Algebra & Number Theory

Volume 13

2019

No. 5

A five-term exact sequence for Kac cohomology

César Galindo and Yiby Morales

msp

ALGEBRA AND NUMBER THEORY 13:5 (2019)
dx.doi.org/10.2140/ant.2019.13.1121

A five-term exact sequence for Kac cohomology

César Galindo and Yiby Morales

Dedicated to Nicolás Andruskiewitsch on the occasion of his 60th birthday

We use relative group cohomologies to compute the Kac cohomology of matched pairs of finite groups. This cohomology naturally appears in the theory of abelian extensions of finite dimensional Hopf algebras. We prove that Kac cohomology can be computed using relative cohomology and relatively projective resolutions. This allows us to use other resolutions, besides the bar resolution, for computations. We compute, in terms of relative cohomology, the first two pages of a spectral sequence which converges to the Kac cohomology and its associated five-term exact sequence. Through several examples, we show the usefulness of the five-term exact sequence in computing groups of abelian extensions.

1. Introduction

Extension theory of groups plays a significant role in the construction and the classification of finite groups. In the same way, the extension theory of Hopf algebras has led to results on the still wide open problem of construction and classification of finite-dimensional semisimple Hopf algebras, [Kashina 2000; Masuoka 1995; Natale 1999; 2001; 2004]. The set of equivalence classes of extensions of a group G by a G-module M is an abelian group with the Baer product of extensions, which is isomorphic to the second cohomology group $H^2(G, M)$. A generalization of this theory to Hopf algebras is obtained for the so-called *abelian extensions*, that is, cleft Hopf algebra extensions of a commutative Hopf algebra K by a cocommutative Hopf algebra H, see [Hofstetter 1994; Kac 1969; Masuoka 1997a; 1999; 2000; Singer 1972].

In this paper, we deal with $H = kF$, a group algebra, and $K = k^G$, the dual of such an algebra, where F and G are finite groups. In this case, each abelian extension has an associated matched pair, that is, a larger group Σ such that G and F are subgroups of Σ satisfying $G \cap F = \{e\}$ and $\Sigma = GF$. The set of equivalence classes of abelian extensions associated to a fixed matched pair, denoted by $\mathrm{Opext}(kF, k^G)$, is an abelian group that can be computed as the second total cohomology of a certain double complex whose cohomology is called *Kac cohomology*.

Obtaining a computation for the $\mathrm{Opext}(kF, k^G)$ of a matched pair of groups can be quite difficult. In fact, there are few general computations in the literature [Masuoka 1997a]. One obstacle for the computation of $\mathrm{Opext}(kF, k^G)$ comes from the fact that it is defined as the cohomology of a very specific

C.G. is partially supported by Faculty of Science of Universidad de los Andes, Convocatoria 2018–2019 para la Financiación de Programas de Investigación, programa "Simetría T (inversión temporal) en categorías de fusión y modulares".
MSC2010: 16T05.
Keywords: Hopf algebras, relative cohomology, abelian extensions of Hopf algebras.

total complex, and the unique "cocycle free" tool is the so-called *Kac exact sequence* (see [Masuoka 1997a] and Corollary 3.10). Perhaps one of the first results that provide a cocycle free description and interpretation of the Kac cohomology is given in [Baaj et al. 2005, Proposition 7.1], where the authors describe the Kac cohomology as the *singular cohomology* of the mapping cone $BG \sqcup BF \to B\Sigma$.

In this paper, we use two different kinds of relative cohomology groups to compute $\operatorname{Opext}(kF, k^G)$: Auslander relative cohomology and Hochschild relative cohomology (see Section 3). We prove that $\operatorname{Opext}(kF, k^G)$ can be computed using Auslander relative cohomology and that Auslander relative cohomology of a matched pair can be computed using relatively projective resolutions. This allows us to use other resolutions, besides the bar resolution, for computing $\operatorname{Opext}(kF, k^G)$. In addition, we compute the first and second page of a spectral sequence which converges to the Kac cohomology. As a consequence, we compute the associated five-term exact sequence, whose second term is $\operatorname{Opext}(kF, k^G)$. In the particular case of a semidirect product, the five-term exact sequence is described in terms of ordinary group cohomology. Finally, doing use of the five-term exact sequence and some nonstandard resolutions, we compute $\operatorname{Opext}(kF, k^G)$ for several families of matched pairs.

The organization of the paper is as follows: In Section 2 we discuss preliminaries on group cohomology and abelian extensions of Hopf algebras. In Section 3 we recall the definitions of Auslander relative cohomology [Auslander and Solberg 1993] and Hochschild relative cohomology [Hochschild 1956]. We prove that $\operatorname{Opext}(kF, k^G)$ can be computed using Auslander Relative cohomology and Auslander relative cohomology of matched pairs can be computed using relatively projective resolutions. In Section 4 we compute the first and second page of a spectral sequence which converges to the Kac cohomology. We also compute, in terms of Hochschild relative cohomology, the associated five-term exact sequence, whose second term is $\operatorname{Opext}(kF, k^G)$. Finally, in Section 5 we compute $\operatorname{Opext}(kF, k^G)$ for several families of matched pairs.

2. Preliminaries

Cohomology of groups. Let G be a group and let M be a G-module. The *n-th cohomology group* of G with coefficients in M is defined as

$$H^n(G, M) = \operatorname{Ext}^n_G(\mathbb{Z}, M).$$

We will use occasionally the *normalized bar resolution* $(\mathbb{Z} \xleftarrow{\epsilon} P_i, \delta)$ of $\mathbb{Z}$ as a trivial G-module. That is

$$P_i = \mathbb{Z}G[G]^i := \mathbb{Z}G[s_1|\cdots|s_i],$$

where $s_i \in G$, $s_i \neq e$ for all i. The differentials are given by

$$\delta([s_1|\cdots|s_{p+1}]) = s_1[s_2|\cdots|s_{p+1}] + \sum_{i=1}^{p}(-1)^i[s_1|\cdots|s_is_{i+1}|\cdots|s_{p+1}] + (-1)^{p+1}[s_1|\cdots|s_p].$$

For a finite cyclic group $C_n = \langle g \rangle$ of order n, we occasionally use a periodic resolution of $\mathbb{Z}$, defined as

$$\cdots \to \mathbb{Z}C_n \xrightarrow{g-1} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{g-1} \mathbb{Z}C_n \xrightarrow{\epsilon} \mathbb{Z}, \tag{2-1}$$

where, $N = \sum_{i=0}^{n-1} g^i$. From this, we have that

$$H^m(C_n, M) = \begin{cases} M^{C_n}/\mathrm{Im}(N) & m = 2k, \\ \mathrm{Ker}(N)/\mathrm{Im}(g-1) & m = 2k+1. \end{cases} \tag{2-2}$$

Resolutions and cohomology for direct products. Let $G = G_1 \times G_2$ be a direct product of groups. Let $(\mathbb{Z} \xleftarrow{\epsilon} P_i, \delta_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon} Q_i, \delta_i')$ be projective resolutions of $\mathbb{Z}$ as a G_1-module and a G_2-module, respectively. Then, the total complex $\mathrm{Tot}(P_i \otimes Q_i)$ is a G-projective resolution of $\mathbb{Z}$. A useful description of the cohomology for direct products is the following. If M is a trivial G-module, then it holds that (see e.g., [Karpilovsky 1985])

$$H^2(G_1 \times G_2, M) \cong H^2(G_1, M) \oplus H^2(G_2, M) \oplus P(G_1, G_2; M), \tag{2-3}$$

where $P(G_1, G_2; M)$ is the abelian group of all pairings from $G_1 \times G_2$ to M. An isomorphism is given by $\bar{\alpha} \mapsto (\bar{\alpha}_1, \bar{\alpha}_2, \phi_\alpha)$, where α_i is the restriction of α to $Q_i \times Q_i$ and

$$\phi_\alpha(x, y) = \mathrm{Alt}(\alpha) = \alpha(x, y) - \alpha(y, x).$$

the inverse isomorphism is defined by $(\bar{\alpha}_1, \bar{\alpha}_2, \phi) \mapsto \bar{\alpha}$ with

$$\alpha((x_1, y_1), (x_2, y_2)) = \alpha_1(x_1, x_2)\alpha_2(y_1, y_2)\phi(x_1, y_2).$$

Second cohomology group and skewsymmetric matrices. Another useful description of the second cohomology group in the case that V is a finite abelian group is provided by using the universal coefficient theorem. Let k be a field and let us consider the group $k^\times$ of units of k as a trivial V-module. There is a short exact sequence

$$0 \to \mathrm{Ext}(V, k^\times) \to H^2(V, k^\times) \xrightarrow{\mathrm{Alt}} \mathrm{Hom}(\wedge^2(V), k^\times) \to 0. \tag{2-4}$$

If V has exponent n and $(k^\times)^n = k^\times$, then $\mathrm{Ext}(V, k^\times) = 0$. Therefore, the map

$$\mathrm{Alt} : H^2(V, k^\times) \to \wedge^2 \hat{V}, \tag{2-5}$$

where $\hat{V} = \mathrm{Hom}(V, k^\times)$, defines an isomorphism $H^2(V, k^\times) \cong \wedge^2 \hat{V}$.

Extensions of Hopf algebras. Let k be a field. A sequence of *finite dimensional* Hopf algebras and Hopf algebra maps

$$(A) : k \to K \xrightarrow{i} A \xrightarrow{\pi} H \to k$$

is called an *extension* of H by K if, i is injective, π is surjective, and $K = A^{\mathrm{co}H}$ (see [Andruskiewitsch and Devoto 1995; Kac 1969; Majid 1990]).

Two extensions (A) and (A') of H by K are said to be *equivalent* if there is an homomorphism $f : A \to A'$ of Hopf algebras such that the following diagram commutes:

$$\begin{tikzcd} & & A \arrow[dd, "f"] \arrow[dr, "\pi"] & & \\ k \arrow[r] & K \arrow[ur, "i"] \arrow[dr, "i'"'] & & H \arrow[r] & k \\ & & A' \arrow[ur, "\pi'"'] & & \end{tikzcd}$$

Matched pairs of groups. Let us recall (see e.g., [Takeuchi 1981]) that a *matched pair of groups* is a collection $(F, G, \triangleright, \triangleleft)$ where G, F are groups and $\triangleright$, $\triangleleft$ are permutation actions

$$G \xleftarrow{\triangleleft} G \times F \xrightarrow{\triangleright} F$$

such that

$$s \triangleright xy = (s \triangleright x)((s \triangleleft x) \triangleright y), \quad st \triangleleft x = (s \triangleleft (t \triangleright x))(t \triangleleft x),$$

for all $s, t \in G$ and $x, y \in F$.

Having groups G and F with a matched pair structure is equivalent to having a group Σ with an exact factorization; the actions $\triangleright$ and $\triangleleft$ are determined by the relations

$$sx = (s \triangleright x)(s \triangleleft x),$$

where $x \in F$ and $s \in G$.

The group Σ associated to a matched pair of groups will be denoted by $F \bowtie G$; it is $F \times G$ with product given by

$$(x, s)(y, t) = (x(s \triangleright y), (s \triangleleft y)t).$$

It is easy to see that the following conditions are equivalent:

(i) The action $\triangleright$ is trivial.

(ii) The action $\triangleleft : G \times F \to G$ is by group automorphisms.

In this case, the associated group $\Sigma = F \ltimes G$ is a semidirect product of groups.

Abelian extensions. Let $(F, G, \triangleright, \triangleleft)$ be a matched pair of groups and let us consider 2-cocycles $\sigma \in Z^2(F, (k^G)^\times)$ and $\tau \in Z^2(G, (k^F)^\times)$. On the vector space

$$k^G \#_{\sigma,\tau} kF := \operatorname{Span}_k\{e_s \# x : s \in G, x \in F\},$$

we can define a unital associative algebra and counital coassociate coalgebra structure by

$$(e_s \# x)(e_t \# y) = \delta_{s \triangleleft x, t}\, \sigma(s; x, y) e_s \# xy,$$
$$\Delta(e_s \# x) = \sum_{s=ab} \tau(a, b; x) e_a \#(b \triangleright x) \otimes e_b \# x.$$

Here the 2-cocycles σ and τ are seen as functions

$$\sigma : G \times F \times F \to k^\times, \quad (s, x, y) \mapsto \sigma(s; x, y),$$
$$\tau : G \times G \times F \to k^\times, \quad (s, t, x) \mapsto \tau(s, t; x).$$

The map Δ is an algebra map if and only if the 2-cocycles satisfy the following compatibility condition

$$\sigma(st; x, y)\tau(s, t; xy) = \sigma(s; t \triangleright x, (t \triangleleft x) \triangleright y)\sigma(t; x, y) \times \tau(s, t; x)\tau(s \triangleleft (t \triangleright x), t \triangleleft x; y),$$

for all $x, y \in G$ and $s, t \in F$. In the case that the 2-cocycles are compatible, the sequence

$$k \to k^G \xrightarrow{i} k^G \#_{\sigma,\tau} kF \xrightarrow{\pi} kF \to k,$$

is a Hopf algebra extension, where $i(e_s) = e_s \# e$ and $\pi(e_s \# x) = x$. These kinds of extensions are called *abelian extensions* of Hopf algebras.

The set of equivalence classes of abelian extensions associated to a fixed matched pair $(F, G, \triangleright, \triangleleft)$ is an abelian group with the Baer product of extensions and will be denoted by $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G)$ (see [Masuoka 2002] for more details).

3. Kac cohomology and relative cohomology

In this section, we recall the definitions of two different kinds of relative group cohomology: the Auslander relative cohomology [Auslander and Solberg 1993] and the Hochschild relative cohomology [Hochschild 1956]. Our aim is to prove that $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G)$ can be computed using Auslander relative cohomology which, in the case of matched pairs, can be computed using relatively projective resolutions. This allows us to use other resolutions besides the bar resolution for computing $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G)$.

Auslander relative cohomology of groups. Let Σ be a group and X a Σ-set. We will denote by Λ_X the kernel of the augmentation map

$$\epsilon_X : \mathbb{Z}[X] \to \mathbb{Z}, \quad x \mapsto 1, \tag{3-1}$$

where $\mathbb{Z}[X]$ is the Σ-module associated to X.

Definition 3.1. Given a Σ-module A, the *n-th cohomology group of Σ relative to X with coefficients in A* is defined by

$$H^k_{\mathscr{A}}(\Sigma, X; A) := \mathrm{Ext}^{k-1}_{\mathbb{Z}\Sigma}(\Lambda_X, A), \quad k \geq 1.$$

Let X be a Σ-set and $\mathcal{R}_X$ a set of representatives of the Σ-orbits in X. Using Shapiro's lemma, we have that

$$\mathrm{Ext}^k_\Sigma(\mathbb{Z}[X], A) = \prod_{x \in \mathcal{R}_X} \mathrm{Ext}^k_\Sigma(\mathbb{Z}[\mathcal{O}(x)], A) \cong \prod_{x \in \mathcal{R}_X} \mathrm{Ext}^k_{\mathrm{St(x)}}(\mathbb{Z}, A),$$

where St(x) denotes the stabilizer of $x \in X$. Hence,

$$\mathrm{Ext}^k_\Sigma(\mathbb{Z}[X], A) = \prod_{x \in \mathcal{R}_X} H^k(\mathrm{St}(x), A).$$

If we apply the functor $\mathrm{Ext}_\Sigma(-, A)$ to the exact sequence of Σ-modules

$$0 \to \Lambda_X \to \mathbb{Z}[X] \to \mathbb{Z} \to 0,$$

we obtain the well-known long exact sequence for relative cohomology

$$\cdots \to H^k(\Sigma, A) \to \prod_{x \in \mathcal{R}_X} H^k(\mathrm{St}(x), A) \to H^{k+1}_{\mathscr{A}}(\Sigma, X, A) \to H^{k+1}(\Sigma, A) \to \cdots \tag{3-2}$$

Hochschild Relative cohomology of groups. Relative cohomology of groups was originally defined by Hochschild [1956] and Adamson [1954]. We follow the description given in [Alperin 1986].

Let U be a G-module and S a subgroup of G. We say that U is *relatively S-projective* if it satisfies the following equivalent properties (see [Alperin 1986, Proposition 1, page 65]):

(i) If $\psi : U \twoheadrightarrow V$ is a surjective G-homomorphism and ψ splits as an S-homomorphism then ψ splits as a G-homomorphism.

(ii) If $\psi : V \twoheadrightarrow W$ is a surjective G-homomorphism and $\phi : U \to W$ is a G-homomorphism, then there is a G-homomorphism $\lambda : U \to V$ with $\psi\lambda = \phi$, provided that there is an S-homomorphisms $\lambda_0 : U \to V$ with the same property.

(iii) U is a direct summand of $U{\downarrow_S}{\uparrow_G}$.

Here, $\downarrow_S$ means the restriction to S, and $\uparrow_G$ the induction to G.

A complex

$$\mathscr{R} : \cdots \to R_3 \xrightarrow{\delta_3} R_2 \xrightarrow{\delta_2} R_1 \xrightarrow{\delta_1} R_0 \xrightarrow{\epsilon} M \to 0$$

of G-modules is called a *relatively S-projective resolution* if:

(1) each G-module R_i is relatively S-projective,

(2) the sequence has a contracting homotopy as S-modules.

Remarks 3.2.

- Since the canonical map $M{\downarrow_S}{\uparrow_G} \to M$ splits as an S-homomorphism, if $\mathscr{T}$ is a projective S-resolution of $M{\downarrow_S}$, then $\mathscr{T}{\uparrow_G}$ is a relatively S-projective resolution of M.
- If S is the trivial subgroup of G, the relatively S-projective resolutions of G-module are the same as projective resolutions of G-modules.

Definition 3.3. Given a relatively S-projective resolution $\mathscr{R}$ of M, the n-th relative S-cohomology group of G is defined by

$$H^m(G, S; M) = H^n(\mathrm{Hom}_G(\mathscr{R}, M), \delta^*).$$

As expected, this definition does not depend on the chosen relatively S-projective resolution of M, (see, e.g., [Hochschild 1956]).

From now on, all relatively projective resolutions are assumed to be free as $\mathbb{Z}$-modules.

Example 3.4 (Standard complex [Snapper 1964]). Let X be a transitive left G-set. Let $C_i = \mathbb{Z}X^{(i+1)}$ be the free $\mathbb{Z}$-module generated by all $(i+1)$-tuples of elements of X. The group G acts diagonally on C_i and the sequence

$$C_*^X := \cdots \to C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where

$$\delta_i(x_1, \dots, x_{r+1}) = \sum_{j=1}^{r+1} (-1)^{j+1} (x_1, \dots, \widehat{x_j}, \dots, x_{r+1}) \tag{3-3}$$

and $\epsilon(x) = 1$ for all $x \in X$, is a complex of G-modules.

If F denotes the stabilizer subgroup of $x_0 \in X$, the complex C_*^X is relatively F-projective resolution of $\mathbb{Z}$ called the *standard complex of* (G, X).

Proposition 3.5. *Let $\Sigma = F \bowtie G$ be a matched pair and $Q := (\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i')$ be the normalized right bar resolution of the trivial G-module $\mathbb{Z}$. Then the group Σ acts on Q_i by*

$$[s_i|\cdots|s_2|s_1]s_0 \cdot (x, s) = [s_i \triangleleft ((s_{i-1} \dots s_0) \triangleright x)|\cdots|s_1 \triangleleft (s_0 \triangleright x)](s_0 \triangleleft x)s. \tag{3-4}$$

and Q is a relatively F-projective resolution of right Σ-modules.

Proof. Since Σ is a matched pair, the right Σ-set of cosets $F\backslash\Sigma$ can be identified with the set G and Σ-action $s \cdot (f, g) = (s \triangleleft f)g$. This Σ-set will be denoted by X. Applying the construction of Example 3.4 to X, we obtain a relatively F-projective resolution $C := C_i \xrightarrow{\epsilon} \mathbb{Z}$, since F is the stabilizer of $e \in G$. The resolution C coincides with the standard G-free resolution of $\mathbb{Z}$ as a trivial G-module. A G-basis of C_i (called bar basis) is given by

$$[s_i|\cdots|s_2|s_1] = (s_i \cdots s_1, \cdots, s_2 s_1, s_1, e).$$

The action of Σ in this basis is given by (3-4). The normalized bar resolution is a quotient of the bar resolution, and it is easy to see that this is also relatively F-projective. □

Remarks 3.6.
- In Proposition 3.5, we may consider the normalized *left bar resolution* $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i')$ for the trivial G-module $\mathbb{Z}$: with action of Σ given by

$$(s, x) \cdot x_0[x_1|\cdots|x_i] = x(s \triangleright x_0)[(s \triangleleft x_0) \triangleright x_1|\cdots|(s \triangleleft x_0 x_1 \dots x_{i-1}) \triangleright x_i]. \tag{3-5}$$

  This is a relatively G-projective resolution.
- The formulas (3-4) and (3-5) appear in [Masuoka 1997a].

Theorem 3.7. *Let $\Sigma = F \bowtie G$ be a matched pair. Let $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i')$ be a relatively F-projective and a relatively G-projective Σ-resolutions of $\mathbb{Z}$, respectively. Then the total complex of the tensor product double complex*

$$P_i \otimes Q_j \quad \textit{for } i, j \geq 0.$$

is a projective Σ-resolution of $\mathbb{Z}$.

Proof. Since P_i is relatively F-projective, $P_i\downarrow_F\uparrow^{\Sigma}= P_i \oplus P_i'$ as Σ-modules. Analogously, $Q_j\downarrow_G\uparrow^{\Sigma}= Q_i \oplus Q_i'$.

Using the fact that (F, G) is an exact factorization of Σ and the Mackey's tensor product theorem (see e.g., [Curtis and Reiner 1990, Theorem 10.18]), we have that

$$\begin{aligned}(P_i \otimes Q_j)\downarrow_{\{e\}}\uparrow^{\Sigma} &\cong P_i\downarrow_F\uparrow^{\Sigma} \otimes Q_j\downarrow_G\uparrow^{\Sigma},\\ &= (P_i \oplus P_i') \otimes (Q_j \oplus Q_j')\\ &= P_i \otimes Q_j \oplus (P_i \otimes Q_j' \oplus P_i' \otimes Q_j \oplus P_i' \otimes Q_j \oplus P_i' \otimes Q_j').\end{aligned}$$

Hence $P_i \otimes Q_j$ is direct summand of the Σ-free module $(P_i \otimes Q_j)\downarrow_{\{e\}}\uparrow^{\Sigma}$, that is, $P_i \otimes Q_j$ is a projective Σ-module.

Finally, since each P_i and Q_j are flat $\mathbb{Z}$-modules, it follows from the Künneth formula that, for $n > 0$, $H_n(\mathrm{Tot}(P_* \otimes Q_*)) = 0$. □

Theorem 3.7 generalizes the results of [Masuoka 1997b; 2003] about the construction of nonstandard free resolutions of $\mathbb{Z}$ associated to a matched pairs of groups. In fact, taking the relatively projective resolutions of Proposition 3.5 and Remarks 3.6 we can obtain the resolutions in [Masuoka 1997b; 2003].

Kac cohomology as relative group cohomology. Using the bijective maps,

$$\begin{aligned}\Sigma/G \to F, &\quad (f, g)G \mapsto f,\\ F\backslash\Sigma \to G, &\quad F(f, g) \mapsto g\end{aligned}$$

we can endow the set F with a left Σ-action $(f, g)x = f(g \triangleright x)$ and G with a right Σ-action $s(f, g) = (s \triangleleft f)g$. From now on, X will denote the left Σ-set defined as the disjoint union $F \sqcup G$, where G is considered a left Σ-set using the inverse. We denote by Λ_X the kernel of the augmentation map (3-1).

Proposition 3.8. *Let $\Sigma = F \bowtie G$ be a matched pair. Let $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i')$ be a relatively F-projective and a relatively G-projective Σ-resolutions of $\mathbb{Z}$, respectively. Let $D_{*,*}$ be the truncated tensor product double complex*

$$D_{i,j} := P_{i+1} \otimes Q_{j+1}, \quad \textit{for } i, j \geq 0. \tag{3-6}$$

Then, the total complex $(\mathrm{Tot}(D_{*,*}), d_i)$ completed with the map

$$P_0 \otimes Q_0 \xleftarrow{-\delta_1 \otimes \delta_1'} \mathrm{Tot}(D_{*,*}),$$

is a projective Σ-resolution of Λ_X.

Proof. Let Λ be the kernel of the map $\epsilon : P_0 \oplus Q_0 \to \mathbb{Z}$ defined by $\epsilon(x \oplus y) = \epsilon_P(x) + \epsilon_Q(y)$. Let us consider the total complex $(\mathrm{Tot}(D_{*,*}), d_i)$ completed with the maps

$$0 \leftarrow \Lambda \xleftarrow{\theta} P_0 \otimes Q_0 \xleftarrow{-\delta_1 \otimes \delta_1'} \mathrm{Tot}(D_{*,*}), \tag{3-7}$$

where $\theta(p\otimes q)=-p\epsilon_Q(q)\oplus\epsilon_P(p)q$. Let us see that $H_n(\mathrm{Tot}(D_{*,*}))=0$ for all $n\geq 1$. Let $B_{*,*}$ be the subcomplex of $P_*\otimes Q_*$ consisting of the first row and first column, that is,

$$\begin{aligned} B_{i,j}&=0 &&\text{for } i,\, j\geq 1,\\ B_{i,j}&=P_i\otimes Q_j &&\text{for } i=0\vee j=0. \end{aligned}$$

Let $S_{*,*}=P_*\otimes Q_*/B_{*,*}$. Note that $\mathrm{Tot}_n(D_{*,*})=\mathrm{Tot}_{n+2}(S_{*,*})$ for all n.

Let us now see that (3-7) is exact in $\mathrm{Tot}_0(D_{*,*})=P_1\otimes Q_1$. The map $d_1:P_2\otimes Q_1\oplus P_1\otimes Q_2\to P_1\otimes Q_1$ in $\mathrm{Tot}(D_{*,*})$ is defined by $d_1(a\oplus b)=(\delta_2\otimes\mathrm{id})(a)-(\mathrm{id}\otimes\delta_2')(b)$, where $a\in P_2\otimes Q_1$ and $b\in P_1\otimes Q_2$. Hence, the composed map is given by

$$(-\delta_1\otimes\delta_1')\circ d_1(a,b)=(-\delta_1\otimes\delta_1')(\delta_2\otimes\mathrm{id})(a)+(\delta_1\otimes\delta_1')(\mathrm{id}\otimes\delta_2')(b)=0.$$

Thus, $\mathrm{Im}(d_1)\subseteq\mathrm{Ker}(-\delta_1\otimes\delta_1')$. Now, since P_i and Q_i are free $\mathbb{Z}$-modules, then

$$\mathrm{Ker}(-\delta_1\otimes\delta_1')=\mathrm{Ker}(\delta_1)\otimes Q_1+P_1\otimes\mathrm{Ker}(-\delta_1')=\mathrm{Im}(\delta_2)\otimes Q_1-P_1\otimes\mathrm{Im}(\delta_2').$$

so $\mathrm{Ker}(-\delta_1\otimes\delta_1')\subseteq\mathrm{Im}(d_1)$. To see the exactness in $P_0\otimes Q_0$, note that

$$\theta\circ(\delta_1\otimes\delta_1')(p\otimes q)=\delta_1(p)\epsilon_Q(\delta_1'(q))\oplus\epsilon_P(\delta_1(p))\delta_1(q)=0.$$

Hence, $\mathrm{Im}(\delta_1\otimes\delta_1')\subseteq\mathrm{Ker}(\theta)$. To see that $\mathrm{Ker}(\theta)\subseteq\mathrm{Im}(\delta_1\otimes\delta_1')$, let us consider the tensor product of the two chain complexes $0\leftarrow\mathbb{Z}\xleftarrow{\epsilon_P}P_i$ and $0\leftarrow\mathbb{Z}\xleftarrow{\epsilon_Q}Q_i$. That is,

$$\begin{tikzcd}
\mathbb{Z}\otimes Q_1 \arrow[d, "-\mathrm{id}\otimes\delta_1'"'] & P_0\otimes Q_1 \arrow[l, "\epsilon_P\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\delta_1'"] & P_1\otimes Q_1 \arrow[l, "\delta_1\otimes\mathrm{id}"'] \arrow[d, "-\mathrm{id}\otimes\delta_1'"] \\
\mathbb{Z}\otimes Q_0 \arrow[d, "-\mathrm{id}\otimes\epsilon_Q"'] & P_0\otimes Q_0 \arrow[l, "\epsilon_P\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\epsilon_Q"] & P_1\otimes Q_0 \arrow[l, "\delta_1\otimes\mathrm{id}"] \arrow[d, "-\mathrm{id}\otimes\epsilon_Q"] \\
\mathbb{Z} & P_0\otimes\mathbb{Z} \arrow[l, "\epsilon_P\otimes\mathrm{id}"] & P_1\otimes\mathbb{Z}. \arrow[l, "\delta_1\otimes\mathrm{id}"]
\end{tikzcd} \tag{3-8}$$

whose total complex is given by the exact sequence

$$0\leftarrow\mathbb{Z}\otimes\mathbb{Z}\xleftarrow{\epsilon}P_0\otimes\mathbb{Z}\oplus\mathbb{Z}\otimes Q_0\xleftarrow{d_1}P_1\otimes\mathbb{Z}\oplus P_0\otimes Q_0\oplus\mathbb{Z}\otimes Q_1\leftarrow\cdots \tag{3-9}$$

which can be written as

$$\mathbb{Z}\xleftarrow{\epsilon}P_0\oplus Q_0\xleftarrow{d_1}P_1\oplus P_0\otimes Q_0\oplus Q_1\leftarrow\cdots \tag{3-10}$$

Note that θ is the restriction of d_1 to $P_0\otimes Q_0$. Suppose that $c\in\mathrm{Ker}(\theta)$. Then, $(\mathrm{id}\otimes\epsilon_Q)(c)=(\epsilon_P\otimes\mathrm{id})(c)=0$. Let b_2, b_3 be the preimages of c under the vertical and horizontal differentials in (3-8) respectively. There is a tuple $b=(b_1,b_2,-b_3,-b_4)$ such that $d_2(b)=0$. Since the total complex of (3-8) is acyclic, there is a tuple $a=(a_1,a_2,a_3,a_4,a_5)$ such that $d_3(a)=b$, and, it can be verified that $a_3\in P_1\otimes Q_1$ satisfies $\theta(a_3)=c$. Therefore, $\mathrm{Ker}(\theta)\subseteq\mathrm{Im}(\delta_1\otimes\delta_1')$.

To see that θ is surjective, note that $\epsilon = \epsilon_P + \epsilon_Q$ in (3-10), and

$$d_1(a \oplus b \oplus c) = (\delta_1(a) + (\mathrm{id} \otimes \epsilon_Q)(b)) \oplus ((\epsilon_P \otimes \mathrm{id})(b) - \delta_1'(c)).$$

Let $p_0 \in d_1(P_1) = \mathrm{Ker}(\epsilon_P)$ and let $q \in Q_0$ such that $\epsilon_Q(q) = 1$. Then,

$$d_1(p_0 \otimes q) = (-\mathrm{id} \otimes \epsilon_Q + \epsilon_P \otimes \mathrm{id})(p_0 \otimes q) = p_0,$$

so $d_1(P_1) \subseteq d_1(P_0 \otimes Q_0)$. Similarly, $d_1(Q_1) \subseteq d_1(P_0 \otimes Q_0)$. Hence,

$$\Lambda = \mathrm{Ker}(\epsilon) = \mathrm{Im}(d_1) = \mathrm{Span}\{d_1(P_1) \cup d_1(P_0 \otimes Q_0) \cup d_1(Q_1)\} = d_1(P_0 \otimes Q_0).$$

Also, the map d_1 restricted to $P_0 \otimes Q_0$ is given by $-\mathrm{id} \otimes \epsilon_Q \oplus \epsilon_P \otimes \mathrm{id} = \theta$, then $\Lambda = \mathrm{Im}(\theta)$ and therefore the sequence (3-7) is exact.

Finally, we see that Λ and Λ_X are isomorphic as Σ-modules. Let us take $P'_* = C^G_*$ and $Q'_* = C^F_*$, the standard resolutions as in Example 3.4, were F and G are consider as Σ-sets. In this case,

$$C^G_0 \oplus C^F_0 = \mathbb{Z}[G] \oplus \mathbb{Z}[F] \cong \mathbb{Z}[G \sqcup F] = \mathbb{Z}[X],$$

and ϵ is the augmentation map (3-1). Then, for this resolution $\Lambda = \Lambda_X$.

If $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i)$ are relatively projective resolutions, there exists homotopy equivalences

$$s : P_* \to C^G_*, \quad l : Q_* \to C^F_*$$

This implies that $P_0 \otimes Q_0 \xleftarrow{-\delta_1 \otimes \delta_1'} \mathrm{Tot}(D_{*,*})$ is homotopically equivalent to $P'_0 \otimes Q'_0 \xleftarrow{-\delta_1 \otimes \delta_1'} \mathrm{Tot}(D_{*,*})$. Hence,

$$\Lambda_X \cong \mathrm{Coker}(P'_1 \otimes Q'_1 \to P'_0 \otimes Q'_0) \cong \mathrm{Coker}(P_1 \otimes Q_1 \to P_0 \otimes Q_0) \cong \Lambda. \qquad \square$$

Theorem 3.9. *Let k be a field and $\Sigma = F \bowtie G$ a matched pair of groups. Then,*

$$\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) \cong H^3_{\mathscr{A}}(\Sigma, X; k^\times),$$

where $k^\times$ is considered as a trivial Σ-module.

Proof. Let $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i')$ be the resolutions in Proposition 3.5. These are the resolutions used in [Masuoka 1997b] to compute $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G)$; they consider the truncated tensor product $D^{i,j}$ of the two resolutions to get

$$\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) \cong H^1(\mathrm{Tot}(\mathrm{Hom}_\Sigma(D^{i,j}))).$$

If Λ is the kernel of the map $\epsilon : P_0 \oplus Q_0 \to \mathbb{Z}$ defined by $\epsilon(x \oplus y) = \epsilon_P(x) + \epsilon_Q(y)$, (here, $P_0 := \mathbb{Z}F = \mathbb{Z}X$ and $Q_0 := \mathbb{Z}Y$ for the Σ-sets $X = F$ and $Y = G$) then, by Proposition 3.8, the total complex of $E_{i,j} := P_{i+1} \otimes Q_{j+1}$ for $i, j \geq 0$ completed in the following way

$$0 \longleftarrow \Lambda \xleftarrow{\theta} P_0 \otimes Q_0 \xleftarrow{-\delta_1 \otimes \delta_1'} \mathrm{Tot}(D_{*,*}),$$

is a resolution of Λ, and $\Lambda = \Lambda_X$. Therefore, if we apply $\mathrm{Hom}_\Sigma(-, k^\times)$ to this total complex, we get the relative cohomology groups $H^n_{\mathscr{A}}(\Sigma, X; A)$. That is

$$H^k(\mathrm{Hom}_\Sigma(\mathrm{Tot}(D_{*,*}))) \cong H^{k+2}_{\mathscr{A}}(\Sigma, X; A). \tag{3-11}$$

In particular, since $\mathrm{Hom}_\Sigma(\mathrm{Tot}(D_{*,*})) \cong \mathrm{Tot}(\mathrm{Hom}_\Sigma(D_{*,*}))$, then,

$$\mathrm{Opext}_{\rhd,\lhd}(kF, k^G) \cong H^1(\mathrm{Tot}(\mathrm{Hom}_\Sigma(D_{*,*}))) \cong H^3_{\mathscr{A}}(\Sigma, X; k^\times), \tag{3-12}$$

which completes the proof. □

As a consequence of Theorem 3.9 and the long exact sequence (3-2) we obtain Kac's exact sequence (see [Masuoka 1997a; Kac 1969]).

Corollary 3.10 (Kac's exact sequence). *For a fixed matched pair of groups* $(F, G, \rhd, \lhd)$, *we have a long exact sequence*

$$\begin{aligned} 0 &\to H^1(F \bowtie G, k^\times) \to H^1(F, k^\times) \oplus H^1(G, k^\times) \to H^3_{\mathscr{A}}(\Sigma, X; k^\times) \\ &\to H^2(F \bowtie G, k^\times) \to H^2(F, k^\times) \oplus H^2(G, k^\times) \to \mathrm{Opext}_{\rhd,\lhd}(kF, k^G) \\ &\to H^3(F \bowtie G, k^\times) \to H^3(F, k^\times) \oplus H^3(G, k^\times) \to H^4_{\mathscr{A}}(\Sigma, X; k^\times). \end{aligned}$$

4. The five-term exact sequence for Kac double complex

The group $\mathrm{Opext}_{\rhd,\lhd}(kF, k^G)$ can be obtained, as described in [Masuoka 1997a], as the first cohomology group of a double cochain complex, which can be computed by means of a spectral sequence. We compute the first pages of the spectral sequence associated to the double cochain complex $D_{*,*}$ in (3-6), which is a particular case of the double cochain complex of Kac. The five-term exact sequence for this spectral sequence will be useful for computing the group $\mathrm{Opext}_{\rhd,\lhd}(kF, k^G)$ for different kinds of matched pairs.

Spectral sequence of a double cochain complex. Through this section we deal with a first quadrant double complex, that is, a double cochain complex $M^{p,q}$ such that $M^{p,q} = \{0\}$ when $p, q < 0$. There is a spectral sequence associated to a first quadrant double complex, whose first pages are obtained taking vertical and horizontal cohomology of the double complex.

Let $M^{*,*}$ be a first quadrant double complex with vertical and horizontal differentials given by δ_v, δ_h. Let $\mathrm{Tot}(M^{*,*})$ be the total complex associated to $M^{*,*}$. There is a spectral sequence $(E_r^{*,*}, d_r)$ with differentials $d_r^{p,q} : E_r^{pq} \to E_r^{p+r,q-r+1}$, which converges to $H^*(\mathrm{Tot}(M^{*,*}))$, whose first pages are given by

$$E_0^{p,q} = M^{p,q}, \quad E_1^{p,q} = H^q(M^{p,*}, d_0), \quad E_2^{p,q} = H^p(E_1^{*,q}, d_1),$$

see [McCleary 2001] for more details. The differentials for each page are given by

$$d_0^{p,q} = d_v, \quad d_1^{p,q} = d_h', \quad d_2^{p,q}(\bar{\alpha}) = \overline{d_h(\gamma)}, \tag{4-1}$$

where d'_h is the differential induced by d_h on $H^q(M^{p,*}, d_0)$ and $\gamma \in M^{p+1,q-1}$ is such that $d_h(\alpha) = d_v(\gamma)$. Associated to the spectral sequence $(E_r^{*,*}, d_r)$, there is a five-term exact sequence

$$0 \to E_2^{1,0} \xrightarrow{i} H^1(\mathrm{Tot}(M^{*,*})) \xrightarrow{p} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{i} H^2(\mathrm{Tot}(M^{*,*})), \tag{4-2}$$

where i is a restriction map, the map p is a projection map.

The five-term exact sequence. Given a matched pair $(F, G\triangleright, \triangleleft)$ we compute a five-term exact sequence to calculate $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G)$.

Theorem 4.1. *Let $\Sigma = F \bowtie G$ be a matched pair and A be a Σ-module. The first and second pages of the spectral sequence associated to the double complex $\mathrm{Hom}_\Sigma(D_{*,*}, A)$, where $D^{i,j} := P_{i+1} \otimes Q_{j+1}$, for $i, j \geq 0$, is the double complex defined in Proposition 3.8, are given by*

$$E_1^{i,n} = H^n(\Sigma, G; \mathrm{Hom}(P_i, A)), \quad E_2^{n,m} = H^m(H^n(\Sigma, G; \mathrm{Hom}(P_*, A))),$$

for $m, n > 0$. The first page does not depend on the resolution Q_* and $E_2^{n,m}$ $(m, n > 0)$ depends neither on the resolution P_* nor the resolution Q_*.

Proof. The double complex $\mathrm{Hom}_\Sigma(D_{*,*}, A)$ with only the vertical differentials is the zeroth page of the spectral sequence

$$E_0^{i,j} := \mathrm{Hom}_\Sigma(P_i \otimes Q_j, A) \cong \mathrm{Hom}_\Sigma(Q_j, \mathrm{Hom}_\mathbb{Z}(P_i, A)). \tag{4-3}$$

Since $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta'_i)$ is a relatively projective resolution of $\mathbb{Z}$, the first page of the spectral sequence is

$$E_1^{i,n} = H^n(\Sigma, G, \mathrm{Hom}(P_i, A)) = F_n(P_i),\ \ i > 0.$$

where $F_n : \Sigma\text{-Mod} \to \mathrm{Ab}$ is the functor given by $H_1^n(\Sigma, G; \mathrm{Hom}(-, A))$. Hence it does not depend on the resolution Q_i. The second page is

$$E_2^{n,m} = H^m(F_n(P_*)), m, n > 0.$$

To see that $E_2^{n,m} = H^m(F_n(P_*))$, $m, n > 0$ do not depend on the resolution, let P'_i be another relatively F-projective resolution of $\mathbb{Z}$. Then, there exists a homotopy equivalence $f : P_i \to P'_i$ as F-modules, that is, there exists $g : P'_i \to P_i$ and $h_i : P_i \to P_{i-1}$ such that

$$\delta_i h + h\delta_i = fg - \mathrm{id}.$$

Since the functor F_n is additive, we get

$$F_n(\delta_i)F_n(h_i) + F_n(h_i)F_n(\delta_i) = F_n(f)F_n(f^{-1}) - F_n(\mathrm{id})$$

for each n, so the map $F_n(f)$ is a homotopy equivalence between the resolutions $F_n(P_i)$ and $F_n(P'_i)$. This means that the second page, which consist on the cohomology groups of the resolutions $F_n(P_i)$ and $F_n(P'_i)$ with the respective induced differentials, is isomorphic to the first one. □

Theorem 4.2. *Let $(F, G, \triangleright, \triangleleft)$ be a matched pair where the action $\triangleright$ is trivial and let k be a field. The spectral sequence in Theorem 4.1 associated to the group $\Sigma = F \bowtie G$ has second page given by*

$$E_2^{p,q} = H^{p+1}(F, H^{q+1}(G, k^\times)), \quad E_2^{p,0} \cong H^{p+1}(F, \hat{G}),$$
$$E_2^{0,q} \cong \mathrm{Der}(F, H^{q+1}(G, k^\times)), \qquad E_2^{0,0} = \mathrm{Der}(F, \hat{G}),$$

for $p \geq 1, q \geq 1$. Therefore, we have the five-term exact sequence:

$$0 \to H^2(F, \hat{G})) \xrightarrow{i} \mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) \xrightarrow{\pi} \mathrm{Der}(F, H^2(G, k^\times)) \xrightarrow{d_2} H^3(F, \hat{G}) \to H^4_{\mathscr{A}}(\Sigma, X, k^\times). \tag{4-4}$$

Proof. According to (4-3), the zeroth page is given by

$$E_0^{i,j} := \mathrm{Hom}_\Sigma(P_i \otimes Q_j, k^\times) \cong \mathrm{Hom}_\Sigma(Q_j, \mathrm{Hom}_\mathbb{Z}(P_i, k^\times)).$$

If we take $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i')$ to be the resolutions in Proposition 3.5, then we have the group isomorphism

$$\mathrm{Hom}_\Sigma(P_i \otimes Q_j, k^\times) \cong \mathrm{Map}_+(G^{q+1} \times F^{p+1}, k^\times),$$

and the vertical and horizontal differentials of the double complex of groups

$$\mathrm{Map}_+(G^{q+1} \times F^{p+1}, k^\times),$$

are respectively given by

$$\begin{aligned}
&\delta_i f(s_{i+1}, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^p} \\
&\quad = f(s_{i+1}, \cdots, s_2; s_1 \triangleright x_1, (s_1 \triangleleft x_1) \triangleright x_2, \cdots\cdots, (s_1 \triangleleft x_1 \cdots x_{p-1}) \triangleright x_p) \\
&\qquad \times \prod_{k=1}^{i} f(s_{i+1}, \cdots, s_{i+1}s_i, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^k} \times f(s_i, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^{q+1}}.
\end{aligned}$$

and

$$\begin{aligned}
&\delta_i' f(s_q, \cdots, s_1; x_1, \cdots, x_{i+1}) \\
&\quad = f(s_q \triangleleft (s_{q-1} \cdots s_1 \triangleright x_1), \cdots\cdots, s_2 \triangleleft (s_1 \triangleright x_1), s_1 \triangleleft x_1; x_2, \cdots, x_{p+1}) \\
&\qquad \times \prod_{k=1}^{i} f(s_q, \cdots, s_1; x_1, \cdots, x_i x_{i+1}, \cdots, x_{i+1})^{(-1)^k} \times f(s_q, \cdots, s_1; x_1, \cdots, x_i)^{(-1)^{i+1}}.
\end{aligned}$$

Since the action $\triangleleft$ is trivial, the vertical differentials are given by

$$\begin{aligned}
&\delta(f)(s_{q+1}, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^p} \\
&\qquad = f(s_{q+1}, \cdots, s_2; x_1, \cdots, x_p) \\
&\qquad\qquad \prod_{i=1}^{q} f(s_{q+1}, \cdots, s_{i+1}s_i, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^i} f(s_q, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^{q+1}}.
\end{aligned}$$

We have that $\mathrm{Map}_+(G^q \times F^p, k^\times) \cong C^q(G, C^p(F, k^\times))$, where $C^q(G, C^p(F, k^\times))$ denotes the group of functions $f : G^q \to C^p(F, k^\times)$ (with a normalization property) and the group G acts trivially on the group $C^p(F, k^\times)$.

Taking vertical cohomology to the zeroth page, we get $H^q(G, C^p(F, k^\times))$. Therefore, the first page $E_1^{*,*}$ of the spectral sequence is given by

$$\begin{aligned} E_1^{p,q} &= H^{q+1}(G, C^{p+1}(F, k^\times)) \cong C^{p+1}(F, H^{q+1}(G, k^\times)) \quad \text{for } q \geq 1 \\ E_1^{p,0} &= \mathrm{Der}(G, C^{p+1}(F, k^\times)) \cong C^{p+1}(F, \mathrm{Der}(G, k^\times)), \end{aligned}$$

where the isomorphisms hold since the vertical differential leaves every element of F fixed. On the other hand, the horizontal differentials are given by

$$\begin{aligned} &\delta'(f)(s_q, \cdots, s_1; x_1, \cdots, x_{p+1}) \\ &\quad = f(s_q \triangleleft x_1, \cdots, s_1 \triangleleft x_1; x_2, \cdots, x_{p+1}) \\ &\qquad\qquad \prod_{i=1}^{p} f(s_q, \cdots, s_1; x_1, \cdots, x_i x_{i+1}, \cdots, x_{p+1})^{(-1)^i} f(s_q, \cdots, s_1; x_1, \cdots, x_p)^{(-1)^{p+1}}, \end{aligned}$$

by differentiating each row by the induced horizontal differentials,

$$\begin{aligned} E_2^{p,q} &= H^{p+1}(F, H^{q+1}(G, k^\times)), & E_2^{p,0} &\cong H^{p+1}(F, \hat{G}) \\ E_2^{0,q} &\cong \mathrm{Der}(F, H^{q+1}(G, k^\times)), & E_2^{0,0} &= \mathrm{Der}(F, \hat{G}) \end{aligned} \tag{4-5}$$

Since $k^\times$ is a trivial G-module, the sequence (4-2) turns into

$$\begin{aligned} 0 \to H^2(F, \mathrm{Der}(G, k^\times)) &\xrightarrow{i} H^1(\mathrm{Tot}(\mathrm{Hom}_\Sigma(P_i \otimes Q_j, k^\times))) \xrightarrow{p} \mathrm{Der}(F, H^2(G, k^\times)) \\ &\xrightarrow{d_2^{0,1}} H^3(F, \mathrm{Der}(G, k^\times)) \xrightarrow{i} H^2(\mathrm{Tot}(\mathrm{Hom}_\Sigma(P_i \otimes Q_j, k^\times))), \end{aligned}$$

From (3-11) and (3-12) we get the five-term exact sequence

$$0 \to H^2(F, \hat{G}) \to \mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) \to \mathrm{Der}(F, H^2(G, k^\times)) \xrightarrow{d_2^{0,1}} H^3(F, \hat{G}) \to H^4(\Sigma, X, k^\times). \quad \square$$

Note that, in the case that $\triangleright$ is a trivial action, the terms $E_2^{p,q}$ with $p \geq 1$, $q \geq 1$ of the second page of the spectral sequence associated to the semidirect product $\Sigma = F \ltimes G$ coincide with the second page of the Lyndon–Hochschild–Serre spectral sequence [Evens 1991].

Corollary 4.3. *Let* $(F, G, \triangleright, \triangleleft)$ *be a matched pair with trivial* $\triangleright$ *action and let* k *be a field. Then*:

(1) *If* $H^2(G, k^\times) = 1$ *then* $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) \cong H^2(F, \hat{G})$.

(2) *If* $(|F|, |\hat{G}|) = 1$, *then* $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) \cong \mathrm{Der}(F, H^2(G, k^\times))$.

(3) *If* G *is a perfect group*, *then* $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) = \mathrm{Der}(F, H^2(G, k^\times))$

(4) *If* $|F| = 2k + 1$ *and* $G = S_n$ *with* $n \geq 4$, *then*, $\mathrm{Opext}_{\triangleright,\triangleleft}(\mathbb{C}F, \mathbb{C}^G) = 0$.

Proof. Part (1) is straightforward.

(2) Since $(|F|, |\hat{G}|) = 1$, then $H^n(F, \hat{G}) = H^n(F, \hat{G}) = \{1\}$ and the result holds.

(3) Since the abelianization of G is trivial, then $\hat{G} \cong \{0\}$. Therefore, $H^2(F, \hat{G}) = H^2(F, \hat{G}) = \{0\}$, similarly $H^3(F, \hat{G}) = 0$, then, $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G) = \mathrm{Der}(F, H^2(G, k^\times))$.

(4) Under the given conditions $(|F|, |\hat{G}|) = 1$, as in (2). So $\mathrm{Opext}_{\triangleright,\triangleleft}(\mathbb{C}F, \mathbb{C}^G) = \mathrm{Der}(F, H^2(G, \mathbb{C}^\times))$. Now, $H^2(S_n, \mathbb{C}^\times) = \mathbb{Z}/2$, for $n \geq 4$ and $\mathrm{Der}(F, H^2(G, \mathbb{C}^\times)) = \mathrm{Hom}(F, \mathbb{Z}/2) = 0$, so the result holds. □

5. Computations

We compute some examples of the group $\mathrm{Opext}_{\triangleright,\triangleleft}(kF, k^G)$ for different semidirect products: the right action $\triangleright$ is trivial, so we denote the group $\mathrm{Opext}_{\triangleright\triangleleft}(kF, k^G)$ by $\mathrm{Opext}_{\triangleleft}(kF, k^G)$). The first calculation generalizes one from Masuoka [1997b].

Theorem 5.1. *Let k be a field. Let G be a group and $\mathbb{Z}/2 \ltimes (G \times G)$ be the semidirect product with $(a, b) \triangleleft 1 = (b, a)$. Then*

$$\mathrm{Opext}_{\triangleleft}(k\mathbb{Z}/2, k^{G\times G}) \cong H^2(G, k^\times) \oplus P_{\mathrm{Sym}}(G, G; k^\times),$$

where $P_{\mathrm{Sym}}(G, G; k^\times)$ is the groups of all symmetric bicharacters of G.

Proof. It follows from (2-2) that $H^n(\mathbb{Z}/2, H^1(G \times G, k^\times)) = 0$ for $n \geq 1$. Then, the sequence (4-4) implies that

$$\mathrm{Opext}_{\triangleleft}(k\mathbb{Z}/2, k^{G\times G}) \cong \mathrm{Der}(\mathbb{Z}/2, H^2(G \times G, k^\times)).$$

According to (2-3), we have $H^2(G \times G, k^\times) \overset{\psi}{\cong} H^2(G, k^\times) \oplus P(G, G, k^\times) \oplus H^2(G, k^\times)$. Given $\alpha \in Z^2(G \times G, k^\times)$, we have $(\bar{\alpha}_1, \bar{\alpha}_2, \phi_{\bar{\alpha}}) = \psi(\bar{\alpha}) \in H^2(G, k^\times) \oplus H^2(G, k^\times) \oplus P(G, G, k^\times)$ given by

$$\alpha_1(x, y) = \alpha((x, e), (y, e)), \quad \alpha_2(x, y) = \alpha((e, x), (e, y)), \quad \phi_{\bar{\alpha}}(x, y) = \frac{\alpha((x, e), (e, y))}{\alpha((e, y), (x, e))}.$$

Hence the induced action of $\bar{1} \in \mathbb{Z}/2$ on $H^2(G, k^\times) \times H^2(G, k^\times) \times P(G, G; k^\times)$ is

$$\begin{aligned} {}^{\bar{1}}(\alpha_1, \alpha_2, \phi_{\bar{\alpha}})(x, y) &= ({}^{\bar{1}}\alpha_1(x, y), {}^{\bar{1}}\alpha_2(x, y), {}^{\bar{1}}\phi_{\bar{\alpha}}(x, y)) \\ &= (\alpha_2(x, y), \alpha_1(x, y), \phi_{\bar{\alpha}}^{-1}(y, x)) \\ &:= (\alpha_2, \alpha_1, (\phi_{\bar{\alpha}}^T)^{-1})(x, y). \end{aligned}$$

Then $\alpha \in \mathrm{Der}(\mathbb{Z}/2, H^2(G \times G, k^\times))$ if and only if $\alpha := \alpha(\bar{1})$ satisfies $\alpha\, {}^{\bar{1}}\alpha = 1$, that is,

$$(\alpha_1, \alpha_2, \phi_\alpha)(\alpha_2, \alpha_1, (\phi_{\bar{\alpha}}^T)^{-1}) = 1 \Leftrightarrow \alpha_1 = \alpha_2^{-1},$$

and $\phi_{\bar{\alpha}}$ is a symmetric bicharacter. □

The following example includes the previous one in the case that $b = c = 1$, $a = 0$.

Theorem 5.2. *Let $\mathbb{Z}/2 \ltimes (\mathbb{Z}/n \oplus \mathbb{Z}/n)$ be the semidirect product where the action of $\mathbb{Z}/2$ on $(\mathbb{Z}/n \oplus \mathbb{Z}/n)$ is defined by the matrix*

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $\mathrm{Det}(A) = -1$. *Let k be a field such that $k^\times/(k^\times)^{2n} = 0$. Then,*

$$\mathrm{Opext}_{\rhd}(k\mathbb{Z}/2, k^{\mathbb{Z}/n\oplus\mathbb{Z}/n}) \cong \frac{\mathrm{Ker}(A-I)}{\mathrm{Im}(A+I)} \oplus \mu_n(k),$$

where $\mu_n(k)$ is the group of n-th roots of unity in k.

Proof. In this case the sequence (4-4) is given by

$$0 \to H^2(\mathbb{Z}/2, \mathrm{Der}(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \xrightarrow{i} \mathrm{Opext}_{\lhd}(k\mathbb{Z}/2, k^{\mathbb{Z}/n\oplus\mathbb{Z}/n})$$
$$\xrightarrow{\pi} \mathrm{Der}(\mathbb{Z}/2, H^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \xrightarrow{d_2^{0,1}} H^3(\mathbb{Z}/2, \mathrm{Der}(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \to H^4(\Sigma, X, A).$$

We will see that:

(i) $H^2(\mathbb{Z}/2, H^1(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \cong \mathrm{Ker}(A-I)/\mathrm{Im}(A+I)$.

(ii) $\mathrm{Der}(\mathbb{Z}/2, H^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \cong \mu_n(k)$.

(iii) $d_2^{0,1} = 0$.

Therefore, $\mathrm{Opext}_{\lhd}(k\mathbb{Z}/2, k^{\mathbb{Z}/n\oplus\mathbb{Z}/n})$ fits in a short exact sequence

$$0 \to \frac{\mathrm{Ker}(A-I)}{\mathrm{Im}(A+I)} \xrightarrow{i} \mathrm{Opext}_{\lhd}(k\mathbb{Z}/2, k^{\mathbb{Z}/n\oplus\mathbb{Z}/n}) \xrightarrow{\pi} \mu_n(k) \to 0, \tag{5-1}$$

moreover, we will see (5-1) is split, so

$$\mathrm{Opext}_{\lhd}(k\mathbb{Z}/2, k^{\mathbb{Z}/n\oplus\mathbb{Z}/n}) \cong \frac{\mathrm{Ker}(A-I)}{\mathrm{Im}(A+I)} \oplus \mu_n(k).$$

(i) It follows immediately from (2-2).

(ii) We identify $\wedge^2(\mathbb{Z}/n \oplus \mathbb{Z}/n)$ with the abelian group of alternating 2×2 matrices over the ring $\mathbb{Z}/n\mathbb{Z}$. Therefore, $\wedge^2(\mathbb{Z}/n \oplus \mathbb{Z}/n) \cong \mathbb{Z}/n$ and

$$H^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times) \cong \mathrm{Hom}(\wedge^2(\mathbb{Z}/n \oplus \mathbb{Z}/n), k^\times) \cong \mu_n(k),$$

where $\mu_n(k)$ is the group of n-th roots unit. Since $A^T M A = -M$ for all $M \in \wedge^2(\mathbb{Z}/n \oplus \mathbb{Z}/n)$ we have that

$$\mathrm{Der}(\mathbb{Z}/2, H^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \cong \mu_n(k).$$

(iii) To compute

$$d_2^{0,1} : \mu_n(k) = \mathrm{Der}(\mathbb{Z}/2, H^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)) \to H^3(\mathbb{Z}/2, \mathrm{Der}(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times)),$$

we follow (4-1). Given $\zeta \in \mu_n(k) = \mathrm{Der}(\mathbb{Z}/2, H^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times))$, we need to find

$$\gamma \in \mathrm{Map}_+((\mathbb{Z}/n \oplus \mathbb{Z}/n) \times (\mathbb{Z}/2)^2, k^\times) \cong C^2(\mathbb{Z}/2, C^1(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times))$$

such that

$$\delta_h(\alpha_\zeta) = \delta_v(\gamma), \tag{5-2}$$

where

$$\alpha_\zeta \in C^2(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times), \quad \alpha_\zeta(x, y) = \zeta^{x_1 y_2} \tag{5-3}$$

and then compute the cohomology class of $\delta_h(\gamma)$ in $H^3(\mathbb{Z}/2, \mathrm{Der}(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times))$. We have that

$$\delta_h(\alpha_\zeta)(1,1)(x, y) = \alpha_\zeta(Ax, Ay)\alpha_\zeta(x, y) = \zeta^{x^T \left(\begin{smallmatrix} ac & bc \\ bc & -ab \end{smallmatrix}\right) y}.$$

This is a bicharacter with associated quadratic form

$$\omega(x, y) = \zeta^{-acx^2 - 2bcxy + aby^2}.$$

Therefore, the cochain $\gamma \in C^2(\mathbb{Z}/2, C^1(\mathbb{Z}/n \oplus \mathbb{Z}/n, k^\times))$ defined by

$$\gamma(1,1) = \zeta^{-(acx^2)/2 - bcxy + (aby^2)/2}, \quad (x, y) \in \mathbb{Z}/n \oplus \mathbb{Z}/n, \tag{5-4}$$

and $\gamma(0,1) = \gamma(1,0) = \gamma(0,0) = 1$, satisfies (5-2).

Finally, the horizontal differential of γ is given by

$$\delta_h(\gamma)(1,1,1) = \gamma(1,1)\gamma(1,0)({}^1(\gamma(1,1))\gamma(0,1))^{-1} = \gamma(1,1)({}^1(\gamma(1,1))^{-1} = 1.$$

Hence, $d_2(\zeta) = 1$.

A section of π in the exact sequence (5-1) is given by cohomology of $s(\zeta) = (\alpha_\zeta, \gamma_\zeta)$, where α_ζ and γ_ζ are given by (5-3) and (5-4), respectively. It is clear from the definition that s is a group homomorphism, that is, (5-1) splits. □

Theorem 5.3. *Let F be an arbitrary group acting on a finite abelian group V with odd order. Suppose that $(k^\times)^n = k^\times$, where n is the exponent of V. Then*

$$\mathrm{Opext}_\triangleleft(kF, k^V) \cong H^2(F, \hat{V}) \oplus \mathrm{Der}(F, \wedge^2 \hat{V}),$$

where $\hat{V} = \mathrm{Hom}(V, k^\times)$.

Proof. By (2-4), we have $H^2(V, k^\times) \cong \wedge^2 \hat{V}$. The sequence (4-4) is given by

$$0 \to H^2(F, \hat{V}) \xrightarrow{i} \mathrm{Opext}_\triangleleft(kF, k^V) \xrightarrow{\pi} \mathrm{Der}(F, \wedge^2 \hat{V}) \xrightarrow{d_2} \to H^3(F, \hat{V}) \to H^4(\Sigma, X, A),$$

where $\Sigma = V \rtimes F$. We will see that $d_2 = 0$ and the resulting short exact sequence splits, hence we get the result.

Let $\alpha \in \mathrm{Der}(F, \wedge^2\hat{V})$, that is, $\alpha : F \to \wedge^2\hat{V}$ such that $\alpha(gh) = {}^g\alpha(h)\alpha(g)$. By (2-5), α can be identified with a map $\alpha : F \to H^2(V, k^\times)$ which can be lifted to a map $\tilde{\alpha} : F \to Z^2(V, k^\times)$ considering that, since V has odd order, the map

$$\mathrm{Alt} : \wedge^2\hat{V} \to \wedge^2\hat{V}$$

given by $\mathrm{Alt}(\phi)(x, y) = \phi(x, y)/\phi(y, x) = \phi(x, y)^2$ is an isomorphism, so we can define the lifting map by $\tilde{\alpha} : F \to Z^2(V, k^\times)$ by

$$\tilde{\alpha}(g) = \alpha(g)^{1/2}.$$

In order to compute $d_2(\alpha)$, we must find a function $\gamma \in C^2(F, C^1(V, k^\times))$ such that

$$\delta(\gamma(g, h)) = \frac{{}^g\tilde{\alpha}(h)\tilde{\alpha}(g)}{\tilde{\alpha}(gh)} = \frac{\widetilde{{}^g\alpha}(h)\tilde{\alpha}(g)}{\tilde{\alpha}(gh)} = \left(\frac{{}^g\alpha(h)\alpha(g)}{\alpha(gh)}\right)^{1/2} = 1.$$

Hence γ can be taken to be the constant cochain and, therefore, $d_2(\alpha) = 1$ for all $\alpha \in \mathrm{Der}(F, \wedge^2\hat{V})$. □

Corollary 5.4. *Let $F = C_m = \langle\sigma\rangle$ be a cyclic group of order m acting on a finite abelian group V with odd order. Suppose that $(k^\times)^n = k^\times$, where n is the exponent of V. Then*

$$\mathrm{Opext}_{\triangleleft}(kF, k^V) \cong \{\psi \in \hat{V} : \sigma\psi = \psi\}/\{N_\sigma\psi : \psi \in \hat{V}\} \oplus \{b \in \wedge^2\hat{V} : N_\sigma b = 0\},$$

where $N_\sigma = 1 + \sigma + \cdots + \sigma^{m-1}$.

An example with nontrivial differential d_2. The next example illustrates the fact that the hypothesis in Theorem 5.3 stating that the order of the order of V must be odd, can not be avoided since otherwise the differential d_2 can be not trivial.

Remarks 5.5. (a) Let G be an elementary abelian p-group of rank n. Once a basis of G is fixed, using the isomorphism (2-5) we can identify $H^2(G, \mathbb{C}^\times)$ with alternating matrices over $\mathbb{Z}/p$. A representative 2-cocycle $\alpha_M \in H^2(G, \mathbb{C}^\times)$ corresponding to a matrix M is defined by

$$\alpha_M(\mathbf{x}, \mathbf{y}) = \exp\left(\frac{2\pi i}{n}\mathbf{x}^{\mathbf{T}}\tilde{M}\mathbf{y}\right), \tag{5-5}$$

where $\tilde{M}$ is the upper triangular part of M.

(b) Let $F = \langle t_1\rangle \oplus \langle t_2\rangle$ be a product of cyclic groups and let M be an left F-module. If $(\mathbb{Z} \xleftarrow{\epsilon} P_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon} P_i')$ are periodic resolutions as in (2-1) for the groups $\langle t_1\rangle$ and $\langle t_2\rangle$ respectively, then the total complex $\mathrm{Tot}(P \otimes P')$ is a free F-resolution of $\mathbb{Z}$. Therefore, given a F-module M, we can compute

$H^*(F, M)$ as the cohomology of total complex of

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
M \arrow[u, "t_2-1"] \arrow[r, "t_1-1"] & M \arrow[u, "t_2-1"] \arrow[r, "N_{t_1}"] & M \arrow[u, "t_2-1"] \arrow[r, "t_1-1"] & \cdots \\
M \arrow[u, "N_{t_2}"] \arrow[r, "t_1-1"] & M \arrow[u, "N_{t_2}"] \arrow[r, "N_{t_1}"] & M \arrow[u, "N_{t_2}"] \arrow[r, "t_1-1"] & \cdots \\
M \arrow[u, "t_2-1"] \arrow[r, "t_1-1"] & M \arrow[u, "t_2-1"] \arrow[r, "N_{t_1}"] & M \arrow[u, "t_2-1"] \arrow[r, "t_1-1"] & \cdots
\end{tikzcd}$$

Since we are mainly interested in $H^2(F, M)$, the second and third differentials $\delta_2 : M \oplus M \to M \oplus M \oplus M$ and $\delta_3 : M \oplus M \oplus M \to M \oplus M \oplus M \oplus M$ of the total complex are given by

$$\delta_2(A, B) = (A + {}^{g_1}A,\, A - {}^{g_2}A - (B - {}^{g_1}B),\, B + {}^{g_2}B), \tag{5-6}$$

$$\delta_3(A, B, C) = ({}^{g_1}A - A,\, {}^{g_2}A - A + B + {}^{g_1}B,\, B + {}^{g_2}B + {}^{g_1}C - C,\, {}^{g_2}C - C). \tag{5-7}$$

Lemma 5.6. *Let $F = \langle t_1, t_2\rangle$, $G = \langle s_1, \dots, s_4\rangle$ be elementary abelian 2-groups of rank 2 and 4, respectively. Consider the (right) action of F on G determined by the matrices*

$$F_1 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 1&1&1&0\\ 0&1&0&1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 1&0&1&0\\ 1&1&0&1 \end{pmatrix}.$$

and the induced left action of F on $H^2(G, \mathbb{C}^\times)$. Then:

(1) *The group $\mathrm{Der}(F, H^2(G, \mathbb{C}^\times))$ is in correspondence with the set of pairs of matrices*

$$A = \begin{pmatrix} 0&0&b&c\\ 0&0&d&e\\ b&d&0&f\\ c&e&f&0 \end{pmatrix}, \quad B = \begin{pmatrix} 0&0&b'&c'\\ 0&0&d'&e'\\ b'&d'&0&f'\\ c'&e'&f'&0 \end{pmatrix}$$

with entries in $\mathbb{Z}/p$, such that

$$\begin{aligned} c' + b' + d' &= 0 \\ c + d + e &= 0 \\ b + e + c' + d' + e &= 0. \end{aligned} \tag{5-8}$$

(2) *The group $H^2(F, H^1(G, \mathbb{C}^\times))$ is isomorphic to $\mathbb{Z}/2^2$.*

Proof. (1) By (5-6), elements in $\mathrm{Der}(F, H^2(G, \mathbb{C}^\times))$ are in correspondence with pairs (A, B) of alternating 4×4 matrices such that

$$\begin{aligned} A + F_1 A F_1^T &= 0 \\ B + F_2 B F_2^T &= 0 \\ A - F_2 A F_2^T - B + F_1 B F_1^T &= 0. \end{aligned} \tag{5-9}$$

The system (5-9) is equivalent to (5-8).

(2) In order to compute $H^2(F, H^1(G, \mathbb{C}^\times))$ we use the canonical identification

$$H^1(G, \mathbb{C}^\times) = \mathrm{Hom}(G, \mathbb{C}^\times) \cong G$$

as left F-modules. By (5-7), we have that $\mathrm{Ker}(\delta_3)$ is in correspondence with 4×3 matrices $S = [n_a, n_b, n_c]$ over $\mathbb{Z}/2$ such that

$$(F_1 - \mathrm{I})n_a = 0, \qquad (F_2 - \mathrm{I})n_c = 0,$$
$$(\mathrm{I} - F_1)n_c = (F_2 + \mathrm{I})n_b, \quad (\mathrm{I} + F_1)n_b = (\mathrm{I} - F_2)n_a.$$

Thus, the space $\mathrm{Ker}(\delta_3)$ corresponds with all 4×3 matrices over $\mathbb{F}_2$ such that $S_{ij} = 0$ for $1 \le i \le 2$ and $1 \le j \le 3$. On the other hand, by (5-6) we have

$$\mathrm{Im}(\delta_2) = \{(l_a + F_1 l_a, l_a - l_b + F_1 l_b - F_2 l_a, l_b + F_2 l_b) : l_a, l_b \in N\},$$

that is, $\mathrm{Im}(\delta_2)$ is in correspondence with all matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1 + x_2 & x_1 + y_1 + y_2 & y_1 \\ x_2 & x_1 + x_2 + y_2 & y_1 + y_2 \end{pmatrix}$$

where $x_i, y_i \in \mathbb{Z}/2$. Hence $H^2(F, H^1(G, \mathbb{C}^\times)) \cong \mathbb{Z}/2^2$. □

Lemma 5.7. *Let $\Sigma = F \ltimes G$ be a semidirect product and let*

$$\cdots \to R_3 \to R_2 \to R_1 \to R_0, \tag{5-10}$$

be a free resolution of a right F-modules M. The action of F on R_i can be extended to an action of Σ by $r \cdot (f, g) = r \cdot f$. With this action, the sequence (5-10) turns out to be a relatively G-projective resolution of the right Σ-module M. □

Theorem 5.8. *Let $F = \langle t_1, t_2 \rangle$ and $G = \langle s_1, \dots, s_4 \rangle$ be elementary abelian 2-groups of rank 2 and 4, respectively. Consider the (right) action of F on G determined by the matrices*

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then $\mathrm{Opext}_\triangleleft(\mathbb{C}F, \mathbb{C}^G) \cong (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/4)^2$.

Proof. The five-term exact sequence (4-4) for this case is

$$0 \to H^2(F, \mathrm{Der}(G, \mathbb{C}^\times)) \xrightarrow{i} \mathrm{Opext}_{\triangleleft}(\mathbb{C}F, \mathbb{C}^G)$$
$$\xrightarrow{\pi} \mathrm{Der}(F, H^2(G, \mathbb{C}^\times)) \xrightarrow{d_2} H^3(F, \mathrm{Der}(G, \mathbb{C}^\times)) \to H^2(\mathrm{Tot}(M^{*,*})).$$

For this computation the relatively F-projective resolution used in Theorem 4.1 $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i, \delta_i')$ will be as in Proposition 3.5, and the relatively G-projective resolution $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i, \delta_i)$ will be the total complex of the tensor product of the cyclic resolutions for $\langle t_1\rangle$ and $\langle t_2\rangle$, considered as a relatively G-projective Σ-resolution of $\mathbb{Z}$ using Lemma 5.7.

The zeroth page of the spectral sequence in Theorem 4.1 is given by

$$E_0^{i,j} = \mathrm{Hom}_{\mathbb{Z}\Sigma}(P_{i+1} \otimes Q_{j+1}, \mathbb{C}^\times) = \mathrm{Hom}_{\mathbb{Z}\Sigma}\left(\bigoplus_{k=1}^{i+2} \mathbb{Z}F \otimes \mathbb{Z}G[G]^{j+1}, \mathbb{C}^\times\right)$$

The horizontal and vertical differentials d_h and d_v are induced by the differentials of the resolutions $(\mathbb{Z} \xleftarrow{\epsilon_P} P_i)$ and $(\mathbb{Z} \xleftarrow{\epsilon_Q} Q_i)$, respectively. Each Σ-module $\mathbb{Z}F \otimes \mathbb{Z}G[G]^{j+1}$ is free with basis

$$\{e \otimes [g_1|\cdots|g_{j+1}] : e \neq g_1, \cdots g_{j+1} \in N\}.$$

Therefore, an element in $E_0^{i,j}$ is defined by a tuple $(h_1, \dots, h_{i+2})$ with $h_k \in C^{j+1}(F, \mathbb{C}^\times)$, where $h_k(f_1, \cdots, f_{j+1}) = 1$ if any of the entries is the identity of F. Similarly, the differentials $d_0^{k,0}$: $\mathrm{Hom}_\Sigma(P^{k+1} \otimes Q^1, \mathbb{C}^\times)$ of the zeroth page, induced by the vertical differentials of the double complex are given by

$$d_0(f)(e \otimes [g_1|g_2]) = f(e \otimes (g_1[g_2] - [g_1 g_2] + [g_1])).$$

Since we are considering $\mathbb{C}^\times$ as a trivial Σ-module, the elements in $E_1^{k,0} = \mathrm{Ker}(d_0^{k,0})$ are in correspondence to tuples $(\chi_1, \dots, \chi_{i+2})$ with $\chi_i \in \hat{G}$.

First, we will compute $\mathrm{Ker}(d_2)$. By Lemma 5.6, the group $E_2^{0,1} = \mathrm{Der}(F, H^2(G, \mathbb{C}^\times))$ is in correspondence with pairs (A, B) of alternating 4×4 matrices satisfying the equations in (5-8). According to Remarks 5.5(a), a representative element for (A, B) in $E_0^{0,1} = \mathrm{Hom}_\Sigma((\mathbb{Z}F \oplus \mathbb{Z}F) \otimes \mathbb{Z}G[G]^2, \mathbb{C}^\times)$ is defined by $\alpha = (\alpha_A, \alpha_B)$, where α_A, α_B are 2-cocycles defined in (5-5).

By (4-1), we have that $d_2(A, B) = \overline{d_h(\gamma)}$ were

$$\gamma \in E_0^{1,0} = \mathrm{Hom}_\Sigma((\mathbb{Z}F \oplus \mathbb{Z}F \oplus \mathbb{Z}F) \otimes \mathbb{Z}G[G]),$$

satisfies $d_h(\alpha_A, \alpha_B) = d_v(\gamma)$.

By (5-6) we have that

$$d_h(\alpha_A, \alpha_B) = (b_{M_1}, b_{M_2}, b_{M_3}) \in E_0^{1,1} = \mathrm{Hom}_\Sigma((\mathbb{Z}F \oplus \mathbb{Z}F \oplus \mathbb{Z}F) \otimes \mathbb{Z}G[G]^2, \mathbb{C}^\times), \tag{5-11}$$

where $b_{M_i}(x,y)=(-1)^{x^T M_i y}$ and

$$M_1=\tilde{A}+F_1\tilde{A}F_1^T=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&b+d&d\\0&0&d&e\end{pmatrix}$$

$$M_2=\tilde{A}-F_2\tilde{A}F_2^T-(\tilde{B}-F_1\tilde{B}F_1^T)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&b+b'+d'&b+d+d'\\0&&b+d+d'&c+e+e'\end{pmatrix}$$

$$M_3=\tilde{B}+F_2\tilde{B}F_2^T=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&b'&c'\\0&0&c'&c'+e'\end{pmatrix}.$$

The cochain $\gamma=(\gamma_{M_1},\gamma_{M_2},\gamma_{M_3})\in E_0^{1,0}=\mathrm{Hom}_\Sigma((\mathbb{Z}F\oplus\mathbb{Z}F\oplus\mathbb{Z}F)\otimes\mathbb{Z}G[G]^2,\mathbb{C}^\times)$ defined by

$$\begin{aligned}\gamma_{M_1}(\boldsymbol{x})&=\exp\bigl(-\tfrac{\pi}{2}((b+d)x_3^2+2dx_3x_4+ex_4^2)\bigr),\\ \gamma_{M_2}(\boldsymbol{x})&=\exp\bigl(-\tfrac{\pi}{2}((b+b'+d')x_3^2+2(b+d+d')x_3x_4+(c+e+e')x_4^2)\bigr),\\ \gamma_{M_3}(\boldsymbol{x})&=\exp\bigl(-\tfrac{\pi}{2}(b'x_3^2+2c'x_3x_4+(c'+e')x_4^2)\bigr),\end{aligned}\tag{5-12}$$

satisfies (5-11). Therefore,

$$\begin{aligned}\delta_h(\gamma_{M_1},\gamma_{M_2},\gamma_{M_3})&=\left(\frac{{}^{t_1}\gamma_{M_1}}{\gamma_{M_1}},\frac{{}^{t_2}\gamma_{M_1}\gamma_{M_2}({}^{t_1}\gamma_{M_2})}{\gamma_{M_1}},\frac{\gamma_{M_2}({}^{t_2}\gamma_{M_2})\,{}^{t_1}\gamma\,\gamma_{M_3}}{\gamma_{M_3}},\frac{{}^{t_2}\gamma_{M_3}}{\gamma_{M_3}}\right)\\ &=(1,\gamma_{M_2}^2,\gamma_{M_2}^2,1)\in\ker(d_1:E_1^{2,0}\to E_1^{3,0}).\end{aligned}$$

Since G is an elementary abelian 2-group, we will use the canonical identification of $\hat{G}$ with G. Under this identification we have that $\gamma_{M_2}^2=(0,0,b+b'+d',c+e+e')$.

The pair (A,B) belongs to $\mathrm{Ker}(d_2)$ if and only if $(0,\gamma_{M_2}^2,\gamma_{M_2}^2,0)$ belongs to the image of $d_1:E_1^{1,0}\to E_1^{2,0}$ if and only if there exists $(\mu_A,\mu_B,\mu_C)\in E_1^{1,0}=F^{\times 3}$ such that $d_h(\mu_A,\mu_B,\mu_C)=(0,\gamma_{M_2}^2,\gamma_{M_2}^2,0)$. This means that

$$(F_1-I)\mu_A=0,\quad (F_2-I)\mu_C=0,\quad (F_2-I)\mu_A+(F_1+I)\mu_B=\gamma_{M_2}^2,\quad (F_1-I)\mu_C+(F_2+I)\mu_B=\gamma_{M_2}^2.$$

From this equation we obtain $b+b'+d'=c+e+e'=0$. Joining these two equations with (5-8) we get a system of equation with 5 free variables, hence $\mathrm{Ker}(d_2)=(\mathbb{Z}/2)^5$.

Hence we have the exact sequence

$$0\to H^2(F,G)\to \mathrm{Opext}_{\triangleleft}(\mathbb{C}F,\mathbb{C}^G)\xrightarrow{\pi}\mathrm{Ker}(d_2)\to 0.\tag{5-13}$$

An element in $\mathrm{Ker}(d_2)$ is represented by a pair of matrices A,B as in Lemma 5.6. Let us assign $c'=1$ and consider the remaining variables zero, and let us call the respective pair of matrices (A_c',B_c').

A section of the sequence (5-13) send (A_c, B_c) to the class of the extension

$$(\alpha_c, \gamma_c) \in H^1(\mathrm{Hom}_\Sigma(\mathrm{Tot}(D_{*,*}, \mathbb{C}^*)) \cong \mathrm{Opext}_\triangleleft(\mathbb{C}F, \mathbb{C}^G)$$

where α_c is the 2-cocycle associated to (A'_c, B'_c) and γ_c is given according to (5-12), by

$$\gamma_{M_1}(\boldsymbol{x}) = \exp\bigl(-\tfrac{\pi}{2}(x_3^2 + x_4^2)\bigr), \quad \gamma_{M_2}(\boldsymbol{x}) = \exp\bigl(-\tfrac{\pi}{2}(x_3^2)\bigr), \quad \gamma_{M_3}(\boldsymbol{x}) = \exp\bigl(-\tfrac{\pi}{2}(x_4^2)\bigr).$$

It can be verified that the class of (α_c, γ_c) has order 4 in $\mathrm{Opext}_{\triangleright,\triangleleft}(\mathbb{C}F, \mathbb{C}^G)$. In the same way, if we take the variable d' to be 1 and consider the remaining variables null we get an element (α_d, γ_d) of order 4. Any other element outside the subgroup $\langle(\alpha_c, \gamma_c), (\alpha_d, \gamma_d)\rangle \cong (\mathbb{Z}/4)^2$ has order 2, otherwise the order of $\mathrm{Opext}_{\triangleright,\triangleleft}(\mathbb{C}F, \mathbb{C}^G)$ could not be 2^7. That is why $\mathrm{Opext}_\triangleleft(\mathbb{C}F, \mathbb{C}^G) \cong (\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/4)^2$. Since $H_n(P) = H_n(Q) = 0$ for $n > 0$, then $H_n(P \otimes Q) = 0$ for $n > 1$ and $H_1(P \otimes Q) = \mathrm{Tor}_1(H_0(P), H_q(Q))$. $\square$

Acknowledgements

We thank Paul Bressler and Bernardo Uribe for kindly answering some questions and for very useful conversations.

References

[Adamson 1954] I. T. Adamson, "Cohomology theory for non-normal subgroups and non-normal fields", *Proc. Glasgow Math. Assoc.* **2** (1954), 66–76. MR Zbl

[Alperin 1986] J. L. Alperin, *Local representation theory*, Cambridge Studies in Advanced Mathematics **11**, Cambridge University Press, 1986. MR Zbl

[Andruskiewitsch and Devoto 1995] N. Andruskiewitsch and J. Devoto, "Extensions of Hopf algebras", *Algebra i Analiz* **7**:1 (1995), 22–61. In Russian; translated in *St. Petersburg Math. J.* **7**:1 (1996), 17–52. MR Zbl

[Auslander and Solberg 1993] M. Auslander and O. . Solberg, "Relative homology and representation theory, I: Relative homology and homologically finite subcategories", *Comm. Algebra* **21**:9 (1993), 2995–3031. MR Zbl

[Baaj et al. 2005] S. Baaj, G. Skandalis, and S. Vaes, "Measurable Kac cohomology for bicrossed products", *Trans. Amer. Math. Soc.* **357**:4 (2005), 1497–1524. MR Zbl

[Curtis and Reiner 1990] C. W. Curtis and I. Reiner, *Methods of representation theory, I: With applications to finite groups and orders*, John Wiley & Sons, New York, 1990. MR Zbl

[Evens 1991] L. Evens, *The cohomology of groups*, The Clarendon Press, Oxford University Press, New York, 1991. MR Zbl

[Hochschild 1956] G. Hochschild, "Relative homological algebra", *Trans. Amer. Math. Soc.* **82** (1956), 246–269. MR Zbl

[Hofstetter 1994] I. Hofstetter, "Extensions of Hopf algebras and their cohomological description", *J. Algebra* **164**:1 (1994), 264–298. MR Zbl

[Kac 1969] G. I. Kac, "Extensions of groups to ring groups", *Math. USSR, Sb.* **5** (1969), 451–474. Zbl

[Karpilovsky 1985] G. Karpilovsky, *Projective representations of finite groups*, Monographs and Textbooks in Pure and Applied Mathematics **94**, Marcel Dekker, New York, 1985. MR Zbl

[Kashina 2000] Y. Kashina, "Classification of semisimple Hopf algebras of dimension 16", *J. Algebra* **232**:2 (2000), 617–663. MR Zbl

[Majid 1990] S. Majid, "Physics for algebraists: noncommutative and noncocommutative Hopf algebras by a bicrossproduct construction", *J. Algebra* **130**:1 (1990), 17–64. MR Zbl

[Masuoka 1995] A. Masuoka, "Self-dual Hopf algebras of dimension p^3 obtained by extension", *J. Algebra* **178**:3 (1995), 791–806. MR Zbl

[Masuoka 1997a] A. Masuoka, "Calculations of some groups of Hopf algebra extensions", *J. Algebra* **191**:2 (1997), 568–588. MR Zbl

[Masuoka 1997b] A. Masuoka, "Corrigendum: "Calculations of some groups of Hopf algebra extensions" [J. Algebra **191** (1997), no. 2, 568–588; MR1448809 (98i:16042)]", *J. Algebra* **197**:2 (1997), 656. MR

[Masuoka 1999] A. Masuoka, "Extensions of Hopf algebras", Trabajo de matemática 41/99, FaMAF, Universidad Nacional de Córdoba (Argentina), 1999. Zbl

[Masuoka 2000] A. Masuoka, "Extensions of Hopf algebras and Lie bialgebras", *Trans. Amer. Math. Soc.* **352**:8 (2000), 3837–3879. MR Zbl

[Masuoka 2002] A. Masuoka, "Hopf algebra extensions and cohomology", pp. 167–209 in *New directions in Hopf algebras*, edited by S. Montgomery and H.-J. Schneider, Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, 2002. MR Zbl

[Masuoka 2003] A. Masuoka, "Cohomology and coquasi-bialgebra extensions associated to a matched pair of bialgebras", *Adv. Math.* **173**:2 (2003), 262–315. MR Zbl

[McCleary 2001] J. McCleary, *A user's guide to spectral sequences*, 2nd ed., Cambridge Studies in Advanced Mathematics **58**, Cambridge University Press, 2001. MR Zbl

[Natale 1999] S. Natale, "On semisimple Hopf algebras of dimension pq^2", *J. Algebra* **221**:1 (1999), 242–278. MR Zbl

[Natale 2001] S. Natale, "On semisimple Hopf algebras of dimension pq^2, II", *Algebr. Represent. Theory* **4**:3 (2001), 277–291. MR Zbl

[Natale 2004] S. Natale, "On semisimple Hopf algebras of dimension pq^r", *Algebr. Represent. Theory* **7**:2 (2004), 173–188. MR Zbl

[Singer 1972] W. M. Singer, "Extension theory for connected Hopf algebras", *J. Algebra* **21** (1972), 1–16. MR Zbl

[Snapper 1964] E. Snapper, "Cohomology of permutation representations, I: Spectral sequences", *J. Math. Mech.* **13** (1964), 133–161. MR Zbl

[Takeuchi 1981] M. Takeuchi, "Matched pairs of groups and bismash products of Hopf algebras", *Comm. Algebra* **9**:8 (1981), 841–882. MR Zbl

Communicated by Susan Montgomery
Received 2018-06-15 Revised 2018-09-18 Accepted 2019-02-22

cn.galindo1116@uniandes.edu.co *Departamento de Matemáticas, Universidad de Los Andes, Bogota, Colombia*

yk.morales964@uniandes.edu.co *Departamento de Matemáticas, Universidad de los Andes, Bogota, Colombia*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2019 is US $/year for the electronic version, and $/year (+$, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 13 No. 5 2019