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Blow-ups and class field theory for curves

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We propose another proof of geometric class field theory for curves by considering blow-ups of symmetric products of curves.

1. Introduction

Geometric class field theory gives a geometric description of the abelian coverings of a curve by using generalized jacobian varieties. Let us recall its precise statement. Let *C* be a projective smooth curve over a perfect field *k*. We assume that *C* is geometrically connected over *k*. Fix a modulus m, i.e., an effective Cartier divisor of *C* and let *U* be its complement in *C*. Denote by $\operatorname{Pic}_{C,\mathfrak{m}}^{0}$ the corresponding generalized jacobian variety. Let $G^{0} \to \operatorname{Pic}_{C,\mathfrak{m}}^{0}$ be an étale isogeny of smooth commutative algebraic groups and $G^{1} \to \operatorname{Pic}_{C,\mathfrak{m}}^{1}$ be a compatible morphism of torsors. We call such a pair ($G^{0} \to \operatorname{Pic}_{C,\mathfrak{m}}^{0}$, $G^{1} \to \operatorname{Pic}_{C,\mathfrak{m}}^{1}$) a *covering of* ($\operatorname{Pic}_{C,\mathfrak{m}}^{0}$, $\operatorname{Pic}_{C,\mathfrak{m}}^{1}$). A covering ($G^{0} \to \operatorname{Pic}_{C,\mathfrak{m}}^{0}$, $G^{1} \to \operatorname{Pic}_{C,\mathfrak{m}}^{1}$) is called *connected abelian* if G^{0} is connected and $G^{0} \to \operatorname{Pic}_{C,\mathfrak{m}}^{0}$ is an abelian isogeny. There is a natural map from *U* to $\operatorname{Pic}_{C,\mathfrak{m}}^{1}$ sending a point of *U* to its associated invertible sheaf with a trivialization. Geometric class field theory states:

Theorem 1.1. Let C be a projective smooth geometrically connected curve over a perfect field k. Fix a modulus \mathfrak{m} of C and denote its complement by U. Let $\operatorname{Pic}_{C,\mathfrak{m}}^0$ be the generalized jacobian variety with modulus \mathfrak{m} . Then a connected abelian covering $(G^0 \to \operatorname{Pic}_{C,\mathfrak{m}}^0, G^1 \to \operatorname{Pic}_{C,\mathfrak{m}}^1)$ pulls back by the natural map $U \to \operatorname{Pic}_{C,\mathfrak{m}}^1$ to a geometrically connected abelian covering of U whose ramification is bounded by \mathfrak{m} . Conversely, every such covering is obtained in this way.

Originally this theorem was proved by M. Rosenlicht [1954]. S. Lang [1956] generalized his results to an arbitrary algebraic variety. Their works are explained in detail in Serre's book [1988].

On the other hand, in 1980s, P. Deligne found another proof for the tamely ramified case by using symmetric powers of curves [Laumon 1990]. The aim of this paper is to complete his proof by considering blow-ups of symmetric powers of curves.

We have learned that Q. Guignard has done similar work [2019].

Actually we prove a variant of Theorem 1.1 now stated.

Theorem 1.2. There is an isomorphism of groups between the subgroup of $H^1(U, \mathbb{Q}/\mathbb{Z})$ consisting of a character χ such that $Sw_P(\chi) \leq n_P - 1$ for all points $P \in \mathfrak{m}$, where n_P is the multiplicity of \mathfrak{m}

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at *P*, and the subgroup of $\mathrm{H}^{1}(\operatorname{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ consisting of ρ which is multiplicative, i.e., the self-external product $\rho \boxtimes 1 + 1 \boxtimes \rho$ on $\operatorname{Pic}_{C,\mathfrak{m}} \times_{k} \operatorname{Pic}_{C,\mathfrak{m}}$ equals to $m^{*}\rho$, the pullback of ρ by the multiplication map $m : \operatorname{Pic}_{C,\mathfrak{m}} \times_{k} \operatorname{Pic}_{C,\mathfrak{m}}$.

The relation between Theorems 1.1 and 1.2 will be explained in Section 4.

When k is algebraically closed, Theorem 1.2 can be stated as follows. Let ρ be a multiplicative element of $\mathrm{H}^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$. Fix a closed point $P \in \mathrm{Pic}_{C,\mathfrak{m}}^1$. The multiplicativity of ρ implies that, for an integer d, the pullback of ρ^d by the multiplication by $dP \operatorname{Pic}_{C,\mathfrak{m}}^0 \to \operatorname{Pic}_{C,\mathfrak{m}}^d$ coincides with ρ^0 . In this way, Theorem 1.2 can be restated as follows:

Theorem 1.3. Assume that k is algebraically closed. Then there is an isomorphism of groups between the subgroup of $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z})$ consisting of a character χ such that $\mathrm{Sw}_P(\chi) \leq n_P - 1$ for all points $P \in \mathfrak{m}$ and the subgroup of $\mathrm{H}^1(\mathrm{Pic}^0_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ consisting of a multiplicative element ρ^0 , i.e., the self-external product $\rho^0 \boxtimes 1 + 1 \boxtimes \rho^0$ on $\mathrm{Pic}^0_{C,\mathfrak{m}} \times_k \mathrm{Pic}^0_{C,\mathfrak{m}}$ equals to $m^* \rho^0$, the pullback of ρ^0 by the multiplication map $m : \mathrm{Pic}^0_{C,\mathfrak{m}} \times_k \mathrm{Pic}^0_{C,\mathfrak{m}} \to \mathrm{Pic}^0_{C,\mathfrak{m}}$.

Here we summarize the construction of this paper. In Section 2, we recall the definition and properties of (refined) Swan conductors, and make a calculation on the Swan conductors of symmetric products of characters. We construct compactifications of the Abel–Jacobi maps $U^{(d)} \rightarrow \text{Pic}_{C,\mathfrak{m}}^d$ and study their properties in Section 3. The main result of this section is that the compactifications can be identified with open subschemes of blow-ups of $C^{(d)}$. In Section 4, we finish the proof of Theorems 1.1 and 1.2 by combining the results in the previous sections.

Throughout this paper, we use the following conventions: We identify an effective Cartier divisor with the associated closed subscheme. For an object defined on a scheme S (e.g., an S-scheme, a locally free sheaf, a vector bundle, and so on) and a S-scheme T, we denote its pullback to T by the same letter, unless there may be ambiguity. We denote the category of S-schemes by Sch/S. For a category C, we call a functor $C^{op} \rightarrow (Set)$, from the opposite category of C to the category of sets (Set), a presheaf on C.

2. Preliminaries

In this section, we recall basic properties of Witt vectors and refined Swan conductors, and calculate the Swan conductors of symmetric products of characters. Fix a prime number p.

Reminder on the refined Swan conductor. Let A be a ring of characteristic p. Let m be an integer ≥ 0 . We denote by $W_{m+1}(A)$ the ring of Witt vectors of length m + 1 with coefficients in A, and write its elements as (a_0, a_1, \ldots, a_m) . Let \mathcal{O}_A be the structure sheaf of rings on the étale topos of Spec(A).

Let *F* be the absolute Frobenius map $\mathcal{O}_A \to \mathcal{O}_A$, i.e., sending $x \mapsto x^p$, and denote the ring homomorphism $W_{m+1}(\mathcal{O}_A) \to W_{m+1}(\mathcal{O}_A)$ induced from *F* by the same letter *F*. The short exact sequence

$$0 \to \mathbb{Z}/p^{m+1}\mathbb{Z} \to W_{m+1}(\mathcal{O}_A) \xrightarrow{F-1} W_{m+1}(\mathcal{O}_A) \to 0$$

of étale sheaves on Spec(A) defines the boundary map

$$\delta_{m+1,A}$$
: $W_{m+1}(A) \to H^1(\operatorname{Spec}(A), \mathbb{Z}/p^{m+1}\mathbb{Z}).$

The boundary map is surjective, hence $W_{m+1}(A)/\operatorname{Im}(F-1) \xrightarrow{\sim} H^1(\operatorname{Spec}(A), \mathbb{Z}/p^{m+1}\mathbb{Z})$, the map it induces, is an isomorphism. The boundary map $\delta_{m+1,A}$ is natural in A. In other words, for a morphism $f: A \to B$ of rings of characteristic p, the diagram

is commutative, where the vertical maps are the canonical ones induced from f.

Let (R, π) be a DVR of equal characteristic p and K be its field of fractions. Let v_R be its normalized valuation. Let m be an integer ≥ 0 . We extend the valuation v_R to $W_{m+1}(K)$ by setting

$$v_R((a_0,\ldots,a_m)) := \min_i \{p^{m-i}v_R(a_i)\}.$$

We define an increasing exhaustive filtration on $W_{m+1}(K)$ by setting, for $n \in \mathbb{Z}$, fil_n $W_{m+1}(K)$ to be the subgroup of $W_{m+1}(K)$ consisting of elements (a_0, \ldots, a_m) such that

$$v_R((a_0,\ldots,a_m))\geq -n.$$

Define an increasing exhaustive filtration $\operatorname{fil}_n \operatorname{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ of $\operatorname{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ by the image of $\operatorname{fil}_n \operatorname{W}_{m+1}(K)$ through the boundary map $\delta_{m+1,K}$.

For any $\chi \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$, the Swan conductor of χ , $Sw_R(\chi)$, is the smallest integer $n \ge 0$ such that $\chi \in \operatorname{fil}_n H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ [Brylinski 1983; Kato 1989]. When *R* is henselian and the residue field is perfect, this is the same as the classical Swan conductor [Kato 1989, Proposition (6.8)].

Lemma 2.1. Let *R* and *K* be as above. Take $\chi \in H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$.

- (1) The subgroup $\operatorname{fil}_0 \operatorname{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ of $\operatorname{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ coincides with the image of the map $\operatorname{H}^1(\operatorname{Spec}(R), \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \operatorname{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$, *i.e.*, the group of unramified characters.
- (2) Let \hat{R} be the completion of R and \hat{K} be its field of fractions. Denote the restriction of χ to \hat{K} by $\hat{\chi}$. Then, the equality $Sw_R(\chi) = Sw_{\hat{R}}(\hat{\chi})$ holds.

Proof. (1) This follows from the commutative diagram

and the fact that the upper horizontal arrow in (2-2) is surjective.

(2) The commutative diagram

implies $\operatorname{Sw}_R(\chi) \ge \operatorname{Sw}_{\hat{R}}(\hat{\chi})$. Let $n = \operatorname{Sw}_{\hat{R}}(\hat{\chi})$. Then there exists a Witt vector $\hat{\alpha} \in \operatorname{fil}_n \operatorname{W}_{m+1}(\hat{K})$ mapping to $\hat{\chi}$. Take $\alpha \in \operatorname{fil}_n \operatorname{W}_{m+1}(K)$ whose components are close enough to those of $\hat{\alpha}$ with respect to the valuation of \hat{K} , so that every component of $\hat{\alpha} - \alpha$ (here α is regarded as an element of $\operatorname{W}_{m+1}(\hat{K})$) is in \hat{R} . Then, $\delta_{m+1,\hat{K}}(\hat{\alpha} - \alpha)$ is an unramified character by (1). Therefore, $\chi - \delta_{m+1,K}(\alpha)$ is unramified. Again by (1), there exists $\beta \in \operatorname{W}_{m+1}(R)$ such that $\chi - \delta_{m+1,K}(\alpha) = \delta_{m+1,K}(\beta)$, hence the assertion.

Next we recall refined Swan conductors.

Define $\widehat{\Omega}_{R}^{1}$ to be the π -adic completion of the absolute differential module Ω_{R}^{1} . Let $\widehat{\Omega}_{K}^{1} := \widehat{\Omega}_{R}^{1} \otimes_{R} K$. The canonical map $\widehat{\Omega}_{R}^{1} \to \widehat{\Omega}_{K}^{1}$ is injective and we usually regard $\widehat{\Omega}_{R}^{1}$ as an *R*-submodule of $\widehat{\Omega}_{K}^{1}$ via this map. The *R*-module $\widehat{\Omega}_{R}^{1}(\log)$ is the *R*-submodule of $\widehat{\Omega}_{K}^{1}$ generated by $\widehat{\Omega}_{R}^{1}$ and $d\log \pi := d\pi/\pi$. From the definition, the following holds:

Lemma 2.2. Assume that R is obtained from a smooth scheme over a perfect field by localizing at a point of codimension one. Let b_1, \ldots, b_n be a lift of a p-basis of the residue field of R to R. Then, $\widehat{\Omega}^1_R(\log)$ is a \widehat{R} -free module with a basis $db_1, \ldots, db_n, d\log \pi$.

For $\omega \in \widehat{\Omega}_{K}^{1}$, define $v_{R}^{\log}(\omega)$ as the largest integer *n* such that $\omega \in \pi^{n} \widehat{\Omega}_{R}^{1}(\log)$ (we formally put $v_{R}^{\log}(0) := \infty$). There is a homomorphism $F^{m}d : W_{m+1}(K) \to \widehat{\Omega}_{K}^{1}$ of groups given by

$$F^m d((a_0,\ldots,a_m)) := \sum_i a_i^{p^{m-i}-1} da_i.$$

Define an increasing exhaustive filtration on $\widehat{\Omega}_{K}^{1}$ by setting

$$\operatorname{fil}_n \widehat{\Omega}^1_K := \{ \omega \in \widehat{\Omega}^1_K \mid v_R^{\log}(\omega) \ge -n \}$$

for $n \in \mathbb{Z}$. From the definitions, the homomorphism $F^m d : W_{m+1}(K) \to \widehat{\Omega}_K^1$ respects their filtrations. In other words, $v_R(\alpha) \le v_R^{\log}(F^m d\alpha)$ hold for all $\alpha \in W_{m+1}(K)$.

Proposition 2.3 [Leal 2018, Proposition 2.8]. Let *n* be an integer ≥ 0 .

(1) There is a unique homomorphism

$$\operatorname{rsw}:\operatorname{fil}_{n}\operatorname{H}^{1}(K,\mathbb{Z}/p^{m+1}\mathbb{Z})\to\operatorname{fil}_{n}\widehat{\Omega}_{K}^{1}/\operatorname{fil}_{\lfloor n/p \rfloor}\widehat{\Omega}_{K}^{1},$$

called the refined Swan conductor, such that the composition

$$\operatorname{fil}_{n} \mathrm{W}_{m+1}(K) \to \operatorname{fil}_{n} \mathrm{H}^{1}(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \operatorname{fil}_{n} \widehat{\Omega}_{K}^{1}/\operatorname{fil}_{\lfloor n/p \rfloor} \widehat{\Omega}_{K}^{1}$$

coincides with $F^m d$.

(2) For $\lfloor \frac{n}{p} \rfloor \leq i \leq n$, the induced map

$$\operatorname{fil}_{n} \operatorname{H}^{1}(K, \mathbb{Z}/p^{m+1}\mathbb{Z})/\operatorname{fil}_{i} \operatorname{H}^{1}(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \operatorname{fil}_{n} \widehat{\Omega}_{K}^{1}/\operatorname{fil}_{i} \widehat{\Omega}_{K}^{1}$$

is injective.

At the end of this subsection, we extend the definition of the Swan conductors for characters in $H^1(K, \mathbb{Q}/\mathbb{Z})$ as follows.

Let *m* be an integer ≥ 0 . We identify the groups $\mathbb{Z}/p^m\mathbb{Z}$ and $\frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$ via the multiplication by $\frac{1}{p^m}$. In this way, we define a filtration on $H^1(K, \frac{1}{p^m}\mathbb{Z}/\mathbb{Z})$ from that of $H^1(K, \mathbb{Z}/p^m\mathbb{Z})$. The natural inclusion $\frac{1}{p^m}\mathbb{Z}/\mathbb{Z} \to \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}$ induces an inclusion

$$\mathrm{H}^{1}(K, \frac{1}{p^{m}}\mathbb{Z}/\mathbb{Z}) \to \mathrm{H}^{1}(K, \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z})$$

of groups.

Lemma 2.4. Let m, n be integers ≥ 0 . The equality

$$\operatorname{fil}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m}} \mathbb{Z}/\mathbb{Z}\right) = \operatorname{H}^{1}\left(K, \frac{1}{p^{m}} \mathbb{Z}/\mathbb{Z}\right) \cap \operatorname{fil}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m+1}} \mathbb{Z}/\mathbb{Z}\right)$$

of subgroups of $\mathrm{H}^{1}(K, \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z})$ holds.

Proof. Fix a separable closure K^s of K. Let $V : W_m(K^s) \to W_{m+1}(K^s)$ be the Verschiebung, i.e., the map sending (a_0, \ldots, a_{m-1}) to $(0, a_0, \ldots, a_{m-1})$. We have the following commutative diagram

$$0 \longrightarrow \frac{1}{p^{m}} \mathbb{Z}/\mathbb{Z} \longrightarrow W_{m}(K^{s}) \xrightarrow{F-1} W_{m}(K^{s}) \longrightarrow 0$$

$$\downarrow \qquad \qquad \qquad \downarrow V \qquad \qquad \qquad \downarrow V$$

$$0 \longrightarrow \frac{1}{p^{m+1}} \mathbb{Z}/\mathbb{Z} \longrightarrow W_{m+1}(K^{s}) \xrightarrow{F-1} W_{m+1}(K^{s}) \longrightarrow 0,$$

$$(2-3)$$

here we identify $\frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$ and $\mathbb{Z}/p^m\mathbb{Z}$ as mentioned above. Taking cohomology groups, we get a commutative diagram

Since the map $V : W_m(K) \to W_{m+1}(K)$ respects the filtrations, the inclusion

$$\operatorname{fil}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m}} \mathbb{Z}/\mathbb{Z}\right) \subset \operatorname{H}^{1}\left(K, \frac{1}{p^{m}} \mathbb{Z}/\mathbb{Z}\right) \cap \operatorname{fil}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m+1}} \mathbb{Z}/\mathbb{Z}\right)$$

holds. To prove the equality, it suffices to show that the morphism

$$\operatorname{Gr}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m}}\mathbb{Z}/\mathbb{Z}\right) \to \operatorname{Gr}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\right)$$

is injective for $n \ge 1$, where $\operatorname{Gr}_n := \operatorname{fil}_n / \operatorname{fil}_{n-1}$. We have the following commutative diagram

$$\operatorname{Gr}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m}} \mathbb{Z}/\mathbb{Z}\right) \xrightarrow{\operatorname{rsw}} \operatorname{Gr}_{n} \widehat{\Omega}_{K}^{1}$$

$$(2-5)$$

$$\operatorname{Gr}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m+1}} \mathbb{Z}/\mathbb{Z}\right).$$

By Proposition 2.3(2), the refined Swan conductors rsw in (2-5) are injective, hence the assertion. \Box

We define a filtration on $\mathrm{H}^{1}(K, \mathbb{Q}_{p}/\mathbb{Z}_{p}) = \bigcup_{m} \mathrm{H}^{1}(K, \frac{1}{p^{m}}\mathbb{Z}/\mathbb{Z})$ by

$$\operatorname{fil}_{n} \operatorname{H}^{1}(K, \mathbb{Q}_{p}/\mathbb{Z}_{p}) = \bigcup_{m} \operatorname{fil}_{n} \operatorname{H}^{1}\left(K, \frac{1}{p^{m}}\mathbb{Z}/\mathbb{Z}\right).$$

Let $\chi \in H^1(K, \mathbb{Q}/\mathbb{Z})$ be a character. Let χ_p be the *p*-primary part of χ and be considered as an element of $H^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$ via the natural decomposition

$$\mathrm{H}^{1}(K, \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{q} \mathrm{H}^{1}(K, \mathbb{Q}_{q}/\mathbb{Z}_{q}),$$

where q runs through all prime numbers. We define the Swan conductor $Sw(\chi)$ to be the smallest integer $n \ge 0$ such that $\chi_p \in fil_n H^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$.

The Swan conductor of a symmetric product. In this subsection, we assume that k is a perfect field of characteristic p.

Let X_1 , X_2 be smooth schemes over k. Let Z_1 and Z_2 be smooth irreducible closed subvarieties of X_1 and X_2 . Let \tilde{X}_1 , \tilde{X}_2 , and $\tilde{X}_1 \times X_2$ be the blow-ups of X_1 , X_2 , and $X_1 \times X_2$ along Z_1 , Z_2 , and $Z_1 \times Z_2$. Denote by R_1 , R_2 , and R_3 the DVRs at the generic points of the exceptional divisor of \tilde{X}_1 , \tilde{X}_2 , and $\tilde{X}_1 \times \tilde{X}_2$. Let K_i be the field of fractions of R_i for i = 1, 2, 3.

Lemma 2.5. (1) The projections $X_1 \times X_2 \rightarrow X_1$ and $X_1 \times X_2 \rightarrow X_2$ induce the extensions R_3/R_1 and R_3/R_2 of DVRs, which preserve uniformizers.

(2) There is a canonical isomorphism

$$\widehat{\Omega}^{1}_{K_{3}} \cong (\widehat{K}_{3} \otimes_{\widehat{K}_{1}} \widehat{\Omega}^{1}_{K_{1}}) \oplus (\widehat{K}_{3} \otimes_{\widehat{K}_{2}} \widehat{\Omega}^{1}_{K_{2}}).$$

This isomorphism respects the filtrations, i.e., via this isomorphism, fil_n $\widehat{\Omega}^1_{K_3}$ coincides with

$$(\hat{R}_3 \otimes_{\hat{R}_1} \operatorname{fil}_n \widehat{\Omega}^1_{K_1}) \oplus (\hat{R}_3 \otimes_{\hat{R}_2} \operatorname{fil}_n \widehat{\Omega}^1_{K_2}).$$

Proof. Let U be the open subscheme of $X_1 \times X_2$ obtained by removing the strict transforms of $Z_1 \times X_2$ and $X_1 \times Z_2$. This is the largest open subscheme where the pull-backs of $Z_1 \times X_2$ and $X_1 \times Z_2$ coincide with the exceptional divisor. By the universality of the blow-ups \tilde{X}_1 and \tilde{X}_2 , the projections $U \to X_1$ and $U \to X_2$ induce morphisms $U \to \tilde{X}_1$ and $U \to \tilde{X}_2$, hence a morphism $U \to \tilde{X}_1 \times \tilde{X}_2$ of $X_1 \times X_2$ -schemes. Denote by D_1 and D_2 the exceptional divisors of \tilde{X}_1 and \tilde{X}_2 . Let $(\tilde{X}_1 \times \tilde{X}_2)'$ be the blow-up of $\tilde{X}_1 \times \tilde{X}_2$ along $D_1 \times D_2$. The morphism $U \to \tilde{X}_1 \times \tilde{X}_2$ lifts to a morphism $U \to (\tilde{X}_1 \times \tilde{X}_2)' \to X_1 \times X_2$ lifts to a morphism $(\tilde{X}_1 \times \tilde{X}_2)' \to X_1 \times X_2$, which implies that U is quasifinite over $(\tilde{X}_1 \times \tilde{X}_2)'$. By Zariski main theorem, the morphism $U \to (\tilde{X}_1 \times \tilde{X}_2)'$ is an open immersion.

Taking an affine open neighborhood of the generic point of the exceptional divisors D_1 and D_2 in \tilde{X}_1 and \tilde{X}_2 , we may assume that $\tilde{X}_1 = \text{Spec}(A_1)$ and $\tilde{X}_2 = \text{Spec}(A_2)$ are affine. We also assume that there are systems of regular parameters $x_1, x_2, \ldots, x_n \in A_1$ and $y_1, y_2, \ldots, y_m \in A_2$ such that the ideal generated by x_1 and y_1 define D_1 and D_2 . The scheme U is canonically isomorphic to $\text{Spec}(A_1 \otimes A_2[\frac{x_1}{y_1}, \frac{y_1}{x_1}])$ and the natural inclusions $A_1, A_2 \to A_1 \otimes A_2[\frac{x_1}{y_1}, \frac{y_1}{x_1}]$ define the projections $U \to \tilde{X}_1, \tilde{X}_2$. The first assertion follows from this calculation. The canonical isomorphism

$$\Omega^1_{X_1 \times X_2} \cong \operatorname{pr}^*_{X_1} \Omega^1_{X_1} \oplus \operatorname{pr}^*_{X_2} \Omega^1_{X_2},$$

where pr_{X_1} and pr_{X_2} are the projections to X_1 and X_2 , gives an isomorphism

$$\widehat{\Omega}_{K_3}^1 \cong (\widehat{K}_3 \otimes_{\widehat{K}_1} \widehat{\Omega}_{K_1}^1) \oplus (\widehat{K}_3 \otimes_{\widehat{K}_2} \widehat{\Omega}_{K_2}^1).$$

The differentials $\frac{dx_1}{x_1}$, $d(\frac{y_1}{x_1})$, dx_2 , ..., dx_n , dy_2 , ..., dy_m form a basis of \hat{R}_3 -module $\widehat{\Omega}^1_{R_3}(\log)$. The second assertion follows from this fact and (1).

Corollary 2.6. Let $\chi_i \in H^1(K_i, \mathbb{Q}/\mathbb{Z})$ for i = 1, 2. Then, the following holds:

$$\operatorname{Sw}_{R_3}(\chi_1 \boxtimes 1 + 1 \boxtimes \chi_2) = \max\{\operatorname{Sw}_{R_1}(\chi_1), \operatorname{Sw}_{R_2}(\chi_2)\}.$$

Proof. Taking the *p*-primary parts of χ_1, χ_2 , and $\chi_1 \boxtimes 1 + 1 \boxtimes \chi_2$, we reduce to the case when $\chi_i \in H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z})$.

First we verify that the morphism

$$\mathrm{H}^{1}(K_{1},\mathbb{Z}/p^{m+1}\mathbb{Z})\oplus\mathrm{H}^{1}(K_{2},\mathbb{Z}/p^{m+1}\mathbb{Z})\to\mathrm{H}^{1}(K_{3},\mathbb{Z}/p^{m+1}\mathbb{Z})$$
(2-6)

respects the filtrations. Since the extensions R_3/R_1 , R_3/R_2 of DVRs preserve uniformizers the morphism $W_{m+1}(K_1) \oplus W_{m+1}(K_2) \rightarrow W_{m+1}(K_3)$ respects the filtrations, which implies the assertion.

To show the corollary, it is enough to prove that the morphism induced from (2-6) by taking Gr_n is injective. This follows from the injectivity of refined Swan conductors (Proposition 2.3) and Lemma 2.5. \Box

Let *S* be a scheme. For a quasiprojective *S*-scheme *X* and a natural number $d \ge 1$, the *d*-th symmetry group \mathfrak{S}_d acts on $X^d := X \times_S X \times_S \cdots \times_S X$ (*d* times) via permutation of coordinates. Define a scheme $X^{(d)} := X^d / \mathfrak{S}_d$. $X^{(d)}$ is called the *d*-th symmetric product of *X*. It is known that, if *X* is smooth of relative dimension 1 over *S*, $X^{(d)}$ is smooth and parametrizes effective Cartier divisors of deg = *d* on *X* [SGA 4₃ 1973, Exposé XVII, Application 1; Polishchuk 2003, 16]. In particular, the formation of $X^{(d)}$ commutes with base change $S' \rightarrow S$.

Let C be a projective smooth geometrically connected curve over k. Let U be a nonempty open subscheme of C.

Let *d* be an integer ≥ 1 . We construct a map $H^1(U, \mathbb{Q}/\mathbb{Z}) \to H^1(U^{(d)}, \mathbb{Q}/\mathbb{Z})$ as follows. First fix a finite abelian group *G*. Let $V \to U$ be a *G*-torsor. Then V^d is a G^d -torsor of U^d . Let *H* be the subgroup of G^d consisting of elements (a_1, \ldots, a_d) satisfying $\sum_{1 \leq i \leq d} a_i = 0$. Then V^d/H is a *G*-torsor of U^d . This torsor has a natural action by the *d*-th symmetry group \mathfrak{S}_d which is equivariant with respect to its action to U^d .

Lemma 2.7. The morphism

$$(V^d/H)/\mathfrak{S}_d \to U^{(d)} \tag{2-7}$$

induced from the map $V^d/H \to U^d$, taking the quotients by \mathfrak{S}_d , is a G-torsor.

Proof. It is sufficient to show that, for every geometric point \bar{x} of U^d , the stabilizer group $(\mathfrak{S}_d)_{\bar{x}}$ at \bar{x} acts trivially on the fiber $(V^d/H)_{\bar{x}}$ over \bar{x} , see [SGA 1 1971, Remarque 5.8].

We may assume that k is algebraically closed and that geometric points considered are k-valued points. Let \bar{x} be a geometric point of U^d . For simplicity, we assume that $\bar{x} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_r, \ldots, x_r)$, where x_1, \ldots, x_r are distinct points and x_i appears d_i times for each i. Then the inertia group $(\mathfrak{S}_d)_{\bar{x}}$ at \bar{x} is isomorphic to $\prod_{1 \le i \le r} \mathfrak{S}_{d_i}$.

For each *i*, take a *k*-valued point e_i of $V \times_U x_i$. From the definition of *H*, the fiber of V^d/H over \bar{x} can be identified with the set

$$\{(e_1, e_1, \dots, e_r, ge_r) \mid g \in G\},$$
(2-8)

on which $(\mathfrak{S}_d)_{\bar{x}}$ acts trivially.

In this way, we construct a *G*-torsor $(V^d/H)/\mathfrak{S}_d$ on $U^{(d)}$. Since this construction is compatible with a morphism of abelian groups $G \to G'$, we obtain a group homomorphism $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(U^{(d)}, \mathbb{Q}/\mathbb{Z})$. We denote by $\chi^{(d)}$ the image of χ via this map. Let *K* be the field of fractions of *U*, $K_{(d)}$ be that of $U^{(d)}$, and K_d be that of U^d . Taking *U* smaller and smaller, we also have a map $\mathrm{H}^1(K, G) \to \mathrm{H}^1(K_{(d)}, G)$ for a finite abelian group *G* and a map $\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(K_{(d)}, \mathbb{Q}/\mathbb{Z})$.

We consider a similar construction on the groups of Witt vectors. Denote by p_i^* the morphism $K \to K_d$ induced by the *i*-th projection $U^d \to U$. Consider the map $\lambda : W_{m+1}(K) \to W_{m+1}(K_d)$ sending a Witt vector α to $p_1^* \alpha + \cdots + p_d^* \alpha$. Since the extension $K_d/K_{(d)}$, induced by the natural projection $U^d \to U^{(d)}$, is finite Galois with the Galois group \mathfrak{S}_d , the \mathfrak{S}_d -fixed part of $W_{m+1}(K_d)$ coincides with $W_{m+1}(K_{(d)})$ (here $W_{m+1}(K_{(d)})$) is considered as a subgroup of $W_{m+1}(K_d)$ via the natural projection $U^d \to U^{(d)}$). Thus the map λ factors through $W_{m+1}(K_{(d)})$. We also denote the induced map $W_{m+1}(K) \to W_{m+1}(K_{(d)})$

1334

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by λ . Note that the diagram

$$W_{m+1}(K) \xrightarrow{\delta_{m+1,K}} H^{1}(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$$

$$\downarrow^{\lambda} \qquad \qquad \downarrow$$

$$W_{m+1}(K_{(d)}) \xrightarrow{\delta_{m+1,K_{(d)}}} H^{1}(K_{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z})$$

is commutative. This follows from the commutativity of pr_i^* and the boundary maps (see the diagram (2-1)), and the injectivity of $H^1(U^{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to H^1(U^d, \mathbb{Z}/p^{m+1}\mathbb{Z})$ (see Lemma 4.2). Also, the canonical morphism $\widehat{\Omega}^1_{K_{(d)}} \otimes_{K_{(d)}} K_d \to \widehat{\Omega}^1_{K_d}$ is an isomorphism and the \mathfrak{S}_d -fixed part of $\widehat{\Omega}^1_{K_d}$ coincides with (the image of) $\widehat{\Omega}^1_{K_{(d)}}$. We define a map $\mu : \widehat{\Omega}^1_K \to \widehat{\Omega}^1_{K_{(d)}}$ similarly to λ . The maps λ and μ commute with $F^m d$.

Let *P* be a closed point of *U*. For simplicity, let us assume that the residue field at *P* is isomorphic to *k*. Let *R* be the DVR of *C* at *P*, and $R_{(d)}$ be the DVR of $K_{(d)}$ at the generic point of the exceptional divisor of the blow-up of $C^{(d)}$ along the point corresponding to the divisor *dP*. We define filtrations on $W_{m+1}(K)$ (resp. $W_{m+1}(K_{(d)})$) and $\widehat{\Omega}_{K}^{1}$ (resp. $\widehat{\Omega}_{K_{(d)}}^{1}$) by *R* (resp. $R_{(d)}$) (see (2-1)).

The following theorem, and corollary are key calculations to prove Theorem 1.2 in Section 4.

Theorem 2.8. Let n be an integer.

(1) The homomorphism

$$\lambda: \mathbf{W}_{m+1}(K) \to \mathbf{W}_{m+1}(K_{(d)})$$

sends fil_n $W_{m+1}(K)$ into fil_{$\lfloor n/d \rfloor$} $W_{m+1}(K_{(d)})$.

(2) The homomorphism

$$\mu:\widehat{\Omega}^1_K\to\widehat{\Omega}^1_{K_{(d)}}$$

sends fil_n $\widehat{\Omega}^1_K$ into fil_{ln/d} $\widehat{\Omega}^1_{K_{(d)}}$. Let j be an integer. The induced map

 $\operatorname{fil}_{(j+1)d-1}\widehat{\Omega}_K^1/\operatorname{fil}_{jd-1}\widehat{\Omega}_K^1\to\operatorname{Gr}_j\widehat{\Omega}_{K_{(d)}}^1$

is injective, here $\operatorname{Gr}_j := \operatorname{fil}_j / \operatorname{fil}_{j-1}$.

Corollary 2.9. Let χ be a character in $H^1(K, \mathbb{Q}/\mathbb{Z})$. The following identity holds:

$$\operatorname{Sw}_{R_{(d)}}(\chi^{(d)}) = \left\lfloor \frac{\operatorname{Sw}_{R}(\chi)}{d} \right\rfloor.$$

Proof of Corollary 2.9. Taking the *p*-primary part of χ and an isomorphism $\frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/p^{m+1}\mathbb{Z}$, we reduce to the case when $\chi \in \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Take $\alpha \in \mathrm{W}_{m+1}(K)$ such that α maps to χ via the boundary map $\mathrm{W}_{m+1}(K) \to \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ and $v_R(\alpha) = -\mathrm{Sw}_R(\chi)$. Since the map $F^m d : \mathrm{W}_{m+1}(K) \to \widehat{\Omega}^1_K$ respects the filtrations, we have $F^m d\alpha \in \mathrm{fil}_{-\mathrm{Sw}_R(\chi)} \widehat{\Omega}^1_K$. On the other hand, by Proposition 2.3(2), we have $F^m d\alpha = \mathrm{rsw}(\chi) \notin \mathrm{fil}_{-1-\mathrm{Sw}_R(\chi)} \widehat{\Omega}^1_K$. By the definition of the filtration on $\widehat{\Omega}^1_K$, the equality $v_R^{\log}(F^m d\alpha) = -\mathrm{Sw}_R(\chi)$ holds.

Daichi Takeuchi

When $Sw_R(\chi) = 0$, χ is unramified since χ is *p*-torsion. Thus $\chi^{(d)}$ is unramified too by the construction of $\chi^{(d)}$, which implies the assertion in this case.

Assume $\operatorname{Sw}_R(\chi) > 0$. Let $r := \lfloor \operatorname{Sw}_R(\chi)/d \rfloor$. From Theorem 2.8(1), the inequality $v_{R_{(d)}}(\lambda(\alpha)) \ge -r$ holds. Thus $\chi^{(d)}$ is contained in fil_r H¹($K_{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z}$), which implies the inequality $\operatorname{Sw}_{R_{(d)}}(\chi^{(d)}) \le r$.

We show that the class of $\chi^{(d)}$ in $\operatorname{Gr}_r \operatorname{H}^1(K_{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is nonzero. Consider the following commutative diagram

which is obtained from Proposition 2.3. It suffices to show that $rsw(\chi^{(d)})$ is nonzero. From the commutativity of (2-9), $rsw(\chi^{(d)})$ coincides with the class containing $F^m d\lambda(\alpha) = \mu(F^m d\alpha)$. Since $v_R^{\log}(F^m d\alpha) = -Sw_R(\chi)$, the class of $F^m d\alpha$ in $fil_{(r+1)d-1} \widehat{\Omega}_K^1/fil_{rd-1} \widehat{\Omega}_K^1$ is nonzero. the assertion follows from Theorem 2.8(2), i.e., the injectivity of μ .

To prove Theorem 2.8, we first collect some basic properties of the DVR $R_{(d)}$ and its module of differentials. Let R_d be the normalization of $R_{(d)}$ in K_d . R_d is a DVR. The natural projection $C^d \to C^{(d)}$ and the *i*-th projection $C^d \to C$ define extensions of DVRs

$$R_{(d)} \hookrightarrow R_d \xleftarrow{\operatorname{pr}_{i}^*} R.$$

Fix a uniformizer t of R. Let S_1, \ldots, S_d be the elementary symmetric polynomials of $\operatorname{pr}_1^* t, \ldots, \operatorname{pr}_d^* t$ in R_d , i.e., S_1, \ldots, S_d satisfy the following identity

$$(T - \mathrm{pr}_1^* t) \cdots (T - \mathrm{pr}_d^* t) = T^d - S_1 T^{d-1} + \cdots + (-1)^d S_d.$$

Lemma 2.10. (1) The residue field of $R_{(d)}$ is isomorphic to $k(S_1/S_d, \ldots, S_{d-1}/S_d)$.

- (2) The elements S_1, \ldots, S_d are uniformizers of $R_{(d)}$.
- (3) The valuations of pr_1^*t, \ldots, pr_d^*t with respect to R_d are the same.

Proof. Since the sequence S_1, \ldots, S_d is a regular system of parameters of the regular local ring of $C^{(d)}$ at the *k*-rational point dP, the exceptional divisor of the blow-up of $C^{(d)}$ is isomorphic to a projective space over *k* with homogeneous coordinates S_1, \ldots, S_d .

- (1) This follows from the considerations above.
- (2) At the generic point of the exceptional divisor, the elements S_1, \ldots, S_d generate the same ideal. Since the exceptional divisor is regular, the assertion follows.

(3) The *d*-th symmetry group \mathfrak{S}_d acts on R_d permuting the pr_i^*t , hence the assertion.

By Lemmas 2.2 and 2.10(1), $\widehat{\Omega}^1_{R_{(d)}}(\log)$ is an $\widehat{R}_{(d)}$ -free module with a basis $dS_1/S_d, \ldots, dS_d/S_d$.

Lemma 2.11. For each integer i, define

$$\omega_i := \frac{d(\mathrm{pr}_1^* t)}{\mathrm{pr}_1^* t^i} + \dots + \frac{d(\mathrm{pr}_d^* t)}{\mathrm{pr}_d^* t^i} \in \widehat{\Omega}_{K_d}^1$$

Let *j* be an integer. Then, the differentials $\omega_{jd+1}, \ldots, \omega_{(j+1)d}$ form an $\hat{R}_{(d)}$ -basis of the $\hat{R}_{(d)}$ -free module $(1/S_d^j)\widehat{\Omega}_{R_{(d)}}^1$ (log).

Proof. To avoid notational confusion, we change the notation d to n in this proof.

Since the differentials ω_j are \mathfrak{S}_n -invariant, they are indeed contained in $\widehat{\Omega}^1_{K_{(n)}}$.

Suppose $j \ge 0$. Define a polynomial $F(T) := (T - pr_1^* t) \cdots (T - pr_n^* t)$. The following equalities hold:

$$-dS_{1}T^{n-1} + \dots + (-1)^{n}dS_{n} = dF = -F\sum_{1 \le i \le n} \frac{d \operatorname{pr}_{i} t}{T - \operatorname{pr}_{i}^{*} t}$$
$$= F\sum_{1 \le i \le n} \frac{1}{\operatorname{pr}_{i}^{*} t} \frac{d \operatorname{pr}_{i}^{*} t}{1 - T/\operatorname{pr}_{i}^{*} t} = F\sum_{r \ge 0} \omega_{r+1}T^{r}.$$

Comparing the coefficients of T^r , we obtain equalities

$$S_n \omega_1 = \pm d S_n$$

$$S_n \omega_2 \pm S_{n-1} \omega_1 = \pm d S_{n-1}$$

$$\vdots$$

$$S_n \omega_{r+1} + (a \text{ linear combination of } \omega_r, \dots, \omega_{r-n}) = 0 \quad (r \ge n)$$

$$\vdots$$

The assertion follows by induction on r.

For the case when j < 0, take F as $(1 - pr_1^* tT) \cdots (1 - pr_n^* tT)$ and argue similarly.

Proof of Theorem 2.8. (1) Let $e_{R_d/R_{(d)}}$ be the ramification index of $R_d/R_{(d)}$. Let $e_{R_d/R}$ be the ramification index of R_d/R induced by pr_i. By Lemma 2.10, $e_{R_d/R}$ is independent of *i*. From the definition of the filtrations, the map pr_i^{*} : W_{m+1}(K) \rightarrow W_{m+1}(K_d) sends fil_n W_{m+1}(K) into fil_{ne_{R_d/R} W_{m+1}(K_d). Since S_d is a uniformizer of $R_{(d)}$ by Lemma 2.10, the equality}

$$de_{R_d/R} = e_{R_d/R_{(d)}}$$

holds. This shows the identity

$$\operatorname{fil}_{\lfloor n/d \rfloor} W_{m+1}(K_{(d)}) = \operatorname{fil}_{ne_{R_d/R}} W_{m+1}(K_d) \cap W_{m+1}(K_{(d)}),$$

hence the assertion.

(2) Note that the map $\mu : \widehat{\Omega}_{K}^{1} \to \widehat{\Omega}_{K_{(d)}}^{1}$ is continuous. The differentials $dt/t^{n+1}, dt/t^{n}, \ldots \in \widehat{\Omega}_{K}^{1}$ map to $\omega_{n+1}, \omega_{n}, \ldots$ via μ , all of which are contained in $\operatorname{fil}_{\lfloor n/d \rfloor} \widehat{\Omega}_{K_{(d)}}^{1}$ by Lemma 2.11. Thus the map μ sends $\operatorname{fil}_{n} \widehat{\Omega}_{K}^{1}$ into $\operatorname{fil}_{\lfloor n/d \rfloor} \widehat{\Omega}_{K_{(d)}}^{1}$. Since the classes of $dt/t^{(j+1)d}, \ldots, dt/t^{jd+1}$ form a k-basis of $\operatorname{fil}_{(j+1)d-1} \widehat{\Omega}_{K}^{1}/\operatorname{fil}_{jd-1} \widehat{\Omega}_{K}^{1}$, the last assertion follows from Lemma 2.11.

3. Generalized jacobians and blow-ups of symmetric powers

In this section, we fix a base scheme *S*. Let *C* be a projective smooth *S*-scheme whose geometric fibers are connected and of dimension 1. Let m be a relative effective Cartier divisor of *C*/*S*, i.e., a closed subscheme of *C* which is finite flat of finite presentation over *S*. We also call m a modulus. Let us denote, for *S*-schemes *T*, the projections $C \times_S T \to T$ by the same symbol pr. In this section, we recall and study the notion of generalized jacobian varieties. Let *d* be an integer and m be a modulus. Let *T* be an *S*-scheme. Consider a datum (\mathcal{L}, ψ) such that:

- \mathcal{L} is an invertible sheaf of deg = d on C_T .
- ψ is an isomorphism $\mathcal{O}_{\mathfrak{m}_T} \to \mathcal{L}|_{\mathfrak{m}_T}$.

We say that two such data (\mathcal{L}, ψ) and (\mathcal{L}', ψ') are isomorphic if there exists an isomorphism of invertible sheaves $f : \mathcal{L} \to \mathcal{L}'$ making the following diagram commute:



For an S-scheme T, define a set

 $\operatorname{Pic}_{C,\mathfrak{m}}^{d,\operatorname{pre}}(T) := \{ \operatorname{isomorphism classes of } (\mathcal{L}, \psi) \text{ defined as above} \}.$

 $\operatorname{Pic}_{C,\mathfrak{m}}^{d,\operatorname{pre}}$ extends in an obvious way to a presheaf on Sch/S, which we also denote by $\operatorname{Pic}_{C,\mathfrak{m}}^{d,\operatorname{pre}}$. Define $\operatorname{Pic}_{C,\mathfrak{m}}^{d}$ as the étale sheafification of $\operatorname{Pic}_{C,\mathfrak{m}}^{d,\operatorname{pre}}$. Their fundamental properties which we use without proofs are:

- (1) $\operatorname{Pic}_{C,\mathfrak{m}}^d$ are represented by *S*-schemes.
- (2) When \mathfrak{m} is faithfully flat over *S*, $\operatorname{Pic}_{C,\mathfrak{m}}^{d,\operatorname{pre}}$ are already étale sheaves.
- (3) $\operatorname{Pic}_{C,\mathfrak{m}}^{0}$ is a smooth commutative group S-scheme with geometrically connected fibers.
- (4) $\operatorname{Pic}_{C,\mathfrak{m}}^{d}$ are $\operatorname{Pic}_{C,\mathfrak{m}}^{0}$ -torsors.

When $\mathfrak{m} = 0$, properties (1) and (3) are proved in [Bosch et al. 1990]. For general \mathfrak{m} , they can be proved similarly as in [Bosch et al. 1990, 9.3], or can be deduced from the case when $\mathfrak{m} = 0$ and Lemma 3.1.

 $\operatorname{Pic}_{C,\mathfrak{m}}^{0}$ is called the generalized jacobian variety of *C* with modulus \mathfrak{m} . When $\mathfrak{m} = 0$, this is the jacobian variety of *C*. In this case, we also denote $\operatorname{Pic}_{C,0}^{d}$ by $\operatorname{Pic}_{C}^{d}$. Let \mathfrak{m} and $\tilde{\mathfrak{m}}$ be moduli such that $\mathfrak{m} \subset \tilde{\mathfrak{m}}$. There exists a natural map from $\operatorname{Pic}_{C,\tilde{\mathfrak{m}}}^{d}$ to $\operatorname{Pic}_{C,\mathfrak{m}}^{d}$, restricting ψ . Since $\tilde{\mathfrak{m}}$ is a finite *S*-scheme, this map is a surjection as a morphism of étale sheaves.

Assume that $C \to S$ has a section. In this case, $\operatorname{Pic}_{C}^{d}$ has an expression as a sheaf as follows [Bosch et al. 1990, 8.1]. Let *T* be an *S*-scheme, and \mathcal{L}_{1} and \mathcal{L}_{2} are invertible sheaves of deg = *d* on C_{T} . Define an equivalence relation on $\operatorname{Pic}_{C}^{d,\operatorname{pre}}$ such that \mathcal{L}_{1} and \mathcal{L}_{2} are equivalent if and only if there exists an invertible sheaf \mathcal{M} on *T* such that $\mathcal{L}_{1} \cong \mathcal{L}_{2} \otimes \operatorname{pr}^{*} \mathcal{M}$. If $C \to S$ has a section, the quotient presheaf of $\operatorname{Pic}_{C}^{d,\operatorname{pre}}$ by

this equivalence relation is an étale sheaf and coincides with the étale sheafification of $\operatorname{Pic}_{C}^{d,\operatorname{pre}}$ via the natural surjection. In particular, the identity map $\operatorname{Pic}_{C}^{d} \to \operatorname{Pic}_{C}^{d}$ corresponds to an equivalence class of invertible sheaves on $C \times_{S} \operatorname{Pic}_{C}^{d}$. In this paper, we call this class the universal class of invertible sheaves of deg = d.

From now on we fix a modulus $\tilde{\mathfrak{m}}$. We call a modulus \mathfrak{m} a submodulus if $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ holds. Until the last paragraph, we treat the case when submoduli considered are everywhere strictly positive on *S*. Let \mathfrak{m} be a submodulus which is everywhere strictly positive. Then, $\operatorname{Pic}_{C,\mathfrak{m}}^d$ has an explicit expression as a sheaf, as explained before.

Denote the genus of C by g. This is a locally constant function on S. We consider a condition on an integer d as below:

$$d \ge \max\{2g - 1 + \deg \tilde{\mathfrak{m}}, \deg \tilde{\mathfrak{m}}\}.$$
(3-1)

When S is quasicompact, such a d always exists. For an integer d and a submodulus m, denote $d_{\mathfrak{m}} := d - \deg \tilde{\mathfrak{m}} + \deg \mathfrak{m}$. If d satisfies (3-1), $d_{\mathfrak{m}}$ satisfies (3-1) with $\tilde{\mathfrak{m}}$ replaced by m.

Fix an integer *d* satisfying (3-1). Let *T* be an *S*-scheme and \mathcal{L} be an invertible sheaf of deg = *d* on C_T . For every usual point $t \in T$, $R^1 \operatorname{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})|_{C_t})$ and $R^1 \operatorname{pr}_*(\mathcal{L}|_{C_t})$ are zero by Serre duality and a degree argument. In this case, $\operatorname{pr}_* \mathcal{L}(-\tilde{\mathfrak{m}})$ and $\operatorname{pr}_* \mathcal{L}$ are locally free sheaves and their formations commute with any base change, i.e., for any morphism of *S*-schemes $f : T' \to T$, the base change morphisms $f^* \operatorname{pr}_* \mathcal{L} \to \operatorname{pr}_* f^* \mathcal{L}$ and $f^* \operatorname{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})) \to \operatorname{pr}_* f^*(\mathcal{L}(-\tilde{\mathfrak{m}}))$ are isomorphisms. Also $R^1 \operatorname{pr}_* f^* \mathcal{L}$ and $R^1 \operatorname{pr}_* f^*(\mathcal{L}(-\tilde{\mathfrak{m}}))$ are zero.

Let \mathfrak{m} be a submodulus. Denote by U the complement of \mathfrak{m} in C. The Abel–Jacobi map $U^{(d_{\mathfrak{m}})} \to \operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$ is a map which sends $D \in U^{(d_{\mathfrak{m}})}$ to $(\mathcal{O}_C(D), \iota_D)$, where ι_D is the one induced from the natural identification $\mathcal{O}_{C\setminus D} \cong \mathcal{O}_C(D)|_{C\setminus D}$. In this section, we define and study a compactification $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ of the Abel–Jacobi map by constructing the following commutative diagram of smooth *S*-schemes:



The S-scheme $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ has, on the one hand, a clear moduli description, and, on the other hand, can be identified by an open subscheme of a blow-up, which will be denoted by $X_{\mathfrak{m}}$, of $C^{(d_{\mathfrak{m}})}$.

Let \mathcal{L} be an invertible sheaf on C_T for an S-scheme T. Denote $\mathcal{L}/(\mathcal{L}(-\mathfrak{m}))$ by $\mathcal{L}_{\mathfrak{m}}$.

For an S-scheme T, consider a pair (\mathcal{L}, ϕ) such that \mathcal{L} is an invertible sheaf of deg = $d_{\mathfrak{m}}$ on C_T and ϕ is an injection $\mathcal{O}_T \to \mathrm{pr}_* \mathcal{L}_{\mathfrak{m}}$ such that the quotient $\mathrm{pr}_* \mathcal{L}_{\mathfrak{m}} / \mathcal{O}_T$ is locally free. Call such pairs (\mathcal{L}, ϕ) and (\mathcal{L}', ϕ') isomorphic if there exists an isomorphism $f : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that the following diagram

Daichi Takeuchi

commutes:



Define $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}(T)$ as the set of isomorphism classes of such pairs. This is an étale sheaf on *Sch/S*. Define a map

$$P_{\mathfrak{m}}^{d_{\mathfrak{m}}} \to \operatorname{Pic}_{C}^{d_{\mathfrak{m}}}$$
 (3-2)

by forgetting ϕ . Let *X* be a scheme, and \mathcal{F} be a locally free sheaf of finite rank on *X*. We use a contra-Grothendieck notation for a projective space. Thus the *X*-scheme $\mathbb{P}(\mathcal{F})$ parametrizes invertible subsheaves of \mathcal{F} .

Lemma 3.1. The sheaf $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ is represented by a proper smooth S-scheme. Assume that $C \to S$ has a section. Let \mathcal{L}' be a representative invertible sheaf of the universal class. Then, as sheaves on Sch/Pic^{d_{\mathfrak{m}}}_C, $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ is isomorphic to the projectivization $\mathbb{P}(\mathrm{pr}_{*}\mathcal{L}'_{\mathfrak{m}})$ of $\mathrm{pr}_{*}\mathcal{L}'_{\mathfrak{m}}$.

Proof. First we consider the case when C(S) is not empty. In this case, $\text{Pic}_{C}^{d_{\mathfrak{m}}}$ has an explicit expression as a sheaf, as explained before.

Via the map (3-2), we regard $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ as a sheaf on $Sch/\operatorname{Pic}_{C}^{d_{\mathfrak{m}}}$. Fix a representative invertible sheaf \mathcal{L}' of the universal class. Let \mathcal{N} be an element of $\mathbb{P}(\operatorname{pr}_{*}\mathcal{L}'_{\mathfrak{m}})(T)$, where T is a $\operatorname{Pic}_{C}^{d_{\mathfrak{m}}}$ -scheme. Let $\phi: \mathcal{O}_{T} \to \operatorname{pr}_{*}((\mathcal{L}' \otimes \operatorname{pr}^{*}\mathcal{N}^{-1})_{\mathfrak{m}})$ be a morphism obtained by tensoring the inclusion $\mathcal{N} \hookrightarrow \operatorname{pr}_{*}\mathcal{L}'_{\mathfrak{m}}$ with \mathcal{N}^{-1} . Then, the assignment $\mathcal{N} \mapsto (\mathcal{L}' \otimes \operatorname{pr}^{*}\mathcal{N}^{-1}, \phi)$ defines a morphism of sheaves on $Sch/\operatorname{Pic}_{C}^{d_{\mathfrak{m}}}$, $\mathbb{P}(\operatorname{pr}_{*}\mathcal{L}'_{\mathfrak{m}}) \to P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$. This is an isomorphism. Indeed, we can construct its inverse as follows. Let T be a $\operatorname{Pic}_{C}^{d_{\mathfrak{m}}}$ -scheme and (\mathcal{L}, ϕ) be an element of $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}(T)$. Let $a: T \to \operatorname{Pic}_{C}^{d_{\mathfrak{m}}}$ be the structure map. Then, there exists an invertible sheaf \mathcal{N} on T such that $\mathcal{L} \otimes \operatorname{pr}^{*}\mathcal{N}$ is isomorphic to $a^{*}\mathcal{L}'$. Such an \mathcal{N} is unique since $C \to S$ has a section. Then, $\mathcal{N} \to \Phi^{\otimes \mathcal{N}}$ pr $_{*}((\mathcal{L} \otimes \operatorname{pr}^{*}\mathcal{N})_{\mathfrak{m}}) \xrightarrow{\sim} \operatorname{pr}_{*} a^{*}\mathcal{L}'_{\mathfrak{m}}$ is an element of $\mathbb{P}(\operatorname{pr}_{*}\mathcal{L}'_{\mathfrak{m}})(T)$.

Next we consider the general case. As the map $C \to S$ has a section étale locally on S, the sheaf $P_m^{d_m}$ is represented, étale locally on S, by a projective space bundle. Since the dual of the canonical line bundle of a projective space bundle is relatively ample, the étale descent is effective.

Let (\mathcal{L}, ϕ) be the universal element on $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$. Define $\mathcal{E}_{\mathfrak{m}}$ as the $\mathcal{O}_{P_{\mathfrak{m}}^{d_{\mathfrak{m}}}}$ - module fitting in the following diagram of sheaves on $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$:



where the bottom horizontal arrow is the pushforward of the quotient map and each square is cartesian. Since all the right arrows are locally split injections and $p : pr_* \mathcal{L} \to pr_* \mathcal{L}_m$ is a surjection of locally free sheaves, \mathcal{E}_m is locally free of finite rank and all the left arrows are locally split injections.

Let $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ be the projectivization of $\mathcal{E}_{\mathfrak{m}}$. As a sheaf on Sch/S, $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ parametrizes triples $(\mathcal{L}, \phi, \mathcal{M})$ such that (\mathcal{L}, ϕ) is an element of $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ and \mathcal{M} is an invertible subsheaf of $\operatorname{pr}_{*}\mathcal{L} \oplus \mathcal{O}_{T}$ such that the following diagram commutes:

$$\begin{array}{cccc}
\mathcal{M} & \longrightarrow & \mathcal{O}_T \\
\downarrow & & \phi \downarrow \\
\mathrm{pr}_* \mathcal{L} & \xrightarrow{p} & \mathrm{pr}_* \mathcal{L}_{\mathfrak{m}},
\end{array}$$
(3-4)

where each arrow from \mathcal{M} is the composition of the inclusion $\mathcal{M} \hookrightarrow \operatorname{pr}_* \mathcal{L} \oplus \mathcal{O}_T$ with the respective projection. This is a proper smooth *S*-scheme.

Lemma 3.2. The map $\operatorname{pr}_*(\mathcal{L}(-\mathfrak{m})) \to \mathcal{E}_{\mathfrak{m}}$ in (3-3) induces a closed immersion $\mathbb{P}(\operatorname{pr}_*\mathcal{L}(-\mathfrak{m})) \hookrightarrow \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$. The closed subspace $\mathbb{P}(\operatorname{pr}_*\mathcal{L}(-\mathfrak{m}))$ is a hyperplane bundle of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$.

Proof. The assertion follows from the exact sequence

$$0 \to \mathrm{pr}_*(\mathcal{L}(-\mathfrak{m})) \to \mathcal{E}_{\mathfrak{m}} \to \mathcal{O}_{P_{\mathfrak{m}}^{d_{\mathfrak{m}}}} \to 0.$$

As a subsheaf of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$, $\mathbb{P}(\mathrm{pr}_{*}\mathcal{L}(-\mathfrak{m}))$ parametrizes triples $(\mathcal{L}, \phi, \mathcal{M})$ such that the first projection $\mathcal{M} \to \mathrm{pr}_{*}\mathcal{L}$ factors through $\mathrm{pr}_{*}\mathcal{L}(-\mathfrak{m})$.

Now we define a map $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})}$ of *S*-schemes taking the homothety class of the left vertical arrow in (3-4).

Let *T* be an *S*-scheme and $(\mathcal{L}, \phi, \mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})(T)$. Since the arrow $\mathcal{E}_{\mathfrak{m}} \to \mathrm{pr}_* \mathcal{L}$ in (3-3) is locally a split injection, the first projection $\mathcal{M} \to \mathrm{pr}_* \mathcal{L}$ is injective and the cokernel is locally free. Since these hold after any base change $t \to T$ from the spectrum of a field, the map $\mathrm{pr}^* \mathcal{M}_t \to \mathcal{L}_t$ is injective for a usual point *t* of *T*. Thus $\mathcal{O}_{C_T} \to \mathcal{L} \otimes \mathrm{pr}^* \mathcal{M}^{-1}$ defines an effective Cartier divisor. Since $\mathrm{deg}(\mathcal{L}^{-1} \otimes \mathrm{pr}^* \mathcal{M})$ equals to $-d_{\mathfrak{m}}$, $\mathrm{Spec}(\mathcal{O}_{C_T}/(\mathcal{L}^{-1} \otimes \mathrm{pr}^* \mathcal{M}))$ is finite flat of finite presentation of $\mathrm{deg} = d_{\mathfrak{m}}$ over *T* by the Riemann–Roch formula.

Let $C^{(d_m)}$ be the d_m -th symmetric product of C, which parametrizes effective Cartier divisors of deg = d_m on C. Define a map

$$\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})} \tag{3-5}$$

sending $(\mathcal{L}, \phi, \mathcal{M})$ to $\operatorname{Spec}(\mathcal{O}_{C_T}/(\mathcal{L}^{-1} \otimes \operatorname{pr}^* \mathcal{M})) \subset C_T$.

Let Z_0 be the closed subscheme of $C^{(d_{\mathfrak{m}})}$ defined by the map $C^{(d-\deg \tilde{\mathfrak{m}})} \to C^{(d_{\mathfrak{m}})}$, adding \mathfrak{m} . Let $X_{\mathfrak{m}}$ be the blow-up of $C^{(d_{\mathfrak{m}})}$ along Z_0 . We now construct an isomorphism $X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$, by which we will identify them.

We define a map

$$h: X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \tag{3-6}$$

as follows.

Let *D* be the universal effective Cartier divisor on $C^{(d_m)}$. Denote $\mathcal{O}_{C \times_S C^{(d_m)}}(D)$ by $\mathcal{O}(D)$ and $\mathcal{O}(D) \otimes \mathcal{O}_{m \times_S C^{(d_m)}}$ by $\mathcal{O}(D)_m$ for short. The composition of the natural maps $\mathcal{O}_{C \times_S C^{(d_m)}} \to \mathcal{O}(D) \to \mathcal{O}(D)_m$ defines a map of locally free sheaves $\mathcal{O}_{C^{(d_m)}} \to \operatorname{pr}_*(\mathcal{O}(D)_m)$ on $C^{(d_m)}$. After a base change $T \to C^{(d_m)}$, this map becomes zero if and only if $T \to C^{(d_m)}$ factors through Z_0 . Thus the image of the dual $(\operatorname{pr}_* \mathcal{O}(D)_m)^{\vee} \to \mathcal{O}_{C^{(d_m)}}$ of this map is the ideal \mathcal{I} defining Z_0 . Let $\mathcal{L} := \mathcal{O}_{C \times_S X_m}(D) \otimes \operatorname{pr}^*(\mathcal{I}\mathcal{O}_{X_m})$. Define $\phi : \mathcal{O}_{X_m} \to \operatorname{pr}_*(\mathcal{O}_{C \times_S X_m}(D) \otimes \operatorname{pr}^*(\mathcal{I}\mathcal{O}_{X_m}))_m$ to be the morphism obtained from the map $(\mathcal{I}\mathcal{O}_{X_m})^{-1} \to \operatorname{pr}_*\mathcal{O}_{C \times_S X_m}(D)_m$ by tensoring $\mathcal{I}\mathcal{O}_{X_m}$. Let $\mathcal{I}\mathcal{O}_{X_m} \to \operatorname{pr}_*\mathcal{L}$ be the map induced from the natural inclusion $\mathcal{O}_{X_m} \to \mathcal{O}_{X_m}$ make the sheaf $\mathcal{I}\mathcal{O}_{X_m}$ into a subsheaf of $\operatorname{pr}_*\mathcal{L} \oplus \mathcal{O}_{X_m}$, which makes the diagram (3-4) commutes. The triple $(\mathcal{L}, \phi, \mathcal{I}\mathcal{O}_{X_m})$ defines a morphism $h : X_m \to \mathbb{P}(\mathcal{E}_m)$. From the construction, h is a morphism over $C^{(d_m)}$.

- **Lemma 3.3.** (1) As a subsheaf of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$, $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \times_{C^{(d_{\mathfrak{m}})}} Z_0$ parametrizes triples $(\mathcal{L}, \phi, \mathcal{M})$ such that the second projection $\mathcal{M} \to \mathcal{O}$ are zero. As closed subspaces of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$, $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \times_{C^{(d_{\mathfrak{m}})}} Z_0$ and $\mathbb{P}(\mathrm{pr}_* \mathcal{L}(-\mathfrak{m}))$ are equal. In particular, $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \times_{C^{(d_{\mathfrak{m}})}} Z_0$ is a smooth divisor of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$.
- (2) Let V be the complement of Z_0 in $C^{(d_m)}$. As a subsheaf of $\mathbb{P}(\mathcal{E}_m)$, $\mathbb{P}(\mathcal{E}_m) \times_{C^{(d_m)}} V$ parametrizes triples $(\mathcal{L}, \phi, \mathcal{M})$ such that the second projection $\mathcal{M} \to \mathcal{O}$ is an isomorphism. The projection $\mathbb{P}(\mathcal{E}_m) \times_{C^{(d_m)}} V \to V$ is an isomorphism and its inverse coincides with the restriction of h to V.

Proof. We are considering the following diagram:



(1) Let $(\mathcal{L}, \phi, \mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})(T)$. This maps into Z_0 via the map $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})}$ if and only if the composition of $\operatorname{pr}^* \mathcal{M} \to \mathcal{L} \to \mathcal{L}_{\mathfrak{m}}$ is zero. Since the right vertical arrow of (3-4) is an injection, this occurs if and only if the second projection $\mathcal{M} \to \mathcal{O}_T$ is zero. The second assertion is obvious from the definition and the expression of $\mathbb{P}(\operatorname{pr}_* \mathcal{L}(-\mathfrak{m}))$ as a subsheaf. The last assertion is verified for $\mathbb{P}(\operatorname{pr}_* \mathcal{L}(-\mathfrak{m}))$ in Lemma 3.2.

(2) Let *T* be an *S*-scheme and $(\mathcal{L}, \phi, \mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})(T)$. Let *t* be a usual point of *T*. By (1), the pullback of the projection $\mathcal{M} \to \mathcal{O}_T$ by $t \hookrightarrow T$ is an isomorphism if and only if the image of *t* by the map

$$T \xrightarrow{(\mathcal{L},\phi,\mathcal{M})} \mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})}$$

is in V.

Let $p : \mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \times_{C^{(d_{\mathfrak{m}})}} V \to V$ be the projection. Since $h : X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ is a $C^{(d_{\mathfrak{m}})}$ -morphism, $p \circ h|_{V}$ is the identity. Let $(\mathcal{L}, \phi, \mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \times_{C^{(d_{\mathfrak{m}})}} V(T)$. Identify \mathcal{M} and \mathcal{O}_{T} by the second projection. By this rigidification, $(\mathcal{L}, \phi, \mathcal{O}_{T})$ is determined by the first projection. Thus p is an injection as a morphism of sheaves. The assertion follows.

After these preparations, we obtain the following:

Theorem 3.4. The morphism $h : X_m \to \mathbb{P}(\mathcal{E}_m)$ in (3-6) is an isomorphism.

Proof. By Lemma 3.3(1) and the universality of blow-ups, there exists a unique map $\mathbb{P}(\mathcal{E}_m) \to X_m$ which is a lift of $\mathbb{P}(\mathcal{E}_m) \to C^{(d_m)}$. Let V be as in Lemma 3.3(2), i.e., the complement of Z_0 in $C^{(d_m)}$. By the lemma, V can be considered as an open subscheme of $\mathbb{P}(\mathcal{E}_m)$. On the other hand, as V and Z_0 are disjoint, V also can be considered as an open subscheme of X_m . Note that the complements of V in $\mathbb{P}(\mathcal{E}_m)$ and X_m are supports of divisors. Therefore, V is schematically dense in both of $\mathbb{P}(\mathcal{E}_m)$ and X_m . Since the morphisms $X_m \to \mathbb{P}(\mathcal{E}_m)$ and $\mathbb{P}(\mathcal{E}_m) \to X_m$ constructed above induce the identity on V, and both schemes are separated over S, the assertion follows.

Let *T* be an *S*-scheme and (\mathcal{L}, ψ) be an element of $\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}(T)$. Define ϕ as the composition $\mathcal{O}_T \to \operatorname{pr}_* \mathcal{O}_{\mathfrak{m}_T} \xrightarrow{\operatorname{pr}_* \psi} \operatorname{pr}_* \mathcal{L}_{\mathfrak{m}}$. Then, the assignment $(\mathcal{L}, \psi) \mapsto (\mathcal{L}, \phi)$ defines a morphism

$$\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}} \to P_{\mathfrak{m}}^{d_{\mathfrak{m}}}.$$
(3-7)

Lemma 3.5. The morphism $\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}} \to P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ in (3-7) is an open immersion. The open subscheme $\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$ parametrizes pairs (\mathcal{L}, ϕ) such that the maps $\mathcal{O}_C \to \mathcal{L}_{\mathfrak{m}}$ obtained from ϕ by adjunction are surjective.

Proof. This morphism is an injection of sheaves. Let (\mathcal{L}, ϕ) be an element of $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}(T)$. This element is in $\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$ if and only if the map $\mathcal{O}_{C_T} \to \mathcal{L}_{\mathfrak{m}}$ obtained from ϕ by adjunction is a surjection. This is an open condition.

Next, we study behavior of various schemes when one replaces the modulus m. Let m be a submodulus and $\mathfrak{m}' := \tilde{\mathfrak{m}} - \mathfrak{m}$. Define a closed immersion $C^{(d-\deg\mathfrak{m}')} \to C^{(d)}$ by adding \mathfrak{m}' . We denote this closed subscheme of $C^{(d)}$ by $Z_{\mathfrak{m}}$. If $\mathfrak{m}_1 \subset \mathfrak{m}_2$, the inclusion $Z_{\mathfrak{m}_1} \subset Z_{\mathfrak{m}_2}$ holds. The closed immersion $Z_{\mathfrak{m}_1} \hookrightarrow Z_{\mathfrak{m}_2}$ is induced by adding $\mathfrak{m}_2 - \mathfrak{m}_1$. This induces a map

$$X_{\mathfrak{m}_1} \hookrightarrow X_{\mathfrak{m}_2} \tag{3-8}$$

of the blow-ups along Z_0 . Let \mathfrak{m}_1 and \mathfrak{m}_2 be submoduli such that $\mathfrak{m}_1 \subset \mathfrak{m}_2$. Define a map $i_{\mathfrak{m}_1,\mathfrak{m}_2} : P_{\mathfrak{m}_1}^{d_{\mathfrak{m}_1}} \to P_{\mathfrak{m}_2}^{d_{\mathfrak{m}_2}}$ by sending (\mathcal{L}_1, ϕ_1) to $(\mathcal{L}_1(\mathfrak{m}_2 - \mathfrak{m}_1), \phi)$, where ϕ is the composition of ϕ_1 and the natural injection $\mathrm{pr}_*(\mathcal{L}_1)_{\mathfrak{m}_1} \to \mathrm{pr}_*\mathcal{L}_1(\mathfrak{m}_2 - \mathfrak{m}_1)_{\mathfrak{m}_2}$. The map $i_{\mathfrak{m}_1,\mathfrak{m}_2}$ is a closed immersion.

Proposition 3.6. (1) Let \mathfrak{m}_1 and \mathfrak{m}_2 be submoduli such that $\mathfrak{m}_1 \subset \mathfrak{m}_2$. As a subsheaf of $P_{\mathfrak{m}_2}^{d_{\mathfrak{m}_2}}$, $P_{\mathfrak{m}_1}^{d_{\mathfrak{m}_1}}$ parametrizes pairs (\mathcal{L}_2, ϕ_2) such that the compositions $\mathcal{O}_C \xrightarrow{\mathrm{pr}^* \phi_2} (\mathcal{L}_2)_{\mathfrak{m}_2} \to (\mathcal{L}_2)_{\mathfrak{m}_2-\mathfrak{m}_1}$ are zero. The commutative diagram



induced by (3-6), (3-8), and the projections $\mathbb{P}(\mathcal{E}_{\mathfrak{m}_i}) \to P_{\mathfrak{m}_i}^{d_{\mathfrak{m}_i}}$ is a cartesian diagram.

Daichi Takeuchi

(2) Assume that a submodulus \mathfrak{m} is the sum $\sum_{i} \mathfrak{m}_{i}$ of submoduli of deg = 1. Let $\mathfrak{m}'_{i} := \sum_{j \neq i} \mathfrak{m}_{j}$. Then, the open subspace $\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$ of $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ is the complement of $P_{\mathfrak{m}'_{i}}^{d_{\mathfrak{m}'_{i}}}$ for all i.

Proof. (1) The first assertion is obvious from the definition of $i_{\mathfrak{m}_1,\mathfrak{m}_2}$. To prove the second assertion, it is enough to show that $\mathcal{E}_{\mathfrak{m}_1} \cong i^*_{\mathfrak{m}_1,\mathfrak{m}_2} \mathcal{E}_{\mathfrak{m}_2}$ by Theorem 3.4. Let (\mathcal{L}_i, ϕ_i) be the universal elements of $P_{\mathfrak{m}_i}^{d_{\mathfrak{m}_i}}$. The pullback of the cartesian diagram



by $i_{\mathfrak{m}_1,\mathfrak{m}_2}$ extends to the diagram



where the two squares are cartesian diagrams, which shows the assertion.

(2) This follows from Lemma 3.5 and (1).

Define the *S*-scheme $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ as the fibered product

$$\begin{array}{ccc}
\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})} &\longrightarrow \operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}} \\
\downarrow & & \downarrow \\
X_{\mathfrak{m}} &\longrightarrow P_{\mathfrak{m}}^{d_{\mathfrak{m}}},
\end{array}$$
(3-9)

where the bottom horizontal map $X_{\mathfrak{m}} \to P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ is the composition $X_{\mathfrak{m}} \xrightarrow{\sim} \mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$. The *S*-scheme $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ is a projective space bundle on $\operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$.

Proposition 3.7. The first projection $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})} \to X_{\mathfrak{m}}$ is an open immersion. Moreover, if \mathfrak{m} is the sum $\sum_{i} \mathfrak{m}_{i}$ of submoduli of deg = 1, $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ coincides with the complement of $X_{\mathfrak{m}'_{i}}$ for all i, where $\mathfrak{m}'_{i} := \sum_{j \neq i} \mathfrak{m}_{j}$.

Proof. These are consequences of Lemma 3.5 and Proposition 3.6.

The Abel–Jacobi map $U^{(d_{\mathfrak{m}})} \to \operatorname{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$ and the canonical open immersion $U^{(d_{\mathfrak{m}})} \to X_{\mathfrak{m}}$ define the following commutative diagram

which induces an $X_{\mathfrak{m}}$ -morphism $U^{(d_{\mathfrak{m}})} \to \tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$. This is an open immersion, since the vertical arrows of (3-9) and the left vertical arrow of (3-10) are open immersions. Combining the previous results, we obtain the following:

Corollary 3.8. As an open subscheme of $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$, $U^{(d_{\mathfrak{m}})}$ is the complement of $\tilde{C}^{(d_{\mathfrak{m}})} \times_{C^{(d_{\mathfrak{m}})}} Z_0$.

Proof. After a finite faithfully flat base change of *S*, we may assume that \mathfrak{m} decomposes the sum $\sum_{i} \mathfrak{m}_{i}$ of submoduli \mathfrak{m}_{i} of deg = 1. Let *V* be the complement of Z_{0} in $C^{(d_{\mathfrak{m}})}$. Since $V \setminus U^{(d_{\mathfrak{m}})}$ is included in $\bigcup_{i} Z_{\mathfrak{m}'_{i}}$, where $\mathfrak{m}'_{i} := \sum_{j \neq i} \mathfrak{m}_{j}$, the assertion follows from Proposition 3.7.

Now assume $\mathfrak{m} = 0$. If *C* has an *S*-valued point *P*, it is well-known that $C^{(d)}$ is a projective space bundle over $\operatorname{Pic}_{C}^{d}$ when $d \ge \max\{2g - 1, 0\}$, where *g* is the genus of *C*. In other words, there exists a locally free sheaf \mathcal{F} of finite rank on $\operatorname{Pic}_{C}^{d}$ such that $C^{(d)}$ is isomorphic to $\mathbb{P}(\mathcal{F})$. Classically this is proved using the Poincaré bundle. On the other hand, using Proposition 3.7, we might prove this fact with an extra condition $d \ge \max\{2g, 1\}$, identifying $\operatorname{Pic}_{C, P}^{d} \cong \operatorname{Pic}_{C}^{d}$ (see [Bosch et al. 1990, 8.2]).

Corollary 3.9. Assume that S is connected noetherian. Let m be a modulus > 0 (resp. = 0) of C and d be a sufficiently large integer. Take a geometric point \bar{x} on $\tilde{C}_{\mathfrak{m}}^{(d)}$ (resp. on $C^{(d)}$) and denote \bar{y} its image to $\operatorname{Pic}_{C,\mathfrak{m}}^{d}$. Then, the morphism of profinite groups $\pi_1(\tilde{C}_{\mathfrak{m}}^{(d)}, \bar{x}) \rightarrow \pi_1(\operatorname{Pic}_{C,\mathfrak{m}}^d, \bar{y})$ (resp. $\pi_1(C^{(d)}, \bar{x}) \rightarrow \pi_1(\operatorname{Pic}_C^d, \bar{y})$) induced from the projection $\tilde{C}_{\mathfrak{m}}^{(d)} \rightarrow \operatorname{Pic}_{C,\mathfrak{m}}^d$ (resp. $C^{(d)} \rightarrow \operatorname{Pic}_C^d$) is an isomorphism.

Proof. When $\mathfrak{m} = 0$, let us also denote $C^{(d)}$ by $\tilde{C}^{(d)}_{\mathfrak{m}}$ for. If $\mathfrak{m} > 0$ (resp. = 0), $\tilde{C}^{(d)}_{\mathfrak{m}}$ is a projective space bundle over $\operatorname{Pic}^{d}_{C,\mathfrak{m}}$ (resp. after the base change from *S* to an étale cover). In any case, the morphism $\tilde{C}^{(d)}_{\mathfrak{m}} \to \operatorname{Pic}^{d}_{C,\mathfrak{m}}$ is proper surjective smooth with geometrically connected fibers. Take a geometric point \bar{s} of $\tilde{C}^{(d)}_{\mathfrak{m},\bar{y}}$ above \bar{x} . Since the scheme $\tilde{C}^{(d)}_{\mathfrak{m},\bar{y}}$ is simply connected, the homotopy exact sequence

$$\pi_1(\tilde{C}^{(d)}_{\mathfrak{m},\bar{y}},\bar{s}) \to \pi_1(\tilde{C}^{(d)}_{\mathfrak{m}},\bar{x}) \to \pi_1(\operatorname{Pic}^d_{C,\mathfrak{m}},\bar{y}) \to 1$$

implies the assertion.

4. Proofs

In this section, we prove Theorems 1.1 and 1.2.

First we need to recall basic results on symmetric products of curves.

Let *C* be a projective smooth geometrically connected curve over a perfect field *k*. Let \mathfrak{m} be a modulus on *C* and write $\mathfrak{m} = n_1 P_1 + \cdots + n_r P_r$, where P_1, \ldots, P_r are distinct closed points of \mathfrak{m} . Denote the complement of \mathfrak{m} in *C* by *U*. Let $d_i := \deg P_i$. Take a positive integer *d* so that $d \ge \deg \mathfrak{m}$.

Daichi Takeuchi

Lemma 4.1. The morphism $\pi : C^{(n_1d_1)} \times \cdots \times C^{(n_rd_r)} \times C^{(d-\deg \mathfrak{m})} \to C^{(d)}$, taking the sum, is étale at the generic point of the closed subvariety $\{n_1P_1\} \times \cdots \times \{n_rP_r\} \times C^{(d-\deg \mathfrak{m})}$ of $C^{(n_1d_1)} \times \cdots \times C^{(n_rd_r)} \times C^{(d-\deg \mathfrak{m})}$.

Proof. We may assume that k is algebraically closed (hence $d_i = 1$ for all i). Since the map $\pi : C^{(n_1)} \times \cdots \times C^{(n_r)} \times C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ is finite flat, it is enough to show that there exists a closed point Q of $n_1P_1 + \cdots + n_rP_r + C^{(d-\deg \mathfrak{m})}$ over which there are $\deg \pi$ points on $C^{(n_1)} \times \cdots \times C^{(n_r)} \times C^{(d-\deg \mathfrak{m})}$. Choose Q as a point corresponding to a divisor $n_1P_1 + \cdots + n_rP_r + P_{r+1} + \cdots + P_{r+d-\deg \mathfrak{m}}$, where $P_1, \ldots, P_{r+d-\deg \mathfrak{m}}$ are distinct points of U(k).

Lemma 4.2. The morphism $\pi_1(U^d) \to \pi_1(U^{(d)})$ induced from the natural projection $U^d \to U^{(d)}$ (base points are omitted) is surjective.

Proof. Since U^d and $U^{(d)}$ are geometrically connected over k, it is enough to show the surjectivity after the base change to an algebraic closure \bar{k} by considering the homotopy exact sequence $1 \to \pi_1(U^d_{\bar{k}}) \to \pi_1(U^d) \to \pi_1(\operatorname{Spec}(k)) \to 1$ and the counterpart of $U^{(d)}$.

Assume that *k* is algebraically closed. Let *V* be a connected finite étale covering of $U^{(d)}$. We show that the pullback $V \times_{U^{(d)}} U^d$ is also connected, which shows the assertion. Note that, since the schemes U^d and $U^{(d)}$ are normal, *V* and $V \times_{U^{(d)}} U^d$ are normal. In particular, $V \times_{U^{(d)}} U^d$ is the disjoint union of integral schemes. Since the map $U^d \to U^{(d)}$ is finite flat, each connected component of $V \times_{U^{(d)}} U^d$ surjects onto *V*. Take a *k*-valued point $P \in U(k)$. Take a *k*-valued point $P' \in V(k)$ over $dP \in U^{(d)}(k)$. Since the fiber of $U^d \to U^{(d)}$ over the point dP consists of one point (P, P, \ldots, P) , the fiber of $V \times_{U^{(d)}} U^d \to V$ over P' also consists of one point. Thus the scheme $V \times_{U^{(d)}} U^d$ has only one connected component.

Proof of Theorem 1.2. Let *C* be a projective smooth geometrically connected curve over a perfect field *k*. Let $\mathfrak{m} = n_1 P_1 + \cdots + n_r P_r$ ($n_i \ge 1$) be a modulus on *C*, and *U* be its complement. Set *A* as the subgroup of $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z})$ consisting of characters χ such that $\mathrm{Sw}_{P_i}(\chi) \le n_i - 1$ for $i = 1, \ldots, r$, and *B* as the subgroup of $\mathrm{H}^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ consisting of multiplicative elements.

We construct a map $\Psi : B \to A$. Take $\rho \in B$. Define χ to be the pullback of ρ^1 by the natural map $U \to \text{Pic}_{C,\mathfrak{m}}^1$. We need to show that the ramification is bounded by \mathfrak{m} . Take a natural number *d* large enough so that *d* satisfies (3-1) for \mathfrak{m} . Consider the following commutative diagram:

By the multiplicativity of ρ , we know that $\pi^* p^* \rho^d = \chi^{\boxtimes d}$. Lemma 4.2 implies that $p^* \rho^d = \chi^{(d)}$. We show that $\operatorname{Sw}_{P_i}(\chi) \leq n_i - 1$. To do this, it is enough to prove that the Swan conductor of $\chi^{(n_i)}$, with respect to the DVR at the generic point of the blow-up of $C^{(n_i)}$ along $n_i P_i$, is zero, by Corollary 2.9. We may assume that *k* is algebraically closed (hence $d_i = 1$). Note that the right vertical arrow *p* in (4-1)

factors through $\tilde{C}_{\mathfrak{m}}^{(d)}$:

$$U^{(d)} \to \tilde{C}^{(d)}_{\mathfrak{m}} \to \operatorname{Pic}^{d}_{C,\mathfrak{m}}$$

Since the pullback of ρ^d by p is $\chi^{(d)}$, we find that $\chi^{(d)}$ is unramified at the generic point of the complement $\tilde{C}_{\mathfrak{m}}^{(d)} \setminus U^{(d)}$. Thus, by Lemma 4.1 and Corollary 3.8, the character

$$\chi^{(n_1)} \boxtimes 1 \boxtimes \cdots \boxtimes 1 + 1 \boxtimes \chi^{(n_2)} \boxtimes \cdots \boxtimes 1 + \cdots + 1 \boxtimes \cdots \boxtimes \chi^{(d-\deg \mathfrak{m})}$$

on $U^{(n_1)} \times \cdots \times U^{(n_r)} \times U^{(d-\deg \mathfrak{m})}$ is unramified at the generic point of the exceptional divisor of the blow-up of $C^{(n_1)} \times \cdots \times C^{(n_r)} \times C^{(d-\deg \mathfrak{m})}$ along $\{n_1P_1\} \times \cdots \times \{n_rP_r\} \times C^{(d-\deg \mathfrak{m})}$. Using Corollary 2.6 repeatedly, the assertion is proved.

Thus the map $B \to A$, pulling back by $U \to \operatorname{Pic}_{C,m}^1$, is well-defined. We denote this map by Ψ .

First we show the injectivity of Ψ . Take ρ from the kernel of Ψ . Since the multiplication map $\operatorname{Pic}_{C,\mathfrak{m}}^{n} \times \operatorname{Pic}_{C,\mathfrak{m}}^{m} \to \operatorname{Pic}_{C,\mathfrak{m}}^{n+m}$ and the two projections $\operatorname{Pic}_{C,\mathfrak{m}}^{n} \times \operatorname{Pic}_{C,\mathfrak{m}}^{m} \to \operatorname{Pic}_{C,\mathfrak{m}}^{n}$, $\operatorname{Pic}_{C,\mathfrak{m}}^{m}$, have geometrically connected fibers, the triviality of two of ρ^{n} , ρ^{m} , ρ^{n+m} implies the triviality of the other. Thus it is enough to show the triviality of ρ^{d} for sufficiently large d. Consider the diagram (4-1). By Lemma 4.2, we know that $p^*\rho^{d}$ is trivial, which implies that ρ^{d} is trivial by Corollary 3.9.

The surjectivity of Ψ is proved as follows. Take $\chi \in A$. Let *d* be an integer satisfying (3-1) for m. Corollary 2.9, Proposition 3.7, and Lemma 4.1 imply that the character $\chi^{(d)}$ extends to a character $\tilde{\chi}^{(d)}$ on $\tilde{C}_{\mathfrak{m}}^{(d)}$. Corollary 3.9 implies that $\tilde{\chi}^{(d)}$ descends to a character ρ^d on $\operatorname{Pic}_{C,\mathfrak{m}}^d$. Let d_1 and d_2 be integers which satisfy (3-1). The commutative diagram

and the fact that the left vertical map has geometrically connected fibers show $q^* \rho^{d_1+d_2} = \rho^{d_1} \boxtimes 1 + 1 \boxtimes \rho^{d_2}$. Fix a nonzero effective Cartier divisor D on U such that deg D satisfies (3-1). Let ξ be the pullback of $\rho^{\deg D}$ by the map $\operatorname{Spec}(k) \to \operatorname{Pic}_{C,\mathfrak{m}}^{\deg D}$, corresponding to the point D. For an arbitrary integer n, take a natural number m so large that the integer $n + m \deg D$ satisfies (3-1). Define $\rho^n := f^* \rho^{n+m \deg D} \cdot a^* \xi^{-m}$, where $f : \operatorname{Pic}_{C,\mathfrak{m}}^n \to \operatorname{Pic}_{C,\mathfrak{m}}^{n+m \deg D}$ is multiplication by $\mathcal{O}_C(mD)$ and $a : \operatorname{Pic}_{C,\mathfrak{m}}^n \to \operatorname{Spec}(k)$ is the structure map. This construction does not depend on m, since the multiplicativity of ρ^n is already verified for large n. By the same reason, the characters ρ^n form a multiplicative element on $\operatorname{Pic}_{C,\mathfrak{m}}$. The equality $\chi = \Psi(\rho)$ follows from the commutative diagram

where g is the composition of the structure map $U \to \operatorname{Spec}(k)$ and the map $\operatorname{Spec}(k) \to U^{(\deg D)}$ corresponding to the divisor D. Indeed, the pullback of $\rho^{\deg D+1}$ by the map $U \to U \times U^{(\deg D)} \to U^{(\deg D+1)} \to \operatorname{Pic}_{C,\mathfrak{m}}^{\deg D+1}$ is $\chi \cdot b^*\xi$, where $b: U \to \operatorname{Spec}(k)$ is the structure map. On the other hand, the pullback of $\rho^{\deg D+1}$ the other way is $\Psi(\rho) \cdot b^*\xi$.

To deduce Theorem 1.1 from Theorem 1.2, first we recall basic facts on torsors.

Assume that k is algebraically closed. Fix a connected commutative algebraic k-group G. Let C(G) be the category as follows. The objects are pairs $(H, \phi : H \to G)$ where H are connected commutative algebraic k-groups and ϕ are abelian isogenies. The morphisms $(H_1, \phi_1 : H_1 \to G) \to (H_2, \phi_2 : H_2 \to G)$ are pairs (f, g) where $f : H_1 \to H_2$ is a morphism of k-group schemes such that $\phi_2 \circ f = \phi_1$ and $g : H_1 \to H_2$ is a compatible morphism of torsors such that $\phi_2 \circ g = \phi_1$. Here we regard H_1 (resp. H_2) itself as an H_1 -torsor (resp. H_2 -torsor) by the multiplication. Note that the kernels of ϕ_i are constant k-schemes since H_i are Galois coverings of G.

Lemma 4.3. Let the notation be as above. Let $(H_i, \phi_i : H_i \to G)$ be objects in $\mathcal{C}(G)$ for i = 1, 2.

- (1) If there exists a morphism $H_1 \rightarrow H_2$ of G-schemes, there exists a unique morphism $f: H_1 \rightarrow H_2$ of k-group schemes with $\phi_2 \circ f = \phi_1$.
- (2) The map

 $\operatorname{Hom}((H_1, \phi_1 \colon H_1 \to G), (H_2, \phi_2 \colon H_2 \to G)) \to \operatorname{Hom}_G(H_1, H_2)$

sending $(f, g) \mapsto g$ is bijective. Here the target is the set of morphisms of G-schemes.

Proof. Let $e_i \in H_i(k)$ be the units.

(1) Uniqueness follows from the fact that H_i are connected étale coverings of G and such an f must send e_1 to e_2 . Let $f: H_1 \to H_2$ be the G-morphism which sends e_1 to e_2 . Such an f does exist since H_2 is Galois over G. We need to show that the diagram



where the vertical maps are the multiplications, is commutative. This follows since $H_1 \times H_1$ is a connected étale covering of $G \times G$ and the two maps send (e_1, e_1) to e_2 .

(2) Injectivity follows since a goup homomorphism $f: H_1 \to H_2$ over *G* is unique if it exists by (1). We show the surjectivity. Thus we assume that there is a group homomorphism $f: H_1 \to H_2$ over *G*. Let $g: H_1 \to H_2$ be a morphism of *G*-schemes. Since H_1 is a connected étale covering of *G*, this is uniquely determined by the image $a := g(e_1)$, which is contained in ker ϕ_2 . Let $g': H_1 \to H_2$ be the compatible morphism of torsors which sends e_1 to a. Since $a \in \ker \phi_2$ and $\phi_2 \circ f = \phi_1$, this is a morphism of *G*-schemes. Thus we have g = g'.

Proof of Theorem 1.1. Let (G^0, G^1) be a connected abelian covering of $(\operatorname{Pic}_{C,\mathfrak{m}}^0, \operatorname{Pic}_{C,\mathfrak{m}}^1)$. Since the *d*-th power of $\operatorname{Pic}_{C,\mathfrak{m}}^1$ is isomorphic to $\operatorname{Pic}_{C,\mathfrak{m}}^d$ as $\operatorname{Pic}_{C,\mathfrak{m}}^0$ -torsors, the *d*-th power G^d of G^1 is naturally equipped with a compatible morphism $G^d \to \operatorname{Pic}_{C,\mathfrak{m}}^d$ of torsors. Let *K* be the kernel of the map $G^0 \to \operatorname{Pic}_{C,\mathfrak{m}}^0$. This is a finite constant group since $G^0 \to \operatorname{Pic}_{C,\mathfrak{m}}^0$ is a Galois isogeny. Take a nontrivial homomorphism $\chi : K \to \mathbb{Q}/\mathbb{Z}$. This defines characters $\rho^d \in \operatorname{H}^1(\operatorname{Pic}_{C,\mathfrak{m}}^d, \mathbb{Q}/\mathbb{Z})$ for all *d*. From the construction, they form a multiplicative element on $\operatorname{Pic}_{C,\mathfrak{m}}^0$. Theorem 1.2 implies that the pullback of ρ^1 by $U \to \operatorname{Pic}_{C,\mathfrak{m}}^1$ is nontrivial and its ramification is bounded by \mathfrak{m} , which shows the first part of Theorem 1.1.

Define the category C_1 as the category of geometrically connected abelian coverings of U whose ramifications are bounded by m and the category C_2 as the category of connected abelian coverings of $(\operatorname{Pic}_{C,\mathfrak{m}}^0, \operatorname{Pic}_{C,\mathfrak{m}}^1)$. We have constructed a functor $\Phi : C_2 \to C_1$. We show that this functor is an equivalence of categories. We only treat the case when k is algebraically closed. General case follows from this special case by using an argument of Galois descent.

Assume that *k* is algebraically closed. Let $C := C(\operatorname{Pic}^0_{C,\mathfrak{m}})$ be the category defined above. In this case, fixing a closed point *P* of *U*, C_2 is isomorphic to *C* via the isomorphism $\operatorname{Pic}^0_{C,\mathfrak{m}} \to \operatorname{Pic}^1_{C,\mathfrak{m}}$ of torsors, tensoring $\mathcal{O}_C(P)$.

We show that the functor $\Phi' : C \to C_1$, pulling back by the morphism $U \to \operatorname{Pic}_{C,\mathfrak{m}}^0$ sending Q to $\mathcal{O}_C(Q-P)$ is an equivalence. Faithfulness is obvious since there only occur connected coverings. To show fullness, let $(G_i, G_i \to \operatorname{Pic}_{C,\mathfrak{m}}^0)$ be elements of C for i = 1, 2 and let $V_i := \Phi'(G_i, G_i \to \operatorname{Pic}_{C,\mathfrak{m}}^0)$. By Lemma 4.3(2) and faithfulness, it is enough to show that, if there is a map $V_1 \to V_2$, there is a map $G_1 \to G_2$. The kernel K_i of $G_i \to \operatorname{Pic}_{C,\mathfrak{m}}^0$ is canonically identified with the Galois group of $V_i \to U$. If there is a map $V_1 \to V_2$, there is a map of abelian groups $h : K_1 \to K_2$, which is independent of the choice of $V_1 \to V_2$. We show the commutativity of the diagram



where the downward diagonals are the canonical surjections. Assume that there is an element $\sigma \in \pi_1(\operatorname{Pic}^0_{C,\mathfrak{m}})$ such that $p_2(\sigma) \neq hp_1(\sigma)$. Take a group homomorphism $\rho^0: K_2 \to \mathbb{Q}/\mathbb{Z}$ such that the images of $p_2(\sigma)$ and $hp_1(\sigma)$ are different. Since the characters $\rho^0 p_2$ and $\rho^0 hp_1$ are multiplicative and are pulled back to the same character via the map $U \to \operatorname{Pic}^0_{C,\mathfrak{m}}$, they are the same character, a contradiction. Thus the diagram (4-2) is commutative, which implies that the quotient group $G_1/\ker h$ of G_1 is isomorphic to G_2 .

For essential surjectivity, we argue as follows. Let $V \in C_1$ be a connected cyclic covering of U. Take a character on U whose kernel corresponds to V. By Theorem 1.3, this character is the pullback of a multiplicative character ρ^0 on $\operatorname{Pic}_{C,\mathfrak{m}}^0$. Let G^0 be an étale covering of $\operatorname{Pic}_{C,\mathfrak{m}}^0$ corresponding to the kernel of ρ^0 . We need to show that G^0 has a group structure. By the definition, G^0 is connected. From the Daichi Takeuchi

multiplicativity of ρ^0 , we know that there is a commutative diagram



Let us denote the map m_G multiplicatively. Let F be the fiber of $G^0 \to \operatorname{Pic}_{C,\mathfrak{m}}^0$ over $1 \in \operatorname{Pic}_{C,\mathfrak{m}}^0$. For distinct points $y_1, y_2 \in F$, the multiplication from right by y_1 and $y_2, G^0 \to G^0$ are distinct. Indeed, Assume that $xy_1 = xy_2$ for all $x \in G^0$. Take a point x in F. The multiplication from left by $x, G^0 \to G^0$ is a $\operatorname{Pic}_{C,\mathfrak{m}}^0$ -morphism and sends y_1 and y_2 to the same point, which implies that $y_1 = y_2$ since G^0 is a connected covering of $\operatorname{Pic}_{C,\mathfrak{m}}^0$.

Thus there exists an element $e \in F$ such that xe = x for all $x \in G^0$. Next we show the commutativity of m_G . This follows from the fact that $G^0 \times G^0$ is a connected covering of $\operatorname{Pic}_{C,\mathfrak{m}}^0 \times \operatorname{Pic}_{C,\mathfrak{m}}^0$ and that the maps $G^0 \times G^0 \to G^0$, $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ send (e, e) to the same point e. The associativity is proved in a similar way. Therefore it is verified that G^0 has a commutative group structure such that $G^0 \to \operatorname{Pic}_{C,\mathfrak{m}}^0$ is a group homomorphism, hence an abelian isogeny. It is easy to show that G^0 is pulled back to V. For a general V, use the fact that V is a connected component of the finite projective limit of cyclic connected coverings which are quotients of V.

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Positivity functions for curves on algebraic varieties BRIAN LEHMANN and JIAN XIAO	1243
The congruence topology, Grothendieck duality and thin groups ALEXANDER LUBOTZKY and TYAKAL NANJUNDIAH VENKATARAMANA	1281
On the ramified class field theory of relative curves QUENTIN GUIGNARD	1299
Blow-ups and class field theory for curves DAICHI TAKEUCHI	1327
Algebraic monodromy groups of <i>l</i> -adic representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ SHIANG TANG	1353
Weyl bound for <i>p</i> -power twist of GL(2) <i>L</i> -functions RITABRATA MUNSHI and SAURABH KUMAR SINGH	1395
Examples of hypergeometric twistor D-modules Alberto Castaño Domínguez, Thomas Reichelt and Christian Sevenheck	1415
Ulrich bundles on K3 surfaces DANIELE FAENZI	1443
Unlikely intersections in semiabelian surfaces DANIEL BERTRAND and HARRY SCHMIDT	1455
Congruences of parahoric group schemes RADHIKA GANAPATHY	1475
An improved bound for the lengths of matrix algebras YAROSLAV SHITOV	1501

