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Weyl bound for p-power twist of GL(2) L-functions

Ritabrata Munshi and Saurabh Kumar Singh

Let f be a cuspidal eigenform (holomorphic or Maass) for the congruence group $\Gamma_0(N)$ with N square-free. Let p be a prime and let χ be a primitive character of modulus p^{3r}. We shall prove the Weyl-type subconvex bound

$$L\left(\tfrac{1}{2}+it, f\otimes\chi\right) \ll_{f,t,\varepsilon} p^{r+\varepsilon},$$

where $\varepsilon > 0$ is any positive real number.

1. Introduction

Bounding automorphic L-functions on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ is a central problem in the analytic theory of L-functions. The functional equation and the Phragmen–Lindelöf principle from complex analysis yield the convexity bound $L\left(\frac{1}{2}+it, \pi\right) \ll C(\pi, t)^{1/4+\varepsilon}$ where $C(\pi, t)$ is the analytic conductor of the L-function, as defined by Iwaniec and Sarnak (see equation (31) of [Iwaniec and Sarnak 1999]), whereas the grand Riemann hypothesis (GRH) implies the Lindelöf hypothesis which predicts the bound $C(\pi, t)^{\varepsilon}$ for any $\varepsilon > 0$. Any bound with exponent smaller than $\frac{1}{4}$ is called a subconvex bound. In this context the Weyl exponent $\frac{1}{6}$, which is one-third of the way down from convexity towards Lindelöf, is a known barrier which has been achieved only for a handful of families. Indeed the only known "arithmetic case" (level aspect) is given by the fundamental work of Conrey and Iwaniec [2000]. For χ_q the quadratic character modulo q (which is square-free and odd) they established: (i) $L\left(\frac{1}{2}+it, \chi_q\right) \ll_t q^{1/6+\varepsilon}$ with polynomial dependence on t. (ii) $L\left(\frac{1}{2}, f\otimes\chi_q\right) \ll q^{1/3+\varepsilon}$ for f a primitive GL(2) cusp form of level dividing q. Note that while the former result is flexible and applies to any point on the critical line, the latter result only applies at the central point, as the nonnegativity of the L-value plays a central role in their argument. Our main objective here is to establish the following Weyl-type bound.

Theorem 1.1. *Let f be a holomorphic Hecke eigenform, or a Maass cusp form for the congruence subgroup $\Gamma_0(N)$ with N square-free. Let χ be a primitive character of modulus p^{3r} where p is a prime and r is a natural number. We have*

$$L\left(\tfrac{1}{2}+it, f\otimes\chi\right) \ll_{f,t,\varepsilon} p^{r+\varepsilon}.$$

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This is the first instance where the Weyl exponent is achieved in the level aspect for a family of L-functions which are not self-dual. Our method is soft and does not rely on the Riemann hypothesis for varieties over finite fields. Indeed the character sums that we encounter are treated in an elementary manner. It is quite surprising that our method could yield such a strong bound. But recently it has been established (see [Munshi 2018; Aggarwal and Singh 2017]) that both the Weyl and the Burgess bound, for both GL(1) and GL(2) L-functions, follow from the method developed in the series [Munshi 2014; 2015a; 2015b; 2015c].

Recall that the twisted automorphic L-function of degree two associated to (f, χ) is defined by the Dirichlet series

$$L(s, f \otimes \chi) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s},$$

where $\mathrm{Re}(s) > 1$ and $\lambda_f(n)$ are the normalized Fourier coefficients of f. These extend to an entire function and satisfy a functional equation relating $L(s, f \otimes \chi)$ to $L(1-s, \bar{f} \otimes \bar{\chi})$. In the family we are considering the form f and the point $\frac{1}{2}+it$ are fixed, but the character χ varies and $p \to \infty$. The above result will be derived as a special case of the following.

Theorem 1.2. *Let f be a holomorphic Hecke eigenform or a Hecke–Maass cusp form for $\Gamma_0(N)$ with N square-free. Let χ be a primitive character of modulus p^r, where p is a prime. We have*

$$L\big(\tfrac{1}{2}+it, f \otimes \chi\big) \ll_{f,\epsilon} p^{1/2(r-\lfloor r/3 \rfloor)+\epsilon},$$

where $\epsilon > 0$ is any positive real number and $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Let us briefly recall the history of subconvexity bounds for L-functions. We will only focus on the results which are related to our case. The convexity bound for the Riemann zeta function is given by $\zeta\big(\frac{1}{2}+it\big) \ll t^{1/4+\varepsilon}$. For a Dirichlet L-function associated with a primitive Dirichlet character χ of modulus q, the convexity bound is given by $L\big(\frac{1}{2}, \chi\big) \ll q^{1/4+\varepsilon}$. The Lindelöf hypothesis asserts that the exponent $\frac{1}{4}+\varepsilon$ can be replaced by ε. The subconvexity bound for $\zeta(s)$ was first proved by Hardy and Littlewood, based on the work of Weyl [1916]. Establishing a bound for exponential sums, it has been proved that (see also [Titchmarsh 1986, page 99, Theorem 5.5])

$$\zeta\big(\tfrac{1}{2}+it\big) \ll t^{1/6} \log^{3/2} t. \tag{1}$$

It was first written down by Landau [1924] in a slightly refined form, and has been generalized to all Dirichlet L-functions. Since then it has been improved by several people. The best known result with exponent $\frac{13}{84} \approx 0.15476$ is due to Bourgain [2017]. On the other hand, the q-aspect subconvexity bound was first proved by Burgess [1963]. Using an ingenious technique of completing short character sums and utilizing the Riemann hypothesis for curves over finite fields (Weil's result), he proved that

$$L(s, \chi) \ll_\epsilon q^{3/16+\varepsilon}, \tag{2}$$

for fixed s with $\mathrm{Re}\, s = \frac{1}{2}$ and for any $\varepsilon > 0$. Heath-Brown [1978] proved the hybrid bound (bound in both parameters q and t together) for Dirichlet L-functions. The Burgess exponent $\frac{3}{16}$, which is one-fourth of the way down from convexity towards the Lindelöf, has come to be realized as a natural barrier. However, stronger bounds can be shown under suitable factorization hypothesis on the modulus. Indeed, in principle the family $L\bigl(\frac{1}{2}, \chi\bigr)$ where χ runs over characters modulo p^r with $r \to \infty$ should behave like the family $\zeta\bigl(\frac{1}{2}+it\bigr)$ with $t \to \infty$. Only recently a suitable p-adic analogue of the van der Corput method has been introduced by Milićević [2016], and he has been able to obtain a sub-Weyl exponent 0.1645 for this family. More precisely, for χ primitive Dirichlet character modulo $q = p^n$ he proved that for any given $\theta > \theta_0 \approx 0.1645$, there is a $j \geq 0$ such that

$$L\bigl(\tfrac{1}{2}, \chi\bigr) \ll p^j q^\theta (\log q)^{1/2}.$$

So it follows that we have a subconvexity exponent which is less than $\frac{1}{6}$ for a prime power modulus $q = p^n$ with $n \geq n_0$, a sufficiently large number.

The t-aspect Weyl exponent for GL(2) L-functions was first proved by Good [1982], for holomorphic modular forms, using the spectral theory of automorphic functions. Jutila [1987] has given an alternative proof, based only on the functional properties of $L(f, s)$ and $L(s, f \otimes \psi)$, where ψ is an additive character. The arguments used in his proof were flexible enough to be adopted for the Maass cusp forms, as was shown by Meurman [1990]. However, the character twist aspect subconvexity bound required some more new ideas. It was first obtained by Duke, Friedlander and Iwaniec using a new form of the circle method and the amplification technique. Assuming χ to be a primitive character of modulus q and $\operatorname{Re} s = \frac{1}{2}$, they obtained (see [Duke et al. 1993, Theorem 1])

$$L(s, f \otimes \chi) \ll_f |s|^2 q^{5/11} \tau^2(q) \log q,$$

where $\tau(q)$ is the divisor function. In the case of a general holomorphic cusp form, Bykovskii [1996] used a trace formula expressing the mean values of cusp form (see [Bykovskii 1996, page 1, line 1]) to derive the Burgess exponent $\frac{3}{8}$. In the case of Maass form subconvexity bound obtained by Harcos. Refining the arguments used in [Bykovskii 1996], Blomer and Harcos [2008] obtained the Burgess exponent $\frac{3}{8}$ for a more general holomorphic or Maass cusp form. To date, the Weyl exponent $\frac{1}{3}$ has only been achieved for quadratic characters, courtesy of the fundamental work of Conrey and Iwaniec [2000], as we have already mentioned above. Extending the above mentioned result of Milićević to GL(2) L-functions, Blomer and Milićević [2015, Theorem 2] obtained

$$L\bigl(\tfrac{1}{2}+it, f \otimes \chi\bigr) \ll_{f,\varepsilon} (1+|t|)^{5/2} p^{7/6} q^{1/3+\varepsilon},$$

where f is a holomorphic or Maass newform for $\mathrm{SL}(2, \mathbb{Z})$, and χ is a primitive character of conductor $q = p^n$, with p an odd prime. This yields a subconvexity bound for $n > 7$ and improves the Burgess exponent as soon as $n > 27$, and the exponent tends to Weyl exponent as $n \to \infty$. Though there is sufficient room for improvements in the above estimate (as the authors themselves comment in [Blomer and Milićević 2015]), it is inherent in their method that the Weyl exponent can never be achieved for any given n. In this paper we propose a different approach which produces an improvement over the known bounds. In Theorem 1.2, we are able to provide an improvement on the Burgess exponent as soon as $n \geq 3$, with the exception when $n = 4, 8$ (in this case our exponent is same as Burgess exponent) and $n = 5$. Of course the most interesting outcome of our result is that we are able to achieve the Weyl exponent when $n \geq 3$ and $n \equiv 0 \pmod 3$. In the next section we briefly explain the method of the proof, which is a rendition of [Munshi 2015a].

2. Sketch of the proof

We start with the approximate functional equation as given in [Iwaniec and Kowalski 2004, Proposition 5.4]. Taking a smooth dyadic subdivision we arrive at

$$L\big(\tfrac{1}{2} + it, f \otimes \chi\big) \ll_{f,A} N^{\varepsilon} \sup_{N \leq P^{1+\varepsilon}} \frac{|S(N)|}{N^{1/2}} + P^{-A},$$

where $P = p^r$ is the modulus of the character and $S(N)$ is a dyadic sum which is given by

$$S(N) := \sum_{n=1}^{\infty} \lambda_f(n) \chi(n) V\left(\frac{n}{N}\right), \tag{3}$$

where $V(x)$ is a smooth bump function supported on the interval $[1, 2]$ and satisfies $V^{(j)}(x) \ll_j (1 + |t|)^j$. As the implied constant in Theorem 1.2 is allowed to depend on t, we can and will from now on assume that $t = 0$. Our method is not sensitive to small perturbations like this. A careful study of the proof shows that the eventual implied constant grows at most polynomially with t. Now trivially estimating, we obtain $S(N) \ll N^{1+\varepsilon}$. We shall examine $S(N)$ in following steps. For simplicity let us focus on the case $r \equiv 0 \pmod 3$.

Step 1 (applying circle method). We shall apply Kloosterman's version of the circle method (see Lemma 3.3) with the conductor lowering mechanism introduced by the first author in [Munshi 2015a]. We obtain the sum

$$S(N) = 2\,\mathrm{Re} \int_0^1 \sum_{1 \leq q \leq Q} \sideset{}{^\star}\sum_{Q < a \leq q+Q} \frac{1}{aqp^{\ell}} \sum_{b(p^{\ell})} \bigg\{ \sum_{n \sim N} \lambda_f(n) e\bigg(\frac{(\bar{a}+bq)n}{p^{\ell} q}\bigg) \bigg\} \bigg\{ \sum_{m \sim N} \chi(m) e\bigg(-\frac{(\bar{a}+bq)m}{p^{\ell} q}\bigg) \bigg\}\, dx,$$

where Q is taken to be $Q = N^{1/2}/p^{\ell/2}$. Trivially estimating after first step, we have $S(N) \ll N^{2+\varepsilon}$.

Step 2 (applying Poisson summation formula). In this step we shall apply the Poisson summation formula to the sum over m. The character χ has conductor p^r and the additive conductor has a size q. Hence the total conductor for the sum over m has size $p^r q$. So the dual sum is essentially supported up to the size qp^r/N. We observe that after application of the Poisson formula we are able to save $N/\sqrt{p^r q}$ from the sum over m. Evaluating the character sum, we also observe that $Q < a \leq Q + q$ can be determined uniquely by the congruence relation $a \equiv \overline{m} p^{r-\ell} \bmod q$. In particular a does not depend on n. The total saving after the first step is given by

$$\frac{N}{\sqrt{p^r q}} \times \sqrt{q} = \frac{N}{p^{r/2}}.$$

Trivially estimating after the second step we obtain $S(N) \ll N p^{r/2}$.

Step 3 (applying Voronoi summation formula). We shall now apply the Voronoi summation formula to the sum over n, which has conductor of size $p^\ell q$. The dual length is essentially supported up to the size $p^{2\ell}q^2/N$. We are able to save $N/p^\ell q$ from the Voronoi summation formula and $p^{\ell/2}$ by assuming square root cancellation in exponential sum over b. Total saving in the third step is

$$\frac{N}{p^\ell q} \times p^{\ell/2} = \frac{N}{p^{\ell/2} q}.$$

Trivially estimating after the third step, we observe that $S(N) \ll N^{1/2} p^{r/2+\varepsilon}$, which shows that we are on the boundary. We are left with a sum of the form:

$$S(N) = \sum_{n \ll p^\ell N^\varepsilon} \lambda_f(n) \Bigg[\sum_{1 \le q \le Q} \sum_{m \ll \frac{N^\varepsilon Q p^r}{N}} \frac{\chi(q)}{aq^2} \mathcal{C}(n,m,q) e\bigg(-n \frac{p^r \overline{p^{2\ell}} \overline{m}}{q} \bigg) \mathcal{I}(x,q,m) \mathcal{J}(x,n,q) \Bigg],$$

where the character sum is given by

$$\mathcal{C}(n,m,q) = \sideset{}{^\star}\sum_{b \bmod p^\ell} \bar{\chi}(m - (\bar{a} + bq) p^{r-\ell}) e\bigg(-n \frac{\overline{a+bq}\,\bar{q}}{p^\ell} \bigg)$$

and the function $\mathcal{J}(x,n,q)$ is of size $O(1)$, and is not oscillatory with respect to n.

Step 4 (Cauchy–Schwarz inequality and Poisson summation formula). To obtain additional savings, we apply the Cauchy–Schwarz inequality to get rid of the Fourier coefficients. But this process also squares the amount we need to save. We now open the absolute square and then interchange the summation over n. Applying the Poisson summation formula we are able to save $Q^2 p^r/N = p^{r-\ell}$ from the diagonal terms and $p^{\ell/2}$ from the nondiagonal terms. We observe that optimal choice for ℓ is given by $\ell = 2r/3$. Substituting the value of ℓ we obtain

$$S(N) \ll \frac{N^{1/2} p^{r/2+\varepsilon}}{p^{\ell/4}} \ll N^{1/2} p^{r/3+\varepsilon}.$$

Upon substituting this bound in the bound we obtained from the approximate functional equation the Weyl bound follows. Observe that when r is not divisible by 3 then we are not allowed to pick the above optimal choice for ℓ, and we have to choose the best possible which is $[2r/3]$. In the following sections we shall provide the proof of the theorem in detail.

3. Preliminaries

To keep the notations simple we will focus on the case of full level. Our argument is robust and is not sensitive to the nature of the fixed form f. We will present our argument in detail for Maass forms of full level. The case of Maass forms is traditionally considered harder. The reader will have no problem to see how the arguments can be adopted in the case of general square-free level with general nebentypus. In principle, our method should work even for levels which are not square-free, but we refrain from including that due to the lack of a suitable Voronoi summation formula. In this section we recall some basic facts about $\mathrm{SL}(2,\mathbb{Z})$ automorphic forms (for details see [Iwaniec 1997; Iwaniec and Kowalski 2004]). Our requirement is minimal, in fact the Voronoi summation formula and the Rankin–Selberg bound (see Lemma 3.2) are all that we shall be using.

Maass cusp forms. Let f be a weight zero Hecke–Maass cusp form with Laplace eigenvalue $\frac{1}{4}+\nu^2$. The Fourier series expansion of f at ∞ is given by

$$f(z)=\sqrt{y}\sum_{n\neq 0}\lambda_f(n)K_{i\nu}(2\pi|n|y)e(nx),$$

where $\lambda_f(1)=1$. Let χ be a primitive Dirichlet character of modulus P. The twisted L-function is defined by

$$L(s,f\otimes\chi):=\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi(n)}{n^s}$$

for $\mathrm{Re}(s)>1$. It extends to an entire function and satisfies the functional equation

$$\Lambda(s,f\otimes\chi)=\varepsilon(f\otimes\chi)\Lambda(1-s,f\otimes\chi),$$

where $|\varepsilon(f\otimes\chi)|=1$ and

$$\Lambda(s,f\otimes\chi)=\left(\frac{P}{\pi}\right)^{s}\Gamma\left(\frac{s+i\nu}{2}\right)\Gamma\left(\frac{s-i\nu}{2}\right)L(s,f\otimes\chi).$$

From the functional equation and Phragmen–Lindelöf principle one can derive the convexity bound

$$L\left(\tfrac{1}{2}+it,f\otimes\chi\right)\ll_{f,t,\epsilon}P^{1/2+\epsilon}.$$

We shall require the following Voronoi summation formula for the Maass form. This was first established by T. Meurman [1988] for full level (for general case see appendix A.4 of [Kowalski et al. 2002]).

Lemma 3.1 (Vornoi summation formula). *Let h be a compactly supported smooth function in the interval $(0,\infty)$. Let $\lambda_f(n)$ be the Fourier coefficient of a weight zero Maass form for the full modular group $\mathrm{SL}(2,\mathbb{Z})$, and a, q be positive integers with $(a,q)=1$. We have*

$$\sum_{n=1}^{\infty}\lambda_f(n)e_q(an)h(n)=\frac{1}{q}\sum_{\pm}\sum_{n=1}^{\infty}\lambda_f(\mp n)e_q(\pm\bar{a}n)H^{\pm}\left(\frac{n}{q^2}\right), \tag{4}$$

where $a\bar{a}\equiv 1 \pmod q$, and

$$H^{-}(y)=\frac{-\pi}{\cosh(\pi\nu)}\int_0^{\infty}h(x)\{Y_{2i\nu}+Y_{-2i\nu}\}(4\pi\sqrt{xy})\,dx,$$
$$H^{+}(y)=4\cosh(\pi\nu)\int_0^{\infty}h(x)K_{2i\nu}(4\pi\sqrt{xy})\,dx,$$

where $Y_{2i\nu}$ and $K_{2i\nu}$ are Bessel functions of first and second kind and $e_q(x)=e^{2\pi ix/q}$.

Remark. When h is supported on the interval $[X, 2X]$ and satisfies $x^j h^{(j)}(x) \ll 1$, then integrating by parts and using the properties of Bessel's function, it is easy to see that the sums on the right hand side of (4) are essentially supported on $n \ll_{f,\varepsilon} q^2(qX)^\varepsilon/X$. For smaller values of n we will use the trivial bound, $H^{\pm}(n/q^2) \ll X$.

Some lemmas. We first recall the Rankin–Selberg bound for Fourier coefficients.

Lemma 3.2. *Let $\lambda_f(n)$ be the normalized Fourier coefficients of a holomorphic cusp form or of a Maass form. Then for any real number $x \geq 1$, we have*

$$\sum_{1 \leq n \leq x} |\lambda_f(n)|^2 \ll_{f,\varepsilon} x^{1+\varepsilon}.$$

Let $\delta : \mathbb{Z} \to \{0, 1\}$ be the Kronecker delta function, which is given by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We have the following lemma, which gives the Fourier–Kloosterman expansion of $\delta(n)$ (see [Iwaniec and Kowalski 2004, page 470, Proposition 20.7]).

Lemma 3.3. *Let $Q \geq 1$ be a real number. We have*

$$\delta(n) = 2\,\mathrm{Re} \int_0^1 \sum_{1 \leq q \leq Q} \sideset{}{^\star}\sum_{Q < q \leq q+Q} \frac{1}{aq} e\left(\frac{n\bar{a}}{q} - \frac{nx}{aq}\right), \tag{6}$$

where $\star$ restricts the summation by $(a, q) = 1$ and $a\bar{a} \equiv 1 \pmod q$.

We also need to estimate the exponential integral of the form

$$\mathfrak{I} = \int_a^b g(x) e(f(x))\, dx, \tag{7}$$

where f and g are smooth real valued function on the interval $[a, b]$. Suppose we have $|f'(x)| \geq B$, $|f^{(j)}(x)| \leq B^{1+\varepsilon}$ for $j \geq 2$ and $|g^{(j)}(x)| \ll_j 1$ on the interval $[a, b]$. Then by making the change of variable

$$u = f(x), \qquad f'(x)\, dx = du,$$

we have

$$\mathfrak{I} = \int_{f(a)}^{f(b)} \frac{g(x)}{f'(x)} e(u)\, du \quad (x = f^{-1}(u)).$$

By applying integration by parts, differentiating $g(x)/f'(x)$ j-times and integrating $e(u)$, we have

$$\mathfrak{I} \ll_{j,\varepsilon} B^{-j+\varepsilon}. \tag{8}$$

This will be used at several places to show that certain exponential integrals are negligibly small in the absence of a stationary phase point. Next we consider the case of stationary phase point(i.e., point where derivative vanishes).

Lemma 3.4. *Suppose f and g are smooth real valued functions on the interval $[a, b]$ satisfying*

$$f^{(i)} \ll \frac{\Theta_f}{\Omega_f^i}, \quad g^{(j)} \ll \frac{1}{\Omega_g^j} \quad \textit{and} \quad f^{(2)} \gg \frac{\Theta_f}{\Omega_f^2}, \tag{9}$$

for $i = 1, 2$ and $j = 0, 1, 2$. Suppose that $g(a) = g(b) = 0$.

(1) *Suppose f' and f'' do not vanish on the interval $[a, b]$. Let $\Lambda = \min_{x\in[a,b]}|f'(x)|$. Then we have*

$$\mathfrak{I} \ll \frac{\Theta_f}{\Omega_f^2 \Lambda^3}\left(1 + \frac{\Omega_f}{\Omega_g} + \frac{\Omega_f^2}{\Omega_g^2}\frac{\Lambda}{\Theta_f/\Omega_f}\right).$$

(2) *Suppose that $f'(x)$ changes sign from negative to positive at $x = x_0$ with $a < x_0 < b$. Let $\kappa = \min\{b - x_0, x_0 - a\}$. Further suppose that bound in (9) holds for $i = 4$. Then we have the following asymptotic expansion*

$$\mathfrak{I} = \frac{g(x_0)e(f(x_0)) + 1/8}{\sqrt{f''(x_0)}} + \left(\frac{\Omega_f^4}{\Theta_f^2 \kappa^3} + \frac{\Omega_f}{\Theta_f^{3/2}} + \frac{\Omega_f^3}{\Theta_f^{3/2}\Omega_g^2}\right).$$

Proof. See Theorem 1 and Theorem 2 of [Huxley 1994]. □

4. Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall establish the following bound.

Proposition 4.1. *We have*

$$S(N) \ll \begin{cases} N^{1+\varepsilon} & \textit{if } 1 \le N \ll p^{2r/3+\varepsilon}, \\ N^{1/2} p^{1/2(r-\lfloor r/3 \rfloor)+\varepsilon} & \textit{if } p^{2r/3} \ll N \ll p^{r+\varepsilon}. \end{cases}$$

Application of the circle method. We first separate the oscillation of Fourier coefficients $\lambda_f(n)$ and $\chi(n)$ using delta symbol. We write

$$S(N) := \sum\sum_{m,n=1}^{\infty} \lambda_f(n)\chi(m)\delta(n-m)V\left(\frac{n}{N}\right)V_1\left(\frac{m}{N}\right),$$

where $V_1(y)$ is another smooth function, supported on the interval $\left[\frac{1}{2}, 3\right]$, $V_1(y) \equiv 1$ for $y \in [1, 2]$ and satisfies $y^j V^{(j)}(y) \ll_j 1$. To analyze sum $S(N)$ we use a conductor lowering mechanism (see [Munshi 2015a] for a discrete version of the conductor lowering method and [Munshi 2015b] for the integral version). The integral equation $n = m$ is equivalent to the congruence $n \equiv m \pmod{p^\ell}$ and the integral equation $(n - m)/p^\ell = 0$, $\ell < r$. This lowers the conductor, as modulus p^ℓ is already present in the character χ. We obtain

$$S(N) := \sum\sum_{\substack{m,n=1 \\ p^\ell|(n-m)}}^{\infty} \lambda_f(n)\chi(m)\delta\left(\frac{n-m}{p^\ell}\right)V\left(\frac{n}{N}\right)V_1\left(\frac{m}{N}\right),$$

Now using Lemma 3.3 for the expression of $\delta(n)$, we have $S(N) = S^{+}(N) + S^{-}(N)$, with

$$S^{\pm}(N) = \int_0^1 \sum_{1\le q\le Q<q\le q+Q}\sum{}^{\star} \frac{1}{aq} \sum_{\substack{m,n=1\\ p^\ell|(n-m)}}^{\infty}\sum \lambda_f(n)\chi(m)e\left(\pm\frac{\bar{a}(n-m)/p^\ell}{q} \mp \frac{x(n-m)/p^\ell}{aq}\right)V\left(\frac{n}{N}\right)V_1\left(\frac{m}{N}\right)dx.$$

We choose $Q = (N/p^\ell)^{1/2}$. We detect the congruence relation $n \equiv m \pmod{p^\ell}$ in the above expression using an exponential sum. We have

$$S^{\pm}(N) = \int_0^1 \sum_{1\le q\le Q<q\le q+Q}\sum{}^{\star} \frac{1}{aqp^\ell} \sum_{b(p^\ell)} \sum_{m,n=1}^{\infty}\sum \lambda_f(n)\chi(m)e\left(\pm\frac{(\bar{a}+bq)(n-m)}{p^\ell q}\right)e\left(\mp\frac{x(n-m)}{ap^\ell q}\right)V\left(\frac{n}{N}\right)V_1\left(\frac{m}{N}\right)dx.$$

We will now analyze the sum $S^{+}(N)$ (analysis of $S^{-}(N)$ is similar). We rearrange the sum as

$$S^{+}(N) = \int_0^1 \sum_{1\le q\le Q<q\le q+Q}\sum{}^{\star} \frac{1}{aqp^\ell} \sum_{b(p^\ell)} \left\{\sum_{n=1}^{\infty}\lambda_f(n)e\left(\frac{(\bar{a}+bq)n}{p^\ell q}\right)e\left(\frac{xn}{p^\ell aq}\right)V\left(\frac{n}{N}\right)\right\} \left\{\sum_{m=1}^{\infty}\chi(m)e\left(-\frac{(\bar{a}+bq)m}{p^\ell q}\right)e\left(\frac{-mx}{p^\ell aq}\right)V_1\left(\frac{m}{N}\right)\right\}dx. \quad (10)$$

Applying Poisson summation formula. We shall apply the Poisson summation formula to the sum over m in (10) as follows. Writing $m = \beta + cp^r q$, $c \in \mathbb{Z}$ and then applying the Poisson summation formula to sum over c, we have

$$\sum_{m=1}^{\infty}\chi(m)e\left(-\frac{(\bar{a}+bq)m}{p^\ell q}\right)e\left(\frac{-mx}{p^\ell aq}\right)V_1\left(\frac{m}{N}\right) = \sum_{\beta(p^r q)}\chi(\beta)e\left(-\frac{(\bar{a}+bq)\beta}{p^\ell q}\right)\sum_{m\in\mathbb{Z}}\int_{\mathbb{R}}V_1\left(\frac{\beta+yp^r q}{N}\right)e\left(\frac{-(\beta+yp^r q)x}{p^\ell aq}\right)e(-my)\,dy.$$

We now substitute the change of variable $(\beta + yp^r q)/N = z$ to obtain

$$= \frac{N}{p^r q}\sum_{m\in\mathbb{Z}}\left\{\sum_{\beta(p^r q)}\chi(\beta)e\left(-\frac{(\bar{a}+bq)\beta}{p^\ell q}+\frac{m\beta}{p^r q}\right)\right\}\int_{\mathbb{R}}V_1(y)e\left(\frac{-Nxz}{p^\ell aq}\right)e\left(\frac{-Nmz}{p^r q}\right)dz$$
$$:= \frac{N}{p^r q}\mathcal{C}(b,q)\mathcal{I}(x,q,m), \quad (11)$$

where $\mathcal{C}(b,q)$ is the character sum and $\mathcal{I}(x,q,m)$ is the integral in the above expression. We now first evaluate the character sum in the following subsection.

Evaluation of the character sum. Writing $q = p^{r_1}q'$ with $(p, q') = 1$, the character sum in (11) can be written as

$$\mathcal{C}(b,q) = \sum_{\beta(p^{r+r_1}q')} \chi(\beta) e\left(\frac{-(\bar{a}+bq)\beta}{p^{\ell+r_1}q'} + \frac{m\beta}{p^{r+r_1}q'}\right).$$

Writing $\beta = \alpha_1 q'\overline{q'} + \alpha_2 p^{r+r_1}\overline{p^{r+r_1}}$, the above character sum splits as

$$\sum_{\alpha_1(p^{r+r_1})} \chi(\alpha_1) e\left(\frac{-(\bar{a}+bq)\alpha_1\overline{q'}}{p^{\ell+r_1}} + \frac{m\alpha_1\overline{q'}}{p^{r+r_1}}\right) \sum_{\alpha_2(q')} e\left(\frac{-(\bar{a}+bq)\alpha_2\overline{p^{r+r_1}}\,p^{r-\ell}}{q'} + \frac{m\alpha_2\overline{p^{r+r_1}}}{q'}\right).$$

Again, writing $\alpha_1 = \beta_1 p^r + \beta_2$, where β_2 is modulo p^r and β_1 modulo p^{r_1}, we obtain

$$\mathcal{C}(b,q) = \sum_{\beta_2(p^r)} \chi(\beta_2) e\left(\frac{-(\bar{a}+bq)p^{r-\ell}\beta_2\overline{q'}}{p^{r+r_1}} + \frac{m\beta_2\overline{q'}}{p^{r+r_1}}\right) \sum_{\beta_1(p^{r_1})} e\left(\frac{-(\bar{a}+bq)\beta_1\overline{q'}\,p^{r-\ell}}{p^{r_1}} + \frac{m\beta_1\overline{q'}}{p^{r_1}}\right)$$
$$\sum_{\alpha_2(q')} e\left(\frac{-(\bar{a}+bq)\alpha_2\overline{p^{r+r_1}}\,p^{r-\ell}}{q'} + \frac{m\alpha_2\overline{p^{r+r_1}}}{q'}\right).$$

From the last two exponential sums, we obtain the congruence relations $m - \bar{a}p^{r-\ell} \equiv 0 \pmod{p^{r_i}}$ and $m - \bar{a}p^{r-\ell} \equiv 0 \pmod{p'}$. Since we have $q = q'p^{r_1}$, we obtain the congruence relation $m - \bar{a}p^{r-\ell} \equiv 0 \pmod 1$, from which $a \pmod q$ can be determined. The sum over β_2 can be written as

$$\sum_{\beta_2(p^r)} \chi(\beta_2) e\left(\frac{(m-(\bar{a}+bq)p^{r-\ell})\beta_2\overline{q'}}{p^{r+r_1}}\right) = \chi(q')\overline{\chi}\left(\frac{m-(\bar{a}+bq)p^{r-\ell}}{p^{r_1}}\right) \sum_{\beta_2(p^r)} \chi(\beta_2) e\left(\frac{\beta_2}{p^r}\right),$$

as $p^{r_1} \mid (m-(\bar{a}+bq)p^{r-\ell})$. We record this into following lemma.

Lemma 4.2. *Let $\mathcal{C}(b,q)$ be as given in* (11)*. We have*

$$\mathcal{C}(b,q) = \begin{cases} q\chi(q')\overline{\chi}\left(\frac{m-(\bar{a}+bq)p^{r-\ell}}{p^{r_1}}\right)\tau_\chi & \text{if } m \equiv \bar{a}p^{r-\ell} \mod q \\ 0 & \text{otherwise,} \end{cases}$$

where $q = q'p^{r_1}$ and τ_χ denotes the Gauss sum.

For simplicity of notation we assume that $q = q'(r_1 = 0)$, as the number of r_1 are bounded by $O(\log p^r)$. Next we consider the integral in (11). Integrating by parts j-times and using $V_1^{(j)}(y) \ll 1$, we have

$$\mathcal{I}(x,q,m) \ll \left(\frac{Nx}{p^\ell aq} + \frac{Nm}{p^r q}\right)^{-j}.$$

We observe that this integral is negligibly small if

$$\left|\frac{Nx}{p^\ell aq} + \frac{Nm}{p^r q}\right| \gg N^\varepsilon.$$

From the above inequality we obtain the effective range of x (in the integral originating from the circle method) as

$$\left|\frac{Nx}{p^\ell aq} + \frac{Nm}{p^r q}\right| \ll N^\varepsilon \Rightarrow \left|x + \frac{ma}{p^{r-\ell}}\right| \ll \frac{qN^\epsilon}{Q}, \tag{12}$$

as $a \asymp Q$ and $N/p^\ell = Q^2$. Again integrating by parts, taking $V_1(y)e((-Nxy)/(p^\ell aq))$ as the first function, we obtain

$$\mathcal{I}(x,q,m) \ll \left(1+\frac{Nx}{p^\ell aq}\right)^j \left(\frac{p^r q}{Nm}\right)^j.$$

Hence the integral is negligibly small if $m \gg (p^r Q N^\varepsilon)/N$. After a first application of the Poisson summation formula we are left with the following expression for $S^+(N)$:

Lemma 4.3. *Let $S^+(N)$ be as given in* (10). *We have*

$$S^+(N) = \int_{x \ll (qN^\epsilon)/Q} \sum_{1\le q\le Q} \frac{1}{aqp^\ell} \sum_{b(p^\ell)} \left\{ \sum_{n=1}^\infty \lambda_f(n) e\left(\frac{(\bar{a}+bq)n}{p^\ell q}\right) e\left(\frac{xn}{p^\ell aq}\right) V\left(\frac{n}{N}\right)\right\} \left\{\frac{\tau_\chi \chi(q) N}{p^r} \sum_{m \ll \frac{N^\varepsilon Q p^r}{N}} \bar{\chi}(m-(\bar{a}+bq)p^{r-\ell})\mathcal{I}(x,q,m)\right\} dx + O_A(p^{-A}), \quad (13)$$

for any real $A > 0$.

Estimating trivially at this stage, we have

$$S^+(N) \ll \sum_{1\le q\le Q} \frac{1}{aqp^\ell} \sum_{b(p^\ell)} \sum_{n=N}^{2N} |\lambda_f(n)| \frac{|\tau_\chi| N}{p^r} \sum_{m \ll \frac{N^\varepsilon Q p^r}{N}} 1 \ll N p^{r/2}.$$

Hence we are able to save $N/p^{r/2}$ from the first application of the Poisson summation formula.

Applying Voronoi summation formula. At this stage we need to differentiate between the holomorphic and the Maass case as the integral transforms appearing in the Voronoi summation formula are different. Nevertheless they are essentially same as far as our argument is concerned. Below we will present the details for the Maass case which is traditionally considered to be harder.

We have $(\bar{a}+bq, q) = 1$. Given a, there exists at most one b mod p^ℓ such that $\bar{a}+bq \equiv 0 \pmod{q^\ell}$. For the rest of b we apply the Voronoi summation formula to the sum over n as follows (The case where $\bar{a}+bq \equiv 0 \pmod{q^\ell}$ is similar and even simpler. We first write $\bar{a}+bq = p^\ell q_1$, and then apply the Voronoi summation formula, which gives us more saving as the conductor is now smaller than qp^ℓ. Also we have a savings of a whole summation over b modulo p^ℓ). We substitute $g(n) = e(-nx/p^\ell aq)V(n/N)$ in Lemma 3.1 to get

$$\sum_{n=1}^\infty \lambda_f(n) e\left(\frac{(\bar{a}+bq)n}{p^\ell q}\right) e\left(\frac{xn}{p^\ell aq}\right) V\left(\frac{n}{N}\right) = \frac{1}{p^\ell q} \sum_{\pm} \sum_{n\ge 1} \lambda_f(\mp n) e\left(\pm n \frac{\overline{\bar{a}+bq}}{p^\ell q}\right) H^\pm\left(\frac{n}{q^2}, \frac{Nx}{aq}\right),$$

where

$$H^-\left(\frac{n}{q^2}, \frac{Nx}{p^\ell aq}\right) = \int_0^\infty e\left(-\frac{xy}{p^\ell aq}\right) V\left(\frac{y}{N}\right) \{Y_{2i\nu} + Y_{-2i\nu}\} \left(\frac{4\pi\sqrt{ny}}{p^\ell q}\right) dy$$

(we have similar expression for $H^{+}(x, y)$). Substituting $y/N = z$, we have

$$H^{-}\left(\frac{n}{q^2}, \frac{Nx}{P_1 aq}\right) = N\int_0^\infty e\left(-\frac{Nxz}{p^\ell aq}\right)V\left(z\right)\{Y_{2i\nu}+Y_{-2i\nu}\}\left(\frac{4\pi\sqrt{nNz}}{p^\ell q}\right)dz := N\mathcal{J}(x,n,q), \quad (14)$$

where $\mathcal{J}(x,n,q)$ denotes the integral in above equation. Pulling out the oscillations, we have the following asymptotic formulae for Bessel functions (see [Kowalski et al. 2002, Lemma C.2]):

$$Y_{\pm 2i\nu}(x) = e^{ix}U_{\pm 2i\nu}(x) + e^{-ix}\overline{U}_{\pm 2i\nu}(x) \quad \text{and} \quad |x^k K_\nu^{(k)}(x)| \ll_{k,\nu} \frac{e^{-x}(1+\log|x|)}{(1+x)^{1/2}}, \quad (15)$$

where the function $U_{\pm 2i\nu}(x)$ satisfies,

$$x^j U_{\pm 2i\nu}^{(j)}(x) \ll_{j,\nu,k} (1+x)^{-1/2}. \quad (16)$$

We also have $J_k(x) = e^{ix}W_k(x) + e^{-ix}\overline{W}(x)$, where

$$x^j W_k^{(j)}(x) \ll_j \frac{1}{(1+x)^{1/2}}.$$

Substituting the above decomposition for $Y_{\pm 2i\nu}(x)$, the first term of the integral in (14) is given by (estimation of the second term is similar)

$$\mathcal{J}^{\pm}(x,n,q) := \int_0^\infty e^{i(-(2\pi Nxy)/(p^\ell aq)\pm i(4\pi\sqrt{nNy})/(p^\ell q))}V(y)U_{2i\nu}^{\pm}\left(\frac{4\pi\sqrt{nNy}}{p^\ell q}\right)dy, \quad (17)$$

where we have denoted $U^{+}(y) := U(y)$ and $U^{-}(y) = \overline{U}(y)$. The integral $\mathcal{J}^{-}(x,n,q)$ has no stationary point. By (8), $\mathcal{J}^{-}(x,n,q)$ is negligibly small. For $\mathcal{J}^{+}(x,n,q)$ we apply the second statement of Lemma 3.4 with

$$f(y) = -\frac{2\pi Nxy}{p^\ell aq} + i\frac{4\pi\sqrt{nNy}}{p^\ell q} \quad \text{and} \quad g(y) = V(y)U_{2i\nu}\left(\frac{4\pi\sqrt{nNy}}{p^\ell q}\right).$$

We have

$$f'(y) = -\frac{2\pi Nx}{p^\ell aq} + \frac{2\pi\sqrt{nN}}{\sqrt{y}\,p^\ell q}, \qquad f''(y) = -\frac{\pi\sqrt{nN}}{y^{3/2}p^\ell q}.$$

We observe that

$$|F''(y_0)| \asymp \frac{\sqrt{nN}}{p^\ell q},$$

where $y_0 = na^2/(Nx^2)$ is the stationary point, which is $y_0 \asymp 1$ as $V(y)$ is supported on the interval [1, 2]. Using $U_{\pm 2i\nu}(x) \ll_\nu (1+x)^{-1/2}$, and applying the second statement of Lemma 3.4, we obtain

$$\mathcal{J}(x,n,q) \ll \frac{p^\ell q}{\sqrt{nN}}, \quad (18)$$

where $\mathcal{J}(x,n,q)$ is given in (14). Also, integrating by parts we have

$$\mathcal{J}(x,n,q) \ll_j \left(\frac{Nx}{p^\ell aq}+1\right)^j \left(\frac{p^\ell q}{\sqrt{nN}}\right)^j.$$

The integral is negligibly small if (note that $x \ll qN^{\epsilon}/Q$)

$$\frac{p^{\ell}q}{\sqrt{nN}} \ll p^{-\varepsilon} \Rightarrow n \gg p^{\varepsilon}p^{\ell}.$$

We record this result in the following lemma. After applying the Poisson and the Voronoi summation formula we have the following expression for $S^{+}(N)$.

Lemma 4.4. *We have*

$$S^{+}(N) = \int_{x \ll (qN^{\varepsilon})/Q} \sum_{1\leq q\leq Q} \frac{1}{qp^{\ell}} \sideset{}{^\star}\sum_{b(p^{\ell})} \frac{\tau_{\chi}\chi(q)N}{p^{r}} \sum_{m \ll \frac{Qp^{r}p^{\varepsilon}}{N}} \tfrac{1}{a}\overline{\chi}(m-(\bar{a}+bq)p^{r-\ell})\mathcal{I}(x,q,m)$$
$$\left\{\frac{N}{p^{\ell}q}\sum_{\pm}\sum_{n\ll p^{\ell}p^{\varepsilon}} \lambda_f(\mp n)e\left(\pm n\frac{\overline{\bar{a}+bq}}{p^{\ell}q}\right)\mathcal{J}(x,n,q)\right\}dx + O_A(p^{-A}).$$

Estimating trivially, we have (assuming square-root cancellation in the character sum over b and Lemma 3.2):

$$\begin{aligned}S^{+}(N) &\ll \sum_{1\leq q\leq Q} \frac{1}{qp^{\ell}}\frac{|\tau_{\chi}|N}{p^{r}} \sum_{m\ll \frac{p^{\varepsilon}Qp^{r}}{N}} \frac{|\mathcal{I}(x,q,m)|}{a}\frac{N}{p^{\ell}q}\sum_{n\ll p^{\ell}N^{\varepsilon}} |\lambda_f(\mp n)\mathcal{J}(x,n,q)| \\ &\qquad \left|\sideset{}{^\star}\sum_{b(p^{\ell})} e\left(\pm n\frac{\overline{\bar{a}+bq}}{p^{\ell}q}\right)\overline{\chi}\left(m-(\bar{a}+bq)p^{r-\ell}\right)\right| \\ &\ll \frac{1}{ap^{\ell}}\frac{p^{r/2}N}{p^{r}}\frac{N^{\varepsilon}Qp^{r}}{N}\times\frac{N}{p^{\ell}q}\sum_{n\ll p^{\ell}}\frac{p^{\ell}q}{\sqrt{nN}}\times p^{\ell/2} \\ &\ll N^{\varepsilon}\sqrt{N}p^{r/2}.\end{aligned}$$

This shows that we are on the boundary. To obtain an additional saving, we shall now apply the Cauchy–Schwarz inequality to the summation over n and then apply the Poisson summation formula. Interchanging the order of summation, we have

$$S^{+}(N) = \frac{N^{2}\tau_{\chi}}{p^{r+2\ell}} \sum_{n\ll p^{\ell}p^{\varepsilon}} \lambda_f(n)\hat{S}_1(n) + O_A(p^{-A}), \tag{19}$$

where

$$\hat{S}_1(n) = \int_{x\ll (qN^{\epsilon})/Q} \sum_{1\leq q\leq Q} \sideset{}{^\star}\sum_{b(p^{\ell})} \sum_{m\ll \frac{p^{\varepsilon}Qp^{r}}{N}}$$
$$\frac{\chi(q)}{aq^{2}}\overline{\chi}(m-(\bar{a}+bq)p^{r-\ell})e\left(-n\frac{\overline{a+bq\bar{q}}}{p^{\ell}}\right)e\left(-n\frac{p^{r}\,\overline{p^{2\ell}}\,\overline{m}}{q}\right)\mathcal{I}(x,q,m)\mathcal{J}(x,n,q)\,dx.$$

5. Proof of Theorem 1.2 — Conclusion

In the previous section we have completed the first three steps of the proof as given in the short sketch. As expected we are at the threshold and any saving will yield subconvexity. We now apply Cauchy–Schwarz to escape from the "trap of involution" and to get rid of the Fourier coefficients.

Applying Cauchy inequality. We split the summation over n into dyadic sum. Applying the Cauchy–Schwarz inequality on the summation over n in (19) and using Lemma 3.2, we have

$$\begin{aligned} S^{+}(N) &\ll \frac{N^2|\tau_\chi|}{p^{r+2\ell}} \sum_{\substack{L\ll P_1\\ L\text{-dyadic}}} \Big\{\sum_{n\ll L}|\lambda_f(n)|^2\Big\}^{1/2}\Big\{\sum_{n\in\mathbb{Z}}|\hat{S}_1(n)|^2 U\Big(\frac{n}{L}\Big)\Big\}^{1/2} \\ &\ll \frac{N^2|\tau_\chi|}{p^{r+2\ell}} \sum_{\substack{L\ll P_1\\ L\text{-dyadic}}} L^{1/2}\{\hat{S}_2(L)\}^{1/2}, \end{aligned} \tag{20}$$

where $P_1 = p^{\ell+\varepsilon}$ and $\hat{S}_2(L)$ is given by (opening the absolute square and pushing the summation over n inside):

$$\begin{aligned} \hat{S}_2(L) := \int_{x\ll (qN^\epsilon)/Q}\int_{x'\ll (q'N^\epsilon)/Q} \sum_{m\ll \frac{N^\varepsilon Qp^r}{N}}\sum_{m'\ll \frac{N^\varepsilon Qp^r}{N}}\sum_{1\le q\le Q}\sum_{1\le q'\le Q} \frac{\chi(q)}{aq^2}\frac{\bar{\chi}(q')}{a'q'^2}\sideset{}{^\star}\sum_{b(p^\ell)}\sideset{}{^\star}\sum_{b'(p^\ell)} \\ \bar{\chi}(m-(\bar{a}+bq)p^{r-\ell})\chi(m'-(\bar{a}+b'q')p^{r-\ell})\mathfrak{I}(x,q,m)\mathfrak{I}(x,q',m')\mathcal{T}\,dx\,dx', \end{aligned} \tag{21}$$

where

$$\mathcal{T} := \sum_{n\in\mathbb{Z}} e\bigg(n\bigg\{-\frac{p^r\,\overline{p^{2\ell}}\,\overline{m}}{q}+\frac{p^r\,\overline{p^{2\ell}}\,\overline{m'}}{q'}-\frac{\overline{a+bq}\,\bar{q}}{p^\ell}+\frac{\overline{a+b'q'}\,\overline{q'}}{p^\ell}\bigg\}\bigg)U\Big(\frac{n}{L}\Big)\mathcal{J}(x,n,q)\mathcal{J}(x,n,q'). \tag{22}$$

Second application of Poisson summation formula. We write a smooth bump function

$$U(n/L)\mathcal{J}(x,n,q)\mathcal{J}(x,n,q') := U_1(n/L),$$

where $\mathcal{J}(x,n,q)$ is as given in (14). Writing $n=\alpha+qq'p^\ell c$, $c\in\mathbb{Z}$ and applying Poisson summation formula to sum over c, we have

$$\mathcal{T} := \sum_{\alpha(qq'p^\ell)} e\bigg(\alpha\bigg\{-\frac{p^r\,\overline{p^{2\ell}}\,\overline{m}}{q}+\frac{p^r\,\overline{p^{2\ell}}\,\overline{m'}}{q'}-\frac{\overline{a+bq}\,\bar{q}}{p^\ell}+\frac{\overline{a+bq'}\,\overline{q'}}{p^\ell}\bigg\}\bigg)\int_{\mathbb{R}}\sum_{n\in\mathbb{Z}}U_1\Big(\frac{\alpha+yqq'p^\ell}{L}\Big)e(-ny)\,dy.$$

We now apply the change of variable $(\alpha+yqq'p^\ell)/L=z$ to get

$$\begin{aligned} \mathcal{T} = \frac{L}{qq'p^\ell}\sum_{n\in\mathbb{Z}}\sum_{\alpha(qq'p^\ell)} e\bigg(\alpha\bigg\{-\frac{p^r\,\overline{p^{2\ell}}\,\overline{m}}{q}+\frac{p^r\,\overline{p^{2\ell}}\,\overline{m'}}{q'}-\frac{\overline{a+bq}\,\bar{q}}{p^\ell}+\frac{\overline{a+bq'}\,\overline{q'}}{p^\ell}+\frac{n}{qq'p^\ell}\bigg\}\bigg) \\ \int_{\mathbb{R}}U_1\Big(y\Big)e\Big(-\frac{nLy}{qq'p^\ell}\Big)\,dy. \end{aligned} \tag{23}$$

We have $U_1(y) = U(y)\mathcal{J}(x, Ly, q)\mathcal{J}(x, Ly, q')$. From the expression of $\mathcal{J}(x, Lu, q)$ in (14) (note that after change of variable we have $u \asymp 1$) and (16), we have

$$\begin{aligned}\frac{\partial}{\partial u}\mathcal{J}(x, Lu, q) &= \int_0^\infty e\Big(-\frac{Nxy}{p^\ell aq}\Big) V(y)\frac{\partial}{\partial u}\{Y_{2i\nu} + Y_{-2i\nu}\}\Big(\frac{4\pi\sqrt{LuNy}}{p^\ell q}\Big)\, dy\\ &= \int_0^\infty e\Big(-\frac{Nxy}{p^\ell aq}\Big) V(y)\frac{1}{u}\frac{4\pi\sqrt{LuNy}}{p^\ell q}\{Y'_{2i\nu} + Y'_{-2i\nu}\}\Big(\frac{4\pi\sqrt{LuNy}}{p^\ell q}\Big)\, dy\\ &\ll 1.\end{aligned}$$

This shows that there is no oscillation in the function $\mathcal{J}(x, Ln, q)$ with respect to variable n. Similarly, higher order derivatives of $\mathcal{J}(x, Ln, q)$ with respect to u are bounded. Also from (18) we have

$$\begin{aligned}\int_{\mathbb{R}} U_1(y) e\Big(-\frac{nLy}{qq'p^\ell}\Big)\, dy &= \int_{\mathbb{R}} U(y)\mathcal{J}(x, Ly, q)\mathcal{J}(x, Ly, q') e\Big(-\frac{nLy}{qq'p^\ell}\Big)\, dy\\ &\ll \frac{p^\ell q}{\sqrt{LN}}\frac{p^\ell q'}{\sqrt{LN}}\int_1^2 U(y)\, dy\\ &\ll \frac{p^{2\ell} qq'}{LN},\end{aligned} \tag{24}$$

as $U(y)$ is supported on the interval $[1, 2]$.

Integrating by parts taking $U_1(y)$ as first function, we observe that the integral in (23) is negligible if $n \gg p^\varepsilon qq'p^\ell/L$. Evaluating the above character sum we get the following congruence relation:

$$-p^r\,\overline{p^{2\ell}}\,\overline{m}\, p^\ell q' + p^r\,\overline{p^{2\ell}}\,\overline{m'}\, p^\ell q - \overline{a + bq}\,\bar{q}qq' + \overline{a + b'q'}\,\overline{q'}qq' + n \equiv 0 \pmod{qq'p^\ell}.$$

Here $\bar{q}$ and $\overline{q'}$ are the inverses of q and q' modulo p^ℓ respectively. We solve the above congruence modulo p^ℓ and modulo qq' respectively to obtain

$$-\overline{a+bq}\,q' + \overline{a+b'q'}\,q + n \equiv 0 \pmod{p^\ell} \quad\text{and}\quad -p^r\,\overline{p^{2\ell}}\,\overline{m}\,p^\ell q' + p^r\,\overline{p^{2\ell}}\,\overline{m'}\,p^\ell q + n \equiv 0 \pmod{qq'}. \tag{25}$$

Writing $n = -p^r\,\overline{p^{2\ell}}\,\overline{m}\,p^\ell q' + p^r\,\overline{p^{2\ell}}\,\overline{m'}\,p^\ell q + jqq'$, we observe that the number of n satisfying the above congruence relation is same as the number of j's. Since we also have $n \ll N^\varepsilon qq'p^\ell/L$, we conclude $j \ll N^\varepsilon$. Hence the number of solutions of n satisfying the above congruence relation modulo qq', and $n \ll qq'N^\varepsilon p^\ell/L$ is bounded by $N^\varepsilon p^\ell/L$. For congruence relation modulo p^ℓ in the above equation, we substitute the change of variable $a + bq = \alpha$ and $a + b'q' = \alpha'$ to obtain

$$\bar{\alpha}q' + \overline{\alpha'}q + n \equiv 0 \pmod{p^\ell}. \tag{26}$$

We record the bound for $\mathcal{T}$ in the following lemma:

Lemma 5.1. *Let $\mathcal{T}$ be as given in* (22)*. We have*

$$\mathcal{T} = L \sideset{}{^\dagger}\sum_{n \ll p^\varepsilon qq'p^\ell/L} \int_{\mathbb{R}} U_1(y) e\Big(-\frac{nLy}{qq'p^\ell}\Big)\, dy,$$

where $\dagger$ in the above summation denotes that n satisfies the congruence relation given in (25)*.*

Substituting the bound for $\mathcal{T}$ in (21) we obtain

$$\begin{aligned}\hat{S}_2(L)=L\int_{x\ll(qN^{\epsilon})/Q}\int_{x'\ll(q'N^{\epsilon})/Q}\sum_{m\ll\frac{N^{\varepsilon}Qp^r}{N}}\sum_{m'\ll\frac{N^{\varepsilon}Qp^r}{N}}\sum_{1\le q\le Q}\sum_{1\le q'\le Q}\frac{\chi(q)}{aq^2}\frac{\overline{\chi}(q')}{a'q'^2}\sum_{n\ll p^{\varepsilon}qq'p^{\ell}/L}^{\dagger}\sum_{\alpha(p^{\ell})}^{\star}\sum_{\alpha'(p^{\ell})}^{\star}\\ \overline{\chi}(m-\alpha p^{r-\ell})\chi(m'-\alpha' p^{r-\ell})\mathfrak{I}(x,q,m)\mathfrak{I}(x,q',m')\int_{\mathbb{R}}U_1(y)e\left(-\frac{nLy}{qq'p^{\ell}}\right)dy\,dx\,dx'\\ :=\hat{S}_2(D)+\hat{S}_2(ND),\end{aligned}\tag{27}$$

where α and α' are related by the congruence relation given in (26), and $\hat{S}_2(D)$ (respectively $\hat{S}_2(ND)$) is contribution of the diagonal terms (respectively the off-diagonal terms). The contribution of the diagonal terms ($\alpha=\alpha'$, $m=m'$ and $q=q'$) is bounded by (using $\mathfrak{I}(x,q,m)\ll 1$, bound from the (24) and sum over n satisfying the congruence relation given in (25) is bounded by $p^{\epsilon}p^{\ell}/N$):

$$\begin{aligned}\hat{S}_2(D)&\ll L\int_{x\ll(qN^{\epsilon})/Q}\int_{x'\ll(q'N^{\epsilon})/Q}\sum_{m\ll\frac{Qp^r}{N}}\sum_{1\le q\le Q}\frac{1}{q^4}\sum_{n\ll p^{\varepsilon}qq'p^{\ell}/L}^{\dagger}\sum_{\alpha(p^{\ell})}\frac{|\mathfrak{I}(x,q,m)|^2}{a^2}\int_{\mathbb{R}}|U_1(y)|\,dy\,dx\,dx'\\ &\ll LN^{\varepsilon}\int_{x\ll q/Q}\frac{Qp^r}{N}\sum_{1\le q\le Q}\sum_{n\ll p^{\varepsilon}qq'p^{\ell}/N}^{\dagger}\frac{1}{a^2q^4}p^{\ell}\frac{p^{2\ell}q^2}{LN}\,dx\\ &\ll\frac{p^{3\ell}N^{\varepsilon}}{N}\frac{Qp^r}{N}\frac{p^{\ell}}{L}\sum_{1\le q\le Q}\frac{1}{a^2q^2}\times\frac{q}{Q}\\ &\ll\frac{p^{3\ell}N^{\varepsilon}}{Q^2N}\frac{p^r}{N}\frac{p^{\ell}}{L},\end{aligned}\tag{28}$$

as $a\asymp Q$. Substituting the value of α' from the congruence relation given in (26), we see that the contribution of the off-diagonal term is given by:

$$\begin{aligned}\hat{S}_2(ND)=L\int_{x\ll(qN^{\epsilon})/Q}\int_{x'\ll(q'N^{\epsilon})/Q}\sum_{m\ll Q}\sum_{m'\ll Q}\sum_{1\le q\le Q}\sum_{1\le q'\le Q}\frac{\chi(q)}{aq^2}\frac{\overline{\chi}(q')}{a'q'^2}\sum_{n\ll p^{\varepsilon}qq'p^{\ell}/N}^{\dagger}\\ \left\{\sum_{\alpha(p^{\ell})}^{\star}\overline{\chi}(m-\alpha p^{r-\ell})\chi(m'+\alpha q\overline{n+q'}p^{r-\ell})\right\}\mathfrak{I}(x,q,m)\mathfrak{I}(x,q',m')\\ \int_{\mathbb{R}}U_1(y)e\left(-\frac{nLy}{qq'p^{\ell}}\right)dy\,dx\,dx'.\end{aligned}\tag{29}$$

Next we evaluate the exponential sum in the above equation.

Evaluation of the character sum. In this subsection we shall prove the following lemma

Lemma 5.2. *Let $\mathcal{A}$ be the character sum given by*

$$\mathcal{A}:=\sum_{\alpha(p^{\ell})}^{\star}\overline{\chi}(m-\alpha p^{r-\ell})\chi(m'+\alpha q\overline{n+q'}p^{r-\ell}),$$

with $\ell=2\lfloor\frac{r}{3}\rfloor$. We have

$$\mathcal{A}\ll p^{\ell/2+\epsilon}.$$

Proof. Applying the change of variable $\alpha = \alpha_1 p^{\ell/2} + \alpha_2$ where α_1 and α_2 run over residue classes modulo $p^{\ell/2}$, the above character sum reduces to

$$\begin{aligned}\mathcal{A} &= \sideset{}{^\star}\sum_{\alpha_2(p^{\ell/2})}\sideset{}{^\star}\sum_{\alpha_1(p^{\ell/2})} \overline{\chi}(m-\alpha_2 p^{r-\ell}-\alpha_1 p^{2(r-\ell)})\chi(m'+(\alpha_1 p^{r-\ell}+\alpha_2)q\overline{n+q'}p^{r-\ell})\\ &= \sideset{}{^\star}\sum_{\alpha_2(p^{\ell/2})}\sideset{}{^\star}\sum_{\alpha_1(p^{\ell/2})} \chi\{(m'+q\overline{n+q'}p^{r-\ell}\alpha_2+q\overline{n+q'}\alpha_1 p^{2(r-\ell)})(\overline{m-\alpha_2 p^{r-\ell}}+(\overline{m-\alpha_2 p^{r-\ell}})^2\alpha_1 p^{2(r-\ell)})\},\end{aligned}$$

as $\overline{m-\alpha_2 p^{r-\ell}-\alpha_1 p^{2(r-\ell)}} = \overline{m-\alpha_2 p^{r-\ell}} + (\overline{m-\alpha_2 p^{r-\ell}})^2\alpha_1 p^{2(r-\ell)} \pmod{p^r}$. Which reduces to

$$\begin{aligned}\mathcal{A} &= \sideset{}{^\star}\sum_{\alpha_2(p^{\ell/2})}\sideset{}{^\star}\sum_{\alpha_1(p^{\ell/2})} \chi(A(\alpha_2)+B(\alpha_2)\alpha_1 p^{2(r-\ell)})\\ &= \sideset{}{^\star}\sum_{\alpha_2(p^{\ell/2})} \chi(A(\alpha_2)) \sideset{}{^\star}\sum_{\alpha_1(p^{\ell/2})} \chi(1+\overline{A(\alpha_2)}B(\alpha_2)\alpha_1 p^{2(r-\ell)}),\end{aligned}$$

where

$$A(\alpha_2) = m'\overline{m-\alpha_2 p^{r-\ell}} + q\overline{n+q'}\alpha_2 p^{r-\ell}\overline{m-\alpha_2 p^{r-\ell}} \quad \text{and} \quad B(\alpha_2) = m'(\overline{m-\alpha_2 p^{r-\ell}})^2 + q\overline{n+q'}\,\overline{m-\alpha_2 p^{r-\ell}}.$$

Note that $(A(\alpha_2),\, p) = 1$, otherwise $\chi(A(\alpha_2)+B(\alpha_2)\alpha_1 p^{2r/3}) = 0$. For a fixed α_2,

$$\chi(1+\overline{A(\alpha_2)}B(\alpha_2)\alpha_1 p^{2(r-\ell)}) := \chi(1+C(\alpha_2)\alpha_1 p^{2(r-\ell)})$$

is an additive character of modulus $p^{\ell/2}$, as we have

$$\chi(1+C(\alpha_2)\alpha_1 p^{2(r-\ell)})\chi(1+C(\alpha_2)\alpha_1' p^{2(r-\ell)}) = \chi(1+C(\alpha_2)(\alpha_1+\alpha_1')p^{2(r-\ell)}),$$

as we have $4(r-\ell) \geq r$. Hence there exists an integer b (uniquely determined modulo $p^{\ell/2}$) such that

$$\chi(1+C(\alpha_2)\alpha_1 p^{2(r-\ell)}) = e\left(\frac{\alpha_1 b C(\alpha_2)}{p^\ell}\right).$$

Executing the sum over α_1 given in Lemma 5.2 we have

$$\mathcal{A} = p^{\ell/2} \sideset{}{^\star}\sum_{\substack{\alpha_2(p^{\ell/2})\\ bC(\alpha_2)\equiv 0 \pmod{p^{\ell/2}}}} \chi(A(\alpha_2)) \ll p^{\ell/2+\varepsilon}. \tag{30}$$

This concludes the proof. □

Substituting the bound for the character sum in (29) and using the bounds of $U_1(y)$ given in (24), we have

$$\begin{aligned}\hat{S}_2(ND) &\ll p^\varepsilon L \sum_{m\ll\frac{Qp^r}{N}}\sum_{m'\ll\frac{Qp^r}{N}}\sum_{1\leq q\leq Q}\sum_{1\leq q'\leq Q}\sideset{}{^\dagger}\sum_{n\ll p^\varepsilon qq'p^\ell/L} \frac{1}{aq^2}\frac{1}{a'q'^2}p^{\ell/2}\frac{p^{2\ell}qq'}{LN}\\ &\ll p^\varepsilon \frac{p^{5\ell/2}}{Q^2N}\left(\frac{Qp^r}{N}\right)^2\frac{p^\ell}{L} \ll p^\varepsilon\frac{p^{3\ell/2}}{Q^2}\left(\frac{p^r}{N}\right)^2\frac{p^\ell}{L},\end{aligned} \tag{31}$$

as $a, a' \asymp Q$, $Q^2 = N/p^\ell$ and dagger on summation over n shows that n satisfies the congruence relation modulo qq' as given in (25). Substituting the bounds for $\hat{S}_2(D)$ and $\hat{S}_2(ND)$ in (27) we have

$$\hat{S}_2(L) \ll p^\varepsilon \frac{p^\ell}{L}\left(\frac{p^{3\ell}}{Q^2N}\frac{p^r}{N} + \frac{p^{3\ell/2}}{Q^2}\frac{p^{2r}}{N^2}\right) \ll p^\varepsilon \frac{p^\ell}{LQ^2N^2} p^{\frac{3\ell}{2}} p^r (p^{3\ell/2} + p^r).$$

Substituting the bound for $\hat{S}_2(L)$ in (20) we obtain

$$\begin{aligned} S_2^+(N) &\ll p^\varepsilon \frac{N^2|\tau_\chi|}{p^{r+2\ell}} \sum_{\substack{L \ll P_1 \\ L\text{-dyadic}}} L^{1/2} \times \frac{p^{\ell/2}}{\sqrt{L}QN} p^{3\ell/4} p^{r/2} (p^{3\ell/4} + p^{r/2}) \\ &\ll p^\varepsilon \frac{N}{Qp^{3\ell/4}} (p^{3\ell/4} + p^{r/2}) \\ &\ll p^\varepsilon N^{1/2}\left(p^{\ell/2} + \frac{p^{r/2}}{p^{\ell/4}}\right) \\ &\ll p^\varepsilon N^{1/2} p^{1/2(r - \lfloor r/3 \rfloor) + \varepsilon}, \end{aligned} \tag{32}$$

as $\ell = 2\lfloor \frac{r}{3} \rfloor$ and $Q = N^{1/2}/p^{\ell/2}$. This proves Proposition 4.1. □

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