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Blow-ups and class field theory for curves

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We propose another proof of geometric class field theory for curves by considering blow-ups of symmetric products of curves.

1. Introduction

Geometric class field theory gives a geometric description of the abelian coverings of a curve by using generalized jacobian varieties. Let us recall its precise statement. Let C be a projective smooth curve over a perfect field k. We assume that C is geometrically connected over k. Fix a modulus $\mathfrak{m}$, i.e., an effective Cartier divisor of C and let U be its complement in C. Denote by $\mathrm{Pic}^0_{C,\mathfrak{m}}$ the corresponding generalized jacobian variety. Let $G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ be an étale isogeny of smooth commutative algebraic groups and $G^1 \to \mathrm{Pic}^1_{C,\mathfrak{m}}$ be a compatible morphism of torsors. We call such a pair $(G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}, G^1 \to \mathrm{Pic}^1_{C,\mathfrak{m}})$ a *covering of* $(\mathrm{Pic}^0_{C,\mathfrak{m}}, \mathrm{Pic}^1_{C,\mathfrak{m}})$. A covering $(G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}, G^1 \to \mathrm{Pic}^1_{C,\mathfrak{m}})$ is called *connected abelian* if G^0 is connected and $G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ is an abelian isogeny. There is a natural map from U to $\mathrm{Pic}^1_{C,\mathfrak{m}}$ sending a point of U to its associated invertible sheaf with a trivialization. Geometric class field theory states:

Theorem 1.1. *Let C be a projective smooth geometrically connected curve over a perfect field k. Fix a modulus $\mathfrak{m}$ of C and denote its complement by U. Let $\mathrm{Pic}^0_{C,\mathfrak{m}}$ be the generalized jacobian variety with modulus $\mathfrak{m}$. Then a connected abelian covering $(G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}, G^1 \to \mathrm{Pic}^1_{C,\mathfrak{m}})$ pulls back by the natural map $U \to \mathrm{Pic}^1_{C,\mathfrak{m}}$ to a geometrically connected abelian covering of U whose ramification is bounded by $\mathfrak{m}$. Conversely, every such covering is obtained in this way.*

Originally this theorem was proved by M. Rosenlicht [1954]. S. Lang [1956] generalized his results to an arbitrary algebraic variety. Their works are explained in detail in Serre's book [1988].

On the other hand, in 1980s, P. Deligne found another proof for the tamely ramified case by using symmetric powers of curves [Laumon 1990]. The aim of this paper is to complete his proof by considering blow-ups of symmetric powers of curves.

We have learned that Q. Guignard has done similar work [2019].

Actually we prove a variant of Theorem 1.1 now stated.

Theorem 1.2. *There is an isomorphism of groups between the subgroup of $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z})$ consisting of a character χ such that $\mathrm{Sw}_P(\chi) \leq n_P - 1$ for all points $P \in \mathfrak{m}$, where n_P is the multiplicity of $\mathfrak{m}$*

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at P, and the subgroup of $\mathrm{H}^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ *consisting of* ρ *which is multiplicative, i.e., the self-external product* $\rho \boxtimes 1 + 1 \boxtimes \rho$ *on* $\mathrm{Pic}_{C,\mathfrak{m}} \times_k \mathrm{Pic}_{C,\mathfrak{m}}$ *equals to* $m^*\rho$*, the pullback of* ρ *by the multiplication map* $m : \mathrm{Pic}_{C,\mathfrak{m}} \times_k \mathrm{Pic}_{C,\mathfrak{m}} \to \mathrm{Pic}_{C,\mathfrak{m}}$.

The relation between Theorems 1.1 and 1.2 will be explained in Section 4.

When k is algebraically closed, Theorem 1.2 can be stated as follows. Let ρ be a multiplicative element of $\mathrm{H}^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$. Fix a closed point $P \in \mathrm{Pic}^1_{C,\mathfrak{m}}$. The multiplicativity of ρ implies that, for an integer d, the pullback of ρ^d by the multiplication by dP $\mathrm{Pic}^0_{C,\mathfrak{m}} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ coincides with ρ^0. In this way, Theorem 1.2 can be restated as follows:

Theorem 1.3. *Assume that k is algebraically closed. Then there is an isomorphism of groups between the subgroup of* $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z})$ *consisting of a character* χ *such that* $\mathrm{Sw}_P(\chi) \leq n_P - 1$ *for all points* $P \in \mathfrak{m}$ *and the subgroup of* $\mathrm{H}^1(\mathrm{Pic}^0_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ *consisting of a multiplicative element* ρ^0*, i.e., the self-external product* $\rho^0 \boxtimes 1 + 1 \boxtimes \rho^0$ *on* $\mathrm{Pic}^0_{C,\mathfrak{m}} \times_k \mathrm{Pic}^0_{C,\mathfrak{m}}$ *equals to* $m^*\rho^0$*, the pullback of* ρ^0 *by the multiplication map* $m : \mathrm{Pic}^0_{C,\mathfrak{m}} \times_k \mathrm{Pic}^0_{C,\mathfrak{m}} \to \mathrm{Pic}^0_{C,\mathfrak{m}}$.

Here we summarize the construction of this paper. In Section 2, we recall the definition and properties of (refined) Swan conductors, and make a calculation on the Swan conductors of symmetric products of characters. We construct compactifications of the Abel–Jacobi maps $U^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ and study their properties in Section 3. The main result of this section is that the compactifications can be identified with open subschemes of blow-ups of $C^{(d)}$. In Section 4, we finish the proof of Theorems 1.1 and 1.2 by combining the results in the previous sections.

Throughout this paper, we use the following conventions: We identify an effective Cartier divisor with the associated closed subscheme. For an object defined on a scheme S (e.g., an S-scheme, a locally free sheaf, a vector bundle, and so on) and a S-scheme T, we denote its pullback to T by the same letter, unless there may be ambiguity. We denote the category of S-schemes by Sch/S. For a category $\mathcal{C}$, we call a functor $\mathcal{C}^{op} \to (Set)$, from the opposite category of $\mathcal{C}$ to the category of sets (Set), a presheaf on $\mathcal{C}$.

2. Preliminaries

In this section, we recall basic properties of Witt vectors and refined Swan conductors, and calculate the Swan conductors of symmetric products of characters. Fix a prime number p.

Reminder on the refined Swan conductor. Let A be a ring of characteristic p. Let m be an integer ≥ 0. We denote by $\mathrm{W}_{m+1}(A)$ the ring of Witt vectors of length $m+1$ with coefficients in A, and write its elements as $(a_0, a_1, \ldots, a_m)$. Let $\mathcal{O}_A$ be the structure sheaf of rings on the étale topos of $\mathrm{Spec}(A)$.

Let F be the absolute Frobenius map $\mathcal{O}_A \to \mathcal{O}_A$, i.e., sending $x \mapsto x^p$, and denote the ring homomorphism $\mathrm{W}_{m+1}(\mathcal{O}_A) \to \mathrm{W}_{m+1}(\mathcal{O}_A)$ induced from F by the same letter F. The short exact sequence

$$0 \to \mathbb{Z}/p^{m+1}\mathbb{Z} \to \mathrm{W}_{m+1}(\mathcal{O}_A) \xrightarrow{F-1} \mathrm{W}_{m+1}(\mathcal{O}_A) \to 0$$

of étale sheaves on $\mathrm{Spec}(A)$ defines the boundary map

$$\delta_{m+1,A} : \mathrm{W}_{m+1}(A) \to \mathrm{H}^1(\mathrm{Spec}(A), \mathbb{Z}/p^{m+1}\mathbb{Z}).$$

The boundary map is surjective, hence $\mathrm{W}_{m+1}(A)/\operatorname{Im}(F-1) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{Spec}(A), \mathbb{Z}/p^{m+1}\mathbb{Z})$, the map it induces, is an isomorphism. The boundary map $\delta_{m+1,A}$ is natural in A. In other words, for a morphism $f : A \to B$ of rings of characteristic p, the diagram

$$\begin{array}{ccc} \mathrm{W}_{m+1}(A) & \xrightarrow{\delta_{m+1,A}} & \mathrm{H}^1(\mathrm{Spec}(A), \mathbb{Z}/p^{m+1}\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{W}_{m+1}(B) & \xrightarrow{\delta_{m+1,B}} & \mathrm{H}^1(\mathrm{Spec}(B), \mathbb{Z}/p^{m+1}\mathbb{Z}) \end{array} \tag{2-1}$$

is commutative, where the vertical maps are the canonical ones induced from f.

Let (R, π) be a DVR of equal characteristic p and K be its field of fractions. Let v_R be its normalized valuation. Let m be an integer ≥ 0. We extend the valuation v_R to $\mathrm{W}_{m+1}(K)$ by setting

$$v_R((a_0, \dots, a_m)) := \min_i \{p^{m-i} v_R(a_i)\}.$$

We define an increasing exhaustive filtration on $\mathrm{W}_{m+1}(K)$ by setting, for $n \in \mathbb{Z}$, $\mathrm{fil}_n \mathrm{W}_{m+1}(K)$ to be the subgroup of $\mathrm{W}_{m+1}(K)$ consisting of elements $(a_0, \dots, a_m)$ such that

$$v_R((a_0, \dots, a_m)) \geq -n.$$

Define an increasing exhaustive filtration $\mathrm{fil}_n \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ of $\mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ by the image of $\mathrm{fil}_n \mathrm{W}_{m+1}(K)$ through the boundary map $\delta_{m+1,K}$.

For any $\chi \in \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$, the Swan conductor of χ, $\mathrm{Sw}_R(\chi)$, is the smallest integer $n \geq 0$ such that $\chi \in \mathrm{fil}_n \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$[Brylinski 1983; Kato 1989]. When R is henselian and the residue field is perfect, this is the same as the classical Swan conductor [Kato 1989, Proposition (6.8)].

Lemma 2.1. *Let R and K be as above. Take $\chi \in \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$.*

(1) *The subgroup $\mathrm{fil}_0 \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ of $\mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$ coincides with the image of the map $\mathrm{H}^1(\mathrm{Spec}(R), \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$, i.e., the group of unramified characters.*

(2) *Let $\hat{R}$ be the completion of R and $\hat{K}$ be its field of fractions. Denote the restriction of χ to $\hat{K}$ by $\hat{\chi}$. Then, the equality $\mathrm{Sw}_R(\chi) = \mathrm{Sw}_{\hat{R}}(\hat{\chi})$ holds.*

Proof. (1) This follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{W}_{m+1}(R) & \xrightarrow{\delta_{m+1,R}} & \mathrm{H}^1(\mathrm{Spec}(R), \mathbb{Z}/p^{m+1}\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{W}_{m+1}(K) & \xrightarrow{\delta_{m+1,K}} & \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \end{array} \tag{2-2}$$

and the fact that the upper horizontal arrow in (2-2) is surjective.

(2) The commutative diagram

$$\begin{array}{ccc} \mathrm{W}_{m+1}(K) & \xrightarrow{\delta_{m+1,K}} & \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{W}_{m+1}(\hat{K}) & \xrightarrow{\delta_{m+1,\hat{K}}} & \mathrm{H}^1(\hat{K}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \end{array}$$

implies $\mathrm{Sw}_R(\chi) \geq \mathrm{Sw}_{\hat{R}}(\hat{\chi})$. Let $n = \mathrm{Sw}_{\hat{R}}(\hat{\chi})$. Then there exists a Witt vector $\hat{\alpha} \in \mathrm{fil}_n \mathrm{W}_{m+1}(\hat{K})$ mapping to $\hat{\chi}$. Take $\alpha \in \mathrm{fil}_n \mathrm{W}_{m+1}(K)$ whose components are close enough to those of $\hat{\alpha}$ with respect to the valuation of $\hat{K}$, so that every component of $\hat{\alpha} - \alpha$ (here α is regarded as an element of $\mathrm{W}_{m+1}(\hat{K})$) is in $\hat{R}$. Then, $\delta_{m+1,\hat{K}}(\hat{\alpha} - \alpha)$ is an unramified character by (1). Therefore, $\chi - \delta_{m+1,K}(\alpha)$ is unramified. Again by (1), there exists $\beta \in \mathrm{W}_{m+1}(R)$ such that $\chi - \delta_{m+1,K}(\alpha) = \delta_{m+1,K}(\beta)$, hence the assertion. □

Next we recall refined Swan conductors.

Define $\widehat{\Omega}^1_R$ to be the π-adic completion of the absolute differential module Ω^1_R. Let $\widehat{\Omega}^1_K := \widehat{\Omega}^1_R \otimes_R K$. The canonical map $\widehat{\Omega}^1_R \to \widehat{\Omega}^1_K$ is injective and we usually regard $\widehat{\Omega}^1_R$ as an R-submodule of $\widehat{\Omega}^1_K$ via this map. The R-module $\widehat{\Omega}^1_R(\log)$ is the R-submodule of $\widehat{\Omega}^1_K$ generated by $\widehat{\Omega}^1_R$ and $d\log\pi := d\pi/\pi$. From the definition, the following holds:

Lemma 2.2. *Assume that R is obtained from a smooth scheme over a perfect field by localizing at a point of codimension one. Let $b_1, \dots, b_n$ be a lift of a p-basis of the residue field of R to R. Then, $\widehat{\Omega}^1_R(\log)$ is a $\hat{R}$-free module with a basis $db_1, \dots, db_n, d\log\pi$.* □

For $\omega \in \widehat{\Omega}^1_K$, define $v^{\log}_R(\omega)$ as the largest integer n such that $\omega \in \pi^n \widehat{\Omega}^1_R(\log)$ (we formally put $v^{\log}_R(0) := \infty$). There is a homomorphism $F^m d : \mathrm{W}_{m+1}(K) \to \widehat{\Omega}^1_K$ of groups given by

$$F^m d((a_0, \dots, a_m)) := \sum_i a_i^{p^{m-i}-1} da_i.$$

Define an increasing exhaustive filtration on $\widehat{\Omega}^1_K$ by setting

$$\mathrm{fil}_n \widehat{\Omega}^1_K := \{\omega \in \widehat{\Omega}^1_K \mid v^{\log}_R(\omega) \geq -n\}$$

for $n \in \mathbb{Z}$. From the definitions, the homomorphism $F^m d : \mathrm{W}_{m+1}(K) \to \widehat{\Omega}^1_K$ respects their filtrations. In other words, $v_R(\alpha) \leq v^{\log}_R(F^m d\alpha)$ hold for all $\alpha \in \mathrm{W}_{m+1}(K)$.

Proposition 2.3 [Leal 2018, Proposition 2.8]. *Let n be an integer ≥ 0.*

(1) *There is a unique homomorphism*

$$\mathrm{rsw} : \mathrm{fil}_n \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{fil}_n \widehat{\Omega}^1_K / \mathrm{fil}_{\lfloor n/p \rfloor} \widehat{\Omega}^1_K,$$

called the refined Swan conductor, such that the composition

$$\mathrm{fil}_n \mathrm{W}_{m+1}(K) \to \mathrm{fil}_n \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{fil}_n \widehat{\Omega}^1_K / \mathrm{fil}_{\lfloor n/p \rfloor} \widehat{\Omega}^1_K$$

coincides with $F^m d$.

(2) *For $\lfloor \frac{n}{p} \rfloor \leq i \leq n$, the induced map*

$$\mathrm{fil}_n \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) / \mathrm{fil}_i \mathrm{H}^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{fil}_n \widehat{\Omega}^1_K / \mathrm{fil}_i \widehat{\Omega}^1_K$$

is injective.

At the end of this subsection, we extend the definition of the Swan conductors for characters in $\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ as follows.

Let m be an integer ≥ 0. We identify the groups $\mathbb{Z}/p^m\mathbb{Z}$ and $\frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$ via the multiplication by $\frac{1}{p^m}$. In this way, we define a filtration on $\mathrm{H}^1\big(K, \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big)$ from that of $\mathrm{H}^1(K, \mathbb{Z}/p^m\mathbb{Z})$. The natural inclusion $\frac{1}{p^m}\mathbb{Z}/\mathbb{Z} \to \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}$ induces an inclusion

$$\mathrm{H}^1\big(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big) \to \mathrm{H}^1\big(K, \tfrac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\big)$$

of groups.

Lemma 2.4. *Let m, n be integers ≥ 0. The equality*

$$\mathrm{fil}_n \mathrm{H}^1\big(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big) = \mathrm{H}^1\big(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big) \cap \mathrm{fil}_n \mathrm{H}^1\big(K, \tfrac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\big)$$

of subgroups of $\mathrm{H}^1\big(K, \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\big)$ holds.

Proof. Fix a separable closure K^s of K. Let $V : \mathrm{W}_m(K^s) \to \mathrm{W}_{m+1}(K^s)$ be the Verschiebung, i.e., the map sending $(a_0, \ldots, a_{m-1})$ to $(0, a_0, \ldots, a_{m-1})$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{1}{p^m}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \mathrm{W}_m(K^s) & \xrightarrow{F-1} & \mathrm{W}_m(K^s) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow V & & \downarrow V & & \\ 0 & \longrightarrow & \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \mathrm{W}_{m+1}(K^s) & \xrightarrow{F-1} & \mathrm{W}_{m+1}(K^s) & \longrightarrow & 0, \end{array} \tag{2-3}$$

here we identify $\frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$ and $\mathbb{Z}/p^m\mathbb{Z}$ as mentioned above. Taking cohomology groups, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{W}_m(K) & \xrightarrow{\delta_{m,K}} & \mathrm{H}^1\big(K, \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big) \\ \downarrow V & & \downarrow \\ \mathrm{W}_{m+1}(K) & \xrightarrow{\delta_{m+1,K}} & \mathrm{H}^1\big(K, \frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\big). \end{array} \tag{2-4}$$

Since the map $V : \mathrm{W}_m(K) \to \mathrm{W}_{m+1}(K)$ respects the filtrations, the inclusion

$$\mathrm{fil}_n \mathrm{H}^1\big(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big) \subset \mathrm{H}^1\big(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\big) \cap \mathrm{fil}_n \mathrm{H}^1\big(K, \tfrac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\big)$$

holds. To prove the equality, it suffices to show that the morphism

$$\operatorname{Gr}_n \mathrm{H}^1\bigl(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\bigr) \to \operatorname{Gr}_n \mathrm{H}^1\bigl(K, \tfrac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\bigr)$$

is injective for $n \geq 1$, where $\operatorname{Gr}_n := \operatorname{fil}_n / \operatorname{fil}_{n-1}$. We have the following commutative diagram

$$\begin{array}{ccc} \operatorname{Gr}_n \mathrm{H}^1\bigl(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\bigr) & \xrightarrow{\ \mathrm{rsw}\ } & \operatorname{Gr}_n \widehat{\Omega}^1_K \\ \downarrow & \nearrow_{\mathrm{rsw}} & \\ \operatorname{Gr}_n \mathrm{H}^1\bigl(K, \tfrac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z}\bigr). & & \end{array} \tag{2-5}$$

By Proposition 2.3(2), the refined Swan conductors rsw in (2-5) are injective, hence the assertion. □

We define a filtration on $\mathrm{H}^1(K, \mathbb{Q}_p/\mathbb{Z}_p) = \bigcup_m \mathrm{H}^1\bigl(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\bigr)$ by

$$\operatorname{fil}_n \mathrm{H}^1(K, \mathbb{Q}_p/\mathbb{Z}_p) = \bigcup_m \operatorname{fil}_n \mathrm{H}^1\bigl(K, \tfrac{1}{p^m}\mathbb{Z}/\mathbb{Z}\bigr).$$

Let $\chi \in \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ be a character. Let χ_p be the p-primary part of χ and be considered as an element of $\mathrm{H}^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$ via the natural decomposition

$$\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_q \mathrm{H}^1(K, \mathbb{Q}_q/\mathbb{Z}_q),$$

where q runs through all prime numbers. We define the Swan conductor $\operatorname{Sw}(\chi)$ to be the smallest integer $n \geq 0$ such that $\chi_p \in \operatorname{fil}_n \mathrm{H}^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$.

The Swan conductor of a symmetric product. In this subsection, we assume that k is a perfect field of characteristic p.

Let X_1, X_2 be smooth schemes over k. Let Z_1 and Z_2 be smooth irreducible closed subvarieties of X_1 and X_2. Let $\tilde{X}_1$, $\tilde{X}_2$, and $\widetilde{X_1 \times X_2}$ be the blow-ups of X_1, X_2, and $X_1 \times X_2$ along Z_1, Z_2, and $Z_1 \times Z_2$. Denote by R_1, R_2, and R_3 the DVRs at the generic points of the exceptional divisor of $\tilde{X}_1$, $\tilde{X}_2$, and $\widetilde{X_1 \times X_2}$. Let K_i be the field of fractions of R_i for $i = 1, 2, 3$.

Lemma 2.5. (1) *The projections $X_1 \times X_2 \to X_1$ and $X_1 \times X_2 \to X_2$ induce the extensions R_3/R_1 and R_3/R_2 of DVRs, which preserve uniformizers.*

(2) *There is a canonical isomorphism*

$$\widehat{\Omega}^1_{K_3} \cong (\hat{K}_3 \otimes_{\hat{K}_1} \widehat{\Omega}^1_{K_1}) \oplus (\hat{K}_3 \otimes_{\hat{K}_2} \widehat{\Omega}^1_{K_2}).$$

This isomorphism respects the filtrations, i.e., via this isomorphism, $\operatorname{fil}_n \widehat{\Omega}^1_{K_3}$ *coincides with*

$$(\hat{R}_3 \otimes_{\hat{R}_1} \operatorname{fil}_n \widehat{\Omega}^1_{K_1}) \oplus (\hat{R}_3 \otimes_{\hat{R}_2} \operatorname{fil}_n \widehat{\Omega}^1_{K_2}).$$

Proof. Let U be the open subscheme of $\widetilde{X_1 \times X_2}$ obtained by removing the strict transforms of $Z_1 \times X_2$ and $X_1 \times Z_2$. This is the largest open subscheme where the pull-backs of $Z_1 \times X_2$ and $X_1 \times Z_2$ coincide with the exceptional divisor. By the universality of the blow-ups $\tilde{X}_1$ and $\tilde{X}_2$, the projections $U \to X_1$ and $U \to X_2$ induce morphisms $U \to \tilde{X}_1$ and $U \to \tilde{X}_2$, hence a morphism $U \to \tilde{X}_1 \times \tilde{X}_2$ of $X_1 \times X_2$-schemes. Denote by D_1 and D_2 the exceptional divisors of $\tilde{X}_1$ and $\tilde{X}_2$. Let $(\tilde{X}_1 \times \tilde{X}_2)'$ be the blow-up of $\tilde{X}_1 \times \tilde{X}_2$ along $D_1 \times D_2$. The morphism $U \to \tilde{X}_1 \times \tilde{X}_2$ lifts to a morphism $U \to (\tilde{X}_1 \times \tilde{X}_2)'$. We claim that this is an open immersion. Indeed, by the universality of the blow-up, the morphism $(\tilde{X}_1 \times \tilde{X}_2)' \to X_1 \times X_2$ lifts to a morphism $(\tilde{X}_1 \times \tilde{X}_2)' \to \widetilde{X_1 \times X_2}$, which implies that U is quasifinite over $(\tilde{X}_1 \times \tilde{X}_2)'$. By Zariski main theorem, the morphism $U \to (\tilde{X}_1 \times \tilde{X}_2)'$ is an open immersion.

Taking an affine open neighborhood of the generic point of the exceptional divisors D_1 and D_2 in $\tilde{X}_1$ and $\tilde{X}_2$, we may assume that $\tilde{X}_1 = \mathrm{Spec}(A_1)$ and $\tilde{X}_2 = \mathrm{Spec}(A_2)$ are affine. We also assume that there are systems of regular parameters $x_1, x_2, \dots, x_n \in A_1$ and $y_1, y_2, \dots, y_m \in A_2$ such that the ideal generated by x_1 and y_1 define D_1 and D_2. The scheme U is canonically isomorphic to $\mathrm{Spec}\big(A_1 \otimes A_2\big[\frac{x_1}{y_1}, \frac{y_1}{x_1}\big]\big)$ and the natural inclusions $A_1, A_2 \to A_1 \otimes A_2\big[\frac{x_1}{y_1}, \frac{y_1}{x_1}\big]$ define the projections $U \to \tilde{X}_1, \tilde{X}_2$. The first assertion follows from this calculation. The canonical isomorphism

$$\Omega^1_{X_1 \times X_2} \cong \mathrm{pr}^*_{X_1} \Omega^1_{X_1} \oplus \mathrm{pr}^*_{X_2} \Omega^1_{X_2},$$

where pr_{X_1} and pr_{X_2} are the projections to X_1 and X_2, gives an isomorphism

$$\widehat{\Omega}^1_{K_3} \cong (\hat{K}_3 \otimes_{\hat{K}_1} \widehat{\Omega}^1_{K_1}) \oplus (\hat{K}_3 \otimes_{\hat{K}_2} \widehat{\Omega}^1_{K_2}).$$

The differentials $\frac{dx_1}{x_1}, d\big(\frac{y_1}{x_1}\big), dx_2, \dots, dx_n, dy_2, \dots, dy_m$ form a basis of $\hat{R}_3$-module $\widehat{\Omega}^1_{R_3}(\log)$. The second assertion follows from this fact and (1). ☐

Corollary 2.6. *Let $\chi_i \in \mathrm{H}^1(K_i, \mathbb{Q}/\mathbb{Z})$ for $i = 1, 2$. Then, the following holds*:

$$\mathrm{Sw}_{R_3}(\chi_1 \boxtimes 1 + 1 \boxtimes \chi_2) = \max\{\mathrm{Sw}_{R_1}(\chi_1), \mathrm{Sw}_{R_2}(\chi_2)\}.$$

Proof. Taking the p-primary parts of χ_1, χ_2, and $\chi_1 \boxtimes 1 + 1 \boxtimes \chi_2$, we reduce to the case when $\chi_i \in \mathrm{H}^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z})$.

First we verify that the morphism

$$\mathrm{H}^1(K_1, \mathbb{Z}/p^{m+1}\mathbb{Z}) \oplus \mathrm{H}^1(K_2, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{H}^1(K_3, \mathbb{Z}/p^{m+1}\mathbb{Z}) \tag{2-6}$$

respects the filtrations. Since the extensions R_3/R_1, R_3/R_2 of DVRs preserve uniformizers the morphism $\mathrm{W}_{m+1}(K_1) \oplus \mathrm{W}_{m+1}(K_2) \to \mathrm{W}_{m+1}(K_3)$ respects the filtrations, which implies the assertion.

To show the corollary, it is enough to prove that the morphism induced from (2-6) by taking Gr_n is injective. This follows from the injectivity of refined Swan conductors (Proposition 2.3) and Lemma 2.5. ☐

Let S be a scheme. For a quasiprojective S-scheme X and a natural number $d \geq 1$, the d-th symmetry group $\mathfrak{S}_d$ acts on $X^d := X \times_S X \times_S \cdots \times_S X$ (d times) via permutation of coordinates. Define a scheme $X^{(d)} := X^d/\mathfrak{S}_d$. $X^{(d)}$ is called the d-th symmetric product of X. It is known that, if X is smooth of relative dimension 1 over S, $X^{(d)}$ is smooth and parametrizes effective Cartier divisors of $\deg = d$ on X [SGA 4_3 1973, Exposé XVII, Application 1; Polishchuk 2003, 16]. In particular, the formation of $X^{(d)}$ commutes with base change $S' \to S$.

Let C be a projective smooth geometrically connected curve over k. Let U be a nonempty open subscheme of C.

Let d be an integer ≥ 1. We construct a map $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(U^{(d)}, \mathbb{Q}/\mathbb{Z})$ as follows. First fix a finite abelian group G. Let $V \to U$ be a G-torsor. Then V^d is a G^d-torsor of U^d. Let H be the subgroup of G^d consisting of elements $(a_1, \ldots, a_d)$ satisfying $\sum_{1 \leq i \leq d} a_i = 0$. Then V^d/H is a G-torsor of U^d. This torsor has a natural action by the d-th symmetry group $\mathfrak{S}_d$ which is equivariant with respect to its action to U^d.

Lemma 2.7. *The morphism*

$$(V^d/H)/\mathfrak{S}_d \to U^{(d)} \tag{2-7}$$

induced from the map $V^d/H \to U^d$, taking the quotients by $\mathfrak{S}_d$, is a G-torsor.

Proof. It is sufficient to show that, for every geometric point $\bar{x}$ of U^d, the stabilizer group $(\mathfrak{S}_d)_{\bar{x}}$ at $\bar{x}$ acts trivially on the fiber $(V^d/H)_{\bar{x}}$ over $\bar{x}$, see [SGA 1 1971, Remarque 5.8].

We may assume that k is algebraically closed and that geometric points considered are k-valued points. Let $\bar{x}$ be a geometric point of U^d. For simplicity, we assume that $\bar{x} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_r, \ldots, x_r)$, where $x_1, \ldots, x_r$ are distinct points and x_i appears d_i times for each i. Then the inertia group $(\mathfrak{S}_d)_{\bar{x}}$ at $\bar{x}$ is isomorphic to $\prod_{1 \leq i \leq r} \mathfrak{S}_{d_i}$.

For each i, take a k-valued point e_i of $V \times_U x_i$. From the definition of H, the fiber of V^d/H over $\bar{x}$ can be identified with the set

$$\{(e_1, e_1, \ldots, e_r, ge_r) \mid g \in G\}, \tag{2-8}$$

on which $(\mathfrak{S}_d)_{\bar{x}}$ acts trivially. □

In this way, we construct a G-torsor $(V^d/H)/\mathfrak{S}_d$ on $U^{(d)}$. Since this construction is compatible with a morphism of abelian groups $G \to G'$, we obtain a group homomorphism $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(U^{(d)}, \mathbb{Q}/\mathbb{Z})$. We denote by $\chi^{(d)}$ the image of χ via this map. Let K be the field of fractions of U, $K_{(d)}$ be that of $U^{(d)}$, and K_d be that of U^d. Taking U smaller and smaller, we also have a map $\mathrm{H}^1(K, G) \to \mathrm{H}^1(K_{(d)}, G)$ for a finite abelian group G and a map $\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(K_{(d)}, \mathbb{Q}/\mathbb{Z})$.

We consider a similar construction on the groups of Witt vectors. Denote by pr_i^* the morphism $K \to K_d$ induced by the i-th projection $U^d \to U$. Consider the map $\lambda : \mathrm{W}_{m+1}(K) \to \mathrm{W}_{m+1}(K_d)$ sending a Witt vector α to $\mathrm{pr}_1^* \alpha + \cdots + \mathrm{pr}_d^* \alpha$. Since the extension $K_d/K_{(d)}$, induced by the natural projection $U^d \to U^{(d)}$, is finite Galois with the Galois group $\mathfrak{S}_d$, the $\mathfrak{S}_d$-fixed part of $\mathrm{W}_{m+1}(K_d)$ coincides with $\mathrm{W}_{m+1}(K_{(d)})$ (here $\mathrm{W}_{m+1}(K_{(d)})$ is considered as a subgroup of $\mathrm{W}_{m+1}(K_d)$ via the natural projection $U^d \to U^{(d)}$). Thus the map λ factors through $\mathrm{W}_{m+1}(K_{(d)})$. We also denote the induced map $\mathrm{W}_{m+1}(K) \to \mathrm{W}_{m+1}(K_{(d)})$

by λ. Note that the diagram

$$\begin{array}{ccc} \mathrm{W}_{m+1}(K) & \xrightarrow{\delta_{m+1,K}} & \mathrm{H}^1(K,\mathbb{Z}/p^{m+1}\mathbb{Z}) \\ \downarrow{\scriptstyle\lambda} & & \downarrow \\ \mathrm{W}_{m+1}(K_{(d)}) & \xrightarrow{\delta_{m+1,K_{(d)}}} & \mathrm{H}^1(K_{(d)},\mathbb{Z}/p^{m+1}\mathbb{Z}) \end{array}$$

is commutative. This follows from the commutativity of pr_i^* and the boundary maps (see the diagram (2-1)), and the injectivity of $\mathrm{H}^1(U^{(d)},\mathbb{Z}/p^{m+1}\mathbb{Z}) \to \mathrm{H}^1(U^d,\mathbb{Z}/p^{m+1}\mathbb{Z})$ (see Lemma 4.2). Also, the canonical morphism $\widehat{\Omega}^1_{K_{(d)}} \otimes_{K_{(d)}} K_d \to \widehat{\Omega}^1_{K_d}$ is an isomorphism and the $\mathfrak{S}_d$-fixed part of $\widehat{\Omega}^1_{K_d}$ coincides with (the image of) $\widehat{\Omega}^1_{K_{(d)}}$. We define a map $\mu : \widehat{\Omega}^1_K \to \widehat{\Omega}^1_{K_{(d)}}$ similarly to λ. The maps λ and μ commute with $F^m d$.

Let P be a closed point of U. For simplicity, let us assume that the residue field at P is isomorphic to k. Let R be the DVR of C at P, and $R_{(d)}$ be the DVR of $K_{(d)}$ at the generic point of the exceptional divisor of the blow-up of $C^{(d)}$ along the point corresponding to the divisor dP. We define filtrations on $\mathrm{W}_{m+1}(K)$ (resp. $\mathrm{W}_{m+1}(K_{(d)})$) and $\widehat{\Omega}^1_K$ (resp. $\widehat{\Omega}^1_{K_{(d)}}$) by R (resp. $R_{(d)}$) (see (2-1)).

The following theorem, and corollary are key calculations to prove Theorem 1.2 in Section 4.

Theorem 2.8. *Let n be an integer.*

(1) *The homomorphism*

$$\lambda : \mathrm{W}_{m+1}(K) \to \mathrm{W}_{m+1}(K_{(d)})$$

sends $\mathrm{fil}_n \mathrm{W}_{m+1}(K)$ *into* $\mathrm{fil}_{\lfloor n/d \rfloor} \mathrm{W}_{m+1}(K_{(d)})$.

(2) *The homomorphism*

$$\mu : \widehat{\Omega}^1_K \to \widehat{\Omega}^1_{K_{(d)}}$$

sends $\mathrm{fil}_n \widehat{\Omega}^1_K$ *into* $\mathrm{fil}_{\lfloor n/d \rfloor} \widehat{\Omega}^1_{K_{(d)}}$. *Let j be an integer. The induced map*

$$\mathrm{fil}_{(j+1)d-1} \widehat{\Omega}^1_K / \mathrm{fil}_{jd-1} \widehat{\Omega}^1_K \to \mathrm{Gr}_j \widehat{\Omega}^1_{K_{(d)}}$$

is injective, here $\mathrm{Gr}_j := \mathrm{fil}_j / \mathrm{fil}_{j-1}$.

Corollary 2.9. *Let χ be a character in $\mathrm{H}^1(K,\mathbb{Q}/\mathbb{Z})$. The following identity holds*:

$$\mathrm{Sw}_{R_{(d)}}(\chi^{(d)}) = \left\lfloor \frac{\mathrm{Sw}_R(\chi)}{d} \right\rfloor.$$

Proof of Corollary 2.9. Taking the p-primary part of χ and an isomorphism $\frac{1}{p^{m+1}}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/p^{m+1}\mathbb{Z}$, we reduce to the case when $\chi \in \mathrm{H}^1(K,\mathbb{Z}/p^{m+1}\mathbb{Z})$. Take $\alpha \in \mathrm{W}_{m+1}(K)$ such that α maps to χ via the boundary map $\mathrm{W}_{m+1}(K) \to \mathrm{H}^1(K,\mathbb{Z}/p^{m+1}\mathbb{Z})$ and $v_R(\alpha) = -\mathrm{Sw}_R(\chi)$. Since the map $F^m d : \mathrm{W}_{m+1}(K) \to \widehat{\Omega}^1_K$ respects the filtrations, we have $F^m d\alpha \in \mathrm{fil}_{-\mathrm{Sw}_R(\chi)} \widehat{\Omega}^1_K$. On the other hand, by Proposition 2.3(2), we have $F^m d\alpha = \mathrm{rsw}(\chi) \notin \mathrm{fil}_{-1-\mathrm{Sw}_R(\chi)} \widehat{\Omega}^1_K$. By the definition of the filtration on $\widehat{\Omega}^1_K$, the equality $v_R^{\log}(F^m d\alpha) = -\mathrm{Sw}_R(\chi)$ holds.

When $\mathrm{Sw}_R(\chi)=0$, χ is unramified since χ is p-torsion. Thus $\chi^{(d)}$ is unramified too by the construction of $\chi^{(d)}$, which implies the assertion in this case.

Assume $\mathrm{Sw}_R(\chi) > 0$. Let $r := \lfloor \mathrm{Sw}_R(\chi)/d \rfloor$. From Theorem 2.8(1), the inequality $v_{R_{(d)}}(\lambda(\alpha)) \geq -r$ holds. Thus $\chi^{(d)}$ is contained in $\mathrm{fil}_r \mathrm{H}^1(K_{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z})$, which implies the inequality $\mathrm{Sw}_{R_{(d)}}(\chi^{(d)}) \leq r$.

We show that the class of $\chi^{(d)}$ in $\mathrm{Gr}_r \mathrm{H}^1(K_{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z})$ is nonzero. Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{fil}_r \mathrm{W}_{m+1}(K_{(d)}) & \longrightarrow & \mathrm{Gr}_r \mathrm{H}^1(K_{(d)}, \mathbb{Z}/p^{m+1}\mathbb{Z}) \\ & \searrow^{F^m d} & \downarrow \mathrm{rsw} \\ & & \mathrm{Gr}_r \widehat{\Omega}^1_{K_{(d)}}, \end{array} \tag{2-9}$$

which is obtained from Proposition 2.3. It suffices to show that $\mathrm{rsw}(\chi^{(d)})$ is nonzero. From the commutativity of (2-9), $\mathrm{rsw}(\chi^{(d)})$ coincides with the class containing $F^m d\lambda(\alpha) = \mu(F^m d\alpha)$. Since $v_R^{\log}(F^m d\alpha) = -\mathrm{Sw}_R(\chi)$, the class of $F^m d\alpha$ in $\mathrm{fil}_{(r+1)d-1} \widehat{\Omega}^1_K / \mathrm{fil}_{rd-1} \widehat{\Omega}^1_K$ is nonzero. the assertion follows from Theorem 2.8(2), i.e., the injectivity of μ. □

To prove Theorem 2.8, we first collect some basic properties of the DVR $R_{(d)}$ and its module of differentials. Let R_d be the normalization of $R_{(d)}$ in K_d. R_d is a DVR. The natural projection $C^d \to C^{(d)}$ and the i-th projection $C^d \to C$ define extensions of DVRs

$$R_{(d)} \hookrightarrow R_d \xleftarrow{\mathrm{pr}_i^*} R.$$

Fix a uniformizer t of R. Let $S_1, \ldots, S_d$ be the elementary symmetric polynomials of $\mathrm{pr}_1^* t, \ldots \mathrm{pr}_d^* t$ in R_d, i.e., $S_1, \ldots, S_d$ satisfy the following identity

$$(T - \mathrm{pr}_1^* t) \cdots (T - \mathrm{pr}_d^* t) = T^d - S_1 T^{d-1} + \cdots + (-1)^d S_d.$$

Lemma 2.10. (1) *The residue field of $R_{(d)}$ is isomorphic to $k\big(S_1/S_d, \ldots, S_{d-1}/S_d\big)$.*

(2) *The elements $S_1, \ldots, S_d$ are uniformizers of $R_{(d)}$.*

(3) *The valuations of $\mathrm{pr}_1^* t, \ldots, \mathrm{pr}_d^* t$ with respect to R_d are the same.*

Proof. Since the sequence $S_1, \ldots, S_d$ is a regular system of parameters of the regular local ring of $C^{(d)}$ at the k-rational point dP, the exceptional divisor of the blow-up of $C^{(d)}$ is isomorphic to a projective space over k with homogeneous coordinates $S_1, \ldots, S_d$.

(1) This follows from the considerations above.

(2) At the generic point of the exceptional divisor, the elements $S_1, \ldots, S_d$ generate the same ideal. Since the exceptional divisor is regular, the assertion follows.

(3) The d-th symmetry group $\mathfrak{S}_d$ acts on R_d permuting the $pr_i^* t$, hence the assertion. □

By Lemmas 2.2 and 2.10(1), $\widehat{\Omega}^1_{R_{(d)}}(\log)$ is an $\hat{R}_{(d)}$-free module with a basis $dS_1/S_d, \ldots, dS_d/S_d$.

Lemma 2.11. *For each integer i, define*

$$\omega_i := \frac{d(\mathrm{pr}_1^* t)}{\mathrm{pr}_1^* t^i} + \cdots + \frac{d(\mathrm{pr}_d^* t)}{\mathrm{pr}_d^* t^i} \in \widehat{\Omega}^1_{K_d}.$$

Let j be an integer. Then, the differentials $\omega_{jd+1}, \ldots, \omega_{(j+1)d}$ form an $\hat{R}_{(d)}$-basis of the $\hat{R}_{(d)}$-free module $(1/S_d^j)\widehat{\Omega}^1_{R_{(d)}}(\log)$.

Proof. To avoid notational confusion, we change the notation d to n in this proof.

Since the differentials ω_j are $\mathfrak{S}_n$-invariant, they are indeed contained in $\widehat{\Omega}^1_{K_{(n)}}$.

Suppose $j \geq 0$. Define a polynomial $F(T) := (T - \mathrm{pr}_1^* t) \cdots (T - \mathrm{pr}_n^* t)$. The following equalities hold:

$$\begin{aligned} -dS_1 T^{n-1} + \cdots + (-1)^n dS_n = dF = -F \sum_{1 \leq i \leq n} \frac{d\,\mathrm{pr}_i^* t}{T - \mathrm{pr}_i^* t} \\ = F \sum_{1 \leq i \leq n} \frac{1}{\mathrm{pr}_i^* t} \frac{d\,\mathrm{pr}_i^* t}{1 - T/\mathrm{pr}_i^* t} = F \sum_{r \geq 0} \omega_{r+1} T^r. \end{aligned}$$

Comparing the coefficients of T^r, we obtain equalities

$$\begin{aligned} S_n \omega_1 &= \pm dS_n \\ S_n \omega_2 \pm S_{n-1} \omega_1 &= \pm dS_{n-1} \\ &\vdots \\ S_n \omega_{r+1} + (\text{a linear combination of } \omega_r, \ldots, \omega_{r-n}) &= 0 \quad (r \geq n) \\ &\vdots \end{aligned}$$

The assertion follows by induction on r.

For the case when $j < 0$, take F as $(1 - \mathrm{pr}_1^* tT) \cdots (1 - \mathrm{pr}_n^* tT)$ and argue similarly. □

Proof of Theorem 2.8. (1) Let $e_{R_d/R_{(d)}}$ be the ramification index of $R_d/R_{(d)}$. Let $e_{R_d/R}$ be the ramification index of R_d/R induced by pr_i. By Lemma 2.10, $e_{R_d/R}$ is independent of i. From the definition of the filtrations, the map $\mathrm{pr}_i^* : \mathrm{W}_{m+1}(K) \to \mathrm{W}_{m+1}(K_d)$ sends $\mathrm{fil}_n \mathrm{W}_{m+1}(K)$ into $\mathrm{fil}_{ne_{R_d/R}} \mathrm{W}_{m+1}(K_d)$. Since S_d is a uniformizer of $R_{(d)}$ by Lemma 2.10, the equality

$$de_{R_d/R} = e_{R_d/R_{(d)}}$$

holds. This shows the identity

$$\mathrm{fil}_{\lfloor n/d \rfloor} \mathrm{W}_{m+1}(K_{(d)}) = \mathrm{fil}_{ne_{R_d/R}} \mathrm{W}_{m+1}(K_d) \cap \mathrm{W}_{m+1}(K_{(d)}),$$

hence the assertion.

(2) Note that the map $\mu : \widehat{\Omega}^1_K \to \widehat{\Omega}^1_{K_{(d)}}$ is continuous. The differentials $dt/t^{n+1}, dt/t^n, \ldots \in \widehat{\Omega}^1_K$ map to $\omega_{n+1}, \omega_n, \ldots$ via μ, all of which are contained in $\mathrm{fil}_{\lfloor n/d \rfloor} \widehat{\Omega}^1_{K_{(d)}}$ by Lemma 2.11. Thus the map μ sends $\mathrm{fil}_n \widehat{\Omega}^1_K$ into $\mathrm{fil}_{\lfloor n/d \rfloor} \widehat{\Omega}^1_{K_{(d)}}$. Since the classes of $dt/t^{(j+1)d}, \ldots, dt/t^{jd+1}$ form a k-basis of $\mathrm{fil}_{(j+1)d-1} \widehat{\Omega}^1_K / \mathrm{fil}_{jd-1} \widehat{\Omega}^1_K$, the last assertion follows from Lemma 2.11. □

3. Generalized jacobians and blow-ups of symmetric powers

In this section, we fix a base scheme S. Let C be a projective smooth S-scheme whose geometric fibers are connected and of dimension 1. Let $\mathfrak{m}$ be a relative effective Cartier divisor of C/S, i.e., a closed subscheme of C which is finite flat of finite presentation over S. We also call $\mathfrak{m}$ a modulus. Let us denote, for S-schemes T, the projections $C \times_S T \to T$ by the same symbol pr. In this section, we recall and study the notion of generalized jacobian varieties. Let d be an integer and $\mathfrak{m}$ be a modulus. Let T be an S-scheme. Consider a datum $(\mathcal{L}, \psi)$ such that:

- $\mathcal{L}$ is an invertible sheaf of $\deg = d$ on C_T.
- ψ is an isomorphism $\mathcal{O}_{\mathfrak{m}_T} \to \mathcal{L}|_{\mathfrak{m}_T}$.

We say that two such data $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are isomorphic if there exists an isomorphism of invertible sheaves $f : \mathcal{L} \to \mathcal{L}'$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{m}_T} & \xrightarrow{\ \psi\ } & \mathcal{L}|_{\mathfrak{m}_T} \\ & {\scriptstyle \psi'} \searrow \quad \swarrow {\scriptstyle f|_{\mathfrak{m}_T}} & \\ & \mathcal{L}'|_{\mathfrak{m}_T} & \end{array}$$

For an S-scheme T, define a set

$$\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}(T) := \{\text{isomorphism classes of } (\mathcal{L}, \psi) \text{ defined as above}\}.$$

$\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ extends in an obvious way to a presheaf on Sch/S, which we also denote by $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$. Define $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ as the étale sheafification of $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$. Their fundamental properties which we use without proofs are:

(1) $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ are represented by S-schemes.

(2) When $\mathfrak{m}$ is faithfully flat over S, $\mathrm{Pic}^{d,\mathrm{pre}}_{C,\mathfrak{m}}$ are already étale sheaves.

(3) $\mathrm{Pic}^{0}_{C,\mathfrak{m}}$ is a smooth commutative group S-scheme with geometrically connected fibers.

(4) $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$ are $\mathrm{Pic}^{0}_{C,\mathfrak{m}}$-torsors.

When $\mathfrak{m} = 0$, properties (1) and (3) are proved in [Bosch et al. 1990]. For general $\mathfrak{m}$, they can be proved similarly as in [Bosch et al. 1990, 9.3], or can be deduced from the case when $\mathfrak{m} = 0$ and Lemma 3.1.

$\mathrm{Pic}^{0}_{C,\mathfrak{m}}$ is called the generalized jacobian variety of C with modulus $\mathfrak{m}$. When $\mathfrak{m} = 0$, this is the jacobian variety of C. In this case, we also denote $\mathrm{Pic}^{d}_{C,0}$ by Pic^{d}_{C}. Let $\mathfrak{m}$ and $\tilde{\mathfrak{m}}$ be moduli such that $\mathfrak{m} \subset \tilde{\mathfrak{m}}$. There exists a natural map from $\mathrm{Pic}^{d}_{C,\tilde{\mathfrak{m}}}$ to $\mathrm{Pic}^{d}_{C,\mathfrak{m}}$, restricting ψ. Since $\tilde{\mathfrak{m}}$ is a finite S-scheme, this map is a surjection as a morphism of étale sheaves.

Assume that $C \to S$ has a section. In this case, Pic^{d}_{C} has an expression as a sheaf as follows [Bosch et al. 1990, 8.1]. Let T be an S-scheme, and $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible sheaves of $\deg = d$ on C_T. Define an equivalence relation on $\mathrm{Pic}^{d,\mathrm{pre}}_{C}$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are equivalent if and only if there exists an invertible sheaf $\mathcal{M}$ on T such that $\mathcal{L}_1 \cong \mathcal{L}_2 \otimes \mathrm{pr}^* \mathcal{M}$. If $C \to S$ has a section, the quotient presheaf of $\mathrm{Pic}^{d,\mathrm{pre}}_{C}$ by this equivalence relation is an étale sheaf and coincides with the étale sheafification of $\mathrm{Pic}_C^{d,\mathrm{pre}}$ via the natural surjection. In particular, the identity map $\mathrm{Pic}_C^d \to \mathrm{Pic}_C^d$ corresponds to an equivalence class of invertible sheaves on $C \times_S \mathrm{Pic}_C^d$. In this paper, we call this class the universal class of invertible sheaves of $\deg = d$.

From now on we fix a modulus $\tilde{\mathfrak{m}}$. We call a modulus $\mathfrak{m}$ a submodulus if $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ holds. Until the last paragraph, we treat the case when submoduli considered are everywhere strictly positive on S. Let $\mathfrak{m}$ be a submodulus which is everywhere strictly positive. Then, $\mathrm{Pic}_{C,\mathfrak{m}}^d$ has an explicit expression as a sheaf, as explained before.

Denote the genus of C by g. This is a locally constant function on S. We consider a condition on an integer d as below:

$$d \geq \max\{2g - 1 + \deg \tilde{\mathfrak{m}}, \deg \tilde{\mathfrak{m}}\}. \tag{3-1}$$

When S is quasicompact, such a d always exists. For an integer d and a submodulus $\mathfrak{m}$, denote $d_{\mathfrak{m}} := d - \deg \tilde{\mathfrak{m}} + \deg \mathfrak{m}$. If d satisfies (3-1), $d_{\mathfrak{m}}$ satisfies (3-1) with $\tilde{\mathfrak{m}}$ replaced by $\mathfrak{m}$.

Fix an integer d satisfying (3-1). Let T be an S-scheme and $\mathcal{L}$ be an invertible sheaf of $\deg = d$ on C_T. For every usual point $t \in T$, $R^1 \mathrm{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})|_{C_t})$ and $R^1 \mathrm{pr}_*(\mathcal{L}|_{C_t})$ are zero by Serre duality and a degree argument. In this case, $\mathrm{pr}_* \mathcal{L}(-\tilde{\mathfrak{m}})$ and $\mathrm{pr}_* \mathcal{L}$ are locally free sheaves and their formations commute with any base change, i.e., for any morphism of S-schemes $f : T' \to T$, the base change morphisms $f^* \mathrm{pr}_* \mathcal{L} \to \mathrm{pr}_* f^* \mathcal{L}$ and $f^* \mathrm{pr}_*(\mathcal{L}(-\tilde{\mathfrak{m}})) \to \mathrm{pr}_* f^*(\mathcal{L}(-\tilde{\mathfrak{m}}))$ are isomorphisms. Also $R^1 \mathrm{pr}_* f^* \mathcal{L}$ and $R^1 \mathrm{pr}_* f^*(\mathcal{L}(-\tilde{\mathfrak{m}}))$ are zero.

Let $\mathfrak{m}$ be a submodulus. Denote by U the complement of $\mathfrak{m}$ in C. The Abel–Jacobi map $U^{(d_{\mathfrak{m}})} \to \mathrm{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}}$ is a map which sends $D \in U^{(d_{\mathfrak{m}})}$ to $(\mathcal{O}_C(D), \iota_D)$, where ι_D is the one induced from the natural identification $\mathcal{O}_{C \setminus D} \cong \mathcal{O}_C(D)|_{C \setminus D}$. In this section, we define and study a compactification $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ of the Abel–Jacobi map by constructing the following commutative diagram of smooth S-schemes:

$$\begin{array}{ccccc}
U^{(d_{\mathfrak{m}})} & \hookrightarrow & \tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})} & \longrightarrow & \mathrm{Pic}_{C,\mathfrak{m}}^{d_{\mathfrak{m}}} \\
\downarrow & & \downarrow & \square & \downarrow \scriptstyle{(3\text{-}7)} \\
X_{\mathfrak{m}} & \xrightarrow{\cong} & \mathbb{P}(\mathcal{E}_{\mathfrak{m}}) & \longrightarrow & P_{\mathfrak{m}}^{d_{\mathfrak{m}}} \\
& \searrow & \downarrow \scriptstyle{(3\text{-}5)} & & \downarrow \scriptstyle{(3\text{-}2)} \\
& & C^{(d_{\mathfrak{m}})} & & \mathrm{Pic}_C^{d_{\mathfrak{m}}}.
\end{array}$$

The S-scheme $\tilde{C}_{\mathfrak{m}}^{(d_{\mathfrak{m}})}$ has, on the one hand, a clear moduli description, and, on the other hand, can be identified by an open subscheme of a blow-up, which will be denoted by $X_{\mathfrak{m}}$, of $C^{(d_{\mathfrak{m}})}$.

Let $\mathcal{L}$ be an invertible sheaf on C_T for an S-scheme T. Denote $\mathcal{L}/(\mathcal{L}(-\mathfrak{m}))$ by $\mathcal{L}_{\mathfrak{m}}$.

For an S-scheme T, consider a pair $(\mathcal{L}, \phi)$ such that $\mathcal{L}$ is an invertible sheaf of $\deg = d_{\mathfrak{m}}$ on C_T and ϕ is an injection $\mathcal{O}_T \to \mathrm{pr}_* \mathcal{L}_{\mathfrak{m}}$ such that the quotient $\mathrm{pr}_* \mathcal{L}_{\mathfrak{m}}/\mathcal{O}_T$ is locally free. Call such pairs $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ isomorphic if there exists an isomorphism $f : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{O}_T & \\ {}^{\phi}\swarrow & & \searrow^{\phi'} \\ \mathrm{pr}_*\,\mathcal{L}_{\mathfrak{m}} & \xrightarrow[\mathrm{pr}_* f]{\qquad} & \mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}}. \end{array}$$

Define $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}(T)$ as the set of isomorphism classes of such pairs. This is an étale sheaf on Sch/S. Define a map

$$P^{d_{\mathfrak{m}}}_{\mathfrak{m}} \to \mathrm{Pic}^{d_{\mathfrak{m}}}_{C} \tag{3-2}$$

by forgetting ϕ. Let X be a scheme, and $\mathcal{F}$ be a locally free sheaf of finite rank on X. We use a contra-Grothendieck notation for a projective space. Thus the X-scheme $\mathbb{P}(\mathcal{F})$ parametrizes invertible subsheaves of $\mathcal{F}$.

Lemma 3.1. *The sheaf $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$ is represented by a proper smooth S-scheme. Assume that $C \to S$ has a section. Let $\mathcal{L}'$ be a representative invertible sheaf of the universal class. Then, as sheaves on $Sch/\mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$, $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$ is isomorphic to the projectivization $\mathbb{P}(\mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}})$ of $\mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}}$.*

Proof. First we consider the case when $C(S)$ is not empty. In this case, $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$ has an explicit expression as a sheaf, as explained before.

Via the map (3-2), we regard $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$ as a sheaf on $Sch/\mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$. Fix a representative invertible sheaf $\mathcal{L}'$ of the universal class. Let $\mathcal{N}$ be an element of $\mathbb{P}(\mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}})(T)$, where T is a $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$-scheme. Let $\phi : \mathcal{O}_T \to \mathrm{pr}_*((\mathcal{L}' \otimes \mathrm{pr}^*\,\mathcal{N}^{-1})_{\mathfrak{m}})$ be a morphism obtained by tensoring the inclusion $\mathcal{N} \hookrightarrow \mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}}$ with $\mathcal{N}^{-1}$. Then, the assignment $\mathcal{N} \mapsto (\mathcal{L}' \otimes \mathrm{pr}^*\,\mathcal{N}^{-1}, \phi)$ defines a morphism of sheaves on $Sch/\mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$, $\mathbb{P}(\mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}}) \to P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$. This is an isomorphism. Indeed, we can construct its inverse as follows. Let T be a $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$-scheme and $(\mathcal{L}, \phi)$ be an element of $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}(T)$. Let $a : T \to \mathrm{Pic}^{d_{\mathfrak{m}}}_{C}$ be the structure map. Then, there exists an invertible sheaf $\mathcal{N}$ on T such that $\mathcal{L} \otimes \mathrm{pr}^*\,\mathcal{N}$ is isomorphic to $a^*\mathcal{L}'$. Such an $\mathcal{N}$ is unique since $C \to S$ has a section. Then, $\mathcal{N} \xrightarrow{\phi \otimes \mathcal{N}} \mathrm{pr}_*((\mathcal{L} \otimes \mathrm{pr}^*\,\mathcal{N})_{\mathfrak{m}}) \xrightarrow{\sim} \mathrm{pr}_*\,a^*\mathcal{L}'_{\mathfrak{m}}$ is an element of $\mathbb{P}(\mathrm{pr}_*\,\mathcal{L}'_{\mathfrak{m}})(T)$.

Next we consider the general case. As the map $C \to S$ has a section étale locally on S, the sheaf $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$ is represented, étale locally on S, by a projective space bundle. Since the dual of the canonical line bundle of a projective space bundle is relatively ample, the étale descent is effective. □

Let $(\mathcal{L}, \phi)$ be the universal element on $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$. Define $\mathcal{E}_{\mathfrak{m}}$ as the $\mathcal{O}_{P^{d_{\mathfrak{m}}}_{\mathfrak{m}}}$- module fitting in the following diagram of sheaves on $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$:

$$\begin{array}{ccc} \mathrm{pr}_*(\mathcal{L}(-\mathfrak{m})) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{E}_{\mathfrak{m}} & \longrightarrow & \mathcal{O}_{P^{d_{\mathfrak{m}}}_{\mathfrak{m}}} \\ \downarrow & & \phi \downarrow \\ \mathrm{pr}_*\,\mathcal{L} & \xrightarrow[p]{\qquad} & \mathrm{pr}_*\,\mathcal{L}_{\mathfrak{m}}, \end{array} \tag{3-3}$$

where the bottom horizontal arrow is the pushforward of the quotient map and each square is cartesian. Since all the right arrows are locally split injections and $p : \mathrm{pr}_* \mathcal{L} \to \mathrm{pr}_* \mathcal{L}_{\mathfrak{m}}$ is a surjection of locally free sheaves, $\mathcal{E}_{\mathfrak{m}}$ is locally free of finite rank and all the left arrows are locally split injections.

Let $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ be the projectivization of $\mathcal{E}_{\mathfrak{m}}$. As a sheaf on Sch/S, $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ parametrizes triples $(\mathcal{L}, \phi, \mathcal{M})$ such that $(\mathcal{L}, \phi)$ is an element of $P_{\mathfrak{m}}^{d_{\mathfrak{m}}}$ and $\mathcal{M}$ is an invertible subsheaf of $\mathrm{pr}_* \mathcal{L} \oplus \mathcal{O}_T$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{O}_T \\ \big\downarrow & & \big\downarrow \phi \\ \mathrm{pr}_* \mathcal{L} & \xrightarrow[p]{} & \mathrm{pr}_* \mathcal{L}_{\mathfrak{m}}, \end{array} \tag{3-4}$$

where each arrow from $\mathcal{M}$ is the composition of the inclusion $\mathcal{M} \hookrightarrow \mathrm{pr}_* \mathcal{L} \oplus \mathcal{O}_T$ with the respecitve projection. This is a proper smooth S-scheme.

Lemma 3.2. *The map* $\mathrm{pr}_*(\mathcal{L}(-\mathfrak{m})) \to \mathcal{E}_{\mathfrak{m}}$ *in* (3-3) *induces a closed immersion* $\mathbb{P}(\mathrm{pr}_* \mathcal{L}(-\mathfrak{m})) \hookrightarrow \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$. *The closed subspace* $\mathbb{P}(\mathrm{pr}_* \mathcal{L}(-\mathfrak{m}))$ *is a hyperplane bundle of* $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$.

Proof. The assertion follows from the exact sequence

$$0 \to \mathrm{pr}_*(\mathcal{L}(-\mathfrak{m})) \to \mathcal{E}_{\mathfrak{m}} \to \mathcal{O}_{P_{\mathfrak{m}}^{d_{\mathfrak{m}}}} \to 0. \qquad \square$$

As a subsheaf of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$, $\mathbb{P}(\mathrm{pr}_* \mathcal{L}(-\mathfrak{m}))$ parametrizes triples $(\mathcal{L}, \phi, \mathcal{M})$ such that the first projection $\mathcal{M} \to \mathrm{pr}_* \mathcal{L}$ factors through $\mathrm{pr}_* \mathcal{L}(-\mathfrak{m})$.

Now we define a map $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})}$ of S-schemes taking the homothety class of the left vertical arrow in (3-4).

Let T be an S-scheme and $(\mathcal{L}, \phi, \mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})(T)$. Since the arrow $\mathcal{E}_{\mathfrak{m}} \to \mathrm{pr}_* \mathcal{L}$ in (3-3) is locally a split injection, the first projection $\mathcal{M} \to \mathrm{pr}_* \mathcal{L}$ is injective and the cokernel is locally free. Since these hold after any base change $t \to T$ from the spectrum of a field, the map $\mathrm{pr}^* \mathcal{M}_t \to \mathcal{L}_t$ is injective for a usual point t of T. Thus $\mathcal{O}_{C_T} \to \mathcal{L} \otimes \mathrm{pr}^* \mathcal{M}^{-1}$ defines an effective Cartier divisor. Since $\deg(\mathcal{L}^{-1} \otimes \mathrm{pr}^* \mathcal{M})$ equals to $-d_{\mathfrak{m}}$, $\mathrm{Spec}(\mathcal{O}_{C_T}/(\mathcal{L}^{-1} \otimes \mathrm{pr}^* \mathcal{M}))$ is finite flat of finite presentation of $\deg = d_{\mathfrak{m}}$ over T by the Riemann–Roch formula.

Let $C^{(d_{\mathfrak{m}})}$ be the $d_{\mathfrak{m}}$-th symmetric product of C, which parametrizes effective Cartier divisors of $\deg = d_{\mathfrak{m}}$ on C. Define a map

$$\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})} \tag{3-5}$$

sending $(\mathcal{L}, \phi, \mathcal{M})$ to $\mathrm{Spec}(\mathcal{O}_{C_T}/(\mathcal{L}^{-1} \otimes \mathrm{pr}^* \mathcal{M})) \subset C_T$.

Let Z_0 be the closed subscheme of $C^{(d_{\mathfrak{m}})}$ defined by the map $C^{(d-\deg \tilde{\mathfrak{m}})} \to C^{(d_{\mathfrak{m}})}$, adding $\mathfrak{m}$. Let $X_{\mathfrak{m}}$ be the blow-up of $C^{(d_{\mathfrak{m}})}$ along Z_0. We now construct an isomorphism $X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$, by which we will identify them.

We define a map

$$h : X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \tag{3-6}$$

as follows.

Let D be the universal effective Cartier divisor on $C^{(d_\mathfrak{m})}$. Denote $\mathcal{O}_{C\times_S C^{(d_\mathfrak{m})}}(D)$ by $\mathcal{O}(D)$ and $\mathcal{O}(D)\otimes\mathcal{O}_{\mathfrak{m}\times_S C^{(d_\mathfrak{m})}}$ by $\mathcal{O}(D)_\mathfrak{m}$ for short. The composition of the natural maps $\mathcal{O}_{C\times_S C^{(d_\mathfrak{m})}}\to\mathcal{O}(D)\to\mathcal{O}(D)_\mathfrak{m}$ defines a map of locally free sheaves $\mathcal{O}_{C^{(d_\mathfrak{m})}}\to\mathrm{pr}_*(\mathcal{O}(D)_\mathfrak{m})$ on $C^{(d_\mathfrak{m})}$. After a base change $T\to C^{(d_\mathfrak{m})}$, this map becomes zero if and only if $T\to C^{(d_\mathfrak{m})}$ factors through Z_0. Thus the image of the dual $(\mathrm{pr}_*\mathcal{O}(D)_\mathfrak{m})^\vee\to\mathcal{O}_{C^{(d_\mathfrak{m})}}$ of this map is the ideal $\mathcal{I}$ defining Z_0. Let $\mathcal{L}:=\mathcal{O}_{C\times_S X_\mathfrak{m}}(D)\otimes\mathrm{pr}^*(\mathcal{I}\mathcal{O}_{X_\mathfrak{m}})$. Define $\phi:\mathcal{O}_{X_\mathfrak{m}}\to\mathrm{pr}_*(\mathcal{O}_{C\times_S X_\mathfrak{m}}(D)\otimes\mathrm{pr}^*(\mathcal{I}\mathcal{O}_{X_\mathfrak{m}}))_\mathfrak{m}$ to be the morphism obtained from the map $(\mathcal{I}\mathcal{O}_{X_\mathfrak{m}})^{-1}\to\mathrm{pr}_*\mathcal{O}_{C\times_S X_\mathfrak{m}}(D)_\mathfrak{m}$ by tensoring $\mathcal{I}\mathcal{O}_{X_\mathfrak{m}}$. Let $\mathcal{I}\mathcal{O}_{X_\mathfrak{m}}\to\mathrm{pr}_*\mathcal{L}$ be the map induced from the natural inclusion $\mathcal{O}_{X_\mathfrak{m}}\to\mathrm{pr}_*\mathcal{O}_{C\times_S X_\mathfrak{m}}(D)$ by tensoring with $\mathcal{I}\mathcal{O}_{X_\mathfrak{m}}$. This map and the natural inclusion $\mathcal{I}\mathcal{O}_{X_\mathfrak{m}}\to\mathcal{O}_{X_\mathfrak{m}}$ make the sheaf $\mathcal{I}\mathcal{O}_{X_\mathfrak{m}}$ into a subsheaf of $\mathrm{pr}_*\mathcal{L}\oplus\mathcal{O}_{X_\mathfrak{m}}$, which makes the diagram (3-4) commutes. The triple $(\mathcal{L},\phi,\mathcal{I}\mathcal{O}_{X_\mathfrak{m}})$ defines a morphism $h:X_\mathfrak{m}\to\mathbb{P}(\mathcal{E}_\mathfrak{m})$. From the construction, h is a morphism over $C^{(d_\mathfrak{m})}$.

Lemma 3.3. (1) *As a subsheaf of $\mathbb{P}(\mathcal{E}_\mathfrak{m})$, $\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}Z_0$ parametrizes triples $(\mathcal{L},\phi,\mathcal{M})$ such that the second projection $\mathcal{M}\to\mathcal{O}$ are zero. As closed subspaces of $\mathbb{P}(\mathcal{E}_\mathfrak{m})$, $\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}Z_0$ and $\mathbb{P}(\mathrm{pr}_*\mathcal{L}(-\mathfrak{m}))$ are equal. In particular, $\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}Z_0$ is a smooth divisor of $\mathbb{P}(\mathcal{E}_\mathfrak{m})$.*

(2) *Let V be the complement of Z_0 in $C^{(d_\mathfrak{m})}$. As a subsheaf of $\mathbb{P}(\mathcal{E}_\mathfrak{m})$, $\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}V$ parametrizes triples $(\mathcal{L},\phi,\mathcal{M})$ such that the second projection $\mathcal{M}\to\mathcal{O}$ is an isomorphism. The projection $\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}V\to V$ is an isomorphism and its inverse coincides with the restriction of h to V.*

Proof. We are considering the following diagram:

$$\begin{array}{ccccc}
\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}Z_0 & \longrightarrow & \mathbb{P}(\mathcal{E}_\mathfrak{m}) & \longleftarrow & \mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}V \\
\downarrow & & \downarrow & & \downarrow \\
Z_0 & \longrightarrow & C^{(d_\mathfrak{m})} & \longleftarrow & V
\end{array}$$

(1) Let $(\mathcal{L},\phi,\mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_\mathfrak{m})(T)$. This maps into Z_0 via the map $\mathbb{P}(\mathcal{E}_\mathfrak{m})\to C^{(d_\mathfrak{m})}$ if and only if the composition of $\mathrm{pr}^*\mathcal{M}\to\mathcal{L}\to\mathcal{L}_\mathfrak{m}$ is zero. Since the right vertical arrow of (3-4) is an injection, this occurs if and only if the second projection $\mathcal{M}\to\mathcal{O}_T$ is zero. The second assertion is obvious from the definition and the expression of $\mathbb{P}(\mathrm{pr}_*\mathcal{L}(-\mathfrak{m}))$ as a subsheaf. The last assertion is verified for $\mathbb{P}(\mathrm{pr}_*\mathcal{L}(-\mathfrak{m}))$ in Lemma 3.2.

(2) Let T be an S-scheme and $(\mathcal{L},\phi,\mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_\mathfrak{m})(T)$. Let t be a usual point of T. By (1), the pullback of the projection $\mathcal{M}\to\mathcal{O}_T$ by $t\hookrightarrow T$ is an isomorphism if and only if the image of t by the map

$$T\xrightarrow{(\mathcal{L},\phi,\mathcal{M})}\mathbb{P}(\mathcal{E}_\mathfrak{m})\to C^{(d_\mathfrak{m})}$$

is in V.

Let $p:\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}V\to V$ be the projection. Since $h:X_\mathfrak{m}\to\mathbb{P}(\mathcal{E}_\mathfrak{m})$ is a $C^{(d_\mathfrak{m})}$-morphism, $p\circ h|_V$ is the identity. Let $(\mathcal{L},\phi,\mathcal{M})$ be an element of $\mathbb{P}(\mathcal{E}_\mathfrak{m})\times_{C^{(d_\mathfrak{m})}}V(T)$. Identify $\mathcal{M}$ and $\mathcal{O}_T$ by the second projection. By this rigidification, $(\mathcal{L},\phi,\mathcal{O}_T)$ is determined by the first projection. Thus p is an injection as a morphism of sheaves. The assertion follows. □

After these preparations, we obtain the following:

Theorem 3.4. *The morphism $h : X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ in (3-6) is an isomorphism.*

Proof. By Lemma 3.3(1) and the universality of blow-ups, there exists a unique map $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to X_{\mathfrak{m}}$ which is a lift of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to C^{(d_{\mathfrak{m}})}$. Let V be as in Lemma 3.3(2), i.e., the complement of Z_0 in $C^{(d_{\mathfrak{m}})}$. By the lemma, V can be considered as an open subscheme of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$. On the other hand, as V and Z_0 are disjoint, V also can be considered as an open subscheme of $X_{\mathfrak{m}}$. Note that the complements of V in $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ and $X_{\mathfrak{m}}$ are supports of divisors. Therefore, V is schematically dense in both of $\mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ and $X_{\mathfrak{m}}$. Since the morphisms $X_{\mathfrak{m}} \to \mathbb{P}(\mathcal{E}_{\mathfrak{m}})$ and $\mathbb{P}(\mathcal{E}_{\mathfrak{m}}) \to X_{\mathfrak{m}}$ constructed above induce the identity on V, and both schemes are separated over S, the assertion follows. ☐

Let T be an S-scheme and $(\mathcal{L}, \psi)$ be an element of $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C,\mathfrak{m}}(T)$. Define ϕ as the composition $\mathcal{O}_T \to \mathrm{pr}_* \mathcal{O}_{\mathfrak{m}_T} \xrightarrow{\mathrm{pr}_* \psi} \mathrm{pr}_* \mathcal{L}_{\mathfrak{m}}$. Then, the assignment $(\mathcal{L}, \psi) \mapsto (\mathcal{L}, \phi)$ defines a morphism

$$\mathrm{Pic}^{d_{\mathfrak{m}}}_{C,\mathfrak{m}} \to P^{d_{\mathfrak{m}}}_{\mathfrak{m}}. \tag{3-7}$$

Lemma 3.5. *The morphism $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C,\mathfrak{m}} \to P^{d_{\mathfrak{m}}}_{\mathfrak{m}}$ in (3-7) is an open immersion. The open subscheme $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C,\mathfrak{m}}$ parametrizes pairs $(\mathcal{L}, \phi)$ such that the maps $\mathcal{O}_C \to \mathcal{L}_{\mathfrak{m}}$ obtained from ϕ by adjunction are surjective.*

Proof. This morphism is an injection of sheaves. Let $(\mathcal{L}, \phi)$ be an element of $P^{d_{\mathfrak{m}}}_{\mathfrak{m}}(T)$. This element is in $\mathrm{Pic}^{d_{\mathfrak{m}}}_{C,\mathfrak{m}}$ if and only if the map $\mathcal{O}_{C_T} \to \mathcal{L}_{\mathfrak{m}}$ obtained from ϕ by adjunction is a surjection. This is an open condition. ☐

Next, we study behavior of various schemes when one replaces the modulus $\mathfrak{m}$. Let $\mathfrak{m}$ be a submodulus and $\mathfrak{m}' := \tilde{\mathfrak{m}} - \mathfrak{m}$. Define a closed immersion $C^{(d-\deg \mathfrak{m}')} \to C^{(d)}$ by adding $\mathfrak{m}'$. We denote this closed subscheme of $C^{(d)}$ by $Z_{\mathfrak{m}}$. If $\mathfrak{m}_1 \subset \mathfrak{m}_2$, the inclusion $Z_{\mathfrak{m}_1} \subset Z_{\mathfrak{m}_2}$ holds. The closed immersion $Z_{\mathfrak{m}_1} \hookrightarrow Z_{\mathfrak{m}_2}$ is induced by adding $\mathfrak{m}_2 - \mathfrak{m}_1$. This induces a map

$$X_{\mathfrak{m}_1} \hookrightarrow X_{\mathfrak{m}_2} \tag{3-8}$$

of the blow-ups along Z_0. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be submoduli such that $\mathfrak{m}_1 \subset \mathfrak{m}_2$. Define a map $i_{\mathfrak{m}_1,\mathfrak{m}_2} : P^{d_{\mathfrak{m}_1}}_{\mathfrak{m}_1} \to P^{d_{\mathfrak{m}_2}}_{\mathfrak{m}_2}$ by sending $(\mathcal{L}_1, \phi_1)$ to $(\mathcal{L}_1(\mathfrak{m}_2 - \mathfrak{m}_1), \phi)$, where ϕ is the composition of ϕ_1 and the natural injection $\mathrm{pr}_*(\mathcal{L}_1)_{\mathfrak{m}_1} \to \mathrm{pr}_* \mathcal{L}_1(\mathfrak{m}_2 - \mathfrak{m}_1)_{\mathfrak{m}_2}$. The map $i_{\mathfrak{m}_1,\mathfrak{m}_2}$ is a closed immersion.

Proposition 3.6. (1) *Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be submoduli such that $\mathfrak{m}_1 \subset \mathfrak{m}_2$. As a subsheaf of $P^{d_{\mathfrak{m}_2}}_{\mathfrak{m}_2}$, $P^{d_{\mathfrak{m}_1}}_{\mathfrak{m}_1}$ parametrizes pairs $(\mathcal{L}_2, \phi_2)$ such that the compositions $\mathcal{O}_C \xrightarrow{\mathrm{pr}^* \phi_2} (\mathcal{L}_2)_{\mathfrak{m}_2} \to (\mathcal{L}_2)_{\mathfrak{m}_2 - \mathfrak{m}_1}$ are zero. The commutative diagram*

$$\begin{array}{ccc} X_{\mathfrak{m}_1} & \longrightarrow & P^{d_{\mathfrak{m}_1}}_{\mathfrak{m}_1} \\ \downarrow & & \downarrow \\ X_{\mathfrak{m}_2} & \longrightarrow & P^{d_{\mathfrak{m}_2}}_{\mathfrak{m}_2} \end{array}$$

induced by (3-6), (3-8), and the projections $\mathbb{P}(\mathcal{E}_{\mathfrak{m}_i}) \to P^{d_{\mathfrak{m}_i}}_{\mathfrak{m}_i}$ is a cartesian diagram.

(2) *Assume that a submodulus* $\mathfrak{m}$ *is the sum* $\sum_i \mathfrak{m}_i$ *of submoduli of* $\deg = 1$. *Let* $\mathfrak{m}'_i := \sum_{j \neq i} \mathfrak{m}_j$. *Then, the open subspace* $\mathrm{Pic}^{d_\mathfrak{m}}_{C,\mathfrak{m}}$ *of* $P^{d_\mathfrak{m}}_\mathfrak{m}$ *is the complement of* $P^{d_{\mathfrak{m}'_i}}_{\mathfrak{m}'_i}$ *for all* i.

Proof. (1) The first assertion is obvious from the definition of $i_{\mathfrak{m}_1,\mathfrak{m}_2}$. To prove the second assertion, it is enough to show that $\mathcal{E}_{\mathfrak{m}_1} \cong i^*_{\mathfrak{m}_1,\mathfrak{m}_2}\mathcal{E}_{\mathfrak{m}_2}$ by Theorem 3.4. Let $(\mathcal{L}_i, \phi_i)$ be the universal elements of $P^{d_{\mathfrak{m}_i}}_{\mathfrak{m}_i}$. The pullback of the cartesian diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathfrak{m}_2} & \longrightarrow & \mathcal{O}_{P^{d_{\mathfrak{m}_2}}_{\mathfrak{m}_2}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{pr}_* \mathcal{L}_2 & \longrightarrow & \mathrm{pr}_*(\mathcal{L}_2)_{\mathfrak{m}_2} \end{array}$$

by $i_{\mathfrak{m}_1,\mathfrak{m}_2}$ extends to the diagram

$$\begin{array}{ccc} i^*_{\mathfrak{m}_1,\mathfrak{m}_2}\mathcal{E}_{\mathfrak{m}_2} & \longrightarrow & \mathcal{O}_{P^{d_{\mathfrak{m}_1}}_{\mathfrak{m}_1}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{pr}_* \mathcal{L}_1 & \longrightarrow & \mathrm{pr}_*(\mathcal{L}_1)_{\mathfrak{m}_1} \\ \big\downarrow & & \big\downarrow \\ \mathrm{pr}_*(\mathcal{L}_1(\mathfrak{m}_2 - \mathfrak{m}_1)) & \longrightarrow & \mathrm{pr}_*(\mathcal{L}_1(\mathfrak{m}_2 - \mathfrak{m}_1))_{\mathfrak{m}_2}, \end{array}$$

where the two squares are cartesian diagrams, which shows the assertion.

(2) This follows from Lemma 3.5 and (1). □

Define the S-scheme $\tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m}$ as the fibered product

$$\begin{array}{ccc} \tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m} & \longrightarrow & \mathrm{Pic}^{d_\mathfrak{m}}_{C,\mathfrak{m}} \\ \big\downarrow & & \big\downarrow \\ X_\mathfrak{m} & \longrightarrow & P^{d_\mathfrak{m}}_\mathfrak{m}, \end{array} \tag{3-9}$$

where the bottom horizontal map $X_\mathfrak{m} \to P^{d_\mathfrak{m}}_\mathfrak{m}$ is the composition $X_\mathfrak{m} \xrightarrow{\sim} \mathbb{P}(\mathcal{E}_\mathfrak{m}) \to P^{d_\mathfrak{m}}_\mathfrak{m}$. The S-scheme $\tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m}$ is a projective space bundle on $\mathrm{Pic}^{d_\mathfrak{m}}_{C,\mathfrak{m}}$.

Proposition 3.7. *The first projection* $\tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m} \to X_\mathfrak{m}$ *is an open immersion. Moreover, if* $\mathfrak{m}$ *is the sum* $\sum_i \mathfrak{m}_i$ *of submoduli of* $\deg = 1$, $\tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m}$ *coincides with the complement of* $X_{\mathfrak{m}'_i}$ *for all* i, *where* $\mathfrak{m}'_i := \sum_{j \neq i} \mathfrak{m}_j$.

Proof. These are consequences of Lemma 3.5 and Proposition 3.6. □

The Abel–Jacobi map $U^{(d_\mathfrak{m})} \to \mathrm{Pic}^{d_\mathfrak{m}}_{C,\mathfrak{m}}$ and the canonical open immersion $U^{(d_\mathfrak{m})} \to X_\mathfrak{m}$ define the following commutative diagram

$$\begin{array}{ccc} U^{(d_\mathfrak{m})} & \longrightarrow & \mathrm{Pic}^{d_\mathfrak{m}}_{C,\mathfrak{m}} \\ \downarrow & & \downarrow \\ X_\mathfrak{m} & \longrightarrow & P^{d_\mathfrak{m}}_\mathfrak{m}, \end{array} \tag{3-10}$$

which induces an $X_\mathfrak{m}$-morphism $U^{(d_\mathfrak{m})} \to \tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m}$. This is an open immersion, since the vertical arrows of (3-9) and the left vertical arrow of (3-10) are open immersions. Combining the previous results, we obtain the following:

Corollary 3.8. *As an open subscheme of $\tilde{C}^{(d_\mathfrak{m})}_\mathfrak{m}$, $U^{(d_\mathfrak{m})}$ is the complement of $\tilde{C}^{(d_\mathfrak{m})} \times_{C^{(d_\mathfrak{m})}} Z_0$.*

Proof. After a finite faithfully flat base change of S, we may assume that $\mathfrak{m}$ decomposes the sum $\sum_i \mathfrak{m}_i$ of submoduli $\mathfrak{m}_i$ of $\deg = 1$. Let V be the complement of Z_0 in $C^{(d_\mathfrak{m})}$. Since $V \setminus U^{(d_\mathfrak{m})}$ is included in $\bigcup_i Z_{\mathfrak{m}'_i}$, where $\mathfrak{m}'_i := \sum_{j\neq i} \mathfrak{m}_j$, the assertion follows from Proposition 3.7. □

Now assume $\mathfrak{m} = 0$. If C has an S-valued point P, it is well-known that $C^{(d)}$ is a projective space bundle over Pic^d_C when $d \geq \max\{2g-1, 0\}$, where g is the genus of C. In other words, there exists a locally free sheaf $\mathcal{F}$ of finite rank on Pic^d_C such that $C^{(d)}$ is isomorphic to $\mathbb{P}(\mathcal{F})$. Classically this is proved using the Poincaré bundle. On the other hand, using Proposition 3.7, we might prove this fact with an extra condition $d \geq \max\{2g, 1\}$, identifying $\mathrm{Pic}^d_{C,P} \cong \mathrm{Pic}^d_C$ (see [Bosch et al. 1990, 8.2]).

Corollary 3.9. *Assume that S is connected noetherian. Let $\mathfrak{m}$ be a modulus > 0 (resp. $= 0$) of C and d be a sufficiently large integer. Take a geometric point $\bar{x}$ on $\tilde{C}^{(d)}_\mathfrak{m}$ (resp. on $C^{(d)}$) and denote $\bar{y}$ its image to $\mathrm{Pic}^d_{C,\mathfrak{m}}$. Then, the morphism of profinite groups $\pi_1(\tilde{C}^{(d)}_\mathfrak{m}, \bar{x}) \to \pi_1(\mathrm{Pic}^d_{C,\mathfrak{m}}, \bar{y})$ (resp. $\pi_1(C^{(d)}, \bar{x}) \to \pi_1(\mathrm{Pic}^d_C, \bar{y})$) induced from the projection $\tilde{C}^{(d)}_\mathfrak{m} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ (resp. $C^{(d)} \to \mathrm{Pic}^d_C$) is an isomorphism.*

Proof. When $\mathfrak{m} = 0$, let us also denote $C^{(d)}$ by $\tilde{C}^{(d)}_\mathfrak{m}$ for. If $\mathfrak{m} > 0$ (resp. $= 0$), $\tilde{C}^{(d)}_\mathfrak{m}$ is a projective space bundle over $\mathrm{Pic}^d_{C,\mathfrak{m}}$ (resp. after the base change from S to an étale cover). In any case, the morphism $\tilde{C}^{(d)}_\mathfrak{m} \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ is proper surjective smooth with geometrically connected fibers. Take a geometric point $\bar{s}$ of $\tilde{C}^{(d)}_{\mathfrak{m},\bar{y}}$ above $\bar{x}$. Since the scheme $\tilde{C}^{(d)}_{\mathfrak{m},\bar{y}}$ is simply connected, the homotopy exact sequence

$$\pi_1(\tilde{C}^{(d)}_{\mathfrak{m},\bar{y}}, \bar{s}) \to \pi_1(\tilde{C}^{(d)}_\mathfrak{m}, \bar{x}) \to \pi_1(\mathrm{Pic}^d_{C,\mathfrak{m}}, \bar{y}) \to 1$$

implies the assertion. □

4. Proofs

In this section, we prove Theorems 1.1 and 1.2.

First we need to recall basic results on symmetric products of curves.

Let C be a projective smooth geometrically connected curve over a perfect field k. Let $\mathfrak{m}$ be a modulus on C and write $\mathfrak{m} = n_1P_1 + \cdots + n_rP_r$, where $P_1, \ldots, P_r$ are distinct closed points of $\mathfrak{m}$. Denote the complement of $\mathfrak{m}$ in C by U. Let $d_i := \deg P_i$. Take a positive integer d so that $d \geq \deg \mathfrak{m}$.

Lemma 4.1. *The morphism $\pi : C^{(n_1d_1)} \times \cdots \times C^{(n_rd_r)} \times C^{(d-\deg \mathfrak{m})} \to C^{(d)}$, taking the sum, is étale at the generic point of the closed subvariety $\{n_1P_1\} \times \cdots \times \{n_rP_r\} \times C^{(d-\deg \mathfrak{m})}$ of $C^{(n_1d_1)} \times \cdots \times C^{(n_rd_r)} \times C^{(d-\deg \mathfrak{m})}$.*

Proof. We may assume that k is algebraically closed (hence $d_i = 1$ for all i). Since the map $\pi : C^{(n_1)} \times \cdots \times C^{(n_r)} \times C^{(d-\deg \mathfrak{m})} \to C^{(d)}$ is finite flat, it is enough to show that there exists a closed point Q of $n_1P_1 + \cdots n_rP_r + C^{(d-\deg \mathfrak{m})}$ over which there are $\deg \pi$ points on $C^{(n_1)} \times \cdots \times C^{(n_r)} \times C^{(d-\deg \mathfrak{m})}$. Choose Q as a point corresponding to a divisor $n_1P_1 + \cdots n_rP_r + P_{r+1} + \cdots + P_{r+d-\deg \mathfrak{m}}$, where $P_1, \dots, P_{r+d-\deg \mathfrak{m}}$ are distinct points of $U(k)$. □

Lemma 4.2. *The morphism $\pi_1(U^d) \to \pi_1(U^{(d)})$ induced from the natural projection $U^d \to U^{(d)}$ (base points are omitted) is surjective.*

Proof. Since U^d and $U^{(d)}$ are geometrically connected over k, it is enough to show the surjectivity after the base change to an algebraic closure $\bar{k}$ by considering the homotopy exact sequence $1 \to \pi_1(U^d_{\bar{k}}) \to \pi_1(U^d) \to \pi_1(\mathrm{Spec}(k)) \to 1$ and the counterpart of $U^{(d)}$.

Assume that k is algebraically closed. Let V be a connected finite étale covering of $U^{(d)}$. We show that the pullback $V \times_{U^{(d)}} U^d$ is also connected, which shows the assertion. Note that, since the schemes U^d and $U^{(d)}$ are normal, V and $V \times_{U^{(d)}} U^d$ are normal. In particular, $V \times_{U^{(d)}} U^d$ is the disjoint union of integral schemes. Since the map $U^d \to U^{(d)}$ is finite flat, each connected component of $V \times_{U^{(d)}} U^d$ surjects onto V. Take a k-valued point $P \in U(k)$. Take a k-valued point $P' \in V(k)$ over $dP \in U^{(d)}(k)$. Since the fiber of $U^d \to U^{(d)}$ over the point dP consists of one point $(P, P, \dots, P)$, the fiber of $V \times_{U^{(d)}} U^d \to V$ over P' also consists of one point. Thus the scheme $V \times_{U^{(d)}} U^d$ has only one connected component. □

Proof of Theorem 1.2. Let C be a projective smooth geometrically connected curve over a perfect field k. Let $\mathfrak{m} = n_1P_1 + \cdots + n_rP_r$ ($n_i \geq 1$) be a modulus on C, and U be its complement. Set A as the subgroup of $\mathrm{H}^1(U, \mathbb{Q}/\mathbb{Z})$ consisting of characters χ such that $\mathrm{Sw}_{P_i}(\chi) \leq n_i - 1$ for $i = 1, \dots, r$, and B as the subgroup of $\mathrm{H}^1(\mathrm{Pic}_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ consisting of multiplicative elements.

We construct a map $\Psi : B \to A$. Take $\rho \in B$. Define χ to be the pullback of ρ^1 by the natural map $U \to \mathrm{Pic}^1_{C,\mathfrak{m}}$. We need to show that the ramification is bounded by $\mathfrak{m}$. Take a natural number d large enough so that d satisfies (3-1) for $\mathfrak{m}$. Consider the following commutative diagram:

$$\begin{array}{ccc} U^d = U \times \cdots \times U & \xrightarrow{\pi} & U^{(d)} \\ \downarrow & & \downarrow p \\ \mathrm{Pic}^1_{C,\mathfrak{m}} \times \cdots \times \mathrm{Pic}^1_{C,\mathfrak{m}} & \longrightarrow & \mathrm{Pic}^d_{C,\mathfrak{m}} \end{array} \tag{4-1}$$

By the multiplicativity of ρ, we know that $\pi^*p^*\rho^d = \chi^{\boxtimes d}$. Lemma 4.2 implies that $p^*\rho^d = \chi^{(d)}$. We show that $\mathrm{Sw}_{P_i}(\chi) \leq n_i - 1$. To do this, it is enough to prove that the Swan conductor of $\chi^{(n_i)}$, with respect to the DVR at the generic point of the blow-up of $C^{(n_i)}$ along n_iP_i, is zero, by Corollary 2.9. We may assume that k is algebraically closed (hence $d_i = 1$). Note that the right vertical arrow p in (4-1) factors through $\tilde{C}_{\mathfrak{m}}^{(d)}$:

$$U^{(d)} \to \tilde{C}_{\mathfrak{m}}^{(d)} \to \mathrm{Pic}^d_{C,\mathfrak{m}}.$$

Since the pullback of ρ^d by p is $\chi^{(d)}$, we find that $\chi^{(d)}$ is unramified at the generic point of the complement $\tilde{C}_{\mathfrak{m}}^{(d)} \setminus U^{(d)}$. Thus, by Lemma 4.1 and Corollary 3.8, the character

$$\chi^{(n_1)} \boxtimes 1 \boxtimes \cdots \boxtimes 1 + 1 \boxtimes \chi^{(n_2)} \boxtimes \cdots \boxtimes 1 + \cdots + 1 \boxtimes \cdots \boxtimes \chi^{(d-\deg \mathfrak{m})}$$

on $U^{(n_1)} \times \cdots \times U^{(n_r)} \times U^{(d-\deg \mathfrak{m})}$ is unramified at the generic point of the exceptional divisor of the blow-up of $C^{(n_1)} \times \cdots \times C^{(n_r)} \times C^{(d-\deg \mathfrak{m})}$ along $\{n_1 P_1\} \times \cdots \times \{n_r P_r\} \times C^{(d-\deg \mathfrak{m})}$. Using Corollary 2.6 repeatedly, the assertion is proved.

Thus the map $B \to A$, pulling back by $U \to \mathrm{Pic}^1_{C,\mathfrak{m}}$, is well-defined. We denote this map by Ψ.

First we show the injectivity of Ψ. Take ρ from the kernel of Ψ. Since the multiplication map $\mathrm{Pic}^n_{C,\mathfrak{m}} \times \mathrm{Pic}^m_{C,\mathfrak{m}} \to \mathrm{Pic}^{n+m}_{C,\mathfrak{m}}$ and the two projections $\mathrm{Pic}^n_{C,\mathfrak{m}} \times \mathrm{Pic}^m_{C,\mathfrak{m}} \to \mathrm{Pic}^n_{C,\mathfrak{m}}, \mathrm{Pic}^m_{C,\mathfrak{m}}$ have geometrically connected fibers, the triviality of two of ρ^n, ρ^m, ρ^{n+m} implies the triviality of the other. Thus it is enough to show the triviality of ρ^d for sufficiently large d. Consider the diagram (4-1). By Lemma 4.2, we know that $p^*\rho^d$ is trivial, which implies that ρ^d is trivial by Corollary 3.9.

The surjectivity of Ψ is proved as follows. Take $\chi \in A$. Let d be an integer satisfying (3-1) for $\mathfrak{m}$. Corollary 2.9, Proposition 3.7, and Lemma 4.1 imply that the character $\chi^{(d)}$ extends to a character $\tilde{\chi}^{(d)}$ on $\tilde{C}_{\mathfrak{m}}^{(d)}$. Corollary 3.9 implies that $\tilde{\chi}^{(d)}$ descends to a character ρ^d on $\mathrm{Pic}^d_{C,\mathfrak{m}}$. Let d_1 and d_2 be integers which satisfy (3-1). The commutative diagram

$$\begin{array}{ccc} U^{(d_1)} \times U^{(d_2)} & \longrightarrow & U^{(d_1+d_2)} \\ \downarrow & & \downarrow \\ \mathrm{Pic}^{d_1}_{C,\mathfrak{m}} \times \mathrm{Pic}^{d_2}_{C,\mathfrak{m}} & \xrightarrow{q} & \mathrm{Pic}^{d_1+d_2}_{C,\mathfrak{m}} \end{array}$$

and the fact that the left vertical map has geometrically connected fibers show $q^*\rho^{d_1+d_2} = \rho^{d_1} \boxtimes 1 + 1 \boxtimes \rho^{d_2}$. Fix a nonzero effective Cartier divisor D on U such that $\deg D$ satisfies (3-1). Let ξ be the pullback of $\rho^{\deg D}$ by the map $\mathrm{Spec}(k) \to \mathrm{Pic}^{\deg D}_{C,\mathfrak{m}}$, corresponding to the point D. For an arbitrary integer n, take a natural number m so large that the integer $n + m \deg D$ satisfies (3-1). Define $\rho^n := f^*\rho^{n+m \deg D} \cdot a^*\xi^{-m}$, where $f : \mathrm{Pic}^n_{C,\mathfrak{m}} \to \mathrm{Pic}^{n+m \deg D}_{C,\mathfrak{m}}$ is multiplication by $\mathcal{O}_C(mD)$ and $a : \mathrm{Pic}^n_{C,\mathfrak{m}} \to \mathrm{Spec}(k)$ is the structure map. This construction does not depend on m, since the multiplicativity of ρ^n is already verified for large n. By the same reason, the characters ρ^n form a multiplicative element on $\mathrm{Pic}_{C,\mathfrak{m}}$. The equality $\chi = \Psi(\rho)$ follows from the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{(\mathrm{id}, g)} & U \times U^{(\deg D)} & \longrightarrow & U^{(\deg D+1)} \\ & & \downarrow & & \downarrow \\ & & \mathrm{Pic}^1_{C,\mathfrak{m}} \times \mathrm{Pic}^{\deg D}_{C,\mathfrak{m}} & \longrightarrow & \mathrm{Pic}^{\deg D+1}_{C,\mathfrak{m}}, \end{array}$$

where g is the composition of the structure map $U \to \mathrm{Spec}(k)$ and the map $\mathrm{Spec}(k) \to U^{(\deg D)}$ corresponding to the divisor D. Indeed, the pullback of $\rho^{\deg D+1}$ by the map $U \to U \times U^{(\deg D)} \to U^{(\deg D+1)} \to \mathrm{Pic}_{C,\mathfrak{m}}^{\deg D+1}$ is $\chi \cdot b^*\xi$, where $b : U \to \mathrm{Spec}(k)$ is the structure map. On the other hand, the pullback of $\rho^{\deg D+1}$ the other way is $\Psi(\rho) \cdot b^*\xi$. □

To deduce Theorem 1.1 from Theorem 1.2, first we recall basic facts on torsors.

Assume that k is algebraically closed. Fix a connected commutative algebraic k-group G. Let $\mathcal{C}(G)$ be the category as follows. The objects are pairs $(H, \phi\colon H \to G)$ where H are connected commutative algebraic k-groups and ϕ are abelian isogenies. The morphisms $(H_1, \phi_1\colon H_1 \to G) \to (H_2, \phi_2\colon H_2 \to G)$ are pairs (f, g) where $f\colon H_1 \to H_2$ is a morphism of k-group schemes such that $\phi_2 \circ f = \phi_1$ and $g\colon H_1 \to H_2$ is a compatible morphism of torsors such that $\phi_2 \circ g = \phi_1$. Here we regard H_1 (resp. H_2) itself as an H_1-torsor (resp. H_2-torsor) by the multiplication. Note that the kernels of ϕ_i are constant k-schemes since H_i are Galois coverings of G.

Lemma 4.3. *Let the notation be as above. Let $(H_i, \phi_i\colon H_i \to G)$ be objects in $\mathcal{C}(G)$ for $i = 1, 2$.*

(1) *If there exists a morphism $H_1 \to H_2$ of G*-schemes, *there exists a unique morphism $f\colon H_1 \to H_2$ of k-group schemes with $\phi_2 \circ f = \phi_1$.*

(2) *The map*

$$\mathrm{Hom}((H_1, \phi_1\colon H_1 \to G), (H_2, \phi_2\colon H_2 \to G)) \to \mathrm{Hom}_G(H_1, H_2)$$

sending $(f, g) \mapsto g$ is bijective. Here the target is the set of morphisms of G-schemes.

Proof. Let $e_i \in H_i(k)$ be the units.

(1) Uniqueness follows from the fact that H_i are connected étale coverings of G and such an f must send e_1 to e_2. Let $f\colon H_1 \to H_2$ be the G-morphism which sends e_1 to e_2. Such an f does exist since H_2 is Galois over G. We need to show that the diagram

$$\begin{array}{ccc} H_1 \times H_1 & \xrightarrow{f \times f} & H_2 \times H_2 \\ \downarrow & & \downarrow \\ H_1 & \xrightarrow{f} & H_2, \end{array}$$

where the vertical maps are the multiplications, is commutative. This follows since $H_1 \times H_1$ is a connected étale covering of $G \times G$ and the two maps send (e_1, e_1) to e_2.

(2) Injectivity follows since a goup homomorphism $f\colon H_1 \to H_2$ over G is unique if it exists by (1). We show the surjectivity. Thus we assume that there is a group homomorphism $f\colon H_1 \to H_2$ over G. Let $g\colon H_1 \to H_2$ be a morphism of G-schemes. Since H_1 is a connected étale covering of G, this is uniquely determined by the image $a := g(e_1)$, which is contained in $\ker\phi_2$. Let $g'\colon H_1 \to H_2$ be the compatible morphism of torsors which sends e_1 to a. Since $a \in \ker\phi_2$ and $\phi_2 \circ f = \phi_1$, this is a morphism of G-schemes. Thus we have $g = g'$. □

Proof of Theorem 1.1. Let (G^0, G^1) be a connected abelian covering of $(\mathrm{Pic}^0_{C,\mathfrak{m}}, \mathrm{Pic}^1_{C,\mathfrak{m}})$. Since the d-th power of $\mathrm{Pic}^1_{C,\mathfrak{m}}$ is isomorphic to $\mathrm{Pic}^d_{C,\mathfrak{m}}$ as $\mathrm{Pic}^0_{C,\mathfrak{m}}$-torsors, the d-th power G^d of G^1 is naturally equipped with a compatible morphism $G^d \to \mathrm{Pic}^d_{C,\mathfrak{m}}$ of torsors. Let K be the kernel of the map $G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}$. This is a finite constant group since $G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ is a Galois isogeny. Take a nontrivial homomorphism $\chi : K \to \mathbb{Q}/\mathbb{Z}$. This defines characters $\rho^d \in \mathrm{H}^1(\mathrm{Pic}^d_{C,\mathfrak{m}}, \mathbb{Q}/\mathbb{Z})$ for all d. From the construction, they form a multiplicative element on $\mathrm{Pic}_{C,\mathfrak{m}}$. Theorem 1.2 implies that the pullback of ρ^1 by $U \to \mathrm{Pic}^1_{C,\mathfrak{m}}$ is nontrivial and its ramification is bounded by $\mathfrak{m}$, which shows the first part of Theorem 1.1.

Define the category $\mathcal{C}_1$ as the category of geometrically connected abelian coverings of U whose ramifications are bounded by $\mathfrak{m}$ and the category $\mathcal{C}_2$ as the category of connected abelian coverings of $(\mathrm{Pic}^0_{C,\mathfrak{m}}, \mathrm{Pic}^1_{C,\mathfrak{m}})$. We have constructed a functor $\Phi : \mathcal{C}_2 \to \mathcal{C}_1$. We show that this functor is an equivalence of categories. We only treat the case when k is algebraically closed. General case follows from this special case by using an argument of Galois descent.

Assume that k is algebraically closed. Let $\mathcal{C} := \mathcal{C}(\mathrm{Pic}^0_{C,\mathfrak{m}})$ be the category defined above. In this case, fixing a closed point P of U, $\mathcal{C}_2$ is isomorphic to $\mathcal{C}$ via the isomorphism $\mathrm{Pic}^0_{C,\mathfrak{m}} \to \mathrm{Pic}^1_{C,\mathfrak{m}}$ of torsors, tensoring $\mathcal{O}_C(P)$.

We show that the functor $\Phi' : \mathcal{C} \to \mathcal{C}_1$, pulling back by the morphism $U \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ sending Q to $\mathcal{O}_C(Q-P)$ is an equivalence. Faithfulness is obvious since there only occur connected coverings. To show fullness, let $(G_i, G_i \to \mathrm{Pic}^0_{C,\mathfrak{m}})$ be elements of $\mathcal{C}$ for $i = 1, 2$ and let $V_i := \Phi'(G_i, G_i \to \mathrm{Pic}^0_{C,\mathfrak{m}})$. By Lemma 4.3(2) and faithfulness, it is enough to show that, if there is a map $V_1 \to V_2$, there is a map $G_1 \to G_2$. The kernel K_i of $G_i \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ is canonically identified with the Galois group of $V_i \to U$. If there is a map $V_1 \to V_2$, there is a map of abelian groups $h : K_1 \to K_2$, which is independent of the choice of $V_1 \to V_2$. We show the commutativity of the diagram

$$\begin{array}{ccc} & \pi_1(\mathrm{Pic}^0_{C,\mathfrak{m}}) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ K_1 & \xrightarrow{\quad h \quad} & K_2 \end{array} \tag{4-2}$$

where the downward diagonals are the canonical surjections. Assume that there is an element $\sigma \in \pi_1(\mathrm{Pic}^0_{C,\mathfrak{m}})$ such that $p_2(\sigma) \neq hp_1(\sigma)$. Take a group homomorphism $\rho^0 : K_2 \to \mathbb{Q}/\mathbb{Z}$ such that the images of $p_2(\sigma)$ and $hp_1(\sigma)$ are different. Since the characters $\rho^0 p_2$ and $\rho^0 hp_1$ are multiplicative and are pulled back to the same character via the map $U \to \mathrm{Pic}^0_{C,\mathfrak{m}}$, they are the same character, a contradiction. Thus the diagram (4-2) is commutative, which implies that the quotient group $G_1/\ker h$ of G_1 is isomorphic to G_2.

For essential surjectivity, we argue as follows. Let $V \in \mathcal{C}_1$ be a connected cyclic covering of U. Take a character on U whose kernel corresponds to V. By Theorem 1.3, this character is the pullback of a multiplicative character ρ^0 on $\mathrm{Pic}^0_{C,\mathfrak{m}}$. Let G^0 be an étale covering of $\mathrm{Pic}^0_{C,\mathfrak{m}}$ corresponding to the kernel of ρ^0. We need to show that G^0 has a group structure. By the definition, G^0 is connected. From the multiplicativity of ρ^0, we know that there is a commutative diagram

$$\begin{array}{ccc} G^0 \times G^0 & \xrightarrow{m_G} & G^0 \\ \downarrow & & \downarrow \\ \mathrm{Pic}^0_{C,\mathfrak{m}} \times \mathrm{Pic}^0_{C,\mathfrak{m}} & \longrightarrow & \mathrm{Pic}^0_{C,\mathfrak{m}}. \end{array}$$

Let us denote the map m_G multiplicatively. Let F be the fiber of $G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ over $1 \in \mathrm{Pic}^0_{C,\mathfrak{m}}$. For distinct points $y_1, y_2 \in F$, the multiplication from right by y_1 and y_2, $G^0 \to G^0$ are distinct. Indeed, Assume that $xy_1 = xy_2$ for all $x \in G^0$. Take a point x in F. The multiplication from left by x, $G^0 \to G^0$ is a $\mathrm{Pic}^0_{C,\mathfrak{m}}$-morphism and sends y_1 and y_2 to the same point, which implies that $y_1 = y_2$ since G^0 is a connected covering of $\mathrm{Pic}^0_{C,\mathfrak{m}}$.

Thus there exists an element $e \in F$ such that $xe = x$ for all $x \in G^0$. Next we show the commutativity of m_G. This follows from the fact that $G^0 \times G^0$ is a connected covering of $\mathrm{Pic}^0_{C,\mathfrak{m}} \times \mathrm{Pic}^0_{C,\mathfrak{m}}$ and that the maps $G^0 \times G^0 \to G^0$, $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ send (e, e) to the same point e. The associativity is proved in a similar way. Therefore it is verified that G^0 has a commutative group structure such that $G^0 \to \mathrm{Pic}^0_{C,\mathfrak{m}}$ is a group homomorphism, hence an abelian isogeny. It is easy to show that G^0 is pulled back to V. For a general V, use the fact that V is a connected component of the finite projective limit of cyclic connected coverings which are quotients of V. □

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