for P, this choice being of course fixed for consistency when used in a formula or a proof. The logarithmic Weil height of P is defined by

$$h(P) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \log \|x_P\|_v, \tag{1-1}$$

this does not depend on the choice of x_P nor on the number field, and satisfies the Northcott property.

• For every $n \ge 1$ and every $i \in \{0, \ldots, n\}$, the i-th coordinate open subset U_i of $\mathbb{P}^n$ is the affine subset defined as

$$U_i = \{(x_0 : \cdots : x_n) \mid x_i \neq 0\}. \tag{1-2}$$

The normalization function $\varphi_i : U_i \to \mathbb{A}^{n+1}$ is then defined by

$$\varphi_i(x_0 : \cdots : x_n) = \left(\frac{x_0}{x_i}, \ldots, 1, \ldots \frac{x_n}{x_i}\right). \tag{1-3}$$

For most of our results, we need to formalize the notion that some families of sets indexed by the places $v \in M_K$ are "uniformly bounded". To this end, we recall some classical definitions (see [Bombieri and Gubler 2006, §2.6]).

Definition 1.1 (M_K-constants and M_K-bounded sets). • An M_K*-constant* is a family $\mathcal{C} = (c_v)_{v \in M_K}$ of real numbers such that $c_v = 0$ except for a finite number of places $v \in M_K$. The set of M_K-constants is stable by finite sum and finite maximum on each coordinate, a fact which we will often use without further mention.

• Let L/K be a finite extension. For an M_K-constant $(c_v)_{v \in M_K}$, we define (with abuse of notation) an M_L-constant $(c_w)_{w \in M_L}$ by $c_w := c_v$ if $w \mid v$. Conversely, if $(c_w)_{w \in M_L}$ is an M_L-constant, we define (again with abuse of notation) $(c_v)_{v \in M_K}$ by $c_v := \max_{w \mid v} c_w$, and get in both cases the inequality

$$\frac{1}{[L:\mathbb{Q}]} \sum_{w \in M_L} n_w c_w \le \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v c_v. \tag{1-4}$$

• If U is an affine variety over K and $E \subset U(\overline{K}) \times M_{\overline{K}}$, a regular function $f \in \overline{K}[U]$ is M_K*-bounded on* E if there is a M_K-constant $\mathcal{C} = (c_v)_{v \in M_K}$ such that for every $(P, w) \in E$ with w above v in M_K,

$$\log|f(P)|_w \le c_v.$$

• An M_K*-bounded subset of* U is, by abuse of definition, a subset E of $U(\overline{K}) \times M_{\overline{K}}$ such that every regular function $f \in \overline{K}[U]$ is M_K-bounded on E.

Remark 1.2. (a) In the projective space $\mathbb{P}^n_K$, for every $i \in \{0, \dots, n\}$, consider the set

$$E_i = \{(P, w) \in \mathbb{P}^n(\overline{K}) \times M_{\overline{K}} : |x_{P,i}|_w = \|x_P\|_w\}. \tag{1-5}$$

The regular functions x_j/x_i $(j \ne i)$ on $\overline{K}[U_i]$ (notation (1-2)) are trivially M_K-bounded (by the zero M_K-constant) on E_i, hence E_i is M_K-bounded in U_i. Notice that the E_i cover $\mathbb{P}^n(\overline{K}) \times M_{\overline{K}}$.

(b) With notations (1-1), (1-2) and (1-3), for a subset E of $U_i(\overline{K})$, if the coordinate functions of U_i are M_K-bounded on $E \times M_{\overline{K}}$, the height $h \circ \varphi_i$ is straightforwardly bounded on E in terms of the involved M_K-constants. This simple observation will be the basis of our finiteness arguments.

The following lemma allows us to split M_K-bounded sets in an affine cover.

Lemma 1.3. *Let U be an affine variety and E an M_K-bounded set. If $(U_j)_{j \in J}$ is a finite affine open cover of U, there exists a cover $(E_j)_{j \in J}$ of E such that every E_j is M_K-bounded in U_j.*

Proof. This is Lemma 2.2.10 together with Remark 2.6.12 of [Bombieri and Gubler 2006]. □

Let us now recall some notions about integral points on schemes and varieties.

For a finite extension L of K, a point $P \in \mathbb{P}^n(L)$ and a nonzero prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$ of residue field $k(\mathfrak{P}) = \mathcal{O}_L/\mathfrak{P}$, the point P extends to a unique morphism $\operatorname{Spec} \mathcal{O}_{L,\mathfrak{P}} \to \mathbb{P}^n_{\mathcal{O}_K}$, and the image of its special point is *the reduction of P modulo* $\mathfrak{P}$, denoted by $P_{\mathfrak{P}} \in \mathbb{P}^n(k(\mathfrak{P}))$. More explicitly, after normalization of the coordinates x_P of P so that they all belong to $\mathcal{O}_{L,\mathfrak{P}}$ and one of them to $\mathcal{O}^*_{L,\mathfrak{P}}$, one has

$$P_{\mathfrak{P}} = (x_{P,0} \bmod \mathfrak{P} : \cdots : x_{P,n} \bmod \mathfrak{P}) \in \mathbb{P}^n_{k(\mathfrak{P})}. \tag{1-6}$$

The following (easy) proposition expresses scheme-theoretic reduction in terms of functions (there will be another in Proposition 3.4). We write it below as it is the inspiration behind the notion of tubular neighborhood in Section 2.

Proposition 1.4. *Let S be a finite set of places of K containing M_K^∞, and $\mathcal{X}$ be a projective scheme on $\mathcal{O}_{K,S}$, seen as a closed subscheme of $\mathbb{P}^n_{\mathcal{O}_{K,S}}$.*

Let $\mathcal{Y}$ be a closed sub-$\mathcal{O}_{K,S}$-scheme of $\mathcal{X}$.

Consider $g_1, \dots, g_s \in \mathcal{O}_{K,S}[X_0, \dots, X_n]$ homogeneous generators of the ideal of definition of $\mathcal{Y}$ in $\mathbb{P}^n_{\mathcal{O}_{K,S_0}}$. For every nonzero prime $\mathfrak{P}$ of $\mathcal{O}_L$ not above S and every point $P \in \mathcal{X}(L)$, the reduction $P_{\mathfrak{P}}$ belongs to $\mathcal{Y}_{\mathfrak{p}}(k(\mathfrak{P}))$ (with $\mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_K$) if and only if $\forall j \in \{1, \dots, s\}$

$$|g_j(x_P)|_{\mathfrak{P}} < \|x_P\|_{\mathfrak{P}}^{\deg g_j}. \tag{1-7}$$

Proof. For every $j \in \{1, \dots, s\}$, by homogeneity of g_j, for a choice x_P of coordinates for P belonging to $\mathcal{O}_{L,\mathfrak{P}}$ with one of them in $\mathcal{O}_{L,\mathfrak{P}}^*$, the inequality (1-7) amounts to

$$g_j(x_{P,0}, \dots, x_{P,n}) = 0 \bmod \mathfrak{P}.$$

On the other hand, the reduction of P modulo $\mathfrak{P}$ belongs to $\mathcal{Y}_{\mathfrak{p}}(\overline{k(\mathfrak{P})})$ if and only if its coordinates satisfy the equations defining $\mathcal{Y}_{\mathfrak{p}}$ in $X_{\mathfrak{p}}$, but these are exactly the equations $g_1, \dots, g_s$ modulo $\mathfrak{p}$. This remark immediately gives the proposition by (1-6). □

2. Definition and properties of tubular neighborhoods

The explicit expression (1-7) is the motivation for our definition of *tubular neighborhood*, at the core of our results. This definition is meant to be used by exclusion; with the same notations as Proposition 1.4, we want to say that a point $P \in X(L)$ is *not* in some tubular neighborhood of $\mathcal{Y}$ if it *never* reduces in $\mathcal{Y}$, whatever the prime ideal $\mathfrak{P}$ of $\mathcal{O}_L$ is.

The main interest of this notion is that it provides us with a convenient alternative to the reduction assumption for the places in S (which are the places where the reduction is not well defined, including the archimedean places), and also allows us to loosen up this reduction hypothesis in a nice fashion. Moreover, as the definition is function-theoretic, we only need to consider the varieties over a base field, keeping in mind that Proposition 1.4 makes the link with reduction at finite places.

Definition 2.1 (tubular neighborhood). Let X be a projective variety over K and Y be a closed K-subscheme of X.

We choose an embedding $X \subset \mathbb{P}^n_K$, a set of homogeneous generators $g_1, \dots, g_s$ in $K[X_0, \dots, X_n]$ of the homogeneous ideal defining Y in $\mathbb{P}^n$ and an M_K-constant $\mathcal{C} = (c_v)_{v \in M_K}$.

The *tubular neighborhood of Y in X associated to $\mathcal{C}$ and $g_1, \dots, g_s$* (the embedding made implicit) is the family $\mathcal{V} = (V_w)_{w \in M_{\overline{K}}}$ of subsets of $X(\overline{K})$ defined as follows.

For every $w \in M_{\overline{K}}$ above some $v \in M_K$, V_w is the set of points $P \in X(\overline{K})$ such that, $\forall j \in \{1, \dots, s\}$,

$$\log|g_j(x_P)|_w < \deg(g_j) \cdot \log\|x_P\|_w + c_v. \tag{2-1}$$

As we said before, this definition will be ultimately used by exclusion:

Definition 2.2. Let X be a projective variety over K and Y be a closed K-subscheme of X.

For any tubular neighborhood $\mathcal{V} = (V_w)_{w \in M_{\overline{K}}}$ of Y, we say that a point $P \in X(\overline{K})$ *does not belong to* $\mathcal{V}$ (and we denote it by $P \notin \mathcal{V}$) if, $\forall w \in M_{\overline{K}}$,

$$P \notin V_w.$$

Remark 2.3. (a) A tubular neighborhood of Y can also be seen as a family of open subsets defined by bounding strictly a global height function relative to Y coming from arithmetic distance functions (see [Vojta 1987], paragraph 2.5 or the original article [Silverman 1987] for more details on arithmetic distance functions). In particular, functoriality of global height functions (Theorem 2.1(h) of [Vojta 1987] for example) implies that if one fixes a second embedding $X \subset \mathbb{P}^m_K$, any tubular neighborhood of Y defined using this embedding can be put between two tubular neighborhoods defined using the original embedding, and conversely. The notion of tubular neighborhood is thus essentially independent of the choice of embedding (which is there to make things as explicit as needed).

(b) Comparing (1-7) and (2-1), for the M_K-constant $\mathcal{C} = 0$ and with the notations of Proposition 1.4, at the finite places w not above S, the tubular neighborhood V_w is exactly the set of points $P \in X(\overline{K})$ reducing in $\mathcal{Y}$ modulo w.

(c) If Y is an ample divisor of X and $\mathcal{V}$ is a tubular neighborhood of Y, one easily sees that if $P \notin \mathcal{V}$ then $h(\psi(P))$ is bounded for some embedding ψ associated to Y, from which we get the finiteness of the set of points P of bounded degree outside of $\mathcal{V}$. This illustrates why such an assumption is only really relevant when Y is of small dimension.

Example 2.4. We have drawn in Figures 1, 2 and 3 three different pictures of tubular neighborhoods in $\mathbb{P}^2(\mathbb{R})$, at the usual archimedean norm. The coordinates are x, y, z, the affine open subset U_z defined by $z \neq 0$, and E_x, E_y, E_z the respective sets such that $|x|, |y|, |z| = \max(|x|, |y|, |z|)$. These different tubular neighborhoods are drawn in U_z, and the contribution of the different parts E_x, E_y and E_z is made clear.

3. Key results

We will now prove the key result for Runge's method, as a consequence of the Nullstellensatz. We only use the projective case in the rest of the paper but the affine case is both necessary for its proof and enlightening for the method we use.

Proposition 3.1 (key proposition). (a) (*Affine version*) *Let U be an affine variety over K, Y a closed subset of U, $g_1, \ldots, g_r \in K[U]$ whose set of common zeroes is Y and $h_1, \ldots, h_s \in K[U]$ all vanishing on Y. For every M_K-bounded set E of U and every M_K-constant $\mathcal{C}_0$, there is an M_K-constant $\mathcal{C}$ such that for every $(P, w) \in E$ with w above $v \in M_K$, one has the following dichotomy*:

$$\max_{1 \le \ell \le r} \log|g_\ell(P)|_w \ge c_v \quad or \quad \max_{1 \le j \le s} \log|h_j(P)|_w < c_{0,v}. \tag{3-1}$$

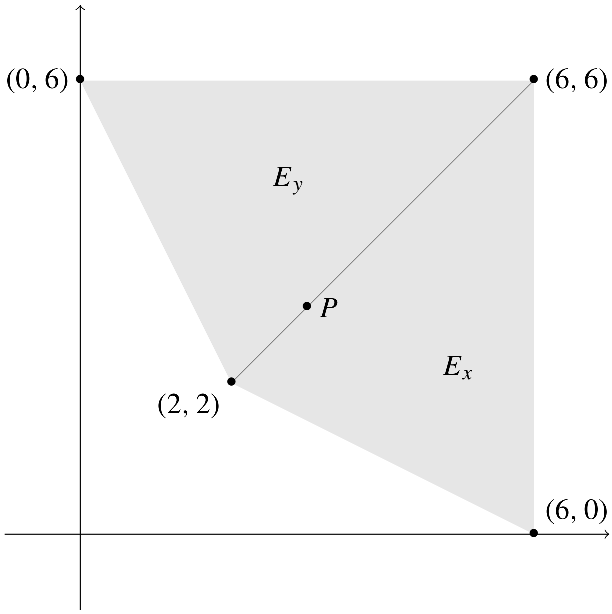


Figure 1. Tubular neighborhood of the point $P = (3:3:1)$ associated to the inequality $\max(|x-3y, y-3z|) < \frac{1}{2}\max(|x|, |y|, |z|)$.

(b) (*Projective version*) *Let X be a normal projective variety over K and $\phi_1, \dots, \phi_r \in K(X)$. Let Y be the closed subset of X defined as the intersection of the supports of the* (*Weil*) *divisors of poles of the ϕ_i. For every tubular neighborhood $\mathcal{V}$ of Y* (Definition 2.1), *there is an M_K-constant $\mathcal{C}$ depending on $\mathcal{V}$ such that for every $w \in M_{\overline{K}}$* (*above $v \in M_K$*) *and every $P \in X(\overline{K})$*,

$$\min_{1 \le \ell \le r} \log|\phi_\ell(P)|_w \le c_v \quad \textit{or} \quad P \in V_w. \tag{3-2}$$

This result has an immediate corollary when $Y = \varnothing$:

Corollary 3.2 [Levin 2008, Lemma 5]. *Let X be a normal projective variety over K and $\phi_1, \dots, \phi_r \in K(X)$ having globally no common pole. Then, there is an M_K-constant $\mathcal{C}$ such that for every $w \in M_{\overline{K}}$* (*above $v \in M_K$*) *and every $P \in X(\overline{K})$*,

$$\min_{1 \le \ell \le r} \log|\phi_\ell(P)|_w \le c_v. \tag{3-3}$$

Remark 3.3. (a) As will become clear in the proof, part (b) is actually part (a) applied to a good cover of X by M_K-bounded subsets of affine open subsets of X (inspired by the natural example of Remark 1.2(a)).

(b) Besides the fact that the results must be uniform in the places (hence the M_K-constants), the principle of (a) and (b) is simple. For (a), we would like to say that if the first part of the dichotomy is not satisfied, the point P must be close to each set of zeroes of the g_ℓ hence to their intersection Y. Consequently, the functions vanishing on Y must be small at P (second part of the dichotomy). This is not immediately true yet (take for example functions vanishing respectively on one hyperbola and one of its axes on the

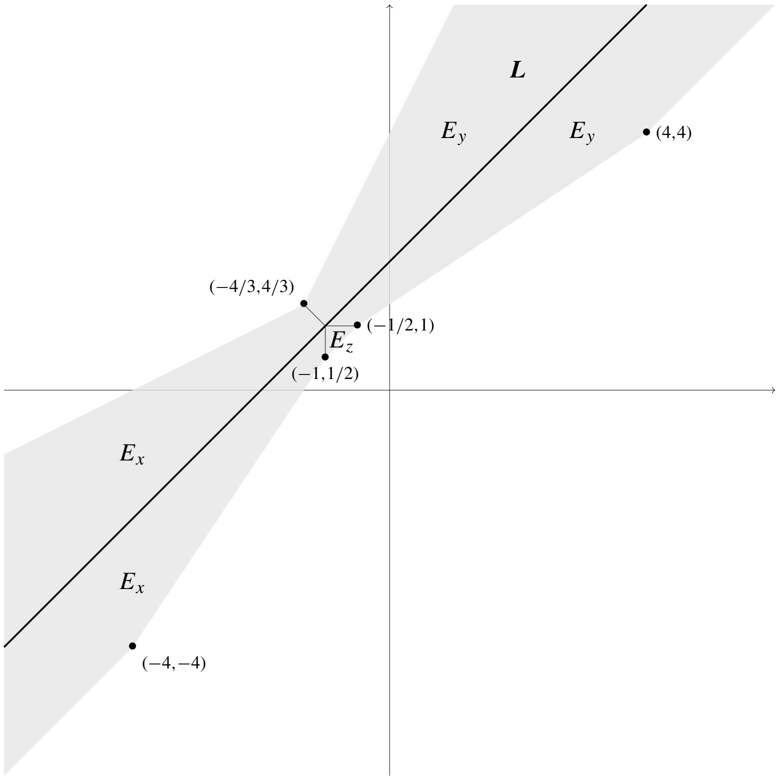


Figure 2. Tubular neighborhood of the line $D : y - x + 2z = 0$ associated to the inequality $\max(|x - y + 2z|) < \frac{1}{2}\max(|x|, |y|, |z|)$. The boundary of the neighborhood is made up with segments between the indicated points.

affine plane). Indeed, one needs to restrict to bounded sets to compactify the situation, which is also why it works in the projective case as the closed sets are then compact.

(c) Corollary 3.2 is the key for Runge's method in the case of curves in Section 4. Notice that Lemma 5 of [Levin 2008] assumed X smooth, but the proof is actually exactly the same for X normal. Moreover, the argument below follows the structure of Levin's proof.

(d) If we replace Y by $Y' \supset Y$ and $\mathcal{V}$ by a tubular neighborhood $\mathcal{V}'$ of Y', the result remains true with the same proof, which is not surprising because tubular neighborhoods of Y' are larger than tubular neighborhoods of Y.

Proof of Proposition 3.1.

(a) By the Nullstellensatz applied to $K[U]$, there are $p \in \mathbb{N}_{\geq 1}$ and regular functions $f_{\ell,m} \in K[U]$ such that for every $m \in \{1, \ldots, s\}$,

$$\sum_{1 \leq \ell \leq r} g_\ell f_{\ell,m} = h_m^p.$$

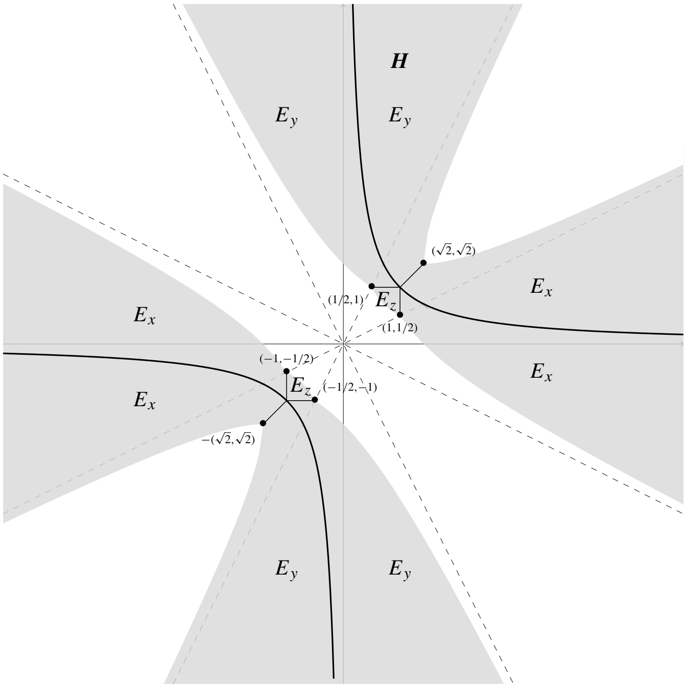


Figure 3. Tubular neighborhood of the hyperbola $H : xy - z^2 = 0$ given by the inequality $|xy - z^2| < \frac{1}{2}\max(|x|, |y|, |z|)$. The boundary is made up with arcs of hyperbola between the indicated points.

As E is M_K-bounded on U, all the $f_{\ell,m}$ are M_K-bounded on E hence there is an auxiliary M_K-constant $\mathcal{C}_1$ such that for all $(P, w) \in E$,

$$\max_{\substack{1\leq\ell\leq r\\1\leq m\leq s}} \log|f_{\ell,m}(P)|_w \leq c_{1,v},$$

therefore

$$|h_m(P)^p|_w = \left|\sum_{1\leq\ell\leq r} g_\ell(P) f_{\ell,m}(P)\right|_w \leq r^{\delta_v} e^{c_{1,v}} \max_{1\leq\ell\leq r} |g_\ell(P)|_w,$$

where δ_v is 1 if v is archimedean and 0 otherwise. For fixed w and P, either $\log|h_m(P)|_w < c_{0,v}$ for all $m \in \{1, \dots, s\}$ (second part of dichotomy (3-1)) or the above inequality applied to some $m \in \{1, \dots, s\}$ gives

$$p \cdot c_{0,v} \leq \delta_v \log(r) + c_{1,v} + \max_{1\leq\ell\leq r} \log|g_{\ell,j}(P)|_w,$$

which is equivalent to

$$\max_{1\leq\ell\leq r} \log|g_\ell(P)|_w \geq p \cdot c_{0,v} - \delta_v \log(r) - c_{1,v}$$

and taking the M_K-constant defined by $c_v := c_{1,v} + \delta_v \log(r) - p \cdot c_{0,v}$ for every $v \in M_K$ gives exactly the first part of (3-1).

(b) We consider X as embedded in some $\mathbb{P}^n_K$ so that $\mathcal{V}$ is exactly the tubular neighborhood of Y in X associated to an M_K-constant $\mathcal{C}_0$ and generators $g_1, \dots, g_s$ for this embedding. Let us define $X_i := X \cap U_i$ for every $i \in \{0, \dots, n\}$ (see notations (1-2), (1-3) and (1-5)). The following argument is designed to make Y appear as a common zero locus of regular functions built from the ϕ_ℓ.

For every $\ell \in \{1, \dots, r\}$, let D_ℓ be the positive Weil divisor of zeroes of ϕ_ℓ on X. For every $i \in \{0, \dots, n\}$, let $I_{\ell,i}$ be the ideal of $K[X_i]$ made up with the regular functions h on the affine variety X_i such that $\operatorname{div}(h) \geq (D_\ell)_{|X_i}$, and we choose generators $h_{\ell,i,1}, \dots, h_{\ell,i,j_{\ell,i}}$ of this ideal. The functions $h_{\ell,i,j}/(\phi_\ell)_{|X_i}$ are then regular on X_i and $\forall j \in \{1, \dots, j_{\ell,i}\}$,

$$\operatorname{div}\left(\frac{h_{\ell,i,j}}{(\phi_\ell)_{|X_i}}\right) \geq (\phi_{\ell,i})_\infty$$

(the divisor of poles of ϕ_ℓ on X_i). By construction of $I_{\ell,i}$, the minimum (prime Weil divisor by prime Weil divisor) of the $\operatorname{div}(h_{\ell,i,j})$ is exactly $(D_\ell)_{|X_i}$; indeed, for every finite family of distinct prime Weil divisors $D'_1, \dots, D'_s, D''$ on X_i, there is a uniformizer h for D'' of order 0 for each of the D'_k, otherwise the prime ideal associated to D'' in X_i would be included in the finite union of the others. This allows us to build for every prime divisor D' of X_i not in the support of $(D_\ell)_{|X_i}$ a function $h \in I_{\ell,i}$ of order 0 along D' (and of the proper order for every D' in the support of $(D_\ell)_{|X_i}$). Consequently, the minimum of the divisors of the $h_{\ell,i,j}/(\phi_\ell)_{|X_i}$, being naturally the minimum of the divisors of the $h/(\phi_\ell)_{|X_i}$ (for $h \in K[X_i]$), is exactly $(\phi_{\ell,i})_\infty$.

Thus, by definition of Y, for fixed i, the set of common zeroes of the regular functions $h_{\ell,i,j}/(\phi_\ell)_{|X_i}$ (for $1 \leq \ell \leq r$ and $1 \leq j \leq j_{\ell,i}$) on X_i is $Y \cap X_i$, so they generate an ideal whose radical is the ideal of definition of $Y \cap X_i$. We apply part (a) of this proposition to the $h_{\ell,i,j}/(\phi_\ell)_{|X_i}$ (for $1 \leq \ell \leq r$ and $1 \leq j \leq j_{\ell,i}$), the $g_j \circ \varphi_i$ (for $1 \leq j \leq s$) and the M_K-constant $\mathcal{C}_0$, which gives us an M_K-constant $\mathcal{C}'_i$ and the following dichotomy on X_i for every $(P, w) \in E_i$:

$$\max_{\substack{1\leq \ell \leq r\\ 1 \leq j \leq s_i}} \log\left|\frac{h_{\ell,i,j}}{\phi_\ell}(P)\right|_w \geq c'_{i,v} \quad \text{or} \quad \max_{1\leq j\leq s} \log|g_j \circ \varphi_i(P)|_w < c_{0,v}.$$

Now, the $h_{\ell,i,j}$ are regular on X_i hence M_K-bounded on E_i, therefore there is a second M_K-constant $\mathcal{C}''_i$ such that for every $(P, w) \in E_i$

$$\max_{\substack{1\leq \ell \leq r\\ 1 \leq j \leq s_i}} \log\left|\frac{h_{\ell,i,j}}{\phi_\ell}(P)\right|_w \geq c'_{i,v} \Longrightarrow \min_{1\leq \ell\leq r} \log|\phi_\ell(P)|_w \leq c''_{i,v}.$$

Taking $\mathcal{C}$ as the maximum of the M_K-constants $\mathcal{C}''_i$, $0 \leq i \leq n$, for every $(P, w) \in X(\overline{K}) \times M_{\overline{K}}$, we choose i such that $(P, w) \in E_i$ and then we have the dichotomy (3-2) by definition of the tubular neighborhood V_w. □

To finish this section, we will give the explicit link between integral points on a projective scheme (relative to a divisor) and integral points relative to rational functions on the scheme. This will also tie our notion of integer points with that of [Levin 2008, Section 2], showing that the two can be treated exactly in the same way.

Proposition 3.4. *Let $\mathcal{X}$ be a normal projective scheme over $\mathcal{O}_{K,S}$.*

(a) *If $\mathcal{Y}$ is an effective Cartier divisor on $\mathcal{X}$ such that $\mathcal{Y}_K$ is an ample (Cartier) divisor of $\mathcal{X}_K$, there is a projective embedding $\psi : \mathcal{X}_K \to \mathbb{P}^n_K$ and an M_K-constant $\mathcal{C}$ such that the pullback by ψ of the hyperplane of equation $x_0 = 0$ in $\mathbb{P}^n_K$ is $\mathcal{Y}_K$, and for any finite extension L of K and any $w \in M_L$ not above S, $\forall P \in (\mathcal{X}\backslash\mathcal{Y})(\mathcal{O}_{L,w})$,*

$$\log\|x_{\psi(P)}\|_w \leq c_v + \log|x_{\psi(P),0}|_w. \tag{3-4}$$

(b) *If $\mathcal{Y}$ is an effective Cartier divisor on $\mathcal{X}$ such that $\mathcal{Y}_K$ is a big (Cartier) divisor of $\mathcal{X}_K$, there is a strict Zariski closed subset Z_K of $\mathcal{X}_K$, a closed immersion $\psi : \mathcal{X}_K\backslash Z_K \to \mathbb{P}^n_K\backslash\{x_0 = 0\}$ and an M_K-constant $\mathcal{C}$ such that for any finite extension L of K and any $w \in M_L$ not above S, formula (3-4) holds outside Z_K.*

Proof of Proposition 3.4. (a) and (b) come from the classical link between integral points in terms of a scheme and integral points in terms of local heights (proven in Lemma 1.4.6 and Proposition 1.4.7 of [Vojta 1987] for instance), combined with the properties of the morphisms associated to (very) ample or big divisors. □

Remark 3.5. (a) This proposition is formulated to avoid the use of local heights, but the idea is exactly that under the hypotheses above, if $P \in (\mathcal{X}\backslash\mathcal{Y})(\mathcal{O}_{L,w})$, the local height at w of P for the divisor $\mathcal{Y}$ is strictly bounded.

(b) The hypotheses on ampleness (or "bigness") are only necessary at the generic fiber. Once again, the auxiliary functions replace the need for a complete understanding of what happens at the finite places.

4. The case of curves revisited

In this section, we reprove the generalization of an old theorem of Runge [1887], obtained by Bombieri [1983, p. 305] (also rewritten as [Bombieri and Gubler 2006, Theorem 9.6.6]), following an idea explained by Bilu in an unpublished note and mentioned for the case $K = \mathbb{Q}$ by [Schoof 2008, Chapter 5]. The aim of this section is to give a general understanding of this idea (quite different from the original proof of Bombieri), as well as explain how it actually gives a *method* to bound heights of integral points on curves. It is also a good start to understand how the intuition behind this result can be generalized to higher dimension, which will be done in the next section.

Proposition 4.1 (Bombieri, 1983). *Let C be a smooth projective algebraic curve defined over a number field K and $\phi \in K(C)$ not constant.*

For any finite extension L/K, let r_L be the number of orbits of the natural action of $\mathrm{Gal}(\overline{L}/L)$ *over the poles of ϕ. For any set of places S_L of L containing M_L^∞, we say that (L, S_L) satisfies the* ***Runge condition*** *if*

$$|S_L| < r_L. \tag{4-1}$$

Then, the union

$$\bigcup_{(L,S_L)} \{P \in C(L) \mid \phi(P) \in \mathcal{O}_{L,S_L}\}, \tag{4-2}$$

where (L, S_L) runs through all the pairs satisfying the Runge condition, is ***finite*** *and can be explicitly bounded in terms of the height $h \circ \phi$.*

Example 4.2. As a concrete example, consider the modular curve $X_0(p)$ for p prime and the j-invariant function. This curve is defined over $\mathbb{Q}$ and j has two rational poles (which are the cusps of $X_0(p)$), hence $r_L = 2$ for any choice of L, and we need to ensure $|M_L^\infty| \leq |S_L| < 2$. The only possibilities satisfying the Runge condition are thus imaginary quadratic fields L with $S_L = \{|\cdot|_\infty\}$.

We proved in [Le Fourn 2016] that for any imaginary quadratic field L and any $P \in X_0(p)(L)$ such that $j(P) \in \mathcal{O}_L$, one has

$$\log|j(P)| \leq 2\pi\sqrt{p} + 6\log(p) + 8.$$

The method for general modular curves is carried out in [Bilu and Parent 2011] and gives explicit estimates on the height for integral points satisfying the Runge condition. This article uses the theory of modular units and implicitly the same proof of Bombieri's result as the one we explain below.

Remark 4.3. (a) The claim of an explicit bound deserves a clarification: it can actually be made explicit when one knows well enough the auxiliary functions involved in the proof below (which is possible in many cases, e.g., for modular curves thanks to the modular units). Furthermore, even as the theoretical proof makes use of M_K-constants and results of Section 3, they are frequently implicit in practical cases.

(b) Despite the convoluted formulation of the proof below and the many auxiliary functions to obtain the full result, its principle is as described in the introduction. It also gives the framework to apply Runge's method to a given couple (C, ϕ).

Proof of Proposition 4.1. We fix K' a finite Galois extension of K on which every pole of ϕ is defined. For any two distinct poles Q and Q' of ϕ, we choose by the Riemann–Roch theorem a function $g_{Q,Q'} \in K'(C)$ whose only pole is Q and which vanishes at Q'. For every point P of $C(\overline{K})$ which is not a pole of ϕ, one has $\mathrm{ord}_P(g_{Q,Q'}) \geq 0$ thus $g_{Q,Q'}$ belongs to the intersection of the discrete valuation rings of $\overline{K}(C)$ containing ϕ and $\overline{K}$ [Hartshorne 1977, proof of Lemma I.6.5], which is exactly the integral closure of $K[\phi]$ in $\overline{K}(C)$ [Atiyah and Macdonald 1969, Corollary 5.22]. Hence, the function $g_{Q,Q'}$ is integral on $K[\phi]$ and up to multiplication by some nonzero integer, we can and will assume it is integral on $\mathcal{O}_K[\phi]$.

For any fixed finite extension L of K included in $\overline{K}$, we define $f_{Q,Q',L} \in L(C)$ the product of the conjugates of $g_{Q,Q'}$ by $\mathrm{Gal}(\overline{L}/L)$. If Q and Q' belong to distinct orbits of poles for $\mathrm{Gal}(\overline{L}/L)$, the set of poles of $f_{Q,Q',L}$ is exactly the orbit of Q by $\mathrm{Gal}(\overline{K}/L)$, and its set of zeroes contains all the orbit of Q' by $\mathrm{Gal}(\overline{K}/L)$. Notice that we thus built only finitely many different functions (even with L running through all finite extensions of K) because each $g_{Q,Q'}$ only has finitely many conjugates in $\mathrm{Gal}(K'/K)$.

Now, let $\mathcal{O}_1, \ldots, \mathcal{O}_{r_L}$ be the orbits of poles of ϕ and denote for any $i \in \{1, \ldots, r_L\}$ by $f_{i,L}$ a product of $f_{Q_i,Q'_j,L}$ where $Q_i \in \mathcal{O}_i$ and Q'_j runs through representatives of the orbits (except $\mathcal{O}_i$). Again, there is a finite number of possible choices, and we obtain a function $f_{i,L} \in L(C)$ having for only poles the orbit $\mathcal{O}_i$ and vanishing at all the other poles of ϕ.

We apply Corollary 3.2 to $f_{i,L}/\phi^k$ and $f_{i,L}$ (for any i) for some k such that $f_{i,L}/\phi^k$ does not have poles at $\mathcal{O}_i$, and take the maximum of the induced M_K-constants (Definition 1.1) for any L and $1 \le i \le r_L$. This gives an M_K-constant $\mathcal{C}_0$ independent of L such that $\forall i \in \{1, \dots, r_L\}$, $\forall w \in M_{\overline{K}}$ and $\forall P \in C(\overline{K})$,

$$\log \min\left(\left| \frac{f_{i,L}}{\phi^k}(P) \right|_w, |f_{i,L}(P)|_w \right) \le c_{0,v} \quad (w \mid v \in M_K).$$

In particular, the result interesting us in this case is that $\forall i \in \{1, \dots, r_L\}$, $\forall w \in M_{\overline{K}}$ and $\forall P \in C(\overline{K})$,

$$|\phi(P)|_w \le 1 \Rightarrow \log|f_{i,L}(P)|_w \le c_{0,v}, \tag{4-3}$$

and we can assume $c_{0,v}$ is 0 for any finite place v by integrality of the $f_{i,L}$ over $\mathcal{O}_K[\phi]$.

Given our construction, we also fix n such that for every $i \in \{1, \dots, r_L\}$, the $\phi f_{i,L}^n$ have poles at $\mathcal{O}_i$ and vanish at all other poles of ϕ. We reapply Corollary 3.2 for every pair $(\phi f_{i,L}^n, \phi f_{j,L}^n)$ with $1 \le i < j \le r_L$, which again by taking the maximum of the induced M_K-constants for all the possible combinations (Definition 1.1) gives an M_K-constant $\mathcal{C}_1$ such that for every $v \in M_K$ and every $(P, w) \in C(\overline{K}) \times M_{\overline{K}}$ with $w \mid v$, the inequality

$$\log|(\phi \cdot f_{i,L}^n)(P)|_w \le c_{1,v} \tag{4-4}$$

is true for all indices i except at most one (depending on the choice of P and w).

Let us now suppose that (L, S_L) is a pair satisfying the Runge condition and $P \in C(L)$ with $\phi(P) \in \mathcal{O}_{L,S_L}$. By integrality on $\mathcal{O}_K[\phi]$, for every $i \in \{1, \dots, r_L\}$, $|f_{i,L}(P)|_w \le 1$ for every place $w \in M_L \backslash S_L$. For every place $w \in S_L$, there is at most one index i not satisfying (4-4) hence by the Runge condition and the pigeon-hole principle, there remains one index i (depending on P) such that $\forall w \in M_L$,

$$\log|\phi(P) f_{i,L}^n(P)|_w \le c_{1,v}. \tag{4-5}$$

With (4-3) and (4-5), we have obtained all the auxiliary results we need to finish the proof. By the product formula,

$$\begin{aligned} 0 &= \sum_{w \in M_L} n_w \log|f_{i,L}(P)|_w \\ &= \sum_{\substack{w \in M_L \\ |\phi(P)|_w > 1}} n_w \log|f_{i,L}(P)|_w + \sum_{w \in M_L^\infty |\phi(P)|_w \le 1} n_w \log|f_{i,L}(P)|_w + \sum_{\substack{w \in M_L \backslash M_L^\infty \\ |\phi(P)|_w \le 1}} n_w \log|f_{i,L}(P)|_w. \end{aligned}$$

Here, the first sum on the right side will be linked to the height $h \circ \phi$ and the third sum is negative by integrality of the $f_{i,L}$, so we only have to bound the second sum. From (4-3) and (1-4), we obtain

$$\sum_{\substack{w \in M_L^\infty \\ |\phi(P)|_w \le 1}} n_w \log|f_{i,L}(P)|_w \le \sum_{\substack{w \in M_L^\infty \\ |\phi(P)|_w \le 1}} n_w c_{0,v} \le [L:K] \sum_{v \in M_K^\infty} n_v c_{0,v}.$$

On another side, by (4-5) (and (1-4) again), we have

$$n \cdot \sum_{\substack{w \in M_L \\ |\phi(P)|_w > 1}} n_w \log|f_{i,L}(P)|_w = \sum_{\substack{w \in M_L \\ |\phi(P)|_w > 1}} n_w \log|\phi f_{i,L}^n(P)|_w - \sum_{\substack{w \in M_L \\ |\phi(P)|_w > 1}} n_w \log|\phi(P)|_w$$
$$\leq \left([L:K] \sum_{v \in M_K} n_v c_{1,v} \right) - [L:\mathbb{Q}] h(\phi(P)).$$

Hence, we obtain

$$0 \leq [L:K] \sum_{v \in M_K} n_v c_{1,v} - [L:\mathbb{Q}] h(\phi(P)) + [L:K] n \sum_{v \in M_K^\infty} n_v c_{0,v},$$

which is equivalent to

$$h(\phi(P)) \leq \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v (c_{1,v} + n c_{0,v}).$$

We thus obtained a bound on $h(\phi(P))$ independent on the choice of (L, S_L) satisfying the Runge condition, and together with the bound on the degree $[L:\mathbb{Q}] \leq 2|S_L| < 2r_L \leq 2r$, we get the finiteness. □

5. The main result: tubular Runge's theorem

We will now present our version of Runge theorem with tubular neighborhoods, which generalizes Theorem 4(b) and (c) of [Levin 2008]. As its complete formulation is quite lengthy, we indicated the different hypotheses by the letter **H** and the results by the letter **R**. The key condition for integral points (generalizing the Runge condition of Proposition 4.1) is indicated by the letters TRC.

We recall that the crucial notion of tubular neighborhood is explained in Definitions 2.1 and 2.2, and we advise the reader to look at the simplified version of this theorem stated in the Introduction to get more insight if necessary.

Theorem 5.1 (tubular Runge's theorem). **(H0)** *Let K be a number field, S_0 a set of places of K containing M_K^∞ and $\mathcal{O}$ the integral closure of $\mathcal{O}_{K,S_0}$ in some finite Galois extension K' of K.*

(H1) *Let $\mathcal{X}$ be a normal projective scheme over $\mathcal{O}_{K,S_0}$ and $D_1, \ldots, D_r$ be effective Cartier divisors on $\mathcal{X}_\mathcal{O} = \mathcal{X} \times_{\mathcal{O}_{K,S_0}} \mathcal{O}$ such that $D_\mathcal{O} = \bigcup_{i=1}^r D_i$ is the scalar extension to $\mathcal{O}$ of some Cartier divisor D on $\mathcal{X}$, and that* Gal(K'/K) *permutes the generic fibers $(D_i)_{K'}$. For every extension L/K, we denote by $\boldsymbol{r_L}$ the number of orbits of $(D_1)_{K'}, \ldots, (D_r)_{K'}$ for the action of* Gal$(K'L/L)$.

(H2) *Let Y be a closed subscheme of $\mathcal{X}_K$ and $\mathcal{V}$ be a tubular neighborhood of Y in $\mathcal{X}_K$. Let $\boldsymbol{m_Y} \in \mathbb{N}$ be the minimal number such that the intersection of any (m_Y+1) of the divisors $(D_i)_{K'}$ amongst the r possible ones is included in $Y_{K'}$.*

*TRC The **tubular Runge condition** for a pair (L, S_L), where L/K is finite and S_L contains all the places above S_0, is*

$$\boldsymbol{m_Y} |\boldsymbol{S_L}| < \boldsymbol{r_L}.$$

Under these hypotheses and notations, the results are the following:

(**R1**) *If* $(D_1)_{K'}, \dots, (D_r)_{K'}$ *are ample divisors, the set*

$$\bigcup_{(L,S_L)} \{P \in (\mathcal{X} \backslash D)(\mathcal{O}_{L,S_L}) \mid P \notin \mathcal{V}\}, \tag{5-1}$$

where (L, S_L) *goes through all the pairs satisfying the tubular Runge condition, is* ***finite***.

(**R2**) *If* $(D_1)_{K'}, \dots, (D_r)_{K'}$ *are big divisors, there exists a proper closed subset* $Z_{K'}$ *of* $\mathcal{X}_{K'}$ *such that the set*

$$\left(\bigcup_{(L,S_L)} \{P \in (\mathcal{X} \backslash D)(\mathcal{O}_{L,S_L}) \mid P \notin \mathcal{V}\} \right) \backslash Z_{K'}(\overline{K}),$$

where (L, S_L) *goes through all the pairs satisfying the tubular Runge condition, is* ***finite***.

Remark 5.2 explains the hypotheses and results of this theorem, and Remark 5.3 compares it with other theorems.

Remark 5.2. (a) The need for the extensions of scalars to K' and $\mathcal{O}$ in (**H0**) and (**H1**) is the analogue of the fact that the poles of ϕ are not necessarily K-rational in the case of curves, hence the assumption that the $(D_i)_{K'}$ are all conjugates by $\mathrm{Gal}(K'/K)$ and the definition of r_L given in (**H1**). It will induce technical additions of the same flavor as the auxiliary functions $f_{Q,Q',L}$ in the proof of Proposition 4.1.

(b) The motivation for the tubular Runge condition is the following: imitating the principle of proof for curves (Remark 4.3(b)), if $P \in (\mathcal{X} \backslash D)(\mathcal{O}_{L,S_L})$, we can say that at the places w of $M_L \backslash S_L$, this point is "w-adically far" from D. Now, the divisors $(D_1)_{K'}, \dots, (D_r)_{K'}$ can intersect (which does not happen for distinct points on curves), so for $w \in S_L$, this point P can be "w-adically close" to many divisors at the same time. More precisely, it can be "w-adically close" to at most m such divisors, where $m = m_\varnothing$, i.e., the largest number such that there are m divisors among $D_1, \dots, D_r$ whose set-theoretic intersection is nonempty. This number is also defined in [Levin 2008] but we found that for our applications, it often makes the Runge condition too strict. Therefore, we allow the use of the closed subset Y in (**H2**), and if we assume that our point P is never too close to Y (i.e., $P \notin \mathcal{V}$), this m goes down to m_Y by definition. Thus, we only need to take out m_Y divisors for each place w in S_L, hence the tubular Runge condition $m_Y|S_L| < r_L$. Actually, one can even mix the Runge conditions, i.e., assume that P is close to Y exactly at s_1 places, and close to one of the divisors (but not from Y) at s_2 places: following along the lines of the proof below, we obtain finiteness given the Runge condition $s_1 m_\varnothing + s_2 m_Y < r_L$ (this is exactly what we do for Theorem 8.2(a)).

(c) The last main difference with the case of curves is the assumption of ample or big divisors, respectively in (**R1**) and (**R2**). In both cases, such an assumption is necessary twice. First, we need it to translate by Proposition 3.4 the integrality condition on schemes to an integrality expression on auxiliary functions (such as in Section 2 of [Levin 2008]) to use the machinery of M_K-constants and the key result (Proposition 3.1).

Then, we need it to ensure that after obtaining a bound on the heights associated to the divisors, it implies finiteness (implicit in Proposition 3.4, see also Remark 3.5(a)).

Remark 5.3. (a) This theorem has some resemblance to the CLZ theorem of [Corvaja et al. 2009] (where our closed subset Y would be the analogue of the $\mathcal{Y}$ in that article), let us point out the differences. In the CLZ theorem, there is no hypothesis of the set of places S_L, no additional hypothesis of integrality (appearing for us under the form of a tubular neighborhood), and the divisors are assumed to be normal crossing divisors, which is replaced in our case by the tubular Runge condition. As for the results themselves, the finiteness formulated by CLZ depends on the set S_L (that is, it is not clear how it would prove that (5-1) is finite). Finally, the techniques employed are greatly different: the CLZ theorem uses Schmidt's subspace theorem (which has not been made effective yet), whereas our method can be made effective if one knows the involved auxiliary functions. It might be possible (and worthy of interest) to build some bridges between the two results, and the techniques involved.

(b) Theorem 5.1 can be seen as a stratification of Runge-like results depending on the dimension of the intersection of the involved divisors: at one extreme, the intersection is empty, and we get back Theorem 4(b) and (c) of [Levin 2008]. At the other extreme, the intersection is a divisor (ample or big), and the finiteness is automatic by (Remark 2.3). Of course, this stratification is not relevant in the case of curves. In another perspective, for a fixed closed subset Y, Theorem 5.1 is more a concentration result of integral points than a finiteness result, as it means that even if we choose a tubular neighborhood $\mathcal{V}$ of Y as small as possible around Y, there is only a finite number of integral points in the set (5-1), i.e., these integral points (ignoring the hypothesis $P \notin \mathcal{V}$) must concentrate around Y (at least at one of the places $w \in M_L$). Specific examples are given in Sections 7 and 8.

Let us now prove Theorem 5.1, following the ideas outlined in Remark 5.2.

Proof of Theorem 5.1. (**R1**) Let us first build the embeddings we need. For every subextension K'' of K'/K, the action of $\mathrm{Gal}(K'/K'')$ on the divisors $(D_1)_{K'}, \dots, (D_r)_{K'}$ has orbits denoted by $O_{K'',1}, \dots, O_{K'',r_{K''}}$. Notice that any $m_Y + 1$ such orbits still have their global intersection included in Y.

For each such orbit, the sum of its divisors is ample by hypothesis and coming from an effective Cartier divisor on $\mathcal{X}_{K''}$, One can then choose by Proposition 3.4 an appropriate embedding $\psi_{K'',i} : \mathcal{X}_{K''} \to \mathbb{P}^{n_i}_{K''}$, whose coordinate functions (denoted by $\phi_{K'',i,j} = (x_j/x_0) \circ \psi_{K'',i} (1 \leq j \leq n_i)$) satisfy Proposition 3.4 on all points of $(\mathcal{X}_{\mathcal{O}} \backslash \overline{O_{K'',i}})$ (where $\overline{O_{K'',i}}$ denotes the Zariski closure of $O_{K'',i}$ in $\mathcal{X}_{\mathcal{O}}$). We will denote by $\mathcal{C}_0$ the maximum of the (induced) M_K-constants obtained from Proposition 3.4 for all possible K''/K and orbits $O_{K'',i} (1 \leq i \leq r_{K''})$. The important point is that for any extension L/K, any $v \in M_K \backslash S_0$, any place $w \in M_L$ above v and any $P \in (\mathcal{X} \backslash D)(\mathcal{O}_{L,w})$, choosing $L' = K' \cap L$, one has

$$\max_{\substack{1 \leq i \leq r_L \\ 1 \leq j \leq n_i}} \log|\phi_{L',i,j}(P)|_w \leq c_{0,v}. \tag{5-2}$$

This is the first step to obtain a bound on the height of one of the $\psi_{K'',i}(P)$. For fixed P, we only have to do so for one of the $i \in \{1, \dots, r_L\}$ as long as the bound is uniform in the choice of (L, S_L) (and P), to obtain finiteness as each $\psi_{K'',i}$ is an embedding. To this end, one only needs to bound the coordinate functions on the places w of S_L, which is what we will do now.

For a subextension K'' of K'/K again, by (**H2**) (see the definition of m_Y), taking any set $\mathcal{I}$ of m_Y+1 couples (i, j), $1 \leq i \leq r_{K''}$, $j \in \{1, \dots, n_i\}$ with m_Y+1 different indices i and considering the rational functions $\phi_{K'',i,j}$, $(i, j) \in \mathcal{I}$, whose common poles are included in Y by hypothesis, we can apply Proposition 3.1 to these functions and the tubular neighborhood $\mathcal{V} = (V_w)_{w \in M_{\overline{K}}}$. Naming as $\mathcal{C}_1$ the maximum of all the (induced) obtained M_K-constants (also for all the possible K''), we just proved that for every subextension K'' of K'/K, every place $w \in M_{\overline{K}}$ (above $v \in M_K$) and any $P \in \mathcal{X}(\overline{K}) \backslash V_w$, the inequality

$$\max_{1 \leq j \leq n_i} \log|\phi_{K'',i,j}(P)|_w \leq c_{1,v} \tag{5-3}$$

is true except for at most m_Y different indices $i \in \{1, \dots, r_{K''}\}$.

Now, let us consider (L, S_L) a pair satisfying the tubular Runge condition $m_Y|S_L| < r_L$ and denote $L' = K' \cap L$ again. For $P \in (\mathcal{X} \backslash D)(\mathcal{O}_{L,S_L})$ not belonging to $\mathcal{V}$, by (5-2), (5-3) and the tubular Runge condition, there remains an index $i \in \{1, \dots, r_L\}$ (dependent on P) such that $\forall w \in M_L$,

$$\max_{1 \leq j \leq n_i} \log|\phi_{L',i,j}(P)|_w \leq \max(c_{0,v}, c_{1,v}) \quad (w \mid v \in M_K).$$

This immediately gives a bound on the height of $\psi_{L',i}(P)$ independent of the choice of pair (L, S_L) (except the fact that $L' = K' \cap L$). As $\psi_{L',i}$ is an embedding and $[L : \mathbb{Q}] \leq 2|S_L| < 2r$, by Northcott's property, P belongs to a finite family of points (depending on i but not on (L, S_L)), and taking the union of these families for $i \in \{1, \dots, r_L\}$, we have proven the finiteness of the set of points

$$\bigcup_{(L,S_L)} \{P \in (\mathcal{X} \backslash D)(\mathcal{O}_{L,S_L}) \mid P \notin \mathcal{V}\},$$

where (L, S_L) goes through all the pairs satisfying the tubular Runge condition.

(**R2**) The proof is the same as for (**R1**) except that we have to exclude a closed subset of $\mathcal{X}_{K'}$ for every big divisor involved, and their union will be denoted by $Z_{K'}$. The arguments above hold for every point $P \notin Z_{K'}(\overline{K})$ (both for the expression of integrality by auxiliary functions, and for the conclusion and finiteness outside of this closed subset), using again Propositions 3.4 and 3.1. □

6. Reminders on Siegel modular varieties

In this section, we recall the classical constructions and results for the Siegel modular varieties, parametrizing principally polarized abelian varieties with a level structure. Most of those results are extracted (or easily deduced) from these general references: Chapter V of [Cornell and Silverman 1986] for the basic notions on abelian varieties, [Debarre 1999] for the complex tori, their line bundles, theta functions and moduli spaces, Chapter II of [Mumford 2007] for the classical complex theta functions, [Mumford 1984] for their links with theta divisors, and Chapter V of [Faltings and Chai 1990] for abelian schemes and their moduli spaces.

Unless specified, all the vectors of $\mathbb{Z}^g$, $\mathbb{R}^g$ and $\mathbb{C}^g$ are assumed to be row vectors.

6A. *Abelian varieties and Siegel modular varieties.*

Definition 6.1 (abelian varieties and polarization). • An *abelian variety* A over a field k is a projective algebraic group over k. Each abelian variety $A_{/k}$ has a dual abelian variety denoted by $\hat{A} = \mathrm{Pic}^0(A/k)$ [Cornell and Silverman 1986, §V.9].

• A *principal polarization* is an isomorphism $\lambda : A \to \hat{A}$ such that there exists a line bundle L on $A_{\bar{k}}$ with $\dim H^0(A_{\bar{k}}, L) = 1$ and λ is the morphism

$$\begin{aligned} \lambda : A_{\bar{k}} &\to \widehat{A_{\bar{k}}} \\ x &\mapsto T_x^* L \otimes L^{-1} \end{aligned}$$

[Cornell and Silverman 1986, §V.13].

• Given a pair (A, λ), for every $n \geq 1$ prime to $\mathrm{char}(k)$, we can define the *Weil pairing*

$$A[n] \times A[n] \to \mu_n(\bar{k}),$$

where $A[n]$ is the n-torsion of $A(\bar{k})$ and μ_n the group of n-th roots of unity in $\bar{k}$. It is alternating and nondegenerate [Cornell and Silverman 1986, §V.16].

• Given a pair (A, λ), for $n \geq 1$ prime to $\mathrm{char}(k)$, a *symplectic level n structure on $A[n]$* is a basis α_n of $A[n]$ in which the matrix of the Weil pairing is

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

• Two triples (A, λ, α_n) and $(A', \lambda', \alpha'_n)$ of principally polarized abelian varieties over K with level n-structures are *isomorphic* if there is an isomorphism of abelian varieties $\phi : A \to A'$ such that $\phi^* \lambda' = \lambda$ and $\phi^* \alpha'_n = \alpha_n$.

In the case of complex abelian varieties, the previous definitions can be made more explicit.

Definition 6.2 (complex abelian varieties and symplectic group). Let $g \geq 1$.

• The *half-superior Siegel space of order g*, denoted by $\mathcal{H}_g$, is the set of matrices

$$\mathcal{H}_g := \{\tau \in M_g(\mathbb{C}) \mid {}^t\tau = \tau \text{ and } \operatorname{Im} \tau > 0\}, \tag{6-1}$$

where $\operatorname{Im} \tau > 0$ means that this symmetric matrix of $M_g(\mathbb{R})$ is positive definite. This space is an open subset of $M_g(\mathbb{C})$.

• For any $\tau \in \mathcal{H}_g$, we define

$$\Lambda_\tau := \mathbb{Z}^g + \mathbb{Z}^g \tau \quad \text{and} \quad A_\tau := \mathbb{C}^g / \Lambda_\tau. \tag{6-2}$$

Let L_τ be the line bundle on A_τ made up as the quotient of $\mathbb{C}^g \times \mathbb{C}$ by the action of Λ_τ defined $\forall p, q \in \mathbb{Z}^g$, by

$$(p\tau + q) \cdot (z, t) = (z + p\tau + q, e^{-i\pi p \tau^t p - 2i\pi p^t z} t). \tag{6-3}$$

Then, L_τ is an ample line bundle on A_τ such that $\dim H^0(A_\tau, L_\tau) = 1$, hence A_τ is a complex abelian variety and L_τ induces a principal polarization denoted by λ_τ on A_τ (see for example [Debarre 1999, Theorem VI.1.3]). We also denote by $\pi_\tau : \mathbb{C}^g \to A_\tau$ the quotient morphism.

- For every $n \geq 1$, the Weil pairing $w_{\tau,n}$ associated to (A_τ, λ_τ) on $A_\tau[n]$ is defined by

$$\begin{aligned} w_{\tau,n} : A_\tau[n] \times A_\tau[n] &\to \mu_n(\mathbb{C}) \\ (\bar{x}, \bar{y}) &\mapsto e^{2i\pi n w_\tau(x,y)} \end{aligned}$$

where $x, y \in \mathbb{C}^g$ have images $\bar{x}$ and $\bar{y}$ by π_τ, and w_τ is the $\mathbb{R}$-bilinear form on $\mathbb{C}^g \times \mathbb{C}^g$ (so that $w_\tau(\Lambda_\tau \times \Lambda_\tau) = \mathbb{Z}$) defined by

$$w_\tau(x, y) := \mathrm{Re}(x) \cdot \mathrm{Im}(\tau)^{-1} \cdot {}^t\,\mathrm{Im}(y) - \mathrm{Re}(y) \cdot \mathrm{Im}(\tau)^{-1} \cdot {}^t\,\mathrm{Im}(x)$$

(also readily checked by making explicit the construction of the Weil pairing).

- Let $(e_1, \dots, e_g)$ be the canonical basis of $\mathbb{C}^g$. The family

$$(\pi_\tau(e_1/n), \dots, \pi_\tau(e_g/n), \pi_\tau(e_1 \cdot \tau/n), \dots, \pi_\tau(e_g \cdot \tau/n)) \tag{6-4}$$

is a symplectic level n structure on (A_τ, λ_τ), denoted by $\alpha_{\tau,n}$.

- Let $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in M_{2g}(\mathbb{Z})$. For any commutative ring A, the *symplectic group of order g over A*, denoted by $\mathrm{Sp}_{2g}(A)$, is the subgroup of $\mathrm{GL}_{2g}(A)$ defined by

$$\mathrm{Sp}_{2g}(A) := \{M \in \mathrm{GL}_{2g}(A) \mid {}^t M J M = J\}, \quad J := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}. \tag{6-5}$$

For every $n \geq 1$, the *symplectic principal subgroup of degree g and level n*, denoted by $\Gamma_g(n)$, is the subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ made up by the matrices congruent to I_{2g} modulo n. For every $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2g}(\mathbb{R})$ and every $\tau \in \mathcal{H}_g$, we define

$$j_\gamma(\tau) = C\tau + D \in \mathrm{GL}_g(\mathbb{C}) \quad \text{and} \quad \gamma \cdot \tau = (A\tau + B)(C\tau + D)^{-1}, \tag{6-6}$$

which defines a left action by biholomorphisms of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathcal{H}_g$, and $(\gamma, \tau) \mapsto j_\gamma(\tau)$ is a left cocycle for this action [Klingen 1990, Proposition I.1].

- For every $g \geq 2$, $n \geq 1$ and $k \geq 1$, a *Siegel modular form of degree g, level n and weight k* is an holomorphic function f on $\mathcal{H}_g$ such that $\forall \gamma \in \Gamma_g(n)$,

$$f(\gamma \cdot z) = \det(j_\gamma(z))^k f(z). \tag{6-7}$$

The reason for this description of the complex abelian varieties is that the (A_τ, λ_τ) defined above make up all the principally polarized complex abelian varieties up to isomorphism. The following results can be found in Chapter VI of [Debarre 1999] except the last point which is straightforward.

Definition-Proposition 6.3 (uniformization of complex abelian varieties). • Every principally polarized complex abelian variety of dimension g with symplectic structure of level n is isomorphic to some triple $(A_\tau, \lambda_\tau, \alpha_{\tau,n})$ where $\tau \in \mathcal{H}_g$.

• For every $n \geq 1$, two triples $(A_\tau, \lambda_\tau, \alpha_{\tau,n})$ and $(A_{\tau'}, \lambda_{\tau'}, \alpha_{\tau',n})$ are isomorphic if and only if there exists $\gamma \in \Gamma_g(n)$ such that $\gamma \cdot \tau = \tau'$, and then such an isomorphism is given by

$$\begin{aligned} A_\tau &\to A_{\tau'} \\ z \bmod \Lambda_\tau &\mapsto z \cdot j_\gamma(\tau)^{-1} \bmod \Lambda_{\tau'} \end{aligned}.$$

• The *Siegel modular variety of degree g and level n* is the quotient $A_g(n)_\mathbb{C} := \Gamma_g(n)\backslash\mathcal{H}_g$. From the previous result, it is the moduli space of principally polarized complex abelian varieties of dimension g with a symplectic level n structure. As a quotient, it also inherits a structure of normal analytic space (with finite quotient singularities) of dimension $g(g+1)/2$, because $\Gamma_g(n)$ acts properly discontinuously on $\mathcal{H}_g$.

• For every positive divisor m of n, the natural morphism $A_g(n)_\mathbb{C} \to A_g(m)_\mathbb{C}$ induced by the identity of $\mathcal{H}_g$ corresponds in terms of moduli to multiplying the symplectic basis $\alpha_{\tau,n}$ by n/m, thus obtaining $\alpha_{\tau,m}$.

• For every $g \geq 1$ and $n \geq 1$, the quotient of $\mathcal{H}_g \times \mathbb{C}$ by the action of $\Gamma_g(n)$ defined as

$$\gamma \cdot (\tau, t) = (\gamma \cdot \tau, t/\det(j_\gamma(z))) \tag{6-8}$$

is a variety over $\mathcal{H}_g$ denoted by L. For a large enough power of k (or if $n \geq 3$), $L^{\otimes k}$ is a line bundle over $A_g(n)_\mathbb{C}$, hence L is a $\mathbb{Q}$-line bundle over $A_g(n)_\mathbb{C}$ called *line bundle of modular forms of weight one* over $A_g(n)_\mathbb{C}$. By definition (6-7), for every $k \geq 1$, the global sections of $L^{\otimes k}$ are the Siegel modular forms of degree g, level n and weight k.

Let us now present the compactification of $A_g(n)_\mathbb{C}$ we will use, that is the Satake compactification (for a complete description of it, see Section 3 of [Namikawa 1980]).

Definition-Proposition 6.4 (Satake compactification). Let $g \geq 1$ and $n \geq 1$. The normal analytic space $A_g(n)_\mathbb{C}$ admits a compactification called *Satake compactification* and denoted by $A_g(n)_\mathbb{C}^S$, satisfying the following properties.

(a) $A_g(n)_\mathbb{C}^S$ is a compact normal analytic space (of dimension $g(g+1)/2$, with finite quotient singularities) containing $A_g(n)_\mathbb{C}$ as an open subset and the boundary $\partial A_g(n)_\mathbb{C} := A_g(n)_\mathbb{C}^S \backslash A_g(n)_\mathbb{C}$ is of codimension g (see [Satake and Cartan 1957] for details).

(b) As a normal analytic space, $A_g(n)_\mathbb{C}^S$ is a projective algebraic variety. More precisely, for $\mathrm{M}_g(n)$ the graded ring of Siegel modular forms of degree g and level n, $A_g(n)_\mathbb{C}^S$ is canonically isomorphic to $\mathrm{Proj}_\mathbb{C} \mathrm{M}_g(n)$ [Cartan 1957, Théorème fondamental].

In particular, one can naturally obtain $A_g(n)_\mathbb{C}^S$ by fixing for some large enough weight k a basis of modular forms of $\mathrm{M}_g(n)$ of weight k and evaluating them all on $A_g(n)_\mathbb{C}$ to embed it in a projective space, so that $A_g(n)_\mathbb{C}^S$ is the closure of the image of the embedding in this projective space.

(c) The $\mathbb{Q}$-line bundle L of modular forms of weight 1 on $A_g(n)_{\mathbb{C}}$ extends naturally to an ample $\mathbb{Q}$-line bundle on $A_g(n)^S_{\mathbb{C}}$ (which is also denoted L); this is a direct consequence of (b).

6B. *Further properties of Siegel modular varieties.* As we are interested in the reduction of abelian varieties on number fields, one needs to have a good model of $A_g(n)_{\mathbb{C}}$ over integer rings, as well as some knowledge of the geometry of $A_g(n)_{\mathbb{C}}$. The integral models below and their properties are given in Chapter V of [Faltings and Chai 1990].

Definition 6.5 (abelian schemes). (a) An *abelian scheme* $A \to S$ is a smooth proper group scheme whose fibers are geometrically connected. It also has a natural *dual* abelian scheme $\hat{A} = \mathrm{Pic}^0(A/S)$, and it is *principally polarized* if it is endowed with an isomorphism $\lambda : A \to \hat{A}$ such that at every geometric point $\bar{s}$ of S, the induced isomorphism $\lambda_{\bar{s}} : A_{\bar{s}} \to \hat{A}_{\bar{s}}$ is a principal polarization of $A_{\bar{s}}$.

(b) A *symplectic structure of level* $n \geq 1$ on a principally polarized abelian scheme (A, λ) over a $\mathbb{Z}[\zeta_n, 1/n]$-scheme S is the datum of an isomorphism of group schemes $A[n] \to (\mathbb{Z}/n\mathbb{Z})^{2g}$, which is symplectic with respect to λ and the canonical pairing on $(\mathbb{Z}/n\mathbb{Z})^{2g}$ given by the matrix J (as in (6-5)).

Definition-Proposition 6.6 (algebraic moduli spaces). For every integers $g \geq 1$ and $n \geq 1$:

(a) The Satake compactification $A_g(n)^S_{\mathbb{C}}$ has an integral model $\mathcal{A}_g(n)^S$ on $\mathbb{Z}[\zeta_n, 1/n]$ which contains as a dense open subscheme the (coarse, if $n \leq 2$) moduli space $\mathcal{A}_g(n)$ over $\mathbb{Z}[\zeta_n, 1/n]$ of principally polarized abelian schemes of dimension g with a symplectic structure of level n. This scheme $\mathcal{A}_g(n)^S$ is normal, proper and of finite type over $\mathbb{Z}[\zeta_n, 1/n]$ [Faltings and Chai 1990, Theorem V.2.5].

(b) For every divisor m of n, we have canonical degeneracy morphisms $\mathcal{A}_g(n)^S \to \mathcal{A}_g(m)^S$ extending the morphisms of Definition-Proposition 6.3.

Before tackling our own problem, let us give some context on the divisors on $A_g(n)^S_{\mathbb{C}}$ to give a taste of the difficulties to overcome.

Definition 6.7 (rational Picard group). For every normal algebraic variety X over a field K, the *rational Picard group* of X is the $\mathbb{Q}$-vector space

$$\mathrm{Pic}(X)_{\mathbb{Q}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Proposition 6.8 (rational Picard groups of Siegel modular varieties). *Let $g \geq 2$ and $n \geq 1$.*

(a) *Every Weil divisor on $A_g(n)_{\mathbb{C}}$ or $A_g(n)^S_{\mathbb{C}}$ is up to some multiple a Cartier divisor, hence their rational Picard group is also their Weil class divisor group tensored by $\mathbb{Q}$.*

(b) *For $g = 3$, the Picard rational groups of $A_3(n)^S_{\mathbb{C}}$ and $A_3(n)_{\mathbb{C}}$ are equal to $\mathbb{Q} \cdot L$ for every $n \geq 1$.*

(c) *For $g = 2$, one has $\mathrm{Pic}_{\mathbb{Q}}(A_2(1)^S_{\mathbb{C}}) = \mathbb{Q} \cdot L$.*

This result has the following immediate corollary, because L is ample on $A_g(n)^S_{\mathbb{C}}$ for every $g \geq 2$ and every $n \geq 1$ (Definition-Proposition 6.4(c)).

Corollary 6.9 (ample and big divisors on Siegel modular varieties). *A $\mathbb{Q}$-divisor on $A_g(n)_{\mathbb{C}}$ or $A_g(n)^S_{\mathbb{C}}$ with $g{=}3$ (or $g{=}2$ and $n{=}1$) is ample if and only if it is big if and only if it is equivalent to $a \cdot L$ with $a > 0$.*

Remark 6.10. We did not mention the case of modular curves (also difficult, but treated by different methods): the point here is that the cases $g \geq 3$ are surprisingly much more uniform because then $\mathrm{Pic}(A_g(n)^S_{\mathbb{C}}) = \mathrm{Pic}(A_g(1)^S_{\mathbb{C}})$. The reason is that some rigidity appears from $g \geq 3$ (essentially by the general arguments of [Borel 1981]), whereas for $g = 2$, the situation seems very complex already for the small levels (see for example $n = 3$ in [Hoffman and Weintraub 2001]).

This is why the ampleness (or bigness) is in general hard to figure out for given divisors of $A_2(n)$, $n > 1$. We consider specific divisors in the following (namely, divisors of zeroes of theta functions), whose ampleness will not be hard to prove.

Proof of Proposition 6.8.

(a) This is true for the $A_g(n)^S_{\mathbb{C}}$ by [Artal Bartolo et al. 2014] as they only have finite quotient singularities (this result actually seems to have been generally assumed a long time ago). Now, as $\partial A_g(n)^S_{\mathbb{C}}$ is of codimension at least 2, the two varieties $A_g(n)^S_{\mathbb{C}}$ and $A_g(n)_{\mathbb{C}}$ have the same Weil and Cartier divisors, hence the same rational Picard groups.

(b) This is a consequence of general results of [Borel 1981] further refined in [Weissauer 1992] (it can even be generalized to every $g \geq 3$).

(c) This comes from the computations of Section III.9 of [Mumford 1983] (for another compactification, called toroidal), from which we extract the result for $A_2(1)_{\mathbb{C}}$ by a classical restriction theorem [Hartshorne 1977, Proposition II.6.5] because the boundary for this compactification is irreducible of codimension 1. The result for $A_2(1)^S_{\mathbb{C}}$ is then the same because the boundary is of codimension 2. □

6C. *Theta divisors on abelian varieties and moduli spaces.* We will now define the useful notions for our integral points problem.

Definition 6.11 (theta divisor on an abelian variety). Let k be an algebraically closed field and A an abelian variety over k.

Let L be an ample symmetric line bundle on A inducing a principal polarization λ on A. A *theta function associated to* (A, L) is a nonzero global section $\vartheta_{A,L}$ of L. The *theta divisor associated to* (A, L), denoted by $\Theta_{A,L}$, is the divisor of zeroes of $\vartheta_{A,L}$, well-defined and independent of our choice because $\dim H^0(A, L) = \deg(\lambda)^2 = 1$.

The theta divisor is in fact determined by the polarization λ itself up to a finite ambiguity, as the result below makes precise.

Proposition 6.12. *Let k be an algebraically closed field and A an abelian variety over k.*

Two ample symmetric line bundles L and L' on A inducing a principal polarization induce the same one if and only if $L' \cong T_x^* L$ for some $x \in A[2]$, and then

$$\Theta_{A,L'} = \Theta_{A,L} + x.$$

Proof. This is a well-known result relying on the properties of the map

$$L \mapsto (\phi_L : x \mapsto T_x^* L \otimes L^{-1})$$

from $\mathrm{Pic}(A)$ to $\mathrm{Hom}(A, \hat{A})$ [Mumford 1970, Corollary 4 p. 60 and Theorem 1 p. 77], and of ample symmetric line bundles. □

When $\mathrm{char}(k) \neq 2$, adding to a principally polarized abelian variety (A, λ) of dimension g the datum α_2 of a symplectic structure of level 2, we can determine an unique ample symmetric line bundle L with the following process called the *Igusa correspondence*, devised in [Igusa 1967]. To any ample symmetric Weil divisor D defining a principal polarization, one can associate bijectively a quadratic form q_D from $A[2]$ to $\{\pm 1\}$ called *even*, which means that the sum of its values on $A[2]$ is 2^g [loc. cit., Theorem 2 and the previous arguments]. On the other hand, the datum α_2 also determines an even quadratic form q_{α_2}, by associating to a $x \in A[2]$ with coordinates $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ in the basis α_2 of $A[2]$ the value

$$q_{\alpha_2}(x) = (-1)^{a^t b}. \tag{6-9}$$

We now only have to choose the unique ample symmetric divisor D such that $q_D = q_{\alpha_2}$ and the line bundle L associated to D.

By construction of this correspondence [loc. cit., p. 823], a point $x \in A[2]$ of coordinates $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ in α_2 automatically belongs to $\Theta_{A,L}$ (with L associated to (A, λ, α_2)) if $a^t b = 1 \bmod 2$. A point of $A[2]$ with coordinates (a, b) such that $a^t b = 0 \bmod 2$ can also belong to $\Theta_{A,L}$ but with even multiplicity.

This allows us to get rid of the ambiguity of choice of an ample symmetric L in the following, as soon as we have a symplectic level 2 structure (or finer) (this result is a reformulation of Theorem 2 of [loc. cit.]).

Definition-Proposition 6.13 (theta divisor canonically associated to a symplectic even level structure). Let $n \geq 2$ even and k algebraically closed such that $\mathrm{char}(k)$ does not divide n.

For (A, λ, α_n) a principally polarized abelian variety of dimension g with symplectic structure of level n (Definition 6.2), there is up to isomorphism an unique ample symmetric line bundle L inducing λ and associated by the Igusa correspondence to the symplectic basis of $A[2]$ induced by α_n. The *theta divisor associated to* (A, λ, α_n), denoted by $\Theta_{A,\lambda,\alpha_n}$, is then the theta divisor associated to (A, L).

The Runge-type theorem we give in Section 7 (Theorem 7.12) focuses on principally polarized abelian surfaces (A, λ) on a number field K whose theta divisor does not contain any n-torsion point of A (except 2-torsion points, as we will see it is automatic). This will imply (Proposition 7.5) that A is not a product of elliptic curves, but this is not a sufficient condition, as pointed out for example in [Boxall and Grant 2000].

We will once again start with the complex case to figure out how such a condition can be formulated on the moduli spaces, using complex theta functions [Mumford 2007, Chapter II].

Definition-Proposition 6.14 (complex theta functions). Let $g \geq 1$.

The holomorphic function Θ on $\mathbb{C}^g \times \mathcal{H}_g$ is defined by the series (uniformly convergent on any compact subset)

$$\Theta(z,\tau) = \sum_{n \in \mathbb{Z}^g} e^{i\pi n \tau^t n + 2i\pi n^t z}. \tag{6-10}$$

For any $a, b \in \mathbb{R}^g$, we also define the holomorphic function $\Theta_{a,b}$ by

$$\Theta_{a,b}(z,\tau) = \sum_{n \in \mathbb{Z}^g} e^{i\pi(n+a)\tau^t(n+a) + 2i\pi(n+a)^t(z+b)}. \tag{6-11}$$

For a fixed $\tau \in \mathcal{H}_g$, one defines $\Theta_\tau : z \mapsto \Theta(z,\tau)$ and similarly for $\Theta_{a,b,\tau}$. These functions have the following properties.

(a) For every $a, b \in \mathbb{Z}^g$,

$$\Theta_{a,b,\tau}(z) = e^{i\pi a\tau^t a + 2i\pi a^t(z+b)}\Theta_\tau(z + a\tau + b). \tag{6-12}$$

(b) For every $p, q \in \mathbb{Z}^g$,

$$\Theta_{a,b,\tau}(z + p\tau + q) = e^{-i\pi p\tau^t p - 2i\pi p^t z + 2i\pi(a^t q - b^t p)}\Theta_{a,b,\tau}(z). \tag{6-13}$$

(c) Let us denote by ϑ and $\vartheta_{a,b}$ the *normalized theta-constants*, which are the holomorphic functions on $\mathcal{H}_g$ defined by

$$\vartheta(\tau) := \Theta(0,\tau) \quad \text{and} \quad \vartheta_{a,b}(\tau) := e^{-i\pi a^t b}\Theta_{a,b}(0,\tau). \tag{6-14}$$

These theta functions satisfy the following modularity property: with the notations of Definition 6.2 and $\forall \gamma \in \Gamma_g(2)$,

$$\vartheta_{a,b}(\gamma \cdot \tau) = \zeta_8(\gamma) e^{i\pi(a,b)^t V_\gamma}\sqrt{j_\gamma(\tau)}\,\vartheta_{(a,b)\gamma}(\tau), \tag{6-15}$$

where $\zeta_8(\gamma)$ (an 8-th root of unity) and $V_\gamma \in \mathbb{Z}^g$ only depend on γ and the determination of the square root of $j_\gamma(\tau)$.

In particular, for every even $n \geq 2$, if $(na, nb) \in \mathbb{Z}^{2g}$, the function $\vartheta_{a,b}^{8n}$ is a Siegel modular form of degree g, level n and weight $4n$, which only depends on $(a,b) \bmod \mathbb{Z}^{2g}$.

Proof. The convergence of these series as well as their functional equations (6-12) and (6-13) are classical and can be found in Section II.1 of [Mumford 2007].

The modularity property (6-15) (also classical) is a particular case of the computations of Section II.5 of [Mumford 2007] (we do not need here the general formula for $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$).

Finally, by natural computations of the series defining $\Theta_{a,b}$, one readily obtains that

$$\vartheta_{a+p,b+q} = e^{2i\pi(a^t q - b^t p)}\vartheta_{a,b}.$$

Therefore, if $(na, nb) \in \mathbb{Z}^{2g}$, the function $\vartheta_{a,b}^n$ only depends on $(a,b) \bmod \mathbb{Z}^{2g}$. Now, putting the modularity formula (6-15) to the power $8n$, one eliminates the eight root of unity and if $\gamma \in \Gamma_g(n)$, one has $(a,b)\gamma = (a,b) \bmod \mathbb{Z}^g$ hence $\vartheta_{a,b}^{8n}$ is a Siegel modular form of weight $4n$ for $\Gamma_g(n)$. □

There is of course an explicit link between the theta functions and the notion of theta divisor, which we explain now with the notations of Definition 6.2.

Proposition 6.15 (theta divisor and theta functions). *Let $\tau \in \mathcal{H}_g$.*

The line bundle L_τ is ample and symmetric on A_τ, and defines a principal polarization on A_τ. It is also the line bundle canonically associated to the 2-structure $\alpha_{\tau,2}$ and its polarization by the Igusa correspondence (Definition-Proposition 6.13).

Furthermore, the global sections of L_τ canonically identify to the multiples of Θ_τ, hence the theta divisor associated to $(A_\tau, \lambda_\tau, \alpha_{\tau,2})$ is exactly the divisor of zeroes of Θ_τ modulo Λ_τ.

Thus, for every $a, b \in \mathbb{R}^g$, the projection of $\pi_\tau(a\tau + b)$ belongs to $\Theta_{A_\tau,\lambda_\tau,\alpha_{\tau,2}}$ if and only if $\vartheta_{a,b}(\tau) = 0$.

Remark 6.16. The proof below that the L_τ is the line bundle associated to $(A_\tau, \lambda_\tau, \alpha_{\tau,2})$ is a bit technical, but one has to suspect that Igusa normalized its correspondence by (6-9) exactly to make it work.

Proof. One can easily see that L_τ is symmetric by writing $[-1]^* L_\tau$ as a quotient of $\mathbb{C}^g \times \mathbb{C}$ by an action of Λ_τ, then figuring out it is the same as (6-3). Then, by simple connectedness, the global sections of L_τ lift by the quotient morphism $\mathbb{C}^g \times \mathbb{C} \to L_\tau$ into functions $z \mapsto (z, f(z))$, and the holomorphic functions f thus obtained are exactly the functions satisfying functional equation (6-13) for $a = b = 0$ because of (6-3), hence the same functional equation as Θ_τ. This identification is also compatible with the associated divisors, hence Θ_{A_τ,L_τ} is the divisor of zeroes of Θ_τ modulo Λ_τ. For more details on the theta functions and line bundles, see [Debarre 1999, Chapters IV, V and Section VI.2].

We now have to check that the Igusa correspondence indeed associates L_τ to $(A_\tau, \lambda_\tau, \alpha_{\tau,2})$. With the notations of the construction of this correspondence [Igusa 1967, pp. 822, 823 and 833], one sees that the meromorphic function ψ_x on A_τ (depending on L_τ) associated to $x \in A_\tau[2]$ has divisor $[2]^* T_x^* \Theta_{A_\tau,L_\tau} - [2]^* \Theta_{A_\tau,L_\tau}$, hence it is (up to a constant) the meromorphic function induced on A_τ by

$$f_x(z) = \frac{\Theta_{a,b,\tau}(2z)}{\Theta_\tau(2z)},$$

where $x = a\tau + b \mod \Lambda_\tau$. Now, the quadratic form q associated to L_τ is defined by the identity

$$f_x(-z) = q(x) f_x(z)$$

for every $z \in \mathbb{C}^g$, but Θ_τ is even hence

$$f_x(-z) = e^{4i\pi a^t b} f_x(z)$$

by (6-12). Now, the coordinates of x in $\alpha_{\tau,2}$ are exactly $(2b, 2a) \mod \mathbb{Z}^{2g}$ by definition, hence $q = q_{\alpha_{\tau,2}}$.

Let us finally make the explicit link between zeroes of theta-constants and theta divisors; using the argument above, the divisor of zeroes of Θ_τ modulo Λ_τ is exactly Θ_{A_τ,L_τ}, hence $\Theta_{A_\tau,\lambda_\tau,\alpha_{\tau,2}}$ by what we just proved for the Igusa correspondence. This implies that for every $z \in \mathbb{C}^g$, $\Theta_\tau(z) = 0$ if and only if $\pi_\tau(z)$ belongs to $\Theta_{A_\tau,\lambda_\tau,\alpha_{\tau,2}}$, and as $\vartheta_{a,b}(\tau)$ is a nonzero multiple of $\Theta(a\tau + b, \tau)$, we finally have that $\vartheta_{a,b}(\tau) = 0$ if and only if $\pi_\tau(a\tau + b)$ belongs to $\Theta_{A_\tau,\lambda_\tau,\alpha_{\tau,2}}$. ◻

7. Applications of the main result on a family of Siegel modular varieties

We now have almost enough definitions to state the problem which we will consider for our Runge-type result (Theorem 7.12). We consider theta divisors on abelian surfaces, and their torsion points.

To make their indexation easier, we use the following notation.

Notation. Until the end of this article, the expression "a couple $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$ (resp. $\mathbb{Z}^4$, $\mathbb{Q}^4$)" is a shorthand to designate the row vector with four coefficients where $a \in (\mathbb{Z}/n\mathbb{Z})^2$ (resp. $\mathbb{Z}^2$, $\mathbb{Q}^2$) make up the first two coefficients and b the last two coefficients.

7A. ***The specific situation for theta divisors on abelian surfaces.*** As an introduction and a preliminary result, let us treat first the case of theta divisors on elliptic curves.

Lemma 7.1 (theta divisor on an elliptic curve). *Let E be an elliptic curve on an algebraically closed field k with $\operatorname{char}(k) \neq 2$ and L an ample symmetric line bundle defining the principal polarization on E.*

The effective divisor $\Theta_{E,L}$ is a 2-torsion point of E with multiplicity one. More precisely, if (e_1, e_2) is the basis of $E[2]$ associated by Igusa correspondence to L (Definition-Proposition 6.13),

$$\Theta_{E,L} = [e_1 + e_2]. \tag{7-1}$$

Remark 7.2. In the complex case, this can simply be obtained by proving that $\Theta_{1/2,1/2,\tau}$ is odd for every $\tau \in \mathcal{H}_1$ hence cancels at 0, and has no other zeroes (by a residue theorem for example), then using Proposition 6.15.

Proof. By the Riemann–Roch theorem on E, the divisor $\Theta_{E,L}$ is of degree 1 because $h^0(E, L) = 1$ (and effective). Now, as explained before when discussing the Igusa correspondence, for $a, b \in \mathbb{Z}$, $ae_1 + be_2$ automatically belongs to $\Theta_{E,L}$ if $ab = 1 \mod 2\mathbb{Z}$, hence $\Theta_{E,L} = [e_1 + e_2]$. □

This allows one to describe the theta divisor of a product of two elliptic curves.

Proposition 7.3 (theta divisor on a product of two elliptic curves). *Let k be an algebraically closed field with $\operatorname{char}(k) \neq 2$.*

Let (A, L) with $A = E_1 \times E_2$ a product of elliptic curves over k and L an ample symmetric line bundle on A inducing the product principal polarization on A. The divisor $\Theta_{A,L}$ is then of the shape

$$\Theta_{A,L} = \{x_1\} \times E_2 + E_1 \times \{x_2\}, \tag{7-2}$$

with $x_i \in E_i[2]$ for $i = 1, 2$. In particular, this divisor has a (unique) singular point of multiplicity two at (x_1, x_2), and:

(a) *There are exactly seven 2-torsion points of A belonging to $\Theta_{A,L}$: the six points given by the coordinates $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^4$ such that $a^t b = 1$ in a basis giving $\Theta_{A,L}$ by the Igusa correspondence, and the seventh point (x_1, x_2).*

(b) *For every even $n \geq 2$ which is nonzero in k, the number of n-torsion (but not 2-torsion) points of A belonging to $\Theta_{A,L}$ is exactly $2(n^2 - 4)$.*

Proof. By construction of (A, L), a global section of (A, L) corresponds to a tensor product of global sections of E_1 and E_2 (with their principal polarizations), hence the shape of $\Theta_{A,L}$ is a consequence of Lemma 7.1.

We readily deduce (a) and (b) from this shape, using that the intersection of the two components of $\Theta_{A,L}$ is a 2-torsion point of even multiplicity for the quadratic form hence different from the six other ones. □

Regarding abelian surfaces which are not products of elliptic curves, we recall below a fundamental result (proven in [Oort and Ueno 1973]).

Proposition 7.4 (shapes of principally polarized abelian surfaces). *Let k be any field.*

A principally polarized abelian surface (A, λ) over k is, after a finite extension of scalars, either the product of two elliptic curves (with its natural product polarization), or the jacobian J of an hyperelliptic curve C of genus 2 (with its canonical principal polarization). In the second case, for the Albanese embedding $\phi_x : C \to J$ with base-point x and an ample symmetric line bundle L over K inducing λ, the divisor $\Theta_{J,L}$ is irreducible, and it is actually a translation of $\phi_x(C)$ by some point of $J(\bar{k})$.

Let us now fix an algebraically closed field k with $\mathrm{char}(k) \neq 2$.

Let C be an hyperelliptic curve of genus 2, and ι its hyperelliptic involution. This curve has exactly six Weierstrass points (the fixed points of ι, by definition), and we fix one of them, denoted by ∞. For the Albanese morphism ϕ_∞, the divisor $\phi_\infty(C)$ is stable by $[-1]$ because the divisor $[x] + [\iota(x)] - 2[\infty]$ is principal for every $x \in C$. As $\Theta_{J,L}$ is also symmetric and a translation of $\phi_\infty(C)$, we know that $\Theta_{J,L} = T_x^*(\phi_\infty(C))$ for some $x \in J[2]$.

This tells us that understanding the points of $\Theta_{J,L}$ amounts to understanding how the curve C behaves when embedded in its jacobian (in particular, how its points add). It is a difficult problem to know which torsion points of J belong to the theta divisor (see [Boxall and Grant 2000] for example), but we will only need to bound their quantity here, with the following result.

Proposition 7.5. *Let k an algebraically closed field with* $\mathrm{char}(k) \neq 2$.

Let C be an hyperelliptic curve of genus 2 over k with jacobian J, and ∞ a fixed Weierstrass point of C. We denote by $\tilde{C}$ the image of C in J by the associated embedding $\phi_\infty : x \mapsto \overline{[x] - [\infty]}$.

(a) *The set $\tilde{C}$ is stable by $[-1]$, and the application*

$$\begin{aligned} \mathrm{Sym}^2(\tilde{C}) &\to J \\ \{P, Q\} &\mapsto P + Q \end{aligned}$$

is the blow-up of J at the origin, in particular it is injective outside the fiber above 0.

(b) *There are exactly six 2-torsion points of J belonging to $\tilde{C}$, and they are equivalently the images of the Weierstrass points and the points of coordinates $(a, b) \in ((\mathbb{Z}/2\mathbb{Z})^2)^2$ such that $a^t b = 1$ in a basis giving $\tilde{C}$ by the Igusa correspondence.*

(c) *For any $n \geq 2$ which is nonzero in k, the number of n-torsion points of J belonging to $\tilde{C}$ is bounded by $\sqrt{2}n^2 + \frac{1}{2}$.*

Remark 7.6. This proposition is not exactly a new result, and its principle can be found (with slightly different formulations) in Theorem 1.3 of [Boxall and Grant 2000] or in Lemma 5.1 of [Pazuki 2013]. The problem of counting (or bounding) torsion points on the theta divisor has interested many people, e.g., [Boxall and Grant 2000] and very recently [Auffarth et al. 2017] in general dimension. Notice that the results above give the expected bound in the case $g = 2$, but we do not know how much we can lower the bound $\sqrt{2}n^2$ in the case of jacobians.

Proof. (a) is a well-known consequence of the Riemann–Roch theorem in genus 2. (b) comes from the construction of the Igusa correspondence, and the definition of Weierstrass points as points P such that $2[P]$ is a canonical divisor. Now, for any $n \geq 2$, let us denote $\tilde{C}[n] := \tilde{C} \cap J[n]$. The summing map from $\tilde{C}[n]^2$ to $J[n]$ has a fiber of cardinal $|\tilde{C}[n]|$ above 0 and at most 2 above any other point of $J[n]$ by (a), hence the inequality of degree two

$$|\tilde{C}[n]|^2 \leq |\tilde{C}[n]| + 2(n^4 - 1),$$

from which we directly obtain (c). □

We can now define the divisors we will consider for our Runge-type theorem.

Definition-Proposition 7.7 (theta divisors on $A_2(n)^S_{\mathbb{C}}$). Let $n \in \mathbb{N}_{\geq 2}$ even.

(a) A couple $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$ is called *regular* if it is *not* of the shape $((n/2)a', (n/2)b')$ with $(a', b') \in ((\mathbb{Z}/2\mathbb{Z})^2)^2$ such that $a'^t b' = 1 \bmod 2$. There are exactly 6 couples (a, b) not satisfying this condition, which we call *singular*.

(b) If $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$ is regular, for every lift $(\tilde{a}, \tilde{b}) \in \mathbb{Z}^4$ of (a, b), the function $\vartheta^{8n}_{\tilde{a}/n,\tilde{b}/n}$ is a *nonzero* Siegel modular form of degree 2, weight $4n$ and level n, independent of the choice of lifts. The *theta divisor associated to* (a, b), denoted by $(D_{n,a,b})_{\mathbb{C}}$, is the Weil divisor of zeroes of this Siegel modular form on $A_2(n)^S_{\mathbb{C}}$.

Remark 7.8. The singular couples correspond to what are called *odd characteristics* by Igusa.

The proof below uses Fourier expansions to figure out which theta functions are nontrivial. One can also prove through Fourier expansions that the Weil divisors $(D_{n,a,b})_{\mathbb{C}}$ and $(D_{n,a',b'})_{\mathbb{C}}$ are distinct (unless $(a, b) = \pm(a', b')$ of course) and it is likely true that they are even set-theoretically pairwise distinct (i.e., even without counting the multiplicities). This is not very important for us since Proposition 7.3 and 7.5 are not modified if some of the divisors taken into account are equal.

Proof of Definition-Proposition 7.7. (a) By construction, for any even $n \geq 2$, the number of singular couples $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$ is the number of couples $(a', b') \in (\mathbb{Z}/2\mathbb{Z})^4$ such that $a'^t b' = 1 \bmod 2$, and we readily see there are exactly six of them, namely

$$(0101), (1010), (1101), (1110), (1011) \text{ and } (0111).$$

For (b) and (c), the modularity of the function comes from Definition-Proposition 6.14(c) hence we only have to prove that it is nonzero when (a, b) is regular. To do this, we will use the Fourier expansion of this modular form (for more details on Fourier expansions of Siegel modular forms, see chapter 4 of [Klingen 1990]), and simply prove that it has nonzero coefficients. This is also how we will prove the $\vartheta_{a,b}$ are distinct.

To shorten the notations, given $(a,b) \in (\mathbb{Z}/n\mathbb{Z})^4$, we consider instead $(\tilde{a}/n, \tilde{b}/n) \in \mathbb{Q}^4$ for some lift $(\tilde{a}, \tilde{b})$ of (a,b) in $\mathbb{Z}^4$) and by abuse of notation we denote it (a,b) for simplicity. Regularity of the couple translates into the fact that (a,b) is different from six possibles values modulo $\mathbb{Z}^4$, namely

$$\left(0,\tfrac{1}{2},0,\tfrac{1}{2}\right), \left(\tfrac{1}{2},0,\tfrac{1}{2},0\right), \left(\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2}\right), \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},0\right), \left(\tfrac{1}{2},0,\tfrac{1}{2},\tfrac{1}{2}\right), \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)$$

by (a), which we will assume now. We also fix $n \in \mathbb{N}$ even such that $(na, nb) \in \mathbb{Z}^4$.

Recall that

$$\vartheta_{a,b}(\tau) = e^{i\pi a^t b} \sum_{k \in \mathbb{Z}^2} e^{i\pi (k+a)\tau^t (k+a) + 2i\pi k^t b} \tag{7-3}$$

by (6-12) and (6-14). Therefore, for any symmetric matrix $S \in M_2(\mathbb{Z})$ such that $S/(2n^2)$ is half-integral (i.e., with integer coefficients on the diagonal, and half-integers otherwise), we have $\forall \tau \in \mathcal{H}_2$,

$$\vartheta_{a,b}(\tau + S) = \vartheta_{a,b}(\tau),$$

because for every $k \in \mathbb{Z}^2$,

$$(k+a)S^t(k+a) \in 2\mathbb{Z}.$$

Hence, the function $\vartheta_{a,b}$ admits a Fourier expansion of the form

$$\vartheta_{a,b}(\tau) = \sum_T a_T e^{2i\pi \operatorname{Tr}(T\tau)},$$

where T runs through all the matrices of $S_2(\mathbb{Q})$ such that $(2n^2)T$ is half-integral. This Fourier expansion is unique, because for any $\tau \in \mathcal{H}_2$ and any T, we have

$$(2n^2)a_T = \int_{[0,1]^4} \vartheta_{a,b}(\tau + x) e^{-2i\pi \operatorname{Tr}(T(\tau+x))} \, dx.$$

In particular, the function $\vartheta_{a,b}$ is zero if and only if all its Fourier coefficients a_T are zero, hence we will directly compute those, which are almost directly given by (7-3). For $a = (a_1, a_2) \in \mathbb{Q}^2$ and $k = (k_1, k_2) \in \mathbb{Z}^2$, let us define

$$T_{a,k} = \begin{pmatrix} (k_1+a_1)^2 & (k_1+a_1)(k_2+a_2) \\ (k_1+a_1)(k_2+a_2) & (k_2+a_2)^2 \end{pmatrix},$$

so that

$$\vartheta_{a,b}(\tau) = e^{i\pi a^t b} \sum_{k \in \mathbb{Z}^2} e^{2i\pi k^t b} e^{i\pi \operatorname{Tr}(T_{a,k}\tau)} \tag{7-4}$$

by construction. It is not yet exactly the Fourier expansion, because we have to gather the $T_{a,k}$ giving the same matrix T (and this is where we will use regularity). Clearly,

$$T_{a,k} = T_{a',k'} \iff (k+a) = \pm(k'+a').$$

If $2a \notin \mathbb{Z}^2$, the function $k \mapsto T_{a,k}$ is injective, so (7-4) is the Fourier expansion of $\vartheta_{a,b}$, with clearly nonzero coefficients, hence $\vartheta_{a,b}$ is nonzero.

If $2a = A \in \mathbb{Z}^2$, for every $k, k' \in \mathbb{Z}^2$, we have $(k+a) = \pm(k'+a)$ if and only if $k = k'$ or $k + k' = A$, so the Fourier expansion of $\vartheta_{a,b}$ is

$$\vartheta_{a,b}(\tau) = \frac{e^{i\pi a^t b}}{2} \sum_T \sum_{\substack{k,k' \in \mathbb{Z}^2 \\ T_{k,a} = T_{k',a} = T}} (e^{2i\pi k^t b} + e^{2i\pi(-A-k)^t b}) e^{i\pi \operatorname{Tr}(T\tau)}. \tag{7-5}$$

Therefore, the coefficients of this Fourier expansion are all zero if and only if, for every $k \in \mathbb{Z}^2$,

$$e^{2i\pi(2k+A)^t b} = -1,$$

i.e., if and only if $b \in (1/2)\mathbb{Z}$ and $(-1)^{4a^t b} = -1$, and this is exactly singularity of the couple (a, b) which proves (b). □

These divisors have the following properties.

Proposition 7.9 (properties of the $(D_{n,a,b})_{\mathbb{C}}$). *Let $n \in \mathbb{N}_{\geq 2}$ even.*

(a) *For every regular $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$, the divisor $(D_{n,a,b})_{\mathbb{C}}$ is ample.*

(b) *For $n = 2$, the ten divisors $(D_{2,a,b})_{\mathbb{C}}$ are set-theoretically pairwise disjoint outside the boundary $\partial A_2(2)_{\mathbb{C}} := A_2(2)^S_{\mathbb{C}} \backslash A_2(2)_{\mathbb{C}}$, and their union is exactly the set of moduli of products of elliptic curves (with any symplectic basis of the 2-torsion).*

(c) *For (A, λ, α_n) a principally polarized complex abelian surface with symplectic structure of level n:*
- *If (A, λ) is a product of elliptic curves, the moduli of (A, λ, α_n) belongs to exactly $n^2 - 3$ divisors $(D_{n,a,b})_{\mathbb{C}}$.*
- *Otherwise, the point (A, λ, α_n) belongs to at most $(\sqrt{2}/2)n^2 + 1/4$ divisors $(D_{n,a,b})_{\mathbb{C}}$.*

Proof. (a) The divisor $(D_{n,a,b})_{\mathbb{C}}$ is by definition the Weil divisor of zeroes of a Siegel modular form of order 2, weight $4n$ and level n, hence of a section of $L^{\otimes 4n}$ on $A_2(n)^S_{\mathbb{C}}$. As L is ample on $A_2(n)^S_{\mathbb{C}}$ (Definition-Proposition 6.4(c)), the divisor $(D_{n,a,b})_{\mathbb{C}}$ is ample.

Now, we know that every complex pair (A, λ) is isomorphic to some (A_τ, λ_τ) with $\tau \in \mathcal{H}_2$ (Definition-Proposition 6.3). If (A, λ) is a product of elliptic curves, the theta divisor of (A, λ, α_2) contains exactly seven 2-torsion points (Proposition 7.3), only one of comes from a regular pair, i.e., (A, λ, α_2) is contained in exactly one of the ten divisors. If (A, λ) is not a product of elliptic curves, it is a jacobian (Proposition 7.4) and the theta divisor of (A, λ, α_2) only contains the six points coming from singular pairs (Proposition 7.5) i.e., (A, λ, α_2) does not belong to any of the ten divisors, which proves (b).

To prove (c), we use the same propositions for general n, keeping in mind that we only count as one the divisors coming from opposite values of (a, b): for products of elliptic curves, this gives $2(n^2-4)/2+1$ divisors (the 1 coming from the even 2-torsion), and for jacobians, this gives $(\sqrt{2}/2)n^2+\frac{1}{4}$ (there are no nontrivial 2-torsion points to consider here). ☐

We will now give the natural divisors extending $(D_{n,a,b})_{\mathbb{C}}$ on the integral models $\mathcal{A}_2(n)$ (Definition-Proposition 6.6).

Definition 7.10. Let $n \in \mathbb{N}_{\geq 2}$ even.

For every regular $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$, the divisor $(D_{n,a,b})_{\mathbb{C}}$ is the geometric fiber at $\mathbb{C}$ of an effective Weil divisor $D_{n,a,b}$ on $\mathcal{A}_2(n)$, such that the moduli of a triple (A, λ, α_n) (on a field k of characteristic prime to n) belongs to $D_{n,a,b}(k)$ if and only if the point of $A[n](\bar{k})$ of coordinates (a, b) for α_n belongs to the theta divisor $\Theta_{A,\lambda,\alpha_n}$ (Definition-Proposition 6.13).

Proof. This amounts to giving an algebraic construction of the $D_{n,a,b}$ satisfying the wanted properties. The following arguments are extracted from Remark I.5.2 of [Faltings and Chai 1990]. Let $\pi : A \to S$ an abelian scheme and $\mathcal{L}$ a symmetric invertible sheaf on A, relatively ample over S and inducing a principal polarization on A. If $s : S \to A$ is a section of A over S, the evaluation at s induces an $\mathcal{O}_S$-module isomorphism between $\pi_*\mathcal{L}$ and $s^*\mathcal{L}$. Now, if s is of n-torsion in A, for $e : S \to A$ the zero section, the sheaf $(s^*\mathcal{L})^{\otimes 2n}$ is isomorphic to $(e^*\mathcal{L})^{\otimes 2n}$, i.e., trivial. We denote by $\omega_{A/S}$ the invertible sheaf on S obtained as the determinant of the sheaf of invariant differential forms on A, and the computations of Theorem I.5.1 and Remark I.5.2 of [Faltings and Chai 1990] give $8\pi_*\mathcal{L} = -4\omega_{A/S}$ in $\mathrm{Pic}(A/S)$. Consequently, the evaluation at s defines (after a choice of trivialization of $(e^*\mathcal{L})^{\otimes 2n}$ and putting to the power $8n$) a section of $\omega_{A/S}^{\otimes 4n}$. Applying this result on the universal abelian scheme (stack if $n \leq 2$) $\mathcal{X}_2(n)$ on $\mathcal{A}_2(n)$, for every $(a, b) \in (\mathbb{Z}/n\mathbb{Z})^4$, the section defined by the point of coordinate (a, b) for the n-structure on $\mathcal{X}_2(n)$ induces a global section $s_{a,b}$ of $\omega_{\mathcal{X}_2(n)/\mathcal{A}_2(n)}^{\otimes 4n}$, and we define $D_{n,a,b}$ as the Weil divisor of zeroes of this section. It remains to check that it satisfies the correct properties.

Let (A, λ, α_n) be a triple over a field k of characteristic prime to n, and L the ample line bundle associated to it by Definition-Proposition 6.13. By construction, its moduli belongs to $D_{n,a,b}$ if and only if the unique (up to constant) nonzero section vanishes at the point of $A[n]$ of coordinates (a, b) in α_n, hence if and only if this point belongs to $\Theta_{A,\lambda,\alpha_n}$.

Finally, we see that the process described above applied to the universal abelian variety $\mathcal{X}_2(n)_{\mathbb{C}}$ of $\mathcal{A}_2(n)_{\mathbb{C}}$ (by means of explicit description of the line bundles as quotients) gives (up to invertible holomorphic functions) the functions $\vartheta_{\tilde{a}/n,\tilde{b}/n}^{8n}$, which proves that $(D_{n,a,b})_{\mathbb{C}}$ is indeed the geometric fiber of $D_{n,a,b}$ (it is easier to see that their complex points are the same, by Proposition 7.9(c) and the above characterization applied to the field $\mathbb{C}$).

If one does not want to use stacks for $n = 2$, one can consider for $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^4$ the divisor $D_{4,2a,2b}$ which is the pullback of $D_{2,a,b}$ by the degeneracy morphism $A_2(4) \to A_2(2)$. ☐

7B. ***Tubular Runge theorems for abelian surfaces and their theta divisors.*** We can now prove a family of tubular Runge theorems for the theta divisors $D_{n,a,b}$ (for even $n \geq 2$).

We will state the case $n = 2$ first because its moduli interpretation is easier but the proofs are the same, as we explain below.

In the following results, the *boundary* of $A_2(n)^S_{\mathbb{C}}$ is defined as $\partial A_2(n)^S_{\mathbb{C}} := A_2(n)^S_{\mathbb{C}} \backslash A_2(n)_{\mathbb{C}}$.

Theorem 7.11 (tubular Runge for products of elliptic curves on $\mathcal{A}_2(2)^S$). *Let U be an open neighborhood of $\partial A_2(2)^S_{\mathbb{C}}$ in $A_2(2)^S_{\mathbb{C}}$ for the natural complex topology.*

For any such U, we define $\mathcal{E}(U)$ the set of moduli P of triples (A, λ, α_2) in $\mathcal{A}_2(2)(\overline{\mathbb{Q}})$ such that (choosing L a number field of definition of the moduli):

- *The abelian surface A has potentially good reduction at every finite place $w \in M_L$ (**tubular condition for finite places**).*
- *For any embedding $\sigma : L \to \mathbb{C}$, the image P_σ of P in $\mathcal{A}_2(2)_{\mathbb{C}}$ is outside of U (**tubular condition for archimedean places**).*
- *The number s_L of nonintegrality places of P, i.e., places $w \in M_L$ such that*
  - *either w is above M_L^∞ or 2,*
  - *or the semistable reduction modulo w of (A, λ) is a product of elliptic curves*

  *satisfies the **tubular Runge condition***

$$s_L < 10.$$

*Then, for every choice of U, the set $\mathcal{E}(U)$ is **finite**.*

Theorem 7.12 (tubular Runge for theta divisors on $\mathcal{A}_2(n)^S$). *Let $n \geq 4$ even.*

Let U be an open neighborhood of $\partial A_2(n)^S_{\mathbb{C}}$ in $A_2(n)^S_{\mathbb{C}}$ for the natural complex topology.

For any such U, we define $\mathcal{E}(U)$ the set of moduli P of triples (A, λ, α_n) in $\mathcal{A}_2(n)(\overline{\mathbb{Q}})$ such that (choosing $L \supset \mathbb{Q}(\zeta_n)$ a number field of definition of the triple):

- *The abelian surface A has potentially good reduction at every place $w \in M_L^\infty$ (**tubular condition for finite places**).*
- *For any embedding $\sigma : L \to \mathbb{C}$, the image P_σ of P in $\mathcal{A}_2(n)_{\mathbb{C}}$ is outside of U (**tubular condition for archimedean places**).*
- *The number s_L of nonintegrality places of P, i.e., places $w \in M_L$ such that*
  - *either w is above M_L^∞ or a prime factor of n,*
  - *or the theta divisor of the semistable reduction modulo w of (A, λ, α_n) contains an n-torsion point which is not one of the six points coming from odd characteristics,*

  *satisfies the **tubular Runge condition***

$$(n^2 - 3)s_L < \frac{n^4}{2} + 2.$$

*Then, for every choice of U, the set of points $\mathcal{E}(U)$ is **finite**.*

Remark 7.13. We put an emphasis on the conditions given in the theorem to make it easier to identify how it is an application of our main result, Theorem 5.1. The tubular conditions (archimedean and finite) mean that our points P do not belong to some tubular neighborhood $\mathcal{V}$ of the boundary. We of course chose the boundary as our closed subset to exclude because of its modular interpretation for finite places. The places above M_L^∞ or a prime factor of n are automatically of nonintegrality for our divisors because the model $\mathcal{A}_2(n)$ is not defined at these places. Finally, the second possibility to be a place of nonintegrality straightforwardly comes from the moduli interpretation of the divisors $D_{n,a,b}$ (Definition 7.10). All this is detailed in the proof below.

To give an example of how we can obtain an explicit result in practice, we prove in Section 8 an explicit (and even theoretically better) version of Theorem 7.11.

It would be more satisfying (and easier to express) to give a tubular Runge theorem for which the divisors considered are exactly the irreducible components parametrizing the products of elliptic curves. Unfortunately, except for $n = 2$, there is a serious obstruction because those divisors are not ample, and there are even reasons to suspect they are not big. We have explained in Remark 6.10 why proving the ampleness for general divisors on $A_2(n)^S_{\mathbb{C}}$ is difficult.

It would also be morally satisfying to give a better interpretation of the moduli of the union of all the $D_{n,a,b}$ (for a fixed $n > 2$), i.e., not in terms of the theta divisor, but maybe of the structure of the abelian surface if possible (nontrivial endomorphisms? isogenous to products of elliptic curves?). As far as the author knows, the understanding of abelian surfaces admitting some nontrivial torsion points on their theta divisor is still very limited.

Finally, to give an idea of the margin the tubular Runge condition gives for $n > 2$ (in terms of the number of places which are not "taken" by the automatic bad places), we can easily see that the number of places of $\mathbb{Q}(\zeta_n)$ which are archimedean or above a prime factor of n is less than $n/2$. Hence, we can find examples of extensions L of $\mathbb{Q}(\zeta_n)$ of degree n such that some points defined on it still can satisfy the tubular Runge condition. This is also where using the full strength of tubular Runge theorem is crucial: for $n = 2$, one can compute that some points of the boundary are contained in 6 different divisors $D_{2,a,b}$, and for general even n, a similar analysis gives that the intersection number $m_\varnothing$ is quartic in n, which leaves a lot less margin for the places of nonintegrality (or even none at all).

Proof of Theorems 7.11 and 7.12. As announced, this result is an application of the tubular Runge theorem (Theorem 5.1) to $\mathcal{A}_2(n)^S_{\mathbb{Q}(\zeta_n)}$ (Definition-Proposition 6.6) and the divisors $D_{n,a,b}$ (Definition 7.10), whose properties will be used without specific mention. We reuse the notations of the hypotheses of Theorem 5.1 to explain carefully how it is applied.

(H0) The field of definition of $A_2(n)^S_{\mathbb{C}}$ is $\mathbb{Q}(\zeta_n)$, and the ring over which our model $\mathcal{A}_2(n)^S$ is built is $\mathbb{Z}[\zeta_n, 1/n]$, hence S_0 is made up with all the archimedean places and the places above prime factors of n. There is no need for a finite extension here as all the $D_{n,a,b}$ are divisors on $\mathcal{A}_2(n)^S$.

(H1) The model $\mathcal{A}_2(n)^S_{\mathbb{C}}$ is indeed normal projective, and we know that the $D_{n,a,b}$ are effective Weil divisors hence Cartier divisors up to multiplication by some constant by Proposition 6.8. For any finite extension L of $\mathbb{Q}(\zeta_n)$, the number of orbits r_L is the number of divisors $D_{n,a,b}$ (as they are divisors on the base model), i.e., $n^4/2+2$ (Proposition 7.9(c)).

(**H2**) The chosen closed subset Y of $\mathcal{A}_2(n)^S_{\mathbb{Q}}(\zeta_n)$ is the boundary, namely

$$\partial \mathcal{A}_2(n)^S_{\mathbb{Q}(\zeta_n)} = \mathcal{A}_2(n)^S_{\mathbb{Q}(\zeta_n)} \backslash \mathcal{A}_2(n)_{\mathbb{Q}(\zeta_n)}.$$

We have to prove that the tubular conditions given above correspond to a tubular neighborhood. To do this, let $\mathcal{Y}$ be the boundary $\mathcal{A}_2(n)^S \backslash \mathcal{A}_2(n)$ and $g_1, \dots, g_s$ homogeneous generators of the ideal of definition of $\mathcal{Y}$ after having fixed a projective embedding of $\mathcal{A}_2(n)$. Let us find an $M_{\mathbb{Q}(\zeta_n)}$-constant such that $\mathcal{E}(U)$ is included in the tubular neighborhood of $\partial \mathcal{A}_2(n)^S_{\mathbb{Q}}(\zeta_n)$ in $A_2(n)^S_{\mathbb{Q}(\zeta_n)}$ associated to $\mathcal{C}$ and $g_1, \dots, g_k$. For the places w not above M_L^∞ or a prime factor of n, the fact that $P = (A, \lambda, \alpha_n)$ does not reduce in Y modulo w is exactly equivalent to A having potentially good reduction at w hence we can choose $c_v = 0$ for the places v of $\mathbb{Q}(\zeta_n)$ not archimedean and not dividing n. For archimedean places, belonging to U for an embedding $\sigma : L \to \mathbb{C}$ implies that $g_1, \dots, g_n$ are small, and we just have to choose c_v strictly larger than the maximum of the norms of the $g_i(U \cap V_j)$ (in the natural affine covering $(V_j)_j$ of the projective space), independent of the choice of $v \in M^\infty_{\mathbb{Q}(\zeta_n)}$. Finally, we have to consider the case of places above a prime factor of n. To do this, we only have to recall that having potentially good reduction can be given by integrality of some quotients of the Igusa invariants at finite places, and these invariants are modular forms on $\Gamma_2(1)$. We can add those who vanish on the boundary to the homogeneous generators $g_1, \dots, g_n$ and consider $c_v = 0$ for these places as well. This is explicitly done in Section 8C for $A_2(2)$.

(**TRC**) As said before, there are $n^4/2+2$ divisors considered, and their generic fibers are ample by Proposition 7.9. Furthermore, by Propositions 7.3 and 7.5, outside the boundary, at most (n^2-3) can have nonempty common intersection, and this exact number is attained only for products of elliptic curves.

This gives the tubular Runge condition

$$(n^2-3)s_L < \frac{n^4}{2} + 2,$$

which concludes the proof.

For $n=2$, the union of the ten $D_{2,a,b}$ is made up with the moduli of products of elliptic curves, and they are pairwise disjoint outside $\partial A_2(2)$ (Proposition 7.9(b)), hence the simply expressed condition $s_L < 10$ in this case. □

8. The explicit Runge result for level two

To finish this paper, we improve and make explicit the finiteness result of Theorem 7.11, as a proof of principle of the method.

Before stating Theorem 8.2, we need some notations. In level two, the auxiliary functions are deduced from the ten even theta constants of characteristic two, namely the functions $\Theta_{m/2}(\tau)$ (notation (6-11)), with the quadruples m going through

$$E = \{(0000), (0001), (0010), (0011), (0100), (0110), (1000), (1001), (1100), (1111)\} \tag{8-1}$$

(see Sections 6C and 7A for details). We recall [van der Geer 1982, Theorem 5.2] that these functions define an embedding

$$\begin{aligned} \psi : A_2(2) &\to \mathbb{P}^9 \\ \bar{\tau} &\mapsto (\Theta^4_{m/2}(\tau))_{m\in E} \end{aligned} \tag{8-2}$$

which induces an isomorphism between $A_2(2)^S_{\mathbb{C}}$ and the subvariety of $\mathbb{P}^9$ (with coordinates indexed by $m \in E$) defined by the linear equations

$$x_{1000} - x_{1100} + x_{1111} - x_{1001} = 0 \tag{8-3}$$

$$x_{0000} - x_{0001} - x_{0110} - x_{1100} = 0 \tag{8-4}$$

$$x_{0110} - x_{0010} - x_{1111} + x_{0011} = 0 \tag{8-5}$$

$$x_{0100} - x_{0000} + x_{1001} + x_{0011} = 0 \tag{8-6}$$

$$x_{0100} - x_{1000} + x_{0001} - x_{0010} = 0 \tag{8-7}$$

(which makes it a subvariety of $\mathbb{P}^4$) together with the quartic equation

$$\left(\sum_{m\in E} x_m^2\right)^2 - 4\sum_{m\in E} x_m^4 = 0. \tag{8-8}$$

Remark 8.1. For the attentive reader, the first linear equation has sign $(+1)$ in x_{1111} whereas it is (-1) in [van der Geer 1982], as there seems to be a typographic mistake there: we found the mistake during our computations in Sage in Section 8C and found the correct sign using Igusa's relations [1964, Lemma 1 combined with the proof of Theorem 1].

There is a natural definition for a tubular neighborhood of $Y = \partial A_2(2)$: for a finite place v, as in Theorem 7.11, we choose V_v as the set of triples $P = \overline{(A, \lambda, \alpha_2)}$ where A has potentially bad reduction modulo v. To complete it with archimedean places, we use the classical fundamental domain for the action of $\mathrm{Sp}_4(\mathbb{Z})$ on $\mathcal{H}_2$ denoted by $\mathcal{F}_2$ (see [Klingen 1990, §I.2], for details). Given some parameter $t \geq \sqrt{3}/2$, the neighborhood $V(t)$ of $\partial A_2(2)^S_{\mathbb{C}}$ in $A_2(2)^S_{\mathbb{C}}$ is made up with the points P whose lift τ in $\mathcal{F}_2$ (for the usual quotient morphism $\mathcal{H}_2 \to A_2(1)_{\mathbb{C}}$) satisfies $\mathrm{Im}(\tau_4) \geq t$, where τ_4 is the lower-right coefficient of τ. We choose $V(t)$ as the archimedean component of the tubular neighborhood for every archimedean place. The reader knowledgeable with the construction of Satake compactification will have already seen such neighborhoods of the boundary.

Notice that for a point $P = \overline{(A, \lambda, \alpha_2)} \in A_2(2)(K)$, the abelian surface A is only defined over a finite extension L of K, but for prime ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of $\mathcal{O}_L$ above the same prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$, the reductions of A modulo $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are of the same type because $P \in A_2(2)(K)$. This justifies what we mean by "semistable reduction of A modulo $\mathfrak{P}$" below.

Theorem 8.2. *Let K be a number field and $P = \overline{(A, \lambda, \alpha_2)} \in A_2(2)(K)$ where A has potentially good reduction at every finite place.*

Let s_P be the number of prime ideals $\mathfrak{P}$ of $\mathcal{O}_K$ such that the semistable reduction of A modulo $\mathfrak{P}$ is a product of elliptic curves. We denote by $h_{\mathcal{F}}$ the stable Faltings height of A.

(a) *If $K = \mathbb{Q}$ or an imaginary quadratic field and*

$$|s_P| < 4$$

then

$$h(\psi(P)) \leq 10.75, \quad h_{\mathcal{F}}(A) \leq 828.$$

(b) *Let $t \geq \sqrt{3}/2$ be a real number. If for every embedding $\sigma : K \to \mathbb{C}$, the point $P_\sigma \in A_2(2)_{\mathbb{C}}$ does not belong to $V(t)$, and*

$$|s_P| + |M_K^\infty| < 10$$

then

$$h(\psi(P)) \leq 4\pi t + 8.44, \quad h_{\mathcal{F}}(A) \leq 2\pi t + 5 + 533 \log(\pi t + 5)$$

Remark 8.3. Previous versions gave a bound $h_{\mathcal{F}}(A) \leq 1070$. This was actually due to an error in comparing the height of $\psi(P)$ and the Faltings height, and this error worsened the bounds, hence the slightly better new bound.

The Runge condition for (b) is a straightforward application of our tubular Runge theorem. For (a), we did not assume anything on the point P at the (unique) archimedean place, which eliminates six divisors when applying Runge's method here, hence the different Runge condition here (see Remark 5.2(b)).

The principle of proof is very simple: we apply Runge's method to bound the height of $\psi(P)$ when P satisfies the conditions of Theorem 7.11, and using the link between this height and Faltings height given in [Pazuki 2012, Corollary 1.3], we know we will obtain a bound of the shape

$$h_{\mathcal{F}}(P) \leq f(t)$$

where f is an explicit function of t, for every point P satisfying the conditions of Theorem 7.11.

At the places of good reduction not dividing 2, the contribution to the height is easy to compute thanks to the theory of algebraic theta functions devised in [Mumford 1966; 1967]. The theory will be sketched in Section 8A, resulting in Proposition 8.4.

For the archimedean places, preexisting estimates due to Streng for Fourier expansions on each of the ten theta functions allow us to make explicit how only one of them can be too small compared to the others, when we are outside of $V(t)$. This is the topic of Section 8B.

For the places above 2, the theory of algebraic theta functions cannot be applied. To bypass the problem, we use Igusa invariants (which behave in a well-known fashion for reduction in any characteristic) and prove that the theta functions are algebraic and "almost integral" on the ring of these Igusa invariants, with explicit coefficients. Combining these two facts in Section 8C, we will obtain Proposition 8.7, a less-sharp avatar of Proposition 8.4, but explicit nonetheless.

Finally, we put together these estimates in Section 8D and obtain the stated bounds on $h \circ \psi$ and the Faltings height.

8A. ***Algebraic theta functions and the places of potentially good reduction outside of 2.*** The goal of this part is the following result.

Proposition 8.4. *Let K be a number field and $\mathfrak{P}$ a maximal ideal of $\mathcal{O}_K$, of residue field $k(\mathfrak{P})$ with characteristic different from 2. Let $P = \overline{(A, \lambda, \alpha_2)} \in A_2(2)(K)$. Then, $\psi(P) \in \mathbb{P}^9(K)$ and:*

(a) *If the semistable reduction of A modulo $\mathfrak{P}$ is a product of elliptic curves, the reduction of $\psi(P)$ modulo $\mathfrak{P}$ has exactly one zero coordinate, in other words every coordinate of $\psi(P)$ has the same $\mathfrak{P}$-adic norm except one which is strictly smaller.*

(b) *If the semistable reduction of A modulo $\mathfrak{P}$ is a jacobian of hyperelliptic curve, the reduction of $\psi(P)$ modulo $\mathfrak{P}$ has no zero coordinate, in other words every coordinate of $\psi(P)$ has the same $\mathfrak{P}$-adic norm.*

To link $\psi(P)$ with the intrinsic behavior of A, we use the theory of algebraic theta functions, devised in [Mumford 1966; 1967] (see also [David and Philippon 2002; Pazuki 2012]). As it is not very useful nor enlightening to go into detail or repeat known results, we only mention them briefly here. In the following, A is an abelian variety of dimension g over a field k and L an ample symmetric line bundle on A inducing a principal polarization λ. We also fix $n \geq 2$ even, assuming that all the points of $2n$-torsion of A are defined over k and $\mathrm{char}(k)$ does not divide n (in particular, we always assume $\mathrm{char}(k) \neq 2$). Let us denote formally the Heisenberg group $\mathcal{G}(\underline{n})$ as the set

$$\mathcal{G}(\underline{n}) := k^* \times (\mathbb{Z}/n\mathbb{Z})^g \times (\mathbb{Z}/n\mathbb{Z})^g$$

equipped with the group law

$$(\alpha, a, b) \cdot (\alpha', a', b') := (\alpha\alpha' e^{(2i\pi/n)a^t b'}, a + a', b + b')$$

(contrary to the convention of [Mumford 1966, p. 294], we identified the dual of $(\mathbb{Z}/n\mathbb{Z})^g$ with itself). Recall that $A[n]$ is exactly the group of elements of $A(\bar{k})$ such that $T_x^*(L^{\otimes n}) \cong L^{\otimes n}$; indeed, it is by definition the kernel of the morphism $\phi_{L^{\otimes n}} = n\phi_L$ from A to $\hat{A}$ (see the references mentioned in the proof of Proposition 6.12).

Proof. Given the datum of a *theta structure* on $L^{\otimes n}$, i.e., an isomorphism $\beta : \mathcal{G}(L^{\otimes n}) \cong \mathcal{G}(\underline{n})$ which is the identity on k^* (see [Mumford 1966, p. 289] for the definition of $\mathcal{G}(L^{\otimes n})$), one has a natural action of $\mathcal{G}(\underline{n})$ on $\Gamma(A, L^{\otimes n})$ (a consequence of Proposition 3 and Theorem 2 of [Mumford 1966]), hence for $n \geq 4$ the following projective embedding of A:

$$\begin{aligned} \psi_\beta : A &\to \mathbb{P}_k^{n^{2g}-1} \\ x &\mapsto (((1, a, b) \cdot (s_0^{\otimes n}))(x))_{a,b \in (\mathbb{Z}/n\mathbb{Z})^g}, \end{aligned} \tag{8-9}$$

where s_0 is a nonzero section of $\Gamma(A, L)$, hence unique up to multiplicative scalar (therefore ψ_β only depends on β). This embedding is not exactly the same as the one defined in [Mumford 1966, p. 298] (it has more coordinates), but the principle does not change at all. One calls *Mumford coordinates of* (A, L) *associated to* β the projective point $\psi_\beta(0) \in \mathbb{P}^{n^{2g}-1}(k)$.

Now, one has the following commutative diagram whose rows are canonical exact sequences [Mumford 1966, Corollary of Theorem 1],

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(L^{\otimes n}) & \longrightarrow & A[n] & \longrightarrow & 0 \\ & & \Big\downarrow{=} & & \Big\downarrow{\beta} & & \Big\downarrow{\alpha_n} & & \\ 0 & \longrightarrow & k^* & \longrightarrow & \mathcal{G}(\underline{n}) & \longrightarrow & (\mathbb{Z}/n\mathbb{Z})^{2g} & \longrightarrow & 0, \end{array}$$

where α_n is a symplectic level n structure on $A[n]$ (Definition 6.1), called *the symplectic level n structure induced by* β. Moreover, for every $x \in A(k)$, the coordinates of $\psi_\beta(x)$ are (up to constant values for each coordinate, only depending on β) the $\vartheta_{A,L}([n]x + \alpha_n^{-1}(a, b))$ (see Definition 6.11). In particular, for any $a, b \in (\mathbb{Z}/n\mathbb{Z})^g$,

$$\psi_\beta(0)_{a,b} = 0 \Leftrightarrow \alpha_n^{-1}(a, b) \in \Theta_{A,L}. \tag{8-10}$$

Furthermore, for two theta structures β and β' on $[n]^*L$ inducing α_n, one sees that $\beta' \circ \beta^{-1}$ is of the shape $(\alpha, a, b) \mapsto (\alpha \cdot f(a, b), a, b)$, where f has values in n-th roots of unity, hence ψ_β and $\psi_{\beta'}$ only differ multiplicatively by n-th roots of unity.

Conversely, given the datum of a symplectic structure α_{2n} on $A[2n]$, there exists an unique *symmetric theta structure* on $[n]^*L$ which is *compatible* with some symmetric theta structure on $[2n]^*L$ inducing α_{2n} [Mumford 1966, p. 317 and Remark 3 p. 319]. We call it the *theta structure on* $[n]^*L$ *induced by* α_{2n}. Thus, we just proved that the datum of a symmetric theta structure on $[n]^*L$ is intermediary between a level $2n$ symplectic structure and a level n symplectic structure (the exact congruence group is easily identified as $\Gamma_g(n, 2n)$ with the notations of [Igusa 1966]).

Now, for a triple (A, L, α_{2n}) (notations of Section 6A), when A is a complex abelian variety, there exists $\tau \in \mathcal{H}_g$ such that this triple is isomorphic to $(A_\tau, L_\tau, \alpha_{\tau,2n})$ (Definition-Proposition 6.3). By definition of L_τ as a quotient (6-3), the sections of $L_\tau^{\otimes n}$ canonically identify to holomorphic functions ϑ on $\mathbb{C}^g$ such that, $\forall p, q \in \mathbb{Z}^g$ and $\forall z \in \mathbb{C}^g$,

$$\vartheta(z + p\tau + q) = e^{-i\pi n\tau^t n - 2i\pi n^t z}\vartheta(z), \tag{8-11}$$

and through this identification one sees (after some tedious computations) that the symmetric theta structure β_τ on $L_\tau^{\otimes n}$ induced by $\alpha_{\tau,2n}$ acts by

$$((\alpha, a, b) \cdot \vartheta)(z) = \alpha \exp\left(\frac{i\pi}{n}\tilde{a}\tau\tilde{a} + \frac{2i\pi}{n}\tilde{a}^t(z + \tilde{b})\right)\vartheta\left(z + \frac{\tilde{a}}{n}\tau + \frac{\tilde{b}}{n}\right),$$

where $\tilde{a}$ and $\tilde{b}$ are lifts of a and b in $\mathbb{Z}^g$ (the result does not depend on this choice by (8-11)). Therefore, by ψ_β and the theta functions with characteristic (formula (6-12)), the Mumford coordinates of (A, L, α_{2n})

(with the induced theta structure β on $L^{\otimes n}$) are *exactly* the projective coordinates

$$(\Theta^n_{\tilde{a}/n,\tilde{b}/n(\tau)}(\tau))_{a,b\in\frac{1}{n}\mathbb{Z}^{2g}/\mathbb{Z}^{2g}} \in \mathbb{P}^{n^{2g}-1}(\mathbb{C}),$$

where the choices of lifts $\tilde{a}$ and $\tilde{b}$ for a and b still do not matter.

In particular, for every $\tau \in \mathcal{H}_2$, the point $\psi(\tau)$ can be intrinsically given as the squares of Mumford coordinates for β_τ, where the six odd characteristics (whose coordinates vanish everywhere) are taken out. The result only depends on the isomorphism class of $(A_\tau, L_\tau, \alpha_{\tau,2})$, as expected.

Finally, as demonstrated in paragraph 6 of [Mumford 1967] (especially the theorem on page 83), the theory of theta structures (and the associated Mumford coordinates) can be extended to abelian schemes (Definition 6.5) (still outside characteristics dividing $2n$), and the Mumford coordinates in this context lead to an embedding of the associated moduli space in a projective space as long as the *type* of the sheaf is a multiple of 8 (which for us amounts to $8 \mid n$). Here, fixing a principally polarized abelian variety A over a number field K and $\mathfrak{P}$ a prime ideal of $\mathcal{O}_K$ not above 2, this theory means that given a symmetric theta structure on (A, L) for $L^{\otimes n}$ where $8 \mid n$, if A has good reduction modulo $\mathfrak{P}$, this theta structure has a natural reduction to a theta structure on the reduction $(A_{\mathfrak{P}}, L_{\mathfrak{P}})$ for $L_{\mathfrak{P}}^{\otimes n}$, and this reduction is compatible with the reduction of Mumford coordinates modulo $\mathfrak{P}$. To link this with the reduction of coordinates of ψ, one just has to extend the number field K of definition of A so that all 8-torsion points of A are defined over K (in particular, the reduction of A modulo $\mathfrak{P}$ is semistable), and consider a symmetric theta structure on $L^{\otimes 8}$. The associated Mumford coordinates then reduce modulo $\mathfrak{P}$, and making use of (8-10) and Propositions 7.3 and 7.5 over the residue field, one of the Mumford coordinates coming from the 2-torsion does not vanish. We can now consider only the coordinates coming from the 2-torsion and it yields Proposition 8.4 (not forgetting the six ever-implicit odd characteristics). □

8B. ***Evaluating the theta functions at archimedean places.*** We denote by $\mathcal{H}_2$ the Siegel half-space of degree 2, and by $\mathcal{F}_2$ the usual fundamental domain of this half-space for the action of $\mathrm{Sp}_4(\mathbb{Z})$ (see [Klingen 1990, §I.2] for details). For $\tau \in \mathcal{H}_2$, we denote by y_4 the imaginary part of the lower-right coefficient of τ.

Proposition 8.5. *For every $\tau \in \mathcal{H}_2$ and a fixed real parameter $t \geq \sqrt{3}/2$, one has*:

(a) *Amongst the ten even characteristics m of E, at most six of them can satisfy*

$$|\Theta_{m/2}(\tau)| < 0.42 \max_{m'\in E} |\Theta_{m'/2}(\tau)|.$$

(b) *If the representative of the orbit of τ in the fundamental domain $\mathcal{F}_2$ satisfies $y_4 \leq t$, at most one of the ten even characteristics m of E can satisfy*

$$|\Theta_{m/2}(\tau)| < 0.747 e^{-\pi t} \max_{m'\in E} |\Theta_{m'/2}(\tau)|.$$

Proof. First, we can assume that $\tau \in \mathcal{F}_2$ as the inequalities (a) and (b) are invariant by the action of $\mathrm{Sp}_4(\mathbb{Z})$, given the complete transformation formula of these theta functions [Mumford 2007, §II.5]. Now, using the Fourier expansions of the ten theta constants (mentioned in the proof of Definition-Proposition 7.7) and isolating their respective dominant terms (such as in [Klingen 1990], proof of Proposition IV.2), we obtain explicit estimates. More precisely, Proposition 7.7 of [Streng 2010] states that, for every $\tau = \left(\begin{smallmatrix}\tau_1 & \tau_2 \\ \tau_2 & \tau_4\end{smallmatrix}\right) \in \mathcal{B}_2$ (which is a domain containing $\mathcal{F}_2$), one has

$$|\Theta_{m/2}(\tau) - 1| < 0.405, \quad m\in\{(0000)(0001),(0010),(0011)\}.$$

$$\left|\frac{\Theta_{m/2}(\tau)}{2e^{i\pi\tau_1/2}} - 1\right| < 0.348, \quad m\in\{(0100),(0110)\}.$$

$$\left|\frac{\Theta_{m/2}(\tau)}{2e^{i\pi\tau_4/2}} - 1\right| < 0.348, \quad m\in\{(1000),(1001)\}.$$

$$\left|\frac{\Theta_{m/2}(\tau)}{2(\varepsilon_m + e^{2i\pi\tau_2})e^{i\pi(\tau_1+\tau_4-2\tau_2)/2}} - 1\right| < 0.438, \quad m\in\{(1100),(1111)\},$$

with $\varepsilon_m = 1$ if $m = (1100)$ and -1 if $m = (1111)$.

Under the assumption that $y_4 \leq t$ (which induces the same bound for $\operatorname{Im}\tau_1$ and $2\operatorname{Im}\tau_2$), we obtain

$$0.595 < |\Theta_{m/2}(\tau)| < 1.405, \quad m\in\{(0000)(0001),(0010),(0011)\}.$$

$$1.304e^{-\pi t/2} < |\Theta_{m/2}(\tau)| < 0.692, \quad m\in\{(0100),(0110),(1000),(1001)\}.$$

$$1.05e^{-\pi t} < |\Theta_{m/2}(\tau)| < 0.855, \quad m=(1100).$$

$$|\Theta_{m/2}(\tau)| < 0.855, \quad m=(1111)$$

Thus, we get (a) with $\frac{0.595}{1.405} > 0.42$, and (b) with $\frac{1.05}{1.405}e^{-\pi t} > 0.747e^{-\pi t}$. □

8C. *Computations with Igusa invariants for the places above 2 case.* In this case, as emphasized before, it is not possible to use Proposition 8.4, as the algebraic theory of theta functions does not work.

We have substituted it in the following way.

Definition 8.6 (auxiliary polynomials). For every $i \in \{1, \dots, 10\}$, let Σ_i be the i-th symmetric polynomial in the ten modular forms $\Theta^8_{m/2}$, $m \in E$ (notation (8-1)). This is a modular form of level $4i$ for the whole modular group $\mathrm{Sp}_4(\mathbb{Z})$.

Indeed, each $\Theta^8_{m/2}$ is a modular form for the congruence subgroup $\Gamma_2(2)$ of weight 4, and they are permuted by the modular action of $\Gamma_2(1)$ [Mumford 2007, §II.5]. The important point is that the Σ_i are then polynomials in the four Igusa modular forms ψ_4, ψ_6, χ_{10} and χ_{12} [Igusa 1967, pp. 848–849]. We can now explain the principle of this paragraph: these four modular forms are linked explicitly with the Igusa invariants (for a given jacobian of an hyperelliptic curve C over a number field K), and the semistable reduction of the jacobian at some place $v \mid 2$ is determined by the integrality (or not) of some quotients of these invariants, hence rational fractions of the modular forms. Now, with the explicit expressions of the Σ_i in terms of ψ_4, ψ_6, χ_{10} and χ_{12}, we can bound these Σ_i by one of the Igusa invariants, and as every $\Theta^8_{m/2}$ is a root of the polynomial

$$P(X) = X^{10} - \Sigma_1 X^9 + \Sigma_2 X^8 - \Sigma_3 X^7 + \Sigma_4 X^6 - \Sigma_5 X^5 + \Sigma_6 X^4 - \Sigma_7 X^4 + \Sigma_8 X^2 - \Sigma_9 X + \Sigma_{10},$$

we can infer an explicit bound above on the $\Theta_{m/2}^8/\lambda$, with a well-chosen normalizing factor λ such that these quotients belong to K. Actually, we will even give an approximate shape of the Newton polygon of the polynomial $\lambda^{10}P(X/\lambda)$, implying that its slopes (except maybe the first one) are bounded above and below, thus giving us a lower bound for each of the $|\Theta_{m/2}|_v/\max_{m'\in E}|\Theta_{m'/2}|_v$, except maybe for one m. The explicit result is the following.

Proposition 8.7. *Let K be a number field, (A, L) a principally polarized jacobian of dimension 2 over K and $\tau \in \mathcal{H}_2$ such that $(A_\tau, L_\tau) \cong (A, L)$.*

Let $\mathfrak{P}$ be a prime ideal of K above 2 such that A has potentially good reduction at $\mathfrak{P}$, and the reduced (principally polarized abelian surface) is denoted by $(A_{\mathfrak{P}}, L_{\mathfrak{P}})$. By abuse of notation, we forget the normalizing factor ensuring that the coordinates $\Theta_{m/2}(\tau)^8$ belong to K.

(a) *If $(A_{\mathfrak{P}}, L_{\mathfrak{P}})$ is the jacobian of a smooth hyperelliptic curve, all the $m \in E$ satisfy*

$$\frac{|\Theta_{m/2}(\tau)^8|_{\mathfrak{P}}}{\max_{m'\in E}|\Theta_{m'/2}(\tau)^8|_{\mathfrak{P}}} \geq |2|_{\mathfrak{P}}^{12}.$$

(b) *If $(A_{\mathfrak{P}}, L_{\mathfrak{P}})$ is a product of elliptic curves, all the $m \in E$ except at most one satisfy*

$$\frac{|\Theta_{m/2}(\tau)^8|_{\mathfrak{P}}}{\max_{m'\in E}|\Theta_{m'/2}(\tau)^8|_{\mathfrak{P}}} \geq |2|_{\mathfrak{P}}^{21}.$$

Proof. The most technical part is computing the Σ_i as polynomials in the four Igusa modular forms. To do this, we worked with Sage in the formal algebra generated by some sums of $\Theta_{m/2}^4$ with explicit relations (namely, $y_0, \dots, y_4$ in the notations of [Igusa 1964, pp. 396–397]). The total computation time, done on a laptop PC, was approximately twelve hours (including the verification of the results). The algorithms and details of their construction is available on a Sage worksheet (in Jupyter format).[1] An approach based on Fourier expansions might be more efficient, but as there is no clear closed formula for the involved modular forms, we privileged computations in this formal algebra. For easier reading, we slightly modified the Igusa modular forms into h_4, h_6, h_{10}, h_{12} defined as

$$\begin{cases} h_4 = 2\cdot\psi_4 = \frac{1}{2}\displaystyle\sum_{m\in E}\Theta_{m/2}^8 \\ h_6 = 2^2\cdot\psi_6 = \displaystyle\sum_{\substack{\{m_1,m_2,m_3\}\subset E\\ \text{syzygous}}} \pm(\Theta_{m_1/2}\Theta_{m_2/2}\Theta_{m_3/2})^4 \\ h_{10} = 2^{15}\cdot\chi_{10} = 2\displaystyle\prod_{m\in E}\Theta_{m/2}^2 \\ h_{12} = 2^{16}\cdot 3\cdot\chi_{12} = \frac{1}{2}\displaystyle\sum_{\substack{C\subset E\\ C \text{ Göpel}}}\prod_{m\in E\setminus C}\Theta_{m/2}^4 \end{cases} \tag{8-12}$$

([Igusa 1967, p. 848] for details on these definitions, notably syzygous triples and Göpel quadruples). The third expression is not explicitly a polynomial in $y_0, \dots, y_4$, but there is such an expression, given

[1] This worksheet can be found at http://msp.org/ant/2019/13-1/ant-v13-n1-x01-Igusainvariants.ipynb.

on page 397 of [Igusa 1964]. We also used to great benefit (both for understanding and computations) Section I.7.1 of [Streng 2010].

Now, the computations in Sage gave us the following formulas (the first and last one being trivial given (8-12), were not computed by the algorithm)

$$\Sigma_1 = 2h_4 \tag{8-13}$$

$$\Sigma_2 = \frac{3}{2}h_4^2 \tag{8-14}$$

$$\Sigma_3 = \frac{29}{2\cdot 3^3}h_4^3 - \frac{1}{2\cdot 3^3}h_6^2 + \frac{1}{2\cdot 3}h_{12} \tag{8-15}$$

$$\Sigma_4 = \frac{43}{2^4\cdot 3^3}h_4^4 - \frac{1}{2\cdot 3^3}h_4h_6^2 + \frac{23}{2\cdot 3}h_4h_{12} + \frac{2}{3}h_6h_{10} \tag{8-16}$$

$$\Sigma_5 = \frac{1}{2^2\cdot 3^3}h_4^5 - \frac{1}{2^3\cdot 3^3}h_4^2h_6^2 + \frac{25}{2^3\cdot 3}h_4^2h_{12} - \frac{1}{2\cdot 3}h_4h_6h_{10} + \frac{123}{2^2}h_{10}^2 \tag{8-17}$$

$$\Sigma_6 = \frac{1}{2^2\cdot 3^6}h_4^6 - \frac{1}{2^2\cdot 3^6}h_4^3h_6^2 + \frac{7}{2\cdot 3^3}h_4^3h_{12} - \frac{1}{2^2\cdot 3}h_4^2h_6h_{10} + \frac{47}{2\cdot 3}h_4h_{10}^2 + \frac{1}{2^4\cdot 3^6}h_6^4 - \frac{5}{2^3\cdot 3^3}h_6^2h_{12} + \frac{43}{2^4\cdot 3}h_{12}^2 \tag{8-18}$$

$$\Sigma_7 = \frac{1}{2\cdot 3^4}h_4^2h_{12} - \frac{1}{2\cdot 3^4}h_4^3h_6h_{10} + \frac{41}{2^3 3^2}h_4^2h_{10}^2 - \frac{1}{2^2\cdot 3^4}h_4h_6^2h_{12} + \frac{11}{2^2\cdot 3^2}h_4h_{12}^2 + \frac{1}{2^2\cdot 3^4}h_6^3h_{10} - \frac{19}{2^2\cdot 3^2}h_6h_{10}h_{12} \tag{8-19}$$

$$\Sigma_8 = \frac{1}{2^2\cdot 3^3}h_4^3h_{10}^2 + \frac{1}{2^2\cdot 3^2}h_4^2h_{12}^2 - \frac{1}{2\cdot 3^2}h_4h_6h_{10}h_{12} + \frac{5}{2^3\cdot 3^3}h_6^2h_{10}^2 - \frac{11}{2^3}h_{10}^2h_{12} \tag{8-20}$$

$$\Sigma_9 = \frac{-5}{2^2\cdot 3^2}h_4h_{10}^2h_{12} + \frac{7}{2^2\cdot 3^3}h_6h_{10}^3 + \frac{1}{3^3}h_{12}^3 \tag{8-21}$$

$$\Sigma_{10} = \frac{1}{2^4}h_{10}^4. \tag{8-22}$$

Remark 8.8. The denominators are always products of powers of 2 and 3. This was predicted by Ichikawa [2009], as all Fourier expansions of $\Theta_{m/2}$ (therefore of the Σ_i) have integral coefficients. Surprisingly, the result of [Ichikawa 2009] would actually be false for a $\mathbb{Z}[1/3]$-algebra instead of a $\mathbb{Z}[1/6]$-algebra, as the expression of Σ_3 (converted as a polynomial in ψ_4, ψ_6, χ_{12}) shows, but this does not provide a counterexample for a $\mathbb{Z}[1/2]$-algebra.

Now, let C be a hyperelliptic curve of genus 2 on a number field K and $\mathfrak{P}$ a prime ideal of $\mathcal{O}_K$ above 2. We will denote by $|\cdot|$ the norm associated to $\mathfrak{P}$ to lighten the notation. Let A be the jacobian of C and J_2, J_4, J_6, J_8, J_{10} the homogeneous Igusa invariants of the curve C, defined as in [Igusa 1960, pp. 621–622] up to a choice of hyperelliptic equation for C. We fix $\tau \in \mathcal{H}_2$ such that A_τ is isomorphic to A, which will be implicit in the following (i.e., h_4 denotes $h_4(\tau)$ for example). By [Igusa 1967, p. 848] applied with our normalization, there is an hyperelliptic equation for C (and we fix it) such that

$$J_2 = \frac{1}{2}\frac{h_{12}}{h_{10}} \tag{8-23}$$

$$J_4 = \frac{1}{2^5 \cdot 3}\left(\frac{h_{12}^2}{h_{10}^2} - 2h_4\right) \tag{8-24}$$

$$J_6 = \frac{1}{2^7 \cdot 3^3}\left(\frac{h_{12}^3}{h_{10}^3} - 6\frac{h_4 h_{12}}{h_{10}} + 4h_6\right) \tag{8-25}$$

$$J_8 = \frac{1}{2^{12} \cdot 3^3}\left(\frac{h_{12}^4}{h_{10}^4} - 12\frac{h_4 h_{12}^2}{h_{10}^2} + 16\frac{h_6 h_{12}}{h_{10}} - 12h_4^2\right) \tag{8-26}$$

$$J_{10} = \frac{1}{2^{13}} h_{10}. \tag{8-27}$$

Let us now figure out the Newton polygons allowing us to bound our theta constants.

(a) If A has potentially good reduction at $\mathfrak{P}$, and this reduction is also a jacobian, by Proposition 3 of [Igusa 1960], the quotients J_2^5/J_{10}, J_4^5/J_{10}^2, J_6^5/J_{10}^3 and J_8^5/J_{10}^4 are all integral at $\mathfrak{P}$. Translating it into quotients of modular forms, this gives

$$\left|\frac{J_2^5}{J_{10}}\right| = |2|^8 \left|\frac{h_{12}^5}{h_{10}^6}\right| \leq 1$$

$$\left|\frac{J_4^5}{J_{10}^2}\right| = |2|^3 \left|\frac{h_{12}^2}{h_{10}^{12/5}} - 2\frac{h_4}{h_{10}^{2/5}}\right|^5 \leq 1$$

$$\left|\frac{J_6^5}{J_{10}^3}\right| = |2|^4 \left|\frac{h_{12}^3}{h_{10}^{18/5}} - 6\frac{h_4 h_{12}}{h_{10}^{8/5}} + 4\frac{h_6}{h_{10}^{3/5}}\right|^5 \leq 1$$

$$\left|\frac{J_8^5}{J_{10}^4}\right| = |2|^{-8} \left|\frac{h_{12}^4}{h_{10}^{24/5}} - 12\frac{h_4 h_{12}^2}{h_{10}^{14/5}} + 16\frac{h_6 h_{12}}{h_{10}^{9/5}} - 12\frac{h_4^2}{h_{10}^{4/5}}\right|^5 \leq 1.$$

By successive bounds on the three first lines, we obtain

$$\left|\frac{h_4}{h_{10}^{2/5}}\right| \leq |2|^{-21/5}, \qquad \left|\frac{h_6}{h_{10}^{3/5}}\right| \leq |2|^{-34/5}, \qquad \left|\frac{h_{12}}{h_{10}^{6/5}}\right| \leq |2|^{-8/5}. \tag{8-28}$$

Using the expressions of the Σ_i ((8-13)–(8-22)), we compute that for every $i \in \{1, \dots, 10\}$, one has $\left|\Sigma_i / h_{10}^{2i/5}\right| \leq |2|^{\lambda_i}$ with the following values of λ_i:

i	10	9	8	7	6	5	4	3	2	1
λ_i	$-\frac{20}{5}$	$-\frac{44}{5}$	$-\frac{83}{5}$	$-\frac{112}{5}$	$-\frac{156}{5}$	$-\frac{125}{5}$	$-\frac{104}{5}$	$-\frac{73}{5}$	$-\frac{47}{5}$	$-\frac{16}{5}$

and for $i = 10$, it is an equality. Therefore, the highest slope of the Newton polygon is at most $\frac{26}{5} \cdot v_{\mathfrak{P}}(2)$, whereas the lowest one is at least $-\frac{34}{5} \cdot v_{\mathfrak{P}}(2)$, which gives part (a) of Proposition 8.7 by the theory of Newton polygons.

(b) If A has potentially good reduction at $\mathfrak{P}$ and the semistable reduction is a product of elliptic curves, defining

$$I_4 = J_2^3 - 25J_4 = \frac{h_4}{2} \tag{8-29}$$

$$I_{12} = -8J_4^3 + 9J_2J_4J_6 - 27J_6^2 - J_2^2J_8 = \frac{1}{2^{10}\cdot 3^3}(2h_4^3 - h_6^2), \tag{8-30}$$

$$P_{48} = 2^{12}\cdot 3^3 h_{10}^4 J_8 = h_{12}^4 - 12h_4h_{12}^2h_{10}^2 + 16h_6h_{12}h_{10}^3 - 12h_4^2h_{10}^4 \tag{8-31}$$

(which as modular forms are of respective weights 4, 12 and 48), by Theorem 1 (parts (V_*) and (V)) of [Liu 1993], we obtain in the same fashion that

$$\left|\frac{h_4}{P_{48}^{1/12}}\right| \leq |2|^{-13/3}, \qquad \left|\frac{h_6}{P_{48}^{1/8}}\right| \leq |2|^{-3}, \qquad \left|\frac{h_{10}}{P_{48}^{5/24}}\right| \leq |2|^{-4/3}. \tag{8-32}$$

Using the Newton polygon for the polynomial of (8-31) defining P_{48}, one deduces quickly that

$$\left|\frac{h_{12}}{P_{48}^{1/4}}\right| \leq |2|^{-7/2}. \tag{8-33}$$

As before, with the explicit expression of the Σ_i, one obtains that the $|\Sigma_i/P_{48}^{i/12}|$ are bounded by $|2|^{\lambda_i}$ with the following values of λ:

i	10	9	8	7	6	5	4	3	2	1
λ_i	$-\frac{28}{3}$	$-\frac{71}{6}$	$\frac{-53}{3}$	$\frac{-55}{3}$	$\frac{-84}{3}$	$\frac{-71}{3}$	$\frac{-64}{3}$	-14	$\frac{-29}{3}$	$\frac{-10}{3}$

(8-34)

This implies directly that the highest slope of the Newton polygon is at most $\frac{16}{3}\cdot v_{\mathfrak{P}}(2)$. Now, for the lowest slope, there is no immediate bound which was expected; in this situation, $\Sigma_{10} = 2^{-4}h_{10}^4$ can be relatively very small compared to $P_{48}^{5/6}$.

As P_{48} is in the ideal generated by h_{10}, h_{12} (in other words, is cuspidal) and dominates all modular forms h_4, h_6, h_{10}, h_{12}, one of h_{10} and h_{12} has to be relatively large enough compared to P_{48}. In practice, we get (with (8-32), (8-33) and (8-31))

$$\left|\frac{h_{12}}{P_{48}^{1/4}}\right| \geq 1 \quad \text{or} \quad \left|\frac{h_{10}}{P_{48}^{5/24}}\right| \geq |2|^{13/6}.$$

Now, if h_{10} is relatively very small (for example, $|h_{10}/P_{48}^{5/24}| \leq |2|^{19/6}|h_{12}/P_{48}^{1/4}|$), we immediately get $|h_{12}/P_{48}^{1/4}| = 1$ and $|\Sigma_9/P_{48}^{3/4}| = 1$. Computing again with these estimates for h_{10} and h_{12}, we obtain that the $|\Sigma_i/P_{48}^{i/12}|$ are bounded by $|2|^{\lambda_i}$ with the following slightly improved values of λ,

i	9	8	7	6	5	4	3	2	1
λ_i	0	$-\frac{32}{3}$	$-\frac{51}{3}$	$\frac{-84}{3}$	$\frac{-71}{3}$	$\frac{-64}{3}$	-14	$\frac{-29}{3}$	$\frac{-10}{3}$

The value at $i = 9$ is exact, hence the second lowest slope is then at least $-\frac{32}{3}\cdot v_{\mathfrak{P}}(2)$.

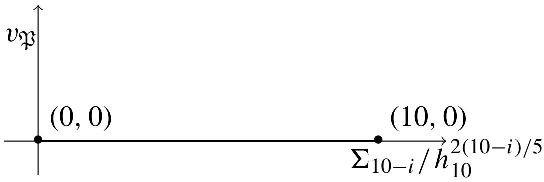


Figure 4. When the reduction of A is a jacobian.

If it is not so small, we have a bound on $v_{\mathfrak{P}}(\Sigma_{10}/P_{48}^{6/5})$, hence the Newton polygon itself is bounded (and looks like the first situation). In practice, one finds that the lowest slope is at least $-\frac{47}{3} \cdot v_{\mathfrak{P}}(2)$, hence all others slopes are at least this value, and this concludes the proof of Proposition 8.7(b). □

Remark 8.9. In characteristics $\neq 2, 3$, Theorem 1 of [Liu 1993] and its precise computations on pages 4 and 5 give the following exact shapes of Newton polygons (notice the different normalization factors).

In particular, when A reduces to a jacobian, the theta coordinates all have the same $\mathfrak{P}$-adic norm and when A reduces to a product of elliptic curves, exactly one of them has smaller norm; in other words, we reproved Proposition 8.4, and the Newton polygons have a very characteristic shape.

The idea behind the computations above is that in cases (a) and (b) (with other normalization factors), the Newton polygons have a shape close to these ones, therefore estimates can be made. It would be interesting to see what the exact shape of the Newton polygons is, to maybe obtain sharper results.

8D. *Wrapping up the estimates and end of the proof.* We can now prove the explicit refined version of Theorem 7.11, namely Theorem 8.2.

Proof of Theorem 8.2. In case (a), one can avoid the tubular assumption for the archimedean place of K; indeed, amongst the ten theta coordinates, there remain 4 which are large enough with no further assumption. As $|s_P| < 4$, there remains one theta coordinate which is never too small (at any place). In practice, normalizing the projective point $\psi(P)$ by this coordinate, one obtains with Propositions 8.5(a) (archimedean places) 8.4 (finite places not above 2) and 8.7 (finite places above 2)

$$h(\psi(P)) \leq -4\log(0.42) + \frac{21/2}{[K:\mathbb{Q}]} \sum_{v \mid 2} n_v \log(2) \leq 10.75$$

after approximation.

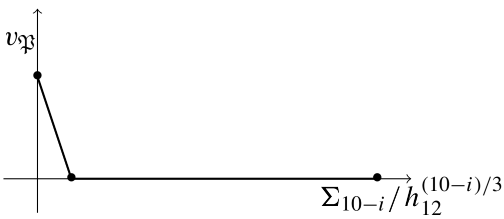


Figure 5. When the reduction of A is a product of elliptic curves.

In case (b), one has to use the tubular neighborhood implicitly given by the parameter t, namely Proposition 8.5(b) for archimedean places, again with Propositions 8.4 and 8.7 for the finite places, hence we get

$$h(\psi(P)) \leq 4\log(e^{\pi t}/0.747) + \frac{21/2}{[K:\mathbb{Q}]}\sum_{v\,|\,2} n_v \log(2) \leq 4\pi t + 8.44$$

after approximation.

Finally, we deduce from there the bounds on the stable Faltings height by Corollary 1.3 of [Pazuki 2012] (with its notations, $h_\Theta(A, L) = h(\psi(P))/4$). □

It would be interesting to give an analogous result for Theorem 7.12, and the estimates for archimedean and finite places not above 2 should not give any particular problem. For finite places above 2, the method outlined above can only be applied if, taking the symmetric polynomials $\Sigma_1, \dots, \Sigma_{f(n)}$ in well-chosen powers $\Theta_{\tilde{a}/n,\tilde{b}/n}(\tau)$ for $\tilde{a}, \tilde{b} \in \mathbb{Z}^g$, we can figure out by other arguments the largest rank k_0 for which Σ_{k_0} is cuspidal but not in the ideal generated by h_{10}. Doing so, we could roughly get back the pictured shape of the Newton polygon when h_{10} is relatively very small (because then Σ_k is relatively very small for $k > k_0$ by construction). Notice that for this process, one needs some way to theoretically bound the denominators appearing in the expressions of the Σ_i in h_4, h_6, h_{10}, h_{12}, but if this works, the method can again be applied.

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