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An improved bound for the lengths of matrix algebras

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Let S be a set of $n \times n$ matrices over a field $\mathbb{F}$. We show that the $\mathbb{F}$-linear span of the words in S of length at most

$$2n \log_2 n + 4n$$

is the full $\mathbb{F}$-algebra generated by S. This improves on the $\frac{n^2}{3} + \frac{2}{3}$ bound by Paz (1984) and an $O(n^{3/2})$ bound of Pappacena (1997).

Let S be a subset of a finite-dimensional associative algebra $\mathcal{A}$ over a field $\mathbb{F}$. An element $a \in \mathcal{A}$ is said to be a *word* of length k in S if there are $a_1, \dots, a_k \in S$ such that $a = a_1 \cdots a_k$. We denote the set of all such words by S^k, and we write $\mathbb{F}S^k$ for the $\mathbb{F}$-linear span of S^k. Similarly, $\mathbb{F}S^{\leqslant k}$ will stand for the $\mathbb{F}$-linear span of all the words in S that have length at most k.

Definition 1. The length $\ell(S)$ is the smallest integer k for which $\mathbb{F}S^{\leqslant k}$ is the full subalgebra generated by S. We also define $\ell(\mathcal{A})$ as the maximum value of $\ell(S)$, where S runs over all subsets of $\mathcal{A}$ that generate $\mathcal{A}$ as an $\mathbb{F}$-algebra.

In our paper, we study the length of $\mathrm{Mat}_n(\mathbb{F})$, the set of $n \times n$ matrices viewed as an algebra over $\mathbb{F}$. A. Paz [1984] proved that $\ell(S) \leqslant \frac{n^2}{3} + \frac{2}{3}$ for all $S \subset \mathrm{Mat}_n(\mathbb{F})$ and proposed the following appealing conjecture.

Conjecture 2. *For all* $S \subset \mathrm{Mat}_n(\mathbb{F})$, *one has* $\ell(S) \leqslant 2n - 2$.

As shown by T. Laffey [1986, page 131], the upper bound in Conjecture 2 should be sharp. This conjecture is known to hold if the size of matrices is at most four [Paz 1984] or if $\mathbb{F}S$ contains a nonderogatory matrix [Guterman et al. 2018]. However, the best known general upper bounds on the lengths of matrix subsets are quite far from the one prescribed by Conjecture 2. It was only in 1997 when a subquadratic estimate was obtained: C. Pappacena proved an $O(n^{3/2})$ upper bound on the length of $\mathrm{Mat}_n(\mathbb{F})$, but no further improvements have been made since then [Guterman et al. 2018; Lambrou and Longstaff 2009; Longstaff et al. 2006]. The main result of this paper is a much stronger $O(n \log n)$ upper bound on the length of $\mathrm{Mat}_n(\mathbb{F})$.

Theorem 3. *For all* $S \subset \mathrm{Mat}_n(\mathbb{F})$, *we have* $\ell(S) \leqslant 2n \log_2 n + 4n - 4$.

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As an additional motivation of our study, we note that the best known upper bounds on a complete set of unitary invariants for $n \times n$ matrices [Laffey 1986] and on the PI degree of semiprime affine algebras of Gelfand–Kirillov dimension one [Pappacena et al. 2003] come from the estimates of $\ell(\mathrm{Mat}_n(\mathbb{F}))$, so the current work also improves our understanding of those invariants.

1. Warm-up

In this section, we explain the idea behind our main construction and illustrate how it works in a simpler setting. We get a small improvement on one of the results of Pappacena [1997], which allows us to prove the $n = 5$ case of Conjecture 2.

We say that a set $S \subset \mathrm{Mat}_n(\mathbb{F})$ is *irreducible* if it generates $\mathrm{Mat}_n(\mathbb{F})$ as the $\mathbb{F}$-algebra. If a set S is not irreducible, and if $\mathbb{F}$ is algebraically closed, then there exist $p \in \{1, \dots, n-1\}$ and $Q \in \mathrm{GL}_n(\mathbb{F})$ such that, for any $A \in S$, we have

$$Q^{-1}AQ = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix} \tag{1-1}$$

where A_{11} is a $p \times p$ matrix (and O is the zero matrix of appropriate dimensions). This is *Burnside's theorem*; see [Radjavi and Rosenthal 2000, Theorem 1.5.1].

Lemma 4 [Markova 2005, Corollary 3]. *Let $\mathcal{A}$ be a matrix algebra whose elements are of the form (1-1), and let $\mathcal{A}_1, \mathcal{A}_2$ be the sets of all A_{11}, A_{22} blocks of matrices in $\mathcal{A}$, respectively. Then $\ell(\mathcal{A}) \leqslant \ell(\mathcal{A}_1) + \ell(\mathcal{A}_2) + 1$.*

We will say that a matrix $Z \in \mathrm{Mat}_n(F)$ is *square-zero* if $Z^2 = 0$. The main idea of the proof of Theorem 3 is to control the product $\lambda\rho(\lambda)$, where $\rho(\lambda)$ is the minimal rank of nonzero square-zero matrices that arise as linear combinations of words of length at most λ. We show in Section 2 below that we can reduce ρ to 1 whilst saving the property $\lambda\rho(\lambda) \in O(n \log n)$, and then we apply Pappacena's technique to deal with low rank matrices; see [Pappacena 1997, Theorem 4.1] and Corollary 7 below. More precisely, let $H \in \mathbb{F}S^{\leqslant\lambda}$ be a square-zero matrix; it can be written as

$$H = \begin{pmatrix} O & O & I_\rho \\ O & O & O \\ O & O & O \end{pmatrix}$$

with respect to some basis. If some matrix A with bottom-left block of small rank $r > 0$ comes as a linear combination of words of length l, then the matrix HAH is square-zero, has rank r, and comes as a linear combination of words of length at most $l + 2\lambda$. As we will see in Claims 13 and 14 below, we can always find an appropriate matrix A to reduce the rank of a square-zero matrix. The following lemma illustrates our approach to the proof of Claim 13.

Lemma 5. *Consider an irreducible set $S \subset \mathbb{F}^{n\times n}$ and a nonzero vector $v \in \mathbb{F}^n$. If $\mathbb{F}S^{\leqslant(n-2)}v \neq \mathbb{F}^n$, then $\mathbb{F}S$ contains a matrix with minimal polynomial of degree n.*

Proof. The sequence

$$\mathbb{F}v = \mathbb{F}S^0 v \subset \mathbb{F}S^{\leqslant 1} v \subset \cdots \subset \mathbb{F}S^{\leqslant k} v = \mathbb{F}^n$$

is strictly increasing [Pappacena 1997, Theorem 4.1], so the assumption of the lemma implies $k = n-1$ and $\dim \mathbb{F}S^{\leqslant t} v - \dim \mathbb{F}S^{\leqslant (t-1)} v = 1$ for all $t \in \{1, \dots, n-1\}$. Therefore, we can set $\mathcal{B}_0 = \{v\}$ and inductively complete $\mathcal{B}_{t-1}$ to a basis $\mathcal{B}_t$ of $\mathbb{F}S^{\leqslant t}$ by adding a single vector v_t. With respect to the basis $\{v, v_1, \dots, v_{n-1}\}$, every matrix in S has the form

$$A = \begin{pmatrix} * & \cdots & \cdots & * & * \\ a_{21} & * & \cdots & * & * \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & * & * \\ 0 & \cdots & 0 & a_{n,n-1} & * \end{pmatrix}$$

with $*$'s denoting the entries we need not specify. Since S is irreducible, all of the $(i+1, i)$ entries are nonzero at some matrix in S, so a generic element of $\mathbb{F}S$ has all of them nonzero — which means that its minimal polynomial has degree n. □

Theorem 6 [Guterman et al. 2018, Theorems 2.4 and 2.5]. *If an irreducible set* $\mathbb{F}S \subset \mathrm{Mat}_n(\mathbb{F})$ *contains a matrix with minimal polynomial of degree* $n-1$ *or* n, *then* $\ell(S) \leqslant 2n-2$.

Lemma 5 and Theorem 6 lead to a tiny improvement of the $r = 1$ case of Theorem 4.1(a) in [Pappacena 1997], which is nevertheless useful to study the case of small n.

Corollary 7. *Let* $S \subset \mathrm{Mat}_n(\mathbb{F})$ *be an irreducible set and* $k \geqslant 2$. *If* $\mathbb{F}S^{\leqslant k}$ *contains a rank-one matrix, then* $\ell(S) \leqslant 2n + k - 4$.

Proof. If $\mathbb{F}S$ contains a matrix with minimal polynomial of degree n, then we are done by Theorem 6. Otherwise, we use Lemma 5 and get

$$\mathbb{F}S^{\leqslant (n-2)} A S^{\leqslant (n-2)} = \sum \mathrm{Mat}_n(\mathbb{F}) \cdot A \cdot \mathrm{Mat}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$$

for any rank-one matrix A. □

We are almost ready to prove the $n = 5$ case of Conjecture 2.

Claim 8. *Assume that the minimal polynomial of every matrix in* $\mathbb{F}S \subset \mathrm{Mat}_n(\mathbb{F})$ *has degree at most* 2. *Then* $\ell(S) \leqslant 2\log_2 n$.

Proof. We denote by w a word in $S^{\ell(S)}$ that is not spanned by shorter words. For any $A, B \in S$, the matrices A^2 and $AB + BA = (A+B)^2 - A^2 - B^2$ belong to $\mathbb{F}S^{\leqslant 1}$, which implies that the letters of w are all different and their permutations do not break the property of w not to be spanned by shorter words. In particular, the products corresponding to the different $2^{\ell(S)}$ subsets of letters of w should be linearly independent, which implies $2^{\ell(S)} \leqslant \dim \mathrm{Mat}_n(\mathbb{F})$. □

Theorem 9. *If* $S \subset \mathrm{Mat}_5(\mathbb{F})$, *then* $\ell(S) \leqslant 8$.

Proof. Since a set of vectors is linearly dependent over $\mathbb{F}$ if it is linearly dependent over the algebraic closure of $\mathbb{F}$, it is sufficient to prove the statement assuming that $\mathbb{F}$ is algebraically closed [Guterman et al. 2018, page 239]. Moreover, Conjecture 2 is known to hold for $n \leqslant 4$ (see [Paz 1984]), so we can use Lemma 4 and assume without loss of generality that S is irreducible. According to Theorem 6 and Claim 8, we can restrict to the case when $\mathbb{F}S$ contains a matrix A with minimal polynomial of degree 3. A straightforward analysis of possible Jordan forms of A shows that the linear span of I, A, A^2 must contain a rank-one matrix, so it remains to apply Corollary 7. □

As said above, the case of $n \leqslant 4$ in Conjecture 2 was considered by Paz [1984], but the case of $n = 5$ remained open until now [Guterman et al. 2018]. Let us mention the works [Lambrou and Longstaff 2009; Longstaff et al. 2006], which cover the case $n \leqslant 6$ under the additional assumption of $\dim \mathbb{F}S \leqslant 2$.

2. The proof of Theorem 3

Let A be an $n \times n$ matrix over a field $\mathbb{F}$, which is assumed to be algebraically closed in this section. We recall that there exists $Q \in \mathrm{GL}_n(\mathbb{F})$ such that $Q^{-1}AQ$ has *rational normal form*, that is, we have $Q^{-1}AQ = \mathrm{diag}(C_{f_1}, \dots, C_{f_k})$, where

$$C_f = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{m-1} \end{pmatrix}$$

is the companion matrix of a polynomial $f = t^m + c_{m-1}t^{m-1} + \cdots + c_0$, and the *invariant factors* $f_1, \cdots, f_k$ satisfy $f_1 | \cdots | f_k$.

Claim 10. *Let δ be the degree of the minimal polynomial of an $n \times n$ matrix A over $\mathbb{F}$. Then the $\mathbb{F}$-linear span of $I, A, \dots, A^{\delta-1}$ contains either a nonzero projector of rank at most n/δ or a nonzero square-zero matrix of rank at most n/δ.*

Proof. We recall that the minimal polynomial φ of A occurs (one or more times) as an invariant factor of A. Let ψ be a polynomial that has degree $\delta - 1$, divides φ and is a multiple of any invariant factor different from φ. Then $\psi(A)$ has equal rank-one matrices in the places of the largest blocks of the rational normal form of A and zeros everywhere else. □

Claim 11. *For any irreducible set $S \subset \mathrm{Mat}_n(\mathbb{F})$, there exist nonzero λ, ρ such that $\lambda\rho \leqslant 2n$ and $\mathbb{F}S^{\leqslant\lambda}$ contains a square-zero matrix of rank ρ.*

Proof. We apply Claim 10 to any nonscalar matrix in S and find a nonzero matrix $P \in \mathbb{F}S^{\leqslant(\delta-1)}$ that has rank at most n/δ and satisfies either $P^2 = P$ or $P^2 = 0$. We are done if $P^2 = 0$; otherwise $H_B = (I - P)BP$ is a square-zero matrix for all B. We can have $H_B = 0$ only when the columns of BP are in the kernel of $I - P$, but this kernel being equal to $\mathrm{Im}\, P$ should then be invariant with respect to B, but since S is irreducible, this obstruction cannot happen for all $B \in S$. □

Claim 12. *Let $A \in \mathbb{F}^{n\times n}$ and $r \in \mathbb{N}$. Assume that the inequality* $\operatorname{rank}(PAQ) \leqslant r$ *holds, with any positive integers p, q, for all matrices $P \in \mathbb{F}^{p\times n}$, $Q \in \mathbb{F}^{n\times q}$ satisfying $PQ = 0$. Then* $\operatorname{rank}(A - \mu I) \leqslant 2r$ *for some $\mu \in \mathbb{F}$.*

Proof. Both the assumption and conclusion are independent of the substitution $A \to C^{-1}AC$, so we can assume that A has rational normal form. We denote the number of diagonal blocks by k and their sizes by $m_1, \dots, m_k$. Since the ranks of the diagonal blocks cannot decrease by more than one upon adding a scalar matrix, and since the characteristic polynomials of these blocks have a common factor, we have $\min_\mu \operatorname{rank}(A - \mu I) = n - k$. We conclude the proof by constructing an identity square submatrix $A' = A[I|J]$ with $I \cap J = \varnothing$ and $|I| = |J| \geqslant 0.5(n - k)$, which would allow us to define P and Q as having the identity matrices at the $I \times I$ and $J \times J$ blocks and completed by an appropriate number of zero columns and rows, respectively. Namely, we pick a family of $\lfloor m_t/2 \rfloor$ nonconsecutive subdiagonal ones from a t-th diagonal block of A, and the union of all such families will be the diagonal of A'. □

Claim 13. *Let $S \subset \mathbb{F}^{n\times n}$, $P \in \mathbb{F}^{p\times n}$, $Q \in \mathbb{F}^{n\times q}$. Let k be the smallest integer such that $PS^kQ \neq 0$. Then, for any $A_1, \dots, A_k \in S$, we have* $\operatorname{rank}(PA_1 \cdots A_kQ) \leqslant n/k$.

Proof. Let $V_0 = \operatorname{Im} Q$ and $V_t = \sum_{M \in S^{\leqslant t}} \operatorname{Im} MQ$. Let $\mathcal{B}_0, \dots, \mathcal{B}_k \subset \mathbb{F}^n$ be vector families such that $\mathcal{B}_0 \cup \dots \cup \mathcal{B}_t$ is a basis of V_t for $t = 0, \dots, k$. Let $\mathcal{C} \subset \mathbb{F}^n$ be such that $\mathcal{B}_0 \cup \dots \cup \mathcal{B}_k \cup \mathcal{C}$ is a basis of $\mathbb{F}^n$. Every matrix $A \in S$ has the form

$$\left(\begin{array}{ccccccc}
 & \mathcal{B}_0 & \mathcal{B}_1 & \cdots & \mathcal{B}_{k-1} & \mathcal{B}_k & \mathcal{C} \\
\mathcal{B}_0 & * & \cdots & \cdots & \cdots & * & * \\
\mathcal{B}_1 & A(1,0) & * & \cdots & \cdots & * & * \\
\mathcal{B}_2 & O & A(2,1) & * & \cdots & * & * \\
\vdots & \vdots & O & \ddots & * & \vdots & \vdots \\
\mathcal{B}_k & \vdots & \vdots & \ddots & A(k,k-1) & * & * \\
\mathcal{C} & O & O & \cdots & O & * & *
\end{array}\right),$$

where the $*$'s stand for entries that we need not specify, and the left column and top row of the matrix above indicate the basis vectors the respective blocks of rows and columns correspond to. We also have $P = (O|\cdots|O|P'|*)$, $Q = (\mathcal{Q}^\top|O|\cdots|O)^\top$ with some matrices P', $\mathcal{Q}$ at the $\mathcal{B}_k$ position of P and the $\mathcal{B}_0$ position of Q, respectively. For $A_1, \dots, A_k \in S$, the matrix $PA_k \cdots A_1Q$ equals $P'A_k(k, k-1) \cdots A_1(1,0)\mathcal{Q}$, so its rank is at most the smallest dimension of any of the matrices $A_k(k, k-1), \dots, A_1(1,0)$, which is $\min_t |\mathcal{B}_t| \leqslant n/k$. □

Claim 14. *Let $S \subset \operatorname{Mat}_n(\mathbb{F})$ be an irreducible set and assume that $\mathbb{F}S^{\leqslant\lambda}$ contains a square-zero matrix H of rank $\rho \geqslant 2$. Then there exist $\rho_1 \in [1, 0.5\rho]$ and*

$$\lambda_1 \leqslant \frac{\lambda\rho}{\rho_1} + \frac{4n(\rho - \rho_1)}{\rho\rho_1}$$

such that $\mathbb{F}S^{\leqslant\lambda_1}$ contains a square-zero matrix of rank equal to ρ_1.

Proof. Let $P \in \mathbb{F}^{p\times\rho}$, $Q \in \mathbb{F}^{\rho\times q}$ be nonzero matrices satisfying $PQ = 0$. We choose a basis such that

$$H = \begin{pmatrix} O & O & I_\rho \\ O & O & O \\ O & O & O \end{pmatrix}$$

and define $P' = (O|O|P)$ and $Q' = (Q^\top|O|O)^\top$. Let k be the smallest integer for which there exist p, q and matrices P', Q' defined as above, and also $A_1, \dots, A_k \in S$ satisfying $P'A_1 \dots A_k Q' \neq 0$ (such an integer k exists because S is irreducible). We write $A = A_1 \cdots A_k$, and we denote by A' the bottom left block of A. Since $PA'Q \neq 0$, the matrix A' is nonscalar, that is, its minimal polynomial has degree $\delta > 1$.

Case 1. Assume $k \leqslant 4n/\rho$. By Claim 10, there is a polynomial ψ of degree at most $(\delta - 1)$ such that $\rho_1 := \operatorname{rank} \psi(A') \in [1, \rho/\delta]$; we see that $H_1 = \psi(HA)H$ is a square-zero matrix of rank ρ_1. It remains to note that H_1 is spanned by words of length at most

$$(\delta - 1)(\lambda + k) + \lambda \leqslant \lambda\delta + (\delta - 1)k \leqslant \lambda\rho/\rho_1 + 4n(\rho/\rho_1 - 1)/\rho.$$

Case 2. Now let $k \geqslant 4n/\rho$. The matrix HAH has A' at the upper right block and zeros everywhere else. According to Claim 13, we have $\operatorname{rank}(PA'Q) \leqslant n/k$ for any choice of p, q and P, Q as above. Using Claim 12, we find a $\mu \in \mathbb{F}$ for which the matrix $H_1 := HAH - \mu H$ satisfies $\rho_1 := \operatorname{rank}(H_1) \leqslant 2n/k$. So we have $\rho_1 \leqslant 0.5\rho$, and H_1 is spanned by words of length at most

$$2\lambda + k \leqslant \lambda\rho/\rho_1 + 2n/\rho_1 \leqslant \lambda\rho/\rho_1 + 4n(1 - \rho_1/\rho)/\rho_1. \qquad \square$$

Proof of Theorem 3. As in the proof of Theorem 9, we can assume without loss of generality that $\mathbb{F}$ is algebraically closed and S is irreducible. Using Claim 11, we find a square-zero matrix of rank $\rho_0 > 0$ in $\mathbb{F}S^{\leqslant\lambda_0}$ with $\lambda_0\rho_0 \leqslant 2n$; if $\rho_0 = 1$, then we apply Corollary 7 and complete the proof. Otherwise, we repeatedly apply Claim 14 and obtain a sequence $(\lambda_0, \rho_0), \dots, (\lambda_\tau, \rho_\tau)$ such that $\rho_\tau = 1$ and for all $t \in \{0, \dots, \tau - 1\}$ it holds that $\rho_{t+1} \in [1, 0.5\rho_t]$,

$$\lambda_{t+1} \leqslant \frac{\lambda_t\rho_t}{\rho_{t+1}} + \frac{4n(\rho_t - \rho_{t+1})}{\rho_t\rho_{t+1}},$$

and every $\mathbb{F}S^{\leqslant\lambda_t}$ contains a square-zero matrix of rank ρ_t. By induction we get

$$\lambda_t \leqslant \frac{\lambda_0\rho_0}{\rho_t} + \frac{4n}{\rho_t}\left(t - \frac{\rho_1}{\rho_0} - \dots - \frac{\rho_t}{\rho_{t-1}}\right),$$

which implies (after the substitution $\alpha_t := \rho_t/\rho_{t-1}$) that

$$\lambda_\tau \leqslant 2n + 4n\left(\tau - \sum_{t=1}^{\tau} \alpha_t\right),$$

and since the minimum value of $\alpha_1 + \dots + \alpha_\tau$ subject to $\alpha_t > 0$ and $\alpha_1 \cdots \alpha_\tau = \rho_0^{-1}$ is attained when $\alpha_1 = \dots = \alpha_\tau = \rho_0^{-1/\tau}$, we get

$$\lambda_\tau \leqslant 2n + 4n\tau(1 - \rho_0^{-1/\tau}).$$

The right-hand side of this inequality is an increasing function of τ, so it attains its maximum at the largest possible value $\tau = \log_2 \rho_0$. We get $\lambda_\tau \leqslant 2n + 2n \log_2 \rho_0$, and it remains to apply Corollary 7. □

The author does not expect his result to be tight even asymptotically, so this paper does not show any effort on improving the $o(n \log n)$ part of the upper bound.

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