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On rational singularities and counting points of schemes over finite rings

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We study the connection between the singularities of a finite type $\mathbb{Z}$-scheme X and the asymptotic point count of X over various finite rings. In particular, if the generic fiber $X_{\mathbb{Q}} = X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q}$ is a local complete intersection, we show that the boundedness of $|X(\mathbb{Z}/p^n\mathbb{Z})|/p^{n\dim X_{\mathbb{Q}}}$ in p and n is in fact equivalent to the condition that $X_{\mathbb{Q}}$ is reduced and has rational singularities. This paper completes a recent result of Aizenbud and Avni.

1. Introduction

1A. ***Motivation.*** Given a finite type $\mathbb{Z}$-scheme X, the study of the quantity $|X(\mathbb{Z}/m\mathbb{Z})|$ and its asymptotic behavior is a fundamental question in number theory. The case when $m = p$, or more generally the quantity $|X(\mathbb{F}_q)|$ with $q = p^n$, has been studied by many authors, most famously by Lang and Weil [1954], Dwork [1960], Grothendieck [1965] and Deligne [1974; 1980]. The Lang–Weil estimates (see [Lang and Weil 1954]) give a good asymptotic description of $|X(\mathbb{F}_q)|$:

$$|X(\mathbb{F}_q)| = q^{\dim X_{\mathbb{F}_q}}(C_X + O(q^{-\frac{1}{2}})),$$

where C_X is the number of top dimension irreducible components of $X_{\overline{\mathbb{F}}_q}$ that are defined over $\mathbb{F}_q$. From these estimates and the fact that

$$|X(F)| = |U(F)| + |(X \setminus U)(F)|, \tag{1-1}$$

for any open subscheme $U \subseteq X$ and any finite field F, one can see that the asymptotics of $|X(\mathbb{F}_{p^n})|$, in p or in n, does not depend on the singularity properties of X. For finite rings, however, (1-1) is no longer true (e.g., $|\mathbb{A}^1(A)| = |A|$ and $|(\mathbb{A}^1 - \{0\})(A)| = |A^{\times}|$) and indeed, the number $|X(\mathbb{Z}/m\mathbb{Z})|$ and its asymptotics have much to do with the singularities of X. The case when $m = p^n$ is a prime power was studied by Borevich and Shafarevich, among others (see the works of Denef [1991], Igusa [2000], du Sautoy and Grunewald [2000], and a recent overview by Mustaţă [2011]).

For a finite ring A, set $h_X(A) := |X(A)|/|A|^{\dim X_{\mathbb{Q}}}$. If $X_{\mathbb{Q}}$ is smooth, one can show that for almost every prime p, we have $h_X(\mathbb{Z}/p^n\mathbb{Z}) = h_X(\mathbb{Z}/p\mathbb{Z})$ for all n, which by the Lang–Weil estimates is uniformly bounded. On the other hand, if $X_{\mathbb{Q}}$ is singular, then $h_X(\mathbb{Z}/p^n\mathbb{Z})$ need not be bounded in n or in p. The goal of this paper is to investigate this phenomena and to complete the main result presented in [Aizenbud and Avni 2018], which we describe next.

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1B. ***Related work.*** Aizenbud and Avni [2018] proved the following:

Theorem 1.1 [Aizenbud and Avni 2018, Theorem 3.0.3]. *Let X be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}}$ is equidimensional and a local complete intersection. Then the following are equivalent*:

(i) *For any n, $\lim_{p\to\infty} h_X(\mathbb{Z}/p^n\mathbb{Z}) = 1$.*

(ii) *There exists a finite set of prime numbers S and a constant C, such that $|h_X(\mathbb{Z}/p^n\mathbb{Z}) - 1| < Cp^{-\frac{1}{2}}$ for any prime $p \notin S$ and any $n \in \mathbb{N}$.*

(iii) *$X_{\overline{\mathbb{Q}}}$ is reduced, irreducible and has rational singularities.*

Definition 1.2 [Aizenbud and Avni 2016, 1.2, Definition II]. Let X and Y be smooth varieties over a field k of characteristic 0. We say that a morphism $\varphi : X \to Y$ is (FRS) if it is flat and any geometric fiber is reduced and has rational singularities. We say that φ is (FRS) at $x \in X(k)$ if there exists a Zariski open neighborhood U of x such that $U \times_Y \{\varphi(x)\}$ is reduced and has rational singularities.

Aizenbud and Avni introduced an analytic criterion for a morphism φ to be (FRS), which played a key role in the proof of Theorem 1.1:

Theorem 1.3 [Aizenbud and Avni 2016, Theorem 3.4]. *Let $\varphi : X \to Y$ be a map between smooth algebraic varieties defined over a finitely generated field k of characteristic* 0, *and let $x \in X(k)$. Then the following conditions are equivalent*:

(a) *φ is* (FRS) *at x.*

(b) *There exists a Zariski open neighborhood $x \in U \subseteq X$, such that for any non-Archimedean local field $F \supseteq k$ and any Schwartz measure m on $U(F)$, the measure $(\varphi|_{U(F)})_*(m)$ has continuous density* (*see Definition 2.5 for the notion of Schwartz measure and continuous density of a measure*).

(c) *For any finite extension k'/k, there exists a non-Archimedean local field $F \supseteq k'$ and a nonnegative Schwartz measure m on $X(F)$ that does not vanish at x such that $\varphi_*(m)$ has continuous density.*

1C. ***Main results.*** In this paper, we strengthen Theorem 1.1 as follows:

Theorem 1.4. *Let X be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}}$ is equidimensional and a local complete intersection. Then* (i), (ii) *and* (iii) *in Theorem 1.1 are also equivalent to*:

(iv) *$X_{\overline{\mathbb{Q}}}$ is irreducible and there exists $C > 0$ such that $h_X(\mathbb{Z}/p^n\mathbb{Z}) < C$ for any prime p and any $n \in \mathbb{N}$.*

(v) *$X_{\overline{\mathbb{Q}}}$ is irreducible and there exists a finite set of primes S, such that for any $p \notin S$, the sequence $n \mapsto h_X(\mathbb{Z}/p^n\mathbb{Z})$ is bounded.*

Remark. In fact, one can drop the demand that $X_{\overline{\mathbb{Q}}}$ is irreducible in conditions (iii), (iv) and (v), such that they will stay equivalent. For a slightly stronger statement, see Theorem 4.1.

There are two main difficulties in the proof of Theorem 1.4. The first one is portrayed in the fact that condition (v) seems a-priori too weak, as it requires the bound on $h_X(\mathbb{Z}/p^n\mathbb{Z})$ to be uniform only in n, while in condition (ii), the demand is that the bound is uniform both in p and in n.

In order to show that condition (v) implies the other conditions, we first reduce to the case when $X_{\mathbb{Q}}$ is a complete intersection in an affine space, and thus can be written as the fiber at 0 of a morphism $\varphi : \mathbb{A}^M_{\mathbb{Q}} \to \mathbb{A}^N_{\mathbb{Q}}$, which is flat above 0. We can then translate condition (iii), i.e., the condition that $X_{\overline{\mathbb{Q}}}$ is reduced and has rational singularities, to the condition that $\varphi : \mathbb{A}^M_{\mathbb{Q}} \to \mathbb{A}^N_{\mathbb{Q}}$ is (FRS) above 0, i.e., at any point $x \in (\varphi^{-1}(0))(\overline{\mathbb{Q}})$. After some technical argument, one can show that condition (v) implies the following:

Condition 1.5. For any finite extensions $k/\mathbb{Q}$ and k'/k, and any $x \in (\varphi^{-1}(0))(k)$, there exists a prime p with $k' \hookrightarrow \mathbb{Q}_p$, $x \in (\varphi^{-1}(0))(\mathbb{Z}_p)$, such that the sequence $n \mapsto \varphi_*(\mu)(p^n \mathbb{Z}_p^N)/p^{-nN}$ is bounded, where μ is the normalized Haar measure on $\mathbb{Z}_p^M$.

Hence, we would like to strengthen Theorem 1.3, such that Condition 1.5 will imply the (FRS) property of φ above 0.

The measure $\varphi_*(m)$ as in Condition 1.5 is said to be bounded with respect to the local basis $\{p^n \mathbb{Z}_p^N\}_n$ for the topology of $\mathbb{Q}_p^N$ at 0 (see Definition 3.1).We introduce the notion of bounded eccentricity of a local basis to the topology of an F-analytic manifold (Section 3A), and prove the following stronger version of Theorem 1.3:

Theorem 1.6. *Let $\varphi : X \to Y$ be a map between smooth algebraic varieties defined over a finitely generated field k of characteristic* 0, *and let $x \in X(k)$. Then* (a), (b), (c) *in Theorem 1.3 are also equivalent to*:

(c′) *For any finite extension k'/k, there exists a non-Archimedean local field $F \supseteq k'$ and a nonnegative Schwartz measure m on $X(F)$ that does not vanish at x, such that $\varphi_*(m)$ is bounded with respect to some local basis $\mathcal{N}$ of bounded eccentricity at $\varphi(x)$.*

We then use Theorem 1.6 and the fact that the local basis $\{p^n \mathbb{Z}_p^N\}_n$ is of bounded eccentricity to show that (v) implies condition (iii).

The second difficulty is to show that if $h_X(\mathbb{Z}/p^n\mathbb{Z})$ is bounded for almost any prime p, then it is in fact bounded for any p. We first prove this for the case that X is a complete intersection in an affine space, denoted (CIA) (Proposition 4.5). We then deal with the case when $X_{\mathbb{Q}}$ is a (CIA), by constructing a finite type $\mathbb{Z}$-scheme $\widehat{X}$, which is a (CIA) and a morphism $\psi : X \longrightarrow \widehat{X}$, such that $\psi_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow \widehat{X}_{\mathbb{Q}}$ is an isomorphism (Lemma 4.6). We prove this case by showing the existence of $c, N \in \mathbb{N}$ such that

$$|X(\mathbb{Z}/p^n\mathbb{Z})| \leq p^{Nc} \cdot |\widehat{X}(\mathbb{Z}/p^n\mathbb{Z})|,$$

(Lemma 4.7). For the general case, we first cover $X_{\mathbb{Q}}$ by affine $\mathbb{Q}$-schemes $\{U_i\}$ such that U_i is a (CIA), and then consider a collection of $\mathbb{Z}$-schemes $\{\widetilde{U}_i\}$, such that $\widetilde{U}_i \simeq U_i$ over $\mathbb{Q}$. Finally, using the explicit construction of $\widetilde{U}_i$ we show that

$$h_X(\mathbb{Z}/p^n\mathbb{Z}) \leq \sum_i h_{\widetilde{U}_i}(\mathbb{Z}/p^n\mathbb{Z}),$$

and since $(\widetilde{U}_i)_{\mathbb{Q}} \simeq U_i$ is a (CIA), we are done by the last case.

2. Preliminaries

In this section, we recall some definitions and facts in algebraic geometry and the theory of F-analytic manifolds, for a non-Archimedean local field F.

2A. ***Preliminaries in algebraic geometry.*** Let A be a commutative ring. A sequence $x_1, \dots, x_r \in A$ is called a *regular sequence* if x_i is not a zero-divisor in $A/(x_1, \dots, x_{i-1})$ for each i, and we have a proper inclusion $(x_1, \dots, x_r) \subsetneq A$. If $(A, \mathfrak{m})$ is a Noetherian local ring then the *depth* of A, denoted $\operatorname{depth}(A)$, is defined to be the length of the longest regular sequence with elements in $\mathfrak{m}$. It follows from Krull's principal ideal theorem that $\operatorname{depth}(A)$ is smaller or equal to $\dim(A)$, the Krull dimension of A. A Noetherian local ring $(A, \mathfrak{m})$ is *Cohen–Macaulay* if $\operatorname{depth}(A) = \dim(A)$. A locally Noetherian scheme X is said to be *Cohen–Macaulay* if for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is Cohen–Macaulay.

Let X be an algebraic variety over a field k. We say that X has a *resolution of singularities*, if there exists a proper morphism $p : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and p is a birational equivalence. A *strong resolution of singularities* of X is a resolution of singularities $p : \widetilde{X} \to X$ which is an isomorphism over the smooth locus of X, denoted X^{sm}. It is a theorem of Hironaka [1964], that any variety X over a field k of characteristic zero admits a strong resolution of singularities $p : \widetilde{X} \to X$.

For the following definition, see [Kempf et al. 1973, I.3 pages 50–51] or [Aizenbud and Avni 2016, Definition 6.1]; a variety X over a field k of characteristic zero is said to have *rational singularities* if for any (or equivalently, for some) resolution of singularities $p : \widetilde{X} \longrightarrow X$, the natural morphism $\mathcal{O}_X \to \mathrm{R}p_*(\mathcal{O}_{\widetilde{X}})$ is a quasi-isomorphism, where $\mathrm{R}p_*$ is the higher direct image. A point $x \in X(k)$ is a *rational singularity* if there exists a Zariski open neighborhood $U \subseteq X$ of x that has rational singularities.

We denote by Ω_X^r the sheaf of differential r-forms on X and by $\Omega_X^r[X]$ (resp. $\Omega_X^r(X)$) the regular (resp. rational) r-forms. The following lemma gives a local characterization of rational singularities:

Lemma 2.1 [Aizenbud and Avni 2016, Proposition 6.2]. *An affine k-variety X has rational singularities if and only if X is Cohen–Macaulay, normal, and for any, or equivalently, some strong resolution of singularities $p : \widetilde{X} \to X$ and any top differential form $\omega \in \Omega_{X^{\mathrm{sm}}}^{\mathrm{top}}[X^{\mathrm{sm}}]$, there exists a top differential form $\widetilde{\omega} \in \Omega_{\widetilde{X}}^{\mathrm{top}}[\widetilde{X}]$ such that $\omega = \widetilde{\omega}|_{X^{\mathrm{sm}}}$.*

Let X be a finite type scheme over a ring R. Then X is called:

(1) A *complete intersection* (CI) if there exists an affine scheme Y, a smooth morphism $Y \to \operatorname{Spec} R$, a closed embedding $X \hookrightarrow Y$ over $\operatorname{Spec} R$, and a regular sequence $f_1, \dots, f_r \in \mathcal{O}_Y(Y)$, such that the ideal of X in Y is generated by the $\{f_i\}$. In this case, we say that X is a *complete intersection in Y*.

(2) A *local complete intersection* (LCI) if there is an open cover $\{U_i\}$ of X such that each U_i is a (CI).

(3) A *complete intersection in an affine space* (CIA) if X is a complete intersection in Y, with $Y = \mathbb{A}_R^n$ an affine space.

(4) A *local complete intersection in an affine space* (LCIA) if there is an open affine cover $\{U_i\}$ of X such that each U_i is a (CIA).

Remark 2.2. For an affine k-variety, the notion of (CIA) is not equivalent to (CI) (e.g., consider X to be any affine smooth k-variety which is not a (CIA)). On the other hand, the notion of (LCI) is equivalent to (LCIA) for finite type k-schemes. We will therefore use the notation (LCI) for both notions.

The following Proposition 2.3 and Proposition 2.4 are a consequence of the above remark and the miracle flatness theorem (e.g., [Vakil 2017, Theorems 26.2.10 and 26.2.11]).

Proposition 2.3. *Let X be k-variety. If X is an* (LCI) *then there exists an open affine cover $\{U_i\}$ of X and morphisms φ_i, ψ_i, where $\varphi_i : \mathbb{A}_k^{m_i} \longrightarrow \mathbb{A}_k^{n_i}$ is flat above* 0, *and $\psi_i : U_i \hookrightarrow \mathbb{A}_k^{m_i}$ is a closed embedding that induces a k-isomorphism $\psi_i : U_i \simeq \varphi_i^{-1}(0)$.*

Proposition 2.4. *Let X be a finite type $\mathbb{Z}$-scheme. If X is a* (CIA) *then there exist $\mathbb{Z}$-morphisms φ, ψ, where $\varphi : \mathbb{A}^m_{\mathbb{Z}} \longrightarrow \mathbb{A}^n_{\mathbb{Z}}$ is flat above* 0, *and $\psi : X \hookrightarrow \mathbb{A}^m_{\mathbb{Z}}$ is a closed embedding that induces a $\mathbb{Z}$-isomorphism $\psi : X \simeq \varphi^{-1}(0)$.*

A commutative Noetherian local ring A is called *Gorenstein* if it has finite injective dimension as an A-module. A locally Noetherian scheme X is said to be *Gorenstein* if all its local rings are Gorenstein. Any locally Noetherian scheme X which is a local complete intersection is also Gorenstein.

2B. ***Some facts on F-analytic manifolds.*** Let X be a d-dimensional smooth algebraic k-variety and $F \supseteq k$ be a non-Archimedean local field, with ring of integers $\mathcal{O}_F$. Then $X(F)$ has a structure of an F-analytic manifold. Given $\omega \in \Omega_X^{\mathrm{top}}(X)$, we can define a measure $|\omega|_F$ on $X(F)$ as follows. For a compact open set $U \subseteq X(F)$ and an F-analytic diffeomorphism ϕ between an open subset $W \subseteq F^d$ and U, we can write $\phi^*\omega = g \cdot dx_1 \wedge \cdots \wedge dx_n$, for some $g : W \to F$, and define

$$|\omega|_F(U) = \int_W |g|_F \, d\lambda,$$

where $|\cdot|_F$ is the normalized absolute value on F and λ is the normalized Haar measure on F^d. Note that this definition is independent of the diffeomorphism ϕ, and that this uniquely defines a measure on $X(F)$.

Definition 2.5. (1) A measure m on $X(F)$ is called *smooth* if every point $x \in X(F)$ has an analytic neighborhood U and an F-analytic diffeomorphism $f : U \to \mathcal{O}_F^d$ such that f_*m is some Haar measure on $\mathcal{O}_F^d$.

(2) A measure on $X(F)$ is called *Schwartz* if it is smooth and compactly supported.

(3) We say that a measure μ on $X(F)$ has *continuous density*, if there is a smooth measure m and a continuous function $f : X(F) \to \mathbb{C}$ such that $\mu = f \cdot m$.

The following proposition characterizes Schwartz measures and measures with continuous density:

Proposition 2.6 [Aizenbud and Avni 2016, Proposition 3.3]. *Let X be a smooth variety over a non-Archimedean local field F.*

(1) *A measure m on $X(F)$ is Schwartz if and only if it is a linear combination of measures of the form $f|\omega|_F$, where f is a Schwartz function* (*i.e., locally constant and compactly supported*) *on $X(F)$, and $\omega \in \Omega_X^{\mathrm{top}}(X)$ has no zeros or poles in the support of f.*

(2) *A measure μ on $X(F)$ has continuous density if and only if for every point $x \in X(F)$ there is an analytic neighborhood U of x, a continuous function $f : U \to \mathbb{C}$, and $\omega \in \Omega_X^{\mathrm{top}}(X)$ with no poles in U such that $\mu = f|\omega|_F$.*

Proposition 2.7 [Aizenbud and Avni 2016, Proposition 3.5]. *Let $\varphi : X \to Y$ be a smooth map between smooth varieties defined over a non-Archimedean local field F.*

(1) *If m is a Schwartz measure on $X(F)$, then φ_*m is a Schwartz measure on $Y(F)$.*

(2) *Assume that $\omega_X \in \Omega_X^{\mathrm{top}}[X]$ and $\omega_Y \in \Omega_Y^{\mathrm{top}}[Y]$, where ω_Y is nowhere vanishing, and that f is a Schwartz function on $X(F)$. Then the measure $\varphi_*(f|\omega_X|_F)$ is absolutely continuous with respect to $|\omega_Y|_F$, and its density at a point $y \in Y(F)$ is $\int_{\varphi^{-1}(y)(F)} f \cdot \left|(\omega_X/\varphi^*\omega_Y)|_{\varphi^{-1}(y)}\right|_F$.*

3. An analytic criterion for the (FRS) property

Our goal in this section is to prove Theorem 1.6, which is a stronger version of Theorem 1.3, and the main ingredient in the proof of the implication (v) $\Rightarrow$ (iii) of Theorem 1.4. As discussed in the introduction, we want to relax condition (c) of Theorem 1.3, and get a weaker condition (c$'$) that is similar to Condition 1.5, such that it will imply the (FRS) property (condition (a) of Theorem 1.3).

Definition 3.1. Let F be a non-Archimedean local field, X be an F-analytic manifold and μ be a measure on X. Let $\mathcal{N}=\{N_i\}_{i\in I}$ be a local basis for the topology of X at a point $x\in X$. We say that μ is *bounded with respect to* $\mathcal{N}$, if there exists a smooth measure λ on X and an open analytic neighborhood U of x, such that $|\mu(N_i)/\lambda(N_i)|$ is uniformly bounded on $\mathcal{N}_U:=\{N_i\in\mathcal{N}\mid N_i\subseteq U\}$.

Let $\varphi: X\to Y$, m and F be as in Theorem 1.3. A possible relaxation (c$'$) of (c), is to require $\varphi_*(m)$ to be bounded with respect to any local basis of the topology of $Y(F)$ at $\varphi(x)$. While this condition is equivalent to (a) and (b) it is still not weak enough for our purpose of proving Theorem 1.4. A much weaker condition (c$''$) is to demand that $\varphi_*(m)$ is bounded with respect to *some* local basis at $\varphi(x)$. Unfortunately, the following example shows that the latter demand is too weak:

Example. Consider the map $\varphi:\mathbb{A}^2_{\mathbb{Q}}\longrightarrow\mathbb{A}_{\mathbb{Q}}$ defined by $(x,y)\longmapsto x^2$. The fiber over 0 is not reduced, and thus φ is not (FRS) over 0. Fix a finite extension $k/\mathbb{Q}$ and embed k in $\mathbb{Q}_p$ for some prime p (see Lemma 4.3). Let λ_1,λ_2 be the normalized Haar measure on $\mathbb{Q}_p,\mathbb{Q}_p^2$ and let $m=1_{\mathbb{Z}_p^2}\cdot\lambda_2$ be a Schwartz measure. Now consider the following collection $\mathcal{N}$ of sets B_n constructed as follows. Define $B_n^1:=\{x\in\mathbb{Z}_p\mid |x|\leq p^{-2n^2}\}$ and $B_n^2:=\{x\in\mathbb{Z}_p\mid |x-a_n|\leq p^{-4n}\}$, where $a_n=p^{2n+1}$. Note that any $x\in B_n^2$ has norm p^{-2n-1} and thus is not a square, so $\varphi^{-1}(B_n^2)=\varnothing$. Denote $B_n=B_n^1\cup B_n^2$ and notice that $\mathcal{N}:=\{B_n\}_{n=1}^{\infty}$ is a local basis at 0 and that:

$$\lim_{n\to\infty}\frac{\varphi_*m(B_n)}{\lambda_1(B_n)}=\lim_{n\to\infty}\frac{m(\varphi^{-1}(B_n^1))}{p^{-2n^2}+p^{-4n}}=\lim_{n\to\infty}\frac{p^{-n^2}}{p^{-2n^2}+p^{-4n}}\to 0.$$

This shows that φ satisfies condition (c$''$) but is not (FRS) at (0, 0).

Luckily, we can relax (c) by demanding that $\varphi_*(m)$ is bounded with respect to *some* local basis at $\varphi(x)$, if this basis is nice enough. In order to define precisely what we mean, we introduce the notion of a local basis of bounded eccentricity.

3A. *Local basis of bounded eccentricity.*

Definition 3.2. Let F be a local field, and λ be a Haar measure on F^n.

(1) A collection of sets $\mathcal{N}=\{N_i\}_{i\in I}$ in F^n is said to have *bounded eccentricity* at $x\in F^n$, if there exists a constant $C>0$ such that $\sup_i(\lambda(B_{\min_i}(x))/\lambda(B_{\max_i}(x)))\leq C$, where $B_{\max_i}(x)$ is the maximal ball around x that is contained in N_i and $B_{\min_i}(x)$ is the minimal ball around x that contains N_i.

(2) We call $\mathcal{N}=\{N_i\}_{i\in I}$ a *local basis of bounded eccentricity* at x, if it is a local basis of the topology of F^n at x, and there exists $\epsilon>0$, such that $\mathcal{N}_\epsilon:=\{N_i\in\mathcal{N}\mid N_i\subseteq B_\epsilon(x)\}$ has bounded eccentricity.

Remark. Note that $\mathcal{N}_\epsilon\neq\varnothing$ for any $\epsilon>0$ since it is a local basis at x.

Lemma 3.3. *Let $\phi: F^n\to F^n$ be an F-analytic diffeomorphism. Let $\mathcal{N}=\{N_i\}_{i\in\alpha}$ be a local basis of bounded eccentricity at $x\in F^n$. Then $\phi(\mathcal{N})$ is a local basis of bounded eccentricity at $\phi(x)$.*

Proof. Let $d\phi_x = A$ be the differential of ϕ at x. Since ϕ is a diffeomorphism, then for any $C > 1$, there exists $\delta, \delta' > 0$ such that for any $y \in B_\delta(x)$:

$$\frac{1}{C} < \frac{|\phi(y) - \phi(x)|_F}{|A \cdot (y - x)|_F} < C,$$

and for any $z \in B_{\delta'}(\phi(x))$ we have:

$$\frac{1}{C} < \frac{|\phi^{-1}(z) - x|_F}{|A^{-1} \cdot (z - \phi(x))|_F} < C.$$

We can choose small enough δ, δ' such that $\mathcal{N}_\delta$ is a collection of sets which has bounded eccentricity and $\phi(\mathcal{N}_\delta) \supseteq \phi(\mathcal{N})_{\delta'}$. We now claim that $\mathcal{M}_{\delta'} := \phi(\mathcal{N})_{\delta'}$ is a collection of sets which has bounded eccentricity at $\phi(x)$. Let $B_{\min_i}(x)$ be the minimal ball that contains $N_i \in \mathcal{N}_\delta$ and $B_{\max_i}(x)$ be the maximal ball that is contained in N_i. Notice that for any $y \in B_{\min_i}(x) \subseteq B_\delta(x)$ we have

$$|\phi(y) - \phi(x)|_F < C \cdot |A \cdot (y - x)|_F \leq C \cdot \|A\| \cdot |y - x|_F \leq C \cdot \min_i \cdot \|A\|,$$

thus $\phi(N_i) \subseteq \phi(B_{\min_i}(x)) \subseteq B_{C \cdot \min_i \cdot \|A\|}(\phi(x))$. Similarly, for any $z \in B_{\max_i /(C \cdot \|A^{-1}\|)}(\phi(x))$ we have that

$$|\phi^{-1}(z) - x|_F < C \cdot |A^{-1} \cdot (z - \phi(x))|_F \leq C \cdot \|A^{-1}\| \cdot \frac{\max_i}{C \cdot \|A^{-1}\|} = \max_i .$$

Therefore, $\phi^{-1}\left(B_{\max_i /(C \cdot \|A^{-1}\|)}(\phi(x))\right) \subseteq B_{\max_i}(x) \subseteq N_i$ and thus $B_{\max_i /(C \cdot \|A^{-1}\|)}(\phi(x)) \subseteq \phi(N_i)$. Thus we get that

$$B_{\max_i /(C \cdot \|A^{-1}\|)}(\phi(x)) \subseteq \phi(N_i) \subseteq B_{C \cdot \min_i \cdot \|A\|}(\phi(x)),$$

for any i. By assumption, there exists some $D > 0$ such that $\lambda(B_{\min_i}(x))/\lambda(B_{\max_i}(x)) < D$ for any set $N_i \in \mathcal{N}_\delta$. Hence:

$$\frac{\lambda\left(B_{C \cdot \min_i \cdot \|A\|}(\phi(x))\right)}{\lambda\left(B_{\max_i /(C \cdot \|A^{-1}\|)}(\phi(x))\right)} \leq C^{2n} \|A\|^n \cdot \|A^{-1}\|^n \cdot D,$$

and $\mathcal{M}_{\delta'}$ has bounded eccentricity. □

Lemma 3.3 implies that the following notion is well defined:

Definition 3.4. Let X be an F-analytic manifold and λ be a Haar measure on F^n.

(1) A local basis $\mathcal{N}$ at $x \in X$ is said to have *bounded eccentricity* if given an F-analytic diffeomorphism ϕ between an open subset $W \subseteq F^n$ and an open neighborhood U of x, we have that

$$\widetilde{\mathcal{N}} = \{\phi^{-1}(N) \mid N \in \mathcal{N},\ N \subseteq U\}$$

is a local basis of bounded eccentricity.

(2) A measure m on X is said to be $\mathcal{N}$*-bounded*, if there exists $\epsilon > 0$ such that:

$$\sup_{N \in \mathcal{N}_\epsilon} \frac{m(N)}{\lambda(N)} < \infty.$$

3B. ***Proof of Theorem 1.6.*** It is easy to see that (c) $\Rightarrow$ (c′). The proof of the implication (c′) $\Rightarrow$ (a) is a variation of the proof of (c) $\Rightarrow$ (a) of Theorem 1.3 (see [Aizenbud and Avni 2016, Section 3.7]). Let k be a finitely generated field of characteristic 0, $\varphi : X \to Y$ be a morphism of smooth k-varieties X, Y and let $x \in X(k)$. Assume that condition (c′) of Theorem 1.6 holds. Let $Z = \varphi^{-1}(\varphi(x))$ and denote by X^S the smooth locus of φ. The following lemma is a slight variation of [Aizenbud and Avni 2016, Claim 3.19]. Since we use the constructions presented in the proof of [loc. cit.], and for the convenience of the reader, we write the full steps and use similar notation as well.

Lemma 3.5. *There exists a Zariski neighborhood U of x such that $Z \cap X^S \cap U$ is a dense subvariety of $Z \cap U$.*

Proof. Let $Z_1, \dots, Z_n$ be the absolutely irreducible components of Z containing x. After restricting to an open neighborhood of x that does not intersect the other irreducible components, it is enough to show that $Z_i \cap X^S$ is Zariski dense in Z_i for any i. Since X^S is open, it is enough to show that $Z_i \cap X^S$ is nonempty for any i.

Assume that $Z_i \cap X^S = \varnothing$ for some i. Then $\dim\ker d\varphi_z > \dim X - \dim Y$ for any $z \in Z_i(\bar{k})$. By the upper semicontinuity of $\dim\ker d\varphi$, there is a nonempty open set $W_i \subseteq Z_i$ and an integer $r \geq 1$ such that $\dim\ker d\varphi|_z = \dim X - \dim Y + r$ for all $z \in W_i(\bar{k})$ and such that $W_i \cap Z_j = \varnothing$ for any $j \neq i$. Let k'/k be a finite extension such that both Z_i, W_i are defined over k' and $W_i^{\mathrm{sm}}(k') \neq \varnothing$. By [Aizenbud and Avni 2016, Lemma 3.14], we can choose k' such that $x \in \overline{W_i^{\mathrm{sm}}(F)}$ for any non-Archimedean local field $F \supseteq k'$.

By our assumption, there exists a non-Archimedean local field $F \supseteq k'$ and a nonnegative Schwartz measure m on $X(F)$ that does not vanish at x and such that $\varphi_* m$ is bounded with respect to some local basis $\mathcal{N}$ (at $\varphi(x)$) of bounded eccentricity. Since $x \in \overline{W_i^{\mathrm{sm}}(F)}$, there exists a point $p \in W_i^{\mathrm{sm}}(F) \cap \operatorname{supp}(m)$.

By the implicit function theorem, there exist neighborhoods $U_X \subseteq X(F)$ and $U_Y \subseteq Y(F)$ of p and $\varphi(x) = \varphi(p)$ respectively, analytic diffeomorphisms $\alpha_X : U_X \to \mathcal{O}_F^{\dim X}$, $\alpha_Y : U_Y \to \mathcal{O}_F^{\dim Y}$ and $\alpha_{Z_i} : U_X \cap W_i^{\mathrm{sm}}(F) \to \mathcal{O}_F^{\dim Z_i}$ such that $\alpha_X(p) = 0$, $\alpha_Y(\varphi(p)) = 0$, and an analytic map $\psi : \mathcal{O}_F^{\dim X} \to \mathcal{O}_F^{\dim Y}$ such that the following diagram commutes:

$$\begin{tikzcd}
U_X \cap W_i^{\mathrm{sm}}(F) \arrow[r, hook] \arrow[d, "\alpha_{Z_i}"'] & U_X \arrow[r, "\varphi|_{U_X}"] \arrow[d, "\alpha_X"] & U_Y \arrow[d, "\alpha_Y"] \\
\mathcal{O}_F^{\dim Z_i} \arrow[r, hook, "j"] & \mathcal{O}_F^{\dim X} \arrow[r, "\psi"] & \mathcal{O}_F^{\dim Y}
\end{tikzcd}$$

where $j : \mathcal{O}_F^{\dim Z_i} \to \mathcal{O}_F^{\dim X}$ is the inclusion to the first $\dim Z_i$ coordinates. After an analytic change of coordinates we may assume that:

$$\ker d\psi_z = \operatorname{span}\{e_1, \dots, e_{\dim X - \dim Y + r}\},$$

for any $z \in \mathcal{O}_F^{\dim Z_i}$. By Lemma 3.3, we have that $\mathcal{M} := \alpha_Y(\mathcal{N})$ is a local basis of bounded eccentricity at $0 \in \mathcal{O}_F^{\dim Y}$. Note that $\mu := (\alpha_X)_*(1_{U_X} \cdot m)$ is a nonnegative Schwartz measure that does not vanish at 0, and that $\psi_*(\mu)$ is $\mathcal{M}$-bounded. By Proposition 2.6, after restricting to a small enough ball around 0 and applying a homothety, we can assume that μ is the normalized Haar measure.

As part of the data, for any $M_j \in \mathcal{M}$ we are given by $B_{\max_j}(0)$ and $B_{\min_j}(0)$, and there exists $\delta, C > 0$ such that for any $M_j \in \mathcal{M}_\delta := \{M_j \in \mathcal{M} \mid M_j \subseteq B_\delta(0)\}$, we have $B_{\max_j}(0) \subseteq M_j \subseteq B_{\min_j}(0)$ and

$\lambda(B_{\min_j})/\lambda(B_{\max_j}) \leq C$. For any $0 < \epsilon < 1$, set

$$A_\epsilon := \left\{(x_1, \dots, x_{\dim X}) \in \mathcal{O}_F^{\dim X} \mid |x_k| < \epsilon^{n_k}\right\},$$

where $n_k = 0$ if $1 \leq k \leq \dim Z_i$; $n_k = 1$ for $\dim Z_i + 1 \leq k \leq \dim X - \dim Y + r$; and $n_k = 2$ for $\dim X - \dim Y + r + 1 \leq k \leq \dim X$.

By choosing δ small enough, we may find a constant $D > 0$ such that $\psi(A_{D\sqrt{\epsilon}}) \subseteq B_\epsilon(0)$ for every $\epsilon < \delta$. In particular, for any $M_j \in \mathcal{M}_\delta$ we get that $\psi(A_{D\cdot\sqrt{\max_j}}) \subseteq B_{\max_j}(0)$, so $\psi^{-1}(B_{\max_j}(0)) \supseteq A_{D\cdot\sqrt{\max_j}}$. Denote $\sqrt{\max_j}$ by ϵ_j and notice that there exists a constant $L > 0$ such that for any j with $M_j \in \mathcal{M}_\delta$, it holds that

$$\begin{aligned}\mu(A_{D\epsilon_j}) &\geq L \cdot (D\epsilon_j)^{\dim X - \dim Y + r - \dim Z_i + 2(\dim Y - r)} \\ &= D' \cdot \epsilon_j^{\dim X + \dim Y - r - \dim Z_i} \\ &\geq D' \cdot \epsilon_j^{2\dim Y - r},\end{aligned}$$

where D' is some positive constant. Altogether, we have:

$$\begin{aligned}\frac{\psi_*(\mu)(M_j)}{\lambda(M_j)} &\geq \frac{\psi_*(\mu)(B_{\max_j}(0))}{\lambda(B_{\min_j}(0))} \geq \frac{1}{C}\frac{\psi_*(\mu)(B_{\max_j}(0))}{\lambda(B_{\max_j}(0))} \\ &\geq \frac{1}{C}\frac{\mu(A_{D\epsilon_j})}{\lambda(B_{\max_j}(0))} \geq \frac{D'}{C}\frac{\epsilon_j^{2\dim Y - r}}{\epsilon_j^{2\dim Y}} \geq \frac{D'}{C}\epsilon_j^{-r}.\end{aligned}$$

Since $\mathcal{M}_\delta$ is a local basis, the above equation is true for arbitrary small ϵ_j, so we have a contradiction to the $\mathcal{M}$-boundedness of $\psi_*(\mu)$. ☐

Corollary 3.6. *We have that φ is flat at x, and that there is a Zariski neighborhood U_0 of x such that $Z \cap U_0$ is reduced and a local complete intersection* (LCI).

Proof. Let $Z_1, \dots, Z_n$ be the absolutely irreducible components of Z containing x. By the previous lemma, each Z_i contains a smooth point of φ, so $\dim_x Z := \max_i \dim Z_i = \dim X - \dim Y$. Hence, we may find a neighborhood U_0 of x such that $\varphi|_{U_0}$ is flat over $\varphi(x)$ (and in particular flat at x). As a consequence, we get that $Z \cap U_0$ is an (LCI), and in particular Cohen–Macaulay. Since $Z \cap X^S \cap U_0$ is dense in $Z \cap U_0$ and $Z \cap X^S = Z^{\mathrm{sm}}$ (see, e.g., [Hartshorne 1977, III.10.2]) it follows that $Z \cap U_0$ is generically reduced. Since $Z \cap U_0$ is also Cohen–Macaulay, it now follows from (e.g., [Vakil 2017, Exercise 26.3.B]) that it is reduced. ☐

Without loss of generality, we assume $X = U_0$. The following lemma implies that φ is (FRS) at x, and thus finishes the proof of Theorem 1.6:

Lemma 3.7. *The element x is a rational singularity of Z.*

Proof. After further restricting to Zariski open neighborhoods of x and $\varphi(x)$, we may assume that X and Y are affine, with $\Omega_X^{\mathrm{top}}, \Omega_Y^{\mathrm{top}}$ free. Fix invertible top forms $\omega_X \in \Omega_X^{\mathrm{top}}[X]$, $\omega_Y \in \Omega_Y^{\mathrm{top}}[Y]$. We may find an invertible section $\eta \in \Omega_Z^{\mathrm{top}}[Z]$, such that $\eta|_{Z^{\mathrm{sm}}} = \omega_X|_{X^S}/\varphi^*(\omega_Y)$ (for more details see the last part of the proof of [Aizenbud and Avni 2016, Theorem 3.4]). We denote $\omega_Z := \eta|_{Z^{\mathrm{sm}}}$.

Fix a finite extension k'/k. By assumption, there exists a non-Archimedean local field $F \supseteq k'$ and a nonnegative Schwartz measure m on $X(F)$ that does not vanish at x, such that $\varphi_*(m)$ is bounded with respect to a local basis $\mathcal{N}$ of bounded eccentricity. Write m as $m = f \cdot |\omega_X|_F$. Since Z is an

(LCI), it is also Gorenstein, so by [Aizenbud and Avni 2016, Corollary 3.15], it is enough to prove that $\int_{X^S\cap Z(F)} f|\omega_Z|_F < \infty$ for any such k'/k and F.

Fix some embedding of X into an affine space, and let d be the metric on $X(F)$ induced from the valuation metric. Define a function $h_\epsilon : X(F) \to \mathbb{R}$ by $h_\epsilon(x') = 1$ if $d(x', (X^S(F))^C) \geq \epsilon$ and $h_\epsilon(x') = 0$ otherwise. Notice that h_ϵ is smooth, and $f \cdot h_\epsilon$ is a Schwartz function whose support lies in $X^S(F)$.

Using Proposition 2.7, we have $\varphi_*(f \cdot h_\epsilon|\omega_X|_F) = g_\epsilon|\omega_Y|_F$, where $g_\epsilon(\varphi(x)) = \int_{X^S\cap Z(F)} f \cdot h_\epsilon|\omega_Z|_F$. Note that f is nonnegative and $f \cdot h_\epsilon$ is monotonically increasing when $\epsilon \to 0$, and converges pointwise to f. By Lebesgue's monotone convergence theorem we have:

$$\int_{X^S\cap Z(F)} f|\omega_Z|_F = \lim_{\epsilon\to 0}\int_{X^S\cap Z(F)} f h_\epsilon|\omega_Z|_F = \lim_{\epsilon\to 0} g_\epsilon(\varphi(x)).$$

It is left to show that $g_\epsilon(\varphi(x))$ is bounded in ϵ and we are done. By our assumption, $\varphi_*(f \cdot |\omega_X|_F)$ is $\mathcal{N}$-bounded, so there exists $\delta > 0$ and $M > 0$ such that for all $N_i \in \mathcal{N}_\delta$,

$$\sup_i \frac{\varphi_*(f|\omega_X|_F)(N_i)}{|\omega_Y|_F(N_i)} < M.$$

Note that we used the fact that for small enough δ, $|\omega_Y|_F$ is just the normalized Haar measure up to homothety. Finally, we obtain:

$$\int_{X^S\cap Z(F)} f|\omega_Z|_F = \lim_{\epsilon\to 0} g_\epsilon(\varphi(x)) = \lim_{\epsilon\to 0}\left(\lim_{i\to\infty} \frac{\varphi_*(f \cdot h_\epsilon|\omega_X|_F)(N_i)}{|\omega_Y|_F(N_i)}\right) \leq \left(\sup_i \frac{\varphi_*(f|\omega_X|_F)(N_i)}{|\omega_Y|_F(N_i)}\right) < M.$$

□

4. Proof of the main theorem

For any prime power $q = p^r$, we denote the unique unramified extension of $\mathbb{Q}_p$ of degree r by $\mathbb{Q}_q$, its ring of integers by $\mathbb{Z}_q$, and the maximal ideal of $\mathbb{Z}_q$ by $\mathfrak{m}_q$. Recall that for a finite type $\mathbb{Z}$-scheme X and a finite ring A, we have defined $h_X(A) := |X(A)|/|A|^{\dim X_{\mathbb{Q}}}$. In this section we prove the following slightly stronger version of Theorem 1.4:

Theorem 4.1. *Let X be a scheme of finite type over $\mathbb{Z}$ such that $X_{\mathbb{Q}}$ is equidimensional and a local complete intersection. Then the following conditions are equivalent*:

(i) *For any $n \in \mathbb{N}$, $\lim_{p\to\infty} h_X(\mathbb{Z}/p^n\mathbb{Z}) = 1$.*

(ii) *There is a finite set S of prime numbers and a constant C, such that $|h_X(\mathbb{Z}/p^n\mathbb{Z}) - 1| < Cp^{-\frac{1}{2}}$ for any prime $p \notin S$ and any $n \in \mathbb{N}$.*

(iii) *$X_{\overline{\mathbb{Q}}}$ is reduced, irreducible and has rational singularities.*

(iv) *$X_{\overline{\mathbb{Q}}}$ is irreducible and there exists $C > 0$ such that $h_X(\mathbb{Z}/p^n\mathbb{Z}) < C$ for any prime p and $n \in \mathbb{N}$.*

(iv′) *$X_{\overline{\mathbb{Q}}}$ is irreducible and for any prime power q, the sequence $n \mapsto h_X(\mathbb{Z}_q/\mathfrak{m}_q^n)$ is bounded.*

(v) *$X_{\overline{\mathbb{Q}}}$ is irreducible and there exists a finite set S of primes, such that for any $p \notin S$, the sequence $n \mapsto h_X(\mathbb{Z}/p^n\mathbb{Z})$ is bounded.*

Moreover, conditions (iii), (iv), (iv′) *and* (v) *are equivalent without demanding that $X_{\overline{\mathbb{Q}}}$ is irreducible.*

We divide the proof of the theorem into two main parts that correspond to the implications (v) ⇒ (iii) (Section 4A) and (iii) ⇒ (iv′) (Section 4B). Theorem 4.1 can then be deduced as follows; the equivalence of conditions (i), (ii) and (iii) was proved in [Aizenbud and Avni 2018, Theorem 3.0.3] (see Theorem 1.1). The implications (ii) $\Rightarrow$ (v) and (iv′) $\Rightarrow$ (v) are trivial, so it follows that conditions (i), (ii), (iii), (iv′) and (v) are equivalent. The implication (iv) $\Rightarrow$ (v) is also trivial. Finally, (iv) follows from the rest of the conditions by first setting $q = p$ in (iv′) and getting that $\{h_X(\mathbb{Z}/p^n\mathbb{Z})\}_{n\in\mathbb{N}}$ is bounded for any prime p, and then by using (ii) to obtain a bound on $\{h_X(\mathbb{Z}/p^n\mathbb{Z})\}_{n\in\mathbb{N}}$ which is uniform over all primes p.

Lemma 4.2 [Aizenbud and Avni 2018, Lemma 3.1.1]. *Let $X = U_1 \cup U_2$ be an open cover of a scheme. Then for any finite local ring A, we have*:

(1) $|X(A)| = |U_1(A)| + |U_2(A)| - |U_1 \cap U_2(A)|$.

(2) $|X(A)| \geq |U_1(A)|$.

The following lemma is a consequence of Chebotarev's density theorem and Hensel's lemma.

Lemma 4.3 [Glazer and Hendel 2018, Lemma 3.15]. *Let X be a finite type $\mathbb{Z}$-scheme and let $x \in X(\overline{\mathbb{Q}})$. Then*:

(1) *There exists a finite extension k of $\mathbb{Q}$, such that $x \in X(k)$.*

(2) *For any finite extension $k/\mathbb{Q}$ as in* (1), *there exist infinitely many primes p with $i_p : k \hookrightarrow \mathbb{Q}_p$ such that $i_{p*}(x) \in X(\mathbb{Z}_p)$, where $i_{p*} : X(k) \hookrightarrow X(\mathbb{Q}_p)$.*

4A. *Boundedness implies rational singularities.*

Theorem 4.4. *Let X be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}}$ is a local complete intersection. Assume that there exists a finite set of primes S, such that for any $p \notin S$, the sequence $n \mapsto h_X(\mathbb{Z}/p^n\mathbb{Z})$ is bounded. Then $X_{\overline{\mathbb{Q}}}$ is reduced and has rational singularities.*

Proof. Step 1: Reduction to the case when $X_{\mathbb{Q}}$ is a complete intersection in an affine space (CIA).

Let $\bigcup_{i=1}^{l} \overline{X}_i$ be an affine cover of $X_{\mathbb{Q}}$, with each $\overline{X}_i$ a (CIA). For any i, there is a finite set S_i of primes, such that $\overline{X}_i$ is defined over $\mathbb{Z}[S_i^{-1}]$ and thus it has a finite type $\mathbb{Z}$-model, denoted X_i. By Lemma 4.2, for each $p \notin S_i$ we have $|X_i(\mathbb{Z}/p^n\mathbb{Z})| \leq |X(\mathbb{Z}/p^n\mathbb{Z})|$ and thus $n \mapsto h_{X_i}(\mathbb{Z}/p^n\mathbb{Z})$ is bounded for each $p \notin S_i \cup S$. By our assumption, this implies that each $(X_i)_{\overline{\mathbb{Q}}}$ is reduced and has rational singularities, and thus also $X_{\overline{\mathbb{Q}}}$.

Step 2: Proof for the case when $X_{\mathbb{Q}}$ is a (CIA).

By Proposition 2.3 we have an inclusion $\overline{\psi} : X_{\mathbb{Q}} \hookrightarrow \mathbb{A}^M_{\mathbb{Q}}$ and a morphism $\overline{\varphi} : \mathbb{A}^M_{\mathbb{Q}} \to \mathbb{A}^N_{\mathbb{Q}}$, flat over 0, such that $\overline{\psi} : X_{\mathbb{Q}} \simeq \overline{\varphi}^{-1}(0)$. As in Step 1, there exists a set S_1 of primes, and morphisms $\varphi : \mathbb{A}^M_{\mathbb{Z}[S_1^{-1}]} \to \mathbb{A}^N_{\mathbb{Z}[S_1^{-1}]}$ and $\psi : X_{\mathbb{Z}[S_1^{-1}]} \hookrightarrow \mathbb{A}^M_{\mathbb{Z}[S_1^{-1}]}$, such that $\varphi_{\mathbb{Q}} = \overline{\varphi}$, $\psi_{\mathbb{Q}} = \overline{\psi}$, φ is flat over 0, and $\psi : X_{\mathbb{Z}[S_1^{-1}]} \simeq \varphi^{-1}(0)$.

It is enough to prove that for any finite extension $k/\mathbb{Q}$ and any $y \in (\varphi^{-1}(0))(k)$, the map $\varphi_k : \mathbb{A}^M_k \to \mathbb{A}^N_k$ is (FRS) at y.

Fix $y \in (\varphi^{-1}(0))(k)$ and let k' be a finite extension of k. By Lemma 4.3, there exists an infinite set of primes T such that for any $p \in T$ we have an inclusion $i_p : k' \hookrightarrow \mathbb{Q}_p$ and $i_{p*}(y) \in \mathbb{Z}_p^M$. Choose $p \in T \setminus (S \cup S_1)$ and consider the local basis of balls $\{p^n\mathbb{Z}_p^N\}_n$ at 0, which clearly has bounded eccentricity. Let μ be the normalized Haar measure on $\mathbb{Z}_p^M$ and notice that μ does not vanish at y. By Theorem 1.6, in order to prove that $\varphi_k : \mathbb{A}^M_k \to \mathbb{A}^N_k$ is (FRS) at y it is enough to show that the sequence

$$n \mapsto \frac{((\varphi_{\mathbb{Z}_p})_*\mu)(p^n\mathbb{Z}_p^N)}{\lambda(p^n\mathbb{Z}_p^N)}$$

is bounded (for any k' and p as above), where λ is the normalized Haar measure on $\mathbb{Q}_p^N$. Consider $\pi_{N,n}:\mathbb{Z}_p^N\to(\mathbb{Z}/p^n\mathbb{Z})^N$ and notice that the following diagram is commutative:

$$\begin{tikzcd}
\mathbb{Z}_p^M \arrow[r, "\varphi_{\mathbb{Z}_p}"] \arrow[d, "\pi_{M,n}"'] & \mathbb{Z}_p^N \arrow[d, "\pi_{N,n}"] \\
(\mathbb{Z}/p^n\mathbb{Z})^M \arrow[r, "\varphi_{\mathbb{Z}/p^n}"] & (\mathbb{Z}/p^n\mathbb{Z})^N
\end{tikzcd}$$

Therefore we have

$$\mu(\varphi_{\mathbb{Z}_p}^{-1}(p^n\mathbb{Z}_p^N))=\mu(\varphi_{\mathbb{Z}_p}^{-1}\circ\pi_{N,n}^{-1}(0))=\mu(\pi_{M,n}^{-1}\circ\varphi_{\mathbb{Z}/p^n}^{-1}(0))=p^{-Mn}\cdot|\varphi_{\mathbb{Z}/p^n}^{-1}(0)|=p^{-Mn}\cdot|X(\mathbb{Z}/p^n\mathbb{Z})|,$$

and hence

$$\frac{((\varphi_{\mathbb{Z}_p})_*\mu)(p^n\mathbb{Z}_p^N)}{\lambda(p^n\mathbb{Z}_p^N)}=\frac{|X(\mathbb{Z}/p^n\mathbb{Z})|}{p^{(M-N)\cdot n}}=h_X(\mathbb{Z}/p^n\mathbb{Z})$$

is bounded and we are done. □

4B. ***Rational singularities implies boundedness.*** In the last section we proved the implication (v) ⇒ (iii) of Theorem 4.1. In this subsection we prove that (iii) implies (iv′). We divide the proof into three cases:

(1) X is a (CIA).

(2) $X_{\mathbb{Q}}$ is a (CIA).

(3) $X_{\mathbb{Q}}$ is an (LCI).

4B1. *Proof for the case that X is a* (CIA).

Proposition 4.5. *If X is a* (CIA), *then* (iii) ⇒ (iv′).

Proof. By Proposition 2.4, there exists an inclusion $X\hookrightarrow\mathbb{A}_{\mathbb{Z}}^M$ and a morphism $\varphi:\mathbb{A}_{\mathbb{Z}}^M\to\mathbb{A}_{\mathbb{Z}}^N$, flat over 0, such that $X\simeq\varphi^{-1}(0)$. Consider $\varphi_{\mathbb{Q}}:\mathbb{A}_{\mathbb{Q}}^M\to\mathbb{A}_{\mathbb{Q}}^N$ and notice that $\varphi_{\mathbb{Q}}$ is (FRS) at any $x\in\varphi_{\mathbb{Q}}^{-1}(0)(\overline{\mathbb{Q}})$, as $X_{\overline{\mathbb{Q}}}$ has rational singularities.

Let μ be the normalized Haar measure on $\mathbb{Z}_q^M$. As in the proof of Step 2 of Theorem 4.4, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}_q^M \arrow[r, "\varphi_{\mathbb{Z}_q}"] \arrow[d, "\pi_{n,M}"'] & \mathbb{Z}_q^N \arrow[d, "\pi_{n,N}"] \\
(\mathbb{Z}_q/\mathfrak{m}_q^n)^M \arrow[r, "\varphi_{\mathbb{Z}_q/\mathfrak{m}_q^n}"] & (\mathbb{Z}_q/\mathfrak{m}_q^n)^N
\end{tikzcd}$$

In order to show that $h_X(\mathbb{Z}_q/\mathfrak{m}_q^n)$ is bounded, it is enough to show that $(\varphi_{\mathbb{Z}_q})_*\mu$ has bounded density with respect to the local basis $\{p^n\mathbb{Z}_q^N\}_n$.

After base change to $\mathbb{Q}_q$, we have a map $\varphi_{\mathbb{Q}_q}:\mathbb{A}_{\mathbb{Q}_q}^M\to\mathbb{A}_{\mathbb{Q}_q}^N$, which is (FRS) at any point $x\in X(\mathbb{Q}_q)$.

For any $t\in\mathbb{N}$, consider the set $U_t=\varphi_{\mathbb{Z}_q}^{-1}(p^t\mathbb{Z}_q^N)$ and note that it is open, closed and compact. We claim that there exists $R\in\mathbb{N}$, such that for any $t>R$ we have that φ is (FRS) at any point $y\in U_t$. Indeed, otherwise we may construct a sequence $x_t\in U_t$ such that φ is not (FRS) at x_t. By a theorem of Elkik [1978] (see also [Aizenbud and Avni 2016, Theorem 6.3]), the (FRS) locus of φ is an open set. After choosing a convergent subsequence $\{x_{t_j}\}$, we obtain that $\varphi_{\mathbb{Q}_q}$ is not (FRS) at the limit $x_0 \in \mathbb{Z}_q^M$. But $\varphi_{\mathbb{Q}_q}(x_0) \in \bigcap_t \varphi_{\mathbb{Q}_q}(U_t) = \{0\}$ so $x_0 \in X(\mathbb{Q}_q)$ and we get a contradiction.

Finally, by Theorem 1.3, the measure $(\varphi_{\mathbb{Z}_q})_*\mu|_{U_R}$ has continuous density, and in particular bounded with respect to the local basis $\{p^n\mathbb{Z}_q^N\}_n$. Hence, from the definition of U_R, we have for $n > R$:

$$h_X(\mathbb{Z}_q/\mathfrak{m}_q^n) = \frac{(\varphi_{\mathbb{Z}_q})_*\mu(p^n\mathbb{Z}_q^N)}{q^{-nN}} = \frac{(\varphi_{\mathbb{Z}_q})_*\mu|_{U_R}(p^n\mathbb{Z}_q^N)}{q^{-nN}} < C,$$

for some constant $C > 0$ and we are done. □

4B2. *Some constructions.* Let X be an affine $\mathbb{Z}$-scheme with a coordinate ring

$$\mathbb{Z}[X] := \mathbb{Z}[x_1, \dots, x_c]/(f_1, \dots, f_m),$$

and fix $K \in \mathbb{N}$.

(1) For any $g \in \mathbb{Z}[x_1, \dots, x_c]$ denote by $g_K \in \mathbb{Q}[x_1, \dots, x_c]$ the function $g_K(x_1, \dots, x_c) := g\left(\frac{x_1}{K}, \dots, \frac{x_l}{K}\right)$.

(2) For any $\varphi : \mathbb{A}_{\mathbb{Z}}^M \to \mathbb{A}_{\mathbb{Z}}^N$ of the form $\varphi = (\varphi_1, \dots, \varphi_N)$, we denote by $\varphi_K : \mathbb{A}_{\mathbb{Q}}^M \to \mathbb{A}_{\mathbb{Q}}^N$ the morphism $\varphi_K := ((\varphi_1)_K, \dots, (\varphi_N)_K)$.

(3) Let $r(K) \in \mathbb{N}$ be minimal such that $K^{r(K)}(f_i)_K$ has integer coefficients for any i. Denote by $\widetilde{X}_K$ the $\mathbb{Z}$-scheme with the following coordinate ring:

$$\mathbb{Z}[\widetilde{X}_K] := \mathbb{Z}[x_1, \dots, x_c]/\left(K^{r(K)}(f_1)_K, \dots, K^{r(K)}(f_m)_K\right).$$

(4) For any $\mathbb{Q}$-morphism $\psi : X_{\mathbb{Q}} \to \mathbb{A}_{\mathbb{Q}}^M$ of the form $\psi = (\psi_1, \dots, \psi_N)$ let $K\psi$ denote $(K \cdot \psi_1, \dots, K \cdot \psi_N)$.

(5) For any affine $\mathbb{Q}$-scheme Z, with $\mathbb{Q}[Z] = \mathbb{Q}[y_1, \dots, y_d]/(g_1, \dots, g_k)$ and a $\mathbb{Q}$-morphism $\phi : Z \to X_{\mathbb{Q}}$, we may define a morphism $K\phi : Z \to (\widetilde{X}_K)_{\mathbb{Q}}$ by $K\phi(y_1, \dots, y_d) := K \cdot \phi(y_1, \dots, y_d)$.

4B3. *Proof for the case that $X_{\mathbb{Q}}$ is a* (CIA). In this case, we have an inclusion $\psi : X_{\mathbb{Q}} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^M$ and a morphism $\varphi : \mathbb{A}_{\mathbb{Q}}^M \to \mathbb{A}_{\mathbb{Q}}^N$, flat over 0, such that $X_{\mathbb{Q}} \simeq \varphi^{-1}(0)$.

Lemma 4.6. *Let X be a finite type $\mathbb{Z}$-scheme, such that $X_{\mathbb{Q}}$ is a* (CIA), *defined by the morphisms φ, ψ as above. Then there exists a $\mathbb{Z}$-scheme $\widehat{X}_{\varphi,\psi}$, which is a* (CIA), *and a $\mathbb{Z}$-morphism $\phi : X \to \widehat{X}_{\varphi,\psi}$, such that $\phi_{\mathbb{Q}}$ is an isomorphism.*

Proof. Let $\mathbb{Z}[X] := \mathbb{Z}[x_1, \dots, x_c]/(f_1, \dots, f_m)$ be the coordinate ring of X. Denote by $S = \{p_1, \dots, p_s\}$ the set of all prime numbers that appear in the denominators of the polynomial maps ψ and φ, and set $P' := \prod_{p_i \in S} p_i$. Let $t \in \mathbb{N}$ be minimal such that $(P')^t\psi$ has integer coefficients. Denote $P := (P')^t$ and notice that $P\psi$ is a $\mathbb{Z}$-morphism. Let $\varphi_P : \mathbb{A}_{\mathbb{Q}}^M \to \mathbb{A}_{\mathbb{Q}}^N$ be as defined in 4B2. Notice that there exists $m \in \mathbb{N}$ such that $P^m\varphi_P$ has coefficients in $\mathbb{Z}$. We now have the following $\mathbb{Z}$-morphisms:

$$X \xrightarrow{P\psi} \mathbb{A}_{\mathbb{Z}}^M \xrightarrow{P^m\varphi_P} \mathbb{A}_{\mathbb{Z}}^N.$$

Set $\widehat{X}_{\varphi,\psi}$ to be the fiber $(P^m\varphi_P)^{-1}(0)$ and notice that $\phi := P\psi$ is a $\mathbb{Z}$-morphism from X to $\widehat{X}_{\varphi,\psi}$, such that $\phi_{\mathbb{Q}}$ is an isomorphism, and $\widehat{X}_{\varphi,\psi}$ is a (CIA). □

Lemma 4.7. *Let X and Y be affine $\mathbb{Z}$-schemes and $\phi : X \to Y$ be a $\mathbb{Z}$-morphism, such that $\phi_{\mathbb{Q}}$ is an isomorphism. Then there exist $c, N \in \mathbb{N}$, such that for any prime power q and any n*:

$$|X(\mathbb{Z}_q/\mathfrak{m}_q^n)| \le q^{N \cdot c} \cdot |Y(\mathbb{Z}_q/\mathfrak{m}_q^n)|.$$

Proof. The morphism ϕ induces a map $\phi_n : X(\mathbb{Z}_q/\mathfrak{m}_q^n) \to Y(\mathbb{Z}_q/\mathfrak{m}_q^n)$. It is enough to show that ϕ_n has fibers of size at most $q^{N\cdot c}$. Assume that $\mathbb{Z}[X] = \mathbb{Z}[x_1, \dots, x_c]/(f_1, \dots, f_m)$. As in Section 4B2, we may choose $K, r(K) \in \mathbb{N}$ such that $\widetilde{X}_K$ is a $\mathbb{Z}$-scheme with a coordinate ring

$$\mathbb{Z}[\widetilde{X}_K] := \mathbb{Z}[x_1, \dots, x_c]/\big(K^{r(K)}(f_1)_K, \dots, K^{r(K)}(f_m)_K\big),$$

and $K\phi^{-1} : Y \to \widetilde{X}_K$ is a $\mathbb{Z}$-morphism. The map $(K\phi^{-1} \circ \phi) : X \to \widetilde{X}_K$ is just coordinatewise multiplication by K. Thus $(K\phi^{-1})_n \circ \phi_n : X(\mathbb{Z}_q/\mathfrak{m}_q^n) \to \widetilde{X}_K(\mathbb{Z}_q/\mathfrak{m}_q^n)$ sends $(a_1, \dots, a_c) \in X(\mathbb{Z}_q/\mathfrak{m}_q^n)$ to $(Ka_1, \dots, Ka_c) \in \widetilde{X}_K(\mathbb{Z}_q/\mathfrak{m}_q^n)$.

For any prime p, let $N(p)$ be the maximal integer such that $p^{N(p)} | K$. Note that the map $(a_1, \dots, a_n) \mapsto (Ka_1, \dots, Ka_n)$ from $(\mathbb{Z}_q/\mathfrak{m}_q^n)^c$ to $(\mathbb{Z}_q/\mathfrak{m}_q^n)^c$ has fibers of size $q^{N(p)\cdot c}$ for $n > N(p)$. Indeed, for $(b_1, \dots, b_c) \in (\mathbb{Z}_q/\mathfrak{m}_q^n)^c$, $(Ka_1, \dots, Ka_c) = (b_1, \dots, b_c)$ if and only if $Ka_i = b_i$ for any $1 \le i \le c$. Since $K/p^{N(p)}$ is invertible in $\mathbb{Z}_q/\mathfrak{m}_q^n$, it is equivalent to demand that $p^{N(p)}a_i = c_i$ for some multiple c_i of b_i by an invertible element. Hence, we can reduce to the case of the map $(a_1, \dots, a_c) \mapsto (p^{N(p)}a_1, \dots, p^{N(p)}a_c)$, which clearly has fibers of size $q^{N(p)\cdot c}$ for $n > N(p)$. Note that for any $y \in Y(\mathbb{Z}_q/\mathfrak{m}_q^n)$ we have $|\phi_n^{-1}(y)| \le \big|((K\phi^{-1})_n \circ \phi_n)^{-1}(x)\big|$, where $x = (K\phi^{-1})_n(y)$. Since the fibers of $(K\phi^{-1})_n \circ \phi_n$ are of size bounded by $q^{N(p)c}$, so are the fibers of ϕ_n. We may take $N := K > N(p)$ and we are done. □

Corollary 4.8. *Let X be a finite type $\mathbb{Z}$-scheme such that $X_{\mathbb{Q}}$ is a* (CIA). *Then condition* (iii) *of Theorem 4.1 implies condition* (iv′).

Proof. By Lemma 4.6, we may choose a $\mathbb{Z}$-scheme $\widehat{X}$, which is a (CIA), and a $\mathbb{Z}$-morphism $\phi : X \to \widehat{X}$, such that $\phi_{\mathbb{Q}}$ is an isomorphism. By Proposition 4.5 and Lemma 4.7, there exists $c, N \in \mathbb{N}$, such that for any prime power q, there exists $C > 0$ such that:

$$h_X(\mathbb{Z}_q/\mathfrak{m}_q^n) = \frac{|X(\mathbb{Z}_q/\mathfrak{m}_q^n)|}{q^{n\dim X_{\mathbb{Q}}}} \le q^{c\cdot N} \cdot \frac{|\widehat{X}(\mathbb{Z}_q/\mathfrak{m}_q^n)|}{q^{n\dim X_{\mathbb{Q}}}} \le q^{c\cdot N} \cdot C,$$

and hence condition (iv′) holds. □

4B4. *Proof for the case when $X_{\mathbb{Q}}$ is an* (LCI). Using Lemma 4.2, we may reduce to the case when X is affine, with coordinate ring $\mathbb{Z}[X] := \mathbb{Z}[x_1, \dots, x_c]/(f_1, \dots, f_m)$. Since $X_{\mathbb{Q}}$ is an (LCI), we have an affine open cover $\{\beta_i : U_i \hookrightarrow X_{\mathbb{Q}}\}_i$ of $X_{\mathbb{Q}}$ with inclusions $\psi_i : U_i \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{M_i}$ and maps $\varphi_i : \mathbb{A}_{\mathbb{Q}}^{M_i} \to \mathbb{A}_{\mathbb{Q}}^{N_i}$, flat over 0, such that $\psi_i : U_i \simeq \varphi_i^{-1}(0)$. We may assume that U_i is isomorphic to a basic open set $D(g_i)$ for $g_i \in \mathbb{Q}[X]$ and $\beta_i^* : \mathbb{Q}[X] \to \mathbb{Q}[X, t]/(g_i t - 1)$ is the natural map. Since $\{D(g_i)\}_i$ is a cover of $X_{\mathbb{Q}}$, there exist $c_i' \in \mathbb{Z}[X]$ and $d_i \in \mathbb{Z}$ such that $\sum c_i' \cdot g_i/d_i = 1$. Thus, by multiplying by all the d_i's, we obtain $\sum c_i g_i = D$ for some $c_i \in \mathbb{Z}[X]$ and $D \in \mathbb{Z}$. Choose large enough $P \in \mathbb{N}$ such that the following algebra

$$\mathbb{Z}[x_1, \dots, x_c, t]/(f_1, \dots, f_m, Pg_i t - D\cdot P)$$

is a coordinate ring of a $\mathbb{Z}$-scheme $\widetilde{U}_i$, for any i. Moreover, notice that $\widetilde{U}_i \simeq U_i$ over $\mathbb{Q}$.

Lemma 4.9. *There exists $N \in \mathbb{N}$, such that for any prime power $q = p^r$ and any $n > N$ we have*

$$|X(\mathbb{Z}_q/\mathfrak{m}_q^n)| \le \sum_i |\widetilde{U}_i(\mathbb{Z}_q/\mathfrak{m}_q^n)|.$$

Proof. Let $N(p)$ be the maximal integer such that $p^{N(p)} \mid D\cdot P$. We first claim that for any $n > N(p)+1$ and $(a_1, \dots, a_c) \in X(\mathbb{Z}_q/\mathfrak{m}_q^n)$, there exists some i such that $Pg_i(a_1, \dots, a_c) \notin \mathfrak{m}_q^{N(p)+1}/\mathfrak{m}_q^n$. Indeed, if

$Pg_i(a_1,\dots,a_c) \in \mathfrak{m}_q^{N(p)+1}/\mathfrak{m}_q^n$ for any i, then $\sum Pg_i(a_1,\dots,a_c)\cdot c_i(a_1,\dots,a_c) = D\cdot P \in \mathfrak{m}_q^{N(p)+1}/\mathfrak{m}_q^n$ and hence $p^{N(p)+1} \mid D\cdot P$ leading to a contradiction. Set $N := D\cdot P+1$ and notice that $N > N(p)+1$ for any prime p. Fix $n > N$ and let i such that $Pg_i(a_1,\dots,a_c) \notin \mathfrak{m}_q^{N(p)+1}/\mathfrak{m}_q^n$. We now claim that the equation $Pg_i(a_1,\dots,a_c)t - PD = 0$ has a solution in $\mathbb{Z}_q/\mathfrak{m}_q^n$. Indeed, if $Pg_i(a_1,\dots,a_c)$ is invertible in $\mathbb{Z}_q/\mathfrak{m}_q^n$, we are done. Otherwise, we have that $Pg_i(a_1,\dots,a_c) = p^l\cdot b \in \mathfrak{m}_q^l/\mathfrak{m}_q^n$ for some $l \leq N(p)$, where b is invertible. Write $PD = p^l\cdot a$. We can rewrite the equation as $p^l\cdot(bt-a) = 0$, which has a solution $d \in \mathbb{Z}_q/\mathfrak{m}_q^n$ since b is invertible. We see that for any $n > N$ and any $(a_1,\dots,a_c) \in X(\mathbb{Z}_q/\mathfrak{m}_q^n)$ there exists i and $d \in \mathbb{Z}_q/\mathfrak{m}_q^n$ such that $(a_1,\dots,a_c,d) \in \widetilde{U}_i(\mathbb{Z}_q/\mathfrak{m}_q^n)$. This implies the lemma. $\square$

Since $(\widetilde{U}_i)_{\mathbb{Q}} \simeq U_i$ is a (CIA) for any i, we obtain

$$h_X(\mathbb{Z}_q/\mathfrak{m}_q^n) = q^{-n\dim X_{\mathbb{Q}}}\cdot|X(\mathbb{Z}_q/\mathfrak{m}_q^n)| \leq \sum_i q^{-n\dim X_{\mathbb{Q}}}\cdot|\widetilde{U}_i(\mathbb{Z}_q/\mathfrak{m}_q^n)| < \sum C_i,$$

where $C_i = \sup_n h_{\widetilde{U}_i}(\mathbb{Z}_q/\mathfrak{m}_q^n)$. The implication (iii) $\Rightarrow$ (iv$'$) of Theorem 4.1 now follows.

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