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The Maillot–Rössler current and the polylogarithm on abelian schemes

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We give a structural proof of the fact that the realization of the degree-zero part of the polylogarithm on abelian schemes in analytic Deligne cohomology can be described in terms of the Bismut–Köhler higher analytic torsion form of the Poincaré bundle. Furthermore, we provide a new axiomatic characterization of the arithmetic Chern character of the Poincaré bundle using only invariance properties under isogenies. For this we obtain a decomposition result for the arithmetic Chow group of independent interest.

Introduction

In an important contribution Maillot and Rössler constructed a Green current $\mathfrak{g}_{\mathcal{A}^\vee}$ for the zero section of an abelian scheme $\mathcal{A}$ which is norm compatible (i.e., $[n]_*\mathfrak{g}_{\mathcal{A}^\vee} = \mathfrak{g}_{\mathcal{A}^\vee}$) and is the push-forward of the arithmetic Chern character of the (canonically metrized) Poincaré bundle. In particular, on the complement of the zero section the Green current $\mathfrak{g}_{\mathcal{A}^\vee}$ is the degree-$(g-1)$ part of the analytic torsion form of the Poincaré bundle. Moreover, certain linear combinations of translates of these currents are even motivic in the sense that their classes in analytic Deligne cohomology are in the image of the regulator from motivic cohomology.

In the special case of a family of elliptic curves, the current $\mathfrak{g}_{\mathcal{A}^\vee}$ is described by a Siegel-function whose usefulness for many arithmetic problems (in particular for special values of L-functions and Iwasawa theory) is well known and one could hope that the Maillot–Rössler current plays a similar role for abelian schemes.

On the other hand the first author has constructed the motivic polylogarithm $\mathrm{pol}^0 \in H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g)$ of the abelian scheme $\mathcal{A}$ without its N-torsion points $\mathcal{A}\setminus\mathcal{A}[N]$ [Kings and Rössler 2017]. Here g is the relative dimension of $\mathcal{A}$. The polylogarithm is also norm-compatible $[n]_*\,\mathrm{pol}^0 = \mathrm{pol}^0$ for n coprime to N, and in the elliptic case it is directly related to Siegel functions and modular units.

It is natural to ask how pol^0 is related to $\mathfrak{g}_{\mathcal{A}^\vee}$. This question was answered completely in [Kings and Rössler 2017], and it turns out that the image of $-2\,\mathrm{pol}^0$ in analytic Deligne cohomology is the Maillot–Rössler current $[N]^*\mathfrak{g}_{\mathcal{A}^\vee} - N^{2g}\mathfrak{g}_{\mathcal{A}^\vee}$. Due to the fact that in analytic Deligne cohomology there is no residue sequence, the proof of this fact in [Kings and Rössler 2017] was much more complicated than

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it should be and proceeded by a reduction to the case of a product of elliptic curves via the moduli space of abelian varieties and an explicit computation.

In this paper we give a much simpler and very structural proof of the identity between the polylogarithm and the Maillot–Rössler current (see Theorem 8). We circumvent the difficulties of the approach in [Kings and Rössler 2017] by working in Betti cohomology instead of analytic Deligne cohomology. As a result one only has to compare the residues of the classes. In fact we achieve much more and give an axiomatic characterization of the Maillot–Rössler current which does not involve the Poincaré bundle (see Theorem 11). More precisely, we prove that any class $\hat{\xi} \in \widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ in the arithmetic Chow group which satisfies that its image in the Chow group $\mathrm{CH}^g(\mathcal{A})_{\mathbb{Q}}$ is the zero section and such that

$$([n]^* - n^{2g})(\hat{\xi}) = 0 \quad \text{in } \widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$$

holds for some $n \geq 2$ is in fact equal to $(-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{[g]}$. This characterization of the Maillot–Rössler current relies on a decomposition into generalized eigenspaces for the action of $[n]^*$ on the arithmetic groups $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$, which might be of independent interest (see Corollary 10).

Here is a short synopsis of our paper. In Section 1 we give some background on motivic cohomology and arithmetic Chow groups. In Section 2 we review the polylogarithm and the Maillot–Rössler current. In Section 3 we carry out the comparison between the Maillot–Rössler current and the polylogarithm. In Section 4 we prove a decomposition of the arithmetic Chow group, and in Section 5 we give an axiomatic characterization of the Maillot–Rössler current.

1. Preliminaries on motivic cohomology, Arakelov theory, and Deligne cohomology

Motivic cohomology. Let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension g, let $\varepsilon : S \to \mathcal{A}$ be the zero section, let $N > 1$ be an integer, and let $\mathcal{A}[N]$ be the finite group scheme of N-torsion points. Here S is smooth over a subfield k of the complex numbers. We will write S_0 for the image of ε in $\mathcal{A}$. We denote by $\mathcal{A}^\vee$ the dual abelian scheme of $\mathcal{A}$ and by $\varepsilon^\vee$ its zero section.

C. Soulé [1985] and A. Beilinson [1985] defined motivic cohomology for any variety V over a field

$$H^i_{\mathcal{M}}(V, j) := \mathrm{Gr}^j_\gamma K_{2j-i}(V) \otimes \mathbb{Q}.$$

Remark. In this paper we work with the above rather old-fashioned definition of motivic cohomology for the compatibility with earlier references. This and the requirement that S is smooth over a base field k are not necessary. The latter condition does no harm as we are mainly interested in the arithmetic Chow groups and Deligne cohomology. For a much more general setting we refer to the paper [Huber and Kings 2018] where also the decomposition of motivic cohomology is considered in an up-to-date fashion.

For any integer $a > 1$ and any $W \subseteq \mathcal{A}$ open subscheme such that

$$j : [a]^{-1}(W) \hookrightarrow W$$

is an open immersion (here $[a] : \mathscr{A} \to \mathscr{A}$ is the a-multiplication on $\mathscr{A}$), the trace map with respect to a is defined as

$$\mathrm{tr}_{[a]} : H^{\cdot}_{\mathcal{M}}(W, *) \xrightarrow{j^*} H^{\cdot}_{\mathcal{M}}([a]^{-1}(W), *) \xrightarrow{[a]_*} H^{\cdot}_{\mathcal{M}}(W, *). \tag{1}$$

For any integer r we let

$$H^{\cdot}_{\mathcal{M}}(W, *)^{(r)} := \{\psi \in H^{\cdot}_{\mathcal{M}}(W, *) \mid (\mathrm{tr}_{[a]} - a^r \,\mathrm{Id})^k \psi = 0 \text{ for some } k \geq 1\}$$

be the generalized eigenspace of $\mathrm{tr}_{[a]}$ of degree (or weight) r. One can prove that there is a decomposition into $\mathrm{tr}_{[a]}$-eigenspaces

$$H^{\cdot}_{\mathcal{M}}(\mathscr{A}, *) \cong \bigoplus_{r=0}^{2g} H^{\cdot}_{\mathcal{M}}(\mathscr{A}, *)^{(r)}$$

which is independent of a and that

$$H^{\cdot}_{\mathcal{M}}(\mathscr{A} \setminus S_0, *)^{(0)} = 0$$

(see Proposition 2.2.1 in [Kings and Rössler 2017]).

Arithmetic varieties. An arithmetic ring is a triple (R, Σ, F_∞) where

- R is an excellent regular Noetherian integral domain,
- Σ is a finite nonempty set of monomorphisms $\sigma : R \to \mathbb{C}$,
- F_∞ is an antilinear involution of the $\mathbb{C}$−algebra $\mathbb{C}^\Sigma := \mathbb{C} \underbrace{\times \cdots \times}_{|\Sigma|} \mathbb{C}$, such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{\delta} & \mathbb{C}^\Sigma \\ {\scriptstyle \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle F_\infty} \\ R & \xrightarrow{\delta} & \mathbb{C}^\Sigma \end{array}$$

  commutes (here by δ we mean the natural map to the product induced by the family of maps Σ).

An arithmetic variety X over R is a scheme of finite type over R, which is flat, quasiprojective, and regular. As usual we write

$$X(\mathbb{C}) := \coprod_{\sigma \in \Sigma} (X \times_{R,\sigma} \mathbb{C})(\mathbb{C}).$$

Note that F_∞ induces an involution $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.

Arithmetic Chow groups. Let $p \in \mathbb{N}$. We denote by

- $E^{p,p}(X_\mathbb{R})$ the $\mathbb{R}$-vector space of smooth real forms ω on $X(\mathbb{C})$ of type (p, p) such that $F_\infty^* \zeta = (-1)^p \omega$,
- $\widetilde{E}^{p,p}(X_\mathbb{R})$ the quotient $E^{p,p}(X_\mathbb{R})/(\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial)$,
- $D^{p,p}(X_\mathbb{R})$ the $\mathbb{R}$-vector space of real currents ζ on $X(\mathbb{C})$ of type (p, p) such that $F_\infty^* \zeta = (-1)^p \zeta$,
- $\widetilde{D}^{p,p}(X_\mathbb{R})$ the quotient $D^{p,p}(X_\mathbb{R})/(\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial)$.

If ω or ζ is a form in $E^{p,p}(X_{\mathbb{R}})$ or a current in $D^{p,p}(X_{\mathbb{R}})$, we write $\tilde{\omega}$ or $\tilde{\zeta}$ for its class in $\widetilde{E}^{p,p}(X_{\mathbb{R}})$ or $\widetilde{D}^{p,p}(X_{\mathbb{R}})$, respectively.

We briefly recall the definition of the arithmetic Chow groups of X, as given in [Gillet and Soulé 1990, §3.3]. Let $Z^q(X)$ denote the group of cycles of codimension q in X and $\mathrm{CH}^q(X)$ denote the q-th Chow group of X. We write $\widehat{Z}^q(X)$ for the subgroup of

$$Z^q(X) \oplus \widetilde{D}^{q-1,q-1}(X_{\mathbb{R}})$$

consisting of pairs $(z, \tilde{h})$ where $z \in Z^q(X)$ and $h \in D^{q-1,q-1}(X_{\mathbb{R}})$ satisfy

$$\mathrm{dd}^c\, h + \delta_z \in E^{q,q}(X_{\mathbb{R}}).$$

By definition, the class $\tilde{h}$ is then a Green current for z. Note that if $\tilde{h}$ is a Green current for z, the form $\mathrm{dd}^c\, h + \delta_z$ is closed.

For any codimension-$(q-1)$ integral subscheme $i : W \hookrightarrow X$ and any $f \in k(W)^*$, one can verify, by means of the Poincaré–Lelong lemma, that the pair

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -i_* \log|f|^2)$$

is an element in $\widehat{Z}^q(X)$. Then the q-th arithmetic Chow group of X is the quotient

$$\widehat{\mathrm{CH}}^q(X) := \widehat{Z}^q(X)/\widehat{R}^q(X)$$

where $\widehat{R}^q(X)$ is the subgroup generated by all pairs $\widehat{\mathrm{div}}(f)$, for any $f \in k(W)^*$ and any $W \subset X$ as above. If $Z^{q,q}(X_{\mathbb{R}}) \subseteq E^{q,q}(X_{\mathbb{R}})$ denotes the subspace of closed forms, we have a well defined map

$$\omega : \widehat{\mathrm{CH}}^q(X) \to Z^{q,q}(X_{\mathbb{R}})$$

sending the class of $(z, \tilde{h})$ to $\mathrm{dd}^c\, h + \delta_z$. Finally we have a map

$$\zeta : \widehat{\mathrm{CH}}^q(X) \to \mathrm{CH}^q(X)$$

sending the class of $(z, \tilde{h})$ to the class of z.

Analytic Deligne cohomology of arithmetic varieties. If X is an arithmetic variety over R we write

$$\mathrm{H}^q_{\mathcal{D}^{\mathrm{an}}}(X_{\mathbb{R}}, \mathbb{R}(p)) := \{\gamma \in \mathrm{H}^q_{\mathcal{D}^{\mathrm{an}}}(X(\mathbb{C}), \mathbb{R}(p)) \mid F_\infty^* \gamma = (-1)^p \gamma\},$$

where $\mathrm{H}^*_{\mathcal{D}^{\mathrm{an}}}(X(\mathbb{C}), \mathbb{R}(p))$ is the analytic Deligne cohomology of the complex manifold $X(\mathbb{C})$, i.e., the hypercohomology of the complex

$$0 \to (2\pi i)^p \mathbb{R} \to \mathcal{O}_{X(\mathbb{C})} \xrightarrow{d} \Omega^1_{X(\mathbb{C})} \to \cdots \to \Omega^{p-1}_{X(\mathbb{C})} \to 0$$

($\Omega^*_{X(\mathbb{C})}$ denotes the de Rham complex of holomorphic forms on $X(\mathbb{C})$). In the following sections we will need the characterization [Burgos 1997, §2]

$$\mathrm{H}^{2p-1}_{\mathcal{D}^{\mathrm{an}}}(X_{\mathbb{R}}, \mathbb{R}(p)) = \{\tilde{x} \in (2\pi i)^{p-1} \widetilde{E}^{p-1,p-1}(X_{\mathbb{R}}) \mid \partial\bar{\partial} x = 0\}. \tag{2}$$

Analytic Deligne cohomology and Betti cohomology. By definition of analytic Deligne cohomology there is a canonical map to Betti cohomology

$$\phi_B : H^{2g-1}_{D^{\mathrm{an}}}((\mathscr{A} \setminus \mathscr{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \to H^{2g-1}_B((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), \mathbb{R}(g)).$$

Later we will need an explicit description of this map: first we compute the group $H^{2g-1}_B((\mathscr{A}\setminus\mathscr{A}[N])(\mathbb{C}), g)$ with the cohomology of the complex of currents $D^*((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g) := (2\pi i)^g D^*((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}))$, so

$$H^{2g-1}_B((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g) = \frac{\{\eta \in D^{2g-1}((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g) \mid d\eta = 0\}}{\{d\omega \mid \omega \in D^{2g-2}((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g)\}}.$$

Lemma 1. *Using the description (2), the map ϕ_B sends the class $\tilde{x}$ of $x \in (2\pi i)^{g-1} E^{g-1,g-1}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}})$ with $\partial\bar{\partial}x = 0$ to*

$$\phi_B(\tilde{x}) = [4\pi i \mathrm{d}^c x]. \tag{3}$$

Proof. This is [Burgos 1997, Theorem 2.6]. □

We also need an explicit description of the connecting homomorphism, which we call the residue homomorphism

$$\mathrm{res}_B : H^{2g-1}_B((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), \mathbb{R}(g)) \to H^{2g}_{B,\mathscr{A}[N]\setminus S_0}((\mathscr{A} \setminus S_0)(\mathbb{C}), \mathbb{R}(g)).$$

For this we compute $H^{2g}_{B,\mathscr{A}[N]\setminus S_0}((\mathscr{A} \setminus S_0)(\mathbb{C}), \mathbb{R}(g))$ with the cohomology of the simple complex of the restriction morphism $D^*((\mathscr{A} \setminus S_0)(\mathbb{C}), g) \to D^*((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g)$ and get

$$\begin{aligned} &H^{2g}_{B,\mathscr{A}[N]\setminus S_0}((\mathscr{A} \setminus S_0)(\mathbb{C}), \mathbb{R}(g)) \\ &\quad = \frac{\{(\xi, \tau) \in D^{2g}((\mathscr{A} \setminus S_0)(\mathbb{C}), g) \oplus D^{2g-1}((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g) \mid d\xi = 0 \text{ and } \xi|_{\mathscr{A}\setminus\mathscr{A}[N]} = d\tau\}}{\{(d\theta, \theta|_{\mathscr{A}\setminus\mathscr{A}[N]} - d\alpha) \mid \theta \in D^{2g-1}((\mathscr{A} \setminus S_0)(\mathbb{C}), g),\ \alpha \in D^{2g-2}((\mathscr{A} \setminus \mathscr{A}[N])(\mathbb{C}), g)\}}. \end{aligned}$$

Note that we are using the simple complex as in [Burgos 1997, §1] (and not the cone in the sense of Verdier) of the restriction morphism to compute cohomology with support. From the definitions one gets immediately:

Lemma 2. *The residue* res_B *sends the class of* η, *which we denote by* $[\eta]$, *to*

$$\mathrm{res}_B([\eta]) = [0, -\eta],$$

where $[0, -\eta]$ *denotes the class of* $(0, -\eta)$.

2. Review of the polylog and the Maillot–Rössler current

***The axiomatic definition of* $\mathbf{pol}^0$.** G. Kings and D. Rössler [2017] have provided a simple axiomatic description of the degree-zero part of the polylogarithm on abelian schemes. We briefly recall it here.

The degree-zero part of the motivic polylogarithm is by definition a class in motivic cohomology

$$\mathrm{pol}^0 \in H^{2g-1}_{\mathscr{M}}(\mathscr{A} \setminus \mathscr{A}[N], g)^{(0)}.$$

To describe it more precisely, consider the residue map along $\mathscr{A}[N]$

$$H^{2g-1}_{\mathscr{M}}(\mathscr{A}\setminus\mathscr{A}[N], g) \to H^0_{\mathscr{M}}(\mathscr{A}[N]\setminus S_0, 0).$$

This map induces an isomorphism

$$\mathrm{res}: H^{2g-1}_{\mathscr{M}}(\mathscr{A}\setminus\mathscr{A}[N], g)^{(0)} \cong H^0_{\mathscr{M}}(\mathscr{A}[N]\setminus S_0, 0)^{(0)}$$

(see Corollary 2.2.2 in [Kings and Rössler 2017]).

Definition 3. The degree-zero part of the polylog pol^0 is the unique element of $H^{2g-1}_{\mathscr{M}}(\mathscr{A}\setminus\mathscr{A}[N], g)^{(0)}$ mapping under res to the fundamental class 1°_N of $\mathscr{A}[N]\setminus S_0$.

We recall now that we have a map $\mathrm{reg}_{\mathrm{an}}$ defined as the composition

$$H^{2g-1}_{\mathscr{M}}(\mathscr{A}\setminus\mathscr{A}[N], g) \xrightarrow{\mathrm{reg}} \mathrm{H}^{2g-1}_{D}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{\mathrm{forget}} \mathrm{H}^{2g-1}_{D^{\mathrm{an}}}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$$

where reg is the regulator map into Deligne–Beilinson cohomology and the second map is the forgetful map from Deligne–Beilinson cohomology to analytic Deligne cohomology.

The Maillot–Rössler current $\mathfrak{g}_{\mathscr{A}^\vee}$**.** V. Maillot and D. Rössler [2015] proved the following theorem.

Theorem 4 [Maillot and Rössler 2015, Theorem 1.1]. *There exists a unique a class of currents* $\mathfrak{g}_{\mathscr{A}^\vee} \in \widetilde{D}^{g-1,g-1}(\mathscr{A}_{\mathbb{R}})$ *which satisfies the following three properties*:

(i) $\mathfrak{g}_{\mathscr{A}^\vee}$ *is a Green current for* S_0,

(ii) $(S_0, \mathfrak{g}_{\mathscr{A}^\vee}) = (-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathscr{P}}))^{[g]}$ *in the group* $\widehat{\mathrm{CH}}^g(\mathscr{A})_{\mathbb{Q}}$,

(iii) $[n]_*\mathfrak{g}_{\mathscr{A}^\vee} = \mathfrak{g}_{\mathscr{A}^\vee}$ *for all* $n > 0$.

Here we take $\overline{\mathscr{P}}$ to be the Poincaré bundle on $\mathscr{A}\times_S\mathscr{A}^\vee$ equipped with a canonical hermitian metric, $p_1 : \mathscr{A}\times_S\mathscr{A}^\vee \to \mathscr{A}$ is the first projection, and $\widehat{\mathrm{CH}}^g(\mathscr{A})$ denotes the g-th arithmetic Chow group of $\mathscr{A}$. The term $\widehat{\mathrm{ch}}(\overline{\mathscr{P}}) \in \bigoplus_i \widehat{\mathrm{CH}}^i(\mathscr{A}\times_S\mathscr{A}^\vee)$ is the arithmetic Chern character of $\overline{\mathscr{P}}$, and $p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathscr{P}}))^{[g]}$ denotes the homogeneous component of degree g of $p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathscr{P}}))$ in the graded ring $\bigoplus_i \widehat{\mathrm{CH}}^i(\mathscr{A})$.

We now consider the arithmetic cycle

$$({}_N S_0, {}_N\,\mathfrak{g}_{\mathscr{A}^\vee}) := ([N]^* - N^{2g})(S_0, \mathfrak{g}_{\mathscr{A}^\vee}).$$

Thanks to the geometry of the Poincaré bundle, one can show that the class of $({}_N S_0, {}_N\,\mathfrak{g}_{\mathscr{A}^\vee})$ in $\widehat{\mathrm{CH}}^g(\mathscr{A})_{\mathbb{Q}}$ is zero [Scarponi 2017, Proposition 5.2]. In particular, $dd^c({}_N\mathfrak{g}_{\mathscr{A}^\vee}|_{\mathscr{A}\setminus\mathscr{A}[N]}) = 0$, and by Theorem 1.2.2(i) in [Gillet and Soulé 1990], there exists a smooth form in the class of currents ${}_N\mathfrak{g}_{\mathscr{A}^\vee}|_{\mathscr{A}\setminus\mathscr{A}[N]}$. Equivalently, ${}_N\mathfrak{g}_{\mathscr{A}^\vee}|_{\mathscr{A}\setminus\mathscr{A}[N]}$ lies in the image of the inclusion

$$\widetilde{E}^{g-1,g-1}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}}) \hookrightarrow \widetilde{D}^{g-1,g-1}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}}).$$

The group $H^{2g-1}_{D^{\mathrm{an}}}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}}, \mathbb{R}(g))$ can be represented by classes in $(2\pi i)^{g-1}\widetilde{E}^{g-1,g-1}((\mathscr{A}\setminus\mathscr{A}[N])_{\mathbb{R}})$ with dd^c equal to zero by (2), so we get:

Lemma 5. *The Maillot–Rössler current defines a class*

$$(2\pi i)^{g-1}({}_N\mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A}\setminus\mathcal{A}[N]} \in H^{2g-1}_{D^{\mathrm{an}}}((\mathcal{A}\setminus\mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

The exact sequence (see the theorem and remark in [Gillet and Soulé 1990, §3.3.5])

$$H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g) \xrightarrow{\mathrm{reg}_{\mathrm{an}}} H^{2g-1}_{D^{\mathrm{an}}}((\mathcal{A}\setminus\mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \xrightarrow{r} \widehat{\mathrm{CH}}^{g}(\mathcal{A}\setminus\mathcal{A}[N])_{\mathbb{Q}}$$

where r sends $\tilde{x}$ to the class of $(0, \tilde{x}/(2\pi i)^{g-1})$, with the vanishing of $({}_N S_0, {}_N\mathfrak{g}_{\mathcal{A}^\vee})$ in $\widehat{\mathrm{CH}}^{g}(\mathcal{A}\setminus\mathcal{A}[N])_{\mathbb{Q}}$, then implies that the Maillot–Rössler current is motivic, i.e.,

$$(2\pi i)^{g-1}({}_N\mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A}\setminus\mathcal{A}[N]} \in \mathrm{reg}_{\mathrm{an}}(H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g)).$$

Since the operator $\mathrm{tr}_{[a]}$ defined in (1) obviously operates on analytic Deligne cohomology and the map $\mathrm{reg}_{\mathrm{an}}$ intertwines this operator with $\mathrm{tr}_{[a]}$, we deduce from Theorem 4(iii) the fact:

Lemma 6. *The Maillot–Rössler current is in the image of the regulator from* $H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g)^{(0)}$:

$$(2\pi i)^{g-1}({}_N\mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A}\setminus\mathcal{A}[N]} \in \mathrm{reg}_{\mathrm{an}}(H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g)^{(0)}).$$

3. The comparison between pol^0 and the class $\mathfrak{g}_{\mathcal{A}^\vee}$

In this section we give an easy conceptual proof of the comparison result between pol^0 and the class $\mathfrak{g}_{\mathcal{A}^\vee}$.

A commutative diagram. The following lemma, proved by Rössler and Kings, is the key for the proof of our comparison result.

Lemma 7 [Kings and Rössler 2017, Lemma 4.2.6]. *The diagram*

$$\begin{tikzcd}
H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g)^{(0)} \arrow[r, "\mathrm{res}", "\simeq"'] \arrow[d, "\mathrm{reg}_{\mathrm{an}}"] & H^{2g}_{\mathcal{M},\mathcal{A}[N]\setminus S_0}(\mathcal{A}\setminus S_0, g)^{(0)} \arrow[dd, "\mathrm{reg}_{\mathrm{B}}"] \\
H^{2g-1}_{D^{\mathrm{an}}}((\mathcal{A}\setminus\mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)) \arrow[d, "\phi_B"] & \\
H^{2g-1}_{B}((\mathcal{A}\setminus\mathcal{A}[N])(\mathbb{C}), \mathbb{R}(g)) \arrow[r, "\mathrm{res}_B"] & H^{2g}_{B,\mathcal{A}[N]\setminus S_0}((\mathcal{A}\setminus S_0)(\mathbb{C}), \mathbb{R}(g))
\end{tikzcd}$$

is commutative, and the map $\mathrm{reg}_{\mathrm{B}}$ *is injective.*

The comparison result. We are now ready to reprove the comparison result of Kings and Rössler.

Theorem 8 [Kings and Rössler 2017]. *We have the equality*

$$-2\cdot\mathrm{reg}_{\mathrm{an}}(\mathrm{pol}^0) = (2\pi i)^{g-1}({}_N\mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A}\setminus\mathcal{A}[N]}.$$

Proof. Let $\psi \in H^{2g-1}_{\mathcal{M}}(\mathcal{A}\setminus\mathcal{A}[N], g)^{(0)}$ be such that $\mathrm{reg}_{\mathrm{an}}(\psi) = -\frac{1}{2}(2\pi i)^{g-1}({}_N\mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A}\setminus\mathcal{A}[N]}$. By Lemma 7, it is sufficient to show that ψ and pol^0 have the same image under $\mathrm{reg}_{\mathrm{B}}\circ\mathrm{res}$.

Now, by definition of pol^0 we have

$$\mathrm{reg}_{\mathrm{B}}(\mathrm{res}(\mathrm{pol}^0)) = \mathrm{reg}_{\mathrm{B}}(1_N^\circ) = [(2\pi i)^g \delta_{1_N^\circ}, 0],$$

and by the description of res_B in Lemma 2 we have

$$\begin{aligned}\mathrm{reg}_{\mathrm{B}}(\mathrm{res}(\psi)) = \mathrm{res}_B(\phi_B(\mathrm{reg}_{\mathrm{an}}(\psi))) &= \mathrm{res}_B([-(2\pi i)^g \mathrm{d}^c({}_N\mathfrak{g}_{\mathscr{A}^\vee}|_{\mathscr{A}\setminus\mathscr{A}[N]})]) \\ &= [0, (2\pi i)^g \mathrm{d}^c({}_N\mathfrak{g}_{\mathscr{A}^\vee})|_{\mathscr{A}\setminus\mathscr{A}[N]}].\end{aligned}$$

The difference $\mathrm{reg}_{\mathrm{B}}(\mathrm{res}((\mathrm{pol}^0 - \psi)))$ is then represented by the pair

$$((2\pi i)^g \delta_{1_N^\circ}, -(2\pi i)^g \mathrm{d}^c({}_N\mathfrak{g}_{\mathscr{A}^\vee})|_{\mathscr{A}\setminus\mathscr{A}[N]}),$$

which is a coboundary, since (by [Scarponi 2017, Proposition 5.2])

$$(2\pi i)^g \delta_{1_N^\circ} + (2\pi i)^g \,\mathrm{dd}^c({}_N\mathfrak{g}_{\mathscr{A}^\vee}|_{\mathscr{A}\setminus S_0}) = 0. \qquad \square$$

4. A decomposition of the arithmetic Chow group

Recall the exact sequence (see the theorem and remark in [Gillet and Soulé 1990, §3.3.5])

$$H^{2p-1}_{\mathcal{M}}(\mathscr{A}, p) \to \widetilde{E}^{p-1,p-1}(\mathscr{A}_{\mathbb{R}}) \to \widehat{\mathrm{CH}}^p(\mathscr{A})_{\mathbb{Q}} \to \mathrm{CH}^p(\mathscr{A})_{\mathbb{Q}} \to 0. \tag{4}$$

The endomorphism $[n]^*$ acts on this sequence, and we want to study the decomposition into generalized eigenspaces. Denote by $E^{p,q}_{\mathscr{A}}$ the sheaf of p, q-forms on $\mathscr{A}(\mathbb{C})$. For the next result observe that we have an isomorphism of sheaves $\pi^*\varepsilon^* E^{p,q}_{\mathscr{A}} \cong E^{p,q}_{\mathscr{A}}$, which identifies the pull-back of sections of $\varepsilon^* E^{p,q}_{\mathscr{A}}$ on the base with the translation invariant differential forms on $\mathscr{A}$. For a $\mathscr{C}^\infty$ section $a : S(\mathbb{C}) \to \mathscr{A}(\mathbb{C})$, we denote by $\tau_a : \mathscr{A}(\mathbb{C}) \to \mathscr{A}(\mathbb{C})$ the translation by a. A differential form ω is translation invariant, if $\tau_a^*\omega = \omega$ for all sections a.

Theorem 9. *Let $n \geq 2$ and $\omega \in \widetilde{E}^{p,q}(\mathscr{A}_{\mathbb{R}})$. Assume that ω is a generalized eigenvector for $[n]^*$ with eigenvalue λ, i.e., $([n]^* - \lambda)^k\omega = 0$, for some $k \geq 1$. Then the form ω is translation invariant. In particular, there is a section $\eta \in \varepsilon^* E^{p,q}_{\mathscr{A}}(S_{\mathbb{R}})$ with $\omega = \pi^*\eta$. Moreover, one has $[n]^*\omega = n^{p+q}\omega$, i.e., ω is an eigenvector with eigenvalue n^{p+q}.*

Proof. The statement that ω is translation invariant does not depend on the complex structure. We use that locally on the base the family of complex tori $\pi : \mathscr{A}(\mathbb{C}) \to S(\mathbb{C})$ is as a $\mathscr{C}^\infty$-manifold of the form $U \times (S^1)^{2g}$, where $U \subset S(\mathbb{C})$ is open and $S^1 = \mathbb{R}/\mathbb{Z}$ is a real torus. In this situation it suffices to show that ω is translation invariant under a dense subset of points of $(S^1)^{2g}$.

We start to prove the following claim: if the form $\eta := ([n]^* - \lambda)\omega$ is translation invariant, then ω is translation invariant.

First note that $\lambda \neq 0$ because $[n]_*[n]^*\omega = n^{2g}\omega$, which implies that $[n]^*$ is injective. As the set $\{a \in (S^1)^{2g} \mid [n^r](a) = 0 \text{ for some } r \geq 0\}$ is dense in $(S^1)^{2g}$, by induction over r it suffices to show that

$\tau_a^*\omega = \omega$ for a with $[n^r](a) = 0$. The case $r = 0$ is trivial because then $a = 0$. Suppose we know that ω is translation invariant for all b with $[n^{r-1}](b) = 0$, and let a be such that $[n^r](a) = 0$. We compute

$$\lambda\tau_a^*\omega = \tau_a^*([n]^*\omega - \eta) = [n]^*\tau_{[n]a}^*\omega - \tau_a^*\eta = [n]^*\omega - \eta = \lambda\omega.$$

As $\lambda \neq 0$, it follows that $\tau_a^*\omega = \omega$. This completes the induction step.

We now show by induction on k that ω with $([n]^* - \lambda)^k\omega = 0$ is translation invariant. For $k = 1$ this follows from the claim by setting $\eta = 0$. Suppose that all forms η with $([n]^* - \lambda)^{k-1}\eta = 0$ are translation invariant. Then $\eta := ([n]^* - \lambda)\omega$ is translation invariant, and it follows from the claim that also ω is translation invariant.

For the final statement we just observe that $[n]^*$ acts via n^{p+q}-multiplication on the bundle $\varepsilon^* E_{\mathscr{A}}^{p,q}$ whose sections identify with the translation invariant forms on $\mathscr{A}(\mathbb{C})$. □

For the next result we have to consider generalized eigenspaces for $[n]^*$, and to distinguish these from the generalized eigenspaces for $[n]_*$, we write

$$V(a) := \{v \in V \mid ([n]^* - n^a)^k v = 0 \text{ for some } k \geq 1\}.$$

Corollary 10. *For each $a = 0, \dots, 2g$ there is an exact sequence*

$$H_{\mathscr{M}}^{2p-1}(\mathscr{A}, p)(a) \to \widetilde{E}^{p-1,p-1}(\mathscr{A}_{\mathbb{R}})(a) \to \widehat{\mathrm{CH}}^p(\mathscr{A})_{\mathbb{Q}}(a) \to \mathrm{CH}^p(\mathscr{A})_{\mathbb{Q}}(a)$$

of generalized $[n]^*$-eigenspaces for the eigenvalue n^a. In particular, for $a \neq 2(p-1)$ one has an injection

$$\widehat{\mathrm{CH}}^p(\mathscr{A})_{\mathbb{Q}}(a) \hookrightarrow \mathrm{CH}^p(\mathscr{A})_{\mathbb{Q}}(a).$$

Proof. The sequence (4) is a sequence of modules under the principal ideal domain $\mathbb{C}[X]$, where X acts as $[n]^*$. Note that taking the torsion submodule

$$TM := \ker(M \to M \otimes_{\mathbb{C}[X]} \operatorname{Quot}\mathbb{C}[X])$$

is a left exact functor on short exact sequences

$$0 \to M' \to M \to M'' \to 0.$$

If M' is torsion, the functor T is even exact. As $H_{\mathscr{M}}^{2p-1}(\mathscr{A}, p)$ is torsion, the exact sequence (4) gives rise to an exact sequence

$$0 \to T\,\mathrm{im}(H_{\mathscr{M}}^{2p-1}(\mathscr{A}, p)) \to T\,\widetilde{E}^{p-1,p-1}(\mathscr{A}_{\mathbb{R}}) \to T\,\widehat{\mathrm{CH}}^p(\mathscr{A})_{\mathbb{Q}} \to T\,\mathrm{CH}^p(\mathscr{A})_{\mathbb{Q}}.$$

As a torsion $\mathbb{C}[X]$-module is the direct sum of its generalized eigenspaces, the first claim follows. The second statement follows from the first, as $\widetilde{E}^{p-1,p-1}(\mathscr{A}_{\mathbb{R}})(a) = 0$ for $a \neq 2(p-1)$ by Theorem 9. □

5. An axiomatic characterization of the Maillot–Rössler current

We want to prove an axiomatic characterization of $(-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{[g]}$. The result is the following:

Theorem 11. *Let $\hat{\xi}$ be an element of $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ satisfying the following two properties*:

- $\zeta(\hat{\xi}) = S_0$ *in* $\mathrm{CH}^g(\mathcal{A})_{\mathbb{Q}}$,
- $([n]^* - n^{2g})^k(\hat{\xi}) = 0$ *in* $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ *for some* $n \geq 2$ *and some* $k \geq 1$.

Then $\hat{\xi} = (-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{[g]} = (S_0, \mathfrak{g}_{\mathcal{A}^\vee})$.

Remark. (1) Notice that, even if $(-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{[g]}$ satisfies the second property for every n [Scarponi 2017, Proposition 6], it is sufficient to ask that this property holds for one integer greater than one to uniquely characterize it.

(2) Notice also that the condition $([n]^* - n^{2g})(\hat{\xi}) = 0$ implies $[n]_*(\hat{\xi}) = \hat{\xi}$, thanks to the projection formula $[n]_*[n]^* = n^{2g}$. In the case of an abelian scheme over the ring of integers of a number field, K. Künnemann [1994] showed that there exists a decomposition of the Arakelov Chow groups as a direct sum of eigenspaces for the pullback $[n]^*$. As a consequence, in this particular case the conditions $([n]^* - n^{2g})(\hat{\xi}) = 0$ and $[n]_*(\hat{\xi}) = \hat{\xi}$ are equivalent, if $\hat{\xi}$ belongs to the Arakelov Chow group.

Proof. By definition $\hat{\xi} \in \widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}(2g)$ and by Corollary 10 one has an injection

$$\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}(2g) \hookrightarrow \mathrm{CH}^g(\mathcal{A})_{\mathbb{Q}}(2g).$$

This shows that $\hat{\xi}$ is uniquely determined by its image in $\mathrm{CH}^g(\mathcal{A})_{\mathbb{Q}}$. As this image is the same as that of $(S_0, \mathfrak{g}_{\mathcal{A}^\vee})$, this shows the theorem. □

Theorems 8 and 11 give us the following axiomatic characterization of $\mathfrak{g}_{\mathcal{A}^\vee}$ and therefore of pol^0.

Theorem 12. *The class $\mathfrak{g}_{\mathcal{A}^\vee}$ is the unique element $\mathfrak{g} \in \widetilde{D}^{g-1,g-1}(\mathcal{A}_{\mathbb{R}})$ such that*

(i) $\mathfrak{g}$ *is a Green current for* S_0,

(ii) $([n]^* - n^{2g})^k(S_0, \mathfrak{g}) = 0$ *in* $\widehat{\mathrm{CH}}^g(\mathcal{A})_{\mathbb{Q}}$ *for some* $n \geq 2$ *and some* $k \geq 1$,

(iii) $[m]_*\mathfrak{g} = \mathfrak{g}$ *for some* $m > 1$.

Furthermore, pol^0 *is the unique element in* $H^{2g-1}_{\mathcal{M}}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)}$ *such that*

$$-2 \cdot \mathrm{reg}_{\mathrm{an}}(\mathrm{pol}^0) = (2\pi i)^{g-1}([N]^*\mathfrak{g}_{\mathcal{A}^\vee} - N^{2g}\mathfrak{g}_{\mathcal{A}^\vee})|_{\mathcal{A}\setminus\mathcal{A}[N]} \in \mathrm{H}^{2g-1}_{D^{\mathrm{an}}}((\mathcal{A} \setminus \mathcal{A}[N])_{\mathbb{R}}, \mathbb{R}(g)).$$

Proof. By Theorem 11 we know that the first two conditions of our theorem are equivalent to the first two conditions in Theorem 4, so that $\mathfrak{g}_{\mathcal{A}^\vee}$ satisfies the three properties above. Suppose now that $\mathfrak{g} \in \widetilde{D}^{g-1,g-1}(\mathcal{A}_{\mathbb{R}})$ is another element satisfying the three properties of our theorem, and let $m > 1$ be such that $[m]_*\mathfrak{g} = \mathfrak{g}$. We want to show that $\mathfrak{g}_{\mathcal{A}^\vee} = \mathfrak{g}$. Since by Theorem 11

$$(S_0, \mathfrak{g}) = (-1)^g p_{1,*}(\widehat{\mathrm{ch}}(\overline{\mathcal{P}}))^{[g]} = (S_0, \mathfrak{g}_{\mathcal{A}^\vee}),$$

the exact sequence (4) implies that the difference $\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}$ belongs to the image of the regulator $H^{2g-1}_{\mathcal{M}}(\mathcal{A}, g) \to \widetilde{E}^{g-1,g-1}(\mathcal{A}_\mathbb{R})$. Since $H^{2g-1}_{\mathcal{M}}(\mathcal{A}, g)$ is a torsion module over $\mathbb{C}[X]$ (with X acting as $[m]^*$), then $\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}$ lies in $T\widetilde{E}^{g-1,g-1}(\mathcal{A}_\mathbb{R})$. The projection formula and Theorem 9 give

$$m^{2g}(\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}) = [m]_*[m]^*(\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}) = m^{2g-2}[m]_*(\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}),$$

i.e., $[m]_*(\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}) = m^2(\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g})$, but property (iii) in our theorem implies that $[m]_*(\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}) = (\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g})$. This is possible only if $\mathfrak{g}_{\mathcal{A}^\vee} - \mathfrak{g}$ is zero.

The second statement is a simple consequence of Theorem 8 and the fact that $\mathrm{reg}_{\mathrm{an}}$ is injective when restricted to $H^{2g-1}_{\mathcal{M}}(\mathcal{A} \setminus \mathcal{A}[N], g)^{(0)}$ [Kings and Rössler 2017, Lemma 4.2.6]. □

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