

Algebra & Number Theory

Volume 13

2019

No. 6

Congruences of parahoric group schemes

Radhika Ganapathy



Congruences of parahoric group schemes

Radhika Ganapathy

Let F be a nonarchimedean local field and let T be a torus over F . With \mathcal{T}^{NR} denoting the Néron–Raynaud model of T , a result of Chai and Yu asserts that the model $\mathcal{T}^{NR} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ is canonically determined by $(\mathrm{Tr}_l(F), \Lambda)$ for $l \gg m$, where $\mathrm{Tr}_l(F) = (\mathfrak{O}_F/\mathfrak{p}_F^l, \mathfrak{p}_F/\mathfrak{p}_F^{l+1}, \epsilon)$ with ϵ denoting the natural projection of $\mathfrak{p}_F/\mathfrak{p}_F^{l+1}$ on $\mathfrak{p}_F/\mathfrak{p}_F^l$, and $\Lambda := X_*(T)$. In this article we prove an analogous result for parahoric group schemes attached to facets in the Bruhat–Tits building of a connected reductive group over F .

1. Introduction

Let F be a nonarchimedean local field, \mathfrak{O}_F its ring of integers, and \mathfrak{p}_F its maximal ideal. Let T be a torus over F . Such a torus is canonically determined by the lattice $\Lambda := X_*(T)$ together with the action of $\Gamma_F = \mathrm{Gal}(F_s/F)$ on it (here F_s is a separable closure of F). For large m , the action of Γ_F on Λ factors through the quotient Γ_F/I_F^m of Γ_F , where I_F^m is the m -th higher ramification subgroup (with upper numbering) of the inertia group I_F . This Galois group depends only on truncated data $\mathrm{Tr}_m(F) := (\mathfrak{O}_F/\mathfrak{p}_F^m, \mathfrak{p}_F/\mathfrak{p}_F^{m+1}, \epsilon)$, where ϵ is the natural projection of $\mathfrak{p}_F/\mathfrak{p}_F^{m+1}$ on $\mathfrak{p}_F/\mathfrak{p}_F^m$, via Deligne’s theory; see (b) below.

Let \mathcal{T}^{NR} denote the Néron–Raynaud model of T (see [Bosch et al. 1990]). The main result of [Chai and Yu 2001] asserts that $\mathcal{T}^{NR} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ is canonically determined by $(\mathrm{Tr}_l(F), \Lambda)$ for $l \gg m$ (see Theorem 8.5 of [Chai and Yu 2001] for the precise statement; the parameters that l depends on are also explicitly determined there). With \mathcal{T} denoting the neutral component of \mathcal{T}^{NR} this also implies that $\mathcal{T} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ is canonically determined by $(\mathrm{Tr}_l(F), \Lambda)$ with l as above. From the point of view of Bruhat–Tits theory, when the connected reductive group is a torus, the model \mathcal{T} can be thought of as its Iwahori (or parahoric) group scheme. The purpose of this article is to prove an analogous result for parahoric group schemes attached to facets in the Bruhat–Tits building of a connected reductive group over F .

Our motivation for proving such a result arises naturally from the question of generalizing Kazhdan’s theory of studying representation theory of split p -adic groups over close local fields to general connected reductive groups. Let us briefly recall the Deligne–Kazhdan correspondence:

- (a) Given a local field F' of characteristic p and an integer $m \geq 1$, there exists a local field F of characteristic 0 such that F' is m -close to F , i.e., $\mathfrak{O}_F/\mathfrak{p}_F^m \cong \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$.

MSC2010: primary 22E50; secondary 11F70.

Keywords: parahoric, close local fields.

(b) Deligne [1984] proved that if $\mathrm{Tr}_m(F) \cong \mathrm{Tr}_m(F')$, then the Galois groups $\mathrm{Gal}(F_s/F)/I_F^m$ and $\mathrm{Gal}(F'_s/F')/I_{F'}^m$ are isomorphic. This gives a bijection

$$\begin{aligned} & \{\text{Iso. classes of cont., complex, f.d. representations of } \mathrm{Gal}(F_s/F) \text{ trivial on } I_F^m\} \\ & \longleftrightarrow \{\text{Iso. classes of cont., complex, f.d. representations of } \mathrm{Gal}(F'_s/F') \text{ trivial on } I_{F'}^m\}. \end{aligned}$$

Moreover, all of the above holds when $\mathrm{Gal}(F_s/F)$ is replaced by W_F , the Weil group of F .

(c) Let G be a split, connected reductive group defined over \mathbb{Z} . For an object X associated to the field F , we will use the notation X' to denote the corresponding object over F' . Kazhdan [1986] proved that given $m \geq 1$, there exists $l \geq m$ such that if F and F' are l -close, then there is an algebra isomorphism $\mathrm{Kaz}_m : \mathcal{H}(G(F), K_m) \rightarrow \mathcal{H}(G(F'), K'_m)$, where K_m is the m -th usual congruence subgroup of $G(\mathfrak{O}_F)$. Hence, when the fields F and F' are sufficiently close, we have a bijection

$$\begin{aligned} & \{\text{Iso. classes of irr. admissible representations } (\Pi, V) \text{ of } G(F) \text{ such that } \Pi^{K_m} \neq 0\} \\ & \longleftrightarrow \{\text{Iso. classes of irr. admissible representations } (\Pi', V') \text{ of } G(F') \text{ such that } \Pi'^{K'_m} \neq 0\}. \end{aligned}$$

These results suggest that, if one understands the representation theory of $\mathrm{Gal}(F_s/F)$ for all local fields F of characteristic 0, then one can use it to understand the representation theory of $\mathrm{Gal}(F'_s/F')$ for a local field F' of characteristic p , and similarly, with an understanding of the representation theory of $G(F)$ for all local fields F of characteristic 0, one can study the representation theory of $G(F')$, for F' of characteristic p . This method has proved useful for studying the local Langlands correspondence for reductive p -adic groups in characteristic p via the corresponding theory in characteristic 0 (see [Badulescu 2002; Lemaire 2001; Ganapathy 2015; Aubert et al. 2016; Ganapathy and Varma 2017]). An obvious observation, that goes into proving the Kazhdan isomorphism, is

$$G(\mathfrak{O}_F)/K_m \cong G(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong G(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m) \cong G(\mathfrak{O}_{F'})/K'_m \quad (1-1)$$

if the fields F and F' are m -close.

A useful variant of the Kazhdan isomorphism is now available for split reductive groups. Let I be the standard Iwahori subgroup of G . It is shown in [Bruhat and Tits 1984] that there is a smooth affine group scheme \mathcal{I} defined over \mathfrak{O}_F with generic fiber $G \times_{\mathbb{Z}} F$ such that $\mathcal{I}(\mathfrak{O}_F) = I$. Define $I_m := \mathrm{Ker}(\mathcal{I}(\mathfrak{O}_F) \rightarrow \mathcal{I}(\mathfrak{O}_F/\mathfrak{p}_F^m))$. In Section 3 of [Ganapathy 2015], a presentation has been written down for this Hecke algebra $\mathcal{H}(G, I_m)$ (extending Theorem 2.1 of [Howe 1985] for GL_n). Furthermore if the fields F and F' are m -close, an argument of J.K. Yu (see Section 3.4.A of [Ganapathy 2015]) gives an isomorphism

$$\beta : I/I_m \rightarrow I'/I'_m. \quad (1-2)$$

Let us note here that unlike (1-1), the above isomorphism is not obvious since the group scheme \mathcal{I} is defined over \mathfrak{O}_F and not over \mathbb{Z} . In fact the above isomorphism is obtained by proving that the reduction $\mathcal{I} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ depends only on $\mathrm{Tr}_m(F)$ and then evaluating it at the $\mathfrak{O}_F/\mathfrak{p}_F^m$ -points. Using the presentation and this isomorphism, one gets an obvious map $\zeta_m : \mathcal{H}(G(F), I_m) \rightarrow \mathcal{H}(G(F'), I'_m)$, when

the fields F and F' are m -close (also see [Lemaire 2001] for GL_n), which was shown in [Ganapathy 2015] to be an isomorphism of rings. Hence we obtain a bijection

$$\begin{aligned} & \{\text{Iso. classes of irr. ad. representations } (\Pi, V) \text{ of } G(F) \text{ with } \Pi^{I_m} \neq 0\} \\ & \longleftrightarrow \{\text{Iso. classes of irr. ad. representations } (\Pi', V') \text{ of } G(F') \text{ with } \Pi'^{I'_m} \neq 0\}. \end{aligned}$$

When one wants to prove the Kazhdan isomorphism or its variant for general connected reductive groups, one is naturally led to consider parahoric subgroups, study the reduction of the underlying parahoric group schemes mod \mathfrak{p}_F^m , and prove that they are determined by truncated data. That is the goal of the present article. Our proof is different from J.K.Yu's approach of proving (1-2) for the Iwahori group scheme of a split p -adic group. We will use the construction of the parahoric group scheme via the Artin–Weil theorem (see [Landvogt 1996]). Let us summarize the main results of this paper.

First, given a split connected reductive group over \mathbb{Z} , one can unambiguously work with this group over an arbitrary field after base change. More generally, given a connected reductive group G over F , we first need to make sense of what it means to give a group G' over F' where F' is suitably close to F . Let us first explain how this is done for quasisplit groups. Let (R, Δ) be a based root datum and let $(G_0, T_0, B_0, \{u_\alpha\}_{\alpha \in \Delta})$ be a pinned, split, connected, reductive \mathbb{Z} -group with based root datum (R, Δ) . We know that the F -isomorphism classes of quasisplit groups G_q that are F -forms of G_0 are parametrized by the pointed cohomology set $H^1(\Gamma_F, \mathrm{Aut}(R, \Delta))$ (see Theorem 3.2). Let $E_{qs}(F, G_0)_m$ be the set of F -isomorphism classes of quasisplit groups G_q that split (and become isomorphic to G_0) over an at most m -ramified extension of F . It is easy to see that this is parametrized by the cohomology set $H^1(\Gamma_F/I_F^m, \mathrm{Aut}(R, \Delta))$. Using the Deligne isomorphism, we prove that there is a bijection $E_{qs}(F, G_0)_m \rightarrow E_{qs}(F', G'_0)_m$, $G_q \rightarrow G'_q$, provided F and F' are m -close. Moreover, with the cocycles chosen compatibly, this will yield data (G_q, T_q, B_q) over F (where T_q is a maximal F -torus and B_q is an F -Borel containing T_q), and correspondingly (G'_q, T'_q, B'_q) over F' , together with an isomorphism $X_*(T_q) \rightarrow X_*(T'_q)$ that is Del_m -equivariant (see Lemma 3.4). It is a simple observation that the maximal F -split subtorus S_q of T_q is a maximal F -split torus in G_q (see Lemma 4.1). We prove that there is a simplicial isomorphism between the apartments $\mathcal{A}_m : \mathcal{A}(S_q, F) \rightarrow \mathcal{A}(S'_q, F')$ if the fields F and F' are m -close (see Proposition 4.4 for precise statement). Let \mathcal{F} be a facet in $\mathcal{A}(S_q, F)$ and $\mathcal{F}' = \mathcal{A}_m(\mathcal{F})$. Then \mathcal{F}' is a facet in $\mathcal{A}(S'_q, F')$. We prove that the parahoric group schemes $\mathcal{P}_{\mathcal{F}} \times_{\mathfrak{D}_F} \mathfrak{D}_F / \mathfrak{p}_F^m$ and $\mathcal{P}_{\mathcal{F}'} \times_{\mathfrak{D}_{F'}} \mathfrak{D}_{F'} / \mathfrak{p}_{F'}^m$ are isomorphic provided F and F' are l -close for $l \gg m$ (see Theorem 4.5 and Proposition 4.10 for precise statements). To prove this theorem, we prove an analogous statement for the root subgroup schemes if the fields F and F' are sufficiently close, invoke the result of Chai–Yu [2001] that the reduction of the (lft) Néron models of the corresponding tori are isomorphic if the fields are sufficiently close, and use the Artin–Weil theorem on obtaining group schemes as solutions to birational group laws.

To move to the general case, we recall that any connected reductive group is an inner form of a quasisplit group, and the F -isomorphism classes of inner forms of G_q is parametrized by the cohomology

set $H^1(\text{Gal}(F_{\text{un}}/F), G_q^{\text{ad}}(F_{\text{un}}))$ (where F_{un} is the maximal unramified extension of F contained in F_s). With G'_q corresponding to G_q as above, we prove in Lemma 5.1 that

$$H^1(\text{Gal}(F_{\text{un}}/F), G_q^{\text{ad}}(F_{\text{un}})) \cong H^1(\text{Gal}(F'_{\text{un}}/F'), G_q^{\text{ad}}(F'_{\text{un}}))$$

as pointed sets if the fields F and F' are m -close using the work of Kottwitz [2014]. Using the work of DeBacker and Reeder [2009] it is further possible to refine the above and obtain an isomorphism at the level of cocycles (see Section 5.A). All the above yields data (G, S, A) where G is a connected reductive group over F that is an inner form of G_q , a maximal F_{un} -split F -torus S that contains a maximal F -split torus A of G , and similarly (G', S', A') over F' , together with a $\text{Gal}(\widehat{F_{\text{un}}}/F)$ -equivariant simplicial isomorphism $\mathcal{A}_{m,*} : \mathcal{A}(S, \widehat{F_{\text{un}}}) \rightarrow \mathcal{A}(S', \widehat{F'_{\text{un}}})$ (see Lemma 6.1). Here $\widehat{F_{\text{un}}}$ denotes the completion of F_{un} . Let $\tilde{\mathcal{F}}_*$ be a $\text{Gal}(\widehat{F_{\text{un}}}/F)$ -invariant facet in $\mathcal{A}(S, \widehat{F_{\text{un}}})$ and let $\tilde{\mathcal{F}}'_* = \mathcal{A}_{m,*}(\tilde{\mathcal{F}}_*)$. We prove that there is a $\text{Gal}(\widehat{F_{\text{un}}}/F)$ -equivariant isomorphism

$$\tilde{p}_{m,*} : \mathcal{P}_{\tilde{\mathcal{F}}_*} \times_{\mathfrak{O}_{\widehat{F_{\text{un}}}}} \mathfrak{O}_{\widehat{F_{\text{un}}}}/\mathfrak{p}_{\widehat{F_{\text{un}}}}^m \rightarrow \mathcal{P}_{\tilde{\mathcal{F}}'_*} \times_{\mathfrak{O}_{\widehat{F'_{\text{un}}}}} \mathfrak{O}_{\widehat{F'_{\text{un}}}}/\mathfrak{p}_{\widehat{F'_{\text{un}}}}^m$$

provided F and F' are l -close (see Proposition 6.2). With $\mathcal{F}_* := (\tilde{\mathcal{F}}_*)^{\text{Gal}(\widehat{F_{\text{un}}}/F)}$ and $\mathcal{F}'_* := (\tilde{\mathcal{F}}'_*)^{\text{Gal}(\widehat{F'_{\text{un}}}/F')}$, the above descends to an isomorphism of group schemes

$$p_{m,*} : \mathcal{P}_{\mathcal{F}_*} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \rightarrow \mathcal{P}_{\mathcal{F}'_*} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m.$$

As a corollary, we obtain that

$$\mathcal{P}_{\mathcal{F}_*}(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong \mathcal{P}_{\mathcal{F}'_*}(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m)$$

as groups provided the fields F and F' are l -close.

2. Some review

Unless otherwise stated, F will denote a nonarchimedean local field, that is, a complete discretely valued field with perfect residue field. Let \mathfrak{O}_F denote its ring of integers, \mathfrak{p}_F its maximal ideal, $\omega = \omega_F$ an additive valuation on F normalized so that $\omega(F) = \mathbb{Z}$, and $\pi = \pi_F$ a uniformizer. Fix a separable closure F_s of F and let $\Gamma_F = \text{Gal}(F_s/F)$.

2.A. Deligne's theory. Let $m \geq 1$. Let I_F be the inertia group of F and I_F^m be its m -th higher ramification subgroup with upper numbering (see Chapter IV of [Serre 1979]). Let us summarize the results of Deligne [1984] that will be used later in this article. Deligne considered the triplet $\text{Tr}_m(F) = (\mathfrak{O}_F/\mathfrak{p}_F^m, \mathfrak{p}_F/\mathfrak{p}_F^{m+1}, \epsilon)$, where ϵ is the natural projection of $\mathfrak{p}_F/\mathfrak{p}_F^{m+1}$ on $\mathfrak{p}_F/\mathfrak{p}_F^m$, and proved that Γ_F/I_F^m is canonically determined by $\text{Tr}_m(F)$. Hence an isomorphism of triplets $\psi_m : \text{Tr}_m(F) \rightarrow \text{Tr}_m(F')$ gives rise to an isomorphism

$$\Gamma_F/I_F^m \xrightarrow{\text{Del}_m} \Gamma_{F'}/I_{F'}^m \quad (2-1)$$

that is unique up to inner automorphisms (see Equation 3.5.1 of [Deligne 1984]). More precisely, given an integer $f \geq 0$, let $\text{ext}(F)^f$ denote the category of finite separable extensions E/F satisfying the

following condition: The normal closure E_1 of E in F_s satisfies $\text{Gal}(E_1/F)^f = 1$. Deligne proved that an isomorphism $\psi_m : \text{Tr}_m(F) \rightarrow \text{Tr}_m(F')$ induces an equivalence of categories $\text{ext}(F)^m \rightarrow \text{ext}(F')^m$. Here is a partial description of the map Del_m (see Section 1.3 of [Deligne 1984]). Let L be a finite totally ramified Galois extension of F satisfying $I(L/F)^m = 1$ (here $I(L/F)$ is the inertia group of L/F). Then $L = F(\alpha)$ where α is a root of an Eisenstein polynomial

$$P(x) = x^n + \pi \sum a_i x^i$$

for $a_i \in \mathfrak{O}_F$. Let $a'_i \in \mathfrak{O}_{F'}$ be such that $a_i \bmod \mathfrak{p}^m \rightarrow a'_i \bmod \mathfrak{p}^m$. So a'_i is well-defined mod \mathfrak{p}^m . Then the corresponding extension L'/F' can be obtained as $L' = F'(\alpha')$ where α' is a root of the polynomial

$$P'(x) = x^n + \pi' \sum a'_i x^i$$

where $\pi \bmod \mathfrak{p}_F^m \rightarrow \pi' \bmod \mathfrak{p}_{F'}^m$. The assumption that $I(L/F)^m = 1$ ensures that the extension L' does not depend on the choice of a'_i , up to a unique isomorphism.

2.B. The main theorem of Chai–Yu. Let T be a torus over F and let K/F be a Galois extension such that T is split over K . Let $\Gamma_{K/F} = \text{Gal}(K/F)$ and let $\Lambda = X_*(T)$, the cocharacter group of T . Then T is determined by the Γ -module Λ up to a canonical isomorphism. With F' denoting another nonarchimedean local field, we will denote the analogous objects over F' with a superscript $'$. We introduce the following series of congruence notation:

- $(\mathfrak{O}_F, \mathfrak{O}_K) \equiv_{\psi_m} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'})$ (level m): this means that ψ_m is an isomorphism $\mathfrak{O}_K/\pi^m \mathfrak{O}_K \rightarrow \mathfrak{O}_{K'}/\pi'^m \mathfrak{O}_{K'}$ and induces an isomorphism $\mathfrak{O}_F/\pi^m \mathfrak{O}_F \rightarrow \mathfrak{O}_{F'}/\pi'^m \mathfrak{O}_{F'}$. We denote this induced isomorphism also by ψ_m . Having chosen the uniformizers, this also induces an isomorphism $\text{Tr}_m(F) \rightarrow \text{Tr}_m(F')$, which we still denote by ψ_m .
- $(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}) \equiv_{\psi_m, \gamma} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'}, \Gamma_{K'/F'})$ (level m): this means $(\mathfrak{O}_F, \mathfrak{O}_K) \equiv_{\psi_m} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'})$ (level m), γ is an isomorphism $\Gamma_{K/F} \rightarrow \Gamma_{K'/F'}$, and ψ_m is $\Gamma_{K/F}$ -equivariant relative to γ .
- $(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}, \Lambda) \equiv_{\psi_m, \gamma, \lambda} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'}, \Gamma_{K'/F'}, \Lambda')$ (level m): this means $(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}) \equiv_{\alpha, \beta} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'}, \Gamma_{K'/F'})$ (level m) and λ is an isomorphism $\Lambda \rightarrow \Lambda'$ which is $\Gamma_{K/F}$ -equivariant relative to γ .

We say that “ X is determined by $(\mathfrak{O}_F/\pi^m \mathfrak{O}_F, \mathfrak{O}_K/\pi^m \mathfrak{O}_K, \Gamma_{K/F}, \Lambda)''$ to mean that if

$$(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}, \Lambda) \equiv_{\psi_m, \gamma, \lambda} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'}, \Gamma_{K'/F'}, \Lambda') \text{ (level } m)$$

then there is a canonical $\Gamma_{K/F}$ -equivariant isomorphism $X \rightarrow X'$ determined by $(\psi_m, \gamma, \lambda)$.

Let \mathcal{T}^{NR} denote the Néron–Raynaud model of T considered in [Chai and Yu 2001]. This is a smooth model of T with connected generic fiber such that $\mathcal{T}^{NR}(\widehat{\mathfrak{O}_{F_{\text{un}}}})$ is the maximal bounded subgroup of $T(\widehat{F_{\text{un}}})$, where $\widehat{F_{\text{un}}}$ is the completion of the maximal unramified extension F_{un} of F contained in F_s . This model is of finite type over \mathfrak{O}_F .

Theorem 2.1 [Chai and Yu 2001, Theorem 8.5]. *Let $m \geq 1$. There exists $l \geq m$ such that the model*

$$\mathcal{T}^{NR} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \text{ is determined by } (\mathfrak{O}_F/\pi^l \mathfrak{O}_F, \mathfrak{O}_K/\pi^l \mathfrak{O}_K, \Gamma_{K/F}, \Lambda).$$

The parameters that l depends on are also explicitly determined in Theorem 8.5 of [Chai and Yu 2001]. Let \mathcal{T} denote the neutral component of \mathcal{T}^{NR} . This is a smooth model over \mathfrak{O}_F with connected generic and special fibers, and is of finite type over \mathfrak{O}_F . Its $\widehat{\mathfrak{O}_{F_{\text{un}}}}$ -points is the Iwahori subgroup of $T(\widehat{F_{\text{un}}})$.

Lemma 2.2. *Let \mathcal{T} , $l \geq m$ as above. Then the model*

$$\mathcal{T} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \text{ is determined by } (\mathfrak{O}_F/\pi^l \mathfrak{O}_F, \mathfrak{O}_K/\pi^l \mathfrak{O}_K, \Gamma_{K/F}, \Lambda).$$

Proof. This lemma follows from Lemma 8.5 of [Chai and Yu 2001] and the observation that the formation of \mathcal{T} commutes with any base change on $\text{Spec}(\mathfrak{O}_F)$, that is,

$$(\mathcal{T}^{NR} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m)^0 = \mathcal{T} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m. \quad \square$$

When the connected reductive group is a torus T , the model \mathcal{T} is its Iwahori (or parahoric) group scheme. We will study congruences of parahoric group schemes attached to facets in the Bruhat–Tits building of a connected reductive group G over F . To this end, let us recall some results from Bruhat–Tits theory and the construction of parahoric group schemes (using Artin–Weil theorem, following [Landvogt 1996]), that will be used later in this article.

Given a connected reductive group G over F , let G^{der} denote the derived subgroup of G , and G^{ad} its adjoint group. Let $\mathcal{B}(G, F)$ denote the reduced Bruhat–Tits building of G over F , that is, the building of G^{der} over F . The building is obtained by gluing together apartments $\mathcal{A}(S, F)$ where S runs over the maximal F -split tori in G . The apartment $\mathcal{A}(S, F)$ is an affine space under $X_*(S^{\text{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$ where $S^{\text{der}} = S \cap G^{\text{der}}$. Let \mathcal{F} be a facet in $\mathcal{B}(G, F)$ and let $P_{\mathcal{F}}$ denote the parahoric subgroup of $G(F)$ attached to \mathcal{F} . Bruhat–Tits show that there exists a smooth affine \mathfrak{O}_F -group scheme $\mathcal{P}_{\mathcal{F}}$ with generic fiber G such that $\mathcal{P}_{\mathcal{F}}(\mathfrak{O}_F) = P_{\mathcal{F}}$. We recall the construction of $\mathcal{P}_{\mathcal{F}}$, following Landvogt [1996]. The parahoric group scheme is first constructed over $\widehat{F_{\text{un}}}$ (note that $G_{\widehat{F_{\text{un}}}}$ is quasisplit), and the model over F is obtained using étale descent.

2.C. Structure of quasisplit groups. Let G denote a quasisplit connected reductive group over F . Let S be a maximal F -split torus in G and let T and N be the centralizer and normalizer of S in G , respectively. Let B be an F -Borel subgroup of G with $T \subset B$. Note that T is a maximal F -torus in G . Further G and T split over F_s and the Galois group Γ_F acts on the group of characters $X^*(T)$ of T , preserves the root system $\Phi(G, T)$ of T in G , and also the base $\tilde{\Delta}$ of $\Phi(G, T)$ associated to the Borel subgroup B . Let $K \subset F_s$ denote the smallest subextension of F_s splitting T (and hence G). Let $\Phi(G, S)$ denote the set of roots of S in G .

2.C.1. Root subgroups U_a , $a \in \Phi(G, S)$. The elements of $\Phi(G, S)$ are restrictions of elements of $\Phi(G, T)$ to S , and the restrictions to S of the elements of $\tilde{\Delta}$ form a basis Δ of $\Phi(G, S)$. Moreover, the elements of $\tilde{\Delta}$ that have the same restriction to S form a single Galois orbit for the action of Γ_F on $\tilde{\Delta}$.

For $\alpha \in \Phi(G, T)$, let \tilde{U}_α be the corresponding root subgroup of G_K . The group $\Gamma_{K/F}$ permutes \tilde{U}_α and $\gamma(\tilde{U}_\alpha) = \tilde{U}_{\gamma(\alpha)}$. Let Σ_α be the stabilizer of \tilde{U}_α and let L_α be the corresponding field of invariants. We say that L_α is the *field of definition* of α . Note that \tilde{U}_α is defined over L_α by Galois descent. Let $\{\tilde{x}_\alpha : \mathbb{G}_{a, L_\alpha} \rightarrow \tilde{U}_\alpha \mid \alpha \in \Phi(G, T)\}$ denote a Chevalley–Steinberg splitting of G . It has the following properties:

- (a) If the restriction a of $\alpha \in \Phi(G, T)$ to S is an indivisible element of $\Phi(G, S)$, then \tilde{x}_α is an L_α -isomorphism of \mathbb{G}_a to \tilde{U}_α and we have $\tilde{x}_{\gamma(\alpha)} = \gamma \circ \tilde{x}_\alpha \circ \gamma^{-1}$ for each $\gamma \in \text{Gal}(K/F)$.
- (b) If the restriction a of $\alpha \in \Phi(G, T)$ to S is divisible, then there exists two distinct roots $\beta, \beta' \in \Phi(G, T)$ of restriction $a/2$ to S such that $\alpha = \beta + \beta'$; we have $L_\beta = L_{\beta'}$, L_β is a quadratic separable extension of L_α and for each $\gamma \in \text{Gal}(K/F)$ there exists $\epsilon = \pm 1$ such that $\gamma \circ \tilde{x}_\alpha(u) \circ \gamma^{-1} = \tilde{x}_{\gamma(\alpha)}(\epsilon u)$; if $\gamma \in \text{Gal}(K/L_\alpha)$, we have $\epsilon = -1$ if and only if γ induces the unique nontrivial automorphism of L_β .

Now we describe all possible structures for the root subgroups U_a , $a \in \Phi(G, S)$. We may and do assume that $a \in \Delta$. Let $\tilde{\Delta}_a$ be the orbit of $\Gamma_{K/F}$ in $\tilde{\Delta}$. Let $\pi : G^a \rightarrow \langle U_a, U_{-a} \rangle$ be the universal cover of the semisimple group generated by U_a and U_{-a} . The classification of Dynkin diagrams gives two possible cases:

Case I. The group G_K^a is isomorphic to a product of the groups SL_2 indexed by $\tilde{\Delta}_a$ and are permuted transitively by $\text{Gal}(K/F)$, the field of definition of the factor of index α is L_α and $G^a \cong \text{Res}_{L_\alpha/F} \text{SL}_2$. Then $U_a \cong \text{Res}_{L_\alpha/F} \tilde{U}_\alpha$ for $\alpha \in \tilde{\Delta}_a$. If $\tilde{x}_\alpha : L_\alpha \rightarrow \tilde{U}_\alpha$, then $x_a = \text{Res}_{L_\alpha/F} \tilde{x}_\alpha$ is a F -isomorphism of $\text{Res}_{L_\alpha/F} \mathbb{G}_a$ to U_a ; the pair (L_α, x_a) is called a pinning of U_a . Via x_a , we obtain an isomorphism of L_α with $U_a(F)$, which we also denote by x_a . If $(\tilde{x}_\beta)_{\beta \in \tilde{\Delta}}$ is an Chevalley–Steinberg splitting of G , then we have for each $u \in L_\alpha$,

$$x_a(u) = \prod_{\beta \in \tilde{\Delta}_a} \tilde{x}_\beta(u_\beta) \quad (2-2)$$

In the above, $\beta = \gamma(\alpha)$ for some $\gamma \in \Gamma_{K/F}$ and $u_\beta := \gamma(u)$. The subgroups U_{-a} and the splitting x_{-a} are obtained using $U_{-\alpha}$ and $\tilde{x}_{-\alpha}$ analogously.

Case II. The group G_K^a is isomorphic to a product of the groups SL_3 indexed by the set I consisting of pairs of two elements $\{\alpha, \bar{\alpha}\}$ of $\tilde{\Delta}_a$ such that $\alpha + \bar{\alpha}$ is a root. We have $L_\alpha = L_{\bar{\alpha}}$, L_α is a quadratic extension of $L_{\alpha+\bar{\alpha}}$. The simple factor \bar{G} of index $\{\alpha, \bar{\alpha}\}$ is defined over $L_{\alpha+\bar{\alpha}}$, split over L_α , and is isomorphic over $L_{\alpha+\bar{\alpha}}$ to the special unitary group of the Hermitian form $h : (x_{-1}, x_0, x_1) \rightarrow \tau(x_{-1})x_1 + \tau(x_0)x_0 + \tau(x_1)x_{-1}$ over L^3 . Here τ is the unique nontrivial element of $\text{Gal}(L_\alpha/L_{\alpha+\bar{\alpha}})$. We denote this simple factor as SU_3 , and then $G^a \cong \text{Res}_{L_{\alpha+\bar{\alpha}}/F} \text{SU}_3$.

Let $H_0(L_\alpha, L_{\alpha+\bar{\alpha}}) := \{(u, v) \in L_\alpha \times L_\alpha \mid v + \tau(v) = u\tau(u)\}$ denote the $L_{\alpha+\bar{\alpha}}$ -group with group law $(u, v) \cdot (\tilde{u}, \tilde{v}) = (u + \tilde{u}, v + \tilde{v} + \tau(u)\tilde{u})$. Then $\zeta : (u, v) \rightarrow \tilde{x}_\alpha(u)\tilde{x}_{\alpha+\bar{\alpha}}(-v)\tilde{x}_{\bar{\alpha}}(\tau(u))$ is an $L_{\alpha+\bar{\alpha}}$ -group isomorphism of $H_0(L_\alpha, L_{\alpha+\bar{\alpha}})$ with the subgroup $\bar{U} = \tilde{U}_\alpha\tilde{U}_{\alpha+\bar{\alpha}}\tilde{U}_{\bar{\alpha}}$ of \bar{G} . Then $U_a = \text{Res}_{L_{\alpha+\bar{\alpha}}/F} \bar{U}$ and $x_a = \text{Res}_{L_{\alpha+\bar{\alpha}}/K} \zeta$ is an F -isomorphism of groups $H(L_\alpha, L_{\alpha+\bar{\alpha}}) = \text{Res}_{L_{\alpha+\bar{\alpha}}/F} H_0(L_\alpha, L_{\alpha+\bar{\alpha}})$ with U_a . Further, for $(u, v) \in L_\alpha \times L_\alpha$,

$$x_a(u, v) = \prod \tilde{x}_\beta(u_\beta)\tilde{x}_{\beta+\bar{\beta}}(-v_\beta)\tilde{x}_{\bar{\beta}}(\tau(u_\beta))$$

In the above, for each β , we choose $\gamma \in \text{Gal}(K/F)$ such that $\beta = \gamma(\alpha)$; then $\bar{\beta} = \gamma(\bar{\alpha})$, $\tilde{x}_\beta = \gamma \circ \tilde{x}_\alpha \circ \gamma^{-1}$, $\tilde{x}_{\bar{\beta}} = \gamma \circ \tilde{x}_{\bar{\alpha}} \circ \gamma^{-1}$, $\tilde{x}_{\beta+\bar{\beta}} = \gamma \circ \tilde{x}_{\alpha+\bar{\alpha}} \circ \gamma^{-1}$, $u_\beta = \gamma(u)$, $v_\beta = \gamma(v)$.

Note that the root subgroup $U_{2a}(K)$ associated to the root $2a$ consists of elements $x_a(0, v)$ where v is an element of $L_\alpha^0 := \{v \in L_\alpha \mid v + \tau(v) = 0\}$, and the map $v \rightarrow x_a(0, v)$ is an F -vector space isomorphism of L_α^0 .

2.C.2. On the splitting extension of the root. Let $a \in \Phi^{\text{red}}(G, S)$ with $2a$ is not a root. We fix a pinning (L_α, x_a) of U_a where $\alpha \in \tilde{\Delta}_a$ as in Case I above. The subset of endomorphisms of the F -vector space U_a of the form $\mu_{x_a}(t) : x_a(u) \rightarrow x_a(tu)$ for $t \in L_\alpha$ does not depend on the choice of (L_α, x_a) (see Section 4.1.8 of [Bruhat and Tits 1984]). This is denoted by L_a and is called the field attached of the root a . It is isomorphic to L_α via the map $t \rightarrow \mu_{x_a}(t)$. Its inverse gives an embedding of $L_a \hookrightarrow K$. A similar definition is obtained when $2a$ is a root in Section 4.1.14 of [Bruhat and Tits 1984].

2.C.3. Valuations. Let $\omega : F \rightarrow \mathbb{R}^\times$ be as in Section 2.A, and we denote its extension to K also as ω . The notion of valuation of root datum was defined in [Bruhat and Tits 1972]. For $\alpha \in \Phi(G, T)$, put

$$\phi_\alpha(\tilde{x}_\alpha(u)) = \omega(u), \quad u \in K^\times.$$

Then $\tilde{\phi} = (\phi_\alpha)_{\alpha \in \Phi(G, T)}$ defines a valuation of the root datum $(T_K, (\tilde{U}_\alpha)_{\alpha \in \Phi(G, T)})$ in the group $G(K)$ (recall that G_K is split). It is shown in [Bruhat and Tits 1984] that $\tilde{\phi}$ descends to $(T, (\tilde{U}_a)_{a \in \Phi(G, S)})$ and defines a valuation on it. We explicitly define $\phi_a : U_a(F) \setminus \{1\} \rightarrow \mathbb{R}$ from $\tilde{\phi}$. For $a \in \Phi(G, S)$, let A (resp. B) be the set of $\alpha \in \Phi(G, T)$ whose restriction to S is a (resp. $2a$). For $u \in U_a(F)$, there exist unique \tilde{u}_α such that $u = \prod_{\alpha \in A \cup B} \tilde{u}_\alpha$ for an arbitrary ordering of $A \cup B$ and we put

$$\phi_a(u) = \inf \left(\inf_{\alpha \in A} \tilde{\phi}_\alpha(\tilde{u}_\alpha), \inf_{\alpha \in B} \frac{1}{2} \tilde{\phi}_\alpha(\tilde{u}_\alpha) \right).$$

This number is independent of the choice of ordering of $A \cup B$. Then $\phi = (\phi_a)_{a \in \Phi(G, S)}$ defines a valuation of root datum on $(T, (U_a)_{a \in \Phi(G, S)})$ (see Section 4.2.2 of [Bruhat and Tits 1984]).

2.D. Parahoric group schemes: quasisplit descent. In this section, we assume that F is also strictly Henselian, that is its residue field is separably closed.

2.D.1. Affine root system and the associated Weyl groups. The apartment $\mathcal{A}(S, F)$ can also be thought of as the set of valuations that are equipollent to $\phi = (\phi_a)_{a \in \Phi(G, S)}$, where ϕ as above. This is an affine space under $X_*(S^{\text{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N(F)$ acts on it by affine transformations (see Section 6.2.2 of [Bruhat and Tits 1972]). Let us denote the point of $\mathcal{A}(S, F)$ corresponding to ϕ as x_0 . For $a \in \Phi(G, S)$, let $\Gamma_a = \phi_a(U_a(F) \setminus \{1\})$ and

$$\tilde{\Gamma}_a = \{\phi_a(u) \mid u \in U_a(F) \setminus \{1\}, \phi_a(u) = \sup \phi_a(u U_{2a}(F))\}.$$

Here we have used the convention that $U_{2a} = 1$ if $2a$ is not a root. Let

$$\Phi^{\text{af}}(G, S) = \{\psi : \mathcal{A}(S, F) \rightarrow \mathbb{R} \mid \psi(\cdot) = a(\cdot - x_0) + l, a \in \Phi(G, S), l \in \tilde{\Gamma}_a\}$$

denote the set of affine roots of S in G . Choosing x_0 allows us to identify $A(S, F)$ with $X_*(S^{\text{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$. With this identification, the vanishing hyperplanes coming from $\Phi(G, S)^{\text{af}}$ makes $\mathcal{A}(S, F)$ into a (poly)simplicial complex. The group generated by reflections through the hyperplanes coming from $\Phi(G, S)^{\text{af}}$ is the affine Weyl group denoted by W^{af} . The extended affine Weyl group is defined as $W^e := N(F)/T(F)_1$ where $T(F)_1$ is the kernel of the Kottwitz homomorphism $\kappa_T : T(F) \rightarrow X^*(\hat{T}^{I_F}) = X_*(T)_{I_F}$ (see [Haines and Rapoport 2008]). With $W := W(G, S)$, the group W^e hence fits into an exact sequence

$$1 \rightarrow X_*(T)_{I_F} \rightarrow W^e \rightarrow W \rightarrow 1.$$

2.D.2. The associated root subgroup schemes. Let us recall the filtrations on root subgroups and the associated root subgroup schemes from Section 4.3 of [Bruhat and Tits 1984]. For $a \in \Phi(G, S)$, let $\phi_a : U_a(F) \rightarrow \mathbb{R} \cup \{\infty\}$ be as above. For $k \in \mathbb{R}$, let $U_{a,k} = \{u \in U_a(F) \mid \phi_a(u) \geq k\}$. Next, let us describe the associated root subgroup schemes.

Case I. Let $a \in \Phi^{\text{red}}(G, S)$ such that $2a \notin \Phi(G, S)$. For $k \in \tilde{\Gamma}_a$, let $L_{a,k} = \{u \in L_a \mid \omega(u) \geq k\}$. Then $L_{a,k}$ is a free \mathfrak{O}_F -module of finite type. Let $\mathcal{L}_{a,k}$ be the canonical smooth \mathfrak{O}_F -group scheme associated to this module. (More precisely, given a free \mathfrak{O}_F -module M of finite type, the functor taking any \mathfrak{O}_F -algebra R to the additive group $R \otimes M$ is representable by a smooth \mathfrak{O}_F -group scheme \mathcal{M} whose affine algebra is identified with the symmetric algebra of the dual of M .) Let $U_{a,k}$ be the image under x_a of $L_{a,k}$ and let $\mathcal{U}_{a,k}$ be the \mathfrak{O}_F -group scheme obtained by transport of structure using x_a . Then $\mathcal{U}_{a,k}$ has generic fiber U_a and $\mathcal{U}_{a,k}(\mathfrak{O}_F) = U_{a,k}$. The definition is extended to $k \in \mathbb{R} \setminus \{0\}$ in Section 4.3.2 of [Bruhat and Tits 1984].

Case II. Let $a \in \Phi^{\text{red}}(G, S)$ with $2a \in \Phi(G, S)$. The root subgroup $U_a \cong \text{Res}_F^{L_{2a}} H_0(L_a, L_{2a})$ via x_a . In order to describe the root subgroup schemes of the filtration $U_{a,k}$, we use an alternate description of $H_0(L_a, L_{2a})$. Recall that L_a^0 is the set of trace 0 elements of L_a . Let L_a^1 denote the set of trace 1 elements in L_a and let

$$(L_a)_{\max}^1 := \{\lambda \in L_a^1 \mid \omega(\lambda) = \sup\{\omega(x) \mid x \in L_a^1\}\}.$$

Note that $(L_a)_{\max}^1 \neq \emptyset$ and when the residue field of L_a is of characteristic $\neq 2$, $1/2 \in (L_a)_{\max}^1$. Let $\lambda \in (L_a)_{\max}^1$ and let $H_0^\lambda := L_a \times L_a^0$ equipped with the action

$$(u, v) \cdot (\tilde{u}, \tilde{v}) = (u + \tilde{u}, v + \tilde{v} - \lambda u \tau(\tilde{u}) + \tau(\lambda) \tau(u) \tilde{u}). \quad (2-3)$$

Then H_0^λ is an algebraic L_{2a} -group and $j_\lambda : (u, v) \rightarrow (u, v - \lambda \tau(u)u)$ is an L_{2a} -group isomorphism of $H_0(L_a, L_{2a})$ onto H_0^λ . Let $H^\lambda = \text{Res}_F^{L_{2a}} H_0^\lambda$.

Let $\gamma = -\frac{1}{2}\omega(\lambda)$. For $k \in \tilde{\Gamma}_a$, let $l = 2k + \frac{1}{e_a}$, and

$$L_{a,k+\gamma} := \{u \in L_a \mid \omega(u) \geq k + \gamma\} \quad \text{and} \quad L_{a,l}^0 := \{u \in L_a^0 \mid \omega(u) \geq l\}.$$

Up to isomorphism, there exists a unique smooth affine \mathfrak{O}_F -group scheme \mathcal{H}_k^λ of finite type with generic fiber H^λ and such that $\mathcal{H}_k^\lambda(\mathfrak{O}_F) = L_{a,k+\gamma} \times L_{a,l}^0$ and a group law, which induces the group law (2-3) on the generic fiber (See Section 4.3.5 of [Bruhat and Tits 1984]). In more detail, let $\mathcal{L}_{a,k+\gamma}$ and $\mathcal{L}_{a,l}^0$ be the

canonical $\mathfrak{O}_{L_{2a}}$ -group schemes associated to $L_{a,k+\gamma}$ and $L_{a,l}^0$. Let $\mathcal{H}_{0,k}^\lambda = \mathcal{L}_{a,k+\gamma} \times \mathcal{L}_{a,l}^0$. The map

$$L_a \times L_a \rightarrow L_a^0, (u, u') \rightarrow \lambda u \tau(\tilde{u}) - \tau(\lambda) \tau(u) \tilde{u}$$

can be extended uniquely to a morphism $\mathcal{L}_{a,k+\gamma} \times \mathcal{L}_{a,k+\gamma} \rightarrow \mathcal{L}_{a,l}^0$. Hence the group law can be extended to $\mathcal{H}_{0,k}^\lambda$. Let $\mathcal{H}_k^\lambda := \text{Res}_F^{\mathfrak{O}_{L_{2a}}} H_{0,k}^\lambda$. By transport of structure using $x_a \circ \text{Res}_F^{L_{2a}} j_\lambda^{-1}$, we obtain the \mathfrak{O}_F -group scheme $\mathcal{U}_{a,k}$. These definitions are extended to $k, l \in \mathbb{R} \setminus \{0\}$ in Section 4.3.8 of [Bruhat and Tits 1984].

Using the isomorphism $v \rightarrow x_a(0, v)$ from $L_a^0 \rightarrow U_{2a}$, we obtain from the scheme \mathcal{L}_k^0 (for $k \in \omega(L_a^0) \setminus \{0\}$), an \mathfrak{O}_F -scheme whose generic fiber is U_{2a} and denote it as $\mathcal{U}_{2a,k}$ (see Section 4.3.7 of [Bruhat and Tits 1984] for further details).

2.D.3. Construction of parahoric group schemes over F . In this section, we recall the construction of parahoric group schemes, following [Landvogt 1996]. Given $x \in \mathcal{A}(S, F)$, let $f_x : \Phi(G, S) \rightarrow \mathbb{R}$ be the function $f_x(a) = -a(x - x_0)$, where x_0 is the unique point arising from quasisplit descent as in Section 2.D.1. Let $U_{a,x} := U_{a,f_x(a)}$. Let $\mathcal{U}_{a,x}$ be the smooth affine group scheme over \mathfrak{O}_F with generic fiber U_a and with $\mathcal{U}_{a,x}(\mathfrak{O}_F) = U_{a,x}$ (as in Section 2.D.2). For $\Psi = \Phi^+(G, S)$ and $\Psi = \Phi^-(G, S)$, Proposition 3.3.2 of [Bruhat and Tits 1984] gives a unique smooth affine \mathfrak{O}_F -group scheme $\mathcal{U}_{\Psi,x}$ of finite type with generic fiber U_Ψ and the property that for every good ordering of Ψ^{red} (See Section 3.1.2 of [Bruhat and Tits 1984]), the F -isomorphism $\prod_{a \in \Psi} U_a \rightarrow U_\Psi$ can be extended to an \mathfrak{O}_F -isomorphism $\prod_{a \in \Psi} \mathcal{U}_{a,x} \rightarrow \mathcal{U}_{\Psi,x}$.

The parahoric subgroup P_x is generated by $\mathcal{T}(\mathfrak{O}_F)$ and the $U_{a,x}$ for $a \in \Phi(G, S)$ (with \mathcal{T} is as in Section 2.B). One of the main results of [Bruhat and Tits 1984] is that there is a unique smooth affine \mathfrak{O}_F -group scheme \mathcal{P}_x with generic fiber G and with $\mathcal{P}_x(\mathfrak{O}_F) = P_x$. We recall the construction of \mathcal{P}_x from [Landvogt 1996]. The idea is to put an \mathfrak{O}_F -birational group law on $\mathcal{U}_{\Phi^+,x} \times \mathcal{T} \times \mathcal{U}_{\Phi^-,x}$ and invoke Artin–Weil theorem (see Chapters 5 and 6 of [Bosch et al. 1990]) to construct \mathcal{P}_x . Let us first introduce some notation. Let $\mathcal{U}_x^\pm = \mathcal{U}_{\Phi^\pm(G,S),x}$ and let $\mathcal{X}_x = \mathcal{U}_x^- \mathcal{T} \mathcal{U}_x^+$. Since its generic fiber $\mathcal{X}_x \times_{\mathfrak{O}_F} F = U^- T U^+$ is an open neighborhood of the 1-section of G , there exists a unique F -birational group law on the generic fiber of \mathcal{X}_x . We want to extend this to \mathcal{X}_x . Since $U^- T U^+$ and $U^+ T U^-$ are both open neighborhoods of the 1-section of G , there exist $f \in F[U^- T U^+]$ and $f' \in F[U^+ T U^-]$ such that $F[U^- T U^+]_f = F[U^+ T U^-]_{f'}$. Without loss of generality, we may assume that $f \in \mathfrak{O}_F[U^- T U^+] \setminus \pi \mathfrak{O}_F[U^- T U^+]$ and $f' \in \mathfrak{O}_F[U^+ T U^-] \setminus \pi \mathfrak{O}_F[U^+ T U^-]$. Proposition 5.16 of [Landvogt 1996] shows that inside $F[U^- T U^+]_f = F[U^+ T U^-]_{f'}$, we have $\mathfrak{O}_F[\mathcal{U}_x^- \mathcal{T} \mathcal{U}_x^+]_f = \mathfrak{O}_F[\mathcal{U}_x^+ \mathcal{T} \mathcal{U}_x^-]_{f'}$. So we will identify $(\mathcal{U}_x^- \mathcal{T} \mathcal{U}_x^+)_f = (\mathcal{U}_x^+ \mathcal{T} \mathcal{U}_x^-)_{f'}$ in the following. By Proposition 5.8 of [Landvogt 1996], we can identify $\mathcal{T} \mathcal{U}_x^+$ and $\mathcal{U}_x^+ \mathcal{T}$ and hence also $\mathcal{T} \mathcal{U}_x^+ \mathcal{U}_x^-$ and $\mathcal{U}_x^+ \mathcal{T} \mathcal{U}_x^-$. In $\mathcal{X}_x \times \mathcal{X}_x = \mathcal{U}_x^- \times (\mathcal{T} \times \mathcal{U}_x^+ \times \mathcal{U}_x^-) \times \mathcal{T} \mathcal{U}_x^+$, we consider the open subscheme

$$\begin{aligned} \mathcal{U}_x^- \times (\mathcal{U}_x^+ \times \mathcal{T} \times \mathcal{U}_x^-)_f \times \mathcal{T} \mathcal{U}_x^+ &= \mathcal{U}_x^- \times (\mathcal{U}_x^- \times \mathcal{T} \times \mathcal{U}_x^+)_{f'} \times \mathcal{T} \mathcal{U}_x^+ \\ &\subset \mathcal{U}_x^- \times \mathcal{U}_x^- \times \mathcal{T} \times \mathcal{U}_x^+ \times \mathcal{T} \mathcal{U}_x^+ \\ &= (\mathcal{U}_x^- \times \mathcal{U}_x^-) \times (\mathcal{T} \times \mathcal{T}) \times (\mathcal{U}_x^+ \times \mathcal{U}_x^+) \\ &\xrightarrow{\text{mult}^3} \mathcal{U}_x^- \times \mathcal{T} \times \mathcal{U}_x^+ \end{aligned}$$

So we obtain a morphism $\mathcal{U}_x^- \times (\mathcal{U}_x^+ \times \mathcal{T} \times \mathcal{U}_x^-)_f \times \mathcal{T}\mathcal{U}_x^+ \rightarrow \mathcal{X}_x$. Since \mathcal{X}_x has irreducible fibers over \mathfrak{O}_F and since $f \notin \pi \mathfrak{O}_F[\mathcal{U}_x^- \mathcal{T}\mathcal{U}_x^+]$, we see that $(\mathcal{U}_x^- \mathcal{T}\mathcal{U}_x^+)_f$ is \mathfrak{O}_F -dense in \mathcal{X}_x (that is, each of its fibers is Zariski dense in the corresponding fiber of \mathcal{X}_x — see Section 2.5 of [Bosch et al. 1990]), and hence $\mathcal{U}_x^- \times (\mathcal{U}_x^+ \times \mathcal{T} \times \mathcal{U}_x^-)_f \times \mathcal{T}\mathcal{U}_x^+$ is \mathfrak{O}_F -dense in $\mathcal{U}_x^- \times (\mathcal{T} \times \mathcal{U}_x^+ \times \mathcal{U}_x^-) \times \mathcal{T}\mathcal{U}_x^+ = \mathcal{X}_x \times \mathcal{X}_x$. Hence we obtain an \mathfrak{O}_F -rational map $m : \mathcal{X}_x \times \mathcal{X}_x \rightarrow \mathcal{X}_x$. By Proposition 5.16 of [Landvogt 1996], m is an \mathfrak{O}_F -birational group law on \mathcal{X}_x . Glue together the schemes G and \mathcal{X}_x along $\mathcal{X}_x \times_{\mathfrak{O}_F} F$ and denote it as \mathcal{Y}_x . As in Proposition 5.17 of [Landvogt 1996], the parahoric group scheme \mathcal{P}_x with group law \bar{m} , together with an open immersion $\mathcal{Y}_x \rightarrow \mathcal{P}_x$ such that the restriction of \bar{m} to \mathcal{Y}_x is m , is obtained by applying Theorem 5.1 of [Bosch et al. 1990] to the scheme \mathcal{Y}_x . The generic fiber of \mathcal{P}_x is G . Let \mathcal{F} be a facet in $\mathcal{A}(S, F)$. Then for $x, y \in \mathcal{F}$, $P_x = P_y$. So we write $P_{\mathcal{F}}$ for the parahoric subgroup attached to the facet \mathcal{F} and denote the underlying group scheme as $\mathcal{P}_{\mathcal{F}}$.

2.E. Parahoric group schemes: étale descent. Let F be a nonarchimedean local field and \widehat{F}_{un} be the completion of the maximal unramified extension $F_{\text{un}}(\subset F_s)$ of F . Let G be a connected reductive group over F . By a theorem of Steinberg (recalled as Theorem 5.2), we know that $G_{F_{\text{un}}}$ is quasisplit. Let A be a maximal F -split torus in G . By Section 5 of [Bruhat and Tits 1984], there is an F -torus S that contains A and is maximal F_{un} -split. Note that $X_*(A) = X_*(S)^{\text{Gal}(\widehat{F}_{\text{un}}/F)}$. Let $\mathcal{A}(A, F)$ denote the apartment of G with respect to A . Let \mathcal{F}_* be a facet in $\mathcal{A}(A, F)$. We fix an algebraic closure $\bar{\kappa}_F$ of the residue field κ_F and identify the Galois groups $\text{Gal}(\widehat{F}_{\text{un}}/F)$ with $\text{Gal}(\bar{\kappa}_F/\kappa_F)$. Let σ denote the Frobenius element of $\text{Gal}(F_{\text{un}}/F)$ under this identification. Then we know that there is a σ -stable facet $\tilde{\mathcal{F}}_*$ in $\mathcal{A}(S, F_{\text{un}})$ such that $\tilde{\mathcal{F}}_*^\sigma = \mathcal{F}_*$ (see Chapter 5 of [Bruhat and Tits 1984]). Since $\tilde{\mathcal{F}}_*$ is stable under the action of σ , the parahoric group scheme $\mathcal{P}_{\tilde{\mathcal{F}}_*}$ is also stable under the action of σ . In this case, the $\mathfrak{O}_{\widehat{F}_{\text{un}}}$ -group scheme $\mathcal{P}_{\tilde{\mathcal{F}}_*}$ admits a unique descent to an \mathfrak{O}_F -group scheme with generic fiber G (see Example B, Section 6.2, [Bosch et al. 1990]). The affine ring of this group scheme is $(\mathfrak{O}_{\widehat{F}_{\text{un}}}[\mathcal{P}_{\tilde{\mathcal{F}}_*}])^{\text{Gal}(\widehat{F}_{\text{un}}/F)}$. This is the parahoric group scheme attached to the facet \mathcal{F}_* of $\mathcal{A}(A, F)$.

3. Quasisplit forms over close local fields

Let G_0 be a split connected reductive group defined over \mathbb{Z} with root datum (R, Δ) . For an extension K/F , let $G_{0,K} := G_0 \times_{\mathbb{Z}} K$.

Let $E(F, G_0)$ be the set of F -isomorphism classes of connected reductive F -algebraic groups G with G_{F_s} isomorphic to G_{0,F_s} . This is in natural bijection with the Galois cohomology set $H^1(\Gamma_F, \text{Aut}(G_{0,F_s}))$. We denote this map

$$E(F, G_0) \rightarrow H^1(\Gamma_F, \text{Aut}(G_{0,F_s})), [G] \rightarrow s_G. \quad (3-1)$$

Lemma 3.1. *Let I_F be the inertia group of F and I_F^m denote the m -th higher ramification subgroup with upper numbering. Let $E(F, G_0)_m$ denote the set of F -isomorphism classes of F -forms G of $G_{0,F}$ such that there exists an at most m -ramified finite extension $L \subset F_s$ (i.e., $\text{Gal}(L/F)^m = 1$) with*

$G \times_F L \cong G_0 \times_{\mathbb{Z}} L$. The bijection (3-1) induces a bijection between $E(F, G_0)_m$ and the cohomology set $H^1(\Gamma_F/I_F^m, (\text{Aut}_{F_s}(G_{0,F_s}))^{I_F^m})$.

Proof. Let $\Omega := (F_s)^{I_F^m}$. Then for every finite extension $F \subset L \subset F_s$, $L \hookrightarrow \Omega$ if and only if $\text{Gal}(L/F)^m = 1$ (see Section 3.5 of [Deligne 1984]). Further we know that $H^1(\text{Aut}(\Omega/F), \text{Aut}_{\Omega}(G_{0,\Omega}))$ classifies isomorphism classes of F -forms $[G]$ with $G \times_F \Omega \cong G_{0,F} \times_F \Omega$. Now simply note that $\text{Aut}(\Omega/F) \cong \Gamma_F/I_F^m$ and $\text{Aut}_{\Omega}(G_{0,\Omega}) = (\text{Aut}_{F_s}(G_{0,F_s}))^{I_F^m}$. \square

3.A. Quasisplit forms. Let $(G_0, T_0, B_0, \{u_{\alpha}\}_{\alpha \in \tilde{\Delta}})$ be a pinned, split, connected, reductive \mathbb{Z} -group with based root datum $(R, \tilde{\Delta})$ where $\{u_{\alpha}\}_{\alpha \in \tilde{\Delta}}$ is a splitting as in Section 3.2.2 of [Bruhat and Tits 1984]. Then $\text{Out}(G_0)$ can be identified with the constant \mathbb{Z} -group scheme associated to the group $\text{Aut}(R, \tilde{\Delta})$. Consider the exact sequence

$$1 \rightarrow \text{Inn}(G_0(F_s)) \rightarrow \text{Aut}(G_{0,F_s}) \rightarrow \text{Aut}(R, \tilde{\Delta}) \rightarrow 1.$$

Let $H = H(G_0, T_0, B_0, \{u_{\alpha}\}_{\alpha \in \tilde{\Delta}})$ be the subgroup of $\text{Aut}(G_{0,F_s})$ consisting of all a such that $a(B_0) = B_0$, $a(T_0) = T_0$ and $\{a \circ u_{\alpha} \mid \alpha \in \tilde{\Delta}\} = \{u_{\alpha} \mid \alpha \in \tilde{\Delta}\}$. Then $H \hookrightarrow \text{Aut}(G_{0,F_s}) \rightarrow \text{Aut}(R, \tilde{\Delta})$ is an isomorphism and $\text{Aut}(G_{0,F_s}) \cong H \rtimes \text{Inn}(G_0(F_s))$. Hence the natural map $H^1(\Gamma_F, \text{Aut}(G_{0,F_s})) \rightarrow H^1(\Gamma_F, \text{Aut}(R, \tilde{\Delta}))$ has a section given by

$$q : H^1(\Gamma_F, \text{Aut}(R, \tilde{\Delta})) \xrightarrow{\sim} H^1(\Gamma_F, H) \rightarrow H^1(\Gamma_F, \text{Aut}(G_{0,F_s})).$$

We now recall the following well-known theorem (see [Conrad 2011], Section 7.2).

Theorem 3.2. *Let $[G] \in E(F, G_0)$. Then s_G lies in the image of*

$$q : H^1(\Gamma_F, \text{Aut}(R, \tilde{\Delta})) \rightarrow H^1(\Gamma_F, \text{Aut}(G_{0,F_s}))$$

if and only if G is quasisplit over F , that is, it has a Borel subgroup defined over F .

Let $E_{qs}(F, G_0) := \{[G] \in E(F, G_0) \mid s_G \in \text{Im}(q)\}$ and $E_{qs}(F, G_0)_m = E_{qs}(F, G_0) \cap E(F, G_0)_m$. Since G_0 is F -split, the action of Γ_F on (G_0, B_0, T_0) is trivial. Hence

$$Z^1(\Gamma_F, \text{Aut}(R, \tilde{\Delta})) = \text{Hom}(\Gamma_F, \text{Aut}(R, \tilde{\Delta})).$$

Lemma 3.3. *We have the following:*

(a) *The class $[G] \in E_{qs}(F, G_0)_m$ if and only if s_G lies in the image of*

$$q : H^1(\Gamma_F/I_F^m, \text{Aut}(R, \tilde{\Delta})^{I_F^m}) \rightarrow H^1(\Gamma_F/I_F^m, \text{Aut}(G_{0,F_s})^{I_F^m}).$$

(b) *The isomorphism $\psi_m : \text{Tr}_m(F) \xrightarrow{\sim} \text{Tr}_m(F')$ induces an isomorphism*

$$\mathfrak{Q}_m : H^1(\Gamma_F/I_F^m, \text{Aut}(R, \tilde{\Delta})) \xrightarrow{\sim} H^1(\Gamma_{F'}/I_{F'}^m, \text{Aut}(R, \tilde{\Delta}))$$

and

$$\mathfrak{Q}_m^c : Z^1(\Gamma_F/I_F^m, \text{Aut}(R, \tilde{\Delta})) \xrightarrow{\sim} Z^1(\Gamma_{F'}/I_{F'}^m, \text{Aut}(R, \tilde{\Delta})).$$

(c) *The isomorphism ψ_m induces a bijection $E_{qs}(F, G_0)_m \rightarrow E_{qs}(F', G_0)_m$, $[G] \rightarrow [G']$, where $s_{G'} = q' \circ \mathfrak{Q}_m(s_G)$.*

Proof. This is clear from Lemma 3.1 and Theorem 3.2. \square

As noted in Lemma 3.3, $Z^1(\text{Gal}(\Omega/F), \text{Aut}(R, \tilde{\Delta})) = \text{Hom}(\text{Gal}(\Omega/F), \text{Aut}(R, \tilde{\Delta}))$ since G_0 is split. Let us fix $s \in Z^1(\text{Gal}(\Omega/F), \text{Aut}(R, \tilde{\Delta})) \cong Z^1(\text{Gal}(\Omega/F), H)$. Let (G, ϕ) be a pair of be a quasisplit connected reductive group over F and $\phi : G_0 \times_{\mathbb{Z}} \Omega \rightarrow G \times_F \Omega$ an Ω -isomorphism such that the Galois action on $G(F_s)$ is given by s . We may and do assume that there is a finite Galois at most m -ramified extension K of F over which ϕ is defined, that is, that $s \in Z^1(\text{Gal}(K/F), \text{Aut}(R, \tilde{\Delta}))$.

More precisely, with $*_F$ denoting the Galois action on $G(K)$, we have

$$\gamma *_F \phi(x) = \phi(s(\gamma)(\gamma \cdot x))$$

for $\gamma \in \text{Gal}(K/F)$ and $x \in G_0(K)$. Then $\phi(T_0) = T$ is a maximal torus of G defined over F and $\phi(B_0) = B$ is a Borel subgroup of G containing T and defined over F . Let $s' \in Z^1(\text{Gal}(K'/F'), \text{Aut}(R, \tilde{\Delta}))$ as in Lemma 3.3. Here K'/F' is determined by K/F via Del_m . Let (G', ϕ') be a pair of quasisplit connected reductive group over F' and $\phi' : G_0 \times_{\mathbb{Z}} K' \rightarrow G' \times_{F'} K'$ such that $\gamma' *_F \phi'(x') = \phi(s'(\gamma')(\gamma' \cdot x'))$, where $\gamma' = \text{Del}_m(\gamma)$. Then $\phi'(T_0) = T'$ and $\phi'(B_0) = B'$ are defined over F' . Note that $X_*(T) \cong X_*(T_0) \cong X_*(T')$ and $X^*(T) \cong X^*(T_0) \cong X^*(T')$ via ϕ and ϕ' .

Recall the notation of Chai–Yu: $(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}) \equiv_{\psi_m, \gamma} (\mathfrak{O}_{F'}, \mathfrak{O}_{K'}, \Gamma_{K'/F'})$ (level m) from Section 2.B. We write

$$(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}, H) \equiv_{\psi_m, \gamma, \mathfrak{O}_m^c} (\mathfrak{O}_{F'}, \mathfrak{O}'_{K'}, \Gamma_{K'/F'}, H') \text{ (level } m)$$

to mean $(\mathfrak{O}_F, \mathfrak{O}_K, \Gamma_{K/F}) \equiv_{\alpha, \beta} (\mathfrak{O}_{F'}, \mathfrak{O}'_{K'}, \Gamma_{K'/F'})$ (level m), H and H' arise from the same \mathbb{Z} -pinned group $(G_0, B_0, T_0, \{u_\alpha\}_{\alpha \in \tilde{\Delta}})$, and the F -quasisplit data (G, B, T) with cocycle s corresponds to the F' -quasisplit data (G', B', T') with cocycle s' via \mathfrak{O}_m^c as in Lemma 3.3 (b) (but applied to K and K' respectively). To abbreviate notation we will write *congruence data* D_m to mean

$$D_m : (\mathfrak{O}, \mathfrak{O}_K, \Gamma_{K/F}, H) \equiv_{\psi_m, \gamma, \mathfrak{O}_m^c} (\mathfrak{O}_{F'}, \mathfrak{O}'_{K'}, \Gamma_{K'/F'}, H') \text{ (level } m).$$

Lemma 3.4. *The congruence data D_m induces isomorphisms:*

$$\begin{aligned} X^*(T)^{\text{Gal}(\Omega/F)} &\cong X^*(T')^{\text{Gal}(\Omega'/F')}, & X_*(T)^{\text{Gal}(\Omega/F)} &\cong X_*(T')^{\text{Gal}(\Omega'/F')}, \\ X^*(T)^{\text{Gal}(\Omega/F)} &\cong X^*(T')^{\text{Gal}(\Omega'/F')}, & X_*(T)^{\text{Gal}(\Omega/F)} &\cong X_*(T')^{\text{Gal}(\Omega'/F')}. \end{aligned}$$

Proof. We know that $\gamma *_F (\phi(x)) = \phi(s(\gamma)(\gamma \cdot x))$ where $s(\gamma) = \phi^{-1} \circ \gamma(\phi)$ takes values in $H = H(G_0, T_0, B_0, \{u_\alpha\}_{\alpha \in \tilde{\Delta}})$. We similarly have $*_{F'}$. This action induces the action on $X_*(T)$ as follows:

$$\gamma *_F (\phi \circ \lambda) = \phi \circ (s(\gamma)(\lambda))$$

where $\gamma \in \text{Gal}(\Omega/F)$ and $\lambda \in X_*(T_0)$, where we now view $s(\gamma)$ as an element of $\text{Aut}(R, \tilde{\Delta})$. By definition $s(\gamma)(\lambda) = s'(\gamma')(\lambda)$ where $\gamma' = \text{Del}_m(\gamma)$. Hence $\gamma' *_F (\phi' \circ \lambda) = \phi' \circ s'(\gamma')(\lambda)$. Now, $X_*(T)^{\text{Gal}(\Omega/F)} = \{\phi \circ \lambda \mid s(\gamma)(\lambda) = \lambda\}$. The lemma is now clear. \square

4. Congruences of parahoric group schemes: quasisplit descent

4.A. Apartment over close local fields. In this section, we additionally assume that F is strictly Henselian. We begin with the following lemma.

Lemma 4.1. *Let T as above and let S be the maximal split subtorus of T . Then S is maximal F -split and $Z_G(S) = T$.*

Proof. Let $S \subset \tilde{S}$ with \tilde{S} maximal F -split. Since G is quasisplit over F , $\tilde{T} = Z_G(\tilde{S})$ is a maximal torus in G and we can assume that $\tilde{T} \subset \tilde{B}$, with \tilde{B} defined over F . Then B and \tilde{B} are $G(F)$ -conjugate, which implies that T and \tilde{T} are $G(F)$ -conjugate. But conjugation by an element of $G(F)$ will preserve the split and anisotropic components of T , which implies that S and \tilde{S} are $G(F)$ -conjugate, which forces $S = \tilde{S}$ to be maximal F -split. It is now clear that $Z_G(S) = T$. \square

Remark 4.2. The torus $S^{\text{der}} := S \cap G^{\text{der}}$ is a maximal F -split torus of G^{der} contained in $T^{\text{der}} := T \cap G^{\text{der}}$.

4.A.1. Compatibility of Chevalley–Steinberg systems. Recall that we have fixed a \mathbb{Z} -pinning $\{u_\alpha\}_{\alpha \in \Delta}$ of G_0 . This, via the Galois action given by the cocycles s and s' , gives rise to a Steinberg splitting $\{x_\alpha\}_{\alpha \in \Delta}$ of G and a Steinberg splitting $\{x'_{\alpha'}\}_{\alpha' \in \Delta'}$ of G' respectively. Let $\Phi_m : \Phi(G, T) \xrightarrow{\sim} \Phi(G', T')$ (since both are isomorphic to $\Phi(G_0, T_0)$). This isomorphism is Del_m -equivariant. Note that with $\gamma \in \text{Gal}(\Omega/F)$ and $\gamma' = \text{Del}_m(\gamma)$, we have that $x_{\gamma(\alpha)} = \gamma \circ x_\alpha \circ \gamma^{-1}$ and $x'_{\gamma'(\alpha')} = \gamma' \circ x'_{\alpha'} \circ \gamma'^{-1}$ where $\alpha' = \Phi_m(\alpha)$. The $\{x_\alpha\}_{\alpha \in \Delta}$ and $\{x'_{\alpha'}\}_{\alpha' \in \Delta'}$ each extend to Chevalley–Steinberg systems on G and G' respectively and continue to have the compatibility with Del_m in the sense described above.

We define

$$e_F := \begin{cases} e_{F/\mathbb{Q}_2} = \omega_F(2) & \text{if } \text{char}(F) = 0 \text{ and residue char}(F) = 2, \\ \infty & \text{otherwise.} \end{cases}$$

We prove the following refinement of Lemma 4.3.3 of [Bruhat and Tits 1984] when the residue characteristic of F is 2, using the additional hypothesis that the extension K/F splitting G is at most m -ramified.

Lemma 4.3. *Let $m \geq 1$ and let F be of residue characteristic 2 with $e_F \geq m$. Let G, B, T as above, where G splits over K with $\text{Gal}(K/F)^m = 1$. Assume that $a, 2a \in \Phi(G, S)$. Consider the separable quadratic extension L_a/L_{2a} inside K . Let $e_a = e_{L_a/F}$, $e_{2a} = e_{L_{2a}/F}$. There exists $t \in L_a$ with $L_a = L_{2a}[t]$ and the coefficients $A, B \in L_{2a}$ of the equation $t^2 + At + B = 0$ satisfied by t have the following properties:*

- (a) $\omega(B) = 0$ or B is a uniformizer of L_{2a} .
- (b) $\omega(B) \leq \omega(A) < \frac{m}{2} + \frac{1}{e_a}$.

In particular $A \neq 0$.

Proof. By Lemma 4.3.3(ii) of [Bruhat and Tits 1984], (a) holds, and $A = 0$ or $\omega(B) \leq \omega(A) < \omega(2)$ or $0 < \omega(B) \leq \omega(A) = \omega(2)$. Since $\text{Gal}(K/F)^m = \text{Gal}(K/F)_{\psi_{K/F}(m)} = 1$ where $\psi_{K/F}$ denotes the inverse of the Herbrand function (See Chapter 4 of [Serre 1979]), we have

$$\text{Gal}(K/L_{2a})^{\psi_{L_{2a}/F}(m)} = \text{Gal}(K/L_{2a})_{\psi_{K/F}(m)} = \text{Gal}(K/L_{2a}) \cap \text{Gal}(K/F)_{\psi_{K/F}(m)} = 1.$$

This implies that $\text{Gal}(L_a/L_{2a})^{\psi_{L_{2a}/F}(m)} = 1$. Using the equivalence of (ii) and (iv) of Lemma A.6.1 of [Deligne 1984], we see that

$$\omega(\tau(t) - t) < \frac{\psi_{L_{2a}/F}(m) + 1}{2e_{2a}} = \frac{\psi_{L_{2a}/F}(m) + 1}{e_a}. \quad (4-1)$$

It is easy to see from the definition that $\psi_{L_{2a}/F}(m) \leq m \cdot e_{2a}$. Hence

$$\omega(\tau(t) - t) < \frac{m}{2} + \frac{1}{e_a}.$$

Now, $\omega(A) = \omega(\tau(t) + t) \geq \min(\omega(\tau(t) - t), \omega(2t))$, and $\omega(2t) = \omega(2) + \omega(t) = e_F + \frac{1}{e_a}$. Since $e_F \geq m > m/2$, we see that

$$\omega(A) = \min(\omega(\tau(t) - t), \omega(2t)) = \omega(\tau(t) - t) < \frac{m}{2} + \frac{1}{e_a} \quad (4-2)$$

and in particular, $A \neq 0$.

Note that when the characteristic of F is 2, the claim that $A \neq 0$ simply follows from the fact that the extension L_a/L_{2a} is separable. \square

Proposition 4.4. *Let G , T and B as in the preceding paragraph. Let $m \geq 1$ and let F, F' be such that $e_F, e_{F'} \geq m$. The congruence data D_m induces a simplicial isomorphism $\mathcal{A}_m : \mathcal{A}(S, F) \rightarrow \mathcal{A}(S', F')$, where (G', B', T') corresponds to the triple (G, B, T) as above and S (resp. S') is the maximal split subtorus of T (resp. T') which is maximal F -split (resp. F' -split) by Lemma 4.1. Furthermore, with W^e as in Section 2.D.1, we also have a group isomorphism $W^e \cong W^{e'}$.*

Proof. The reduced apartment $\mathcal{A}(S, F)$ is an affine space under $X_*(S^{\text{der}}) \otimes_{\mathbb{Z}} \mathbb{R}$. Using Lemma 3.4, we see that D_m induces a unique bijection $\mathcal{A}_m : \mathcal{A}(S, F) \rightarrow \mathcal{A}(S', F')$ such that $x_0 \rightarrow x'_0$ (where x_0, x'_0 are as in Section 2.D.1 arising from Chevalley–Steinberg systems chosen compatibly as in Section 4.A.1).

It remains to observe that \mathcal{A}_m is a simplicial isomorphism. Recall that the elements of $\Phi(G, S)$ are restrictions to S of the elements of $\Phi(G, T)$ and two elements of $\Phi(G, T)$ restrict to the same element of $\Phi(G, S)$ if and only if they lie in the same $\text{Gal}(K/F)$ -orbit. Further, with $\tilde{\Delta}$ denoting a base of $\Phi(G, T)$, the elements $\alpha|_S, \alpha \in \tilde{\Delta}$ form a base Δ of $\Phi(G, S)$. Let $\Phi_m : \Phi(G, T) \xrightarrow{\sim} \Phi(G', T')$ (since both are isomorphic to $\Phi(G_0, T_0)$). This isomorphism is Del_m -equivariant. Hence the obvious map $\Phi(G, S) \rightarrow \Phi(G', S'), \alpha|_S \rightarrow \Phi_m(\alpha)|_{S'}$, which we also denote as Φ_m , is an isomorphism of the relative root systems. (In more detail, since S and S' have the same rank, we have a isomorphism of \mathbb{R} -vector spaces $X^*(S) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow X^*(S') \otimes_{\mathbb{Z}} \mathbb{R}$. Further, we have a bijection between $\Delta \rightarrow \Delta'$; this is because if $\Phi_m(\alpha)|_{S'} = \Phi_m(\beta)|_{S'}$, then there is $\eta' \in \text{Gal}(\Omega'/F')$ with $\eta' \cdot \Phi_m(\alpha) = \Phi_m(\beta)$. Then $\eta \cdot \alpha = \beta$ where $\eta' = \text{Del}_m(\eta)$. Finally note that $\langle \Phi_m(\alpha)|_{S'}, \Phi_m(\beta)|_{S'} \rangle = \langle \Phi_m(\alpha), \Phi_m(\beta) \rangle = \langle \alpha, \beta \rangle = \langle \alpha|_S, \beta|_S \rangle$).

The vanishing hyperplanes with respect to the affine roots $\Phi^{\text{af}}(G, S)$ gives the simplicial structure on $\mathcal{A}(S, F)$. Recall that

$$\Phi^{\text{af}}(G, F) = \{\psi : \mathcal{A}(S, F) \rightarrow \mathbb{R} \mid \psi(\cdot) = a(\cdot - x_0) + l, a \in \Phi(G, S), l \in \tilde{\Gamma}_a\}.$$

For any $a \in \Phi(G, S)$, let $a' = \Phi_m(a)$. Let $L_{a'} \subset K'$ denote splitting extension of the root a' obtained by Del_m . Since F is strictly Henselian, the extensions L_a/F and $L_{a'}/F'$ are totally ramified. To prove that the bijection Φ_m extends to a bijection $\Phi_m^{\text{af}}: \Phi^{\text{af}}(G, F) \rightarrow \Phi^{\text{af}}(G', F')$ making \mathcal{A}_m a simplicial isomorphism, we simply have to observe that for each $a \in \Phi(G, S)$, $\tilde{\Gamma}_a = \tilde{\Gamma}_{a'}$. By Section 4.3.4 of [Bruhat and Tits 1984], we have the following:

Case I. Suppose $a \in \Phi^{\text{red}}(G, S)$, $2a \notin \Phi(G, S)$. Then $\Gamma_a = \tilde{\Gamma}_a = \frac{1}{e_a}\mathbb{Z}$.

Case II. Suppose $a, 2a \in \Phi(G, S)$.

(a) Suppose L_a/L_{2a} is ramified and the residue characteristic of F is not 2. Then

$$\tilde{\Gamma}_a = \frac{1}{e_a}\mathbb{Z} \quad \text{and} \quad \tilde{\Gamma}_{2a} = \frac{1}{e_a} + \frac{1}{e_{2a}}\mathbb{Z}.$$

(b) Suppose L_a/L_{2a} is ramified and the residue characteristic of F is 2. By Lemma 4.3, $A \neq 0$. Then

$$\tilde{\Gamma}_a = \frac{1}{2e_a} + \frac{1}{e_a}\mathbb{Z} \quad \text{and} \quad \tilde{\Gamma}_{2a} = \frac{1}{e_{2a}}\mathbb{Z}.$$

Since $e_a = e_{a'}$, $e_{2a} = e_{2a'}$, and the valuations ω and ω' are normalized so that $\omega(F) = \omega'(F') = \mathbb{Z}$, we have $\tilde{\Gamma}_a = \tilde{\Gamma}_{a'}$ for all $a \in \Phi(G, S)$. \square

4.B. Congruences of parahoric group schemes: strictly Henselian case. In this section, we additionally assume that F is strictly Henselian.

Theorem 4.5. Let $m \geq 1$ and let F and F' be such that $e_F, e_{F'} \geq 2m$. Let l be as in Lemma 2.2 and let D_l and G, S, T, B as in the beginning of this section. Let $\mathcal{F} \in \mathcal{A}(S, F)$ and $\mathcal{F}' = \mathcal{A}_m(\mathcal{F})$ as in Proposition 4.4. Let $\mathcal{P}_{\mathcal{F}}$ be the parahoric group scheme over \mathfrak{O}_F attached to \mathcal{F} by Bruhat–Tits, and let $\mathcal{P}_{\mathcal{F}'}$ be the group scheme attached to \mathcal{F}' over $\mathfrak{O}_{F'}$. Then the congruence data D_l induces an isomorphism of group schemes

$$\tilde{p}_m: \mathcal{P}_{\mathcal{F}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \rightarrow \mathcal{P}_{\mathcal{F}'} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m \times_{\psi_m^{-1}} \mathfrak{O}_F/\mathfrak{p}_F^m.$$

In particular, $\mathcal{P}_{\mathcal{F}}(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong \mathcal{P}_{\mathcal{F}'}(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m)$ as groups.

To prove this theorem, we will study the reduction of root subgroup schemes mod \mathfrak{p}_F^m and prove that they are determined by congruence data, use the result of Chai–Yu that the reduction of the Néron model of the torus is determined by congruence data, study the reduction of \mathfrak{O}_F -birational group laws in Section 2.D.3, and invoke the Artin–Weil theorem to obtain the corresponding result for parahoric group schemes in Section 4.B.1.

The following lemma is easy.

Lemma 4.6. Let M be a free \mathfrak{O}_F -module of finite type and let $A = \text{Sym}_{\mathfrak{O}_F}(M^\vee)$ be the symmetric algebra of M^\vee , where $M^\vee := \text{Hom}_{\mathfrak{O}_F}(M, \mathfrak{O}_F)$. Then

$$A \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m \cong \text{Sym}_{\mathfrak{O}_F/\mathfrak{p}_F^m}(M^\vee \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m) \cong \text{Sym}_{\mathfrak{O}_F/\mathfrak{p}_F^m}(\text{Hom}_{\mathfrak{O}_F/\mathfrak{p}_F^m}(M \otimes_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m, \mathfrak{O}_F/\mathfrak{p}_F^m)).$$

Lemma 4.7. Let $m \geq 1$, let F and F' be such that $e_F, e_{F'} \geq 2m$ and let D_m as before. Let $a \in \Phi(G, S)$ and $k \in \mathbb{R}$. Let $\mathcal{U}_{a,k}$ be the \mathfrak{O}_F -group scheme in Section 2.D.2. Let $a' = \Phi_m(a) \in \Phi(G', S')$ and let $\mathcal{U}'_{a',k}$

be the $\mathfrak{D}_{F'}$ -group scheme in Section 2.D.2. Then the congruence data D_m induces an isomorphism of group schemes

$$\mathcal{U}_{a,k} \times_{\mathfrak{D}_F} \mathfrak{D}_F/\mathfrak{p}_F^m \cong \mathcal{U}_{a',k} \times_{\mathfrak{D}_{F'}} \mathfrak{D}_{F'}/\mathfrak{p}_{F'}^m \times_{\psi_m^{-1}} \mathfrak{D}_F/\mathfrak{p}_F^m.$$

In particular,

$$\mathcal{U}_{a,k}(\mathfrak{D}_F/\mathfrak{p}_F^m) \cong \mathcal{U}_{a',k}(\mathfrak{D}_{F'}/\mathfrak{p}_{F'}^m).$$

Proof. We will stick to the notation in Section 2.D.2.

Case I. Suppose $a \in \Phi^{\text{red}}(G, S)$, $2a \notin \Phi(G, S)$. The affine ring representing $\mathcal{U}_{a,k}$ is isomorphic to $\text{Sym}_{\mathfrak{D}_F} L_{a,k}^\vee$. Note that $L_{a,k} = \mathfrak{p}_{L_a}^{[k/e]}$. Since \mathfrak{p}_{L_a} is a free \mathfrak{D}_F -module of rank equal to $[L_a : F]$, it is clear that the data D_m induces an isomorphism of $L_{a,k} \otimes_{\mathfrak{D}_F} \mathfrak{D}_F/\mathfrak{p}_F^m$ and $L_{a',k} \otimes_{\mathfrak{D}_{F'}} \mathfrak{D}_{F'}/\mathfrak{p}_{F'}^m$, and we are done by the previous lemma.

Case II. Suppose $a, 2a \in \Phi(G, S)$. Since F is strictly Henselian, the extension L_a/L_{2a} is totally ramified. Let $L_a = L_{2a}(t)$, where $t^2 + At + B = 0$ with A, B satisfying Lemma 4.3.3 of [Bruhat and Tits 1984]. When:

- The residue characteristic of F is not 2, we take $\lambda = \frac{1}{2}$ (See Lemma 4.3.3(ii) of [loc. cit.]).
- The residue characteristic of F is 2, we take $\lambda = tA^{-1}$ (using Lemma 4.3.3(ii) of [loc. cit.] and Lemma 4.3).

Then the affine ring representing the scheme \mathcal{H}_0^λ is

$$\text{Sym}_{\mathfrak{D}_{L_{2a}}} L_{a,k+\gamma}^\vee \otimes_{\mathfrak{D}_{L_{2a}}} \text{Sym}_{\mathfrak{D}_{L_{2a}}} (L_{a,l}^0)^\vee \cong \text{Sym}_{\mathfrak{D}_{L_{2a}}} ((L_{a,k+\gamma} \times L_{a,l}^0)^\vee),$$

where $l = 2k + \frac{1}{e_a}$. We describe $L_{a,l}^0$.

- (a) If the residue characteristic of F is not 2, then using that $\omega(2) = 0$ in Lemma 4.3.3 of [loc. cit.], we see that $A = 0$. Then $L_a^0 = \{x \in L_a \mid \tau(x) + x = 0\} = \{yt \mid y \in L_{2a}\}$ and

$$L_{a,l}^0 = \{yt \mid y \in L_{2a}, \omega(yt) \geq l\} = \{yt \mid y \in L_{2a}, \omega(y) \geq 2k\}.$$

- (b) If the residue characteristic of F is 2, then:

- (i) If $\text{char}(F) = 2$, then $L_a^0 = L_{2a}$ and $L_{a,l}^0 = \{y \in L_{2a} \mid \omega(y) \geq l\}$.
- (ii) If $\text{char}(F) = 0$, then $L_a^0 = \{y(1 - 2tA^{-1}) \mid y \in L_{2a}\}$. By Lemma 4.3, we have

$$\omega(2tA^{-1}) = e_F + \frac{1}{e_a} - \omega(A) > e_F - \frac{m}{2} \geq m$$

since $e_F \geq 2m$. Hence $1 - 2tA^{-1} \in 1 + \mathfrak{p}_{L_a}^{me_a}$, and $L_{a,l}^0 = \{y(1 - 2tA^{-1}) \mid y \in L_{2a}, \omega(y) \geq l\}$.

Let $L_{a'} \subset \Omega'$ be obtained from L_a via the Deligne isomorphism Del_m . Then $L_{a'}$ is the splitting extension of the root a' (and similarly we obtain $L_{2a'}$). We may and do assume that $L_{a'} = L_{2a'}(t')$, where $t'^2 + A't' + B' = 0$, with A', B' satisfying:

- $\omega(A) = \omega'(A')$ and $A \bmod \mathfrak{p}_{L_{2a}}^{me_{2a}} \xrightarrow{\psi_m} A' \bmod \mathfrak{p}_{L_{2a'}}^{me_{2a}}$.
- $\omega(B) = \omega'(B')$ and $B \bmod \mathfrak{p}_{L_{2a}}^{me_{2a}} \xrightarrow{\psi_m} B' \bmod \mathfrak{p}_{L_{2a'}}^{me_{2a}}$.

Then $t \bmod \mathfrak{p}_{L_a}^{me_a} \xrightarrow{\psi_m} t' \bmod \mathfrak{p}_{L'_{2a'}}^{me_a}$. It is now easy to check that the map ψ_m induces isomorphisms

$$\begin{aligned} L_{a,k+\gamma} \otimes_{\mathfrak{D}_{L_{2a}}} \mathfrak{D}_{L_{2a}}/\mathfrak{p}_{L_{2a}}^{me_{2a}} &\cong L_{a',k+\gamma}^0 \otimes_{\mathfrak{D}_{L'_{2a'}}} \mathfrak{D}_{L'_{2a'}}/\mathfrak{p}_{L'_{2a'}}^{me_{2a}} \\ L_{a,l}^0 \otimes_{\mathfrak{D}_{L_{2a}}} \mathfrak{D}_{L_{2a}}/\mathfrak{p}_{L_{2a}}^{me_{2a}} &\cong L_{a',l}^0 \otimes_{\mathfrak{D}_{L'_{2a'}}} \mathfrak{D}_{L'_{2a'}}/\mathfrak{p}_{L'_{2a'}}^{me_{2a}}. \end{aligned}$$

In the above, we have used that when the residue characteristic of F is 2, $1 - 2tA^{-1} \equiv 1 \bmod \mathfrak{p}_{L_a}^{me_a}$. Consequently, D_m induces an isomorphism of the reduction of the respective affine rings $\bmod \mathfrak{p}_{L_{2a}}^{me_{2a}}$. To see that this is an isomorphism of group schemes, we observe that reducing the map

$$\begin{aligned} j : L_{a,k} \times L_{a,l}^0 \times L_{a,k} \times L_{a,l}^0 &\rightarrow L_{a,k} \times L_{a,l}^0 \\ ((x, y), (x', y')) &\rightarrow (x + x', y + y' - \lambda x \tau(x') + \lambda x' \tau(x)) \end{aligned}$$

$\bmod \mathfrak{p}_{L_{2a}}^{me_{2a}}$ is ψ_m -equivariant. Finally $\mathcal{H}^\lambda = \text{Res}_{\mathfrak{D}_F}^{\mathfrak{D}_{L_{2a}}} \mathcal{H}_0^\lambda$ and the result now follows from [Bosch et al. 1990, page 192].

The lemma for $\mathcal{U}_{2a,k}$ follows using that

$$L_{a,k}^0 \otimes_{\mathfrak{D}_{L_{2a}}} \mathfrak{D}_{L_{2a}}/\mathfrak{p}_{L_{2a}}^{me_{2a}} \cong L_{a',k}^0 \otimes_{\mathfrak{D}_{L'_{2a'}}} \mathfrak{D}_{L'_{2a'}}/\mathfrak{p}_{L'_{2a'}}^{me_{2a}}$$

and [Bosch et al. 1990, page 192]. \square

The following corollary is an obvious consequence of the previous lemma.

Corollary 4.8. *With assumptions of Lemma 4.7, and with $\mathcal{F}' = \mathcal{A}_m(\mathcal{F})$ where \mathcal{F} is a facet in $\mathcal{A}(S, F)$, let $\mathcal{U}_{a,\mathcal{F}}$ (resp. $\mathcal{U}_{a',\mathcal{F}'}$) be the smooth root subgroup scheme over \mathfrak{D}_F (resp. $\mathfrak{D}_{F'}$) as in Section 2.D.3. The congruence data D_m induces an isomorphism*

$$\mathcal{U}_{a,\mathcal{F}} \times_{\mathfrak{D}_F} \mathfrak{D}_F/\mathfrak{p}_F^m \cong \mathcal{U}_{a',\mathcal{F}'} \times_{\mathfrak{D}_{F'}} \mathfrak{D}_{F'}/\mathfrak{p}_{F'}^m \times_{\psi_m^{-1} \mathfrak{D}_F/\mathfrak{p}_F^m} \mathfrak{D}_F/\mathfrak{p}_F^m.$$

In particular, $\mathcal{U}_{a,\mathcal{F}}(\mathfrak{D}_F/\mathfrak{p}_F^m) \cong \mathcal{U}_{a',\mathcal{F}'}(\mathfrak{D}_{F'}/\mathfrak{p}_{F'}^m)$ as groups.

4.B.1. Proof of Theorem 4.5. For a scheme X defined over a local ring R with maximal ideal \mathfrak{m} , we will denote $X^{(m)} := X \times_R R/\mathfrak{m}^m$. Let l be as in Lemma 2.2. We want to prove that D_l induces an isomorphism of $\mathfrak{D}_F/\mathfrak{p}_F^m$ -group schemes $\mathcal{P}_{\mathcal{F}}^{(m)} \cong \mathcal{P}_{\mathcal{F}'}^{(m)} \times_{\psi_m^{-1} \mathfrak{D}_F/\mathfrak{p}_F^m} \mathfrak{D}_F/\mathfrak{p}_F^m$. Let $\mathcal{X}_{\mathcal{F}}, \mathcal{X}_{\mathcal{F}'}$ be as in Section 2.D.3. Let $m^{(m)}$ be the $\mathfrak{D}_F/\mathfrak{p}_F^m$ -birational group law on $\mathcal{X}_{\mathcal{F}}^{(m)}$ and similarly $n^{(m)}$ on $\mathcal{X}_{\mathcal{F}'}^{(m)}$. Note that via D_l , we also have that

$$(\mathfrak{D}_F, \mathfrak{D}_K, \Gamma_{K/F}, \Lambda) \equiv_{\psi_\epsilon, \gamma, \lambda} (\mathfrak{D}_{F'}, \mathfrak{D}_{K'}, \Gamma_{K'/F'}, \Lambda') \text{ (level } l)$$

as in the notation of Chai–Yu of Section 2.B, where $\Lambda = X_*(T)$, $\Lambda' = X_*(T')$; so the result of Lemma 2.2 holds. We know by Lemmas 2.2 and 4.8 that

$$\mathcal{X}_{\mathcal{F}}^{(m)} \cong \mathcal{X}_{\mathcal{F}'}^{(m)} \times_{\psi_m^{-1} \mathfrak{D}_F/\mathfrak{p}_F^m} \mathfrak{D}_F/\mathfrak{p}_F^m \quad (4-3)$$

as $\mathfrak{D}_F/\mathfrak{p}_F^m$ -schemes. Further, by these lemmas, we also have that the $\mathfrak{D}_F/\mathfrak{p}_F^m$ -birational group laws $n^{(m)} \times_{\psi_m^{-1} \mathfrak{D}_F/\mathfrak{p}_F^m} m^{(m)}$ and $m^{(m)}$ on $\mathcal{X}_{\mathcal{F}}^{(m)}$ are equivalent. Since $\mathcal{Y}_{\mathcal{F}}$ is the \mathfrak{D}_F -scheme obtained by gluing G and $\mathcal{X}_{\mathcal{F}}$ along $\mathcal{X}_{\mathcal{F}} \times_{\mathfrak{D}_F} F$, we have that $\mathcal{Y}_{\mathcal{F}}^{(m)}$ is isomorphic to $\mathcal{X}_{\mathcal{F}}^{(m)}$ as $\mathfrak{D}_F/\mathfrak{p}_F^m$ -schemes. Now, $\mathcal{P}_{\mathcal{F}}^{(m)}$ with

group law $\bar{m}^{(m)}$, and $\mathcal{P}_{\mathcal{F}'}^{(m)} \times_{\psi_m^{-1}} \mathfrak{D}_F/\mathfrak{p}_F^m$ with group law $\bar{n}^{(m)} \times_{\psi_m^{-1}} \mathfrak{D}_F/\mathfrak{p}_F^m$, are both smooth, separated $\mathfrak{D}_F/\mathfrak{p}_F^m$ -group schemes that are faithfully flat and of finite type. Recall that the restriction of \bar{m} to $\mathcal{Y}_{\mathcal{F}}$ is m , and similarly for \bar{n} . Hence the group laws $\bar{m}^{(m)}$ and $\bar{n}^{(m)} \times_{\psi_m^{-1}} \mathfrak{D}_F/\mathfrak{p}_F^m$ have the same restriction to $\mathcal{Y}_{\mathcal{F}}^{(m)}$. Following the proof of uniqueness of Artin–Weil theorem (see Proposition 3, Section 5.1 of [Bosch et al. 1990]), we obtain that the group schemes $\mathcal{P}_{\mathcal{F}}^{(m)}$ and $\mathcal{P}_{\mathcal{F}'}^{(m)} \times_{\psi_m^{-1}} \mathfrak{D}_F/\mathfrak{p}_F^m$ are isomorphic. \square

4.C. Congruences of parahoric group schemes: descending from $G_{\widehat{F_{\text{un}}}}$ to G_F . In this section, F denotes a nonarchimedean local field and $\widehat{F_{\text{un}}}$ denotes the completion of the maximal unramified extension F_{un} of F . Let A be a maximal F -split torus in G , S maximal F_{un} -split F -torus that contains A . Let $T = Z_G(S)$. Note that $X_*(S) = X_*(T)^{\text{Gal}(\Omega/F_{\text{un}})}$ and $X_*(A) = X_*(T)^{\text{Gal}(\Omega/F)}$.

Lemma 4.9. *The simplicial isomorphism*

$$\mathcal{A}_m : \mathcal{A}(S, \widehat{F_{\text{un}}}) \rightarrow \mathcal{A}(S', \widehat{F'_{\text{un}}})$$

of Proposition 4.4 is Del_m -equivariant.

Proof. This is clear from the proof of Proposition 4.4, Section 4.A.1, and Lemma 3.4. \square

Let $\sigma \in \text{Gal}(\widehat{F_{\text{un}}}/F)$ be as in Section 2.E. Let \mathcal{F} be a facet in $X_*(A)$. Then \mathcal{F} corresponds to a σ -stable facet $\tilde{\mathcal{F}}$ in $X_*(S)$. Note that Del_m induces isomorphisms

$$\text{Gal}(\widehat{F_{\text{un}}}/F) \cong \text{Gal}(F_s/F)/I_F \cong \text{Gal}(F'_s/F')/I_{F'} \cong \text{Gal}(\widehat{F'_{\text{un}}}/F').$$

Let $\sigma' = \text{Del}_m(\sigma)$ under this isomorphism. Let $\tilde{\mathcal{F}}' = \mathcal{A}_m(\tilde{\mathcal{F}})$ and $\mathcal{F}' = \tilde{\mathcal{F}}'^{\sigma'}$.

Proposition 4.10. *The isomorphism*

$$\tilde{p}_m : \mathcal{P}_{\tilde{\mathcal{F}}} \times_{\mathfrak{D}_{\widehat{F_{\text{un}}}}} \mathfrak{D}_{\widehat{F_{\text{un}}}}/\mathfrak{p}_{\widehat{F_{\text{un}}}}^m \rightarrow \mathcal{P}_{\tilde{\mathcal{F}}'} \times_{\mathfrak{D}_{\widehat{F'_{\text{un}}}}} \mathfrak{D}_{\widehat{F'_{\text{un}}}}/\mathfrak{p}_{\widehat{F'_{\text{un}}}}^m$$

has the property that $\sigma' \circ \tilde{p}_m = \tilde{p}_m \circ \sigma$.

Proof. Recall that the cocycle s_G has been chosen to take values in $\text{Aut}(H)$ and $s_G \rightarrow s_{G'}$ via Lemma 3.3. Further, \mathcal{T} is defined over \mathfrak{D}_F and $\mathcal{T}_{\widehat{F_{\text{un}}}} = \mathcal{T} \times_{\mathfrak{D}_F} \mathfrak{D}_{\widehat{F_{\text{un}}}}$. From this it is clear that $\sigma' \circ \tilde{p}_m = \tilde{p}_m \circ \sigma$ on $\mathcal{T} \times_{\mathfrak{D}_{\widehat{F_{\text{un}}}}} \mathfrak{D}_{\widehat{F_{\text{un}}}}/\mathfrak{p}_{\widehat{F_{\text{un}}}}^m$. In addition, using the fact that Chevalley–Steinberg systems on G and G' have been chosen compatibly (see Section 4.A.1), it is easy to see that $\sigma' \circ \tilde{p}_m = \tilde{p}_m \circ \sigma$ on $\mathcal{U}_{\tilde{\mathcal{F}}} \times_{\mathfrak{D}_{\widehat{F_{\text{un}}}}} \mathfrak{D}_{\widehat{F_{\text{un}}}}/\mathfrak{p}_{\widehat{F_{\text{un}}}}^m$. This completes the proof of the proposition. \square

5. Inner forms of quasisplit groups over close local fields

Let F be a nonarchimedean local field and let G be a connected reductive group over F . Then there is a quasisplit group G_q defined over F such that G is an inner form of G_q . In particular, the F -isomorphism class of G is determined by an element in $H^1(\Gamma_F, G_q^{\text{ad}}(F_s))$. Moreover if $[G] \in E(F, G_0)_m$ then $[G_q] \in E(F, G_0)_m$ and $[G]$ is determined by an element of $H^1(\text{Aut}(\Omega/F), G_q^{\text{ad}}(\Omega))$ (Recall that $\Omega = (F_s)^{I_F^m}$). Let s_{G_q} be the element of $H^1(\Gamma_F/I_F^m, \text{Aut}(R, \Delta)^{I_F^m})$ that determines (G_q, B_q, T_q) , up to

F -isomorphisms. Let G_q^{der} be the derived subgroup of G_q and let $G_q^{\text{ad}}, G_q^{\text{sc}}$ denote the corresponding adjoint and simply connected groups. Then the groups $G_q^{\text{der}}, G_q^{\text{ad}}, G_q^{\text{sc}}$ are quasisplit (if S_q is a maximal F -split torus in G_q whose centralizer T_q is a maximal torus, then $S_q \cap G_q^{\text{der}}$ is a maximal F -split torus of G_q^{der} and $Z_{G_q^{\text{der}}}(S_q \cap G_q^{\text{der}}) = T_q \cap G_q^{\text{der}}$ is a maximal torus of G_q^{der} , similarly for G_q^{ad} and G_q^{sc}) and are in fact forms of $G_0^{\text{der}}, G_0^{\text{ad}}$ and G_0^{sc} respectively (to see this note that $G_q^{\text{der}} \times_F \Omega \cong (G_q \times_F \Omega)^{\text{der}}$ and $Z(G_q) \times_F \Omega \cong Z(G_q \times_F \Omega)$). Using Proposition 13.1(1) of [Kottwitz 2014] and the fact that G_q^{ad} has trivial center, we have a canonical bijection

$$\kappa_{G_q} : H^1(\text{Aut}(\Omega/F), G_q^{\text{ad}}(\Omega)) \rightarrow (X_*(T_q^{\text{ad}})/X_*(T_q^{\text{sc}}))_{\text{Aut}(\Omega/F)}.$$

Let $E_i(F, G_q)_m$ denote the F -isomorphism classes of inner forms of G_q that split over an at most m -ramified extension of F . Let (G'_q, B'_q, T'_q) correspond to the cocycle $q' \circ \Omega_m(s_{G_q})$ and let $E_i(F', G'_q)_m$ be the corresponding object over F' .

Lemma 5.1. *The congruence data D_m induces an isomorphism*

$$\mathfrak{I}_m : (X_*(T_q^{\text{ad}})/X_*(T_q^{\text{sc}}))_{\text{Aut}(\Omega/F)} \xrightarrow{\sim} (X_*(T'^{\text{ad}}_q)/X_*(T'^{\text{sc}}_q))_{\text{Aut}(\Omega'/F')}.$$

In particular, D_m induces a bijection $E_i(F, G_q)_m \rightarrow E_i(F', G'_q)_m$, $[G] \rightarrow [G']$ where $s_{G'} = \kappa_{G'_q}^{-1} \circ \mathfrak{I}_m \circ \kappa_{G_q}(s_G)$.

Proof. Note that $X_*(T_q) \cong X_*(T_0) \cong X_*(T'_q)$ as \mathbb{Z} -modules and the Galois action on $X_*(T_q)$ is determined by the cocycle s_{G_q} (and similarly for $X_*(T_q^{\text{ad}}), X_*(T_q^{\text{sc}})$). Now the lemma is obvious by Lemma 3.3. \square

To proceed, we need to prove a version of Lemma 5.1 at the level of cocycles. To do this, we will use some results from Section 2 of [DeBacker and Reeder 2009].

Steinberg's vanishing theorem. Let G be a connected, reductive F -group. Steinberg's vanishing theorem asserts that

Theorem 5.2 [Steinberg 1965, Theorem 56]. $H^1(\text{Gal}(F_s/F_{\text{un}}), G(F_s)) = 1$.

As a corollary of this theorem, we obtain that the natural surjection from $\text{Gal}(F_s/F) \rightarrow \text{Gal}(F_{\text{un}}/F)$ induces an isomorphism

$$H^1(\text{Gal}(F_{\text{un}}/F), G(F_{\text{un}})) \cong H^1(\text{Gal}(F_s/F), G(F_s)).$$

5.A. Congruence data for inner forms: a comparison of cocycles. Let A_q be a maximal F -split torus in G_q and let S_q be a maximal F_{un} -split F -torus in G_q that contains A_q . Let $T_q = Z_{G_q}(S_q)$. Then T_q is a maximal torus in $G_{q, F_{\text{un}}}$ with maximal F_{un} -split torus S_q . Let C_q be an σ -stable alcove in $\mathcal{A}(S_q, F_{\text{un}})$.

Let P_{C_q} be the Iwahori subgroup of $G_q^{\text{ad}}(F_{\text{un}})$ attached to C_q . Let $\Omega_{C_q}^{\text{ad}} \subset \tilde{W}^{\text{ad}} := X_*(T_q^{\text{ad}})_{I_F} \rtimes W$ consist of elements which preserve the alcove C_q . Here I_F is the inertia subgroup of F and $W = W(G_{q, F_{\text{un}}}, S_{q, F_{\text{un}}})$. Then

$$\Omega_{C_q}^{\text{ad}} \cong (X_*(T_q^{\text{ad}})/X_*(T_q^{\text{sc}}))_{I_F} \quad (5-1)$$

by Lemma 15 of [Haines and Rapoport 2008]. Let $P_{C_q}^*$ be the normalizer in G_q^{ad} of P_{C_q} . Let $N_{C_q}^{\text{ad}} = N_{G_q^{\text{ad}}}(S_q^{\text{ad}})(F_{\text{un}}) \cap P_{C_q}^*$. Then $\Omega_{C_q}^{\text{ad}}$ is the image of $N_{C_q}^{\text{ad}}$ in \tilde{W}^{ad} and $\Omega_{C_q}^{\text{ad}} \cong P_{C_q}^*/P_{C_q}$.

The following lemma is proved in Sections 2.3 and 2.4 of [DeBacker and Reeder 2009]. Although the authors assume that $G_{q,F_{\text{un}}}$ is split in the beginning of Section 2.3 of [DeBacker and Reeder 2009], this assumption is not needed in their proof of the following lemma. They use that when $G_{q,F_{\text{un}}}$ is split, $\Omega_{C_q}^{\text{ad}} \cong X_*(T_q^{\text{ad}})/X_*(T_q^{\text{sc}})$ in Corollary 2.4.2 and Corollary 2.4.3; one should instead use (5-1) when $G_{q,F_{\text{un}}}$ is not necessarily split.

Lemma 5.3 [DeBacker and Reeder 2009, Corollary 2.4.3]. *We have isomorphisms*

$$H^1(\text{Gal}(F_{\text{un}}/F), \Omega_{C_q}^{\text{ad}}) \cong H^1(\text{Gal}(F_{\text{un}}/F), N_{C_q}^{\text{ad}}) \cong H^1(\text{Gal}(F_{\text{un}}/F), G_q^{\text{ad}}(F_{\text{un}})).$$

Let c be a cocycle in $Z^1(\text{Gal}(F_{\text{un}}/F), \Omega_{C_q}^{\text{ad}})$. By Lemma 2.1.2 of [DeBacker and Reeder 2009], since $\Omega_{C_q}^{\text{ad}}$ is finite, we have

$$Z^1(\text{Gal}(F_{\text{un}}/F), \Omega_{C_q}^{\text{ad}}) = \Omega_{C_q}^{\text{ad}}.$$

Let G be the inner form of G_q determined by c . Let $c(\sigma) = w_\sigma$. Write $w_\sigma = (\lambda, w)$ with $\lambda \in X_*(T^{\text{ad}})_{I_F}$ and $w \in W$. Let $K \subset F_s$ denote the finite at most m -ramified extension of F_{un} over which $G_{q,F_{\text{un}}}$ splits. Let $t = Nm(\tilde{\lambda}(\pi_K))$ where $Nm : T_q^{\text{ad}}(K) \rightarrow T_q^{\text{ad}}(F_{\text{un}})$ and $\tilde{\lambda} \rightarrow \lambda$ under the usual surjection $X_*(T_q^{\text{ad}}) \rightarrow X_*(T_q^{\text{ad}})_I$. Let $\tilde{w} \in N_{G_q}(S_q)(F_{\text{un}})$ be the representative of w chosen using the Chevalley–Steinberg system we fixed in Section 4.A.1.

Let $m_\sigma = t\tilde{w}$. Since w_σ stabilizes C_q , it follows that $m_\sigma P_{C_q} m_\sigma^{-1} = P_{C_q}$. Hence $m_\sigma \in P_{C_q}^*$. Therefore $\tilde{c}(\sigma) = m_\sigma \in Z^1(\text{Gal}(F_{\text{un}}/F), N_{C_q}^{\text{ad}})$. Denoting

$$G(F_{\text{un}}) \rightarrow G_q(F_{\text{un}}), \quad g_* \rightarrow g,$$

the new action of σ on an element $g_* \in G(F_{\text{un}})$, which we denote by σ_* , is given by

$$\sigma_* \cdot g_* = (\tilde{c}(\sigma)(\sigma \cdot g))_*$$

(Here $\sigma \cdot g$ denotes the action of σ on $g \in G_q(F_{\text{un}})$). Note that $c(\sigma) \in G_q^{\text{ad}}(F_{\text{un}}) = \text{Inn}(G_q)(F_{\text{un}})$. The maximal F_{un} -split torus S_q of G_q gives a maximal F_{un} -split, F_{un} -torus S in G . Let $X_*(S) \rightarrow X_*(S_q)$, $\tau_* \rightarrow \tau$. For $\tau_* \in X_*(S)$, $\sigma_* \cdot \tau_* = (w_\sigma(\sigma \cdot \tau))_*$. Since S_q is defined over F , $\sigma \cdot \tau \in X_*(S_q)$. Since $w_\sigma \in \Omega_{C_q}^{\text{ad}}$, we see that $X_*(S)$ is stable under the action of σ , and hence S is defined over F .

Lemma 5.4. *Let A be the F -split torus of G determined by the \mathbb{Z} -module $X_*(S)^{\sigma_*}$. Then A is a maximal F -split torus in G .*

Proof. Consider the reduced apartment $\mathcal{A}(S_q, \widehat{F_{\text{un}}})$. We view this as an apartment in the reduced building of $G(\widehat{F_{\text{un}}})$ and denote it as $\mathcal{A}(S, \widehat{F_{\text{un}}})$. The action σ_* on $x_* \in \mathcal{A}(S, \widehat{F_{\text{un}}})$ given by $\sigma_* \cdot x_* = (w_\sigma(\sigma \cdot x))_*$. Let C_* denote the alcove in $\mathcal{A}(S, \widehat{F_{\text{un}}})$ corresponding to C_q . Then $\sigma_* \cdot C_* = (w_\sigma(\sigma \cdot C_q))_*$. Since $\sigma \cdot C_q = C_q$ and since $w_\sigma \in \Omega_{C_q}^{\text{ad}}$, we see that C_* is a σ_* -stable alcove in $\mathcal{A}(S, \widehat{F_{\text{un}}})$. In particular, $\mathcal{A}(S, \widehat{F_{\text{un}}})$ is σ_* -stable. By Proposition 5.1.14 of [Bruhat and Tits 1984], $C_*^{\sigma_*}$ is an alcove in the affine space $\mathcal{A}(A, F)$. Since $\mathcal{A}(A, F)$ contains a facet of maximal possible dimension, we see that A is maximal F -split in G . \square

Let (G'_q, T'_q, B'_q, S'_q) correspond to (G_q, T_q, B_q, S_q) via congruence data D_m as in Section 4. By Lemma 3.4, we have

$$\Omega_{C'_q}^{\text{ad}} \cong \Omega_{C_q}^{\text{ad}}.$$

Let $w_{\sigma'} \in \Omega_{C'_q}^{\text{ad}}$ be the image of w_σ under this isomorphism. This isomorphism gives rise to a bijection of pointed sets

$$\begin{aligned} \mathfrak{I}_m : Z^1(\text{Gal}(F_{\text{un}}/F), \Omega_{C_q}^{\text{ad}}) &\rightarrow Z^1(\text{Gal}(F'_{\text{un}}/F'), \Omega_{C'_q}^{\text{ad}}), \\ c &\rightarrow c' \end{aligned} \quad (5-2)$$

where $c'(\sigma') = w_{\sigma'}$. Let $m_{\sigma'} = t' \tilde{w}'$ where $w_{\sigma'} = (\lambda', w') \in X_*(T_q^{\text{ad}})_{I_{F'}} \rtimes W'$. Here $t' = Nm(\tilde{\lambda}'(\pi'_{K'}))$ where $Nm : T_q^{\text{ad}'}(K') \rightarrow T_q^{\text{ad}'}(F'_{\text{un}})$ and $\tilde{\lambda}' \rightarrow \lambda'$ under the usual surjection $X_*(T_q^{\text{ad}'}) \rightarrow X_*(T_q^{\text{ad}'})_{I_{F'}}$, and $\tilde{\lambda} \rightarrow \tilde{\lambda}'$ under the isomorphism $X_*(T_q^{\text{ad}}) \cong X_*(T_q^{\text{ad}'})$. Also \tilde{w}' is the representative of w chosen using the Chevalley–Steinberg system fixed in Section 4.A.1. Let $\tilde{c}' \in Z^1(\text{Gal}(F'_{\text{un}}/F'), N_{C'_q}^{\text{ad}'})$ be the cocycle with $\tilde{c}'(\sigma') = m_{\sigma'}$.

Let G' be the inner form of G'_q determined by c' (or \tilde{c}'). Let S' be the maximal F'_{un} -split, F_{un} -torus of G' corresponding to S'_q but with the action of σ' given by the cocycle \tilde{c}' . More precisely, for $g'_* \in G'(F'_{\text{un}})$,

$$\sigma'_* \cdot g'_* = (\tilde{c}'(\sigma') \cdot (\sigma' \cdot g'_*))_*$$

where $\sigma' = \text{Del}_m(\sigma)$ as before, and $\sigma' \cdot g'$ denotes the action of σ' on $G'_q(F'_{\text{un}})$.

As in Lemma 5.4, we see that S' is an F' -torus that is maximal F'_{un} -split and whose split component A' is a maximal F' -split torus in G' .

Corollary 5.5. *With $G \rightarrow G'$ as above, the F -rank of G is equal to the F' -rank of G' .*

Proof. This is because $\text{rank}(S) = \text{rank}(S')$ and the isomorphism $X_*(S) \rightarrow X_*(S')$ is σ_* -equivariant. Hence $\text{rank}(A) = \text{rank}(A')$ by Lemma 5.4. \square

6. Congruences of parahoric group schemes: étale descent

The following lemma is easy.

Lemma 6.1. *The σ -equivariant isomorphism $\tilde{\mathcal{A}}_m : \mathcal{A}(S_q, \widehat{F_{\text{un}}}) \rightarrow \mathcal{A}(S'_q, \widehat{F'_{\text{un}}})$ induces a σ_* -equivariant isomorphism $\tilde{\mathcal{A}}_{m,*} : \mathcal{A}(S, \widehat{F_{\text{un}}}) \rightarrow \mathcal{A}(S', \widehat{F'_{\text{un}}})$.*

Now let $\tilde{\mathcal{F}}_*$ be σ_* -invariant facet in $\mathcal{A}(S, \widehat{F_{\text{un}}})$ and let $\tilde{\mathcal{F}}'_* = \tilde{\mathcal{A}}_{m,*}(\tilde{\mathcal{F}}_*)$. Let $\mathcal{F}_* = \tilde{\mathcal{F}}_*^{\sigma_*}$ and $\mathcal{F}'_* = \tilde{\mathcal{F}}'^{\sigma'_*}$.

Proposition 6.2. *Let $m \geq 1$, F, F' nonarchimedean local fields with $e_F, e_{F'} \geq 2m$. Let l as in Theorem 4.5, let D_l be the congruence data of level l , and let (G'_q, T'_q, B'_q, S'_q) correspond to (G_q, T_q, B_q, S_q) via D_l . Let $\tilde{p}_m : \mathcal{P}_{\tilde{\mathcal{F}}} \times_{\mathfrak{D}_{\widehat{F_{\text{un}}}}} \mathfrak{D}_{\widehat{F_{\text{un}}}} / \mathfrak{p}_{\widehat{F_{\text{un}}}}^m \rightarrow \mathcal{P}_{\tilde{\mathcal{F}}'} \times_{\mathfrak{D}_{\widehat{F'_{\text{un}}}}} \mathfrak{D}_{\widehat{F'_{\text{un}}}} / \mathfrak{p}_{\widehat{F'_{\text{un}}}}^m$ denote the σ -equivariant isomorphism of Theorem 4.5 and Proposition 4.10. Let $c \rightarrow c'$ via \mathfrak{I}_m (see (5-2)). The isomorphism \tilde{p}_m induces a σ_* -equivariant isomorphism $\tilde{p}_{m,*} : \mathcal{P}_{\tilde{\mathcal{F}}_*} \times_{\mathfrak{D}_{\widehat{F_{\text{un}}}}} \mathfrak{D}_{\widehat{F_{\text{un}}}} / \mathfrak{p}_{\widehat{F_{\text{un}}}}^m \rightarrow \mathcal{P}_{\tilde{\mathcal{F}}'_*} \times_{\mathfrak{D}_{\widehat{F'_{\text{un}}}}} \mathfrak{D}_{\widehat{F'_{\text{un}}}} / \mathfrak{p}_{\widehat{F'_{\text{un}}}}^m$.*

Proof. We begin by understanding the action of σ_* on an element of $P_{\tilde{F}_*}$ more explicitly. Recall that

$$P_{\tilde{F}} = \langle \mathcal{U}_{\tilde{F}}^+(\mathfrak{S}_{F_{\text{un}}}), \mathcal{T}(\mathfrak{S}_{F_{\text{un}}}), \mathcal{U}_{\tilde{F}}^-(\mathfrak{S}_{F_{\text{un}}}) \rangle$$

Let $g \in P_{\tilde{F}}$. Then $\sigma_* \cdot g_* = (m_\sigma(\sigma \cdot g)m_\sigma^{-1})_*$. Let $b_0 \in \Phi^{\text{red}}(G_q, S_q)$ such that $2b_0$ is not a root. Let $y \in U_{b_0, \tilde{F}}$. Fix $\beta_0|_{S_q} = b_0$, fix the pinning (L_{β_0}, x_{b_0}) and write $y = x_{b_0}(u_0)$ for $u_0 \in L_{\beta_0}$ (As explained in Section 2.C.2, $L_{b_0} \cong L_{\beta_0} \hookrightarrow K$). Let $\tilde{\sigma}$ denote a lift of σ to Γ_F and let $\beta = \tilde{\sigma} \cdot \beta_0$, $b = \sigma \cdot b_0$. Then we obtain a pinning (L_β, x_b) from the pinning (L_{β_0}, x_{b_0}) via $\tilde{\sigma}$ and we have $\sigma \cdot x_{b_0}(u_0) = x_b(\tilde{\sigma} \cdot u_0)$; this follows using properties of Chevalley–Steinberg system recalled in Section 2.C.1 (a), (b). Let $u = \tilde{\sigma} \cdot u_0$. Then $u \in L_\beta$. We need to compute $\tilde{w}x_b(u)\tilde{w}^{-1}$. We will first compute $\tilde{s}_a x_b(u)\tilde{s}_a^{-1}$ for $a \in \Delta$. Note that

$$\tilde{s}_a = \prod_{\alpha \in \tilde{\Delta}_a} \tilde{s}_\alpha \quad (6-1)$$

and that $L_{s_a \cdot b} = L_b$. Now for $\alpha_1, \beta_1 \in \Phi(G_q, T_q)$, we have $\tilde{s}_{\alpha_1} x_{\beta_1}(z)\tilde{s}_{\alpha_1}^{-1} = x_{s_{\alpha_1}(\beta_1)}(d_{\alpha_1, \beta_1} z)$ for all $z \in K$, with $d_{\alpha_1, \beta_1} = \pm 1$. Using the properties of Chevalley–Steinberg system recalled in Section 2.C.1 (a), (b), we have

$$d_{\alpha_1, \beta_1} = d_{\gamma(\alpha_1), \gamma(\beta_1)} \quad \forall \gamma \in \text{Gal}(K/\widehat{F_{\text{un}}}). \quad (6-2)$$

With β as above, note that $\beta|_{S_q} = b$. Let

$$d_{a,b} := \prod_{\alpha \in \tilde{\Delta}_a} d_{\alpha, \beta}$$

This notation is justified since (6-2) implies that the definition of $d_{a,b}$ does not depend on the choice of β . Using the definition of x_b in (2-2), a simple calculation yields that $\tilde{s}_a x_b(u)\tilde{s}_a^{-1} = x_{s_a(b)}(d_{a,b}u)$. Since we chose our Chevalley–Steinberg systems compatibly (see Section 4.A.1), we evidently have $d_{a,b} = d_{a',b'}$ for all $a \in \Delta$, $b \in \Phi(G_q, S_q)$. Iterating this process, we see that $\tilde{w}x_b(u)\tilde{w}^{-1} = x_{w \cdot b}(d_{w,b}u)$ where $d_{w,b} = \pm 1$ and $d_{w,b} = d_{w',b'}$.

Suppose $b_0 \in \Phi(G_q, S_q)$ such that $2b_0$ is a root. Let $\beta_0, \bar{\beta}_0|_{S_q} = b_0$. Fix the pinning $(L_{\beta_0}, L_{\beta_0 + \bar{\beta}_0}, x_{b_0})$ and write $y = x_{b_0}(u_0, v_0)$, with $u_0, v_0 \in L_{\beta_0}$ (Recall that $L_{b_0} \cong L_{\beta_0} \subset K$). Let $\beta = \tilde{\sigma} \cdot \beta_0$, $\bar{\beta} = \tilde{\sigma} \cdot \bar{\beta}_0$ and $b = \sigma \cdot b_0$. We then obtain a pinning $(L_\beta, L_{\beta + \bar{\beta}}, x_b)$ via $\tilde{\sigma}$ and $\sigma \cdot x_{b_0}(u_0, v_0) = x_{\sigma \cdot b_0}(\tilde{\sigma} \cdot u_0, \tilde{\sigma} \cdot v_0)$ where $\tilde{\sigma}$ as before. Let $u = \tilde{\sigma} \cdot u_0$, $v = \tilde{\sigma} \cdot v_0$. Then $u, v \in L_\beta$. We need to compute $\tilde{s}_a x_b(u, v)\tilde{s}_a^{-1}$ where s_a is as in (6-1). Let

$$d_{a,b} := \prod_{\alpha \in \tilde{\Delta}_a} d_{\alpha, \beta}, \quad d_{a,2b} := \prod_{\alpha \in \tilde{\Delta}_a} d_{\alpha, \beta + \bar{\beta}}$$

Again, the definitions of $d_{a,b}$ and $d_{a,2b}$ do not depend on the choice of β by (6-2).

Then a simple calculation yields

$$\tilde{s}_a x_b(u, v)\tilde{s}_a^{-1} = x_{s_a(b)}(d_{a,b}u, d_{a,2b}v).$$

Proceeding as in the previous case, we have $\tilde{w}x_b(u, v)\tilde{w}^{-1} = x_{w \cdot b}(d_{w,b}u, d_{w,2b}v)$ where $d_{w,b}, d_{w,2b} = \pm 1$ and $d_{w,b} = d_{w',b'}$ and $d_{w,2b} = d_{w',2b'}$.

Recall that $t = Nm(\tilde{\lambda}(\pi_K)) \in T_q^{\text{ad}}(F_{\text{un}})$. Then for each $\gamma \in \text{Gal}(K/F_{\text{un}})$, $\gamma \cdot t = t$. Let $c \in \Phi^{\text{red}}(G_q, S_q)$ with $2c$ not a root. Let $\chi \in \Phi(G_q, T_q)$ with $\chi|_{S_q} = c$. Note that χ factors through T_q^{ad} . Fixing the pinning (L_χ, x_c) we have that $\chi : T \rightarrow G_m$ is defined over L_χ and $\chi(t) \in L_\chi^\times$. A simple calculation yields $tx_c(u)t^{-1} = x_c(\chi(t)u)$ for each $u \in L_\chi$. If $c, 2c$ are roots, then with $\chi, \bar{\chi}$ such that $\chi|_{S_q} = \bar{\chi}|_{S_q} = c$ and fixing the pinning $(L_\chi, L_{\chi+\bar{\chi}}, x_c)$, it follows that $tx_c(u, v)t^{-1} = x_c(\chi(t)u, (\chi + \bar{\chi})(t)v)$. Hence, if $2b_0$ is not a root then

$$\sigma_* \cdot (x_{b_0}(u_0))_* = (x_{w \cdot b}(d_{w,b}\chi(t)u))_*$$

where $\chi|_{S_q} = w \cdot b$. If $2b_0$ is a root, then

$$\sigma_* \cdot (x_{b_0}(u_0, v_0))_* = (x_{w \cdot b}(d_{w,b}\chi(t)u, d_{w,2b}(\chi + \bar{\chi})(t)v))_*$$

where $\chi, \chi' \in \Phi(G_q, T_q)$ are such that $\chi, \bar{\chi}|_{S_q} = w \cdot b$. It is easy to calculate $\sigma_* \cdot (x_{2b_0}(0, v_0))_*$ using the observations above. For $x \in \mathcal{T}_q(\widehat{\mathfrak{O}_{F_{\text{un}}}})$,

$$\sigma_* \cdot x_* = (w(\sigma \cdot x)w^{-1})_*.$$

Combining these observations with the fact that \tilde{p}_m is σ -equivariant (see Proposition 4.10), it follows that the map $\tilde{p}_{m,*}$ has the property that $\tilde{p}_{m,*} \circ \sigma_* = \sigma'_* \circ \tilde{p}_{m,*}$ (in this verification, we choose σ' to correspond to $\tilde{\sigma}$ via Del_m). \square

Corollary 6.3. *The isomorphism $\tilde{p}_{m,*}$ induces an isomorphism of group schemes*

$$p_{m,*} : \mathcal{P}_{\mathcal{F}_*} \times_{\mathfrak{O}_F} \mathfrak{O}_F / \mathfrak{p}_F^m \rightarrow \mathcal{P}_{\mathcal{F}'} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'} / \mathfrak{p}_{F'}^m \times_{\psi_m^{-1}} \mathfrak{O}_F / \mathfrak{p}_F^m.$$

In particular $\mathcal{P}_{\mathcal{F}_}(\mathfrak{O}_F / \mathfrak{p}_F^m)$ and $\mathcal{P}_{\mathcal{F}'}(\mathfrak{O}_{F'} / \mathfrak{p}_{F'}^m)$ are isomorphic as groups.*

Proof. This follows from Proposition 6.2 and étale descent [Bosch et al. 1990, Example B, Section 6.2]. \square

Acknowledgments

I would like to express my gratitude to J. K. Yu for introducing me to questions related to this article and for the insightful discussions during my graduate school years. I thank the referees for the helpful comments and suggestions, particularly regarding removing the assumption on residue characteristic that was made in an earlier version of this article. I thank Dipendra Prasad and K. V. Shuddhodan for the interesting discussions and for their encouragement.

References

- [Aubert et al. 2016] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld, “The local Langlands correspondence for inner forms of SL_n ”, *Res. Math. Sci.* **3** (2016), art. id. 32. MR Zbl
- [Badulescu 2002] A. I. Badulescu, “Correspondance de Jacquet–Langlands pour les corps locaux de caractéristique non nulle”, *Ann. Sci. École Norm. Sup. (4)* **35**:5 (2002), 695–747. MR Zbl
- [Bosch et al. 1990] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, *Ergebnisse der Mathematik* (3) **21**, Springer, 1990. MR Zbl

- [Bruhat and Tits 1972] F. Bruhat and J. Tits, “Groupes réductifs sur un corps local”, *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5–251. MR Zbl
- [Bruhat and Tits 1984] F. Bruhat and J. Tits, “Groupes réductifs sur un corps local, II: Schémas en groupes, existence d’une donnée radicielle valuée”, *Inst. Hautes Études Sci. Publ. Math.* **60** (1984), 197–376. MR Zbl
- [Chai and Yu 2001] C.-L. Chai and J.-K. Yu, “Congruences of Néron models for tori and the Artin conductor”, *Ann. of Math.* (2) **154**:2 (2001), 347–382. MR Zbl
- [Conrad 2011] B. Conrad, “Reductive group schemes”, preprint, SGA3 Summer School, 2011, Available at <https://tinyurl.com/conreduct>.
- [DeBacker and Reeder 2009] S. DeBacker and M. Reeder, “Depth-zero supercuspidal L -packets and their stability”, *Ann. of Math.* (2) **169**:3 (2009), 795–901. MR Zbl
- [Deligne 1984] P. Deligne, “Les corps locaux de caractéristique p , limites de corps locaux de caractéristique 0”, pp. 119–157 in *Representations of reductive groups over a local field*, Hermann, Paris, 1984. MR Zbl
- [Ganapathy 2015] R. Ganapathy, “The local Langlands correspondence for GSp_4 over local function fields”, *Amer. J. Math.* **137**:6 (2015), 1441–1534. MR Zbl
- [Ganapathy and Varma 2017] R. Ganapathy and S. Varma, “On the local Langlands correspondence for split classical groups over local function fields”, *J. Inst. Math. Jussieu* **16**:5 (2017), 987–1074. MR Zbl
- [Haines and Rapoport 2008] T. Haines and M. Rapoport, “Appendix: On parahoric subgroups”, 2008. Appendix to G. Pappas and M. Rapoport, “Twisted loop groups and their affine flag varieties”, *Adv. Math.* **219**:1 (2008), 118–198. MR Zbl
- [Howe 1985] R. Howe, *Harish-Chandra homomorphisms for p -adic groups*, CBMS Regional Conf. Series in Math. **59**, Amer. Math. Soc., Providence, RI, 1985. MR Zbl
- [Kazhdan 1986] D. Kazhdan, “Representations of groups over close local fields”, *J. Analyse Math.* **47** (1986), 175–179. MR Zbl
- [Kottwitz 2014] R. Kottwitz, “ $B(G)$ for all local and global fields”, preprint, 2014. arXiv
- [Landvogt 1996] E. Landvogt, *A compactification of the Bruhat–Tits building*, Lecture Notes in Math. **1619**, Springer, 1996. MR Zbl
- [Lemaire 2001] B. Lemaire, “Représentations génériques de GL_N et corps locaux proches”, *J. Algebra* **236**:2 (2001), 549–574. MR Zbl
- [Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Math. **67**, Springer, 1979. MR Zbl
- [Steinberg 1965] R. Steinberg, “Regular elements of semisimple algebraic groups”, *Inst. Hautes Études Sci. Publ. Math.* **25** (1965), 49–80. MR Zbl

Communicated by Michael Rapoport

Received 2018-11-06 Revised 2019-04-16 Accepted 2019-05-25

radhikag@iisc.ac.in

School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India

Algebra & Number Theory

msp.org/ant

EDITORS

MANAGING EDITOR

Bjorn Poonen
Massachusetts Institute of Technology
Cambridge, USA

EDITORIAL BOARD CHAIR

David Eisenbud
University of California
Berkeley, USA

BOARD OF EDITORS

Richard E. Borcherds	University of California, Berkeley, USA	Martin Olsson	University of California, Berkeley, USA
Antoine Chambert-Loir	Université Paris-Diderot, France	Raman Parimala	Emory University, USA
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Jonathan Pila	University of Oxford, UK
Brian D. Conrad	Stanford University, USA	Anand Pillay	University of Notre Dame, USA
Samit Dasgupta	University of California, Santa Cruz, USA	Michael Rapoport	Universität Bonn, Germany
Hélène Esnault	Freie Universität Berlin, Germany	Victor Reiner	University of Minnesota, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Peter Sarnak	Princeton University, USA
Hubert Flenner	Ruhr-Universität, Germany	Joseph H. Silverman	Brown University, USA
Sergey Fomin	University of Michigan, USA	Michael Singer	North Carolina State University, USA
Edward Frenkel	University of California, Berkeley, USA	Christopher Skinner	Princeton University, USA
Wee Teck Gan	National University of Singapore	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Andrew Granville	Université de Montréal, Canada	J. Toby Stafford	University of Michigan, USA
Ben J. Green	University of Oxford, UK	Pham Huu Tiep	University of Arizona, USA
Joseph Gubeladze	San Francisco State University, USA	Ravi Vakil	Stanford University, USA
Roger Heath-Brown	Oxford University, UK	Michel van den Bergh	Hasselt University, Belgium
Craig Huneke	University of Virginia, USA	Akshay Venkatesh	Institute for Advanced Study, USA
Kiran S. Kedlaya	Univ. of California, San Diego, USA	Marie-France Vignéras	Université Paris VII, France
János Kollár	Princeton University, USA	Kei-Ichi Watanabe	Nihon University, Japan
Philippe Michel	École Polytechnique Fédérale de Lausanne	Melanie Matchett Wood	University of Wisconsin, Madison, USA
Susan Montgomery	University of Southern California, USA	Shou-Wu Zhang	Princeton University, USA
Shigefumi Mori	RIMS, Kyoto University, Japan		

PRODUCTION

production@msp.org
Silvio Levy, Scientific Editor


See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2019 is US \$385/year for the electronic version, and \$590/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

Algebra & Number Theory

Volume 13 No. 6 2019

Positivity functions for curves on algebraic varieties BRIAN LEHMANN and JIAN XIAO	1243
The congruence topology, Grothendieck duality and thin groups ALEXANDER LUBOTZKY and TYAKAL NANJUNDIAH VENKATARAMANA	1281
On the ramified class field theory of relative curves QUENTIN GUIGNARD	1299
Blow-ups and class field theory for curves DAICHI TAKEUCHI	1327
Algebraic monodromy groups of l -adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ SHIANG TANG	1353
Weyl bound for p -power twist of $\mathrm{GL}(2)$ L -functions RITABRATA MUNSHI and SAURABH KUMAR SINGH	1395
Examples of hypergeometric twistor \mathcal{D} -modules ALBERTO CASTAÑO DOMÍNGUEZ, THOMAS REICHELT and CHRISTIAN SEVENHECK	1415
Ulrich bundles on K3 surfaces DANIELE FAENZI	1443
Unlikely intersections in semiabelian surfaces DANIEL BERTRAND and HARRY SCHMIDT	1455
Congruences of parahoric group schemes RADHIKA GANAPATHY	1475
An improved bound for the lengths of matrix algebras YAROSLAV SHITOV	1501



1937-0652(2019)13:6;1-7