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Crystalline comparison isomorphisms in p-adic Hodge theory: the absolutely unramified case

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We construct the crystalline comparison isomorphisms for proper smooth formal schemes over an absolutely unramified base. Such isomorphisms hold for étale cohomology with nontrivial coefficients, as well as in the relative setting, i.e., for proper smooth morphisms of smooth formal schemes. The proof is formulated in terms of the proétale topos introduced by Scholze, and uses his primitive comparison theorem for the structure sheaf on the proétale site. Moreover, we need to prove the Poincaré lemma for crystalline period sheaves, for which we adapt the idea of Andreatta and Iovita. Another ingredient for the proof is the geometric acyclicity of crystalline period sheaves, whose computation is due to Andreatta and Brinon.

Notation

- Let p be a prime number.
- Let k be a p-adic field, i.e., a discretely valued complete nonarchimedean extension of $\mathbb{Q}_p$, whose residue field κ is a perfect field of characteristic p. (We often assume k to be absolutely unramified in this paper.)
- Let $\bar{k}$ be a fixed algebraic closure of k. Set $\mathbb{C}_p := \hat{\bar{k}}$ the p-adic completion of $\bar{k}$. The p-adic valuation v on $\mathbb{C}_p$ is normalized so that $v(p) = 1$. Write the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ as G_k.

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- For a (commutative unitary) ring A, let $A\langle T_1, \dots, T_d\rangle$ be the PD-envelope of the polynomial ring $A[T_1, \dots, T_d]$ with respect to the ideal $(T_1, \dots, T_d) \subset A[T_1, \dots, T_d]$ (with the requirement that the PD-structure be compatible with the one on the ideal (p)) and then let $A\{\langle T_1, \dots, T_d\rangle\}$ be its p-adic completion.
- We use the symbol $\simeq$ to denote canonical isomorphisms, and sometimes quasiisomorphisms. The symbol $\approx$ is frequently used for almost isomorphisms with respect to some almost-setting that will be fixed later.

1. Introduction

Let k be a discretely valued complete nonarchimedean field over $\mathbb{Q}_p$, which is absolutely unramified.

Consider a rigid analytic variety over k, or more generally an adic space X over $\mathrm{Spa}(k, \mathcal{O}_k)$ which admits a proper smooth formal model $\mathcal{X}$ over $\mathrm{Spf}\,\mathcal{O}_k$, whose special fiber is denoted by $\mathcal{X}_0$. Let $\mathbb{L}$ be a lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$. On the one hand, we have the p-adic étale cohomology $H^i(X_{\bar{k}}, \mathbb{L})$ which is a finitely generated $\mathbb{Z}_p$-module carrying a continuous $G_k = \mathrm{Gal}(\bar{k}/k)$-action. On the other hand, one may consider the crystalline cohomology $H^i_{\mathrm{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E})$ with the coefficient $\mathcal{E}$ being a filtered (convergent) F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. At least in the case that X comes from a scheme and the coefficients $\mathbb{L}$ and $\mathcal{E}$ are trivial, it was Grothendieck's problem of the *mysterious functor* to find a comparison between the two cohomology theories. This problem was later formulated as the *crystalline conjecture* by Fontaine [1982].

In the past decades, the crystalline conjecture was proved in various generalities, by Fontaine and Messing, Kato, Tsuji, Niziol, Faltings, Andreatta and Iovita, Beilinson and Bhatt. Among them, the first proof for the whole conjecture was given by Faltings [1989]. Along this line, Andreatta and Iovita introduced the Poincaré lemma for the crystalline period sheaf $\mathbb{B}_{\mathrm{cris}}$ on the Faltings site, a sheaf-theoretic generalization of Fontaine's period ring B_{cris}. Both the approach of Fontaine and Messing and that of Faltings, Andreatta and Iovita use an intermediate topology, namely the syntomic topology and the Faltings topology, respectively. The approach of Faltings, Andreatta and Iovita, however, has the advantage that it works for nontrivial coefficients $\mathbb{L}$ and $\mathcal{E}$.

More recently, Scholze [2013] introduced the proétale site $X_{\text{proét}}$, which allows him to construct the de Rham comparison isomorphism for any proper smooth adic space over a discretely valued complete nonarchimedean field over $\mathbb{Q}_p$, with coefficients being lisse $\mathbb{Z}_p$-sheaves on $X_{\text{proét}}$. (The notion of lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ and that on $X_{\text{proét}}$ are equivalent.) Moreover, his approach is direct and flexible enough to attack the relative version of the de Rham comparison isomorphism, i.e., the comparison for a proper smooth morphism between two smooth adic spaces.

It seems that to deal with nontrivial coefficients in a comparison isomorphism, one is forced to work over analytic bases. For the generality and some technical advantages provided by the proétale topology, we adapt Scholze's approach to give a proof of the crystalline conjecture for proper smooth formal schemes over $\mathrm{Spf}\,\mathcal{O}_k$, with nontrivial coefficients, in both absolute and relative settings. Meanwhile, we point out that the method adopted in our proof is rather different from that in [Bhatt et al. 2018], in which the authors develop a new cohomology that allows them to prove a strong integral comparison theorem (for trivial coefficients).

Let us explain our construction of crystalline comparison isomorphism (in the absolutely unramified case) in more details. First of all, Scholze is able to prove the finiteness of the étale cohomology of a proper smooth adic space over $\mathbb{C}_p = \hat{\bar{k}}$ with coefficient $\mathbb{L}'$ being an $\mathbb{F}_p$-local system. Consequently, he shows the following "primitive comparison", an almost (with respect to the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$) isomorphism

$$H^i(X_{\mathbb{C}_p,\text{ét}}, \mathbb{L}') \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathbb{C}_p}/p \xrightarrow{\sim} H^i(X_{\mathbb{C}_p,\text{ét}}, \mathbb{L}' \otimes_{\mathbb{F}_p} \mathcal{O}_X^+/p).$$

With some more efforts, one can produce the primitive comparison isomorphism in the crystalline case:

Theorem 1.1 (see Theorem 4.3). *For $\mathbb{L}$ a lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$, we have a functorial isomorphism of B_{cris}-modules*

$$H^i(X_{\bar{k},\text{ét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \xrightarrow{\sim} H^i(X_{\bar{k},\text{proét}}, \mathbb{L} \otimes \mathbb{B}_{\text{cris}}). \tag{1A.1}$$

compatible with G_k-action, filtration, and Frobenius.

It seems to us that such a result alone may have interesting arithmetic applications, since it works for any lisse $\mathbb{Z}_p$-sheaves, without the crystalline condition needed for comparison theorems.

Following Faltings, we say a lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$ on the proétale site $X_{\text{proét}}$ is *crystalline* if there exists a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$ together with an isomorphism of $\mathcal{O}\mathbb{B}_{\text{cris}}$-modules

$$\mathcal{E} \otimes_{\mathcal{O}_X^{\text{ur}}} \mathcal{O}\mathbb{B}_{\text{cris}} \simeq \mathbb{L} \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\text{cris}}, \tag{1A.2}$$

which is compatible with connection, filtration and Frobenius. Here, $\mathcal{O}_X^{\text{ur}}$ is the pullback to $X_{\text{proét}}$ of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$ and $\mathcal{O}\mathbb{B}_{\text{cris}}$ is the crystalline period sheaf of $\mathcal{O}_X^{\text{ur}}$-module with connection ∇ such that $\mathcal{O}\mathbb{B}_{\text{cris}}^{\nabla=0} = \mathbb{B}_{\text{cris}}$. When this holds, we say the lisse $\mathbb{Z}_p$-sheaf $\mathbb{L}$ and the filtered F-isocrystal $\mathcal{E}$ are *associated*.

We illustrate the construction of the crystalline comparison isomorphism briefly. Firstly, we prove a Poincaré lemma for the crystalline period sheaf $\mathbb{B}_{\text{cris}}$ on $X_{\text{proét}}$. It follows from the Poincaré lemma (Corollary 2.17) that the natural morphism from $\mathbb{B}_{\text{cris}}$ to the de Rham complex $\text{dR}(\mathcal{O}\mathbb{B}_{\text{cris}})$ of $\mathcal{O}\mathbb{B}_{\text{cris}}$ is a quasiisomorphism, which is compatible with filtration and Frobenius. When $\mathbb{L}$ and $\mathcal{E}$ are associated, the natural morphism

$$\mathbb{L} \otimes_{\mathbb{Z}_p} \text{dR}(\mathcal{O}\mathbb{B}_{\text{cris}}) \to \text{dR}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}$$

is an isomorphism compatible with Frobenius and filtration. Therefore we find a quasiisomorphism

$$\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{cris}} \simeq \text{dR}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}.$$

From this we deduce

$$R\Gamma(X_{\bar{k},\text{proét}}, \mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{cris}}) \xrightarrow{\sim} R\Gamma(X_{\bar{k},\text{proét}}, \text{dR}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}).$$

Via the natural morphism of topoi $\overline{w} : X_{\bar{k},\text{proét}}^{\sim} \to \mathcal{X}_{\text{ét}}^{\sim}$, one has

$$R\Gamma(X_{\bar{k},\text{proét}}, \text{dR}(\mathcal{E}) \otimes \mathcal{O}\mathbb{B}_{\text{cris}}) \simeq R\Gamma(\mathcal{X}_{\text{ét}}, \text{dR}(\mathcal{E}) \widehat{\otimes}_{\mathcal{O}_k} B_{\text{cris}}))$$

for which we have used the fact that the natural morphism

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}B_{\mathrm{cris}} \to R\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}$$

is an isomorphism (compatible with extra structures), which is a result of Andreatta and Brinon.

Combining the isomorphisms above, we obtain the desired crystalline comparison isomorphism.

Theorem 1.2 (see Theorem 4.5). *Let $\mathbb{L}$ be a lisse $\mathbb{Z}_p$-sheaf on X and $\mathcal{E}$ be a filtered F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$ which are associated as in* (1A.2). *Then there is a natural isomorphism of B_{cris}-modules*

$$H^i(X_{\bar{k},\mathrm{\acute{e}t}},\mathbb{L})\otimes B_{\mathrm{cris}} \xrightarrow{\sim} H^i_{\mathrm{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{E})\otimes_k B_{\mathrm{cris}}$$

which is compatible with G_k-action, filtration and Frobenius.

After obtaining a refined version of the acyclicity of crystalline period sheaf $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ in Appendix A, we achieve the crystalline comparison in the relative setting, which reduces to Theorem 1.2 when $\mathcal{Y} = \mathrm{Spf}\,\mathcal{O}_k$.

Theorem 1.3 (see Theorem 5.5). *Let $f:\mathcal{X}\to\mathcal{Y}$ be a proper smooth morphism of smooth formal schemes over* $\mathrm{Spf}\,\mathcal{O}_k$, *with $f_k: X\to Y$ the generic fiber and f_{cris} the morphism between the crystalline topoi. Let $\mathbb{L}$ and $\mathcal{E}$ be as in Theorem 1.2. Suppose that $R^if_{k*}\mathbb{L}$ is a lisse $\mathbb{Z}_p$-sheaf on Y. Then it is crystalline and is associated to the filtered F-isocrystal $R^if_{\mathrm{cris}\,*}\mathcal{E}$.*

2. Crystalline period sheaves

Let k be a discretely valued nonarchimedean extension of $\mathbb{Q}_p$, with κ its residue field. Let X be a locally noetherian adic space over $\mathrm{Spa}(k,\mathcal{O}_k)$. For the fundamentals on the proétale site $X_{\mathrm{pro\acute{e}t}}$, we refer to [Scholze 2013].

The following terminology and notation will be used frequently throughout the paper. We shall fix once for all an algebraic closure $\bar{k}$ of k, and consider $X_{\bar{k}} := X\times_{\mathrm{Spa}(k,\mathcal{O}_k)}\mathrm{Spa}(\bar{k},\mathcal{O}_{\bar{k}})$ as an object of $X_{\mathrm{pro\acute{e}t}}$ (see the paragraph after the proof of Proposition 3.13 in [loc. cit.]). As in [loc. cit., Definition 4.3], an object $U\in X_{\mathrm{pro\acute{e}t}}$ lying above $X_{\bar{k}}$ is called an *affinoid perfectoid* (lying above $X_{\bar{k}}$) if U has a proétale presentation $U=\varprojlim U_i\to X$ by affinoids $U_i=\mathrm{Spa}(R_i,R_i^+)$ above $X_{\bar{k}}$ such that, with R^+ the p-adic completion of $\varinjlim R_i^+$ and $R=R^+[1/p]$, the pair (R,R^+) is a perfectoid affinoid $(\hat{\bar{k}},\mathcal{O}_{\hat{\bar{k}}})$-algebra. Write $\hat{U}=\mathrm{Spa}(R,R^+)$. By [loc. cit., Proposition 4.8, Lemma 4.6], the set of affinoid perfectoids lying above $X_{\bar{k}}$ of $X_{\mathrm{pro\acute{e}t}}$ forms a basis for the topology.

2A. ***Period sheaves and their acyclicities.*** Following [loc. cit.], let

$$\nu: X^{\sim}_{\mathrm{pro\acute{e}t}}\to X^{\sim}_{\mathrm{\acute{e}t}}$$

be the morphism of topoi, which, on the underlying sites, sends an étale morphism $U\to X$ to the proétale morphism from U (viewed as a constant projective system) to X. Consider $\mathcal{O}_X^+=\nu^{-1}\mathcal{O}^+_{X_{\mathrm{\acute{e}t}}}$ and $\mathcal{O}_X=\nu^{-1}\mathcal{O}_{X_{\mathrm{\acute{e}t}}}$, the (uncompleted) structural sheaves on $X_{\mathrm{pro\acute{e}t}}$. More concretely, for $U=\varprojlim U_i$ a qcqs (quasicompact and quasiseparated) object of $X_{\text{proét}}$, one has $\mathcal{O}_X(U) = \varinjlim \mathcal{O}_X(U_i) = \varinjlim \mathcal{O}_{X_{\text{ét}}}(U_i)$ [Lemma 3.16]. Set

$$\hat{\mathcal{O}}_X^+ := \varprojlim_n \mathcal{O}_X^+/p^n, \quad \hat{\mathcal{O}}_X := \hat{\mathcal{O}}_X^+\big[\tfrac{1}{p}\big], \quad \text{and} \quad \mathcal{O}_X^{\flat+} := \varprojlim_{x\mapsto x^p} \mathcal{O}_X^+/p.$$

For $U \in X_{\text{proét}}$ an affinoid perfectoid lying above $X_{\bar{k}}$ with $\hat{U} = \operatorname{Spa}(R, R^+)$, by [loc. cit., Lemmas 4.10 and 5.10], we have

$$\hat{\mathcal{O}}_X^+(U) = R^+, \quad \hat{\mathcal{O}}_X(U) = R, \quad \text{and} \quad \mathcal{O}_X^{\flat+}(U) = R^{\flat+} := \varprojlim_{x\mapsto x^p} R^+/p.$$

Denote

$$R^{\flat+} \to R^+, \quad x = (x_0, x_1, \ldots) \mapsto x^\sharp := \lim_{n\to\infty} \hat{x}_n^{p^n},$$

for $\hat{x}_n$ any lifting from R^+/p to R^+. We have the multiplicative bijection induced by projection $R^+ \to R^+/p$:

$$\varprojlim_{x\mapsto x^p} R^+ \xrightarrow{\sim} R^{\flat+},$$

whose inverse sends $x \in R^{\flat+}$ to $(x^\sharp, (x^{1/p})^\sharp, \ldots)$. Put $\mathbb{A}_{\text{inf}} := W(\mathcal{O}_X^{\flat+})$ and $\mathbb{B}_{\text{inf}} = \mathbb{A}_{\text{inf}}[1/p]$. As $R^{\flat+}$ is a perfect ring, $\mathbb{A}_{\text{inf}}(U) = W(R^{\flat+})$ has no p-torsion. In particular, $\mathbb{A}_{\text{inf}}$ has no p-torsion and it is a subsheaf of $\mathbb{B}_{\text{inf}}$.

Following Fontaine, define as in [loc. cit., Definition 6.1] a natural morphism

$$\theta\colon \mathbb{A}_{\text{inf}} \to \hat{\mathcal{O}}_X^+ \tag{2A.1}$$

which, on an affinoid perfectoid U with $\hat{U} = \operatorname{Spa}(R, R^+)$, is given by

$$\theta(U)\colon \mathbb{A}_{\text{inf}}(U) = W(R^{\flat+}) \to \hat{\mathcal{O}}_X^+(U) = R^+, \quad \sum_{n=0}^{\infty} p^n[x_n] \mapsto \sum_{n=0}^{\infty} p^n x_n^\sharp \tag{2A.2}$$

with $x_n \in R^{\flat+}$. As (R, R^+) is a perfectoid affinoid algebra, $\theta(U)$ is known to be surjective (see [Brinon 2008, 5.1.2]). Therefore, θ is also surjective.

Definition 2.1. Let X be a locally noetherian adic space over $\operatorname{Spa}(k, \mathcal{O}_k)$ as above.

(1) Define $\mathbb{A}_{\text{cris}}$ to be the p-adic completion of the PD-envelope $\mathbb{A}_{\text{cris}}^0$ of $\mathbb{A}_{\text{inf}}$ with respect to the ideal sheaf $\ker(\theta) \subset \mathbb{A}_{\text{inf}}$, and define $\mathbb{B}_{\text{cris}}^+ := \mathbb{A}_{\text{cris}}[1/p]$.

(2) For $r \in \mathbb{Z}_{\geq 0}$, set $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0 := \ker(\theta)^{[r]} \subset \mathbb{A}_{\text{cris}}^0$ to be the r-th divided power ideal, and $\operatorname{Fil}^{-r} \mathbb{A}_{\text{cris}}^0 = \mathbb{A}_{\text{cris}}^0$. The family $\{\operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0 : r \in \mathbb{Z}\}$ gives a descending filtration of $\mathbb{A}_{\text{cris}}^0$.

(3) For $r \in \mathbb{Z}$, define $\operatorname{Fil}^r \mathbb{A}_{\text{cris}} \subset \mathbb{A}_{\text{cris}}$ to be the image of the following morphism of sheaves (we shall see below that this map is injective):

$$\varprojlim_n (\operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0)/p^n \to \varprojlim_n \mathbb{A}_{\text{cris}}^0/p^n = \mathbb{A}_{\text{cris}}, \tag{2A.3}$$

and define $\operatorname{Fil}^r \mathbb{B}_{\text{cris}}^+ = \operatorname{Fil}^r \mathbb{A}_{\text{cris}}[1/p]$.

Let $p^\flat = (p_i)_{i\geq 0}$ be a fixed family of elements of $\bar{k}$ such that $p_0 = p$ and that $p_{i+1}^p = p_i$ for any $i \geq 0$. Set $\xi := [p^\flat] - p$, which can be seen as a section of the restriction $\mathbb{A}_{\inf}|_{X_{\bar{k}}}$ of the proétale sheaf $\mathbb{A}_{\inf}$ to $X_{\text{proét}}/X_{\bar{k}}$.

Proposition 2.2. *We have* $\ker(\theta)|_{X_{\bar{k}}} = (\xi) \subset \mathbb{A}_{\inf}|_{X_{\bar{k}}}$. *Furthermore,* $\xi \in \mathbb{A}_{\inf}|_{X_{\bar{k}}}$ *is not a zero-divisor.*

Proof. As the set of affinoid perfectoids U lying above $X_{\bar{k}}$ forms a basis for the topology of $X_{\text{proét}}/X_{\bar{k}}$, we only need to check that, for any such U, $\xi \in \mathbb{A}_{\inf}(U)$ is not a zero-divisor and that the kernel of $\theta(U)\colon \mathbb{A}_{\inf}(U) \to \hat{\mathcal{O}}_X^+(U)$ is generated by ξ. Write $\hat{U} = \operatorname{Spa}(R, R^+)$. Then $\mathbb{A}_{\inf}(U) = W(R^{\flat+})$ and $\hat{\mathcal{O}}_X^+(U) = R^+$, hence we reduce our statement to (the proof of) [Scholze 2013, Lemma 6.3]. □

Corollary 2.3. (1) *We have* $\mathbb{A}_{\text{cris}}^0|_{X_{\bar{k}}} = \mathbb{A}_{\inf}|_{X_{\bar{k}}}[\xi^n/n! : n \in \mathbb{N}] \subset \mathbb{B}_{\inf}|_{X_{\bar{k}}}$. *In particular,* $\mathbb{A}_{\text{cris}}^0$ *and* $\mathbb{A}_{\text{cris}}$ *have no p-torsion. Moreover, for every* $r \geq 0$, $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0|_{X_{\bar{k}}} = \mathbb{A}_{\inf}|_{X_{\bar{k}}}[\xi^n/n! : n \geq r]$ *and* $\operatorname{gr}^r \mathbb{A}_{\text{cris}}^0|_{X_{\bar{k}}} = \hat{\mathcal{O}}_X^+ \cdot (\xi^r/r!) \xrightarrow{\sim} \hat{\mathcal{O}}_X^+|_{X_{\bar{k}}}$.

(2) *The morphism* (2A.3) *is injective, hence* $\varprojlim_n \operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0/p^n \xrightarrow{\sim} \operatorname{Fil}^r \mathbb{A}_{\text{cris}}$. *Moreover, for* $r \geq 0$, $\operatorname{gr}^r \mathbb{A}_{\text{cris}}|_{X_{\bar{k}}} \xrightarrow{\sim} \hat{\mathcal{O}}_X^+|_{X_{\bar{k}}}$.

Proof. The first three statements in (1) are clear from Proposition 2.2. In particular, for $r \geq 0$ we have the following exact sequence

$$0 \to \operatorname{Fil}^{r+1} \mathbb{A}_{\text{cris}}^0|_{X_{\bar{k}}} \to \operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0|_{X_{\bar{k}}} \to \hat{\mathcal{O}}_X^+|_{X_{\bar{k}}} \to 0, \tag{2A.4}$$

where the second map sends $a\xi^r/r!$ to $\theta(a)$. This gives the last assertion of (1).

As $\hat{\mathcal{O}}_X^+$ has no p-torsion, an induction on r shows that the cokernel of the inclusion $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0 \subset \mathbb{A}_{\text{cris}}^0$ has no p-torsion. As a result, the morphism (2A.3) is injective and $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}$ is the p-adic completion of $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}^0$. Since $\hat{\mathcal{O}}_X^+$ is p-adically complete, we deduce from (2A.4) also the following short exact sequence after passing to p-adic completions:

$$0 \longrightarrow \operatorname{Fil}^{r+1} \mathbb{A}_{\text{cris}}|_{X_{\bar{k}}} \longrightarrow \operatorname{Fil}^r \mathbb{A}_{\text{cris}}|_{X_{\bar{k}}} \longrightarrow \hat{\mathcal{O}}_X^+|_{X_{\bar{k}}} \longrightarrow 0 \tag{2A.5}$$

giving the last part of (2). □

Let $\epsilon = (\epsilon^{(i)})_{i\geq 0}$ be a sequence of elements of $\bar{k}$ such that $\epsilon^{(0)} = 1$, $\epsilon^{(1)} \neq 1$ and $(\epsilon^{(i+1)})^p = \epsilon^{(i)}$ for all $i \geq 0$. Then $1 - [\epsilon]$ is a well-defined element of the restriction $\mathbb{A}_{\inf}|_{X_{\bar{k}}}$ to $X_{\text{proét}}/X_{\bar{k}}$ of $\mathbb{A}_{\inf}$. Moreover $1 - [\epsilon] \in \ker(\theta)|_{X_{\bar{k}}} = \operatorname{Fil}^1 \mathbb{A}_{\text{cris}}|_{X_{\bar{k}}}$. Let

$$t := \log([\epsilon]) = -\sum_{n=1}^{\infty} \frac{(1-[\epsilon])^n}{n}, \tag{2A.6}$$

which is well-defined in $\mathbb{A}_{\text{cris}}|_{X_{\bar{k}}}$ since $\operatorname{Fil}^1 \mathbb{A}_{\text{cris}}$ is a PD-ideal.

Definition 2.4. Let X be a locally noetherian adic space over $\operatorname{Spa}(k, \mathcal{O}_k)$. Define $\mathbb{B}_{\text{cris}} = \mathbb{B}_{\text{cris}}^+[1/t]$. For $r \in \mathbb{Z}$, set $\operatorname{Fil}^r \mathbb{B}_{\text{cris}} = \sum_{s\in\mathbb{Z}} t^{-s} \operatorname{Fil}^{r+s} \mathbb{B}_{\text{cris}}^+ \subset \mathbb{B}_{\text{cris}}$.

Remark 2.5. We shall see in Corollary 2.24 that t is not a zero-divisor in $\mathbb{A}_{\text{cris}}$ and in $\mathbb{B}_{\text{cris}}^+$, so $\mathbb{B}_{\text{cris}}^+ \subset \mathbb{B}_{\text{cris}}$.

Before investigating these period sheaves in details, we first study them over a perfectoid affinoid $(\hat{\bar{k}}, \mathcal{O}_{\hat{\bar{k}}})$-algebra (R, R^+). Consider

$$\mathbb{A}_{\mathrm{inf}}(R, R^+) := W(R^{\flat +}), \quad \mathbb{B}_{\mathrm{inf}}(R, R^+) := \mathbb{A}_{\mathrm{inf}}(R, R^+)[1/p],$$

and define the morphism

$$\theta_{(R,R^+)} : \mathbb{A}_{\mathrm{inf}}(R, R^+) \to R^+$$

in the same way as in (2A.2). It is known to be surjective as (R, R^+) is perfectoid. The element ξ generates $\ker(\theta_{(R,R^+)})$ and is not a zero-divisor in $\mathbb{A}_{\mathrm{inf}}(R, R^+)$. Let $\mathbb{A}_{\mathrm{cris}}(R, R^+)$ be the p-adic completion of the PD-envelope of $\mathbb{A}_{\mathrm{inf}}(R, R^+)$ with respect to $\ker(\theta_{(R,R^+)})$. So $\mathbb{A}_{\mathrm{cris}}(R, R^+)$ is the p-adic completion of

$$\mathbb{A}^0_{\mathrm{cris}}(R, R^+) := \mathbb{A}_{\mathrm{inf}}(R, R^+)\left[\frac{\xi^n}{n!} : n \in \mathbb{N}\right] \subset \mathbb{B}_{\mathrm{inf}}(R, R^+).$$

For r an integer, let $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}(R, R^+) \subset \mathbb{A}^0_{\mathrm{cris}}(R, R^+)$ be the r-th PD-ideal, i.e., the ideal generated by $\xi^n/n!$ for $n \geq \max\{r, 0\}$. Let $\mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}(R, R^+) \subset \mathbb{A}_{\mathrm{cris}}(R, R^+)$ be the closure (for the p-adic topology) of $\mathrm{Fil}^r \mathbb{A}^0_{\mathrm{cris}}(R, R^+)$ inside $\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Finally, put $\mathbb{B}^+_{\mathrm{cris}}(R, R^+) := \mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p]$, $\mathbb{B}_{\mathrm{cris}}(R, R^+) := \mathbb{B}^+_{\mathrm{cris}}(R, R^+)[1/t]$, and for $r \in \mathbb{Z}$, set

$$\mathrm{Fil}^r \mathbb{B}^+_{\mathrm{cris}}(R, R^+) := \mathrm{Fil}^r \mathbb{A}_{\mathrm{cris}}(R, R^+)\left[\tfrac{1}{p}\right] \quad \text{and} \quad \mathrm{Fil}^r \mathbb{B}_{\mathrm{cris}}(R, R^+) := \sum_{s\in\mathbb{Z}} t^{-s} \mathrm{Fil}^{r+s} \mathbb{B}^+_{\mathrm{cris}}(R, R^+).$$

In particular, taking $R^+ = \mathcal{O}_{\mathbb{C}_p}$ with $\mathbb{C}_p$ the p-adic completion of the fixed algebraic closure $\bar{k}$ of k, we get Fontaine's rings A_{cris}, B^+_{cris}, B_{cris} as in [Fontaine 1994]. Write $\mathbb{C}_p^\flat$ the tilt of $\mathbb{C}_p$, which is an algebraically closed nonarchimedean field of characteristic p. The maximal ideal of its ring of integers $\mathcal{O}^\flat_{\mathbb{C}_p}$ is generated by $[p^\flat]^{1/p^N}$ for all $N \in \mathbb{N}$. Let $\mathcal{I} \subset A_{\mathrm{cris}}$ be the ideal generated by

$$\{[\epsilon]^{1/p^N} - 1, [p^\flat]^{1/p^N} : N \in \mathbb{N}\} \subset A_{\mathrm{cris}}.$$

By [Brinon 2008, Lemme 6.3.1], we have $\mathcal{I} \subset \mathcal{I}^2 + p^n \cdot A_{\mathrm{cris}}$ for any $n \in \mathbb{N}_{>0}$. In particular, $\mathcal{I} \cdot (A_{\mathrm{cris}}/p^n) = (\mathcal{I} \cdot (A_{\mathrm{cris}}/p^n))^2$. In the following, when working with algebras (or modules) over A_{cris}/p^n, we consider the almost-setting with respect to the ideal $\mathcal{I} \cdot (A_{\mathrm{cris}}/p^n) \subset A_{\mathrm{cris}}/p^n$. When $n = 1$, as $\epsilon^{1/p^N} - 1 \in \mathcal{O}^\flat_{\mathbb{C}_p}$ is contained in the maximal ideal, $\mathcal{I} \cdot (A_{\mathrm{cris}}/p)$ is the same as the ideal generated by $\{[p^\flat]^{1/p^N} : N \in \mathbb{N}\}$. So the almost-setting adopted here for A_{cris}/p-modules is the same as the one used by Scholze [2013] (see the paragraph before Theorem 6.5 for his convention).

Lemma 2.6. *Let X be a locally noetherian adic space over $(k, \mathcal{O}_k)$. Let $\mathcal{F}$ be a p-adically complete sheaf of A_{cris}-modules on $X_{\mathrm{proét}}$, flat over $\mathbb{Z}_p$. Set $\mathcal{F}_n = \mathcal{F}/p^n$, $n \in \mathbb{Z}_{\geq 1}$. Assume that, for any affinoid perfectoid U above $X_{\bar{k}}$,*

(a) *there exists a p-adically complete A_{cris}-module $F(U)$, flat over $\mathbb{Z}_p$, equipped with a morphism of A_{cris}-modules $\alpha_U : F(U) \to \mathcal{F}(U)$ such that the composed morphism*

$$\alpha_{U,1} : F(U)/p \xrightarrow{\alpha_U \bmod p} \mathcal{F}(U)/p \to \mathcal{F}_1(U)$$

is an almost isomorphism; and

(b) *the A_{cris}/p-module $H^i(U, \mathcal{F}_1)$ is almost zero for any $i > 0$.*

Then, for an affinoid perfectoid U as above, $n \geq 1$ and $i > 0$,

(1) *the composed morphism*

$$\alpha_{U,n} : F(U)/p^n \xrightarrow{\alpha_U \bmod p^n} \mathcal{F}(U)/p^n \to \mathcal{F}_n(U)$$

is an almost isomorphism, and $H^i(U, \mathcal{F}_n)$ is almost zero;

(2) *we have $\mathcal{I} \cdot R^1 \varprojlim \mathcal{F}_n(U) = 0$, and that $\ker(\alpha_U)$ and $\mathrm{coker}(\alpha_U)$ are killed by $\mathcal{I}^2$. Furthermore, $\mathcal{I}^2 \cdot R^i \varprojlim \mathcal{F}_n = 0$, and $\mathcal{I}^{2i+1} \cdot H^i(U, \mathcal{F}) = 0$.*

Proof. Let U be a affinoid perfectoid lying above $X_{\bar{k}}$. For $n \in \mathbb{Z}_{\geq 1}$, let $F(U)_n = F(U)/p^n$. Since $\mathcal{F}$ and $F(U)$ are flat over $\mathbb{Z}_p$, we have exact sequences

$$0 \to \mathcal{F}_1 \xrightarrow{\cdot p^n} \mathcal{F}_{n+1} \xrightarrow{\mathrm{can}} \mathcal{F}_n \to 0, \quad \text{and} \quad 0 \to F(U)_1 \xrightarrow{\cdot p^n} F(U)_{n+1} \xrightarrow{\mathrm{can}} F(U)_n \to 0,$$

from which we deduce exact sequences

$$H^i(U, \mathcal{F}_1) \to H^i(U, \mathcal{F}_{n+1}) \to H^i(U, \mathcal{F}_n), \quad i \geq 0,$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(U)_1 & \xrightarrow{\cdot p^n} & F(U)_{n+1} & \longrightarrow & F(U)_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{U,1}} & & \downarrow{\scriptstyle \alpha_{U,n+1}} & & \downarrow{\scriptstyle \alpha_{U,n}} & & \\
0 & \longrightarrow & \mathcal{F}_1(U) & \xrightarrow{\cdot p^n} & \mathcal{F}_{n+1}(U) & \longrightarrow & \mathcal{F}_n(U). & &
\end{array}$$

So, by induction on n, (1) follows from conditions (a) and (b) above and the fact that $\mathcal{I} \cdot (A_{\mathrm{cris}}/p^n) = \mathcal{I}^2 \cdot (A_{\mathrm{cris}}/p^n)$ for any $n \in \mathbb{N}$ [Brinon 2008, Lemme 6.3.1]. In particular, the collection $(\alpha_{U,n})_{n \in \mathbb{N}}$ gives a morphism of projective systems

$$(F(U)_n)_{n \in \mathbb{N}} \to (\mathcal{F}_n(U))_{n \in \mathbb{N}}$$

whose kernel and cokernel are killed by $\mathcal{I}$. Passing to limits relative to n, we find $\mathcal{I} \cdot R^1 \varprojlim \mathcal{F}_n(U) = 0$, and that $\ker(\alpha_U)$ and $\mathrm{coker}(\alpha_U)$ are killed by $\mathcal{I}^2$, giving also the first part of (2).

To go further, let Sh (resp. PreSh) denote the category of sheaves (resp. of presheaves) on $X_{\text{proét}}$, and let $\mathrm{Sh}^{\mathbb{N}}$ (resp. $\mathrm{PreSh}^{\mathbb{N}}$) denote the category of projective systems of sheaves (resp. projective systems of presheaves) indexed by $\mathbb{N}$ on $X_{\text{proét}}$. The projective limit functor $\varprojlim : \mathrm{Sh}^{\mathbb{N}} \to \mathrm{Sh}$ factors as

$$\mathrm{Sh}^{\mathbb{N}} \xrightarrow{\sigma} \mathrm{PreSh}^{\mathbb{N}} \xrightarrow{\varprojlim'} \mathrm{PreSh} \xrightarrow{a} \mathrm{Sh},$$

where the first functor σ is induced from the natural inclusion $\mathrm{Sh} \subset \mathrm{PreSh}$, the second is the projective limit functor of presheaves, and the third takes a presheaf to its associated sheaf. Let $\tau := \varprojlim' \circ \sigma$. Since the functor a is exact, $R\varprojlim = a\circ R\tau$. In particular, for each i, $R^i\varprojlim \mathcal{F}_n$ is the associated sheaf of the presheaf $R^i\tau(\mathcal{F}_\bullet)$, where we denote by $\mathcal{F}_\bullet$ the projective system $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Let

$$0\to I(0)_\bullet\to I(1)_\bullet\to\cdots$$

be an injective resolution of $\mathcal{F}_\bullet$ in $\mathrm{Sh}^{\mathbb{N}}$, with $I(j)_\bullet=(I(j)_n)_{n\in\mathbb{N}}$. For each i, $R^i\sigma(\mathcal{F}_\bullet)$ is the i-th cohomology of this complex in $\mathrm{PreSh}^{\mathbb{N}}$. On the other hand, for each n, this resolution gives an injective resolution of $\mathcal{F}_n$ in the category of sheaves on $X_{\text{proét}}$ [Jannsen 1988, (1.1) Proposition]

$$0\to I(0)_n\to I(1)_n\to\cdots.$$

So, for U an affinoid perfectoid lying above $X_{\bar{k}}$, $H^i(U,\mathcal{F}_n)$ is the i-th cohomology group of the induced complex

$$0\to I(0)_n(U)\to I(1)_n(U)\to\cdots,$$

which is annihilated by $\mathcal{I}$ when $i>0$ by (1). Varying n, we find

$$\mathcal{I}\cdot R^i\sigma(\mathcal{F}_\bullet)(U)=\mathcal{I}\cdot H^i(\mathcal{U},\mathcal{F}_\bullet)=0,\quad \text{for } i>0. \tag{2A.7}$$

On the other hand, as infinite products exist and are exact functors in PreSh, by [Jannsen 1988, (1.6) Proposition], we have an exact sequence of presheaves for each $i\in\mathbb{Z}$:

$$0\to R^1\varprojlim{}' R^{i-1}\sigma(\mathcal{F}_\bullet)\to R^i\tau(\mathcal{F}_\bullet)\to \varprojlim{}' R^i\sigma(\mathcal{F}_\bullet)\to 0.$$

The latter gives an exact sequence of abelian groups

$$0\to (R^1\varprojlim{}' R^{i-1}\sigma(\mathcal{F}_\bullet))(U)\to R^i\tau(\mathcal{F}_\bullet)(U)\to (\varprojlim{}' R^i\sigma(\mathcal{F}_\bullet))(U)\to 0.$$

We claim that $\mathcal{I}^2\cdot R^i\tau(\mathcal{F}_\bullet)(U)=0$ for $i\geq 1$. Indeed, when $i\geq 2$, our claim follows from (2A.7). When $i=1$, by what we have shown in the first paragraph, $\mathcal{I}\cdot(R^1\varprojlim{}'\sigma(\mathcal{F}_\bullet))(U)=\mathcal{I}\cdot R^1\varprojlim(\mathcal{F}_n(U))=0$. Combining (2A.7), we get $\mathcal{I}^2\cdot R^1\tau(\mathcal{F}_\bullet)(U)=0$, as claimed. Since $R^i\varprojlim\mathcal{F}_n$ is the associated sheaf of $R^i\tau(\mathcal{F}_\bullet)$, we deduce $\mathcal{I}^2\cdot R^i\varprojlim\mathcal{F}_n=0$ when $i>0$. This proves the second part of (2).

Now, because $\mathcal{I}^2\cdot R^i\varprojlim\mathcal{F}_n=0$ for $i>0$, for the spectral sequence below

$$E_2^{i,j}=H^i(U,R^j\varprojlim\mathcal{F}_n)\Longrightarrow H^{i+j}(U,R\varprojlim\mathcal{F}_n),$$

one checks that $\mathcal{I}^2\cdot E_\infty^{i,j}=0$ for $j>0$, $E_\infty^{i,0}=E_{i+1}^{i,0}$, and the surjection $E_2^{i,0}\to E_\infty^{i,0}$ has kernel killed by $\mathcal{I}^{2i-2}$. It follows that the canonical map

$$H^i(U,\mathcal{F})=H^i(U,\varprojlim\mathcal{F}_n)\to H^i(U,R\varprojlim\mathcal{F}_n)$$

has kernel annihilated by $\mathcal{I}^{2i-2}$ and cokernel annihilated by $\mathcal{I}^{2i}$. Using the short exact sequence (see Lemma 4.1)

$$0\to R^1\varprojlim H^{i-1}(U,\mathcal{F}_n)\to H^i(U,R\varprojlim\mathcal{F}_n)\to\varprojlim H^i(U,\mathcal{F}_n)\to 0,$$

and that $R^1 \varprojlim H^{i-1}(U, \mathcal{F}_n)$ is annihilated by $\mathcal{I}$, one deduces that the morphism

$$H^i(U, \mathcal{F}) = H^i(U, \varprojlim \mathcal{F}_n) \to \varprojlim H^i(U, \mathcal{F}_n), \quad i \geq 0$$

is an isomorphism up to $\mathcal{I}^{2i}$-torsion, i.e., its kernel and cokernel are killed by $\mathcal{I}^{2i}$. In particular, $H^i(U, \mathcal{F})$ is killed by $\mathcal{I}^{2i+1}$ when $i > 0$, as wanted. □

Lemma 2.7. *Let X be a locally noetherian adic space over $(k, \mathcal{O}_k)$. Let $U \in X_{\text{proét}}$ be an affinoid perfectoid above $X_{\bar{k}}$ with $\hat{U} = \operatorname{Spa}(R, R^+)$. Then there is a natural filtered morphism $\mathbb{A}_{\text{cris}}(R, R^+) \to \mathbb{A}_{\text{cris}}(U)$ of A_{cris}-algebras, inducing an almost isomorphism* $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}(R, R^+)/p^n \to (\operatorname{Fil}^r \mathbb{A}_{\text{cris}}/p^n)(U)$ *for any $r \geq 0$ and $n \geq 1$. Moreover, $H^i(U, \operatorname{Fil}^r \mathbb{A}_{\text{cris}}/p^n)^a = 0$ for any $i > 0$.*

Proof. As U is affinoid perfectoid, $\hat{\mathcal{O}}_X^+(U) = R^+$, $\mathcal{O}_X^{\flat+}(U) = R^{\flat+}$ and $\theta(U) = \theta_{(R,R^+)}$. In particular, $\mathbb{A}_{\text{inf}}(U) = \mathbb{A}_{\text{inf}}(R, R^+)$, and the natural morphism

$$\mathbb{A}_{\text{inf}}(R, R^+) = \mathbb{A}_{\text{inf}}(U) \to \mathbb{A}_{\text{cris}}(U)$$

sends $\ker(\theta_{(R,R^+)})$ into $\operatorname{Fil}^1 \mathbb{A}_{\text{cris}}(U)$. As $\operatorname{Fil}^1 \mathbb{A}_{\text{cris}}(U) \subset \mathbb{A}_{\text{cris}}(U)$ has a PD-structure, the morphism above induces a map $\mathbb{A}^0_{\text{cris}}(R, R^+) \to \mathbb{A}_{\text{cris}}(U)$, respecting the filtrations on both sides. Passing to p-adic completions, we obtain the required filtered morphism $\mathbb{A}_{\text{cris}}(R, R^+) \to \mathbb{A}_{\text{cris}}(U)$ of A_{cris}-algebras.

In particular, for each $r \geq 0$, we have a natural morphism

$$\operatorname{Fil}^r \mathbb{A}_{\text{cris}}(R, R^+) \to \operatorname{Fil}^r \mathbb{A}_{\text{cris}}(U).$$

Composing its reduction modulo p^n with $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}(U)/p^n \to (\operatorname{Fil}^r \mathbb{A}_{\text{cris}}/p^n)(U)$, we get a morphism

$$\operatorname{Fil}^r \mathbb{A}_{\text{cris}}(R, R^+)/p^n \to (\operatorname{Fil}^r \mathbb{A}_{\text{cris}}/p^n)(U) \tag{2A.8}$$

for all $n \geq 1$. We need to show that this is an almost isomorphism of A_{cris}/p^n-modules, and that $H^i(U, \operatorname{Fil}^r \mathbb{A}_{\text{cris}}/p^n)^a = 0$ for $i > 0$. Using Lemma 2.6(1), one reduces to the case where $n = 1$. Then, we claim that it suffices to prove this when $r = 0$. Indeed, from the exact sequence (2A.4) and the fact that $\hat{\mathcal{O}}_X^+$ is p-torsion free, we deduce a short exact sequence for each $r \geq 0$:

$$0 \to (\operatorname{Fil}^{r+1} \mathbb{A}_{\text{cris}}|_{X_{\bar{k}}})/p \to (\operatorname{Fil}^r \mathbb{A}_{\text{cris}}|_{X_{\bar{k}}})/p \to (\mathcal{O}_X^+|_{X_{\bar{k}}})/p \to 0. \tag{2A.9}$$

We have a short exact sequence for $\operatorname{Fil}^r \mathbb{A}_{\text{cris}}(R, R^+)/p$ obtained in a similar way:

$$0 \to \operatorname{Fil}^{r+1} \mathbb{A}_{\text{cris}}(R, R^+)/p \to \operatorname{Fil}^r(R, R^+)/p \to R^+/p \to 0.$$

As U is affinoid perfectoid, by [Scholze 2013, Lemma 4.10], the natural morphism

$$R^+/p = \hat{\mathcal{O}}_X^+(U)/p \to (\mathcal{O}_X^+/p)(U)$$

is an almost isomorphism: recall that the almost-setting adopted here for A_{cris}/p-modules is the same as the one used by Scholze [2013]. So we have a commutative diagram with exact rows, such that the right vertical map is an almost isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Fil}^{r+1}\mathbb{A}_{\mathrm{cris}}/p)(U)^a \arrow[r] & (\mathrm{Fil}^r\mathbb{A}_{\mathrm{cris}}/p)(U)^a \arrow[r] & (\mathcal{O}_X^+/p)(U)^a & \\
0 \arrow[r] & \mathrm{Fil}^{r+1}\mathbb{A}_{\mathrm{cris}}(R,R^+)^a/p \arrow[r] \arrow[u] & \mathrm{Fil}^r(R,R^+)^a/p \arrow[r] \arrow[u] & (R^+/p)^a \arrow[r] \arrow[u, "\approx"] & 0.
\end{tikzcd}$$

In particular, the upper row of the diagram above is right exact. On the other hand, combined with [Scholze 2013, Lemma 4.10], the long exact sequence associated with (2A.9) gives an isomorphism

$$H^i(U,\mathrm{Fil}^{r+1}\mathbb{A}_{\mathrm{cris}}/p)^a \xrightarrow{\sim} H^i(U,\mathrm{Fil}^r\mathbb{A}_{\mathrm{cris}}/p)^a,\quad \forall\, i\geq 1.$$

Therefore, our claim follows by induction on $r\geq 0$.

So, it remains to prove the second part of our lemma when $r=0$ and $n=1$. Denote by α_1 the map (2A.8) in this case. Recall the following identification of $\mathbb{A}_{\mathrm{cris}}(R,R^+)/p$ (see [Brinon 2008, Proposition 6.1.2])

$$\mathbb{A}_{\mathrm{cris}}(R,R^+)/p \xrightarrow{\sim} (R^{\flat+}/(p^\flat)^p)[\delta_i : i\in\mathbb{N}]/(\delta_i^p : i\in\mathbb{N}),$$

with δ_i being the image of $\xi^{[p^{i+1}]}$. Similarly, restricting to $X_{\mathrm{pro\acute{e}t}}/X_{\bar{k}}$, we have

$$\mathbb{A}_{\mathrm{cris}}/p \xrightarrow{\sim} (\mathcal{O}_X^{\flat+}/(p^\flat)^p)[\delta_i : i\in\mathbb{N}]/(\delta_i^p : i\in\mathbb{N}).$$

In particular, $\mathbb{A}_{\mathrm{cris}}/p$ is a direct sum of copies of $\mathcal{O}_X^{\flat+}/(p^\flat)^p$ on $X_{\mathrm{pro\acute{e}t}}/X_{\bar{k}}$. Under these identifications, the morphism α_1 is induced by

$$R^{\flat+}/(p^\flat)^p = \mathcal{O}_X^{\flat+}(U)/(p^\flat)^p \to (\mathcal{O}_X^{\flat+}/(p^\flat)^p)(U).$$

Since U is qcqs, to conclude the proof, it suffices to show $H^i(U,\mathcal{O}_X^{\flat+}/(p^\flat)^p)^a=0$ for $i>0$, and that the morphism above is an almost isomorphism. Both of these two assertions follow from [Scholze 2013, Lemma 4.10]. □

Corollary 2.8. *Keep the notation of Lemma 2.7. In particular, U is an affinoid perfectoid of $X_{\mathrm{pro\acute{e}t}}$ lying above $X_{\bar{k}}$, with $\hat{U}=\mathrm{Spa}(R,R^+)$.*

(1) *For any $r\in\mathbb{N}$, there is a natural morphism $\mathrm{Fil}^r\mathbb{A}_{\mathrm{cris}}(R,R^+)\to\mathrm{Fil}^r\mathbb{A}_{\mathrm{cris}}(U)$ of A_{cris}-modules whose kernel and cokernel are killed by $\mathcal{I}^2$. Moreover, $\mathcal{I}^2\cdot R^i\varprojlim_n \mathrm{Fil}^r\mathbb{A}_{\mathrm{cris}}/p^n=0$ and $\mathcal{I}^{2i+1}\cdot H^i(U,\mathrm{Fil}^r\mathbb{A}_{\mathrm{cris}})=0$ for $i>0$.*

(2) *The natural morphisms in* (1) *induce isomorphisms*

$$\mathbb{B}_{\mathrm{cris}}(R,R^+)\xrightarrow{\sim}\mathbb{B}_{\mathrm{cris}}(U),\quad \textit{and}\quad \mathrm{Fil}^r\mathbb{B}_{\mathrm{cris}}(R,R^+)\xrightarrow{\sim}\mathrm{Fil}^r\mathbb{B}_{\mathrm{cris}}(U)$$

for all $r\in\mathbb{Z}$. Moreover, $H^i(U,\mathbb{B}_{\mathrm{cris}})=H^i(U,\mathrm{Fil}^r\mathbb{B}_{\mathrm{cris}})=0$ for $i\geq 1$.

Proof. (1) This follows directly from Lemma 2.6 and Lemma 2.7.

(2) As U is qcqs, inverting t, we deduce from (1) a morphism of B_{cris}-modules $\mathbb{B}_{\mathrm{cris}}(R,R^+)\to\mathbb{B}_{\mathrm{cris}}(U)$, with kernel and cokernel killed by $\mathcal{I}^2$. Moreover, the B_{cris}-module $H^i(U,\mathbb{B}_{\mathrm{cris}})$ is annihilated by $\mathcal{I}^{2i+1}$

for $i > 0$. Note that t is divisible by $[\epsilon]-1$ in A_{cris} (see for example the proof of Theorem A.12), so the assertions for $\mathbb{B}_{\mathrm{cris}}$ follow as $\mathcal{I}\cdot B_{\mathrm{cris}} = B_{\mathrm{cris}}$.

To prove our assertions for $\mathrm{Fil}^r\,\mathbb{B}_{\mathrm{cris}}$, observe first that the following two properties hold. For $s\in\mathbb{N}$, (a) the canonical map $\mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+)\to\mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}}(U)$ is an isomorphism; and (b) $H^i(U,\mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}})=0$. Indeed, over $X_{\mathrm{pro\acute{e}t}}/X_{\bar{k}}$, we have $\mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}}=\hat{\mathcal{O}}_X\cdot\xi^{[s]}$ by (2A.5). Similarly, $\mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+)=R\cdot\xi^{[s]}$. Therefore the two properties above follow from [Scholze 2013, Lemma 4.10].

Now, let us begin the proof for $\mathrm{Fil}^r\,\mathbb{B}_{\mathrm{cris}}$. Twisting by t^{-r} if necessary, we shall assume $r=0$. Inverting p, we get from (1) a morphism of B^+_{cris}-modules:

$$\alpha_s : \mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+)\to\mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(U)$$

whose kernel and cokernel are killed by $\mathcal{I}^2$. Passing to direct limits (with respect to multiplication-by-t), we deduce a natural map of B^+_{cris}-modules, denoted by β:

$$\mathrm{Fil}^0\,\mathbb{B}_{\mathrm{cris}}(R,R^+)=\varinjlim_{s\geq 0}\mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+)\to\mathrm{Fil}^0\,\mathbb{B}_{\mathrm{cris}}(U)=\varinjlim_{s\geq 0}\mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(U),$$

whose kernel and cokernel are killed by $\mathcal{I}^2$, hence by t^2. One needs to show that this map is an isomorphism. The injectivity of β is clear as $\ker(\beta)\subset\mathbb{B}_{\mathrm{cris}}(R,R^+)$ is t-torsion free. So it is enough to check its surjectivity. Note that we have the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^{s+1}\,\mathbb{B}^+_{\mathrm{cris}}(U) \arrow[r] & \mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(U) \arrow[r] & \mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}}(U) & \\
0 \arrow[r] & \mathrm{Fil}^{s+1}\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+) \arrow[r] \arrow[u, "\alpha_{s+1}"] & \mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+) \arrow[r] \arrow[u, "\alpha_s"] & \mathrm{gr}^s\,\mathbb{B}^+_{\mathrm{cris}}(R,R^+) \arrow[r] \arrow[u, "\simeq"] & 0.
\end{tikzcd}$$

Here the right vertical map is an isomorphism because of the property (a) above. Then, by the snake lemma, the inclusion $\mathrm{Fil}^{s+1}\,\mathbb{B}^+_{\mathrm{cris}}(U)\subset\mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(U)$ induces an isomorphism $\mathrm{coker}(\alpha_{s+1})\xrightarrow{\sim}\mathrm{coker}(\alpha_s)$. So we get identification $\mathrm{coker}(\alpha_s)\xrightarrow{\sim}\mathrm{coker}(\alpha_0)=:C$ induced by the inclusion $\mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}(U)\subset\mathrm{Fil}^0\,\mathbb{B}^+_{\mathrm{cris}}(U)=\mathbb{B}^+_{\mathrm{cris}}(U)$ for all $s\geq 0$. With these identifications, we have

$$\mathrm{coker}(\beta)=\varinjlim_{s\geq 0}\mathrm{coker}(\alpha_s)=\varinjlim_{s\geq 0}C$$

where, in the last direct limit, the transition maps are multiplication-by-t. Since $C=\mathrm{coker}(\alpha_0)$ is killed by t^2, necessarily $\mathrm{coker}(\beta)=0$. In other words, β is surjective, thus is an isomorphism.

Finally, it remains to show $H^i(U,\mathrm{Fil}^0\,\mathbb{B}_{\mathrm{cris}})=0$ when $i>0$. For $s\in\mathbb{N}$, from the commutative diagram

$$\begin{tikzcd}
\mathrm{Fil}^0\,\mathbb{B}^+_{\mathrm{cris}} \arrow[r, "\cdot t^s"] & \mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}}\ , \\
\mathrm{Fil}^s\,\mathbb{B}^+_{\mathrm{cris}} \arrow[u, hook, "\mathrm{canonical}"] \arrow[ur, "\cdot t^s"'] &
\end{tikzcd}$$

we get a commutative diagram of cohomology groups

$$\begin{tikzcd} H^i(U, \mathrm{Fil}^0 \mathbb{B}_{\mathrm{cris}}^+) \arrow[r, "\cdot t^s"] & H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+) \\ H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+) \arrow[u] \arrow[ur, "\cdot t^s"'] & \end{tikzcd} \tag{2A.10}$$

We claim that, for $i > 0$, the vertical map above is surjective. To see this, it suffices to check the surjectivity of the map

$$H^i(U, \mathrm{Fil}^{s+1} \mathbb{B}_{\mathrm{cris}}^+) \to H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+)$$

induced by the inclusion $\mathrm{Fil}^{s+1} \mathbb{B}_{\mathrm{cris}}^+ \subset \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+$ for any $s \geq 0$. So, one only needs to show that $H^i(U, \mathrm{gr}^s \mathbb{B}_{\mathrm{cris}}^+) = 0$ for $i > 0$, as claimed by the property (b) above. Thus the vertical map in (2A.10) is surjective. On the other hand, the B_{cris}^+-module $H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+)$ is killed by $\mathcal{I}^{2i+1}$ and t is a multiple of $[\epsilon] - 1 \in \mathcal{I}$, so the map

$$H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+) \to H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+), \quad x \mapsto t^s x$$

is zero whenever $s \geq 2i + 1$. Thus, the horizontal map in (2A.10) is trivial when $s \geq 2i + 1$. We conclude $H^i(U, \mathrm{Fil}^0 \mathbb{B}_{\mathrm{cris}}) = \varinjlim_s H^i(U, \mathrm{Fil}^s \mathbb{B}_{\mathrm{cris}}^+) = 0$ for $i > 0$. □

2B. ***Period sheaves with connections.*** In this section, assume that the p-adic field k is *absolutely unramified*. Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Set $X := \mathcal{X}_k$ the generic fiber of $\mathcal{X}$, viewed as an adic space over $\mathrm{Spa}(k, \mathcal{O}_k)$. For any étale morphism $\mathcal{Y} \to \mathcal{X}$, by taking the generic fiber, we obtain an étale morphism $\mathcal{Y}_k \to X$ of adic spaces, hence an object of the proétale site $X_{\mathrm{pro\acute{e}t}}$. In this way, we get a morphism of sites $\mathcal{X}_{\mathrm{\acute{e}t}} \to X_{\mathrm{pro\acute{e}t}}$, with the induced morphism of topoi

$$w \colon X_{\mathrm{pro\acute{e}t}}^{\sim} \to \mathcal{X}_{\mathrm{\acute{e}t}}^{\sim}.$$

Let $\mathcal{O}_{\mathcal{X}_{\mathrm{\acute{e}t}}}$ denote the structural sheaf of the étale site $\mathcal{X}_{\mathrm{\acute{e}t}}$: for any étale morphism $\mathcal{Y} \to \mathcal{X}$ of formal schemes over $\mathcal{O}_k$, $\mathcal{O}_{\mathcal{X}_{\mathrm{\acute{e}t}}}(\mathcal{Y}) = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Define $\mathcal{O}_X^{\mathrm{ur}+} := w^{-1}\mathcal{O}_{\mathcal{X}_{\mathrm{\acute{e}t}}}$ and $\mathcal{O}_X^{\mathrm{ur}} := w^{-1}\mathcal{O}_{\mathcal{X}_{\mathrm{\acute{e}t}}}[1/p]$. Thus $\mathcal{O}_X^{\mathrm{ur}+}$ is the associated sheaf of the presheaf $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}$:

$$X_{\mathrm{pro\acute{e}t}} \ni U \mapsto \varinjlim_{(\mathcal{Y}, a)} \mathcal{O}_{\mathcal{X}_{\mathrm{\acute{e}t}}}(\mathcal{Y}) =: \widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U),$$

where the limit runs through all pairs $(\mathcal{Y}, a)$ with $\mathcal{Y} \in \mathcal{X}_{\mathrm{\acute{e}t}}$ and $a \colon U \to \mathcal{Y}_k$ a morphism making the following diagram commutative:

$$\begin{tikzcd} U \arrow[r] \arrow[dr, "a"'] & X = \mathcal{X}_k \\ & \mathcal{Y}_k \arrow[u] \end{tikzcd} \tag{2B.1}$$

The morphism $a\colon U \to \mathcal{Y}_k$ induces a map $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to \mathcal{O}_X(U)$. There is then a morphism of presheaves $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}} \to \mathcal{O}_X^+$, whence a morphism of sheaves

$$\mathcal{O}_X^{\mathrm{ur}+} \to \mathcal{O}_X^+. \tag{2B.2}$$

Recall $\mathbb{A}_{\mathrm{inf}} := W(\mathcal{O}_X^{\flat+})$. Set $\mathcal{O}\mathbb{A}_{\mathrm{inf}} := \mathcal{O}_X^{\mathrm{ur}+} \otimes_{\mathcal{O}_k} \mathbb{A}_{\mathrm{inf}}$ and

$$\theta_X\colon \mathcal{O}\mathbb{A}_{\mathrm{inf}} \to \hat{\mathcal{O}}_X^+ \tag{2B.3}$$

to be the map induced from $\theta\colon \mathbb{A}_{\mathrm{inf}} \to \hat{\mathcal{O}}_X^+$ of (2A.1) by extension of scalars.

Definition 2.9. Consider the following sheaves on $X_{\mathrm{proét}}$.

(1) Let $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ be the p-adic completion of the PD-envelope $\mathcal{O}\mathbb{A}_{\mathrm{cris}}^0$ of $\mathcal{O}\mathbb{A}_{\mathrm{inf}}$ with respect to the ideal sheaf $\ker(\theta_X) \subset \mathcal{O}\mathbb{A}_{\mathrm{inf}}$, $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ := \mathcal{O}\mathbb{A}_{\mathrm{cris}}[1/p]$, and $\mathcal{O}\mathbb{B}_{\mathrm{cris}} := \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+[1/t]$ with $t = \log([\epsilon])$ defined in (2A.6).

(2) For $r \geq 0$ an integer, define $\mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}^0 \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}^0$ to be the r-th PD-ideal $\ker(\theta_X)^{[r]}$, and $\mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ the image of the canonical map

$$\varprojlim \mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}^0/p^n \to \varprojlim \mathcal{O}\mathbb{A}_{\mathrm{cris}}^0/p^n = \mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

Also set $\mathrm{Fil}^{-r}\, \mathcal{O}\mathbb{A}_{\mathrm{cris}} = \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ for $r > 0$.

(3) For any integer r, set $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ := \mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}[1/p]$ and $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}} := \sum_{s\in\mathbb{Z}} t^{-s}\, \mathrm{Fil}^{r+s}\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$.

Remark 2.10. As $t^p = p! \cdot t^{[p]}$ in $A_{\mathrm{cris}} = \mathbb{A}_{\mathrm{cris}}(\hat{\bar{k}}, \mathcal{O}_{\hat{\bar{k}}})$, one can also define $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ as

$$\sum_{s\in\mathbb{N}} t^{-s}\, \mathrm{Fil}^{r+s}\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}.$$

A similar observation holds for $\mathrm{Fil}^r\, \mathbb{B}_{\mathrm{cris}}$.

Remark 2.11. (1) We shall see later that $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ has neither p-torsion (Corollary 2.16) nor t-torsion (Corollary 2.24). So $\mathcal{O}\mathbb{A}_{\mathrm{cris}} \subset \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ \subset \mathcal{O}\mathbb{B}_{\mathrm{cris}}$.

(2) The morphism θ_X of (2B.3) extends to a surjective morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}}^0 \to \hat{\mathcal{O}}_X^+$ with kernel $\mathrm{Fil}^1\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}^0$, hence a morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \hat{\mathcal{O}}_X^+$. Let us denote them again by θ_X. As $\hat{\mathcal{O}}_X^+$ is p-adically complete and has no p-torsion, using the snake lemma and passing to limits one can deduce the following short exact sequence

$$0 \to \varprojlim_n (\mathrm{Fil}^1\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}^0/p^n) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}} \xrightarrow{\theta_X} \hat{\mathcal{O}}_X^+ \to 0.$$

In particular, $\mathrm{Fil}^1\, \mathcal{O}\mathbb{A}_{\mathrm{cris}} = \ker(\theta_X)$.

Definition 2.12. Consider the following sheaves on $X_{\mathrm{proét}}$.

(1) Let $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}$ be the p-adic completion of the sheaf of PD polynomial rings $\mathbb{A}_{\mathrm{cris}}^0\langle u_1, \dots, u_d\rangle \subset \mathbb{B}_{\mathrm{inf}}[u_1, \dots, u_d]$. Set $\mathbb{B}_{\mathrm{cris}}^+\{\langle u_1, \dots, u_d\rangle\} := \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}[1/p]$ and $\mathbb{B}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\} := \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}[1/t]$.

(2) For r an integer, let $\mathrm{Fil}^r\, \mathbb{A}^0_{\mathrm{cris}}\langle u_1, \dots, u_d\rangle \subset \mathbb{A}^0_{\mathrm{cris}}\langle u_1, \dots, u_d\rangle$ be the ideal sheaf

$$\sum_{i_1,\dots,i_d\geq 0} \mathrm{Fil}^{r-(i_1+\cdots+i_d)}\, \mathbb{A}^0_{\mathrm{cris}}\cdot u_1^{[i_1]}\cdots u_d^{[i_d]} \subset \mathbb{A}^0_{\mathrm{cris}}\langle u_1, \dots, u_d\rangle,$$

and $\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}) \subset \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}$ the image of the morphism

$$\varprojlim_n (\mathrm{Fil}^r\, \mathbb{A}^0_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}/p^n) \to \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}.$$

The family $\{\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}) : r \in \mathbb{Z}\}$ gives a descending filtration of $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}$. Inverting p, we obtain $\mathrm{Fil}^r(\mathbb{B}^+_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\})$. Finally set

$$\mathrm{Fil}^r(\mathbb{B}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}) := \sum_{s\in\mathbb{Z}} t^{-s}\, \mathrm{Fil}^{r+s}(\mathbb{B}^+_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}).$$

To describe $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ more explicitly, assume that $\mathcal{X}$ is *small*, i.e., there is an étale morphism $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}) =: \mathcal{T}^d$ of formal schemes over $\mathcal{O}_k$, where we have used $\{-, \dots, -\}$ to denote convergent power series. Let $\mathbb{T}^d$ denote the generic fiber of $\mathcal{T}^d$ and $\tilde{\mathbb{T}}^d$ be obtained from $\mathbb{T}^d$ by adding a compatible system of p^n-th roots of T_i for $1 \leq i \leq d$ and $n \geq 1$

$$\tilde{\mathbb{T}}^d := \mathrm{Spa}(k\{T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\}, \mathcal{O}_k\{T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\}).$$

Set $\tilde{X} := X \times_{\mathbb{T}^d} \tilde{\mathbb{T}}^d$. Let $T_i^\flat \in \mathcal{O}_X^{\flat +}|_{\tilde{X}}$ be the element $(T_i, T_i^{1/p}, \dots, T_i^{1/p^n}, \dots)$. Then $\theta_X(T_i \otimes 1 - 1 \otimes [T_i^\flat]) = 0$, giving an $\mathbb{A}_{\mathrm{cris}}$-linear morphism

$$\alpha : \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}} \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\tilde{X}}, \quad u_i \mapsto T_i \otimes 1 - 1 \otimes [T_i^\flat]. \tag{2B.4}$$

Clearly, α respects the filtrations on both sides.

Proposition 2.13. *The morphism α of* (2B.4) *is an isomorphism. Moreover, α is strictly compatible with the filtrations on both sides, i.e., the inverse of the isomorphism α respects also the filtrations of both sides.*

Lemma 2.14. *Let $\bar{k}$ be an algebraic closure of k. Then $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}_{\bar{k}}}$ has an $\mathcal{O}_X^{\mathrm{ur}\,+}|_{\tilde{X}_{\bar{k}}}$-algebra structure, sending T_i to $u_i + [T_i^\flat]$, such that the composition*

$$\mathcal{O}_X^{\mathrm{ur}\,+}|_{\tilde{X}_{\bar{k}}} \to \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}_{\bar{k}}} \xrightarrow{\theta'|_{\tilde{X}_{\bar{k}}}} \hat{\mathcal{O}}_X^+|_{\tilde{X}_{\bar{k}}}$$

is the map (2B.2) *composed with $\mathcal{O}_X^+ \to \hat{\mathcal{O}}_X^+$. Here $\theta' : \mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\} \to \hat{\mathcal{O}}_X^+$ is induced from the map $\mathbb{A}_{\mathrm{cris}} \xrightarrow{\theta} \hat{\mathcal{O}}_X^+$ by sending u_i's to* 0.

Proof. Let U be an affinoid perfectoid lying above $\tilde{X}_{\bar{k}}$, and $\mathcal{Y} \in \mathcal{X}_{\text{ét}}$, equipped with a map $a : U \to \mathcal{Y}_k$ as in (2B.1). We shall first construct a morphism of $\mathcal{O}_k$-algebras

$$\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \to (\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\})(U) \tag{2B.5}$$

sending T_i to $u_i + [T_i^\flat]$. As our construction is functorial and as $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}$ is a sheaf, shrinking U and $\mathcal{Y}$ if necessary, we may and we do assume $\mathcal{Y} = \mathrm{Spf}(A)$ affine. Then, the map $a : U \to \mathcal{Y}_k$ gives us a morphism of $\mathcal{O}_k$-algebras $a^\# : A \to R^+$. Moreover, U being qcqs, $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\}(U) =$

$\varprojlim_n((\mathbb{A}_{\mathrm{cris}}/p^n)(U)\langle u_1,\dots,u_d\rangle)$. Consequently, the morphisms $\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n \to (\mathbb{A}_{\mathrm{cris}}/p^n)(U)$ for all $n\geq 1$ in Lemma 2.7 induce a natural filtered map

$$\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\} \to \mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}(U). \tag{2B.6}$$

Therefore, to obtain (2B.5), it suffices to construct a natural map

$$A=\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \to \mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}. \tag{2B.7}$$

of $\mathcal{O}_k$-algebras mapping T_i to $u_i+[T_i^\flat]$. To do so, composing the map $\mathcal{Y}\to\mathcal{X}$ with $\mathcal{X}\to\mathcal{T}^d$, we obtain an étale morphism $b:\mathcal{Y}\to\mathcal{T}^d$ of p-adic formal schemes, whence an étale morphism

$$b^\#:\mathcal{O}_k\{T_1^{\pm 1},\dots,T_d^{\pm 1}\}\to A$$

of $\mathcal{O}_k$-algebras. On the other hand, u_i has divided powers in $\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}$, so $[T_i^\flat]+u_i\in\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}$ is invertible. This allows to define a map

$$f:\mathcal{O}_k\{T_1^{\pm 1},\dots,T_d^{\pm 1}\}\to \mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}$$

of $\mathcal{O}_k$-algebras, sending each T_i to $u_i+[T_i^\flat]$. Let f_n be its reduction modulo p^n for $n\geq 1$. Then, we have the following diagram, which is commutative without the dotted map:

$$\begin{tikzcd}
R^+/p^n & & A/p^n \arrow[ll, "a^\# \bmod p^n"'] \arrow[dll, dotted, "\exists!\ g_n"'] \\
\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}/p^n \arrow[u, "\theta'_{(R,R^+)} \bmod p^n"] & & \mathcal{O}_k\{T_1^{\pm 1},\dots,T_d^{\pm 1}\}/p^n. \arrow[ll, "f_n"'] \arrow[u, "b^\# \bmod p^n"']
\end{tikzcd} \tag{2B.8}$$

Here, the left vertical map is the reduction modulo p^n of the ring homomorphism

$$\theta'_{(R,R^+)}:\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\dots,u_d\rangle\}\to R^+$$

which sends each u_i to 0 and extends the usual map $\theta_{(R,R^+)}:\mathbb{A}_{\mathrm{cris}}(R,R^+)\to R^+$. Since $\ker(\theta'_{(R,R^+)})$ has PD-structure, the left vertical map of (2B.8) has a nilpotent kernel. Then, by the étaleness of $b^\#$, we get a unique dotted map g_n, making the whole diagram (2B.8) commutative. These g_n's are compatible with each other, and the limit $\varprojlim_n g_n$ gives the morphism (2B.7) and thus (2B.5). Since $\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U)=\varinjlim\mathcal{O}_{\mathcal{Y}}(\mathcal{Y})$, where the direct limit runs through the diagrams (2B.1), we get from (2B.5) a morphism of $\mathcal{O}_k$-algebras, sending T_i to $u_i+[T_i^\flat]$:

$$\widetilde{\mathcal{O}_X^{\mathrm{ur}+}}(U)\to\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\dots,u_d\rangle\}(U),$$

whose construction is functorial with respect to affinoid perfectoid $U\in X_{\mathrm{pro\acute{e}t}}$ lying above $\tilde{X}_{\bar{k}}$. As such affinoid perfectoids form a basis of the topology on $X_{\mathrm{pro\acute{e}t}}/\tilde{X}_{\bar{k}}$, by passing to the associated sheaf, we obtain the required morphism of sheaves of $\mathcal{O}_k$-algebras $\mathcal{O}_X^{\mathrm{ur}}|_{\tilde{X}_{\bar{k}}}\to\mathbb{A}_{\mathrm{cris}}|_{\tilde{X}_{\bar{k}}}\{\langle u_1,\dots,u_d\rangle\}$ sending T_i to $u_i+[T_i^\flat]$. The last statement follows from the assignment $\theta'(U_i)=0$ and the fact that $\theta([T_i^\flat])=T_i$. $\square$

Proof of Proposition 2.13. As $\tilde{X}_{\bar{k}} \to \tilde{X}$ is a covering in $X_{\text{proét}}$, it is enough to show that $\alpha|_{\tilde{X}_{\bar{k}}}$ is an isomorphism. By Lemma 2.14, there exists a morphism of sheaves of $\mathcal{O}_k$-algebras $\mathcal{O}_X^{\text{ur}+}|_{\tilde{X}_{\bar{k}}} \to \mathbb{A}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}_{\bar{k}}}$ sending T_i to $u_i + [T_i^\flat]$. By extension of scalars, we find the morphism

$$\beta : \mathcal{O}\mathbb{A}_{\text{inf}}|_{\tilde{X}_{\bar{k}}} = (\mathcal{O}_X^{\text{ur}+} \otimes_{\mathcal{O}_k} \mathbb{A}_{\text{inf}})|_{\tilde{X}_{\bar{k}}} \to \mathbb{A}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}_{\bar{k}}}$$

which maps $T_i \otimes 1$ to $u_i + [T_i^\flat]$. Consider the composite (with θ' as in Lemma 2.14)

$$\theta'|_{\tilde{X}_{\bar{k}}} \circ \beta \colon \mathcal{O}\mathbb{A}_{\text{inf}}|_{\tilde{X}_{\bar{k}}} \to \mathbb{A}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}_{\bar{k}}} \to \hat{\mathcal{O}}_X^+|_{\tilde{X}_{\bar{k}}},$$

which is $\theta_X|_{\tilde{X}_{\bar{k}}}$ by Lemma 2.14. Therefore, $\beta(\ker(\theta_X|_{\tilde{X}_{\bar{k}}})) \subset \ker(\theta'|_{\tilde{X}_{\bar{k}}})$. Since $\ker(\theta'|_{\tilde{X}_{\bar{k}}})$ has a PD-structure, β extends to the PD-envelope $\mathcal{O}\mathbb{A}^0_{\text{cris}}|_{\tilde{X}_{\bar{k}}}$ of the source, and thus to $\mathcal{O}\mathbb{A}_{\text{cris}}|_{\tilde{X}_{\bar{k}}}$ as $\mathbb{A}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}$ is p-adically complete. Thus we obtain the morphism below, still denoted by β, sending $T_i \otimes 1$ to $u_i + [T_i^\flat]$:

$$\beta \colon \mathcal{O}\mathbb{A}_{\text{cris}}|_{\tilde{X}_{\bar{k}}} \to \mathbb{A}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}_{\bar{k}}}.$$

The morphism β above preserves Fil^1, hence all the Fil^r's. One shows that β and α are inverse to each other, giving our proposition. □

Corollary 2.15. *Keep the notations above. There are natural $\mathbb{B}^+_{\text{cris}}$-linear and $\mathbb{B}_{\text{cris}}$-linear* isomorphisms *sending u_i to $T_i \otimes 1 - 1 \otimes [T_i^\flat]$,*

$$\mathbb{B}^+_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}} \xrightarrow{\sim} \mathcal{O}\mathbb{B}^+_{\text{cris}}|_{\tilde{X}} \quad \textit{and} \quad \mathbb{B}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}|_{\tilde{X}} \xrightarrow{\sim} \mathcal{O}\mathbb{B}_{\text{cris}}|_{\tilde{X}}$$

which is strictly compatible with filtrations on both sides.

Corollary 2.16. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Then $\mathcal{O}\mathbb{A}_{\text{cris}}$ has no p-torsion. In particular, $\mathcal{O}\mathbb{A}_{\text{cris}} \subset \mathcal{O}\mathbb{B}^+_{\text{cris}}$.*

Proof. This is a local question on X. Hence we may and do assume there is an étale morphism $X \to \text{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. So we reduces ourselves to the corresponding statement for $\mathbb{A}_{\text{cris}}\{\langle u_1, \dots, u_d\rangle\}$. As the latter is the p-adic completion of $\mathbb{A}^0_{\text{cris}}\langle u_1, \dots, u_d\rangle$, one reduces further to the fact that $\mathbb{A}^0_{\text{cris}}$ has no p-torsion, as shown in Corollary 2.3 (1). □

An important feature of $\mathcal{O}\mathbb{A}_{\text{cris}}$ is that it has an $\mathbb{A}_{\text{cris}}$-linear connection on it. To see this, set $\Omega^{1,\text{ur}+}_{X/k} := w^{-1}\Omega^1_{\mathcal{X}_{\text{ét}}/\mathcal{O}_k}$, which is locally free of finite rank over $\mathcal{O}_X^{\text{ur}+}$. Let

$$\Omega^{i,\text{ur}+}_{X/k} := \wedge^i_{\mathcal{O}_X^{\text{ur}+}} \Omega^{1,\text{ur}+}_{X/k} \quad \text{and} \quad \Omega^{i,\text{ur}}_{X/k} := \Omega^{1,\text{ur}+}_{X/k}[1/p] \quad \forall i \geq 0.$$

Then $\mathcal{O}\mathbb{A}_{\text{inf}}$ admits a unique $\mathbb{A}_{\text{inf}}$-linear connection $\nabla \colon \mathcal{O}\mathbb{A}_{\text{inf}} \to \mathcal{O}\mathbb{A}_{\text{inf}} \otimes_{\mathcal{O}_X^{\text{ur}+}} \Omega^{1,\text{ur}+}_{X/k}$ induced from the usual one on $\mathcal{O}_{\mathcal{X}_{\text{ét}}}$. This connection extends uniquely to $\mathcal{O}\mathbb{A}_{\text{cris}}$

$$\nabla \colon \mathcal{O}\mathbb{A}_{\text{cris}} \to \mathcal{O}\mathbb{A}_{\text{cris}} \otimes_{\mathcal{O}_X^{\text{ur}+}} \Omega^{1,\text{ur}+}_{X/k},$$

which is $\mathbb{A}_{\mathrm{cris}}$-linear. Inverting p (resp. t), we get also a $\mathbb{B}^+_{\mathrm{cris}}$-linear (resp. $\mathbb{B}_{\mathrm{cris}}$-linear) connection on $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ (resp. on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$):

$$\nabla\colon \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/k} \quad \text{and} \quad \nabla\colon \mathcal{O}\mathbb{B}_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/k}.$$

From Proposition 2.13, we obtain

Corollary 2.17 (crystalline Poincaré lemma). *Let $\mathcal{X}$ be a smooth formal scheme of dimension d over $\mathcal{O}_k$. Then there is an exact sequence of proétale sheaves*

$$0 \to \mathbb{A}_{\mathrm{cris}} \to \mathcal{O}\mathbb{A}_{\mathrm{cris}} \xrightarrow{\nabla} \mathcal{O}\mathbb{A}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}+}} \Omega^{1,\mathrm{ur}+}_{X/k} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{O}\mathbb{A}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}+}} \Omega^{d,\mathrm{ur}+}_{X/k} \to 0,$$

which is strictly exact with respect to the filtration giving $\Omega^{i,\mathrm{ur}+}_{X/k}$ degree i. In particular, the connection ∇ is integrable and satisfies Griffiths transversality with respect to the filtration on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, i.e., $\nabla(\mathrm{Fil}^i\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}) \subset \mathrm{Fil}^{i-1}\, \mathcal{O}\mathbb{A}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}+}} \Omega^{1,\mathrm{ur}+}_{X/k}$. Moreover, the similar results for $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ hold.

Proof. It suffices to prove the assertion for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. The question is local on $\mathcal{X}$, so we assume there is an étale morphism $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Under the isomorphism (2B.4) of Proposition 2.13, $\mathrm{Fil}^i\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}|_{\tilde{X}}$ is the p-adic completion of

$$\sum_{i_1,\dots i_d \geq 0} \mathrm{Fil}^{i-(i_0+\cdots+i_d)}\, \mathbb{A}_{\mathrm{cris}}|_{\tilde{X}} u_1^{[i_1]} \cdots u_d^{[i_d]}$$

with $T_i \otimes 1 - 1 \otimes [T_i^\flat]$ sent to u_i. Moreover $\nabla(u_i^{[n]}) = u_i^{[n-1]} \otimes dT_i$ for any $i, n \geq 1$, since the connection ∇ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is $\mathbb{A}_{\mathrm{cris}}$-linear. The strict exactness and Griffiths transversality then follow. □

Using Proposition 2.13, we can establish an acyclicity result for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ as in Lemma 2.7. Let $\mathcal{U} = \mathrm{Spf}(A)$ be an affine open subset of $\mathcal{X}$, admitting an étale morphism to $\mathcal{T}^d = \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Let U be the generic fiber, and set $\tilde{U} := U \times_{\mathbb{T}^d} \tilde{\mathbb{T}}^d$. Let V be an affinoid perfectoid of $X_{\mathrm{proét}}$ lying above $\tilde{U}_{\bar{k}}$. Write $\hat{V} = \mathrm{Spa}(R, R^+)$. Let $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ be the p-adic completion of the PD-envelope $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}(R, R^+)$ of $A \otimes_{\mathcal{O}_k} W(R^{\flat +})$ with respect to the kernel of the following morphism induced from $\theta_{(R,R^+)}$ by extending scalars to A:

$$\theta_A\colon A \otimes_{\mathcal{O}_k} W(R^{\flat +}) \to R^+.$$

Set $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(R, R^+) := \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p]$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) := \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(R, R^+)[1/t]$. For $r \in \mathbb{Z}$, define $\mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ to be the closure inside $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ for the p-adic topology of the r-th PD-ideal of $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}(R, R^+)$. Finally, set $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(R, R^+) := \mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)[1/p]$ and $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) := \sum_{s \in \mathbb{Z}} t^{-s}\, \mathrm{Fil}^{r+s}\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(R, R^+)$.

Lemma 2.18. *There is a natural filtered morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)$ of $R^+ \otimes_{\mathcal{O}_k} A_{\mathrm{cris}}$-algebras, inducing an almost isomorphism $\mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \xrightarrow{\sim} (\mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n)(V)$ for each $r, n \geq 0$. Moreover, $H^i(V, \mathrm{Fil}^r\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n)$ is almost zero whenever $i > 0$.*

Proof. Let ι be the composition of the two natural ring homomorphisms below

$$A \otimes_{\mathcal{O}_k} W(R^{\flat +}) \to A \otimes_{\mathcal{O}_k} \mathbb{A}_{\mathrm{cris}}(V) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(V).$$

Then, $\theta_X(V)\circ\iota=\theta_A$, with $\theta_X(V)$ the map obtained by taking sections at V of $\theta_X:\mathcal{O}\mathbb{A}_{\mathrm{cris}}\to\hat{\mathcal{O}}_X^+$. So $\iota(\ker(\theta_A))\subset\ker(\theta_X(V))$. As $\ker(\theta_X(V))$ has PD-structure, ι extends to $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}(R,R^+)$ and the resulting morphism $\mathcal{O}\mathbb{A}^0_{\mathrm{cris}}(R,R^+)\to\mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)$ respects the filtrations. Passing to p-adic completions, we obtain a filtered morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)\to\mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)$ of $A\otimes_{\mathcal{O}_k}A_{\mathrm{cris}}$-algebras, still noted by ι in the following.

Let $\alpha_{(R,R^+)}:\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\}\to\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)$ be the $\mathbb{A}_{\mathrm{cris}}(R,R^+)$-linear map, mapping u_i to $T_i\otimes1-1\otimes[T_i^\flat]$. As in Proposition 2.13, it is an isomorphism, strictly compatible with the filtrations. Here $\mathrm{Fil}^r(\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\})$ is defined to be the p-adic completion of

$$\sum_{i_1,\ldots,i_d\geq0}\mathrm{Fil}^{r-(i_1+\cdots+i_d)}\,\mathbb{A}^0_{\mathrm{cris}}(R,R^+)u_1^{[i_1]}\cdots u_d^{[i_d]}\subset\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\}.$$

Therefore, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\} \arrow[r, "\simeq", "\alpha_{(R,R^+)}"'] \arrow[d, "(2B.6)"'] & \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+) \arrow[d, "\iota"] \\
\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\ldots,u_d\rangle\}(V) \arrow[r, "\simeq", "\alpha(V)"'] & \mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)
\end{tikzcd},$$

whose horizontal arrows are filtered isomorphisms. To prove the first part of our lemma, it suffices to show that (2B.6) induces an almost isomorphism

$$\mathrm{Fil}^r\,\mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle u_1,\ldots,u_d\rangle\}/p^n\to(\mathrm{Fil}^r\,\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\ldots,u_d\rangle\}/p^n)(V)\tag{2B.9}$$

for any $r,n\geq0$. By definition, the left-hand side of the morphism above is

$$\bigoplus_{i_1,\ldots,i_d\geq0}(\mathrm{Fil}^{r-i_1-\cdots-i_d}\,\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)\cdot u_1^{[i_1]}\cdots u_d^{[i_d]},$$

while the right-hand side is given by (recall V is qcqs)

$$\bigoplus_{i_1,\ldots,i_d\geq0}(\mathrm{Fil}^{r-i_1-\cdots-i_d}\,\mathbb{A}_{\mathrm{cris}}/p^n)(V)\cdot u_1^{[i_1]}\cdots u_d^{[i_d]}.$$

Under these descriptions, the map (2B.9) is induced from the natural maps

$$\mathrm{Fil}^{r-i_1-\cdots-i_d}\,\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\to(\mathrm{Fil}^{r-i_1-\cdots-i_d}\,\mathbb{A}_{\mathrm{cris}}/p^n)(V),\quad i_1,\ldots,i_d\geq0.$$

So, that (2B.9) is an almost isomorphism follows from Lemma 2.7.

It remains to prove $H^i(V,\mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n)^a=0$ for $i>0$ and $n,r\geq0$. Using the isomorphism α of Proposition 2.13, we are reduced to the similar statement for $\mathrm{Fil}^r\,\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\ldots,u_d\rangle\}/p^n$. As V is qcqs, we have

$$H^i(V,\mathrm{Fil}^r\,\mathbb{A}_{\mathrm{cris}}\{\langle u_1,\ldots,u_d\rangle\}/p^n)\simeq\bigoplus_{i_1,\ldots,i_d\geq0}H^i(V,\mathrm{Fil}^{r-i_1-\cdots-i_d}\,\mathbb{A}_{\mathrm{cris}}/p^r),$$

which vanishes by Lemma 2.7. $\square$

Corollary 2.19. *Keep the notation of Lemma 2.18. Then the filtered morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)$ of Lemma 2.18 induces an isomorphism of $R^+ \otimes_{\mathcal{O}_k} B_{\mathrm{cris}}$-modules $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$, strictly compatible with filtrations. Moreover, $H^i(V, \mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = H^i(V, \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = 0$ for any $i > 0$ and $r \in \mathbb{Z}$.*

Proof. The proof is the same as that of Corollary 2.8(2). Indeed, as $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ and $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ are p-adically complete and flat over $\mathbb{Z}_p$, by Lemmas 2.6 and 2.18, the filtered morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)$ induces a filtered morphism $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \to \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$ of $A \otimes_{\mathcal{O}_k} B_{\mathrm{cris}}$-modules, with kernel and cokernel killed by $\mathcal{I}^2$. Therefore, the latter is an isomorphism since $\mathcal{I} \cdot (A \otimes_{\mathcal{O}_k} B_{\mathrm{cris}}) = A \otimes_{\mathcal{O}_k} B_{\mathrm{cris}}$. To prove the statements for $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}$, as in the proof of Corollary 2.8, it suffices to establish the similar properties (a) and (b) hold for $\mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$. Let $s \geq 0$ be an integer. By Corollary 2.15, one checks that, over $X_{\mathrm{pro\acute{e}t}}/\tilde{U}_{\bar{k}}$, $\mathrm{gr}^s\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}$ is a free module over $\hat{\mathcal{O}}_X$, with a basis given by the images of the elements

$$\xi^{i_1} u_1^{i_1} \cdots u_d^{i_d}, \qquad \text{where } i_0, \dots, i_d \in \mathbb{N} \text{ and } i_0 + \cdots + i_d = s.$$

Similar observation holds for $\mathrm{gr}^s\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(R, R^+)$. Consequently, by [Scholze 2013, Lemma 4.10], the canonical map $\mathrm{gr}^s\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(R, R^+) \to \mathrm{gr}^s\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}(V)$ is an isomorphism and $H^i(V, \mathrm{gr}^r\, \mathcal{O}\mathbb{B}^+_{\mathrm{cris}}) = 0$ for $i > 0$, as wanted. ☐

2C. ***Frobenius on crystalline period sheaves.*** We keep the notations in the previous subsection. So k is *absolutely unramified* and $\mathcal{X}$ is a smooth formal scheme of dimension d over $\mathcal{O}_k$. We want to endow Frobenius endomorphisms on the crystalline period sheaves.

On $\mathbb{A}_{\mathrm{inf}} = W(\mathcal{O}_X^{\flat +})$, we have the Frobenius map

$$\varphi\colon \mathbb{A}_{\mathrm{inf}} \to \mathbb{A}_{\mathrm{inf}}, \qquad (a_0, a_1, \dots, a_n, \dots) \mapsto (a_0^p, a_1^p, \dots, a_n^p, \dots).$$

Then for any $a \in \mathbb{A}_{\mathrm{inf}}$, we have $\varphi(a) \equiv a^p \mod p$. Thus, $\varphi(\xi) = \xi^p + p \cdot b$ with $b \in \mathbb{A}_{\mathrm{inf}}|_{X_{\bar{k}}}$. In particular $\varphi(\xi) \in \mathbb{A}^0_{\mathrm{cris}}|_{X_{\bar{k}}}$ has all divided powers. As a consequence we obtain a Frobenius φ on $\mathbb{A}^0_{\mathrm{cris}}$ extending that on $\mathbb{A}_{\mathrm{inf}}$. By continuity, φ extends to $\mathbb{A}_{\mathrm{cris}}$ and thus to $\mathbb{B}^+_{\mathrm{cris}}$. Note that $\varphi(t) = \log([\epsilon^p]) = pt$. Consequently φ is extended to $\mathbb{B}_{\mathrm{cris}}$ by setting $\varphi\left(\frac{1}{t}\right) = \frac{1}{pt}$.

To endow a Frobenius on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, we first assume that the Frobenius of $\mathcal{X}_0 = \mathcal{X} \otimes_{\mathcal{O}_k} \kappa$ lifts to a morphism σ on $\mathcal{X}$, which is compatible with the Frobenius on $\mathcal{O}_k$. Then for $\mathcal{Y} \in \mathcal{X}_{\mathrm{\acute{e}t}}$, consider the following diagram:

$$\begin{tikzcd}
\mathcal{Y}_\kappa \arrow[rrr, hook] & & & \mathcal{Y} \arrow[d, "\text{étale}"] \\
\mathcal{Y}_\kappa \arrow[u, "\text{absolute Frobenius}"] \arrow[r, hook] & \mathcal{Y} \arrow[urr, dotted, "\exists\sigma_{\mathcal{Y}}"] \arrow[r] & \mathcal{X} \arrow[r, "\sigma"] & \mathcal{X}.
\end{tikzcd}$$

As the right vertical map is étale, there is a unique dotted morphism above making the diagram commute. When $\mathcal{Y}$ varies in $\mathcal{X}_{\mathrm{\acute{e}t}}$, the $\sigma_{\mathcal{Y}}$'s give rise to a σ-semilinear endomorphism on $\mathcal{O}_{\mathcal{X}_{\mathrm{\acute{e}t}}}$ whence a σ-semilinear endomorphism φ on $\mathcal{O}_X^{\mathrm{ur}\,+}$.

Remark 2.20. In general $\mathcal{X}$ does not admit a lifting of Frobenius. But as $\mathcal{X}$ is smooth over $\mathcal{O}_k$, for each open subset $\mathcal{U} \subset \mathcal{X}$ admitting an étale morphism $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$, a similar argument as above shows that there exists a unique lifting of Frobenius on $\mathcal{U}$ mapping T_i to T_i^p.

We deduce from above a Frobenius on $\mathcal{O}\mathbb{A}_{\mathrm{inf}} = \mathcal{O}_X^{\mathrm{ur}+} \otimes_{\mathcal{O}_k} \mathbb{A}_{\mathrm{inf}}$ given by $\varphi \otimes \varphi$. Abusing notation, we will denote it again by φ. An argument similar to the previous paragraphs shows that φ extends to $\mathcal{O}\mathbb{A}_{\mathrm{cris}}^0$, hence to $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ by continuity, and finally to $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$. Moreover, under the isomorphism (2B.4), the Frobenius on $\mathbb{A}_{\mathrm{cris}}\{\langle u_1, \dots, u_d\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}$ sends u_i to $\varphi(u_i) = \sigma(T_i) - [T_i^\flat]^p$.

Lemma 2.21. *Assume as above that the Frobenius of $\mathcal{X}_0 = \mathcal{X} \otimes_{\mathcal{O}_k} \kappa$ lifts to a morphism σ on $\mathcal{X}$ compatible with the Frobenius on $\mathcal{O}_k$. The Frobenius φ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is horizontal with respect to the connection $\nabla\colon \mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \mathcal{O}\mathbb{A}_{\mathrm{cris}} \otimes \Omega_{X/k}^{1,\mathrm{ur}+}$. Similar assertions hold for $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$ and for $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$.*

Proof. We need to check $\nabla \circ \varphi = (\varphi \otimes d\sigma) \circ \nabla$ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. It is enough to do this locally. Thus we assume there exists an étale morphism $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Recall the isomorphism (2B.4). By $\mathbb{A}_{\mathrm{cris}}$-linearity, it suffices to check the equality on the $u_i^{[n]}$'s. We have

$$(\nabla \circ \varphi)(u_i^{[n]}) = \nabla(\varphi(u_i)^{[n]}) = \varphi(u_i)^{[n-1]} \nabla(\varphi(u_i))$$

Meanwhile, $\varphi(u_i) - \sigma(T_i) = -[T_i^\flat]^p \in \mathbb{A}_{\mathrm{cris}}$, hence $\nabla(\varphi(u_i)) = d\sigma(T_i)$. Thus

$$((\varphi \otimes d\sigma) \circ \nabla)(u_i^{[n]}) = (\varphi \otimes d\sigma)(u_i^{[n-1]} \otimes dT_i) = \varphi(u_i)^{[n-1]} \otimes d\sigma(T_i) = (\nabla \circ \varphi)(u_i^{[n]}),$$

as desired. □

The Frobenius on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ above depends on the initial lifting of Frobenius on $\mathcal{X}$. For different choices of liftings of Frobenius on $\mathcal{X}$, it is possible to compare explicitly the resulting Frobenius endomorphisms on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ with the help of the connection on it, at least when the formal scheme $\mathcal{X}$ admits an étale morphism to $\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$.

Lemma 2.22. *Assume there is an étale morphism $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Let σ_1 and σ_2 be two Frobenius liftings on $\mathcal{X}$, and let φ_1 and φ_2 be the induced Frobenius maps on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, respectively. Then we have the following relation on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$:*

$$\varphi_2 = \sum_{(n_1,\dots,n_d)\in\mathbb{N}^d} \left(\prod_{i=1}^d (\sigma_2(T_i) - \sigma_1(T_i))^{[n_i]}\right)\left(\varphi_1 \circ \left(\prod_{i=1}^d N_i^{n_i}\right)\right) \tag{2C.1}$$

where the N_i are the endomorphisms of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ such that $\nabla = \sum_{i=1}^d N_i \otimes dT_i$.

Proof. To simplify the notations, we will use the multiindex: for $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$, set $\underline{N}^{\underline{m}} := \prod_{i=1}^d N_i^{m_i}$ and $|\underline{m}| := \sum_i m_i$. Let us first observe that the series on the right-hand side of (2C.1) gives a well-defined map on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. As $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is p-adically complete, it suffices to show that this is the case for $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n$ for any $n \geq 1$. Identify $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n$ with $(\mathbb{A}_{\mathrm{cris}}/p^n)\langle u_1, \dots, u_d\rangle$ using Proposition 2.13. Thus, a local section a of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n$ can locally be written as a *finite* sum

$$a=\sum_{\underline{m}\in\mathbb{N}^d} b_{\underline{m}}\cdot\underline{u}^{[\underline{m}]},\qquad b_{\underline{m}}\in\mathbb{A}_{\mathrm{cris}}/p^n.$$

A calculation shows

$$\underline{N}^{\underline{l}}(a)=\sum_{\underline{m}\geq\underline{l}} b_{\underline{m}}\underline{u}^{[\underline{m}-\underline{l}]}=\sum_{\underline{m}\in\mathbb{N}^d} b_{\underline{m}+\underline{l}}\underline{u}^{[\underline{m}]}.$$

As there are only finitely many nonzero coefficients $b_{\underline{m}}$, $\underline{N}^{\underline{l}}(a)=0$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n$ when $|\underline{l}|\gg 0$. Meanwhile, note that $\sigma_2(T_i)-\sigma_1(T_i)\in p\mathcal{O}_X^{\mathrm{ur}+}$, hence their divided powers lie in $\mathcal{O}_X^{\mathrm{ur}+}$. Therefore the series of the right-hand side of (2C.1) applied to a does make sense and gives a well-defined additive map on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}/p^n$. Consequently, the series on the right-hand side of (2C.1) gives a well-defined additive map on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, which is also semilinear relative to the Frobenius on $\mathbb{A}_{\mathrm{cris}}$.

It remains to verify the formula (2C.1). Since both sides of (2C.1) are semilinear relative to the Frobenius of $\mathbb{A}_{\mathrm{cris}}$, it suffices to check the equality for the $\underline{u}^{[\underline{m}]}$'s. In fact, we have

$$\begin{aligned}\Bigg(\sum_{(n_1,\dots,n_d)\in\mathbb{N}^d}\Bigg(\prod_{i=1}^d(\sigma_2(T_i)-\sigma_1(T_i))^{[n_i]}\Bigg)\Bigg(\varphi_1\circ\Bigg(\prod_{i=1}^d N_i^{n_i}\Bigg)\Bigg)\Bigg)&(u^{[\underline{m}]})\\
&=\sum_{\underline{n}\in\mathbb{N}^d}(\sigma_2(\underline{T})-\sigma_1(\underline{T}))^{[\underline{n}]}(\varphi_1(\underline{N}^{\underline{n}}(\underline{u}^{[\underline{m}]})))\\
&=\sum_{\underline{n}\in\mathbb{N}^d}(\varphi_2(\underline{u})-\varphi_1(\underline{u}))^{[\underline{n}]}(\varphi_1(\underline{N}^{\underline{n}}(\underline{u}^{[\underline{m}]})))\\
&=\sum_{\underline{n}\in\mathbb{N}^d|\underline{n}\leq\underline{m}}(\varphi_2(\underline{u})-\varphi_1(\underline{u}))^{[\underline{n}]}\cdot\varphi_1(\underline{u})^{[\underline{m}-\underline{n}]}\\
&=(\varphi_2(\underline{u})-\varphi_1(\underline{u})+\varphi_1(\underline{u}))^{[\underline{m}]}\\
&=\varphi_2(u^{[\underline{m}]}).\end{aligned}$$

This finishes the proof. □

2D. ***Comparison with de Rham period sheaves.*** Let X be a locally noetherian adic space over $\mathrm{Spa}(k,\mathcal{O}_k)$ and recall the map θ in (2A.1). Set $\mathbb{B}_{\mathrm{dR}}^+=\varprojlim W(\mathcal{O}_X^{\flat+})[1/p]/(\ker\theta)^n$, and $\mathbb{B}_{\mathrm{dR}}=\mathbb{B}_{\mathrm{dR}}^+[1/t]$. For $r\in\mathbb{Z}$, let $\mathrm{Fil}^r\,\mathbb{B}_{\mathrm{dR}}=t^r\mathbb{B}_{\mathrm{dR}}^+$. By its very definition, the filtration on $\mathbb{B}_{\mathrm{dR}}$ is decreasing, separated and exhaustive. On the other hand, we can define the de Rham period sheaves with connection $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ (see the erratum to [Scholze 2013, Definition 6.8(iii)]). The filtration on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ is decreasing, separated and exhaustive. Moreover, as in [Brinon 2008, 5.2.8, 5.2.9], one shows that

$$\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+\cap\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}=\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+,$$

implying that the filtration on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is also decreasing, separated and exhaustive.

In the rest of this subsection, assume that $k/\mathbb{Q}_p$ is absolutely unramified.

Proposition 2.23. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$.*

(1) *There are injective filtered morphisms $\mathbb{B}_{\mathrm{cris}}^+ \hookrightarrow \mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ \hookrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$. In this way, we view $\mathbb{B}_{\mathrm{cris}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$ respectively as a subsheaf of rings of $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$.*

(2) *For any $i \in \mathbb{N}$, one has $\mathrm{Fil}^i\, \mathbb{B}_{\mathrm{cris}}^+ = \mathrm{Fil}^i\, \mathbb{B}_{\mathrm{dR}}^+ \cap \mathbb{B}_{\mathrm{cris}}^+$ and $\mathrm{Fil}^i\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ = \mathrm{Fil}^i\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \cap \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$. In particular, the filtrations on $\mathbb{B}_{\mathrm{cris}}^+$ and on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$ are decreasing, separated and exhaustive. Furthermore, the filtered morphisms in* (1) *induce isomorphisms $\mathrm{gr}^i\, \mathbb{B}_{\mathrm{cris}}^+ \xrightarrow{\sim} \mathrm{gr}^i\, \mathbb{B}_{\mathrm{dR}}^+$, and $\mathrm{gr}^i\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ \xrightarrow{\sim} \mathrm{gr}^i\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$.*

Proof. (1) Recall that $\mathbb{B}_{\mathrm{dR}}^+$ is a sheaf of $\mathbb{Q}_p$-algebras, so the natural morphism $\mathbb{A}_{\mathrm{inf}} = W(\mathcal{O}_X^{\flat+}) \to \mathbb{B}_{\mathrm{dR}}^+$ extends to the PD-envelope $\mathbb{A}_{\mathrm{cris}}^0$ of $\mathbb{A}_{\mathrm{inf}}$ with respect to the kernel of the map θ in (2A.1). The resulting map $\mathbb{A}_{\mathrm{cris}}^0 \to \mathbb{B}_{\mathrm{dR}}^+$ respects the filtrations. On the other hand, for each $n \in \mathbb{N}$, the composite

$$\mathbb{A}_{\mathrm{cris}}^0 \to \mathbb{B}_{\mathrm{dR}}^+ \to \mathbb{B}_{\mathrm{dR}}^+/\,\mathrm{Fil}^n\, \mathbb{B}_{\mathrm{dR}}^+ = W(\mathcal{O}_X^{\flat+})[1/p]/(\ker(\theta))^n$$

extends to the p-adic completion $\mathbb{A}_{\mathrm{cris}}$ of $\mathbb{A}_{\mathrm{cris}}^0$. Indeed, this is because the image of the composite above is contained in $\frac{1}{p^n}(W(\mathcal{O}_X^{\flat+})/(\ker(\theta))^n) \subset \mathbb{B}_{\mathrm{dR}}^+/\,\mathrm{Fil}^n\, \mathbb{B}_{\mathrm{dR}}^+$ and the latter is p-adically complete. On passing to the projective limit relative to n, we obtain a filtered morphism $\mathbb{A}_{\mathrm{cris}} \to \mathbb{B}_{\mathrm{dR}}^+$, whence the required filtered morphism $\mathbb{B}_{\mathrm{cris}}^+ \to \mathbb{B}_{\mathrm{dR}}^+$ by inverting $p \in \mathbb{A}_{\mathrm{cris}}$.

To define a natural filtered morphism $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$, observe that $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ is an algebra over $\mathcal{O}_X \otimes_{\mathcal{O}_k} W(\mathcal{O}_X^{\flat+})$, so we have a natural morphism

$$\mathcal{O}\mathbb{A}_{\mathrm{inf}} = \mathcal{O}_X^{\mathrm{ur}\,+} \otimes_{\mathcal{O}_k} W(\mathcal{O}_X^{\flat+}) \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+,$$

which extends to the PD-envelope $\mathcal{O}\mathbb{A}_{\mathrm{cris}}^0$ of $\mathcal{O}\mathbb{A}_{\mathrm{inf}}$ relative to the kernel of the map θ_X in (2B.3). For $n \in \mathbb{N}$, consider the composed morphism

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}^0 \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+ \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+/\,\mathrm{Fil}^n\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+. \tag{2D.1}$$

As above, it extends to $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, the p-adic completion of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}^0$. To check this assertion, assume that $\mathcal{X}$ is affine and étale over $\mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm1}, \dots, T_d^{\pm1}\})$, and let $\tilde{X}$ be the affinoid perfectoid obtained by joining to X a compatible family of p^n-th roots of T_i for $n \in \mathbb{N}$ and $1 \le i \le d$. It suffices to show that, for any affinoid perfectoid V above $\tilde{X}_{\bar{k}}$, the restriction to V of (2D.1) extends in a uniquely way to a morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}}|_V \to (\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+/\,\mathrm{Fil}^n\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)|_V$, with image contained in a $W(\mathcal{O}_X^{\flat+})|_V$-submodule of finite type. By [Scholze 2013, Proposition 6.10], we have $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+|_V = \mathbb{B}_{\mathrm{dR}}^+|_V[[u_1, \dots, u_d]]$, with $u_i = T_i \otimes 1 - 1 \otimes [T_i^\flat]$. So

$$(\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+/\,\mathrm{Fil}^n\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)|_V = \bigoplus_{\underline{m} \in \mathbb{N}^d,\, |\underline{m}| \le n} (W(\mathcal{O}_X^{\flat+})[1/p]/(\xi)^{n-|\underline{m}|})|_V \cdot \underline{u}^{\underline{m}}.$$

Through this identification, the image of (2D.1) (restricted to V) is contained in

$$\frac{1}{p^a}\Bigg(\bigoplus_{\underline{m} \in \mathbb{N}^d,\, |\underline{m}| \le n} (W(\mathcal{O}_X^{\flat+})/(\xi)^{n-|\underline{m}|})|_V \cdot \underline{u}^{\underline{m}} \Bigg) \subset (\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+/\,\mathrm{Fil}^n\, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+)|_V$$

for some $a \in \mathbb{N}$. Since the latter is p-adically complete, the restriction to V of (2D.1) extends to $\mathcal{O}\mathbb{A}_{\mathrm{cris}}|_V$, and the image of this extension is contained in the $W(\mathcal{O}_X^{\flat+})$-submodule of finite type above. If we have two such extensions, the images of both extensions are contained in some $W(\mathcal{O}_X^{\flat+})$-submodule of finite type of the form above for some a (large enough). As the latter is p-adically complete, these two extensions must coincide. This proves our assertion. So (2D.1) extends to a morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+/\operatorname{Fil}^n \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$. Passing to the projective limit relative to n, we obtain the required filtered morphism $\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$.

The two morphisms constructed above are compatible with the isomorphisms in Corollary 2.15 and its de Rham analogue [Scholze 2013, Proposition 6.10]. To finish the proof of (1), we only need to show the morphism $\mathbb{B}_{\mathrm{cris}}^+ \to \mathbb{B}_{\mathrm{dR}}^+$ constructed above is injective. This can be done in the same way as [Brinon 2008, Proposition 6.2.1], and we omit the detail here.

(2) By (1), the corresponding statement for $\mathbb{B}_{\mathrm{cris}}^+$ follows from the fact that the natural induced map

$$\operatorname{gr}^r \mathbb{B}_{\mathrm{cris}}^+|_{X_{\bar{k}}} = \hat{\mathcal{O}}_{X_{\bar{k}}} \cdot (\xi^r/r!) \to \operatorname{gr}^r \mathbb{B}_{\mathrm{dR}}^+|_{X_{\bar{k}}} = \hat{\mathcal{O}}_{X_{\bar{k}}} \cdot \xi^r$$

is an isomorphism. To show the statements for $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$, assume that $\mathcal{X}$ admits an étale map to $\operatorname{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Then we conclude by Corollary 2.15 and its de Rham analogue, and by what we have just shown for $\mathbb{B}_{\mathrm{cris}}^+$. □

Corollary 2.24. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. Then, over $X_{\mathrm{proét}}/X_{\bar{k}}$, the sheaves of A_{cris}-modules $\mathbb{A}_{\mathrm{cris}}$, $\mathbb{B}_{\mathrm{cris}}^+$, $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$ have no t-torsion.*

Proof. As $\mathbb{A}_{\mathrm{cris}}$ and $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ have no p-torsion, they are included respectively in $\mathbb{B}_{\mathrm{cris}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+$. Hence, to prove our corollary, by Proposition 2.23, it is enough to show that, over $X_{\mathrm{proét}}/X_{\bar{k}}$, $\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ have no t-torsion. These two statements are contained in [Scholze 2013, Remarks 6.2 and 6.9]. □

Corollary 2.25. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$.*

(1) *There are natural inclusions $\mathbb{B}_{\mathrm{cris}} \hookrightarrow \mathbb{B}_{\mathrm{dR}}$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}} \hookrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}}$.*

(2) *For any $i \in \mathbb{Z}$, we have $\operatorname{Fil}^i \mathbb{B}_{\mathrm{cris}} = \mathbb{B}_{\mathrm{cris}} \bigcap \operatorname{Fil}^i \mathbb{B}_{\mathrm{dR}}$ and $\operatorname{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}} = \mathcal{O}\mathbb{B}_{\mathrm{cris}} \bigcap \operatorname{Fil}^i \mathcal{O}\mathbb{B}_{\mathrm{dR}}$. In particular, the filtrations on $\mathbb{B}_{\mathrm{cris}}$ and on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ are decreasing, separated and exhaustive. Furthermore, the inclusions in* (1) *induce isomorphisms $\operatorname{gr}^i \mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \operatorname{gr}^i \mathbb{B}_{\mathrm{dR}}$ and $\operatorname{gr}^i \mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \operatorname{gr}^i \mathcal{O}\mathbb{B}_{\mathrm{dR}}$.*

Corollary 2.26. *Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$, with X its generic fiber. Then $w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$.*

Proof. Let $\nu : X_{\mathrm{proét}}^{\sim} \to X_{\text{ét}}^{\sim}$ and $\nu' : X_{\text{ét}}^{\sim} \to \mathcal{X}_{\text{ét}}^{\sim}$ the natural morphisms of topoi. Then $w = \nu' \circ \nu$. Therefore $\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p] \xrightarrow{\sim} \nu'_*\mathcal{O}_{X_{\text{ét}}} \xrightarrow{\sim} \nu'_*\nu_*\mathcal{O}_X = w_*\mathcal{O}_X$. By [Scholze 2013, Corollary 6.19], the natural map $\mathcal{O}_{X_{\text{ét}}} \to \nu_*\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ is an isomorphism. Thus, $w_*\mathcal{O}\mathbb{B}_{\mathrm{dR}} = \nu'_*(\nu_*\mathcal{O}\mathbb{B}_{\mathrm{dR}}) \simeq \nu'_*\mathcal{O}_{X_{\text{ét}}} \simeq \mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$. On the other hand, we have the injection of $\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$-algebras $w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} \hookrightarrow w_*\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Thereby $\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p] \xrightarrow{\sim} w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}$. □

3. Crystalline cohomology and proétale cohomology

In this section, assume that k is *absolutely unramified*. Let σ denote the Frobenius on $\mathcal{O}_k$ and on k, lifting the Frobenius of the residue field κ. The ideal $(p) \subset \mathcal{O}_k$ is endowed with a PD-structure and $\mathcal{O}_k$ becomes a PD-ring in this way.

3A. ***A reminder on convergent F-isocrystals.*** Let $\mathcal{X}_0$ be a κ-scheme of finite type. Let us begin with some general definitions about crystals on the small crystalline site $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ endowed with étale topology. For basics of crystals, we refer to [Berthelot 1996; Berthelot and Ogus 1978]. Recall that a *crystal of $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-modules* is an $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-module $\mathbb{E}$ on $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ such that (i) for any object $(U, T) \in (\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$, the restriction $\mathbb{E}_T$ of $\mathbb{E}$ to the étale site of T is a coherent $\mathcal{O}_T$-module; and (ii) for any morphism $u : (U', T') \to (U, T)$ in $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$, the canonical morphism $u^*\mathbb{E}_T \xrightarrow{\sim} \mathbb{E}_{T'}$ is an isomorphism.

Remark 3.1. Let $\mathcal{X}_0$ be the closed fiber of a smooth formal scheme $\mathcal{X}$ over $\mathcal{O}_k$. The category of crystals on $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ is equivalent to that of coherent $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{M}$ equipped with an integrable and quasinilpotent connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^1_{\mathcal{X}/\mathcal{O}_k}$. Here the connection ∇ is said to be *quasinilpotent* if its reduction modulo p is quasinilpotent in the sense of [Berthelot and Ogus 1978, Definition 4.10].

The absolute Frobenius $F\colon \mathcal{X}_0 \to \mathcal{X}_0$ is a morphism over the Frobenius σ on $\mathcal{O}_k = W(\kappa)$, hence it induces a morphism of topoi, still denoted by F:

$$F\colon (\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\mathrm{cris}} \to (\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\mathrm{cris}}.$$

An *F-crystal* on $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ is a crystal $\mathbb{E}$ equipped with a morphism $\varphi\colon F^*\mathbb{E} \to \mathbb{E}$ of $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-modules, which is nondegenerate, i.e., there exists a map $V : \mathbb{E} \to F^*\mathbb{E}$ of $\mathcal{O}_{\mathcal{X}_0/\mathcal{O}_k}$-modules such that $\varphi V = V\varphi = p^m$ for some $m \in \mathbb{N}$. In the following, we will denote by $F\text{-}\mathrm{Cris}(\mathcal{X}_0, \mathcal{O}_k)$ the category of F-crystals on $\mathcal{X}_0/\mathcal{O}_k$.

Before discussing isocrystals, let us observe the following facts.

Remarks 3.2. (1) Let $\mathcal{X}$ be a quasicompact smooth formal scheme over $\mathcal{O}_k$. Let X^{rig} be its rigid generic fiber, which is a rigid analytic variety over k. Let $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}[1/p])$ denote the category of coherent $\mathcal{O}_{\mathcal{X}}[1/p]$-modules on $\mathcal{X}$, or equivalently, the category of coherent sheaves on $\mathcal{X}$ up to isogeny. Denote by $\mathrm{Coh}(X^{\mathrm{rig}})$ the category of coherent sheaves on X^{rig}. Then, we have the functor below, obtained by taking the rigid generic fiber of a coherent $\mathcal{O}_{\mathcal{X}}[1/p]$-module

$$\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}[1/p]) \to \mathrm{Coh}(X^{\mathrm{rig}}).$$

This is an equivalence of categories. Indeed, it is a consequence of the fact that any coherent sheaf on X^{rig} extends to a coherent sheaf on $\mathcal{X}$ [Lütkebohmert 1990, Lemma 2.2].

(2) Let Y be a rigid analytic variety. Huber [1994, Proposition 4.3] constructed from Y an adic space Y^{ad}, together with a locally coherent morphism $\rho : (|Y^{\mathrm{ad}}|, \mathcal{O}_{Y^{\mathrm{ad}}}) \to (|Y|, \mathcal{O}_Y)$ of ringed sites, satisfying some universal property. We call Y^{ad} the *associated adic space* of Y. The morphism ρ gives rise to an equivalence between the category of sheaves on the Grothendieck site associated to Y and that of sheaves

on the sober topological space Y^{ad} [Huber 1996, 1.1.11]. Moreover, under this equivalence, the notion of coherent sheaves on the ringed site $Y = (|Y|, \mathcal{O}_Y)$ is the same as the one on $Y^{\mathrm{ad}} = (|Y^{\mathrm{ad}}|, \mathcal{O}_{Y^{\mathrm{ad}}})$ since in the case of $Y = \operatorname{Sp} A$, with A a complete topologically finitely generated Tate algebra over k, both of them are naturally equivalent to that of finite A-modules (see [Scholze 2013, Theorem 9.1]).

Let $\mathcal{X}_0$ be a κ-scheme of finite type, embedded as a closed subscheme into a smooth formal scheme $\mathcal{P}$ over $\mathcal{O}_k$. Let P be the adic generic fiber of $\mathcal{P}$ and $]\mathcal{X}_0[_{\mathcal{P}} \subset P$ the preimage of the closed subset $\mathcal{X}_0 \subset \mathcal{P}$ under the specialization map. Following [Berthelot 1996, 2.3.2(i)] (with Remarks 3.2(2) in mind), the *realization on $\mathcal{P}$ of a convergent isocrystal* on $\mathcal{X}_0/\mathcal{O}_k$ is a coherent $\mathcal{O}_{]\mathcal{X}_0[_{\mathcal{P}}}$-module $\mathcal{E}$ equipped with an integrable and *convergent* connection $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{]\mathcal{X}_0[_{\mathcal{P}}}} \Omega^1_{]\mathcal{X}_0[_{\mathcal{P}}/k}$ (we refer to [loc. cit., 2.2.5] for the definition of convergent connections). Being a coherent $\mathcal{O}_{]\mathcal{X}_0[_{\mathcal{P}}}$-module with integrable connection, $\mathcal{E}$ is locally free of finite rank by [loc. cit., 2.2.3(ii)]. The category of realizations on $\mathcal{P}$ of convergent isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ is denoted by $\operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P})$, where the morphisms are morphisms of $\mathcal{O}_{]\mathcal{X}_0[_{\mathcal{P}}}$-modules which commute with connections.

Let $\mathcal{X}_0 \hookrightarrow \mathcal{P}'$ be a second embedding of $\mathcal{X}_0$ into a smooth formal scheme $\mathcal{P}'$ over $\mathcal{O}_k$, and assume there exists a morphism $u\colon \mathcal{P}' \to \mathcal{P}$ of formal schemes inducing identity on $\mathcal{X}_0$. The generic fiber of u gives a morphism of adic spaces $u_k\colon\,]\mathcal{X}_0[_{\mathcal{P}'} \to\,]\mathcal{X}_0[_{\mathcal{P}}$, hence a natural functor

$$u_k^*\colon \operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P}) \to \operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P}'), \quad (\mathcal{E}, \nabla) \mapsto (u_k^*\mathcal{E}, u_k^*\nabla).$$

By [loc. cit., 2.3.2(i)], the functor u_k^* is an equivalence of categories. Furthermore, for a second morphism $v\colon \mathcal{P}' \to \mathcal{P}$ of formal schemes inducing identity on $\mathcal{X}_0$, the two equivalence u_k^*, v_k^* are canonically isomorphic [loc. cit., 2.2.17(i)]. The category of *convergent isocrystal on $\mathcal{X}_0/\mathcal{O}_k$*, denoted by $\operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$, is defined as

$$\operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k) := 2-\varinjlim_{\mathcal{P}} \operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{P}),$$

where the limit runs through all smooth formal embeddings $\mathcal{X}_0 \hookrightarrow \mathcal{P}$ of $\mathcal{X}_0$.

Remark 3.3. In general, $\mathcal{X}_0$ does not necessarily admit a global formal embedding. In this case, the category of convergent isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ can still be defined by a gluing argument (see [loc. cit., 2.3.2(iii)]). But the definition recalled above will be enough for our purpose.

As for the category of crystals on $\mathcal{X}_0/\mathcal{O}_k$, the Frobenius morphism $F\colon \mathcal{X}_0 \to \mathcal{X}_0$ induces a natural functor (see [loc. cit., 2.3.7] for the construction):

$$F^*\colon \operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k) \to \operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k).$$

A *convergent F-isocrystal* on $\mathcal{X}_0/\mathcal{O}_k$ is a convergent isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$ equipped with an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ in $\operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$. The category of convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ will be denoted in the following by $F\text{-}\operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$.

Remark 3.4. The category $F\text{-}\operatorname{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$ has the isogeny category $F\text{-}\operatorname{Cris}(\mathcal{X}_0/\mathcal{O}_k) \otimes \mathbb{Q}$ of F-crystals $\mathbb{E}$ on $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ as a full subcategory. To explain this, assume for simplicity that $\mathcal{X}_0$ is the closed fiber of a smooth formal scheme $\mathcal{X}$ over $\mathcal{O}_k$. So $]\mathcal{X}_0[_{\mathcal{X}} = X$, the generic fiber of $\mathcal{X}$. Let $(\mathcal{M}, \nabla)$ be the $\mathcal{O}_{\mathcal{X}}$-module with integrable and quasinilpotent connection associated to the F-crystal $\mathbb{E}$ (Remark 3.1). Let $\mathbb{E}^{\mathrm{an}}$ denote the generic fiber of $\mathcal{M}$, which is a coherent (hence locally free by [loc. cit., 2.3.2(ii)]) $\mathcal{O}_{X_{\mathrm{an}}}$-module equipped with an integrable connection $\nabla^{\mathrm{an}} \colon \mathbb{E}^{\mathrm{an}} \to \mathbb{E}^{\mathrm{an}} \otimes \Omega^1_{X_{\mathrm{an}}/k}$, which is nothing but the generic fiber of ∇. Because of the F-crystal structure on $\mathbb{E}$, the connection ∇^{an} is convergent ([loc. cit., 2.4.1]). In this way we obtain an F-isocrystal $\mathbb{E}^{\mathrm{an}}$ on $\mathcal{X}_0/\mathcal{O}_k$, whence a natural functor

$$(-)^{\mathrm{an}} \colon F\text{-}\mathrm{Cris}(\mathcal{X}_0/\mathcal{O}_k) \otimes \mathbb{Q} \to F\text{-}\mathrm{Isoc}(\mathcal{X}_0/\mathcal{O}_k), \quad \mathbb{E} \mapsto \mathbb{E}^{\mathrm{an}}.$$

By [loc. cit., 2.4.2], this analytification functor is fully faithful, and for $\mathcal{E}$ a convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, there exists an integer $n \geq 0$ and an F-crystal $\mathbb{E}$ such that $\mathcal{E} \xrightarrow{\sim} \mathbb{E}^{\mathrm{an}}(n)$, where for $\mathcal{F} = (\mathcal{F}, \nabla, \varphi \colon F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F})$ an F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, $\mathcal{F}(n)$ denotes the Tate twist of $\mathcal{F}$, given by $\left(\mathcal{F}, \nabla, \frac{\varphi}{p^n} \colon F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}\right)$ [loc. cit., 2.3.8(i)].

Our next goal is to give a more explicit description of the Frobenius morphisms on convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$. From now on, assume for simplicity that $\mathcal{X}_0$ *is the closed fiber of a smooth formal scheme* $\mathcal{X}$ and we identify convergent isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ with their realizations on $\mathcal{X}$. Let X be the adic generic fiber of $\mathcal{X}$. The proof of the following lemma is obvious.

Lemma 3.5. *Assume that the Frobenius $F \colon \mathcal{X}_0 \to \mathcal{X}_0$ can be lifted to a morphism $\sigma \colon \mathcal{X} \to \mathcal{X}$ compatible with the Frobenius on $\mathcal{O}_k$. Still denote by σ the endomorphism on X induced by σ. Then there is an equivalence of categories between*

(1) *the category $F\text{-}\mathrm{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$ of convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$; and*

(2) *the category $\mathrm{Mod}^{\sigma,\nabla}_{\mathcal{O}_X}$ of $\mathcal{O}_{X_{\mathrm{an}}}$-vector bundles $\mathcal{E}$ equipped with an integrable and convergent connection ∇ and an $\mathcal{O}_{X_{\mathrm{an}}}$-linear horizontal isomorphism $\varphi \colon \sigma^*\mathcal{E} \to \mathcal{E}$.*

Consider two liftings of Frobenius σ_i $(i = 1, 2)$ on $\mathcal{X}$. By the lemma above, for $i = 1, 2$, both categories $\mathrm{Mod}^{\sigma_i,\nabla}_{\mathcal{O}_X}$ are naturally equivalent to the category of convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$:

$$\mathrm{Mod}^{\sigma_1,\nabla}_{\mathcal{O}_X} \xleftarrow{\sim} F\text{-}\mathrm{Isoc}(\mathcal{X}_0/\mathcal{O}_k) \xrightarrow{\sim} \mathrm{Mod}^{\sigma_2,\nabla}_{\mathcal{O}_X}.$$

Therefore we deduce an equivalence of categories

$$F_{\sigma_1,\sigma_2} \colon \mathrm{Mod}^{\sigma_1,\nabla}_{\mathcal{O}_X} \to \mathrm{Mod}^{\sigma_2,\nabla}_{\mathcal{O}_X}. \tag{3A.1}$$

When our formal scheme $\mathcal{X}$ is small, we can explicitly describe this equivalence. Assume there is an étale morphism $\mathcal{X} \to \mathcal{T}^d = \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. So $\Omega^1_{X_{\mathrm{an}}/k}$ is a free $\mathcal{O}_{X_{\mathrm{an}}}$-module with a basis given by dT_i $(i = 1, \dots, d)$. In the following, for ∇ a connection on an $\mathcal{O}_{X_{\mathrm{an}}}$-module $\mathcal{E}$, let N_i be the endomorphism of $\mathcal{E}$ (as an abelian sheaf) such that $\nabla = \sum_{i=1}^d N_i \otimes dT_i$.

Lemma 3.6 [Brinon 2008, Proposition 7.2.3]. *Assume that $\mathcal{X} = \mathrm{Spf}(A)$ is affine, admitting an étale morphism $\mathcal{X} \to \mathcal{T}^d$ as above. Let $(\mathcal{E}, \nabla, \varphi_1) \in \mathrm{Mod}^{\sigma_1,\nabla}_{\mathcal{O}_X}$, with $(\mathcal{E}, \nabla, \varphi_2)$ the corresponding object of*

$\mathrm{Mod}_{\mathcal{O}_X}^{\sigma_2,\nabla}$ *under the equivalence* F_{σ_1,σ_2}. *Then on* $\mathcal{E}(X)$ *we have*

$$\varphi_2=\sum_{(n_1,\dots,n_d)\in\mathbb{N}^d}\Bigl(\prod_{i=1}^{d}(\sigma_2(T_i)-\sigma_1(T_i))^{[n_i]}\Bigr)\Bigl(\varphi_1\circ\Bigl(\prod_{i=1}^{d}N_i^{n_i}\Bigr)\Bigr).$$

Furthermore, φ_1 *and* φ_2 *coincide on* $\mathcal{E}(X)^{\nabla=0}$.

More generally, i.e., without assuming the existence of Frobenius lifts to $\mathcal{X}$, for $(\mathcal{E},\nabla)$ an $\mathcal{O}_{X_{\mathrm{an}}}$-module with integrable and convergent connection, *a compatible system of Frobenii on* $\mathcal{E}$ consists of, for any open subset $\mathcal{U}\subset\mathcal{X}$ equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, a horizontal isomorphism $\varphi_{(\mathcal{U},\sigma_{\mathcal{U}})}\colon \sigma_{\mathcal{U}}^*\mathcal{E}|_{\mathcal{U}_k}\to\mathcal{E}|_{\mathcal{U}_k}$ satisfying the following condition: for $\mathcal{V}\subset\mathcal{X}$ another open subset equipped with a lifting of Frobenius $\sigma_{\mathcal{V}}$, the functor

$$F_{\sigma_{\mathcal{U}},\sigma_{\mathcal{V}}}\colon\ \mathrm{Mod}_{\mathcal{O}_{\mathcal{U}_k\cap\mathcal{V}_k}}^{\sigma_{\mathcal{U}},\nabla}\to\mathrm{Mod}_{\mathcal{O}_{\mathcal{U}_k\cap\mathcal{V}_k}}^{\sigma_{\mathcal{V}},\nabla}$$

sends $(\mathcal{E}|_{\mathcal{U}_k\cap\mathcal{V}_k},\nabla,\varphi_{(\mathcal{U},\sigma_{\mathcal{U}})}|_{\mathcal{U}_k\cap\mathcal{V}_k})$ to $(\mathcal{E}|_{\mathcal{U}_k\cap\mathcal{V}_k},\nabla,\varphi_{(\mathcal{V},\sigma_{\mathcal{V}})}|_{\mathcal{U}_k\cap\mathcal{V}_k})$. We denote a compatible system of Frobenii on $\mathcal{E}$ by the symbol φ, when no confusion arises. Let $\mathrm{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ be the category of $\mathcal{O}_{X_{\mathrm{an}}}$-vector bundles equipped with an integrable and convergent connection, and with a compatible system of Frobenii. The morphism in $\mathrm{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ are the morphisms of $\mathcal{O}_{X_{\mathrm{an}}}$-modules which commute with the connections, and with the Frobenius morphisms on any open subset $\mathcal{U}\subset\mathcal{X}$ equipped with a lifting of Frobenius.

Remark 3.7. Let $\mathcal{E}$ be a convergent isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. To define a compatible system of Frobenii on $\mathcal{E}$, we only need to give, for a cover $\mathcal{X}=\bigcup_i\mathcal{U}_i$ of $\mathcal{X}$ by open subsets $\mathcal{U}_i$ equipped with a lifting of Frobenius σ_i, a family of horizontal isomorphisms $\varphi_i:\sigma_i^*\mathcal{E}|_{U_i}\xrightarrow{\sim}\mathcal{E}|_{U_i}$ such that $\varphi_i|_{U_i\cap U_j}$ corresponds to $\varphi_j|_{U_i\cap U_j}$ under the functor $F_{\sigma_i,\sigma_j}\colon\ \mathrm{Mod}_{\mathcal{O}_{U_i\cap U_j}}^{\sigma_i,\nabla}\to\mathrm{Mod}_{\mathcal{O}_{U_i\cap U_j}}^{\sigma_j,\nabla}$ (Here $U_\bullet:=\mathcal{U}_{\bullet,k}$). Indeed, for $\mathcal{U}$ any open subset equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, one can first use the functor $F_{\sigma_i,\sigma_{\mathcal{U}}}$ of (3A.1) applied to $(\mathcal{E}|_{U_i},\nabla|_{U_i},\varphi_i)|_{U_i\cap U}$ to obtain a horizontal isomorphism $\varphi_{\mathcal{U},i}\colon(\sigma_{\mathcal{U}}^*(\mathcal{E}|_{\mathcal{U}}))|_{U_i\cap U}\to\mathcal{E}|_{U_i\cap U}$. From the compatibility of the φ_i's, we deduce $\varphi_{\mathcal{U},i}|_{U\cap U_i\cap U_j}=\varphi_{\mathcal{U},j}|_{U\cap U_i\cap U_j}$. Consequently we can glue the $\varphi_{\mathcal{U},i}$ $(i\in I)$ to get a horizontal isomorphism $\varphi_{\mathcal{U}}\colon\sigma_{\mathcal{U}}^*(\mathcal{E}|_U)\to\mathcal{E}|_U$. One checks that these $\varphi_{\mathcal{U}}$ give the desired compatible system of Frobenii on $\mathcal{E}$.

Let $\mathcal{E}$ be a convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. For $\mathcal{U}\subset\mathcal{X}$ an open subset equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, the restriction $\mathcal{E}|_{\mathcal{U}_k}$ gives rise to a convergent F-isocrystal on $\mathcal{U}_0/k$. Thus there exists a ∇-horizontal isomorphism $\varphi_{(\mathcal{U},\sigma_{\mathcal{U}})}\colon\sigma_{\mathcal{U}}^*\mathcal{E}|_{\mathcal{U}_k}\to\mathcal{E}|_{\mathcal{U}_k}$. Varying $(\mathcal{U},\sigma_{\mathcal{U}})$ we obtain a compatible system of Frobenii φ on $\mathcal{E}$. In this way, $(\mathcal{E},\nabla,\varphi)$ becomes an object of $\mathrm{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$. Directly from the definition, we have the following

Corollary 3.8. *The natural functor* $F\text{-}\mathrm{Isoc}(\mathcal{X}_0/\mathcal{O}_k)\to\mathrm{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ *is an equivalence of categories.*

In the following, denote by $\mathrm{FMod}_{\mathcal{O}_X}^{\sigma,\nabla}$ the category of quadruples $(\mathcal{E},\nabla,\varphi,\mathrm{Fil}^\bullet(\mathcal{E}))$ with $(\mathcal{E},\nabla,\varphi)\in\mathrm{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ and a decreasing, separated and exhaustive filtration $\mathrm{Fil}^\bullet(\mathcal{E})$ on $\mathcal{E}$ by locally free direct summands, such that ∇ satisfies Griffiths transversality with respect to $\mathrm{Fil}^\bullet(\mathcal{E})$, i.e., $\nabla(\mathrm{Fil}^i(\mathcal{E}))\subset\mathrm{Fil}^{i-1}(\mathcal{E})\otimes_{\mathcal{O}_{X_{\mathrm{an}}}}\Omega^1_{X_{\mathrm{an}}/k}$. The morphisms are the morphisms in $\mathrm{Mod}_{\mathcal{O}_X}^{\sigma,\nabla}$ which respect the filtrations. We call the objects in

$\mathrm{FMod}_{\mathcal{O}_X}^{\sigma,\nabla}$ *filtered (convergent) F-isocrystals* on $\mathcal{X}_0/\mathcal{O}_k$. By analogy with the category $F\text{-}\mathrm{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$ of F-isocrystals, we also denote the category of filtered F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ by $FF\text{-}\mathrm{Isoc}(\mathcal{X}_0/\mathcal{O}_k)$.

3B. *Lisse $\hat{\mathbb{Z}}_p$-sheaves and filtered F-isocrystals.* Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$ with X its adic generic fiber. Define $\hat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^n$ and $\hat{\mathbb{Q}}_p := \hat{\mathbb{Z}}_p[1/p]$ as sheaves on $X_{\text{proét}}$. Recall that a *lisse $\mathbb{Z}_p$-sheaf* on $X_{\text{ét}}$ is an inverse system of abelian sheaves $\mathbb{L}_\bullet = (\mathbb{L}_n)_{n\in\mathbb{N}}$ on $X_{\text{ét}}$ such that each $\mathbb{L}_n$ is locally a constant sheaf associated to a finitely generated $\mathbb{Z}/p^n$-modules, and such that $\mathbb{L}_\bullet$ is isomorphic in the procategory to such an inverse system for which $\mathbb{L}_{n+1}/p^n \simeq \mathbb{L}_n$. A *lisse $\hat{\mathbb{Z}}_p$-sheaf* on $X_{\text{proét}}$ is a sheaf of $\hat{\mathbb{Z}}_p$-modules on $X_{\text{proét}}$, which is locally isomorphic to $\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}_p} M$ where M is a finitely generated $\mathbb{Z}_p$-module. By [Scholze 2013, Proposition 8.2] these two notions are equivalent via the functor $\nu^{-1}\colon X_{\text{ét}}^{\sim} \to X_{\text{proét}}^{\sim}$. In the following, we use frequently the natural morphism of topoi

$$w\colon X_{\text{proét}}^{\sim} \xrightarrow{\nu} X_{\text{ét}}^{\sim} \to \mathcal{X}_{\text{ét}}^{\sim}.$$

Before defining crystalline sheaves, let us make the following observation.

Remarks 3.9. (1) Let $\mathcal{M}$ be a crystal on $\mathcal{X}_0/\mathcal{O}_k$, viewed as a coherent $\mathcal{O}_{\mathcal{X}}$-module admitting an integrable connection. Then $w^{-1}\mathcal{M}$ is a coherent $\mathcal{O}_X^{\mathrm{ur}+}$-module with an integrable connection $w^{-1}\mathcal{M} \to w^{-1}\mathcal{M}\otimes_{\mathcal{O}_X^{\mathrm{ur}+}} \Omega_{X/k}^{1,\mathrm{ur}+}$. If furthermore $\mathcal{M}$ is an F-crystal, then $w^{-1}\mathcal{M}$ inherits a system of Frobenii: for any open subset $\mathcal{U} \subset \mathcal{X}$ equipped with a lifting of Frobenius $\sigma_{\mathcal{U}}$, there is naturally an endomorphism of $w^{-1}\mathcal{M}|_U$ which is semilinear with respect to the Frobenius $w^{-1}\sigma_{\mathcal{U}}$ on $\mathcal{O}_X^{\mathrm{ur}+}|_U$ (here $U := \mathcal{U}_k$). Indeed, the Frobenius structure on $\mathcal{M}$ gives a horizontal $\mathcal{O}_{\mathcal{U}}$-linear morphism $\sigma_{\mathcal{U}}^*\mathcal{M}|_{\mathcal{U}} \to \mathcal{M}|_{\mathcal{U}}$, or equivalently, a $\sigma_{\mathcal{U}}$-semilinear morphism $\varphi_{\mathcal{U}}\colon \mathcal{M}|_{\mathcal{U}} \to \mathcal{M}|_{\mathcal{U}}$ (as $\sigma_{\mathcal{U}}$ is the identity map on the underlying topological space). So we obtain a natural endomorphism $w^{-1}\varphi_{\mathcal{U}}$ of $w^{-1}\mathcal{M}|_U$, which is $w^{-1}\sigma_{\mathcal{U}}$-semilinear.

(2) Let $\mathcal{E}$ be a convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. By Remark 3.4, there exists an F-crystal $\mathcal{M}$ on $\mathcal{X}_0/\mathcal{O}_k$ and $n \in \mathbb{N}$ such that $\mathcal{E} \simeq \mathcal{M}^{\mathrm{an}}(n)$. By (1), $w^{-1}\mathcal{M}$ is a coherent $\mathcal{O}_X^{\mathrm{ur}+}$-module equipped with an integrable connection and a compatible system of Frobenii φ. Inverting p, we get an $\mathcal{O}_X^{\mathrm{ur}}$-module $w^{-1}\mathcal{M}[1/p]$ equipped with an integrable connection and a system of Frobenii φ/p^n, which does not depend on the choice of the formal model $\mathcal{M}$ or the integer n. For this reason, abusing notation, let us denote $w^{-1}\mathcal{M}[1/p]$ by $w^{-1}\mathcal{E}$, which is equipped with an integrable connection and a system of Frobenii inherited from $\mathcal{E}$. If furthermore $\mathcal{E}$ has a descending filtration $\{\mathrm{Fil}^i\,\mathcal{E}\}$ by locally direct summands, by Remarks 3.2(1), each $\mathrm{Fil}^i\,\mathcal{E}$ has a coherent formal model $\mathcal{E}_i^+$ on $\mathcal{X}$. Then $\{w^{-1}\mathcal{E}_i^+[1/p]\}$ gives a descending filtration by locally direct summands on $w^{-1}\mathcal{E}$.

Definition 3.10. We say a lisse $\hat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ on $X_{\text{proét}}$ is *crystalline* if there exists a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$, together with an isomorphism of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$-modules

$$w^{-1}\mathcal{E}\otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathbb{L}\otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \tag{3B.1}$$

which is compatible with connection, filtration and Frobenius. In this case, we say that the lisse $\hat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ and the filtered F-isocrystal $\mathcal{E}$ are *associated*.

Remark 3.11. The Frobenius compatibility of the isomorphism (3B.1) means the following. Take any open subset $\mathcal{U} \subset \mathcal{X}$ equipped with a lifting of Frobenius $\sigma : \mathcal{U} \to \mathcal{U}$. By the discussion in Section 2C, we know that $\mathcal{O}\mathbb{B}_{\mathrm{cris}}|_{\mathcal{U}_k}$ is naturally endowed with a Frobenius φ. Meanwhile, as $\mathcal{E}$ is an F-isocrystal, by Remarks 3.9 $w^{-1}\mathcal{E}|_{\mathcal{U}_k}$ is endowed with a $w^{-1}\sigma$-semilinear Frobenius, still denoted by φ. Now the required Frobenius compatibility means that when restricted to any such $\mathcal{U}_k$, we have $\varphi \otimes \varphi = \mathrm{id} \otimes \varphi$ via the isomorphism (3B.1).

Definition 3.12. For $\mathbb{L}$ a lisse $\hat{\mathbb{Z}}_p$-sheaf and $i \in \mathbb{Z}$, set

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) := w_*(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \quad \text{and} \quad \mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L}) := w_*(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathrm{Fil}^i\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

All of them are $\mathcal{O}_{\mathcal{X}}[1/p]$-modules, and the $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ give a separated exhaustive decreasing filtration on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ (as the same holds for the filtration on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$; see Corollary 2.25).

Next we shall compare the notion of crystalline sheaves with other related notions considered in [Brinon 2008, Chapitre 8; Faltings 1989; Scholze 2013]. We begin with the following characterization of crystalline sheaves, which is more closely related to the classical definition of crystalline representations by Fontaine [1982] (see also [Brinon 2008, Chapitre 8]).

Proposition 3.13. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{pro\acute{e}t}}$. Then $\mathbb{L}$ is crystalline if and only if the following two conditions are verified*:

(1) *The $\mathcal{O}_{\mathcal{X}}[1/p]$-modules $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ and $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$, $i \in \mathbb{Z}$, are all coherent.*

(2) *The adjunction morphism $w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \to \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ is an isomorphism of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$-modules.*

Before proving this proposition, let us express locally the sheaf $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) = w_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})$ as the Galois invariants of some Galois module. Consider $\mathcal{U} = \mathrm{Spf}(A) \subset \mathcal{X}$ a connected affine open subset admitting an étale map $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Write U the generic fiber of $\mathcal{U}$. As $\mathcal{U}$ is smooth and connected, A is an integral domain. Fix an algebraic closure Ω of $\mathrm{Frac}(A)$, and let $\bar{A}$ be the union of finite and normal A-algebras B contained in Ω such that $B[1/p]$ is étale over $A[1/p]$. Write $G_U := \mathrm{Gal}(\bar{A}[1/p]/A[1/p])$, which is nothing but the fundamental group of $U = \mathcal{U}_k$. Let U^{univ} be the profinite étale cover of U corresponding to $(\bar{A}[1/p], \bar{A})$. One checks that U^{univ} is affinoid perfectoid (over the completion of $\bar{k}$). As $\mathbb{L}$ is a lisse $\hat{\mathbb{Z}}_p$-sheaf on X, its restriction to U corresponds to a continuous $\mathbb{Z}_p$-representation $V_U(\mathbb{L}) := \mathbb{L}(U^{\mathrm{univ}})$ of G_U. Write $\widehat{U^{\mathrm{univ}}} = \mathrm{Spa}(R, R^+)$, where (R, R^+) is the p-adic completion of $(\bar{A}[1/p], \bar{A})$.

Lemma 3.14. *Keep the notation above. Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on X. Then there exist natural isomorphisms of $A[1/p]$-modules*

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \xrightarrow{\sim} (V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+))^{G_U} =: D_{\mathrm{cris}}(V_U(\mathbb{L}))$$

and, for any $r \in \mathbb{Z}$,

$$(\mathrm{Fil}^r\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L}))(\mathcal{U}) \xrightarrow{\sim} (V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+))^{G_U}.$$

Moreover, the $A[1/p]$-module $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U})$ is projective of rank at most that of $V_U(\mathbb{L}) \otimes \mathbb{Q}_p$ over $\mathbb{Q}_p$.

Proof. As $\mathbb{L}$ is a lisse $\hat{\mathbb{Z}}_p$-sheaf, it becomes constant restricted to U^{univ}. In other words, we have $\mathbb{L}|_{U^{\mathrm{univ}}} \simeq V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p|_{U^{\mathrm{univ}}}$. For $i \geq 0$ an integer, denote by $U^{\mathrm{univ},i}$ the $(i+1)$-fold product of U^{univ} over U. Then $U^{\mathrm{univ},i} \simeq U^{\mathrm{univ}} \times G_U^i$, and it is again an affinoid perfectoid. We claim that there exists a natural identification

$$H^0(U^{\mathrm{univ},i}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = \mathrm{Map}_{\mathrm{cont}}(G_U^i, V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))[1/t],$$

where for T, T' two topological spaces, $\mathrm{Map}_{\mathrm{cont}}(T, T')$ denotes the set of continuous maps from T to T'. To see this, write $\widehat{U^{\mathrm{univ},i}} = \mathrm{Spa}(R_i, R_i^+)$. Then, by Corollary 2.19, $H^0(U^{\mathrm{univ},i}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R_i, R_i^+)$, which is also

$$(\varprojlim_n V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R_i, R_i^+)/p^n)[1/t].$$

Since $V_U(\mathbb{L})$ is of finite type over $\mathbb{Z}_p$, it suffices to show that, for all $n \in \mathbb{N}$, $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R_i, R_i^+)/p^n$ can be identified with

$$\mathrm{Map}_{\mathrm{cont}}(G_U^i, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n) = \varinjlim_N \mathrm{Map}_{\mathrm{cont}}(G_U^i/N, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)$$

where N runs through the set of open normal subgroups of G_U^i. Since both $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ and $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R_i, R_i^+)$ are flat over $\mathbb{Z}_p$, one reduces to the case where $n = 1$, and thus to showing $R_i^{\flat+}/(\underline{p}^p) = \mathrm{Map}_{\mathrm{cont}}(G_U^i, R^{\flat+}/(\underline{p}^p))$ by the explicit descriptions of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p$ and $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R_i, R_i^+)/p$. As $R_i^{\flat+}$ and $R^{\flat+}$ are flat over $\mathcal{O}_{\mathbb{C}_p}^{\flat}$, we finally reduces to showing $R_i^+/p = \mathrm{Map}_{\mathrm{cont}}(G_U^i, R^+/p)$. But this last assertion is clear, giving our claim.

Consider the following spectral sequence associated to the cover $U^{\mathrm{univ}} \to U$:

$$E_1^{i,j} = H^j(U^{\mathrm{univ},i}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \Longrightarrow H^{i+j}(U, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

As $E_1^{i,j} = 0$ for $j \geq 1$ (Corollary 2.19), we have $E_2^{n,0} = E_\infty^{n,0} \simeq H^n(U, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})$. Thus, by the discussion in the paragraph above, we deduce a natural isomorphism

$$H^j(U, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} H^j_{\mathrm{cont}}(G_U, V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))[1/t]$$

where the right-hand side is the continuous group cohomology. Taking $j = 0$, we obtain our first assertion. The isomorphism concerning $\mathrm{Fil}^r\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}$ can be proved in the same way. The last assertion follows from the first isomorphism and [Brinon 2008, Proposition 8.3.1], which gives the assertion for the right-hand side. □

Corollary 3.15. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\mathrm{proét}}$ which satisfies the condition* (1) *of Proposition 3.13. Let $\mathcal{U} = \mathrm{Spf}(A)$ be a small connected affine open subset of $\mathcal{X}$. Write $U = \mathcal{U}_k$. Then for any $V \in X_{\mathrm{proét}}/U$, we have*

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} (w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})(V).$$

Proof. By Lemma 3.14, the $A[1/p]$-module $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U})$ is projective of finite type, hence it is a direct summand of a finite free $A[1/p]$-module. As $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is coherent over $\mathcal{O}_{\mathcal{X}}[1/p]$ and as $\mathcal{U}$ is affine, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}}$ is then a direct summand of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]|_{\mathcal{U}}$-module. The isomorphism in our corollary then follows, since we have similar isomorphism when $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}}$ is replaced by a free $\mathcal{O}_{\mathcal{X}}[1/p]|_{\mathcal{U}}$-module. □

Corollary 3.16. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf verifying the condition* (1) *of Proposition 3.13. Then the condition* (2) *of Proposition 3.13 holds for $\mathbb{L}$ if and only if for any small affine connected open subset $\mathcal{U} = \mathrm{Spf}(A) \subset \mathcal{X}$* (*with $U := \mathcal{U}_k$*), *the G_U-representation $V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline in the sense that the following natural morphism is an isomorphism* [Brinon 2008, Chapitre 8]

$$D_{\mathrm{cris}}(V_U(\mathbb{L})) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+),$$

where $G_U, U^{\mathrm{univ}}, \widehat{U^{\mathrm{univ}}} = \mathrm{Spa}(R, R^+)$ are as in the paragraph before Lemma 3.14.

Proof. If $\mathbb{L}$ satisfies in addition the condition (2) of Proposition 3.13, combining it with Corollary 3.15, we find

$$\begin{aligned}\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(U^{\mathrm{univ}}) &\xrightarrow{\sim} (w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})(U^{\mathrm{univ}})\\ &\xrightarrow{\sim} (\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})(U^{\mathrm{univ}})\\ &= V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(U^{\mathrm{univ}}).\end{aligned}$$

So, by Corollary 2.19 and Lemma 3.14, the G_U-representation $V_U(\mathbb{L}) \otimes \mathbb{Q}_p$ is crystalline.

Conversely, assume that for any small connected affine open subset $\mathcal{U} = \mathrm{Spf}(A)$ of $\mathcal{X}$, the G_U-representation $V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline. Together with Lemmas 2.18 and 3.14, we get

$$\mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(U^{\mathrm{univ}}) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(U^{\mathrm{univ}})$$

and the similar isomorphism after replacing U^{univ} by any $V \in X_{\text{proét}}/U^{\mathrm{univ}}$. Using Corollary 3.15, we deduce $(w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})(V) \xrightarrow{\sim} (\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})(V)$ for any $V \in X_{\text{proét}}/U^{\mathrm{univ}}$, i.e.,

$$(w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})|_{U^{\mathrm{univ}}} \xrightarrow{\sim} (\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})|_{U^{\mathrm{univ}}}.$$

When the small opens $\mathcal{U}$'s run through a cover of $\mathcal{X}$, the U^{univ}'s form a cover of X in $X_{\text{proét}}$. Therefore, $w^{-1}\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}$, as desired. □

Lemma 3.17. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on X satisfying the two conditions of Proposition 3.13. Then* (*the analytification of*) $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ *has a natural structure of filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$.*

Proof. First of all, the $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$'s ($i \in \mathbb{Z}$) endow a separated exhaustive decreasing filtration on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ by Corollary 2.25, and the connection on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) = w_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}})$ can be given by the composite of

$$w_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{w_*(\mathrm{id} \otimes \nabla)} w_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \Omega^{1,\mathrm{ur}}_{X/k}) \xrightarrow{\sim} w_*(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \otimes_{\mathcal{O}_{\mathcal{X}}[1/p]} \Omega^1_{\mathcal{X}/\mathcal{O}_k}[1/p]$$

where the last isomorphism is the projection formula. That the connection satisfies the Griffiths transversality with respect to the filtration $\mathrm{Fil}^\bullet\, \mathbb{D}_{\mathrm{cris}}$ follows from the analogous assertion for $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ (Corollary 2.17).

Now consider the special case where $\mathcal{X} = \mathrm{Spf}(A)$ is affine connected admitting an étale map $\mathcal{X} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$, such that $\mathcal{X}$ is equipped with a lifting of Frobenius σ. As in the paragraph before Lemma 3.14, let X^{univ} be the universal profinite étale cover of X (which is an affinoid perfectoid). Write $\widehat{X^{\mathrm{univ}}} = \mathrm{Spa}(R, R^+)$ and G_X the fundamental group of X. As $\mathcal{X}$ is affine, the category $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}[1/p])$ is equivalent to the category of finite type $A[1/p]$-modules. Under this equivalence, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ corresponds to $D_{\mathrm{cris}}(V_X(\mathbb{L})) := (V_X(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+))^{G_X}$, denoted by D for simplicity. So D is a projective $A[1/p]$-module of finite type (Lemma 3.14) equipped with a connection $\nabla\colon D \to D \otimes \Omega^1_{A[1/p]/k}$. Under the same equivalence, $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ corresponds to $\mathrm{Fil}^i\, D := (V_X(\mathbb{L}) \otimes \mathrm{Fil}^i\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+))^{G_X}$, by Lemma 3.14 again. By the same proof as in [Brinon 2008, 8.3.2], the graded quotient $\mathrm{gr}^i(D)$ is a projective module. In particular, $\mathrm{Fil}^i\, D \subset D$ is a direct summand. Therefore, each $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$. Furthermore, since $\mathcal{X}$ admits a lifting of Frobenius σ, we get from Section 2C a σ-semilinear endomorphism φ on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(X^{\mathrm{univ}}) \simeq \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$, whence a σ-semilinear endomorphism on D, still denoted by φ. Via Lemma 2.21, one checks that the Frobenius φ on D is horizontal with respect to its connection. Thus $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is endowed with a horizontal σ-semilinear morphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \to \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$, always denoted by φ in the following.

To finish the proof in the special case, one needs to show that $(\mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \nabla, \varphi)$ gives an F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. As D is of finite type over $A[1/p]$, there is some $n \in \mathbb{N}$ such that $D = D^+[1/p]$ with $D^+ := (V_X(\mathbb{L}) \otimes_{\mathbb{Z}_p} t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))^{G_X}$. The connection on $t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ induces a connection $\nabla^+\colon D^+ \to D^+ \otimes_A \Omega^1_{A/\mathcal{O}_k}$ on D^+, compatible with that of $D_{\mathrm{cris}}(V_X(\mathbb{L}))$. Moreover, let N_i be the endomorphism of D^+ so that $\nabla^+ = \sum_{i=1}^d N_i \otimes dT_i$. Then for any $a \in D^+$, $\underline{N}^{\underline{m}}(a) \in p \cdot D^+$ for all but finitely many $\underline{m} \in \mathbb{N}^d$ (as this holds for the connection on $t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}$, seen in the proof of Lemma 2.22). Similarly, the Frobenius on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$ induces a map (note that the Frobenius on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$ sends t to $p \cdot t$)

$$\varphi\colon D^+ \to (V_X(\mathbb{L}) \otimes p^{-n}t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))^{G_X}.$$

Thus $\psi := p^n\varphi$ gives a well-defined σ-semilinear morphism on D^+. One checks that ψ is horizontal with respect to the connection ∇^+ on D^+ and it induces an R^+-linear isomorphism $\sigma^* D^+ \xrightarrow{\sim} D^+$. As a result, the triple (D^+, ∇, ψ) will define an F-crystal on $\mathcal{U}_0/\mathcal{O}_k$, once we know D^+ is of finite type over A. The required finiteness of D^+ is explained in [Andreatta and Iovita 2013, Proposition 3.6], and for the sake of completeness we recall briefly their proof here. As D is projective of finite type (Lemma 3.14), it is a direct summand of a finite free $A[1/p]$-module T. Let $T^+ \subset T$ be a finite free A-submodule of T such that $T^+[1/p] = T$. Then we have the inclusion $D \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \hookrightarrow T^+ \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$. As $V_X(\mathbb{L})$ is of finite type over $\mathbb{Z}_p$ and $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) = \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)[1/t]$, there exists $m \in \mathbb{N}$ such that the $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$-submodule $V_X(\mathbb{L}) \otimes t^{-n}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$ of $V_X(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \simeq D \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$ is contained in $T^+ \otimes_A t^{-m}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$. By taking G_U-invariants and using the fact that A is noetherian, we are reduced to showing that $A' := (t^{-m}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+))^{G_X}$ is of finite type over A. From the construction, A' is p-adically separated and $A \subset A' \subset A[1/p] = (\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+))^{G_X}$. As A is normal, we deduce $p^N A' \subset A$ for some $N \in \mathbb{N}$. Thus $p^N A'$ and hence A' are of finite type over A. As a result, (D^+, ∇, ψ)

defines an F-crystal $\mathcal{D}^+$ on $\mathcal{U}_0/\mathcal{O}_k$. As $D = D^+[1/p]$ and $\nabla = \nabla^+[1/p]$, the connection ∇ on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is convergent; this is standard and we refer to [Berthelot 1996, 2.4.1] for detail. Consequently, the triple $(\mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \nabla, \varphi)$ is an F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, which is isomorphic to $\mathcal{D}^{+,\mathrm{an}}(n)$. This finishes the proof in the special case.

In the general case, consider a covering $\mathcal{X} = \bigcup_i \mathcal{U}_i$ of $\mathcal{X}$ by connected small affine open subsets such that each $\mathcal{U}_i$ admits a lifting of Frobenius σ_i and an étale morphism to some torus over $\mathcal{O}_k$. By the special case, each $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \subset \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is locally a direct summand, and the connection on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is convergent [Berthelot 1996, 2.2.8]. Furthermore, each $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}_i}$ is equipped with a Frobenius φ_i, and over $\mathcal{U}_i \cap \mathcal{U}_j$, the two Frobenii φ_i, φ_j on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})|_{\mathcal{U}_i \cap \mathcal{U}_j}$ are related by the formula in Lemma 3.6 as it is the case for φ_i, φ_j on $\mathcal{O}\mathbb{B}_{\mathrm{cris}}|_{U_i \cap U_j}$ (Lemma 2.22). So these local Frobenii glue together to give a compatible system of Frobenii φ on $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ and the analytification of the quadruple $(\mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \mathrm{Fil}^\bullet\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L}), \nabla, \varphi)$ is a filtered F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, as wanted. □

Proof of Proposition 3.13. If a lisse $\hat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ on X is associated to a filtered F-isocrystal $\mathcal{E}$ on X, then we just have to show $\mathcal{E} \simeq \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$. By assumption, we have $\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$. Then

$$w_*(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \simeq w_*(w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}) \simeq \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]} w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathcal{E}$$

where the second isomorphism has used Remark 3.11, and the last isomorphism is by the isomorphism $w_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \mathcal{O}_{\mathcal{X}_{\text{ét}}}[1/p]$ from Corollary 2.26.

Conversely, let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf verifying the two conditions of our proposition. By Lemma 3.17, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is naturally a filtered F-isocrystal. To finish the proof, we need to show that the isomorphism in (2) is compatible with the extra structures. Only the compatibility with filtrations needs verification. This is a local question, hence we shall assume $\mathcal{X} = \mathrm{Spf}(A)$ is a small connected affine formal scheme. As $\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ is coherent over $\mathcal{O}_{\mathcal{X}}[1/p]$ and is a direct summand of $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})$, the same proof as that of Corollary 3.15 gives a natural isomorphism

$$\mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{X}) \otimes_{A[1/p]} \mathrm{Fil}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} (w^{-1}\, \mathrm{Fil}^i\, \mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathrm{Fil}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}})(V)$$

for any $V \in X_{\text{proét}}$. Consequently, the isomorphism in Corollary 3.15 is strictly compatible with filtrations. Thus, we reduce to showing that, for an affinoid perfectoid $V \in X_{\text{proét}}/X^{\mathrm{univ}}$, the isomorphism $D_{\mathrm{cris}}(V_X(\mathbb{L})) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} V_X(\mathbb{L}) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$ is strictly compatible with the filtrations in the sense that its inverse respects also the filtrations on both sides, or equivalently, the induced morphisms between the graded quotients are isomorphisms:

$$\bigoplus_{i+j=n} (\mathrm{gr}^i\, D_{\mathrm{cris}}(V_X(\mathbb{L})) \otimes_{A[1/p]} \mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)) \to V_X(\mathbb{L}) \otimes \mathrm{gr}^n\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V). \tag{3B.2}$$

When $V = X^{\mathrm{univ}}$, this follows from [Brinon 2008, 8.4.3]. For the general case, write $\widehat{X^{\mathrm{univ}}} = \mathrm{Spa}(R, R^+)$ and $\hat{V} = \mathrm{Spa}(R_1, R_1^+)$. By [Scholze 2013, Corollary 6.15] and Corollary 2.25,

$$\mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \xi^j \hat{\mathcal{O}}_X[U_1/\xi, \ldots, U_d/\xi] \subset \mathrm{gr}^\bullet\, \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq \hat{\mathcal{O}}_X[\xi^{\pm 1}, U_1, \ldots, U_d],$$

where ξ and all U_i have degree 1. So $\mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(X^{\mathrm{univ}}) \simeq \xi^j R[U_1/\xi, \dots, U_d/\xi]$ and $\mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \simeq \xi^j R_1[U_1/\xi, \dots, U_d/\xi]$. As a result, the natural morphism $\mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(X^{\mathrm{univ}}) \otimes_R R_1 \xrightarrow{\sim} \mathrm{gr}^j\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)$ is an isomorphism. The required isomorphism (3B.2) for general V then follows from the special case for X^{univ}. □

Let $\mathrm{Lis}^{\mathrm{cris}}_{\hat{\mathbb{Z}}_p}(X)$ denote the category of lisse crystalline $\hat{\mathbb{Z}}_p$-sheaves on X, and $\mathrm{Lis}^{\mathrm{cris}}_{\hat{\mathbb{Q}}_p}(X)$ the corresponding isogeny category. The functor

$$\mathbb{D}_{\mathrm{cris}} \colon \mathrm{Lis}^{\mathrm{cris}}_{\hat{\mathbb{Q}}_p}(X) \to FF\text{-}\mathrm{Iso}(\mathcal{X}_0/\mathcal{O}_k), \quad \mathbb{L} \mapsto \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$$

allows us to relate $\mathrm{Lis}^{\mathrm{cris}}_{\hat{\mathbb{Q}}_p}(X)$ to the category $FF\text{-}\mathrm{Iso}(\mathcal{X}_0/\mathcal{O}_k)$ of filtered convergent F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$, thanks to Proposition 3.13. A filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$ is called *admissible* if it lies in the essential image of the functor above. The full subcategory of admissible filtered F-isocrystals on $\mathcal{X}_0/\mathcal{O}_k$ will be denoted by $FF\text{-}\mathrm{Iso}(\mathcal{X}_0/\mathcal{O}_k)^{\mathrm{adm}}$.

Theorem 3.18. *The functor $\mathbb{D}_{\mathrm{cris}}$ above induces an equivalence of categories*

$$\mathbb{D}_{\mathrm{cris}} \colon \mathrm{Lis}^{\mathrm{cris}}_{\hat{\mathbb{Q}}_p}(X) \xrightarrow{\sim} FF\text{-}\mathrm{Iso}(\mathcal{X}_0/\mathcal{O}_k)^{\mathrm{adm}}.$$

A quasiinverse of $\mathbb{D}_{\mathrm{cris}}$ is given by

$$\mathbb{V}_{\mathrm{cris}} : \mathcal{E} \mapsto \mathrm{Fil}^0(w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1}$$

where φ denotes the compatible system of Frobenii on $\mathcal{E}$ as before.

Proof. Observe first that, for $\mathcal{E}$ a filtered convergent F-isocrystal, the local Frobenii on $\mathcal{E}^{\nabla=0}$ glue to give a unique σ-semilinear morphism on $\mathcal{E}^{\nabla=0}$ (Lemma 3.6). In particular, the abelian sheaf $\mathbb{V}_{\mathrm{cris}}(\mathcal{E})$ is well defined. Assume moreover $\mathcal{E}$ is admissible, and let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf such that $\mathcal{E} \simeq \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$. So $\mathbb{L}$ and $\mathcal{E}$ are associated by Proposition 3.13. Hence $\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \simeq w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}$, and we find

$$\begin{aligned}
\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathbb{Q}}_p &\xrightarrow{\sim} \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathrm{Fil}^0(\mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1} \\
&\xrightarrow{\sim} \mathrm{Fil}^0(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1} \\
&\xrightarrow{\sim} \mathrm{Fil}^0(w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}})^{\nabla=0,\varphi=1} \\
&= \mathbb{V}_{\mathrm{cris}}(\mathcal{E}),
\end{aligned}$$

where the first isomorphism following from the fundamental exact sequence (by Lemma 2.7 and [Brinon 2008, Corollary 6.2.19])

$$0 \to \mathbb{Q}_p \to \mathrm{Fil}^0\, \mathbb{B}_{\mathrm{cris}} \xrightarrow{1-\varphi} \mathbb{B}_{\mathrm{cris}} \to 0.$$

In particular, $\mathbb{V}_{\mathrm{cris}}(\mathcal{E})$ is the associated $\hat{\mathbb{Q}}_p$-sheaf of a lisse $\hat{\mathbb{Z}}_p$-sheaf. Thus $\mathbb{V}_{\mathrm{cris}}(\mathcal{E}) \in \mathrm{Lis}^{\mathrm{cris}}_{\hat{\mathbb{Q}}_p}(X)$ and the functor $\mathbb{V}_{\mathrm{cris}}$ is well defined. Furthermore, as we can recover the lisse $\hat{\mathbb{Z}}_p$-sheaf up to isogeny, it follows that $\mathbb{D}_{\mathrm{cris}}$ is fully faithful, and a quasiinverse on its essential image is given by $\mathbb{V}_{\mathrm{cris}}$. □

Remark 3.19. Using [Brinon 2008, Theorem 8.5.2], one can show that the equivalence above is an equivalence of tannakian category.

Next we compare Definition 3.10 with the "associatedness" defined in [Faltings 1989]. Let $\mathcal{E}$ be a filtered convergent F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$, and $\mathcal{M}$ an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{M}^{\mathrm{an}} = \mathcal{E}(-n)$ for some $n \in \mathbb{N}$ (see Remark 3.4 for the notations). Let $\mathcal{U} = \mathrm{Spf}(A)$ be a small connected affine open subset of $\mathcal{X}$, equipped with a lifting of Frobenius σ. Write $U = \mathrm{Spa}(A[1/p], A)$ the generic fiber of $\mathcal{U}$. As before, let $\bar{A}$ be the union of all finite normal A-algebras (contained in some fixed algebraic closure of $\mathrm{Frac}(A)$) which are étale over $A[1/p]$. Let $G_U := \mathrm{Gal}(\bar{A}[1/p]/A[1/p])$ and (R, R^+) the p-adic completion of $(\bar{A}[1/p], \bar{A})$. Then (R, R^+) is an perfectoid affinoid algebra over $\mathbb{C}_p = \hat{\bar{k}}$. So we can consider the period ring $\mathbb{A}_{\mathrm{cris}}(R, R^+)$. Moreover the composite of the following two natural morphisms

$$\mathbb{A}_{\mathrm{cris}}(R, R^+) \xrightarrow{\theta} R^+ \xrightarrow{\mathrm{can}} R^+/pR^+, \tag{3B.3}$$

defines a p-adic PD-thickening of $\mathrm{Spec}(R^+/pR^+)$. Evaluate our F-crystal $\mathcal{M}$ at it and write the resulting $\mathbb{A}_{\mathrm{cris}}(R, R^+)$-module as $\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(R, R^+))$. As an element of G_U defines a morphism of the PD-thickening (3B.3) in the big crystalline site of $\mathcal{X}_0/\mathcal{O}_k$ and $\mathcal{M}$ is a crystal, $\mathcal{M}(\mathbb{A}(R, R^+))$ is endowed naturally with an action of G_U. Similarly, the Frobenius on the crystal $\mathcal{M}$ gives a Frobenius ψ on $\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(R, R^+))$. Set $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R, R^+)) := \mathcal{M}(\mathbb{A}_{\mathrm{cris}}(R, R^+))[1/t]$, which is a $\mathbb{B}_{\mathrm{cris}}(R, R^+)$-module of finite type endowed with a Frobenius $\varphi = \psi/p^n$ and an action of G_U.

On the other hand, as $\mathcal{U}$ is small, there exists a morphism $\alpha\colon A \to \mathbb{A}_{\mathrm{cris}}(R, R^+)$ of $\mathcal{O}_k$-algebras, whose composite with $\theta : \mathbb{A}_{\mathrm{cris}}(R, R^+) \to R^+$ is the inclusion $A \subset R^+$. For example, consider an étale morphism $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Let (T_i^{1/p^n}) be a compatible system of p^n-th roots of T_i inside $\bar{A} \subset R^+$, and $T_i^\flat$ the corresponding element of $R^{\flat +} := \varprojlim_{x \mapsto x^p} R^+/pR^+$. Then one can take α as the unique morphism of $\mathcal{O}_k$-algebras $A \to \mathbb{A}_{\mathrm{cris}}(R, R^+)$ sending T_i to $[T_i^\flat]$, such that its composite with the projection $\mathbb{A}_{\mathrm{cris}}(R, R^+) \to R^+/pR^+$ is just the natural map $A \to R^+/pR^+$ (such a morphism exists as A is étale over $\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$ and because of (3B.3); see the proof of Lemma 2.14 for a similar situation). Now we fix such a morphism α. So we obtain a morphism of PD-thickenings from $\mathcal{U}_0 \hookrightarrow \mathcal{U}$ to the one defined by (3B.3). Consequently we get a natural isomorphism $\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(R, R^+)) \simeq \mathcal{M}(\mathcal{U}) \otimes_{A,\alpha} \mathbb{A}_{\mathrm{cris}}(R, R^+)$, whence

$$\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R, R^+)) \simeq \mathcal{E}(\mathcal{U}) \otimes_{A[1/p],\alpha} \mathbb{B}_{\mathrm{cris}}(R, R^+),$$

here $\mathcal{E}(\mathcal{U}) := \mathcal{M}(\mathcal{U})[1/p]$. Using this isomorphism, we define the filtration on $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R, R^+))$ as the tensor product of the filtration on $\mathcal{E}(\mathcal{U})$ and that on $\mathbb{B}_{\mathrm{cris}}(R, R^+)$.

Remark 3.20. It is well-known that the filtration on $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R, R^+))$ does not depend on the choice of α. More precisely, let α' be a second morphism $A \to \mathbb{A}_{\mathrm{cris}}(R, R^+)$ of $\mathcal{O}_k$-algebras whose composite with $\mathbb{A}_{\mathrm{cris}}(R, R^+) \to R^+$ is the inclusion $A \subset R^+$. Fix an étale morphism $\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_k\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\})$. Denote $\beta = (\alpha, \alpha') : A \otimes_{\mathcal{O}_k} A \to \mathbb{A}_{\mathrm{cris}}(R, R^+)$ and by the same notation the corresponding map on schemes, and write $p_1, p_2 : \mathrm{Spec}\, A \times \mathrm{Spec}\, A \to \mathrm{Spec}\, A$ the two projections. We have a canonical isomorphism

$(p_2\circ\beta)^*\mathcal{E}\xrightarrow{\sim}(p_1\circ\beta)^*\mathcal{E}$, as $\mathcal{E}$ is a crystal. In terms of the connection ∇ on $\mathcal{E}$, this gives (see [Berthelot 1996, 2.2.4]) the following $\mathbb{B}_{\mathrm{cris}}(R,R^+)$-linear isomorphism

$$\eta\colon \mathcal{E}(\mathcal{U})\otimes_{A[1/p],\alpha}\mathbb{B}_{\mathrm{cris}}(R,R^+)\to\mathcal{E}(\mathcal{U})\otimes_{A[1/p],\alpha'}\mathbb{B}_{\mathrm{cris}}(R,R^+)$$

sending $e\otimes 1$ to $\sum_{\underline{n}\in\mathbb{N}^d}\underline{N}^{\underline{n}}(e)\otimes(\alpha(\underline{T})-\alpha'(\underline{T}))^{[\underline{n}]}$, with $\underline{N}$ the endomorphism of $\mathcal{E}$ such that $\nabla=\underline{N}\otimes d\underline{T}$. Here we use the multiindex to simplify the notations, and note that $\alpha(T_i)-\alpha'(T_i)\in\mathrm{Fil}^1\mathbb{A}_{\mathrm{cris}}(R,R^+)$ hence the divided power $(\alpha(T_i)-\alpha'(T_i))^{[n_i]}$ is well defined. Moreover, the series converge since the connection on $\mathcal{M}$ is quasinilpotent. Now as the filtration on $\mathcal{E}$ satisfies Griffiths transversality, the isomorphism η is compatible with the tensor product filtrations on both sides. Since the inverse η^{-1} can be described by a similar formula (one just switches α and α'), it is also compatible with filtrations on both sides. Hence the isomorphism η is strictly compatible with the filtrations, and the filtration on $\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R,R^+))$ does not depend on the choice of α.

Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on X, and write as before $V_U(\mathbb{L})$ the $\mathbb{Z}_p$-representation of G_U corresponding to the lisse sheaf $\mathbb{L}|_U$. Following [Faltings 1989], we say a filtered convergent F-isocrystal $\mathcal{E}$ *on* $\mathcal{X}_0/\mathcal{O}_k$ *is associated to* $\mathbb{L}$ *in the sense of Faltings* if, for all small open subset $\mathcal{U}\subset\mathcal{X}$, there is a functorial filtered isomorphism

$$\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R,R^+))\xrightarrow{\sim}V_U(\mathbb{L})\otimes_{\mathbb{Q}_p}\mathbb{B}_{\mathrm{cris}}(R,R^+),\tag{3B.4}$$

which is compatible with G_U-action and Frobenius.

Proposition 3.21. *If $\mathcal{E}$ is associated to $\mathbb{L}$ in the sense of Faltings then $\mathbb{L}$ is crystalline* (*not necessarily associated to $\mathcal{E}$*) *and there is an isomorphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})\simeq\mathcal{E}$ compatible with filtration and Frobenius. Conversely*, *if $\mathbb{L}$ is crystalline and if there is an isomorphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{L})\simeq\mathcal{E}$ of $\mathcal{O}_{X^{\mathrm{an}}}$-modules compatible with filtration and Frobenius*, *then $\mathbb{L}$ and $\mathcal{E}$ are associated in the sense of Faltings.*

Before giving the proof of Proposition 3.21, we observe first the following commutative diagram in which the left vertical morphisms are all PD-morphisms:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\mathrm{can}"'] & A/pA \arrow[d] \\
\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+) \arrow[r, "\theta_A"] & R^+/pR^+ \arrow[d, equal] \\
\mathbb{A}_{\mathrm{cris}}(R,R^+) \arrow[u, "\mathrm{can}"] \arrow[r, "\theta"] & R^+/pR^+.
\end{tikzcd}$$

Therefore, we have isomorphisms

$$\mathcal{M}(\mathcal{U})\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)\xrightarrow{\sim}\mathcal{M}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+))\xleftarrow{\sim}\mathcal{M}(\mathbb{A}_{\mathrm{cris}}(R,R^+))\otimes_{\mathbb{A}_{\mathrm{cris}}(R,R^+)}\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+),$$

where the second term in the first row denotes the evaluation of the crystal $\mathcal{M}$ at the PD-thickening defined by the PD-morphism θ_A in the commutative diagram above. Inverting t, we obtain a natural isomorphism

$$\mathcal{E}(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} \mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R, R^+)) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+), \tag{3B.5}$$

where the last tensor product is taken over $\mathbb{B}_{\mathrm{cris}}(R, R^+)$. This isomorphism is clearly compatible with Galois action and Frobenius. By a similar argument as in Remark 3.20 one checks that (3B.5) is also strictly compatible with the filtrations. Furthermore, using the identification

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle u_1, \dots, u_d\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+), \quad u_i \mapsto T_i \otimes 1 - 1 \otimes [T_i^\flat]$$

we obtain a section s of the canonical map $\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$:

$$s\colon \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathbb{A}_{\mathrm{cris}}(R, R^+), \quad u_i \mapsto 0,$$

which is again a PD-morphism. Composing it with the inclusion $A \subset \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$, we get a morphism $\alpha_0\colon R^+ \to \mathbb{A}_{\mathrm{cris}}(R, R^+)$ whose composite with the projection $\mathbb{A}_{\mathrm{cris}}(R, R^+) \to R^+$ is the inclusion $A \subset R^+$.

Proof of Proposition 3.21. Now assume that $\mathcal{E}$ is associated with $\mathbb{L}$ in the sense of Faltings. Extending scalars to $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$ of the isomorphism (3B.4) and using the identification (3B.5), we obtain a functorial isomorphism, compatible with filtration, G_U-action, and Frobenius:

$$V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} \mathcal{E}(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+).$$

Therefore, $V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a crystalline G_U-representation (Corollary 3.16), and we get by Lemma 3.14 an isomorphism $\mathcal{E}(\mathcal{U}) \xrightarrow{\sim} \mathbb{D}_{\mathrm{cris}}(\mathbb{L})(\mathcal{U})$ compatible with filtrations and Frobenius. As such small open subsets $\mathcal{U}$ form a basis for the Zariski topology of $\mathcal{X}$, we find an isomorphism $\mathcal{E} \xrightarrow{\sim} \mathbb{D}_{\mathrm{cris}}(\mathbb{L})$ compatible with filtrations and Frobenius, and that $\mathbb{L}$ is crystalline in the sense of Definition 3.10 (Corollary 3.16).

Conversely, assume $\mathbb{L}$ is crystalline with $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}) \simeq \mathcal{E}$ compatible with filtrations and Frobenius. As in the proof of Corollary 3.16, we have a functorial isomorphism

$$\mathcal{E}(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+)$$

which is compatible with filtration, Galois action and Frobenius. Pulling it back via the section $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \to \mathbb{B}_{\mathrm{cris}}(R, R^+)$ obtained from s by inverting p, we obtain a functorial isomorphism

$$\mathcal{E}(\mathbb{B}_{\mathrm{cris}}(R, R^+)) \simeq \mathcal{E}(\mathcal{U}) \otimes_{A[1/p],\alpha_0} \mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{cris}}(R, R^+),$$

which is again compatible with Galois action, Frobenius and filtrations. Therefore $\mathbb{L}$ and $\mathcal{E}$ are associated in the sense of Faltings. □

Finally we compare Definition 3.10 with its de Rham analogue considered in [Scholze 2013].

Proposition 3.22. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on X and $\mathcal{E}$ a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$. Assume that $\mathbb{L}$ and $\mathcal{E}$ are associated as defined in Definition 3.10, then $\mathbb{L}$ is de Rham in the sense of Scholze*

[2013, Definition 8.3]. *More precisely, if we view $\mathcal{E}$ as a filtered module with integrable connection on X (namely we forget the Frobenius), there exists a natural filtered isomorphism compatible with connections*

$$\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}}.$$

Proof. Let $\mathcal{U} = \mathrm{Spf}(R^+) \subset \mathcal{X}$ be a connected affine open subset, and denote by U (resp. U^{univ}) the generic fiber of $\mathcal{U}$ (resp. the universal étale cover of U). Let V be a affinoid perfectoid lying above U^{univ}. As $\mathbb{L}$ and $\mathcal{E}$ are associated, there exits a filtered isomorphism compatible with connections and Frobenius

$$\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} w^{-1}\mathcal{E} \otimes_{\mathcal{O}_X^{\mathrm{ur}}} \mathcal{O}\mathbb{B}_{\mathrm{cris}}.$$

Evaluate this map at $V \in X_{\mathrm{pro\acute{e}t}}$ and use the fact that the $A[1/p]$-module $\mathcal{E}(\mathcal{U})$ is projective, we deduce a filtered isomorphism compatible with all extra structures:

$$V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{E}(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{cris}}(V).$$

Taking tensor product $-\otimes_{\mathcal{O}\mathbb{B}_{\mathrm{cris}}(V)} \mathcal{O}\mathbb{B}_{\mathrm{dR}}(V)$ on both sides, we get a filtered isomorphism compatible with connection

$$V_U(\mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathcal{E}(\mathcal{U}) \otimes_{A[1/p]} \mathcal{O}\mathbb{B}_{\mathrm{dR}}(V).$$

Again, as $\mathcal{E}(\mathcal{U})$ is a projective $A[1/p]$-module and as $\mathcal{E}$ is coherent, the isomorphism above can be rewritten as

$$(\mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathcal{O}\mathbb{B}_{\mathrm{dR}})(V) \xrightarrow{\sim} (\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{dR}})(V),$$

which is clearly functorial in $\mathcal{U}$ and in V. Varying $\mathcal{U}$ and V, we deduce that $\mathbb{L}$ is de Rham, giving our proposition. □

3C. ***From proétale site to étale site.*** Let $\mathcal{X}$ be a smooth formal scheme over $\mathcal{O}_k$. For $\mathcal{O} = \mathcal{O}_{\mathcal{X}}$, $\mathcal{O}_{\mathcal{X}}[1/p]$, $\mathcal{O}_X^{\mathrm{ur}+}$, $\mathcal{O}_X^{\mathrm{ur}}$ and a sheaf of $\mathcal{O}$-modules $\mathcal{F}$ with connection, we denote the de Rham complex of $\mathcal{F}$ as

$$\mathrm{dR}(\mathcal{F}) = (0 \to \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}} \Omega^1 \xrightarrow{\nabla} \cdots).$$

Let $\overline{w}$ be the composite of natural morphisms of topoi (here we use the same notation to denote the object in $X^{\sim}_{\mathrm{pro\acute{e}t}}$ represented by $X_{\bar{k}} \in X_{\mathrm{pro\acute{e}t}}$)

$$X^{\sim}_{\mathrm{pro\acute{e}t}}/X_{\bar{k}} \to X^{\sim}_{\mathrm{pro\acute{e}t}} \xrightarrow{w} \mathcal{X}^{\sim}_{\mathrm{\acute{e}t}}.$$

The following lemma is a global reformulation of the main results of [Andreatta and Brinon 2013]. As we shall prove a more general result later (Lemma 5.3), let us omit the proof here.

Lemma 3.23. *Let $\mathcal{X}$ be smooth formal scheme over $\mathcal{O}_k$. Then the natural morphism below is an isomorphism in the filtered derived category*:

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_{\mathcal{O}_k} B_{\mathrm{cris}} \to R\overline{w}_*(\mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

Here $\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}B_{\mathrm{cris}} := (\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}A_{\mathrm{cris}})[1/t]$ *with*

$$\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}A_{\mathrm{cris}} := \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_k} A_{\mathrm{cris}}/p^n,$$

and $\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}B_{\mathrm{cris}}$ *is filtered by the subsheaves*

$$\mathcal{O}\widehat{\otimes}_{\mathcal{O}_k} \mathrm{Fil}^r\, B_{\mathrm{cris}} := \varinjlim_{n\in\mathbb{N}} t^{-n}(\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k} \mathrm{Fil}^{r+n}\, A_{\mathrm{cris}}), \quad r\in\mathbb{Z},$$

with $\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k} \mathrm{Fil}^{r+n}\, A_{\mathrm{cris}} := \varprojlim_n \mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_k} \mathrm{Fil}^{r+n}\, A_{\mathrm{cris}}/p^n$.

Corollary 3.24. *Let* $\mathcal{X}$ *be a smooth formal scheme over* $\mathcal{O}_k$. *Let* $\mathbb{L}$ *be a crystalline lisse* $\hat{\mathbb{Z}}_p$*-sheaf associated with a filtered convergent* F*-isocrystal* $\mathcal{E}$. *Then there exists a natural quasiisomorphism in the filtered derived category*

$$R\overline{w}_*(\mathbb{L}\otimes_{\hat{\mathbb{Z}}_p}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{dR}(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}B_{\mathrm{cris}}).$$

If moreover $\mathcal{X}$ *is endowed with a lifting of Frobenius* σ, *then the isomorphism above is also compatible with the Frobenii deduced from* σ *on both sides.*

Proof. Using the Poincaré lemma (Corollary 2.17), we get first a quasiisomorphism which is strictly compatible with filtrations

$$\mathbb{L}\otimes\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathbb{L}\otimes\mathrm{dR}(\mathcal{O}\mathbb{B}_{\mathrm{cris}}) = \mathrm{dR}(\mathbb{L}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}}).$$

As $\mathbb{L}$ and $\mathcal{E}$ are associated, there is a filtered isomorphism $\mathbb{L}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} w^{-1}\mathcal{E}\otimes_{\mathcal{O}_X^{\mathrm{ur}}}\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ compatible with connection and Frobenius, from which we get the quasiisomorphisms in the filtered derived category

$$\mathbb{L}\otimes\mathbb{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{dR}(\mathbb{L}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{dR}(w^{-1}\mathcal{E}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}}). \tag{3C.1}$$

On the other hand, as $R^j\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} = 0$ for $j > 0$ (Lemma 3.23), we obtain using projection formula that $R^j\overline{w}_*(w^{-1}\mathcal{E}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}}) = \mathcal{E}\otimes R^j\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}} = 0$ (note that $\mathcal{E}$ is locally a direct factor of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]$-module, hence one can apply projection formula here). In particular, each component of $\mathrm{dR}(w^{-1}\mathcal{E}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}})$ is $\overline{w}_*$-acyclic. Therefore,

$$\mathrm{dR}(\mathcal{E}\otimes\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} \overline{w}_*(\mathrm{dR}(w^{-1}\mathcal{E}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}})) \xrightarrow{\sim} R\overline{w}_*(\mathrm{dR}(w^{-1}\mathcal{E}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}})).$$

Combining this with Lemma 3.23, we deduce the following quasiisomorphisms in the filtered derived category

$$\mathrm{dR}(\mathcal{E}\otimes\mathcal{O}_{\mathcal{X}}\widehat{\otimes}B_{\mathrm{cris}}) \xrightarrow{\sim} \mathrm{dR}(\mathcal{E}\otimes\overline{w}_*\mathcal{O}\mathbb{B}_{\mathrm{cris}}) \xrightarrow{\sim} R\overline{w}_*(\mathrm{dR}(w^{-1}\mathcal{E}\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris}})). \tag{3C.2}$$

The desired quasiisomorphism follows from (3C.1) and (3C.2). When moreover $\mathcal{X}$ admits a lifting of Frobenius σ, one checks that both quasiisomorphisms are compatible with Frobenius, hence the last part of our corollary. □

Remark 3.25. Recall that G_k denotes the absolute Galois group of k. Each element of G_k defines a morphism of $U_{\bar{k}}$ in the proétale site $X_{\text{proét}}$ for any $\mathcal{U} \in \mathcal{X}_{\text{proét}}$ with $U := \mathcal{U}_k$. Therefore, the object $R\overline{w}_*(\mathbb{L} \otimes \mathbb{B}_{\text{cris}})$ comes with a natural Galois action of G_k. With this Galois action, one checks that the quasiisomorphism in Corollary 3.24 is also Galois equivariant.

Assume moreover that $\mathcal{X}$ is proper over $\mathcal{O}_k$. Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, and $\mathcal{M}$ an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ (viewed as a coherent $\mathcal{O}_{\mathcal{X}}$-module equipped with an integrable connection) such that $\mathcal{E} \simeq \mathcal{M}^{\text{an}}(n)$ for some $n \in \mathbb{N}$ (Remark 3.4). The crystalline cohomology group $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M})$ is an $\mathcal{O}_k$-module of finite type endowed with a Frobenius ψ. In the following, the crystalline cohomology (or more appropriately, the rigid cohomology) of the convergent F-isocrystal $\mathcal{E}$ is defined as

$$H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) := H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M})[1/p].$$

It is a finite dimensional k-vector space equipped with the Frobenius ψ/p^n. Moreover, let $u = u_{\mathcal{X}_0/\mathcal{O}_k}$ be the morphism of topoi

$$(\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\text{cris}} \to \mathcal{X}^{\sim}_{\text{ét}}$$

such that $u_*(\mathcal{F})(\mathcal{U}) = H^0_{\text{cris}}(\mathcal{U}_0/\mathcal{O}_k, \mathcal{F})$ for $\mathcal{U} \in \mathcal{X}_{\text{ét}}$. With the étale topology replaced by the Zariski topology, this is precisely the morphism $u_{\mathcal{X}_0/\hat{S}}$ (with $\hat{S} = \operatorname{Spf}(\mathcal{O}_k)$) considered in [Berthelot and Ogus 1978, Theorem 7.23]. By [loc. cit.], there exists a natural quasiisomorphism in the derived category

$$Ru_*\mathcal{M} \xrightarrow{\sim} \mathrm{dR}(\mathcal{M}), \tag{3C.3}$$

which induces an isomorphism $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{M}) \xrightarrow{\sim} H^i(\mathcal{X}, \mathrm{dR}(\mathcal{M}))$. Thereby

$$H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) \xrightarrow{\sim} H^i(\mathcal{X}, \mathrm{dR}(\mathcal{E})). \tag{3C.4}$$

On the other hand, the de Rham complex $\mathrm{dR}(\mathcal{E})$ of $\mathcal{E}$ is filtered by its subcomplexes

$$\mathrm{Fil}^r\, \mathrm{dR}(\mathcal{E}) := (\mathrm{Fil}^r\, \mathcal{E} \xrightarrow{\nabla} \mathrm{Fil}^{r-1}\, \mathcal{E} \otimes \Omega^1_{X/k} \xrightarrow{\nabla} \cdots).$$

So the hypercohomology $H^i(\mathcal{X}, \mathrm{dR}(\mathcal{E}))$ has a descending filtration given by

$$\mathrm{Fil}^r\, H^i(\mathcal{X}, \mathrm{dR}(\mathcal{E})) := \mathrm{Im}(H^i(\mathcal{X}, \mathrm{Fil}^r\, \mathrm{dR}(\mathcal{E})) \to H^i(\mathcal{X}, \mathrm{dR}(\mathcal{E}))).$$

Consequently, through the isomorphism (3C.4), the k-space $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E})$ is endowed naturally with a decreasing filtration.

Theorem 3.26. *Assume that the smooth formal scheme $\mathcal{X}$ is proper over $\mathcal{O}_k$. Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$ and $\mathbb{L}$ a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. Assume that $\mathcal{E}$ and $\mathbb{L}$ are associated. Then there is a natural filtered isomorphism*

$$H^i(X_{\bar{k},\text{proét}}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{B}_{\text{cris}}) \xrightarrow{\sim} H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) \otimes_k B_{\text{cris}} \tag{3C.5}$$

of B_{cris}-modules, which is compatible with Frobenius and Galois action.

Proof. By Corollary 3.24, we have the natural Galois equivariant quasiisomorphism in the filtered derived category:

$$R\Gamma(X_{\bar{k},\text{proét}}, \mathbb{L}\otimes\mathbb{B}_{\text{cris}}) = R\Gamma(\mathcal{X}, R\overline{w}_*(\mathbb{L}\otimes\mathbb{B}_{\text{cris}})) \xrightarrow{\sim} R\Gamma(\mathcal{X}, \mathrm{dR}(\mathcal{E}\otimes\mathcal{O}_{\mathcal{X}}\widehat{\otimes}B_{\text{cris}})).$$

We claim that the natural morphism in the filtered derived category

$$R\Gamma(\mathcal{X}, \mathrm{dR}(\mathcal{E}))\otimes_{\mathcal{O}_k} A_{\text{cris}} \to R\Gamma(\mathcal{X}_{\text{ét}}, \mathrm{dR}(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}A_{\text{cris}})) \tag{3C.6}$$

is an isomorphism. Let $\mathcal{M}$ be an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ with $\mathcal{E}=\mathcal{M}^{\text{an}}(n)$. Then, the similar natural morphism below is an isomorphism:

$$R\Gamma(\mathcal{X}, \mathrm{dR}(\mathcal{M}))\otimes_{\mathcal{O}_k} A_{\text{cris}} \to R\Gamma(\mathcal{X}_{\text{ét}}, \mathrm{dR}(\mathcal{M}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}A_{\text{cris}})). \tag{3C.7}$$

Indeed, as A_{cris} is flat over $\mathcal{O}_k$, $\mathcal{M}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}A_{\text{cris}}\simeq\mathcal{M}\widehat{\otimes}_{\mathcal{O}_k}A_{\text{cris}}$. So $\mathrm{dR}(\mathcal{M}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}A_{\text{cris}}) = \mathrm{dR}(\mathcal{M}\widehat{\otimes}_{\mathcal{O}_k}A_{\text{cris}})$, and (3C.7) is an isomorphism by Lemma B.2. Thus, to prove our claim, it suffices to check that (3C.6) induces quasiisomorphisms on gradeds. Further filtering the de Rham complex by its naive filtration, we are reduced to checking the following isomorphism for $\mathcal{A}$ a coherent $\mathcal{O}_{\mathcal{X}}$-module:

$$R\Gamma(\mathcal{X},\mathcal{A})\otimes_{\mathcal{O}_k}\mathcal{O}_{\mathbb{C}_p} \xrightarrow{\sim} R\Gamma(\mathcal{X},\mathcal{A}\otimes_{\mathcal{O}_{\mathcal{X}}}\widehat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{\mathbb{C}_p}) \simeq R\Gamma(\mathcal{X},\mathcal{A}\widehat{\otimes}_{\mathcal{O}_k}\mathcal{O}_{\mathbb{C}_p}),$$

which holds because again $\mathcal{O}_{\mathbb{C}_p}$ is flat over $\mathcal{O}_k$ (Lemma B.2). Consequently, inverting t we obtain an isomorphism in the filtered derived category

$$R\Gamma(\mathcal{X}, \mathrm{dR}(\mathcal{E}))\otimes_k B_{\text{cris}} \to R\Gamma(\mathcal{X}_{\text{ét}}, \mathrm{dR}(\mathcal{E}\otimes_{\mathcal{O}_{\mathcal{X}}}\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_{\mathcal{O}_k}B_{\text{cris}})).$$

Thus we get a Galois equivariant quasiisomorphism in the filtered derived category

$$R\Gamma(X_{\bar{k},\text{proét}}, \mathbb{L}\otimes\mathbb{B}_{\text{cris}}) \xrightarrow{\sim} R\Gamma(\mathcal{X}, \mathrm{dR}(\mathcal{E}))\otimes_k B_{\text{cris}}.$$

Combining it with (3C.4), we obtain the isomorphism (3C.5) verifying the required properties except for the Frobenius compatibility.

To check the Frobenius compatibility, it suffices to check that the restriction to $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{E})\hookrightarrow H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{E})\otimes_k B_{\text{cris}}$ of the inverse of (3C.5) is Frobenius-compatible. Let $\mathcal{M}$ be an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ with $\mathcal{E}=\mathcal{M}^{\text{an}}(n)$. Via the identification $H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{E}) = H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k,\mathcal{M})[1/p]$, the restriction map in question is induced from the following composed morphism at the level of derived category:

$$Ru_*\mathcal{M}[1/p] \xrightarrow{\sim} \mathrm{dR}(\mathcal{M})[1/p] \xrightarrow{\sim} \mathrm{dR}(\mathcal{E}) \to \mathrm{dR}(\mathcal{E}\otimes\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_k B_{\text{cris}}) \xrightarrow{\sim} R\overline{w}_*(\mathbb{L}\otimes\mathbb{B}_{\text{cris}}),$$

where the first morphism is (3C.3), and the last one is the inverse in the derived category of the quasiisomorphism in Corollary 3.24. Let us denote by θ the composite of these morphisms. Let ψ and φ be the induced Frobenius on $Ru_*\mathcal{M}$ and $R\overline{w}_*(\mathbb{L}\otimes\mathbb{B}_{\text{cris}})$, respectively. One needs to check that $\varphi\circ\theta=\frac{1}{p^n}\theta\circ\psi$. This can be done locally on $\mathcal{X}$. So let $\mathcal{U}\subset\mathcal{X}$ be a small open subset equipped with a lifting of Frobenius σ. Thus $\mathcal{M}|_{\mathcal{U}}$ and $\mathcal{E}|_{\mathcal{U}}$ admit naturally a Frobenius, which we denote by $\psi_{\mathcal{U}}$ and $\varphi_{\mathcal{U}}$, respectively. Then all the morphisms above except the second one are Frobenius-compatible (see Corollary 3.24 for the last quasiisomorphism). But by definition, under the identification $\mathcal{M}[1/p]|_{\mathcal{U}} \simeq \mathcal{E}|_{\mathcal{U}}$, the Frobenius $\varphi_{\mathcal{U}}$ on $\mathcal{E}$ corresponds exactly to $\psi_{\mathcal{U}}/p^n$ on $\mathcal{M}[1/p]$. This gives the desired equality $\varphi \circ \theta = \frac{1}{p^n}\theta \circ \psi$ on $\mathcal{U}$, from which the Frobenius compatibility in (3C.5) follows. □

4. Primitive comparison on the proétale site

Let $\mathcal{X}$ be a proper smooth formal scheme over $\mathcal{O}_k$, with X (resp. $\mathcal{X}_0$) its generic (resp. closed) fiber. Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. In this section, we will construct a primitive comparison isomorphism for any lisse $\hat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ on the proétale site $X_{\text{proét}}$ (Theorem 4.3). In particular, this primitive comparison isomorphism also holds for noncrystalline lisse $\hat{\mathbb{Z}}_p$-sheaves, which may lead to interesting arithmetic applications. On the other hand, in the case that $\mathbb{L}$ is crystalline, such a result and Theorem 3.26 together give rise to the crystalline comparison isomorphism between étale cohomology and crystalline cohomology (Theorem 4.5).

We shall begin with some preparations.

Lemma 4.1. *Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a projective system of abelian sheaves on a site T. Then for any object $Y \in T$ and any $i \in \mathbb{Z}$, there exists a natural exact sequence*

$$0 \to R^1 \varprojlim H^{i-1}(Y, \mathcal{F}_n) \to H^i(Y, R\varprojlim \mathcal{F}_n) \to \varprojlim H^i(Y, \mathcal{F}_n) \to 0.$$

Proof. This is essentially [Jannsen 1988, (1.6) Proposition]. Let Sh denote the category of abelian sheaves on T and $\mathrm{Sh}^{\mathbb{N}}$ the category of projective systems of abelian sheaves indexed by $\mathbb{N}$. Let Ab denote the category of abelian groups. Consider the functor

$$\tau\colon\ \mathrm{Sh}^{\mathbb{N}} \to \mathrm{Ab}, \quad (\mathcal{G}_n) \mapsto \varprojlim \Gamma(Y, \mathcal{G}_n).$$

Then τ is left-exact, and we can consider its right derived functor $R\tau(\mathcal{F}_n)$. By [Jannsen 1988, (1.6) Proposition], we have a short exact sequence for each $i \in \mathbb{Z}$

$$0 \to R^1 \varprojlim H^{i-1}(Y, \mathcal{F}_n) \to R^i\tau(\mathcal{F}_n) \to \varprojlim H^i(Y, \mathcal{F}_n) \to 0.$$

One the other hand, write τ as the composite of the following two functors

$$\mathrm{Sh}^{\mathbb{N}} \xrightarrow{\varprojlim} \mathrm{Sh} \xrightarrow{\Gamma(Y,-)} \mathrm{Ab}\ .$$

The functor $\varprojlim\colon \mathrm{Sh}^{\mathbb{N}} \to \mathrm{Sh}$ admits an exact left-adjoint given by sending a sheaf to its associated constant projective system, so it sends injectives to injectives. Thus $R\Gamma(Y, R\varprojlim \mathcal{F}_n) \simeq R\tau(\mathcal{F}_n)$, and $H^i(Y, R\varprojlim \mathcal{F}_n) \simeq R^i\tau(\mathcal{F}_n)$ for each i. Together with the short exact sequence above, we obtain our lemma. □

Lemma 4.2. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\bar{k},\text{proét}}$ and $\mathbb{L}_n := \mathbb{L}/p^n$ for $n \in \mathbb{N}$. Then, for $i \in \mathbb{Z}$, $H^i(X_{\bar{k},\text{proét}}, \mathbb{L}) \xrightarrow{\sim} \varprojlim_n H^i(X_{\bar{k},\text{proét}}, \mathbb{L}_n)$. Moreover, $H^i(X_{\bar{k},\text{proét}}, \mathbb{L})$ is a $\mathbb{Z}_p$-module of finite type, and $H^i(X_{\bar{k},\text{proét}}, \mathbb{L}) = 0$ whenever $i \notin [0, 2\dim(X)]$.*

Proof. All the cohomology groups below are computed in the proétale site, so we shall omit the subscript "proét" from the notations.

Thanks to [Scholze 2013, Proposition 8.2], $R^j \varprojlim_n \mathbb{L}_n = 0$ for $j > 0$. So $\mathbb{L} \xrightarrow{\sim} R \varprojlim_n \mathbb{L}_n$. Furthermore, by [loc. cit., Theorem 5.1], $H^{i-1}(X_{\bar{k}}, \mathbb{L}_n)$ is finite for all $n \in \mathbb{N}$. So $R^1 \varprojlim H^{i-1}(X_{\bar{k}}, \mathbb{L}_n) = 0$. Consequently, by Lemma 4.1, the morphism

$$H^i(X_{\bar{k}}, \mathbb{L}) \to \varprojlim H^i(X_{\bar{k}}, \mathbb{L}_n)$$

is an isomorphism, giving the first part of our lemma. In particular, $H^i(X_{\bar{k}}, \mathbb{L}) = 0$ whenever $i \notin [0, 2\dim(X)]$ according to [loc. cit., Theorem 5.1].

Let $\mathbb{L}_{\mathrm{tor}}$ be the torsion subsheaf of $\mathbb{L}$. The remaining part of our lemma follows from the corresponding statements for $\mathbb{L}_{\mathrm{tor}}$ and for $\mathbb{L}/\mathbb{L}_{\mathrm{tor}}$. Therefore, we assume that $\mathbb{L}$ is either of torsion or locally free of finite rank. In the first case, we reduce to the finiteness statement of Scholze [2013, Theorem 5.1]. So it suffices to consider the case where $\mathbb{L}$ is locally free. Then, we have the exact sequence

$$0 \to \mathbb{L} \xrightarrow{p^n} \mathbb{L} \to \mathbb{L}_n \to 0,$$

inducing the following short exact sequence

$$0 \to H^i(X_{\bar{k}}, \mathbb{L})/p^n \to H^i(X_{\bar{k}}, \mathbb{L}_n) \to H^{i+1}(X_{\bar{k}}, \mathbb{L})[p^n] \to 0. \tag{4A.1}$$

By the first part of our lemma, $H^{i+1}(X_{\bar{k}}, \mathbb{L}) \xrightarrow{\sim} \varprojlim H^{i+1}(X_{\bar{k}}, \mathbb{L}_n)$ is a pro-p abelian group, hence it does not contain any element infinitely divisible by p. Thus, $\varprojlim(H^{i+1}(X_{\bar{k}}, \mathbb{L})[p^n]) = 0$ (the transition map is multiplication by p). From the exactness of (4A.1), we deduce a canonical isomorphism

$$\varprojlim(H^i(X_{\bar{k}}, \mathbb{L})/p^n) \xrightarrow{\sim} \varprojlim H^i(X_{\bar{k}}, \mathbb{L}_n).$$

So $H^i(X_{\bar{k}}, \mathbb{L}) \xrightarrow{\sim} \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L})/p^n$, and $H^i(X_{\bar{k}}, \mathbb{L})$ is p-adically complete. Thus it can be generated as a $\mathbb{Z}_p$-module by a family of elements whose images in $H^i(X_{\bar{k}}, \mathbb{L})/p$ generate it as an $\mathbb{F}_p$-vector space. The latter is finite dimensional over $\mathbb{F}_p$: recall the inclusion $H^i(X_{\bar{k}}, \mathbb{L})/p \hookrightarrow H^i(X_{\bar{k}}, \mathbb{L}_1)$ by (4A.1). So the $\mathbb{Z}_p$-module $H^i(X_{\bar{k}}, \mathbb{L})$ is of finite type, as desired. □

The primitive form of the crystalline comparison isomorphism on the proétale site is as follows.

Theorem 4.3. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. Then the natural morphism of B_{cris}-modules below is a filtered isomorphism*

$$H^i(X_{\bar{k},\text{proét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \xrightarrow{\sim} H^i(X_{\bar{k},\text{proét}}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{cris}}), \tag{4A.2}$$

compatible with Galois action and Frobenius.

Proof. In the following, all the cohomologies are computed on the proétale site, so we omit the subscript "proét" from the notations.

If $\mathbb{L}$ is of torsion, our theorem is obvious since both sides of (4A.2) are trivial. Therefore, it suffices to consider the case where $\mathbb{L}$ is locally a free lisse $\hat{\mathbb{Z}}_p$-module. Let $\mathbb{L}_n = \mathbb{L}/p^n$, $n \in \mathbb{N}$. So, we have the short exact sequence (4A.1), from which we deduce a short exact sequence of projective systems since A_{cris} is flat over $\mathbb{Z}_p$

$$0 \to (H^i(X_{\bar{k}}, \mathbb{L}) \otimes A_{\mathrm{cris}}/p^n)_n \to (H^i(X_{\bar{k}}, \mathbb{L}_n) \otimes A_{\mathrm{cris}})_n \to ((H^{i+1}(X_{\bar{k}}, \mathbb{L})[p^n]) \otimes A_{\mathrm{cris}})_n \to 0.$$

Because $H^{i+1}(X_{\bar{k}}, \mathbb{L})$ is a finite $\mathbb{Z}_p$-modules, $\varprojlim_n (H^{i+1}(X_{\bar{k}}, \mathbb{L})[p^n] \otimes A_{\mathrm{cris}}) = 0$ (the transition map is multiplication by p). Thus, we get

$$H^i(X_{\bar{k}}, \mathbb{L}) \otimes A_{\mathrm{cris}} \simeq \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L}) \otimes A_{\mathrm{cris}}/p^n \xrightarrow{\sim} \varprojlim_n (H^i(X_{\bar{k}}, \mathbb{L}_n) \otimes A_{\mathrm{cris}}). \tag{4A.3}$$

Here we have the first identification since $H^i(X_{\bar{k}}, \mathbb{L}) \otimes A_{\mathrm{cris}}$ is p-adically complete thanks to the fact that $H^i(X_{\bar{k}}, \mathbb{L})$ is a finite $\mathbb{Z}_p$-module.

Next, we claim that, for all $i \geq 0$, the canonical map of A_{cris}/p^n-modules

$$H^i(X_{\bar{k}}, \mathbb{L}_n) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \to H^i(X_{\bar{k}}, \mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}}) \tag{4A.4}$$

is an almost isomorphism. Since A_{cris} and $\mathbb{A}_{\mathrm{cris}}$ are flat respectively over $\mathbb{Z}_p$ and $\hat{\mathbb{Z}}_p$, by induction on n, it suffices to show that the natural map of A_{cris}/p-modules

$$H^i(X_{\bar{k}}, \mathbb{K}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \to H^i(X_{\bar{k}}, \mathbb{K} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}})$$

is an almost isomorphism, where $\mathbb{K}$ is an $\mathbb{F}_p$-local system on $X_{\text{proét}}$. In this case, one can rewrite the morphism above as

$$H^i(X_{\bar{k}}, \mathbb{K}) \otimes_{\mathbb{F}_p} A_{\mathrm{cris}}/p \to H^i(X_{\bar{k}}, \mathbb{K} \otimes_{\mathbb{F}_p} \mathbb{A}_{\mathrm{cris}}/p). \tag{4A.5}$$

Recall the following identification of A_{cris}/p (see [Brinon 2008, Proposition 6.1.2])

$$A_{\mathrm{cris}}/p \xrightarrow{\sim} (\mathcal{O}_{\mathbb{C}_p}^{\flat}/(p^{\flat})^p)[\delta_i : i \in \mathbb{N}]/(\delta_i^p : i \in \mathbb{N}),$$

with δ_i the image of $\xi^{[p^{i+1}]}$. Similarly, on $X_{\text{proét}}/X_{\bar{k}}$, we have

$$\mathbb{A}_{\mathrm{cris}}/p \xrightarrow{\sim} (\mathcal{O}_X^{\flat +}/(p^{\flat})^p)[\delta_i : i \in \mathbb{N}]/(\delta_i^p : i \in \mathbb{N}).$$

As $X_{\bar{k}}$ is qcqs, to show that (4A.5) is an almost isomorphism, it suffices to check that it is the case for the map

$$H^i(X_{\bar{k}}, \mathbb{K}) \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathbb{C}_p}^{\flat}/(p^{\flat})^p \to H^i(X_{\bar{k}}, \mathbb{K} \otimes_{\mathbb{F}_p} \mathcal{O}_X^{\flat +}/(p^{\flat})^p).$$

Using the $p^{\flat}$-adic filtration on $\mathcal{O}_{\mathbb{C}_p}^{\flat}/(p^{\flat})^p$ and on $\mathcal{O}_X^{\flat +}/(p^{\flat})^p$, one reduces further to showing that the natural map

$$H^i(X_{\bar{k}}, \mathbb{K}) \otimes_{\mathbb{F}_p} \mathcal{O}_{\mathbb{C}_p}^{\flat}/(p^{\flat}) \to H^i(X_{\bar{k}}, \mathbb{K} \otimes_{\mathbb{F}_p} \mathcal{O}_X^{\flat +}/(p^{\flat})),$$

is an almost isomorphism. But this is proved in [Scholze 2013, Theorem 5.1] since $\mathcal{O}_{\mathbb{C}_p}^{\flat}/p^{\flat} \simeq \mathcal{O}_{\mathbb{C}_p}/p$ and $\mathcal{O}_X^{\flat +}/p^{\flat} \simeq \mathcal{O}_X^{+}/p$: recall that the almost-setting adopted here for A_{cris}/p-modules is the same as the one used by Scholze. Consequently, the map (4A.4) is an almost isomorphism. Varying n in (4A.4), we obtain a morphism of projective systems of A_{cris}-modules, with kernel and cokernel killed by $\mathcal{I}$. Passing to limits relative to n and using (4A.3), one deduces a natural morphism of A_{cris}-modules

$$H^i(X_{\bar{k}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \simeq \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L}_n) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \to \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}}),$$

with kernel and cokernel killed by $\mathcal{I}^2$. Moreover, we have $\mathcal{I} \cdot R^1 \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris}}) = 0$ since $R^1 \varprojlim (H^i(X_{\bar{k}}, \mathbb{L}_n) \otimes A_{\mathrm{cris}}) = 0$.

Then, we claim that the A_{cris}-module $R^j \varprojlim(\mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}})$ is killed by $\mathcal{I}^2$ for $j > 0$. The question being local on $X_{\text{proét}}$, we may and do assume $\mathbb{L} = M \otimes_{\mathbb{Z}_p} \hat{\mathbb{Z}}_p$ with M a finitely generated free $\mathbb{Z}_p$-module (recall that we have assumed that $\mathbb{L}$ is locally free over $\hat{\mathbb{Z}}_p$). So our claim in this case follows from Lemma 2.6. As a result, in the spectral sequence below

$$E_2^{i,j} = H^i(X_{\bar{k}}, R^j \varprojlim(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris}})) \Rightarrow H^{i+j}(X_{\bar{k}}, R \varprojlim(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris}})),$$

we have $\mathcal{I}^2 \cdot E_2^{i,j} = 0$ for $j > 0$ and $E_\infty^{i,0} = E_{i+1}^{i,0}$. Moreover, the natural surjection $E_2^{i,0} \to E_\infty^{i,0}$ has kernel killed by $\mathcal{I}^{2i-2}$. It follows that the canonical map

$$H^i(X_{\bar{k}}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}}) \to H^i(X_{\bar{k}}, R \varprojlim(\mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}}))$$

has kernel killed by $\mathcal{I}^{2i-2}$ and cokernel killed by $\mathcal{I}^{2i}$. On the other hand, by Lemma 4.1, the kernel of the canonical surjective morphism

$$H^i(X_{\bar{k}}, R \varprojlim_n(\mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}})) \to \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}})$$

is $R^1 \varprojlim_n H^{i-1}(X_{\bar{k}}, \mathbb{L}_n \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}})$, thus is killed by $\mathcal{I}$ by what we have shown above. Therefore, the kernel and cokernel of the composed map

$$H^i(X_{\bar{k}}, \mathbb{L} \otimes \mathbb{A}_{\mathrm{cris}}) \to H^i(X_{\bar{k}}, R \varprojlim_n(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris}})) \to \varprojlim_n H^i(X_{\bar{k}}, \mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris}})$$

are killed by $\mathcal{I}^{2i}$. Finally, from the commutative square

$$\begin{tikzcd} H^i(X_{\bar{k}}, \mathbb{L}) \otimes A_{\mathrm{cris}} \arrow[r, "\sim", "(4A.3)"'] \arrow[d] & \varprojlim(H^i(X_{\bar{k}}, \mathbb{L}_n) \otimes A_{\mathrm{cris}}) \arrow[d, "\varprojlim_n (4A.4)"', "\text{iso. up to } \mathcal{I}^2"] \\ H^i(X_{\bar{k}}, \mathbb{L} \otimes \mathbb{A}_{\mathrm{cris}}) \arrow[r, "\text{iso. up to } \mathcal{I}^{2i}"] & \varprojlim H^i(X_{\bar{k}}, \mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris}}) \end{tikzcd} ,$$

we deduce that the natural map below has kernel and cokernel killed by $\mathcal{I}^{2i+2}$, hence by t^{2i+2}

$$H^i(X_{\bar{k}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \to H^i(X_{\bar{k}}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{A}_{\mathrm{cris}}).$$

Inverting t, we get the required isomorphism (4A.2).

We still need to check that (4A.2) is compatible with the extra structures. Clearly only the strict compatibility with filtrations needs verification, and it suffices to check this on gradeds. So we reduce to showing that the natural morphism is an isomorphism:

$$H^i(X_{\bar{k}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p(j) \to H^i(X_{\bar{k}}, \mathbb{L} \otimes \hat{\mathcal{O}}_X(j)).$$

Twisting, one reduces to $j = 0$, which is given by the following lemma. □

Lemma 4.4. *Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\bar{k},\text{proét}}$. Then the following natural morphism is an isomorphism*:

$$H^i(X_{\bar{k},\text{proét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\sim} H^i(X_{\bar{k},\text{proét}}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathcal{O}}_X),$$

where $\hat{\mathcal{O}}_X$ is the completed structural sheaf of $X_{\bar{k},\text{proét}}$ and $\mathbb{C}_p = \hat{\bar{k}}$.

Proof. It suffices to show that the natural morphism of $\mathcal{O}_{\mathbb{C}_p}$-modules

$$H^i(X_{\bar{k},\text{proét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} \to H^i(X_{\bar{k},\text{proét}}, \mathbb{L} \otimes_{\hat{\mathbb{Z}}_p} \hat{\mathcal{O}}_X^+)$$

has kernel and cokernel annihilated by some power of $\mathcal{I}\mathcal{O}_{\mathbb{C}_p}$. The proof is similar to that of the first part of Theorem 4.3, so we omit the details here. □

Recall that the notion of lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ and lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$ are equivalent. Combining Theorem 3.26 and Theorem 4.3, we finally deduce the following crystalline comparison theorem:

Theorem 4.5. *Let $\mathcal{X}$ be a proper smooth formal scheme over $\mathcal{O}_k$, with X (resp. $\mathcal{X}_0$) its generic (resp. closed) fiber. Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$, associated to a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$. Then there exists a functorial filtered isomorphism*

$$H^i(X_{\bar{k},\text{ét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \xrightarrow{\sim} H^i_{\text{cris}}(\mathcal{X}_0/\mathcal{O}_k, \mathcal{E}) \otimes_{\mathcal{O}_k} B_{\text{cris}}$$

of B_{cris}-modules, compatible with Galois action and Frobenius.

5. Comparison isomorphism in the relative setting

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth morphism between two smooth formal schemes over $\mathrm{Spf}(\mathcal{O}_k)$ of relative dimension $d \geq 0$. The induced morphism between the generic fibers will be denoted by $f_k\colon X \to Y$. We shall denote by $w_{\mathcal{X}}$ and $w_{\mathcal{Y}}$ the natural morphism of topoi $X^{\sim}_{\text{proét}} \to \mathcal{X}^{\sim}_{\text{ét}}$ and $Y^{\sim}_{\text{proét}} \to \mathcal{Y}^{\sim}_{\text{ét}}$. By abuse of notation, the morphism of topoi $X^{\sim}_{\text{proét}} \to Y^{\sim}_{\text{proét}}$ will be still denoted by f_k.

Let $\nabla_{X/Y} : \mathcal{O}\mathbb{A}_{\text{cris},X} \to \mathcal{O}\mathbb{A}_{\text{cris},X} \otimes_{\mathcal{O}_X^{\text{ur}+}} \Omega^{1,\text{ur}+}_{X/Y}$ be the natural relative derivation, where $\Omega^{1,\text{ur}+}_{X/Y} := w^*_{\mathcal{X}}\Omega^1_{\mathcal{X}/\mathcal{Y}}$.

Proposition 5.1. (1) (*relative Poincaré lemma*) *The following sequence of proétale sheaves is exact and strict with respect to the filtration giving $\Omega^{i,\text{ur}+}_{X/Y}$ degree i*:

$$0 \to \mathbb{A}_{\text{cris},X} \widehat{\otimes}_{f_k^{-1}\mathbb{A}_{\text{cris},Y}} f_k^{-1}\mathcal{O}\mathbb{A}_{\text{cris},Y} \to \mathcal{O}\mathbb{A}_{\text{cris},X} \xrightarrow{\nabla_{X/Y}} \mathcal{O}\mathbb{A}_{\text{cris},X} \otimes_{\mathcal{O}_X^{\text{ur}+}} \Omega^{1,\text{ur}+}_{X/Y} \xrightarrow{\nabla_{X/Y}} \cdots$$
$$\cdots \xrightarrow{\nabla_{X/Y}} \mathcal{O}\mathbb{A}^+_{\text{cris},X} \otimes_{\mathcal{O}_X^{\text{ur}+}} \Omega^{d,\text{ur}+}_{X/Y} \to 0.$$

In particular, the connection $\nabla_{X/Y}$ is integrable and satisfies Griffiths transversality with respect to the filtration, i.e., $\nabla_{X/Y}(\mathrm{Fil}^i\, \mathcal{O}\mathbb{A}_{\text{cris},X}) \subset \mathrm{Fil}^{i-1}\, \mathcal{O}\mathbb{A}_{\text{cris},X} \otimes \Omega^{1,\text{ur}+}_{X/Y}$.

(2) *Suppose the Frobenius on $\mathcal{X}_0$ (resp. $\mathcal{Y}_0$) lifts to a Frobenius σ_X (resp. σ_Y) on the formal scheme $\mathcal{X}$ (resp. $\mathcal{Y}$) and they commute with f. Then the induced Frobenius φ_X on $\mathcal{O}\mathbb{A}_{\mathrm{cris},X}$ is horizontal with respect to $\nabla_{X/Y}$.*

Proof. The proof is routine (see Proposition 2.13), so we omit the detail here. □

For the relative version of the crystalline comparison, we shall need the following primitive comparison in the relative setting.

Proposition 5.2. *Let $f:\mathcal{X}\to\mathcal{Y}$ be a proper smooth morphism between two smooth formal schemes over $\mathcal{O}_k$. Let $\mathbb{L}$ be a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. Suppose that $R^i f_{k*}\mathbb{L}$ is a lisse $\hat{\mathbb{Z}}_p$-sheaf for all $i\geq 0$. Then, the canonical morphism*

$$(R^i f_{k*}\mathbb{L})\otimes_{\hat{\mathbb{Z}}_p}\mathbb{B}_{\mathrm{cris},Y}\to R^i f_{k*}(\mathbb{L}\otimes_{\hat{\mathbb{Z}}_p}\mathbb{B}_{\mathrm{cris},X}) \tag{5A.1}$$

is a filtered isomorphism, compatible with Frobenius. Similarly, the natural morphism

$$(R^i f_{k*}\mathbb{L})\otimes_{\hat{\mathbb{Z}}_p}\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}\to R^i f_{k*}(\mathbb{L}\otimes_{\hat{\mathbb{Z}}_p}\mathbb{A}_{\mathrm{cris},X}\widehat{\otimes}_{f_k^{-1}\mathbb{A}_{\mathrm{cris},Y}}f_k^{-1}\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}[1/t]) \tag{5A.2}$$

is a filtered isomorphism, compatible with Frobenius and connections.

Proof. The proof is similar to that of Theorem 4.3, so we shall only give a sketch here. As in the proof of Theorem 4.3, to show our proposition, it suffices to consider the case where $\mathbb{L}$ is locally free over $\hat{\mathbb{Z}}_p$. Let $\mathbb{L}_n=\mathbb{L}/p^n$, $n\in\mathbb{N}$. Since $\mathbb{L}$ has no p-torsion, we have short exact sequences of lisse $\hat{\mathbb{Z}}_p$-sheaves on Y

$$0\to R^i f_{k*}(\mathbb{L})/p^n\to R^i f_{k*}(\mathbb{L}_n)\to R^{i+1}f_{k*}(\mathbb{L})[p^n]\to 0,\quad n\in\mathbb{N},$$

inducing an exact sequence of projective systems as $\mathbb{A}_{\mathrm{cris},Y}$ is flat over $\hat{\mathbb{Z}}_p$

$$0\to (R^i f_{k*}(\mathbb{L})\otimes\mathbb{A}_{\mathrm{cris},Y}/p^n)_n\to (R^i f_{k*}(\mathbb{L}_n)\otimes\mathbb{A}_{\mathrm{cris},Y})_n\to (R^{i+1}f_{k*}(\mathbb{L})[p^n]\otimes\mathbb{A}_{\mathrm{cris},Y})_n\to 0.$$

Because $R^{i+1}f_{k*}(\mathbb{L})$ is a lisse $\hat{\mathbb{Z}}_p$-module, there exists $N\in\mathbb{N}$ such that p^N kills $R^{i+1}f_{k*}(\mathbb{L})[p^n]$ for all n. Thus, the composed transition map in the last projective system of the sequence above is zero:

$$R^{i+1}f_{k*}(\mathbb{L})[p^{n+N}]\otimes\mathbb{A}_{\mathrm{cris},Y}\to R^{i+1}f_{k*}(\mathbb{L})[p^n]\otimes\mathbb{A}_{\mathrm{cris},Y},\quad x\mapsto p^N x.$$

Therefore, $R^j\varprojlim_n(R^{i+1}f_{k*}(\mathbb{L})[p^n]\otimes\mathbb{A}_{\mathrm{cris},Y})=0$ for every $j\in\mathbb{Z}$, and thus

$$R^j\varprojlim_n(R^i f_{k*}(\mathbb{L})\otimes\mathbb{A}_{\mathrm{cris},Y}/p^n)\xrightarrow{\sim}R^j\varprojlim_n(R^i f_{k*}(\mathbb{L}_n)\otimes\mathbb{A}_{\mathrm{cris},Y})$$

for all $j\in\mathbb{Z}$. In particular,

$$R^i f_{k*}(\mathbb{L})\otimes\mathbb{A}_{\mathrm{cris},Y}\simeq\varprojlim_n(R^i f_{k*}(\mathbb{L})\otimes\mathbb{A}_{\mathrm{cris},Y}/p^n)\xrightarrow{\sim}\varprojlim_n(R^i f_{k*}(\mathbb{L}_n)\otimes\mathbb{A}_{\mathrm{cris},Y}), \tag{5A.3}$$

and $R^j\varprojlim_n(R^i f_{k*}(\mathbb{L}_n)\otimes\mathbb{A}_{\mathrm{cris},Y})\simeq R^j\varprojlim_n(R^i f_{k*}(\mathbb{L})\otimes\mathbb{A}_{\mathrm{cris},Y}/p^n)$ is killed by $\mathcal{I}^2$ whenever $j>0$ as this is the case for $R^j\varprojlim_n(\mathbb{A}_{\mathrm{cris},Y}/p^n)$ by Corollary 2.8 and as $R^i f_{k*}(\mathbb{L})$ is a lisse $\hat{\mathbb{Z}}_p$-sheaf.

Next, with the help of [Scholze 2013, Corollary 5.11], the same argument as in the proof of Theorem 4.3 yields almost isomorphisms

$$R^i f_{k*}(\mathbb{L}_n) \otimes \mathbb{A}_{\mathrm{cris},Y} \xrightarrow{\sim} R^i f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}), \quad n \in \mathbb{N}. \tag{5A.4}$$

So, the kernel and the cokernel of the natural map below are killed by $\mathcal{I}^2$:

$$\varprojlim_n (R^i f_{k*}(\mathbb{L}_n) \otimes \mathbb{A}_{\mathrm{cris},Y}) \to \varprojlim_n R^i f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}). \tag{5A.5}$$

Moreover, for $j > 0$, from (5A.4), we find

$$\mathcal{I}^4 \cdot R^j \varprojlim_n R^i f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}) = 0$$

since $\mathcal{I}^2 \cdot R^j \varprojlim_n (R^i f_{k*}(\mathbb{L}_n) \otimes \mathbb{A}_{\mathrm{cris},Y}) = 0$ as observed at the end of the last paragraph. Therefore, by a standard argument using the spectral sequence

$$E_2^{a,b} = R^a \varprojlim_n R^b f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}) \Longrightarrow \mathcal{H}^{a+b}(R \varprojlim_n R f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X})),$$

one checks that the kernel and the cokernel of the map

$$\mathcal{H}^i(R \varprojlim_n R f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X})) \simeq R^i f_{k*}(R \varprojlim_n (\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X})) \to E_2^{0,i} = \varprojlim_n R^i f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X})$$

are killed by some power of $\mathcal{I}$. On the other hand, as shown in the proof of Theorem 4.3, if $j > 0$ then $\mathcal{I}^2 \cdot R^j \varprojlim_n (\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}) = 0$. So the kernel and the cokernel of

$$R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X}) \to R^i f_{k*}(R \varprojlim_n (\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}))$$

are killed by some power of $\mathcal{I}$. Consequently, the kernel and cokernel of the map

$$R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X}) \to \varprojlim_n R^i f_{k*}(\mathbb{L}_n \otimes \mathbb{A}_{\mathrm{cris},X}) \tag{5A.6}$$

are killed by some power of $\mathcal{I}$. Combining the morphisms (5A.3), (5A.5) and (5A.6), we deduce that the kernel and the cokernel of the map

$$R^i f_{k*}(\mathbb{L}) \otimes \mathbb{A}_{\mathrm{cris},Y} \to R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X})$$

are killed by some power of $\mathcal{I}$, thus also killed by some power of t. Inverting t, we obtain the desired isomorphism (5A.1).

We need to verify the compatibility of the isomorphism (5A.1) with the extra structures. It clearly respects Frobenius structures. To check the strict compatibility with respect to filtrations, by taking grading quotients, we just need to show that for each $r \in \mathbb{N}$, the natural morphism

$$R^i f_{k*}\mathbb{L} \otimes \hat{\mathcal{O}}_Y(r) \to R^i f_{k*}(\mathbb{L} \otimes \hat{\mathcal{O}}_X(r))$$

is an isomorphism: it is a local question, hence it suffices to show this after restricting the latter morphism to $Y_{\bar{k}}$. As $\hat{\mathcal{O}}_X(r)|_{X_{\bar{k}}} \simeq \hat{\mathcal{O}}_X|_{X_{\bar{k}}}$ and $\hat{\mathcal{O}}_Y(r)|_{Y_{\bar{k}}} \simeq \hat{\mathcal{O}}_Y|_{Y_{\bar{k}}}$, we then reduce to the case where $r = 0$. The proof of the latter statement is similar as above, so we omit the details here.

Finally, using Proposition 2.13, a similar proof as above shows that the map (5A.2) is a filtered isomorphism compatible with Frobenius and connections. □

For a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F}$ with an $\mathcal{O}_{\mathcal{Y}}$-linear connection $\nabla\colon \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{\mathcal{X}/\mathcal{Y}}$, we denote the de Rham complex of $\mathcal{F}$ as

$$\mathrm{dR}_{X/Y}(\mathcal{F}) := (\cdots \to 0 \to \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{Y}}} \Omega^1_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\nabla} \cdots).$$

The same rule applies if we consider an $\mathcal{O}_X^{\mathrm{un}}$-module endowed with an $\mathcal{O}_Y^{\mathrm{un}}$-linear connection, etc.

In the lemma below, assume $\mathcal{Y} = \mathrm{Spf}(A)$ is affine and is étale over a torus $\mathcal{S} = \mathrm{Spf}(\mathcal{O}_k\{S_1^{\pm 1}, \dots, S_\delta^{\pm 1}\})$. For each $1 \le j \le \delta$, let $(S_j^{1/p^n})_{n\in\mathbb{N}}$ be a compatible family of p-power roots of S_j. As in Proposition 2.13, set

$$\tilde{Y} := (Y \times_{\mathcal{S}_k} \mathrm{Spa}(k\{S_1^{\pm 1/p^n}, \dots, S_\delta^{\pm 1/p^n}\}, \mathcal{O}_k\{S_1^{\pm 1/p^n}, \dots, S_\delta^{\pm 1/p^n}\}))_{n\in\mathbb{N}} \in Y_{\mathrm{pro\acute{e}t}}.$$

Lemma 5.3. *Let $V \in Y_{\mathrm{pro\acute{e}t}}$ be an affinoid perfectoid which is proétale over $\tilde{Y}_{\bar{k}}$, with $\hat{V} = \mathrm{Spa}(R, R^+)$. Let w_V be the composite of natural morphisms of topoi*

$$w_V : X^{\sim}_{\mathrm{pro\acute{e}t}}/X_V \to X^{\sim}_{\mathrm{pro\acute{e}t}} \xrightarrow{w} \mathcal{X}^{\sim}_{\mathrm{\acute{e}t}}.$$

(1) *For any $j > 0$, we have $R^j w_{V*}\mathcal{O}\mathbb{B}_{\mathrm{cris}} = 0$, and the natural morphism*

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \to w_{V*}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})$$

is an isomorphism.

(2) *For any $r \in \mathbb{Z}$ and any $j > 0$, we have $R^j w_{V*}(\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris}}) = 0$. Moreover, the natural morphism*

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \to w_{V*}(\mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris},X})$$

is an isomorphism.

Here $\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) := (\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V))[1/t]$ with

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) := \varprojlim (\mathcal{O}_{\mathcal{X}} \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)/p^n),$$

and

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) := \varinjlim_{n\in\mathbb{N}} t^{-n}(\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathrm{Fil}^{r+n} \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)).$$

In particular, if we filter $\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$ using $\{\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathrm{Fil}^r \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)\}_{r\in\mathbb{Z}}$, the natural morphism

$$\mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \to Rw_{V*}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})$$

is an isomorphism in the filtered derived category.

Proof. Recall first that, for $\mathcal{F}$ a proétale sheaf on X and for $j \geq 0$, $R^j w_{V*}\mathcal{F}$ is the associated sheaf on $\mathcal{X}_{\text{ét}}$ of the presheaf sending $\mathcal{U} \in \mathcal{X}_{\text{ét}}$ to $H^i(\mathcal{U}_V, \mathcal{F})$, where $\mathcal{U}_V := \mathcal{U}_k \times_X X_V$. Take $\mathcal{U} = \mathrm{Spf}(B) \in \mathcal{X}_{\text{ét}}$ to be affine such that the composition $\mathcal{U} \to \mathcal{X} \to \mathcal{Y}$ can be factored as

$$\mathcal{U} \to \mathcal{T} \to \mathcal{Y},$$

where the first morphism is étale and $\mathcal{T} := \mathrm{Spf}(A\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\})$. Write $\mathcal{T}_V = \mathcal{T}_k \times_Y V$. Then $\mathcal{T}_V = \mathrm{Spa}(S, S^+)$ with $S^+ = R\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}$ and $S = S^+[1/p]$. Write $\mathcal{U}_V = \mathrm{Spa}(\tilde{S}, \tilde{S}^+)$. For each $1 \leq i \leq d$, let $(T_i^{1/p^n})_{n\in\mathbb{N}}$ be a compatible family of p-power roots of T_i. Set

$$S_\infty^+ := R^+\{T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\}, \quad \tilde{S}_\infty^+ := B \widehat{\otimes}_{A\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}} S_\infty^+, \quad S_\infty := S_\infty^+[1/p] \quad \text{and} \quad \tilde{S}_\infty := \tilde{S}_\infty^+[1/p].$$

Then (S_∞, S_∞^+) and $(\tilde{S}_\infty, \tilde{S}_\infty^+)$ are affinoid perfectoid algebras over $(\hat{\bar{k}}, \mathcal{O}_{\hat{\bar{k}}})$. Let $\tilde{\mathcal{U}}_V \in X_{\text{proét}}$ (resp. $\tilde{\mathcal{T}}_V \in \mathcal{T}_{k\,\text{proét}}$) be the affinoid perfectoid corresponding to $(\tilde{S}_\infty, \tilde{S}_\infty^+)$ (resp. to (S_∞, S_∞^+)). We have the following commutative diagram of ringed spaces

$$\begin{tikzcd}
\hat{\tilde{\mathcal{U}}}_V = \mathrm{Spa}(\tilde{S}_\infty, \tilde{S}_\infty^+) \arrow[r, "\Gamma"] \arrow[d] & \tilde{\mathcal{U}}_V = \mathrm{Spa}(\tilde{S}, \tilde{S}^+) \arrow[r] \arrow[d] & \mathcal{U} = \mathrm{Spf}(B) \arrow[d] \\
\hat{\tilde{\mathcal{T}}}_V = \mathrm{Spa}(S_\infty, S_\infty^+) \arrow[r] & \tilde{\mathcal{T}}_V = \mathrm{Spa}(S, S^+) \arrow[r] \arrow[d] & \mathcal{T} \arrow[d] \\
 & \hat{V} = \mathrm{Spa}(R, R^+) \arrow[r] & \mathcal{Y} = \mathrm{Spf}(A).
\end{tikzcd}$$

The morphism $\tilde{\mathcal{U}}_V \to \mathcal{U}_V$ is a profinite Galois cover, with Galois group $\Gamma \simeq \mathbb{Z}_p(1)^d$. For $q \in \mathbb{N}$, let $\tilde{\mathcal{U}}_V^q$ be the $(q+1)$-fold fiber product of $\tilde{\mathcal{U}}_V$ over $\mathcal{U}_V$. So $\tilde{\mathcal{U}}_V^q \simeq \tilde{\mathcal{U}}_V \times \Gamma^q$ is an affinoid perfectoid.

(1) As in the proof of Lemma 3.14, there is a natural isomorphism of B_{cris}-modules

$$H^q(\Gamma, \mathcal{O}\mathbb{A}_{\text{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))[1/t] \xrightarrow{\sim} H^q(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\text{cris},X}),$$

where the first group is the continuous group cohomology and $\mathcal{O}\mathbb{A}_{\text{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)$ is endowed with the p-adic topology. So, by Theorem A.12, $H^q(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\text{cris},X}) = 0$ whenever $q > 0$, and there is a natural isomorphism

$$B \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\text{cris}}(R, R^+) \xrightarrow{\sim} H^0(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\text{cris},X}). \tag{5A.7}$$

On the other hand, V being affinoid perfectoid with $\hat{V} = \mathrm{Spa}(R, R^+)$, the maps

$$\mathcal{O}\mathbb{A}_{\text{cris}}(R, R^+)/p^n \to \mathcal{O}\mathbb{A}_{\text{cris},Y}(V)/p^n, \quad n \in \mathbb{N},$$

and thus the maps

$$B \otimes_A \mathcal{O}\mathbb{A}_{\text{cris}}(R, R^+)/p^n \to B \otimes_A \mathcal{O}\mathbb{A}_{\text{cris},Y}(V)/p^n, \quad n \in \mathbb{N},$$

are almost isomorphisms (Lemma 2.18). Passing to projective limits, it follows that the kernel and the cokernel of the induced map

$$B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to B\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)$$

are killed by $\mathcal{I}^2$, hence also by t^2. Inverting t, we deduce $B\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} B\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$, and an isomorphism from (5A.7)

$$\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) = B\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \xrightarrow{\sim} H^0(\mathcal{U}_V, \mathcal{O}\mathbb{B}_{\mathrm{cris},X}).$$

To conclude the proof of (1), it remains to check that the canonical morphism

$$\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \to (\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V))(\mathcal{U})$$

is an isomorphism. In fact, we have

$$\begin{aligned}\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) &= \varprojlim_n \mathcal{O}_{\mathcal{X}}(\mathcal{U}) \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)/p^n \\ &\xrightarrow{\sim} \varprojlim_n((\mathcal{O}_{\mathcal{X}} \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)/p^n)(\mathcal{U})) \\ &\xrightarrow{\sim} (\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V))(\mathcal{U}).\end{aligned}$$

Therefore, as $\mathcal{U}$ is quasicompact and quasiseparated, we find

$$\begin{aligned}\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) &= (\mathcal{O}_{\mathcal{X}}(\mathcal{U})\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V))[1/t] \\ &\xrightarrow{\sim} ((\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V))(\mathcal{U}))[1/t] \\ &\xrightarrow{\sim} (\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V)[1/t])(\mathcal{U}) \\ &= (\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V))(\mathcal{U}),\end{aligned}$$

as required.

(2) We shall only prove (2) when $r = 0$; the general case can be deduced by twisting. As in (1), there exists a natural isomorphism

$$\varinjlim_{s\geq 0} H^q(\Gamma, \mathrm{Fil}^s\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)) \to H^q(\mathcal{U}_V, \mathrm{Fil}^0\, \mathcal{O}\mathbb{B}_{\mathrm{cris},X}).$$

By definition, the first group is $H^q(\Gamma, \mathrm{Fil}^0\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$ computed in Appendix A. So, according to Proposition A.14 and Corollary A.16, $H^q(\mathcal{U}_V, \mathrm{Fil}^0\, \mathcal{O}\mathbb{B}_{\mathrm{cris},X}) = 0$ if $q > 0$, and we have an isomorphism

$$B\widehat{\otimes}\, \mathrm{Fil}^0\, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) := \varinjlim_{s\geq 0} B\widehat{\otimes}\, \mathrm{Fil}^s\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} H^0(\mathcal{U}_V, \mathrm{Fil}^0\, \mathcal{O}\mathbb{B}_{\mathrm{cris},X}).$$

To go further, one has to identify $B\widehat{\otimes}\mathrm{Fil}^0\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R,R^+)$ with $B\widehat{\otimes}\mathrm{Fil}^0\mathcal{O}\mathbb{B}_{\mathrm{cris}}(V) := \varinjlim_{s\geq 0} B\widehat{\otimes}\mathrm{Fil}^s\mathcal{O}\mathbb{A}_{\mathrm{cris}}(V)$. As V is an affinoid perfectoid proétale over $\tilde{Y}_{\bar{k}}$, by Lemmas 2.6 and 2.18, the kernels and cokernels of the natural maps

$$\mathrm{Fil}^s\, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathrm{Fil}^s\, \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V), \quad s \in \mathbb{N},$$

are killed by $\mathcal{I}^2$, and $\mathcal{I}^3 \cdot H^1(V, \mathrm{Fil}^s \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}) = 0$. In particular, the kernels and cokernels of

$$\mathrm{gr}^s \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathrm{gr}^s \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V), \quad s \in \mathbb{N},$$

are killed by some power of $\mathcal{I}(A_{\mathrm{cris}}/\ker(\theta)) = \mathcal{I}\mathcal{O}_{\mathbb{C}_p}$, thus by p^{1/p^N} for every $N \in \mathbb{N}$. Moreover, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B\widehat{\otimes}_A \mathrm{Fil}^{s+1} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \arrow[r] \arrow[d] & B\widehat{\otimes}_A \mathrm{Fil}^{s} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \arrow[r] \arrow[d] & B\widehat{\otimes}_A \mathrm{gr}^{s} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \arrow[r] \arrow[d, "\text{iso. up to } p\text{-torsion}"] & 0 \\
0 \arrow[r] & B\widehat{\otimes}_A \mathrm{Fil}^{s+1} \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) \arrow[r] & B\widehat{\otimes}_A \mathrm{Fil}^{s} \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) \arrow[r] & B\widehat{\otimes}_A \mathrm{gr}^{s} \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}(V) &
\end{tikzcd}$$

By the observations above, the kernels and cokernels of the first two vertical maps are killed by some power of $\mathcal{I}$, thus by some power of t, and the last vertical map becomes an isomorphism after inverting p. So, by a similar argument as in the proof of Corollary 2.8 (2), we find $B\widehat{\otimes}_A \mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} B\widehat{\otimes}_A \mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$. We get finally a natural isomorphism

$$B\widehat{\otimes}_A \mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \xrightarrow{\sim} H^0(\mathcal{U}_V, \mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{cris},X}).$$

The remaining part of (2) can be done similarly as in the last part of the proof of (1), so we omit the details here. □

From now on, assume $f: \mathcal{X} \to \mathcal{Y}$ is a proper smooth morphism (between smooth formal schemes) over $\mathcal{O}_k$. Its closed fiber gives rise to a morphism between the crystalline topoi,

$$f_{\mathrm{cris}}: (\mathcal{X}_0/\mathcal{O}_k)^{\sim}_{\mathrm{cris}} \to (\mathcal{Y}_0/\mathcal{O}_k)^{\sim}_{\mathrm{cris}}.$$

Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$, and $\mathcal{M}$ an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{E} \simeq \mathcal{M}^{\mathrm{an}}(n)$ for some $n \in \mathbb{N}$ (see Remark 3.4). Then $\mathcal{M}$ can be viewed naturally as a coherent $\mathcal{O}_{\mathcal{X}}$-module endowed with an integrable and quasinilpotent $\mathcal{O}_k$-linear connection $\mathcal{M} \to \mathcal{M} \otimes \Omega^1_{\mathcal{X}/\mathcal{O}_k}$.

In the following we consider the higher direct image $R^i f_{\mathrm{cris}\,*}\mathcal{M}$ of the crystal $\mathcal{M}$. One can determine the value of this abelian sheaf on $\mathcal{Y}_0/\mathcal{O}_k$ at the p-adic PD-thickening $\mathcal{Y}_0 \hookrightarrow \mathcal{Y}$ in terms of the relative de Rham complex $\mathrm{dR}_{X/Y}(\mathcal{M})$ of $\mathcal{M}$. To state this, take $\mathcal{V} = \mathrm{Spf}(A)$ an affine open subset of $\mathcal{Y}$, and put $\mathcal{X}_A := f^{-1}(\mathcal{V})$. We consider A as a PD-ring with the canonical divided power structure on $(p) \subset A$. In particular, we can consider the crystalline site $(\mathcal{X}_{A,0}/A)_{\mathrm{cris}}$ of $\mathcal{X}_{A,0} := \mathcal{X} \times_{\mathcal{Y}} \mathcal{V}_0$ relative to A. By [Berthelot 1996, Lemme 3.2.2], the latter can be identified naturally to the open subset of $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ whose objects are objects (U, T) of $(\mathcal{X}_0/\mathcal{O}_k)_{\mathrm{cris}}$ such that $f(U) \subset \mathcal{V}_0$ and such that there exists a morphism $\alpha: T \to \mathcal{V}_n := \mathcal{V} \otimes_A A/p^{n+1}$ for some $n \in \mathbb{N}$, making the square below commute

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, "\mathrm{can}"'] & T \arrow[d, "\alpha"] \\
\mathcal{V}_0 \arrow[r, hook] & \mathcal{V}_n.
\end{tikzcd}$$

Using [Berthelot 1996, Corollaire 3.2.3] and a limit argument, one finds

$$R^i f_{\mathrm{cris}\,*}(\mathcal{M})(\mathcal{V}_0,\mathcal{V}) \xrightarrow{\sim} H^i_{\mathrm{cris}}(\mathcal{X}_{A,0}/A,\mathcal{M})$$

where we denote again by $\mathcal{M}$ the restriction of $\mathcal{M}$ to $(\mathcal{X}_{A,0}/A)_{\mathrm{cris}}$. Let

$$u = u_{\mathcal{X}_{A,0}/A} : (\mathcal{X}_{A,0}/A)^{\sim}_{\mathrm{cris}} \to \mathcal{X}^{\sim}_{A\,\text{ét}}$$

be the morphism of topoi such that $u_*(\mathcal{F})(\mathcal{U}) = H^0_{\mathrm{cris}}(\mathcal{U}_0/A,\mathcal{F})$ for $\mathcal{U} \in \mathcal{X}_{A\,\text{ét}}$. By [Berthelot and Ogus 1978, Theorem 7.23], there is a natural quasiisomorphism $Ru_*\mathcal{M} \xrightarrow{\sim} \mathrm{dR}_{X/Y}(\mathcal{M})$ in the derived category, inducing an isomorphism

$$H^i_{\mathrm{cris}}(\mathcal{X}_{A,0}/A,\mathcal{M}) \xrightarrow{\sim} H^i(\mathcal{X}_A,\mathrm{dR}_{X/Y}(\mathcal{M})).$$

Passing to associated sheaves, we deduce $R^i f_{\mathrm{cris}\,*}(\mathcal{M})_{\mathcal{Y}} \xrightarrow{\sim} R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{M}))$. On the other hand, as $f\colon \mathcal{X}\to\mathcal{Y}$ is proper and smooth, $R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E}))$, viewed as a coherent sheaf on the adic space Y, is the i-th relative convergent cohomology of $\mathcal{E}$ with respect to the morphism $f_0\colon \mathcal{X}_0\to\mathcal{Y}_0$. Thus, by [Berthelot 1986, Théorème 5] (see also [Tsuzuki 2003, Theorem 4.1.4]), if we invert p, the $\mathcal{O}_{\mathcal{Y}}[1/p]$-module $R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E})) \simeq R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{M}))[1/p]$, together with the Gauss–Manin connection and the natural Frobenius structure inherited from $R^i f_{\mathrm{cris}\,*}(\mathcal{M})_{\mathcal{Y}} \simeq R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{M}))$, is a convergent F-isocrystal on $\mathcal{Y}_0/\mathcal{O}_k$, denoted by $R^i f_{\mathrm{cris}\,*}(\mathcal{E})$ in the following (this is an abuse of notation, a more appropriate notation should be $R^i f_{0\,\mathrm{conv}\,*}(\mathcal{E})$). Using the filtration on $\mathcal{E}$, one sees that $R^i f_{\mathrm{cris}\,*}(\mathcal{E})$ has naturally a filtration, and it is well-known that this filtration satisfies Griffiths transversality with respect to the Gauss–Manin connection.

Proposition 5.4. *Let $\mathcal{X}\to\mathcal{Y}$ be a proper smooth morphism of smooth p-adic formal schemes over $\mathcal{O}_k$. Let $\mathcal{E}$ be a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_k$ and $\mathbb{L}$ a lisse $\hat{\mathbb{Z}}_p$-sheaf on $X_{\text{proét}}$. Assume that $\mathcal{E}$ and $\mathbb{L}$ are associated.*

(1) *For every $i\in\mathbb{Z}$, $R^i f_{\mathrm{cris}\,*}\mathcal{E}$ is a filtered convergent F-isocrystal on $\mathcal{Y}_0/\mathcal{O}_k$, with, for $r\in\mathbb{Z}$, gradeds $R^i f_*(\mathrm{gr}^r\,\mathrm{dR}_{X/Y}(\mathcal{E}))$.*

(2) *There is a natural filtered isomorphism of $\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}$-modules*

$$R^i f_{k*}(\mathbb{L}\otimes_{\hat{\mathbb{Z}}_p}\mathbb{A}_{\mathrm{cris},X}\widehat{\otimes}_{f_k^{-1}\mathbb{A}_{\mathrm{cris},Y}} f_k^{-1}\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}[1/t]) \xrightarrow{\sim} w_{\mathcal{Y}}^{-1}(R^i f_{\mathrm{cris}\,*}(\mathcal{E}))\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris},Y} \tag{5A.8}$$

which is compatible with Frobenius and connection.

Proof. (1) We have observed above that $R^i f_{\mathrm{cris}\,*}\mathcal{E}$ is naturally a convergent F-isocrystal. To complete the proof of (1), it suffices to check that the filtration on $R^i f_{\mathrm{cris}\,*}\mathcal{E}$ is given by locally direct summands. By Proposition 3.22, the lisse $\hat{\mathbb{Z}}_p$-sheaf $\mathbb{L}$ is de Rham with associated filtered $\mathcal{O}_X$-module with integrable connection $\mathcal{E}$. Therefore the Hodge-to-de Rham spectral sequence

$$E_1^{i,j} = R^{i+j} f_*(\mathrm{gr}^i(\mathrm{dR}_{X/Y}(\mathcal{E}))) \Rightarrow R^{i+j} f_*(\mathrm{dR}_{X/Y}(\mathcal{E}))$$

degenerates at E_1. Moreover, $E_1^{i,j}$, the relative Hodge cohomology of $\mathcal{E}$ in [Scholze 2013, Theorem 8.8], is a locally free $\mathcal{O}_Y$-module of finite rank for all i, j by [loc. cit.]. Thereby the filtration on $R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E})) = R^i f_{\mathrm{cris}\,*}(\mathcal{E})$, which is the same as the one induced by the spectral sequence above, is given by locally direct summands, with gradeds $R^i f_*(\mathrm{gr}^r(\mathrm{dR}_{X/Y}(\mathcal{E})))$, $r \in \mathbb{Z}$.

(2) Using Proposition 5.1(1) and the fact that $\mathbb{L}$ and $\mathcal{E}$ are associated, we have the following filtered isomorphisms compatible with connection:

$$\begin{aligned} R^i f_{k*}(\mathbb{L} \otimes \mathbb{A}_{\mathrm{cris},X} \widehat{\otimes} f_k^{-1} \mathcal{O}\mathbb{A}_{\mathrm{cris},Y}[1/t]) &\xrightarrow{\sim} R^i f_{k*}(\mathbb{L} \otimes \mathrm{dR}_{X/Y}(\mathcal{O}\mathbb{B}_{\mathrm{cris},X})) \\ &\xrightarrow{\sim} R^i f_{k*}(\mathrm{dR}_{X/Y}(\mathbb{L} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})) \\ &\xrightarrow{\sim} R^i f_{k*}(\mathrm{dR}_{X/Y}(w_{\mathcal{X}}^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})). \end{aligned} \tag{5A.9}$$

On the other hand, the morphism below given by adjunction respects the connections on both sides:

$$w_{\mathcal{Y}}^{-1} R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E})) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y} \to R^i f_{k*}(\mathrm{dR}_{X/Y}(w_{\mathcal{X}}^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})). \tag{5A.10}$$

We claim that (5A.10) is a filtered isomorphism. This is a local question, we may and do assume that $\mathcal{Y} = \mathrm{Spf}(A)$ is affine and is étale over some torus over $\mathcal{O}_k$. Let $V \in Y_{\mathrm{pro\acute{e}t}}$ be an affinoid perfectoid proétale over $\tilde{Y}_{\bar{k}}$. As $R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E})) = R^i f_{\mathrm{cris}\,*}(\mathcal{E})$ is a locally free $\mathcal{O}_{\mathcal{Y}}[1/p]$-module on $\mathcal{Y}$, we have

$$(w_{\mathcal{Y}}^{-1} R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E})) \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},Y})(V) \simeq H^i(\mathcal{X}, \mathrm{dR}_{X/Y}(\mathcal{E})) \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V).$$

So we only need to check that the natural morphism below is a filtered isomorphism

$$H^i(\mathcal{X}, \mathrm{dR}_{X/Y}(\mathcal{E})) \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V) \to H^i(X_V, \mathrm{dR}_{X/Y}(w_{\mathcal{X}}^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})).$$

By Lemma 5.3, one has further identifications strictly compatible with filtrations:

$$\begin{aligned} H^i(X_V, \mathrm{dR}_{X/Y}(w_{\mathcal{X}}^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X})) &\simeq H^i(\mathcal{X}, Rw_{V*}(\mathrm{dR}_{X/Y}(w_{\mathcal{X}}^{-1}\mathcal{E} \otimes \mathcal{O}\mathbb{B}_{\mathrm{cris},X}))) \\ &\simeq H^i(\mathcal{X}, \mathrm{dR}_{X/Y}(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)). \end{aligned}$$

Write $\hat{V} = \mathrm{Spa}(R, R^+)$. So $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \xrightarrow{\sim} \mathcal{O}\mathbb{B}_{\mathrm{cris},Y}(V)$ by Corollary 2.19. Thus, to prove our claim, it suffices to show that the canonical morphism

$$H^i(\mathcal{X}, \mathrm{dR}_{X/Y}(\mathcal{E})) \otimes_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+(R, R^+) \to H^i(\mathcal{X}, \mathrm{dR}_{X/Y}(\mathcal{E} \otimes \mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}^+(R, R^+)))$$

is a filtered isomorphism. One only needs to check this on the gradeds. Since the gradeds of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}^+(R, R^+)$ are finite free R-modules, we are reduced to showing that for every $r \in \mathbb{Z}$, the natural map

$$H^i(\mathcal{X}, \mathrm{gr}^r \mathrm{dR}_{X/Y}(\mathcal{E})) \otimes_A R \to H^i(\mathcal{X}, \mathrm{gr}^r(\mathrm{dR}_{X/Y}(\mathcal{E})) \otimes \mathcal{O}_{\mathcal{X}} \widehat{\otimes}_A R) \tag{5A.11}$$

is an isomorphism. This follows from Proposition B.3 by taking $B = R^+$. More precisely, let $\mathcal{F}$ be a bounded complex of coherent sheaves on $\mathcal{X}$, such that $\mathcal{F}[1/p] = \mathrm{gr}^r \mathrm{dR}_{X/Y}(\mathcal{E})$. So, for each term $\mathcal{F}^i$ of $\mathcal{F}$, $\mathcal{F}^i[1/p]$ is locally a direct factor of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]$-module. In particular, the kernel and cokernel of the natural map $\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A R^+\to\mathcal{F}\widehat{\otimes}_A R^+$ is killed by some bounded power of p. So

$$H^i(\mathcal{X},\mathcal{F}\otimes\mathcal{O}_{\mathcal{X}}\widehat{\otimes}_A R^+)[1/p]\xrightarrow{\sim}H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A R^+)[1/p].\tag{5A.12}$$

Moreover, by (1), $R^i f_*\operatorname{gr}^r\mathrm{dR}_{X/Y}(\mathcal{E})$ is a graded piece of the filtered isocrystal

$$R^i f_{\mathrm{cris}\,*}(\mathcal{E})=R^i f_*(\mathrm{dR}_{X/Y}(\mathcal{E})).$$

As $\mathcal{Y}=\mathrm{Spf}(A)$ is affine, taking global sections, we see that $H^i(\mathcal{X},\mathrm{dR}_{X/Y}(\mathcal{E}))$ is projective over $A[1/p]$, and $H^i(\mathcal{X},\operatorname{gr}^r\mathrm{dR}_{X/Y}(\mathcal{E}))$ is locally a direct factor of $H^i(\mathcal{X},\mathrm{dR}_{X/Y}(\mathcal{E}))$. Therefore,

$$H^i(\mathcal{X},\operatorname{gr}^r(\mathrm{dR}_{X/Y}(\mathcal{E})))=H^i(\mathcal{X},\mathcal{F})[1/p]$$

is flat over $A[1/p]$. By Proposition B.3, the kernel and the cokernel of the map

$$H^i(\mathcal{X},\mathcal{F})\otimes_A R^+\to H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A R^+)$$

are killed by some power of p. Inverting p and combining (5A.12), we obtain that (5A.11) is an isomorphism, completing the proof of our claim.

Composing the isomorphisms in (5A.9) with the inverse of (5A.10), we get the desired filtered isomorphism (5A.8) that is compatible with connections on both sides. It remains to check the Frobenius compatibility of (5A.8). For this, we may and do assume again that $\mathcal{Y}=\mathrm{Spf}(A)$ is affine and is étale over some torus over $\mathcal{O}_k$, and let $V\in Y_{\text{proét}}$ some affinoid perfectoid proétale over $\tilde{Y}$. In particular, A admits a lifting of the Frobenius on $\mathcal{Y}_0$, denoted by σ. Let $\mathcal{M}$ be an F-crystal on $\mathcal{X}_0/\mathcal{O}_k$ such that $\mathcal{E}=\mathcal{M}^{\mathrm{an}}(n)$ for some $n\in\mathbb{N}$ (Remark 3.4). Then the crystalline cohomology $H^i_{\mathrm{cris}}(\mathcal{X}_0/A,\mathcal{M})$ is endowed with a Frobenius which is σ-semilinear. We just need to check the Frobenius compatibility of composition of the maps below (here the last one is induced by the inverse of (5A.8)):

$$\begin{aligned}H^i_{\mathrm{cris}}(\mathcal{X}_0/A,\mathcal{M})\to H^i_{\mathrm{cris}}(\mathcal{X}_0/A,\mathcal{M})[1/p]\xrightarrow{\sim}H^i(\mathcal{X},\mathrm{dR}_{X/Y}(\mathcal{E}))\\ \to(w_{\mathcal{Y}}^{-1}(R^i f_{\mathrm{cris}\,*}(\mathcal{E}))\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris},Y})(V)\to H^i(X_V,\mathbb{L}\otimes\mathbb{A}_{\mathrm{cris},X}\widehat{\otimes}f_k^{-1}\mathcal{O}\mathbb{A}_{\mathrm{cris},Y}[1/t]),\end{aligned}$$

which can be done in the same way as in the proof of Theorem 3.26. □

The relative crystalline comparison theorem then can be stated as follows:

Theorem 5.5. *Let $\mathbb{L}$ be a crystalline lisse $\hat{\mathbb{Z}}_p$-sheaf on X associated to a filtered F-isocrystal $\mathcal{E}$ on $\mathcal{X}_0/\mathcal{O}_k$. Assume that, for any $i\in\mathbb{Z}$, $R^i f_{k*}\mathbb{L}$ is a lisse $\hat{\mathbb{Z}}_p$-sheaf on Y. Then $R^i f_{k*}\mathbb{L}$ is crystalline and is associated to the filtered convergent F-isocrystal $R^i f_{\mathrm{cris}\,*}\mathcal{E}$.*

Proof. We have seen in Proposition 5.4(1) that $R^i f_{\mathrm{cris}\,*}\mathcal{E}$ is a filtered convergent F-isocrystal on $\mathcal{Y}_0/\mathcal{O}_k$. To complete the proof, we need to find filtered isomorphisms that are compatible with Frobenius and connections

$$R^i f_{k*}(\mathbb{L})\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris},Y}\xrightarrow{\sim}w_{\mathcal{Y}}^{-1}R^i f_{\mathrm{cris}\,*}(\mathcal{E})\otimes\mathcal{O}\mathbb{B}_{\mathrm{cris},Y},\quad i\in\mathbb{Z}.$$

For this, it suffices to combine Proposition 5.2 and Proposition 5.4(2). □

Appendix A: Geometric acyclicity of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$

In this section, we extend the main results of [Andreatta and Brinon 2013] to the setting of perfectoids. The generalization is rather straightforward. Although one may see here certain difference from the arguments in [loc. cit.], the strategy and technique are entirely theirs.

Let $f: \mathcal{X} = \mathrm{Spf}(B) \to \mathcal{Y} = \mathrm{Spf}(A)$ be a smooth morphism of smooth affine formal schemes over $\mathcal{O}_k$. Write X and Y for the generic fiber of $\mathcal{X}$ and $\mathcal{Y}$. By abuse of notation the morphism $X \to Y$ induced from f is still denoted by f.

Assume that $\mathcal{Y}$ is étale over the torus $\mathcal{S} := \mathrm{Spf}(\mathcal{O}_k\{S_1^{\pm 1}, \dots, S_\delta^{\pm 1}\})$ defined over $\mathcal{O}_k$ and that the morphism $f: \mathcal{X} \to \mathcal{Y}$ can factor as

$$\mathcal{X} \xrightarrow{\text{étale}} \mathcal{T} \to \mathcal{Y},$$

where $\mathcal{T} = \mathrm{Spf}(C)$ is a torus over $\mathcal{Y}$ and the first morphism $\mathcal{X} \to \mathcal{T}$ is étale.

Write $C = A\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}$. For each $1 \le i \le d$ (resp. each $1 \le j \le \delta$), let $\{T_i^{1/p^n}\}_{n \in \mathbb{N}}$ (resp. $\{S_j^{1/p^n}\}_{n \in \mathbb{N}}$) be a compatible family of p-power roots of T_i (resp. of S_j). As in Proposition 2.13, we denote by $\tilde{Y}$ the following fiber product over the generic fiber $\mathcal{S}_k$ of $\mathcal{S}$:

$$\tilde{Y} = Y \times_{\mathcal{S}_k} \mathrm{Spa}(k\{S_1^{\pm 1/p^\infty}, \dots, S_\delta^{\pm 1/p^\infty}\}, \mathcal{O}_k\{S_1^{\pm 1/p^\infty}, \dots, S_\delta^{\pm 1/p^\infty}\}).$$

Let $V \in Y_{\mathrm{proét}}$ be an affinoid perfectoid over $\tilde{Y}_{\bar{k}}$ with $\hat{V} = \mathrm{Spa}(R, R^+)$. Let $T_V = \mathrm{Spa}(S, S^+)$ be the base change $\mathcal{T}_k \times_Y V$ and $X_V = \mathrm{Spa}(\tilde{S}, \tilde{S}^+)$ the base change $X \times_Y V$. Thus $S^+ = R^+\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\}$ and $S = S^+[1/p]$. Set

$$S_\infty^+ = R^+\{T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\}, \quad \tilde{S}_\infty^+ := B \widehat{\otimes}_C S_\infty^+,$$

$S_\infty := S_\infty^+[1/p]$ and $\tilde{S}_\infty := \tilde{S}_\infty^+[1/p]$. Then (S_∞, S_∞^+) and $(\tilde{S}_\infty, \tilde{S}_\infty^+)$ are affinoid perfectoids and

$$S_\infty^{\flat +} = R^{\flat +}\{(T_1^\flat)^{\pm 1/p^\infty}, \dots, (T_d^\flat)^{\pm 1/p^\infty}\},$$

where $T_i^\flat := (T_i, T_i^{1/p}, T_i^{1/p^2}, \dots) \in S_\infty^{\flat +}$. The inclusions $S^+ \subset S_\infty^+$ and $\tilde{S}^+ \subset \tilde{S}_\infty^+$ define two profinite Galois covers. Their Galois groups are the same, denoted by Γ, which is a profinite group isomorphic to $\mathbb{Z}_p(1)^d$. One can summarize these notations in the following commutative diagram

$$\begin{tikzcd}
\tilde{S}_\infty^+ & \tilde{S}^+ \arrow[l, "\Gamma"'] & B \arrow[l] \\
S_\infty^+ \arrow[u] \arrow[d, equal] & S^+ \arrow[l, "\Gamma"'] \arrow[u] \arrow[d, equal] & C \arrow[l] \arrow[u, "\text{étale}"'] \arrow[d, equal] \\
R^+\{T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\} & R^+\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\} \arrow[l, "\Gamma"'] & A\{T_1^{\pm 1}, \dots, T_d^{\pm 1}\} \arrow[l] \\
 & R^+ \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

The group Γ acts naturally on the period ring $\mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)$ and on its filtration $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)$. The aim of this appendix is to compute the group cohomology

$$H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)) := H^q_{\mathrm{cont}}(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))[1/t]$$

and

$$H^q(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)) := \varinjlim_{n\geq |r|} H^q_{\mathrm{cont}}\big(\Gamma, \tfrac{1}{t^n}\,\mathrm{Fil}^{r+n}\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)\big)$$

for $q, r \in \mathbb{Z}$.

In the following, we will omit systematically the subscript "cont" whenever there is no confusion arising. Moreover, we shall use multiindices to simplify the notation: for example, for $\underline{a} = (a_1, \ldots, a_d) \in \mathbb{Z}[1/p]^d$, $T^{\underline{a}} := T_1^{a_1} \cdot T_2^{a_2} \cdots T_d^{a_d}$.

A1. *Cohomology of $\mathcal{O}\mathbb{B}_{\mathrm{cris}}$.* We will first compute $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$ up to $(1-[\epsilon])^\infty$-torsion for all $q, n \in \mathbb{N}$ (Corollary A.9). From these computations, we deduce from the results about the cohomology groups $H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$, $q \in \mathbb{Z}$ (Theorem A.12).

Lemma A.1. *For $n \in \mathbb{Z}_{\geq 1}$, there are natural isomorphisms*

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \otimes_{W(R^{\flat+})/p^n} W(S_\infty^{\flat+})/p^n \xrightarrow{\sim} \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$$

and

$$(\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n \otimes \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)\langle u_1, \ldots, u_d\rangle \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n,$$

sending u_i to $T_i - [T_i^\flat]$. Here the tensor product in the second isomorphism above is taken over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$. Moreover, the natural morphisms

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n, \quad \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$$

are both injective.

Proof. Recall $\xi = [p^\flat] - p$. We know that $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$ is the p-adic completion of

$$\mathbb{A}^0_{\mathrm{cris}}(S_\infty, S_\infty^+) := W(S_\infty^{\flat+})\left[\frac{\xi^m}{m!} \mid m = 0, 1, \ldots\right] = \frac{W(S_\infty^{\flat+})[X_0, X_1, \ldots]}{(m!X_m - \xi^m : m \in \mathbb{Z}_{\geq 0})}.$$

Note that we have the same expression with R in place of S_∞. We then have

$$\mathbb{A}^0_{\mathrm{cris}}(S_\infty, S_\infty^+) \xleftarrow{\sim} W(S_\infty^{\flat+}) \otimes_{W(R^{\flat+})} \frac{W(R^{\flat+})[X_0, X_1, \ldots]}{(m!X_m - \xi^m \mid m \in \mathbb{Z}_{\geq 0})} = W(S_\infty^{\flat+}) \otimes_{W(R^{\flat+})} \mathbb{A}^0_{\mathrm{cris}}(R, R^+).$$

The first isomorphism follows.

Secondly, as V lies above $\tilde{Y}_{\bar{k}}$, by Proposition 2.13 we have

$$\mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle w_1, \ldots, w_\delta\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+), \quad w_j \mapsto S_j - [S_j^\flat]$$

where $S_j^\flat := (S_j, S_j^{1/p}, S_j^{1/p^2}, \ldots) \in R^{\flat+}$. Similarly, we have

$$\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)\{\langle u_1, \ldots, u_d, w_1, \ldots, w_\delta\rangle\} \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+), \quad u_i \mapsto T_i - [T_i^\flat],\ w_j \mapsto S_j - [S_j^\flat].$$

Thus (the isomorphisms below are all the natural ones)

$$\begin{aligned}\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n} &\xleftarrow{\sim} \left(\frac{\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n}\right)\langle u_1, \dots, u_d, w_1, \dots, w_\delta\rangle \\ &\xleftarrow{\sim} \frac{\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n} \otimes_{\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n} \left(\frac{\mathbb{A}_{\mathrm{cris}}(R, R^+)}{p^n}\langle u_1, \dots, u_d, w_1, \dots, w_\delta\rangle\right) \\ &\xrightarrow{\sim} \frac{\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n} \otimes_{\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n} \left(\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)}{p^n}\langle u_1, \dots, u_d\rangle\right) \\ &\xrightarrow{\sim} \left(\frac{\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n} \otimes_{\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n} \frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)}{p^n}\right)\langle u_1, \dots, u_d\rangle.\end{aligned}$$

So our second isomorphism is obtained.

Next we prove that the natural morphism $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is injective. When $n = 1$, we are reduced to showing the injectivity of

$$\frac{(R^{\flat+}/(p^\flat)^p)[X_1, X_2, \dots]}{(X_1^p, X_2^p, \dots)} \to \frac{(S^{\flat+}/(p^\flat)^p)[X_1, X_2, \dots]}{(X_1^p, X_2^p, \dots)},$$

or equivalently the injectivity of

$$R^{\flat+}/(p^\flat)^p \to S^{\flat+}/(p^\flat)^p = (R^{\flat+}/(p^\flat)^p)[(T_1^\flat)^{\pm 1/p^\infty}, \dots, (T_d^\flat)^{\pm 1/p^\infty}],$$

which is clear. The general case follows easily since $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$ is p-torsion free. One deduces also the injectivity of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,^+)/p^n \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ by using the natural isomorphisms

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \simeq (\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)\langle w_1, \dots, w_\delta\rangle,$$

and

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n \simeq (\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)\langle u_1, \dots, u_d, w_1, \dots, w_\delta\rangle.$$

This concludes the proof of our lemma. □

Proposition A.2. *$\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is free over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$ with a basis given by $\{[T^\flat]^{\underline{a}} \mid \underline{a} \in \mathbb{Z}[1/p]^d\}$.*

Proof. By Lemma A.1, $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is generated over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$ by elements of the form $[x]$ with $x \in S_\infty^{\flat+} = R^{\flat+}\{(T_1^\flat)^{\pm 1/p^\infty}, \dots, (T_d^\flat)^{\pm 1/p^\infty}\}$. Write $B_n \subset \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ for the $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$-submodule generated by elements of the form $[x]$ with $x \in \mathsf{S} := R^{\flat+}[(T_1^\flat)^{\pm 1/p^\infty}, \dots, (T_d^\flat)^{\pm 1/p^\infty}] \subset S_\infty^{\flat+}$. We claim that $B_n = \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$.

Since $S_\infty^{\flat+}$ is the $p^\flat$-adic completion of S, for each $x \in S^{\flat+}$ we can write $x = y_0 + p^\flat x'$ with $x' \in \mathsf{S}$. Iteration yields

$$x = y_0 + p^\flat y_1 + \cdots + (p^\flat)^{p-1} y_{p-1} + (p^\flat)^p x''$$

with $y_i \in \mathsf{S}$ and $x'' \in S_\infty^{\flat+}$. Then in $W(S_\infty^{\flat+})$

$$\begin{aligned}[x] &\equiv [y_0] + [p^\flat][y_1] + \cdots + [(p^\flat)^{p-1}][y_{p-1}] + [(p^\flat)^p][x''] \bmod pW(S_\infty^{\flat+}) \\ &\equiv [y_0] + \xi[y_1] + \cdots + \xi^{p-1}[y_{p-1}] + \xi^p[x''] \bmod pW(S_\infty^{\flat+}).\end{aligned}$$

As $\xi \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$ has divided power, $\xi^p = p! \cdot \xi^{[p]} \in p\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$. So we obtain in $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$

$$[x] \equiv [y_0] + \xi[y_1] + \cdots + \xi^{p-1}[y_{p-1}] \bmod p\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

For any $\alpha \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n = \mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \otimes_{W(R^{\flat+})/p^n} W(S_\infty^{\flat+})/p^n$, we may write

$$\alpha = \sum_{i=0}^{m} \lambda_i [x_i] + p\alpha', \quad x_i \in S_\infty^{\flat+}, \lambda_i \in \mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n, \alpha' \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

The observation above tells us that one can write

$$\alpha = \beta_0 + p\alpha'', \quad \beta_0 \in B_n, \alpha'' \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

By iteration again, we find

$$\alpha = \beta_0 + p\beta_1 + \cdots + p^{n-1}\beta_{n-1} + p^n\tilde{\alpha}, \quad \beta_0, \ldots, \beta_{n-1} \in B_n, \tilde{\alpha} \in \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

Thus

$$\alpha = \beta_0 + p\beta_1 + \cdots + p^{n-1}\beta_{n-1} \in B_n \subset \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n.$$

This shows the claim, i.e., $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is generated over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$ by the elements of the form $[x]$ with $x \in \mathsf{S} = R^{\flat+}[(T_1^\flat)^{\pm 1/p^\infty}, \ldots, (T_d^\flat)^{\pm 1/p^\infty}] \subset S_\infty^{\flat+}$. Furthermore, as for any $x, y \in S_\infty^{\flat+}$

$$[x+y] \equiv [x] + [y] \bmod pW(S_\infty^{\flat+}),$$

a similar argument shows that $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ is generated over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$ by the family of elements $\{[T^\flat]^{\underline{a}} \mid \underline{a} \in \mathbb{Z}[1/p]^d\}$.

It remains to show the freeness of the family $\{[T^\flat]^{\underline{a}} \mid \underline{a} \in \mathbb{Z}[1/p]^d\}$ over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$. For this, suppose there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$ and distinct elements $\underline{a}_1, \ldots, \underline{a}_m \in \mathbb{Z}[1/p]^d$ such that

$$\sum_{i=1}^{m} \lambda_i [T^\flat]^{\underline{a}_i} \in p^n\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

One needs to prove $\lambda_i \in p^n\mathbb{A}_{\mathrm{cris}}(R, R^+)$ for each i. Modulo p we find

$$\sum_{i=1}^{m} \bar{\lambda}_i \cdot (T^\flat)^{\underline{a}_i} = 0 \quad \text{in } \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p,$$

with $\bar{\lambda}_i \in \mathbb{A}_{\mathrm{cris}}(R, R^+)/p$ the reduction modulo p of λ_i. On the other hand, the family of elements $\{(T^\flat)^{\underline{a}} : \underline{a} \in \mathbb{Z}[1/p]^d\}$ in

$$\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p \simeq \frac{S^{\flat+}/((p^\flat)^p)[\delta_2, \delta_3, \ldots]}{(\delta_2^p, \delta_3^p, \ldots)} \simeq \frac{R^{\flat+}/((p^\flat)^p)[(T_1^\flat)^{\pm 1/p^\infty}, \ldots, (T_d^\flat)^{\pm 1/p^\infty}, \delta_2, \delta_3, \ldots]}{(\delta_2^p, \delta_3^p, \ldots)}$$

is free over $\mathbb{A}_{\mathrm{cris}}(R, R^+)/p \simeq (R^{\flat+}/((p^\flat)^p)[\delta_2, \delta_3, \ldots])/(\delta_2^p, \delta_3^p, \ldots)$. Therefore, $\bar{\lambda}_i = 0$, or equivalently, $\lambda_i = p\lambda_i'$ for some $\lambda_i' \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$. In particular,

$$\sum_{i=1}^m \lambda_i [T^\flat]^{\underline{a}_i} = p \cdot \left(\sum_{i=1}^m \lambda_i' [T^\flat]^{\underline{a}_i} \right) \in p^n \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

But $\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$ is p-torsion free, which implies that

$$\sum_{i=1}^m \lambda_i' [T^\flat]^{\underline{a}_i} \in p^{n-1} \mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+).$$

This way, we may find $\lambda_i = p^n \tilde{\lambda}_i$ for some $\tilde{\lambda}_i \in \mathbb{A}_{\mathrm{cris}}(R, R^+)$, which concludes the proof of the freeness. □

Recall that Γ is the Galois group of the profinite cover (S_∞, S_∞^+) of (S, S^+). Let $\epsilon = (\epsilon^{(0)}, \epsilon^{(1)}, \ldots) \in \mathcal{O}_{\hat{\bar{k}}}^\flat$ be a compatible system of p-power roots of unity such that $\epsilon^{(0)} = 1$ and that $\epsilon^{(1)} \neq 1$. Let $\{\gamma_1, \ldots, \gamma_d\}$ be a family of generators such that for each $1 \le i \le d$, γ_i acts trivially on the variables T_j for any index j different from i and that $\gamma_i(T_i^\flat) = \epsilon T_i^\flat$.

Lemma A.3. *Let $1 \le i \le d$ be an integer. Then one has $\gamma_i([T_i^\flat]^{p^n}) = [T_i^\flat]^{p^n}$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$.*

Proof. By definition, $\gamma_i([T_i^\flat]^{p^n}) = [\epsilon]^{p^n}[T_i^\flat]^{p^n}$ in $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)$. So our lemma follows from the fact that $[\epsilon]^{p^n} - 1 = \exp(p^n t) - 1 = \sum_{r \ge 1} p^{nr} t^{[r]} \in p^n A_{\mathrm{cris}}$. □

Let A_n be the $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$-subalgebra of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ generated by $[T_i^\flat]^{\pm p^n}$ for $1 \le i \le d$. The previous lemma shows that Γ acts trivially on A_n. Furthermore, by the second isomorphism of Lemma A.1 and by Proposition A.2, we have

$$\frac{\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)}{p^n} \xrightarrow{\sim} \Bigg(\bigoplus_{\underline{a} \in \mathbb{Z}[1/p]^d \cap [0, p^n)^d} A_n [T^\flat]^{\underline{a}} \Bigg) \langle u_1, \ldots, u_d \rangle, \quad T_i - [T_i^\flat] \mapsto u_i.$$

Transport the Galois action of Γ on $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ to the right-hand side of this isomorphism. It follows that

$$\gamma_i(u_i) = u_i + (1 - [\epsilon])[T_i^\flat].$$

Therefore,

$$\gamma_i(u_i^{[n]}) = u_i^{[n]} + \sum_{j=1}^n [T_i^\flat]^j (1 - [\epsilon])^{[j]} u_i^{[n-j]}.$$

For other index $j \neq i$, $\gamma_i(u_j) = u_j$ and hence $\gamma_i(u_j^{[n]}) = u_j^{[n]}$ for any n. Set

$$\mathbf{X}_n := \bigoplus_{\underline{a} \in (\mathbb{Z}[1/p] \cap [0, p^n))^d \setminus \mathbb{Z}^d} A_n[T^\flat]^{\underline{a}}, \quad \text{and} \quad \mathbf{A}_n := \bigoplus_{\underline{a} \in \mathbb{Z}^d \cap [0, p^n)^d} A_n[T^\flat]^{\underline{a}}.$$

Then we have the following decomposition, which respects the Γ-actions:

$$\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n = \mathbf{X}_n\langle u_1, \dots, u_d\rangle \oplus \mathbf{A}_n\langle u_1, \dots, u_d\rangle.$$

The following result can be proven similarly as [Andreatta and Brinon 2013, Proposition 16].

Proposition A.4. *For any $q \in \mathbb{N}$, $H^q(\Gamma, \mathbf{X}_n\langle u_1, \dots, u_d\rangle)$ is killed by $(1 - [\epsilon]^{1/p})^2$.*

The computation of $H^q(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$ is more subtle. Note that we have the following decomposition

$$\mathbf{A}_n\langle u_1, \dots, u_d\rangle = \bigotimes_{i=1}^{d} (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[[T_i^\flat]^{\pm 1}]\langle u_i\rangle,$$

where the tensor products above are taken over $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n$.

We shall first treat the case where $d = 1$. We set $T := T_1$, $u := u_1$ and $\gamma := \gamma_1$. Let $\mathcal{A}_n^{(m)}$ be the A_n-submodule of $\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n$ generated by the $u^{[m+a]}/[T^\flat]^a$'s with $m + a \geq 0$ and $0 \leq a < p^n$. Then

$$\mathbf{A}_n\langle u\rangle = A_n[[T^\flat]]\langle u\rangle = \sum_{m > -p^n} \mathcal{A}_n^{(m)}.$$

Consider the following complex:

$$A_n[[T^\flat]]\langle u\rangle \xrightarrow{\gamma - 1} A_n[[T^\flat]]\langle u\rangle, \tag{A1.1}$$

which computes $H^q(\Gamma, \mathbf{A}_n\langle u\rangle) = H^q(\Gamma, A_n[[T^\flat]]\langle u\rangle)$.

Again, the following lemma can be proven in a completely analogous way as is done in the proof of [Andreatta and Brinon 2013, Proposition 20].

Lemma A.5. *The cokernel of* (A1.1), *and hence $H^q(\Gamma, \mathbf{A}_n\langle u\rangle)$ for any $q > 0$, are killed by* $1 - [\epsilon]$.

One still needs to compute $H^0(\Gamma, \mathbf{A}_n\langle u\rangle)$. Note first that we have the following isomorphism

$$(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[T^{\pm 1}]\langle u\rangle \xrightarrow{\sim} \mathbf{A}_n\langle u\rangle = (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[[T^\flat]]\langle u\rangle$$

sending T to $u + [T^\flat]$. Endow an action of Γ on $(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[T^{\pm 1}]\langle u\rangle$ via the isomorphism above. So $H^0(\Gamma, \mathbf{A}_n\langle u\rangle)$ is naturally isomorphic to the kernel of the morphism

$$\gamma - 1 \colon (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[T^{\pm 1}]\langle u\rangle \to (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[T^{\pm 1}]\langle u\rangle,$$

and there is a natural injection

$$C \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n = (\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n)[T^{\pm 1}] \hookrightarrow H^0(\Gamma, \mathbf{A}_n\langle u\rangle).$$

The proof of [Andreatta and Brinon 2013, Lemme 29] applies to this case. We have

Lemma A.6. *The cokernel of the last map is killed by* $1-[\epsilon]$.

Now we are ready to prove

Proposition A.7. *For any* $d>0$, $n>0$ *and* $q>0$, $H^q(\Gamma, \mathbf{A}_n\langle u_1,\dots,u_d\rangle)$ *is killed by* $(1-[\epsilon])^{2d-1}$. *Moreover, the natural morphism*

$$C\otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n \to H^0(\Gamma, \mathbf{A}_n\langle u_1,\dots,u_d\rangle),\quad T_i\mapsto u_i+[T_i^\flat] \tag{A1.2}$$

is injective with cokernel killed by $(1-[\epsilon])^{2d-1}$.

Proof. Recall that we have the decomposition

$$\mathbf{A}_n\langle u_1,\dots,u_d\rangle=\bigotimes_{i=1}^{d}(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[[T_i^\flat]^{\pm 1}]\langle u_i\rangle.$$

We shall proceed by induction on d. The case $d=1$ comes from the previous two lemmas. For integer $d>1$, one uses the Hochschild–Serre spectral sequence

$$E_2^{i,j}=H^i(\Gamma/\Gamma_1, H^j(\Gamma_1,\mathbf{A}_n\langle u_1,\dots,u_d\rangle))\Longrightarrow H^{i+j}(\Gamma,\mathbf{A}_n\langle u_1,\dots,u_d\rangle).$$

Using the decomposition above, the group $H^j(\Gamma_1,\mathbf{A}_n\langle u_1,\dots,u_d\rangle)$ is isomorphic to

$$H^j(\Gamma_1,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n[[T_1^\flat]^{\pm1}]\langle u_1\rangle)\otimes(\otimes_{i=2}^d(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[[T_i^\flat]^{\pm1}]\langle u_i\rangle).$$

So by the calculation done for the case $d=1$, we find that, up to $(1-[\epsilon])$-torsion, $H^j(\Gamma_1, A_n\langle u_1,\dots,u_d\rangle)$ is zero when $j>0$, and is equal to

$$(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R+)/p^n)[T_1^{\pm1}]\otimes(\otimes_{i=2}^d(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[[T_i^\flat]^{\pm1}]\langle u_i\rangle)$$

when $j=0$. Thus, up to $(1-[\epsilon])$-torsion, $E_2^{i,j}=0$ when $j>0$ and $E_2^{i,0}$ is equal to

$$(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R+)/p^n)[T_1^{\pm1}]\otimes H^i(\Gamma/\Gamma_1,(\otimes_{i=2}^d(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[[T_i^\flat]^{\pm1}]\langle u_i\rangle)).$$

Using the induction hypothesis, we get that, up to $(1-[\epsilon])^{2(d-1)-1+1}$-torsion, $E_2^{i,0}=0$ when $i>0$ and

$$E_2^{0,0}=(\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n)[T_1^{\pm1},\dots,T_d^{\pm1}]=C\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n.$$

As $E_2^{i,j}=0$ for $j>1$, we have short exact sequence

$$0\to E_\infty^{q,0}\to H^q(\Gamma,\mathbf{A}_n\langle u_1,\dots,u_d\rangle)\to E_\infty^{q-1,1}\to 0.$$

By what we have shown above, $E_\infty^{q-1,1}$ is killed by $(1-[\epsilon])$ (as this is already the case for $E_2^{q-1,1}$), and $E_\infty^{q,0}$ is killed by $(1-[\epsilon])^{2d-2}$ for $q>0$ (as this is the case for $E_2^{q,0}$), thus $H^q(\Gamma,\mathbf{A}_n\langle u_1,\dots,u_d\rangle)$ is killed by $(1-[\epsilon])^{2d-1}$. For $q=0$, $H^0(\Gamma,\mathbf{A}_n\langle u_1,\dots,u_d\rangle)\simeq E_\infty^{0,0}=E_2^{0,0}$. So the cokernel of the natural injection

$$C\otimes_A\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R,R^+)/p^n\to H^0(\Gamma,\mathbf{A}_n\langle u_1,\dots,u_d\rangle)$$

is killed by $(1-[\epsilon])^{2d-2}$, hence by $(1-[\epsilon])^{2d-1}$. □

Remark A.8. With more efforts, one may prove that $H^q(\Gamma, \mathbf{A}_n\langle u_1, \dots, u_d\rangle)$ is killed by $(1-[\epsilon])^d$ for $q>0$ [Andreatta and Brinon 2013, Proposition 21], and that the cokernel of the morphism (A1.2) is killed by $(1-[\epsilon])^2$ [Andreatta and Brinon 2013, Proposition 30].

Corollary A.9. *For any $n \geq 0$ and any $q>0$, $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$ is killed by $(1-[\epsilon])^{2d}$. Moreover, the natural morphism*

$$C\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$$

is injective, with cokernel killed by $(1-[\epsilon])^{2d}$.

Recall that we want to compute $H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$. For this, one needs

Lemma A.10. *Keep the notations above and assume that the morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is étale; thus $V = \mathrm{Spa}(R, R^+)$ and $X \times_Y V = \mathrm{Spa}(\tilde{S}, \tilde{S}^+) = \mathrm{Spa}(\tilde{S}_\infty, \tilde{S}_\infty^+)$. The natural morphism*

$$B\widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)$$

is an isomorphism.

Proof. By Lemma 2.18 we are reduced to showing that the natural map (here $w_j = S_j - [S_j^\flat]$)

$$B\widehat{\otimes}_A \mathbb{A}_{\mathrm{cris}}(R, R^+)\{\langle w_1, \dots, w_\delta\rangle\} \to \mathbb{A}_{\mathrm{cris}}(\tilde{S}, \tilde{S}^+)\{\langle w_1, \dots, w_\delta\rangle\}$$

is an isomorphism. Since both sides of the previous maps are p-adically complete and without p-torsion, we just need to check that its reduction modulo p

$$B/p \otimes_{A/p} (\mathbb{A}_{\mathrm{cris}}(R, R^+)/p)\langle w_1, \dots, w_\delta\rangle \to (\mathbb{A}_{\mathrm{cris}}(\tilde{S}, \tilde{S}^+)/p)\langle w_1, \dots, w_\delta\rangle$$

is an isomorphism. Note that we have the following expression

$$(\mathbb{A}_{\mathrm{cris}}(R, R^+)/p)\langle w_1, \dots, w_\delta\rangle \simeq \frac{(R^{\flat +}/((p^\flat)^p))[\delta_m, w_i, Z_{im}]_{1\leq i\leq \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^m, Z_{im}^p)_{1\leq i\leq \delta, m\in\mathbb{N}}},$$

and the similar expression for $(\mathbb{A}_{\mathrm{cris}}(\tilde{S}, \tilde{S}^+)/p)\langle w_1, \dots, w_\delta\rangle$, where δ_m is the image of $\gamma^{m+1}(\xi)$ with $\gamma : x \mapsto x^p/p$. One sees that both sides of the morphism above are $p^\flat$-adically complete (in fact $p^\flat$ is nilpotent). Moreover $R^{\flat +}$ has no $p^\flat$-torsion. So we are reduced to showing that the morphism

$$B/p \otimes_{A/p} \frac{(R^{\flat +}/p^\flat)[\delta_m, w_i, Z_{im}]_{1\leq i\leq \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\leq i\leq \delta, m\in\mathbb{N}}} \to \frac{(\tilde{S}^{\flat +}/p^\flat)[\delta_m, w_i, Z_{im}]_{1\leq i\leq \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\leq i\leq \delta, m\in\mathbb{N}}}$$

is an isomorphism. But $R^{\flat +}/p^\flat \simeq R^+/p$ and $\tilde{S}^{\flat +}/p^\flat \simeq \tilde{S}^+/p$, so we just need to show that the following morphism is an isomorphism:

$$\alpha : B/p \otimes_{A/p} \frac{(R^{+}/p)[\delta_m, w_i, Z_{im}]_{1\leq i\leq \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\leq i\leq \delta, m\in\mathbb{N}}} \to \frac{(\tilde{S}^{+}/p)[\delta_m, w_i, Z_{im}]_{1\leq i\leq \delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\leq i\leq \delta, m\in\mathbb{N}}}.$$

To see this, we consider the following diagram

$$\begin{tikzcd}
 & & \frac{\tilde{S}^+/p[\delta_m, w_i, Z_{im}]_{1\le i\le\delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le\delta, m\in\mathbb{N}}} . \\
B/p \arrow[r] \arrow[urr, bend left] & B/p\otimes_{A/p} \frac{R^+/p[\delta_m, w_i, Z_{im}]_{1\le i\le\delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le\delta, m\in\mathbb{N}}} \arrow[ur, "\alpha"] & \\
A/p \arrow[u, "\text{étale}"] \arrow[r] & \frac{R^+/p[\delta_m, w_i, Z_{im}]_{1\le i\le\delta, m\in\mathbb{N}}}{(\delta_m^p, w_i^p, Z_{im}^p)_{1\le i\le\delta, m\in\mathbb{N}}} \arrow[u, "\text{étale}"] \arrow[uur, bend right, "\text{étale}"'] &
\end{tikzcd}$$

It follows that α is étale. To see that α is an isomorphism, we just need to show that this is the case after modulo some nilpotent ideals of both sides of α. Hence we are reduced to showing that the natural morphism

$$B/p\otimes_{A/p} R^+/p \to \tilde{S}^+/p$$

is an isomorphism, which is clear from the definition. $\square$

Apply the previous lemma to the étale morphism $f\colon \mathcal{X}\to\mathcal{T}$, we find a canonical Γ-equivariant isomorphism

$$B\widehat{\otimes}_C\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+) \xrightarrow{\sim} \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+).$$

In particular, we find

$$H^q(\Gamma, B\widehat{\otimes}_C\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \xrightarrow{\sim} H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)).$$

Now consider the following spectral sequence

$$E_2^{i,j} = R^i \varprojlim_n H^j(\Gamma, B\otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n) \Longrightarrow H^{i+j}(\Gamma, B\widehat{\otimes}_C\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))$$

which induces a short exact sequence for each i:

$$\begin{aligned}0\to R^1\varprojlim_n H^{i-1}(\Gamma, B\otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n) &\to H^i(\Gamma, B\widehat{\otimes}_C\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))\\ &\to \varprojlim_n H^i(\Gamma, B\otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n)\to 0. \quad \text{(A1.3)}\end{aligned}$$

As B is flat over C, it can be written as a filtered limit of finite free C-modules by Lazard's theorem [1969, Théorème 1.2] and, as Γ acts trivially on B, the following natural morphism is an isomorphism for each i:

$$B\otimes_C H^i(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n) \xrightarrow{\sim} H^i(\Gamma, B\otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty)/p^n).$$

Therefore, for $i \geq 1$, $H^i(\Gamma, B \otimes_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)$ is killed by $(1-[\epsilon])^{2d}$ by Corollary A.9. Moreover, by the same corollary, we know that the following morphism is injective with cokernel killed by $(1-[\epsilon])^{2d}$:

$$C \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n).$$

Thus the same holds if we apply the functor $B \otimes_C -$ to the morphism above

$$B \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to B \otimes_C H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n).$$

Passing to limits we obtain an injective morphism

$$B \widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to \varprojlim_n (B \otimes_C H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n)),$$

whose cokernel is killed by $(1-[\epsilon])^{2d}$, and that

$$R^1 \varprojlim_n (B \otimes_C H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)/p^n))$$

is killed by $(1-[\epsilon])^{2d}$. As a result, using the short exact sequence (A1.3), we deduce that for $i \geq 1$, $H^i(\Gamma, B \widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \simeq H^i(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$ is killed by $(1-[\epsilon])^{4d}$, and that the canonical morphism

$$B \widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, B \widehat{\otimes}_C \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) \simeq H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$$

is injective with cokernel killed by $(1-[\epsilon])^{2d}$. One can summarize the calculations above as follows:

Proposition A.11. (i) *For any $n \geq 0$ and $q > 0$, $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)/p^n)$ is killed by $(1-[\epsilon])^{2d}$, and the natural morphism*

$$B \otimes_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)/p^n \to H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))/p^n$$

is injective with cokernel killed by $(1-[\epsilon])^{2d}$.

(ii) *For any $q > 0$, $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))$ is killed by $(1-[\epsilon])^{4d}$ and the natural morphism*

$$B \widehat{\otimes}_A \mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$$

is injective, with cokernel killed by $(1-[\epsilon])^{2d}$.

Theorem A.12. *Keep the notations above. Then $H^q(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)) = 0$ for $q \geq 1$, and the natural morphism*

$$B \widehat{\otimes}_A \mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, \mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+))$$

is an isomorphism.

Proof. By the previous proposition, we just need to remark that to invert $1-[\epsilon]$ one just needs to invert t, as

$$t = \log([\epsilon]) = -\sum_{n\geq 1}(n-1)! \cdot (1-[\epsilon])^{[n]} = -(1-[\epsilon])\sum_{n\geq 1}(n-1)! \cdot \frac{(1-[\epsilon])^{[n]}}{1-[\epsilon]}.$$

Here, by [Andreatta and Brinon 2013, Lemme 18], $(1-[\epsilon])^{[n]}/(1-[\epsilon]) \in \ker(A_{\mathrm{cris}} \xrightarrow{\theta} \hat{\mathcal{O}}_{\bar{k}})$, hence the last summation above converges in A_{cris}. □

A2. *Cohomology of* $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}$. With Proposition A.11 in hand, the following results on the cohomology of $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}$ can be shown in exactly the same way as is done in [Andreatta and Brinon 2013, Section 5]. We thus only state these results, and refer to [loc. cit.] for the detailed proofs.

Lemma A.13 [Andreatta and Brinon 2013, Proposition 32]. *Let $q \in \mathbb{N}_{>0}$, and $n \in \mathbb{Z}_{\geq 4d+r}$. The A_{cris}-module $H^q(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))$ is killed by t^n.*

Proof. Let $\mathrm{gr}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}} := \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}/\mathrm{Fil}^{r+1}\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}$. As $\theta(1-[\epsilon]) = 0$, $\mathrm{gr}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ is killed by $1-[\epsilon]$. So using the tautological short exact sequence below

$$0 \to \mathrm{Fil}^{r+1}\,\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to \mathrm{gr}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}} \to 0$$

and by induction on the integer $r \geq 0$, one shows that $H^q(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}})$ is killed by $(1-[\epsilon])^{4d+r}$; the $r = 0$ case being Proposition A.11(ii). So the multiplication-by-t^n with $n \geq 4d+r$ is zero for $H^q(\Gamma, \mathcal{O}\mathbb{A}_{\mathrm{cris}})$. □

Then, as in [Andreatta and Brinon 2013, Proposition 34], we have

Proposition A.14. $H^q(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(\tilde{S}_\infty, \tilde{S}_\infty^+)) = 0$ *for any* $q > 0$.

It remains to compute the Γ-invariants of $\mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S_\infty, S_\infty)$. We shall first show

$$H^0(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+)) = B\widehat{\otimes}_A \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$$

in the way of the proof of [Andreatta and Brinon 2013, Proposition 41].

Proposition A.15. *For each $r \in \mathbb{N}$, the natural injective map*

$$\iota_r\colon B\widehat{\otimes}_A \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(S_\infty, S_\infty^+))$$

is an isomorphism.

Corollary A.16. *The natural morphism*

$$B\widehat{\otimes}_A \mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) \to H^0(\Gamma, \mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(S_\infty, S_\infty^+))$$

is an isomorphism, where

$$B\widehat{\otimes}_A \mathrm{Fil}^r\,\mathcal{O}\mathbb{B}_{\mathrm{cris}}(R, R^+) := \varinjlim_{n\geq 0} B\widehat{\otimes}_A \mathrm{Fil}^{r+n}\,\mathcal{O}\mathbb{A}_{\mathrm{cris}}(R, R^+)$$

with transition maps given by multiplication by t.

Appendix B: Base change for cohomology of formal schemes

The aim of this appendix is to establish a base change result for cohomology of formal schemes (Proposition B.3) that is used in the proof of relative comparison theorem.

Lemma B.1. *Let A be a noetherian ring that is p-adically complete, i.e., the natural morphism $A \to \varprojlim A/p^n$ is an isomorphism. Let B be a flat A-module, and F a finite A-module. Then, if B is p-adically complete, so is $F \otimes_A B$.*

Proof. Let $B_n := B/p^n$. So $B \xrightarrow{\sim} \varprojlim_n B_n$. Since B is A-flat, B_n is flat over A/p^n. Moreover, the canonical map $B_{n+1} \to B_n$ is clearly surjective. Therefore, by [Stacks 2005–, Lemma 0912], for F a finite A-module, we have $F \otimes B \simeq \varprojlim_n (F \otimes B_n) \simeq \varprojlim_n (F \otimes B)/p^n$. In other words, $F \otimes B$ is p-adically complete. □

Lemma B.2. *Let A be a p-adically complete noetherian ring, and B a flat A-module that is p-adically complete. Let $\mathcal{X} \to \mathrm{Spf}(A)$ be a proper morphism of p-adic formal schemes. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then, for every $i \in \mathbb{Z}$, the natural morphism*

$$H^i(\mathcal{X}, \mathcal{F}) \otimes_A B \to H^i(\mathcal{X}, \mathcal{F} \widehat{\otimes}_A B)$$

is an isomorphism.

Proof. If $\mathcal{F}$ is a coherent sheaf annihilated by some power of p, $\mathcal{F} \widehat{\otimes}_A B = \mathcal{F} \otimes_A B$. Then, our proposition follows from the standard flat base change result. Indeed, by a theorem of Lazard [1969, Théorème 1.2], B can be written as a filtered inductive limit of finite free A-modules. As $\mathcal{X}$ is quasicompact and quasiseparated, one only needs to prove our assertion when B is finite and free over A. But in this case our assertion is obvious. In general, let $\mathcal{G} \subset \mathcal{F}$ be the subsheaf formed by elements killed by some power of p. As $\mathcal{F}$ is coherent and as $\mathcal{X}$ is quasicompact, $\mathcal{G}$ is killed by a large power of p. Therefore $\mathcal{G}$ and the quotient $\mathcal{F}/\mathcal{G}$ are coherent sheaves on $\mathcal{X}$. Since $\mathcal{F}/\mathcal{G}$ is p-torsion free, the same holds for $(\mathcal{F}/\mathcal{G}) \otimes_A B$ as B is flat over A. Using the tautological exact sequence

$$0 \to \mathcal{G} \otimes_A B \to \mathcal{F} \otimes_A B \to (F/\mathcal{G}) \otimes_A B \to 0, \tag{B0.1}$$

we get a short exact sequence of projective systems

$$0 \to (\mathcal{G} \otimes_A B/p^n)_{n\geq 0} \to (\mathcal{F} \otimes_A B/p^n)_{n\geq 0} \to ((\mathcal{F}/\mathcal{G}) \otimes_A B/p^n)_{n\geq 0} \to 0.$$

Because $p^n \mathcal{G} = 0$ and thus $p^n(\mathcal{G} \otimes_A B) = 0$ for large n, we find

$$\mathcal{G} \otimes_A B/p^{n+1} \xrightarrow{\sim} \mathcal{G} \otimes_A B/p^n, \quad \text{for } n \gg 0.$$

So $R^1 \varprojlim_n ((\mathcal{G} \otimes_A B)/p^n) = 0$. Passing to projective limits in (B0.1), we obtain a short exact sequence

$$0 \to \mathcal{G} \widehat{\otimes}_A B \to \mathcal{F} \widehat{\otimes}_A B \to (\mathcal{F}/\mathcal{G}) \widehat{\otimes}_A B \to 0,$$

from which we get a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(\mathcal{X},\mathcal{G})\otimes B \arrow[r] \arrow[d, "\simeq"] & H^i(\mathcal{X},\mathcal{F})\otimes B \arrow[r] \arrow[d, "\mathrm{can}"] & H^i(\mathcal{X},\mathcal{F}/\mathcal{G})\otimes B \arrow[r] \arrow[d, "\mathrm{can}"] & \cdots \\
\cdots \arrow[r] & H^i(\mathcal{X},\mathcal{G}\widehat{\otimes}B) \arrow[r] & H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}B) \arrow[r] & H^i(\mathcal{X},(\mathcal{F}/\mathcal{G})\widehat{\otimes}B) \arrow[r] & \cdots .
\end{tikzcd}$$

Consequently, to prove our proposition for $\mathcal{F}$, it suffices to show it for $\mathcal{F}/\mathcal{G}$. Therefore, replacing $\mathcal{F}$ by $\mathcal{F}/\mathcal{G}$ if needed, we assume that $\mathcal{F}$ is p-torsion free.

Let $\mathcal{F}_n = \mathcal{F}/p^n$. By flat base change, for all $n \geq 0$, the natural morphisms

$$H^i(\mathcal{X},\mathcal{F}_n)\otimes_A B \to H^i(\mathcal{X},\mathcal{F}_n\otimes_A B), \quad i\in\mathbb{Z},$$

are isomorphisms. Passing to projective limits, we obtain isomorphisms

$$\alpha\colon \varprojlim_n (H^i(\mathcal{X},\mathcal{F}_n)\otimes_A B) \to \varprojlim_n H^i(\mathcal{X},\mathcal{F}_n\otimes_A B), \quad i\in\mathbb{Z}.$$

As A is noetherian, the projective system $(H^{i-1}(\mathcal{X},\mathcal{F}_n))_{n\geq 0}$ satisfies the (ML)-condition [EGA III_1 1961, Corollaire 3.4.4], hence so does $(H^{i-1}(\mathcal{X},\mathcal{F}_n)\otimes_A B)_{n\geq 0}$. Thus

$$R^1\varprojlim_n (H^{i-1}(\mathcal{X},\mathcal{F}_n)\otimes_A B) = R^1\varprojlim_n H^{i-1}(\mathcal{X},\mathcal{F}_n\otimes_A B) = 0.$$

Using the set of affine open formal subschemes of $\mathcal{X}$ and [Scholze 2013, Lemma 3.18], one checks that $R^j\varprojlim_n(\mathcal{F}_n\otimes_A B)=0$ whenever $j>0$. So $H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A B)\simeq H^i(\mathcal{X}, R\varprojlim(\mathcal{F}_n\otimes_A B))$, and the natural morphism

$$\beta\colon H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}_A B)\to \varprojlim H^i(\mathcal{X},\mathcal{F}_n\otimes_A B).$$

is surjective with kernel isomorphic to $R^1\varprojlim_n H^{i-1}(\mathcal{X},\mathcal{F}_n\otimes_A B)=0$ (Lemma 4.1). In other words, β is an isomorphism.

Next, consider the tautological exact sequence (recall that $\mathcal{F}$ has no p-torsion)

$$0\to\mathcal{F}\xrightarrow{p^n}\mathcal{F}\to\mathcal{F}_n\to 0.$$

We obtain a short exact sequence

$$0\to H^i(\mathcal{X},\mathcal{F})/p^n\to H^i(\mathcal{X},\mathcal{F}_n)\to H^{i+1}(\mathcal{X},\mathcal{F})[p^n]\to 0,$$

and thus the one below as B is flat over A:

$$0\to H^i(\mathcal{X},\mathcal{F})\otimes_A B/p^n\xrightarrow{\gamma_n} H^i(\mathcal{X},\mathcal{F}_n)\otimes_A B\to H^{i+1}(\mathcal{X},\mathcal{F})[p^n]\otimes_A B\to 0.$$

Because $H^{i+1}(\mathcal{X},\mathcal{F})$ is an A-module of finite type and A is noetherian, the A-submodule $H^{i+1}(\mathcal{X},\mathcal{F})_{p\text{-tor}}\subset H^{i+1}(\mathcal{X},\mathcal{F})$ of elements killed by some power of p is finitely generated over A. In particular, there exists

$a \in \mathbb{N}$ such that p^a kills $H^{i+1}(\mathcal{X}, \mathcal{F})_{p\text{-tor}}$ and thus $H^{i+1}(\mathcal{X}, \mathcal{F})[p^n]$ for all $n \in \mathbb{N}$. It follows that the transition map below is trivial for every n:

$$H^{i+1}(\mathcal{X}, \mathcal{F})[p^{n+a}] \to H^{i+1}(\mathcal{X}, \mathcal{F})[p^n], \quad x \mapsto p^a x.$$

Thus the projective systems $(H^{i+1}(\mathcal{X}, \mathcal{F})[p^n])_n$ and $(H^{i+1}(\mathcal{X}, \mathcal{F})[p^n] \otimes_A B)_n$ satisfy the (ML)-condition. So one deduces an isomorphism $\gamma := \varprojlim \gamma_n$

$$\gamma \colon H^i(\mathcal{X}, \mathcal{F}) \otimes_A B = \varprojlim (H^i(\mathcal{X}, \mathcal{F}) \otimes_A B/p^n) \xrightarrow{\sim} \varprojlim (H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B).$$

Here we have the first equality because $H^{i+1}(\mathcal{X}, \mathcal{F})$ is of finite type over A, and B is flat and p-adically complete (Lemma B.1).

Finally from the commutative diagram

$$\begin{tikzcd} H^i(\mathcal{X}, \mathcal{F}) \otimes_A B \arrow[r, "\mathrm{can}"] \arrow[d, "\gamma"', "\simeq"] & H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes}_A B) \arrow[d, "\simeq"', "\beta"] \\ \varprojlim (H^i(\mathcal{X}, \mathcal{F}_n) \otimes_A B) \arrow[r, "\alpha", "\simeq"'] & \varprojlim H^i(\mathcal{X}, \mathcal{F}_n \otimes_A B), \end{tikzcd}$$

we obtain that the upper horizontal morphism is an isomorphism, as required. □

Proposition B.3. *Let A be a p-adically complete noetherian ring. Let B be an A-module that is p-adically complete. Assume that A and B are flat over $\mathbb{Z}_p$. Let $\mathcal{X} \to \mathrm{Spf}(A)$ be a proper flat morphism between p-adic formal schemes. Let $\mathcal{F}$ be a bounded complex of coherent sheaves on $\mathcal{X}$, such that for every term $\mathcal{F}^i$ of $\mathcal{F}$, $\mathcal{F}^i[1/p]$ is locally a direct factor of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]$-module.*

(1) *For every $i \in \mathbb{Z}$, there is a natural map*

$$H^i(R\Gamma(\mathcal{X}, \mathcal{F}) \otimes_A^L B) \to H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes}_A B)$$

whose kernel and cokernel are killed by some power of p.

(2) *If moreover the finite $A[1/p]$-modules $H^j(\mathcal{X}, \mathcal{F})[1/p]$, $j \in \mathbb{Z}$, are flat over $A[1/p]$, the kernel and the cokernel of the natural map*

$$H^i(\mathcal{X}, \mathcal{F}) \otimes B \to H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes} B)$$

are annihilated by some power of p. In particular, we have isomorphisms

$$H^i(\mathcal{X}, \mathcal{F}) \otimes B[1/p] \xrightarrow{\sim} H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes} B[1/p]), \quad \forall\, i \in \mathbb{Z}.$$

Proof. (1) We claim first that B has a resolution $B^\bullet \to B$ by p-adically complete and flat A-modules. Indeed, let $F^\bullet \to B$ be a resolution of B by free A-modules. As A is flat over $\mathbb{Z}_p$, each F^i is p-torsion free. Since B is flat over $\mathbb{Z}_p$, it is also p-torsion free. Therefore, the induced complex

$$\cdots \to F^{-1}/p \to F^0/p \to B/p \to 0 \to \cdots, \tag{B0.2}$$

and thus the complex

$$\cdots \to \hat{F}^{-1} \to \hat{F}^{0} \to B \to 0 \to \cdots$$

are exact. Therefore, we get a resolution $B^\bullet := \hat{F}^\bullet$ of B by flat A-modules [Stacks 2005–, Lemma 06LE] that are p-adically complete [Stacks 2005–, Lemma 05GG]. In particular, by Lemma B.2, we obtain an isomorphism in the derived category

$$R\Gamma(\mathcal{X}, \mathcal{F}) \otimes_A B^\bullet \xrightarrow{\sim} R\Gamma(\mathcal{X}, \mathcal{F}\widehat{\otimes} B^\bullet). \tag{B0.3}$$

Consider the morphism $R\Gamma(\mathcal{X}, \mathcal{X}) \otimes^L B \to R\Gamma(\mathcal{X}, \mathcal{F}\widehat{\otimes} B)$ in the derived category of abelian sheaves on $\mathcal{X}$ defined by the commutative diagram below

$$\begin{tikzcd} R\Gamma(\mathcal{X}, \mathcal{F}) \otimes^L B \arrow[r] & R\Gamma(\mathcal{X}, \mathcal{F}\widehat{\otimes} B) \\ R\Gamma(\mathcal{X}, \mathcal{F}) \otimes B^\bullet \arrow[u, "\simeq"] \arrow[r, "\simeq", "\text{(B0.3)}"'] & R\Gamma(\mathcal{X}, \mathcal{F}\widehat{\otimes} B^\bullet). \arrow[u] \end{tikzcd} \tag{B0.4}$$

Here the left vertical map is an isomorphism as $B^\bullet \to B$ is a flat resolution of B. To complete the proof of (1), it remains to show that the upper horizontal map of (B0.4) induces a morphism

$$H^i(R\Gamma(\mathcal{X}, \mathcal{F}) \otimes^L_A B) \to H^i(\mathcal{X}, \mathcal{F}\widehat{\otimes}_A B) \tag{B0.5}$$

whose kernel and cokernel are annihilated by some power of p. Using the naive truncations of $\mathcal{F}$ and by induction on the length of the bounded complex $\mathcal{F}$, we reduce to the case where $\mathcal{F}$ is a complex concentrated in degree 0, i.e., a coherent sheaf on $\mathcal{X}$, such that $\mathcal{F}[1/p]$ is locally a direct factor of a finite free $\mathcal{O}_{\mathcal{X}}[1/p]$-module.

Consider the complex

$$\cdots \to \mathcal{F}\widehat{\otimes}_A B^{-1} \to \mathcal{F}\widehat{\otimes}_A B^{0} \to \mathcal{F}\widehat{\otimes}_A B \to 0 \to \cdots . \tag{B0.6}$$

We claim that there exists $N \in \mathbb{N}$ such that p^N kills all its cohomologies. This is a local question on $\mathcal{X}$, so assume that $\mathcal{F}[1/p]$ is a direct factor of the finite free $\mathcal{O}_{\mathcal{X}}[1/p]$-module $\mathcal{O}^d_{\mathcal{X}}[1/p]$. Because $\mathcal{F}$ is coherent, there exist morphisms $f : \mathcal{F} \to \mathcal{O}^d_{\mathcal{X}}$ and $g : \mathcal{O}^d_{\mathcal{X}} \to \mathcal{F}$ with $g \circ f = p^N \cdot \mathrm{id}_{\mathcal{F}}$ for some $N \in \mathbb{N}$. By the functoriality of the complex (B0.6) relative to $\mathcal{F}$, we have the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{F}\widehat{\otimes}_A B^{-1} \arrow[r] \arrow[d, "f\widehat{\otimes}\mathrm{id}_{B^{-1}}"] & \mathcal{F}\widehat{\otimes}_A B^{0} \arrow[r] \arrow[d, "f\widehat{\otimes}\mathrm{id}_{B^{0}}"] & \mathcal{F}\widehat{\otimes}_A B \arrow[r] \arrow[d, "f\widehat{\otimes}\mathrm{id}_{B}"] & 0 \arrow[r] \arrow[d, "0"] & \cdots . \\ \cdots \arrow[r] & \mathcal{O}^d_{\mathcal{X}}\widehat{\otimes}_A B^{-1} \arrow[r] \arrow[d, "g\widehat{\otimes}\mathrm{id}_{B^{-1}}"] & \mathcal{O}^d_{\mathcal{X}}\widehat{\otimes}_A B^{0} \arrow[r] \arrow[d, "g\widehat{\otimes}\mathrm{id}_{B^{0}}"] & \mathcal{O}^d_{\mathcal{X}}\widehat{\otimes}_A B \arrow[r] \arrow[d, "g\widehat{\otimes}\mathrm{id}_{B}"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & \mathcal{F}\widehat{\otimes}_A B^{-1} \arrow[r] & \mathcal{F}\widehat{\otimes}_A B^{0} \arrow[r] & \mathcal{F}\widehat{\otimes}_A B \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Because $(g\widehat{\otimes}\,\mathrm{id}_B)\circ(f\widehat{\otimes}\,\mathrm{id}_B)=p^N$ and $(g\widehat{\otimes}\,\mathrm{id}_{B^i})\circ(f\widehat{\otimes}\,\mathrm{id}_{B^i})=p^N$ for every i, to prove our claim, it suffices to show that the complex in the second row

$$\cdots\to\mathcal{O}^d_{\mathcal{X}}\widehat{\otimes}_A B^{-1}\to\mathcal{O}^d_{\mathcal{X}}\widehat{\otimes}_A B^{0}\to\mathcal{O}^d_{\mathcal{X}}\widehat{\otimes}_A B\to 0\to\cdots \tag{B0.7}$$

is exact. Since $\mathcal{X}$ is flat over A, we obtain from (B0.2) a similar exact sequence

$$\cdots\to\mathcal{O}^d_{\mathcal{X}}\otimes_A B^{-1}/p\to\mathcal{O}^d_{\mathcal{X}}\otimes_A B^{0}/p\to\mathcal{O}^d_{\mathcal{X}}\otimes_A B/p\to 0\to\cdots.$$

Because the sheaves $\mathcal{O}^d_{\mathcal{X}}\otimes_A B^i$'s and $\mathcal{O}^d_{\mathcal{X}}\otimes_A B$ are p-torsion free, we deduce as above the exactness of (B0.7), completing the proof of our claim.

Let C be the complex of abelian sheaves concentrated in degrees ≤ 0 given by the following distinguished triangle

$$\mathcal{F}\widehat{\otimes}B^\bullet\to\mathcal{F}\widehat{\otimes}B\to C\xrightarrow{+1}.$$

By what we have shown above, there exists some $N\in\mathbb{N}$ such that $p^N\cdot\mathcal{H}^i(C)=0$ for every $i\in\mathbb{Z}$. In particular, the cohomology groups of $R\Gamma(\mathcal{X},C)$, which is also the mapping cone of the right vertical map of (B0.4), are annihilated by some power of p. Consequently, the kernel and the cokernel of the map (B0.5) are killed by some power of p, as required by (1).

(2) Consider the spectral sequence

$$E_2^{a,b}=\mathrm{Tor}_A^{-a}(H^b(\mathcal{X},\mathcal{F}),B)\Rightarrow H^{a+b}(R\Gamma(\mathcal{X},\mathcal{F})\otimes^L_A B).$$

Observe that, for M a finite A-module such that $M[1/p]$ is flat, thus locally free, over $A[1/p]$, and $\mathrm{Tor}_A^{-a}(M,B)$ is killed by some power of p whenever $a<0$. In particular, $E_2^{a,b}$ is killed by some power of p for $a<0$. Thus, the kernel and the cokernel of

$$E_2^{0,i}=H^i(\mathcal{X},\mathcal{F})\otimes B\to H^i(R\Gamma(\mathcal{X},\mathcal{F})\otimes^L B)$$

are annihilated by some power of p. Combining the first statement of our proposition, we obtain that the natural map

$$H^i(\mathcal{X},\mathcal{F})\otimes B\to H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}B)$$

has its kernel and cokernel killed by some power of p. Inverting p. we deduce

$$H^i(\mathcal{X},\mathcal{F})\otimes B[1/p]\xrightarrow{\sim}H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}B)[1/p]\simeq H^i(\mathcal{X},\mathcal{F}\widehat{\otimes}B[1/p]),$$

as desired in (2). □

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Pseudorepresentations of weight one are unramified

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We prove that the determinant (pseudorepresentation) associated to the Hecke algebra of Katz modular forms of weight one and level prime to p is unramified at p.

1. Introduction

Let p be prime and let $N \geq 5$ be prime to p. Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$ with uniformizer ϖ. Let $X_1(N)$ be the modular curve considered as a smooth proper curve over $\operatorname{Spec}(\mathcal{O})$ and let ω be the pushforward of the relative dualizing sheaf along the universal elliptic curve. The coherent cohomology group $H^0(X_1(N), \omega)$ may be identified with the space of modular forms of weight one with coefficients in $\mathcal{O}$. For general m, one knows that the map:

$$H^0(X_1(N), \omega) \to H^0(X_1(N), \omega/\varpi^m)$$

need not be surjective. This was first observed by Mestre for $N = 1429$ and $p = 2$ (see [Edixhoven 2006, Appendix A]) and many examples for larger p have been subsequently computed by Buzzard [2014] and Schaeffer [2015]. In particular, if $\boldsymbol{T}$ denotes the subring of

$$\operatorname{End}_{\mathcal{O}} \varinjlim H^0(X_1(N), \omega/\varpi^m) = \operatorname{End}_{\mathcal{O}} H^0(X_1(N), \omega \otimes K/\mathcal{O}),$$

generated by Hecke operators T_l and $\langle l \rangle$ for $(l, N) = 1$, then $\boldsymbol{T}$ may be bigger than the classical Hecke algebra acting on the space $H^0(X_1(N), \omega \otimes \mathbb{C})$ of classical modular forms of weight one. Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$. Let $G_{\mathbb{Q},N}$ be the absolute Galois group of the maximal extension of $\mathbb{Q}$ unramified outside $N\infty$. Our main theorem is as follows:

Theorem 1.1. *Let $\boldsymbol{T} \subset \operatorname{End}_{\mathcal{O}} H^0(X_1(N), \omega \otimes K/\mathcal{O})$ denote the algebra generated by Hecke operators T_l and $\langle l \rangle$ for all l prime to N. There is a degree $d = 2$ determinant:*[1]

$$\boldsymbol{D} : \boldsymbol{T}[G_{\mathbb{Q}}] \to \boldsymbol{T}, \quad P(\boldsymbol{D}, \sigma) = X^2 - T(\sigma)X + D(\sigma),$$

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[1] A notion of pseudorepresentation which works in all characteristics, see Section 2.

which is unramified outside $N\infty$ — equivalently, which factors through $\boldsymbol{T}[G_{\mathbb{Q},N}]$ — such that for all primes $l \nmid N$, including $l = p$, one has

$$T(\mathrm{Frob}_l) = T_l \quad \textit{and} \quad D(\mathrm{Frob}_l) = \langle l \rangle.$$

The ring $\boldsymbol{T}$ is a finite $\mathcal{O}$-algebra and is moreover a semilocal ring, and thus is a direct sum $\bigoplus \boldsymbol{T}_{\mathfrak{m}}$ of its completions at maximal ideals $\mathfrak{m}$. For each maximal ideal $\mathfrak{m}$ of $\boldsymbol{T}$, the residual determinant $\bar{P} : \mathcal{O}[G_{\mathbb{Q}}] \to \boldsymbol{T}_{\mathfrak{m}}/\mathfrak{m} = k$ arises from to a semisimple Galois representation $\bar{\rho}$ over $\bar{k}$ [Chenevier 2014, Theorem A]. If this representation is irreducible, then P itself also arises from a genuine representation, which, by a theorem of Carayol [1994], takes values in $\boldsymbol{T}_{\mathfrak{m}}$. It follows from Theorem 1.1 that the corresponding representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\boldsymbol{T}_{\mathfrak{m}})$$

is unramified at p. For $p > 2$, this is a consequence of Theorem 3.11 of [Calegari and Geraghty 2018]. Hence the main interest of this result is to residually reducible representations. However, the result is new even for absolutely irreducible representations when $p = 2$ (although there are significant partial results by Wiese [2014]). Although the proof of Theorem 1.1 is similar to that of Theorem 3.11 of [Calegari and Geraghty 2018], it is more direct, and does not rely on any explicit analysis of the ordinary deformation rings of Snowden [2018]. Hence this paper can also be seen as providing a simplification of the proof of Theorem 3.11 of [Calegari and Geraghty 2018]. (See also Remark 3.3.)

The existence of the determinant without any condition at p is an easy consequence of the corresponding result in higher weight: first consider the action of $\boldsymbol{T}$ on $H^0(X_1(N), \omega/\varpi^m)$ and then multiply by a suitable power of the Hasse invariant which is Hecke equivariant. Hence the main content of this theorem is that the determinant is unramified at p.

2. Determinants

In this paper, we will use the term "pseudorepresentation" as a catch-all to refer to various types of generalized representations. The first pseudorepresentations were introduced by Wiles [1988] for 2-dimensional representations; these were later generalized to any dimension by Taylor [1991]. Following Roquier [1996], we will call Taylor-style pseudorepresentations "pseudocharacters," because of their resemblance to the trace of a representation. In this paper, we will mainly consider the pseudorepresentations of Chenevier [2014] called "determinants." These are more general and flexible than pseudocharacters, and in particular allow us to treat the case where $p = d = 2$. We shall only be concerned with determinants of degree $d = 2$.

We begin by recalling the notion of a determinant [Chenevier 2014, page 223]. Let G be a group and A be a ring. Let d be a positive integer. If M is a free, rank-d A-module equipped with a linear G action, then one may consider the family of characteristic polynomials associated to the elements of $A[G]$ acting on M. This family of polynomials is highly interdependent and is a robust invariant of the representation M. Informally, a degree d determinant is a pseudorepresentation containing the information of a family of polynomials which satisfies the collection of common relations shared by all families of degree d characteristic polynomials. If B is an A-algebra, one can extend the action of $A[G]$ on M to an action of $B[G]$ on $M \otimes_A B$, and also obtain corresponding characteristic polynomials over B for elements in $B[G]$. Chenevier's definition of a determinant follows from the following two insights. First, the data of the characteristic polynomials for elements in $B[G]$ as one ranges over all A-algebras B is equivalent to that of the literal determinants of the elements of $B[G]$ acting on $M \otimes_A B$ as one ranges over all A-algebras B: the characteristic polynomial of an element $m \in B[G]$ is, by definition, the determinant of the endomorphism $X - m$ acting on $M \otimes_A B[X]$. Second, relations in families of characteristic polynomials arise via compatibilities of the determinant map. The literal determinants of the elements of $B[G]$ acting on $M \otimes_A B$ can be organized as a series of set theoretic maps $\det : B[G] \to B$, one for each A-algebra B, which satisfy the following compatibilities:

(1) The maps det are natural in B.

(2) $\det(1) = 1$ and the element $\det(xy) = \det(x)\det(y)$ for all $x, y \in B[G]$.

(3) $\det(bx) = b^d \det(x)$, where $b \in B$ and d is equal to the rank of M.

A *determinant* is simply a family of maps which are compatible in these three ways.

Definition 2.1. Let A be a ring,[2] G be a topological group, and d be a positive integer. A degree d determinant is a continuous A-valued polynomial law $\boldsymbol{D} : A[G] \to A$,[3] which is multiplicative and homogeneous of degree d. If B is an A-algebra and $m \in B[G]$, we call $P(\boldsymbol{D}, m)(X) := \boldsymbol{D}(X - m) \in B[X]$ the characteristic polynomial of m.

Given a determinant $\boldsymbol{D} : A[G] \to A$ and an A-algebra B, the restriction of $\boldsymbol{D}$ to the category of B-algebras defines a determinant $\boldsymbol{D}_B : B[G] \to B$ on B. We call $\boldsymbol{D}_B$ the base change of $\boldsymbol{D}$ to B.

Determinants of degree $d = 2$. Given a determinant $\boldsymbol{D} : A[G] \to A$ of degree 2, the corresponding characteristic polynomials $P(\boldsymbol{D}, m) \in B[X]$ for $m \in B[G]$ have degree 2 and can be written in the form

$$P(m) = P(\boldsymbol{D}, m) = X^2 - T(m)X + D(m),$$

for maps $T, D : B[G] \to B$. Note that the family of maps $D : B[G] \to B$ as B ranges over all A-algebras is precisely the data which defines the polynomial law $\boldsymbol{D}$. In practice, our groups G will always be Galois groups with the usual profinite topology, and our rings A will either be p-adically complete semilocal $W(k)$-algebras with the p-adic topology or p-adic fields with the p-adic topology. We insist that all

[2] All rings considered in this note will carry a Hausdorff topology, and, with the exception of group rings, will be commutative. Our terminology will suppress these topological and algebraic considerations. We use the terms *module* and *algebra* to denote a Hausdorff topological module and a commutative, Hausdorff topological algebra, respectively.

[3] An A-valued polynomial law between two A-modules M and N is by definition a natural transformation $N \otimes_A B \to M \otimes_A B$ on the category of commutative A-algebras B. A polynomial law is called multiplicative if $\boldsymbol{D}(1) = 1$ and $\boldsymbol{D}(xy) = \boldsymbol{D}(x)\boldsymbol{D}(y)$ for all $x, y \in A[G] \otimes B$, and is called homogeneous of degree d, if $\boldsymbol{D}(xb) = b^d \boldsymbol{D}(x)$ for all $x \in A[G] \otimes B$ and $b \in B$. A polynomial law is called continuous if its characteristic polynomial map on G given by $g \mapsto P(\boldsymbol{D}, g)$ is continuous.

Galois representations and all determinants considered in this paper are continuous with respect to the topologies on G and A.

In residue characteristic different from 2 and degree 2, one can recover D from T via the identity

$$D(\sigma) = \frac{T(\sigma)^2 - T(\sigma^2)}{2}.$$

On the other hand, for any p, one can recover T from D by the formula

$$T(\sigma) = D(\sigma + 1) - D(\sigma) - 1.$$

We have the following characterization of determinants of degree 2.

Lemma 2.2 [Chenevier 2014, Lemma 7.7]. *The set of determinants of G over A of degree 2 are in bijection with maps (T, D) from G to A satisfying the following two conditions*:

(1) *$D : G \to A^\times$ is a homomorphism.*

(2) *$T : G \to A$ is a function with $T(1) = 2$ and such that, for all $g, h \in G$*:

  (a) *$T(gh) = T(hg)$.*

  (b) *$D(g)T(g^{-1}h) - T(g)T(h) + T(gh) = 0$.*

In light of this lemma, we shall (from now on) regard a determinant $\boldsymbol{D}$ of G over A of degree 2 as precisely given by a pair of functions (T, D) satisfying the equations above. Given $g \in G$, we have a corresponding characteristic polynomial $P(g) = X^2 - T(g)X + D(g)$. By abuse of notation, we shall denote the pair (T, D) by $P = (T, D)$. By [Chenevier 2014, Lemma 7.7], the functions T and D extend to functions from $A[G]$ to A. In the case of T, this extension is the linear extension, and in the case of D, it can be constructed explicitly by using the equation for $D(xt + ys)$ given below. Note that D as a function of $A[G]$ determines T and hence P and hence $\boldsymbol{D}$, but D as a function of G (in general) does not. Under this equivalence, the base change of a determinant $P := (T, D)$ to an A-algebra B corresponds to the determinant $f \circ P := (f \circ T, f \circ D)$ obtained by post-composing the functions T and D with the structure homomorphism $f : A \to B$.

If A is an algebraically closed field, then (T, D) may be realized as the trace and (classical) determinant of an actual semisimple representation [Chenevier 2014, Theorem A].

There is a well-defined notion of the kernel of P (see [Chenevier 2014, Section 1.4]), which in our case has the following simple description:

Lemma 2.3. *The kernel of a determinant $P = (T, D)$ of degree 2 consist of the elements $x \in A[G]$ satisfying the following two conditions*:

(1) *$T(xy) = 0$ for all $y \in A[G]$.*

(2) *$D(x) = 0$.*

Proof. For polynomial laws of degree 2, we have (see Example 7.6 of [Chenevier 2014])

$$D(xt + ys) = D(x)t^2 + (T(x)T(y) - T(xy))ts + D(y)s^2. \tag{1}$$

As follows from Section 1.4 of [Chenevier 2014], we may compute the $x \in \ker(P)$ by finding the x for which this expression is independent of t. Taking $y = 1$ yields the equalities $T(x) = 0$ and $D(x) = 0$. Returning to the case of general y, we then deduce that $T(xy) = T(x)T(y) = 0$. $\square$

Suppose that H is a subgroup of G such that $[h] - 1 \in \ker(P)$ for all $h \in H$. In this case, by abuse of notation, we say that $\ker(P)$ contains H. If $\ker(P)$ contains H, then $[ghg^{-1}] - 1 \in \ker(P)$ for any $g \in G$, and (compare Lemma 7.14 of [Chenevier 2014]) the determinant P factors through $A[G/N]$, where N is the normal closure of H. (That is, the functions T and D on $A[G]$ depend only on their image in the quotient $A[G/N]$.) In particular, to show that a determinant on $\mathcal{O}[G_{\mathbb{Q}}]$ is unramified at a prime l (for example $l = p$), it suffices to show that the kernel contains some (any) choice of inertia subgroup I_l at l, or equivalently:

Lemma 2.4. *$I_l = H \subset G = G_{\mathbb{Q}}$ lies in the kernel of P if and only if:*

(1) *$T(hg) = T(g)$ for all $h \in H = I_l$ and $g \in G = G_{\mathbb{Q}}$.*

(2) *$D(h - 1) = 0$ for all $h \in H = I_l$.*

Ordinary determinants. Let $\mathcal{O}$ be the ring of integers of a finite extension $[K : \mathbb{Q}_p] < \infty$, let ϖ be a uniformizer of $\mathcal{O}$, and suppose that $\mathcal{O}/\varpi = k$. Let $\overline{P} = (\overline{T}, \overline{D}) : G_{\mathbb{Q}} \to k$ be a degree 2 determinant which is unramified outside Np. In practice, it will always be taken to be modular of level $\Gamma_1(N)$. Let us fix, once and for all, an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$, and hence inclusions

$$I_p \subset D_p \subset G_{\mathbb{Q}},$$

where I_p is the inertia group of $\mathbb{Q}_p$ and $D_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the decomposition group. Let us also fix a Frobenius element $\phi \in D_p$. There is a natural projection $D_p \to D_p/I_p \simeq \hat{\mathbb{Z}}$ whose image is topologically generated by the image of ϕ. Let $\epsilon : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ be the cyclotomic character; we may choose ϕ so that $\epsilon(\phi) = 1$. Enlarging k if necessary, let $\bar{\alpha}$ and $\bar{\beta}$ be the roots of the quadratic polynomial

$$X^2 - \overline{T}(\phi)X + \overline{D}(\phi) = 0$$

over k. We do not assume that these are necessarily distinct.

There are a number of slightly different definitions of ordinary Galois representations in the literature. Let us say that a 2-dimensional representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is ordinary if the underlying 2-dimensional vector space V admits a two step filtration $0 \subsetneq V' \subsetneq V$ such that the action of $G_{\mathbb{Q}_p}$ on $V'' = V/V'$ is unramified. (This coincides, for example, with the definition of ordinary in [Skinner and Wiles 1997].) We furthermore say that ρ is ordinary of weight n if the action of $G_{\mathbb{Q}_p}$ on V' is via an unramified twist of ϵ^{n-1}. By abuse of notation, if $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ is a global Galois representation, we say that it is ordinary if $\rho|_{G_{\mathbb{Q}_p}}$ is ordinary (respectively, ordinary of weight n). When a representation is ordinary, various relations are imposed on its associated determinant. We collect several of these relations common to all ordinary 2-dimensional representations of weight n, and then define that a determinant $P = (T, D) : A[G_{\mathbb{Q}}] \to A$ of degree 2 to be an "ordinary determinant of weight n" if and only if it satisfies these conditions. Our definition includes the auxiliary data of an "eigenvalue" $\alpha \in A^\times$ of the Frobenius element ϕ. This "eigenvalue" satisfies some relations shared by every value which occurs as the eigenvalue of ϕ on a choice of unramified quotient of $\rho|_{G_{\mathbb{Q}_p}}$ in an 2-dimensional ordinary representation of weight n. We will be interested in deformations of $\overline{P}$ to Artinian local rings $(A, \mathfrak{m})$ which are ordinary of weight n.

Definition 2.5. Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field k. An ordinary determinant $P : A[G_{\mathbb{Q}}] \to A$ of degree 2 and weight n with eigenvalue $\alpha \in A^\times$ consists of a pair (P, α) where $P = (T, D) : A[G_{\mathbb{Q}}] \to A$ is a degree $d = 2$ determinant satisfying the following properties:

(1) $P(h) = (X - 1)(X - \psi(h))$ for all $h \in I_p$, where $\psi = \epsilon^{n-1}$.

(2) α is a root of $X^2 - T(\phi)X + D(\phi)$.

(3) For all $h \in I_p$, $(h - \psi(h))(\phi - \alpha) \in \ker(P)$. Equivalently, for all $g \in G_{\mathbb{Q}}$ and $h \in I_p$,

$$T(g(h - \psi(h))(\phi - \alpha)) = T(gh\phi) - \psi(h)T(g\phi) - T(gh)\alpha + T(g)\psi(h)\alpha = 0.$$

The first two conditions of this definition are self-explanatory. The last may be somewhat surprising to the reader; note that it involves a condition on general elements $g \in G_{\mathbb{Q}}$ rather than simply being a condition on the decomposition group. This turns out to be necessary, because the determinant (or pseudocharacter) associated to the decomposition group of a locally reducible representation does not know which character comes from the quotient and which comes from the submodule. The idea behind this definition, as we shall see shortly below, is to capture the notion that the product $(h - \psi(h))(\phi - \alpha)$ is *identically* zero, rather than just of the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$. There is presumably a close relationship between this definition and the definition of ordinary pseudocharacters given by Wake and Wang-Erickson [2017] (see also Section 7.3 of [Wang-Erickson 2018]), although in our context it is important that we can work in non-p distinguished situations by choosing an eigenvalue of Frobenius, which amounts to a partial resolution of the corresponding deformation rings (presumably such modifications could also be adapted to [Wang-Erickson 2018]). On the other hand, we do exploit the crucial idea due to Wang-Erickson that the notion of ordinarity for pseudorepresentations should be a global rather than local condition. The following lemma provides a justification for the final condition above, and the proof provides a motivation for its definition.

Lemma 2.6. *Suppose that f is a classical modular eigenform of level $\Gamma_0(p) \cap \Gamma_1(N)$ with nebentypus character χ of weight $n \geq 2$ with coefficients in $\mathcal{O}$, and suppose that α is the U_p-eigenvalue of f. Assume that f is ordinary (equivalently, that α has trivial valuation). Then the associated determinant $P_f : \mathcal{O}[G_{\mathbb{Q}}] \to \mathcal{O}$ is ordinary with eigenvalue α, weight n, and is unramified outside Np.*

Note that f in Lemma 2.6 need not be new at either the prime p or primes dividing N.

Proof. Since $\mathcal{O}$ has characteristic zero, there is a Galois representation (via [Deligne 1971])

$$\rho_f : G_{\mathbb{Q}, Np} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

associated to f. The determinant $P_f = (T_f, D_f)$ is (by definition) the determinant associated to the representation ρ_f. Since ρ_f factors through $G_{\mathbb{Q},Np}$, this determinant is unramified at primes outside Np. Let $\lambda_\alpha : G_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_p^\times$ denote the unramified character which sends Frob_p to α. We collect the following facts concerning the Galois representation ρ_f:

Fact 2.7. *The representation ρ_f has the following properties*:[4]

(1) *The representation ρ_f is unramified outside Np. The trace and* (*classical*) *determinant of $\rho_f(\mathrm{Frob}_l)$ are equal to $a_l(f)$ and $l^{n-1}\chi(l)$ respectively. The* (*classical*) *determinant of ρ_f is the character $\chi\epsilon^{n-1}$, where χ is unramified outside N.*

(2) *If f is old at level p, and the corresponding eigenform g of level $\Gamma_1(N)$ has T_p eigenvalue a_p, then α is the unit root of $X^2 - a_pX + p^{n-1}\chi(p)$, and*
$$\rho_f|_{D_p} = \rho_g|_{D_p} \sim \begin{pmatrix} \epsilon^{n-1}\lambda_\alpha^{-1}\chi & * \\ 0 & \lambda_\alpha \end{pmatrix}.$$

(3) *If f is new at level p, then $n = 2$,*
$$\rho_f|_{D_p} \sim \begin{pmatrix} \epsilon\lambda_\alpha & * \\ 0 & \lambda_\alpha \end{pmatrix},$$
and $\chi|_{D_p} \simeq \lambda_\alpha^2$.

Using these properties, we see that the required conditions for P_f to be ordinary with eigenvalue α are easily met with the possible exception of the final condition. For this, note that from the explicit descriptions above there exists a basis such that
$$\rho_f|_{I_p} = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}, \quad \rho_f(\phi) = \begin{pmatrix} \chi(\phi)\alpha^{-1} & * \\ 0 & \alpha \end{pmatrix},$$
where $\det(\rho_f) = \epsilon^{n-1}\chi$. We find, with $h \in I_p$, that, in $M_2(\mathcal{O})$,
$$(\rho_f(h) - \psi(h))(\rho_f(\phi) - \alpha) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
It follows that
$$T_f(s(h - \psi(h))(\phi - \alpha)) = \mathrm{tr}(\rho_f(s)(\rho_f(h) - \psi(h))(\rho_f(\phi) - \alpha)) = 0$$
for all $s \in G_{\mathbb{Q}}$. □

[4] The fact that ρ_f is unramified outside Np already follows from the original construction of Deligne [1971]. Since the nebentypus character has conductor dividing N, the corresponding Galois representation χ is certainly unramified outside N. The second claim follows immediately from [Wiles 1988, Theorem 2]. Consider the third claim, so we are assuming that f is new at p. If one writes $\chi = \chi_p\chi_N$ where χ_p and χ_N are characters corresponding to the identification $(\mathbb{Z}/Np\mathbb{Z})^\times = (\mathbb{Z}/p\mathbb{Z})^\times \oplus (\mathbb{Z}/N\mathbb{Z})^\times$, then (by assumption) χ_p is trivial. It follows (see Section 1 of [Atkin and Li 1978]) that f is an eigenform for operator W_p with eigenvalue $\lambda_p(f)$ satisfying $\lambda_p^2(f) = \chi(p)$ [Atkin and Li 1978, Proposition 1.1]. On the other hand, by [Atkin and Li 1978, Theorem 2.1], we deduce that $\alpha^2 = \lambda_p^2(f)p^{n-2} = \chi(p)p^{n-2}$. Under our assumption that α is a p-adic unit, this can only occur when the weight $n = 2$. When $n = 2$, however, we can appeal to [Darmon et al. 1997, Theorem 3.1(e)] which gives a detailed description of the local properties of Galois representations associated to ordinary forms. Finally, the identification of $\chi|_{D_p}$ with λ_α^2 follows either by considering determinants or the identity $\alpha^2 = \chi(p)$ discussed above.

We now fix our choice of $\overline{P}$. Let $\overline{P} : k[G_{\mathbb{Q}}] \to k$ be the determinant associated to a mod ϖ weight one eigenform g of level $\Gamma_1(N)$, i.e., the determinant associated to the Galois representation classically attached to g [Gross 1990, Proposition 11.1]. Suppose that g has nebentypus character χ and T_p-eigenvalue a_p, and let $\bar{\alpha}$ and $\bar{\beta}$ be the roots of $X^2 - a_p X + \chi(p)$, which we assume (enlarging $\mathcal{O}$ if necessary) are k-rational.

Lemma 2.8. *Let $n \equiv 1 \bmod (p-1)$ be an integer. The determinant $\overline{P}$ is ordinary of degree 2 and weight n with eigenvalue $\bar{\alpha}$ and is unramified outside Np. If $n > 1$, there is an eigenform f of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight n which is ordinary at p for which the U_p-eigenvalue of f is congruent to $\bar{\alpha}$ mod ϖ such that $\overline{P} = \overline{P}_f$.*

By symmetry, the result holds with $\bar{\alpha}$ replaced by $\bar{\beta}$.

Proof. Since ϵ^{n-1} is trivial mod ϖ for $n \equiv 1 \bmod (p-1)$, if $\overline{P}$ is ordinary with eigenvalue $\bar{\alpha}$ for one such n, it is ordinary for all such n. Suppose we can construct an eigenform h modulo ϖ of level $\Gamma_1(N)$ of weight p and such that the T_p-eigenvalue of h is congruent to $\bar{\alpha}$ mod ϖ, and such that $\overline{P}_h = \overline{P}$. By multiplying by powers of the Hasse invariant, we deduce that there also exists such a form in any weight $n \equiv 1 \bmod (p-1)$ such that $n > 1$. All mod ϖ modular forms in weights $n > 1$ and level $\Gamma_1(N)$ lift to characteristic zero. (This follows as in [Katz 1973, Theorem 1.7.1], the running assumption that $N \geq 5$ guaranteeing that $X_1(N)$ is a fine moduli space.) Moreover, using the Deligne–Serre lifting lemma [Deligne and Serre 1974, Lemme 6.11], one can always choose a lift h which is an eigenform for all the Hecke operators. The lifted form h of weight $\Gamma_1(N)$ has weight $n > 1$ and T_p-eigenvalue $\bar{\alpha}$ mod ϖ. But now the ordinary stabilization f of h of level $\Gamma_1(N) \cap \Gamma_0(p)$ has U_p-eigenvalue $\bar{\alpha}$ mod ϖ, and $\overline{P}_f = \overline{P}_h = \overline{P}$, as required. Finally, we deduce from Lemma 2.6 applied to f that $\overline{P}$ is ordinary with eigenvalue $\bar{\alpha}$ (of weight n and unramified outside Np). Thus it remains to construct h from g.

If A is the Hasse invariant, then Ag is a modular form mod ϖ of level $\Gamma_1(N)$ and weight p which is an eigenform for all Hecke operators except for T_p, and moreover has the same eigenvalues as g. The same is true of $T_p(Ag)$ and also $A(T_p g)$ (the latter is just $a_p Ag$). On the level of q-expansions, there are equalities $Ag = g$ and $AT_p(g) - T_p(Ag) = Vg$ respectively. Hence $h = g - \bar{\beta} Vg$ is a weight p modular eigenform mod ϖ of level $\Gamma_1(N)$ with $\overline{P}_f = \overline{P}$ and with T_p-eigenvalue $\bar{\alpha}$. To see that $T_p h = \bar{\alpha} h$, note (compare [Gross 1990, Section 4, especially (4.7)]) that $T_p(Vg) = Ag$ and $T_p Ag = a_p Ag - \chi(p) Vg$, and hence

$$\begin{aligned} T_p h &= T_p(Ag - \bar{\beta} Vg) \\ &= a_p Ag - \chi(p) Vg - \bar{\beta} Ag \\ &= (\bar{\alpha} + \bar{\beta}) Ag - \bar{\alpha}\bar{\beta} Vg - \bar{\beta} Ag \\ &= \bar{\alpha}(Ag - \bar{\beta} Vg) = \bar{\alpha} h. \end{aligned}$$

□

Let $R = R^{\mathrm{univ}}$ denote the universal deformation ring of $\overline{P}$ (compare [Chenevier 2014, Proposition 7.59]) unramified outside Np. It prorepresents the functor which, for Artinian local $W(k)$-algebras $(A, \mathfrak{m})$ with residue field $A/\mathfrak{m} = k$, consists of determinants $P = (T, D)$ valued in A whose mod $\mathfrak{m}$ reduction is $\overline{P}$.

Let $P^{\mathrm{univ}} = (T^{\mathrm{univ}}, D^{\mathrm{univ}})$ denote the corresponding universal determinant. We define a mild variant on this ring by considering such determinants: $P = (T, D) : A[G_{\mathbb{Q}}] \to A$ *together* with a root α of $X^2 - T(\phi)X + D(\phi)$. The result is an extension $\tilde{R}$ of R given by

$$\tilde{R} = R[\alpha]/(\alpha^2 - T^{\mathrm{univ}}(\phi)\alpha + D^{\mathrm{univ}}(\phi)).$$

The ring R is a local $W(k)$-algebra, but the ring $\tilde{R}$ is a semilocal $W(k)$-algebra with either one or two maximal ideals. It has 2 maximal ideals precisely when the polynomial $\alpha^2 - \overline{T}(\phi)\alpha + \overline{D}(\phi) \in k[\alpha]$ is separable.

Definition 2.9. Let $\tilde{D}^{\dagger}_n(A)$ denote the functor which, for Artinian local rings $(A, \mathfrak{m})$ with residue field $A/\mathfrak{m} = k$, consists of ordinary determinants (P, α_0) of weight n unramified outside Np, where P is a deformation of $\overline{P}$ to A, and $n \equiv 1 \bmod (p-1)$ is a positive integer.

Note that elements in $\tilde{D}^{\dagger}_n(k)$ are in bijection with choices of $\bar{\alpha} \in k$ so that $\overline{P}$ is ordinary of weight n with eigenvalue $\bar{\alpha}$. By Lemma 2.8, such a choice of eigenvalue exists. Furthermore since $\bar{\alpha}$ is a root $X^2 - \overline{T}(\phi)X + \overline{D}(\phi)$, the size of $\tilde{D}^{\dagger}_n(k)$ is at most 2. For each root $\bar{\alpha} \in k$ of $X^2 - \overline{T}(\phi)X + \overline{D}(\phi)$, consider the subfunctor $\tilde{D}^{\dagger,\bar{\alpha}}_n(A) \subseteq \tilde{D}^{\dagger}_n(A)$ consisting of pairs with (P, α_0) such that $\alpha_0 \equiv \bar{\alpha} \bmod \mathfrak{m}$. The functor $\tilde{D}^{\dagger}_n$ decomposes as the coproduct

$$\tilde{D}^{\dagger}_n(A) = \coprod_{(\overline{P},\bar{\alpha}) \in \tilde{D}^{\dagger}_n(k)} \tilde{D}^{\dagger,\bar{\alpha}}_n(A),$$

and each of the subfunctors $\tilde{D}^{\dagger,\bar{\alpha}}_n$ are prorepresented by a (potentially trivial) Noetherian local $W(k)$-algebra $\tilde{R}^{\dagger,\bar{\alpha}}_n$. By abuse of terminology, we will say $\tilde{D}^{\dagger}_n$ is prorepresented by the semilocal ring

$$\tilde{R}^{\dagger}_n := \bigoplus_{(\overline{P},\bar{\alpha}) \in \tilde{D}^{\dagger}_n(k)} \tilde{R}^{\dagger,\bar{\alpha}}_n.$$

Explicitly, if $\tilde{P}^{\mathrm{univ}}$ is the base change of P^{univ} to the R-algebra $\tilde{R}$, then $\tilde{R}^{\dagger}_n$ is the quotient of $\tilde{R}$ by the ideal generated by all the relations which obstruct P^{univ} from being ordinary of weight n with eigenvalue α. The universal determinant $P^{\dagger,\mathrm{univ}}_n$ is base change of $\tilde{P}^{\mathrm{univ}}$ to $\tilde{R}^{\dagger}_n$ and the universal eigenvalue is α.

The determinant $P^{\dagger,\mathrm{univ}}$ itself is valued in the subring $R^{\dagger}_n$ of $\tilde{R}^{\dagger}_n$, which is the image of $R \subset \tilde{R}$. However, the element α will not, in general, lie in $R^{\dagger}_n$. The extra data of α records, implicitly, the "choice" of realizing the corresponding determinant as ordinary. (The same determinant P can in principle be realized as an ordinary determinant (P, α) for different values of α.)

The following result is the key proposition which allows us to prove that certain ordinary determinants are unramified. The idea is that, given a representation which is ordinary, the more the representation is ramified, the more the choice of ordinary eigenvalue α is pinned down by the Galois representation, because the ramification structure gives a partial filtration on the representation which mirrors the ordinary filtration. The extreme case, in which α cannot be distinguished from the other root $\alpha^{-1}D(\phi)$ of the characteristic polynomial of ϕ, should only occur when the representation is unramified. While these claims are obvious for $\overline{\mathbb{Q}}_p$-valued representations, the key property of our definition is that one can prove this for any quotient of $\tilde{R}_n^\dagger$.

Proposition 2.10. *Let $\tilde{R}_n^\dagger \to \tilde{S}$ be a surjective homomorphism of $W(k)$-algebras, and let S denote the image of $R_n^\dagger$ in $\tilde{S}$. Suppose that $\tilde{S}/S$ is a free S-module of rank one, or equivalently, that the annihilator of $\tilde{S}/S$ as an S-module is trivial. Then the corresponding determinant P valued in S is unramified.*

Proof. We first verify that $D(h-1)=0$ without any assumptions. From first condition of Definition 2.5 we see that $D(h)=\psi(h)$ and $T(h)=1+\psi(h)$, and thus, from (1) in the proof of Lemma 2.3, we deduce that

$$D(h-1)=D(h)-(T(h)T(1)-T(h))+D(1)=\psi(h)-(\psi(h)+1)+1=0.$$

We now turn to the second condition of Lemma 2.4. The module $\tilde{S}/S$ is a cyclic S-module generated by α, so it is free if and only if the annihilator of α is trivial. We have by definition the identity (for $s \in G_{\mathbb{Q}}$ and $h \in I_p$ and $\psi = \epsilon^{n-1}$)

$$T(sh\phi)-\psi(h)T(s\phi)-T(sh)\alpha+T(s)\psi(h)\alpha=0.$$

We may rearrange this to obtain the identity

$$\alpha(T(sh)-T(s)\psi(h))=T(sh\phi)-\psi(h)T(s\phi).$$

Note that the value $T(s)$ for any $s \in G_{\mathbb{Q}}$ lands in S, as does the image of any element of $W(k)$, and hence it follows that

$$\alpha(T(sh)-T(s)\psi(h))=0 \in \tilde{S}/S.$$

Take g to be the identity, so $T(sh)=T(h)=1+\psi(h)$ and $T(s)=2$. Then we deduce that

$$\alpha(1-\psi(h))=0 \in \tilde{S}/S$$

for all $h \in I_p$. If $\tilde{S}/S$ is free, then its annihilator of α is trivial, and thus $\psi(h)=1$ for all h. But we then deduce for the same reason that $T(sh)-T(s)\psi(h)=T(sh)-T(s)=0$ for all $s \in G_{\mathbb{Q}}$ and $h \in I_p$, from which it follows by Lemma 2.4 (note that $T(sh)=T(hs)$) that I_p is contained in the kernel. □

3. Galois Deformations

By Lemma 2.8, our fixed determinant $\overline{P}=(\overline{T},\overline{D}):k[G_{\mathbb{Q}}]\to k$ is associated to an ordinary mod ϖ eigenform of level $\Gamma_1(N)\cap\Gamma_0(p)$ in each weight $n \geq 2$ satisfying $n \equiv 1 \bmod p-1$. Given our choice of Frobenius element $\phi \in D_p \subset G_{\mathbb{Q}}$, recall that $\bar{\alpha}$ and $\bar{\beta}$ are the roots of the polynomial

$$\overline{P}(\phi)=X^2-\overline{T}(\phi)X+\overline{D}(\phi).$$

We start by considering determinants arising from forms of higher weight.

Lemma 3.1. *Let $n \geq 2$ be an integer such that $n \equiv 1 \bmod p-1$. Let $\tilde{\boldsymbol{T}}_n$ denote the $\mathcal{O}$-algebra of endomorphisms of*

$$M_n(\Gamma_0(p) \cap \Gamma_1(N), \mathcal{O})$$

generated by Hecke operators T_l and $\langle l \rangle$ for l prime to Np, together with U_p. Let $\mathfrak{m}$ denote the ideal of $\tilde{\boldsymbol{T}}_n$ generated by ϖ and by any lift in $\tilde{\boldsymbol{T}}_n$ of the following elements of $\tilde{\boldsymbol{T}}_n/\varpi$: the operators $T_l - \overline{T}(\mathrm{Frob}_l)$ and $\langle l \rangle l^{n-1} - \overline{D}(\mathrm{Frob}_l)$ for $(l, Np) = 1$, and $(U_p - \bar{\alpha})(U_p - \bar{\beta})$. Assume that $\overline{P}$ is associated to an ordinary mod ϖ eigenform of level $\Gamma_1(N) \cap \Gamma_0(p)$ and weight n with U_p-eigenvalue congruent to either $\bar{\alpha}$ or $\bar{\beta}$ modulo ϖ, so that $\mathfrak{m}$ is a proper ideal. Then there exists a canonical surjection of semilocal rings

$$\tilde{R}_n^{\dagger} \to \tilde{\boldsymbol{T}}_{n,\mathfrak{m}}$$

sending $\alpha \in \tilde{R}_n^{\dagger}$ to U_p.

Remark 3.2. If $\boldsymbol{T}_n \subset \tilde{\boldsymbol{T}}_n$ denotes the subring generated by the all the Hecke operators except U_p, then $\mathfrak{m} \cap \boldsymbol{T}_n$ is maximal. However, $\mathfrak{m}$ itself need not be maximal. Throughout the rest of the paper, we let $\tilde{\boldsymbol{T}}_{n,\mathfrak{m}}$ denote the completion $\tilde{\boldsymbol{T}}_{n,\mathfrak{m}} := \operatorname{proj\,lim} \tilde{\boldsymbol{T}}_n/\mathfrak{m}^r$ — it need not be a local ring. The Hecke algebra $\tilde{\boldsymbol{T}}_{n,\mathfrak{m}}$ is nonlocal precisely when $\bar{\alpha} \neq \bar{\beta}$ and when $\overline{P}$ is associated to an ordinary mod ϖ eigenform with U_p-eigenvalue congruent to $\bar{\alpha}$ mod ϖ and is *also* associated to an eigenform with U_p-eigenvalue congruent to $\bar{\beta}$ mod ϖ. In that case, the ideals $\mathfrak{m}_{\bar{\alpha}}$ and $\mathfrak{m}_{\bar{\beta}}$ obtained by adjoining any lift of $U_p - \bar{\alpha}$ or $U_p - \bar{\beta}$ respectively from $\tilde{\boldsymbol{T}}_n/\varpi$ to $\mathfrak{m}$ are both maximal, and there is an isomorphism $\tilde{\boldsymbol{T}}_{n,\mathfrak{m}} \cong \tilde{\boldsymbol{T}}_{n,\mathfrak{m}_{\bar{\alpha}}} \oplus \tilde{\boldsymbol{T}}_{n,\mathfrak{m}_{\bar{\beta}}}$. Working with semilocal rings allows us to treat the cases $\bar{\alpha} = \bar{\beta}$ and $\bar{\alpha} \neq \bar{\beta}$ simultaneously. If M is a module for $\tilde{\boldsymbol{T}}_n$, then, when $\mathfrak{m}$ is not maximal, there is also a corresponding identification $M_{\mathfrak{m}} := \operatorname{proj\,lim} M/\mathfrak{m}^r = M_{\mathfrak{m}_{\bar{\alpha}}} \oplus M_{\mathfrak{m}_{\bar{\beta}}}$.

Proof of Lemma 3.1. Consider an embedding $K \to L$, where L is a field which contains the eigenvalues of all elements of $\tilde{\boldsymbol{T}}_n$. The Hecke algebra $\tilde{\boldsymbol{T}}_n$ acts faithfully on $M_n(\Gamma_0(p) \cap \Gamma_1(N), L)$. Recall that $\boldsymbol{T}_n \subset \tilde{\boldsymbol{T}}_n$ denotes the subring generated by Hecke operators away from Np (i.e., without U_p). For each newform h which contributes to $M_n(\Gamma_0(p) \cap \Gamma_1(N), L)$, there is a corresponding vector space $V(h) \subset M_n(\Gamma_0(p) \cap \Gamma_1(N), L)$ generated by h together with the old forms associated to h. (The space $V(h)$ can also be identified with the invariants $\pi^{\Gamma_1(N) \cap \Gamma_0(p)}$, where π is the smooth admissible $\mathrm{GL}_2(\mathbf{A}^{(\infty)})$-representation over L generated by h.) There is a $\boldsymbol{T}_n$-equivariant isomorphism

$$M_n(\Gamma_0(p) \cap \Gamma_1(N), L) \simeq \bigoplus_g V(h),$$

where $\boldsymbol{T}_n$ acts on $V(h)$ through scalars corresponding to the homomorphism $\eta_h : \boldsymbol{T}_n \to L$ sending T_l to $a_l(h)$ and $\langle l \rangle$ to $l^{n-1}\chi(l)$ where χ is the nebentypus character of h. Let us now consider the action of the operator U_p. For each map $\eta_h : \boldsymbol{T}_n \to L$ (which corresponds to a fixed Galois representation ρ_h) one of the following two things happens:

(1) The newform h has level $\Gamma_0(p)$ at p, in which case U_p acts on $V(h)$ via a scalar.

(2) The newform h has level $\Gamma_0(1)$ at p, in which case U_p acts on $V(h)$ and satisfies the identity $U_p^2 - a_p U_p + p^{n-1}\chi(p) = 0$.

In particular, the algebra $\tilde{\boldsymbol{T}}_n$ will always acts semisimply in the first case and act semi-simply in the second case as long as the corresponding polynomial $X^2 - a_p X + p^{n-1}\chi(p)$ has distinct roots. This is known in general only under the assumption of the Tate conjecture (see [Coleman and Edixhoven 1998]), but it can certainly only fail to happen when $a_p^2 = 4p^{n-1}\chi(p)$, which would force the (multiple) eigenvalue of U_p to have positive valuation (since $n \geq 2$). In particular, such forms do not contribute to $M_n(\Gamma_0(p) \cap \Gamma_1(N), \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L$, because (since $\mathfrak{m}$ contains the preimage of $(U_p - \bar{\alpha})(U_p - \bar{\beta})$ for nonzero $\bar{\alpha}$ and $\bar{\beta}$) the element U_p acts invertibly on this space. (Recall, following Remark 3.2, that when $\mathfrak{m}$ is contained in two primes, $M_n(\Gamma_0(p) \cap \Gamma_1(N), \mathcal{O})_{\mathfrak{m}}$ is simply the direct sum of $M_n(\Gamma_0(p) \cap \Gamma_1(N), \mathcal{O})_{\mathfrak{m}_{\bar{\alpha}}}$ and $M_n(\Gamma_0(p) \cap \Gamma_1(N), \mathcal{O})_{\mathfrak{m}_{\bar{\beta}}}$.) It follows that there is an injection

$$i_n : \tilde{\boldsymbol{T}}_{n,\mathfrak{m}} \hookrightarrow \bigoplus_f L,$$

where the sum ranges over all $\tilde{\boldsymbol{T}}_n$-eigenforms $f \in M_n(\Gamma_0(p) \cap \Gamma_1(N), L)$ such that $\bar{P}_f = \bar{P}$ and the U_p-eigenvalue is congruent either to $\bar{\alpha}$ or $\bar{\beta}$. We identify $\tilde{\boldsymbol{T}}_{n,\mathfrak{m}}$ with its image under i_n. For each of the forms f above, denote the U_p-eigenvalue by $\alpha(f)$. By Lemma 2.6, the determinants P_f are ordinary with eigenvalues $\alpha(f)$, weight n, and unramified outside Np. Hence, for each form f there is a homomorphism

$$i_f : \tilde{R}_n^{\dagger} \to L$$

such that $i_f \circ P^{\dagger,\mathrm{univ}} = P_f$ and which maps α to $\alpha(f)$. Taking the direct sum of the maps i_f, we obtain a homomorphism

$$j_n : \tilde{R}_n^{\dagger} \to \bigoplus_f L$$

under which α maps to U_p, $T(\mathrm{Frob}_l)$ maps to T_l, and $D(\mathrm{Frob}_l)$ maps to $\langle l \rangle$. We conclude that j_n factors through a surjective homomorphism

$$\tilde{R}_n^{\dagger} \to \tilde{\boldsymbol{T}}_{n,\mathfrak{m}}$$

under which α maps to U_p. □

We are now ready to prove the main theorem.

Proof of Theorem 1.1. Recall that $\boldsymbol{T} = \boldsymbol{T}_1$ is the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}} H^0(X_1(N), \omega \otimes K/\mathcal{O})$ generated by T_l and $\langle l \rangle$ for $(l, N) = 1$. This ring contains T_p, but the element T_p in weight one is also generated by the other Hecke operators (see, for example, Lemma 3.1 of [Calegari 2018]). For each positive integer m, let $\boldsymbol{T}(m)$ denote the image $\boldsymbol{T}$ in $\mathrm{End}_{\mathcal{O}} H^0(X_1(N), \omega/\varpi^m)$. The ring $\boldsymbol{T} \cong \varprojlim \boldsymbol{T}(m)$. Therefore, to prove Theorem 1.1, it suffices to construct for each $m > 0$ a degree $d = 2$ determinant

$$\boldsymbol{D}_m : \boldsymbol{T}(m)[G_{\mathbb{Q}}] \to \boldsymbol{T}(m), \quad P(\boldsymbol{D}_m, \sigma) = X^2 - T_m(\sigma)X + D_m(\sigma),$$

which is unramified outside $N\infty$, and such that for all primes $l \nmid N$ (including $l = p$) the characteristic polynomial of Frob_l satisfies

$$T_m(\mathrm{Frob}_l) = T_l \quad \text{and} \quad D_m(\mathrm{Frob}_l) = \langle l \rangle.$$

In the remainder of the proof, we will assume that $m > 0$ is fixed, and will denote by an abuse of notation $\boldsymbol{T}(m)$ by $\boldsymbol{T}$.

There is a decomposition $\boldsymbol{T} = \bigoplus \boldsymbol{T}_\mathfrak{m}$ over the maximal ideals $\mathfrak{m}$ of $\boldsymbol{T}$. Hence, it suffices to construct the desired determinant after completing at a maximal ideal $\mathfrak{m}$ of $\boldsymbol{T}$. Let $\overline{P}$ denote our fixed modular residual determinant, which we have assumed is supported in weight one, and let $\mathfrak{m}$ denote the maximal ideal which is the kernel of the corresponding map $\boldsymbol{T} \to k$. Let $\tilde{\boldsymbol{T}}_n$ denote the Hecke algebra of Lemma 3.1 in weight $n := 1 + p^{m-1}(p-1)$ which contains U_p (and has coefficients in $\mathcal{O}$). By abuse of notation, we also let $\mathfrak{m}$ denote the ideal of $\tilde{\boldsymbol{T}}_n$ defined in Lemma 3.1. By Lemma 2.8, this ideal is proper.

By Lemma 3.16 of [Calegari and Geraghty 2018], there is a surjective map

$$\tilde{R}_n^\dagger \twoheadrightarrow \tilde{\boldsymbol{T}}_{n,\mathfrak{m}} \twoheadrightarrow \tilde{S} := \boldsymbol{T}_\mathfrak{m}[U_p]/(U_p^2 - T_p U_p + \langle p \rangle) \tag{2}$$

(which sends T_l and $\langle l \rangle$ to T_l and $\langle l \rangle$ respectively, and sends U_p to U_p, where U_p in $\tilde{S}$ is viewed as a formal variable satisfying the given quadratic relation). Although the running assumption in Section 3 of [Calegari and Geraghty 2018] is that $p > 2$, the proof of [loc. cit., Lemma 3.16] applies (as written with no changes necessary) with $p = 2$. The image S of $R_n^\dagger \subset \tilde{R}_n^\dagger$ is generated by the values of T and D on Frobenius elements, which land inside the ring $\boldsymbol{T}_\mathfrak{m}$ (in fact, they generate the ring $\boldsymbol{T}_\mathfrak{m}$). But $\tilde{S}$ is free of rank two over $\boldsymbol{T}_\mathfrak{m}$, and thus $\tilde{S}/S$ has no annihilator. Consequently, the corresponding determinant in $\boldsymbol{T}_\mathfrak{m}$ is unramified by Proposition 2.10. To show that $T(\mathrm{Frob}_p) = T_p$ and $D(\mathrm{Frob}_p) = \langle p \rangle$, it suffices to show that $T(\phi) = T_p$ and $D(\phi) = \langle p \rangle$. The image of α in $\boldsymbol{T}_\mathfrak{m}[U_p]/(U_p^2 - T_p U_p + \langle p \rangle)$ was U_p, which satisfies the equation $X^2 - T_p X + \langle p \rangle = 0$. Yet α also satisfies the equation $X^2 - T(\phi)X + D(\phi) = 0$. Since this algebra is free of rank two over $\boldsymbol{T}_\mathfrak{m}$, these quadratics must be the same, and hence $T(\phi) = T_p$ and $D(\phi) = \langle p \rangle$. $\square$

Remark 3.3. The proof above relies on [Calegari and Geraghty 2018, Lemma 3.16]. We also note, however, that the content of this lemma is simply an alternate form of doubling which is a already implicit in the work of Wiese [2014].

Remark 3.4. One should also be able to apply the methods of this paper in the case $l \neq p$ when l exactly divides N, where now one wants to capture in this context the notion of a determinant "admitting an unramified quotient line" when restricted to the inertia group I_l at l (compare Section 1.8 of [Wake and Wang-Erickson 2018]).

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On the p-typical de Rham–Witt complex over $W(k)$

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Hesselholt and Madsen (2004) define and study the (absolute, p-typical) de Rham–Witt complex in mixed characteristic, where p is an odd prime. They give as an example an elementary algebraic description of the de Rham–Witt complex over $\mathbb{Z}_{(p)}$, $W_{\cdot}\Omega^{\bullet}_{\mathbb{Z}_{(p)}}$. The main goal of this paper is to construct, for k a perfect ring of characteristic $p > 2$, a Witt complex over $A = W(k)$ with an algebraic description which is completely analogous to Hesselholt and Madsen's description for $\mathbb{Z}_{(p)}$. Our Witt complex is not isomorphic to the de Rham–Witt complex; instead we prove that, in each level, the de Rham–Witt complex over $W(k)$ surjects onto our Witt complex, and that the kernel consists of all elements which are divisible by arbitrarily high powers of p. We deduce an explicit description of $W_n\Omega^{\bullet}_A$ for each $n \geq 1$. We also deduce results concerning the de Rham–Witt complex over certain p-torsion-free perfectoid rings.

Introduction

Fix an odd prime p and a $\mathbb{Z}_{(p)}$-algebra R. Hesselholt and Madsen [2004] define the (absolute, p-typical) de Rham–Witt complex over R to be the initial object in the category of Witt complexes over R. Their definition generalizes the de Rham–Witt complex of Bloch, Deligne and Illusie, which was defined for $\mathbb{F}_p$-algebras. The goal of this paper is to define a Witt complex $E^{\bullet}_{\cdot}$ over $A = W(k)$, where k is a perfect ring of characteristic p, and to use this Witt complex to describe the de Rham–Witt complex over $W(k)$ and also to study the de Rham–Witt complex over certain perfectoid rings B.

Among many other conditions, the de Rham–Witt complex $W_{\cdot}\Omega^{\bullet}_R$ is a prosystem of differential graded rings. There is an isomorphism $W_n(R) \to W_n\Omega^0_R$, so the degree zero piece of the de Rham–Witt complex is well-understood. For each positive integer n and for every degree d, there is a surjective morphism of differential graded rings

$$\Omega^d_{W_n(R)} \twoheadrightarrow W_n\Omega^d_R,$$

and so it is easy to write down elements of $W_n\Omega^d_R$. On the other hand, especially in the degree one case $d = 1$, it is often difficult to determine which of these elements in $W_n\Omega^1_A$ are nonzero. The author is not aware of a complete algebraic description of the (absolute, p-typical) de Rham–Witt complex in mixed characteristic for any examples other than $\mathbb{Z}_{(p)}$ and polynomial algebras over this ring. One of the goals of the current paper is to give a complete algebraic description of the de Rham–Witt complex over $A = W(k)$, where k is a perfect ring of odd characteristic p. For example, we prove that in the de Rham–Witt complex over $W(k)$, the element $dV^n(1)$ is a nontrivial p^n-torsion element for every integer $n \geq 1$. It is easy to

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see, using the relation $pdV = Vd$, that this element is indeed p^n-torsion, but showing that this element is nonzero takes much more work.

To better analyze relations within the de Rham–Witt complex, we first define in Section 3 a Witt complex $E^\bullet_\cdot$ over $A = W(k)$ which has a simple algebraic description as a $W(k)$-module. The proof that $E^\bullet_\cdot$ is indeed a Witt complex over $W(k)$ is one of the major parts of this paper. It is not isomorphic to the de Rham–Witt complex over $W(k)$; see Remark 3.11. Instead, in each level n and in each positive degree $d \geq 1$, our Witt complex $E^\bullet_\cdot$ is the quotient of the de Rham–Witt complex by the $W(k)$-submodule consisting of all elements which are divisible by arbitrarily large powers of p. In the language of [Hesselholt 2015, Remark 4.8], our Witt complex $E^\bullet_\cdot$ is the p-typical de Rham–Witt complex over $W(k)$ relative to the p-typical λ-ring $(W(k), s_\varphi)$, where s_φ is the ring homomorphism $W(k) \to W(W(k))$ recalled in Proposition 2.1 below.

Our description of $E^\bullet_\cdot$, which we define for each $W(k)$ with k a perfect ring of odd characteristic p, is completely modeled after Hesselholt and Madsen's description of $W_n\Omega^1_{\mathbb{Z}_{(p)}}$ [2004, Example 1.2.4]. They show that for all $n \geq 1$, there is an isomorphism of $\mathbb{Z}_{(p)}$-modules

$$W_n\Omega^1_{\mathbb{Z}_{(p)}} \cong \prod_{i=0}^{n-1} \mathbb{Z}/p^i\mathbb{Z} \cdot dV^i(1). \tag{0.1}$$

This shows that $W_n\Omega^1_{\mathbb{Z}_{(p)}}$ is nonzero if $n \geq 2$. The proof in [Hesselholt and Madsen 2004] involves the topological Hochschild spectrum $T(\mathbb{Z}_{(p)})$. The results below provide an alternative (and elementary) proof that $W_n\Omega^1_{\mathbb{Z}_{(p)}}$ is nonzero if $n \geq 2$.

Of course an elementary algebraic proof of the isomorphism in (0.1) could be given by directly verifying that the stated groups satisfy all the necessary relations to form a Witt complex. It is this approach we follow in the current paper for the case $A = W(k)$, where k is a perfect ring of odd characteristic p. Moreover, we prove that, for such A and for every $n \geq 1$, there is a surjective map

$$W_n\Omega^1_A \twoheadrightarrow \prod_{i=0}^{n-1} A/p^iA \cdot dV^i(1) =: E^1_n, \tag{0.2}$$

and we prove that the kernel of this map consists of all elements of $W_n\Omega^1_A$ which are divisible by arbitrarily large powers of p.

The groups $E^\bullet_n$ in a Witt complex over A are in particular $W_n(A)$-modules, and the $W_n(A)$-module structure we define is also analogous to the description for $\mathbb{Z}_{(p)}$. In the de Rham–Witt complex over $\mathbb{Z}_{(p)}$, and in fact in any Witt complex, for integers $i, j \geq 1$, one has

$$V^j(1)\,dV^i(1) = p^j\,dV^i(1). \tag{0.3}$$

This alone does not completely determine the $W_n(A)$-module structure, but for our specific case $A = W(k)$, there is a ring homomorphism $s_\varphi : A \to W(A)$, and we require that for all $a \in A$ and $x \in E^1_n$, we have $s_\varphi(a)x = a \cdot x$. Here the product $s_\varphi(a)x$ on the left side refers to the $W_n(A)$-module structure we wish to define, and the product $a \cdot x$ on the right side refers to the A-module structure on E_n^1 that is apparent from the description in (0.2). This requirement completely determines our $W_n(A)$-module structure.

With these prerequisites in mind, the verification that our complex is a Witt complex is largely straightforward. The most difficult step is proving that our complex satisfies

$$Fd[a] = [a]^{p-1}\, d[a] \in E_n^1$$

for every $a \in A$ and for every integer $n \geq 1$. The difficulty, which arises repeatedly in what follows, lies in the fact that the multiplicative Teichmüller lift $[\cdot] : A \to W(A)$ is not related in a simple way to our ring homomorphism lift $s_\varphi : A \to W(A)$.

Once we know that our complex $E_\cdot^\bullet$ is a Witt complex over A, we attain relatively easily a complete algebraic description of the de Rham–Witt complex $W.\Omega_A^\bullet$. See Section 4 for the proofs of the following results, as well as for a more complete (but longer) description of $W_n\Omega_A^1$ (Corollary 4.10).

Theorem A. *Let k denote a perfect ring of odd characteristic p and let $A = W(k)$.*

(1) *Fix an integer $n \geq 1$. Let $S_n \subseteq W_n\Omega_A^1$ denote $\bigcap_{j=1}^\infty p^j W_n\Omega_A^1$, the $W_n(A)$-submodule of all elements which are infinitely p-divisible. Then we have an isomorphism of abelian groups*

$$W_n\Omega_A^1/S_n \cong \prod_{i=0}^{n-1} A/p^i A.$$

(2) *Fix integers $n \geq 1$ and $d \geq 2$. Then we have an isomorphism of abelian groups*

$$W_n\Omega_A^d \cong \prod_{i=0}^{n-1} \Omega_A^d.$$

In Section 5, we turn to describing the de Rham–Witt complex over the quotient ring A/xA, for an element $x \in A$; this is done with the purpose of applying it in the case that A/xA is a perfectoid ring B, and $A = W(B^\flat)$ is the ring of Witt vectors of the tilt of B. Our complete algebraic description of $W_n\Omega_A^1$ makes extensive use of the ring homomorphisms $s_\varphi : A \to W_n(A)$, and in general we have no such ring homomorphisms $B \to W_n(B)$, so our algebraic description of $W_n\Omega_B^1$ is less complete. However, for a certain class of perfectoid rings, we are able to completely describe the kernel of the restriction map $W_{n+1}\Omega_B^1 \to W_n\Omega_B^1$. We phrase the following theorem in slightly more generality, to include also the case $W(k)$ which is proved earlier.

Theorem B. *Let p denote an odd prime. Let S denote either $W(k)$ for k a perfect ring of characteristic p, or else let S denote a p-torsion-free perfectoid ring for which there exists some nonzero p-power torsion element $\omega \in \Omega_S^1$. In either of these cases, the following is a short exact sequence of $W_{n+1}(S)$-modules:*

$$0 \to S \xrightarrow{(-d,p^n)} \Omega_S^1 \oplus S \xrightarrow{V^n + dV^n} W_{n+1}\Omega_S^1 \xrightarrow{R} W_n\Omega_S^1 \to 0.$$

See Propositions 4.7 and 6.12 for the proofs, and also for a description of the module structures. The existence of an element ω as described in the statement is guaranteed, for example, whenever $\zeta_p \in S$ and $d\zeta_p \neq 0$.

One motivation for studying the de Rham–Witt complexes we consider in this paper is our hope to adapt results from [Hesselholt 2006]. That paper concerns the de Rham–Witt complex over the ring of integers in an algebraic closure of a mixed characteristic local field, and we hope to perform a similar analysis in the context of perfectoid rings. Our proofs for perfectoid rings will be modeled after Hesselholt's proof for $\mathcal{O}_{\overline{\mathbb{Q}}_p}$, and our proofs will use an induction argument that requires a precise description of the kernel of restriction $W_{n+1}\Omega^1_B \to W_n\Omega^1_B$. We will pursue this direction in joint work with Irakli Patchkoria.

A second, but indirect, motivation for the current paper is the recent remarkable work of Bhatt, Morrow and Scholze [2016], which makes use of the de Rham–Witt complex in mixed characteristic. Currently this is only a philosophical motivation, however, because they study the *relative* de Rham–Witt complex of Langer and Zink [2004], whereas we study the absolute de Rham–Witt complex of [Hesselholt and Madsen 2003; 2004; Hesselholt 2005]. Our work is not directly relevant to the work of Bhatt, Morrow and Scholze, but it could potentially be relevant to generalizations of their work which involved the *absolute* de Rham–Witt complex.

0.1. ***Notation.*** Throughout this paper, $p > 2$ denotes an odd prime, k is a perfect ring of characteristic p, W denotes p-typical Witt vectors, and $A = W(k)$. To distinguish between the Witt vector Frobenius on $A = W(k)$ and on $W(A)$, we write φ for the Witt vector Frobenius on A and we write F for the Witt vector Frobenius on $W(A)$ and on $W.\Omega^\bullet_A$. Rings in this paper are assumed to be commutative and to have unity, and ring homomorphisms are assumed to map unity to unity. We write Ω^1_R for the R-module of absolute Kähler differentials, i.e., $\Omega^1_R = \Omega_{R/\mathbb{Z}}$ in the notation of [Matsumura 1989, Section 25]. The de Rham–Witt complex we consider is the absolute, p-typical de Rham–Witt complex defined in [Hesselholt and Madsen 2004, Introduction].

1. Background on Witt complexes and the de Rham–Witt complex

Fix k, a perfect ring of odd characteristic p and let $A = W(k)$. The main goal of this paper is to construct a certain Witt complex over A, and to use this Witt complex to deduce properties of the de Rham–Witt complex over A. Similar properties are proven in the work of Hesselholt [2005; 2006] and Hesselholt and Madsen [2003; 2004]; the main difference between our results and these earlier results is that our proofs use only algebra. The only aspect of the current paper which is not elementary is our proof that $\Omega^1_{W(k)}$ has no nontrivial p-torsion (Proposition 2.7), which uses the cotangent complex. The current paper does not use any notions from algebraic topology, such as the spectrum $TR^{\cdot}_{\bullet}$.

The current paper does, however, use many standard facts about (p-typical) Witt vectors $W(R)$ and the (p-typical, absolute) de Rham–Witt complex $W.\Omega^\bullet_R$, and it is written with the assumption that the reader is familiar with their basic properties, including the case R is not characteristic p. For background on Witt vectors, we refer to [Illusie 1979] or to the brief introduction given in Section 1 of [Hesselholt and Madsen 2004]. A thorough treatment of Witt vectors is given in Section 1 of [Hesselholt 2015], but those results are framed in the context of big Witt vectors instead of p-typical Witt vectors.

We work in this section over an arbitrary $\mathbb{Z}_{(p)}$-algebra R, where p is an odd prime. We now recall the basic properties of Witt complexes and the de Rham–Witt complex which we will use. Our reference is [Hesselholt and Madsen 2004].

The de Rham–Witt complex over R (or, more generally, any Witt complex over R) is a prosystem of differential graded rings. The index indicating the position in the prosystem is a positive integer $n = 1, 2, \ldots$ which we refer to as the *level*. The index indicating the degree in the differential graded ring is a nonnegative integer $d = 0, 1, \ldots$ which we refer to as the *degree*. We write E_n^d for the level n, degree d component of a Witt complex $E_\cdot^\bullet$.

Definition 1.1 [Hesselholt and Madsen 2004, Introduction]. Fix an odd prime p and a $\mathbb{Z}_{(p)}$-algebra R. A *Witt complex* over R is the following:

(1) A prodifferential graded ring $E_\cdot^\bullet$ and a strict map of prorings
$$\lambda : W_\cdot(R) \to E_\cdot^\bullet.$$

(2) A strict map of prograded rings
$$F : E_\cdot^\bullet \to E_{\cdot-1}^\bullet$$
such that $F\lambda = \lambda F$ and for all $r \in R$, we have
$$Fd\lambda([r]) = \lambda([r]^{p-1})d\lambda([r]).$$

(3) A strict map of graded $E_\cdot^\bullet$-modules
$$V : F_* E_{\cdot-1}^\bullet \to E_\cdot^\bullet.$$
(In other words,
$$V(F(\omega)\eta) = \omega V(\eta) \text{ for all } \omega \in E_\cdot^\bullet,\ \eta \in E_{\cdot-1}^\bullet,$$
and similarly for multiplication on the right.) The map V must further satisfy $V\lambda = \lambda V$ and
$$FdV = d, \quad FV = p.$$

Remark 1.2. In this paper we never consider the prime $p = 2$. See [Hesselholt 2015, Definition 4.1] for a definition of Witt complex which can be used for all primes, or [Costeanu 2008] for a careful treatment of the 2-typical de Rham–Witt complex. One subtlety is that for $p = 2$, the differential does not necessarily satisfy $d \circ d = 0$.

The following theorem defines the de Rham–Witt complex over R as the initial object in the category of Witt complexes over R. Its existence is proved in [Hesselholt and Madsen 2004].

Theorem 1.3 [Hesselholt and Madsen 2004, Theorem A]. *Let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime. There is an initial object $W.\Omega^\bullet_R$ in the category of Witt complexes over R. We call this complex the de Rham–Witt complex over R. Moreover, for every $d \geq 0$ and $n \geq 1$, the canonical map*

$$\Omega^d_{W_n(R)} \to W_n\Omega^d_R$$

is surjective.

The following result, like our last result, is proved in [Hesselholt and Madsen 2004]. It describes the degree 0 piece and the level 1 piece of the de Rham–Witt complex, respectively.

Theorem 1.4. *Let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime.*

(1) [loc. cit., Remark 1.2.2] *The canonical map $\lambda : W_n(R) \to W_n\Omega^0_R$ is an isomorphism for all $n \geq 1$.*

(2) [loc. cit., Theorem D and the first sentence of the proof of Proposition 5.1.1] *The canonical map $\Omega^\bullet_R \to W_1\Omega^\bullet_R$ is an isomorphism.*

Two of the main results of this paper are Propositions 4.7 and 6.12. The main content of these propositions describes, for suitable rings R, the intersection

$$V^n(\Omega^1_R) \cap dV^n(R) \subseteq W_{n+1}\Omega^1_R.$$

Our next proposition, which is true for every $\mathbb{Z}_{(p)}$-algebra R, identifies

$$V^n(\Omega^1_R) + dV^n(R) \subseteq W_{n+1}\Omega^1_R$$

as the kernel of restriction.

Proposition 1.5. *Let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime. Fix integers $d \geq 1$ and $n \geq 1$. Then ω is in the kernel of restriction*

$$W_{n+1}\Omega^d_R \to W_n\Omega^d_R$$

if and only if there exist $\alpha \in \Omega^d_R$ and $\beta \in \Omega^{d-1}_R$ such that

$$\omega = V^n(\alpha) + dV^n(\beta).$$

The difficult part is the *only if* direction. See [Hesselholt and Madsen 2003, Lemma 3.2.4] for a proof in terms of the log de Rham–Witt complex. We recall the idea of that proof. (See also the proof of Proposition 5.7 below for similar arguments.) For every n, d, define

$${}'W_n\Omega^d_R := W_{n+1}\Omega^d_R/(V^n(\Omega^d_R) + dV^n(\Omega^{d-1}_R)).$$

One then shows that ${}'W.\Omega^\bullet_R$ is an initial object in the category of Witt complexes over R, and hence in particular that the natural map

$${}'W_n\Omega^d_R \to W_n\Omega^d_R \tag{1.6}$$

is an isomorphism. That natural map is induced by restriction $W_{n+1}\Omega^d_R \to W_n\Omega^d_R$, so our proposition follows from the injectivity of the map in (1.6).

The following results we recall from [Hesselholt and Madsen 2004] have significantly easier proofs than the previous results we have cited; the proofs of the relations in Proposition 1.7 below are just a few lines of computation.

Proposition 1.7 [Hesselholt and Madsen 2004, Lemma 1.2.1]. *Again let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime. The following equalities hold in every Witt complex over R:*

$$dF = pFd, \quad Vd = pdV, \quad V(x_0 dx_1 \cdots dx_m) = V(x_0)dV(x_1)\cdots dV(x_m).$$

2. Results on $W(A)$ and Ω^1_A when $A = W(k)$

Let k denote a perfect ring of odd characteristic p and let $A = W(k)$. In this paper, we study the de Rham–Witt complex over A. In this section, we prove several preliminary results about the degree zero case, $W(A)$, and the level one case, Ω^1_A. Special thanks are due to Bhargav Bhatt and Lars Hesselholt for their assistance with the Ω^1_A proofs.

The following result allows us to view the ring $W(A)$ as an A-algebra. This is a key fact. This is also a similarity between the case $A = W(k)$ and the case $A = \mathbb{Z}_{(p)}$, after which our results are modeled: the ring $W(A)$ is an A-algebra and the ring $W(\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$-algebra. This is also the main reason our methods don't easily translate to more general rings such as ramified extensions of $\mathbb{Z}_p$.

Recall that, to avoid confusion, we write the Witt vector Frobenius differently on $A = W(k)$ from how we write it on $W(A) = W(W(k))$: we write $\varphi : A \to A$ and $F : W(A) \to W(A)$ for these Witt vector Frobenius maps. The map φ is a ring isomorphism, but the map F is not an isomorphism.

Proposition 2.1 [Illusie 1979, (0.1.3.16)]. *Let k denote a perfect ring of characteristic p, let $A = W(k)$, and let $\varphi : A \to A$ denote the Witt vector Frobenius. Then there is a unique ring homomorphism*

$$s_\varphi : A \to W(A)$$

satisfying $F \circ s_\varphi = s_\varphi \circ \varphi$ and such that for all $a \in A$, the ghost components of $s_\varphi(a)$ are $(a, \varphi(a), \varphi^2(a), \ldots)$.

Proof. The ring A is p-torsion free, so this result follows from [Illusie 1979, (0.1.3.16)], provided we know that the ring homomorphism $\varphi : A \to A$ satisfies $\varphi(a) \equiv a^p \bmod pA$ for all $a \in A$. This last congruence is in fact true more generally for any ring $W(R)$ of p-typical Witt vectors. We recall the short proof from [Illusie 1979, Section 0.1.4]. For arbitrary $a \in W(R)$, write $a = [r_0] + V(a_+)$, where $r_0 \in R$ and $a_+ \in W(R)$. We then have

$$\varphi(a) = [r_0]^p + pa_+ \equiv [r_0]^p \bmod pW(R) \equiv ([r_0] + V(a_+))^p \bmod pW(R),$$

where the last congruence uses that $V(x)V(y) = pV(xy) \in pW(R)$ for Witt vectors $x, y \in W(R)$. □

Lemma 2.2. *For every $x \in W(A)$, there exist unique elements $a_0, a_1, \ldots \in A$ for which*

$$x = \sum_{i=0}^{\infty} s_\varphi(a_i)V^i(1) \in W(A). \tag{2.3}$$

Proof. We have

$$\sum_{i=0}^{\infty} s_\varphi(a_i)V^i(1) = \sum_{i=0}^{\infty} V^i(F^i(s_\varphi(a_i))) = \sum_{i=0}^{\infty} V^i(s_\varphi(\varphi^i(a_i))),$$

so the result now follows from the fact that $\varphi : A \to A$ is an isomorphism and that the first component of $s_\varphi(a) \in W(A)$ is a. □

Lemma 2.4. *If $x \in W(A)$ is given as in* (2.3), *then*

$$V(x) = \sum_{i=1}^{\infty} s_\varphi(\varphi^{-1}(a_{i-1}))V^i(1) \in W(A).$$

Proof. This follows from the formula $V(F(x)y) = xV(y)$, for $x, y \in W(A)$ and from the fact that $F(s_\varphi(a_i)) = s_\varphi(\varphi(a_i))$. □

The following result gives explicit formulas for the elements $a_i \in A$ appearing in (2.3) in the specific case that x is a Teichmüller lift of some element $a \in A$. The main technical difficulty of this paper involves studying congruences involving these coefficients.

Lemma 2.5. *In the specific case $x = [a] \in W(A)$ is the Teichmüller lift of an element $a \in A$, then the terms a_i from* (2.3) *are given by the formulas $a_0 = a$ and $a_i = \varphi^{-i}((a^{p^i} - (\varphi(a))^{p^{i-1}})/p^i)$ for $i \geq 1$.*

Proof. This follows using induction on i, by comparing the ghost components of the two sides of (2.3). (Notice that the ghost map is injective because A is p-torsion free.) To simplify the proof, notice that a finite sum

$$s_\varphi(a_0) + s_\varphi(a_1)V(1) + \cdots + s_\varphi(a_n)V^n(1),$$

has ghost components which stabilize in the following pattern

$$(w_0, \ldots, w_{n-1}, w_n, \varphi(w_n), \varphi^2(w_n), \ldots).$$ □

When we define our Witt complex $E^\bullet_\cdot$ in Section 3, we will express E^1_n in terms of quotients A/p^iA. The groups E^1_n in a Witt complex over A always possess a $W_n(A)$-module structure, and the following lemma describes the $W_n(A)$-module structure we put on A/p^iA; notice that this module structure is not the one induced by the obvious projection map $W_n(A) \to A$.

Lemma 2.6. *Let $n, i \geq 1$ be integers and consider the map $W_n(A) \to A/p^iA$ given by*

$$\sum_{j=0}^{n-1} s_\varphi(a_j)V^j(1) \mapsto \sum_{j=0}^{n-1} a_j p^j.$$

This is a surjective ring homomorphism with kernel the ideal in $W_n(A)$ generated by

$$\{p^i, V^j(1) - p^j \mid 0 \leq j \leq n-1\}.$$

Proof. If we view $W_n(A)$ as an A-module via s_φ, then it's clear that the map is a surjective A-module homomorphism. To prove it's a ring homomorphism, we use the formula $V^j(1)V^i(1) = p^j V^i(1)$ for $j \le i$.

We now prove the statement about the kernel. Clearly the proposed elements are in the kernel; we now show an arbitrary element in the kernel is generated by the proposed elements. Assume $\sum_{j=0}^{n-1} s_\varphi(a_j)V^j(1)$ is in the kernel. This means that there exists $a \in A$ such that

$$p^i a = \sum_{j=0}^{n-1} a_j p^j \in A.$$

Applying s_φ to both sides, we find

$$p^i s_\varphi(a) = \sum_{j=0}^{n-1} s_\varphi(a_j)p^j \in W_n(A),$$

and thus

$$p^i s_\varphi(a) + \sum_{j=0}^{n-1} s_\varphi(a_j)(V^j(1) - p^j) = \sum_{j=0}^{n-1} s_\varphi(a_j)V^j(1),$$

which completes the proof. □

This concludes our collection of preliminary results on Witt vectors over $A = W(k)$. We now turn our attention to $\Omega^1_{W(k)}$. We thank Bhargav Bhatt and Lars Hesselholt for their help with the remainder of this section. Our first result, Proposition 2.7, is the most important. It says that multiplication by p is bijective on $\Omega^1_{W(k)}$; we will use this result repeatedly. By contrast, the results from Proposition 2.9 to the end of this section are closer to "reality-checks". For example, Corollary 2.10 below shows that $\Omega^1_{W(k)}$ is not the zero-module.

Proposition 2.7. *Let k denote a perfect ring of characteristic p. Then multiplication by p: $\Omega^1_{W(k)} \to \Omega^1_{W(k)}$ is a bijection.*

Remark 2.8. The proof below is due to Bhargav Bhatt. The tools used in the proof (the cotangent complex and, more generally, the language of derived categories) do not appear elsewhere in this paper, so the reader (or author) who is not comfortable with them is advised to treat the proof of Proposition 2.7 as a black box. See also the proof of [Hesselholt and Madsen 2003, Lemma 2.2.4] for a proof of a related result.

Before giving Bhatt's proof, we point out an elementary argument for surjectivity. The Witt vector Frobenius $\varphi : W(k) \to W(k)$ is surjective on one hand, and on the other hand, $\varphi(a) \equiv a^p \bmod pW(k)$ for every $a \in W(k)$. So for every $a \in W(k)$, we can find $a_0, a_1 \in W(k)$ such that $a = a_0^p + pa_1$. Thus every $da \in \Omega^1_{W(k)}$ is divisible by p, and hence multiplication by p on $\Omega^1_{W(k)}$ is surjective. We are not aware of a similarly elementary proof of injectivity.

Proof. Let $L_{W(k)/\mathbb{Z}}$ denote the cotangent complex. Because $\mathbb{Z} \to W(k)$ is flat, we have

$$L_{W(k)/\mathbb{Z}} \otimes^L_{\mathbb{Z}} \mathbb{F}_p \cong L_{k/\mathbb{F}_p}$$

by [Stacks 2005–, Tag 08QQ]. The right-hand side is zero, because the Frobenius automorphism on k induces a map on $L_{k/\mathbb{F}_p}$ which is simultaneously zero and an isomorphism. Thus the left-hand side is also 0. This implies that multiplication by p on $L_{W(k)/\mathbb{Z}}$ is a quasiisomorphism. In particular, multiplication by p is an isomorphism on $H^0(L_{W(k)/\mathbb{Z}}) \cong \Omega^1_{W(k)}$, which completes the proof. □

Throughout this paper, k denotes a perfect *ring* of characteristic p. We prove Corollary 2.10 below for $W(k)$ by deducing it from Proposition 2.9, which concerns the case of $W(k')$, where k' is a perfect *field* of characteristic p.

Proposition 2.9. *Let k' denote a perfect field of characteristic p. Let $\{x_\alpha\}_{\alpha \in A} \subseteq W(k')$ denote elements such that $\{x_\alpha\}_{\alpha\in A}$ is a transcendence basis for $W(k')[1/p]$ over $\mathbb{Q}$. Then $\{dx_\alpha\}_{\alpha\in A}$ is a basis for $\Omega^1_{W(k')}$ as a $W(k')[1/p]$-vector space.*

Proof. By Proposition 2.7, we have

$$\Omega^1_{W(k')} \cong \Omega^1_{W(k')} \otimes_{W(k')} W(k')[1/p] \cong \Omega^1_{W(k')[1/p]} \cong \Omega^1_{W(k')[1/p]/\mathbb{Q}}.$$

Thus it suffices to prove that if $\{x_\alpha\}_{\alpha\in A}$ is a transcendence basis for a field $K/\mathbb{Q}$, then $\{dx_\alpha\}_{\alpha\in A}$ is a K-basis for $\Omega^1_{K/\mathbb{Q}}$. The result now follows by [Matsumura 1989, Theorem 26.5]. □

Corollary 2.10. *Let k denote a perfect ring of characteristic p. Then the $W(k)$-module $\Omega^1_{W(k)}$ is nonzero.*

Proof. Let $\mathfrak{m} \subseteq k$ denote a maximal ideal. Then $k \to k/\mathfrak{m}$ is a surjection from k onto a perfect field of characteristic p; write $k' = k/\mathfrak{m}$. The induced map $W(k) \to W(k')$ is a surjective ring homomorphism, so $\Omega^1_{W(k)} \to \Omega^1_{W(k')}$ is a surjective $W(k)$-module homomorphism. Because $W(k')$ is uncountable, the field $W(k')[1/p]$ is transcendental over $\mathbb{Q}$, so our result follows from Proposition 2.9. □

Corollary 2.11. *For every integer $n \geq 1$, the $W_n(W(k))$-module $W_n\Omega^1_{W(k)}$ is nonzero.*

Proof. Begin with any nonzero element $\alpha \in \Omega^1_{W(k)}$. We then have $p^{n-1}\alpha \neq 0$ by Proposition 2.7, but on the other hand, $p^{n-1}\alpha = F^{n-1}V^{n-1}(\alpha)$, and so $V^{n-1}(\alpha) \in W_n\Omega^1_{W(k)}$ is nonzero. □

3. A p-adically separated Witt complex over $W(k)$

Let k denote a perfect ring of odd characteristic p and let $A = W(k)$. We are going to define a Witt complex over A. Our definition is modeled after [Hesselholt and Madsen 2004, Example 1.2.4], which gives a completely analogous description of the de Rham–Witt complex over $\mathbb{Z}_{(p)}$.

As an abelian group, we define

$$E_n^0 := W_n(A) \text{ for all } n \geq 1,$$
$$E_n^1 := \prod_{i=0}^{n-1} A/p^i A \cdot dV^i(1) \text{ for all } n \geq 1,$$
$$E_n^d := 0 \text{ for all } n \geq 1,\ d \geq 2;$$

here $dV^i(1)$ should be viewed as a formal basis symbol. The ring structure on $E_n^\bullet$ is obvious with the exception of the multiplication $E_n^0 \times E_n^1 \to E_n^1$, and for this we use the ring homomorphisms from Lemma 2.6 to give $A/p^i A$ the structure of a $W_n(A)$-module. (We note again that the module structure does not arise from the restriction map $W_n(A) \to W_1(A) = A$.) Define $\lambda : W_n(A) \to E_n^0$ to be the identity map and equip E_n^0 with the usual ring structure and with the usual maps R, F, V.

Recalling Lemma 2.2, which guarantees that each element in $W_n(A)$ corresponds to a unique expression $\sum_{i=0}^{n-1} s_\varphi(a_i) \cdot V^i(1)$, we define $d : E_n^0 \to E_n^1$ by the formula

$$d\left(\sum_{i=0}^{n-1} s_\varphi(a_i) V^i(1)\right) = \sum_{i=1}^{n-1} a_i \cdot dV^i(1).$$

Define $R : E_{n+1}^1 \to E_n^1$ by the formula

$$R\left(\sum_{i=0}^{n} a_i \cdot dV^i(1)\right) = \sum_{i=0}^{n-1} a_i \cdot dV^i(1).$$

Define $F : E_{n+1}^1 \to E_n^1$ by the formula

$$F\left(\sum_{i=1}^{n} a_i \cdot dV^i(1)\right) = \sum_{i=0}^{n-1} \varphi(a_{i+1}) \cdot dV^i(1).$$

Define $V : E_n^1 \to E_{n+1}^1$ by the formula

$$V\left(\sum_{i=1}^{n-1} a_i \cdot dV^i(1)\right) = \sum_{i=1}^{n-1} p\varphi^{-1}(a_i) \cdot dV^{i+1}(1).$$

We emphasize that this last definition means in particular that $V(dV^i(1)) = p \cdot dV^{i+1}(1)$.

Remark 3.1. We use the dot $\cdot$ in the notation $A/p^i A \cdot dV^i(1)$ to help distinguish between this $A/p^i A$-module structure and the $W_n(A)$-module structure, which we write without the dot. For example, if we let $\pi_{n,i} : W_n(A) \to A/p^i A$ denote the ring homomorphism from Lemma 2.6, then we would write $x dV^i(1) = \pi_{n,i}(x) \cdot dV^i(1)$. This distinction isn't mathematically important, but we find it helps to reinforce whether we are multiplying by elements in $A/p^i A$ or by elements in $W_n(A)$ or $W(A)$.

Before proving that $E_{\cdot}^{\bullet}$ is a Witt complex, we make a preliminary calculation that does not involve Witt vectors. This calculation will be used to verify that

$$Fd([a]) = [a]^{p-1}d([a]) \in E_n^1 \tag{3.2}$$

holds for all $n \geq 1$, which is the most difficult step in our verification that $E_{\cdot}^{\bullet}$ is a Witt complex.

Remark 3.3. In (3.2), we are being less careful with notation than Hesselholt and Madsen [2004]. In their notation, this equation would be written

$$Fd([a]_{n+1}) = ([a]_n)^{p-1}d([a]_n) \in E_n^1,$$

where the subscripts are indicating $[a]_n \in W_n(A)$ and $[a]_{n+1} \in W_{n+1}(A)$.

Lemma 3.4. *Continue to let $A = W(k)$, where k is a perfect ring of odd characteristic p, and let $\varphi : A \to A$ denote the Witt vector Frobenius. Fix $a \in A$. Then for every $i \geq 1$, we have*

$$\tfrac{1}{p^{i+1}}(a^{p^{i+1}} - \varphi(a)^{p^i}) \equiv \tfrac{1}{p^i}(a^{p^i} - \varphi(a)^{p^{i-1}})a^{p^i(p-1)} \bmod p^i A. \tag{3.5}$$

Proof. The only fact we will use about $\varphi : A \to A$ is that for every $a \in A$, there exists $x \in A$ such that $\varphi(a) = a^p + px$. Multiplying both sides of (3.5) by p^{i+1} and applying the binomial theorem to the powers of $\varphi(a) = a^p + px$, we reduce immediately to proving that

$$\sum_{j=1}^{p^i} \binom{p^i}{j} (a^p)^{p^i-j}(px)^j \equiv pa^{p^i(p-1)} \sum_{j=1}^{p^{i-1}} \binom{p^{i-1}}{j} (a^p)^{p^{i-1}-j}(px)^j \bmod p^{2i+1}A.$$

By distributing the $a^{p^i(p-1)}$ term on the right side, this simplifies to proving that

$$\sum_{j=1}^{p^i} \binom{p^i}{j} a^{p^{i+1}-pj}(px)^j \equiv p \sum_{j=1}^{p^{i-1}} \binom{p^{i-1}}{j} a^{p^{i+1}-pj}(px)^j \bmod p^{2i+1}A.$$

By comparing the coefficients of the $a^m x^n$ monomials, it suffices then to prove the following two claims:

- For every j in the range $1 \leq j \leq p^{i-1}$, we have
$$p^j \binom{p^i}{j} \equiv p^{j+1} \binom{p^{i-1}}{j} \bmod p^{2i+1}.$$
- For every j in the range $p^{i-1} + 1 \leq j \leq p^i$, we have
$$p^j \binom{p^i}{j} \equiv 0 \bmod p^{2i+1}.$$

To prove the first claim, we rewrite it as

$$p^j \left(\binom{p^i}{j} - p \binom{p^{i-1}}{j} \right) \equiv 0 \bmod p^{2i+1}.$$

The left side equals 0 if $j = 1$, so we may assume $j \geq 2$ and simplify the expression as

$$p^j \tfrac{p^i}{j!}((p^i - 1)\cdots(p^i - j + 1) - (p^{i-1} - 1)\cdots(p^{i-1} - j + 1)) \equiv 0 \bmod p^{2i+1}.$$

The term inside the parentheses is the difference of two terms which are congruent modulo p^{i-1}, hence the term inside the parentheses is divisible by p^{i-1}. Thus it suffices to show that for every $j \geq 2$ we have

$$p^j \tfrac{p^{2i-1}}{j!} \equiv 0 \bmod p^{2i+1}.$$

Thus it suffices to show that for every $j \geq 2$, we have $j - v_p(j!) \geq 2$, where v_p denotes the p-adic valuation. Because $p \geq 3$, the inequality is true if $j = 2$. For the case $j \geq 3$, again using $p \geq 3$, we compute

$$j - v_p(j!) \geq j - \left(\tfrac{j}{p} + \tfrac{j}{p^2} + \cdots\right) = j - j\tfrac{1}{p(p-1)} \geq j - \tfrac{j}{6} = \tfrac{5j}{6} \geq \tfrac{15}{6} \geq 2,$$

which completes the proof of the first claim.

To prove the second claim, we first treat the case $j = p^i$. Then we need to show that $p^i \geq 2i + 1$, which is true because $p \geq 3$ and $i \geq 1$. For the case $p^{i-1} + 1 \leq j < p^i$, we know the binomial coefficient in the expression has p-adic valuation at least one, so it suffices to prove that $j + 1 \geq 2i + 1$. Thus it suffices to prove that $p^{i-1} + 2 \geq 2i + 1$. Again this holds because $p \geq 3$ and $i \geq 1$. □

Remark 3.6. Lemma 3.4 is false in general if $p = 2$. For example, it is already false in the case $A = \mathbb{Z}_2$, $\varphi = \mathrm{id}$, $a = 2$, and $i = 1$.

We can now state our main theorem of this section; all the main results of this paper are dependent on the following result.

Theorem 3.7. *Let k be a perfect ring of characteristic $p > 2$, and let $A = W(k)$. The complex $E_\cdot^\bullet$ defined above is a Witt complex over A.*

Proof. Many of the required properties are obvious; the main difficulty is proving that for all $a \in A$ and all $n \geq 2$, we have

$$Fd([a]) = [a]^{p-1}d([a]) \in E^1_{n-1}. \tag{3.8}$$

We postpone this verification to the end of the proof.

The following properties are clear:

- For each n, $E_n^\bullet$ is a ring.
- The maps R are ring homomorphisms.
- The map λ is a ring homomorphism that commutes with R.
- The maps F, V commute with λ.
- The maps d, F, V are additive.
- The maps d, F, V commute with R.
- The composition FV is equal to multiplication by p.

Next we check that d verifies the Leibniz rule. Because d is additive and because $ds_\varphi(a) = 0$ for all a, it suffices to prove that for all $1 \le j \le i$, we have

$$d(V^i(1)V^j(1)) = V^i(1)dV^j(1) + V^j(1)dV^i(1).$$

Using the definition of our multiplication $E_n^0 \times E_n^1 \to E_n^1$ and using $V(x)V(y) = pV(xy)$, we see that both sides are equal to $(p^j + p^i A) \cdot dV^i(1)$.

Next we check that F is multiplicative. The only part which isn't obvious is to show that if $x \in E_n^0$ and $y \in E_n^1$, then we have

$$F(xy) = F(x)F(y).$$

Because we already know F is additive, it suffices to check this in the special cases $x = x_1 := s_\varphi(a)$, $x = x_2 := V^i(1)$ with $i \ge 1$, and $y = (b + p^j A) \cdot dV^j(1)$, where $j \ge 1$. We have $F(x_1) = s_\varphi(\varphi(a))$, $F(x_2) = pV^{i-1}(1)$, and $F(y) = (\varphi(b) + p^{j-1}A) \cdot dV^{j-1}(1)$. On the other hand, $x_1 y = (ab + p^j A) \cdot dV^j(1)$ and $F(x_1 y) = (\varphi(ab) + p^{j-1}A) \cdot dV^{j-1}(1) = F(x_1)F(y)$. We also have $x_2 y = (p^i b + p^j A) \cdot dV^j(1)$ and $F(x_2 y) = (p^i\varphi(b) + p^{j-1}A) \cdot dV^{j-1}(1) = F(x_2)F(y)$.

We next check that for all $x \in E_{n+1}^\bullet$ and $y \in E_n^\bullet$, we have

$$V(F(x)y) = xV(y). \tag{3.9}$$

This is obvious if x, y are both in degree zero or both in degree one, thus we only need to consider the case that one of them is degree zero and the other is degree one. It suffices to consider the case that the degree one term has the form $dV^j(1)$ and the degree zero term has the form $s_\varphi(a)V^i(1)$. If $x = dV(1)$ and $y = s_\varphi(a)V^i(1)$, then both sides of (3.9) are zero. If $x = dV^j(1)$ with $j \ge 2$ and $y = s_\varphi(a)V^i(1)$, we compute

$$V(F(x)y) = V(p^i a \cdot dV^{j-1}(1)) = p^{i+1}\varphi^{-1}(a) \cdot dV^j(1) = (s_\varphi(\varphi^{-1}(a))V^{i+1}(1))dV^j(1) = xV(y).$$

If $x = s_\varphi(a)$ and $y = dV^j(1)$, then we compute

$$V(F(x)y) = V(\varphi(a) \cdot dV^j(1)) = ps_\varphi(a)dV^{j+1}(1) = xV(y).$$

If $x = s_\varphi(a)V^i(1)$ with $i \ge 1$ and $y = dV^j(1)$, then we compute

$$\begin{aligned} V(F(x)y) &= V(s_\varphi(\varphi(a))pV^{i-1}(1)dV^j(1)) \\ &= V(\varphi(a)p^i \cdot dV^j(1)) \\ &= ap^{i+1} \cdot dV^{j+1}(1) \\ &= pxdV^{j+1}(1) \\ &= xV(y). \end{aligned}$$

To prove $FdV = d$, we begin with a term $x = s_\varphi(a)V^i(1) \in E_n^0$ and compute

$$\begin{aligned} FdV(x) &= Fd(s_\varphi(\varphi^{-1}(a))V^{i+1}(1)) \\ &= F(\varphi^{-1}(a)\cdot dV^{i+1}(1)) \\ &= a\cdot dV^i(1) \\ &= dx, \end{aligned}$$

as required.

To complete the proof, it remains to prove (3.8). For fixed $n \geq 2$, we compute

$$Fd[a] = Fd\left(\sum_{i=0}^{n-1} s_\varphi(a_i)V^i(1)\right),$$

where the a_i are given by the formulas in Lemma 2.5. We then compute further

$$= \sum_{i=1}^{n-1} F(a_i \cdot dV^i(1)) = \sum_{i=2}^{n-1} \varphi(a_i)\cdot dV^{i-1}(1) = \sum_{i=1}^{n-2} \varphi(a_{i+1})\cdot dV^i(1).$$

For the other side of (3.8), we have

$$[a]^{p-1}d[a] = \left(\sum_{j=0}^{n-2} s_\varphi(a_j)V^j(1)\right)^{p-1} d\left(\sum_{i=0}^{n-2} s_\varphi(a_i)V^i(1)\right) = \sum_{i=1}^{n-2}\left(a_i\left(\sum_{j=0}^{n-2} a_j p^j\right)^{p-1}\right)\cdot dV^i(1).$$

We are finished if we can prove that, for every i in the range $1 \leq i \leq n-2$, we have

$$\varphi(a_{i+1}) \equiv a_i\left(\sum_{j=0}^{n-2} a_j p^j\right)^{p-1} \mod p^i A,$$

which is clearly equivalent to proving

$$\varphi(a_{i+1}) \equiv a_i\left(\sum_{j=0}^{i} a_j p^j\right)^{p-1} \mod p^i A.$$

Because φ is an isomorphism, it suffices to prove

$$\varphi^{i+1}(a_{i+1}) \equiv \varphi^i(a_i)\left(\sum_{j=0}^{i} \varphi^i(a_j)p^j\right)^{p-1} \mod p^i A.$$

Recall our definition of the a_j terms:

$$[a] = \sum_{j=0}^{\infty} s_\varphi(a_j)V^j(1) \in W(A).$$

Comparing the ghost components of the two sides, we have $a^{p^i} = \sum_{j=0}^{i} \varphi^i(a_j) p^j$ for every $i \geq 0$. Thus we are finished if we can prove

$$\varphi^{i+1}(a_{i+1}) \equiv \varphi^i(a_i) a^{p^i(p-1)} \bmod p^i A.$$

By Lemma 2.5, we have reduced to showing

$$\frac{a^{p^{i+1}} - (\varphi(a))^{p^i}}{p^{i+1}} \equiv \frac{a^{p^i} - (\varphi(a))^{p^{i-1}}}{p^i} a^{p^i(p-1)} \bmod p^i A,$$

which was proved in Lemma 3.4. This completes the proof of (3.8), and this also completes the proof that $E_{\cdot}^{\bullet}$ is a Witt complex over A. □

Corollary 3.10. *For every integer n, the ring $E_n^{\bullet}$ is p-adically separated.*

Proof. This follows immediately from our definition of $E_n^{\bullet}$: in degree zero, $E_n^0 = W_n(A)$, which is p-adically separated because A is p-adically separated. In degree one, we have $p^{n-1} E_n^1 = 0$, and hence E_n^1 is also p-adically separated. □

Remark 3.11. Our Witt complex $E_{\cdot}^{\bullet}$ is not isomorphic to the de Rham–Witt complex $W.\Omega_A^{\bullet}$. For example, $E_1^1 = 0$, while on the other hand it was shown in Corollary 2.10 that $W_1\Omega_A^1 = \Omega_A^1 \neq 0$. Nor is our Witt complex isomorphic to the relative de Rham–Witt complex of Langer and Zink [2004]: in their Witt complex, one always has $dV(1) = 0$. Following the language of [Hesselholt 2015, Remark 4.8], our Witt complex $E_{\cdot}^{\bullet}$ is the p-typical de Rham–Witt complex over A relative to the p-typical λ-ring (A, s_φ): this follows from the fact that the elements $s_\varphi(\alpha)$ for $\alpha \in \Omega_A^1$ are all zero in E_n^d, and that the differential map $E_{\cdot}^{\bullet} \to E_{\cdot}^{\bullet+1}$ is A-linear.

4. Applications to the de Rham–Witt complex over $A = W(k)$

Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. In this section, we use our p-adically separated Witt complex $E_{\cdot}^{\bullet}$ from Section 3 to give an explicit description (as an A-module) of the de Rham–Witt complex over A.

Remark 4.1. In this section we describe the de Rham–Witt complex over $A = W(k)$ as an A-module. The level n piece of the de Rham–Witt complex over A is always a $W_n(A)$-module. We warn that the $W_n(A)$-module structure does not factor through restriction $W_n(A) \to W_1(A) \cong A$. For example, multiplication by $V(1)$ is nonzero.

As $W.\Omega_A^{\bullet}$ is by definition the initial object in the category of Witt complexes over A, we get a natural map $W.\Omega_A^{\bullet} \to E_{\cdot}^{\bullet}$. The following key result identifies the kernel of this map in degree one.

Proposition 4.2. *Fix any integer $n \geq 1$, and let $S_n \subseteq W_n\Omega_A^1$ be the $W_n(A)$-submodule $\bigcap_{j=1}^{\infty} p^j W_n\Omega_A^1$. The natural map $\eta : W_n\Omega_A^1 \to E_n^1$ induces an isomorphism $W_n\Omega_A^1/S_n \cong E_n^1$.*

Proof. Because E_n^1 is p-adically separated, we see that S_n is contained in the kernel of the map $W_n\Omega_A^1 \to E_n^1$. Consider the composition

$$\Omega^1_{W_n(A)} \to W_n\Omega^1_A \to E^1_n.$$

From our explicit description of E_n^1, we see that this composition is surjective. We will now show that the kernel of this composition is generated as a $W_n(A)$-module by elements of the form

- $ds_\varphi(a)$,
- $(V^j(1) - p^j)dV^i(1)$, and
- $p^i dV^i(1)$.

It is clear that these groups of elements are all in the kernel.

Consider now an arbitrary element $\omega \in \Omega^1_{W_n(A)}$ which is in the kernel; we must show that ω can be expressed as a $W_n(A)$-linear combination of the above elements. Viewing $\Omega^1_{W_n(A)}$ as an A-module via s_φ, we have that an arbitrary element in $\Omega^1_{W_n(A)}$ can be expressed as an A-linear combination of the elements $V^i(1)ds_\varphi(a)$ and $V^j(1)dV^i(1)$ with $0 \le j \le i \le n-1$. Thus we may write

$$\omega = \sum_{i=0}^{n-1} s_\varphi(b_i)V^i(1)ds_\varphi(a_i) + \sum_{0\le j\le i\le n-1} s_\varphi(a_{j,i})V^j(1)dV^i(1),$$

for some elements $b_i, a_i, a_{j,i} \in A$. Because the above itemized elements are all also in the kernel, we deduce that the element

$$\omega' := \sum_{0\le j< i\le n-1} p^j s_\varphi(a_{j,i})dV^i(1)$$

must also be in the kernel. From the explicit description of E_n^1, because ω' is in the kernel of the composition, we have that for each fixed i, we have $\sum_j p^j a_{j,i} \in p^i A$. Thus, for each fixed i, we have that $\sum_j p^j s_\varphi(a_{j,i})dV^i(1)$ is a $W_n(A)$-multiple of $p^i dV^i(1)$. This proves that ω', and hence also ω, is in the $W_n(A)$-submodule generated by the above elements.

We are finished, because $\Omega^1_{W_n(A)} \to W_n\Omega^1_A$ is surjective, and because the images of the above elements in $W_n\Omega^1_A$ are all in the submodule S_n. In fact, the images of the second and third groups of elements are equal to 0 in $W_n\Omega^1_A$: this follows from the identities $p^i dV^i = V^i d$ and

$$V(1)dV^i(1) = V(FdV^i(1)) = V(dV^{i-1}(1)) = pdV^i(1),$$

which hold in every Witt complex. □

The following is modeled after [Hesselholt and Madsen 2003, Section 3.2].

Lemma 4.3. *Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. For every $j \ge 1$, the map*

$$h_j : A \to \Omega^1_A \oplus A, \quad a \mapsto (-da, p^j a),$$

is an A-module homomorphism, where the left-hand side has its A-module structure induced by φ^j and where the right-hand side has component-wise addition and A-module multiplication defined by

$$x \cdot (\alpha, a) = \big(\varphi^j(x)\alpha - \tfrac{1}{p^j} a d\varphi^j(x), \varphi^j(x)a\big).$$

Remark 4.4. For any element $z \in \Omega^1_A$, the term $\frac{1}{p^j} z$ makes sense in Ω^1_A, because multiplication by p is a bijection on Ω^1_A.

Proof. We first check that the right-hand side is actually an A-module with respect to the structure we described. It's clear that $(x_1 + x_2) \cdot (\alpha, a) = x_1 \cdot (\alpha, a) + x_2 \cdot (\alpha, a)$ and that $x \cdot ((\alpha_1, a_1) + (\alpha_2, a_2)) = x \cdot (\alpha_1, a_1) + x \cdot (\alpha_2, a_2)$. Next we compute

$$\begin{aligned} x_1 \cdot (x_2 \cdot (\alpha, a)) &= x_1 \cdot \big(\varphi^j(x_2)\alpha - \tfrac{1}{p^j} a d\varphi^j(x_2), \varphi^j(x_2)a\big) \\ &= \big(\varphi^j(x_1)\big(\varphi^j(x_2)\alpha - \tfrac{1}{p^j} a d\varphi^j(x_2)\big) - \tfrac{1}{p^j}\varphi^j(x_2) a d\varphi^j(x_1), \varphi^j(x_1)\varphi^j(x_2)a\big) \\ &= \big(\varphi^j(x_1x_2)\alpha - \tfrac{1}{p^j} a\varphi^j(x_1) d\varphi^j(x_2) - \tfrac{1}{p^j} a\varphi^j(x_2) d\varphi^j(x_1), \varphi^j(x_1x_2)a\big) \\ &= (x_1x_2) \cdot (\alpha, a). \end{aligned}$$

Notice that so far the $1/p^j$ factor has played no role.

Next we check that the proposed map is an A-module homomorphism; this is where the $1/p^j$ factor becomes important. The map is clearly additive. We then check that, on one hand,

$$\varphi^j(x)a \mapsto (-d(\varphi^j(x)a), p^j\varphi^j(x)a) = (-\varphi^j(x)d(a) - ad(\varphi^j(x)), p^j\varphi^j(x)a),$$

and on the other hand,

$$x \cdot (-da, p^j a) = \big(-\varphi^j(x)da - \tfrac{1}{p^j} p^j ad(\varphi^j(x)), \varphi^j(x)p^j a\big).$$ □

Let M_j denote the cokernel of the A-module homomorphism h_j from Lemma 4.3. (This module is the analogue of what is denoted ${}_hW_n\omega^i_{(R,M)}$ in [Hesselholt and Madsen 2003, Section 3.2].) We are going to describe the de Rham–Witt complex over A in terms of these modules M_j. First we describe an A-module homomorphism $\Omega^1_A \to W_n\Omega^1_A$.

Given any ring homomorphism $R \to S$, there is an induced R-module homomorphism $\Omega^1_R \to \Omega^1_S$. In what follows, we will often use the following special case. Let $s_\varphi : A \to W(A)$ be the ring homomorphism described in Proposition 2.1. For every $n \geq 1$, composing s_φ with the restriction map induces a ring homomorphism $s_\varphi : A \to W_n(A)$ and hence an A-module homomorphism $s_\varphi : \Omega^1_A \to \Omega^1_{W_n(A)} \to W_n\Omega^1_A$. If we want to be explicit about the codomain, we write $s_{\varphi,n}$ instead of s_φ.

Lemma 4.5. *For every integer $n \geq 2$, the two A-module homomorphisms $s_{\varphi,n-1} \circ \varphi \circ \frac{1}{p}$ and $F \circ s_{\varphi,n}$ mapping $\Omega^1_A \to W_{n-1}\Omega^1_A$ are equal.*

Proof. It suffices to prove the images of a term $a_0 da_1$ are equal, and this follows from the relationships $dF = pFd$ and $s_\varphi \circ \varphi = F \circ s_\varphi$. □

Lemma 4.6. *Fix integers $n \geq j \geq 1$ and let M_j be the cokernel of the A-module homomorphism h_j from Lemma 4.3. Consider $W_{n+1}\Omega^1_A$ as an A-module using the map $s_\varphi : A \to W(A)$. The map*

$$\begin{aligned} M_j &\to W_{n+1}\Omega^1_A, \\ (\alpha, a) &\mapsto V^j(s_\varphi(\alpha)) + dV^j(s_\varphi(a)) \end{aligned}$$

is an A-module homomorphism.

Proof. The map is clearly well-defined, because of the relation $p^j dV^j = V^j d$. We have

$$\begin{aligned} x \cdot (\alpha, a) &= \big(\varphi^j(x)\alpha - \tfrac{1}{p^j} a d\varphi^j(x), \varphi^j(x)a\big) \\ &\mapsto V^j \circ s_\varphi\big(\varphi^j(x)\alpha - \tfrac{1}{p^j} a d\varphi^j(x)\big) + dV^j \circ s_\varphi(\varphi^j(x)a) \\ &\qquad = V^j(F^j(s_\varphi(x))s_\varphi(\alpha)) - V^j\big(\tfrac{1}{p^j} s_\varphi(a) dF^j(s_\varphi(x))\big) + dV^j(F^j(s_\varphi(x))s_\varphi(a)) \\ &\qquad = V^j(F^j(s_\varphi(x))s_\varphi(\alpha)) - V^j(s_\varphi(a)F^j ds_\varphi(x)) + d(s_\varphi(x)V^j(s_\varphi(a))) \\ &\qquad = s_\varphi(x)V^j(s_\varphi(\alpha)) - V^j(s_\varphi(a))ds_\varphi(x) + V^j(s_\varphi(a))ds_\varphi(x) + s_\varphi(x)dV^j(s_\varphi(a)) \\ &\qquad = s_\varphi(x)(V^j(s_\varphi(\alpha)) + dV^j(s_\varphi(a))). \end{aligned}$$

□

Proposition 4.7. *Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. Fix any integer $n \geq 1$, and let M_n be the cokernel of the A-module homomorphism from Lemma 4.3. Consider $W_n\Omega^1_A$ and $W_{n+1}\Omega^1_A$ as A-modules via the ring homomorphism $s_\varphi : A \to W(A)$. We have a short exact sequence of A-modules*

$$0 \to M_n \to W_{n+1}\Omega^1_A \xrightarrow{R} W_n\Omega^1_A \to 0, \tag{4.8}$$

where the first map is given by

$$(\alpha, a) \mapsto V^n(s_\varphi(\alpha)) + dV^n(s_\varphi(a)).$$

Proof. Using Lemma 4.6, we see that these are maps of A-modules. Then using Proposition 1.5, we reduce to proving that the map $M_n \to W_{n+1}\Omega^1_A$ is injective. Assume $\alpha \in \Omega^1_A$ and $a \in A$ satisfy $V^n(s_\varphi(\alpha)) + dV^n(s_\varphi(a)) = 0 \in W_{n+1}\Omega^1_A$. Then, because α is divisible by arbitrarily large powers of p, we have that $dV^n(s_\varphi(a))$ is divisible by arbitrarily large powers of p. Write $a' = \varphi^{-n}(a)$. We have

$$dV^n(s_\varphi(a)) = d(s_\varphi(a')V^n(1)) = s_\varphi(a')dV^n(1) + V^n(1)ds_\varphi(a').$$

The term $ds_\varphi(a')$ is divisible by arbitrarily large powers of p, so this implies $s_\varphi(a')dV^n(1)$ is divisible by arbitrarily large powers of p. Thus by Corollary 3.10, the image of $s_\varphi(a')dV^n(1)$ is equal to 0 in E^1_{n+1}, but then by our definition of E^1_{n+1}, we have that a' is divisible by p^n, and hence so is $a = \varphi^n(a')$.

Write $a = p^n a_0$. We then have

$$0 = V^n(s_\varphi(\alpha)) + dV^n(s_\varphi(p^n a_0)) = V^n(s_\varphi(\alpha + da_0)).$$

By Proposition 2.7, the map $p^n : \Omega^1_A \to \Omega^1_A$ is injective. Because $p^n = F^n V^n$, we have that V^n is also injective. This shows that $\alpha = -da_0$, as claimed. □

Remark 4.9. Proposition 4.7 is the main result of this section. The exactness claimed is mostly analogous to [Hesselholt and Madsen 2003, Proposition 3.2.6]; the most interesting part of our result is the fact that the map $A \to \Omega^1_A \oplus A$ surjects onto the kernel of the map $\Omega^1_A \oplus A \to W_{n+1}\Omega^1_A$. This result is difficult to prove because in general it is difficult to prove that elements in the de Rham–Witt complex are nonzero. See [Hesselholt 2005, Proposition 2.2.1] for a result proving this same exactness in the context of the log de Rham–Witt complex over the ring of integers in an algebraic closure of a local field. See also [Illusie 1979, Théorème I.3.8] for a version of this result which is valid in characteristic p.

Using induction, we are able to give the following explicit description of $W_n\Omega^1_A$. The key fact used by the construction is that the maps $\Omega^1_A \oplus A \to W_j\Omega^1_A$ given by $(\alpha, a) \mapsto V^{j-1}(\alpha) + dV^{j-1}(a)$ can be extended to maps into $W_n\Omega^1_A$ using $s_\varphi : A \to W(A)$.

Corollary 4.10. *Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. View $W_{n+1}\Omega^1_A$ as an A-module using the ring homomorphism $s_\varphi : A \to W(A)$. Let $M_0 = \Omega^1_A$, and for every $j \geq 1$, let $M_j = (\Omega^1_A \oplus A)/h_j(A)$ be the cokernel of the A-module homomorphism $h_j : a \mapsto (-da, p^j a)$ from Lemma 4.3. For every integer $n \geq 2$, the map*

$$\prod_{j=0}^{n} M_j \to W_{n+1}\Omega^1_A$$

induced by

$$M_0 \to W_{n+1}\Omega^1_A,$$
$$\alpha_0 \mapsto s_\varphi(\alpha_0)$$

and

$$M_j \to W_{n+1}\Omega^1_A \quad \textit{for } j \geq 1,$$
$$(\alpha_j, a_j) \mapsto V^j(s_\varphi(\alpha_j)) + dV^j(s_\varphi(a_j))$$

is an isomorphism of A-modules.

Proof. We know that the map is a homomorphism of A-modules by Lemma 4.6. For every integer $n \geq 1$, consider the complex

$$\begin{tikzcd} 0 \arrow[r] & M_n \arrow[r] \arrow[d] & \prod_{j=0}^{n} M_j \arrow[r] \arrow[d] & \prod_{j=0}^{n-1} M_j \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M_n \arrow[r] & W_{n+1}\Omega^1_A \arrow[r] & W_n\Omega^1_A \arrow[r] & 0. \end{tikzcd}$$

The top row is clearly exact. The bottom row is exact by (4.8). The right-hand vertical map is an isomorphism by induction. Thus we are finished by the five lemma. □

Similar, but easier, arguments work also for degrees $d \geq 2$. Our applications involve degree $d = 1$, so we indicate the results more briefly.

Proposition 4.11. *For every $d \geq 2$, $n \geq 1$, we have an exact sequence of A-modules*

$$0 \to \Omega^d_A \xrightarrow{V^n} W_{n+1}\Omega^d_A \to W_n\Omega^d_A \to 0,$$

where the A-module structure on Ω^d_A is given by $a \cdot \alpha := F^n(a)\alpha$, and where the A-module structure on the other two pieces is induced by $s_\varphi : A \to W(A)$.

Proof. The map $V^n : \Omega^d_A \to W_{n+1}\Omega^d_A$ is injective because $F^n \circ V^n = p^n$ is injective on Ω^d_A. We must also show that if $\omega \in W_{n+1}\Omega^d_A$ is in the kernel of R, then we can find $\alpha \in \Omega^d_A$ such that $\omega = V^n(\alpha)$. We know that there exist $\alpha \in \Omega^d_A$ and $\beta \in \Omega^{d-1}_A$ such that

$$V^n(\alpha) + dV^n(\beta) = \omega.$$

But now we are finished, because we can write $\beta = p^n\beta_0$ for some $\beta_0 \in \Omega^{d-1}_A$. (This is where we use that $d \geq 2$.) □

We can deduce the following corollary in the same way as we deduced Corollary 4.10.

Corollary 4.12. *For every $d \geq 2$ and every $n \geq 1$, we have an isomorphism of A-modules*

$$\prod_{i=0}^{n-1} \Omega^d_A \cong W_n\Omega^d_A,$$

where the A-module structure on the i-th piece is given by $a \cdot \alpha_i := \varphi^i(a)\alpha_i$.

Remark 4.13. Much of the author's intuition for the de Rham–Witt complex comes from the cases treated in [Illusie 1979], such as the description of the de Rham–Witt complex over $\mathbb{F}_p[t_1, \ldots, t_r]$ given in [loc. cit., Section I.2]. In this case, the de Rham–Witt complex is 0 in degrees $d > r$. We remark that the absolute, mixed characteristic de Rham–Witt complex we are studying is very different. Consider the easiest case of our setup, $A = \mathbb{Z}_p = W(\mathbb{F}_p)$. Then Ω^1_A is infinite-dimensional as a $\mathbb{Q}_p$-vector space by Proposition 2.9. Thus $\Omega^d_A := \bigwedge^d \Omega^1_A$ is nonzero for all degrees d. Thus in particular $W_n\Omega^d_A$ is nonzero for all integers $d \geq 0$ and $n \geq 1$.

Remark 4.14. Corollaries 4.10 and 4.12 give an explicit description of the A-module structure of the Witt complex $W.\Omega^\bullet_A$. (Notice that for a general ring $B \neq W(k)$, we cannot expect a B-algebra structure on $W.\Omega^\bullet_B$.) It seems worthwhile to describe the entire Witt complex structure, at least for degrees $d = 0, 1$, in terms of the description from Corollary 4.10. Similar descriptions could be given for higher degrees.

- We already know the A-module structure, so to describe the $W_n(A)$-algebra structure on $W_n\Omega^1_A$, it suffices by Lemma 2.2 to describe the effect of multiplication by $V^j(1)$ on $\prod M_i$. It sends all M_i with $i \leq j$ into the M_j component, via the formulas

$$\begin{aligned} V^j(1) \cdot \alpha &= (\varphi^j(\alpha), 0) \in M_j \text{ for } \alpha \in M_0 = \Omega^1_A, && \text{and} \\ V^j(1) \cdot (\alpha_i, a_i) &= (p^i\varphi^{j-i}(\alpha_i) + \varphi^{j-i}(da_i), 0) \in M_j, && \text{for } (\alpha_i, a_i) \in M_i, \text{ where } i \leq j. \end{aligned}$$

  When $i \geq j$, multiplication by $V^j(1)$ acts on the M_i component as multiplication by p^j.

- To describe the differential $d : W_n(A) \to \prod M_i$, it suffices by Lemma 2.2 to note that $d : s_\varphi(a) \mapsto da \in M_0 = \Omega^1_A$ and that $d : s_\varphi(a_j)V^j(1) \mapsto (0, \varphi^j(a_j)) \in M_j$ for $j \geq 1$.
- The restriction map $R : \prod_{i=0}^{n} M_i \to \prod_{i=0}^{n-1} M_i$ is the obvious projection map.
- To describe the map $V : \prod_{i=0}^{n} M_i \to \prod_{i=0}^{n+1} M_i$, we note that

$$\begin{aligned} V &: \alpha \mapsto (\alpha, 0) \in M_1, && \text{where } \alpha \in M_0 = \Omega^1_A, \text{ and} \\ V &: (\alpha_i, a_i) \mapsto (\alpha_i, pa_i) \in M_{i+1}, && \text{where } (\alpha_i, a_i) \in M_i. \end{aligned}$$

- To describe the map $F : \prod_{i=0}^{n+1} M_i \to \prod_{i=0}^{n} M_i$, we note that

$$\begin{aligned} F &: \alpha \mapsto \varphi(\alpha) \in M_0, && \text{for } \alpha \in M_0 = \Omega^1_A \\ F &: (\alpha_1, a_1) \mapsto p\alpha_1 + da_1 \in M_0, && \text{for } (\alpha_1, a_1) \in M_1, \text{ and} \\ F &: (\alpha_i, a_i) \mapsto (p\alpha_i, a_i) \in M_{i-1}, && \text{for } (\alpha_i, a_i) \in M_i. \end{aligned}$$

Corollary 4.15. *For every $n \geq 1$, the p-torsion submodule of $W_n\Omega^1_A$ is isomorphic to the free A/p-module of rank $n-1$ generated by $p^{j-1}dV^j(1)$, for $j = 1, \ldots, n-1$.*

Proof. Using the fact that multiplication by p is a bijection on Ω^1_A, we see that the p-torsion module in $M_j = (\Omega^1_A \oplus A)/h_j(A)$ is a free A/pA-module of rank 1 generated by $(0, p^{j-1})$. Then from Corollary 4.10, we see that these elements together generate the p-torsion submodule of $W_n\Omega^1_A$. In the factor $M_j \cong (\Omega^1_A \oplus A)/h_j(A)$, a representative (α, a) has element a uniquely determined modulo p^jA. This shows that we have a relation

$$\sum dV^j(p^{j-1}\varphi^j(a)) = 0$$

only if each $a \in pA$. This shows that the proposed elements are free generators, which completes the proof. □

5. The de Rham–Witt complex over A/xA

As usual, let p denote an odd prime, let k denote a perfect ring of characteristic p, and let $A = W(k)$. There are two natural ways to lift elements from A to $W(A)$: the first is our ring homomorphism s_φ, and the second is the multiplicative Teichmüller map. So far in this paper, we have made extensive use of the ring homomorphism s_φ. In this section and the next, we make more frequent use of the Teichmüller map. The reason is that we will be studying the kernel of the natural ring homomorphism $W(A) \to W(A/xA)$ for $x \in A$, and $[x]$ is in this kernel whereas $s_\varphi(x)$ in general is not. For example, $[p]$ is in the kernel of $W(\mathbb{Z}_p) \to W(\mathbb{Z}_p/p\mathbb{Z}_p)$, whereas $s_\varphi(p) = p$ is not.

The exactness in (4.8) above is very useful for making induction arguments involving the de Rham–Witt complex. For example, our proof of Corollary 4.10 was dependent on our Witt complex $E^\bullet_\cdot$ only because $E^\bullet_\cdot$ was used to prove exactness in (4.8). The goal of the remainder of the paper is to prove exactness of the corresponding sequence for the de Rham–Witt complex over a certain class of perfectoid rings. See [Hesselholt 2006, Proposition 2.2.1; Hesselholt and Madsen 2003, Theorem 3.3.8] for related results. In future joint work with Irakli Patchkoria, we hope to use this exact sequence to provide algebraic proofs of results similar to Hesselholt's p-adic Tate module computation [2006, Proposition 2.3.2]. In this section we prove general results concerning $W_n\Omega^1_{(A/xA)}$ that are valid for arbitrary $x \in A$. In Section 6, we specialize to a certain class of perfectoid rings, in which case we can prove stronger results, including the analogue of the exact sequence in (4.8).

Fix an element $x \in A$. For every integer $n \geq 1$, we have a surjective A-module homomorphism $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$, and Corollary 4.10 gives an explicit description of the domain. We will give explicit A-module generators for the kernel. Unfortunately, this kernel is not generated as an A-module by elements which are homogeneous with respect to the direct sum decomposition from Corollary 4.10.

First we consider the case of level $n = 1$, which will be used repeatedly.

Lemma 5.1. *The kernel of the A-module homomorphism $\Omega^1_A \to \Omega^1_{(A/xA)}$ is generated by $x\alpha$ for $\alpha \in \Omega^1_A$ together with the element dx.*

Proof. This follows immediately from the usual right exact sequence of (A/xA)-modules

$$xA/x^2A \to \Omega^1_A \otimes_A (A/xA) \to \Omega^1_{(A/xA)} \to 0 \tag{5.2}$$

[Matsumura 1989, Theorem 25.2], where the left-most map is given by $xa \mapsto d(xa) \otimes 1$. $\square$

Next we identify the kernel in the degree zero case, $W.\Omega^0_A \to W.\Omega^0_{(A/xA)}$.

Lemma 5.3. *Let K^0 denote the kernel of the ring homomorphism $W(A) \to W(A/xA)$ induced by the projection $A \to A/xA$. Then K^0 consists precisely of elements of the form*

$$\sum_{k=0}^{\infty} s_\varphi(a_k) V^k([x]),$$

where $a_k \in A$.

Proof. It's clear that these elements are in the kernel. We now prove that an arbitrary element in the kernel can be written in this way. Working one level at a time, it suffices to show that if $V^k(y_k)$ is in the kernel, then we can find $a_k \in A$ and $y_{k+1} \in W(A)$ such that

$$V^k(y_k) = s_\varphi(a_k)V^k([x]) + V^{k+1}(y_{k+1}).$$

(Note that this also implies that $V^{k+1}(y_{k+1})$ is in the kernel.) Because

$$s_\varphi(a_k)V^k([x]) = V^k(F^k(s_\varphi(a_k))[x]) = V^k(s_\varphi(\varphi^k(a_k))[x])$$

and $\varphi : A \to A$ is surjective, we can find such elements a_k and y_{k+1}. $\square$

We now do the same thing for the degree one case. In this case, the ring $W(A) \cong \varprojlim W_n(A)$ from Lemma 5.3 gets replaced by the $W(A)$-module, $\varprojlim W_n\Omega^1_A$. Corollary 4.10 leads to an explicit description of this inverse limit as an A-module.

More concretely, we give generators for the kernels of the A-module homomorphisms $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$, and we choose these generators so they are compatible under restriction maps for varying

$n \geq 1$. We view these generators as elements in $\varprojlim W_n\Omega^1_A$. The main work involves studying, for particular choices of A and x, the A-submodule of $\varprojlim W_n\Omega^1_A$ generated by these elements in the kernel. Because these elements involve the Teichmüller lift $[x]$, they do not have a simple description in terms of our decomposition of $W_n\Omega^1_A$ given in Corollary 4.10.

Definition 5.4. Let $M_0 = \Omega^1_A$ and for each integer $j \geq 1$, let M_j be the cokernel of the A-module homomorphism in Lemma 4.3. Let M denote the A-module

$$M = \prod_{j=0}^{\infty} M_j.$$

Let $K^1 \subseteq M$ denote the A-submodule consisting of all elements of the form

$$\sum_{k=0}^{\infty} (V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])),$$

where $\alpha_k \in \Omega^1_A$ and where $a_k \in A$; here, to make sense of such an expression as an element in M, we use the structures described in Remark 4.14.

Remark 5.5. (1) By Corollary 4.10, M is isomorphic as an A-module to $\varprojlim W_n\Omega^1_A$.

(2) The A-module K^1 depends on our choice of element x, but that element is fixed throughout this section, so we write simply K^1 and not more suggestive notation such as K^1_x.

We will use K^1 from Definition 5.4 to describe the kernel of $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$; namely, we will show that this kernel is the image of K^1 under the restriction map $R_n : M \to W_n\Omega^1_A$.

Lemma 5.6. *For $n \geq 1$, write R_n for the restriction map $W(A) \to W_n(A)$ and also for the restriction map $M \to W_n\Omega^1_A$. The A-submodule of $W_n\Omega^\bullet_A$ generated by $R_n(K^0)$ and $R_n(K^1)$ and all higher degree terms ($W_n\Omega^d_A$ for $d \geq 2$) is an ideal in the ring $W_n\Omega^\bullet_A$.*

Proof. We have to show that the A-module generated by these elements is closed under multiplication by elements in $W_n\Omega^\bullet_A$. Consider an element $V^k([x])m$, where $m \in W_n\Omega^1_A$. This can be rewritten as $V^k([x]m_0)$, where $m_0 = F^k(m)$. The element m_0 can be written (not uniquely) as

$$m_0 = s_\varphi(\alpha_0) + \sum_{i=1}^{n-k-1} (V^i(s_\varphi(\alpha_i)) + dV^i(s_\varphi(a_i))),$$

and so

$$\begin{aligned}[x]m_0 &= [x]s_\varphi(\alpha_0) + \sum_{i=1}^{n-k-1} ([x]V^i(s_\varphi(\alpha_i)) + [x]dV^i(s_\varphi(a_i)))\\ &= [x]s_\varphi(\alpha_0) + \sum_{i=1}^{n-k-1} (V^i([x]^{p^i}s_\varphi(\alpha_i)) + dV^i([x]^{p^i}s_\varphi(a_i)) - V^i(s_\varphi(a_i)[x]^{p^i-1}d[x])).\end{aligned}$$

(Here we used the formula $Fd[x] = [x]^{p-1}d[x]$.) And so

$$V^k([x]m_0) \in R_n(K^1).$$

Now we consider degree 1 terms in our A-module. We first consider a term $V^k([x]s_\varphi(\alpha))$ and then below we consider $dV^k([x])$. We can write an arbitrary element $y \in W_n(A)$ as $\sum_{i=0}^{n-1} s_\varphi(y_i)V^i(1)$, thus it suffices to show that

$$V^k([x]s_\varphi(\alpha))V^i(1) \in R_n(K^1).$$

If $i \leq k$, we have

$$V^k([x]s_\varphi(\alpha))V^i(1) = V^k([x]s_\varphi(p^i\alpha)) \in R_n(K^1).$$

If $i > k$, we have

$$V^k([x]s_\varphi(\alpha))V^i(1) = V^i(F^{i-k}([x]s_\varphi(\alpha))) = V^i([x]^{p^{i-k}} s_\varphi\big(\tfrac{1}{p^{i-k}}\varphi^{i-k}(\alpha)\big)) \in R_n(K^1).$$

Similarly, we find

$$dV^k([x])V^i(1) = V^i(dV^{k-i}([x]) = p^i dV^k([x]) \quad \text{for } i \leq k$$

and

$$dV^k([x])V^i(1) = V^i(F^{i-k}d[x]) = V^i([x]m) \quad \text{for } i > k \text{ and } m \in W_n\Omega^1_A.$$

It was shown in the degree zero portion of our proof that this latter element is in $R_n(K^1)$. ☐

Proposition 5.7. *Define $G^\bullet_\cdot$ by*

$$G^0_n := W_n(A)/R_n(K^0), \quad G^1_n := W_n\Omega^1_A/R_n(K^1), \quad G^d_n := 0 \text{ for } d \geq 2.$$

Equipped with the structure maps inherited from $W_\cdot\Omega^\bullet_A$, this is a Witt complex over A/xA.

Proof. The main thing to verify is that all of the necessary maps are well-defined. All the various relations required of a Witt complex will then hold automatically since they hold in $W_n\Omega^\bullet_A$.

The fact that $G^\bullet_n$ is a ring follows from Lemma 5.6. Define $\lambda : W_n(A/xA) \to G^0_n$ to be the unique map such that the composition $W_n(A) \to W_n(A/xA) \to G^0_n$ is the projection map; this is possible by Lemma 5.3. To define the differential $d : G^0_n \to G^1_n$, we check that $d(s_\varphi(a)V^k([x])) \in R_n(K^1)$, which follows because

$$d(s_\varphi(a)V^k([x])) = s_\varphi(a)dV^k([x]) + V^k([x]F^k ds_\varphi(a)) = s_\varphi(a)dV^k([x]) + V^k([x]s_\varphi\big(\tfrac{1}{p^k}d\varphi^k(a)\big)),$$

where the last equality holds by Lemma 4.5. Because $R \circ R_n = R_{n-1}$, it is clear that the restriction map R is well-defined. The fact that V is well-defined follows from $VdV^k = pdV^{k+1}$ and the fact that K^1 is closed under multiplication by arbitrary elements in $W(A)$.

To check that F is well-defined on G_n^1, we need to show that $F(R_n(K^1)) \subseteq R_{n-1}(K^1)$, which means that we need to evaluate F on elements

$$\sum_{k=0}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])).$$

The result is immediate from the de Rham–Witt relations, but we need to be careful to treat the $k=0$ case separately from the $k>0$ case. We have

$$F([x]s_\varphi(\alpha_0)) = [x]^p s_\varphi\bigl(\tfrac{1}{p}\varphi(\alpha_0)\bigr) \quad \text{and} \quad F(s_\varphi(a_0)d[x]) = s_\varphi(\varphi(a_0))[x]^{p-1}d[x],$$

and these elements are in $R_{n-1}(K^1)$ by Lemma 5.6. For $k \geq 1$, we have

$$F(V^k([x]s_\varphi(\alpha_k))) \quad \text{and} \quad F(s_\varphi(a_k)dV^k([x])) \in R_{n-1}(K^1),$$

because $FV = p$ and $FdV = d$. □

Proposition 5.8. *We have an isomorphism of A-modules*

$$W_n\Omega_A^1/R_n(K^1) \cong W_n\Omega_{(A/xA)}^1.$$

Proof. Viewing $W_n\Omega^1_{(A/xA)}$ as a Witt complex over A, we have a map of $W_n(A)$-modules $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$ which induces a map $f : (W_n\Omega^1_A)/R_n(K^1) \to W_n\Omega^1_{(A/xA)}$. Similarly, $G^\bullet_\cdot$ is a Witt complex over A/xA by Proposition 5.7, so we have a map of $W_n(A/xA)$-modules $g : W_n\Omega^1_{(A/xA)} \to (W_n\Omega^1_A)/R_n(K^1)$. We claim that the compositions gf and fg are both the identity map.

Because the maps f and g arise from maps of Witt complexes, the two triangles in the following diagram commute.

$$\begin{tikzcd} W_n\Omega^1_A/R_n(K^1) \arrow[rr, bend left, "f"] & & W_n\Omega^1_{(A/xA)} \arrow[ll, bend left, "g"] \\ & \Omega^1_{W_n(A/xA)} \arrow[ul, two heads] \arrow[ur, two heads] & \end{tikzcd}$$

Then, because the diagonal maps are both surjective, a diagram chase shows that fg and gf are both the identity map. □

We conclude this section with a technical result about K^1 that will be used in the following section. We include it in this section because it is valid in a more general context than what we consider in Section 6.

Notation 5.9. For every integer $n \geq 1$, let P_n denote the property

- P_n: If $z \in K^1$ and $R_n(z) = 0$, then we can write

$$z = \sum_{k=n}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])).$$

Proposition 5.10. *Assume that $x \notin pA$. If property P_1 holds, then for every integer $n \geq 1$, the property P_n also holds.*

Proof. We prove this using induction on n. Thus assume we know that property P_{n-1} holds for some $n \geq 2$, and assume we have $z \in K^1$ such that $R_n(z) = 0$. By our induction hypothesis, we can assume

$$z = \sum_{k=n-1}^{\infty} (V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])).$$

The terms for $k \geq n$ do not affect the conclusion, so we can in fact assume

$$\begin{aligned} z &= V^{n-1}([x]s_\varphi(\alpha)) + s_\varphi(a)dV^{n-1}([x]) \\ &= V^{n-1}([x]s_\varphi(\alpha)) + dV^{n-1}([x]F^{n-1}(s_\varphi(a))) - V^{n-1}([x]F^{n-1}(ds_\varphi(a))) \\ &= V^{n-1}\bigl([x]s_\varphi\bigl(\alpha - \tfrac{1}{p^{n-1}}d(\varphi^{n-1}(a))\bigr)\bigr) + dV^{n-1}([x]s_\varphi(\varphi^{n-1}(a))). \end{aligned}$$

Using Proposition 4.7, because we are assuming $R_n(z) = 0$ and that x is not divisible by p, we have that a must be divisible by p^{n-1}, and we find

$$z = V^{n-1}\bigl([x]s_\varphi\bigl(\alpha - \tfrac{1}{p^{n-1}}d(\varphi^{n-1}(a))\bigr) + d([x]s_\varphi(\varphi^{n-1}(a/p^{n-1})))\bigr).$$

The fact that $R_n(z) = 0$ implies that

$$[x]s_\varphi\bigl(\alpha - \tfrac{1}{p^{n-1}}d(\varphi^{n-1}(a))\bigr) + d([x]s_\varphi(\varphi^{n-1}(a/p^{n-1})))$$

satisfies the assumption in property P_1. Hence we have

$$\begin{aligned} z &= V^{n-1}\Bigl(\sum_{k=1}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x]))\Bigr) \\ &= \sum_{k=1}^{\infty}(V^{k+n-1}([x]s_\varphi(\alpha_k)) + s_\varphi(\varphi^{1-n}(a_k))V^{n-1}(dV^k([x]))) \\ &= \sum_{k=1}^{\infty}(V^{k+n-1}([x]s_\varphi(\alpha_k)) + s_\varphi(\varphi^{1-n}(p^{n-1}a_k))dV^{k+n-1}([x])). \end{aligned}$$

This completes the proof of property P_n. □

Lemma 5.11. *An element $z \in M$ can be written in the form*

$$z = \sum_{k=n}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x]))$$

if and only if

$$z \in V^n(K^1) + dV^n(K_0).$$

In particular, property P_n is equivalent to the following:

- *If $z \in K^1$ and $R_n(z) = 0$, then we have*

$$z \in V^n(K^1) + dV^n(K_0).$$

Proof. This follows from the same sorts of manipulations as in the above proofs. The most difficult of these manipulations is showing that

$$s_\varphi(a_n) dV^n([x]) \in V^n(K^1) + dV^n(K_0).$$

Using Lemma 4.5 and the Leibniz rule, one checks that

$$s_\varphi(a_n) dV^n([x]) = V^n\big([x] s_\varphi\big(\varphi^n\big(\tfrac{-1}{p^n} d(a_n)\big)\big)\big) + dV^n(s_\varphi(\varphi^n(a_n))[x]) \in V^n(K^1) + dV^n(K_0). \qquad \square$$

Similar manipulations show the following.

Lemma 5.12. *For every integer $n \geq 1$, we have that $V^n(K^1) + dV^n(K_0) \subseteq M$ is a $W(A)$-submodule.*

Proof. It's clear that the collection of elements of the form

$$z = \sum_{k=n-1}^{\infty} (V^k([x] s_\varphi(\alpha_k)) + s_\varphi(a_k) dV^k([x]))$$

forms an A-module, so we reduce to proving that $V^n(K^1) + dV^n(K_0)$ is closed under multiplication by $V^i(1)$, for $i \geq 1$. Consider first the case $i \geq n$. We have

$$V^i(1) V^n(K^1) = V^i(p^n F^{i-n}(K^1)) \subseteq V^n(K^1), \quad V^i(1) dV^n(K^0) = V^i(F^{i-n} d(K^0)) \subseteq V^n(K^1).$$

Next we consider the case $i < n$. We have

$$V^i(1) V^n(K^1) = V^n(p^i K^1) \subseteq V^n(K^1),$$
$$V^i(1) dV^n(K^0) = d(V^i(1) V^n(K^0)) - V^n(K^0) dV^i(1) \subseteq dV^n(K^0). \qquad \square$$

We cannot expect property P_1 to hold in general, as the following example shows. In the next section we will prove that property P_1 (and hence property P_n for every n) holds when A/xA is a perfectoid ring satisfying Assumption 6.2 below.

Example 5.13. Consider the ring $A = \mathbb{Z}_p$ and the element $x = p \in \mathbb{Z}_p$. Clearly

$$d[p] \in W_2 \Omega^1_{\mathbb{Z}_p}$$

restricts to $dp = 0$ in $\Omega^1_{\mathbb{Z}_p}$. On the other hand, because

$$[p] \equiv p + V(p^{p-1} - 1) \bmod V^2(W(\mathbb{Z}_p)),$$

we have

$$d[p] = -dV(1) \in W_2 \Omega^1_{\mathbb{Z}_p}.$$

The exactness of sequence (4.8) shows this element cannot be written as a $\mathbb{Z}_p$-linear combination of terms in $V(p\Omega^1_{\mathbb{Z}_p}) = V(\Omega^1_{\mathbb{Z}_p})$ and $dV(p) = Vd1 = 0$.

6. Applications to the de Rham–Witt complex over perfectoid rings

As usual, p in this section denotes an odd prime. The term *perfectoid* was originally used in the context of algebras over a field, but we work with the more general notion of *perfectoid ring* which has since been defined; see Definition 6.1 below. Examples of rings satisfying our definition of perfectoid include the p-adic completion of $\mathbb{Z}_p[\zeta_{p^\infty}]$, the p-adic completion of $\mathbb{Z}_p[p^{1/p^\infty}]$, and $\mathbb{O}_{\mathbb{C}_p}$.

Throughout this section, we let B denote a perfectoid ring satisfying Assumption 6.2 below, and we let $A = W(B^\flat)$, where

$$B^\flat := \varprojlim_{x \mapsto x^p} (B/pB)$$

is the tilt of B. The ring $B^\flat$ is a perfect ring of characteristic p. Let $\theta : A = W(B^\flat) \to B$ denote the map θ_1 from [Bhatt et al. 2016, Section 3]. This is the "usual" θ map from p-adic Hodge theory. We will not need the definition of θ; we will only need that it is surjective and its kernel is a principal ideal (by our definition of perfectoid). Throughout this section, $x \in A$ denotes a fixed choice of generator for this principal ideal.

We now explicitly state our definition of perfectoid.

Definition 6.1 [Bhatt et al. 2016, Definition 3.5]. A commutative ring B is called *perfectoid* if it is π-adically complete and separated for some element $\pi \in B$ such that π^p divides p, the Frobenius map $B/pB \to B/pB$ is surjective, and the kernel of $\theta : W(B^\flat) \to B$ is principal.

Assumption 6.2. We further assume that our perfectoid ring B is p-torsion free and that there exists a p-power torsion element $\omega \in \Omega^1_B$ such that the annihilator of ω is contained in $p^n B$ for some integer $n \geq 1$.

Remark 6.3. (1) Assumption 6.2 is satisfied, for example, if the perfectoid ring B is contained in $\mathbb{O}_{\mathbb{C}_p}$ and contains ζ_p. We do not know an elementary argument for this. Fontaine [1981/82, Théorème 1′] gives an elementary argument to show that $d\zeta_p$ is nonzero in Ω^1_R, where $R = \mathbb{O}_{\overline{\mathbb{Q}}_p}$. Bhargav Bhatt has shown us an argument involving the cotangent complex (which was used above in the proof of Proposition 2.7) to deduce that $d\zeta_p \in \Omega^1_{\mathbb{O}_{\mathbb{C}_p}}$ is nonzero. Once one knows that $d\zeta_p \neq 0$, an elementary argument shows that Assumption 6.2 is satisfied. We hope to consider the question, "How restrictive is Assumption 6.2?", in later applications.

(2) Our proofs in this section work for any quotient A/xA satisfying Assumption 6.2, but we do not know any interesting examples where A/xA is not perfectoid. In particular, see the next point.

(3) We have been careful throughout this paper to work with $W(k)$ where k is a perfect ring, instead of restricting our attention to the case where k is a perfect field. That generality is essential for Assumption 6.2 to be reasonable, because when k is a perfect field, the only p-torsion free quotient of $W(k)$ is the zero ring.

The entire goal of this section is to prove Proposition 6.12 below, which identifies the kernel of restriction $W_{n+1}\Omega^1_B \to W_n\Omega^1_B$ in terms of B and Ω^1_B. Using a spectral sequence argument, our result will follow easily from property P_n described in Notation 5.9. By Proposition 5.10, it will suffice to prove property P_1, which loosely says that if an element in $W_n\Omega^1_A$ is in both $\ker\theta$ and in the kernel of restriction R_1 to Ω^1_A, then the element can be written as $V(\alpha)+dV(a)$, where both α and a are in $\ker\theta$. We now begin the proof that property P_1 holds.

We will apply the following lemma to our fixed $x \in A$ which generates $\ker\theta$, but it also holds for arbitrary $x \in A$.

Lemma 6.4. *Choose $y \in W(A)$ such that $[x] = s_\varphi(x) + V(y)$. Then we have*

$$[x]^p = s_\varphi(\varphi(x)) + py$$

Proof. Apply F to both sides of $[x] = s_\varphi(x) + V(y)$. □

Property P_1 concerns elements which are both in the kernel of $W_n(\theta) : W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$ and also in the kernel of restriction $R_1 : W_n\Omega^1_A \to \Omega^1_A$. The following lemma considers the case of a particular element which is obviously in this intersection.

Lemma 6.5. *We have $[x]ds_\varphi(x) - s_\varphi(x)d[x] \in V(K^1) + dV(K^0)$.*

Proof. We use the notation from Lemma 6.4. We compute

$$\begin{aligned}
[x]ds_\varphi(x) - s_\varphi(x)d[x] &= (s_\varphi(x) + V(y))ds_\varphi(x) - s_\varphi(x)d(s_\varphi(x) + V(y)) \\
&= V(y)ds_\varphi(x) - s_\varphi(x)dV(y) \\
&= V(yFds_\varphi(x)) - d(s(x)V(y)) + V(y)ds_\varphi(x) \\
&= V(2yFds_\varphi(x)) - dV(yF(s_\varphi(x))) \\
&= V(2yFds_\varphi(x)) - dV(y[x]^p - py^2).
\end{aligned}$$

Because the term $dV(y[x]^p) \in dV(K^0)$, we reduce to showing the following element is in $V(K^1)$.

$$\begin{aligned}
V(2yFds_\varphi(x)) + dV(py^2) &= V(2yFds_\varphi(x) + 2ydy) \\
&= V(2y(Fds_\varphi(x) + dy)) \\
&= V(2y(Fd([x] - V(y)) + dy)) \\
&= V(2y([x]^{p-1}d[x] - dy + dy)) \in V(K^1).
\end{aligned}$$

This completes the proof. □

Lemma 6.6. *If $x\alpha_1 = 0 \in \Omega^1_A$, then $[x]s_\varphi(\alpha_1) \in V(K^1)$.*

Proof. The key idea is that, because multiplication by p is a bijection on Ω^1_A, we also have that $x\alpha_1/p^N = 0 \in \Omega^1_A$ for every integer $N \geq 1$. Applying Frobenius to both sides, we have $\varphi(x)\varphi(\alpha_1)/p^N = 0 \in \Omega^1_A$. We will apply this observation in the case $N = 2$.

Use the same notation as in Lemma 6.4. We have

$$\begin{aligned}[x]s_\varphi(\alpha_1) &= s_\varphi(x)s_\varphi(\alpha_1) + V(y)s_\varphi(\alpha_1) \\ &= V(y)s_\varphi(\alpha_1) \\ &= V(yF(s_\varphi(\alpha_1))) \\ &= V\left(ys_\varphi\left(\frac{\varphi(\alpha_1)}{p}\right)\right) \\ &= V\left(pys_\varphi\left(\frac{\varphi(\alpha_1)}{p^2}\right)\right) \\ &= V\left(([x]^p - s_\varphi(\varphi(x)))s_\varphi\left(\frac{\varphi(\alpha_1)}{p^2}\right)\right) \\ &= V\left([x]^p s_\varphi\left(\frac{\varphi(\alpha_1)}{p^2}\right)\right) \in V(K^1).\end{aligned}$$

□

Using Assumption 6.2, we have a p-power torsion element $\omega \in \Omega^1_B$ with annihilator contained in $p^n B$ for some integer $n \geq 1$. For every integer $r \geq 1$, the following lemma enables us to produce a p-power torsion element $\eta \in \Omega^1_B$ with annihilator contained in $p^{n+r} B$.

Lemma 6.7. *Assume $\omega \in \Omega^1_B$ is such that* $\operatorname{Ann}\omega \subseteq p^n B$, *where $n \geq 1$ is an integer. If $\eta \in \Omega^1_B$ is an element such that $p^r\eta = \omega$ for some integer $r \geq 1$, then* $\operatorname{Ann}\eta \subseteq p^{n+r} B$.

Proof. It suffices to prove this in the case $r = 1$, so let $\eta \in \Omega^1_B$ be such that $p\eta = \omega$. Let $b \in \operatorname{Ann}\eta$. Then in particular $b \in \operatorname{Ann}\omega$, so we can write $b = p^n b_0$ for some $b_0 \in B$. Then we know

$$0 = b\eta = p^n b_0 \eta = p^{n-1} b_0 \omega,$$

and hence $p^{n-1}b_0 \in p^n B$. Assumption 6.2 requires that B is p-torsion free, so we deduce that $b_0 \in pB$, and hence $b \in p^{n+1}B$, as required. □

The following is the most important of the preliminary results in this section. If we could prove Proposition 6.8 without using the element ω from Assumption 6.2, then the results of this section would hold for all p-torsion free perfectoid rings.

Proposition 6.8. *If $a\,dx \in x\Omega^1_A$, then $a \in xA$.*

Proof. Our hypothesis implies $a\frac{dx}{p^N} \in \ker\theta$ for every integer $N \geq 0$, and we will show this implies $\theta(a) \in \bigcap p^r B = 0$.

Fix an integer $N \geq 1$. Because $\theta : A \to B$ is surjective, we know the induced map $\Omega^1_A \to \Omega^1_B$ is surjective. Let $\omega_A \in \Omega^1_A$ map to the element $\omega \in \Omega^1_B$ described in Assumption 6.2. Because ω is p-power torsion, we know that $p^m\omega_A \in x\Omega^1_A + A\,dx$ for some integer $m \geq 1$. Thus, for every integer $N \geq 1$, we can write $(1/p^{N-m})\omega_A = x\alpha_N + a_N\frac{dx}{p^N}$ for some $\alpha_N \in \Omega^1_A$ and $a_N \in A$.

Consider now the element $a\,dx \in x\Omega^1_A$ from the statement of this proposition. We deduce that $a\frac{dx}{p^N} \in x\Omega^1_A$ for every integer $N \geq 1$, so $a\frac{dx}{p^N} \in \ker\theta$ for every integer $N \geq 1$. If we multiply by the element a_N from the previous paragraph, we know that $aa_N \frac{dx}{p^N}$ is in $\ker\theta$ for every integer $N \geq 1$. If we apply θ to $aa_N \frac{dx}{p^N}$, we see that $\theta(a) \in B$ is in the annihilator of some element η satisfying $p^{N-m}\eta = \omega$. Thus, by Lemma 6.7, we have that $\theta(a) \in B$ is divisible by arbitrarily large powers of p. Thus $a \in xA$, as required. □

Remark 6.9. Proposition 6.8 implies that for our particular rings A and A/xA, the left-most map in the exact sequence (5.2) is injective.

Proposition 6.10. *If $x\alpha + adx = 0 \in \Omega^1_A$, then $[x]s_\varphi(\alpha) + s_\varphi(a)d[x] \in V(K^1) + dV(K^0)$.*

Proof. We have $adx = -x\alpha$, so by Proposition 6.8, we know that $a = xa_1$ for some $a_1 \in A$, and thus our assumption means $x(\alpha + a_1 dx) = 0 \in \Omega^1_A$. By Lemma 6.6, we know that $[x](s_\varphi(\alpha) + s_\varphi(a_1)d(s_\varphi(x))) \in V(K^1)$. Thus it suffices to show that

$$[x]s_\varphi(a_1)ds_\varphi(x) - s_\varphi(x)s_\varphi(a_1)d[x] \in V(K^1) + dV(K^0).$$

Thus, by Lemma 5.12, it suffices to show that

$$[x]ds_\varphi(x) - s_\varphi(x)d[x] \in V(K^1) + dV(K^0).$$

So we are done by Lemma 6.5. □

Consider now an arbitrary element $y \in K^1$,

$$y = \sum_{k=0}^{\infty} (V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])),$$

and assume it restricts to 0 in level one, i.e., assume $R_1(y) = 0 \in \Omega^1_A$. This means that

$$x\alpha_0 + a_0 dx = 0 \in \Omega^1_A.$$

Then Proposition 6.10 shows that property P_1 from Notation 5.9 holds. We immediately deduce the following from Proposition 5.10.

Corollary 6.11. *For every $n \geq 1$, property P_n from Notation 5.9 holds.*

The following result is the main result of this section. It is modeled after [Hesselholt and Madsen 2003, Proposition 3.2.6]. Compare also Proposition 4.7.

Proposition 6.12. *Let B be a perfectoid ring satisfying Assumption 6.2. For every integer $n \geq 1$, we have a short exact sequence of $W_{n+1}(B)$-modules*

$$0 \to B \to \Omega^1_B \oplus B \to W_{n+1}\Omega^1_B \xrightarrow{R} W_n\Omega^1_B \to 0, \tag{6.13}$$

where the maps and $W_{n+1}(B)$-module structure are defined as follows. The map $B \to \Omega^1_B \oplus B$ is given by $b \mapsto (-db, p^n b)$. The map $\Omega^1_B \oplus B \to W_{n+1}\Omega^1_B$ is given by $(\beta, b) \mapsto V^n(\beta) + dV^n(b)$. The $W_{n+1}(B)$-module structure on B is given by F^n. The $W_{n+1}(B)$-module structure on $\Omega^1_B \oplus B$ is given by

$$y \cdot (\omega, b) = (F^n(y)\omega - bF^n(dy),\ F^n(y)b),\ \text{where } y \in W_{n+1}(B).$$

The $W_{n+1}(B)$-module structure on $W_n\Omega^1_B$ is induced by restriction.

Proof. Consider the following short exact sequence of chain complexes (the chain complexes are written horizontally, and the short exact sequences are written vertically):

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u] & \Omega^1_B \oplus B \arrow[r] \arrow[u] & W_{n+1}\Omega^1_B \arrow[r] \arrow[u] & W_n\Omega^1_B \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "\theta"] & \Omega^1_A \oplus A \arrow[r] \arrow[u, "\theta"] & W_{n+1}\Omega^1_A \arrow[r] \arrow[u, "W_{n+1}(\theta)"] & W_n\Omega^1_A \arrow[r] \arrow[u, "W_n(\theta)"] & 0 \\
0 \arrow[r] & xA \arrow[r] \arrow[u] & R_1(K^1) \oplus R_1(K^0) \arrow[r] \arrow[u] & R_{n+1}(K^1) \arrow[r] \arrow[u] & R_n(K^1) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

For convenience, write these chain complexes as $0 \to K_\bullet \to A_\bullet \to B_\bullet \to 0$, where we consider the complexes concentrated in degrees 0 to 3. We must show that $H_n(B_\bullet) \cong 0$ for all n. It's trivial that $H_0(B_\bullet) \cong 0$ and $H_3(B_\bullet) \cong 0$. Using Proposition 1.5, we have also that $H_1(B_\bullet) \cong 0$. This leaves $H_2(B_\bullet)$.

Consider now the long exact sequence in homology [Weibel 1994, Theorem 1.3.1] associated to the above short exact sequence of chain complexes. By Proposition 4.7, we have that $H_n(A_\bullet) \cong 0$ for all n. It follows that $H_2(B_\bullet) \cong H_1(K_\bullet)$. We will finish the proof by showing that $H_1(K_\bullet) \cong 0$.

Consider an element in $R_{n+1}(K^1)$ which restricts to 0 in $W_n\Omega^1_A$. By Corollary 6.11, we know that this element can be written as $V^n([x]s_\varphi(\alpha_n)) + s_\varphi(a_n)dV^n([x])$, for some $\alpha_n \in \Omega^1_A$ and some $a_n \in A$. By Lemma 5.11, such an element lies in $V^n(K^1) + dV^n(K^0)$, and hence is in the image of the map

$$R_1(K^1) \oplus R_1(K^0) \xrightarrow{V^n + dV^n} R_{n+1}(K^1).$$

This shows that $H_1(K_\bullet) \cong 0$, and hence that $H_2(B_\bullet) \cong 0$, as required. □

Example 6.14. As in Example 5.13, the analogue of exactness in (6.13) does not hold for arbitrary quotients of a ring $A = W(k)$. For example, exactness does not hold for $B = \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. In this case, not even the left-most map $B \to \Omega^1_B \oplus B$ is injective. More significantly, we know $W_{n+1}\Omega^1_{(\mathbb{Z}/p\mathbb{Z})}$ is zero for all n, so $dV^n(1) = 0 \in W_{n+1}\Omega^1_{(\mathbb{Z}/p\mathbb{Z})}$ for all $n \geq 1$. By contrast, Proposition 6.12 shows that $dV^n(1) \neq 0$ for all perfectoid rings B satisfying Assumption 6.2.

Remark 6.15. Assume B is a ring for which the sequence in Equation (6.13) is exact. Assume $B_0 \subseteq B$ is a subring satisfying the following two properties:

(1) We have $p^n B \cap B_0 = p^n B_0$.

(2) The B_0-module homomorphism $\Omega^1_{B_0} \to \Omega^1_B$ is injective.

It then follows that the analogue of (6.13) for B_0 is also exact. In foreseeable applications, verifying the first condition will be trivial, but in general it may be difficult to verify the second condition. For example, if B is $\mathbb{O}_{\mathbb{C}_p}$ and B_0 is the valuation ring in an algebraic extension of $\mathbb{Q}_p$, it is not clear whether we should expect the second condition to hold. For this reason, this remark might be more useful in the context of [Hesselholt 2006, Proposition 2.2.1], which shows exactness of a log analogue of (6.13) when $B = \mathbb{O}_{\overline{\mathbb{Q}}_p}$.

Remark 6.16. In this section and the previous section, we have been working with an explicit quotient of the de Rham–Witt complex over $A = W(k)$. Perhaps similar results could be attained by working with an explicit quotient of the de Rham–Witt complex over the polynomial algebra $A[t]$. An explicit description of the de Rham–Witt complex over $A[t]$ is given, in terms of the de Rham–Witt complex over A, in [Hesselholt and Madsen 2004, Theorem B].

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Coherent Tannaka duality and algebraicity of Hom-stacks

Jack Hall and David Rydh

We establish Tannaka duality for noetherian algebraic stacks with affine stabilizer groups. Our main application is the existence of $\underline{\mathrm{Hom}}$-stacks in great generality.

1. Introduction

Classically, Tannaka duality reconstructs a group from its category of finite-dimensional representations [Tannaka 1939]. Various incarnations of Tannaka duality have been studied for decades. The focus of this article is a recent formulation for algebraic stacks [Lurie 2004] which we now recall.

Let X be a noetherian algebraic stack. We denote its abelian category of coherent sheaves by $\mathsf{Coh}(X)$. If $f\colon T \to X$ is a morphism of noetherian algebraic stacks, then there is an induced pullback functor

$$f^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(T).$$

It is well-known that f^* has the following three properties:

(i) f^* sends $\mathcal{O}_X$ to $\mathcal{O}_T$.

(ii) f^* preserves the tensor product of coherent sheaves.

(iii) f^* is a right exact functor of abelian categories.

Hence, there is a functor

$$\mathrm{Hom}(T, X) \to \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(T)),$$
$$(f\colon T \to X) \mapsto (f^*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(T)),$$

where the right-hand side denotes the category with objects the functors $F\colon \mathsf{Coh}(X) \to \mathsf{Coh}(T)$ satisfying conditions (i)–(iii) above and morphisms given by natural isomorphisms of functors.

If X has affine diagonal (e.g., X is the quotient of a variety by an affine algebraic group), then the functor above is known [Lurie 2004] to be fully faithful with image consisting of *tame* functors. Even

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though tameness of a functor is a difficult condition to verify, Lurie was able to establish some striking applications to algebraization problems.

Various stacks of singular curves [Alper and Kresch 2016, Section 4.1] and log stacks can fail to have affine, quasiaffine, or even separated diagonals. In particular, for applications in moduli theory, the results of [Lurie 2004] are insufficient. The main result of this article is the following theorem, which besides removing Lurie's hypothesis of affine diagonal, obviates tameness.

Theorem 1.1. *Let X be a noetherian algebraic stack with affine stabilizers. If T is an algebraic stack that is locally the spectrum of a G-ring* (*e.g.*, *locally excellent*), *then the functor*:

$$\mathrm{Hom}(T, X) \to \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(T))$$

is an equivalence.

That X has *affine stabilizers* means that $\mathrm{Aut}(x)$ is affine for every field k and point $x\colon \operatorname{Spec} k \to X$; equivalently, the diagonal of X has affine fibers. Examples of algebraic stacks that are locally the spectrum of a G-ring include those that are locally of finite type over a field, over $\mathbb{Z}$, over a complete local noetherian ring, or over an excellent ring (see Remark 7.2). We also wish to emphasize that we do not assume that the diagonal of X is separated in Theorem 1.1. The restriction to stacks with affine stabilizers is a necessary condition for the equivalence in Theorem 1.1 (see Theorem 10.1).

Theorem 1.1 is a consequence of Theorem 8.4, which also gives various refinements in the nonnoetherian situation and when X has quasiaffine diagonal or is Deligne–Mumford.

Main applications. In work with J. Alper [2015], Theorem 1.1 is applied to resolve Alper's conjecture on the local quotient structure of algebraic stacks [Alper 2010]. A more immediate application of Theorem 1.1 is the following algebraicity result for Hom-stacks, generalizing all previously known results and answering [Abramovich et al. 2011, Question 1.4].

Theorem 1.2. *Let $Z \to S$ and $X \to S$ be morphisms of algebraic stacks such that $Z \to S$ is proper and flat of finite presentation*, *and $X \to S$ is locally of finite presentation*, *quasiseparated*, *and has affine stabilizers. Then*

(i) *the stack $\underline{\mathrm{Hom}}_S(Z, X)\colon T \mapsto \mathrm{Hom}_S(Z \times_S T, X)$*, *is algebraic*;

(ii) *the morphism $\underline{\mathrm{Hom}}_S(Z, X) \to S$ is locally of finite presentation*, *quasiseparated*, *and has affine stabilizers*; *and*

(iii) *if $X \to S$ has affine* (*or quasiaffine or separated*) *diagonal*, *then so has $\underline{\mathrm{Hom}}_S(Z, X) \to S$.*

Theorem 1.2 has already seen applications to log geometry [Wise 2016], an area which provides a continual source of stacks that are neither Deligne–Mumford nor have separated diagonals. In general, the condition that X has affine stabilizers is necessary (see Theorem 10.4). That the Hom-stacks above are quasiseparated is nontrivial, and is established in Appendix B. The main result in Appendix B is a substantial generalization of the strongest boundedness result in the existing literature [Olsson 2006b, Proposition 5.10].

There are analogous algebraicity results for Weil restrictions (that is, restrictions of scalars).

Theorem 1.3. *Let $f\colon Z \to S$ and $g\colon X \to Z$ be morphisms of algebraic stacks such that f is proper and flat of finite presentation and $f \circ g$ is locally of finite presentation, quasiseparated and has affine stabilizers. Then*

(i) *the stack $f_*X = \boldsymbol{R}_{Z/S}(X)\colon T \mapsto \mathrm{Hom}_Z(Z \times_S T, X)$ is algebraic*;

(ii) *the morphism $\boldsymbol{R}_{Z/S}(X) \to S$ is locally of finite presentation, quasiseparated and has affine stabilizers*; *and*

(iii) *if g has affine* (*or quasiaffine or separated*) *diagonal, then so has $f_*X \to S$.*

When Z has finite diagonal and X has quasifinite and separated diagonal, Theorems 1.2 and 1.3 were proved in [Hall and Rydh 2014, Theorems 3 and 4]. In Corollary 9.2, we also excise the finite presentation assumptions on $X \to S$ in Theorems 1.2 and 1.3, generalizing the results of [Hall and Rydh 2015b, Theorem 2.3 and Corollary 2.4] for stacks with quasifinite diagonal.

Application to descent. If X has quasiaffine diagonal, then it is well-known that it is a stack for the fpqc topology [Laumon and Moret-Bailly 2000, Corollaire 10.7]. In general, it is only known that algebraic stacks satisfy effective descent for fppf coverings. Nonetheless, using that QCoh is a stack for the fpqc-topology and Tannaka duality, we are able to establish the following result.

Corollary 1.4. *Let X be a quasiseparated algebraic stack with affine stabilizers. Let $\pi\colon T' \to T$ be an fpqc covering such that T is locally the spectrum of a G-ring and T' is locally noetherian. Then X satisfies effective descent for π.*

Application to completions. Another application concerns completions.

Corollary 1.5. *Let A be a noetherian ring and let $I \subseteq A$ be an ideal. Assume that A is complete with respect to the I-adic topology. Let X be a noetherian algebraic stack and consider the natural morphism*

$$X(A) \to \varprojlim X(A/I^n)$$

of groupoids. This morphism is an equivalence if either

(i) *X has affine stabilizers and A is a G-ring* (*e.g., excellent*); *or*

(ii) *X has quasiaffine diagonal*; *or*

(iii) *X is Deligne–Mumford.*

Using methods from derived algebraic geometry, Corollary 1.5(ii) was recently proved for nonnoetherian complete rings A [Bhatt 2016; Bhatt and Halpern-Leistner 2017]. That X has affine stabilizers in Corollary 1.5 is necessary (see Theorem 10.5).

On the proof of Tannaka duality. We will discuss the proof of Theorem 8.4, the refinement of Theorem 1.1. The reason for this is that it is much more convenient from a technical standpoint to consider the problem in the setting of quasicoherent sheaves on potentially nonnoetherian algebraic stacks.

So let T and X be algebraic stacks and let $\mathsf{QCoh}(T)$ and $\mathsf{QCoh}(X)$ denote their respective abelian categories of quasicoherent sheaves. We will assume that X is quasicompact and quasiseparated. Our principal concern is the properties of the functor

$$\begin{aligned} \omega_X(T)\colon \operatorname{Hom}(T, X) &\to \operatorname{Hom}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(T)), \\ (f\colon T \to X) &\mapsto (f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)), \end{aligned}$$

where the right-hand side denotes the additive functors $F\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ satisfying

(i) $F(\mathcal{O}_X) = \mathcal{O}_T$,

(ii) F preserves the tensor product, and

(iii) F is right exact and preserves (small) direct sums.

We call such F *cocontinuous tensor functors*.

An algebraic stack X has the *resolution property* if every quasicoherent sheaf is a quotient of a direct sum of vector bundles. In Theorem 4.11 we establish the equivalence of $\omega_X(T)$ when X has affine diagonal and the resolution property. This result has appeared in various forms in the work of others [Schäppi 2012, Theorem 1.3.2; Savin 2006; Brandenburg 2014, Corollary 5.7.12] and forms an essential stepping stone in the proof of our main theorem (Theorem 8.4).

In general, there are stacks — even schemes — that do not have the resolution property. Indeed, if X has the resolution property, then X has at least affine diagonal [Totaro 2004, Proposition 1.3]. Our proof uses the following three ideas to overcome this problem:

(i) If $U \subseteq X$ is a quasicompact open immersion and $\mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is a tensor functor, then there is an induced tensor functor $\mathsf{QCoh}(U) \to \mathsf{QCoh}(V)$ where $V \subseteq T$ is the "inverse image of U". The proof of this is based on ideas of Brandenburg and Chirvasitu [2014]. (Section 5)

(ii) If X is an infinitesimal neighborhood of a stack with the resolution property, then $\omega_X(T)$ is an equivalence for all T. (Section 6)

(iii) There is a constructible stratification of X into stacks with affine diagonal and the resolution property (Proposition 8.2). We deduce the main theorem by induction on the number of strata using formal gluings [Moret-Bailly 1996; Hall and Rydh 2016]. This step uses special cases of Corollaries 1.4 and 1.5. (Sections 7 and 8)

In the third step, we assume that our functors preserve sheaves of finite type.

Open questions. Concerning (ii), it should be noted that we do not know the answers to the following two questions.

Question 1.6. If X_0 has the resolution property and $X_0 \hookrightarrow X$ is a nilpotent closed immersion, then does X have the resolution property?

The question has an affirmative answer if X_0 is cohomologically affine, e.g., $X_0 = B_k G$ where G is a linearly reductive group scheme over k. The question is open if $X_0 = B_k G$ where G is not linearly reductive, even if $X = B_{k[\epsilon]} G_\epsilon$ where G_ϵ is a deformation of G over the dual numbers [Conrad 2010].

Question 1.7. If $X_0 \hookrightarrow X$ is a nilpotent closed immersion and $\omega_{X_0}(T)$ is an equivalence, is then $\omega_X(T)$ an equivalence?

Step (ii) answers neither of these questions but uses a special case of the first question (Lemma 6.2) and the conclusion (Main Lemma 6.1) is a special case of the second question.

The following technical question also arose in this research.

Question 1.8. Let X be an algebraic stack with quasicompact and quasiseparated diagonal and affine stabilizers. Let k be a field. Is every morphism $\operatorname{Spec} k \to X$ affine?

If X étale-locally has quasiaffine diagonal, then Question 1.8 has an affirmative answer (Lemma 4.7). This makes finding counterexamples extraordinarily difficult and thus very interesting. This question arose because if $\operatorname{Spec} k \to X$ is nonaffine, then $\omega_X(\operatorname{Spec} k)$ is not fully faithful (Theorem 10.2). This explains our restriction to natural isomorphisms in Theorem 1.1. Note that every morphism $\operatorname{Spec} k \to X$ as in Question 1.8 is at least quasiaffine [Rydh 2011a, Theorem B.2]. We do not know the answer to the question even if X has separated diagonal and is of finite type over a field.

On the applications. Let T be a noetherian algebraic stack that is locally the spectrum of a G-ring, and let Z be a closed substack defined by a coherent ideal $J \subseteq \mathcal{O}_T$. Let $Z^{[n]}$ be the closed substack defined by J^{n+1}. Assume that the natural functor $\mathsf{Coh}(T) \to \varprojlim_n \mathsf{Coh}(Z^{[n]})$ is an equivalence of categories. Then an immediate consequence of Tannaka duality (Theorem 1.1) is that

$$\operatorname{Hom}(T, X) \to \varprojlim \operatorname{Hom}(Z^{[n]}, X)$$

is an equivalence of categories for every noetherian algebraic stack X with affine stabilizers. This applies in particular if A is excellent and I-adically complete and $T = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A/I$; this gives Corollary 1.5. More generally, it also applies if T is proper over $\operatorname{Spec} A$ and $Z = T \times_{\operatorname{Spec} A} \operatorname{Spec} A/I$ (Grothendieck's existence theorem). This latter case is fed into Artin's criterion to prove Theorem 1.2 (the remaining hypotheses have largely been verified elsewhere).

There are also nonproper stacks T satisfying $\mathsf{Coh}(T) \to \varprojlim_n \mathsf{Coh}(Z^{[n]})$, such as global quotient stacks with proper good moduli spaces (see [Geraschenko and Zureick-Brown 2015; Alexeev and Brion 2005] for some special cases). This featured in the resolution of Alper's conjecture [Alper et al. 2015].

Such statements, and their derived versions, were also recently considered by Halpern-Leistner and Preygel [2014]. There, they considered variants of our Theorem 1.2. For their algebraicity results, their assumption was similar to assuming that $\mathsf{Coh}(T) \to \varprojlim_n \mathsf{Coh}(Z^{[n]})$ was an equivalence (though they also considered other derived versions), and that $X \to S$ was locally of finite presentation with affine diagonal.

Relation to other work. As mentioned in the beginning of the introduction (Section 1), Lurie identifies the image of $\omega_X(T)$ with the *tame* functors when X is quasicompact with affine diagonal [Lurie 2004]. Tameness means that faithful flatness of objects is preserved. This is a very strong assumption that makes it possible to directly pull back a smooth presentation of X to a smooth covering of T and deduce the result by descent. Note that every tensor functor preserves coherent flat objects — these are vector bundles and hence dualizable — but this does not imply that flatness of quasicoherent objects are preserved. Lurie's methods also work for nonnoetherian T.

Brandenburg and Chirvasitu [2014], have shown that $\omega_X(T)$ is an equivalence for every quasicompact and quasiseparated scheme X, also for nonnoetherian T. The key idea of their proof is the tensor localization that we have adapted in Section 5. Using this technique, we give a slightly simplified proof of their theorem in Theorem 5.10.

When X has quasiaffine diagonal, derived variants of Theorem 1.1 have recently been considered by various authors [Fukuyama and Iwanari 2013; Bhatt 2016; Bhatt and Halpern-Leistner 2017]. Specifically, they were concerned with symmetric monoidal ∞-functors $G\colon D(X) \to D(T)$ between stable ∞-categories of quasicoherent sheaves. These functors are assumed to preserve derived tensor products, connective complexes (i.e., are right t-exact) and pseudocoherent complexes. Hence, such a functor induces a right-exact tensor functor $H^0(G)\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ preserving sheaves of finite type. When T is locally noetherian, our main result (Theorem 8.4) thus implies that of [Bhatt and Halpern-Leistner 2017, Theorem 1.4]. When X has finite stabilizers, the right t-exactness is sometimes automatic [Fukuyama and Iwanari 2013; Bhatt 2016; Ben-Zvi 2010].

Conversely, given a tensor functor $F\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$, it is not obvious how to derive it to a symmetric monoidal ∞-functor $\mathsf{L}F\colon D(X) \to D(T)$ without additional assumptions. In particular, without additional assumptions, our results cannot be deduced from the derived variants. If, however, we assume

(i) there are enough flat quasicoherent sheaves on X,

(ii) F takes exact sequences of flat quasicoherent sheaves to exact sequences,

(iii) $D(X)$ and $D(T)$ are compactly generated,

(iv) X and T have affine diagonals or are noetherian and affine-pointed,

then $\mathsf{L}F$ exists. Indeed, the first two conditions permit us to derive F in the usual way to a symmetric monoidal ∞-functor $D(\mathsf{QCoh}(X)) \to D(\mathsf{QCoh}(T))$. The last two conditions, combined with [Hall et al. 2018, Theorem 1.2], establish the equivalences $D(X) \simeq D(\mathsf{QCoh}(X))$ and $D(T) \simeq D(\mathsf{QCoh}(T))$. Condition (i) is known to hold when X has the resolution property or is a scheme with affine diagonal. Condition (ii) is part of the tameness assumption in [Lurie 2004]. Condition (iii) is known to hold if X has quasifinite and separated diagonal or étale-locally has the resolution property in characteristic 0 [Hall and Rydh 2017].

We do not address the Tannaka *recognition* problem, i.e., which symmetric monoidal categories arise as the category of quasicoherent sheaves on an algebraic stack. For gerbes, this has been done in characteristic zero by Deligne [1990, Théorème 7.1]. For stacks with the resolution property, this has been done by Schäppi [2018, Theorem 1.4; 2015, Theorems 1.2.2 and 5.3.10]. Similar results from the derived perspective have been considered by Wallbridge [2012] and Iwanari [2018].

2. Symmetric monoidal categories

A *symmetric monoidal category* is the data of a category $\boldsymbol{C}$, a tensor product $\otimes_{\boldsymbol{C}} \colon \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$, and a unit $\mathcal{O}_{\boldsymbol{C}}$ that together satisfy various naturality, commutativity, and associativity properties [Mac Lane 1971, VII.7]. A symmetric monoidal category $\boldsymbol{C}$ is *closed* if for any $M \in \boldsymbol{C}$ the functor $- \otimes_{\boldsymbol{C}} M \colon \boldsymbol{C} \to \boldsymbol{C}$ admits a right adjoint, which we denote as $\mathcal{H}om_{\boldsymbol{C}}(M, -)$.

Example 2.1. Let A be a commutative ring; then the category of A-modules, $\mathsf{Mod}(A)$, together with its tensor product $\otimes_A$, is a symmetric monoidal category with unit A. In fact, $\mathsf{Mod}(A)$ is even closed: the right adjoint to $- \otimes_A M$ is the A-module $\mathrm{Hom}_A(M, -)$. If A is noetherian, then the subcategory of finite A-modules, $\mathsf{Coh}(A)$, is also a closed symmetric monoidal category.

A functor $F \colon \boldsymbol{C} \to \boldsymbol{D}$ between symmetric monoidal categories is *lax symmetric monoidal* if for each M and M' of $\boldsymbol{C}$ there are natural maps $F(M) \otimes_{\boldsymbol{D}} F(M') \to F(M \otimes_{\boldsymbol{C}} M')$ and $\mathcal{O}_{\boldsymbol{D}} \to F(\mathcal{O}_{\boldsymbol{C}})$ that are compatible with the symmetric monoidal structure. If these maps are both isomorphisms, then F is *symmetric monoidal*. Note that if $F \colon \boldsymbol{C} \to \boldsymbol{D}$ is a symmetric monoidal functor, then a right adjoint $G \colon \boldsymbol{D} \to \boldsymbol{C}$ to F is always lax symmetric monoidal.

Example 2.2. Let $\phi \colon A \to B$ be a ring homomorphism. The functor $- \otimes_A B \colon \mathsf{Mod}(A) \to \mathsf{Mod}(B)$ is symmetric monoidal. It admits a right adjoint, $\mathsf{Mod}(B) \to \mathsf{Mod}(A)$, which is given by the forgetful functor. This forgetful functor is lax monoidal, but not monoidal.

If $\boldsymbol{C}$ is a symmetric monoidal category, then a *commutative $\boldsymbol{C}$-algebra* consists of an object A of $\boldsymbol{C}$ together with a multiplication $m \colon A \otimes_{\boldsymbol{C}} A \to A$ and a unit $e_A \colon \mathcal{O}_{\boldsymbol{C}} \to A$ with the expected properties [Mac Lane 1971, VII.3]. Let $\mathsf{CAlg}(\boldsymbol{C})$ denote the category of commutative $\boldsymbol{C}$-algebras. The category $\mathsf{CAlg}(\boldsymbol{C})$ is naturally endowed with a symmetric monoidal structure that makes the forgetful functor $\mathsf{CAlg}(\boldsymbol{C}) \to \boldsymbol{C}$ symmetric monoidal.

Example 2.3. If A is a ring, then $\mathsf{CAlg}(\mathsf{Mod}(A))$ is the category of commutative A-algebras.

The following observation will be used frequently: if $F \colon \boldsymbol{C} \to \boldsymbol{D}$ is a lax symmetric monoidal functor and A is a commutative $\boldsymbol{C}$-algebra, then $F(A)$ is a commutative $\boldsymbol{D}$-algebra.

3. Abelian tensor categories

An *abelian tensor category* is a symmetric monoidal category that is abelian and the tensor product is right exact and preserves finite direct sums in each variable (i.e., preserves all finite colimits in each variable).

Recall that an abelian category is *Grothendieck* if it is closed under small direct sums, filtered colimits are exact, and it has a generator [Stacks Project, Tag 079A]. Also, recall that a functor $F\colon \boldsymbol{C} \to \boldsymbol{D}$ between two Grothendieck abelian categories is *cocontinuous* if it is right-exact and preserves small direct sums, equivalently, it preserves all small colimits.

A *Grothendieck abelian tensor category* is an abelian tensor category such that the underlying abelian category is Grothendieck abelian and the tensor product is cocontinuous in each variable. By the special adjoint functor theorem [Kashiwara and Schapira 2006, Proposition 8.3.27(iii)], if $\boldsymbol{C}$ is a Grothendieck abelian tensor category, then it is also closed.

Example 3.1. Let A be a ring. Then $\mathsf{Mod}(A)$ is a Grothendieck abelian tensor category. If A is noetherian, then $\mathsf{Coh}(A)$ is an abelian tensor category but not Grothendieck abelian — it is not closed under small direct sums.

A *tensor functor* $F\colon \boldsymbol{C} \to \boldsymbol{D}$ is an additive symmetric monoidal functor between abelian tensor categories. Let GTC be the 2-category of Grothendieck abelian tensor categories and cocontinuous tensor functors. By the special adjoint functor theorem, if $F\colon \boldsymbol{C} \to \boldsymbol{D}$ is a cocontinuous tensor functor, then F admits a (lax symmetric monoidal) right adjoint.

Example 3.2. Let T be a ringed site. The category of $\mathcal{O}_T$-modules $\mathsf{Mod}(T)$ is a Grothendieck abelian tensor category with unit $\mathcal{O}_T$ and the internal Hom is the functor $\mathcal{H}om_{\mathcal{O}_T}(M, -)$ [Kashiwara and Schapira 2006, Sections 18.1–2].

Example 3.3. Let X be an algebraic stack. The category of quasicoherent sheaves $\mathsf{QCoh}(X)$ is a Grothendieck abelian tensor category with unit $\mathcal{O}_X$ [Stacks Project, Tag 0781]. The internal Hom is $\mathrm{QC}(\mathcal{H}om_{\mathcal{O}_X}(M, -))$, where QC denotes the quasicoherator (the right adjoint to the inclusion of the category of quasicoherent sheaves in the category of lisse-étale $\mathcal{O}_X$-modules). If X is an algebraic stack, then $\mathsf{CAlg}(\mathsf{QCoh}(X))$ is the symmetric monoidal category of quasicoherent $\mathcal{O}_X$-algebras.

If $f\colon X \to Y$ is a morphism of algebraic stacks, then the resulting functor $f^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is a cocontinuous tensor functor. If f is flat, then f^* is exact. We always denote the right adjoint of f^* by $f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$. If f is quasicompact and quasiseparated, then f_* coincides with the pushforward of lisse-étale $\mathcal{O}_X$-modules [Olsson 2007, Lemma 6.5(i)]. In particular, if f is quasicompact and quasiseparated, then $f_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ preserves directed colimits (work smooth-locally on Y and then apply [Stacks Project, Tag 0738]) and is lax symmetric monoidal.

Definition 3.4. Given abelian tensor categories $\boldsymbol{C}$ and $\boldsymbol{D}$, we let $\mathrm{Hom}_{c\otimes}(\boldsymbol{C}, \boldsymbol{D})$ and $\mathrm{Hom}_{r\otimes}(\boldsymbol{C}, \boldsymbol{D})$ denote the categories of cocontinuous and right exact tensor functors, respectively, and natural transformations. The transformations are required to be natural with respect to both homomorphisms and the symmetric monoidal structure. We let $\mathrm{Hom}_{c\otimes,\simeq}(\boldsymbol{C}, \boldsymbol{D})$ and $\mathrm{Hom}_{r\otimes,\simeq}(\boldsymbol{C}, \boldsymbol{D})$ denote the groupoids of cocontinuous and right exact tensor functors, respectively, together with natural isomorphisms.

We conclude this section with some useful facts for the paper. We first consider modules over algebras, which are addressed, for example, in Brandenburg's thesis [2014, Section 5.3] in even greater generality.

3.1. ***Modules over an algebra in tensor categories.*** Let $\boldsymbol{C}$ be a Grothendieck abelian tensor category and let A be a commutative $\boldsymbol{C}$-algebra. Define $\mathsf{Mod}_{\boldsymbol{C}}(A)$ to be the category of A-modules. Objects are pairs (M, a), where $M \in \boldsymbol{C}$ and $a\colon A \otimes_{\boldsymbol{C}} M \to M$ is an action of A on M. Morphisms $\phi\colon (M, a) \to (M', a')$ in $\mathsf{Mod}_{\boldsymbol{C}}(A)$ are those $\boldsymbol{C}$-morphisms $\phi\colon M \to M'$ that preserve the respective actions. We identify A with $(A, m) \in \mathsf{Mod}_{\boldsymbol{C}}(A)$ where $m\colon A \otimes_{\boldsymbol{C}} A \to A$ is the multiplication. It is straightforward to show that $\mathsf{Mod}_{\boldsymbol{C}}(A)$ is a Grothendieck abelian tensor category, with tensor product $\otimes_A$ and unit A, and the natural forgetful functor $\mathsf{Mod}_{\boldsymbol{C}}(A) \to \boldsymbol{C}$ preserves all limits and colimits [Kashiwara and Schapira 2006, Section 4.3].

If $s\colon A \to B$ is a $\boldsymbol{C}$-algebra homomorphism, then there is a natural cocontinuous tensor functor

$$s^*\colon \mathsf{Mod}_{\boldsymbol{C}}(A) \to \mathsf{Mod}_{\boldsymbol{C}}(B), \quad (M, a) \mapsto (B \otimes_A M, B \otimes_A a).$$

Suppose $f^*\colon \boldsymbol{C} \to \boldsymbol{D}$ is a cocontinuous tensor functor with right adjoint $f_*\colon \boldsymbol{D} \to \boldsymbol{C}$. If A is a commutative $\boldsymbol{C}$-algebra, then there is a natural induced cocontinuous tensor functor

$$f_A^*\colon \mathsf{Mod}_{\boldsymbol{C}}(A) \to \mathsf{Mod}_{\boldsymbol{D}}(f^*A), \quad (M, a) \mapsto (f^*M, f^*a).$$

Noting that $\epsilon\colon f^*f_*\mathcal{O}_{\boldsymbol{D}} \to \mathcal{O}_{\boldsymbol{D}}$ is a $\boldsymbol{D}$-algebra homomorphism, there is a natural induced cocontinuous tensor functor

$$\bar{f}^*\colon \mathsf{Mod}_{\boldsymbol{C}}(f_*\mathcal{O}_{\boldsymbol{D}}) \xrightarrow{f^*_{f_*\mathcal{O}_{\boldsymbol{D}}}} \mathsf{Mod}_{\boldsymbol{D}}(f^*f_*\mathcal{O}_{\boldsymbol{D}}) \xrightarrow{\epsilon^*} \mathsf{Mod}_{\boldsymbol{D}}(\mathcal{O}_{\boldsymbol{D}}) = \boldsymbol{D}.$$

Moreover, if we let $\eta\colon \mathcal{O}_{\boldsymbol{C}} \to f_*f^*\mathcal{O}_{\boldsymbol{C}} = f_*\mathcal{O}_{\boldsymbol{D}}$ denote the unit, then $f^* = \bar{f}^*\eta^*$. We have the following striking characterization of module categories.

Proposition 3.5 [Brandenburg 2014, Proposition 5.3.1]. *Let $\boldsymbol{C}$ be a Grothendieck abelian tensor category and let A be a commutative algebra in $\boldsymbol{C}$. Then for every Grothendieck abelian tensor category $\boldsymbol{D}$, there is an equivalence of categories*

$$\mathrm{Hom}_{c\otimes}(\mathsf{Mod}_{\boldsymbol{C}}(A), \boldsymbol{D}) \simeq \{(F, h)\colon F \in \mathrm{Hom}_{c\otimes}(\boldsymbol{C}, \boldsymbol{D}),\ h \in \mathrm{Hom}_{\mathsf{CAlg}(\boldsymbol{D})}(F(A), \mathcal{O}_{\boldsymbol{D}})\},$$

where a morphism $(F, h) \to (F', h')$ is a natural transformation $\alpha\colon F \to F'$ such that $h = h' \circ \alpha(A)$.

The following corollary is immediate (see [Brandenburg 2014, Corollary 5.3.7]).

Corollary 3.6. *Let $p\colon Y' \to Y$ be an affine morphism of algebraic stacks. Let X be an algebraic stack and let $g^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ be a cocontinuous tensor functor. If X' is the affine X-scheme $\mathcal{S}\mathrm{pec}_X(g^*p_*\mathcal{O}_{Y'})$ with structure morphism $p'\colon X' \to X$, then there is a 2-cocartesian diagram in* GTC:

$$\begin{tikzcd} \mathsf{QCoh}(X') & \mathsf{QCoh}(Y') \arrow[l, "g'^*"'] \\ \mathsf{QCoh}(X) \arrow[u, "p'^*"] & \mathsf{QCoh}(Y). \arrow[l, "g^*"] \arrow[u, "p^*"'] \end{tikzcd}$$

Moreover, the natural transformation $g^*p_* \Rightarrow p'_*g'^*$ is an isomorphism.

Note that if g^* comes from a morphism $g\colon X \to Y$, then $X' \cong X \times_Y Y'$.

3.2. ***Inverse limits of abelian tensor categories.*** We will now briefly discuss some inverse limits that will be crucial when we apply Tannaka duality to establish the algebraicity of Hom-stacks in Theorem 1.2. The following notation will be useful.

Notation 3.7. Let $i\colon Z \to X$ be a closed immersion of algebraic stacks defined by a quasicoherent ideal I. For each integer $n \geq 0$, we let $i^{[n]}\colon Z^{[n]} \to X$ denote the closed immersion defined by the quasicoherent ideal I^{n+1}, which we call the *n-th infinitesimal neighborhood of Z.*

Let X be a noetherian algebraic stack and $i: Z \to X$ be a closed immersion. Let $\mathsf{Coh}(X,Z)$ denote the category $\varprojlim_n \mathsf{Coh}(Z^{[n]})$. The arguments of [Stacks Project, Tag 087X] easily extend to establish the following:

(i) $\mathsf{Coh}(X, Z)$ is an abelian tensor category with
  (a) unit: $\{\mathcal{O}_{Z^{[n]}}\}$,
  (b) tensor product: $\{M_n\}_{n\geq 0} \otimes \{N_n\}_{n\geq 0} = \{M_n \otimes_{\mathcal{O}_{Z^{[n]}}} N_n\}_{n\geq 0}$,
  (c) addition: $\{f_n\colon M_n \to N_n\}_{n\geq 0} + \{g_n\colon M_n \to N_n\}_{n\geq 0} = \{f_n + g_n\}_{n\geq 0}$, and
  (d) cokernels: $\operatorname{coker}(\{f_n\colon M_n \to N_n\}_{n\geq 0}) = \{\operatorname{coker} f_n\}_{n\geq 0}$.

(ii) If U is a noetherian algebraic stack and $p\colon U \to X$ is a flat morphism, then the pullback $\mathsf{Coh}(X, Z) \to \mathsf{Coh}(U, U \times_X Z)$ is an exact tensor functor.

(iii) Exactness in $\mathsf{Coh}(X, Z)$ may be checked on a flat, noetherian covering of X.

Computing $\ker(\{f_n\colon M_n \to N_n\}_{n\geq 0})$ is more involved without additional flatness assumptions. The problem is that in general the system of kernels $\{\ker f_n\}_{n\geq 0}$ is not an adic system; that is, the morphism $(\ker f_{n+1}) \otimes_{\mathcal{O}_{Z^{[n+1]}}} \mathcal{O}_{Z^{[n]}} \to \ker(f_n)$ need not be an isomorphism. As shown in the proof of [Stacks Project, Tag 087X], $\ker\{f_n\}_{n\geq 0}$ ends up being the stable value of the $\ker f_n$ (in the sense of the Artin–Rees lemma).

The abelian category $\mathsf{Coh}(X, Z)$ is also the limit of $\mathsf{Coh}(Z^{[n]})$ as an abelian tensor category. This is made precise by the following lemma.

Lemma 3.8. $\mathsf{Coh}(X, Z)$ *is the limit of the inverse system of categories* $\{\mathsf{Coh}(Z^{[n]})\}_{n\geq 0}$ *in the* 2*-category of abelian tensor categories with right exact tensor functors and natural isomorphisms of tensor functors.*

Proof. It remains to verify that for every abelian tensor category $\boldsymbol{C}$, a functor $F\colon \boldsymbol{C} \to \mathsf{Coh}(X, Z)$ is a right exact tensor functor if and only if the induced functors $q_n \circ F\colon \boldsymbol{C} \to \mathsf{Coh}(X, Z) \to \mathsf{Coh}(Z^{[n]})$ are right exact tensor functors. This follows from the description of the abelian tensor structure of $\mathsf{Coh}(X, Z)$ in (a)–(d) above. $\square$

4. Tensorial algebraic stacks

Let T and X be algebraic stacks. There is an induced functor

$$\omega_X(T)\colon \operatorname{Hom}(T, X) \to \operatorname{Hom}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(T))$$

that takes a morphism f to f^*. We also let $\operatorname{Hom}^{\mathrm{ft}}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(T))$ denote the full subcategory of functors that preserve sheaves of finite type. Similarly, we let $\operatorname{Hom}_{c\otimes,\simeq}(\mathsf{QCoh}(X), \mathsf{QCoh}(T))$ denote the subcategory with all objects but only natural isomorphisms of functors. Clearly, $\omega_X(T)$ factors through all of these subcategories and we let $\omega_{X,\simeq}(T)$, $\omega_X^{\mathrm{ft}}(T)$ and $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ denote the respective factorizations. Note that when X and T are locally noetherian, the natural functor:

$$\mathrm{Hom}_{r\otimes}(\mathsf{Coh}(X), \mathsf{Coh}(T)) \to \mathrm{Hom}_{c\otimes}^{\mathrm{ft}}(\mathsf{QCoh}(X), \mathsf{QCoh}(T))$$

is an equivalence of categories. Thus, Theorem 1.1 says that $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ is an equivalence.

Since $\mathsf{QCoh}(-)$ is a stack in the fpqc topology, the target categories of the functors ω_X, $\omega_{X,\simeq}$, ω_X^{ft} and $\omega_{X,\simeq}^{\mathrm{ft}}$ are stacks in the fpqc topology when varying T — for an elaborate proof of this, see [Liu and Tseng 2012, Theorem 1.1]. The source categories $\mathrm{Hom}(T, X)$ are groupoids and, when varying T, form a stack for the fppf topology in general and for the fpqc topology when X has quasiaffine diagonal [Laumon and Moret-Bailly 2000, Corollaire 10.7].

Definition 4.1. Let T and X be algebraic stacks. We say that a tensor functor $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is *algebraic* if it arises from a morphism of algebraic stacks $f\colon T \to X$. If $f, g\colon T \to X$ are morphisms, then a natural transformation $\tau\colon f^* \Rightarrow g^*$ of tensor functors is *realizable* if it is induced by a 2-morphism $f \Rightarrow g$. We say that X is *tensorial* if $\omega_X(T)$ is an equivalence for every algebraic stack T, or equivalently, for every affine scheme T [Brandenburg 2014, Definition 3.4.4].

We begin with a descent lemma.

Lemma 4.2. *Let X be an algebraic stack. Let $p\colon T' \to T$ be a morphism of algebraic stacks that is covering for the fpqc topology. Let $T'' = T' \times_T T'$ and $T''' = T' \times_T T' \times_T T'$. Assume that p is a morphism of effective descent for X (e.g., p is flat and locally of finite presentation).*

(i) *Let $f_1, f_2\colon T \to X$ be morphisms and let $\tau, \tau'\colon f_1 \Rightarrow f_2$ be 2-morphisms. If $p^*\tau = p^*\tau'\colon f_1 \circ p \Rightarrow f_2 \circ p$ then $\tau = \tau'$.*

(ii) *Let $f_1, f_2\colon T \to X$ be morphisms and let $\gamma\colon f_1^* \Rightarrow f_2^*$ be a natural transformation. If $p^*\gamma\colon p^* f_1^* \Rightarrow p^* f_2^*$ is realizable and $\omega_X(T'')$ is faithful, then γ is realizable.*

(iii) *Let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. If $p^* f^*$ is algebraic, $\omega_{X,\simeq}(T'')$ is fully faithful and $\omega_X(T''')$ is faithful, then f^* is algebraic.*

Proof. It is sufficient to observe that $\mathrm{Hom}(-, X)$ is a stack in groupoids for the covering p and $\mathrm{Hom}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(-))$ is an fpqc stack in categories, so the result boils down to a straightforward and general result for a 1-morphism of such stacks. ☐

Lemma 4.3. *Let $\boldsymbol{C} \subset \mathsf{AlgSt}$ be a full 2-subcategory of algebraic stacks, such that if $p\colon T' \to T$ is representable and smooth with $T \in \boldsymbol{C}$, then $T' \in \boldsymbol{C}$. For example, $\boldsymbol{C}$ could be the 2-category of locally noetherian algebraic stacks or the 2-category of algebraic stacks that are locally the spectra of G-rings. Let $\omega \in \{\omega_X, \omega_{X,\simeq}, \omega_X^{\mathrm{ft}}, \omega_{X,\simeq}^{\mathrm{ft}}\}$. If $\omega(T)$ is faithful (or fully faithful or an equivalence) for every affine scheme T in $\boldsymbol{C}$, then $\omega(T)$ is faithful (or fully faithful or an equivalence, respectively) for every algebraic stack T in $\boldsymbol{C}$.*

Proof. Note that ω is faithful if and only if ω_X is faithful and that if ω is fully faithful, then so is $\omega_{X,\simeq}$. The conclusion of the lemma holds for disjoint unions of affine schemes in $\boldsymbol{C}$ since $\mathsf{QCoh}\big(\coprod_i T_i\big) = \prod_i \mathsf{QCoh}(T_i)$. We then deduce the conclusion for every stack T in $\boldsymbol{C}$ with affine diagonal. This follows from Lemma 4.2 applied to a presentation $T' \to T$ where T' is a disjoint union of affine schemes since T'' and T''' then also are disjoint unions of affine schemes. We may then similarly deduce the conclusion for every stack T in $\boldsymbol{C}$ with affine double diagonal since T'' and T''' then have affine diagonals. Finally we deduce the conclusion for every stack T in $\boldsymbol{C}$ since the triple diagonal of T is an isomorphism. ☐

We next recall two basic lemmas on tensorial stacks. The first is the combination of [Brandenburg 2014, Corollaries 5.3.4 and 5.6.4].

Lemma 4.4. *Let $q\colon X' \to X$ be a quasiaffine morphism of algebraic stacks. If T is an algebraic stack and $\omega_X(T)$ is faithful, fully faithful or an equivalence; then so is $\omega_{X'}(T)$. In particular, if X is tensorial, then so is X'.*

Proof. Since q is the composition of a quasicompact open immersion followed by an affine morphism, it suffices to treat these two cases separately. When q is affine the result is an easy consequence of Proposition 3.5. If q is a quasicompact open immersion, then the counit $q^*q_* \to \mathrm{id}_{\mathsf{QCoh}(X')}$ is an isomorphism; the result now follows from [Brandenburg and Chirvasitu 2014, Proposition 2.3.6]. ☐

The second lemma is well-known (e.g., it is a very special case of [Brandenburg and Chirvasitu 2014, Theorem 3.4.2]).

Lemma 4.5. *Every quasiaffine scheme is tensorial.*

Proof. By Lemma 4.4, it is sufficient to prove that $X = \operatorname{Spec}\mathbb{Z}$ is tensorial, which is well-known. We refer the interested reader to [Brandenburg and Chirvasitu 2014, Corollary 2.2.4] or [Brandenburg 2014, Corollary 5.2.3]. ☐

Lacking an answer to Question 1.8 in general, we are forced to make the following definition to treat natural transformations that are not isomorphisms.

Definition 4.6. An algebraic stack X is *affine-pointed* if every morphism $\operatorname{Spec} k \to X$, where k is a field, is affine.

Note that if X is affine-pointed, then it has affine stabilizers. The following lemma shows that many algebraic stacks with affine stabilizers that are encountered in practice are affine-pointed.

Lemma 4.7. *Let X be an algebraic stack.*

(i) *If X has quasiaffine diagonal, then X is affine-pointed.*

(ii) *Let $g\colon V \to X$ be a quasifinite and faithfully flat morphism of finite presentation (not necessarily representable). If V is affine-pointed, then X is affine-pointed.*

Proof. Throughout, we fix a field k and a morphism $x\colon \operatorname{Spec} k \to X$.

For (i), since k is a field, every extension in $\mathsf{QCoh}(\operatorname{Spec} k)$ is split; thus x_* is cohomologically affine [Alper 2013, Definition 3.1]. Since X has quasiaffine diagonal, this property is preserved after pulling back x along a smooth morphism $p\colon U \to X$, where U is an affine scheme [Alper 2013, Proposition 3.10(vii)]. By Serre's criterion [EGA II 1961, 5.2.2], the morphism $\operatorname{Spec} k \times_X U \to U$ is affine; and this case follows.

For (ii), the pullback of g along x gives a quasifinite and faithfully flat morphism $g_0\colon V_0 \to \operatorname{Spec} k$. Since V_0 is discrete with finite stabilizers, there exists a finite surjective morphism $W_0 \to V_0$ where W_0 is a finite disjoint union of spectra of fields. By assumption $W_0 \to V_0 \to V$ is affine; hence so is $V_0 \to V$ (by Chevalley's theorem [Rydh 2015, Theorem 8.1] applied smooth-locally on V). By descent, $\operatorname{Spec} k \to X$ is affine and the result follows. □

The following lemma highlights the benefits of affine-pointed stacks.

Lemma 4.8. *Let $f_1, f_2\colon T \to X$ be morphisms of algebraic stacks and let $\gamma\colon f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If X is affine-pointed, then the induced maps of topological spaces $|f_1|, |f_2|\colon |T| \to |X|$ coincide.*

Proof. It suffices to prove that if $T = \operatorname{Spec} k$, where k is a field, then γ is realizable. Since X is affine-pointed, the morphisms f_1 and f_2 are affine. Also, the natural transformation γ induces, by adjunction, a morphism of quasicoherent $\mathcal{O}_X$-algebras $\gamma^\vee(\mathcal{O}_T)\colon (f_2)_*\mathcal{O}_T \to (f_1)_*\mathcal{O}_T$. In particular, $\gamma^\vee(\mathcal{O}_T)$ induces a morphism $v\colon T \to T$ over X. We are now free to replace X by T, f_2 by id_T, and f_1 by v. Since T is affine, the result now follows from Lemma 4.5. □

We can now prove the following proposition (generalizing Lurie's corresponding result for an algebraic stack with affine diagonal).

Proposition 4.9. *Let X be an algebraic stack.*

(i) *If T is an algebraic stack and X has quasiaffine diagonal, then the functor $\omega_X(T)$ is fully faithful.*

(ii) *Let T be a quasiaffine scheme and let $f_1, f_2\colon T \to X$ be quasiaffine morphisms.*
   (a) *If $\alpha, \beta\colon f_1 \Rightarrow f_2$ are 2-morphisms and $\alpha^* = \beta^*$ as natural transformations $f_1^* \Rightarrow f_2^*$, then $\alpha = \beta$.*
   (b) *Let $\gamma\colon f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If γ is an isomorphism or X is affine-pointed, then γ is realizable.*

Proof. For (i), we may assume that T is an affine scheme (Lemma 4.3). Then every morphism $T \to X$ is quasiaffine and the result follows by (ii) and Lemma 4.7(i).

For (ii), there are quasicompact open immersions $i_k\colon T \hookrightarrow V_k$ over X, where $V_k := \mathcal{S}\mathrm{pec}_X((f_k)_*\mathcal{O}_T)$ and $k = 1, 2$. Let $v_k\colon V_k \to X$ be the induced 1-morphism.

We first treat (a). The hypotheses imply that $\alpha_* = \beta_*$ as natural isomorphisms of functors from $(f_2)_*$ to $(f_1)_*$. In particular, α_* and β_* induce the same 1-morphism from V_1 to V_2 over X. Since i_1 and i_2 are open immersions, they are monomorphisms; hence $\alpha = \beta$.

We now treat (b). The natural transformation $\gamma\colon f_1^* \Rightarrow f_2^*$ uniquely induces a natural transformation of lax symmetric monoidal functors $\gamma^\vee\colon (f_2)_* \Rightarrow (f_1)_*$. In particular, there is an induced morphism of quasicoherent $\mathcal{O}_X$-algebras $\gamma^\vee(\mathcal{O}_T)\colon (f_2)_*\mathcal{O}_T \to (f_1)_*\mathcal{O}_T$; hence a morphism of algebraic stacks $g\colon V_1 \to V_2$ over X. Note that $\gamma^\vee$ uniquely induces a natural transformation of lax symmetric monoidal functors $(i_2)_* \Rightarrow g_*(i_1)_*$, and by adjunction we have a uniquely induced natural transformation of tensor functors $\gamma'\colon (g\circ i_1)^* \Rightarrow i_2^*$.

Replacing X by V_2, f_1 by $g\circ i_1$, f_2 by i_2, and γ by γ', we may assume that f_2 is a quasicompact open immersion such that $\mathcal{O}_X \to (f_2)_*\mathcal{O}_T$ is an isomorphism.

If γ is an isomorphism, then f_1 is also a quasicompact open immersion. Let Z_1 and Z_2 denote closed substacks of X whose complements are $f_1(T)$ and $f_2(T)$, respectively. Then $f_2^*\mathcal{O}_{Z_2} \cong 0$; indeed, by definition we have $Z_2 \cap f_2(T) = \varnothing$. In particular, the isomorphism $\gamma(\mathcal{O}_Z)$ implies that $f_1^*\mathcal{O}_{Z_2} \cong f_2^*\mathcal{O}_{Z_2} \cong 0$; hence, $f_1(T) \subseteq f_2(T)$. Arguing similarly, we obtain the reverse inclusion and we see that $f_1(T) = f_2(T)$. Since f_1 and f_2 are open immersions, we obtain the result when γ is assumed to be an isomorphism.

Otherwise, Lemma 4.8 implies that f_1 factors through $f_2(T) \subseteq X$. We may now replace X by T and γ with $(f_2)_*(\gamma)$ [Brandenburg and Chirvasitu 2014, Proposition 2.3.6]. Then X is quasiaffine and the result follows from Lemma 4.5. □

From Proposition 4.9(b), we obtain an analogue of Lemma 4.8 for natural isomorphisms of functors when X has affine stabilizers (as opposed to affine-pointed).

Corollary 4.10. *Let $f_1, f_2\colon T \to X$ be morphisms of algebraic stacks and let $\gamma\colon f_1^* \simeq f_2^*$ be a natural isomorphism of cocontinuous tensor functors. If X has affine stabilizers and quasicompact diagonal, then the induced maps of topological spaces $|f_1|, |f_2|\colon |T| \to |X|$ coincide.*

Proof. It suffices to prove the result when $T = \operatorname{Spec} k$, where k is a field. Since X has affine stabilizers and quasicompact diagonal the morphisms f_1 and f_2 are quasiaffine [Rydh 2011a, Theorem B.2]. The result now follows from Proposition 4.9(b). □

The following result, in a slightly different context, was proved by Schäppi [2012, Theorem 1.3.2]. Using the Totaro–Gross theorem, we can simplify Schäppi's arguments in the algebro-geometric setting.

Theorem 4.11. *Let X be a quasicompact and quasiseparated algebraic stack with affine stabilizers. If X has the resolution property, then it is tensorial.*

Proof. By Totaro–Gross [Gross 2017], there is a quasiaffine morphism $g\colon X \to \mathrm{BGL}_{N,\mathbb{Z}}$. By Lemma 4.4, it is enough to prove that $X = \mathrm{BGL}_{N,\mathbb{Z}}$ is tensorial.

We must prove that $\omega_X(T)$ is an equivalence for every algebraic stack T. Since X is quasicompact with affine diagonal, the functor $\omega_X(T)$ is fully faithful for every T (Proposition 4.9). Thus, it remains to prove that for every algebraic stack T, every cocontinuous tensor functor $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is algebraic. To this end, we note the following. Let Y be an algebraic stack. Then

(i) the dualizable objects in $\mathsf{QCoh}(Y)$ are the vector bundles [Brandenburg 2014, Proposition 4.7.5]; and

(ii) every tensor functor $g^*\colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(T)$ preserves dualizable objects and exact sequences of dualizable objects [Brandenburg 2014, Definition 4.7.1 and Lemma 4.7.10].

Now let $p\colon \operatorname{Spec}\mathbb{Z} \to \mathrm{BGL}_{N,\mathbb{Z}}$ be the universal GL_N-bundle and let $\mathcal{A} = p_*\mathbb{Z}$ be the regular representation. There is an exact sequence

$$0 \to \mathcal{O}_{\mathrm{BGL}_{N,\mathbb{Z}}} \to \mathcal{A} \to \mathcal{Q} \to 0$$

of flat quasicoherent sheaves. Write $\mathcal{A}$ as the directed colimit of its subsheaves $\mathcal{A}_\lambda$ of finite type containing the unit and let $\mathcal{Q}_\lambda = \mathcal{A}_\lambda/\mathcal{O}_{\mathrm{BGL}_{N,\mathbb{Z}}} \subseteq \mathcal{Q}$. Then $\mathcal{A}_\lambda$ and $\mathcal{Q}_\lambda$ are vector bundles.

Thus, let $f^*\colon \mathsf{QCoh}(\mathrm{BGL}_{N,\mathbb{Z}}) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. Then by (i) and (ii) above there are exact sequences of vector bundles:

$$0 \to \mathcal{O}_T \to f^*\mathcal{A}_\lambda \to f^*\mathcal{Q}_\lambda \to 0.$$

Since f^* is cocontinuous, we also obtain an exact sequence

$$0 \to \mathcal{O}_T \to f^*\mathcal{A} \to f^*\mathcal{Q} \to 0$$

of flat quasicoherent sheaves. In particular, $f^*\mathcal{A}$ is a faithfully flat algebra.

Let $V = \mathcal{S}\mathrm{pec}_T(f^*\mathcal{A})$; then $r\colon V \to T$ is faithfully flat. By Corollary 3.6, we have a cocartesian diagram

$$\begin{tikzcd} \mathsf{QCoh}(V) & \mathsf{QCoh}(\operatorname{Spec}\mathbb{Z}) \arrow[l, "f'^*"'] \\ \mathsf{QCoh}(T) \arrow[u, "r^*"] & \mathsf{QCoh}(X). \arrow[l, "f^*"'] \arrow[u, "p^*"'] \end{tikzcd}$$

Since $\operatorname{Spec}\mathbb{Z}$ is tensorial (Lemma 4.5), the functor f'^* is algebraic. Thus, $f'^*p^* \simeq r^*f^*$ is algebraic. Descent along $r\colon V \to T$ (Lemma 4.2(iii)) implies that f^* is algebraic. $\square$

5. Tensor localizations

The goal of this section is to prove the following theorem.

Theorem 5.1. *Let X be a quasicompact and quasiseparated algebraic stack. Let $i\colon Z \to X$ be a finitely presented closed immersion defined by an ideal sheaf I. Let $j\colon U \to X$ be the open complement of Z. Let T be an algebraic stack and let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. Let $i_T\colon Z_T \to T$ be the closed immersion defined by the ideal $I_T := \mathrm{Im}(f^*I \to \mathcal{O}_T)$. Let $j_T\colon U_T \to T$ denote the complement of Z_T.*

(i) *There exists an essentially unique cocontinuous tensor functor*

$$f_U^*\colon \mathsf{QCoh}(U) \to \mathsf{QCoh}(U_T),$$

such that there is an isomorphism of tensor functors $j_T^*f^* \simeq f_U^*j^*$.

(ii) *For each integer $n \geq 0$,*

$$f_{Z^{[n]}}^* := (i_T^{[n]})^* f^* (i^{[n]})_*\colon \mathsf{QCoh}(Z^{[n]}) \to \mathsf{QCoh}(Z_T^{[n]})$$

is a cocontinuous tensor functor and there is a canonical isomorphism of tensor functors $(i_T^{[n]})^* f^* \simeq f^*_{Z^{[n]}}(i^{[n]})^*$. *Moreover*, $f^*(i^{[n]})_* \simeq (i_T^{[n]})_* f^*_{Z^{[n]}}$.

In addition, if f^* *preserves sheaves of finite type, then the same is true of* f_U^* *and* $f^*_{Z^{[n]}}$ *for all* $n \geq 0$.

Theorem 5.1 features in a key way in the proof of our main theorem (Theorem 8.4), which we prove via stratifications and formal gluings. From this context, we hope that the long and technical statement of Theorem 5.1 should appear to be quite natural. While Theorem 5.1(ii) follows easily from the results of Section 3.1, Theorem 5.1(i) is more subtle. It turns out, however, that it is a consequence of a more general result about Grothendieck abelian tensor categories (Theorem 5.8), which is what we will spend most of this section proving.

Let $\boldsymbol{C}$ be a Grothendieck abelian category. A *Serre subcategory* is a full nonempty subcategory $\boldsymbol{K} \subseteq \boldsymbol{C}$ closed under subquotients and extensions. Serre subcategories are abelian and the inclusion functor is exact. A Serre subcategory is *localizing* if it is also closed under small direct sums in $\boldsymbol{C}$, equivalently, it is closed under small colimits in $\boldsymbol{C}$.

If $\boldsymbol{K} \subseteq \boldsymbol{C}$ is a Serre subcategory, then there is a quotient $\boldsymbol{Q}$ of $\boldsymbol{C}$ by $\boldsymbol{K}$ and an exact functor $q^* : \boldsymbol{C} \to \boldsymbol{Q}$, which is universal for exact functors out of $\boldsymbol{C}$ that vanish on $\boldsymbol{K}$ [Gabriel 1962, Chapitre III]. Note that $\boldsymbol{K}$ is localizing if and only if the quotient $q^* : \boldsymbol{C} \to \boldsymbol{Q}$ is a *localization*, that is, q^* admits a right adjoint $q_* : \boldsymbol{Q} \to \boldsymbol{C}$; it follows that $\boldsymbol{Q}$ is Grothendieck abelian, q^* is cocontinuous, q_* is fully faithful and $q^* q_* \simeq \mathrm{id}_{\boldsymbol{Q}}$. This statement follows by combining the Gabriel–Popescu theorem (e.g., [Bucur and Deleanu 1968, Theorem 6.25]) with [Bucur and Deleanu 1968, Proposition 6.21].

Let $\boldsymbol{C}$ be a Grothendieck abelian tensor category and let $\boldsymbol{K} \subseteq \boldsymbol{C}$ be a Serre subcategory. We say that $\boldsymbol{K}$ is a *tensor ideal* if $\boldsymbol{K}$ is closed under tensor products with objects in $\boldsymbol{C}$. If $\boldsymbol{K}$ is also localizing, then we say that $\boldsymbol{K}$ is a *localizing tensor ideal*.

If $f^* : \boldsymbol{C} \to \boldsymbol{D}$ is an exact cocontinuous tensor functor between Grothendieck abelian tensor categories, then $\ker(f^*)$ is a localizing tensor ideal. Conversely, if $\boldsymbol{K} \subseteq \boldsymbol{C}$ is a localizing tensor ideal, then the quotient $\boldsymbol{Q} = \boldsymbol{C}/\boldsymbol{K}$ is a Grothendieck abelian tensor category, the localization $q^* : \boldsymbol{C} \to \boldsymbol{Q}$ is an exact cocontinuous tensor functor and $\ker(q^*) = \boldsymbol{K}$; in this situation, we will refer to q^* as a *tensor localization*.

Example 5.2. Let $f : X \to Y$ be a morphism of algebraic stacks. If f is flat, then f^* is exact. If f is a quasicompact flat monomorphism (e.g., a quasicompact open immersion), then $\mathsf{QCoh}(X)$ is the quotient of $\mathsf{QCoh}(Y)$ by $\ker(f^*)$. This follows from the fact that the counit $f^* f_* \to \mathrm{id}$ is an isomorphism so that f_* is a section of f^* [Gabriel 1962, Proposition III.2.5].

Definition 5.3. Let $\boldsymbol{C}$ be a Grothendieck abelian tensor category. For $M \in \boldsymbol{C}$ let $\varphi_M : \mathcal{O}_{\boldsymbol{C}} \to \mathcal{H}om_{\boldsymbol{C}}(M, M)$ denote the adjoint to the canonical isomorphism $\mathcal{O}_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} M \to M$. Let the annihilator $\mathrm{Ann}_{\boldsymbol{C}}(M)$ of M be the kernel of φ_M, which we consider as an ideal of $\mathcal{O}_{\boldsymbol{C}}$.

Example 5.4. Let X be an algebraic stack and $\mathcal{F} \in \mathsf{QCoh}(X)$. Then $\mathrm{Ann}_{\mathsf{QCoh}(X)}(\mathcal{F}) = \mathrm{QC}(\mathrm{Ann}_{\mathsf{Mod}(X)}(\mathcal{F}))$. In particular, if $\mathcal{F}$ is of finite type, then $\mathrm{Ann}_{\mathsf{QCoh}(X)}(\mathcal{F}) = \mathrm{Ann}_{\mathsf{Mod}(X)}(\mathcal{F})$.

Recall that an object $c \in \boldsymbol{C}$ is *finitely generated* if the natural map

$$\varinjlim_{\lambda} \operatorname{Hom}_{\boldsymbol{C}}(c, d_\lambda) \to \operatorname{Hom}_{\boldsymbol{C}}(c, \varinjlim_{\lambda} d_\lambda)$$

is bijective for every direct system $\{d_\lambda\}_\lambda$ in $\boldsymbol{C}$ with monomorphic bonding maps. A category $\boldsymbol{C}$ is *locally finitely generated* if it is cocomplete (all small colimits exist) and has a set $\mathcal{A}$ of finitely generated objects such that every object c of $\boldsymbol{C}$ is a directed colimit of objects from $\mathcal{A}$.

Example 5.5. Let X be a quasicompact and quasiseparated algebraic stack. The finitely generated objects in $\mathsf{QCoh}(X)$ are the quasicoherent sheaves of finite type. Thus $\mathsf{QCoh}(X)$ is locally finitely generated [Rydh 2016].

We also require the following definition.

Definition 5.6. Let $q^* \colon \boldsymbol{C} \to \boldsymbol{Q}$ be a tensor localization. Then it is *supported* if $q^*(\mathcal{O}_{\boldsymbol{C}} / \operatorname{Ann}(K)) \cong 0$ for every finitely generated object K of $\boldsymbol{C}$ such that $q^*(K) \cong 0$.

The notion of a supported tensor localization is very natural.

Example 5.7. If $f \colon X \to Y$ is a flat monomorphism of quasicompact and quasiseparated algebraic stacks, then the tensor localization $f^* \colon \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ of Example 5.2 is supported. Indeed, if M is a quasi-coherent $\mathcal{O}_Y$-module of finite type in the kernel of f^*, then $f^* \operatorname{Ann}_{\mathsf{QCoh}(Y)}(M) = \operatorname{Ann}_{\mathsf{QCoh}(X)}(f^*M) = \mathcal{O}_X$.

We now have our key result, which also generalizes [Brandenburg and Chirvasitu 2014, Lemma 3.3.6].

Theorem 5.8. *Let $\boldsymbol{C}$ be a locally finitely generated Grothendieck abelian tensor category. Let $q^* \colon \boldsymbol{C} \to \boldsymbol{Q}$ be a supported tensor localization. Let $\boldsymbol{D}$ be a Grothendieck abelian tensor category. If $f^* \colon \boldsymbol{C} \to \boldsymbol{D}$ is a cocontinuous tensor functor such that $f^*(K) \cong 0$ for every finitely generated object K of $\boldsymbol{C}$ such that $q^*(K) \cong 0$, then f^* factors essentially uniquely through a cocontinuous tensor functor $g^* \colon \boldsymbol{Q} \to \boldsymbol{D}$. If f^* preserves finitely generated objects, then so does g^*.*

Note that Theorem 5.8 is trivial if f^* is exact. The challenge is to use the symmetric monoidal structure to deduce this also when f^* is merely right-exact. The proof we give is a straightforward generalization of [Brandenburg and Chirvasitu 2014, Lemma 3.3.6]. First, we will see how Theorem 5.8 implies Theorem 5.1.

Proof of Theorem 5.1. For (ii), note that $(i^{[n]})_*$ identifies $\mathsf{QCoh}(Z^{[n]})$ with the category of modules over the algebra $A_n = \mathcal{O}_X / I^{n+1}$. The algebra $f^* A_n$ is $\mathcal{O}_T / I_T^{n+1}$ and $(f_{Z^{[n]}})^* = (f_{A_n})^*$ in the terminology of Section 3.1.

For (i), recall that $\mathsf{QCoh}(X)$ is locally finitely generated (Example 5.5) and that $j^* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(U)$ is a supported localization (Example 5.7). If $K \in \mathsf{QCoh}(X)$ is finitely generated and $j^*K = 0$, then $I^m K = 0$ for sufficiently large m. Thus, the natural map $I^m \otimes_{\mathcal{O}_X} K \to \mathcal{O}_X \otimes_{\mathcal{O}_X} K \cong K$ is zero. Applying $j_T^* f^*$, the map becomes the identity since $j_T^* f^*(I^m) \to j_T^* f^*(\mathcal{O}_X) = \mathcal{O}_{U_T}$ is an isomorphism. It follows that $j_T^* f^* K = 0$. We may thus apply Theorem 5.8 and deduce that $j_T^* f^*$ factors via j^* and a tensor functor $f_U^* \colon \mathsf{QCoh}(U) \to \mathsf{QCoh}(U_T)$. $\square$

To prove Theorem 5.8 we require the following lemma.

Lemma 5.9 [Brandenburg and Chirvasitu 2014, Lemma 3.3.2]. *Let $f^*\colon \boldsymbol{C} \to \boldsymbol{D}$ be a cocontinuous tensor functor. If $I \subseteq \mathcal{O}_{\boldsymbol{C}}$ is an $\mathcal{O}_{\boldsymbol{C}}$-ideal such that $f^*(\mathcal{O}_{\boldsymbol{C}}/I) \cong 0$, then $f^*(I) \to f^*(\mathcal{O}_{\boldsymbol{C}})$ is an isomorphism.*

Proof. Since f^* is right-exact and $f^*(\mathcal{O}_{\boldsymbol{C}}/I) = 0$, it follows that $f^*(I) \to f^*(\mathcal{O}_{\boldsymbol{C}}) = \mathcal{O}_{\boldsymbol{D}}$ is surjective. Let $J = f^*(I)$ and let $\varphi\colon J \to \mathcal{O}_{\boldsymbol{D}}$ denote the surjection. The multiplication $I \otimes_{\boldsymbol{C}} I \to I$ factors through $I \otimes_{\boldsymbol{C}} \mathcal{O}_{\boldsymbol{C}}$ and $\mathcal{O}_{\boldsymbol{C}} \otimes_{\boldsymbol{C}} I$ and gives rise to the commutative diagram:

$$\begin{tikzcd}
J \otimes_{\boldsymbol{D}} J \arrow[r, two heads, "\mathrm{id}_J \otimes \varphi"] \arrow[d, two heads, "\varphi \otimes \mathrm{id}_J"'] & J \otimes_{\boldsymbol{D}} \mathcal{O}_{\boldsymbol{D}} \arrow[d, "\cong"] \\
\mathcal{O}_{\boldsymbol{D}} \otimes_{\boldsymbol{D}} J \arrow[r, "\cong"] & J.
\end{tikzcd}$$

Let η_F denote the unit of the adjunction between $- \otimes_{\boldsymbol{D}} F$ and $\mathcal{H}om_{\boldsymbol{D}}(F, -)$. Then we obtain the commutative diagram:

$$\begin{tikzcd}
J \arrow[r, "\eta_J(J)"] \arrow[d, two heads, "\varphi"'] & \mathcal{H}om_{\boldsymbol{D}}(J, J \otimes J) \arrow[r, "\mathcal{H}om(-, \mathrm{id}_J \otimes \varphi)"] \arrow[d, "\mathcal{H}om(-, \varphi \otimes \mathrm{id}_J)"] & \mathcal{H}om_{\boldsymbol{D}}(J, J \otimes \mathcal{O}_{\boldsymbol{D}}) \arrow[d, "\cong"] \\
\mathcal{O}_{\boldsymbol{D}} \arrow[r, "\eta_J(\mathcal{O}_{\boldsymbol{D}})"] & \mathcal{H}om_{\boldsymbol{D}}(J, \mathcal{O}_{\boldsymbol{D}} \otimes J) \arrow[r, "\cong"] & \mathcal{H}om_{\boldsymbol{D}}(J, J).
\end{tikzcd}$$

But the top row also factors as

$$J \xrightarrow{\eta_{\mathcal{O}_{\boldsymbol{D}}}(J)} \mathcal{H}om_{\boldsymbol{D}}(\mathcal{O}_{\boldsymbol{D}}, J \otimes \mathcal{O}_{\boldsymbol{D}}) \xrightarrow{\mathcal{H}om(\varphi, -)} \mathcal{H}om_{\boldsymbol{D}}(J, J \otimes \mathcal{O}_{\boldsymbol{D}})$$

which is injective since $\eta_{\mathcal{O}_{\boldsymbol{D}}}$ is an isomorphism and φ is surjective. It follows that $J \to \mathcal{H}om_{\boldsymbol{D}}(J, J)$ is injective, hence so is $\varphi\colon J \to \mathcal{O}_{\boldsymbol{D}}$. $\square$

Proof of Theorem 5.8. If $K \in \boldsymbol{C}$, since $\boldsymbol{C}$ is locally finitely generated, it may be written as a directed colimit $K = \varinjlim_\lambda K_\lambda$, where $K_\lambda \subseteq K$ and K_λ is finitely generated. If $K \in \ker(q^*)$, then $q^* K_\lambda \subseteq q^* K \cong 0$. In particular, $\boldsymbol{K} := \ker(q^*) \subseteq \ker(f^*)$.

Let $0 \to K \to M \to N \to Q \to 0$ be an exact sequence in $\boldsymbol{C}$ with $K, Q \in \boldsymbol{K}$. We have to prove that $f^*(M \to N)$ is an isomorphism in $\boldsymbol{D}$. Let N_0 be the image of M in N. By right-exactness, we have an exact sequence $f^*(K) \to f^*(M) \to f^*(N_0) \to 0$. Since $f^*(K) = 0$, we have that $f^*(M) = f^*(N_0)$. We may thus replace M with N_0 and assume that $K = 0$ and $M \to N$ is injective.

Write N as the directed colimit of finitely generated subobjects $N_\lambda^\circ \subseteq N$. Let $N_\lambda = M + N_\lambda^\circ \subseteq N$ and $I_\lambda = \mathrm{Ann}(N_\lambda/M)$. By definition, we have that $I_\lambda \otimes N_\lambda/M \to N_\lambda/M$ is zero; hence $I_\lambda \otimes N_\lambda \to N_\lambda$ factors through M.

Note that $N_\lambda/M = N_\lambda^\circ/(N_\lambda^\circ \cap M)$ is a quotient of a finitely generated object and a subobject of Q, so $\mathcal{O}_{\boldsymbol{C}}/I_\lambda \in \boldsymbol{K}$ since q^* is supported. We conclude that $f^*(I_\lambda) \to f^*(\mathcal{O}_{\boldsymbol{C}})$ is an isomorphism using

Lemma 5.9. Now consider the commutative diagrams:

$$\begin{tikzcd} I_\lambda \otimes M \arrow[r] \arrow[d] & M \arrow[d] \\ I_\lambda \otimes N_\lambda \arrow[r] \arrow[ur] & N_\lambda \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f^*(M) \arrow[r, "\cong"] \arrow[d] & f^*(M) \arrow[d] \\ f^*(N_\lambda) \arrow[r, "\cong"] \arrow[ur] & f^*(N_\lambda), \end{tikzcd}$$

where the right diagram is obtained by applying f^* to the left diagram. It follows that $f^*(M) \to f^*(N_\lambda)$ is an isomorphism. Since f^* is cocontinuous, it follows that $f^*(M) \to f^*(N) = \varinjlim f^*(N_\lambda)$ is an isomorphism.

This proves that $f^* = g^* q^*$ where $g^* = f^* q_*$. It is readily verified that g^* is cocontinuous (it preserves small direct sums and is right-exact). If $M \in \boldsymbol{Q}$ is a finitely generated object, then we may find a finitely generated object $N \in \boldsymbol{C}$ such that $M = q^* N$. Indeed, by assumption $q_* M$ is a filtered colimit of finitely generated objects. It follows that there is a finitely generated subobject $N \subseteq q_* M$ such that $q^* N \to M$ is an isomorphism. If f^* preserves finitely generated objects, then $g^* M = f^* N$ is finitely generated. □

To show how powerful tensor localization is, we can quickly prove that tensoriality is local for the Zariski topology — even for stacks.

Theorem 5.10. *Let X be a quasicompact and quasiseparated algebraic stack. Let $X = \bigcup_{k=1}^n X_k$ be an open covering by quasicompact open substacks. If every X_k is tensorial, then so is X.*

Proof. Let $j_k \colon X_k \to X$ denote the open immersion and let I_k be an ideal of finite type defining a closed substack complementary to X_k [Rydh 2016, Proposition 8.2].

Let T be an algebraic stack. First we will show that $\omega_X(T)$ is fully faithful. Thus, let $f, g \colon T \to X$ be two morphisms and suppose that we are given a natural transformation of cocontinuous tensor functors $\gamma \colon f^* \Rightarrow g^*$. Then $f^*(\mathcal{O}_X/I_k) \twoheadrightarrow g^*(\mathcal{O}_X/I_k)$ so there is an inclusion $f^{-1}(X_k) \subseteq g^{-1}(X_k)$ for every k. Let $T_k = f^{-1}(X_k)$, let $j_{k,T} \colon T_k \to T$ denote the corresponding open immersion and let $f_k, g_k \colon T_k \to X_k$ denote the restrictions of f and g. Since $(f_k)^* = j_{k,T}^* f^* (j_k)_*$ and $(g_k)^* = j_{k,T}^* g^* (j_k)_*$, we obtain a natural transformation $\gamma_k \colon f_k^* \Rightarrow g_k^*$, hence a unique 2-isomorphism $f_k \Rightarrow g_k$. Since $T = \bigcup_{k=1}^N T_k$, it follows by fppf-descent, that $\omega_X(T)$ is faithful (Lemma 4.2(i)). As this holds for all T, we also have that $\omega_X(T_k \cap T_{k'})$ is faithful and it follows by fppf-descent that $\omega_X(T)$ is full (Lemma 4.2(ii)).

For essential surjectivity, let $f^* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor. The surjection $\mathcal{O}_T \twoheadrightarrow f^*(\mathcal{O}_X/I_k)$ defines a closed subscheme and we let $j_{k,T} \colon T_k \to T$ denote its open complement. By Theorem 5.1(i), $j_{k,T}^* f^*$ factors via j_k^* and a tensor functor $f_k^* \colon \mathsf{QCoh}(X_k) \to \mathsf{QCoh}(T_k)$. The latter is algebraic by assumption; hence, so is $j_{k,T}^* f^* = f_k^* j_k^*$.

Finally, since $\mathcal{O}_X/I_1 \otimes \cdots \otimes \mathcal{O}_X/I_n = 0$, it follows that $f^*(\mathcal{O}_X/I_1) \otimes \cdots \otimes f^*(\mathcal{O}_X/I_n) = 0$ so $T = \bigcup_{k=1}^n T_k$ is an open covering. We conclude that f^* is algebraic by fppf descent (Lemma 4.2(iii)). □

Combining Theorem 5.10 with Lemma 4.5 we obtain a short proof of the main result of [Brandenburg and Chirvasitu 2014].

Corollary 5.11 (Brandenburg–Chirvasitu). *Every quasicompact and quasiseparated scheme is tensorial.*

6. The main lemma

The main result of this section is the following technical lemma, which proves that the tensorial property extends over nilpotent thickenings of quasicompact algebraic stacks with affine stabilizers having the resolution property.

Main Lemma 6.1. *Let $i\colon X_0 \to X$ be a closed immersion of algebraic stacks defined by a quasicoherent ideal I such that $I^n = 0$ for some integer $n > 0$. Suppose that X_0 is quasicompact and quasiseparated with affine stabilizers. If X_0 has the resolution property, then X is tensorial.*

We have another lemma that will be crucial for proving Main Lemma 6.1.

Lemma 6.2. *Consider a 2-cocartesian diagram of algebraic stacks*:

$$\begin{tikzcd} U_0 \arrow[r, hook, "i"] \arrow[d, "p_0"'] & U \arrow[d, "p"] \\ X_0 \arrow[r, hook, "j"] & X, \end{tikzcd}$$

such that the following conditions are satisfied.

(i) *i is a nilpotent closed immersion*;

(ii) *U_0 is an affine scheme*; *and*

(iii) *X_0 is quasicompact and quasiseparated with affine stabilizers.*

If X_0 has the resolution property, *then so has X.*

Proof. Note that X_0 has affine diagonal by the Totaro–Gross theorem; hence p_0 is affine. By [Hall 2017, Proposition A.2], the square is a geometric pushout. In particular, j is a nilpotent closed immersion, p is affine, and the natural map $\mathcal{O}_X \to p_*\mathcal{O}_U \times_{p_*i_*\mathcal{O}_{U_0}} j_*\mathcal{O}_{X_0}$ is an isomorphism. By the Totaro–Gross theorem [Gross 2017, Corollary 5.9], there exists a vector bundle V_0 on X_0 such that the total space of the frame bundle of V_0 is quasiaffine. Let $E_0 = p_0^*V_0$; then, since U_0 is affine, there exists a vector bundle E on U equipped with an isomorphism $\alpha\colon i^*E \to E_0$. Let V be the quasicoherent $\mathcal{O}_X$-module $p_*E \times_\alpha j_*V_0$. By [Ferrand 2003, Théorème 2.2(iv)], V is a vector bundle on X and there is an isomorphism $j^*V \cong V_0$. By [Gross 2017, Proposition 5.7], it follows that X has the resolution property. ☐

Proof of Main Lemma 6.1. We prove the result by induction on $n > 0$. The case $n = 1$ is Theorem 4.11. So we let $n > 1$ be an integer and we will assume that if $W_0 \hookrightarrow W$ is any closed immersion of algebraic stacks defined by an ideal J such that $J^{n-1} = 0$ and W_0 has the resolution property, then W is tensorial. We now fix a closed immersion of algebraic stacks $i\colon X_0 \to X$ defined by an ideal I such that $I^n = 0$ and X_0 has the resolution property. It remains to prove that X is tensorial.

We observe that the Totaro–Gross theorem [Gross 2017, Corollary 5.9] implies that X_0 has affine diagonal; thus, X has affine diagonal. We have seen that $\omega_X(T)$ is fully faithful (Proposition 4.9) so it remains to prove that $\omega_X(T)$ is essentially surjective. By descent, it suffices to prove that if T is an affine scheme and $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ is a cocontinuous tensor functor, then there exists an étale and surjective morphism $c\colon T' \to T$ such that $c^* f^*$ is algebraic (Lemma 4.2(iii)).

By Corollary 3.6, there is a 2-cocartesian diagram in GTC

$$\begin{tikzcd} \mathsf{QCoh}(T_0) & \mathsf{QCoh}(X_0) \arrow[l, "f_0^*"'] \\ \mathsf{QCoh}(T) \arrow[u, "k^*"] & \mathsf{QCoh}(X), \arrow[l, "f^*"'] \arrow[u, "i^*"'] \end{tikzcd}$$

where $k\colon T_0 \to T$ is the closed immersion defined by the image K of f^*I in $\mathcal{O}_T$. In particular, $K^n = 0$. Since X_0 has the resolution property, f_0^* is given by a morphism of algebraic stacks $f_0\colon T_0 \to X_0$ (Theorem 4.11).

Let $p\colon U \to X$ be a smooth and surjective morphism, where U is an affine scheme; then, p is affine. The pullback of p along the morphism $i \circ f_0\colon T_0 \to X$ results in a smooth and affine surjective morphism of schemes $q_0\colon V_0 \to T_0$. By [EGA IV$_4$ 1967, 17.16.3(ii)], there exists an affine étale and surjective morphism $c_0\colon T_0' \to T_0$ such that the pullback $q_0'\colon V_0' \to T_0'$ of q_0 to T_0' admits a section. By [EGA IV$_4$ 1967, 18.1.2], there exists a unique affine étale morphism $c\colon T' \to T$ lifting $c_0\colon T_0' \to T_0$. After replacing T with T' and f^* with $c^* f^*$, we may thus assume that q_0 admits a section (Lemma 4.2(iii)).

Let $X' = \mathcal{S}\mathrm{pec}_X(f_*\mathcal{O}_T)$. Let $I' = I(f_*\mathcal{O}_T)$ be the $\mathcal{O}_{X'}$-ideal generated by I and let $X_0' = V(I')$. Then X' is a quasicompact stack with affine diagonal, $X_0' \to X'$ is a closed immersion defined by an ideal whose n-th power vanishes and X_0' has the resolution property. Let $f'^* = \bar{f}^*\colon \mathsf{QCoh}(X') \to \mathsf{QCoh}(T)$ be the resulting tensor functor.

Since f'^* is right-exact, it follows that $K = \mathrm{im}(f'^*I' \to \mathcal{O}_T)$. Also, $I' \subseteq f'_*K \subseteq \mathcal{O}_{X'}$. Thus $V(f'_*K) \subseteq X_0'$, so has the resolution property. Note that $f'^*I' \to f'^*f'_*K \to K$ is surjective. Since f'_* is lax symmetric monoidal, for each integer $l \geq 1$ the morphism $(f'_*K)^{\otimes l} \to \mathcal{O}_{X'}$ factors through $f'_*(K^{\otimes l}) \to \mathcal{O}_{X'}$. In particular, $(f'_*(K^l))^2 \subseteq f'_*(K^{l+1})$ and $(f'_*K)^n = 0$. We may thus replace X by X', X_0 by $V(f'_*K)$, f^* by f'^*, I by f_*K and assume henceforth that

(i) $\mathcal{O}_X \to f_*\mathcal{O}_T$ is an isomorphism,

(ii) $I = f_*K$ for some $\mathcal{O}_T$-ideal K with $K^n = 0$,

(iii) $f_*(K^l)^2 \subseteq f_*(K^{l+1})$ for each integer $l \geq 1$,

(iv) $(f_*K)^l \subseteq f_*(K^l)$ for $l \geq 1$, and

(v) $q_0\colon V_0 \to T_0$ admits a section.

For each integer $l \geq 0$ let $I_l = f_*(K^{l+1})$, which is a quasicoherent sheaf of ideals on X. Let $i_l\colon X_l \to X$ be the closed immersion defined by I_l and let $k_l\colon T_l \to T$ be the closed immersion defined by K^{l+1}. Since $f^*f_*(K^{l+1}) \to f^*\mathcal{O}_X = \mathcal{O}_T$ factors through K^{l+1}, it follows that $k_l^* f^*(i_l)_*(\mathcal{O}_{X_l}) = \mathcal{O}_{T_l}$. Hence, $f_l^* = k_l^* f^*(i_l)_*\colon \mathsf{QCoh}(X_l) \to \mathsf{QCoh}(T_l)$ is a tensor functor and $k_l^* f^* \simeq f_l^*(i_l)^*$ (Theorem 5.1(ii)).

By condition (iv), we see that $i_l\colon X_0 \to X_l$ is a closed immersion of algebraic stacks defined by an ideal whose $(l+1)$-th power is zero. In particular, if $l < n-1$, then X_l is tensorial by the inductive hypothesis. Thus, the tensor functor f_l^* is given by an affine morphism $f_l\colon T_l \to X_l$.

We will now prove by induction on $l \geq 0$ that X_l has the resolution property. Since $X_{n-1} = X$, the result will then follow from Theorem 4.11. Note that (iii) implies that the closed immersion $X_l \to X_{l+1}$ is a square zero extension of X_l by I_l/I_{l+1}. Let $m = n-2$.

Claim 1. If $M \in \mathsf{QCoh}(T_m)$, then the natural map $f_*(k_m)_*M \to p_*p^*f_*(k_m)_*M$ is split injective.

Proof of Claim 1. Form the cartesian diagram of algebraic stacks:

$$\begin{tikzcd} V_0 \arrow[r] \arrow[d, "q_0"'] & V_m \arrow[r, "g_m"] \arrow[d, "q_m"'] & U_m \arrow[r, "u_m"] \arrow[d, "p_m"'] & U \arrow[d, "p"] \\ T_0 \arrow[r] & T_m \arrow[r, "f_m"] & X_m \arrow[r, "i_m"] & X. \end{tikzcd}$$

Now observe that $f_*(k_m)_*M \cong (i_m)_*(f_m)_*M$. Since f_m^* is given by a morphism $f_m\colon T_m \to X_m$, there are natural isomorphisms

$$\begin{aligned} p_*p^*f_*(k_m)_*M &\cong p_*p^*(i_m)_*(f_m)_*M \\ &\cong p_*(u_m)_*p_m^*(f_m)_*M \\ &\cong p_*(u_m)_*(g_m)_*q_m^*M \\ &\cong (i_m)_*(f_m)_*(q_m)_*q_m^*M. \end{aligned}$$

Hence, it remains to prove that the natural map $M \to (q_m)_*q_m^*M$ is split injective. But q_m is affine, so $(q_m)_*q_m^*M \cong (q_m)_*\mathcal{O}_{V_m} \otimes_{\mathcal{O}_{T_m}} M$. Thus, we are reduced to proving that $\mathcal{O}_{T_m} \to (q_m)_*\mathcal{O}_{V_m}$ is split injective. By (v), q_0 admits a section. Since q_m is smooth and T_m is affine, the section that q_0 admits lifts to a section of q_m. This implies that the morphism $\mathcal{O}_{T_m} \to (q_m)_*\mathcal{O}_{V_m}$ is split injective. $\square$

Claim 2. If $0 \leq l < n-1$, then the natural maps $I_l/I_{l+1} \to p_*p^*(I_l/I_{l+1})$ are split injective.

Proof of Claim 2. If $N \in \mathsf{QCoh}(T_m)$, then $f_*(k_m)_*N = (i_m)_*(f_m)_*N$. Since f_m is an affine morphism, it follows that $f_*(k_m)_*\colon \mathsf{QCoh}(T_m) \to \mathsf{QCoh}(X)$ is exact. If P is one of the modules K^{l+1}, K^{l+2}, or K^{l+1}/K^{l+2}, then $K^{m+1}P = 0$, so the natural map $P \to (k_m)_*k_m^*P$ is an isomorphism. In particular, $f_*P \cong f_*(k_m)_*k_m^*P$. Hence, $I_l/I_{l+1} \cong f_*(k_m)_*k_m^*(K^{l+1}/K^{l+2})$ and the claim now follows from Claim 1. $\square$

So we let $l \geq 0$ be an integer, which we assume to be $< n-1$. We will assume that X_l has the resolution property and we will now prove that X_{l+1} has the resolution property. Retaining the notation of Claim 1, there is a 2-commutative diagram of algebraic stacks:

$$\begin{tikzcd} U_l \arrow[r] \arrow[dd, "p_l"] & U_{l+1} \arrow[d] \\ & \tilde{X}_{l+1} \arrow[d] \\ X_l \arrow[r] \arrow[ur] & X_{l+1}, \end{tikzcd}$$

where both the inner and outer squares are 2-cartesian and the inner square is 2-cocartesian. Let $Q_l = I_l/I_{l+1}$. The morphism $X_l \to \tilde{X}_{l+1}$ is a square zero extension of X_l by $(p_l)_* p_l^* Q_l \cong p_* p^* Q_l$ and the morphism $\tilde{X}_{l+1} \to X_{l+1}$ is the morphism of X_l-extensions given by the natural map $Q_l \to (p_l)_* p_l^* Q_l$. By Claim 2, the morphism $Q_l \to (p_l)_* p_l^* Q_l$ is split injective and so there is an induced splitting $X_{l+1} \to \tilde{X}_{l+1}$ which is affine. By [Gross 2017, Proposition 4.3(i)], it remains to prove that $\tilde{X}_{l+1}$ has the resolution property, which is just Lemma 6.2. □

7. Formal gluings

Let T be an algebraic stack, let $i\colon Z \hookrightarrow T$ be a finitely presented closed immersion and let $j\colon U \to T$ denote its complement. A *flat Mayer–Vietoris square* is a cartesian square of algebraic stacks

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "\pi_U"'] \arrow[dr, phantom, "\square"] & T' \arrow[d, "\pi"] \\ U \arrow[r, "j"] & T \end{tikzcd}$$

such that π is flat and the induced morphism $\pi_Z\colon T' \times_T Z \to Z$ is an isomorphism [Moret-Bailly 1996; Hall and Rydh 2016].

Given an affine noetherian scheme $T = \operatorname{Spec} A$ and a closed subscheme $Z = V(I)$ we obtain a flat Mayer–Vietoris square as follows: let $U = T \setminus Z$, let $T' = \operatorname{Spec} \hat{A}$, where $\hat{A}$ is the I-adic completion of A, and let $U' = U \times_T T'$. We call such squares *formal gluings*. While we will state our results more generally, the flat Mayer–Vietoris squares of relevance to this article will always be formal gluings.

If $F\colon \mathsf{AlgSt}^{\mathrm{op}} \to \mathrm{Cat}$ is a pseudofunctor, then there is a natural functor

$$\Phi_F\colon F(T) \to F(T') \times_{F(U')} F(U).$$

Here AlgSt denotes the 2-category of algebraic stacks. For the purposes of this paper, it is enough to consider pseudofunctors defined on affine schemes, that is, fibered categories over affine schemes.

In this article, we will only consider two examples of pseudofunctors F. Let X be an algebraic stack.

(i) We may view X as a pseudofunctor via the 2-Yoneda lemma: $T \mapsto \operatorname{Hom}(T, X) \simeq X(T)$. Note that the flat Mayer–Vietoris square is a pushout (i.e., cocartesian) if and only if Φ_X is an equivalence.

(ii) There is also the pseudofunctor $X_\otimes(-) = \operatorname{Hom}_\otimes(\mathsf{QCoh}(X), \mathsf{QCoh}(-))$. Note that $\Phi_{X_\otimes}$ is an equivalence if and only if quasicoherent sheaves can be glued along the flat Mayer–Vietoris square.

The following theorem follows from the main results of [Hall and Rydh 2016] (and almost from [Moret-Bailly 1996]).

Theorem 7.1. *Consider a flat Mayer–Vietoris square as above. Then*

(i) $\Phi_{X_\otimes}$ *is an equivalence of categories*;

(ii) Φ_X *is fully faithful*; *and is an equivalence if*

(a) Δ_X *is quasiaffine*; *or*
(b) *X is Deligne–Mumford*; *or*
(c) *T is locally the spectrum of a G-ring* (*see Remark 7.2*).

Proof. By [Hall and Rydh 2016, Theorem B(1)] (or one of [Moret-Bailly 1996, 0.3] and [Ferrand and Raynaud 1970, Appendice] when π is affine), there is an equivalence

$$\mathsf{QCoh}(T) \to \mathsf{QCoh}(T') \times_{\mathsf{QCoh}(U')} \mathsf{QCoh}(U).$$

Thus, we have (i). Claims (ii) and (a) are [Hall and Rydh 2016, Theorem B(3)] and claims (b) and (c) are [Hall and Rydh 2016, Theorems E and A] respectively. Under some additional assumptions: π is affine, Δ_X is quasicompact and separated, and in (c) T' is locally noetherian; claims (a)–(c) also follow from [Moret-Bailly 1996, 6.2 and 6.5.1]. $\square$

Remark 7.2. Recall that a noetherian ring A is *excellent* [Matsumura 1989, page 260; 1980, Chapter 13; EGA IV_2 1965, 7.8.2], if:

(i) A is a G-ring, that is, $A_{\mathfrak{p}} \to \hat{A}_{\mathfrak{p}}$ has geometrically regular fibers for every prime ideal $\mathfrak{p} \subset A$.

(ii) The regular locus $\operatorname{Reg} B \subseteq \operatorname{Spec} B$ is open for every finitely generated A-algebra B.

(iii) A is universally catenary.

If (i) and (ii) hold, then we say that A is *quasiexcellent*. All G-ring assumptions in this paper originate from [Moret-Bailly 1996; Hall and Rydh 2016] via Theorem 7.1. The assumptions are used to guarantee that the formal fibers are geometrically regular so that Néron–Popescu desingularization applies. Note that whereas being a G-ring and being quasiexcellent are local for the smooth topology [Matsumura 1989, 32.2], excellency does not descend even for finite étale coverings [EGA IV_4 1967, 18.7.7].

Corollary 7.3. *Let X be an algebraic stack. Let A be a ring and let $I \subset A$ be a finitely generated ideal. Let $T = \operatorname{Spec} A$, $Z = V(I)$ and $U = T \setminus Z$. Let $i\colon Z \to T$ and $j\colon U \to T$ be the resulting immersions.*

(i) *Let $f_1, f_2\colon T \to X$ be morphisms of algebraic stacks.*
  (a) *Assume that* $\ker(\mathcal{O}_T \to j_*\mathcal{O}_U) \cap \bigcap_{n=0}^{\infty} I^n = 0$. *Let $\alpha, \beta\colon f_1 \Rightarrow f_2$ be 2-morphisms. If $\alpha_U = \beta_U$ and $\alpha_{Z^{[n]}} = \beta_{Z^{[n]}}$ for all n, then $\alpha = \beta$.*
  (b) *Assume that T is noetherian and that $\omega_X(T)$ is faithful for all noetherian T. Let $t\colon f_1^* \Rightarrow f_2^*$ be a natural transformation of cocontinuous tensor functors. If $j^*(t)$ and $(i^{[n]})^*(t)$ are realizable for all n, then t is realizable.*

(ii) *Assume either* (*a*) *T is the spectrum of a G-ring*, *or* (*b*) *T is noetherian and X has quasiaffine or unramified diagonal. Further*, *assume that $\omega_{X,\simeq}(T)$ is fully faithful for all noetherian T. Let $f^*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ be a cocontinuous tensor functor that* preserves sheaves of finite type. *If j^*f^* and $(i^{[n]})^*f^*$ are algebraic for all n, then f^* is algebraic.*

The assumption in (a) says that the filtration $\{\varnothing \hookrightarrow Z \hookrightarrow T\}$ is *separating* (Definition A.1). This is automatic if T is noetherian (Lemma A.2).

Proof of Corollary 7.3. First, we show (ii). By assumption, the induced functor $(i^{[n]})^*f^*$ comes from a morphism $f^{[n]}\colon Z^{[n]} \to X$. Pick an étale cover $q\colon \tilde{Z} \to Z$ such that $f^{[0]} \circ q\colon \tilde{Z} \to X$ has a lift $g\colon \tilde{Z} \to W$, where $p\colon W \to X$ is a smooth morphism and W is affine. Descent (Lemma 4.2(iii)) implies that we are free to replace T with an étale cover, so we may assume that $f^{[0]}$ also has a lift $g\colon Z \to W$ [EGA IV$_4$ 1967, 18.1.1].

Since p is smooth, we may choose compatible lifts $g^{[n]}\colon Z^{[n]} \to W$ of $f^{[n]}$ for all n. But W is affine, so there is an induced morphism $\hat{g}\colon \hat{T} \to W$, where $\hat{T} = \operatorname{Spec} \hat{A}$ and $\hat{A}$ denotes the completion of A at the ideal I. Let $\hat{f} = p \circ \hat{g}$. Then $(i^{[n]})^*\hat{f}^* = (f^{[n]})^* = (i^{[n]})^*f^*$ for all n. Since $\mathsf{Coh}(\hat{T}) = \varprojlim_n \mathsf{Coh}(Z_n)$ (Lemma 3.8), it follows that $\hat{f}^* \simeq \pi^*f^*$ where $\pi\colon \hat{T} \to T$ is the completion morphism. Indeed, this last equivalence may be verified after restricting both sides to quasicoherent $\mathcal{O}_X$-modules of finite type (Example 5.5) and both sides send quasicoherent $\mathcal{O}_X$-modules of finite type to $\mathsf{Coh}(\hat{T})$.

Let $\hat{j}\colon \hat{U} \to \hat{T}$ be the pullback of j along π; then we obtain a flat Mayer–Vietoris square:

$$\begin{tikzcd} \hat{U} \arrow[r, "\hat{j}"] \arrow[d, "\pi_U"'] \arrow[dr, phantom, "\square"] & \hat{T} \arrow[d, "\pi"] \\ U \arrow[r, "j"] & T. \end{tikzcd}$$

Since U and $\hat{U}$ are noetherian, $\omega_{X,\simeq}(U)$ and $\omega_{X,\simeq}(\hat{U})$ are fully faithful. Thus, there is an essentially unique morphism of algebraic stacks $h\colon U \to X$ such that $h^* \simeq j^*f^*$. But there are isomorphisms:

$$\hat{j}^*\hat{f}^* \simeq \hat{j}^*\pi^*f^* \simeq \pi_U^*j^*f^* \simeq \pi_U^*h^*,$$

so $\hat{f} \circ \hat{j} \simeq h \circ \pi_U$. That f^* is algebraic now follows from Theorem 7.1.

For (b), we proceed similarly. Consider the representable morphism $E \to T$ given by the equalizer of f_1 and f_2. Then 2-isomorphisms between f_1 and f_2 correspond to T-sections of E. By assumption, we have compatible sections $\tau_U \in E(U)$ and $\tau^{[n]} \in E(Z^{[n]})$ for all n. Choose an étale presentation $E' \to E$ by an affine scheme E'. We may replace T with an étale cover (Lemma 4.2(ii)) and thus assume that $\tau^{[0]}$ lifts to E'. In particular, there are compatible lifts of all the $\tau^{[n]}$ to E'. Since E' is affine, we get an induced morphism $\hat{T} \to E'$; thus, a morphism $\hat{T} \to E$. Equivalently, we get a 2-isomorphism between $f_1 \circ \pi$ and $f_2 \circ \pi$. The induced 2-isomorphism between $\pi^*f_1^*$ and $\pi^*f_2^*$ equals π^*t since it coincides on the truncations. We may now apply Theorem 7.1 to deduce that t is realized by a 2-morphism $\tau\colon f_1 \Rightarrow f_2$.

For (a), we consider the representable morphism $r\colon R \to T$ given by the equalizer of α and β. It suffices to prove that r is an isomorphism. Note that r is always a monomorphism and locally of finite presentation. By assumption, there are compatible sections of r over U and $Z^{[n]}$ for all n, thus r_U and $r_{Z^{[n]}}$ are isomorphisms for all n. By Proposition A.3, r is an isomorphism. □

Remark 7.4. We do not know if the condition that f^* preserves sheaves of finite type in (ii) is necessary. We do know that for any sheaf F of finite type, the restrictions of f^*F to U and $Z^{[n]}$ are coherent but this does not imply that f^*F is coherent. For example, if $A = k[\![x]\!]$, and $I = (x)$, then the A-module $k((x))/k[\![x]\!]$ is not finitely generated but becomes 0 after tensoring with $A/(x^n)$ or A_x.

8. Tannaka duality

In this section, we prove our general Tannaka duality result (Theorem 8.4) and as a consequence also establish Theorem 1.1. To accomplish this, we consider the following refinement of [Hall and Rydh 2015a, Definition 2.4].

Definition 8.1. Let X be a quasicompact algebraic stack. A *finitely presented filtration of X* is a sequence of finitely presented closed immersions $\varnothing = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_r \hookrightarrow X$ such that $|X_r| = |X|$. The *strata* of the filtration are the locally closed finitely presented substacks $Y_k := X_k \setminus X_{k-1}$.

Stacks that have affine stabilizers can be stratified into stacks with the resolution property.

Proposition 8.2. *Let X be an algebraic stack. The following are equivalent*:

(i) *X is quasicompact and quasiseparated with affine stabilizers.*

(ii) *X has a finitely presented filtration (X_k) with strata of the form $Y_k = [U_k/\operatorname{GL}_{N_k}]$ where U_k is quasiaffine.*

(iii) *X has a finitely presented filtration (X_k) with strata Y_k that are quasicompact with affine diagonal and the resolution property.*

Proof. That (i)$\Longrightarrow$(ii) is [Hall and Rydh 2015a, Proposition 2.6(i)]. That (ii)$\Longleftrightarrow$(iii) is the Totaro–Gross theorem [Gross 2017]. That (iii)$\Longrightarrow$(i) is straightforward. $\square$

When in addition X is noetherian or, more generally, X has finitely presented inertia, this result is due to Kresch [1999, Proposition 3.5.9] and Drinfeld and Gaitsgory [2013, Proposition 2.3.4]. They construct stratifications by quotient stacks of the form $[V_k/\operatorname{GL}_{N_k}]$, where each V_k is quasiprojective and the action is linear. This implies that the strata have the resolution property. When X has finitely presented inertia the situation is simpler since X can be stratified into gerbes [Rydh 2016, Corollary 8.4], something which is not possible in general.

Remark 8.3. Drinfeld and Gaitsgory [2013, Definition 1.1.7] introduced the notion of a QCA stack. These are (derived) algebraic stacks that are quasicompact and quasiseparated with affine stabilizers and finitely presented inertia. The condition on the inertia is presumably only used for [Drinfeld and Gaitsgory 2013, Proposition 2.3.4] and could be excised using Proposition 8.2.

We now state and prove the main result of the paper.

Theorem 8.4. *Let T and X be algebraic stacks and consider the functor*

$$\omega_X(T)\colon \operatorname{Hom}(T, X) \to \operatorname{Hom}_{c\otimes}(\mathsf{QCoh}(X), \mathsf{QCoh}(T))$$

and its variants $\omega_X^{\mathrm{ft}}(T)$, $\omega_{X,\simeq}(T)$ and $\omega_{X,\simeq}^{\mathrm{ft}}(T)$ (see Section 4). Assume that X is quasicompact and quasiseparated with affine stabilizers and that T is locally noetherian. Then

(i) *$\omega_{X,\simeq}(T)$ is fully faithful*;

(ii) *$\omega_X(T)$ is fully faithful if X is affine-pointed*;

(iii) $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ *is an equivalence of groupoids if either*

(a) T *is locally the spectrum of a G-ring*; (*Remark 7.2*)

(b) X *has quasiaffine diagonal*; *or*

(c) X *is Deligne–Mumford.*

In particular, $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ *is an equivalence if T is locally excellent and X has affine stabilizers, and* $\omega^{\mathrm{ft}}_{X}(T)$ *is an equivalence if T is locally noetherian and X either has quasiaffine diagonal or is Deligne–Mumford.*

When X has quasiaffine diagonal, we have already seen that $\omega_X(T)$ is fully faithful without any noetherian assumptions on T (Proposition 4.9(i)). In some situations, we can also prove faithfulness for nonnoetherian T, see Proposition 8.5 below.

Proof of Theorem 8.4. We will first prove that $\omega_X(T)$ is faithful, then prove that $\omega_{X,\simeq}(T)$ and $\omega_X(T)$ are fully faithful (the latter under the assumption that X is affine-pointed) and finally prove that $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ is an equivalence under the assumptions in (iii). By Lemma 4.3, it is enough to prove these results when $T = \operatorname{Spec} A$ is affine.

Stratification on X. Choose a filtration (X_k) as in Proposition 8.2. We will prove the theorem by induction on the number of strata r. If $r = 0$, then $X = \varnothing$ and there is nothing to prove. If $r \geq 1$, then $U := X \setminus X_1$ has a filtration of length $r - 1$; thus by induction the theorem holds over U.

Let $I \subseteq \mathcal{O}_X$ be the ideal defining $Z := X_1$. Let $i^{[n]} \colon Z^{[n]} \hookrightarrow X$ be the closed substack defined by I^{n+1} and let $j \colon U \to X$ be its complement. The filtration was chosen such that Z has the resolution property. Thus $\omega_{Z^{[n]}}(T)$ is an equivalence of categories for every $n \geq 0$ by the Main Lemma 6.1. In particular, the theorem holds over $Z^{[n]}$ for every $n \geq 0$.

Setup. For faithfulness, pick two maps $f_1, f_2 \colon T \to X$ and two 2-isomorphisms $\tau_1, \tau_2 : f_1 \Rightarrow f_2$ and assume that $\omega_X(T)(\tau_1) = \omega_X(T)(\tau_2)$. We need to prove that $\tau_1 = \tau_2$.

For fullness of $\omega_{X,\simeq}(T)$ (resp. $\omega_X(T)$), pick two maps $f_1, f_2 \colon T \to X$ and a natural isomorphism (resp. transformation) $\gamma \colon f_1^* \Rightarrow f_2^*$ of cocontinuous tensor functors. We need to prove that γ is realizable.

For essential surjectivity, pick a cocontinuous tensor functor $f^* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(T)$ preserving sheaves of finite type. We need to prove that f^* is algebraic.

Pulled-back stratification on T. For faithfulness and fullness, let $I_T = \operatorname{Im}(f_2^* I \to f_2^* \mathcal{O}_X = \mathcal{O}_T)$, which is a finitely generated ideal because f_2 is a morphism. For essential surjectivity, let $I_T = \operatorname{Im}(f^* I \to f^* \mathcal{O}_X = \mathcal{O}_T)$, which is a finitely generated ideal since f^* is assumed to preserve finite type objects. Let $i_T^{[n]} \colon Z_T^{[n]} \hookrightarrow T$ be the finitely presented closed immersion defined by I_T^{n+1} and let $j_T : U_T \hookrightarrow T$ be its complement, a quasicompact open immersion.

Result holds on strata. For faithfulness and fullness, we have that $U_T = f_1^{-1}(U) = f_2^{-1}(U)$; for faithfulness this is obvious and for fullness of $\omega_{X,\simeq}(T)$ and $\omega_X(T)$ this follows from Corollary 4.10 and Lemma 4.8 (when X affine-pointed), respectively. We also have that $Z_T^{[n]} = f_2^{-1}(Z^{[n]}) \hookrightarrow f_1^{-1}(Z^{[n]})$. Thus, by the inductive assumption and the case $r = 1$, after restricting to either U_T or $Z_T^{[n]}$ we have that $\tau_1 = \tau_2$ (for faithfulness), and that γ is realizable (for fullness).

For essential surjectivity, Theorem 5.1 produces for every $n \geq 0$ essentially unique cocontinuous tensor functors $f_U^*\colon \mathsf{QCoh}(U) \to \mathsf{QCoh}(U_T)$ and $f_{Z^{[n]}}^*\colon \mathsf{QCoh}(Z^{[n]}) \to \mathsf{QCoh}(Z_T^{[n]})$ such that $j_T^* f^* \simeq f_U^* j^*$ and $(i_T^{[n]})^* f^* \simeq (f_{Z^{[n]}})^*(i^{[n]})^*$. By the inductive assumption, f_U^* is algebraic and the case $r = 1$ implies that $f_{Z^{[n]}}^*$ is algebraic for each $n \geq 0$. In particular, $j_T^* f^*$ and $(i_T^{[n]})^* f^*$ is algebraic for each $n \geq 0$.

Formal gluing. The result now follows from Corollary 7.3 which uses the noetherian assumption on T. □

We also have some partial results in the nonnoetherian situation.

Proposition 8.5. *Let $f_1, f_2\colon T \to X$ be morphisms of algebraic stacks. Assume that X is quasicompact and quasiseparated with affine stabilizers. Further assume that either*

(i) *T has no embedded associated points*; *or*

(ii) *f_2 factors as $T \to S \to X$ where S is locally noetherian and $\pi : T \to S$ is flat.*

Then $\omega_X(T)\colon$ $\mathrm{Hom}(f_1, f_2) \to \mathrm{Hom}(f_1^*, f_2^*)$ is injective. In particular, *if T has no embedded associated points*, *then $\omega_X(T)$ is faithful.*

Proof. The proof is identical with the proof of faithfulness in Theorem 8.4. We only have to argue that Corollary 7.3(a) applies. That is, we have to show that the filtration $\{\varnothing \hookrightarrow Z_T \hookrightarrow T\}$ is separating.

If T has no embedded associated points, then the stratification $\varnothing \subset Z_T \subset T$ is separating by Lemma A.2. If f_2 factors as in (ii), then the stratification $\{\varnothing \hookrightarrow Z_T \hookrightarrow T\}$ is the pull-back along π of a stratification $\{\varnothing \hookrightarrow Z_S \hookrightarrow S\}$, hence separating by Lemma A.2. □

We conclude with the proof of Theorem 1.1.

Proof of Theorem 1.1. First, observe that if Y is a noetherian algebraic stack, then $\mathsf{QCoh}(Y)$ may be identified as the ind-category of $\mathsf{Coh}(Y)$ [Lurie 2004, 3.9–10]. Essentially by definition, this induces an equivalence of categories:

$$\mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(T)) \to \mathrm{Hom}^{\mathrm{ft}}_{c\otimes,\simeq}(\mathsf{QCoh}(X), \mathsf{QCoh}(T)).$$

It is thus enough to prove that $\omega^{\mathrm{ft}}_{X,\simeq}(T)$ is an equivalence of groupoids, which follows from Theorem 8.4. □

9. Applications

In this section, we address the applications outlined in the introduction.

Proof of Corollary 1.4. Let $T' \to T$ be an fpqc covering where T is an algebraic stack, locally the spectrum of a G-ring, and T' is a locally noetherian algebraic stack. Since X is an fppf-stack, we may assume that T and T' are affine and that $T' \to T$ is faithfully flat. Let $T'' = T' \times_T T'$. Since X has affine stabilizers, the functor $\omega_{X,\simeq}(T)$ is an equivalence, the functor $\omega_{X,\simeq}(T')$ is fully faithful and the functor $\omega_X(T'')$ is faithful for morphisms $T'' \to T' \to X$ (Theorem 8.4 and Proposition 8.5). Since $\mathrm{Hom}_{c\otimes,\simeq}(\mathsf{QCoh}(X), \mathsf{QCoh}(-))$ is an fpqc stack, it follows that $T' \to T$ is a morphism of effective descent for X. □

Proof of Corollary 1.5. As A is noetherian, $\mathsf{Coh}(A) = \varprojlim_n \mathsf{Coh}(A/I^n)$. Thus, by Theorem 8.4,

$$\begin{aligned} X(A) &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(A)) \\ &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \varprojlim \mathsf{Coh}(A/I^n)) \\ &\cong \varprojlim \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(X), \mathsf{Coh}(A/I^n)) \\ &\cong \varprojlim X(A/I^n). \end{aligned}$$

□

Proof of Theorem 1.2. First, we prove (i); that is, the stack $\underline{\mathrm{Hom}}_S(Z, X)$ is algebraic. We begin with the following standard reductions: we can assume that S is affine; $X \to S$ is quasicompact, so is of finite presentation; and S is of finite type over $\operatorname{Spec} \mathbb{Z}$.

Since S is now assumed to be excellent, we can prove the algebraicity of $\underline{\mathrm{Hom}}_S(Z, X)$ using a variant of Artin's criterion for algebraicity due to the first author [Hall 2017, Theorem A]. Hence, it is sufficient to prove that $\underline{\mathrm{Hom}}_S(Z, X)$ is

[1] a stack for the étale topology;

[2] limit preserving, equivalently, locally of finite presentation;

[3] homogeneous, that is, satisfies a strong version of the Schlessinger–Rim criteria;

[4] effective, that is, formal deformations can be algebraized;

[5] the automorphisms, deformations, and obstruction functors are coherent.

The main result of this article provides a method to prove [4] in maximum generality, which we address first. Thus, let $T = \operatorname{Spec} B \to S$, where $(B, \mathfrak{m})$ is a complete local noetherian ring. Let $T_n = \operatorname{Spec}(B/\mathfrak{m}^{n+1})$. Since $Z \to S$ is proper, for every noetherian algebraic stack W with affine stabilizers there are equivalences

$$\begin{aligned} \mathrm{Hom}(Z \times_S T, W) &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(W), \mathsf{Coh}(Z \times_S T)) && \text{(Theorem 1.1)} \\ &\cong \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(W), \varprojlim \mathsf{Coh}(Z \times_S T_n)) && \text{[Olsson 2005, Theorem 1.4]} \\ &\cong \varprojlim \mathrm{Hom}_{r\otimes,\simeq}(\mathsf{Coh}(W), \mathsf{Coh}(Z \times_S T_n)) && \text{(Lemma 3.8)} \\ &\cong \varprojlim \mathrm{Hom}(Z \times_S T_n, W) && \text{(Theorem 1.1).} \end{aligned}$$

Since X and S have affine stabilizers, it follows that

$$\mathrm{Hom}_S(Z \times_S T, X) \cong \varprojlim \mathrm{Hom}_S(Z \times_S T_n, X);$$

that is, the stack $\underline{\mathrm{Hom}}_S(Z, X)$ is effective and so satisfies [4].

The remainder of Artin's conditions are routine, so we will just sketch the arguments and provide pointers to the literature where they are addressed in more detail. Condition [1] is just étale descent and [2] is standard — see, for example, [Laumon and Moret-Bailly 2000, Proposition 4.18]. For conditions [3] and [5], it will be convenient to view $\underline{\mathrm{Hom}}_S(Z, X)$ as a substack of another moduli problem. This lets us avoid having to directly discuss the deformation theory of nonrepresentable morphisms of algebraic stacks.

If $W \to S$ is a morphism of algebraic stacks, let $\underline{\mathrm{Rep}}_{W/S}$ denote the S-groupoid that assigns to each S-scheme T the category of *representable* morphisms of algebraic stacks $V \to W \times_S T$ such that the composition $V \to W \times_S T \to T$ is proper, flat and of finite presentation. There is a morphism of S-groupoids

$$\Gamma\colon \underline{\mathrm{Hom}}_S(Z, X) \to \underline{\mathrm{Rep}}_{Z\times_S X/S},$$

which is given by sending a T-morphism $f\colon Z\times_S T \to X\times_S T$ to its graph $\Gamma(f)\colon Z\times_S T \to (Z\times_S X)\times_S T$. It is readily seen that Γ is formally étale since $Z \to S$ is flat. Hence, it is sufficient to verify conditions [3] and [5] for $\underline{\mathrm{Rep}}_{Z\times_S X/S}$ [Hall 2017, Lemmas 1.5(9), 6.3 and 6.11]. That $\underline{\mathrm{Rep}}_{Z\times_S X/S}$ is homogeneous follows immediately from [Hall 2017, Lemma 9.3]. A description of the automorphism, deformation and obstruction functors of $\underline{\mathrm{Rep}}_{Z\times_S X/S}$ in terms of the cotangent complex are given on [Hall 2017, page 173], which mostly follows from the results of [Olsson 2006a]. That these functors are coherent is [Hall 2014, Theorem C]. This completes the proof of (i).

We now address (ii) and (iii), that is, the separation properties of the algebraic stack $\underline{\mathrm{Hom}}_S(Z, X)$ relative to S. Let T be an affine scheme. Let Z_T and X_T denote $Z\times_S T$ and $X\times_S T$, respectively. Suppose we are given two T-morphisms $f_1, f_2\colon Z_T \to X_T$ and consider $Q := \underline{\mathrm{Isom}}_{Z_T}(f_1, f_2) = X \times_{X\times_S X} Z_T$. Then $Q \to Z_T$ is representable and of finite presentation. If $\pi: Z_T \to T$ denotes the structure morphism, then π_*Q is an algebraic space which is locally of finite presentation, being the pull-back of the diagonal of $\underline{\mathrm{Hom}}_S(Z, X)$ along the morphism $T \to \underline{\mathrm{Hom}}_S(Z, X) \times_S \underline{\mathrm{Hom}}_S(Z, X)$ corresponding to (f_1, f_2).

Let P be one of the properties: affine, quasiaffine, separated, quasiseparated. Assume that Δ_X has P; then $Q \to Z_T$ has P. We claim that the induced morphism $\pi_*Q \to T$ has P. For the properties affine and quasiaffine, this is [Hall and Rydh 2015b, Theorem 2.3(i),(ii)]. For quasiseparated (resp. separated), this is [Hall and Rydh 2015b, Theorem 2.3(ii),(iv)] applied to the quasiaffine morphism (resp. closed immersion) $Q \to Q\times_Z Q$ and the Weil restriction $\pi_*Q \to \pi_*Q\times_T \pi_*Q = \pi_*(Q\times_Z Q)$. In particular, we have proved that $\underline{\mathrm{Hom}}_S(Z, X)$ is algebraic and locally of finite presentation with quasiseparated diagonal over S.

Now by Theorem B.1, $\Delta_{\underline{\mathrm{Hom}}_S(Z,X)/S} = \underline{\mathrm{Hom}}_S(Z, \Delta_{X/S})$ is of finite presentation, so $\underline{\mathrm{Hom}}_S(Z, X)$ is also quasiseparated. It remains to prove that it has affine stabilizers. To see this, we may assume that T is the spectrum of an algebraically closed field. In this situation, either π_*Q is empty or $f_1 \simeq f_2$; it suffices to treat the latter case. In the latter case, $T \to X\times_S X$ factors through the diagonal $\Delta_{X/S}\colon X \to X\times_S X$, so it is sufficient to prove that $\underline{\mathrm{Hom}}_S(Z, I_{X/S})$, where $I_{X/S}\colon X\times_{X\times_S X} X \to X$ is the inertia stack, has affine fibers. But $I_{X/S}$ defines a group over X with affine fibers, and the result follows from Theorem B.1. □

Lemma 9.1. *Let $f\colon Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks. For any morphism $g\colon X \to Z$ of algebraic stacks, the forgetful morphism $f_*X \to \underline{\mathrm{Hom}}_S(Z, X)$ is an open immersion.*

Proof. It is sufficient to prove that if T is an affine S-scheme and $h\colon Z\times_S T \to X\times_S T$ is a T-morphism, then the locus of points where $g_T \circ h\colon Z\times_S T \to Z\times_S T$ is an isomorphism is open on T.

First, consider the diagonal of $g_T \circ h$. This morphism is proper and representable and the locus on T where this map is a closed immersion is open [Rydh 2011b, Lemma 1.8(iii)]. We may thus assume that $g_T \circ h$ is representable. Repeating the argument on $g_T \circ h$, we may assume that $g_T \circ h$ is a closed immersion. That the locus in T where $g_T \circ h$ is an isomorphism is open now follows easily by studying the étale locus of $g_T \circ h$, see [Olsson 2006b, Lemma 5.2]. The result follows. □

Proof of Theorem 1.3. That $f_*X \to S$ is algebraic, locally of finite presentation, with quasicompact and quasiseparated diagonal and affine stabilizers follows from Theorem 1.2 and Lemma 9.1. The additional separation properties of f_*X follows from [Hall and Rydh 2015b, Theorem 2.3(i),(ii) and (iv)] applied to the diagonal and double diagonal of $X \to Z$. □

As claimed in the introduction, we now extend [Hall and Rydh 2015b, Theorem 2.3 and Corollary 2.4]. The statement of the following corollary uses the notion of a morphism of algebraic stacks that is *locally of approximation type* [Hall and Rydh 2015b, Section 1]. A trivial example of a morphism locally of approximation type is a quasiseparated morphism that is locally of finite presentation. It is hoped that every quasiseparated morphism of algebraic stacks is locally of approximation type, but this is currently unknown. It is known, however, that morphisms of algebraic stacks that have quasifinite and locally separated diagonal are locally of approximation type [Rydh 2015]. In particular, all quasiseparated morphisms of algebraic stacks that are relatively Deligne–Mumford are locally of approximation type.

Corollary 9.2. *Let $f : Z \to S$ be a proper and flat morphism of finite presentation between algebraic stacks.*

(i) *Let $h : X \to S$ be a morphism of algebraic stacks with affine stabilizers that is quasiseparated and locally of approximation type. Then $\underline{\mathrm{Hom}}_S(Z, X)$ is algebraic, quasiseparated and locally of approximation type with affine stabilizers. If h is locally of finite presentation, then so is $\underline{\mathrm{Hom}}_S(Z, X) \to S$. If the diagonal of h is affine (or quasiaffine, or separated), then so is the diagonal of $\underline{\mathrm{Hom}}_S(Z, X) \to S$.*

(ii) *Let $g : X \to Z$ be a morphism of algebraic stacks such that $f \circ g : X \to S$ has affine stabilizers and is quasiseparated and locally of approximation type. Then the S-stack f_*X is algebraic, quasiseparated and locally of approximation type with affine stabilizers. If g is locally of finite presentation, then so is $f_*X \to S$. If the diagonal of g is affine (or quasiaffine, or separated), then so is the diagonal of $f_*X \to S$.*

Proof. For (i), we may immediately reduce to the situation where S is an affine scheme. Since f is quasicompact, we may further assume that h is quasicompact. By [Hall and Rydh 2015b, Lemma 1.1], there is an fppf covering $\{S_i \to S\}$ such that each S_i is affine and $X \times_S S_i \to S_i$ factors as $X \times_S S_i \to X_i^0 \to S_i$, where $X_i^0 \to S_i$ is of finite presentation and $X \times_S S_i \to X_i^0$ is affine. Combining the results of [Hall and Rydh 2015a, Theorem 2.8] with [Rydh 2015, Theorems D and 7.10], we can arrange so that each $X_i^0 \to S$ has affine stabilizers (or has one of the other desired separation properties).

Thus, we may now replace S by S_i and may assume that $X \to S$ factors as $X \xrightarrow{q} X_0 \to S$, where q is affine and $X_0 \to S$ is of finite presentation with the appropriate separation condition. By Theorem 1.2, the stack $\underline{\mathrm{Hom}}_S(Z, X_0)$ is algebraic and locally of finite presentation with the appropriate separation condition. By [Hall and Rydh 2015b, Theorem 2.3(i)], the morphism $\underline{\mathrm{Hom}}_S(Z, X) \to \underline{\mathrm{Hom}}_S(Z, X_0)$ is representable by affine morphisms; the result follows.

For (ii) we argue exactly as in the proof of Theorem 1.3. □

10. Counterexamples

In this section we give four counterexamples (Theorems 10.1, 10.2, 10.4, and 10.5):

- In Theorems 1.1 and 8.4, and Proposition 8.5 it is necessary that X has affine stabilizer groups.
- In Theorem 8.4(ii), it is necessary that X is affine-pointed.
- In Theorem 1.2, it is necessary that X has affine stabilizer groups.
- In Corollary 1.5, it is necessary that X has affine stabilizer groups.

For this section, the following definition will be important. Let k be a field and let G be an algebraic group scheme over k; we say that G is *antiaffine* if $\Gamma(G, \mathcal{O}_G) = k$ [Brion 2009]. Abelian varieties are always antiaffine, but there are many other antiaffine group schemes [Brion 2009, Section 2]. Antiaffine group schemes are always smooth, connected, and commutative [Demazure and Gabriel 1970, Corollaire III.3.8.3]. In general, there is always a largest antiaffine k-subgroup scheme G_{ant} contained in the center of G such that the resulting quotient G/G_{ant} is affine. In fact, $G_{\mathrm{ant}} = \ker(G \to \operatorname{Spec} \Gamma(G, \mathcal{O}_G))$; in particular, if G is not affine, then G_{ant} is nontrivial [Demazure and Gabriel 1970, Théorème III.3.8.2].

Theorem 10.1. *Let X be a quasiseparated algebraic stack. If k is an algebraically closed field and $x\colon \operatorname{Spec} k \to X$ is a point with nonaffine stabilizer, then $\operatorname{Aut}(x) \to \operatorname{Aut}_{\otimes}(x^*)$ is not injective. In particular, $\omega_X(\operatorname{Spec} k)$ is not faithful and X is not tensorial.*

Proof. By assumption, the stabilizer group scheme G_x of x is not affine. Let $H = (G_x)_{\mathrm{ant}}$ be the largest antiaffine subgroup of G_x; then H is a nontrivial antiaffine group scheme over k and the quotient group scheme G_x/H is affine [Demazure and Gabriel 1970, Section III.3.8]. The induced morphism $B_k H \to B_k G_x \to X$ is thus quasiaffine by [Rydh 2011a, Theorem B.2].

By [Brion 2009, Lemma 1.1], the morphism $p\colon \operatorname{Spec} k \to B_k H$ induces an equivalence of Grothendieck abelian tensor categories $p^*\colon \mathsf{QCoh}(B_k H) \to \mathsf{QCoh}(\operatorname{Spec} k)$. Since $\operatorname{Aut}(p) = H(k) \neq \{\mathrm{id}_p\} = \operatorname{Aut}_{\otimes}(p^*)$, the functor $\omega_{B_k H}(\operatorname{Spec} k)$ is not faithful. Hence $\omega_X(\operatorname{Spec} k)$ is not faithful by Lemma 4.4. □

We also have the following theorem.

Theorem 10.2. *Let X be a quasicompact and quasiseparated algebraic stack with affine stabilizers. If k is a field and $x_0\colon \operatorname{Spec} k \to X$ is a nonaffine morphism, then there exists a field extension K/k and a point $y\colon \operatorname{Spec} K \to X$ such that $\operatorname{Isom}(y, x) \to \operatorname{Hom}_{\otimes}(y^*, x^*)$ is not surjective, where x denotes the K-point corresponding to x_0. In particular, $\omega_X(\operatorname{Spec} K)$ is not full.*

Proof. To simplify notation, we let $x = x_0$. Since X has quasicompact diagonal, x is quasiaffine [Rydh 2011a, Theorem B.2]. By Lemma 4.4, we may replace X by $\mathcal{S}\mathrm{pec}_X(x_*k)$ and consequently assume that x is a quasicompact open immersion and $\mathcal{O}_X \to x_*k$ is an isomorphism. In particular, x is a section to a morphism $f\colon X \to \operatorname{Spec} k$. Since x is not affine, it follows that there exists a closed point y disjoint from the image of x. In particular, there is a field extension K/k and a k-morphism $y\colon \operatorname{Spec} K \to X$ whose image is a closed point disjoint from x.

We now base change the entire situation by $\operatorname{Spec} K \to \operatorname{Spec} k$. This results in two morphisms x_K, $y_K\colon \operatorname{Spec} K \to X \otimes_k K$, where x_K is a quasicompact open immersion such that $\mathcal{O}_{X\otimes_k K} \cong (x_K)_* K$ and y_K has image a closed point disjoint from the image of x_K. We replace X, k, x, and y by $X \otimes_k K$, K, x_K, and y_K respectively.

Let $\mathcal{G}_y \subseteq X$ be the residual gerbe associated to y, which is a closed immersion. We define a natural transformation $\gamma^\vee\colon x_* \Rightarrow y_*$ at k to be the composition $x_* k \cong \mathcal{O}_X \twoheadrightarrow \mathcal{O}_{\mathcal{G}_y} \to y_* k$ and extend to all of $\mathsf{QCoh}(\operatorname{Spec} k)$ by taking colimits. By adjunction, there is an induced natural transformation $\gamma\colon y^* \to x^*$. A simple calculation shows that γ is a natural transformation of cocontinuous tensor functors. Since its adjoint $\gamma^\vee$ is not an isomorphism, γ is not an isomorphism; thus γ is not realizable. The result follows. $\square$

The following lemma is a variant of [Bhatt 2016, Example 4.12], which B. Bhatt communicated to the authors.

Lemma 10.3. *Let k be an algebraically closed field and let G/k be an antiaffine group scheme of finite type. Let Z/k be a regular scheme with a closed subscheme C that is a nodal curve over k. Then there is a compatible system of G-torsors $E_n \to C^{[n]}$ such that there does not exist a G-torsor $E \to Z$ that restricts to the E_n's.*

Proof. Recall that G is smooth, connected and commutative [Demazure and Gabriel 1970, Section III.3.8]. Furthermore, by Chevalley's theorem, there is an extension $0 \to H \to G \to A \to 0$, where A is an abelian variety (of positive dimension) and H is affine. Let $x_A \in A(k)$ be an element of infinite order and let $x \in G(k)$ be any lift of x_A.

Let $\tilde{C}$ be the normalization of C. Let $F_0 \to C$ be the G-torsor obtained by gluing the trivial G-torsor on $\tilde{C}$ along the node by translation by x. Note that the induced A-torsor $F_0/H \to C$ is not torsion as it is obtained by gluing along the nontorsion element x_A.

We may now lift $F_0 \to C$ to G-torsors $F_n \to C^{[n]}$. Indeed, the obstruction to lifting F_{n-1} to F_n lies in $\operatorname{Ext}^1_{\mathcal{O}_C}(\mathsf{L}g_0^* L^\bullet_{BG/k}, I^n/I^{n+1})$, where $g_0\colon C \to BG$ is the morphism corresponding to $F_0 \to C$ and I is the ideal defining C in Z. Since G is smooth, the cotangent complex $L^\bullet_{BG/k}$ is concentrated in degree 1 and since C is a curve, it has cohomological dimension 1. It follows that the obstruction group is zero.

Now given a G-torsor $F \to Z$, there is an induced A-torsor $F/H \to Z$. Since Z is regular, the torsor $F/H \to Z$ is torsion in $H^1(Z, A)$ [Raynaud 1970, XIII 2.4 and 2.6]. Thus, $F/H \to Z$ cannot restrict to $F_0/H \to C$ and the result follows. $\square$

We now have the following theorem, which is a counterexample to [Aoki 2006a, Theorem 1.1; 2006b, Case I].

Theorem 10.4. *Let $X \to S$ be a quasiseparated morphism of algebraic stacks. If k is an algebraically closed field and $x\colon \operatorname{Spec} k \to X$ is a point with nonaffine stabilizer, then there exists a morphism $\mathbb{A}^1_k \to S$*

and a proper and flat family of curves $Z \to \mathbb{A}^1_k$, where Z is regular, such that $\underline{\mathrm{Hom}}_{\mathbb{A}^1_k}(Z, X \times_S \mathbb{A}^1_k)$ is not algebraic.

Proof. Let Q be the stabilizer group scheme of x and let G be the largest antiaffine subgroup scheme of Q; thus, G is a nontrivial antiaffine group scheme over k and the quotient group scheme Q/G is affine [Demazure and Gabriel 1970, Section III.3.8].

Let Z be a proper family of curves over $T = \mathbb{A}^1_k = \operatorname{Spec} k[t]$ with regular total space and a nodal curve C as the fiber over the origin; for example, take $Z = \operatorname{Proj}_T(k[t][x, y, z]/(y^2z - x^2z - x^3 - tz^3))$ over T. Let $T_n = V(t^{n+1})$, $\hat{T} = \operatorname{Spec} \hat{\mathcal{O}}_{T,0}$, $Z_n = Z \times_T T_n$, and $\hat{Z} = Z \times_T \hat{T}$. We now apply Lemma 10.3 to C in $\hat{Z}$ and G. Since $Z_n = C^{[n]}$, this produces an element in

$$\varprojlim_n \underline{\mathrm{Hom}}_T(Z, BG_T)(T_n) = \varprojlim_n \mathrm{Hom}(Z_n, BG)$$

that does not lift to

$$\underline{\mathrm{Hom}}_T(Z, BG_T)(\hat{T}) = \mathrm{Hom}(\hat{Z}, BG).$$

This shows that $\underline{\mathrm{Hom}}_T(Z, BG_T)$ is not algebraic.

By [Rydh 2011a, Theorem B.2], the morphism x factors as $\operatorname{Spec} k \to BQ \to \mathcal{Q} \to X$, where $\mathcal{Q}$ is the residual gerbe, $\mathcal{Q} \to X$ is quasiaffine and $BQ \to \mathcal{Q}$ is affine. Since Q/G is affine, it follows that the induced morphism $BG \to BQ \to X$ is quasiaffine. By [Hall and Rydh 2015b, Theorem 2.3(ii)], the induced morphism $\underline{\mathrm{Hom}}_T(Z, BG_T) \to \underline{\mathrm{Hom}}_T(Z, X \times_S T)$ is quasiaffine. In particular, if $\underline{\mathrm{Hom}}_T(Z, X \times_S T)$ is algebraic, then $\underline{\mathrm{Hom}}_T(Z, BG_T)$ is algebraic, which is a contradiction. The result follows. □

The following theorem extends [Bhatt 2016, Example 4.12].

Theorem 10.5. *Let X be an algebraic stack with quasicompact diagonal. If X does not have affine stabilizers, then there exists a noetherian two-dimensional regular ring A, complete with respect to an ideal I, such that $X(A) \to \varprojlim X(A/I^n)$ is not an equivalence of categories.*

Proof. Let $x \in |X|$ be a point with nonaffine stabilizer group. Arguing as in the proof of Theorem 10.4, there exists an algebraically closed field k, an antiaffine group scheme G/k of finite type and a quasiaffine morphism $BG \to X$. An easy calculation shows that it is enough to prove the theorem for $X = BG$.

Let $A_0 = k[x, y]$ and let A be the completion of A_0 along the ideal $I = (y^2 - x^3 - x^2)$. Then $Z = \operatorname{Spec} A$ and $C = \operatorname{Spec} A/I$ satisfies the conditions of Lemma 10.3 and we obtain an element in $\varprojlim_n X(A/I^n)$ that does not lift to $X(A)$. □

Appendix A: Monomorphisms and stratifications

In this appendix, we introduce some notions and results needed for the faithfulness part of Theorem 8.4 when T is not noetherian. This is essential for the proof of Corollary 1.4.

We recall Definition 8.1: let X be a quasicompact algebraic stack. A *finitely presented filtration of* X is a sequence of finitely presented closed immersions $\varnothing = X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_r \hookrightarrow X$ such that $|X_r| = |X|$. The *strata* of the filtration are the locally closed finitely presented substacks $Y_k := X_k \setminus X_{k-1}$.

As in Notation 3.7, the n-th *infinitesimal neighborhood of* X_k is the finitely presented closed immersion $X_k^{[n]} \hookrightarrow X$ which is given by the ideal I_k^{n+1} where $X_k \hookrightarrow X$ is given by I_k. The n-th *infinitesimal neighborhood of the stratum* Y_k is the locally closed finitely presented substack $Y_k^{[n]} := X_k^{[n]} \setminus X_{k-1}$.

Definition A.1. A finitely presented filtration (X_k) of X is *separating* if the family $\{j_k^n : Y_k^{[n]} \to X\}_{k,n}$ is separating [EGA IV$_3$ 1966, 11.9.1]; that is, if the intersection $\bigcap_{k,n} \ker(\mathcal{O}_X \to (j_k^n)_* \mathcal{O}_{Y_k^{[n]}})$ is zero as a lisse-étale sheaf.

Lemma A.2. *Every finitely presented filtration* (X_k) *on* X *is separating if either*

(i) X *is noetherian*; *or*

(ii) X *has no embedded* (*weakly*) *associated point.*

If X *is noetherian with a filtration* (X_k) *and* $X' \to X$ *is flat*, *then* $(X_k \times_X X')$ *is a separating filtration on* X'.

Proof. As the question is smooth-local, we can assume that X and X' are affine schemes. If X is noetherian, then by primary decomposition there exists a separating family $\coprod_{i=1}^m \operatorname{Spec} A_i \to X$ where the A_i are artinian. As every $\operatorname{Spec} A_i$ factors through some $Y_k^{[n]}$, it follows that (X_k) is separating. In general, $\{\operatorname{Spec} \mathcal{O}_{X,x} \to X\}_{x \in \operatorname{Ass}(X)}$ is separating [Lazard 1964, 1.2, 1.5, 1.6]. If x is a nonembedded associated point, then $\operatorname{Spec} \mathcal{O}_{X,x}$ is a one-point scheme and factors through some $Y_k^{[n]}$ and the first claim follows.

For the last claim, we note that a finite number of the infinitesimal neighborhoods of the strata suffices in the noetherian case and that flat morphisms preserve kernels and finite intersections. ☐

Proposition A.3. *Let* X *be an algebraic stack with a finitely presented filtration* (X_k). *Let* $f : Z \to X$ *be a morphism locally of finite type. If* $f|_{Y_k^{[n]}}$ *is an isomorphism for every* k *and* n, *then* f *is a surjective closed immersion. If in addition* (X_k) *is separating*, *then* f *is an isomorphism.*

Proof. First note that f is a surjective and quasicompact monomorphism.

We will prove that f is a closed immersion by induction on the number of strata r. If $r = 0$, then $X = \varnothing$ and there is nothing to prove. If $r = 1$, then $X = X_1^{[n]} = Y_1^{[n]}$ for sufficiently large n and the result follows. If $r \geq 2$, then let $U = X \setminus X_1$. By the induction hypothesis, $f|_U$ is a surjective closed immersion. We may also assume that $X_1 \neq \varnothing$ and $|X_1| \neq |X|$ since these cases are trivial.

Verifying that f is a closed immersion is local on X, so we may assume that X, and hence Z, are schemes [Laumon and Moret-Bailly 2000, Théorème A.2]. Then f is a closed immersion if and only if f is proper [EGA IV$_4$ 1967, 18.12.6]. By the valuative criterion for properness, we may thus assume that $X = \operatorname{Spec} V$ is the spectrum of a henselian valuation ring. Note that $f|_U$ is now an isomorphism since X is reduced.

By [EGA IV$_4$ 1967, 18.12.3], there is a decomposition $Z = Z_1 \amalg Z_2$, where $Z_1 \to X$ is a closed immersion and $Z_2 \cap f^{-1}(\mathfrak{m}) = \varnothing$, where $\mathfrak{m}$ is the maximal ideal of V. It remains to prove that $Z_2 = \varnothing$.

Since X_1 is local, hence connected, and $f|_{X_1}$ is an isomorphism, it follows that $Z_2 \cap f^{-1}(X_1) = \varnothing$. Similarly, since U is integral, hence connected, and $f|_U$ is an isomorphism, it follows that either $Z_2 = \varnothing$ or $Z_2 = U$. In the former case we are done. We will now show that the latter case is impossible.

Since V is a valuation ring and $X_1 \subseteq \operatorname{Spec} V$ is a finitely presented closed immersion, there is an $a \in V$ such that $X_1 = \operatorname{Spec} V/(a)$ and $U = D(a)$. Write $Z_1 = \operatorname{Spec} V/J$ for some ideal J. Since the closed immersion $Z_1 \hookrightarrow X$ is an isomorphism over $X_1^{[n]}$ for all $n \geq 0$, we have that $J \subseteq (a^n)$ for all $n \geq 0$. The condition that $Z_1 \cap D(a) = \varnothing$ is equivalent to $a^n \in J$ for all $n \gg 0$. That is, $J = (a^n)$ for all $n \gg 0$. This implies that $(a^{n+1}) = (a^n)$ is an equality for all $n \gg 0$, which is absurd since a is neither zero nor a unit. Hence, $Z_2 = \varnothing$ and f is a surjective closed immersion.

For general X, if (X_k) is separating, then the schematic image of f contains all the $Y_k^{[n]}$, and hence equals X by definition. This proves the last claim. □

The following example illustrates that a closed immersion $f\colon Z \to X$ as in Proposition A.3 need not be an isomorphism even if f is of finite presentation.

Example A.4. Let $A = k[x, z_1, z_2, \ldots]/(xz_1, \{z_k - xz_{k+1}\}_{k\geq 1}, \{z_i z_j\}_{i,j\geq 1})$ and $B = A/(z_1)$. Then $A/(x^n) = k[x]/(x^n) = B/(x^n)$ and $A_x = k[x]_x = B_x$ but the surjection $A \to B$ is not an isomorphism.

Appendix B: A relative boundedness result for Hom stacks

Here we prove the following relative boundedness result for Hom stacks.

Theorem B.1. *Let $f\colon Z \to S$ be a proper, flat and finitely presented morphism of algebraic stacks. Let X and Y be algebraic stacks that are locally of finite presentation and quasiseparated over S and have affine stabilizers over S. Let $g\colon X \to Y$ be a finitely presented S-morphism. If g has affine fibers, then*

$$\underline{\operatorname{Hom}}_S(Z, g)\colon \underline{\operatorname{Hom}}_S(Z, X) \to \underline{\operatorname{Hom}}_S(Z, Y)$$

is of finite presentation. If in addition $g\colon X \to Y$ is a group, then $\underline{\operatorname{Hom}}_S(Z, g)$ is a group with affine fibers.

Theorem B.1 is used in Theorems 1.2 and 1.3 to establish the quasicompactness of the diagonal of Hom-stacks and Weil restrictions. Prior to Theorem B.1, the strongest boundedness result is due to Olsson [2006b, Proposition 5.10]. There it is assumed that g is finite and f is representable.

Without using Theorem B.1, the proof of Theorem 1.2 give the algebraicity of the Hom-stacks and that they have quasiseparated diagonals. In the setting of Theorem B.1, we may conclude that $\underline{\operatorname{Hom}}_S(Z, g)$ is a quasiseparated morphism of algebraic stacks that are locally of finite presentation over S. It remains to prove that the morphism $\underline{\operatorname{Hom}}_S(Z, g)$ is quasicompact.

Preliminary reductions. If W and T are algebraic stacks over S, let $W_T = W \times_S T$; similarly for morphisms between stacks over S. We will use this notation throughout this appendix.

As the question is local on S, we may assume that S is an affine scheme. We may also assume that X and Y are of finite presentation over S since it is enough to prove the theorem after replacing Y with an open quasicompact substack and X with its inverse. By standard approximation results, we may then assume that S is of finite type over $\operatorname{Spec} \mathbb{Z}$. For the remainder of this article, all stacks will be of finite presentation over S and hence excellent with finite normalization.

By noetherian induction on S, to prove that $\underline{\mathrm{Hom}}_S(Z, g)$ is quasicompact, we may assume that S is integral and replace S with a suitable dense open subscheme. Moreover, we may also replace $Z \to S$ with the pull-back along a dominant map $S' \to S$. Recall that there exists a field extension $K'/K(S)$ such that $(Z_{K'})_{\mathrm{red}}$ is geometrically reduced and $(Z_{K'})_{\mathrm{norm}}$ is geometrically normal over K'. After replacing S with a dense open subset of the normalization in K', we may thus assume that

(i) $Z_{\mathrm{red}} \to S$ is flat with geometrically reduced fibers; and

(ii) $Z_{\mathrm{norm}} \to S$ is flat with geometrically normal fibers;

since these properties are constructible [EGA IV_3 1966, 9.7.7(iii) and 9.9.4(iii)].

We now prove three reduction results. Throughout, we will assume the following:

- Z is proper and flat over S.
- X and Y are finitely presented algebraic stacks over S with affine stabilizers.
- $g\colon X \to Y$ is a representable morphism over S.

Our first reduction result is similar to [Olsson 2006b, Lemma 5.11].

Lemma B.2. *If $\underline{\mathrm{Hom}}_{S'}(Z', g_{S'})$ is quasicompact for every scheme S', morphism $S' \to S$ and nilimmersion $Z' \to Z_{S'}$ such that $Z' \to S'$ is proper and flat with geometrically reduced fibers, then $\underline{\mathrm{Hom}}_S(Z, g)$ is quasicompact.*

Proof. Assume that the condition holds. To prove that $\underline{\mathrm{Hom}}_S(Z, g)$ is quasicompact, we may assume that S is integral. We may also assume that $Z_{\mathrm{red}} \to S$ has geometrically reduced fibers. Pick a sequence of square-zero nilimmersions $Z_{\mathrm{red}} = Z_0 \hookrightarrow Z_1 \hookrightarrow \cdots \hookrightarrow Z_n = Z$. After replacing S with a dense open subset, we may assume that all the $Z_i \to S$ are flat. Thus, it suffices to show that if $j\colon Z_0 \to Z$ is a square-zero closed immersion where Z_0 is flat over S and $\underline{\mathrm{Hom}}_S(Z_0, g)$ is quasicompact, then $\underline{\mathrm{Hom}}_S(Z, g)$ is quasicompact. Now argue as in [Olsson 2006b, Lemma 5.11], but this time using the deformation theory of [Olsson 2006a, Theorem 1.5] and the semicontinuity theorem of [Hall 2014, Theorem A]. □

Before we proceed, we make the following observation: fix an S-scheme T and an S-morphism $y\colon Z_T \to Y$. This corresponds to a map $T \to \underline{\mathrm{Hom}}_S(Z, Y)$. The pullback of $\underline{\mathrm{Hom}}_S(Z, g)$ along this map is isomorphic to the Weil restriction $\boldsymbol{R}_{Z_T/T}(X \times_{g,Y,y} Z_T)$, which we will denote as $H_{Z/S,g}(y)$. Note that our hypotheses guarantee that $H_{Z/S,g}(y)$ is locally of finite type and quasiseparated over T.

The second reduction is for a (partial) normalization.

Lemma B.3. *If $\underline{\mathrm{Hom}}_{S'}(Z', g_{S'})$ is quasicompact for every scheme S', morphism $S' \to S$ and finite morphism $Z' \to Z_{S'}$ such that $Z' \to S'$ is proper and flat with geometrically normal fibers, then $\underline{\mathrm{Hom}}_S(Z, g)$ is quasicompact.*

Proof. By Lemma B.2, we may assume that $Z \to S$ is flat with geometrically reduced fibers. We will use induction on the maximal fiber dimension d of $Z \to S$. After modifying S, we may assume that $W := Z_{\mathrm{norm}} \to S$ is flat with geometrically normal fibers. Let $Z_0 \hookrightarrow Z$ and $W_0 \hookrightarrow W$ be the closed substacks given by the conductor ideal of $W \to Z$.

After replacing S with a dense open subset, we may assume that $Z_0 \to S$ and $W_0 \to S$ are flat and that $W \to Z$ is an isomorphism over an open subset $U \subseteq Z$ that is dense in every fiber. In particular, since $Z_0 \cap U = \varnothing$, the dimensions of the fibers of $Z_0 \to S$ are strictly smaller than d. Thus, by induction we may assume that $\underline{\mathrm{Hom}}_S(Z_0, g)$ is quasicompact. But

$$\begin{tikzcd} W_0 \arrow[r, hook, "i"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\square"] & W \arrow[d, "h"] \\ Z_0 \arrow[r, hook, "j"] & Z \end{tikzcd}$$

is a bicartesian square and remains so after arbitrary base change over S since $W_0 \to S$ is flat. Indeed, that it is cartesian is [Hall 2017, Lemma A.3(i)]. That it is cocartesian and the commutes with arbitrary base change over S follows from the arguments of [Hall 2017, Lemma A.4, A.8] and the existence of pinchings of algebraic spaces [Kollár 2012, Theorem 38].

It remains to prove that $H_{Z/S,g}(y) \to T$ is quasicompact, where T is an integral scheme of finite type over S and $y \colon Z_T \to Y$ is a morphism. The bicartesian square above implies that

$$H_{Z/S,g}(y) \simeq H_{Z_0/S,g}(yj) \times_{H_{W_0/S,g}(yhi)} H_{W/S,g}(yh).$$

The result follows, since $H_{Z_0/S,g}(yj)$ and $H_{W/S,g}(yh)$ are quasicompact and $H_{W_0/S,g}(yhi)$ is quasiseparated. □

We have the following variant of h-descent [Rydh 2010, Theorem 7.4].

Lemma B.4. *Let S be an algebraic stack, let T be an algebraic S-stack and let $g \colon T' \to T$ be a universally subtrusive (e.g., proper and surjective) morphism of finite presentation such that g is flat over an open substack $U \subseteq T$. If T is weakly normal in U (e.g., T normal and U open dense), then for every representable morphism $X \to S$, the following sequence of sets is exact*:

$$X(T) \longrightarrow X(T') \rightrightarrows X(T' \times_T T')$$

where $X(T) = \mathrm{Hom}_S(T, X)$ etc.

Proof. It is enough to prove that given a morphism $f \colon T' \to X$ such that $f \circ \pi_1 = f \circ \pi_2 \colon T' \times_T T' \to X$, there exists a unique morphism $h \colon T \to X$ such that $f = h \circ g$. By fppf-descent over U, there is a unique $h|_U \colon U \to X$ such that $f|_{g^{-1}(U)} = h|_U \circ g|_{g^{-1}(U)}$. Consider the morphism $\tilde{g} \colon \tilde{T}' = T' \amalg U \to T$. The morphism $\tilde{f} = (f, h|_U) \colon \tilde{T}' \to X$ satisfies $\tilde{f} \circ \tilde{\pi}_1 = \tilde{f} \circ \tilde{\pi}_2$ where $\tilde{\pi}_i$ denotes the projections of $\tilde{T}' \times_X \tilde{T}' \to \tilde{T}'$. By assumption, $\tilde{g}$ is universally subtrusive and weakly normal. Thus, by h-descent [Rydh 2010, Theorem 7.4], we have an exact sequence

$$X(T) \longrightarrow X(\tilde{T}') \rightrightarrows X((\tilde{T}' \times_T \tilde{T}')_{\mathrm{red}}).$$

Indeed, by smooth descent we can assume that S, T and $\tilde{T}'$ are schemes so that [Rydh 2010, Theorem 7.4] applies. We conclude that $\tilde{f}$ comes from a unique morphism $h \colon T \to X$. □

We now have our last general reduction result.

Proposition B.5. *Let $w\colon W \to Z$ be a proper surjective morphism over S. Assume that $Z \to S$ has geometrically normal fibers and $W \to S$ is flat. If $\underline{\mathrm{Hom}}_S(W, g)$ is quasicompact, then so is $\underline{\mathrm{Hom}}_S(Z, g)$.*

Proof. We may assume that S is an integral scheme. After replacing S with an open subscheme, we may also assume that $W \to Z$ is flat over an open subset $U \subseteq Z$ that is dense in every fiber over S and $W \times_Z W$ is flat over S. It remains to prove that $H_{Z/S,g}(y) \to T$ is quasicompact, where T is an integral scheme of finite type over S and $y\colon Z_T \to Y$ is a morphism. By assumption, $H_{W/S,g}(yw) \to T$ is quasicompact. Now consider the sequence

$$H_{Z/S,g}(y) \longrightarrow H_{W/S,g}(yw) \rightrightarrows H_{W\times_Z W/S,g}(yv),$$

where $v\colon W \times_Z W \to Z$ is the natural map. There is a canonical morphism $\varphi\colon H_{Z/S,g}(y) \to E$, where E denotes the equalizer of the parallel arrows. Since $H_{W/S,g}(yw)$ is quasicompact (and $H_{W\times_V W/S,g}(yv)$ is quasiseparated), the equalizer E is quasicompact. It is thus enough to show that φ is quasicompact. Thus, pick a scheme T' and a morphism $T' \to E$ and let us show that $H_{Z/S,g}(y) \times_E T'$ is quasicompact.

By noetherian induction on T', we may assume that T' is normal. The morphism $T' \to E$ gives an element of $\mathrm{Hom}_Y(W_{T'}, X)$ such that the two images in $\mathrm{Hom}_Y(W_{T'} \times_{Z_{T'}} W_{T'}, X)$ coincide. Noting that $Z_{T'}$ is normal, Lemma B.4 applies to $W_{T'} \to Z_{T'}$ and gives a unique element in $\mathrm{Hom}_Y(Z_{T'}, X) = \mathrm{Hom}_T(T', H_{Z/S,g}(y))$. Thus, the morphism $\varphi_{T'}\colon H_{Z/S,g}(y) \times_E T' \to T'$ has a section. Repeating the argument with $T' = \operatorname{Spec} \kappa(t')$ for every point $t' \in T'$, we see that $\varphi_{T'}$ is injective, so the section is surjective. It follows that $H_{Z/S,g}(y) \times_E T'$ is quasicompact. □

Proof of the main result.

Proof of Theorem B.1. As usual, we may assume that S is an affine integral scheme. By Lemma B.3, we may in addition assume that $Z \to S$ has geometrically normal fibers. Let $W \to Z$ be a proper surjective morphism with W a projective S-scheme [Olsson 2005]. By replacing S with a dense open, we may assume that $W \to S$ is flat. By Proposition B.5, we may replace Z with W and assume that Z is a (projective) scheme. Repeating the first reduction, we may still assume that $Z \to S$ has geometrically normal fibers.

As before, it remains to prove that $H_{Z/S,g}(y) \to T$ is quasicompact, where T is an integral S-scheme of finite type and $y\colon Z_T \to Y$ is an S-morphism. Hence, it suffices to prove the following claim.

Claim. Let S be integral. If $Z \to S$ is projective with geometrically normal fibers and $q\colon Q \to Z$ is representable with affine fibers, then $\boldsymbol{R}_{Z/S}(Q) \to S$ is quasicompact.

Proof of Claim. Let $\overline{Q} = \operatorname{Spec}_Z(q_*\mathcal{O}_Q)$ and let $Q \to \overline{Q} \to Z$ be the induced factorization. Since $Q \to Z$ has affine fibers, $Q \to \overline{Q}$ is an isomorphism over an open dense subset $U \subseteq Z$. After replacing S with a dense open subscheme, we may assume that U is dense in every fiber over S. Since $\boldsymbol{R}_{Z/S}(\overline{Q}) \to S$ is affine [Hall and Rydh 2015b, Theorem 2.3(i)], it is enough to prove that $\boldsymbol{R}_{Z/S}(Q) \to \boldsymbol{R}_{Z/S}(\overline{Q})$ is quasicompact. We may thus replace Q, Z, U and S with $Q \times_{\overline{Q}} (Z \times_S \boldsymbol{R}_{Z/S}(\overline{Q}))$, $Z \times_S \boldsymbol{R}_{Z/S}(\overline{Q})$, $U \times_S \boldsymbol{R}_{Z/S}(\overline{Q})$ and $\boldsymbol{R}_{Z/S}(\overline{Q})$. We may thus assume that $Q \to Z$ is an isomorphism over U. □

Since Q is an algebraic space, there exists a finite surjective morphism $\tilde{Q} \to Q$ such that $\tilde{Q}$ is a scheme. In particular, there is a finite field extension $L/K(U)$ such that the normalization of Q in L is a scheme. Take a splitting field L'/L and let Z' be the normalization of Z in L'. Then $Q' := (Q \times_Z Z')_{\mathrm{norm}} = Q_{\mathrm{norm}/L'}$ is a scheme. By replacing S with a normalization in an extension of $K(S)$ and shrinking, we may assume that $Z' \to S$ and $Q' \to S$ are flat with geometrically normal fibers. By Proposition B.5, it is enough to prove that $\boldsymbol{R}_{Z'/S}(Q \times_Z Z')$ is quasicompact.

There is a natural morphism $\boldsymbol{R}_{Z'/S}(Q') \to \boldsymbol{R}_{Z'/S}(Q \times_Z Z')$, which we claim is surjective. To see this, we may assume that S is the spectrum of an algebraically closed field. Then Z' and Q' are normal and any section $Z' \to Q \times_Z Z'$ lifts uniquely to a section $Z' \to Q'$. Indeed, $Z' \times_{Q \times_Z Z'} Q' \to Z'$ is finite and an isomorphism over U, hence has a canonical section. We can thus replace Q and Z with Q' and Z' and assume that Q is a scheme.

Since Q is a scheme, it is locally separated; hence, there is a U-admissible blow-up $Z' \to Z$ such that the strict transform $Q' \to Z'$ of $Q \to Z$ is étale [Raynaud and Gruson 1971, Théorème 5.7.11]. After shrinking S, we may assume that $Z' \to S$ is flat. Then since $U \subseteq Z'$ remains dense after arbitrary pull-back over S, we have that

$$\boldsymbol{R}_{Z'/S}(Q \times_Z Z') = \boldsymbol{R}_{Z'/S}(Q').$$

Replacing $Q \to Z$ with $Q' \to Z'$ (Proposition B.5), we may thus assume that $Q \to Z$ in addition is étale.

Finally, we note that the étale morphism $Q \to Z$ corresponds to a constructible sheaf on $Z_{\text{Ét}}$ and that $\boldsymbol{R}_{Z/S}(Q)$ is nothing but the étale sheaf $f_{\text{Ét},*}Q$. By a special case of the proper base change theorem [SGA 4_3 1973, XIV.1.1], $f_{\text{Ét},*}Q$ is constructible, so $\boldsymbol{R}_{Z/S}(X) \to S$ is of finite presentation.

For the second part of the theorem on groups: let T be the spectrum of an algebraically closed field and let $y\colon Z_T \to Y$ be a morphism. By the first part $H_{Z/S,g}(y) \simeq \boldsymbol{R}_{Z_T/T}(Q)$ is then a group scheme G of finite type over T, where $Q = X \times_Y Z_T$. Let $K = G_{\mathrm{ant}}$ be the largest antiaffine subgroup of G; it is normal, connected and smooth and the quotient G/K is affine [Demazure and Gabriel 1970, Section III.3.8].

The universal family $G \times_T Z_T \to Q$ is a group homomorphism and induces a group homomorphism $K \times_T Z_T \to Q$. It is enough to show that this factors through the unit section of $Q \to Z_T$, because this forces $K = 0$. Note that for every stack $W \to T$, the pull-back $K \times_T W \to W$ is an antiaffine group in the sense that the push-forward of $\mathcal{O}_{K \times_T W}$ is $\mathcal{O}_W$ (flat base change).

We will now use the results on finitely presented filtrations in Appendix A Since $Q \to Z_T$ has affine fibers, there is a finitely presented filtration $(Z_{T,i})$ of Z_T with strata $V_i = Z_{T,i} \setminus Z_{T,i-1}$ such that $Q \times_{Z_T} V_i \to V_i$ are affine for every i. By Chevalley's theorem, $Q \times_{Z_T} V_i^{[n]} \to V_i^{[n]}$ is affine for every i and $n \geq 0$. Since $K \times_T V_i^{[n]} \to V_i^{[n]}$ is antiaffine, it follows that $K \times_T V_i^{[n]} \to Q \times_{Z_T} V_i^{[n]}$ factors through the unit section $V_i^{[n]} \to Q \times_{Z_T} V_i^{[n]}$ for every i and n.

Let E be the equalizer of $K \times_T Z_T \to Q$ and the constant map $K \times_T Z_T \to Q$ to the unit. The above discussion shows that the monomorphism $E \to K \times_T Z_T$ is an isomorphism over every infinitesimal neighborhood $V_i^{[n]}$ of every stratum V_i, hence an isomorphism (Proposition A.3, using that the filtration is separating since Z_T is noetherian). $\square$

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Supercuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ distinguished by a Galois involution

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Let F/F_0 be a quadratic extension of nonarchimedean locally compact fields of residual characteristic $p \neq 2$ and let σ denote its nontrivial automorphism. Let R be an algebraically closed field of characteristic different from p. To any cuspidal representation π of $\mathrm{GL}_n(\mathrm{F})$, with coefficients in R, such that $\pi^\sigma \simeq \pi^\vee$ (such a representation is said to be σ-selfdual) we associate a quadratic extension D/D_0, where D is a tamely ramified extension of F and D_0 is a tamely ramified extension of F_0, together with a quadratic character of $\mathrm{D}_0^\times$. When π is supercuspidal, we give a necessary and sufficient condition, in terms of these data, for π to be $\mathrm{GL}_n(\mathrm{F}_0)$-distinguished. When the characteristic ℓ of R is not 2, denoting by ω the nontrivial R-character of $\mathrm{F}_0^\times$ trivial on F/F_0-norms, we prove that any σ-selfdual supercuspidal R-representation is either distinguished or ω-distinguished, but not both. In the modular case, that is when $\ell > 0$, we give examples of σ-selfdual cuspidal nonsupercuspidal representations which are not distinguished nor ω-distinguished. In the particular case where R is the field of complex numbers, in which case all cuspidal representations are supercuspidal, this gives a complete distinction criterion for arbitrary complex cuspidal representations, as well as a purely local proof, for cuspidal representations, of the dichotomy and disjunction theorem due to Kable and Anandavardhanan, Kable and Tandon, when $p \neq 2$.

1. Introduction

1A. Let F/F_0 be a separable quadratic extension of nonarchimedean locally compact fields of residual characteristic p and let σ denote its nontrivial automorphism. Let $\mathbf{G}$ be a connected reductive group defined over F_0, let G denote the locally profinite group $\mathbf{G}(\mathrm{F})$ equipped with the natural action of σ and $\mathrm{G}^\sigma = \mathbf{G}(\mathrm{F}_0)$ be the σ-fixed points subgroup. The study of those irreducible (smooth) complex representations of G which are G^σ-distinguished, that is which carry a nonzero G^σ-invariant linear form, goes back to the 1980's. We refer to [Harder et al. 1986; Jacquet and Ye 1996] for the initial motivation for distinguished representations in a global context and to [Hakim 1991; Flicker 1991] in a nonarchimedean context.

1B. In this work, we will consider the case where $\mathbf{G}$ is the general linear group GL_n for $n \geq 1$. We thus have $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ and $\mathrm{G}^\sigma = \mathrm{GL}_n(\mathrm{F}_0)$. In this case, it is well-known (see [Prasad 1990; 2001; Flicker 1991]) that any distinguished irreducible complex representation π of G is *σ-selfdual*, that is, the contragredient $\pi^\vee$ of π is isomorphic to $\pi^\sigma = \pi \circ \sigma$, and the space $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1)$ of all G^σ-invariant linear forms on π has dimension 1. Also, the central character of π is trivial on $\mathrm{F}_0^\times$. This gives us two

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necessary conditions for an irreducible complex representation of G to be distinguished and it is natural to ask whether or not they are sufficient.

1C. First, let us consider the case where F/F_0 is replaced by a quadratic extension $\boldsymbol{k}/\boldsymbol{k}_0$ of finite fields of characteristic p. In this case, Gow [1984] proved that an irreducible complex representation of $\mathrm{GL}_n(\boldsymbol{k})$ is $\mathrm{GL}_n(\boldsymbol{k}_0)$-distinguished if and only if π is σ-selfdual. (Note that the latter condition automatically implies that the central character is trivial on $\boldsymbol{k}_0^\times$.) Besides, if $p \neq 2$, the σ-selfdual irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$ are those which arise from some irreducible representation of the unitary group $\mathrm{U}_n(\boldsymbol{k}/\boldsymbol{k}_0)$ by base change (see Kawanaka [1977]).

1D. We now go back to the nonarchimedean setting of Paragraph 1B and consider a σ-selfdual irreducible complex representation π of G whose central character is trivial on $\mathrm{F}_0^\times$. When π is cuspidal and F/F_0 is unramified, Prasad [2001] proved that, if $\omega = \omega_{\mathrm{F}/\mathrm{F}_0}$ denotes the nontrivial character of $\mathrm{F}_0^\times$ trivial on F/F_0-norms, then π is either distinguished or ω-distinguished, the latter case meaning that the complex vector space $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, \omega \circ \det)$ is nonzero.

When $p \neq 2$ and π is an *essentially tame* cuspidal representation, that is, when the number of unramified characters χ of G such that $\pi\chi \simeq \pi$ is prime to p, Hakim and Murnaghan [2002] gave sufficient conditions for π to be distinguished. These conditions are stated in terms of admissible pairs [Howe 1977], which parametrize essentially tame cuspidal complex representations of G [Howe 1977; Bushnell and Henniart 2005a]. Note that they assume F has characteristic 0, but their approach also works in characteristic p.

When π is a discrete series representation and F has characteristic 0, Kable [2004] proved that if π is σ-selfdual, then it is either distinguished or ω-distinguished: this is the *Dichotomy Theorem*. In addition, Anandavardhanan, Kable and Tandon [2004] proved that π can't be both distinguished and ω-distinguished: this is the *Disjunction Theorem*. The proofs use global arguments, which is why F was assumed to have characteristic 0, and the Asai L-function of π. However, these results still hold when F has characteristic $p \neq 2$, as is explained in [Anandavardhanan et al. 2018, Appendix A]. Note that:

- The disjunction theorem implies that the sufficient conditions given by Hakim and Murnaghan in [Hakim and Murnaghan 2002] in the essentially tame cuspidal case are necessary conditions as well.
- The dichotomy theorem implies that, when n is odd, any σ-selfdual discrete series representation of G with central character trivial on $\mathrm{F}_0^\times$ is automatically distinguished. Indeed, an ω-distinguished irreducible representation has a central character whose restriction to $\mathrm{F}_0^\times$ is ω^n.

When $p \neq 2$ and π is cuspidal of level zero — in particular is essentially tame — Coniglio-Guilloton [2016] gave a necessary and sufficient condition of distinction in terms of admissible pairs. Her proof is purely local and does not use the disjunction theorem. (In fact, she considers the more general case where $\mathbf{G}$ is an inner form of GL_n over F_0 and representations of level zero of $\mathbf{G}(\mathrm{F})$ whose local Jacquet–Langlands transfer to $\mathrm{GL}_n(\mathrm{F})$ is cuspidal.)

If one takes the classification of distinguished cuspidal representations of general linear groups for granted, and assuming F has characteristic 0, Anandavardhanan and Rajan [2005; Anandavardhanan 2008] classified all distinguished discrete series representations of G in terms of the distinction of their cuspidal support (see also [Matringe 2009]) and Matringe [2011; 2014] classified distinguished generic, as well as distinguished unitary, representations of G in terms of the Langlands classification. This provides a class of representations for which the dichotomy and disjunction theorems do not hold: some σ-selfdual generic irreducible representations are nor distinguished nor ω-distinguished, and some are both.

Max Gurevich [2015] extended the dichotomy theorem to the class of *ladder* representations, which contains all discrete series representations: a σ-selfdual ladder representation of G is either distinguished or ω-distinguished, but it may be both (see [Gurevich 2015, Theorem 4.6]). Here again, F is assumed to have characteristic 0.

Finally, one can deduce from these works the connection between distinction for generic irreducible representations of G and base change from a quasisplit unitary group; see [Gan and Raghuram 2013, Theorem 6.2; Anandavardhanan and Prasad 2018, Theorem 2.3; Gurevich et al. 2018].

1E. The discussion above leaves us with an open problem about cuspidal representations: find a distinction criterion for an *arbitrary* σ-selfdual cuspidal representation π, with no assumption on the characteristic of F, on the ramification of F/F_0, on n nor on the torsion number of π (that is, the number of unramified characters χ such that $\pi\chi \simeq \pi$).

In this paper, *assuming that* $p \neq 2$, we propose an approach which allows us to generalize both Hakim and Murnaghan's and Coniglio's distinction criteria to *all* cuspidal irreducible complex representations of G and which works:

- With no assumption on the characteristic of F (apart from the assumption "$p \neq 2$").
- With purely local methods.
- Not only for complex representations, but more generally for representations with coefficients in an algebraically closed field of arbitrary characteristic $\ell \neq p$.

We thus give a complete solution to the problem above for cuspidal complex representations when $p \neq 2$. We actually do more: we solve this problem in the larger context of *supercuspidal* representations with coefficients in an algebraically closed field of arbitrary characteristic $\ell \neq p$.

1F. First, let us say a word about the third item above. The theory of smooth representations of $\mathrm{GL}_n(\mathrm{F})$ with coefficients in an algebraically closed field of characteristic $\ell \neq p$ has been initiated by Vignéras [1996; 1998] in view to extend the local Langlands programme to representations with coefficients in a field — or a ring — as general as possible (see for instance [Vignéras 2001; Helm and Moss 2018]). Inner forms have also been taken into account [Mínguez and Sécherre 2014a; Sécherre and Stevens 2016] and the congruence properties of the local Jacquet–Langlands correspondence have been studied in [Dat 2012b; Mínguez and Sécherre 2017]. It is thus natural to extend the study of distinguished representations to this wider context, where the field of complex numbers is replaced by a more general field. Very little has been done about distinction of modular representations so far: a first study can be found in [Sécherre and Venketasubramanian 2017].

An important phenomenon in the modular case, that is when $\ell > 0$, is that a cuspidal representation π may occur as a subquotient of a proper parabolically induced representation (see [Vignéras 1989, Corollaire 5]). When this is not the case, that is when π does not occur as a subquotient of a proper parabolically induced representation, π is said to be *supercuspidal*.

1G. *From now on, we fix an algebraically closed field* R *of arbitrary characteristic* $\ell \neq p$, *and consider irreducible smooth representations of* G *with coefficients in* R. Note that ℓ can be 0.

We first notice that, as in the complex case, any distinguished irreducible representation π of G with coefficients in R is σ-selfdual and $\mathrm{Hom}_{G^\sigma}(\pi, 1)$ is 1-dimensional (see Theorem 4.1).

We prove that, if $\ell \neq 2 \neq p$, the dichotomy and disjunction theorems hold for all *supercuspidal* representations with coefficients in R. In particular, when R is the field of complex numbers, in which case any cuspidal representation is supercuspidal, we get a purely local proof of the dichotomy and disjunction theorems for cuspidal representations in the case where $p \neq 2$.

When $\ell = 2 \neq p$, in which case there is no character of order 2 on $F_0^\times$, the dichotomy theorem takes a simplified form: any σ-selfdual supercuspidal representation is distinguished. Let us summarize this first series of results in the theorem below.

Theorem 1.1 (Theorem 10.8). *Suppose that* $p \neq 2$ *and let* π *be a* σ*-selfdual supercuspidal irreducible* R*-representation of* G.

(1) *If* $\ell = 2$, *then* π *is distinguished.*

(2) *If* $\ell \neq 2$, *then* π *is either distinguished or* ω*-distinguished*, *but not both.*

In the modular case, for $\ell > 2$, we give examples of σ-selfdual cuspidal, nonsupercuspidal representations which are not distinguished nor ω-distinguished (see Remarks 7.5 and 2.8).

From now on, until the end of this introduction, we will assume that $p \neq 2$.

1H. The dichotomy and disjunction theorem stated in Theorem 1.1 relies on a distinction criterion, which we state in Theorem 1.2. The basic idea is that we canonically associate to any σ-selfdual supercuspidal representation π of G a finite extension D of F equipped with an F_0-involution extending σ and a quadratic character δ_0 of the fixed points of $D^\times$; it is these data which govern the distinction of π. The character δ_0 refines the information given by the central character of π in the sense that they coincide on $F_0^\times$, the latter one being not enough in general to determine whether π is distinguished or not. To state our distinction criterion, let us write D_0 for the fixed points subfield of D.

Theorem 1.2 (Theorem 10.9). *A* σ*-selfdual supercuspidal* R*-representation of* G *is distinguished if and only if either*

(1) $\ell = 2$, *or*

(2) $\ell \neq 2$ *and*

(a) *if* F/F_0 *is ramified*, D/D_0 *is unramified and* $\mathrm{D}_0/\mathrm{F}_0$ *has odd ramification order*, *then the character* δ_0 *is nontrivial*,

(b) *otherwise*, δ_0 *is trivial.*

Even in the complex case, this is the first time a necessary and sufficient distinction criterion is exhibited for an *arbitrary* cuspidal representation of $\mathrm{GL}_n(\mathrm{F})$ in odd residual characteristic, in the spirit of [Hakim and Murnaghan 2002; 2008; Coniglio-Guilloton 2016].

1I. The starting point of our strategy for proving Theorem 1.2 is to use Bushnell and Kutzko's construction of cuspidal representations of G via compact induction. This construction, elaborated in the complex case by Bushnell and Kutzko [1993], has been extended to the modular case by Vignéras [1996] and Minguez and Sécherre [2014b]. There is a family of pairs $(\mathbf{J}, \boldsymbol{\lambda})$, made of certain compact mod center open subgroups $\mathbf{J}$ of G and certain irreducible representations $\boldsymbol{\lambda}$ of $\mathbf{J}$, such that:

- For any such pair $(\mathbf{J}, \boldsymbol{\lambda})$, the compact induction of $\boldsymbol{\lambda}$ to G is irreducible and cuspidal.
- Any irreducible cuspidal representation of G occurs in this way, for a pair $(\mathbf{J}, \boldsymbol{\lambda})$ uniquely determined up to G-conjugacy.

Such pairs are called *extended maximal simple types* in [Bushnell and Kutzko 1993], which we will abbreviate to *types* for simplicity. We need to give more details about the structure of these types:

(1) The group $\mathbf{J}$ has a unique maximal compact subgroup J, and a unique maximal normal pro-p-group J^1.

(2) There is a group isomorphism $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ for some divisor m of n and finite extension $\boldsymbol{l}$ of the residual field $\boldsymbol{k}$ of F.

(3) The restriction of $\boldsymbol{\lambda}$ to J^1 is made of copies of a single irreducible representation η, which extends (not uniquely, nor canonically) to $\mathbf{J}$.

(4) Given a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, there is a unique irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$, and $\boldsymbol{\rho}$ restricts irreducibly to J.

(5) The representation of $\mathrm{GL}_m(\boldsymbol{l})$ obtained by restricting $\boldsymbol{\rho}$ to J is cuspidal.

The integer m, called the *relative degree* of π, is uniquely determined by π. There is another type-theoretical invariant called the *tame parameter field* of π: this is a tamely ramified extension T of F, uniquely determined up to F-isomorphism, whose degree divides n/m and whose residual field is $\boldsymbol{l}$ (see [Bushnell and Henniart 2014] for more details). Note that π if essentially tame if and only if $[\mathrm{T}:\mathrm{F}] = n/m$.

1J. Now consider a σ-selfdual cuspidal R-representation π of G. The starting point of all our work is [Anandavardhanan et al. 2018, Theorem 4.1], which asserts that among all the types contained in π, there is a type $(\mathbf{J}, \boldsymbol{\lambda})$ which is σ-selfdual, that is $\mathbf{J}$ is σ-stable and $\boldsymbol{\lambda}^\vee$ is isomorphic to $\boldsymbol{\lambda}^\sigma$. Moreover, the tame parameter field T of π is equipped with an F_0-involution. If T_0 denotes the fixed points subfield of T, then T/T_0 is a quadratic extension, uniquely determined up to F_0-isomorphism. The invariants m and T/T_0 associated with π will play a central role in what follows.

First, the following result says that the distinction of π can be detected by a σ-selfdual type.

Theorem 1.3 (Theorem 6.1). *Let π be a σ-selfdual cuspidal representation of* G. *Then π is distinguished if and only if it contains a distinguished σ-selfdual type, that is a σ-selfdual type* $(\mathbf{J}, \boldsymbol{\lambda})$ *such that* $\mathrm{Hom}_{\mathbf{J}\cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$ *is nonzero.*

The proof of this theorem — which occupies Section 6 — is the most technical part of the paper: starting with a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ contained in π and $g \in \mathrm{G}$, one has to prove that, if the type $(\mathbf{J}^g, \boldsymbol{\lambda}^g)$ is distinguished, then it is σ-selfdual, that is $\sigma(g)g^{-1} \in \mathbf{J}$. First, one determines the set of the $g \in \mathrm{G}$ such that η^g is distinguished, as well as the dimension of the space of invariant linear forms (Paragraphs 6A–6C); then, one analyzes the distinction of $\boldsymbol{\kappa}^g$ (Paragraphs 6D and 6E); one obtains the final statement by using the cuspidality of the representation of $\mathrm{GL}_m(\boldsymbol{l})$ induced by $\boldsymbol{\rho}$ (see Theorem 6.21).

1K. When T is unramified over T_0, the σ-selfdual types contained in π form a single G^σ-conjugacy class. When T is ramified over T_0, the σ-selfdual types contained in π form $\lfloor m/2 \rfloor + 1$ different G^σ-conjugacy classes, characterized by an integer $i \in \{0, \dots, \lfloor m/2 \rfloor\}$ called the *index* of the class. Since the space $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1)$ has dimension 1, only one of these conjugacy classes can contribute to distinction: we prove that it is the one with maximal index. This gives us the following refinement of Theorem 1.3.

Proposition 1.4 (Corollary 6.24 and Proposition 7.1). *Let π be a σ-selfdual cuspidal representation of* G. *Let m be its relative degree and* T/T_0 *be its associated quadratic extension.*

(1) *If* T *is unramified over* T_0, *then π is distinguished if and only if any of its σ-selfdual types is distinguished.*

(2) *If* T *is ramified over* T_0, *then π is distinguished if and only if any of its σ-selfdual types of index* $\lfloor m/2 \rfloor$ *is distinguished.*

Note that this proposition is proved in [Anandavardhanan et al. 2018] in a different manner, based on a result of Ok [1997]. However, the proof given in the present article is more likely to generalize to other situations.

When T/T_0 is ramified, one can be more precise (see Proposition 7.1): if π is distinguished, m is either even or equal to 1. It is not difficult to construct σ-selfdual cuspidal representations of G such that T/T_0 is ramified and $m > 1$ is odd: such cuspidal representations are not distinguished nor ω-distinguished (see Remark 7.5).

In the case where T/T_0 is unramified, m is odd for any supercuspidal σ-selfdual representation (see Proposition 9.8). This does not hold for σ-selfdual cuspidal representations (which is easy to see), and this does not even hold for distinguished cuspidal representations: Kurinczuk and Matringe recently proved that, when F/F_0 is unramified and $n = \ell = 2$, any σ-selfdual nonsupercuspidal cuspidal representation of $\mathrm{GL}_2(\mathrm{F})$ of level zero (thus of relative degree 2) is distinguished.

1L. As in the previous paragraph, π is a σ-selfdual cuspidal R-representation of G. The following definition will be convenient to us (see Remark 10.2 for the connection with the usual notion of a generic representation).

Definition 1.5 (Definition 10.1). A σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ in π is *generic* if either T/T_0 is unramified, or T/T_0 is ramified and this type has index $\lfloor m/2 \rfloor$.

Proposition 1.4 thus says that, up to G^σ-conjugacy, a σ-selfdual cuspidal representation π contains a unique generic σ-selfdual type, and that π is distinguished if and only if such a type is distinguished (see Theorem 10.3). This uniqueness property is crucial to the proof of the disjunction part of Theorem 1.1.

Let us fix a generic σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ in π. Recall that, by construction, $\boldsymbol{\lambda}$ can be decomposed (noncanonically) as $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. However, not any of these decompositions are suitable for our purpose. It is not difficult to prove that $\boldsymbol{\kappa}$ can be chosen to be σ-selfdual, but this is not enough: we need to prove that $\boldsymbol{\kappa}$ can be chosen to be both σ-selfdual and distinguished. The strategy of the proof depends on the ramification of T over T_0. This is why we treat separately the ramified and the unramified cases, in Sections 7 and 9, respectively.

The easiest case is when T/T_0 is ramified. Using the fact that m is either even or equal to 1, we prove that $\boldsymbol{\kappa}$ can be chosen to be distinguished by adapting a result of Matringe [2012] (which we do in Paragraph 2C).

When T/T_0 is unramified, the existence of a distinguished $\boldsymbol{\kappa}$ is more difficult to establish. Our proof requires π to be supercuspidal, since in that case m is known to be odd, thus $\mathrm{GL}_m(\boldsymbol{l})$ has $\mathrm{GL}_m(\boldsymbol{l}_0)$-distinguished supercuspidal representations in characteristic 0, where $\boldsymbol{l}_0$ is the residual field of T_0 (see the proof of Proposition 9.4).

In both cases, a distinguished $\boldsymbol{\kappa}$ is automatically σ-selfdual, and π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished. Considering $\boldsymbol{\rho}$ as a (σ-selfdual) level zero type, we are then reduced to the level zero case, which has been treated by Coniglio-Guilloton in the complex case. We thus have to extend her results to the modular case, which we know how to do when π is supercuspidal only.

To summarize, we need the assumption that π is supercuspidal in Theorems 1.1 and 1.2 for two reasons: for the existence of a distinguished $\boldsymbol{\kappa}$ in the case when T/T_0 is unramified and for the level zero case.

1M. To study the distinction of $\boldsymbol{\rho}$ when π is supercuspidal, we use admissible pairs of level zero as in Coniglio-Guilloton [2016]. We attach to $\boldsymbol{\rho}$ a pair $(\mathrm{D}/\mathrm{T}, \delta)$ made of an unramified extension of degree m equipped with an involutive T_0-algebra homomorphism, nontrivial on T, denoted by σ, together with a character δ of $\mathrm{D}^\times$ such that $\delta \circ \sigma = \delta^{-1}$. (See Paragraphs 5C and 5E although the result is presented in a different way there.)

However, the distinguished representation $\boldsymbol{\kappa}$ of Paragraph 1L is not unique in general, thus neither $\boldsymbol{\rho}$ nor δ are. Write D_0 for the σ-fixed points of D and δ_0 for the restriction of δ to $\mathrm{D}_0^\times$. This is a quadratic character, trivial on D/D_0-norms. We prove in Proposition 10.5 that the pair $(\mathrm{D}/\mathrm{D}_0, \delta_0)$ is uniquely determined by π up to F_0-isomorphism. This is the one occurring in our distinction criterion Theorem 1.2.

It remains to explain our strategy to prove the distinction criterion for $\boldsymbol{\rho}$, in the modular case, in terms of the character δ_0, as well as the dimension of the space of invariant linear forms. This depends on the ramification of T/T_0.

The easiest case is when T/T_0 is unramified. In this case, we are reduced to studying the distinction of supercuspidal representations of $\mathrm{GL}_m(\boldsymbol{l})$ by $\mathrm{GL}_m(\boldsymbol{l}_0)$. That any distinguished irreducible representation is

σ-selfdual follows from a finite and ℓ-modular version of Theorem 4.1 (see Remark 4.3). For the converse statement, we use a lifting argument to characteristic 0, based on the fact that any σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation has a σ-selfdual $\overline{\mathbf{Q}}_\ell$-lift, where $\overline{\mathbf{Q}}_\ell$ is an algebraic closure of the field of ℓ-adic numbers and $\overline{\mathbf{F}}_\ell$ is its residual field. This does not hold for σ-selfdual nonsupercuspidal representations: by Remark 2.8, there are σ-selfdual cuspidal representations, with m even, which are not distinguished.

In the case where T/T_0 is ramified, we are reduced to studying the distinction of supercuspidal representations of $\mathrm{GL}_m(\boldsymbol{l})$ by either $\mathrm{GL}_1(\boldsymbol{l})$ if $m = 1$, or $\mathrm{GL}_r(\boldsymbol{l}) \times \mathrm{GL}_r(\boldsymbol{l})$ if $m = 2r$ is even. It is more difficult, as we do not have an analogue of Theorem 4.1. Our proof relies on the structure of the projective envelope of a supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$, as well as a lifting argument to characteristic 0. We prove that a supercuspidal representation is distinguished if and only if it is selfdual. Unlike the complex case, one can find σ-selfdual cuspidal representations, with $m > 1$ odd, which are not distinguished (see Remark 2.18).

In both cases, we prove that a σ-selfdual supercuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$ is distinguished if any only if it admits a distinguished lift to characteristic 0. We conclude by the following theorem.

Theorem 1.6 (Theorem 10.11). *Let π be a σ-selfdual supercuspidal representation of the group* G *with coefficients in $\overline{\mathbf{F}}_\ell$.*

(1) *The representation π admits a σ-selfdual supercuspidal lift to $\overline{\mathbf{Q}}_\ell$.*

(2) *Let $\tilde{\pi}$ be a σ-selfdual lift of π, and suppose that $\ell \neq 2$. Then $\tilde{\pi}$ is distinguished if and only if π is distinguished.*

1N. In this paragraph, we discuss the assumption $p \neq 2$. In Section 2, in which we study the finite field case, we assume that $p \neq 2$ in Paragraphs 2C and 2D only; see Remark 2.11. Note that neither Gow's results [1984] nor the modular version established in Paragraph 2B require p to be odd. The same is true of the results of Prasad and Flicker — as well as their modular version proved in Section 4 — asserting that any distinguished irreducible representation π of G is σ-selfdual and $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1)$ has dimension 1.

From Paragraph 5D the assumption $p \neq 2$ is crucial (as in [Hakim and Murnaghan 2002; 2008] and to a lesser extent [Coniglio-Guilloton 2016]). We use at many places, in particular in Section 6 and in the proof of the σ-selfdual type Theorem 5.10, the fact that the first cohomology set of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_0)$ in a pro-p-group is trivial. I do not know whether or not Theorem 5.10 still holds when $p = 2$.

I also do not now whether the dichotomy and disjunction theorems hold when F has characteristic 2. The only exception is Prasad's dichotomy theorem [2001] for cuspidal complex representations when F/F_0 is unramified, which remains the only known distinction criterion for cuspidal representations in arbitrary residual characteristic. Note that Prasad's approach does not work in the modular case, for [Prasad 2001, Theorem 1] does not hold in characteristic $\ell > 0$.

1O. Finally, let us mention that the methods developed in this paper are expected to generalize to other groups. The distinction of supercuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ by a unitary group is currently being explored by Jiandi Zou in his ongoing PhD thesis at Université de Versailles St-Quentin.

2. The finite field case

The aim of this section is to extend to the modular case some results about distinction of cuspidal representations of GL_n over a finite field which are know in the complex case only. These results will be needed in Sections 8 and 9, but this section can also be read independently from the rest of the article.

In this section, $\boldsymbol{k}$ is a finite field whose characteristic is an arbitrary prime number p. In Paragraphs 2C and 2D, we will assume that p is odd. Let q denote the cardinality of $\boldsymbol{k}$.

Let R be an algebraically closed field of characteristic different from p, denoted ℓ. (Note that ℓ can be 0.) We say that we are in the "modular case" when we consider the case where $\ell > 0$. By *representation* of a finite group, we mean a representation on an R-vector space.

Given a representation π of a finite group G, we write $\pi^\vee$ for the contragredient of π. Given a subgroup H of G, we say that π is H-*distinguished* if the space $\mathrm{Hom}_{\mathrm{H}}(\pi, 1)$ is nonzero, where 1 denotes the trivial character of H.

Given $n \geq 1$, an irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$ is said to be *cuspidal* if it has no nonzero invariant vector under the unipotent radical of any proper parabolic subgroup or, equivalently, if it does not occur as a subrepresentation of any proper parabolically induced representation. It is *supercuspidal* if it does not occur as a subquotient of a proper parabolically induced representation. When R has characteristic 0, any cuspidal representation is supercuspidal.

When $\ell > 0$, we denote by $\overline{\mathbf{Q}}_\ell$ an algebraic closure of the field of ℓ-adic numbers, by $\overline{\mathbf{Z}}_\ell$ the ring of ℓ-adic integers in $\overline{\mathbf{Q}}_\ell$ and by $\overline{\mathbf{F}}_\ell$ the residue field of $\overline{\mathbf{Z}}_\ell$. We refer to [Serre 1977, Section 15] for a definition of the reduction mod ℓ of a $\overline{\mathbf{Q}}_\ell$-representation of a finite group.

2A. ***Parametrization of supercuspidal representations.*** For the results stated in this paragraph, we refer to [Green 1955; Dipper 1985; Dipper and James 1986; James 1986] (see also [Vignéras 1996, III.2; Mínguez and Sécherre 2015, Section 2.6]).

Let $\boldsymbol{t}/\boldsymbol{k}$ be an extension of degree $n \geq 1$ of finite fields of characteristic p. Fix a $\boldsymbol{k}$-embedding of $\boldsymbol{t}$ in the matrix algebra $\mathrm{M}_n(\boldsymbol{k})$, and consider $\boldsymbol{t}^\times$ as a maximal torus in $\mathrm{GL}_n(\boldsymbol{k})$. An R-character ξ of $\boldsymbol{t}^\times$ is said to be $\boldsymbol{k}$-*regular* if the characters $\xi, \xi^q, \dots, \xi^{q^{n-1}}$ are all distinct.

Let ξ be a $\boldsymbol{k}$-regular R-character of $\boldsymbol{t}^\times$. By Green [1955] when R has characteristic 0 and James [1986] when R has positive characteristic $\ell \neq p$, there is a supercuspidal irreducible representation ρ_ξ of $\mathrm{GL}_n(\boldsymbol{k})$, uniquely determined up to isomorphism, such that

$$\operatorname{tr} \rho_\xi(g) = (-1)^{n-1} \cdot \sum_{\gamma} \xi^\gamma(g) \tag{2-1}$$

for all $g \in \boldsymbol{t}^\times$ with irreducible characteristic polynomial, where γ runs over $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{k})$. This induces a surjective map

$$\xi \mapsto \rho_\xi \tag{2-2}$$

between $\boldsymbol{k}$-regular characters of $\boldsymbol{t}^\times$ and isomorphism classes of supercuspidal irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$, whose fibers are $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{k})$-orbits.

Suppose that R is the field $\overline{\mathbf{Q}}_\ell$. In the next proposition, we record the main properties of the reduction mod ℓ of supercuspidal $\overline{\mathbf{Q}}_\ell$-representations of $\mathrm{GL}_n(\boldsymbol{k})$.

Proposition 2.1 [Dipper 1985; Dipper and James 1986; James 1986]. *Let ξ be a $\boldsymbol{k}$-regular $\overline{\mathbf{Q}}_\ell$-character of $\boldsymbol{t}^\times$ and ρ be the supercuspidal irreducible representation which corresponds to it.*

(1) *The reduction mod ℓ of ρ, denoted $\bar{\rho}$, is irreducible and cuspidal.*

(2) *The representation $\bar{\rho}$ is supercuspidal if and only if the reduction mod ℓ of ξ, denoted $\bar{\xi}$, is $\boldsymbol{k}$-regular.*

(3) *When $\bar{\xi}$ is $\boldsymbol{k}$-regular, the supercuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation of $\mathrm{GL}_n(\boldsymbol{k})$ which corresponds to it is $\bar{\rho}$.*

Moreover, for any cuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation π of $\mathrm{GL}_n(\boldsymbol{k})$, there is a supercuspidal irreducible $\overline{\mathbf{Q}}_\ell$-representation ρ of $\mathrm{GL}_n(\boldsymbol{k})$ whose reduction mod ℓ is π. Such a representation ρ is said to be a *lift* of π.

Remark 2.2. Here are a couple of additional properties which we will use at various places.

(1) The identity (2-1) shows that, if ξ is a $\boldsymbol{k}$-regular character of $\boldsymbol{t}^\times$ and ρ is the supercuspidal representation corresponding to ξ by (2-2), then its contragredient $\rho^\vee$ corresponds to ξ^{-1}.

(2) Let $\iota : \mathrm{R} \to \mathrm{R}'$ be an embedding of algebraically closed fields of characteristic $\ell \neq p$. Then any irreducible R'-representation π' of $\mathrm{GL}_n(\boldsymbol{k})$ is isomorphic to $\pi \otimes \mathrm{R}'$ for a uniquely determined irreducible R-representation π of $\mathrm{GL}_n(\boldsymbol{k})$, which follows from the fact that, since $\mathrm{GL}_n(\boldsymbol{k})$ is finite, the trace of π' takes values in $\iota(\mathrm{R})$. Given a subgroup H of $\mathrm{GL}_n(\boldsymbol{k})$, the representation π is cuspidal (respectively supercuspidal, H-distinguished) if and only if π' is cuspidal (respectively supercuspidal, H-distinguished). Moreover, by (2-1), if π is supercuspidal and corresponds to the $\boldsymbol{k}$-regular R-character ξ, then π' corresponds to the $\boldsymbol{k}$-regular R'-character $\xi' = \iota \circ \xi$.

2B. ***The Galois case.*** Recall that p is an arbitrary prime number. Let $\boldsymbol{k}/\boldsymbol{k}_0$ be a quadratic extension of finite fields of characteristic p. Write σ for the nontrivial $\boldsymbol{k}_0$-automorphism of $\boldsymbol{k}$, and q_0 for the cardinality of $\boldsymbol{k}_0$. We thus have $q_0^2 = q$.

If π is an irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$, we write π^σ for the representation $\pi \circ \sigma$, and we say that π is *σ-selfdual* if π^σ, $\pi^\vee$ are isomorphic.

Lemma 2.3. *Let $n \geq 1$ be a positive integer.*

(1) *If there is a σ-selfdual supercuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$, then n is odd.*

(2) *Suppose that R has characteristic 0 and n is odd. Then there is a σ-selfdual supercuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$.*

Proof. Let ξ be a $\boldsymbol{k}$-regular character of $\boldsymbol{t}^\times$, and let ρ denote the supercuspidal irreducible representation of $\mathrm{GL}_n(\boldsymbol{k})$ corresponding to it by (2-2). The identity (2-1) shows that ρ^σ corresponds to ξ^{q_0}. Indeed, for all $g \in \boldsymbol{t}^\times \subseteq \mathrm{GL}_n(\boldsymbol{k})$ with irreducible characteristic polynomial, $\sigma(g)$ and g^{q_0} have the same characteristic polynomial, thus they are conjugate in $\mathrm{GL}_n(\boldsymbol{k})$. It follows that

$$\operatorname{tr} \rho^\sigma(g) = \operatorname{tr} \rho(g^{q_0}) = (-1)^{n-1} \cdot \sum_\gamma \xi^\gamma(g^{q_0}) = (-1)^{n-1} \cdot \sum_\gamma (\xi^{q_0})^\gamma(g)$$

for all $g \in \boldsymbol{t}^\times$ with irreducible characteristic polynomial. Thus ρ is σ-selfdual if and only if

$$\xi^{-q_0} = \xi^{q_0^{2i}}, \quad \text{for some } i \in \{0, \dots, n-1\}. \tag{2-3}$$

Exponentiating to $-q_0$ again gives us the equality $\xi^q = \xi^{q^{2i}}$. The $\boldsymbol{k}$-regularity assumption on ξ implies that n divides $2i-1$, thus n is odd. Besides, since $0 \le i \le n-1$, we have $n = 2i-1$. It follows that

$$\rho \text{ is } \sigma\text{-selfdual} \Leftrightarrow \xi^{-1} = \xi^{q_0^n}. \tag{2-4}$$

This is also equivalent to ξ being trivial on $\boldsymbol{t}_0^\times$, where $\boldsymbol{t}_0$ is the subfield of $\boldsymbol{t}$ with q_0^n elements.

Conversely, suppose that R has characteristic 0 and n is odd. Let ξ be an R-character of $\boldsymbol{t}^\times$ of order q_0^n+1, which exists since $\boldsymbol{t}^\times$ has order $q^n - 1 = (q_0^n-1)(q_0^n+1)$. It is thus trivial on $\boldsymbol{t}_0^\times$. On ther other hand, q_0 has order $2n$ mod q_0^n+1, thus q has order n mod q_0^n+1. It follows that the character ξ is $\boldsymbol{k}$-regular. □

Remark 2.4. When R has characteristic $\ell > 0$, the group $\mathrm{GL}_n(\boldsymbol{k})$ may have σ-selfdual cuspidal (nonsupercuspidal) representations for n even, and it may have no σ-selfdual supercuspidal representation for n odd.

(1) For an example of σ-selfdual cuspidal nonsupercuspidal representation for n even, let e be the order of q mod ℓ, and suppose that $n = e\ell^u$ for some integer $u \ge 0$. Then, by [Mínguez and Sécherre 2015, Théorème 2.4], the unique generic subquotient π of the representation induced from the trivial character of a Borel subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ is cuspidal and σ-selfdual. One may choose q, ℓ and u such that n is even. For instance, this is the case when $n = 2$ and $\ell \neq 2$ divides $q+1$.

(2) The group $\mathrm{GL}_n(\boldsymbol{k})$ may even have no supercuspidal representation at all: this is the case, for instance, when $n = 3$, $q = 2$ and $\ell = 7$.

Lemma 2.5. *Let ρ be a supercuspidal representation of* $\mathrm{GL}_n(\boldsymbol{k})$ *for some odd integer $n \ge 1$. The following assertions are equivalent*:

(1) *The representation ρ is σ-selfdual.*

(2) *The representation ρ is* $\mathrm{GL}_n(\boldsymbol{k}_0)$*-distinguished.*

(3) *The space* $\mathrm{Hom}_{\mathrm{GL}_n(\boldsymbol{k}_0)}(\rho, 1)$ *has dimension* 1.

Proof. When R has characteristic 0, this is due to Gow [1984]. Suppose that R has characteristic $\ell > 0$ prime to q. We postpone to Section 4 the proof of the fact that (2) implies (1) and is equivalent to (3), since the proof of Theorem 4.1 works in both the finite and nonarchimedean cases (see Remark 4.3). Here we prove that (1) implies (2). For this, we use the following general lemma.

Lemma 2.6. *Let* G *be a finite group and* H *be a subgroup of* G. *Let* π *be an irreducible representation of* G *on a* $\bar{\mathbf{Q}}_\ell$*-vector space* V *such that* $\mathrm{Hom}_{\mathsf{H}}(\pi, 1)$ *is nonzero. Let* $\mathrm{L} \subseteq \mathrm{V}$ *be a* G*-stable* $\bar{\mathbf{Z}}_\ell$*-lattice. Then* $\mathrm{L} \otimes \bar{\mathbf{F}}_\ell$ *has at least one irreducible* G*-subquotient* τ *such that* $\mathrm{Hom}_{\mathsf{H}}(\tau, 1) \neq \{0\}$.

Proof. Let φ be a nonzero H-invariant linear form on V. The image of L by φ, denoted M, is a $\bar{\mathbf{Z}}_\ell$-lattice in $\bar{\mathbf{Q}}_\ell$. Reducing mod the maximal ideal of $\bar{\mathbf{Z}}_\ell$ gives a nonzero H-invariant $\bar{\mathbf{F}}_\ell$-linear map $\bar{\varphi}$ from $\mathrm{L} \otimes \bar{\mathbf{F}}_\ell$ to $\mathrm{M} \otimes \bar{\mathbf{F}}_\ell \simeq \bar{\mathbf{F}}_\ell$. Let W be the largest subrepresentation of $\mathrm{L} \otimes \bar{\mathbf{F}}_\ell$ contained in the kernel of $\bar{\varphi}$. Then any irreducible subrepresentation τ of $(\mathrm{L} \otimes \bar{\mathbf{F}}_\ell)/\mathrm{W}$ satisfies the required condition. □

Let ξ be a $\boldsymbol{k}$-regular character of $\boldsymbol{t}^\times$ parametrizing some σ-selfdual supercuspidal representation ρ of $\mathrm{GL}_n(\boldsymbol{k})$. By (2-4), we have $\xi^{-1} = \xi^{q_0^n}$. Let us fix a field embedding $\iota : \bar{\mathbf{F}}_\ell \to \mathrm{R}$. Since ξ has finite image, there is a $\boldsymbol{k}$-regular character $\tilde{\xi}$ of $\boldsymbol{t}^\times$ with values in $\bar{\mathbf{Z}}_\ell$ such that:

- The character $\tilde{\xi}$ satisfies the identity $\tilde{\xi}^{-1} = \tilde{\xi}^{q_0^n}$.
- One has $\xi = \iota \circ \xi_0$ where ξ_0 is the reduction mod ℓ of $\tilde{\xi}$.

The character $\tilde{\xi}$ corresponds to a σ-selfdual supercuspidal $\bar{\mathbf{Q}}_\ell$-representation $\tilde{\rho}$ of $\mathrm{GL}_n(\boldsymbol{k})$. Let V denote the vector space of $\tilde{\rho}$ and fix a $\mathrm{GL}_n(\boldsymbol{k})$-stable $\bar{\mathbf{Z}}_\ell$-lattice L in V. By Paragraph 2A, the representation of $\mathrm{GL}_n(\boldsymbol{k})$ on the $\bar{\mathbf{F}}_\ell$-vector space $\mathrm{L} \otimes \bar{\mathbf{F}}_\ell$ is isomorphic to the supercuspidal representation ρ_0 corresponding to ξ_0, and it is distinguished by Lemma 2.6. The result now follows from Remark 2.2, which tells us that $\rho_0 \otimes \mathrm{R}$ is distinguished and isomorphic to ρ. □

Remark 2.7. If R is the field $\bar{\mathbf{F}}_\ell$, we proved that the representation ρ is distinguished if and only if it has a distinguished lift to $\bar{\mathbf{Q}}_\ell$.

Remark 2.8. We give an example of a σ-selfdual cuspidal nonsupercuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$ which is not distinguished. With the notation of Remark 2.4, assume that $n = e = 2$. Thus π is a σ-selfdual cuspidal (nonsupercuspidal) representation of $\mathrm{GL}_2(\boldsymbol{k})$. Let $\tilde{\pi}$ be an ℓ-adic lift of π (see Remark 2.2), and decompose its restriction to $\mathrm{GL}_2(\boldsymbol{k}_0)$ as a direct sum

$$\mathrm{V}_1 \oplus \cdots \oplus \mathrm{V}_r$$

of irreducible components. Since the order of $\mathrm{GL}_2(\boldsymbol{k}_0)$ is prime to ℓ, reduction mod ℓ preserves irreducibility, and the restriction of π to $\mathrm{GL}_2(\boldsymbol{k}_0)$ is semisimple. It follows that π decomposes as $\mathrm{W}_1 \oplus \cdots \oplus \mathrm{W}_r$, where W_i is irreducible and is the reduction mod ℓ of V_i for each $i = 1, \ldots, r$. Suppose that π is distinguished. Then W_i is the trivial character for some i. Thus V_i is a character, and it must be trivial since $\mathrm{GL}_2(\boldsymbol{k}_0)$ has no nontrivial character of order a power of ℓ, which implies that $\tilde{\pi}$ is distinguished. This is impossible, since $n = 2$ is even.

Remark 2.9. More generally, the argument of Remark 2.8 shows that, if H is a subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ whose order is prime to ℓ, then a cuspidal representation π of $\mathrm{GL}_n(\boldsymbol{k})$ is H-distinguished if and only if any ℓ-adic lift of π is H-distinguished.

2C. ***A mirabolic interlude.*** This paragraph is inspired from Matringe [2012]. We assume that $p \neq 2$. Let G denote the group $\mathrm{GL}_n(\boldsymbol{k})$ for some $n \geq 2$. Write P for the mirabolic subgroup of G, which is made of all matrices in G whose last row is $(0 \cdots 0\,1)$. Let U be the unipotent radical of P, and G' be the image of $\mathrm{GL}_{n-1}(\boldsymbol{k})$ in G under the group homomorphism

$$g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

We thus have $\mathrm{P} = \mathrm{G}'\mathrm{U}$, and we write P' for the mirabolic subgroup of G'. Let N be the maximal unipotent subgroup of G made of all upper triangular unipotent matrices, and ψ be a nontrivial character of $\boldsymbol{k}$. We still write ψ for the nondegenerate character

$$x \mapsto \psi(x_{1,2} + \cdots + x_{n-1,n})$$

of N. We have a functor

$$\pi \mapsto \mathrm{Ind}_{\mathrm{P}'\mathrm{U}}^{\mathrm{P}}(\pi \otimes \psi)$$

denoted Φ^+, from R-representations of P' to R-representations of P, where $\pi \otimes \psi$ is the representation of $\mathrm{P}'\mathrm{U}$ defined by $xu \mapsto \pi(x)\psi(u)$ for all $x \in \mathrm{P}'$ and $u \in \mathrm{U}$.

Given integers $r \geq s \geq 0$ such that $r + s = n$, let $\mathrm{H}_{r,s}$ be the subgroup of G defined in [Matringe 2012]. It is the conjugate of the Levi subgroup $\mathrm{GL}_r(\boldsymbol{k}) \times \mathrm{GL}_s(\boldsymbol{k})$ of G under the permutation matrix $w_{r,s}$ defined by the permutation

$$\begin{pmatrix} 1 & \cdots & t & \cdots & t+i & \cdots & r & r+1 & \cdots & r+1+j & \cdots & n-1 & n \\ 1 & \cdots & t & \cdots & t+2i & \cdots & n-1 & t+1 & \cdots & t+1+2j & \cdots & n-2 & n \end{pmatrix}$$

where $t = r - s + 1$. If $s \geq 1$, let $\mathrm{H}'_{r,s}$ be the subgroup $\mathrm{G}' \cap \mathrm{H}_{r,s}$ (denoted $\mathrm{H}_{r,s-1}$ in [Matringe 2012]).

Lemma 2.10. *Let π be a representation of P', and χ be a character of $\mathrm{H}_{r,s}$. Suppose that the vector space $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Phi^+(\pi), \chi)$ is nonzero. Then it is isomorphic to $\mathrm{Hom}_{\mathrm{P}'\cap\mathrm{H}'_{r,s}}(\pi, \chi)$.*

Proof. Given $g \in \mathrm{G}$ and a representation τ of a subgroup H of G, we will write $\mathrm{H}^g = g^{-1}\mathrm{H}g$, and τ^g for the representation $x \mapsto \tau(gxg^{-1})$ of H^g. Applying the Mackey formula, and since G' normalizes U, the restriction of $\Phi^+(\pi)$ to $\mathrm{P} \cap \mathrm{H}_{r,s}$ decomposes as the direct sum

$$\bigoplus_g \mathrm{Ind}_{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi^g \otimes \psi^g)$$

where g ranges over a set of representatives of $(\mathrm{P}'\mathrm{U}, \mathrm{P} \cap \mathrm{H}_{r,s})$-double cosets in P. Since $\mathrm{P} = \mathrm{G}'\mathrm{U}$, we may assume g ranges over a set of representatives of $(\mathrm{P}', \mathrm{H}'_{r,s})$-double cosets in G'. For each g, let us write the following isomorphism of representations of $\mathrm{P} \cap \mathrm{H}_{r,s}$

$$\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi^g \otimes \psi^g) \simeq \mathrm{Ind}_{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}((\pi^g \otimes \psi^g) \otimes \mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}(1)). \tag{2-5}$$

Since the induced representation $\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}(1)$ canonically surjects onto the trivial character of $\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}$ by Frobenius reciprocity, the right-hand side of (2-5) surjects onto

$$\mathrm{Ind}_{\mathrm{P}\cap\mathrm{H}_{r,s}\cap\mathrm{P}'^g\mathrm{U}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi^g\otimes\psi^g).$$

On the other hand, since π^g is trivial on $\mathrm{U}\cap\mathrm{H}_{r,s}$, the left-hand side of (2-5) is a sum of finitely many copies of $\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\psi^g)$. It follows that, if $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Phi^+(\pi),\chi)$ is nonzero, then there is a $g\in\mathrm{G}'$ such that

$$\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\mathrm{Ind}_{\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\psi^g),\chi)\neq\{0\}.$$

By Frobenius reciprocity, this implies that $\psi^g=\chi$ on $\mathrm{U}\cap\mathrm{H}_{r,s}$. Since we assumed that $p\neq 2$, the field $\boldsymbol{k}$ has at least three elements, thus any character of $\mathrm{H}_{r,s}\simeq\mathrm{GL}_r(\boldsymbol{k})\times\mathrm{GL}_s(\boldsymbol{k})$ is trivial on unipotent elements. We thus get $\psi^g=1$ on $\mathrm{U}\cap\mathrm{H}_{r,s}$. By [Matringe 2012, Lemma 3.1], this implies that $g\in\mathrm{P}'\mathrm{H}'_{r,s}$, that is, we may assume that $g=1$. We thus have

$$\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Phi^+(\pi),\chi)=\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\mathrm{Ind}_{\mathrm{P}'\mathrm{U}\cap\mathrm{H}_{r,s}}^{\mathrm{P}\cap\mathrm{H}_{r,s}}(\pi\otimes\psi),\chi)\simeq\mathrm{Hom}_{\mathrm{P}'\mathrm{U}\cap\mathrm{H}_{r,s}}(\pi\otimes\psi,\chi).$$

The result now follows from the fact that $\mathrm{P}'\mathrm{U}\cap\mathrm{H}_{r,s}=\mathrm{P}'\cap\mathrm{H}'_{r,s}$. (This latter equality has been pointed out to me by N. Matringe.) □

Remark 2.11. (1) If χ is assumed to be trivial on unipotent elements, or if $\boldsymbol{k}$ has at least three elements, then Lemma 2.10 holds without assuming that $p\neq 2$.

(2) If $\boldsymbol{k}$ has cardinality 2, the group $\mathrm{GL}_2(\boldsymbol{k})$ has a cuspidal character. Thus, when $r\geq 2$, the group $\mathrm{H}_{r,2}$ has a character which is nontrivial on $\mathrm{U}\cap\mathrm{H}_{r,2}$.

Now write G'' for the copy of $\mathrm{GL}_{n-2}(\boldsymbol{k})$ in the upper left block of G' and P'' for the mirabolic subgroup of G''.

Lemma 2.12. *Let π' be a representation of P'', and χ' be a character of $\mathrm{H}'_{r,s}$. Suppose that $s\geq 1$. If $\mathrm{Hom}_{\mathrm{P}'\cap\mathrm{H}'_{r,s}}(\Phi^+(\pi'),\chi')$ is nonzero, then it is isomorphic to $\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,s-1}}(\pi',\chi')$.*

Proof. It is similar to that of Lemma 2.10, replacing [Matringe 2012, Lemma 3.1] by [Matringe 2012, Lemma 3.2]. □

Let Γ denote the mirabolic representation of P. Recall that it is defined as the representation of P induced from the character ψ of N.

Lemma 2.13. *Let $n\geq 2$ and $r\geq s\geq 0$ be such that $r+s=n$. Let χ be a character of $\mathrm{H}_{r,s}$. If the space $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Gamma,\chi)$ is nonzero, then $r=s$.*

Proof. If $s=0$, then $\mathrm{H}_{r,s}=\mathrm{G}$ and the result follows from the fact that the representation Γ is irreducible of dimension greater than 1.

Suppose that $s\geq 1$ and that the space $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,s}}(\Gamma,\chi)$ is nonzero. The mirabolic representation Γ is isomorphic to $\Phi^+(\Gamma')$, where Γ' denotes the mirabolic representation of P'. By Lemma 2.10, the space $\mathrm{Hom}_{\mathrm{P}'\cap\mathrm{H}'_{r,s}}(\Gamma',\chi')$ is nonzero, where χ' is the restriction of χ to $\mathrm{H}'_{r,s}$. Now identify Γ' with $\Phi^+(\Gamma'')$, where

Γ'' is the mirabolic representation of P''. By Lemma 2.12, the space $\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,s-1}}(\Gamma'', \chi'')$ is nonzero, where χ'' is the restriction of χ' to $\mathrm{H}_{r-1,s-1}$. By induction on n, the fact that $\mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,s-1}}(\Gamma'', \chi'')$ is nonzero implies that $r-1 = s-1$, thus $r = s$. □

Proposition 2.14. *Let $n \geq 2$ and $r, s \geq 0$ be such that $r+s = n$. Let ρ be a cuspidal representation of* G, *and χ be a character of* $\mathrm{M} = (\mathrm{GL}_r \times \mathrm{GL}_s)(\boldsymbol{k})$. *Suppose* $\mathrm{Hom}_{\mathrm{M}}(\rho, \chi)$ *is nonzero. Then $r = s$.*

Proof. Conjugating M and χ if necessary, we may assume that $r \geq s$. The result then follows from Lemma 2.13 and the fact that the restriction of ρ to P is isomorphic to Γ. This latter fact is well-known when R has characteristic 0, and is given by [Vignéras 1996, III.1] when R is equal to $\overline{\mathbf{F}}_\ell$. For an arbitrary R of characteristic $\ell > 0$, fix a field embedding of $\overline{\mathbf{F}}_\ell$ in R and write ρ as $\rho_0 \otimes \mathrm{R}$ for some cuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation ρ_0 of G as in Remark 2.2. Since the restriction of ρ_0 to P is isomorphic to Γ_0, the mirabolic $\overline{\mathbf{F}}_\ell$-representation of P, the restriction of ρ to P is isomorphic to $\Gamma_0 \otimes \mathrm{R} \simeq \Gamma$. □

Remark 2.15. Suppose that $r = s \geq 1$. Putting Lemmas 2.10 and 2.12 together, we get

$$\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,r}}((\Phi^+)^2(\pi), 1) \simeq \mathrm{Hom}_{\mathrm{P}''\cap\mathrm{H}_{r-1,r-1}}(\pi, 1)$$

for any representation π of P''. By induction, we get an isomorphism $\mathrm{Hom}_{\mathrm{P}\cap\mathrm{H}_{r,r}}(\Gamma, 1) \simeq \mathrm{R}$.

Corollary 2.16. *Suppose that $n = 2r$ for some $r \geq 1$, and let ρ be a cuspidal representation of* $\mathrm{GL}_n(\boldsymbol{k})$. *Then the* R*-vector space* $\mathrm{Hom}_{(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})}(\rho, 1)$ *has dimension at most* 1.

Proof. This follows from Remark 2.15, together with the fact that the restriction of ρ to P is isomorphic to Γ. □

2D. *The Levi case.* In this paragraph, we consider the supercuspidal irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$ distinguished by some maximal Levi subgroup. As in Paragraph 2C, we assume that $p \neq 2$.

Lemma 2.17. *Let $n \geq 1$ be a positive integer.*

(1) *If there is a selfdual supercuspidal irreducible representation of* $\mathrm{GL}_n(\boldsymbol{k})$, *then either $n = 1$ or n is even.*

(2) *Suppose that* R *has characteristic* 0, *and that either $n = 1$ or n is even. Then there exists a selfdual supercuspidal irreducible representation of* $\mathrm{GL}_n(\boldsymbol{k})$.

Proof. If $n = 1$, the trivial character of $\boldsymbol{k}^\times$ is selfdual and supercuspidal. Suppose that $n \geq 2$. Let us fix an extension $\boldsymbol{t}$ of $\boldsymbol{k}$ of degree n, and identify $\boldsymbol{t}^\times$ with a maximal torus in $\mathrm{GL}_n(\boldsymbol{k})$. We consider the Green–James parametrization (2-2) of isomorphism classes of supercuspidal irreducible representations of $\mathrm{GL}_n(\boldsymbol{k})$ by $\boldsymbol{k}$-regular characters of $\boldsymbol{t}^\times$.

Given a $\boldsymbol{k}$-regular character ξ of $\boldsymbol{t}^\times$, let ρ denote the representation corresponding to it. Recall (see Remark 2.2) that $\rho^\vee$ corresponds to ξ^{-1}. Thus ρ is selfdual if and only if

$$\xi^{-1} = \xi^{q^i}, \quad \text{for some } i \in \{0, \dots, n-1\}. \tag{2-6}$$

Taking the contragredient again gives us the equality $\xi = \xi^{q^{2i}}$. The regularity assumption on ξ implies that n divides $2i$ and, since $0 \le i \le n-1$, we get $n = 2i$. It follows that

$$\rho_\xi \text{ is selfdual} \Leftrightarrow \xi^{-1} = \xi^{q^{n/2}}. \tag{2-7}$$

This is also equivalent to ξ being trivial on $\boldsymbol{t}'^{\times}$, where $\boldsymbol{t}'$ is the subfield of $\boldsymbol{t}$ with cardinality $q^{n/2}$.

Conversely, suppose that R has characteristic 0 and $n = 2r$ for some $r \ge 1$. Let us consider an R-character ξ of $\boldsymbol{t}^{\times}$ of order q^r+1, which exists since $\boldsymbol{t}^{\times}$ has order $q^n-1 = (q^r-1)(q^r+1)$. It is trivial on $\boldsymbol{t}'^{\times}$. On ther other hand, q has order $n = 2r$ mod q^r+1, which implies that ξ is $\boldsymbol{k}$-regular. □

Remark 2.18. When R has characteristic $\ell > 0$, the group $\mathrm{GL}_n(\boldsymbol{k})$ may have selfdual cuspidal (nonsupercuspidal) representations even if n is odd and > 1. Indeed, let e be the order of q mod ℓ, and suppose that $n = e\ell^u$ for some $u \ge 0$. The unique generic irreducible subquotient of the representation induced from the trivial character of a Borel subgroup of $\mathrm{GL}_n(\boldsymbol{k})$ is then both cuspidal (see Remark 2.4) and selfdual. One then can choose q, ℓ and u such that n is odd and > 1. For instance, this is the case when $\ell \ne 2$ divides $q-1$ and $n = \ell$.

Also, as in Remark 2.4, the group $\mathrm{GL}_n(\boldsymbol{k})$ may even have no supercuspidal representation at all, which is the case, for instance, when $n = q = 2$ and $\ell = 3$.

Lemma 2.19. *Suppose that $n = 2r$ with $r \ge 1$, and let ρ be a supercuspidal representation of $\mathrm{GL}_n(\boldsymbol{k})$. The following assertions are equivalent*:

(1) *The representation ρ is selfdual.*

(2) *The representation ρ is $(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})$-distinguished.*

(3) *The* R*-vector space $\mathrm{Hom}_{(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})}(\rho, 1)$ has dimension* 1.

Proof. When R has characteristic 0, this is [Hakim and Murnaghan 2002, Proposition 6.1] (see also [Coniglio-Guilloton 2016, Lemme 3.4.10]). Suppose now R has prime characteristic ℓ not dividing q. To prove that (1) implies (2), we follow the same lifting argument as in the proof of Lemma 2.5.

We now prove that (2) implies (1). Let us write $\mathrm{G} = \mathrm{GL}_n(\boldsymbol{k})$ and $\mathrm{H} = (\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{k})$. Let ρ be an H-distinguished supercuspidal representation of G. If one fixes a field embedding of $\overline{\mathbf{F}}_\ell$ in R, then Remark 2.2 tells us that ρ is isomorphic to $\rho_0 \otimes \mathrm{R}$ for some distinguished supercuspidal irreducible $\overline{\mathbf{F}}_\ell$-representation ρ_0 of G. Since ρ is selfdual if and only if ρ_0 is, we are thus reduced to proving the result in the case where R is equal to $\overline{\mathbf{F}}_\ell$, which we assume now.

Since ρ is distinguished, its contragredient $\rho^\vee$ has a nonzero H-invariant vector. We thus have a nonzero homomorphism $i : \overline{\mathbf{Z}}_\ell[\mathrm{H}\backslash\mathrm{G}] \to \rho^\vee$. Let us consider a projective envelope $f : \mathrm{P} \to \rho^\vee$ of $\rho^\vee$ in the category of $\overline{\mathbf{Z}}_\ell[\mathrm{G}]$-modules. Since $\rho^\vee$ is supercuspidal, it has the following properties (see for instance [Vignéras 1996, III.2.9]):

- The representation $\mathrm{P} \otimes \overline{\mathbf{Q}}_\ell$ is isomorphic to the direct sum of all the $\overline{\mathbf{Q}}_\ell$-lifts of $\rho^\vee$.
- There are ℓ^a such lifts, where a is the ℓ-adic valuation of q^n-1.

- The representation $\mathrm{P}\otimes\overline{\mathbf{F}}_\ell$ is indecomposable of length ℓ^a and has a unique irreducible quotient, and all its irreducible components are isomorphic to $\rho^\vee$.

By projectivity of P, the homomorphism i gives rise to a nonzero homomorphism

$$j : \mathrm{P} \to \overline{\mathbf{Z}}_\ell[\mathrm{H}\backslash\mathrm{G}] \tag{2-8}$$

such that $i \circ j = f$. Inverting ℓ, we get a nonzero homomorphism from $\mathrm{P}\otimes\overline{\mathbf{Q}}_\ell$ to $\overline{\mathbf{Q}}_\ell[\mathrm{H}\backslash\mathrm{G}]$. It follows that $\rho^\vee$ has at least one H-distinguished lift. Thanks to the characteristic 0 case, such a lift is selfdual. Reducing mod ℓ, it follows that ρ is selfdual.

We now go back to the case of a general R. The fact that (2) implies (3) is a particular case of Corollary 2.16. However, we are going to give another proof here, which works for supercuspidal representations only but is more likely to generalize to other situations.

Let V be the maximal direct summand of $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}]$ in the block of ρ. This means that $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}]$ decomposes as a direct sum $\mathrm{V}\oplus\mathrm{V}'$ where all irreducible subquotients of V are isomorphic to ρ, and no irreducible subquotient of V' is isomorphic to ρ. Besides, since ρ is selfdual, we have

$$\dim \mathrm{Hom}_\mathrm{H}(\rho, 1) = \dim \mathrm{Hom}_\mathrm{H}(1, \rho) = \dim \mathrm{Hom}_\mathrm{G}(\mathrm{R}[\mathrm{H}\backslash\mathrm{G}], \rho) = \dim \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \rho).$$

We thus have to prove that the cosocle of V is isomorphic to ρ.

Lemma 2.20. *The* R*-algebra* $\mathrm{A} = \mathrm{End}_\mathrm{G}(\mathrm{V})$ *is commutative.*

Proof. Since the convolution algebra $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ decomposes as $\mathrm{End}_\mathrm{G}(\mathrm{V})\oplus\mathrm{End}_\mathrm{G}(\mathrm{V}')$, it suffices to prove that $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ is commutative. For $x \in \mathrm{G}$, let f_x be the characteristic function in $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ of the double coset $\mathrm{H}x\mathrm{H}$. For $x, y \in \mathrm{G}$, one has

$$f_x * f_y = \sum_{z\in\mathrm{H}\backslash\mathrm{G}/\mathrm{H}} a(x, y, z) f_z$$

where $a(x, y, z) \in \mathrm{R}$ is the image of the cardinality of $(\mathrm{H}x\mathrm{H}\cap z\mathrm{H}y^{-1}\mathrm{H})/\mathrm{H}$ in R. When R has characteristic 0, the algebra $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ is known to be commutative since $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}]$ is multiplicity free as an R[G]-module, thus

$$\mathrm{card}(\mathrm{H}x\mathrm{H}\cap z\mathrm{H}y^{-1}\mathrm{H})/\mathrm{H} = \mathrm{card}(\mathrm{H}y\mathrm{H}\cap z\mathrm{H}x^{-1}\mathrm{H})/\mathrm{H} \tag{2-9}$$

for all $x, y, z \in \mathrm{G}$. Now if R has characteristic $\ell > 0$, reducing (2-9) mod ℓ gives us a congruence relation which tells us that the algebra $\mathrm{R}[\mathrm{H}\backslash\mathrm{G}/\mathrm{H}]$ is commutative. □

It remains to prove that the cosocle of V is multiplicity free. Let $m \geq 1$ be the multiplicity of ρ in the cosocle of V and Q be the projective indecomposable R[G]-module associated with ρ. It has length ℓ^a, has a unique irreducible quotient, and all its irreducible components are isomorphic to ρ. Write $\mathrm{V} = \mathrm{V}_1 \oplus\cdots\oplus \mathrm{V}_m$ where $\mathrm{V}_1, \dots, \mathrm{V}_m$ are indecomposable R[G]-modules with cosocle isomorphic to ρ. There is a nilpotent endomorphism $\mathrm{N} \in \mathrm{End}_\mathrm{G}(\mathrm{Q})$ such that

$$\mathrm{End}_\mathrm{G}(\mathrm{Q}) = \mathrm{R}[\mathrm{N}]$$

with $N^{\ell^a} = 0$ and $N^{\ell^a - 1} \neq 0$. Therefore each V_i is isomorphic to the quotient of Q by the image of N^{k_i} for some $k_i \geq 0$. Reordering if necessary, we may assume that $\mathrm{Hom}(V_1, V_i)$ is nonzero for all $i \geq 1$. Suppose that $m \geq 2$, and define two endomorphisms $u, u' \in A$ by:

(1) The endomorphisms u, u' are trivial on V_i for all $i \geq 2$.

(2) The restriction of u to V_1 is the identity on V_1.

(3) The restriction of u' to V_1 coincides with some nonzero homomorphism in $\mathrm{Hom}(V_1, V_2)$.

Then $uu' = 0$ and $u'u \neq 0$, thus A is not commutative. Thus $m = 1$. □

Remark 2.21. If R is the field $\overline{\mathbf{F}}_\ell$, we proved that ρ is distinguished if and only if it has a distinguished lift to $\overline{\mathbf{Q}}_\ell$ (see Remark 2.7).

Remark 2.22. If we only assume ρ to be cuspidal in Lemmas 2.17 and 2.19, then the lifting argument may not work, that is, ξ may not have a σ-selfdual $\boldsymbol{k}$-regular lift $\tilde{\xi}$. Besides, the structure of the projective envelope of ρ is more complicated when ρ is cuspidal nonsupercuspidal.

3. Notation and basic definitions in the nonarchimedean case

Let F/F_0 be a separable quadratic extension of locally compact nonarchimedean local fields of residual characteristic p. Apart from Section 4, we will assume that $p \neq 2$.

Write σ for the nontrivial F_0-automorphism of F. Write $\mathcal{O}$ for the ring of integers of F and $\mathcal{O}_0$ for that of F_0. Write $\boldsymbol{k}$ for the residue field of F and $\boldsymbol{k}_0$ for that of F_0. The involution σ induces a $\boldsymbol{k}_0$-automorphism of $\boldsymbol{k}$, still denoted σ, which generates $\mathrm{Gal}(\boldsymbol{k}/\boldsymbol{k}_0)$.

As in Section 2, let R be an algebraically closed field of characteristic ℓ different from p. (Note that ℓ can be 0.) We say we are in the "modular case" when we consider the case where $\ell > 0$.

We fix once and for all a character

$$\psi_0 : F_0 \to R^\times \tag{3-1}$$

trivial on the maximal ideal of $\mathcal{O}_0$ but not on $\mathcal{O}_0$, and define $\psi = \psi_0 \circ \mathrm{tr}_{F/F_0}$.

When $\ell \neq 2$, we write

$$\omega = \omega_{F/F_0} : F_0^\times \to R^\times \tag{3-2}$$

for the character of $F_0^\times$ whose kernel is the subgroup of F/F_0-norms.

Let G be the locally profinite group $G = \mathrm{GL}_n(F)$, with $n \geq 1$, equipped with the involution σ acting componentwise. Its σ-fixed points is the closed subgroup $G^\sigma = \mathrm{GL}_n(F_0)$. We will identify the center of G with $F^\times$ and that of G^σ with $F_0^\times$.

By *representation* of a locally profinite group, we always mean a smooth representation on an R-module. Given a representation π of a closed subgroup H of G, we write $\pi^\vee$ for the smooth contragredient of π and π^σ for the representation $\pi \circ \sigma$ of the subgroup $\sigma(H)$. We say that π is *σ-selfdual* if H is σ-stable and π^σ, $\pi^\vee$ are isomorphic. If $g \in G$, we write $H^g = \{g^{-1}hg \mid h \in H\}$ and π^g for the representation $x \mapsto \pi(gxg^{-1})$ of H^g. If χ is a character of H, we write $\pi\chi$ for the representation $g \mapsto \chi(g)\pi(g)$.

If μ is a character of $\mathrm{H}\cap\mathrm{G}^\sigma$, we say that π is *μ-distinguished* if the space $\mathrm{Hom}_{\mathrm{H}\cap\mathrm{G}^\sigma}(\pi,\mu)$ is nonzero. If μ is the trivial character, we will simply say that π is *$\mathrm{H}\cap\mathrm{G}^\sigma$-distinguished*, or just *distinguished*. If $\mathrm{H}=\mathrm{G}$ and ϕ is a character of $\mathrm{F}_0^\times$, we will abbreviate $\phi\circ\det$-*distinguished* to *ϕ-distinguished*.

An irreducible representation of G is said to be *cuspidal* if all its proper Jacquet modules are trivial or, equivalently, if it does not occur as a subrepresentation of a proper parabolically induced representation. It is said to be *supercuspidal* if it does not occur as a subquotient of a proper parabolically induced representation (by [Dat 2012a, Corollaire B.1.3], this is equivalent to not occurring as a subquotient of the parabolic induction of any *irreducible* representation of a proper Levi subgroup of G). When R has characteristic 0, any cuspidal representation is supercuspidal.

4. A modular version of theorems of Prasad and Flicker

In this section, the residue characteristic p is arbitrary. We prove the following theorem, which is well-known in the complex case. Note that, in the modular case, any irreducible representation of G has a central character by [Vignéras 1996, II.2.8].

Theorem 4.1. *Let π be a distinguished irreducible representation of* G. *Then*:

1. *The central character of π is trivial on $\mathrm{F}_0^\times$.*
2. *The contragredient representation $\pi^\vee$ is distinguished.*
3. *The space* $\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi,1)$ *has dimension* 1.
4. *The representations π^σ and $\pi^\vee$ are isomorphic, that is, π is σ-selfdual.*

Remark 4.2. In the complex case, this theorem was first proved under the assumption that the characteristic of F is not 2, which was required in the proof of [Flicker 1991, Proposition 10]. Later, Prasad [2001, Section 4] gave an argument which only requires F/F_0 to be separable quadratic.

Proof. Property (1) is straightforward. Property (2) follows from an argument of Gelfand and Kazhdan (see [Sécherre and Venketasubramanian 2017, Proposition 8.4] in the modular case). For properties (3) and (4), we follow the proofs of Prasad [1990] and Flicker [1991]. The reference for the basic results in the theory of modular representations of p-adic reductive groups which we use in the proof is [Vignéras 1996].

Write $\mathcal{C}_c^\infty(\mathrm{G})$ for the space of locally constant, compactly supported R-valued functions on G, and fix an R-valued Haar measure on G, that is, a nonzero R-linear form on $\mathcal{C}_c^\infty(\mathrm{G})$ invariant under left translation by G.

Let W denote the vector space of π, and $l:\mathrm{W}\to\mathrm{R}$ be a nonzero G^σ-invariant linear form. For any $f\in\mathcal{C}_c^\infty(\mathrm{G})$, define a linear form on W by

$$\pi(f)l : w\mapsto \int_{\mathrm{G}} f(x)l(\pi(x)w)\,dx.$$

Since f is smooth, the linear form $\pi(f)l$ on W is smooth. This defines a nonzero homomorphism $\mathrm{L}:\mathcal{C}_c^\infty(\mathrm{G})\to\mathrm{W}^\vee$. It is G-equivariant under right translation and G^σ-invariant under left translation. Since

W is irreducible, it is surjective. Similarly, given a nonzero G^σ-invariant linear form $m : W^\vee \to R$, we obtain a surjective right G-equivariant and left G^σ-invariant homomorphism M from $\mathcal{C}_c^\infty(G)$ to $W^{\vee\vee} \simeq W$ (see [Vignéras 1996, Proposition I.4.18] for the latter isomorphism). We now define

$$B(f, g) = \langle M(f), L(g)\rangle \in R$$

for all $f, g \in \mathcal{C}_c^\infty(G)$. This defines a right G-invariant and left $G^\sigma \times G^\sigma$-invariant linear form B on the space $\mathcal{C}_c^\infty(G) \otimes \mathcal{C}_c^\infty(G) \simeq \mathcal{C}_c^\infty(G \times G)$. As in [Prasad 1990, Lemma 4.2] (and with [Prasad 2001, Lemma 4.1], which extends the result of [Flicker 1991, Proposition 10] to the case where F has arbitrary characteristic) we have

$$B(f, g) = B(g \circ \sigma, f \circ \sigma) \tag{4-1}$$

for all $f, g \in \mathcal{C}_c^\infty(G)$. It follows that the kernel of L is equal to $\{f \circ \sigma \mid f \in \mathrm{Ker}(M)\}$. Thus, if l' is any nonzero G^σ-invariant linear form on W, with associated homomorphism L', then L, L' have the same kernel. Since $\pi^\vee$ is admissible (by [Vignéras 1996, II.2.8]), Schur's lemma applies (see [Vignéras 1996, I.6.9]) thus one has $l' = cl$ for some $c \in R^\times$. Thus $\mathrm{Hom}_{G^\sigma}(W, 1)$ has dimension 1.

As in [Prasad 1990, Lemma 4.2], the bilinear form B corresponds to the G^σ-biinvariant linear form D on $\mathcal{C}_c^\infty(G)$ defined by

$$D(h) = m(\pi(h)l)$$

for all $h \in \mathcal{C}_c^\infty(G)$. The correspondence between B and D is given by

$$D(h) = B(k), \quad \text{with } k : (x, y) \mapsto h(xy^{-1}).$$

Note that (4-1) gives $D(h) = D(h \circ \sigma \circ \iota)$ with $\iota : x \mapsto x^{-1}$ on G. Replacing π by $\pi^* = \pi^{\vee\sigma}$ and exchanging the roles played by l, m we get a linear form

$$D^* : h \mapsto l(\pi^*(h)m).$$

Since we have $l(\pi^*(h)m) = m(\pi(h \circ \sigma \circ \iota)l)$, it follows that $D^* = D$. In order to deduce Property (4), it remains to prove that D determines π entirely. For any $\xi \in W^\vee$ we define the function

$$c_\xi : x \mapsto m(\pi^\vee(x)\xi) = m(\xi \circ \pi(x^{-1}))$$

on G. Then $\xi \mapsto c_\xi$ is an embedding of $W^\vee$ in the space $\mathcal{C}^\infty(G^\sigma\backslash G)$ of smooth R-valued functions on $G^\sigma\backslash G$. For $y \in G$ and $h \in \mathcal{C}_c^\infty(G)$, let ${}^y h$ denote the function $x \mapsto h(xy)$. Since L and M are surjective, there is a function h such that $\pi(h)l$ is nonzero. Then $y \mapsto D({}^y h)$ is a nonzero function in the space $\mathcal{C}^\infty(G^\sigma\backslash G)$, generating a subrepresentation isomorphic to $W^\vee$. Indeed, it is equal to $c_{\pi(h)l}$. It thus follows from the equality $D^* = D$ that we have $\pi^\sigma \simeq \pi^\vee$, as expected. □

Remark 4.3. The same results hold — and the same argument works — when F/F_0 is replaced by a quadratic extension of finite fields of arbitrary characteristic. It suffices to replace [Prasad 2001, Lemma 4.1] by [Gow 1984, Lemma 3.5].

5. Preliminaries on simple types

From now on, until the end of this article, we will assume that $p \neq 2$. This assumption is not needed in Paragraphs 5A–5C, but we assume it from now on for simplicity.

We assume the reader is familiar with the language of simple types. We recall the main results on simple strata, characters and types [Bushnell and Kutzko 1993; Bushnell and Henniart 1996; 2014; Mínguez and Sécherre 2014b] that we will need. Part of these preliminaries can also be found in [Anandavardhanan et al. 2018].

5A. ***Simple strata and characters.*** Let $[\mathfrak{a}, \beta]$ be a simple stratum in the F-algebra $\mathrm{M}_n(\mathrm{F})$ of $n \times n$ matrices with entries in F for some $n \geq 1$. Recall that $\mathfrak{a}$ is a hereditary order in $\mathrm{M}_n(\mathrm{F})$ and β is a matrix in $\mathrm{M}_n(\mathrm{F})$ such that:

(1) The F-algebra $\mathrm{E} = \mathrm{F}[\beta]$ is a field, whose degree over F is denoted d.

(2) The multiplicative group $\mathrm{E}^\times$ normalizes $\mathfrak{a}$.

The centralizer of E in $\mathrm{M}_n(\mathrm{F})$, denoted B, is an E-algebra isomorphic to $\mathrm{M}_m(\mathrm{E})$, with $n = md$. The intersection $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is a hereditary order in B.

Write $\mathfrak{p}_\mathfrak{a}$ for the Jacobson radical of $\mathfrak{a}$, and $\mathrm{U}^1(\mathfrak{a})$ for the compact open pro-p-subgroup $1 + \mathfrak{p}_\mathfrak{a}$ of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. We recall the following useful *simple intersection property* [Bushnell and Kutzko 1993, Theorem 1.6.1]: for all $x \in \mathrm{B}^\times$, we have

$$\mathrm{U}^1(\mathfrak{a})x\mathrm{U}^1(\mathfrak{a}) \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b})x\mathrm{U}^1(\mathfrak{b}). \tag{5-1}$$

Associated with $[\mathfrak{a}, \beta]$, there are compact open subgroups

$$\mathrm{H}^1(\mathfrak{a}, \beta) \subseteq \mathrm{J}^1(\mathfrak{a}, \beta) \subseteq \mathrm{J}(\mathfrak{a}, \beta)$$

of $\mathfrak{a}^\times$ and a finite set $\mathcal{C}(\mathfrak{a}, \beta)$ of characters of $\mathrm{H}^1(\mathfrak{a}, \beta)$ called *simple characters*, depending on the choice of the character ψ fixed in Section 3. Write $\mathbf{J}(\mathfrak{a}, \beta)$ for the compact mod center subgroup generated by $\mathrm{J}(\mathfrak{a}, \beta)$ and the normalizer of $\mathfrak{b}$ in $\mathrm{B}^\times$.

Proposition 5.1 [Bushnell and Henniart 2014, 2.1]. *We have the following properties*:

(1) *The group* $\mathrm{J}(\mathfrak{a}, \beta)$ *is the unique maximal compact subgroup of* $\mathbf{J}(\mathfrak{a}, \beta)$.

(2) *The group* $\mathrm{J}^1(\mathfrak{a}, \beta)$ *is the unique maximal normal pro-p-subgroup of* $\mathrm{J}(\mathfrak{a}, \beta)$.

(3) *The group* $\mathrm{J}(\mathfrak{a}, \beta)$ *is generated by* $\mathrm{J}^1(\mathfrak{a}, \beta)$ *and* $\mathfrak{b}^\times$, *and we have*

$$\mathrm{J}(\mathfrak{a}, \beta) \cap \mathrm{B}^\times = \mathfrak{b}^\times, \quad \mathrm{J}^1(\mathfrak{a}, \beta) \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b}). \tag{5-2}$$

(4) *The normalizer of any simple character* $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ *in* G *is equal to* $\mathbf{J}(\mathfrak{a}, \beta)$.

(5) *The intertwining set of any* $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ *in* G *is equal to* $\mathrm{J}^1(\mathfrak{a}, \beta)\mathrm{B}^\times\mathrm{J}^1(\mathfrak{a}, \beta)$.

By [Bushnell and Kutzko 1993, Theorem 3.4.1], the quotient $\mathrm{J}^1(\mathfrak{a},\beta)/\mathrm{H}^1(\mathfrak{a},\beta)$ is a finite $\boldsymbol{k}$-vector space, and the map

$$(x,y)\mapsto\langle x,y\rangle=\theta(xyx^{-1}y^{-1}) \tag{5-3}$$

makes it into a nondegenerate symplectic space. More precisely, if $\mathfrak{h}^1(\mathfrak{a},\beta)$ and $\mathfrak{j}^1(\mathfrak{a},\beta)$ are the sub-$\mathcal{O}$-lattices of $\mathfrak{a}$ such that $\mathrm{H}^1(\mathfrak{a},\beta)=1+\mathfrak{h}^1(\mathfrak{a},\beta)$ and $\mathrm{J}^1(\mathfrak{a},\beta)=1+\mathfrak{j}^1(\mathfrak{a},\beta)$, then we have

$$\langle 1+u,1+v\rangle=\psi\circ\mathrm{tr}(\beta(uv-vu)) \tag{5-4}$$

for all $u,v\in\mathfrak{j}^1(\mathfrak{a},\beta)$ [Bushnell and Henniart 2005b, Proposition 6.1], where tr denotes the trace map of $\mathrm{M}_n(\mathrm{F})$.

Let $[\mathfrak{a}',\beta']$ be another simple stratum in $\mathrm{M}_{n'}(\mathrm{F})$ for some $n'\geq 1$, and suppose that there is an F-algebra isomorphism $\varphi:\mathrm{F}[\beta]\to\mathrm{F}[\beta']$ such that $\varphi(\beta)=\beta'$. Then there is a canonical bijective map

$$\mathcal{C}(\mathfrak{a},\beta)\to\mathcal{C}(\mathfrak{a}',\beta') \tag{5-5}$$

called the *transfer map* [Bushnell and Kutzko 1993, Theorem 3.6.14].

When the hereditary order $\mathfrak{b}=\mathfrak{a}\cap\mathrm{B}$ is a maximal order in B, we say that the simple stratum $[\mathfrak{a},\beta]$ and the simple characters in $\mathcal{C}(\mathfrak{a},\beta)$ are *maximal*. When this is the case, then, given a homomorphism of E-algebras $\mathrm{B}\simeq\mathrm{M}_m(\mathrm{E})$ identifying $\mathfrak{b}$ with the standard maximal order, there are group isomorphisms

$$\mathrm{J}(\mathfrak{a},\beta)/\mathrm{J}^1(\mathfrak{a},\beta)\simeq\mathfrak{b}^\times/\mathrm{U}^1(\mathfrak{b})\simeq\mathrm{GL}_m(\boldsymbol{l}) \tag{5-6}$$

where $\boldsymbol{l}$ is the residue field of E.

5B. ***Types and cuspidal representations.*** Let us write $\mathrm{G}=\mathrm{GL}_n(\mathrm{F})$ for some $n\geq 1$. A family of pairs $(\mathbf{J},\boldsymbol{\lambda})$ called *extended maximal simple types*, made of a compact mod center, open subgroup $\mathbf{J}$ of G and an irreducible representation $\boldsymbol{\lambda}$ of $\mathbf{J}$, has been constructed in [Bushnell and Kutzko 1993] (see also [Mínguez and Sécherre 2014b] in the modular case).

Given an extended maximal simple type $(\mathbf{J},\boldsymbol{\lambda})$ in G, there are a maximal simple stratum $[\mathfrak{a},\beta]$ in $\mathrm{M}_n(\mathrm{F})$ and a maximal simple character $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ such that $\mathbf{J}(\mathfrak{a},\beta)=\mathbf{J}$ and θ is contained in the restriction of $\boldsymbol{\lambda}$ to $\mathrm{H}^1(\mathfrak{a},\beta)$. Such a simple character is said to be *attached to* $\boldsymbol{\lambda}$. By [Bushnell and Kutzko 1993, Proposition 5.1.1] (or [Mínguez and Sécherre 2014b, Proposition 2.1] in the modular case), the group $\mathrm{J}^1(\mathfrak{a},\beta)$ carries, up to isomorphism, a unique irreducible representation η whose restriction to $\mathrm{H}^1(\mathfrak{a},\beta)$ contains θ. It is called the *Heisenberg representation* associated to θ and has the following properties:

(1) The restriction of η to $\mathrm{H}^1(\mathfrak{a},\beta)$ is made of $(\mathrm{J}^1(\mathfrak{a},\beta):\mathrm{H}^1(\mathfrak{a},\beta))^{1/2}$ copies of θ.

(2) The representation η extends to $\mathbf{J}$.

For any representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, there is, up to isomorphism, a unique irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on $\mathrm{J}^1(\mathfrak{a},\beta)$ such that $\boldsymbol{\lambda}\simeq\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Through (5-6), the restriction of $\boldsymbol{\rho}$ to the maximal compact subgroup $\mathrm{J}=\mathrm{J}(\mathfrak{a},\beta)$ identifies with a cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$.

Remark 5.2. The reader familiar with the theory of simple types will have noticed that we did not introduce the notion of beta-extension. Since $\mathrm{GL}_m(\boldsymbol{l})$ is not isomorphic to $\mathrm{GL}_2(\mathbb{F}_2)$ (as p is not 2), any character of $\mathrm{GL}_m(\boldsymbol{l})$ factors through the determinant. It follows that, if $[\mathfrak{a}, \beta]$ is a maximal simple stratum, any representation of J extending η is a beta-extension.

We have the following additional property, which follows from [Mínguez and Sécherre 2014b, Lemme 2.6].

Proposition 5.3. *Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η, and write J^1 for the maximal normal pro-p-subgroup of $\mathbf{J}$. The map*

$$\boldsymbol{\xi} \mapsto \boldsymbol{\kappa} \otimes \boldsymbol{\xi}$$

induces a bijection between isomorphism classes of irreducible representations $\boldsymbol{\xi}$ of $\mathbf{J}$ trivial on J^1 and isomorphism classes of irreducible representations of $\mathbf{J}$ whose restriction to J^1 contains η.

We now give the classification of cuspidal irreducible representations of G in terms of extended maximal simple types (see [Bushnell and Kutzko 1993, 6.2, 8.4] and [Mínguez and Sécherre 2014b, Section 3] in the modular case).

Proposition 5.4 [Bushnell and Kutzko 1993; Mínguez and Sécherre 2014b]. *Let π be a cuspidal representation of* G.

(1) *There is an extended maximal simple type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\boldsymbol{\lambda}$ occurs as a subrepresentation of the restriction of π to $\mathbf{J}$. It is uniquely determined up to* G-*conjugacy.*

(2) *Compact induction defines a bijection between the* G-*conjugacy classes of extended maximal simple types and the isomorphism classes of cuspidal representations of* G.

From now on, we will abbreviate *extended maximal simple type* to *type*.

5C. *Supercuspidal representations.* Let π be a cuspidal representation of G. By Proposition 5.4, it contains a type $(\mathbf{J}, \boldsymbol{\lambda})$. Fix an irreducible representation $\boldsymbol{\kappa}$ as in Proposition 5.3 and let $\boldsymbol{\rho}$ be the corresponding representation of $\mathbf{J}$ trivial on its maximal normal pro-p-subgroup J^1.

Fix a maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$. Write $\mathrm{E} = \mathrm{F}[\beta]$ and let ρ be the cuspidal representation of $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ induced by $\boldsymbol{\rho}$. We record the following fact.

Fact 5.5 [Mínguez and Sécherre 2014a, Proposition 6.10]. *The representation π is supercuspidal if and only if ρ is supercuspidal.*

Now suppose that π is supercuspidal, thus ρ is also supercuspidal. We show how to parametrize $\boldsymbol{\rho}$ by an "admissible pair of level zero". This will be needed in Sections 8 and 10.

First, let $\boldsymbol{t}$ be an extension of degree m of $\boldsymbol{l}$, and identify $\boldsymbol{t}^\times$ with a maximal torus in $\mathrm{GL}_m(\boldsymbol{l})$. We have the correspondence (2-2) between $\boldsymbol{l}$-regular characters of $\boldsymbol{t}^\times$ and isomorphism classes of supercuspidal irreducible representations of $\mathrm{GL}_m(\boldsymbol{l})$.

Definition 5.6 [Howe 1977; Bushnell and Henniart 2005a]. An *admissible pair of level zero* over E is a pair $(\mathrm{K}/\mathrm{E}, \xi)$ made of a finite unramified extension K of E and a tamely ramified character $\xi : \mathrm{K}^\times \to \mathrm{R}^\times$ which does not factor through $\mathrm{N}_{\mathrm{K}/\mathrm{L}}$ for any field L such that $\mathrm{E} \subseteq \mathrm{L} \subsetneq \mathrm{K}$. Its *degree* is $[\mathrm{K} : \mathrm{E}]$.

If $(\mathrm{K}'/\mathrm{E}, \xi')$ is another admissible pair of level zero over E, it is said to be *isomorphic* to $(\mathrm{K}/\mathrm{E}, \xi)$ if there is an isomorphism of E-algebras $\varphi : \mathrm{K}' \to \mathrm{K}$ such that $\xi' = \xi \circ \varphi$.

Recall (see Paragraph 5A) that, if we write $\mathrm{B}^\times$ for the centralizer of E in G, then $\mathbf{J} = (\mathbf{J} \cap \mathrm{B}^\times)\mathrm{J}^1$, thus the group $\mathbf{J}/\mathrm{J}^1$ is isomorphic to $(\mathbf{J} \cap \mathrm{B}^\times)/(\mathrm{J}^1 \cap \mathrm{B}^\times)$. In particular, the image of $\mathrm{E}^\times$ in $\mathbf{J}/\mathrm{J}^1$ is central. Since $\boldsymbol{\rho}$ is trivial on J^1, the automorphism $\boldsymbol{\rho}(x)$ is thus a scalar for all $x \in \mathrm{E}^\times$.

Definition 5.7. An admissible pair $(\mathrm{K}/\mathrm{E}, \xi)$ of level zero and degree m is *attached to* $\boldsymbol{\rho}$ if:

(1) Writing $\boldsymbol{t}$ for the residue field of K, the $\boldsymbol{l}$-regular character of $\boldsymbol{t}^\times$ induced by the restriction of ξ to the units of the ring of integers of K corresponds to ρ via (2-2).

(2) One has $\boldsymbol{\rho}(x) = \xi(x) \cdot \mathrm{id}$ for all $x \in \mathrm{E}^\times$, where id is the identity on the space of $\boldsymbol{\rho}$.

The following proposition is a refinement of the property of the map (2-2).

Proposition 5.8. *The attachment relation defines a bijection*

$$(\mathrm{K}/\mathrm{E}, \xi) \mapsto \boldsymbol{\rho}(\mathrm{K}/\mathrm{E}, \xi) \tag{5-7}$$

between isomorphism classes of admissible pairs of level zero over E *and isomorphism classes of irreducible representations of* $\mathbf{J}$ *trivial on* J^1 *whose restriction to* J *defines a supercuspidal representation of* $\mathrm{GL}_m(\boldsymbol{l})$ *through* (5-6).

Remark 5.9. As in Remark 2.2, let us fix an embedding $\iota : \mathrm{R} \to \mathrm{R}'$ of algebraically closed fields of characteristic ℓ, and let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level zero over E such that ξ takes values in R. Then $(\mathrm{K}/\mathrm{E}, \iota \circ \xi)$ is an admissible pair of level zero over E, and

$$\boldsymbol{\rho}(\mathrm{K}/\mathrm{E}, \iota \circ \xi) = \boldsymbol{\rho}(\mathrm{K}/\mathrm{E}, \xi) \otimes \mathrm{R}'.$$

This refines the last assertion of Remark 2.2.

5D. ***The σ-selfdual type theorem.*** Let us fix an integer $n \geq 1$ and write $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$. We recall the first main result of [Anandavardhanan et al. 2018].

Theorem 5.10 [Anandavardhanan et al. 2018, Theorem 4.1]. *Let* π *be a cuspidal representation of* G. *It is* σ*-selfdual if and only if it contains a type* $(\mathbf{J}, \boldsymbol{\lambda})$ *such that* $\mathbf{J}$ *is* σ*-stable and* $\boldsymbol{\lambda}^\sigma \simeq \boldsymbol{\lambda}^\vee$.

Remark 5.11. More precisely (see [Anandavardhanan et al. 2018, Corollary 4.21]), any σ-selfdual cuspidal representation contain a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ with the additional property that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ for some maximal simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_n(\mathrm{F})$ such that:

(1) The hereditary order $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$.

(2) The element β has the block diagonal form

$$\beta = \begin{pmatrix} \beta_0 & & \\ & \ddots & \\ & & \beta_0 \end{pmatrix} = \beta_0 \otimes 1 \in \mathrm{M}_d(\mathrm{F}) \otimes_{\mathrm{F}} \mathrm{M}_m(\mathrm{F}) = \mathrm{M}_n(\mathrm{F})$$

for some $\beta_0 \in \mathrm{M}_d(\mathrm{F})$, where d is the degree of β over F and $n = md$; the centralizer B of $\mathrm{E} = \mathrm{F}[\beta]$ in $\mathrm{M}_n(\mathrm{F})$ thus identifies with $\mathrm{M}_m(\mathrm{E})$, equipped with the involution σ acting componentwise.

(3) The order $\mathfrak{b} = \mathfrak{a} \cap \mathrm{B}$ is the standard maximal order of $\mathrm{M}_m(\mathrm{E})$.

Such a type will be useful in the discussion following Proposition 5.17.

Remark 5.12. If $(\mathbf{J}, \boldsymbol{\lambda})$ is any σ-selfdual type, then there is a maximal simple stratum $[\mathfrak{a}, \beta]$ in $\mathrm{M}_n(\mathrm{F})$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$, the order $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$ (see [Anandavardhanan et al. 2018, Corollary 4.24]). Such a maximal simple stratum will be said to be *σ-selfdual*.

Remark 5.13. Let π be a σ-selfdual cuspidal representation of G. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π, let $[\mathfrak{a}, \beta]$ be a simple stratum such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ and let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be the maximal simple character attached to $\boldsymbol{\lambda}$. Then $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$.

Let π be a σ-selfdual cuspidal representation of G. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π and fix a σ-selfdual simple stratum $[\mathfrak{a}, \beta]$ as in Remark 5.12. Then $\mathrm{E} = \mathrm{F}[\beta]$ is σ-stable. We denote by E_0 the field of σ-fixed points in E, by T the maximal tamely ramified subextension of E over F and by T_0 the intersection $\mathrm{T} \cap \mathrm{E}_0$. Also write $d = [\mathrm{E} : \mathrm{F}]$ and $n = md$.

Proposition 5.14 [Anandavardhanan et al. 2018, Proposition 4.30]. *The integer*

$$m(\pi) = m = n/d \tag{5-8}$$

and the F_0*-isomorphism class of the quadratic extension* T/T_0 *only depend on* π, *and not on the choice of the σ-selfdual simple stratum* $[\mathfrak{a}, \beta]$ *as in Remark 5.12.*

The integer m defined by (5-8) it called the *relative degree* of π. We record a list of properties of the field extension T/F.

Lemma 5.15. (1) *The canonical homomorphism of* $\mathrm{T}_0 \otimes_{\mathrm{F}_0} \mathrm{F}$*-modules*

$$\mathrm{T}_0 \otimes_{\mathrm{F}_0} \mathrm{F} \to \mathrm{T}$$

is an isomorphism.

(2) *If* F/F_0 *is unramified*, *then* T/T_0 *is unramified and* T/F *has odd residual degree.*

(3) *The extension* T/T_0 *is ramified if and only if* F/F_0 *is ramified and* $\mathrm{T}_0/\mathrm{F}_0$ *has odd ramification order.*

Proof. Assertion (1) is [Anandavardhanan et al. 2018, Lemma 4.10]. We now prove (2) and (3).

First, suppose that F/F_0 is unramified. We remark that

$$f(T/F) \cdot f(F/F_0) = f(T/T_0) \cdot f(T_0/F_0)$$

is even. Since F does not embed in T_0 as an F_0-algebra, T_0 has odd residue degree over F_0. It follows that $f(T/T_0) = 2$ and that T has odd residue degree over F.

Suppose F/F_0 is ramified, and let ϖ be a uniformizer of F such that $\varpi_0 = \varpi^2$ is a uniformizer of F_0. Let e_0 be the ramification order of T_0/F_0, and let t_0 be a uniformizer of T_0 such that

$$\varpi_0 = t_0^{e_0} \zeta_0$$

for some root of unity $\zeta_0 \in F_0^\times$ of order prime to p. Let a be the greatest integer smaller than or equal to $e_0/2$, and write $x = \varpi t_0^{-a}$. We have $\sigma(x) = -x$, thus $x \notin T_0$ and $x^2 \in T_0$.

If e_0 is odd, then $x^2 = \zeta_0 t_0$ is a uniformizer of T_0, whereas x is a uniformizer of T, thus T is ramified over T_0.

If e_0 is even, then $x^2 = \zeta_0$. It follows that x is a root of unity of order prime to p which is in T but not in T_0, thus T is unramified over T_0. □

Remark 5.16. (1) The extensions E/E_0 and T/T_0 have the same ramification order.

(2) The extension E/E_0 is ramified if and only if F/F_0 is ramified and E_0/F_0 has odd ramification order.

The first property comes from [Anandavardhanan et al. 2018, Remark 4.22], and the second one follows from the first one together with Lemma 5.15.

We now recall the classification of all σ-selfdual types contained in a given σ-selfdual cuspidal representation of G (see [Anandavardhanan et al. 2018, Proposition 4.31]).

Proposition 5.17. *Let π be a σ-selfdual cuspidal representation of* G, *and let* T/T_0 *denote the quadratic extension associated to it.*

(1) *If* T *is unramified over* T_0, *the σ-selfdual types contained in π form a single G^σ-conjugacy class.*

(2) *If* T *is ramified over* T_0, *the σ-selfdual types contained in π form exactly* $\lfloor m/2 \rfloor + 1$ *different G^σ-conjugacy classes.*

One can give a more precise description in the ramified case. Suppose that T is ramified over T_0, and let $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$ be a σ-selfdual type in π satisfying the conditions of Remark 5.11. Let us fix a uniformizer t of E. For $i = 0, \dots, \lfloor m/2 \rfloor$, let t_i denote the diagonal matrix

$$\mathrm{diag}(t, \dots, t, 1, \dots, 1) \in B^\times = GL_m(E)$$

where t occurs i times. Then the pairs $(\mathbf{J}_i, \boldsymbol{\lambda}_i) = (\mathbf{J}_0^{t_i}, \boldsymbol{\lambda}_0^{t_i})$, for $i = 0, \dots, \lfloor m/2 \rfloor$, form a set of representatives of the G^σ-conjugacy classes of σ-selfdual types in π.

Definition 5.18. The integer i is called the *index* of the G^σ-conjugacy class of $(\mathbf{J}_i, \boldsymbol{\lambda}_i)$. It does not depend on the choice of $(\mathbf{J}_0, \boldsymbol{\lambda}_0)$, nor on that of t.

Let $[\mathfrak{a}, \beta]$ be a simple stratum as in Remark 5.11 such that $\mathbf{J}_0 = \mathbf{J}(\mathfrak{a}, \beta)$. If one identifies the quotient $\mathrm{J}(\mathfrak{a}, \beta)^{t_i}/\mathrm{J}^1(\mathfrak{a}, \beta)^{t_i}$ with $\mathrm{GL}_m(\boldsymbol{l})$ via

$$\mathrm{J}(\mathfrak{a}, \beta)^{t_i}/\mathrm{J}^1(\mathfrak{a}, \beta)^{t_i} \simeq \mathrm{J}(\mathfrak{a}, \beta)/\mathrm{J}^1(\mathfrak{a}, \beta) \simeq \mathrm{U}(\mathfrak{b})/\mathrm{U}^1(\mathfrak{b}) \simeq \mathrm{GL}_m(\boldsymbol{l})$$

then σ acts on $\mathrm{GL}_m(\boldsymbol{l})$ by conjugacy by the diagonal element

$$\delta_i = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1) \in \mathrm{GL}_m(\boldsymbol{l})$$

where -1 occurs i times, and $(\mathrm{J}(\mathfrak{a}, \beta)^{t_i} \cap \mathrm{G}^\sigma)/(\mathrm{J}^1(\mathfrak{a}, \beta)^{t_i} \cap \mathrm{G}^\sigma)$ identifies with the Levi subgroup $(\mathrm{GL}_i \times \mathrm{GL}_{m-i})(\boldsymbol{l})$ of $\mathrm{GL}_m(\boldsymbol{l})$.

5E. *Admissible pairs and σ-selfduality.* Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in G. Fix a σ-selfdual maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ as in Remark 5.12, and a decomposition of $\boldsymbol{\lambda}$ of the form $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ as in Paragraph 5B. Write $\mathrm{E} = \mathrm{F}[\beta]$ as usual.

Proposition 5.19. *Suppose that the representation $\boldsymbol{\rho}$ is σ-selfdual, and let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level zero attached to it. There is a unique involutive E_0-automorphism $\hat{\sigma}$ of K such that $\xi \circ \hat{\sigma} = \xi^{-1}$ and $\hat{\sigma}$ coincides with σ on E.*

Proof. Let K' denote the extension of E given by the field K equipped with the map $x \mapsto \sigma(x)$ from E to K. Then the pair $(\mathrm{K}'/\mathrm{E}, \xi)$ is admissible of level zero, and it is attached to $\boldsymbol{\rho}^\sigma$. On the other hand, $(\mathrm{K}/\mathrm{E}, \xi^{-1})$ is admissible of level zero, attached to $\boldsymbol{\rho}^\vee$. Since $\boldsymbol{\rho}$ is σ-selfdual, there is an E-algebra isomorphism $\hat{\sigma} : \mathrm{K} \to \mathrm{K}'$ such that $\xi \circ \hat{\sigma} = \xi^{-1}$. We thus have

$$\xi \circ \hat{\sigma}^2 = \xi^{-1} \circ \hat{\sigma} = \xi$$

and $\hat{\sigma}^2$ is an E-algebra automorphism of K. By admissibility of $(\mathrm{K}/\mathrm{E}, \xi)$, the latter automorphism is trivial, thus $\hat{\sigma}$ satisfies the required conditions. Uniqueness follows by admissibility again. □

For simplicity, we will write σ for the involutive automorphism given by Proposition 5.19. Let K_0 be the field of σ-fixed points of K. The following lemma will be useful in Section 10.

Lemma 5.20. *If E/E_0 is ramified and m is even, then K/K_0 is unramified.*

Proof. Write $m = 2r$ for some $r \geq 1$. Let t be a uniformizer of E such that $\sigma(t) = -t$ and let $\zeta \in \mathrm{K}$ be a root of unity of order $c^m - 1$, where c is the cardinality of $\boldsymbol{l}$. We thus have $\mathrm{E} = \mathrm{E}_0[t]$ and $\mathrm{K} = \mathrm{E}[\zeta]$. Since σ is involutive, it induces an involutive $\boldsymbol{l}$-automorphism of $\boldsymbol{t}$, the residual field of K. If the latter were trivial, the relation $\xi \circ \sigma = \xi^{-1}$ would imply that the character $\bar{\xi}$ of $\boldsymbol{t}^\times$ induced by ξ is quadratic, contradicting the fact that it is $\boldsymbol{l}$-regular. The automorphism of $\boldsymbol{t}$ induced by σ is thus the r-th power of the Frobenius automorphism. Now consider the element

$$\alpha = \zeta^{(c^r+1)/2}.$$

It has order $2(c^r - 1)$, thus $\sigma(\alpha) = -\alpha$. Since α^2 has order $c^r - 1$, the extension of E_0 it generates is unramified and has degree r. We thus have $\mathrm{E}_0[\alpha^2, t\alpha] \subseteq \mathrm{K}_0$ and their degrees are equal. Now we deduce that $\mathrm{K} = \mathrm{K}_0[\alpha] = \mathrm{K}_0[\zeta]$ is unramified over K_0. □

5F. The following lemma will be useful in Sections 7 and 9, when we investigate decompositions of σ-selfdual types of the form $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$ which behave well under σ.

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a maximal simple character such that $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$. Let $\mathbf{J}$ be its normalizer in G, let J^1 be the maximal normal pro-p-subgroup of $\mathbf{J}$ and η be the irreducible representation of J^1 containing θ.

Lemma 5.21. *Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η. There is a unique character μ of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$. It satisfies the identity $\mu \circ \sigma = \mu$.*

Proof. Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η. By uniqueness of the Heisenberg representation, the fact that $\theta \circ \sigma = \theta^{-1}$ implies that $\eta^{\sigma\vee}$ is isomorphic to η. Thus $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^{\sigma\vee}$ are representations of $\mathbf{J}$ extending η. There is a character μ of $\mathbf{J}$ trivial on J^1 such that we have $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$. It satisfies $\mu \circ \sigma = \mu$. It is unique by Proposition 5.3. □

6. The distinguished type theorem

In this section we prove the following result, which is our first main theorem. It will be refined by Theorem 10.3 in Section 10. Recall that $p \neq 2$ until the end of this article.

Theorem 6.1. *Let π be a σ-selfdual cuspidal representation of* G. *Then π is distinguished if and only if it contains a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\mathrm{Hom}_{\mathbf{J}\cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$ is nonzero.*

Remark 6.2. If π is distinguished, it follows easily from the multiplicity 1 property in Theorem 4.1 that the distinguished σ-selfdual types $(\mathbf{J}, \boldsymbol{\lambda})$ occurring in π form a single G^σ-conjugacy class (see Remark 6.23).

Remark 6.3. Theorem 6.1 is proved in [Anandavardhanan et al. 2018] in a different manner than the one we give here, although both proofs use the σ-selfdual type Theorem 5.10. The proof given in [Anandavardhanan et al. 2018] is based on a result of Ok [1997], proved by Ok for complex representations and extended to the modular case in [Anandavardhanan et al. 2018, Appendix B]. However, the proof we give here is more likely to generalize to other situations.

Let π be a σ-selfdual cuspidal representation. Theorem 5.10 tells us that it contains a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$, and Proposition 5.4 tells us that π is compactly induced from $\boldsymbol{\lambda}$. A simple application of the Mackey formula gives us

$$\mathrm{Hom}_{\mathrm{G}^\sigma}(\pi, 1) \simeq \prod_{g} \mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1) \tag{6-1}$$

where g ranges over a set of representatives of $(\mathbf{J}, \mathrm{G}^\sigma)$-double cosets in G.

Remark 6.4. It follows from Theorem 4.1 that there is at most one double coset $\mathbf{J}g\mathrm{G}^\sigma$ such that the space $\mathrm{Hom}_{\mathbf{J}^g \cap \mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ is nonzero, and that this space has dimension at most 1. Thus the product in (6-1) is actually a direct sum.

In this section, our main task (achieved in Paragraph 6E) is to prove that, if $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ is nonzero, then $\sigma(g)g^{-1}\in\mathbf{J}$. Theorem 6.1 will follow easily from there (see Paragraph 6F).

We may assume that $\mathbf{J}=\mathbf{J}(\mathfrak{a},\beta)$ for a maximal simple stratum $[\mathfrak{a},\beta]$ satisfying the conditions of Remark 5.11. The extension $\mathrm{E}=\mathrm{F}[\beta]$, its centralizer B and the maximal order $\mathfrak{b}=\mathfrak{a}\cap\mathrm{B}$ are thus stable by σ. We write $d=[\mathrm{E}:\mathrm{F}]$ and $n=md$. We identify B with the E-algebra $\mathrm{M}_m(\mathrm{E})$ equipped with the involution σ acting componentwise, and $\mathfrak{b}$ with its standard maximal order.

We write $\mathrm{E}_0=\mathrm{E}^\sigma$, the field of σ-invariant elements of E, and fix once and for all a uniformizer t of E such that

$$\sigma(t)=\begin{cases} t & \text{if E is unramified over } \mathrm{E}_0,\\ -t & \text{if E is ramified over } \mathrm{E}_0.\end{cases} \tag{6-2}$$

We also write $\mathrm{J}=\mathrm{J}(\mathfrak{a},\beta)$, $\mathrm{J}^1=\mathrm{J}^1(\mathfrak{a},\beta)$ and $\mathrm{H}^1=\mathrm{H}^1(\mathfrak{a},\beta)$. Recall that $\mathbf{J}=\mathrm{E}^\times\mathrm{J}$.

We denote by T the maximal tamely ramified subextension of E over F, and set $\mathrm{T}_0=\mathrm{T}\cap\mathrm{E}_0$.

We insist on the fact that, throughout this section, we assume that the stratum $[\mathfrak{a},\beta]$ *satisfies the conditions of Remark 5.11.*

6A. ***Double cosets contributing to the distinction of $\boldsymbol{\theta}$.*** Let $\theta\in\mathcal{C}(\mathfrak{a},\beta)$ be the maximal simple character occurring in the restriction of $\boldsymbol{\lambda}$ to H^1. Suppose that $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g,1)$ is nonzero for some double coset $\mathbf{J}g\mathrm{G}^\sigma$. Restricting to $\mathrm{H}^{1g}\cap\mathrm{G}^\sigma$, we deduce that the character θ^g is trivial on $\mathrm{H}^{1g}\cap\mathrm{G}^\sigma$.

In this paragraph, we look for the double cosets $\mathbf{J}g\mathrm{G}^\sigma\subseteq\mathrm{G}$ such that the character θ^g is trivial on $\mathrm{H}^{1g}\cap\mathrm{G}^\sigma$. For this, let us introduce the following general lemma.

Lemma 6.5. *Let τ be an involution of* G, *let* H *be a τ-stable open pro-p-subgroup of* G *and let χ be a character of* H *such that $\chi\circ\tau=\chi^{-1}$. For any $g\in\mathrm{G}$, the character χ^g is trivial on $\mathrm{H}^g\cap\mathrm{G}^\tau$ if and only if $\tau(g)g^{-1}$ intertwines χ.*

Proof. Write K for the τ-stable subgroup $\mathrm{H}^g\cap\tau(\mathrm{H}^g)$, which contains $\mathrm{H}^g\cap\mathrm{G}^\tau$. Let A be the quotient of K by $\overline{[\mathrm{K},\mathrm{K}]}$, the closure of the derived subgroup of K. This is a τ-stable commutative pro-p-group. Given $x\in\mathrm{K}$, write x' for its image in A. For any $b\in\mathrm{A}$, we have

$$b=\sqrt{b\tau(b)}\cdot\sqrt{b\tau(b)^{-1}}$$

where $b\mapsto\sqrt{b}$ is the inverse of the automorphism $b\mapsto b^2$ of A. Thus, for any $x\in\mathrm{K}$, there are $y,z\in\mathrm{K}$ such that $x=yz$ and $\tau(y')=y'$ and $\tau(z')=z'^{-1}$.

Since $\tau(z)=z^{-1}h$ for some $h\in\overline{[\mathrm{K},\mathrm{K}]}$, we have

$$\chi^g(\tau(z))=\chi^g(z^{-1}h)=\chi^g(z)^{-1}. \tag{6-3}$$

On the other hand, since $\tau(y)=yk$ for some $k\in\overline{[\mathrm{K},\mathrm{K}]}$, the element $y^{-1}\tau(y)$ defines a 1-cocycle in the τ-stable pro-p-group $\overline{[\mathrm{K},\mathrm{K}]}$. Since $p\neq 2$, this cocycle is a coboundary, which implies

$$y\in(\mathrm{H}^g\cap\mathrm{G}^\tau)\overline{[\mathrm{K},\mathrm{K}]}. \tag{6-4}$$

Now suppose that χ^g is trivial on $\mathrm{H}^g \cap \mathrm{G}^\tau$. Then (6-3) and (6-4) imply that

$$\chi^g(\tau(x)) = \chi^g(\tau(z)) = \chi^g(z)^{-1} = \chi^g(x)^{-1}, \quad \text{for all } x \in \mathrm{K}. \tag{6-5}$$

Besides, (6-5) is *equivalent* to χ^g being trivial on $\mathrm{H}^g \cap \mathrm{G}^\tau$. On the other hand, we have

$$\chi^g \circ \tau = (\chi \circ \tau)^{\tau(g)} = (\chi^{-1})^{\tau(g)} = (\chi^{\tau(g)})^{-1} \tag{6-6}$$

on K by assumption on χ. If we set $\gamma = \tau(g)g^{-1}$, then (6-5) is equivalent to

$$\chi(h) = \chi^\gamma(h) \quad \text{for all } h \in \mathrm{H} \cap \gamma^{-1}\mathrm{H}\gamma.$$

This amounts to saying that γ intertwines χ. □

Proposition 6.6. *Let $g \in \mathrm{G}$. Then the character θ^g is trivial on $\mathrm{H}^{1g} \cap \mathrm{G}^\sigma$ if and only if we have $\sigma(g)g^{-1} \in \mathrm{JB}^\times\mathrm{J}$.*

Proof. This follows from Lemma 6.5 applied to the simple character θ of H^1 and the involution σ, together with the fact that the intertwining set of θ is $\mathrm{JB}^\times\mathrm{J}$ by Proposition 5.1(5). □

6B. *The double coset lemma.* We now prove the following fundamental lemma.

Lemma 6.7. *Let $g \in \mathrm{G}$. Then $\sigma(g)g^{-1} \in \mathrm{JB}^\times\mathrm{J}$ if and only if $g \in \mathrm{JB}^\times\mathrm{G}^\sigma$.*

Proof. Write $\gamma = \sigma(g)g^{-1}$. If $g \in \mathrm{JB}^\times\mathrm{G}^\sigma$, one verifies immediately that $\gamma \in \mathrm{JB}^\times\mathrm{J}$. Conversely, suppose that $\gamma \in \mathrm{J}c\mathrm{J}$ for some $c \in \mathrm{B}^\times$. We will first show that the double coset representative c can be chosen nicely.

Lemma 6.8. *There is a $b \in \mathrm{B}^\times$ such that $\gamma \in \mathrm{J}b\mathrm{J}$ and $b\sigma(b) = 1$.*

Proof. Recall that $\mathrm{B}^\times$ has been identified with $\mathrm{GL}_m(\mathrm{E})$ and $\mathrm{U} = \mathrm{J} \cap \mathrm{B}^\times = \mathfrak{b}^\times$ is its standard maximal compact subgroup. By the Cartan decomposition, $\mathrm{B}^\times$ decomposes as the disjoint union of the double cosets

$$\mathrm{U} \cdot \mathrm{diag}(t^{a_1}, \dots, t^{a_m}) \cdot \mathrm{U}$$

where $a_1 \geq \cdots \geq a_m$ ranges over nonincreasing sequences of m integers, and $\mathrm{diag}(\lambda_1, \dots, \lambda_m)$ denotes the diagonal matrix of $\mathrm{B}^\times$ with eigenvalues $\lambda_1, \dots, \lambda_m \in \mathrm{E}^\times$. We thus may assume that $c = \mathrm{diag}(t^{a_1}, \dots, t^{a_m})$ for a uniquely determined sequence of integers $a_1 \geq \cdots \geq a_m$.

The fact that $\sigma(\gamma) = \gamma^{-1}$ implies that we have $c \in \mathrm{J}c^{-1}\mathrm{J} \cap \mathrm{B}^\times$. Using the simple intersection property (5-1) together with the fact that $\mathrm{J} = \mathrm{UJ}^1$ and $\mathrm{J}^1 \subseteq \mathrm{U}^1(\mathfrak{a})$, we have $\mathrm{J}c^{-1}\mathrm{J} \cap \mathrm{B}^\times = \mathrm{U}c^{-1}\mathrm{U}$. The uniqueness of the Cartan decomposition of $\mathrm{B}^\times$ thus implies that the sequences $a_1 \geq \cdots \geq a_m$ and $-a_m \geq \cdots \geq -a_1$ are equal. We thus have $a_i + a_{m+1-i} = 0$ for all $i \in \{1, \dots, m\}$. Now write $\kappa = \sigma(t)t^{-1} \in \{-1, 1\}$ and choose signs $\kappa_1, \dots, \kappa_m \in \{-1, 1\}$ such that $\kappa_i\kappa_{m+1-i} = \kappa^{a_i}$ for all i. This is always possible since $a_{(m+1)/2} = 0$ when m is odd. Then the antidiagonal element

$$b = \begin{pmatrix} & & \kappa_1 t^{a_1} \\ & \cdot^{\cdot^{\cdot}} & \\ \kappa_m t^{a_m} & & \end{pmatrix} \in \mathrm{B}^\times \tag{6-7}$$

satisfies the required conditions $b\sigma(b) = 1$ and $\gamma \in \mathrm{J}b\mathrm{J}$. □

Now write $\gamma = x'bx$ with $x, x' \in \mathrm{J}$ and $b \in \mathrm{B}^\times$. Replacing g by $\sigma(x')^{-1}g$ does not change the double coset $\mathrm{J}g\mathrm{G}^\sigma$ but changes γ into $bx\sigma(x')$. From now on, we will thus assume that

$$\gamma = bx, \quad b\sigma(b) = 1, \quad x \in \mathrm{J}, \quad b \text{ is of the form (6-7).} \tag{6-8}$$

Write K for the group $\mathrm{J} \cap b^{-1}\mathrm{J}b$. Since $\sigma(b) = b^{-1}$ and J is σ-stable, we have $x \in \mathrm{K}$.

Lemma 6.9. *The map $\delta : k \mapsto b^{-1}\sigma(k)b$ is an involutive group automorphism of* K.

Proof. This follows from an easy calculation using the fact that $b\sigma(b) = 1$. □

Let $b_1 > \cdots > b_r$ be the unique decreasing sequence of integers such that

$$\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_r\}$$

and m_j denote the multiplicity of b_j in $(a_1, \ldots, a_m)$, for $j \in \{1, \ldots, r\}$. We have $m_j = m_{r+1-j}$ for all j, and $m_1 + \cdots + m_r = m$. These integers define a standard Levi subgroup

$$\mathrm{M} = \mathrm{GL}_{m_1 d}(\mathrm{F}) \times \cdots \times \mathrm{GL}_{m_r d}(\mathrm{F}) \subseteq \mathrm{G}. \tag{6-9}$$

Write P for the standard parabolic subgroup of G generated by M and upper triangular matrices. Let N and N^- denote the unipotent radicals of P and its opposite parabolic subgroup with respect to M, respectively. Since b has the form (6-7), it normalizes M and we have

$$\mathrm{K} = (\mathrm{K} \cap \mathrm{N}^-) \cdot (\mathrm{K} \cap \mathrm{M}) \cdot (\mathrm{K} \cap \mathrm{N}), \quad \mathrm{K} \cap \mathrm{P} = \mathrm{J} \cap \mathrm{P}, \quad \mathrm{K} \cap \mathrm{N}^- \subseteq \mathrm{J}^1 \cap \mathrm{N}^-.$$

We have similar properties for the subgroup $\mathrm{V} = \mathrm{K} \cap \mathrm{B}^\times = \mathrm{U} \cap b^{-1}\mathrm{U}b$ of $\mathrm{B}^\times$, that is

$$\mathrm{V} = (\mathrm{V} \cap \mathrm{N}^-) \cdot (\mathrm{V} \cap \mathrm{M}) \cdot (\mathrm{V} \cap \mathrm{N}), \quad \mathrm{V} \cap \mathrm{P} = \mathrm{U} \cap \mathrm{P}, \quad \mathrm{V} \cap \mathrm{N}^- \subseteq \mathrm{U}^1 \cap \mathrm{N}^-,$$

where $\mathrm{U}^1 = \mathrm{J}^1 \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b})$. Note that this subgroup V is stable by δ.

Lemma 6.10. *The subset*

$$\mathrm{K}^1 = (\mathrm{K} \cap \mathrm{N}^-) \cdot (\mathrm{J}^1 \cap \mathrm{M}) \cdot (\mathrm{K} \cap \mathrm{N})$$

is a δ-stable normal pro-p-subgroup of K, *and we have* $\mathrm{K} = \mathrm{V}\mathrm{K}^1$.

Proof. To prove that K^1 is a subgroup of K, it is enough to prove that one has the containment $(\mathrm{K} \cap \mathrm{N}) \cdot (\mathrm{K} \cap \mathrm{N}^-) \subseteq \mathrm{K}^1$. Let $\mathfrak{j}^1$ be the sub-$\mathcal{O}$-lattice of $\mathfrak{a}$ such that $\mathrm{J}^1 = 1 + \mathfrak{j}^1$ and let $\mathfrak{j} = \mathfrak{b} + \mathfrak{j}^1$, thus $\mathrm{J} = \mathfrak{j}^\times$. A simple computation shows that $\mathrm{K} \cap \mathrm{N}^- \subseteq (1 + t\mathfrak{j}) \cap \mathrm{N}^-$ and

$$(\mathrm{K} \cap \mathrm{N}) \cdot (\mathrm{K} \cap \mathrm{N}^-) \subseteq (\mathrm{K} \cap \mathrm{N}^-) \cdot ((1 + t\mathfrak{j}) \cap \mathrm{M}) \cdot (\mathrm{K} \cap \mathrm{N}).$$

The expected result thus follows from the fact that $t\mathfrak{j} \subseteq \mathfrak{j}^1$. Besides, K^1 is a δ-stable pro-p-group.

Since $\mathrm{V} \cap \mathrm{M}$ normalizes $\mathrm{K} \cap \mathrm{N}^-$, $\mathrm{K} \cap \mathrm{N}$ and $\mathrm{K}^1 \cap \mathrm{M}$, we have $(\mathrm{V} \cap \mathrm{M})\mathrm{K}^1 = \mathrm{K}$ whence K^1 is normal in K and $\mathrm{K} = \mathrm{V}\mathrm{K}^1$, as expected. □

The subgroup K^1 is useful in the following lemma. Note that we have $x\delta(x) = 1$.

Lemma 6.11. *Let $y \in \mathrm{K}$ be such that $y\delta(y) = 1$. There are $k \in \mathrm{K}$ and $v \in \mathrm{V}$ such that*:

(1) *The element v is diagonal in $\mathrm{B}^\times$ with eigenvalues in $\{-1, 1\}$ and it satisfies $v\delta(v) = 1$.*

(2) *One has $\delta(k)yk^{-1} \in v\mathrm{K}^1$.*

Proof. Let $\mathrm{V}^1 = \mathrm{V} \cap \mathrm{K}^1 = \mathrm{K}^1 \cap \mathrm{B}^\times$. We have

$$\mathrm{V}^1 = (\mathrm{V} \cap \mathrm{N}^-) \cdot (\mathrm{U}^1 \cap \mathrm{M}) \cdot (\mathrm{U} \cap \mathrm{N}).$$

We thus have canonical δ-equivariant group isomorphisms

$$\mathrm{K}/\mathrm{K}^1 \simeq \mathrm{V}/\mathrm{V}^1 \simeq (\mathrm{U} \cap \mathrm{M})/(\mathrm{U}^1 \cap \mathrm{M}). \tag{6-10}$$

By (6-9), we have $\mathrm{M} \cap \mathrm{B}^\times = \mathrm{GL}_{m_1}(\mathrm{E}) \times \cdots \times \mathrm{GL}_{m_r}(\mathrm{E})$, thus the right-hand side of (6-10) identifies with $\mathcal{M} = \mathrm{GL}_{m_1}(\boldsymbol{l}) \times \cdots \times \mathrm{GL}_{m_r}(\boldsymbol{l})$, where $\boldsymbol{l}$ denotes the residue field of E. Besides, since b is given by (6-7), the involution δ acts on $\mathcal{M}$ as

$$(g_1, \dots, g_r) \mapsto (\sigma(g_r), \dots, \sigma(g_1)).$$

Write $y = vy'$ for some $v \in \mathrm{V}$ and $y' \in \mathrm{K}^1$. The simple intersection property (5-1) gives us

$$\delta(v)^{-1} = \delta(y')vy' \in \mathrm{V} \cap \mathrm{K}^1 v \mathrm{K}^1 = \mathrm{V}^1 v \mathrm{V}^1.$$

Thus there is $u \in \mathrm{V}^1$ such that $vu\delta(vu) \in \mathrm{V}^1$. Replacing (v, y') by $(vu, u^{-1}y')$, we may and will assume that $y = vy'$ with $v\delta(v) \in \mathrm{V}^1$.

We now compute the first cohomology set of δ in $\mathcal{M}$. Let $w = (w_1, \dots, w_r)$ denote the image of y in $\mathcal{M}$. We have $w\delta(w) = 1$, that is

$$\sigma(w_j) = w_{r+1-j}^{-1}, \qquad \text{for all } j \in \{1, \dots, r\}.$$

If r is even, one can find an element $a \in \mathcal{M}$ such that $w = \delta(a)a^{-1}$. If r is odd, say $r = 2s - 1$, one can find an element $a \in \mathcal{M}$ such that

$$\delta(a)wa^{-1} = (1, \dots, 1, w_s, 1, \dots, 1)$$

and we have $w_s\sigma(w_s) = 1$. If E/E_0 is unramified, then $\boldsymbol{l}$ is quadratic over the residue field of E_0, and it follows from the triviality of the first cohomology set of σ in $\mathrm{GL}_{m_s}(\boldsymbol{l})$ that $w = \sigma(c)c^{-1}$ for some $c \in \mathcal{M}$. In these two cases, we thus may find $k \in \mathrm{K}$ such that $\delta(k)xk^{-1} \in \mathrm{K}^1$.

It remains to treat the case where r is odd and E/E_0 is ramified. In this case, we have $w_s^2 = 1$, thus w_s is conjugate in $\mathrm{GL}_{m_s}(\boldsymbol{l})$ to a diagonal element with eigenvalues 1 and -1. Let i denote the multiplicity of -1. Let

$$v \in \mathrm{U} \cap \mathrm{M} = \mathrm{GL}_{m_1}(\mathcal{O}_\mathrm{E}) \times \cdots \times \mathrm{GL}_{m_r}(\mathcal{O}_\mathrm{E})$$

(here $\mathcal{O}_\mathrm{E}$ is the ring of integers of E) be a diagonal matrix with eigenvalues 1 and -1, such that -1 occurs with multiplicity i and only in the s-th block. Then $v\delta(v) = 1$ and there is $k \in \mathrm{K}$ such that $\delta(k)yk^{-1} \in v\mathrm{K}^1$. ☐

Applying Lemma 6.11 to x gives us $k \in \mathrm{K}$, $v \in \mathrm{V}$ such that $bv\sigma(bv) = 1$ and $\delta(k)xk^{-1} \in v\mathrm{K}^1$. Besides, bv is antidiagonal of the form (6-7) and $\sigma(k)\gamma k^{-1} \in bv\mathrm{K}^1$. Therefore, replacing g by kg, which does not change the double coset $\mathrm{J}g\mathrm{G}^\sigma$, we will assume that γ can be written

$$\gamma = bx, \quad b\sigma(b) = 1, \quad x \in \mathrm{J}^1, \quad b \text{ is of the form (6-7).} \tag{6-11}$$

Comparing with (6-8), we now have a stronger condition on x, that is $x\delta(x) = 1$ and $x \in \mathrm{K}^1$.

Since K^1 is a δ-stable pro-p-group and p is odd, the first cohomology set of δ in K^1 is trivial. Thus $x = \delta(y)y^{-1}$ for some $y \in \mathrm{K}^1$, hence $\gamma = \sigma(y)by^{-1}$. Since $b\sigma(b) = 1$ and the first cohomology set of σ in $\mathrm{B}^\times$ is trivial, one has $b = \sigma(h)h^{-1}$ for some $h \in \mathrm{B}^\times$. Thus $g \in yh\mathrm{G}^\sigma \subseteq \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$, and Lemma 6.7 is proved. □

6C. *Contribution of the Heisenberg representation.* Let η be the Heisenberg representation of J^1 associated to θ (see Paragraph 5B). In this paragraph, we prove the following result.

Proposition 6.12. *Given $g \in \mathrm{G}$, we have*

$$\dim \mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1) = \begin{cases} 1 & \text{if } g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$ is nonzero. Restricting to $\mathrm{H}^{1g} \cap \mathrm{G}^\sigma$, the character θ^g is trivial on $\mathrm{H}^{1g} \cap \mathrm{G}^\sigma$, and Proposition 6.6 together with Lemma 6.7 give us $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. Conversely, assume that $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. Since the dimension of $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$ does not change when g varies in a given $(\mathbf{J}, \mathrm{G}^\sigma)$-double coset, we may and will assume that we have $g \in \mathrm{B}^\times$. Thus we have $\gamma = \sigma(g)g^{-1} \in \mathrm{B}^\times$ as well.

Lemma 6.13. *The map $\tau : x \mapsto \gamma^{-1}\sigma(x)\gamma$ is an involutive automorphism of* G *and, for any subgroup* $\mathrm{H} \subseteq \mathrm{G}$, *we have* $\mathrm{H}^g \cap \mathrm{G}^\sigma = (\mathrm{H} \cap \mathrm{G}^\tau)^g$.

Proof. This follows from an easy calculation using the fact that $\sigma(\gamma) = \gamma^{-1}$. □

Our goal is thus to prove that the space $\mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{G}^\tau}(\eta, 1)$ has dimension 1. By Paragraph 5B, the representation of J^1 induced from θ decomposes as the direct sum of $(\mathrm{J}^1 : \mathrm{H}^1)^{1/2}$ copies of the representation η. The space

$$\mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{G}^\tau}(\mathrm{Ind}_{\mathrm{H}^1}^{\mathrm{J}^1}(\theta), 1) \tag{6-12}$$

thus decomposes as the direct sum of $(\mathrm{J}^1 : \mathrm{H}^1)^{1/2}$ copies of $\mathrm{Hom}_{\mathrm{J}^1\cap\mathrm{G}^\tau}(\eta, 1)$. Applying Frobenius reciprocity and the Mackey formula, the space (6-12) is isomorphic to

$$\mathrm{Hom}_{\mathrm{J}^1}(\mathrm{Ind}_{\mathrm{H}^1}^{\mathrm{J}^1}(\theta), \mathrm{Ind}_{\mathrm{J}^1\cap\mathrm{G}^\tau}^{\mathrm{J}^1}(1)) \simeq \bigoplus_{x\in\mathrm{X}} \mathrm{Hom}_{\mathrm{H}^1}(\theta, \mathrm{Ind}_{\mathrm{H}^1\cap(\mathrm{J}^1\cap\mathrm{G}^\tau)^x}^{\mathrm{H}^1}(1))$$

where X is equal to $\mathrm{J}^1/(\mathrm{J}^1 \cap \mathrm{G}^\tau)\mathrm{H}^1$ (recall that H^1 is normal in J^1 and $\mathrm{J}^1/\mathrm{H}^1$ is abelian). Since J^1 normalizes θ, this is isomorphic to

$$\bigoplus_{x\in\mathrm{X}} \mathrm{Hom}_{\mathrm{H}^1}(\theta, \mathrm{Ind}_{\mathrm{H}^1\cap\mathrm{G}^\tau}^{\mathrm{H}^1}(1)) \simeq \bigoplus_{x\in\mathrm{X}} \mathrm{Hom}_{\mathrm{H}^1\cap\mathrm{G}^\tau}(\theta, 1). \tag{6-13}$$

Since $\mathrm{Hom}_{\mathrm{H}^1\cap\mathrm{G}^\tau}(\theta, 1)$ has dimension 1, the right-hand side of (6-13) has dimension the cardinality of X. It thus remains to prove that X has cardinality $(\mathrm{J}^1 : \mathrm{H}^1)^{1/2}$, or equivalently

$$(\mathrm{J}^1 \cap \mathrm{G}^\tau : \mathrm{H}^1 \cap \mathrm{G}^\tau) = (\mathrm{J}^1 : \mathrm{H}^1)^{1/2}. \tag{6-14}$$

Now consider the groups $\mathrm{J}^1 \cap \mathrm{J}^{1\gamma}$ and $\mathrm{H}^1 \cap \mathrm{H}^{1\gamma}$, which are both stable by τ.

Lemma 6.14. *We have $\theta(\tau(x)) = \theta(x)^{-1}$ for all $x \in \mathrm{H}^1 \cap \mathrm{H}^{1\gamma}$.*

Proof. Given $x \in \mathrm{H}^1 \cap \mathrm{H}^{1\gamma}$, and using the fact that $\theta \circ \sigma = \theta^{-1}$ on H^1, we have

$$\theta(\tau(x))^{-1} = \theta \circ \sigma(\tau(x)) = \theta^\gamma(x) = \theta(x)$$

since $\gamma \in \mathrm{B}^\times$ intertwines θ. $\square$

Let us write $\mathbb{V}$ for the $\boldsymbol{k}$-vector space $(\mathrm{J}^1 \cap \mathrm{J}^{1\gamma})/(\mathrm{H}^1 \cap \mathrm{H}^{1\gamma})$ equipped with both the involution τ and the symplectic form $(x, y) \mapsto \langle x, y\rangle$ induced by (5-3). We write $\mathbb{V}^+ = \{v \in \mathbb{V} \mid \tau(v) = v\}$ and $\mathbb{V}^- = \{v \in \mathbb{V} \mid \tau(v) = -v\}$. We have the decomposition $\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-$ since $p \neq 2$.

Lemma 6.15. *There is a group isomorphism $\mathbb{V}^+ \simeq (\mathrm{J}^1 \cap \mathrm{G}^\tau)/(\mathrm{H}^1 \cap \mathrm{G}^\tau)$.*

Proof. First note that we have the containment $(\mathrm{J}^1 \cap \mathrm{G}^\tau)(\mathrm{H}^1 \cap \mathrm{H}^{1\gamma})/(\mathrm{H}^1 \cap \mathrm{H}^{1\gamma}) \subseteq \mathbb{V}^+$. The lemma will follow if we prove that this containment is an equality. Let $x \in \mathrm{J}^1 \cap \mathrm{J}^{1\gamma}$ be such that $x(\mathrm{H}^1 \cap \mathrm{H}^{1\gamma}) \in \mathbb{V}^+$. One thus has $x^{-1}\tau(x) \in \mathrm{H}^1 \cap \mathrm{H}^{1\gamma}$. Since $\mathrm{H}^1 \cap \mathrm{H}^{1\gamma}$ is a τ-stable pro-p-group and $p \neq 2$, there is an $h \in \mathrm{H}^1 \cap \mathrm{H}^{1\gamma}$ such that $x^{-1}\tau(x) = h^{-1}\tau(h)$. The expected result follows. $\square$

We are now going to prove that $\mathbb{V}^+$ and $\mathbb{V}^-$ have the same dimension.

Lemma 6.16. *The subspaces $\mathbb{V}^+$ and $\mathbb{V}^-$ are totally isotropic.*

Proof. Indeed, thanks to Lemma 6.14, first note that

$$\langle \tau(x), y\rangle = \langle \tau(y), x\rangle, \quad x, y \in \mathbb{V}. \tag{6-15}$$

If $x, y \in \mathbb{V}^+$, then we get $\langle x, y\rangle = \langle x, y\rangle^{-1}$, thus $\langle x, y\rangle = 1$ since $p \neq 2$. If $x, y \in \mathbb{V}^-$, then we get $\langle x^{-1}, y\rangle = \langle x, y^{-1}\rangle^{-1}$. But $\langle x^{-1}, y\rangle = \langle x, y\rangle^{-1} = \langle x, y^{-1}\rangle$. It follows again that $\langle x, y\rangle = 1$. $\square$

Let $\mathbb{W}$ denote the kernel of the symplectic form $(x, y) \mapsto \langle x, y\rangle$ on $\mathbb{V}$, that is

$$\mathbb{W} = \{w \in \mathbb{V} \mid \langle w, v\rangle = 1 \text{ for all } v \in \mathbb{V}\}.$$

Let $\mathbb{Y}$ and $\mathbb{Y}'$ denote the images of $\mathrm{H}^1 \cap \mathrm{J}^{1\gamma}$ and $\mathrm{J}^1 \cap \mathrm{H}^{1\gamma}$ in $\mathbb{V}$, respectively.

Lemma 6.17. *The subspaces $\mathbb{Y}$ and $\mathbb{Y}'$ are both contained in $\mathbb{W}$, and we have $\mathbb{W} = \mathbb{Y} \oplus \mathbb{Y}'$.*

Proof. One easily verifies that τ stabilizes $\mathbb{W}$ and exchanges $\mathbb{Y}$ and $\mathbb{Y}'$. First note that $\mathbb{Y} \subseteq \mathbb{W}$, since $\langle x, y\rangle = 1$ for any $x \in \mathrm{H}^1$ and $y \in \mathrm{J}^1$. By applying τ, and thanks to (6-15), we deduce that $\mathbb{Y}'$ is also contained in $\mathbb{W}$. Now, thanks to (5-4), we have

$$\langle 1 + x, 1 + y\rangle = \psi \circ \mathrm{tr}(\beta(xy - yx))$$

for all $x, y \in \mathrm{j}^1 \cap \mathrm{j}^{1\gamma}$, where j^1 is the sub-$\mathcal{O}$-lattice of $\mathfrak{a}$ such that $\mathrm{J}^1 = 1 + \mathrm{j}^1$. Let a_β denote the endomorphism of F-algebras $x \mapsto \beta x - x\beta$ of $\mathrm{M}_n(\mathrm{F})$. Given a subset $\mathrm{S} \subseteq \mathrm{M}_n(\mathrm{F})$, write S^* for the set of $a \in \mathrm{M}_n(\mathrm{F})$ such that $\psi(\mathrm{tr}(as)) = 1$ for all $s \in \mathrm{S}$. Then the set of $x \in \mathrm{j}^1 \cap \mathrm{j}^{1\gamma}$ such that $\langle 1+x, 1+y\rangle = 1$ for all $y \in \mathrm{j}^1 \cap \mathrm{j}^{1\gamma}$ is equal to

$$\begin{aligned}\mathrm{j}^1 \cap \mathrm{j}^{1\gamma} \cap a_\beta(\mathrm{j}^1 \cap \mathrm{j}^{1\gamma})^* &= \mathrm{j}^1 \cap \mathrm{j}^{1\gamma} \cap (a_\beta(\mathrm{j}^1) \cap a_\beta(\mathrm{j}^1)^\gamma)^* \\ &= \mathrm{j}^1 \cap \mathrm{j}^{1\gamma} \cap (a_\beta(\mathrm{j}^1)^* + a_\beta(\mathrm{j}^1)^{*\gamma}) \\ &= \mathrm{j}^{1\gamma} \cap (\mathrm{j}^1 \cap a_\beta(\mathrm{j}^1)^*) + \mathrm{j}^1 \cap (\mathrm{j}^1 \cap a_\beta(\mathrm{j}^1)^*)^\gamma.\end{aligned}$$

We now claim that

$$\mathrm{j}^1 \cap a_\beta(\mathrm{j}^1)^* = \mathfrak{h}^1. \tag{6-16}$$

To see this, look at the case where $g = 1$. On the one hand, for $x \in \mathrm{j}^1$, we have $\langle 1+x, 1+y\rangle = 1$ for all $y \in \mathrm{j}^1$ if and only if $x \in \mathrm{j}^1 \cap a_\beta(\mathrm{j}^1)^*$. On the other hand, the symplectic form (5-3) on the space $\mathrm{J}^1/\mathrm{H}^1$ is nondegenerate. We thus have $\mathrm{j}^1 \cap a_\beta(\mathrm{j}^1)^* \subseteq \mathfrak{h}^1$ and the other containment follows from the fact that ψ is trivial on the maximal ideal of $\mathcal{O}$.

We now go back to our general situation with $g \in \mathrm{B}^\times$. Applying (6-16) to j^1 and $\mathrm{j}^{1\gamma}$, we get

$$\mathrm{j}^1 \cap \mathrm{j}^{1\gamma} \cap a_\beta(\mathrm{j}^1 \cap \mathrm{j}^{1\gamma})^* = \mathrm{j}^{1\gamma} \cap \mathfrak{h}^1 + \mathfrak{h}^{1\gamma} \cap \mathrm{j}^1.$$

The result follows. □

Corollary 6.18. *The subspaces $\mathbb{W}^+ = \mathbb{W} \cap \mathbb{V}^+$ and $\mathbb{W}^- = \mathbb{W} \cap \mathbb{V}^-$ have the same dimension and we have $\mathbb{W} = \mathbb{W}^+ \oplus \mathbb{W}^-$.*

Proof. The map $x \mapsto x + \tau(x)$ is an isomorphism from $\mathbb{Y}$ to $\mathbb{W}^+$, and the map $x \mapsto x - \tau(x)$ is an isomorphism from $\mathbb{Y}$ to $\mathbb{W}^-$. Thanks to Lemma 6.17 and the fact that $\mathbb{Y}$, $\mathbb{Y}'$ have the same dimension, we thus have

$$\dim \mathbb{W}^+ + \dim \mathbb{W}^- = 2 \cdot \dim \mathbb{Y} = \dim \mathbb{W},$$

which ends the proof of the corollary. □

Now consider the nondegenerate symplectic space $\mathbb{V}/\mathbb{W}$. It decomposes into the direct sum of two totally isotropic subspaces $(\mathbb{V}^+ + \mathbb{W})/\mathbb{W}$ and $(\mathbb{V}^- + \mathbb{W})/\mathbb{W}$. We thus have

$$\begin{gathered}\max(\dim((\mathbb{V}^+ + \mathbb{W})/\mathbb{W}), \dim((\mathbb{V}^- + \mathbb{W})/\mathbb{W})) \leq \tfrac{1}{2} \cdot \dim(\mathbb{V}/\mathbb{W}), \\ \dim((\mathbb{V}^+ + \mathbb{W})/\mathbb{W}) + \dim((\mathbb{V}^- + \mathbb{W})/\mathbb{W}) = \dim(\mathbb{V}/\mathbb{W}).\end{gathered}$$

These spaces thus have the same dimension and are maximal totally isotropic. Corollary 6.18 now implies that $\mathbb{V}^+$ and $\mathbb{V}^-$ have the same dimension.

In order to deduce the equality (6-14), and thanks to Lemma 6.15, it remains to prove that[1]

$$(\mathrm{J}^1 \cap \mathrm{J}^{1\gamma} : \mathrm{H}^1 \cap \mathrm{H}^{1\gamma}) = (\mathrm{J}^1 : \mathrm{H}^1),$$

which follows from [Bushnell and Kutzko 1993, Lemma 5.1.10]. This ends the proof of Proposition 6.12. □

[1] I thank Jiandi Zou for pointing out this argument to me, which was missing in a former version of the proof.

6D. ***Contribution of the mixed Heisenberg representation.*** Let $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. We saw in Paragraph 6B (see (6-11)) that, changing g without changing the double coset $\mathrm{J}g\mathrm{G}^\sigma$, we may assume that $g \in \mathrm{B}^\times$ and that $\gamma = \sigma(g)g^{-1}$ can be written $\gamma = bu$ with b of the form (6-7) and $u \in \mathrm{U}^1 = \mathrm{J}^1 \cap \mathrm{B}^\times$. We write τ for the involution defined by Lemma 6.13 and $\mathrm{U} = \mathrm{J} \cap \mathrm{B}^\times$.

We have a standard Levi subgroup M of G defined by (6-9) and parabolic subgroups P, P^- of G with Levi component M, opposite to each other and with unipotent radicals N, N^- respectively. There is a unique standard hereditary order $\mathfrak{b}_\mathrm{m} \subseteq \mathfrak{b}$ such that

$$\mathfrak{b}_\mathrm{m}^\times = (\mathrm{U}^1 \cap \mathrm{N}^-)\cdot(\mathrm{U} \cap \mathrm{P}).$$

Since $u \in \mathrm{U}^1 \subseteq \mathrm{U}^1(\mathfrak{b}_\mathrm{m})$ and thanks to the specific form of b, one verifies that

$$\mathrm{U}^1(\mathfrak{b}_\mathrm{m}) = (\mathrm{U}^1 \cap \mathrm{P}^-)\cdot(\mathrm{U} \cap \mathrm{N}) = (\mathrm{U} \cap \mathrm{U}^{1\gamma})\mathrm{U}^1. \tag{6-17}$$

Let $\mathfrak{a}_\mathrm{m} \subseteq \mathfrak{a}$ be the unique hereditary order of $\mathrm{M}_n(\mathrm{F})$ normalized by $\mathrm{E}^\times$ such that $\mathfrak{a}_\mathrm{m} \cap \mathrm{B} = \mathfrak{b}_\mathrm{m}$. This gives us a simple stratum $[\mathfrak{a}_\mathrm{m}, \beta]$. Let $\theta_\mathrm{m} \in \mathcal{C}(\mathfrak{a}_\mathrm{m}, \beta)$ be the transfer of θ (see (5-5)) and η_m be the Heisenberg representation on $\mathrm{J}^1_\mathrm{m} = \mathrm{J}^1(\mathfrak{a}_\mathrm{m}, \beta)$ associated with θ_m (by [Bushnell and Kutzko 1993, Proposition 5.1.1; Mínguez and Sécherre 2014b, Proposition 2.1]).

Let S^1 be the pro-p-subgroup $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\mathrm{J}^1 \subseteq \mathrm{J}$. By [Bushnell and Kutzko 1993, Proposition 5.1.15], there is an irreducible representation μ of the group S^1, unique up to isomorphism, extending η and such that

$$\mathrm{Ind}_{\mathrm{S}^1}^{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})}(\mu) \simeq \mathrm{Ind}_{\mathrm{J}^1_\mathrm{m}}^{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})}(\eta_\mathrm{m}). \tag{6-18}$$

In this paragraph, we prove the following result.

Proposition 6.19. *We have* $\dim \mathrm{Hom}_{\mathrm{S}^{1g}\cap\mathrm{G}^\sigma}(\mu^g, 1) = 1$.

Proof. Since μ extends η, the space $\mathrm{Hom}_{\mathrm{S}^{1g}\cap\mathrm{G}^\sigma}(\mu^g, 1)$ is contained in the 1-dimensional space $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$. It is thus enough to prove that $\mathrm{Hom}_{\mathrm{S}^{1g}\cap\mathrm{G}^\sigma}(\mu^g, 1)$ is nonzero. Equivalently, by Lemma 6.13, it is enough to prove that $\mathrm{Hom}_{\mathrm{S}^1\cap\mathrm{G}^\tau}(\mu, 1)$ is nonzero.

First note that, since $\mathfrak{b}_\mathrm{m}$ is σ-stable, $\mathfrak{a}_\mathrm{m}$ is σ-stable as well. We have

$$\sigma(\mathrm{H}^1(\mathfrak{a}_\mathrm{m}, \beta)) = \mathrm{H}^1(\sigma(\mathfrak{a}_\mathrm{m}), \sigma(\beta)) = \mathrm{H}^1(\mathfrak{a}_\mathrm{m}, -\beta) = \mathrm{H}^1(\mathfrak{a}_\mathrm{m}, \beta)$$

thus $\mathrm{H}^1_\mathrm{m} = \mathrm{H}^1(\mathfrak{a}_\mathrm{m}, \beta)$ is σ-stable. By an argument similar to the one used in [Anandavardhanan et al. 2018, Paragraph 4.6], it then follows that $\theta_\mathrm{m} \circ \sigma = (\theta_\mathrm{m})^{-1}$.

Since γ intertwines θ_m by Proposition 5.1(5), it follows from Proposition 6.6 that the character θ_m^g is trivial on $\mathrm{H}^{1g}_\mathrm{m} \cap \mathrm{G}^\sigma$, thus $\mathrm{Hom}_{\mathrm{J}^{1g}_\mathrm{m}\cap\mathrm{G}^\sigma}(\eta^g_\mathrm{m}, 1) = \mathrm{Hom}_{\mathrm{J}^1_\mathrm{m}\cap\mathrm{G}^\tau}(\eta_\mathrm{m}, 1)$ is nonzero. Inducing to $\mathrm{U}^1(\mathfrak{a}_\mathrm{m})$, we get

$$\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})\cap\mathrm{G}^\tau}(\mathrm{Ind}_{\mathrm{J}^1_\mathrm{m}}^{\mathrm{U}^1(\mathfrak{a}_\mathrm{m})}(\eta_\mathrm{m}), 1) \neq \{0\}.$$

Applying the Frobenius reciprocity and the Mackey formula, it follows that there is a $x \in \mathrm{U}^1(\mathfrak{a}_\mathrm{m})$ such that

$$\mathrm{Hom}_{\mathrm{S}^{1x}\cap\mathrm{G}^\tau}(\mu^x, 1) \neq \{0\}. \tag{6-19}$$

We claim that $x \in \mathrm{S}^1(\mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{G}^\tau)$. Restricting (6-19) to the subgroup $\mathrm{H}^{1x} \cap \mathrm{G}^\tau$ and applying Proposition 6.6, we get

$$\sigma(xg)g^{-1}x^{-1} = \sigma(x)\gamma x^{-1} \in \mathrm{J}^1\mathrm{B}^\times\mathrm{J}^1 \cap \mathrm{U}^1(\mathfrak{a}_\mathrm{m})\gamma\mathrm{U}^1(\mathfrak{a}_\mathrm{m}).$$

Write $\sigma(x)\gamma x^{-1} = jcj'$ for some $j, j' \in \mathrm{J}^1$ and $c \in \mathrm{B}^\times$. Since $\gamma \in \mathrm{B}^\times$ and $\mathrm{J}^1 \subseteq \mathrm{U}^1(\mathfrak{a}_\mathrm{m})$, the simple intersection property (5-1) implies that $c \in \mathrm{U}^1(\mathfrak{a}_\mathrm{m})\gamma\mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{B}^\times = \mathrm{U}^1(\mathfrak{b}_\mathrm{m})\gamma\mathrm{U}^1(\mathfrak{b}_\mathrm{m})$. Therefore we have $\sigma(x)\gamma x^{-1} = \sigma(s)\gamma s'$ for some $s, s' \in \mathrm{S}^1$. If we let $y = s^{-1}x$, then we have $\sigma(y)\gamma y^{-1} = \gamma l$ for some $l \in \mathrm{S}^1$, that is $\tau(y)y^{-1} = l$. Since the first cohomology set of τ in $\mathrm{S}^1 \cap \mathrm{S}^{1\gamma}$ is trivial, we get $l = \tau(h)h^{-1}$ for some $h \in \mathrm{S}^1$. This gives us

$$x \in \mathrm{U}^1(\mathfrak{a}_\mathrm{m}) \cap \mathrm{S}^1(\mathrm{G}^\sigma)^{g^{-1}}$$

and the claim follows from the fact that $\mathrm{S}^1 \subseteq \mathrm{U}^1(\mathfrak{a}_\mathrm{m})$.

Putting the claim and (6-19) together, we deduce that $\mathrm{Hom}_{\mathrm{S}^1\cap\mathrm{G}^\tau}(\mu, 1)$ is nonzero. □

6E. *The double coset theorem.* Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η as in Paragraph 5B. There is an irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$, unique up to isomorphism, which is trivial on the subgroup J^1 and satisfies $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. We have the following lemma.

Lemma 6.20. *Let $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$.*

(1) *There is a unique character χ of $\mathbf{J}^g \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^{1g} \cap \mathrm{G}^\sigma$ such that*

$$\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1) = \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}^g, \chi^{-1}).$$

(2) *The canonical linear map*

$$\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1) \otimes \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}^g, \chi) \to \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$$

is an isomorphism.

Proof. Let us fix a nonzero linear form $\mathscr{E} \in \mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$. The choice of $\boldsymbol{\kappa}$ defines an action of $\mathbf{J}^g \cap \mathrm{G}^\sigma$ on the space $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$, which has dimension 1 by Proposition 6.12. This determines a unique character χ of $\mathbf{J}^g \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^{1g} \cap \mathrm{G}^\sigma$ such that

$$\mathscr{E} \circ \boldsymbol{\kappa}^g(x) = \chi(x)^{-1} \cdot \mathscr{E}$$

for all $x \in \mathbf{J}^g \cap \mathrm{G}^\sigma$. This gives us the first part of the lemma.

Given $\mathscr{L} \in \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ and w in the space of $\boldsymbol{\rho}$, the linear form $v \mapsto \mathscr{L}(v \otimes w)$ defined on the space of η is in $\mathrm{Hom}_{\mathrm{J}^{1g}\cap\mathrm{G}^\sigma}(\eta^g, 1)$. By Proposition 6.12 it is thus of the form $\mathscr{F}(w)\mathscr{E}$ for a unique $\mathscr{F}(w) \in \mathrm{R}$. We have $\mathscr{L} = \mathscr{E} \otimes \mathscr{F}$ and $\mathscr{F} \in \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}^g, \chi)$. □

Theorem 6.21. *Let $g \in \mathrm{G}$ and suppose $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ is nonzero. Then $\sigma(g)g^{-1} \in \mathbf{J}$.*

Proof. We know from Proposition 6.6 and Lemma 6.7 that $g \in \mathrm{J}\mathrm{B}^\times\mathrm{G}^\sigma$. We thus may assume that $g \in \mathrm{B}^\times$ and $\gamma = \sigma(g)g^{-1}$ is as in Paragraph 6D. In particular, we have a standard hereditary order $\mathfrak{b}_\mathrm{m} \subseteq \mathfrak{b}$ and an involution τ.

Let us fix an irreducible representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, and let χ be the character given by Lemma 6.20. The restriction of $\boldsymbol{\kappa}$ to J, denoted κ, is an irreducible representation of J extending η. It follows from Remark 5.2 that κ is a beta-extension of η, and from [Bushnell and Kutzko 1993, Theorem 5.2.3] that κ extends μ. Proposition 6.19 thus implies

$$\mathrm{Hom}_{\mathrm{S}^{1g}\cap\mathrm{G}^\sigma}(\mu^g, 1) = \mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}^g, \chi^{-1})$$

and χ is trivial on $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})^g\cap\mathrm{G}^\sigma$. By Lemma 6.20, the space $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}^g, \chi)$ is nonzero.

Write $\mathrm{U}=\mathrm{J}\cap\mathrm{B}^\times$ and $\mathrm{U}^1=\mathrm{J}^1\cap\mathrm{B}^\times$. Since $g\in\mathrm{B}^\times$, we have $\mathrm{J}^g\cap\mathrm{G}^\sigma=(\mathrm{U}^g\cap\mathrm{G}^\sigma)(\mathrm{J}^{1g}\cap\mathrm{G}^\sigma)$. Let ρ be the restriction of $\boldsymbol{\rho}$ to J. Then $\mathrm{Hom}_{\mathrm{U}^g\cap\mathrm{G}^\sigma}(\rho^g, \chi)$ is nonzero. Lemma 6.13 implies

$$\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau}(\rho, 1) = \mathrm{Hom}_{\mathrm{U}^1(\mathfrak{b}_\mathrm{m})^g\cap\mathrm{G}^\sigma}(\rho^g, 1) \neq \{0\}. \tag{6-20}$$

We now describe more carefully the subgroup $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})$.

Lemma 6.22. *We have* $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})=(\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau)\mathrm{U}^1$.

Proof. We follow the proof of [Hakim and Murnaghan 2008, Proposition 5.20]. According to (6-17) it is enough to prove that $\mathrm{U}\cap\mathrm{U}^{1\gamma}$ is contained in $(\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau)\mathrm{U}^1$. Let $x\in\mathrm{U}\cap\mathrm{U}^{1\gamma}$ and define $y=x^{-1}\tau(x)^{-1}x\tau(x)$. Then $y\in\mathrm{U}^1\cap\mathrm{U}^{1\gamma}$ and $y\tau(y)=1$. Since the first cohomology set of τ in $\mathrm{U}^1\cap\mathrm{U}^{1\gamma}$ is trivial, we get $y=z\tau(z)^{-1}$ for some $z\in\mathrm{U}^1\cap\mathrm{U}^{1\gamma}$. Define $x'=x\tau(x)\tau(z)$. Then $x'\in\mathrm{U}^1(\mathfrak{b}_\mathrm{m})\cap\mathrm{G}^\tau$ and we have $x\in x'\mathrm{U}^1$. □

Since ρ is trivial on U^1, Lemma 6.22 and (6-20) together imply that $\mathrm{Hom}_{\mathrm{U}^1(\mathfrak{b}_\mathrm{m})}(\rho, 1)$ is nonzero. Since $\mathrm{U}^1(\mathfrak{b}_\mathrm{m})/\mathrm{U}^1$ is a unipotent subgroup of $\mathrm{U}/\mathrm{U}^1\simeq\mathrm{GL}_m(\boldsymbol{l})$, the fact that the representation ρ is cuspidal (see Paragraph 5B) implies that $\mathfrak{b}_\mathrm{m}=\mathfrak{b}$, that is $\gamma\in\mathrm{U}\subseteq\mathbf{J}$. □

Lemma 4.25 of [Anandavardhanan et al. 2018] gives a detailed account of the elements $g\in\mathrm{G}$ such that $\sigma(g)g^{-1}\in\mathbf{J}$.

6F. *Proof of Theorem 6.1.* Let π be a σ-selfdual cuspidal representation of G, and $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π given by Theorem 5.10. If the space $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$ is nonzero, then (6-1) implies that π is distinguished.

Conversely, suppose that π is distinguished and that $(\mathbf{J}, \boldsymbol{\lambda})$ has been chosen as in Remark 5.11 as it may be. Then the space $\mathrm{Hom}_{\mathbf{J}^g\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}^g, 1)$ is nonzero for some $g\in\mathrm{G}$. By Theorem 6.21, one has $\sigma(g)g^{-1}\in\mathbf{J}$. Thus $\mathbf{J}^g$ is σ-stable, and

$$(\boldsymbol{\lambda}^g)^\sigma = (\boldsymbol{\lambda}^\sigma)^{\sigma(g)} \simeq (\boldsymbol{\lambda}^\vee)^g = (\boldsymbol{\lambda}^g)^\vee$$

thus the type $(\mathbf{J}^g, \boldsymbol{\lambda}^g)$ is σ-selfdual.

Remark 6.23. Let $(\mathbf{J}, \boldsymbol{\lambda})$ and $(\mathbf{J}', \boldsymbol{\lambda}')$ be two distinguished σ-selfdual types in π. Since they both occur in π, there is a $g\in\mathrm{G}$ such that $\mathbf{J}'=\mathbf{J}^g$ and $\boldsymbol{\lambda}'\simeq\boldsymbol{\lambda}^g$. Thanks to the multiplicity 1 property of Theorem 4.1, the formula (6-1) tells us that the double cosets $\mathbf{J}\mathrm{G}^\sigma$ and $\mathbf{J}g\mathrm{G}^\sigma$ are equal, which implies that $g\in\mathbf{J}\mathrm{G}^\sigma$. Thus a distinguished cuspidal representation π contains, up to G^σ-conjugacy, a unique distinguished σ-selfdual type.

Recall that Proposition 5.14 associates to any σ-selfdual cuspidal representation of G a quadratic extension T/T_0.

Corollary 6.24. *Let π be a σ-selfdual cuspidal representation of* G, *and suppose that* T/T_0 *is unramified. Then π is distinguished if and only if any σ-selfdual type in π is distinguished.*

Proof. This follows from Theorem 6.1 together with Proposition 5.17, which says that the representation π contains, up to G^σ-conjugacy, a unique σ-selfdual type. □

When T/T_0 is ramified, Proposition 5.17 tells us that π contains more than one G^σ-conjugacy class of σ-selfdual types as soon as its relative degree m is at least 2. In the next section, we will see that the G^σ-conjugacy class of index $\lfloor m/2 \rfloor$ (see Definition 5.18) is the only one which may contribute to the distinction of π.

7. The cuspidal ramified case

As usual, write $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ for some $n \geq 1$. To any σ-selfdual cuspidal representation of G, one can associate a quadratic extension T/T_0 and its relative degree m (see Proposition 5.14). In this section, we will consider the case where T/T_0 is ramified.

7A. The first main result of this section is the following proposition, which we will prove in Paragraph 7C.

Proposition 7.1. *Let π be a σ-selfdual cuspidal representation of* G *with quadratic extension* T/T_0 *and relative degree m. Suppose* T/T_0 *is ramified. Then π is distinguished if and only if*

(1) *either $m = 1$ or m is even*, *and*

(2) *any σ-selfdual type of index $\lfloor m/2 \rfloor$ contained in π is distinguished.*

Remark 7.2. Proposition 7.1 refines Theorem 6.1 by saying that, if T/T_0 is ramified, then the G^σ-conjugacy class of σ-selfdual types of index $\lfloor m/2 \rfloor$ contained in π is the only one which may contribute to the distinction of π. See [Anandavardhanan et al. 2018, Proposition 5.5] for a characterization of this class in terms of Whittaker data. See also Definition 10.1 and Remark 10.2 below.

Remark 7.3. Proposition 7.1 is proved in [Anandavardhanan et al. 2018] in a different manner from the one we give here (see Remark 6.3 above and [Anandavardhanan et al. 2018, Corollary 6.6 and Remark 6.7]).

Remark 7.4. If we assume π to be *supercuspidal* in Proposition 7.1, then m is automatically either even or equal to 1, even if π is not distinguished (see Proposition 8.1).

Remark 7.5. However, if π is nonsupercuspidal in Proposition 7.1, then its relative degree m need not be either even nor equal to 1. Let k be a divisor of n and τ be a σ-selfdual supercuspidal representation of $\mathrm{GL}_{n/k}(\mathrm{F})$. Assume R has characteristic $\ell > 0$, let ν be the unramified character "absolute value of the determinant" and let $e(\tau)$ be the smallest integer $i \geq 1$ such that $\tau\nu^i \simeq \tau$. Suppose that $k = e(\tau)\ell^u$

for some $u \geq 0$. Then [Mínguez and Sécherre 2014a, Théorème 6.14] tells us that the unique generic irreducible subquotient π of the normalized parabolically induced representation

$$\tau \times \tau\nu \times \cdots \times \tau\nu^{k-1}$$

is cuspidal, and that it is σ-selfdual since τ is. If $k > 1$ and $m(\pi) = km(\tau)$ is odd, then π is a σ-selfdual cuspidal representation which is not distinguished nor ω-distinguished. (For instance, this is the case when τ is the trivial character of $\mathrm{F}^\times$ and $k = n = \ell$ where $\ell \neq 2$ divides $q-1$, which gives $m(\tau) = e(\tau) = 1$).

7B. ***Existence of σ-selfdual extensions of the Heisenberg representation.*** We now go back to our usual notation. Let $[\mathfrak{a}, \beta]$ be a maximal simple stratum in $\mathrm{M}_n(\mathrm{F})$ such that $\mathfrak{a}$ is σ-stable and $\sigma(\beta) = -\beta$. Write E for the extension $\mathrm{F}[\beta]$, and suppose that it is ramified over the field E_0 of σ-fixed points in E. Let d be the degree $[\mathrm{E} : \mathrm{F}]$ and write $n = md$.

Let $\boldsymbol{l}$ denote the residue field of E. Let us notice once and for all that, since $p \neq 2$, any character of $\mathrm{GL}_m(\boldsymbol{l})$ is of the form $\alpha \circ \det$, for some character α of $\boldsymbol{l}^\times$.

The following lemma generalizes [Coniglio-Guilloton 2016, Lemme 3.4.6] (which is concerned with complex representations and χ trivial only).

Lemma 7.6. *Let χ be a character of $(\mathrm{GL}_i \times \mathrm{GL}_{m-i})(\boldsymbol{l})$ for some $i \in \{0, \cdots, \lfloor m/2 \rfloor\}$. Suppose there is a χ-distinguished cuspidal representation of $\mathrm{GL}_m(\boldsymbol{l})$. Then either $m = 1$ or $m = 2i$.*

Proof. If $m \geq 2$, the result follows from Proposition 2.14. Note that, if $m = 1$, then χ is the unique χ-distinguished irreducible representation of $\mathrm{GL}_1(\boldsymbol{l})$. □

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a maximal simple character such that $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$, and let $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ be its normalizer in G. Let η be the Heisenberg representation of $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$ containing θ and write $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$.

Lemma 7.7. *There is a σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η.*

Proof. Conjugating by a suitable element in G, we may assume that the stratum $[\mathfrak{a}, \beta]$ satisfies the conditions of Remark 5.11. Indeed, if it doesn't, there is a $g \in \mathrm{G}$ such that θ^g is σ-selfdual and $[\mathfrak{a}^g, \beta^g]$ satisfies these conditions. This implies that $\gamma = \sigma(g)g^{-1}$ normalizes θ, that is $\gamma \in \mathbf{J}$. Now, assuming the lemma to be true for $[\mathfrak{a}^g, \beta^g]$, there exists a σ-selfdual representation $\boldsymbol{\kappa}'$ of $\mathbf{J}^g$ extending η^g. Define a representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ by $\boldsymbol{\kappa}^g = \boldsymbol{\kappa}'$. Then $\boldsymbol{\kappa}$ extends η, and it is σ-selfdual since $\gamma \in \mathbf{J}$. From now on, we will assume that $[\mathfrak{a}, \beta]$ satisfies the conditions of Remark 5.11. We will identify J/J^1 with $\mathrm{GL}_m(\boldsymbol{l})$, on which σ acts trivially.

Suppose first that R has characteristic 0. Let $\boldsymbol{\kappa}$ be a representation of $\mathbf{J}$ extending η, let μ be the character of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ given by Lemma 5.21 and χ be the character of $\mathbf{J} \cap \mathrm{G}^\sigma$ associated with $\boldsymbol{\kappa}$ by Lemma 6.20. We claim that there is a character ν of $\mathbf{J}$ trivial on J^1 such that $(\nu \circ \sigma)\nu = \mu$. Indeed, $\boldsymbol{\kappa}\nu$ will then extend η and be σ-selfdual. We have

$$\mathrm{Hom}_{\mathrm{J}^1 \cap \mathrm{G}^\sigma}(\eta, 1) = \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\boldsymbol{\kappa}, \chi^{-1}) \simeq \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\chi, \boldsymbol{\kappa}^{\sigma\vee}) \simeq \mathrm{Hom}_{\mathbf{J} \cap \mathrm{G}^\sigma}(\chi\mu^{-1}, \boldsymbol{\kappa})$$

where the isomorphism in the middle follows from the fact that σ acts trivially on $\mathbf{J}\cap\mathrm{G}^\sigma$ and by duality. Since R has characteristic 0 and $\mathbf{J}\cap\mathrm{G}^\sigma$ is compact, the latter space is isomorphic to $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa},\chi\mu^{-1})$. By uniqueness of χ, it follows that the restriction of μ to $\mathbf{J}\cap\mathrm{G}^\sigma$ is χ^2. Restricting to $\mathrm{J}\cap\mathrm{G}^\sigma$ and writing $\mu=\varphi\circ\det$ and $\chi=\alpha\circ\det$ as characters of $\mathrm{GL}_m(\boldsymbol{l})$ for suitable characters φ, α of $\boldsymbol{l}^\times$, we get $\varphi=\alpha^2$. Let ν be the unique character of J which is trivial on J^1 and equal to $\alpha\circ\det$ as a character of $\mathrm{GL}_m(\boldsymbol{l})$. Since $\mathbf{J}$ is generated by t and J, it remains to extend ν to $\mathbf{J}$ by fixing a scalar $\nu(t)\in\mathrm{R}^\times$ such that $\nu(t)^2=\nu(-1)\mu(t)$.

Suppose now that R is equal to $\overline{\mathbf{F}}_\ell$. As in the proof of Lemma 2.5, we use a lifting and reduction argument. Note that reducing finite-dimensional smooth $\overline{\mathbf{Q}}_\ell$-representations of profinite groups is the same as for finite groups (for which we referred to [Serre 1977, Section 15]). The simple character θ lifts to a simple character $\tilde\theta$ with values in $\overline{\mathbf{Z}}_\ell$, defined with respect to the same simple stratum as θ, and such that $\tilde\theta\circ\sigma=\tilde\theta^{-1}$. By the characteristic 0 case, there is a σ-selfdual $\overline{\mathbf{Q}}_\ell$-representation $\tilde{\boldsymbol{\kappa}}$ of $\mathbf{J}$ extending the irreducible $\overline{\mathbf{Q}}_\ell$-representation $\tilde\eta$ of J^1 associated with $\tilde\theta$. The reduction mod ℓ of $\tilde\eta$ is a representation of J^1 containing θ, of the same dimension as η: it is thus isomorphic to η itself. Let $\tilde\kappa$ denote the restriction of $\tilde{\boldsymbol{\kappa}}$ to J. Its reduction mod ℓ, denoted κ, is a σ-selfdual representation of J extending η, and which extends to some representation $\boldsymbol{\kappa}$ of $\mathbf{J}$. Since κ is σ-selfdual, the representation $\boldsymbol{\kappa}^{\sigma\vee}$ is isomorphic to $\boldsymbol{\kappa}\mu$ for some character μ of $\mathbf{J}$ trivial on J. Since $\mathbf{J}$ is generated by J and t, there is a character ν of $\mathbf{J}$ trivial on J such that $(\nu\circ\sigma)\nu=\mu$, thus $\boldsymbol{\kappa}\nu$ is σ-selfdual.

Finally, suppose that R has characteristic $\ell>0$, and fix an embedding $\iota:\overline{\mathbf{F}}_\ell\to\mathrm{R}$. Since θ has finite image, there is a simple $\overline{\mathbf{F}}_\ell$-character θ_0 defined with respect to the same simple stratum as θ such that $\theta_0\circ\sigma=\theta_0^{-1}$ and $\theta=\iota\circ\theta_0$. Let $\boldsymbol{\kappa}_0$ be a σ-selfdual $\overline{\mathbf{F}}_\ell$-representation of $\mathbf{J}$ extending the irreducible $\overline{\mathbf{F}}_\ell$-representation η_0 of J^1 associated with θ_0. The irreducible representations η and $\eta_0\otimes\mathrm{R}$ both contain θ. By uniqueness of the Heisenberg representation, they are isomorphic. It follows that $\boldsymbol{\kappa}=\boldsymbol{\kappa}_0\otimes\mathrm{R}$ is a σ-selfdual R-representation of J extending η. □

7C. ***Proof of Proposition 7.1.*** Let $(\mathbf{J},\boldsymbol{\lambda})$ be a σ-selfdual type, with associated simple character the character θ of Paragraph 7B.

Lemma 7.8. *If* $(\mathbf{J},\boldsymbol{\lambda})$ *is distinguished, then either*

(1) $m=1$, *or*

(2) $m=2r$ *for some* $r\geq 1$, *and* $(\mathbf{J},\boldsymbol{\lambda})$ *has index* r.

Proof. Let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η provided by Lemma 7.7. Let $\boldsymbol{\rho}$ be the unique irreducible representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}\simeq\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ and i be the index of $(\mathbf{J},\boldsymbol{\lambda})$. Lemma 6.20 tells us that $\boldsymbol{\rho}$ is χ-distinguished for some character χ of $\mathbf{J}\cap\mathrm{G}^\sigma$ trivial on $\mathrm{J}^1\cap\mathrm{G}^\sigma$. Restricting $\boldsymbol{\rho}$ to J and identifying J/J^1 with $\mathrm{GL}_m(\boldsymbol{l})$, we get a cuspidal representation ρ of $\mathrm{GL}_m(\boldsymbol{l})$ and a character χ of $(\mathrm{GL}_i\times\mathrm{GL}_{m-i})(\boldsymbol{l})$ such that ρ is χ-distinguished. The result follows from Lemma 7.6. □

Let π be a σ-selfdual cuspidal representation of G, and suppose that the quadratic extension T/T_0 associated with it by Proposition 5.14 is ramified. Let $(\mathbf{J},\boldsymbol{\lambda})$ be a σ-selfdual type contained in π. By Remark 5.12, we may assume that it is defined with respect to a σ-selfdual simple stratum. By Remark 5.16, E is ramified over E_0. We can thus apply the results of Paragraph 7B and Lemma 7.8. Proposition 7.1 now follows from Theorem 6.1 together with Lemma 7.8.

7D. *Existence of distinguished extensions of the Heisenberg representation.* The second main result of this section is the following proposition.

Proposition 7.9. *Let π be a σ-selfdual cuspidal representation of* G *with ramified quadratic extension* T/T_0. *Assume that $m=1$ or m is even, and let $(\mathbf{J},\boldsymbol{\lambda})$ be a σ-selfdual type in π of index $\lfloor m/2\rfloor$. Let J^1 be the maximal normal pro-p-subgroup of $\mathbf{J}$ and η be an irreducible component of the restriction of $\boldsymbol{\lambda}$ to* J^1.

(1) *There is a distinguished representation of $\mathbf{J}$ extending η, and any such representation of $\mathbf{J}$ is σ-selfdual.*

(2) *Let $\boldsymbol{\kappa}$ be a distinguished representation of $\mathbf{J}$ extending η, and let $\boldsymbol{\rho}$ be the unique representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}\simeq\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Then π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished.*

We start with the following lemma, which slightly refines part (1) of the proposition.

Lemma 7.10. *Let $(\mathbf{J},\boldsymbol{\lambda})$ be as in Proposition 7.9.*

(1) *There is a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η.*

(2) *If $\ell=2$ or if m is even, such a distinguished representation $\boldsymbol{\kappa}$ is unique.*

(3) *Any distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η is σ-selfdual.*

Remark 7.11. If $\ell\neq 2$ and $m=1$, there are exactly two distinguished representations of $\mathbf{J}$ extending η, twisted of each other by the unique nontrivial character of $\mathbf{J}$ trivial on $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$. (See the proof below, which shows that $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$ has index 2 in $\mathbf{J}$.)

Proof. Let J be the maximal compact subgroup of $\mathbf{J}$, and J^1 be its maximal normal pro-p-subgroup. As usual, we fix a maximal simple stratum $[\mathfrak{a},\beta]$ defining $(\mathbf{J},\boldsymbol{\lambda})$ such that $\mathfrak{a}$ is σ-stable and $\sigma(\beta)=-\beta$, and write $\mathrm{E}=\mathrm{F}[\beta]$ and $\boldsymbol{l}$ for its residue field. We will identify J/J^1 with $\mathrm{GL}_m(\boldsymbol{l})$ equipped with an involution whose fixed points is $(\mathrm{GL}_i\times\mathrm{GL}_{m-i})(\boldsymbol{l})$ where $i=\lfloor m/2\rfloor$.

Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η. By Lemma 6.20, there is a character χ of $\mathbf{J}\cap\mathrm{G}^\sigma$ trivial on $\mathrm{J}^1\cap\mathrm{G}^\sigma$ associated to $\boldsymbol{\kappa}$. We claim that χ extends to a character ϕ of $\mathbf{J}$ trivial on J^1. It will then follow that $\boldsymbol{\kappa}\phi$ is distinguished and extends η.

Suppose first that $m=1$. We then have canonical group isomorphisms

$$(\mathrm{J}\cap\mathrm{G}^\sigma)/(\mathrm{J}^1\cap\mathrm{G}^\sigma)\simeq\mathrm{J}/\mathrm{J}^1\simeq\boldsymbol{l}^\times. \tag{7-1}$$

Thus there is a unique character ϕ of J trivial on J^1 which coincides with χ on $\mathrm{J}\cap\mathrm{G}^\sigma$. Since $\mathbf{J}$ is generated by t and J, and since t normalizes ϕ, this character extends to a character of $\mathbf{J}$ trivial on J^1.

Lemma 7.12. *Suppose that $m=1$. Then $\mathbf{J}\cap\mathrm{G}^\sigma$ is generated by $\mathrm{J}\cap\mathrm{G}^\sigma$ and t^2.*

Proof. Since we have $\mathbf{J} = \mathrm{E}^\times \mathrm{J}^1$ when $m = 1$, we may consider the exact sequence of σ-groups

$$1 \to \mathrm{U}^1_\mathrm{E} \to \mathrm{E}^\times \times \mathrm{J}^1 \to \mathbf{J} \to 1.$$

Taking σ-invariants and since the first cohomology group $\mathrm{H}^1(\sigma, \mathrm{U}^1_\mathrm{E})$ is trivial, $\mathbf{J} \cap \mathrm{G}^\sigma$ is generated by $\mathrm{E}_0^\times$ and $\mathrm{J}^1 \cap \mathrm{G}^\sigma$. The result follows from (7-1) and the fact that t^2 is a uniformizer of E_0. □

It follows from Lemma 7.12 that χ can be extended to a character ϕ of J trivial on J^1. Since we must have $\phi(t)^2 = \chi(t^2)$ in the field R of characteristic ℓ, there are at most two such characters, with uniqueness if and only if $\ell = 2$.

Suppose now that $m = 2r$ for some $r \geq 1$, and consider the element

$$w = \begin{pmatrix} & \mathrm{id}_r \\ \mathrm{id}_r & \end{pmatrix} \in \mathfrak{b}^\times \subseteq \mathrm{GL}_m(\mathrm{E})$$

where id_r is the identity matrix in $\mathrm{GL}_r(\mathrm{E})$.

Lemma 7.13. *Suppose that $m = 2r$. The group $\mathbf{J} \cap \mathrm{G}^\sigma$ is generated by $\mathrm{J} \cap \mathrm{G}^\sigma$ and tw.*

Proof. First, notice that $t' = tw$ is σ-invariant. Any $x \in \mathbf{J}$ can be written $x = t'^k y$ for unique $k \in \mathbb{Z}$ and $y \in \mathrm{J}$. We thus have $x \in \mathbf{J} \cap \mathrm{G}^\sigma$ if and only if $y \in \mathrm{J} \cap \mathrm{G}^\sigma$. □

Since $\boldsymbol{\kappa}$ and $\mathbf{J} \cap \mathrm{G}^\sigma$ are normalized by w, we have $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, \chi^{-1}) = \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, (\chi^w)^{-1})$, and the uniqueness of χ implies that $\chi^w = \chi$. First, consider the character of

$$(\mathrm{J} \cap \mathrm{G}^\sigma)/(\mathrm{J}^1 \cap \mathrm{G}^\sigma) \simeq (\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{l})$$

defined by χ and write it $(\alpha_1 \circ \det) \otimes (\alpha_2 \circ \det)$ for some characters α_1, α_2 of $\boldsymbol{l}^\times$. The identity $\chi^w = \chi$ implies that $\alpha_1 = \alpha_2$, thus there is a unique character ϕ of J trivial on J^1 which coincides with χ on $\mathrm{J} \cap \mathrm{G}^\sigma$. By Lemma 7.13, there is a unique character ϕ of $\mathbf{J}$ trivial on J^1 extending χ. This proves (1) and (2).

Now let $\boldsymbol{\kappa}$ be a distinguished representation of $\mathbf{J}$ extending η. It satisfies $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ for some character μ of $\mathbf{J}$ trivial on J^1 such that $\mu \circ \sigma = \mu$ (see Lemma 5.21). Since $\boldsymbol{\kappa}$ is distinguished, μ is trivial on $\mathbf{J} \cap \mathrm{G}^\sigma$. We will prove that $\boldsymbol{\kappa}$ is σ-selfdual, that is, that the character μ is trivial.

Suppose first that $m = 1$. Thus $(\mathbf{J}, \boldsymbol{\kappa})$ is a distinguished type in G. Let π denote the cuspidal irreducible representation of G compactly induced from $\boldsymbol{\kappa}$. It is distinguished, thus σ-selfdual by Theorem 4.1. It follows that $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ are both contained in π, thus μ is trivial.

Suppose now that $m = 2r$. Since μ is trivial on $(\mathrm{GL}_r \times \mathrm{GL}_r)(\boldsymbol{l})$, it must be trivial on $\mathrm{GL}_m(\boldsymbol{l})$. Since tw is σ-invariant, we have $\mu(tw) = 1$. Thus μ is trivial. This proves (3). □

For part (2) of Proposition 7.9, it suffices to fix a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η and to consider the canonical isomorphism

$$\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, 1) \otimes \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \to \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$$

(compare with Lemma 6.20).

Proposition 7.9 reduces the problem of the distinction of π to that of $\boldsymbol{\rho}$. In the next section, we investigate the distinction of $\boldsymbol{\rho}$ *in the case where* π *is supercuspidal.*

8. The supercuspidal ramified case

In this section, we investigate the distinction of σ-selfdual *supercuspidal* representations of G in the case where T/T_0 ramified.

8A. *The relative degree.* Let π be a σ-selfdual cuspidal representation of G such that T/T_0 is ramified. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type contained in π and let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η given by Lemma 7.7. This defines a σ-selfdual irreducible representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on J^1. Let J denote the maximal compact subgroup of $\mathbf{J}$ and ρ denote the cuspidal representation of $\mathrm{J}/\mathrm{J}^1 \simeq \mathrm{GL}_m(\boldsymbol{l})$ induced by $\boldsymbol{\rho}$.

Since $\boldsymbol{\rho}$ is σ-selfdual, the representation ρ is selfdual. Applying Fact 5.5 together with Lemma 2.17, we get the following lemma mentioned in Remark 7.4.

Proposition 8.1. *Let* π *be a* σ*-selfdual supercuspidal representation of* G *such that* T/T_0 *is ramified. Then its relative degree* m *is either even or equal to* 1.

8B. *Distinction criterion in the ramified case.* Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type of index $\lfloor m/2 \rfloor$ contained in π. We fix a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η given by Proposition 7.9. It is σ-selfdual, thus the representation $\boldsymbol{\rho}$ of $\mathbf{J}$ trivial on J^1 which correspond to this choice is σ-selfdual. By Proposition 7.9 again, π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished. We now investigate the distinction of $\boldsymbol{\rho}$. For this, we will use the admissible pairs of level zero introduced in Paragraphs 5C and 5E.

Let us fix a σ-selfdual maximal simple stratum $[\mathfrak{a}, \beta]$ such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$. Write $\mathrm{E} = \mathrm{F}[\beta]$. Let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level zero attached to $\boldsymbol{\rho}$ in the sense of Definition 5.7. Since $\boldsymbol{\rho}$ is σ-selfdual, Proposition 5.19 tells us that there is a unique involutive E_0-automorphism of K, which we denote by σ, which is nontrivial on E and satisfies $\xi \circ \sigma = \xi^{-1}$. Let K_0 be the σ-fixed points of K and $\mathrm{E}_0 = \mathrm{K}_0 \cap \mathrm{E}$.

Lemma 8.2. *The representation* $\boldsymbol{\rho}$ *is distinguished if and only if at least one of the following conditions is fulfilled*:

(1) $\ell = 2$.

(2) $m = 1$ *and* $\boldsymbol{\rho}$ *is trivial on* $\mathrm{E}_0^\times$.

(3) m *is even and* ξ *is nontrivial on* $\mathrm{K}_0^\times$.

Remark 8.3. Note that case (3) cannot happen when $\ell = 2$.

Proof. The case $m = 1$ is clear. Let us suppose that $m = 2r$ for some $r \geq 1$. The case where the characteristic of R is 0 is given by [Hakim and Murnaghan 2002, Proposition 6.3]. Suppose R has characteristic $\ell > 0$ and fix an embedding $\iota : \overline{\mathbf{F}}_\ell \to \mathrm{R}$. Since $\xi \circ \sigma = \xi^{-1}$, the image of ξ is finite, thus contained in the image of $\overline{\mathbf{F}}_\ell$ in R. Indeed, the restriction of ξ to the units of $\mathrm{K}^\times$ has finite image and $\xi(t)$ has order at most 4 since $\xi(\sigma(t)) = \xi(t)^{-1}$ and $\sigma(t) \in \{-t, t\}$. There is thus a $\overline{\mathbf{F}}_\ell$-character ξ_0 of $\mathrm{K}^\times$ such that $\xi_0 \circ \sigma = \xi_0^{-1}$ and $\xi = \iota \circ \xi_0$. In particular, $(\mathrm{K}/\mathrm{E}, \xi_0)$ is an admissible pair of level zero. Let $\boldsymbol{\rho}_0$ be the σ-selfdual $\overline{\mathbf{F}}_\ell$-representation attached to it. By Remark 5.9, the representation $\boldsymbol{\rho}$ is isomorphic to $\boldsymbol{\rho}_0 \otimes \mathrm{R}$. It thus suffices to prove the lemma when R is equal to $\overline{\mathbf{F}}_\ell$, which we assume now.

We consider the canonical $\overline{\mathbf{Q}}_\ell$-lift $\tilde{\xi}$ of ξ, which has the same finite order as ξ. It satisfies the identity $\tilde{\xi} \circ \sigma = \tilde{\xi}^{-1}$, and the pair $(\mathrm{K}/\mathrm{E}, \tilde{\xi})$ is admissible of level zero. Attached to it, there is thus a σ-selfdual $\overline{\mathbf{Q}}_\ell$-representation $\tilde{\boldsymbol{\rho}}$ of $\mathbf{J}$ trivial on J^1. Note that the kernel of $\tilde{\boldsymbol{\rho}}$ has finite index, since it contains J^1 and t^4, thus $\tilde{\boldsymbol{\rho}}$ can be considered as a representation of a finite group. From Proposition 2.1, one checks easily that its reduction mod ℓ is $\boldsymbol{\rho}$. Note that the restriction of ξ to $\mathrm{K}_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{\mathrm{K}/\mathrm{K}_0}$.

Suppose ξ is nontrivial on $\mathrm{K}_0^\times$. Then the same holds for $\tilde{\xi}$, and the characteristic 0 case tells us that $\tilde{\boldsymbol{\rho}}$ is distinguished. As in the proof of Lemma 2.5, by applying Lemma 2.6, reducing mod ℓ a nonzero invariant form on $\tilde{\boldsymbol{\rho}}$ gives us a nonzero invariant form on $\boldsymbol{\rho}$, which is thus distinguished.

Suppose now that ξ is trivial on $\mathrm{K}_0^\times$. Then the same holds for $\tilde{\xi}$. Let $\tilde{\alpha}$ denote the unramified ℓ-adic character of $\mathrm{K}^\times$ of order 2. Then $(\mathrm{K}/\mathrm{E}, \tilde{\xi}\tilde{\alpha})$ is an admissible pair of level zero. It is attached to $\tilde{\boldsymbol{\rho}}\tilde{\varphi}$ where $\tilde{\varphi}$ is the unramified ℓ-adic character of $\mathbf{J}$ of order 2. Since $\tilde{\xi}\tilde{\alpha}$ is nontrivial on $\mathrm{K}_0^\times$, the representation $\tilde{\boldsymbol{\rho}}\tilde{\varphi}$ is distinguished. Thus $\boldsymbol{\rho}$ is φ-distinguished, where φ is the reduction mod ℓ of $\tilde{\varphi}$.

If $\ell = 2$, then $\boldsymbol{\rho}$ is distinguished. Suppose now that $\ell \neq 2$. If $\boldsymbol{\rho}$ were both φ-distinguished and distinguished, one would have two linearly independent linear forms in $\mathrm{Hom}_{\mathrm{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1)$, and this would contradict Lemma 2.19. The result follows. □

The field extension E of F is not uniquely determined by π, unlike its maximal tamely ramified extension T. To remedy this, let D be the maximal tamely ramified subextension of K/F. Write $\mathrm{D}_0 = \mathrm{D} \cap \mathrm{K}_0$, and let δ_0 be the restriction of ξ to $\mathrm{D}_0^\times$.

Since $\xi \circ \sigma = \xi^{-1}$ the character δ_0 is quadratic, either trivial or (if $\ell \neq 2$) equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$. We will see in Proposition 10.5 that, up to F_0-equivalence, D/D_0 and δ_0 are determined by π.

Theorem 8.4. *Let π be a σ-selfdual supercuspidal representation of* G. *Suppose that* T/T_0 *is ramified. Let m be its relative degree and δ_0 be the quadratic character of* $\mathrm{D}_0^\times$ *associated to it.*

(1) *The representation π is distinguished if and only if at least one of the following conditions is fulfilled*:
   (a) $\ell = 2$.
   (b) $m = 1$ *and δ_0 is trivial.*
   (c) *m is even and δ_0 is nontrivial.*

(2) *Suppose that $\ell \neq 2$. Then π is ω-distinguished if and only if either*
   (a) $m = 1$ *and δ_0 is nontrivial, or*
   (b) *m is even and δ_0 is trivial.*

Remark 8.5. If R has characteristic 2, then π is always distinguished. If R has characteristic not 2, then π is either distinguished or ω-distinguished, but not both.

Proof. By Proposition 7.9 and Lemma 8.2, it suffices to compare the restriction of ξ to $\mathrm{K}_0^\times$ with δ_0 when $\ell \neq 2$.

Suppose first that $m = 1$ and δ_0 is trivial. Since the restriction of $\boldsymbol{\rho}$ to $\mathrm{E}_0^\times$ is equal to either 1 or $\omega_{\mathrm{E}/\mathrm{E}_0}$, its restriction to $\mathrm{T}_0^\times$ is either 1 or $\omega_{\mathrm{T}/\mathrm{T}_0}$, respectively. Since δ_0 is trivial, we are in the first case, that is, the restriction of $\boldsymbol{\rho}$ to $\mathrm{E}_0^\times$ is trivial.

Suppose now that $m \neq 1$ and ξ is nontrivial on $\mathrm{K}_0^\times$. We want to prove that δ_0 is nontrivial. The restriction of ξ to $\mathrm{K}_0^\times$ is equal to $\omega_{\mathrm{K}/\mathrm{K}_0}$. Thus δ_0 is equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$.

Now suppose that R has characteristic different from 2 and let χ be an unramified character of $\mathrm{F}^\times$ extending ω. Note that the twisted representation $\pi' = \pi(\chi^{-1} \circ \det)$ is supercuspidal and σ-selfdual and that the character associated with π' is $\delta_0' = \delta_0(\chi^{-1} \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})|_{\mathrm{D}_0^\times}$, where $\mathrm{N}_{\mathrm{K}/\mathrm{F}}$ is the norm map from K to F. Suppose first that $m = 1$. Then

$$\begin{aligned}\pi \text{ is } \omega\text{-distinguished} &\Leftrightarrow \pi' \text{ is distinguished}\\ &\Leftrightarrow \text{the character } \delta_0' \text{ is trivial}\\ &\Leftrightarrow \text{the character } \delta_0 \text{ coincides with } \chi \circ \mathrm{N}_{\mathrm{E}/\mathrm{F}} \text{ on } \mathrm{T}_0^\times.\end{aligned}$$

Suppose now that $m \neq 1$. Then

$$\begin{aligned}\pi \text{ is } \omega\text{-distinguished} &\Leftrightarrow \pi' \text{ is distinguished}\\ &\Leftrightarrow \text{the character } \delta_0' \text{ is nontrivial}\\ &\Leftrightarrow \text{the character } \delta_0 \text{ coincides with } (\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})\omega_{\mathrm{D}/\mathrm{D}_0} \text{ on } \mathrm{D}_0^\times.\end{aligned}$$

The restriction of $\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}}$ to $\mathrm{D}_0^\times$ is $\omega \circ \mathrm{N}_{\mathrm{D}_0/\mathrm{F}_0} = \omega_{\mathrm{D}/\mathrm{D}_0}$ to the power of $[\mathrm{K}_0 : \mathrm{D}_0]$, which is a p-power with p odd. This gives us the expected result. □

Remark 8.6. If π is as in Theorem 8.4 and $m > 1$, its central character ω_π is always trivial on $\mathrm{F}_0^\times$. Indeed, since π and $\boldsymbol{\lambda}$ have the same central character, we can express ω_π as the product $\omega_{\boldsymbol{\kappa}}\omega_{\boldsymbol{\rho}}$, where $\omega_{\boldsymbol{\kappa}}$ and $\omega_{\boldsymbol{\rho}}$ are the central characters of $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$ on $\mathrm{F}^\times$, respectively. Since $\boldsymbol{\kappa}$ is distinguished, its central character is trivial on $\mathrm{F}_0^\times$, thus ω_π and δ_0 coincide on $\mathrm{F}_0^\times$. If δ_0 is trivial then ω_π is trivial on $\mathrm{F}_0^\times$. Now assume that δ_0 is equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$. Since D/D_0 is unramified by Lemma 5.20, its restriction to $\mathrm{F}_0^\times$ is trivial if and only if $e(\mathrm{D}_0/\mathrm{F}_0)$ is even, which is the case since $e(\mathrm{D}_0/\mathrm{T}_0) = 2$ when m is even.

Remark 8.7. On the other hand, since T/T_0 is ramified, the restriction of $\omega_{\mathrm{T}/\mathrm{T}_0}$ to $\mathrm{F}_0^\times$ is trivial if and only if $\boldsymbol{k}_0^\times$ is contained in the subgroup $\boldsymbol{l}^{\times 2}$ of squares in $\boldsymbol{l}^\times$, that is, if and only if $f(\mathrm{T}_0/\mathrm{F}_0)$ is even. It follows that, if $m = 1$ and $f(\mathrm{T}_0/\mathrm{F}_0)$ is odd (which is equivalent to n being odd by Lemma 5.15 and since m is either 1 or even) then π is distinguished if and only if ω_π is trivial on $\mathrm{F}_0^\times$.

9. The supercuspidal unramified case

Let π be a σ-selfdual cuspidal representation of $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$ for $n \geq 1$. In this section, we investigate the case where the quadratic extension T/T_0 is unramified. By Corollary 6.24, the representation π is distinguished if and only if any of the σ-selfdual types contained in π is distinguished.

9A. ***Existence of σ-selfdual extensions of the Heisenberg representation.*** Let $[\mathfrak{a}, \beta]$ be a maximal simple stratum as in Remark 5.11. Write $\mathrm{E} = \mathrm{F}[\beta]$ and suppose that it is unramified over $\mathrm{E}_0 = \mathrm{E}^\sigma$. Let us write $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$, $\mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$ and $\mathrm{J}^1 = \mathrm{J}^1(\mathfrak{a}, \beta)$.

We may and will identify J/J^1 with the group $\mathrm{GL}_m(\boldsymbol{l})$, denoted $\mathcal{G}$, equipped with the residual involution $\sigma \in \mathrm{Gal}(\boldsymbol{l}/\boldsymbol{l}_0)$, where $\boldsymbol{l}$ and $\boldsymbol{l}_0$ are the residue fields of E and E_0, respectively.

Lemma 9.1. *The group $\mathbf{J} \cap \mathrm{G}^\sigma$ is generated by t and $\mathrm{J} \cap \mathrm{G}^\sigma$.*

Proof. Any $x \in \mathbf{J}$ can be written $x = t^m y$ for unique $m \in \mathbb{Z}$ and $y \in \mathrm{J}$. Since t is σ-invariant, we have $x \in \mathbf{J} \cap \mathrm{G}^\sigma$ if and only if $y \in \mathrm{J} \cap \mathrm{G}^\sigma$. □

Lemma 9.2. *Any character of $\mathbf{J} \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^1 \cap \mathrm{G}^\sigma$ extends to a character of $\mathbf{J}$ trivial on J^1.*

Proof. Let χ be a character of $\mathbf{J} \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^1 \cap \mathrm{G}^\sigma$. Since J^1 is a pro-p-group, the first cohomology group of σ in J^1 is trivial. The subgroup $\mathcal{G}^\sigma$ thus identifies with $(\mathrm{J} \cap \mathrm{G}^\sigma)/(\mathrm{J}^1 \cap \mathrm{G}^\sigma)$. The restriction of χ to $\mathrm{J} \cap \mathrm{G}^\sigma$ thus induces a character of $\mathcal{G}^\sigma$, which can be written $\alpha_0 \circ \det$ for some character α_0 of $\boldsymbol{l}_0^\times$. Let α be a character of $\boldsymbol{l}^\times$ extending α_0, and let ϕ be the character of J trivial on J^1 inducing the character $\alpha \circ \det$ of $\mathcal{G}$. Since $\mathrm{J} = \mathfrak{b}^\times \mathrm{J}^1$, the element t acts trivially on J/J^1 by conjugacy, thus normalizes ϕ. We thus may extend ϕ to $\mathbf{J}$ by setting $\phi(t) = \chi(t)$. Lemma 9.1 implies that ϕ extends χ and it is trivial on J^1. □

Let $\theta \in \mathcal{C}(\mathfrak{a}, \beta)$ be a maximal simple character such that $\mathrm{H}^1(\mathfrak{a}, \beta)$ is σ-stable and $\theta \circ \sigma = \theta^{-1}$. Let η denote the Heisenberg representation of θ on the group J^1.

Lemma 9.3. *There is a σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η.*

Proof. Let $\boldsymbol{\kappa}$ be an irreducible representation of $\mathbf{J}$ extending η. By Lemma 5.21, there is a character μ of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\kappa}^{\sigma\vee} \simeq \boldsymbol{\kappa}\mu$ and $\mu \circ \sigma = \mu$. We claim that there is a character ν of $\mathbf{J}$ trivial on J^1 such that $(\nu \circ \sigma)\nu = \mu$. Indeed, if this is the case, the representation $\boldsymbol{\kappa}\nu$ extends η and is σ-selfdual.

Consider first μ as a character of $\mathcal{G}$ and write $\mu = \varphi \circ \det$ for some character φ of $\boldsymbol{l}^\times$. Then we have $\varphi \circ \sigma = \varphi$, thus there is a character α of $\boldsymbol{l}^\times$ such that $(\alpha \circ \sigma)\alpha = \varphi$. Choosing such a α, there exists a unique character ν of J inducing $\alpha \circ \det$ on $\mathcal{G}$. Since $\mathbf{J}$ is generated by t and J, it remains to extend ν to $\mathbf{J}$ by choosing a scalar $\nu(t) \in \mathrm{R}^\times$ such that $\nu(t)^2 = \mu(t)$. □

9B. ***Existence of distinguished extensions of the Heisenberg representation.*** Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type, with associated simple character the character θ of Paragraph 9A.

In this paragraph, we suppose that m is odd.

Proposition 9.4. *Suppose that m is odd. There is a σ-selfdual distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η.*

Proof. We first assume that R has characteristic 0. By Lemma 9.3, there is a σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η. Let χ denote the character of $\mathbf{J} \cap \mathrm{G}^\sigma$ trivial on $\mathrm{J}^1 \cap \mathrm{G}^\sigma$ associated to $\boldsymbol{\kappa}$ by Lemma 6.20. Since m is odd, Lemma 2.3 implies that $\mathcal{G}$ possesses a σ-selfdual supercuspidal representation ρ. Let $\boldsymbol{\rho}$ be the unique representation of $\mathbf{J}$ trivial on J^1 such that $t \in \mathrm{Ker}(\boldsymbol{\rho})$ and whose restriction to J is the inflation of ρ. This representation $\boldsymbol{\rho}$ is σ-selfdual. By Lemma 2.5, it is also distinguished. Now let $\boldsymbol{\lambda}$ be the σ-selfdual type $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$ on $\mathbf{J}$. The natural isomorphism

$$\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\kappa},\chi)\otimes\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\rho},1)\to\mathrm{Hom}_{\mathbf{J}\cap G^\sigma}(\boldsymbol{\lambda},\chi)$$

thus shows that $\boldsymbol{\lambda}$ is χ-distinguished.

By Lemma 9.2, there exists a character ϕ of $\mathbf{J}$ trivial on J^1 extending χ. The representation $\boldsymbol{\lambda}'=\boldsymbol{\lambda}\phi^{-1}$ is thus a distinguished type. Let π' be the cuspidal representation of G compactly induced from $(\mathbf{J},\boldsymbol{\lambda}')$. It is distinguished, thus σ-selfdual by Theorem 4.1. Since $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}'^{\sigma\vee}\simeq\boldsymbol{\lambda}'\phi(\phi\circ\sigma)$ are both contained in π', it follows that $\phi(\phi\circ\sigma)$ is trivial. This implies that $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\phi^{-1}$ is both σ-selfdual and distinguished.

Now assume that R has characteristic $\ell>0$. We then reduce to the characteristic 0 case as in the proof of Lemma 7.7. □

Remark 9.5. I don't know whether Proposition 9.4 holds when m is even.

Corollary 9.6. (1) *Any distinguished representation of* $\mathbf{J}$ *extending* η *is* σ*-selfdual.*

(2) *If* $\ell=2$, *any* σ*-selfdual representation of* $\mathbf{J}$ *extending* η *is distinguished.*

Proof. Let us fix a distinguished σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η given by Proposition 9.4. Let $\boldsymbol{\kappa}'$ be a distinguished representation of $\mathbf{J}$ extending η. Then $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\phi$ for some character ϕ of $\mathbf{J}$ trivial on $(\mathbf{J}\cap G^\sigma)J^1$. Thus $\phi(t)=1$ and ϕ induces the character $\alpha\circ\det$ on $\mathcal{G}$, where α is a character of $\boldsymbol{l}^\times$ trivial on $\boldsymbol{l}_0^\times$, or equivalently $\alpha^{q_0+1}=1$. Thus we have $\phi(\phi\circ\sigma)=1$. This implies that $\boldsymbol{\kappa}'$ is σ-selfdual, which proves the first assertion.

Now suppose that $\boldsymbol{\kappa}'$ is a σ-selfdual representation of $\mathbf{J}$ extending η. Then $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\xi$ for some character ξ of $\mathbf{J}$ such that $\xi(\xi\circ\sigma)$ is trivial. Thus $\xi(t)\in\{-1,1\}$ and there is a character ν of $\boldsymbol{l}^\times$ such that ξ induces $\nu\circ\det$ on $\mathcal{G}$ and $\nu^{q_0+1}=1$. It follows that ξ is trivial on $(J\cap G^\sigma)J^1$. Thus, if $\ell=2$, the representation $\boldsymbol{\kappa}'$ is distinguished. □

Remark 9.7. Let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η. Then the character χ of $\mathbf{J}\cap G^\sigma$ associated to $\boldsymbol{\kappa}$ by Lemma 6.20 is quadratic and unramified.

9C. *Distinction criterion in the unramified case.* Let π be a σ-selfdual supercuspidal representation of G. Associated to it by Proposition 5.14, there is a quadratic extension T/T_0. We assume that T is unramified over T_0.

Recall that, by Theorem 6.1 and Proposition 5.17, the representation π is distinguished if and only any of its σ-selfdual types is distinguished. The following result is the analogue of Proposition 7.9.

Proposition 9.8. *Let* π *be a* σ*-selfdual supercuspidal representation of* G, *with unramified quadratic extension* T/T_0 *and relative degree* m. *Let* $(\mathbf{J},\boldsymbol{\lambda})$ *be a* σ*-selfdual type in* π. *Let* J^1 *be the maximal normal pro-*p*-subgroup of* $\mathbf{J}$ *and* η *be an irreducible component of the restriction of* $\boldsymbol{\lambda}$ *to* J^1.

(1) *The integer m is odd.*

(2) *There is a distinguished representation of* $\mathbf{J}$ *extending* η, *and any such extension of* η *is* σ*-selfdual.*

(3) *Let* $\boldsymbol{\kappa}$ *be a distinguished representation of* $\mathbf{J}$ *extending* η, *and let* $\boldsymbol{\rho}$ *be the unique representation of* $\mathbf{J}$ *trivial on* J^1 *such that* $\boldsymbol{\lambda} \simeq \boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. *Then* π *is distinguished if and only if* $\boldsymbol{\rho}$ *is distinguished.*

Proof. We may and will assume that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$ for some maximal simple stratum $[\mathfrak{a}, \beta]$ as in Remark 5.11. Following Remark 5.16, the extension E is unramified over E_0. We thus may apply the results of Paragraph 9A.

Let $\boldsymbol{\kappa}$ be a σ-selfdual representation of $\mathbf{J}$ extending η, the existence of which is given by Lemma 9.3, and let $\boldsymbol{\rho}$ be the irreducible representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. Since $\boldsymbol{\lambda}$ and $\boldsymbol{\kappa}$ are σ-selfdual, $\boldsymbol{\rho}$ is σ-selfdual. Its restriction to J induces a cuspidal irreducible representation of $\mathrm{GL}_m(\boldsymbol{l})$, denoted ρ. Since π is supercuspidal, ρ is also supercuspidal by Fact 5.5. Lemma 2.3 implies that m is odd. We thus apply Proposition 9.4, which gives us a σ-selfdual distinguished representation extending η.

Part (2) of the proposition is given by Corollary 9.6. For (3), it suffices to fix a distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η and to consider the canonical isomorphism

$$\mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\kappa}, 1) \otimes \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \to \mathrm{Hom}_{\mathbf{J}\cap\mathrm{G}^\sigma}(\boldsymbol{\lambda}, 1)$$

as in the ramified case. □

Remark 9.9. If one relaxes the supercuspidality assumption on π (that is, we only assume π to be σ-selfdual cuspidal with T/T_0 unramified), then its relative degree m need not be odd, in which case our proof of Proposition 9.8(2) doesn't apply (see Remarks 2.4 and 9.5). Unlike the ramified case, I thus don't know whether there is a distinguished and σ-selfdual extension $\boldsymbol{\kappa}$ of η when π is not supercuspidal.

Remark 9.10. In both the ramified and unramified cases, the distinguished representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η is not unique in general, so neither is $\boldsymbol{\rho}$. If $\boldsymbol{\kappa}$ is a distinguished representation of $\mathbf{J}$ extending η, the other ones are exactly the $\boldsymbol{\kappa}\phi$ where ϕ ranges over the set of characters of $\mathbf{J}$ trivial on $(\mathbf{J} \cap \mathrm{G}^\sigma)\mathrm{J}^1$.

From now, we will thus assume that $\boldsymbol{\kappa}$ is a distinguished σ-selfdual representation of $\mathbf{J}$ extending η. Proposition 9.8 reduces the problem of the distinction of π to that of $\boldsymbol{\rho}$. We now investigate the distinction of $\boldsymbol{\rho}$.

Let ρ be the representation of $\mathrm{GL}_m(\boldsymbol{l})$ defined by restricting $\boldsymbol{\rho}$ to J. It is σ-selfdual. By Fact 5.5, it is also supercuspidal.

Let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level 0 attached to $\boldsymbol{\rho}$ in the sense of Definition 5.7. Since $\boldsymbol{\rho}$ is σ-selfdual, Proposition 5.19 tells us that there is a unique involutive E_0-automorphism of K, which we denote by σ, which is nontrivial on E and satisfies $\xi \circ \sigma = \xi^{-1}$. Let K_0 be the σ-fixed points of K and $\mathrm{E}_0 = \mathrm{K}_0 \cap \mathrm{E}$.

Lemma 9.11. *The representation* $\boldsymbol{\rho}$ *is distinguished if and only if it is trivial on* $\mathrm{E}_0^\times$.

Proof. Note that $\boldsymbol{\rho}(x) = \xi(x)\cdot\mathrm{id}$ for all $x \in \mathrm{E}^\times$, thus $\boldsymbol{\rho}$ is trivial on $\mathrm{E}_0^\times$ if and only if ξ is. The representation ρ is σ-selfdual, thus distinguished (see Lemma 2.5). We thus have

$$\mathrm{Hom}_{\mathbf{J}\cap \mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \subseteq \mathrm{Hom}_{\mathrm{J}\cap \mathrm{G}^\sigma}(\boldsymbol{\rho}, 1) \simeq \mathrm{Hom}_{\mathrm{GL}_m(\boldsymbol{l}_0)}(\rho, 1)$$

where the space on the right-hand side is nonzero (and has dimension 1). Since $\mathbf{J}\cap \mathrm{G}^\sigma$ is generated by $\mathrm{J}\cap \mathrm{G}^\sigma$ and t, we deduce that $\boldsymbol{\rho}$ is distinguished if and only if t acts trivially on the space $\mathrm{Hom}_{\mathrm{J}\cap \mathrm{G}^\sigma}(\boldsymbol{\rho}, 1)$, that is, if and only if $\xi(t)$ is trivial. The result follows from the fact that, since ξ is σ-selfdual, it is trivial on the E/E_0-norms in $\mathrm{E}_0^\times$, thus on the units of E_0. □

Let ε_0 denote the restriction of the character ξ to $\mathrm{T}_0^\times$.

Lemma 9.12. *The character ε_0 is quadratic and unramified.*

Proof. As has been said in the proof of Lemma 9.11, the character ξ is trivial on the subgroup of E/E_0-norms in $\mathrm{E}_0^\times$, since $\boldsymbol{\rho}$ is σ-selfdual. Thus the restriction of ξ to $\mathrm{E}_0^\times$ is either trivial or (if $\ell \neq 2$) equal to $\omega_{\mathrm{E}/\mathrm{E}_0}$. We get the expected result by restricting to $\mathrm{T}_0^\times$, since E is unramified over E_0 and $e(\mathrm{E}_0/\mathrm{T}_0)$ is a p-power with p odd. □

We will see below (Remark 10.7) that the character ε_0 is uniquely determined by π.

Theorem 9.13. *Let π be a σ-selfdual supercuspidal representation of* G. *Suppose that* T *is unramified over* T_0.

(1) *The representation π is distinguished if and only if ε_0 is trivial.*

(2) *Suppose that the characteristic of* R *is not* 2. *Then π is ω-distinguished if and only if ε_0 is nontrivial.*

Remark 9.14. If R has characteristic 2, then π is always distinguished. If R has characteristic not 2, then π is either distinguished or ω-distinguished, but not both.

Proof. By Proposition 9.8, the representation π is distinguished if and only if $\boldsymbol{\rho}$ is distinguished. Lemma 9.11 tells us that it is distinguished if and only if $\xi(t) = 1$. The restriction of ξ to $\mathrm{E}_0^\times$ is a quadratic unramified character. Since the ramification index of E_0 over T_0 is odd (for it is a p-power), ξ is trivial on $\mathrm{E}_0^\times$ if and only if it is trivial on $\mathrm{T}_0^\times$. The first assertion is proven.

Now suppose that R has characteristic different from 2, and let χ be an unramified character of $\mathrm{F}^\times$ extending ω. Note that the twisted representation $\pi' = \pi(\chi^{-1}\circ\det)$ is supercuspidal and σ-selfdual, and that the character associated with π' is $\varepsilon_0' = \varepsilon_0(\chi^{-m}\circ \mathrm{N}_{\mathrm{E}/\mathrm{F}})|_{\mathrm{T}_0^\times}$ where $\mathrm{N}_{\mathrm{E}/\mathrm{F}}$ is the norm map from E to F. Thus

$$\begin{aligned}\pi \text{ is } \omega\text{-distinguished} &\Leftrightarrow \pi' \text{ is distinguished}\\ &\Leftrightarrow \text{the character } \varepsilon_0' \text{ is trivial on } \mathrm{T}_0^\times\\ &\Leftrightarrow \text{the character } \varepsilon_0 \text{ coincides with } \chi^m\circ \mathrm{N}_{\mathrm{E}/\mathrm{F}} \text{ on } \mathrm{T}_0^\times.\end{aligned}$$

The restriction of $\chi\circ \mathrm{N}_{\mathrm{E}/\mathrm{F}}$ to $\mathrm{T}_0^\times$ is equal to $\omega\circ \mathrm{N}_{\mathrm{T}_0/\mathrm{F}_0} = \omega_{\mathrm{T}/\mathrm{T}_0}$ to the power of $[\mathrm{E}_0 : \mathrm{T}_0]$, which is a p-power. The second assertion then follows from the fact that p and m are odd. □

Corollary 9.15. *Let π be a supercuspidal representation of* G. *Suppose that* T/T_0 *is unramified*, *and that the ramification index of* T/F *is odd. Then π is distinguished if and only if it is σ-selfdual and its central character is trivial on* $\mathrm{F}_0^\times$.

Proof. Suppose that π is σ-selfdual and that its central character ω_π is trivial on $\mathrm{F}_0^\times$. By using a σ-selfdual type $(\mathbf{J}, \boldsymbol{\lambda})$ contained in π as above, we can express ω_π as the product $\omega_{\boldsymbol{\kappa}}\omega_{\boldsymbol{\rho}}$, where $\omega_{\boldsymbol{\kappa}}$ and $\omega_{\boldsymbol{\rho}}$ are the central characters of $\boldsymbol{\kappa}$ and $\boldsymbol{\rho}$ on $\mathrm{F}^\times$, respectively. Since $\boldsymbol{\kappa}$ is distinguished, its central character is trivial on $\mathrm{F}_0^\times$, thus ω_π and ε_0 coincide on $\mathrm{F}_0^\times$. It remains to prove that ε_0 is trivial if and only if it is trivial on $\mathrm{F}_0^\times$.

Suppose ε_0 is trivial on $\mathrm{F}_0^\times$. By Lemma 9.12, it is unramified, thus $\varepsilon_0^{e(\mathrm{T}_0/\mathrm{F}_0)}$ is trivial on $\mathrm{T}_0^\times$. Since $e(\mathrm{T}/\mathrm{F})$ is odd, $e(\mathrm{T}_0/\mathrm{F}_0)$ is odd too, and the expected result follows from the fact that ε_0 is quadratic. □

Remark 9.16. In particular, when n is odd and F is unramified over F_0, a supercuspidal representation of G is distinguished if and only if it is σ-selfdual and its central character is trivial on $\mathrm{F}_0^\times$. This has been proved by Prasad [2001] when R has characteristic 0. Note that, since m and p are odd here, n is odd if and only if $[\mathrm{T} : \mathrm{F}]$ is odd.

Remark 9.17. Note that, in the proof of Prasad [2001, Theorem 4], the isomorphism of π with $\pi^{\sigma\vee}$ gives an element $g \in \mathrm{G}$ which has the property that $g\sigma(g) \in \mathrm{J} = \mathrm{J}(\mathfrak{a}, \beta)$, but g has *a priori* no reason to normalize J. Anyway, g can be chosen in the maximal compact open subgroup $\mathfrak{a}^\times$ which contains J (which derives from [Bushnell and Kutzko 1993, Theorem 3.5.11]), thus the group generated by g and J will indeed be compact mod center.

10. Statement of the final results

In this section we put together the main results of Sections 7–9. Let π be a σ-selfdual supercuspidal representation of G. Associated to it, there are its relative degree m and the quadratic extension T/T_0. It is convenient to introduce the following definition, which comes from [Anandavardhanan et al. 2018].

Definition 10.1. A σ-selfdual type in π is said to be *generic* if either T/T_0 is unramified or T/T_0 is ramified and this type has index $\lfloor m/2 \rfloor$.

Remark 10.2. It is proved in [Anandavardhanan et al. 2018, Proposition 5.5] that a σ-selfdual type is generic in the sense of Definition 10.1 if and only if there are a σ-stable maximal unipotent subgroup N of G and a σ-selfdual nondegenerate character ψ_{N} of N such that $\mathrm{Hom}_{\mathbf{J}\cap\mathrm{N}}(\boldsymbol{\lambda}, \psi_{\mathrm{N}})$ is nonzero.

Definition 10.1 is convenient to us because of the following result, which subsumes Propositions 7.1 and 8.1 (compare with Theorem 6.1).

Theorem 10.3. *A σ-selfdual cuspidal representation of* G *is distinguished if and only if any of its generic σ-selfdual types is distinguished.*

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a *generic* σ-selfdual type contained in π. Let $[\mathfrak{a}, \beta]$ be a σ-selfdual simple stratum such that $\mathbf{J} = \mathbf{J}(\mathfrak{a}, \beta)$. The restriction of $\boldsymbol{\lambda}$ to the maximal normal pro-p-subgroup J^1 is made of copies of a single irreducible representation η. We fix a distinguished σ-selfdual representation $\boldsymbol{\kappa}$ of $\mathbf{J}$ extending η, the existence of which is given by Propositions 7.9 and 9.8. Let $\boldsymbol{\rho}$ be the representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa}\otimes\boldsymbol{\rho}$. Let $(\mathrm{K}/\mathrm{E},\xi)$ be a admissible pair of level 0 attached to $\boldsymbol{\rho}$ and σ be the involution of K given by Proposition 5.19. Let K_0 be the field of σ-fixed points of K. We thus have $\mathrm{K}\simeq\mathrm{K}_0\otimes_{\mathrm{F}_0}\mathrm{F}$.

Definition 10.4. Let D be the maximal tamely ramified subextension of K/F. Write $\mathrm{D}_0=\mathrm{D}\cap\mathrm{K}_0$, and let δ_0 be the restriction of ξ to $\mathrm{D}_0^\times$.

It follows immediately from the definition that $\mathrm{D}_0/\mathrm{F}_0$ is tamely ramified and the character δ_0 is quadratic, either trivial or (if $\ell\neq 2$) equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$.

Proposition 10.5. *The quadratic extension* D/D_0 *and the character* δ_0 *are uniquely determined by* π *up to* F_0*-equivalence. That is, if* $\mathrm{D}'/\mathrm{D}'_0$ *and* δ'_0 *are another quadratic extension and character associated to* π*, then there is an* F_0*-isomorphism* $\varphi:\mathrm{D}\to\mathrm{D}'$ *such that* $\varphi(\mathrm{D}_0)=\mathrm{D}'_0$ *and* $\delta_0=\delta'_0\circ\varphi$.

Proof. Start with a generic σ-selfdual type contained in π. Since it is unique up to G^σ-conjugacy, we may assume this is $(\mathbf{J},\boldsymbol{\lambda})$. Fix a σ-selfdual stratum $[\mathfrak{a}',\beta']$ such that $\mathbf{J}=\mathbf{J}(\mathfrak{a}',\beta')$. By [Anandavardhanan et al. 2018, Lemma 4.29], we may assume that the maximal tamely ramified subextension of $\mathrm{E}'=\mathrm{F}[\beta']$ over F is equal to T. Fix a distinguished σ-selfdual representation $\boldsymbol{\kappa}'$ of $\mathbf{J}$ extending η, let $\boldsymbol{\rho}'$ be the representation of $\mathbf{J}$ trivial on J^1 corresponding to this choice and $(\mathrm{K}'/\mathrm{E}',\xi')$ be an admissible pair of level 0 attached to $\boldsymbol{\rho}'$. This gives us a quadratic extension $\mathrm{D}'/\mathrm{D}'_0$ and a character δ'_0 of $\mathrm{D}_0'^\times$.

First, suppose that $[\mathfrak{a}',\beta']=[\mathfrak{a},\beta]$ and $\mathrm{K}'=\mathrm{K}$. We have $\boldsymbol{\kappa}'=\boldsymbol{\kappa}\phi$ for some character ϕ of $\mathbf{J}$ trivial on $(\mathbf{J}\cap\mathrm{G}^\sigma)\mathrm{J}^1$, thus $\boldsymbol{\rho}'$ is isomorphic to $\boldsymbol{\rho}\phi^{-1}$. Thus ξ' is E-isomorphic to $\xi\alpha^{-1}$ for some tamely ramified character α of $\mathrm{K}^\times$ trivial on $\mathrm{K}_0^\times$. Restricting to D_0, we get $\delta'_0=\delta_0$.

We now go back to the general case. By the previous argument, we may assume that $\boldsymbol{\kappa}'=\boldsymbol{\kappa}$, thus $\boldsymbol{\rho}'=\boldsymbol{\rho}$. Since D and D' are both unramified of same degree m over T, they are T-isomorphic. Let us fix a T-isomorphism $f:\mathrm{D}\to\mathrm{D}'$. Write σ' for the involutive automorphism of K' given by Proposition 5.19.

Lemma 10.6. *We have* $\sigma'\circ f=f\circ\sigma$.

Proof. Let us identify the residual fields of K and D, denoted $\boldsymbol{t}$, and those of E and T, denoted $\boldsymbol{l}$. Note that, if φ is any T-automorphism of D, then it commutes with σ since φ and $\sigma\circ\varphi\circ\sigma^{-1}$ are both in Gal(D/T) and have the same image in $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{l})$.

We now consider the pair $(\mathrm{D}/\mathrm{T},\xi|_{\mathrm{D}^\times})$. Since D/T is unramified of degree m, it is admissible of level 0. Moreover, the $\boldsymbol{l}$-regular character of $\boldsymbol{t}^\times$ it induces is $\mathrm{Gal}(\boldsymbol{t}/\boldsymbol{l})$-conjugate to the one induced by $(\mathrm{K}/\mathrm{E},\xi)$, which doesn't depend on the identifications of residual fields we have made. We have a similar result for $(\mathrm{D}/\mathrm{T},\xi'\circ f)$ and $(\mathrm{K}'/\mathrm{E}',\xi')$. Since $\boldsymbol{\rho}'=\boldsymbol{\rho}$ we deduce that $\xi'\circ f=\xi\circ\varphi$ for some $\varphi\in\mathrm{Gal}(\mathrm{D}/\mathrm{T})$. Let α be the T-automorphism $\sigma'\circ f\circ\sigma^{-1}\circ f^{-1}$ of D'. We have

$$\xi'\circ\alpha=\xi'^{-1}\circ f\circ\sigma^{-1}\circ f^{-1}=\xi^{-1}\circ\varphi\circ\sigma^{-1}\circ f^{-1}=\xi\circ\varphi\circ f^{-1}=\xi'.$$

It follows from admissibility of ξ' that α is trivial, as expected. ☐

Lemma 10.6 implies that D_0 and D'_0 are T_0-isomorphic. We thus now may assume that $\mathrm{D} = \mathrm{D}'$ and $\mathrm{D}_0 = \mathrm{D}'_0$, thus K, K′ have the same maximal unramified subextension D over T and there is an automorphism $\varphi \in \mathrm{Gal}(\mathrm{D}/\mathrm{T})$ such that $\xi'(x) = \xi \circ \varphi(x)$ for all $x \in \mathrm{D}^\times$. Restricting to $\mathrm{D}_0^\times$, we deduce that $\delta'_0 = \delta_0$. □

Remark 10.7. In particular, the character ε_0 of Paragraph 9C, which is the restriction of δ_0 to $\mathrm{T}_0^\times$, is uniquely determined by π.

We state the dichotomy and disjunction theorem.

Theorem 10.8. *Let π be a σ-selfdual supercuspidal representation of* G. *Let ℓ be the characteristic of* R.

(1) *If $\ell \neq 2$, then π is either distinguished or ω-distinguished, but not both.*

(2) *If $\ell = 2$, then π is always distinguished.*

Proof. See Remarks 8.5 and 9.14. □

We now state the distinction criterion theorem.

Theorem 10.9. *Let π be a σ-selfdual supercuspidal representation of* G. *Attached to it, there are the quadratic extensions* T/T_0 *and* D/D_0 *and the character* δ_0.

(1) *Suppose that n is odd. Then π is distinguished if and only if its central character is trivial on* $\mathrm{F}_0^\times$.

(2) *If $\ell \neq 2$,* T/T_0 *is ramified and* D/D_0 *is unramified, then π is distinguished if and only if the character δ_0 is nontrivial.*

(3) *Otherwise, π is distinguished if and only if δ_0 is trivial.*

Proof. Item (1) is an immediate consequence of Theorem 10.8 as explained in Paragraph 1D. If $\ell \neq 2$, a σ-selfdual supercuspidal representation π is either distinguished or ω-distinguished. In the latter case, the restriction of its central character to $\mathrm{F}_0^\times$ is ω^n, which is trivial if and only if n is even. See also Remarks 8.7 and 9.16.

For the remaining items, see Theorems 8.4 and 9.13; it suffices to check that, if T/T_0 is unramified, then δ_0 is trivial if and only if its restriction ε_0 to $\mathrm{T}_0^\times$ is trivial, which follows from the fact that m is odd in that case. □

Remark 10.10. The following conditions are equivalent:

(1) D/D_0 is ramified.

(2) T/T_0 is ramified and $m = 1$.

Indeed this follows from Remark 5.16, Lemma 5.20 and Proposition 8.1. The following conditions are thus also equivalent:

(1) T/T_0 is ramified and D/D_0 is unramified.

(2) F/F_0 is ramified, $\mathrm{T}_0/\mathrm{F}_0$ has odd ramification order and D/D_0 is unramified.

(3) F/F_0 is ramified, $\mathrm{T}_0/\mathrm{F}_0$ has odd ramification order and $m \neq 1$.

We now state the distinguished lift theorem. For the notion of the reduction mod ℓ of an integral irreducible $\overline{\mathbf{Q}}_\ell$-representation of G, we refer to [Vignéras 1996; 2004]. If π is an irreducible $\overline{\mathbf{F}}_\ell$-representation of G, we say an integral irreducible $\overline{\mathbf{Q}}_\ell$-representation of G is a *lift* of π if its reduction mod ℓ is irreducible and isomorphic to π.

Theorem 10.11. *Let π be a σ-selfdual supercuspidal $\overline{\mathbf{F}}_\ell$-representation of* G.

(1) *The representation π admits a σ-selfdual supercuspidal lift to $\overline{\mathbf{Q}}_\ell$.*

(2) *Let $\tilde{\pi}$ be a σ-selfdual lift of π, and suppose that $\ell \neq 2$. Then $\tilde{\pi}$ is distinguished if and only if π is distinguished.*

Proof. Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-selfdual type in π. Let η be the Heisenberg representation contained in the restriction of $\boldsymbol{\lambda}$ to J^1, with associated simple character θ. As in the proof of Lemma 7.7, let $\tilde{\theta}$ be the lift of θ with values in $\overline{\mathbf{Q}}_\ell$ and $\tilde{\eta}$ be the associated Heisenberg representation, whose reduction mod ℓ is isomorphic to η. Propositions 7.9 and 9.4 tell us that there is a distinguished σ-selfdual representation $\tilde{\boldsymbol{\kappa}}$ of $\mathbf{J}$ which extends $\tilde{\eta}$. Its reduction mod ℓ, denoted $\boldsymbol{\kappa}$, is a σ-selfdual representation of $\mathbf{J}$ extending η and it is distinguished thanks to Lemma 2.6. (Note that, as in the proof of Lemma 8.2, the fact that $\boldsymbol{\kappa}$ is σ-selfdual implies that it has finite image; it thus can be considered as a representation of a finite group.) Let $\boldsymbol{\rho}$ be the irreducible representation of $\mathbf{J}$ trivial on J^1 such that $\boldsymbol{\lambda}$ is isomorphic to $\boldsymbol{\kappa} \otimes \boldsymbol{\rho}$. It is σ-selfdual.

The representation $\boldsymbol{\rho}$ admits a σ-selfdual $\overline{\mathbf{Q}}_\ell$-lift $\tilde{\boldsymbol{\rho}}$ on $\mathbf{J}$ trivial on J^1. Indeed, let $(\mathrm{K}/\mathrm{E}, \xi)$ be an admissible pair of level 0 attached to $\boldsymbol{\rho}$. Then, as in the proof of Lemma 8.2, the canonical $\overline{\mathbf{Q}}_\ell$-lift $\tilde{\xi}$ of ξ defines a pair $(\mathrm{K}/\mathrm{E}, \tilde{\xi})$ which is admissible of level 0, and the $\overline{\mathbf{Q}}_\ell$-representation $\tilde{\boldsymbol{\rho}}$ of $\mathbf{J}$ trivial on J^1 which is attached to it is both σ-selfdual and a lift of $\boldsymbol{\rho}$. The representation $\tilde{\boldsymbol{\kappa}} \otimes \tilde{\boldsymbol{\rho}}$ is thus a σ-selfdual ℓ-adic type whose reduction mod ℓ is $\boldsymbol{\lambda}$. Inducing $\tilde{\boldsymbol{\kappa}} \otimes \tilde{\boldsymbol{\rho}}$ to G, we get a σ-selfdual supercuspidal lift $\tilde{\pi}$ of π.

Suppose that $\ell \neq 2$ and let $\tilde{\omega}$ be the canonical ℓ-adic lift of ω, that is, the $\overline{\mathbf{Q}}_\ell$-character of $\mathrm{F}_0^\times$ of kernel $\mathrm{N}_{\mathrm{F}/\mathrm{F}_0}(\mathrm{F}^\times)$. By Theorem 10.8, since the representation $\tilde{\pi}$ is σ-selfdual, it is either distinguished or $\tilde{\omega}$-distinguished, but not both. Using the reduction argument of invariant linear forms as in Lemma 2.6, we see that if $\tilde{\pi}$ is distinguished (respectively, $\tilde{\omega}$-distinguished), then π is distinguished (respectively, ω-distinguished). By Theorem 10.8 applied to π, this is an equivalence. □

We end with the following result, which is useful in [Anandavardhanan et al. 2018].

Proposition 10.12. *Suppose that π is a distinguished supercuspidal representation of* G *and that $\ell \neq 2$. Then π has no ω-distinguished unramified twist if and only if* D/D_0 *is ramified, that is, if and only if* T/T_0 *is ramified and $m = 1$.*

Proof. Consider an unramified twist $\pi' = \pi(\chi \circ \det)$ of π, where χ is an unramified character of $\mathrm{F}^\times$. We are looking for a χ such that π' is ω-distinguished. First, π' is σ-selfdual if and only if $\pi\chi(\chi \circ \sigma) \simeq \pi$, that is, if and only if

$$(\chi(\chi \circ \sigma))^{t(\pi)} = \chi^{2t(\pi)} = 1 \tag{10-1}$$

where $t(\pi)$ denotes the torsion number of π, that is the number of unramified characters α of G such that $\pi\alpha \simeq \pi$. By [Mínguez and Sécherre 2014b, Section 3.4], we have $t(\pi) = f(\mathrm{K}/\mathrm{F}) = f(\mathrm{D}/\mathrm{F})$. Now the quadratic character associated with π' is $\delta'_0 = \delta_0(\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})|_{\mathrm{D}_0^\times}$ and we have

$$(\chi \circ \mathrm{N}_{\mathrm{K}/\mathrm{F}})|_{\mathrm{D}_0^\times} = (\chi \circ \mathrm{N}_{\mathrm{D}_0/\mathrm{F}_0})^{[\mathrm{K}:\mathrm{D}]}. \tag{10-2}$$

By Theorem 10.11, the representation π' is ω-distinguished if and only if the character (10-2) is equal to $\omega_{\mathrm{D}/\mathrm{D}_0}$. If D is ramified over D_0, this is not possible since χ is unramified. If D/D_0 is unramified, choosing an unramified character χ of order $f(\mathrm{D}_0/\mathrm{F}_0)$ gives us an ω-distinguished twist π'. □

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A vanishing result for higher smooth duals

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In this paper we prove a general vanishing result for Kohlhaase's higher smooth duality functors S^i. If G is any unramified connected reductive p-adic group, K is a hyperspecial subgroup, and V is a Serre weight, we show that $S^i(\mathrm{ind}_K^G V)=0$ for $i>\dim(G/B)$, where B is a Borel subgroup and the dimension is over $\mathbb{Q}_p$. This is due to Kohlhaase for $\mathrm{GL}_2(\mathbb{Q}_p)$, in which case it has applications to the calculation of S^i for supersingular representations. Our proof avoids explicit matrix computations by making use of Lazard theory, and we deduce our result from an analogous statement for graded algebras via a spectral sequence argument. The graded case essentially follows from Koszul duality between symmetric and exterior algebras.

1. Introduction

Dual representations are ubiquitous in the classical Langlands program. For one thing they appear in the functional equation for automorphic L-functions. In the p-adic Langlands program, and its counterpart modulo p, dual representations play a prominent role as well. Say π is a smooth representation of some p-adic reductive group G, having coefficients in a subfield $E\subset\bar{\mathbb{F}}_p$. Then the full linear dual $\pi^\vee=\mathrm{Hom}_E(\pi,E)$ lives in a completely different category of (pseudocompact) modules over the Iwasawa

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algebra $\Lambda(G) = E[G] \otimes_{E[K]} E[\![K]\!]$ where $K \subset G$ is an arbitrary compact open subgroup. One can of course take the smooth vectors of $\pi^\vee$ and get a smooth representation $S^0(\pi) = (\pi^\vee)^\infty$ — but this is often zero! In fact $S^0(\pi) = 0$ for all irreducible admissible π which are infinite-dimensional. Moreover, the smooth dual functor S^0 is not exact. Kohlhaase [2017] extended S^0 to a δ-functor consisting of certain higher smooth duality functors

$$S^i : \mathrm{Rep}_E^\infty(G) \to \mathrm{Rep}_E^\infty(G)$$

defined as $\Sigma^i \circ (\cdot)^\vee$ where Σ^i extends the smooth vectors functor $(\cdot)^\infty$ ("stable cohomology"), and he establishes foundational results on the functorial properties of the S^i. Some aspects of this are analogous to (and in fact rely on) the theory of Schneider and Teitelbaum [2005] for locally analytic representations in which they (among other things) produce an involutive functor from $D^b_{\mathrm{adm}}(\mathrm{Rep}_{\mathcal{E}}^\infty(G))$ to itself, which is compatible with a natural duality on the derived category of coadmissible modules over the locally analytic distribution algebra $D(G, \mathcal{E})$. Here $\mathcal{E}/\mathbb{Q}_p$ is a p-adic field and D^b_{adm} is the bounded derived category of complexes with admissible cohomology, see [Schneider and Teitelbaum 2005, Corollary 3.7] and the diagram at the bottom of page 317 there. Kohlhaase does not mention it explicitly, but one can easily prove the analogous result for mod p coefficients E and define a functor S from $D^b(\mathrm{Rep}_E^\infty(G))$ to itself which becomes involutive on the subcategory $D^b_{\mathrm{adm}}(\mathrm{Rep}_E^\infty(G))$. Moreover, there is a spectral sequence starting from $S^i(h^j(V^\bullet))$ converging to the cohomology of $S(V^\bullet)$. In fact Emerton gave a series of lectures at the Institut Henri Poincaré in March 2010 in which he introduced a derived duality functor $\mathbb{D}_G$ similar to S which goes into the proof of the Poincaré duality spectral sequence for completed (Borel–Moore) homology. His derived category "$D^b(G)$" is a bit subtler however.

Schneider [2015, Theorem 9] proved that the unbounded derived category $D(\mathrm{Rep}_E^\infty(G))$ is equivalent to the derived category of differential graded modules over a certain DGA variant of the Hecke algebra $\mathcal{H}_I^\bullet$, defined relative to a torsion free pro-p group $I \subset G$ (and a choice of injective resolution). A key ingredient was [Schneider 2015, Proposition 6] which shows that $\mathrm{ind}_I^G 1$ is a (compact) generator of $D(\mathrm{Rep}_E^\infty(G))$. From this point of view, to understand S on the whole derived category of G we should first understand the higher smooth duals of the generator $S^i(\mathrm{ind}_I^G 1)$. In fact this ties to a question posed by Harris [2016, Question 4.5] as to whether there is any relation between Kohlhaase's S^i and E-linear duality on $D(\mathcal{H}_I^\bullet)$. We know $S^i = 0$ for $i > \dim(G)$. (Here and throughout the paper $\dim(\cdot)$ refers to the dimension as a $\mathbb{Q}_p$-manifold.) Some of our main motivation for writing this paper was to show that at least for the generator $\mathrm{ind}_I^G 1$ one can improve this bound significantly and show that $S^i(\mathrm{ind}_I^G 1) = 0$ for $i > \dim(G/B)$ where B is a Borel subgroup. This follows from our main result (Theorem 1.1) below by taking $V = \mathrm{ind}_I^K 1$ there. In fact an easy inductive argument shows more generally that

$$S^i(\pi) = 0 \quad \forall i > \dim(G/B) + \ell$$

if there is an exact sequence of length ℓ of the form

$$0 \to (\mathrm{ind}_I^G 1)^{\oplus r_\ell} \to \cdots \to (\mathrm{ind}_I^G 1)^{\oplus r_1} \to (\mathrm{ind}_I^G 1)^{\oplus r_0} \to \pi \to 0$$

for some $r_i \in \mathbb{Z}_{>0}$. Of course this bound on i is trivial unless $\ell \leq \dim(B)$. We do not know in which generality such resolutions exist for a general group G.[1] For groups of semisimple rank one, Kohlhaase [2018] has constructed a class of representations to which our bound applies. We also do not know whether the bound $\dim(G/B)$ is sharp for $\pi = \mathrm{ind}_I^G 1$ in the sense that $S^{\dim(G/B)}(\mathrm{ind}_I^G 1)$ is nonzero — even in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$.

We now state the main result of this paper which we alluded to above. Let $F/\mathbb{Q}_p$ be a finite extension, and $G_{/F}$ a connected reductive group with a Borel subgroup B. We assume that G is unramified[2] and choose a hyperspecial maximal compact subgroup $K \subset G$ along with a finite-dimensional smooth representation $K \to \mathrm{GL}(V)$ with coefficients in some algebraic extension $E/\mathbb{F}_p$. Let $\mathrm{ind}_K^G V$ be the compact induction. Then we have the following vanishing result for its higher smooth duals S^i.

Theorem 1.1. $S^i(\mathrm{ind}_K^G V) = 0$ *for all* $i > \dim(G/B)$.

What we actually prove is a slightly stronger result on the vanishing of the transition maps. Namely, if $N \lhd K$ has an Iwahori factorization and acts trivially on V, then the restriction map

$$\mathrm{Ext}^i_{\Lambda(N)}(E, (\mathrm{ind}_K^G V)^\vee) \to \mathrm{Ext}^i_{\Lambda(N^{p^m})}(E, (\mathrm{ind}_K^G V)^\vee)$$

vanishes for $i > \dim(G/B)$ as long as m is greater than some constant depending only on i and V, N (and an auxiliary filtration on N). Here $(-)^\vee$ denotes Pontryagin duality, and $\Lambda(N) = E[\![N]\!]$ is the completed group algebra over E. Note that the *set* of p^m-powers N^{p^m} is a group for m large enough for any p-valuable group N, see [Schneider 2011, Remark 26.9].

We have been unable to show that $S^{\dim(G/B)}(\mathrm{ind}_K^G V) \neq 0$ but we believe this should be true under a suitable regularity condition on the weight V, see Section 12. We hope to address this in future work, and to say more about the action of Hecke operators on $S^i(\mathrm{ind}_K^G V)$ for all i. Let us add that the bound $\dim(G/B)$ is not sharp for all V. Indeed $S^i(\mathrm{ind}_K^G 1) = 0$ for all $i > 0$, see Remark 11.2.

For $\mathrm{GL}_2(\mathbb{Q}_p)$ Theorem 1.1 amounts to [Kohlhaase 2017, Theorem 5.11], which is one of the main results of that paper. There is a small difference coming from the center $Z \simeq \mathbb{Q}_p^\times$ though. He assumes V is an irreducible representation of $K = \mathrm{GL}_2(\mathbb{Z}_p)$ which factors through $\mathrm{GL}_2(\mathbb{F}_p)$, extends the central character of V to Z by sending $p \mapsto 1$, and considers $\mathrm{ind}_{KZ}^G V$ instead. The latter carries a natural Hecke operator $T = T_V$ whose cokernel π_V is an irreducible supersingular representation. As V varies this gives all supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (with p acting trivially), see [Barthel and Livné 1994, Proposition 4] and [Breuil 2003, Theorem 1.1]. The short exact sequence

$$0 \to \mathrm{ind}_{KZ}^G V \xrightarrow{T} \mathrm{ind}_{KZ}^G V \to \pi_V \to 0$$

gives rise to a long exact sequence of higher smooth duals

$$\cdots \to S^i(\pi_V) \to S^i(\mathrm{ind}_{KZ}^G V) \xrightarrow{S^i(T)} S^i(\mathrm{ind}_{KZ}^G V) \to S^{i+1}(\pi_V) \to \cdots .$$

[1]Ollivier [2014, Theorem 1.1] has constructed analogous resolutions when π is a principal series representation of GL_n.

[2]Experts have informed us this restriction is likely to be unnecessary (with K special and B a minimal parabolic in general).

Since $S^i(\mathrm{ind}_{KZ}^G V) = 0$ for $i > 1$ it follows immediately that $S^3(\pi_V) = S^4(\pi_V) = \cdots = 0$. Kohlhaase [2017, page 45] takes this a step further and shows that also $S^2(\pi_V) = 0$ by checking that $S^1(T)$ is bijective, which is also a key ingredient in his long verification that $S^1(\pi_V) \simeq \pi_{\check{V}}$. The latter is another of his main results, [Kohlhaase 2017, Theorem 5.13]. Essentially what this shows is that via the mod p local Langlands correspondence $\rho \rightsquigarrow \pi(\rho)$ the dual Galois representation corresponds to S^1. More precisely $\pi(\rho^\vee) \otimes \omega_{\mathrm{cyc}} = S^1(\pi(\rho))$ for Galois representations $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ (not a twist of an extension of ω_{cyc} by 1). See the conclusion in [Kohlhaase 2017, Remark 5.15].

Similar descriptions of the supersingular representations exist for other rank one groups. Abdellatif [2014] classifies the supersingulars of $\mathrm{SL}_2(\mathbb{Q}_p)$ by decomposing $\pi_V|_{\mathrm{SL}_2(\mathbb{Q}_p)}$ into irreducibles, and Kozioł [2016] was able to bootstrap from this case and deal with the unramified unitary group $U(1,1)$ over $\mathbb{Q}_p$. As an application of our Theorem 1.1 there is forthcoming work of Jake Postema for his UC San Diego Ph.D. thesis in which he calculates all the S^i for supersingular representations of both $\mathrm{SL}_2(\mathbb{Q}_p)$ and $U(1,1)$, and explores how S^1 flips the members of an L-packet.

Let us give a roadmap of our proof of Theorem 1.1 and then point out some analogies and discrepancies with Kohlhaases's approach. First of all $(\mathrm{ind}_K^G V)^\vee$ is the full induction $I_K^G \check{V}$ and we need to understand the restriction maps between the various $\mathrm{Ext}^i_{\Lambda(N)}(E, I_K^G \check{V})$ for N deep enough. We start with a normal subgroup $N \lhd K$ which has an Iwahori factorization (relative to B). This means among other things that multiplication defines a homeomorphism

$$(N \cap \bar{U}) \times (N \cap T) \times (N \cap U) \xrightarrow{\sim} N, \tag{1.2}$$

where $B = TU$. Moreover, conjugation by $s \in S^+$ contracts and expands the rightmost and leftmost factors respectively. Here $S \subset T$ is a maximal split subtorus, and S^+ is the monoid of elements $s \in S$ satisfying $|\alpha(s)| \leq 1$ for all roots $\alpha > 0$. Essentially by the Cartan decomposition, we get an isomorphism of $E[K]$-modules

$$\mathrm{Ext}^i_{\Lambda(N)}(E, I_K^G \check{V}) \xrightarrow{\sim} \prod_{s \in S^+/S(\mathcal{O})} \mathrm{ind}_{NK^s}^K \mathrm{Ext}^i_{\Lambda(N \cap K^s)}(E, \check{V}^s),$$

compatible with restriction maps on either side. Here $K^s := K \cap s^{-1}Ks$, and $\check{V}^s$ denotes the representation of K^s on $\check{V}$ obtained by $\kappa *_s \check{v} := (s\kappa s^{-1})\check{v}$. It therefore suffices to understand the restriction maps between the various $\mathrm{Ext}^i_{\Lambda(N \cap K^s)}(E, \check{V}^s)$ well enough, and control what happens when we vary s. For the sake of exposition let us first outline the argument in the case $s = 1$. Here the goal is to show the vanishing of

$$\mathrm{Ext}^i_{\Lambda(N)}(E, \check{V}) \to \mathrm{Ext}^i_{\Lambda(N')}(E, \check{V})$$

for $i > \dim(G/B)$ and $N' \subset N$ sufficiently small. To run the argument below for general s it is important *not* to assume N acts trivially on V here (otherwise the map would vanish for $i > 0$ and $N' = N^p$). The point is that s contracts $N \cap U$ but expands $N \cap \bar{U}$. Thus we can safely assume $N \cap U$ acts trivially on V, but we do not know this for the factor $N \cap \bar{U}$ (after conjugating by s).

The basic idea is to prove the analogous vanishing result for certain graded algebras $\operatorname{gr}\Lambda(N)$ and then employ a spectral sequence argument elaborated upon in the Appendix. This is based on Lazard theory. There is a natural way to endow N with a p-valuation ω which behaves well with respect to the Iwahori factorization, and although (1.2) is not an isomorphism of groups it does become one once we pass to graded groups. Consequently, there is an isomorphism of graded algebras

$$\operatorname{gr}\Lambda(N\cap\bar{U})\otimes\operatorname{gr}\Lambda(N\cap T)\otimes\operatorname{gr}\Lambda(N\cap U)\xrightarrow{\sim}\operatorname{gr}\Lambda(N),$$

and one obtains a Künneth formula for $\operatorname{Ext}^i_{\operatorname{gr}\Lambda(N)}(E,\operatorname{gr}\check{V})$ expressing it as a direct sum

$$\bigoplus_{i_1+i_2+i_3=i}\operatorname{Ext}^{i_1}_{\operatorname{gr}\Lambda(N\cap\bar{U})}(E,\operatorname{gr}\check{V})\otimes\operatorname{Ext}^{i_2}_{\operatorname{gr}\Lambda(N\cap T)}(E,E)\otimes\operatorname{Ext}^{i_3}_{\operatorname{gr}\Lambda(N\cap U)}(E,E).$$

The (i_2,i_3)-factors can be dealt with using the Koszul resolution, which shows for instance that

$$\operatorname{Ext}^{i_2}_{\operatorname{gr}\Lambda(N\cap T)}(E,E)\simeq\bigwedge^{i_2}(E\otimes_{\mathbb{F}_p[\pi]}\operatorname{gr}\Lambda(N\cap T))^*$$

where π is the usual p-power operator from Lazard theory. We have tacitly perturbed ω if necessary to make the Lie algebra $\operatorname{gr}N$ abelian. The restriction map $\operatorname{Ext}^i_{\operatorname{gr}\Lambda(N)}(E,\operatorname{gr}\check{V})\to\operatorname{Ext}^i_{\operatorname{gr}\Lambda(N^p)}(E,\operatorname{gr}\check{V})$ therefore vanishes on the (i_1,i_2,i_3)-summand unless $i_2=i_3=0$. So we only need to worry about the summand with $i_1=i$. But the factor $\operatorname{Ext}^i_{\operatorname{gr}\Lambda(N\cap\bar{U})}(E,\operatorname{gr}\check{V})$ itself vanishes in the range $i>\dim(\bar{U})$ by well-known results on the cohomological dimension. This is where the bound $\dim(G/B)=\dim(\bar{U})$ comes from. Altogether this shows that

$$\operatorname{Ext}^i_{\operatorname{gr}\Lambda(N)}(E,\operatorname{gr}\check{V})\xrightarrow{0}\operatorname{Ext}^i_{\operatorname{gr}\Lambda(N^p)}(E,\operatorname{gr}\check{V})\quad\forall i>\dim(G/B).$$

Since $\Lambda(N)$ is a filtered algebra, its bar resolution inherits a filtration which results in a convergent spectral sequence (see the Appendix where this is explained in detail) which is functorial in N,

$$E_1^{i,j}=\operatorname{Ext}^{i,j}_{\operatorname{gr}\Lambda(N)}(E,\operatorname{gr}\check{V})\Rightarrow\operatorname{Ext}^{i+j}_{\Lambda(N)}(E,\check{V}).$$

One deduces that the restriction map $\operatorname{Ext}^i_{\Lambda(N)}(E,\check{V})\to\operatorname{Ext}^i_{\Lambda(N^p)}(E,\check{V})$ is zero on the graded pieces, that is it takes Fil^j to Fil^{j+1} for all j. This filtration is both exhaustive and separated, so doing this repeatedly shows that $\operatorname{Ext}^i_{\Lambda(N)}(E,\check{V})\to\operatorname{Ext}^i_{\Lambda(N^{p^m})}(E,\check{V})$ vanishes for large enough m. In fact we can control how large (and this is crucial when we vary s). We may assume $\operatorname{gr}\Lambda(N)$ is Koszul (even a polynomial algebra) in which case we understand the internal grading $\operatorname{Ext}^{i,j}_{\operatorname{gr}\Lambda(N)}$ better and it suffices to take $m>\mu+i$ where μ is the first index for which $\operatorname{Fil}^{\mu}\Lambda(N)\check{V}=0$. We end with a few remarks on the case of arbitrary $s\in S^+$. The argument above still works, and shows that

$$\operatorname{Ext}^i_{\Lambda(N\cap K^s)}(E,\check{V}^s)\xrightarrow{0}\operatorname{Ext}^i_{\Lambda(N^{p^m}\cap K^s)}(E,\check{V}^s)\quad\forall i>\dim(G/B)$$

as long as $m>\mu+i$. It is of course extremely important that our invariant μ is independent of s. Getting this lower bound on m which is uniform in $s\in S^+$ is indeed a key point of the whole paper.

To guide the reader we now point out some similarities and differences between our paper and [Kohlhaase 2017]. Overall Kohlhaase argues more from the point of view of [Dixon et al. 1991] whereas we lean more towards [Lazard 1965]. His groups G_n are analogous to our K^s (the difference being the center Z). On the other hand, where we choose to work with N^{p^m} he is considering the full congruence subgroups $K_m = \ker(\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Z}/p^m\mathbb{Z}))$. The first few formal steps, using the Cartan and Mackey decompositions, are identical. In Kohlhaase the problem is reduced to understanding the restriction maps between $\mathrm{Ext}^i_{\Lambda(\mathcal{U}_{m,n})}(E, \check{V})$ where $\mathcal{U}_{m,n} = \alpha^{-n}(G_n \cap K_m)\alpha^n$ corresponds to our $K \cap sN^{p^m}s^{-1}$. The key point where Kohlhaase uses he is in the $\mathrm{GL}_2(\mathbb{Q}_p)$ situation is to show that these groups $\mathcal{U}_{m,n}$ are uniform pro-p groups (certain finitely generated torsion-free groups $\mathcal{U}$ for which $[\mathcal{U}, \mathcal{U}] \subset \mathcal{U}^p$ — at least for $p > 2$). Based on the Iwahori factorization of $\mathcal{U}_{m,n}$ this becomes case-by-case explicit matrix computations exhibiting various commutators as p-powers (see [Kohlhaase 2017, page 40]). One of the goals of our paper was to do this is a more conceptual way using Lazard's theory of p-valuations. Knowing $\mathcal{U}_{m,n}$ is uniform (for $m \geq 2$) Kohlhaase invokes [Dixon et al. 1991, Theorem 7.24] to get an isomorphism $\mathrm{gr}\,\Lambda(\mathcal{U}_{m,n}) \simeq E[X_1, X_2, X_3, X_4]$, where one puts the $\mathfrak{m}$-adic filtration on $\Lambda(\mathcal{U}_{m,n})$. This isomorphism is also a fundamental result in Lazard theory where more generally one identifies $\mathrm{gr}\,\Lambda(\mathcal{U})$ with $U(\mathrm{gr}\,\mathcal{U})$ for any p-valued group $\mathcal{U}$. At this point our calculations on the "graded side" more or less agree. However, to pass from the graded situation and deduce vanishing of the restriction maps for $\Lambda(\mathcal{U}_{m,n})$ itself Kohlhaase employs a result of Grünenfelder [1979] to the effect that there are functorial isomorphisms

$$\mathrm{Ext}^i_{\Lambda(\mathcal{U}_{m,n})}(E, \check{V}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{gr}\,\Lambda(\mathcal{U}_{m,n})}(E, \mathrm{gr}\,\check{V}) \tag{1.3}$$

dual to an analogue for Tor_i which supposedly is [Grünenfelder 1979, Theorem 3.3']. We have not been able to understand the details of Grünenfelder's paper, and we have been unsuccessful[3] in establishing the isomorphism (1.3) directly for g weights V; even for $i = 0$. Instead we use a spectral sequence similar to the one used in [Björk 1987, page 72] of the form

$$E_1^{i,j} = \mathrm{Ext}^{i,j}_{\mathrm{gr}\,\Lambda(\mathcal{U})}(E, \mathrm{gr}\,\check{V}) \Rightarrow \mathrm{Ext}^{i+j}_{\Lambda(\mathcal{U})}(E, \check{V}).$$

However, [Björk 1987] does not contain quite what we need. To get a spectral sequence which is functorial in $\mathcal{U}$ we have to work with bar resolutions instead of the resolutions arising from "good" filtrations as in [Björk 1987]. This spectral sequence and its applications are discussed at great length in the Appendix, which also treats the Koszul case where the sequence simplifies due to the vanishing of most $E_1^{i,j}$.

2. Notation

Let p be a prime. Fix a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Choose a uniformizer ϖ and let $k = \mathcal{O}/\varpi\mathcal{O}$ be the residue field, with cardinality $q = p^f$. We normalize the absolute value $|\cdot|$ on F such that $|\varpi| = q^{-1}$. Thus $|p| = q^{-e}$ where e is the ramification index.

[3]In fact the first paragraph of [Polo and Tilouine 2002, Sectection 3.4] indicates there are counterexamples to [Grünenfelder 1979, Theorem 3.3'].

We take $G_{/F}$ to be a connected reductive F-group, assumed to be unramified. The latter means G is quasisplit (over F) and $G \times_F F'$ is split for some finite unramified extension F'/F. Such groups admit integral models, see [Tits 1979, 3.8]. More precisely G extends to a smooth affine group scheme $G_{/\mathcal{O}}$ for which $G \times_{\mathcal{O}} k$ is connected and reductive. We fix such a model once and for all and let $K = G(\mathcal{O})$ be the corresponding hyperspecial maximal compact subgroup of $G(F)$. When there is no risk of confusion we will just write G instead of $G(F)$; similarly for any linear algebraic F-group.

Choose a maximal F-split subtorus $S \subset G$ and let $T = Z_G(S)$ which is then a maximal F-subtorus. They extend naturally to smooth $\mathcal{O}$-subgroup schemes $S \subset T$ of G (which reduce to a maximal k-split torus and its centralizer in the special fiber). Let $\Phi = \Phi(G, S) \subset X^*(S)$ be the relative root system, which we interchangeably view in the generic or special fiber. Once and for all we choose a system of positive roots Φ^+ and let $B = T \ltimes U$ be the corresponding Borel subgroup, which also extends to a closed $\mathcal{O}$-subgroup scheme of $G_{/\mathcal{O}}$. We let $\bar{B} = T \ltimes \bar{U}$ denote the opposite Borel subgroup ($B \cap \bar{B} = T$).

3. The dual of compact induction

Throughout the paper we let $E/\mathbb{F}_p$ be a fixed algebraic extension which will serve as the coefficient field of our representations. We start with a finite-dimensional smooth E-representation V of $K = G(\mathcal{O})$. Note that $K_r = \ker(G(\mathcal{O}) \to G(\mathcal{O}/\varpi^r\mathcal{O}))$ necessarily acts trivially for r sufficiently large. Let r_V be the smallest $r \geq 1$ for which $K \to \mathrm{GL}(V)$ factors through K/K_r. In applications V will typically be a Serre weight, meaning it arises from an absolutely irreducible rational representation of $G \times_{\mathcal{O}} E$ (via some choice of embedding $k \hookrightarrow E$) but this restriction is unnecessary.

Let $\check{V} = \mathrm{Hom}_E(V, E)$ be the full E-linear dual with the contragredient K-action. This is again a smooth representation, and obviously $r_{\check{V}} = r_V$. We let $\langle \cdot, \cdot \rangle_V$ denote the natural pairing $\check{V} \times V \to E$.

Our conventions on induced representations differ from [Kohlhaase 2017, page. 38]. We let G act by right translations, whereas Kohlhaase lets G act by left translations.

Definition 3.1. $I_K^G V$ is the full induction of *all* functions $f : G \to V$ such that $f(\kappa g) = \kappa f(g)$ for all $\kappa \in K$ and $g \in G$. We let $\mathrm{Ind}_K^G V$ be the subrepresentation of smooth functions f (i.e., invariant under right-translation by an open subgroup). Finally $\mathrm{ind}_K^G V \subset \mathrm{Ind}_K^G V$ consists of smooth functions f which are compactly supported (so $\mathrm{supp}(f)$ is the union of finitely many cosets Kg).

Remark 3.2. Upon choosing a set of coset representatives $(g_i)_{i \in I}$ for $K\backslash G$, the map $f \mapsto (f(g_i))_{i \in I}$ defines isomorphisms of E-vector spaces

$$I_K^G V \xrightarrow{\sim} \prod_{i \in I} V \quad \text{and} \quad \mathrm{ind}_K^G V \xrightarrow{\sim} \bigoplus_{i \in I} V.$$

Pontryagin duality $(\cdot)^\vee$ sets up an antiequivalence between the category of smooth representations $\mathrm{Rep}_E^\infty(G)$ and the category $\mathrm{Mod}_{\Lambda(G)}^{\mathrm{pc}}$ of pseudocompact E-vector spaces M with a jointly continuous action $G \times M \to M$, see [Kohlhaase 2017, Theorem 1.5]. The $E[G]$-module structure of any such M extends to a $\Lambda(G)$-module structure, where $\Lambda(G) := E[G] \otimes_{E[K]} E[\![K]\!]$ is the Iwasawa algebra. Here

$E[\![K]\!] = \varprojlim_N E[K/N]$ is the completed group algebra, with $N \lhd K$ ranging over the open normal subgroups.

The compact induction $\mathrm{ind}_K^G V$ is a (nonadmissible) object of $\mathrm{Rep}_E^\infty(G)$. We first identify its Pontryagin dual with $I_K^G \check{V}$. This is [Kohlhaase 2017, Lemma 5.9]; we include the argument for convenience.

Lemma 3.3. *There is an isomorphism $I_K^G \check{V} \xrightarrow{\sim} (\mathrm{ind}_K^G V)^\vee$ of pseudocompact $\Lambda(G)$-modules.*

Proof. We define a G-equivariant pairing $I_K^G \check{V} \times \mathrm{ind}_K^G V \to E$ by the following (finite) sum

$$\langle \check{f}, f \rangle := \sum_{g \in K \backslash G} \langle \check{f}(g), f(g) \rangle_V.$$

The resulting map $I_K^G \check{V} \to (\mathrm{ind}_K^G V)^\vee$ corresponds to the natural map $\prod_{i \in I} \check{V} \to (\bigoplus_{i \in I} V)^\vee$ after choosing coset representatives $(g_i)_{i \in I}$. The latter is trivially an isomorphism (of E-vector spaces). □

Consequently, the compact induction $\mathrm{ind}_K^G V$ has smooth dual the smooth induction of $\check{V}$,

$$S^0(\mathrm{ind}_K^G V) \simeq \mathrm{Ind}_K^G \check{V}.$$

The ultimate goal of this paper is to get a better understanding of the higher smooth duals $S^i(\mathrm{ind}_K^G V)$, see [Kohlhaase 2017, Definition 3.12]. In particular, for which $i > 0$ do they vanish? For $G = \mathrm{GL}_2(\mathbb{Q}_p)$ Kohlhaase has shown that $S^i(\mathrm{ind}_K^G V) = 0$ for $i > 1$, see [Kohlhaase 2017, Theorem 5.11]. We extend his result to arbitrary G.

4. Stable cohomology and the functors S^i

We fix an open $N \lhd K$ for now and let it act trivially on E, which we view as an object of $\mathrm{Mod}_{\Lambda(N)}^{\mathrm{pc}}$. The module structure on E comes from the augmentation map $\epsilon : \Lambda(N) = E[\![N]\!] \to E$. For other objects M of $\mathrm{Mod}_{\Lambda(N)}^{\mathrm{pc}}$ we will explore how $\mathrm{Ext}_{\Lambda(N)}^i(E, M)$ varies when we shrink N. The subscript in $\mathrm{Ext}_{\Lambda(N)}^i$ signifies we take extensions in the abelian category $\mathrm{Mod}_{\Lambda(N)}^{\mathrm{pc}}$ (not the category of abstract left $\Lambda(N)$-modules). Eventually we will be focusing on $M = I_K^G \check{V}$ (which is not coadmissible).

To make the functorial properties of $\mathrm{Ext}_{\Lambda(N)}^i(E, M)$ more transparent we prefer to fix a concrete projective resolution of E which itself is functorial in N. We take the topological bar resolution $B_\bullet \Lambda(N)$, see [Lazard 1965, V(1.2.9)]. Recall that

$$B_n \Lambda(N) = \Lambda(N)^{\hat{\otimes}(n+1)} = \Lambda(N) \hat{\otimes}_E \cdots \hat{\otimes}_E \Lambda(N) \simeq \Lambda(N \times \cdots \times N)$$

is a projective object of $\mathrm{Mod}_{\Lambda(N)}^{\mathrm{pc}}$ by [Brumer 1966, Corollary 3.3], and together they form a resolution

$$\cdots \xrightarrow{d_3} B_2 \Lambda(N) \xrightarrow{d_2} B_1 \Lambda(N) \xrightarrow{d_1} \Lambda(N) \xrightarrow{\epsilon} E \to 0$$

whose differentials d_n are defined by a standard formula we will not state here. Note that K acts on $B_\bullet \Lambda(N)$ as follows. For each element $\kappa \in K$ consider the conjugation map $\kappa(\cdot)\kappa^{-1}$ on N. It induces an automorphism $\kappa : \Lambda(N) \xrightarrow{\sim} \Lambda(N)$ and in turn an automorphism of each $B_n \Lambda(N)$ which we henceforth view as an $E[K]$-module this way.

Notation 4.1. When we write $R\,\mathrm{Hom}_{\Lambda(N)}(E, M)$ and $\mathrm{Ext}^i_{\Lambda(N)}(E, M)$ we will always mean those obtained from the bar resolution $B_\bullet\Lambda(N)$. i.e.,

$$R\,\mathrm{Hom}_{\Lambda(N)}(E, M) := \mathrm{Hom}_{\Lambda(N)}(B_\bullet\Lambda(N), M), \quad \mathrm{Ext}^i_{\Lambda(N)}(E, M) := h^i(\mathrm{Hom}_{\Lambda(N)}(B_\bullet\Lambda(N), M)).$$

Note that if M carries an $E[K]$-module structure, then $R\,\mathrm{Hom}_{\Lambda(N)}(E, M)$ becomes a complex of $E[K/N]$-modules

$$0 \to \mathrm{Hom}_{\Lambda(N)}(\Lambda(N), M) \to \mathrm{Hom}_{\Lambda(N)}(B_1\Lambda(N), M) \to \mathrm{Hom}_{\Lambda(N)}(B_2\Lambda(N), M) \to \cdots$$

(and each $\mathrm{Ext}^i_{\Lambda(N)}(E, M)$ is naturally an $E[K/N]$-module). Perhaps it is worthwhile to really flesh out the $E[K/N]$-module structure of $\mathrm{Hom}_{\Lambda(N)}(B_n\Lambda(N), M)$ here. Since $N \lhd K$ there is a K-action on $\Lambda(N)$ an therefore also on $B_n\Lambda(N)$ induces by conjugation $n \mapsto \kappa n \kappa^{-1}$ on N. We let $\kappa \in K$ act on an element $\phi \in \mathrm{Hom}_{\Lambda(N)}(B_n\Lambda(N), M)$ in the usual way $(\kappa\phi)(x) = \kappa\phi(\kappa^{-1}x)$. It is trivial to check that $\kappa\phi$ is again $\Lambda(N)$-linear, and clearly $\kappa\phi = \phi$ for $\kappa \in N$ — thus the K-action factors through K/N.

We will now discuss how these vary when we shrink N. So let $N' \subset N$ be an open subgroup. The induced map $\Lambda(N') \hookrightarrow \Lambda(N)$ is a homeomorphism onto its image and $\Lambda(N)$ becomes a free module over $\Lambda(N')$ of rank $[N : N']$, see [Schneider 2011, Corollary 19.4] for instance. In particular, viewing $B_\bullet\Lambda(N)$ as a complex of $\Lambda(N')$-modules still gives a projective resolution of E.

Lemma 4.2. *The natural map $B_\bullet\Lambda(N') \to B_\bullet\Lambda(N)$ is a homotopy equivalence.*

Proof. Basic homological algebra. As observed, both $B_\bullet\Lambda(N')$ and $B_\bullet\Lambda(N)$ are projective resolutions of E (viewed as a $\Lambda(N')$-module). Therefore *any* map between the resolutions $B_\bullet\Lambda(N') \to B_\bullet\Lambda(N)$ must be a homotopy equivalence. Indeed any extension to either resolution of the identity map $E \to E$ is homotopic to the identity map of complexes, see [Gelfand and Manin 1996, Theorem 3, page 141]. □

As a result, there is a natural restriction map $\mathrm{res}_{N,N'} : R\,\mathrm{Hom}_{\Lambda(N)}(E, M) \to R\,\mathrm{Hom}_{\Lambda(N')}(E, M)$ defined as the composition

$$\mathrm{Hom}_{\Lambda(N)}(B_\bullet\Lambda(N), M) \xrightarrow{\mathrm{inc}} \mathrm{Hom}_{\Lambda(N')}(B_\bullet\Lambda(N), M) \xrightarrow{\mathrm{qis}} \mathrm{Hom}_{\Lambda(N')}(B_\bullet\Lambda(N'), M).$$

The first map is just the inclusion of a subcomplex; the second is the quasiisomorphism (in fact homotopy equivalence) obtained by composing with the map of Lemma 4.2. Taking cohomology h^i yields restriction maps

$$\mathrm{res}^i_{N,N'} : \mathrm{Ext}^i_{\Lambda(N)}(E, M) \to \mathrm{Ext}^i_{\Lambda(N')}(E, M).$$

These are K-equivariant if M is an $E[K]$-module and $N, N' \lhd K$. As N varies we get a direct system, and passing to the limit gives the stable cohomology groups of M, see [Kohlhaase 2017, page 18].

Definition 4.3. We introduce the complex $R\Sigma(M) := \varinjlim_N R\,\mathrm{Hom}_{\Lambda(N)}(E, M)$ and denote its cohomology groups by

$$\Sigma^i(M) := h^i(R\Sigma(M)) = \varinjlim_N \mathrm{Ext}^i_{\Lambda(N)}(E, M).$$

(Note that $\varinjlim_N$ is exact so it commutes with h^i.) The limit is over all open subgroups $N \subset G$.

Suppose M is an object of $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)}$ (so it carries a G-action). For each element $g \in G$ the conjugation map $g(\cdot)g^{-1}$ induces an isomorphism $\Lambda(N) \xrightarrow{\sim} \Lambda(gNg^{-1})$. The induced map between the bar resolutions (together with the automorphism $g : M \to M$) then gives an isomorphism

$$R\operatorname{Hom}_{\Lambda(N)}(E, M) \xrightarrow{\sim} R\operatorname{Hom}_{\Lambda(gNg^{-1})}(E, M)$$

which is the identity if $g \in N$. This way $R\Sigma(M)$ becomes a complex of smooth $E[G]$-modules and the $\Sigma^i(\cdot)$ form a δ-functor $\mathrm{Mod}^{\mathrm{pc}}_{\Lambda(G)} \to \mathrm{Rep}^{\infty}_E(G)$. Note that Σ^0 is just the functor $M \mapsto M^{\infty} = \bigcup_N M^N$ which associates the subspace of smooth vectors.

Following [Kohlhaase 2017, Definition 3.12] we define the higher smooth duals $S^i : \mathrm{Rep}^{\infty}_E(G) \to \mathrm{Rep}^{\infty}_E(G)$ by composing the Σ^i with Pontryagin duality $(\cdot)^{\vee}$.

Definition 4.4. We introduce the complex $S(\pi) := \varinjlim_N R\operatorname{Hom}_{\Lambda(N)}(E, \pi^{\vee})$ and denote its cohomology groups by

$$S^i(\pi) := h^i(S(\pi)) = \varinjlim_N \operatorname{Ext}^i_{\Lambda(N)}(E, \pi^{\vee}).$$

Note that $S^0(\pi) = \operatorname{Hom}_E(\pi, E)^{\infty}$ is the smooth dual of π and the S^i form a δ-functor.

5. First reductions

We now specialize to the case $\pi = \operatorname{ind}_K^G V$. Our goal is to make the complex $R\operatorname{Hom}_{\Lambda(N)}(E, I_K^G \check{V})$ more explicit and understand the restriction maps as we shrink N. For now we fix an open $N \lhd K$.

First we decompose $I_K^G \check{V}$ as a K-representation using the Cartan decomposition: Let $X_*(S)^+$ be the set of dominant coweights (relative to B). That is, the set of cocharacters $\mu \in X_*(S)$ such that $\langle \mu, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. The map $\mu \mapsto \mu(\varpi)$ gives an isomorphism $X_*(S) \xrightarrow{\sim} S/S(\mathcal{O})$ which clearly restricts to a bijection $X_*(S)^+ \xrightarrow{\sim} S^+/S(\mathcal{O})$ where S^+ denotes the monoid

$$S^+ = \{s \in S : |\alpha(s)| \leq 1 \; \forall \alpha \in \Phi^+\}.$$

The Cartan decomposition reads $G = \bigsqcup_s KsK$ with $s \in S^+$ running over a fixed set of representatives for $S^+/S(\mathcal{O})$. (To fix ideas take $s = \mu(\varpi)$ for a choice of uniformizer ϖ and let $\mu \in X_*(S)^+$ vary.)

Definition 5.1. Let $K^s := K \cap s^{-1}Ks$ act on $\check{V}$ by the rule $\kappa *_s \check{v} := (s\kappa s^{-1})\check{v}$. We denote the resulting representation of K^s by $\check{V}^s$.

With this notation we have the following Mackey factorization of $I_K^G \check{V}$.

Lemma 5.2. *There is an isomorphism of $E[K]$-modules $I_K^G \check{V} \xrightarrow{\sim} \prod_s \operatorname{ind}_{K^s}^K \check{V}^s$ given by $f \mapsto (f_s)$ where the function f_s is defined by the formula $f_s(\kappa) := f(s\kappa)$.*

Proof. This is trivial to check; see [Kohlhaase 2017, page 39] but beware that our conventions on induced representations are different. □

Consequently,

$$R\operatorname{Hom}_{\Lambda(N)}(E, I_K^G \check{V}) \xrightarrow{\sim} \prod_s R\operatorname{Hom}_{\Lambda(N)}(E, \operatorname{ind}_{K^s}^K \check{V}^s).$$

Let us fix an $s \in S^+$ and work out the s-th factor above. We consider the compact open subgroup NK^s of K (noting that $K^s \subset K$ normalizes N) and do the induction in stages as $\mathrm{ind}_{NK^s}^{K} \circ \mathrm{ind}_{K^s}^{NK^s}$ by transitivity. This allows us to write

$$R\,\mathrm{Hom}_{\Lambda(N)}(E, \mathrm{ind}_{K^s}^{K} \check{V}^s) = \mathrm{ind}_{NK^s}^{K} R\,\mathrm{Hom}_{\Lambda(N)}(E, \mathrm{ind}_{K^s}^{NK^s} \check{V}^s),$$

where the induction on the right-hand side means we are inducing each term of the complex. Here we have used the following simple observation (with $\mathcal{U} = B_\bullet\Lambda(N)$ and $\mathcal{V} = \mathrm{ind}_{K^s}^{NK^s} \check{V}^s$):

Lemma 5.3. *Suppose $\mathcal{V}$ is a $\Lambda(H)$-module where $H \subset K$ is an open subgroup containing $N \lhd K$. Let $\mathcal{U}$ be a $\Lambda(N)$-module. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\Lambda(N)}(\mathcal{U}, \mathrm{ind}_H^K \mathcal{V}) \xrightarrow{\sim} \mathrm{ind}_H^K \mathrm{Hom}_{\Lambda(N)}(\mathcal{U}, \mathcal{V})$$

of $E[K/N]$-modules. (Here $\mathrm{ind}_H^K \mathcal{V} \simeq E[K] \otimes_{E[H]} \mathcal{V} \simeq \Lambda(K) \otimes_{\Lambda(H)} \mathcal{V}$.)

Proof. Sending $\gamma : \mathcal{U} \to \mathrm{ind}_H^K \mathcal{V}$ to the function $f_\gamma : K \to \mathrm{Hom}_{\Lambda(N)}(\mathcal{U}, \mathcal{V})$ defined as $f_\gamma(\kappa)(u) = \gamma(u)(\kappa)$ does the job. (Here $\kappa \in K$ and $u \in \mathcal{U}$.) We leave the details to the reader. $\square$

The restriction map gives an isomorphism of $E[N]$-modules $\mathrm{ind}_{K^s}^{NK^s} \check{V}^s \xrightarrow{\sim} \mathrm{ind}_{N\cap K^s}^{N} \check{V}^s$. We transfer the K^s-action on the source to $\mathrm{ind}_{N\cap K^s}^{N} \check{V}^s$. More explicitly, for a function $f : N \to \check{V}^s$ in the latter space and $\kappa \in K^s$ we have $(\kappa f)(n) = \kappa \cdot f(\kappa^{-1} n \kappa)$. Now, by Frobenius reciprocity (and Lemma 4.2 applied to the inclusion $N \cap K^s \subset N$) we get a quasiisomorphism

$$R\,\mathrm{Hom}_{\Lambda(N)}(E, \mathrm{ind}_{N\cap K^s}^{N} \check{V}^s) \xrightarrow{\sim} R\,\mathrm{Hom}_{\Lambda(N\cap K^s)}(E, \check{V}^s).$$

N acts trivially on this complex, and the K^s-action is the natural one on the target sending an element $\eta : B_\bullet\Lambda(N \cap K^s) \to \check{V}^s$ to its conjugate $\kappa\eta\kappa^{-1}$ for $\kappa \in K^s$ (by unwinding the Frobenius reciprocity map).

Lemma 5.4. $$R\,\mathrm{Hom}_{\Lambda(N\cap K^s)}(E, \check{V}^s) \xrightarrow{\sim} R\,\mathrm{Hom}_{\Lambda(s(N\cap K^s)s^{-1})}(E, \check{V}).$$

Proof. Send an η as above to its composition with the map $B_\bullet\Lambda(s(N\cap K^s)s^{-1}) \xrightarrow{\sim} B_\bullet\Lambda(N\cap K^s)$ induced by the conjugation map $s^{-1}(\cdot)s$. One easily checks the resulting map is $\Lambda(s(N \cap K^s)s^{-1})$-linear for the natural action of $s(N \cap K^s)s^{-1} = K \cap sNs^{-1} \subset K$ on $\check{V}$. $\square$

Remark 5.5. The target in Lemma 5.4 carries a natural action of $K^{s^{-1}} = K \cap sKs^{-1}$ since it contains $s(N \cap K^s)s^{-1}$ as a normal subgroup. The map $s^{-1}(\cdot)s$ gives an isomorphism $K^s \xrightarrow{\sim} K^{s^{-1}}$ under which the $K^{s^{-1}}$-action on the target corresponds to the K^s-action on the source.

We conclude that there is a natural quasiisomorphism

$$R\,\mathrm{Hom}_{\Lambda(N)}(E, I_K^G \check{V}) \xrightarrow{\sim} \prod_{s\in S^+/S(\mathcal{O})} \mathrm{ind}_{NK^s}^{K} R\,\mathrm{Hom}_{\Lambda(N\cap K^s)}(E, \check{V}^s)$$

which is compatible with restriction in the following sense. For an open subgroup $N' \subset N$ the restriction map $\mathrm{res}_{N,N'}$ (with $M = I_K^G \check{V}$) corresponds to $\prod_s \varrho_s$ where

$$\varrho_s : \mathrm{ind}_{NK^s}^{K} R\,\mathrm{Hom}_{\Lambda(N\cap K^s)}(E, \check{V}^s) \to \mathrm{ind}_{N'K^s}^{K} R\,\mathrm{Hom}_{\Lambda(N'\cap K^s)}(E, \check{V}^s)$$

is composition with the restriction map

$$\mathrm{res}_{N\cap K^s, N'\cap K^s} : R\,\mathrm{Hom}_{\Lambda(N\cap K^s)}(E, \check{V}^s) \to R\,\mathrm{Hom}_{\Lambda(N'\cap K^s)}(E, \check{V}^s).$$

6. Iwahori factorization and p-valuations

The next step is to understand the individual complexes $R\,\mathrm{Hom}_{\Lambda(K\cap sNs^{-1})}(E, \check{V})$ for $s \in S^+$ appearing in the factorization of $R\,\mathrm{Hom}_{\Lambda(N)}(E, I_K^G \check{V})$, see Lemma 5.4.

We will now assume $N \lhd K$ has Iwahori factorization (with respect to B). This means three things:

(i) $(N \cap \bar{U}) \times (N \cap T) \times (N \cap U) \xrightarrow{\sim} N$ is a homeomorphism,

(ii) $s(N \cap U)s^{-1} \subset N \cap U$,

(iii) $s(N \cap \bar{U})s^{-1} \supset N \cap \bar{U}$,

for all $s \in S^+$. It is well-known that K has a neighborhood basis at the identity consisting of such groups. In fact one can take $N = K_r$ for large enough r, see [Casselman 1995, Proposition 1.4.4] and [Iwahori and Matsumoto 1965, Theorem 2.5], or the very readable account [Rabinoff 2003, Proposition 3.12].

In this section we fix an $s \in S^+$. Conjugating (i) by s and then intersecting with K results in a decomposition of the group of interest,

$$(K \cap sNs^{-1} \cap \bar{U}) \times (N \cap T) \times (K \cap sNs^{-1} \cap U) \xrightarrow{\sim} K \cap sNs^{-1}.$$

Here the rightmost factor is contained in $N \cap U$ by (ii), and the leftmost factor contains $N \cap \bar{U}$ by (iii). The homeomorphism in (i) is not an isomorphism of groups. Our first goal in this section is to show that it nevertheless becomes a group isomorphism after passing to the graded groups defined by a suitable p-valuation.

Recall that a p-valuation on a group N is a function $\omega : N\backslash\{1\} \to (1/(p-1), \infty)$ satisfying the axioms

- $\omega(x^{-1}y) \geq \min\{\omega(x), \omega(y)\}$,
- $\omega([x, y]) \geq \omega(x) + \omega(y)$,
- $\omega(x^p) = \omega(x) + 1$,

for all $x, y \in N$. Here the commutator is $[x, y] = xyx^{-1}y^{-1}$. This notion was introduced by Lazard [1965] and his theory is elegantly exposed by Schneider [2011] which we will use as our main reference. A p-valued group (N, ω) is said to be saturated if all $x \in N$ satisfying the inequality $\omega(x) > p/(p-1)$ lie in N^p, see [Schneider 2011, page 187]. If so the set of p^n-powers N^{p^n} is a subgroup.

Lemma 6.1. *There are arbitrarily small N which admit a p-valuation ω such that one has an equality*

$$\omega(n) = \min\{\omega(\bar{u}), \omega(t), \omega(u)\}$$

for all $n \in N$ with Iwahori factorization $n = \bar{u}tu$ as in (i) above, and (N, ω) is saturated.

Proof. Consider the smooth affine group scheme $U_{/\mathcal{O}}$ and let $N = K_r$ for some r. Note that

$$N \cap U = \ker(U(\mathcal{O}) \to U(\mathcal{O}/\varpi^r \mathcal{O})).$$

By a general observation of Serre this carries a natural p-valuation ω_U for which $(N \cap U, \omega_U)$ is saturated when r is large enough, see [Serre 1965, Chapter IV.9; 1966, Section 2.3] or the more recent [Huber et al. 2011, Lemma 2.2.2] (and the pertaining discussion on page 239). The construction goes via the formal group law given by choosing coordinates $\hat{U} \simeq \mathrm{Spf}(\mathcal{O}[[T_1, \ldots, T_\nu]])$ for the formal completion of U at the identity, see [Demazure 1972, Section II.10].

The same comments apply to the group schemes $\bar{U}$ and T, which results in the two saturated p-valued groups $(N \cap \bar{U}, \omega_{\bar{U}})$ and $(N \cap T, \omega_T)$. Choosing ordered bases gives homeomorphisms

$$\bar{\phi} : N \cap \bar{U} \xrightarrow{\sim} \mathbb{Z}_p^a, \quad \psi : N \cap T \xrightarrow{\sim} \mathbb{Z}_p^b, \quad \phi : N \cap U \xrightarrow{\sim} \mathbb{Z}_p^a,$$

all sending 1_N to the zero-vector. Composing $\bar{\phi} \times \psi \times \phi$ with the inverse of the multiplication map in (i) gives a global chart $\Phi : N \xrightarrow{\sim} \mathbb{Z}_p^d$ sending $1_N \mapsto 0$, where $d = 2a + b$. It follows that $N_m = \Phi^{-1}(p^m \mathbb{Z}_p^d)$ is a subgroup for $m \gg 0$ chosen large enough for the formal group law to have coefficients in $\mathbb{Z}_p$, see the proof of [Schneider 2011, Theorem 27.1] which also verifies the following defines a p-valuation on N_m,

$$\tilde{\omega}(n) = \delta + \max\{\ell : n \in N_{m+\ell}\} = \delta + \min\{v(x_1), \ldots, v(x_d)\}, \quad \Phi(n) = (p^m x_1, \ldots, p^m x_d).$$

Here $\delta = 1$ if $p > 2$, and $\delta = 2$ if $p = 2$. Note that the multiplication map

$$(N_m \cap \bar{U}) \times (N_m \cap T) \times (N_m \cap U) \xrightarrow{\sim} N_m$$

is trivially a homeomorphism since its composition with Φ is the restriction of $\bar{\phi} \times \psi \times \phi$, and for the same reason $(N_m, \tilde{\omega})$ clearly satisfies the properties in the lemma. It remains to check properties (ii) and (iii) for N_m. We will only do (ii); the argument for (iii) is similar by replacing $s \in S^+$ with s^{-1}. For any $u \in N_m \cap U$ we have to check that $sus^{-1} \in N_m$. This follows from the fact that $(N \cap U, \omega_U)$ is saturated, which implies

$$(N \cap U)^{p^m} = \phi^{-1}(p^m \mathbb{Z}_p^a) = N_m \cap U$$

by [Schneider 2011, Corollary 26.12]. □

7. Review of Lazard theory

Let (N, ω) be a p-valued group as in the previous Lemma 6.1 of the last section. In this section we review some general constructions and results of Lazard, partly to set up more notation. For any $v \in \mathbb{R}_{>0}$ we let

$$N_v = \{n \in N : \omega(n) \geq v\}, \quad N_{v+} = \{n \in N : \omega(n) > v\}.$$

Both are normal subgroups of N, and $\mathrm{gr}_v N = N_v/N_{v+}$ is a central subgroup of N/N_{v+} (in particular $\mathrm{gr}_v N$ is abelian). We form the associated graded abelian group

$$\mathrm{gr}\, N = \bigoplus_{v>0} \mathrm{gr}_v N$$

which has additional structure. First of all $\mathrm{gr}\, N$ is an $\mathbb{F}_p$-vector space since $\omega(x^p) = \omega(x) + 1$. Moreover, the commutator $[x, y] = xyx^{-1}y^{-1}$ defines a Lie bracket on $\mathrm{gr}\, N$ which turn it into a graded Lie algebra ($\mathrm{gr}_v N \times \mathrm{gr}_{v'} N \to \mathrm{gr}_{v+v'} N$). Finally, $\mathrm{gr}\, N$ naturally becomes a module over the one-variable polynomial ring $\mathbb{F}_p[\pi]$ as follows. The indeterminate π acts on $\mathrm{gr}\, N$ as the degree one map $\pi : \mathrm{gr}_v N \to \mathrm{gr}_{v+1} N$ given by $\pi(nN_{v+}) = n^p N_{(v+1)+}$ where $n \in N_v$. Since (N, ω) is of finite rank $\mathrm{gr}\, N$ is a free $\mathbb{F}_p[\pi]$-module of said rank; which equals $\dim N$. We refer to [Schneider 2011, Sections 23–25] where all of the above is explained in great detail.

The same remarks apply to the p-valued groups $(N \cap \bar{U}, \omega)$ etc., and because of the formula for ω in Lemma 6.1 we deduce that multiplication defines a homeomorphism

$$(N \cap \bar{U})_v \times (N \cap T)_v \times (N \cap U)_v \xrightarrow{\sim} N_v$$

and similarly for N_{v+}. Since $\mathrm{gr}_v N$ is abelian we conclude that there is an isomorphism of $\mathbb{F}_p$-vector spaces

$$\mathrm{gr}_v(N \cap \bar{U}) \oplus \mathrm{gr}_v(N \cap T) \oplus \mathrm{gr}_v(N \cap U) \xrightarrow{\sim} \mathrm{gr}_v N,$$

and summing up over v gives an analogous decomposition of $\mathrm{gr}\, N$. Further inspection reveals that this direct sum decomposition

$$\mathrm{gr}(N \cap \bar{U}) \oplus \mathrm{gr}(N \cap T) \oplus \mathrm{gr}(N \cap U) \xrightarrow{\sim} \mathrm{gr}\, N \tag{7.1}$$

preserves the $\mathbb{F}_p[\pi]$-module structures, and thus becomes an isomorphism of graded Lie algebras over $\mathbb{F}_p[\pi]$. Here we may assume the Lie bracket is trivial by the perturbation argument given at the very end of Section 8. We will exploit the ensuing factorization of the universal enveloping algebra of $\mathrm{gr}\, N \otimes_{\mathbb{F}_p[\pi]} E$ below.

Let $\mathcal{W} = W(E)$ be the ring of Witt vectors (a complete DVR with residue field E in which p remains prime) and consider the completed group algebra $\mathcal{W}[\![N]\!]$. As explained in [Schneider 2011, Section 28] the p-valuation ω defines a function $\tilde{\omega} : \mathcal{W}[\![N]\!] \backslash \{0\} \to \mathbb{R}_{\geq 0}$ which extends ω in the sense that $\tilde{\omega}(n-1) = \omega(n)$ holds for all $n \in N$. If we fix an ordered basis $(n_1, \dots, n_d)$ for (N, ω) it is explicitly given by the formula

$$\tilde{\omega}(\lambda) = \inf_{\alpha}\left(v(c_\alpha) + \sum_{i=1}^{d} \alpha_i \omega(n_i) \right), \quad \lambda = \sum_{\alpha} c_\alpha \boldsymbol{b}^\alpha.$$

Here $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $\boldsymbol{b}^\alpha = \boldsymbol{b}_1^{\alpha_1} \cdots \boldsymbol{b}_d^{\alpha_d}$ where $\boldsymbol{b}_i = n_i - 1$. We emphasize that the function $\tilde{\omega}$ is independent of the choice of basis however, see [Schneider 2011, Corollary 28.4]. This gives rise to a filtration of $\mathcal{W}[\![N]\!]$ as follows. For $v \geq 0$ we let

$$J_v = \mathcal{W}[\![N]\!]_v = \{\lambda : \tilde{\omega}(\lambda) \geq v\}, \quad J_{v+} = \mathcal{W}[\![N]\!]_{v+} = \{\lambda : \tilde{\omega}(\lambda) > v\}.$$

These two-sided ideals form a fundamental system of open neighborhoods at zero, and can be made very explicit. For instance $\mathcal{W}[\![N]\!]_v$ is the smallest closed $\mathcal{W}$-submodule containing all elements of the form $p^{\mu}(\nu_1 - 1)\cdots(\nu_t - 1)$ with $\mu + \sum_{i=1}^{t} \omega(\nu_i) \geq v$, see [Schneider 2011, Theorem 28.3(ii)]. The graded completed group algebra is then defined as

$$\operatorname{gr} \mathcal{W}[\![N]\!] = \bigoplus_{v \geq 0} \operatorname{gr}_v \mathcal{W}[\![N]\!], \quad \operatorname{gr}_v \mathcal{W}[\![N]\!] = J_v / J_{v+}.$$

This is naturally an algebra over $\operatorname{gr} \mathcal{W}$ (formed with respect to the filtration $p^n \mathcal{W}$). Note that $\operatorname{gr} \mathbb{Z}_p \simeq \mathbb{F}_p[\pi]$ via the identification $p + p^2 \mathbb{Z}_p \leftrightarrow \pi$, and this is how we view $\operatorname{gr} \mathcal{W}$ as an $\mathbb{F}_p[\pi]$-algebra below. For each $v > 0$ there is a natural homomorphism $\mathcal{L}_v : \operatorname{gr}_v N \to \operatorname{gr}_v \mathcal{W}[\![N]\!]$ sending $nN_{v+} \mapsto (n-1) + J_{v+}$, and one of the main results of Lazard is that $\mathcal{L} = \oplus_{v>0} \mathcal{L}_v$ extends to an isomorphism of graded $\operatorname{gr} \mathcal{W}$-algebras

$$\tilde{\mathcal{L}} : \operatorname{gr} \mathcal{W} \otimes_{\mathbb{F}_p[\pi]} U(\operatorname{gr} N) \xrightarrow{\sim} \operatorname{gr} \mathcal{W}[\![N]\!].$$

See [Schneider 2011, Theorem 28.3(i)]. We are not assuming ω is $\mathbb{Z}$-valued, and this flexibility will be important later when we perturb ω to make the Lie algebra $\operatorname{gr} N$ abelian, see [Schneider 2011, Lemma 26.13(i)]. However, we may and do assume that ω takes values in $\frac{1}{A}\mathbb{Z}$ for some $A \in \mathbb{Z}_{>0}$ (see [Schneider 2011, Corollary 33.3]). Then we reindex and let $\operatorname{Fil}^i \mathcal{W}[\![N]\!] := J_{\frac{i}{A}}$, which defines a ring filtration of $\mathcal{W}[\![N]\!]$ indexed by integers $i \geq 0$.

We will employ the analogous results for the reduction $\Lambda(N) = E \otimes_{\mathcal{W}} \mathcal{W}[\![N]\!]$. We endow $\Lambda(N)$ with the quotient filtration $\operatorname{Fil}^i \Lambda(N)$ defined as the image of $\operatorname{Fil}^i \mathcal{W}[\![N]\!]$ under the quotient map $\mathcal{W}[\![N]\!] \twoheadrightarrow \Lambda(N)$. The associated graded E-algebra $\operatorname{gr} \Lambda(N) = \oplus_{i \geq 0} \operatorname{Fil}^i \Lambda(N) / \operatorname{Fil}^{i+1} \Lambda(N)$ is isomorphic to $E \otimes_{\operatorname{gr} \mathcal{W}} \operatorname{gr} \mathcal{W}[\![N]\!]$. The tensor product $E \otimes_{\operatorname{gr} \mathcal{W}} \tilde{\mathcal{L}}$ therefore induces an isomorphism

$$\bar{\mathcal{L}} : U(E \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N) = E \otimes_{\mathbb{F}_p[\pi]} U(\operatorname{gr} N) \xrightarrow{\sim} \operatorname{gr} \Lambda(N).$$

Here $\mathbb{F}_p[\pi] \to E$ takes $\pi \mapsto 0$. If we tensor (7.1) by E over $\mathbb{F}_p[\pi]$ and take universal enveloping algebras the Kronecker product gives an isomorphism

$$U(E \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr}(N \cap \bar{U})) \otimes_E U(E \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr}(N \cap T)) \otimes_E U(E \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr}(N \cap U)) \xrightarrow{\sim} U(E \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} N)$$

which via $\bar{\mathcal{L}}$ gives the main take-away from this section; namely that there is a natural isomorphism of graded E-algebras

$$\operatorname{gr} \Lambda(N \cap \bar{U}) \otimes_E \operatorname{gr} \Lambda(N \cap T) \otimes_E \operatorname{gr} \Lambda(N \cap U) \xrightarrow{\sim} \operatorname{gr} \Lambda(N). \tag{7.2}$$

In the next section we will extend this to s-conjugates ($s \in S^+$) and invoke a Künneth formula.

8. Application of the Künneth formula

The isomorphism (7.2) extends easily to arbitrary $s \in S^+$. Having fixed (N, ω) as in Lemma 6.1 we define a p-valuation ω_s on $K \cap sNs^{-1}$ by the formula $\omega_s(sns^{-1}) = \omega(n)$. It is compatible with the Iwahori factorization of $K \cap sNs^{-1}$ as in Lemma 6.1. Defining $\operatorname{gr}\Lambda(K \cap sNs^{-1})$ and so on relative to ω_s the arguments leading up to (7.2) therefore yield more generally an isomorphism of graded E-algebras

$$\operatorname{gr}\Lambda(K \cap sNs^{-1} \cap \bar{U}) \otimes_E \operatorname{gr}\Lambda(N \cap T) \otimes_E \operatorname{gr}\Lambda(K \cap sNs^{-1} \cap U) \xrightarrow{\sim} \operatorname{gr}\Lambda(K \cap sNs^{-1}).$$

For $s = 1$ one recovers (7.2). The shuffle product (see [Loday 1992, Proposition 4.2.4] for example) gives a homotopy equivalence relating bar resolutions

$$B_\bullet \operatorname{gr}\Lambda(K \cap sNs^{-1} \cap \bar{U}) \otimes_E B_\bullet \operatorname{gr}\Lambda(N \cap T) \otimes_E B_\bullet \operatorname{gr}\Lambda(K \cap sNs^{-1} \cap U) \to B_\bullet \operatorname{gr}\Lambda(K \cap sNs^{-1}),$$

and in turn we have a Künneth formula in the form of a quasiisomorphism

$$\begin{aligned} &R\operatorname{Hom}_{\operatorname{gr}\Lambda(K\cap sNs^{-1})}(E, \operatorname{gr}\check{V}) \xrightarrow{\sim} \\ &\quad R\operatorname{Hom}_{\operatorname{gr}\Lambda(K\cap sNs^{-1}\cap\bar{U})}(E, \operatorname{gr}\check{V}) \otimes_E R\operatorname{Hom}_{\operatorname{gr}\Lambda(N\cap T)}(E, E) \otimes_E R\operatorname{Hom}_{\operatorname{gr}\Lambda(K\cap sNs^{-1}\cap U)}(E, E). \end{aligned} \tag{8.1}$$

Here $\operatorname{gr}\check{V}$ is defined with respect to ω_s. That is we first filter $\check{V}$ by $\operatorname{Fil}^i \check{V} := \operatorname{Fil}^i \Lambda(K \cap sNs^{-1})\check{V}$ and let $\operatorname{gr}\check{V}$ be the associated graded module. We have taken N small enough that it acts trivially on V and therefore $\operatorname{gr}\check{V}$ factors as an external tensor product $\operatorname{gr}\check{V} \boxtimes E \boxtimes E$ where the two E's denote the trivial modules over $\operatorname{gr}\Lambda(N \cap T)$ and $\operatorname{gr}\Lambda(K \cap sNs^{-1} \cap U)$ respectively. Note that $K \cap sNs^{-1} \cap U \subset N$ by property (ii) of an Iwahori factorization. Taking cohomology h^i results in E-vector space isomorphisms

$$\begin{aligned} &\operatorname{Ext}^i_{\operatorname{gr}\Lambda(K\cap sNs^{-1})}(E, \operatorname{gr}\check{V}) \xrightarrow{\sim} \\ &\quad \bigoplus_{a+b+c=i} \operatorname{Ext}^a_{\operatorname{gr}\Lambda(K\cap sNs^{-1}\cap\bar{U})}(E, \operatorname{gr}\check{V}) \otimes_E \operatorname{Ext}^b_{\operatorname{gr}\Lambda(N\cap T)}(E, E) \otimes_E \operatorname{Ext}^c_{\operatorname{gr}\Lambda(K\cap sNs^{-1}\cap U)}(E, E) \end{aligned} \tag{8.2}$$

compatible with restriction maps when shrinking N. Note that by perturbing ω (i.e., replacing it by $\omega_C = \omega - C$ for sufficiently small $C > 0$) we may assume all the graded algebras above are polynomial rings over E in a number of variables, see [Schneider 2011, Lemma 26.13]. This will allow us to control some of the factors in the Künneth formula using Koszul duality.

9. Restriction and the Koszul dual

We first deal with the two factors $\operatorname{Ext}^b_{\operatorname{gr}\Lambda(N\cap T)}(E, E)$ and $\operatorname{Ext}^c_{\operatorname{gr}\Lambda(K\cap sNs^{-1}\cap U)}(E, E)$ in the Künneth formula (8.2). This can be done uniformly so in this section we let H denote one of the two groups $N \cap T$ or $K \cap sNs^{-1} \cap U$ equipped with the p-valuation ω_s. (Note that $\omega_s = \omega$ in the case $H = N \cap T$.) We assume that ω has been chosen in such a way that $\operatorname{gr}\Lambda(H)$ is a polynomial ring over E in a number of variables, or more canonically a symmetric algebra

$$\operatorname{gr}\Lambda(H) \simeq S(E \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} H)$$

via the mod p Lazard isomorphism $\bar{\mathcal{L}}$ discussed in Section 7. This can be ensured by perturbing ω if necessary as noted above. Then the Yoneda algebra $\bigoplus_{j\geq 0}\mathrm{Ext}^j_{\mathrm{gr}\,\Lambda(H)}(E,E)$ is the Koszul dual which in this case is simply the exterior algebra $\bigwedge^\bullet(E\otimes_{\mathbb{F}_p[\pi]}\mathrm{gr}\,H)^*$ of the E-linear dual, see [Beilinson et al. 1996, Theorem 1.2.5]. In particular

$$\mathrm{Ext}^j_{\mathrm{gr}\,\Lambda(H)}(E,E)\simeq\bigwedge^j(E\otimes_{\mathbb{F}_p[\pi]}\mathrm{gr}\,H)^*.$$

For any choice of H we get a saturated p-valued group (H,ω_s), see the proof of Lemma 6.1. In particular the set of p^m-powers $H^{p^m}=H_{(m+1/(p-1))+}$ is a subgroup, and in fact they form a fundamental system of open neighborhoods of the identity as m varies, see [Schneider 2011, Proposition 26.15].

Lemma 9.1. *Let $n\geq 1$. Then $\mathrm{Ext}^n_{\mathrm{gr}\,\Lambda(H^{p^m})}(E,E)\xrightarrow{0}\mathrm{Ext}^n_{\mathrm{gr}\,\Lambda(H^{p^{m+1}})}(E,E)$ for all $m\in\mathbb{Z}_{\geq 0}$.*

Proof. One reduces to the case $m=0$ by replacing H by H^{p^m}. We then have to show the vanishing of

$$\bigwedge^n(E\otimes_{\mathbb{F}_p[\pi]}\mathrm{gr}\,H)^*\to\bigwedge^n(E\otimes_{\mathbb{F}_p[\pi]}\mathrm{gr}\,H^p)^*$$

for $n>0$ and we clearly may assume that $n=1$. In other words, we are to check the vanishing of the dual map

$$E\otimes_{\mathbb{F}_p[\pi]}\mathrm{gr}\,H^p=\mathrm{gr}\,H^p/\pi\,\mathrm{gr}\,H^p\to\mathrm{gr}\,H/\pi\,\mathrm{gr}\,H=E\otimes_{\mathbb{F}_p[\pi]}\mathrm{gr}\,H.$$

This is trivial to verify. Indeed $\mathrm{gr}_{v+1}H^p=\pi\,\mathrm{gr}_v H$ holds for all $v>0$ as follows straight from the definition of the π-action on $\mathrm{gr}\,H$, which proves the claim. □

The above result is the key to establishing hypothesis (A.8) in the Appendix.

10. Invoking the appendix

We will apply Theorem A.11 of the Appendix to the decreasing chain of subalgebras of $\Lambda(K\cap sNs^{-1})$ given by the p^m-powers of N,

$$A^{(m)}:=\Lambda(K\cap sN^{p^m}s^{-1}).$$

Note that by the proof of Lemma 6.1 we know that N^{p^m} inherits an Iwahori factorization, and therefore

$$(K\cap sN^{p^m}s^{-1}\cap\bar{U})\times(N\cap T)^{p^m}\times(sNs^{-1}\cap U)^{p^m}\xrightarrow{\sim}K\cap sN^{p^m}s^{-1}.$$

Here we have used that both $N\cap T$ and $N\cap U$ are saturated to move the p^m-powers outside. (Indeed, suppose $x\in sN^{p^m}s^{-1}\cap U$. Then $s^{-1}xs\in N^{p^m}\cap U = (N\cap U)^{p^m}$, because $N\cap U$ is saturated. We conclude that $x\in(sNs^{-1}\cap U)^{p^m}$ since $s\in S^+\subset T$ normalizes the unipotent radical U.) Also, in the third factor we have deliberately written $sNs^{-1}\cap U$ instead of $K\cap sNs^{-1}\cap U$ (they are the same since $s(N\cap U)s^{-1}$ lies in $N\cap U$ and therefore in K). The upshot is the Künneth formula (8.2) also applies to each of the graded algebras $\mathrm{gr}\,A^{(m)}$.

Lemma 10.1. *Hypothesis* (A.8) *of the Appendix is fulfilled for all $n > \dim U$. That is, the restriction map*

$$\mathrm{Ext}^n_{\mathrm{gr}\, A^{(m)}}(E, \mathrm{gr}\, \check{V}) \to \mathrm{Ext}^n_{\mathrm{gr}\, A^{(m+1)}}(E, \mathrm{gr}\, \check{V})$$

vanishes for all m.

Proof. To make the argument more transparent we will only give the details for $m = 0$. In the Künneth formula (8.2) for $\mathrm{Ext}^n_{\mathrm{gr}\, A^{(0)}}(E, \mathrm{gr}\, \check{V})$ we consider the restriction map to $A^{(1)}$ on the (a, b, c)-summand. When $b > 0$ or $c > 0$ we get zero by Lemma 9.1. What is left is to see what happens to restriction on the $(n, 0, 0)$-summand

$$\mathrm{Ext}^n_{\mathrm{gr}\, \Lambda(K \cap sNs^{-1} \cap \bar{U})}(E, \mathrm{gr}\, \check{V}).$$

However, this summand itself is zero when $n > \mathrm{rank}(K \cap sNs^{-1} \cap \bar{U}) = \dim \bar{U} = \dim U$, see [Lazard 1965, V.2.2] and [Schneider 2011, Theorem 27.1]. ☐

Theorem A.11 applies and yields the following key result.

Proposition 10.2. *Let $n > \dim U$ be arbitrary. Then the restriction map*

$$\mathrm{Ext}^n_{\Lambda(K \cap sNs^{-1})}(E, \check{V}) \to \mathrm{Ext}^n_{\Lambda(K \cap sN^{p^m}s^{-1})}(E, \check{V})$$

vanishes for all $m > \mathrm{amp}(\check{V}) + n$. (Here $\mathrm{amp}(\cdot)$ is the amplitude introduced in the Appendix.)

Proof. We may arrange for $\mathrm{gr}\, \Lambda(K \cap sNs^{-1})$ to be Koszul (e.g., a polynomial algebra) by perturbing ω if necessary. We checked in Lemma 10.1 that hypothesis (A.8) is satisfied for $n > \dim U$ so Theorem A.11 applies and gives the vanishing of $\mathrm{Ext}^n_A(E, \check{V}) \to \mathrm{Ext}^n_{A^{(m)}}(E, \check{V})$ as long as $m > \mathrm{amp}(\check{V}) + n$. Here we use the notation from the Appendix. In particular $A = A^{(0)}$, see the paragraph containing (A.8). ☐

11. Proof of the main result

By Lemma 5.4 we may reformulate Proposition 10.2 as saying that the map

$$\mathrm{Ext}^n_{\Lambda(N \cap K^s)}(E, \check{V}^s) \to \mathrm{Ext}^n_{\Lambda(N^{p^m} \cap K^s)}(E, \check{V}^s)$$

vanishes for $n > \dim U$ and $m > \mathrm{amp}(\check{V}^s) + n$. The amplitude $\mathrm{amp}(\check{V})$ in Proposition 10.2 is computed relative to ω_s, and therefore coincides with $\mathrm{amp}(\check{V}^s)$ which is relative to the ω from N. Note that all our Λ's satisfy $\mathrm{Fil}^0\, \Lambda = \Lambda$ so $\nu \geq 0$ (in the notation of the appendix); in other words the amplitude $\mathrm{amp}(M)$ is at most μ (the first index for which $\mathrm{Fil}^\mu\, M = 0$).

Lemma 11.1. $\mathrm{amp}(\check{V}^s)$ *is uniformly bounded in $s \in S^+$.*

Proof. We recall that $\mathrm{amp}(\check{V}^s)$ is relative to the filtration $\mathrm{Fil}^i\, \check{V}^s := \mathrm{Fil}^i\, \Lambda(N \cap K^s)\check{V}$ where the filtration of $\Lambda(N \cap K^s)$ is defined with respect to (the restriction of) ω. As $N \cap K^s \subset N$ it is enough to observe that $\mathrm{Fil}^i\, \Lambda(N)\check{V} = 0$ for $i \geq A$ for some $A = A_{N,V} > 0$ depending only on N and V (not s); where again $\mathrm{Fil}^i\, \Lambda(N)$ is relative to ω. Assuming N is pro-p the vanishing for $i \geq A$ follows from Nakayama's lemma by comparing $\mathrm{Fil}^i\, \Lambda(N)\check{V}$ to the filtration $\mathfrak{m}^i_{\Lambda(N)}\check{V}$ which must be stationary since V is finite-dimensional, see [Schneider 2011, Remark 28.1]. ☐

From the last paragraph of Section 5 we have the isomorphism

$$\mathrm{Ext}^n_{\Lambda(N)}(E, I_K^G \check{V}) \xrightarrow{\sim} \prod\nolimits_{s\in S^+/S(\mathcal{O})} \mathrm{ind}^K_{NK^s} \mathrm{Ext}^n_{\Lambda(N\cap K^s)}(E, \check{V}^s)$$

which is compatible with the restriction maps on either side. We conclude that

$$\mathrm{Ext}^n_{\Lambda(N)}(E, I_K^G \check{V}) \to \mathrm{Ext}^n_{\Lambda(N^{p^m})}(E, I_K^G \check{V})$$

vanishes for $n > \dim U$ and $m > A_{V,N} + n$ arbitrary (here $A_{V,N}$ is the uniform bound for $\mathrm{amp}(\check{V}^s)$ found in the proof of Lemma 11.1). In particular this proves the vanishing of $S^n(\mathrm{ind}_K^G V)$ for $n > \dim U$, which is our main Theorem 1.1 from the introduction.

Remark 11.2. For the trivial weight $V = E$ one can strengthen this significantly. Indeed $\forall n > 0$ the map

$$\mathrm{Ext}^n_{\Lambda(K\cap sNs^{-1})}(E, E) \to \mathrm{Ext}^n_{\Lambda(K\cap sN^ps^{-1})}(E, E)$$

vanishes for N small. Thus $S^n(\mathrm{ind}_K^G 1) = 0$ for $n > 0$ as mentioned in the introduction. The reason is that one can arrange for $K \cap sNs^{-1}$ to be equi-p-valued (when equipped with the valuation ω_s) by shrinking N and therefore by Lazard's computation of its mod p cohomology algebra [Lazard 1965, page 183] we have

$$\mathrm{Ext}^n_{\Lambda(K\cap sNs^{-1})}(E, E) = H^n(K\cap sNs^{-1}, E) = \bigwedge^n \mathrm{Hom}(K\cap sNs^{-1}, E).$$

By saturation elements of $\mathrm{Hom}(K\cap sNs^{-1}, E)$ vanish upon restriction to $K \cap sN^ps^{-1}$.

12. A few unaddressed questions

In this section we sharpen the expectation that $S^{\dim(G/B)}(\mathrm{ind}_K^G V)$ is g for sufficiently nondegenerate weights V. We first recall the notion of regularity introduced in [Henniart and Vigneras 2012]. For a weight V one defines M_V to be the (unique) largest standard Levi subgroup for which $M_V(k)$ preserves the line $V^{U(k)}$.

Definition 12.1. Let $P \supset B$ be a parabolic subgroup defined over F with standard Levi factor M. We say that V is M-regular if $M_V \subset M$.

For instance, all weights are G-regular. For GL_2 the weight V is T-regular exactly when $\dim V > 1$. We believe T-regularity is enough to guarantee nonvanishing, but we have no evidence.

Question 12.2. Is $S^{\dim(G/B)}(\mathrm{ind}_K^G V) \neq 0$ for all T-regular weights V?

We have not been able to show this even in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. Using [Henniart and Vigneras 2012, Theorem 1.2] and [Kohlhaase 2017, Theorem 4.7(ii)] one can at least prove the nonvanishing of $S^{\dim(G/B)}$ on principal series representations $\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G(V),\chi} E$ where the eigensystems $\chi : \mathcal{H}_G(V) \to E$ factor through the Satake homomorphism $\mathcal{H}_G(V) \to \mathcal{H}_T(V_{\bar{U}(k)})$.

Initially we had hoped to prove the following more precise bound in Theorem 1.1: Suppose $P \supset B$ is a standard parabolic subgroup defined over F with standard Levi factor $M_P \supset T$ and unipotent radical U_P. Let V be a weight for which $M_P(k)$ preserves the line $V^{U(k)}$. In other words $M_P \subset M_V$.

Question 12.3. Is it true that $S^i(\mathrm{ind}_K^G V) = 0$ for all $i > \dim(U_P)$?

This is consistent with our results for $P = G$ and $P = B$. What seems to go wrong if one tries to mimic our argument is that in the Iwahori factorization of N relative to P, the middle factor $N \cap M_P$ is not fixed under conjugation by $s \in S^+$.

If Question 12.3 has a positive answer, it seems natural to strengthen Question 12.2 as follows.

Question 12.4. Is $S^{\dim(U_P)}(\mathrm{ind}_K^G V) \neq 0$ for weights V with $M_V = M_P$?

This more refined question was brought to our attention by Niccolò Ronchetti.

Appendix: A spectral sequence for Ext over filtered rings

Let k be a field and A an augmented k-algebra with augmentation map $\epsilon : A \to k$.[4] We assume A is filtered; meaning it comes with a decreasing filtration $A = \mathrm{Fil}^0 A \supset \mathrm{Fil}^1 A \supset \cdots$ by two-sided ideals $\mathrm{Fil}^i A$ which satisfy the following two properties:

(1) $\mathrm{Fil}^i A \times \mathrm{Fil}^j A \to \mathrm{Fil}^{i+j} A$ for all $i, j \in \mathbb{N}$.

(2) $\epsilon(\mathrm{Fil}^1 A) = 0$.

The associated graded k-algebra $\mathrm{gr}\, A = \oplus_{i\geq 0} \mathrm{gr}^i A = \oplus_{i \geq 0} \mathrm{Fil}^i A/\mathrm{Fil}^{i+1} A$ inherits an augmentation map ϵ by projecting to the first component $A/\mathrm{Fil}^1 A$. Note that a filtration-preserving morphism of augmented k-algebras $A' \to A$ induces a morphism of graded augmented k-algebras $\mathrm{gr}\, A' \to \mathrm{gr}\, A$.

The filtered bar resolution. Below we will study the functorial properties of $\mathrm{Ext}^i_A(k, -)$ as we vary A. For that we will need a projective resolution of k which is functorial in A; the bar resolution $B_\bullet A$,

$$\cdots \xrightarrow{d_3} A \otimes_k A \otimes_k A \xrightarrow{d_2} A \otimes_k A \xrightarrow{d_1} A \xrightarrow{\epsilon} k \to 0.$$

Let $B_n A = A^{\otimes(n+1)}$ be the $(n+1)$-fold tensor product, and define the differentials by letting $d_0 = \epsilon$ and for $n > 0$ let $d_n : B_n A \to B_{n-1} A$ be the map which takes $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to

$$(-1)^n (a_0 \otimes \cdots \otimes a_{n-1})\epsilon(a_n) + \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n).$$

Note that $B_n A$ is a free left A-module (via the first factor a_0) of rank $\dim_k A^{\otimes n}$. It is easily checked that $d_{n-1} \circ d_n = 0$, and in fact the complex is exact (taking $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ to $1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n$ defines a contracting homotopy). We conclude that $B_\bullet A$ is a resolution of k by free left A-modules (usually of infinite rank) which is functorial in A; a morphism of augmented k-algebras $A' \to A$ induces a morphism of complexes $B_\bullet A' \to B_\bullet A$ in the obvious way. This is the reason we choose to work with the bar resolution throughout, as opposed to any projective resolution of k.

[4] We apologize for the change of notation. Now k is the coefficient field (formerly E); not the residue field of F.

Since our algebra A is filtered, $B_\bullet A$ is filtered as well by endowing each $B_n A$ with the tensor product filtration. That is

$$\mathrm{Fil}^i\, B_n A = \sum_{i_0+i_1+\cdots+i_n=i} \mathrm{Fil}^{i_0} A \otimes_k \mathrm{Fil}^{i_1} A \otimes_k \cdots \otimes_k \mathrm{Fil}^{i_n} A.$$

This is a (not necessarily free) A-submodule. It is trivial to check that $d_n : \mathrm{Fil}^i\, B_n A \to \mathrm{Fil}^i\, B_{n-1} A$, i.e., $B_\bullet A$ is a filtered complex (if $i_n > 0$ we have $\epsilon(a_n) = 0$, and otherwise $i_0 + \cdots + i_{n-1} = i$). Let $\mathrm{gr}\, B_\bullet A$ be the associated graded complex whose n-th term is $\mathrm{gr}\, B_n A = \oplus_{i\geq 0} \mathrm{gr}^i\, B_n A$. The next result identifies it with the bar resolution for $\mathrm{gr}\, A$ which is graded in a natural way.

Lemma A.1. $\mathrm{gr}\, B_\bullet A = B_\bullet(\mathrm{gr}\, A)$; *more precisely for any $i \geq 0$ the natural map*

$$\mathrm{gr}^i\, B_n(\mathrm{gr}\, A) = \bigoplus_{i_0+i_1+\cdots+i_n=i} \mathrm{gr}^{i_0} A \otimes_k \mathrm{gr}^{i_1} A \otimes_k \cdots \otimes_k \mathrm{gr}^{i_n} A \to \mathrm{gr}^i\, B_n A$$

is an isomorphism of left $\mathrm{gr}\, A$*-modules commuting with the differentials.*

Proof. For each i choose a k-subspace $\Delta^i \subset \mathrm{Fil}^i A$ such that $\mathrm{Fil}^i A = \Delta^i \oplus \mathrm{Fil}^{i+1} A$. For the rest of the proof fix an $i \geq 0$ and decompose A as a direct sum of subspaces $A = \Delta^0 \oplus \cdots \oplus \Delta^i \oplus \mathrm{Fil}^{i+1} A$. We introduce a more uniform notation by letting $D^j \subset \mathrm{Fil}^j A$ denote Δ^j when $j \leq i$ and $D^{i+1} = \mathrm{Fil}^{i+1} A$. Then $B_n A$ decomposes as

$$B_n A = \Bigl(\bigoplus_{i_0+\cdots+i_n\leq i} \Delta^{i_0} \otimes_k \cdots \otimes_k \Delta^{i_n}\Bigr) \oplus \Bigl({\bigoplus_{i_0+\cdots+i_n> i}}' D^{i_0} \otimes_k \cdots \otimes_k D^{i_n}\Bigr)$$

where the prime in the second $\bigoplus'$ indicates we are only summing over $i_0, \dots, i_n \leq i+1$ with sum $> i$. Clearly this sum $\bigoplus'$ is contained in $\mathrm{Fil}^{i+1}\, B_n A$. Conversely, noting that $\mathrm{Fil}^j A = \Delta^j \oplus \cdots \oplus \Delta^i \oplus \mathrm{Fil}^{i+1} A$ for $j \leq i+1$ it follows immediately that $\mathrm{Fil}^{i+1}\, B_n A$ lies in $\bigoplus'$. Indeed, if $j_0 + \cdots + j_n = i+1$,

$$\mathrm{Fil}^{j_0} A \otimes_k \cdots \otimes_k \mathrm{Fil}^{j_n} A = \bigoplus_{j_0\leq i_0\leq i+1, \dots, j_n \leq i_n \leq i+1} D^{i_0} \otimes_k \cdots \otimes_k D^{i_n},$$

and $i_0 + \cdots + i_n \geq i+1$ for i's in that range. We conclude that for each $i \geq 0$ we have

$$B_n A = \Bigl(\bigoplus_{i_0+\cdots+i_n\leq i} \Delta^{i_0} \otimes_k \cdots \otimes_k \Delta^{i_n}\Bigr) \oplus \mathrm{Fil}^{i+1}\, B_n A.$$

In particular $\mathrm{gr}^i\, B_n A$ is the direct sum over $i_0 + \cdots + i_n = i$. Since $\Delta^i \simeq \mathrm{gr}^i A$ this finishes the proof. □

Ronchetti pointed me to the reference [Sjödin 1973] which contains results of the same flavor. For instance our Lemma A.1 above appears to be closely related to [Sjödin 1973, Lemma 10]. Furthermore our Lemma A.2 below is exactly [Sjödin 1973, Lemma 16] — in a different notation. Instead of comparing notations we found it easier and more convenient to just include the proofs in this appendix.

***Graded R* Hom.** For a left A-module M we consider the complex $R\operatorname{Hom}_A(k, M) = \operatorname{Hom}_A(B_\bullet A, M)$ of k-vector spaces (of A-modules if A is commutative). More precisely

$$\cdots \to 0 \to \operatorname{Hom}_A(A, M) \xrightarrow{\partial_1} \operatorname{Hom}_A(A \otimes_k A, M) \xrightarrow{\partial_2} \operatorname{Hom}_A(A \otimes_k A \otimes_k A, M) \xrightarrow{\partial_3} \cdots$$

whose i-th cohomology group is $\operatorname{Ext}^i_A(k, M)$. Observe that a k-algebra map $A' \to A$ gives a morphism of complexes $R\operatorname{Hom}_A(k, M) \to R\operatorname{Hom}_{A'}(k, M)$ which we will refer to as the restriction map (at least when $A' \subset A$ is a subalgebra). Taking cohomology yields maps $\operatorname{Ext}^i_A(k, M) \to \operatorname{Ext}^i_{A'}(k, M)$ for each i.

Now suppose M is equipped with a decreasing filtration by A-submodules $\operatorname{Fil}^i M \supset \operatorname{Fil}^{i+1} M \supset \cdots$ such that $\operatorname{Fil}^i A \times \operatorname{Fil}^j M \to \operatorname{Fil}^{i+j} M$ for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ (for more flexibility we allow filtrations of M to be indexed by $\mathbb{Z}$); we will often take $\operatorname{Fil}^i M := (\operatorname{Fil}^i A)M$ but this restriction is unnecessary. We let $\operatorname{gr} M = \oplus_{i\in\mathbb{Z}} \operatorname{gr}^i M$ be the associated graded $\operatorname{gr} A$-module.

In this situation $R\operatorname{Hom}_A(k, M)$ becomes a filtered complex by defining ($\forall s \in \mathbb{Z}$)

$$\operatorname{Fil}^s \operatorname{Hom}_A(B_n A, M) := \{\phi \in \operatorname{Hom}_A(B_n A, M) : \phi(\operatorname{Fil}^i B_n A) \subset \operatorname{Fil}^{i+s} M \;\forall i \geq 0\}.$$

(This is a decreasing filtration by k-subspaces, compatible with the differentials ∂.) We will always assume the filtration on M satisfies $\operatorname{Fil}^i M = 0$ for $i \gg 0$ and $\operatorname{Fil}^i M = M$ for $i \ll 0$. Then clearly also $\operatorname{Fil}^s \operatorname{Hom}_A(B_n A, M) = 0$ for $s \gg 0$; however it may not be exhaustive. We let

$$^*\operatorname{Hom}_A(B_n A, M) := \bigcup_{s\in\mathbb{Z}} \operatorname{Fil}^s \operatorname{Hom}_A(B_n A, M) = \{\phi : \phi(\operatorname{Fil}^i B_n A) = 0 \;\forall i \gg 0\}.$$

This is all of $\operatorname{Hom}_A(B_n A, M)$ if the filtration on A is finite, i.e., $\operatorname{Fil}^i A = 0$ for i sufficiently large. Similarly, we let

$$R\operatorname{Hom}_{\operatorname{gr} A}(k, \operatorname{gr} M) := \operatorname{Hom}_{\operatorname{gr} A}(B_\bullet(\operatorname{gr} A), \operatorname{gr} M) \simeq \operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr} B_\bullet A, \operatorname{gr} M),$$

with i-th cohomology group $\operatorname{Ext}^i_{\operatorname{gr} A}(k, \operatorname{gr} M)$. For a fixed $n \geq 0$ and $s \in \mathbb{Z}$ we consider the subspace of homogeneous degree s maps

$$\operatorname{Hom}^s_{\operatorname{gr} A}(\operatorname{gr} B_n A, \operatorname{gr} M) := \{\psi \in \operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr} B_n A, \operatorname{gr} M) : \psi(\operatorname{gr}^i B_n A) \subset \operatorname{gr}^{i+s} M \;\forall i \geq 0\}.$$

They clearly form a direct sum in $\operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr} B_n A, \operatorname{gr} M)$ which we will denote by

$$^*\operatorname{Hom}_{\operatorname{gr} A}(\operatorname{gr} B_n A, \operatorname{gr} M) := \bigoplus_{s\in\mathbb{Z}} \operatorname{Hom}^s_{\operatorname{gr} A}(\operatorname{gr} B_n A, \operatorname{gr} M) = \{\psi : \psi(\operatorname{gr}^i B_n A) = 0 \;\forall i \gg 0\}.$$

(For the inclusion $\supset$ write $\psi = \sum \psi_s$ where ψ_s is defined as the projection $\pi_{i+s} \circ \psi$ on $\operatorname{gr}^i B_n A$. One easily checks ψ_s is $\operatorname{gr} A$-linear and the vanishing condition on ψ guarantees that $\psi_s = 0$ for $|s| \gg 0$.)

Lemma A.2. *For every $s \in \mathbb{Z}$ there is a natural isomorphism of k-vector spaces*

$$\operatorname{gr}^s \operatorname{Hom}_A(B_n A, M) \xrightarrow{\sim} \operatorname{Hom}^s_{\operatorname{gr} A}(\operatorname{gr} B_n A, \operatorname{gr} M).$$

Proof. Any $\phi \in \mathrm{Fil}^s \operatorname{Hom}_A(B_n A, M)$ defines a $\psi \in \operatorname{Hom}^s_{\mathrm{gr}\,A}(\mathrm{gr}\, B_n A, \mathrm{gr}\, M)$ in the obvious way; and $\psi = 0$ exactly when $\phi \in \mathrm{Fil}^{s+1} \operatorname{Hom}_A(B_n A, M)$. In other words, there is a k-linear injection

$$\mathrm{gr}^s \operatorname{Hom}_A(B_n A, M) \hookrightarrow \operatorname{Hom}^s_{\mathrm{gr}\,A}(\mathrm{gr}\, B_n A, \mathrm{gr}\, M).$$

To show this is surjective observe that $B_n A = A \otimes_k V$ where V is a filtered k-vector space ($A^{\otimes n}$) such that $\mathrm{Fil}^i\, B_n A = \sum_{p+q=i} F^p A \otimes_k \mathrm{Fil}^q\, V$ for all $i \geq 0$. Therefore, as filtered k-vector spaces,

$$\operatorname{Hom}_A(B_n A, M) \simeq \operatorname{Hom}_k(V, M).$$

Similarly $\mathrm{gr}\, B_n A = B_n(\mathrm{gr}\, A) = \mathrm{gr}\, A \otimes_k \mathrm{gr}\, V$ as graded k-vector spaces by Lemma A.1. Thus

$$\operatorname{Hom}_{\mathrm{gr}\,A}(\mathrm{gr}\, B_n A, \mathrm{gr}\, M) \simeq \operatorname{Hom}_k(\mathrm{gr}\, V, \mathrm{gr}\, M),$$

and it remains to observe that $\mathrm{gr}^s \operatorname{Hom}_k(V, M) \hookrightarrow \operatorname{Hom}^s_k(\mathrm{gr}\, V, \mathrm{gr}\, M)$ is an isomorphism. Indeed, as in the proof of A.1, we may choose a sequence of k-subspaces $\nabla^i \subset \mathrm{Fil}^i\, V$ such that $V = \nabla^0 \oplus \cdots \oplus \nabla^i \oplus \mathrm{Fil}^{i+1}\, V$. Let $\psi \in \operatorname{Hom}^s_k(\mathrm{gr}\, V, \mathrm{gr}\, M)$. Since the filtration on M is assumed to be finite, we may choose i large enough that $\psi(\mathrm{gr}^{i+1}\, V) = 0$. Now define a preimage $\phi \in \mathrm{Fil}^s \operatorname{Hom}_k(V, M)$ of ψ as follows. Let $\phi = 0$ on $\mathrm{Fil}^{i+1}\, V$, and on $\nabla^j \simeq \mathrm{gr}^j\, V$ for $j = 0, \dots, i$ take ϕ to be a lift $\nabla^j \to \mathrm{Fil}^{j+s}\, M$ of the given $\psi : \mathrm{gr}^j\, V \to \mathrm{gr}^{j+s}\, M$. □

A topological variant. We now assume A is a pseudocompact k-algebra. More precisely, we assume the ideals $\mathrm{Fil}^i\, A$ are open and form a neighborhood basis at 0, and $A \xrightarrow{\sim} \varprojlim A/\mathrm{Fil}^i\, A$ as topological rings with Artinian (discrete) quotients $A/\mathrm{Fil}^i\, A$. We let $\mathcal{C}_A$ denote the abelian category of pseudocompact A-modules (i.e., complete Hausdorff topological left A-modules which are inverse limits of discrete finite length A-modules). As is well-known, $\mathcal{C}_A$ has exact inverse limits and enough projectives. Similarly we let $\mathcal{C}_{\mathrm{gr}\,A}$ denote the abelian category of $\mathbb{Z}$-graded $\mathrm{gr}\, A$-modules, with morphisms $\operatorname{Hom}_{\mathcal{C}_{\mathrm{gr}\,A}} = {}^*\operatorname{Hom}_{\mathrm{gr}\,A}$ (the sums of homogeneous maps as above). Any projective $\mathrm{gr}\, A$-module with a $\mathbb{Z}$-grading is projective in $\mathcal{C}_{\mathrm{gr}\,A}$ and vice versa (see [Fossum and Foxby 1974, page 289] for commutative rings).

The bar resolution has a topological variant $\hat{B}_\bullet A$ obtained by replacing $\otimes_k$ with completed tensor products $\hat{\otimes}_k$ everywhere. For instance $\hat{B}_1 A = A \hat{\otimes}_k A = \varprojlim_{i,j} A/\mathrm{Fil}^i\, A \otimes_k A/\mathrm{Fil}^j\, A$, see [Brumer 1966, page 446]. More generally $\hat{B}_n A = A^{\hat{\otimes}(n+1)}$. The differential d_n extends to $\hat{B}_n A$ by continuity and this defines a resolution of k in $\mathcal{C}_A$,

$$\cdots \xrightarrow{d_3} A \hat{\otimes}_k A \hat{\otimes}_k A \xrightarrow{d_2} A \hat{\otimes}_k A \xrightarrow{d_1} A \xrightarrow{\epsilon} k \to 0.$$

Moreover, $\hat{B}_n A$ is projective in $\mathcal{C}_A$ by [Brumer 1966, Corollary 3.3] (being an inverse limit of free $A/\mathrm{Fil}^i\, A$-modules).

Furthermore, suppose M is a discrete finite length A-module with exhaustive and separated filtration $\mathrm{Fil}^i\, M$ as above.

Lemma A.3. *We have the following natural isomorphisms of complexes*:

(1) $R\operatorname{Hom}_{\mathcal{C}_A}(k, M) := \operatorname{Hom}_{\mathcal{C}_A}(\hat{B}_\bullet A, M) = {}^*\operatorname{Hom}_A(B_\bullet A, M)$.

(2) $R\operatorname{Hom}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M) := \operatorname{Hom}_{\mathcal{C}_{\mathrm{gr}\,A}}(\mathrm{gr}\, B_\bullet A, \mathrm{gr}\, M) = {}^*\operatorname{Hom}_{\mathrm{gr}\,A}(gr\, B_\bullet A, \mathrm{gr}\, M)$.

Proof. Part (2) follows straight from the definition of morphisms in $\mathcal{C}_{\mathrm{gr}\,A}$. For part (1) we have to show that $\phi \in \mathrm{Hom}_A(B_nA, M)$ extends to a continuous map $\hat{B}_nA \to M$ if and only if $\phi(\mathrm{Fil}^i\, B_nA) = 0$ for large i. Note that $\hat{B}_nA$ is the completion of B_nA for the topology defined by the submodules

$$J_i := \ker(A^{\otimes(n+1)} \to (A/\mathrm{Fil}^i\, A)^{\otimes(n+1)}) = (\mathrm{Fil}^i\, A \otimes_k \cdots \otimes_k A) + \cdots + (A \otimes_k \cdots \otimes_k \mathrm{Fil}^i\, A).$$

Obviously $J_i \subset \mathrm{Fil}^i\, B_nA$, and conversely $\mathrm{Fil}^{i(n+1)}\, B_nA \subset J_i$. If $i_0 + \cdots + i_n = i(n+1)$ we must have some $i_j \geq i$ and thus $\mathrm{Fil}^{i_0}\, A \otimes_k \mathrm{Fil}^{i_1}\, A \otimes_k \cdots \otimes_k \mathrm{Fil}^{i_n}\, A$ is contained in J_i. This proves the lemma. □

Note that if A is Noetherian then the cohomology of $R\,\mathrm{Hom}_{\mathcal{C}_A}(k, M)$ computes $\mathrm{Ext}^i_A(k, M)$, and similarly for $R\,\mathrm{Hom}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M)$ (assuming $\mathrm{gr}\, A$ is Noetherian): k has a resolution by finite free A-modules, and any A-linear map $A^r \to A^s$ is automatically continuous.

***The spectral sequence for* Ext.** Combining Lemmas A.2 and A.3 we arrive at the following result.

Theorem A.4. *With notation as above, there is a natural isomorphism of graded complexes*

$$\mathrm{gr}\, R\,\mathrm{Hom}_{\mathcal{C}_A}(k, M) \xrightarrow{\sim} R\,\mathrm{Hom}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M).$$

We extract a spectral sequence relating the Ext-functors over A and $\mathrm{gr}\, A$.

Corollary A.5. *Assume* $\dim_k \mathrm{Ext}^n_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M)$ *is finite for all n, and zero for n sufficiently large. Then there is a convergent spectral sequence, which is functorial in A,*

$$E_1^{i,j} = \mathrm{Ext}^{i,j}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M) \Rightarrow \mathrm{Ext}^{i+j}_{\mathcal{C}_A}(k, M).$$

(*The definition of the internal grading* $\mathrm{Ext}^{i,j}_{\mathcal{C}_{\mathrm{gr}\,A}}$ *is recalled in the proof—see* (A.6) *below.*)

Proof. Consider the spectral sequence of the filtered complex $K = {}^*\mathrm{Hom}_A(B_\bullet A, M)$, see [Gelfand and Manin 1996, page 203]. It consists of bigraded groups $E_r = \oplus_{i,j} E_r^{i,j}$ starting with $E_1^{i,j} = H^{i+j}(\mathrm{gr}^i\, K)$ where $\mathrm{gr}\, K = \oplus_{i\in\mathbb{Z}} \mathrm{gr}^i\, K$ is the associated graded complex, which in our case equals ${}^*\mathrm{Hom}_{\mathrm{gr}\,A}(gr\,B_\bullet A, \mathrm{gr}\, M)$ by Theorem A.4. Its cohomology $\mathrm{Ext}^{i+j}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M)$ is graded and $E_1^{i,j}$ sits as the i-th graded piece;

$$E_1^{i,j} = \mathrm{gr}^i\, \mathrm{Ext}^{i+j}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M) =: \mathrm{Ext}^{i,j}_{\mathcal{C}_{\mathrm{gr}\,A}}(k, \mathrm{gr}\, M). \tag{A.6}$$

On the infinite sheet we have groups $E_\infty = \oplus_{i,j} E_\infty^{i,j}$ with $E_\infty^{i,j} = \mathrm{gr}^i\, H^{i+j}(K)$, where the filtration on $H^n(K) = \mathrm{Ext}^n_{\mathcal{C}_A}(k, M)$ is given by the images of the maps $H^n(\mathrm{Fil}^i\, K) \to H^n(K)$. Convergence follows easily from the finiteness assumptions: $E_1^{i,j}$ is nonzero only for finitely many pairs (i, j), so eventually the incoming and outgoing differentials vanish at (i, j) for $r \gg 0$. □

In most situations of interest $\mathrm{gr}\, A$ will be Noetherian, and k of finite projective dimension over $\mathrm{gr}\, A$. Then the finiteness conditions in A.5 are automatically satisfied for all finite-dimensional M.

Following [Polo and Tilouine 2002], Ronchetti has recently established a spectral sequence akin to the one in Corollary A.5 above, albeit in a more restricted setting, see [Ronchetti 2018, Theorem 3]. He considers $\mathbb{Z}_p[\![U(\mathcal{O})]\!]$ filtered by powers of the augmentation ideal and relates $\mathrm{Ext}^*_{\mathrm{gr}\,\mathbb{Z}_p[\![U(\mathcal{O})]\!]}(\mathrm{gr}\,\mathbb{Z}_p, \mathrm{gr}\, M)$ to $\mathrm{Ext}^*_{\mathbb{Z}_p[\![U(\mathcal{O})]\!]}(\mathbb{Z}_p, M)$ for finitely generated $\mathbb{Z}_p$-modules M with a continuous $B(\mathcal{O})$-action.

An application. Let $A' \to A$ be a filtration-preserving map of augmented k-algebras, both assumed to be pseudocompact. As pointed out earlier this gives rise to a morphism of filtered complexes $R\operatorname{Hom}_{\mathcal{C}_A}(k, M) \to R\operatorname{Hom}_{\mathcal{C}_{A'}}(k, M)$, and a fortiori a map of spectral sequences — which we will assume converge.

Corollary A.7. *Fix an n and suppose the restriction map $\operatorname{Ext}^n_{\mathcal{C}_{\operatorname{gr} A}}(k, \operatorname{gr} M) \to \operatorname{Ext}^n_{\mathcal{C}_{\operatorname{gr} A'}}(k, \operatorname{gr} M)$ is zero. Then the map $\operatorname{Ext}^n_{\mathcal{C}_A}(k, M) \to \operatorname{Ext}^n_{\mathcal{C}_{A'}}(k, M)$ is zero on all graded pieces. That is, for all $i \in \mathbb{Z}$ it maps*

$$\operatorname{Fil}^i \operatorname{Ext}^n_{\mathcal{C}_A}(k, M) \to \operatorname{Fil}^{i+1} \operatorname{Ext}^n_{\mathcal{C}_{A'}}(k, M).$$

Proof. The map between the first sheets $f_1^{i,j} : E_1^{i,j} \to E_1'^{i,j}$ comes from the restriction maps

$$\operatorname{Ext}^{i+j}_{\mathcal{C}_{\operatorname{gr} A}}(k, \operatorname{gr} M) \to \operatorname{Ext}^{i+j}_{\mathcal{C}_{\operatorname{gr} A'}}(k, \operatorname{gr} M)$$

by taking the i-th graded piece. Analogously the map $f^n : \operatorname{Ext}^n_{\mathcal{C}_A}(k, M) \to \operatorname{Ext}^n_{\mathcal{C}_{A'}}(k, M)$ is the obvious map obtained by taking the cohomology of the map of filtered complexes above $K \to K'$. The map f^n preserves filtrations and the induced map on graded pieces ($n = i + j$)

$$E_\infty^{i,j} = \operatorname{gr}^i \operatorname{Ext}^{i+j}_{\mathcal{C}_A}(k, M) \xrightarrow{f^{i+j}} \operatorname{gr}^i \operatorname{Ext}^{i+j}_{\mathcal{C}_{A'}}(k, M) = E_\infty'^{i,j}$$

coincides with the map between the infinite sheets $f_\infty^{i,j}$ which is zero since $f_1^{i,j} = 0$ by assumption. □

Now suppose instead of a single A' we have a whole system of pseudocompact filtered k-algebras $A^{(m)}$ along with filtration-preserving maps of augmented k-algebras

$$A =: A^{(0)} \leftarrow A^{(1)} \leftarrow \cdots \leftarrow A^{(m)} \leftarrow \cdots .$$

(Typically the $A^{(m)}$ will be subalgebras of A.) Suppose furthermore that all the graded restriction maps vanish for a given n. i.e.,

$$\operatorname{Ext}^n_{\mathcal{C}_{\operatorname{gr} A}}(k, \operatorname{gr} M) \xrightarrow{0} \operatorname{Ext}^n_{\mathcal{C}_{\operatorname{gr} A^{(1)}}}(k, \operatorname{gr} M) \xrightarrow{0} \cdots \xrightarrow{0} \operatorname{Ext}^n_{\mathcal{C}_{\operatorname{gr} A^{(m)}}}(k, \operatorname{gr} M) \xrightarrow{0} \cdots . \tag{A.8}$$

We arrive at one of the main results in this appendix, which we will strengthen in the next section.

Corollary A.9. *Under the assumption* (A.8) *for a given n,*

$$\varinjlim_m \operatorname{Ext}^n_{\mathcal{C}_{A^{(m)}}}(k, M) = 0.$$

Proof. It suffices to show that any $c \in \operatorname{Ext}^n_{\mathcal{C}_A}(k, M)$ maps to 0 in $\operatorname{Ext}^n_{\mathcal{C}_{A^{(m)}}}(k, M)$ for m large enough (how large may depend on c). Keep the notation $K = {}^*\operatorname{Hom}_A(B_\bullet A, M)$. By definition of ${}^*\operatorname{Hom}$ the filtration on K is exhaustive; $K = \bigcup_{s \in \mathbb{Z}} \operatorname{Fil}^s K$. Also, by definition of the graded cohomology $H^n(\operatorname{Fil}^i K) \twoheadrightarrow \operatorname{Fil}^i H^n(K)$, so the filtration on $H^n(K) = \operatorname{Ext}^n_{\mathcal{C}_A}(k, M)$ is also exhaustive. Thus any c lies in $\operatorname{Fil}^i \operatorname{Ext}^n_{\mathcal{C}_A}(k, M)$ for some i. By repeated use of Corollary A.7 restriction to $A^{(m)}$ takes

$$\operatorname{Fil}^i \operatorname{Ext}^n_{\mathcal{C}_A}(k, M) \to \operatorname{Fil}^{i+m} \operatorname{Ext}^n_{\mathcal{C}_{A^{(m)}}}(k, M).$$

It suffices to observe the target is zero as long as $\mathrm{Fil}^{i+m} M = 0$ (indeed $\mathrm{Fil}^{\mu} M = 0$ implies $\mathrm{Fil}^{\mu} K = 0$, and hence $\mathrm{Fil}^{\mu} H^n(K) = 0$). As long as we take $m \geq \mu - i$ the class c restricts to zero (note that i may depend on c). □

The next and last section of this appendix deals with the uniformity of m. Under favorable circumstances one can choose an m which is independent of c and thereby show that the restriction map $\mathrm{Ext}^n_{\mathcal{C}_A}(k, M) \to \mathrm{Ext}^n_{\mathcal{C}_{A(m)}}(k, M)$ vanishes; still assuming (A.8) of course. This is a key point on which our paper relies.

Minimal resolutions and Koszul algebras. We assume $\mathrm{gr}\, A$ is Noetherian and $k \xrightarrow{\sim} \mathrm{gr}^0 A$. Then the trivial module k admits a minimal projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to k \to 0.$$

Here each $P_n \simeq \mathrm{gr}\, A \otimes_k V_n$ for some graded finite-dimensional vector space V_n, say $\dim_k V_n = b_n$. Choosing a basis of homogeneous elements for V_n we can identify $P_n \simeq (\mathrm{gr}\, A)^{b_n}$ in such a way that the standard basis vectors are homogeneous elements of various degrees. Then the minimal resolution becomes

$$\cdots \xrightarrow{T_3} (\mathrm{gr}\, A)^{b_2} \xrightarrow{T_2} (\mathrm{gr}\, A)^{b_1} \xrightarrow{T_1} (\mathrm{gr}\, A)^{b_0} = \mathrm{gr}\, A \xrightarrow{\epsilon} k \to 0,$$

where the differentials are left-multiplication by certain matrices $T_i \in M_{b_{i-1}\times b_i}(\mathrm{gr}\, A)$. Minimality refers to the fact that the entries of all the T_i in fact lie in the augmentation ideal $(\mathrm{gr}\, A)_+ = \oplus_{i>0} \mathrm{gr}^i A$. In other words that $P_\bullet \otimes_{\mathrm{gr}\, A} k$ has 0-differentials so its terms give the cohomology. Similarly for $\mathrm{Hom}_{\mathrm{gr}\, A}(P_\bullet, k)$. Recall that $\mathrm{gr}\, A$ is said to be a *Koszul* algebra if the entries of the T_i all lie in $\mathrm{gr}^1 A$. A convenient reference for all this is [Krähmer 2011]. Note however that our bigrading $\mathrm{Ext}^{i,j}_{\mathcal{C}_{\mathrm{gr}\, A}}$ differs from his internal grading; our $\mathrm{Ext}^{i,j}$ would be $\mathrm{Ext}^{i+j,i}$ in Krähmer's notation.

The minimal resolution is a resolution as graded modules, so all the T_i preserve the grading. Let us focus on the first T_1 for a second; a row vector with b_1 entries in $(\mathrm{gr}\, A)_+$. Let $e \in (\mathrm{gr}\, A)^{b_1}$ be one of the standard basis vectors. It has some degree $|e| = d$. Since T_1 is of degree zero $|T_1(e)| = d$. However, $T_1(e)$ is one of the entries of T_1 which are in $(\mathrm{gr}\, A)_+$. We infer that $d > 0$. In other words that $(\mathrm{gr}\, A)^{b_1}$ has a basis consisting of elements of degree ≥ 1 (of degree *equal* to 1 if $\mathrm{gr}\, A$ is Koszul). More generally we obtain the following.

Lemma A.10. *$P_n \simeq (\mathrm{gr}\, A)^{b_n}$ has a $\mathrm{gr}\, A$-basis consisting of homogeneous elements all of degree $\geq n$ (all of degree exactly n if $\mathrm{gr}\, A$ is Koszul).*

Proof. Induction on n. We did the case $n = 1$ above. Let $n > 1$ and consider $T_n : (\mathrm{gr}\, A)^{b_n} \to (\mathrm{gr}\, A)^{b_{n-1}}$. Let $e \in (\mathrm{gr}\, A)^{b_n}$ be the j-th standard basis vector, of some degree $|e| = d$. Then $T_n(e)$ is the j-th column of the matrix $T_n = (a_{ij})$. That is

$$T_n(e) = a_{1j}e_1 + a_{2j}e_2 + \cdots + a_{b_{n-1},j}e_{b_{n-1}}$$

where all the e_i's have degree $\geq n-1$ (with equality in the Koszul case) by induction and $a_{ij} \in (\operatorname{gr} A)_+$. This shows $T_n(e)$ is a sum of terms of degree $\geq n$. Since we know it is homogeneous, $|e| = |T_n(e)| \geq n$. In the Koszul case all terms of $T_n(e)$ have degree exactly n by induction since all $a_{ij} \in \operatorname{gr}^1 A$. □

Now let M be a filtered A-module as above, with associated graded $\operatorname{gr} A$-module $\operatorname{gr} M$. Since $P_\bullet$ and $\operatorname{gr} B_\bullet A$ are homotopy equivalent as graded resolutions of k, the grading of $\operatorname{Ext}^n_{\mathcal{C}_{\operatorname{gr} A}}(k, \operatorname{gr} M)$ is also determined by that of $P_\bullet$. Thus by its minimality we have

$$E_1^{i,j} = \operatorname{Ext}^{i,j}_{\mathcal{C}_{\operatorname{gr} A}}(k, \operatorname{gr} M) = H^{i+j}(\operatorname{Hom}^i_{\operatorname{gr} A}(P_\bullet, \operatorname{gr} M)) = \operatorname{Hom}^i_{\operatorname{gr} A}(P_{i+j}, \operatorname{gr} M).$$

Let μ be the smallest integer for which $\operatorname{Fil}^\mu M = 0$, and let ν be the largest integer for which $\operatorname{Fil}^\nu M = M$. In particular $\operatorname{gr}^s M = 0$ when s lies outside the interval $[\nu, \mu]$. We define the *amplitude* of M as the length of the filtration, i.e., $\operatorname{amp}(M) = \mu - \nu$.

Theorem A.11. *Suppose* $\operatorname{gr} A$ *is a* Koszul *algebra and the vanishing hypothesis* (A.8) *is satisfied for a given $n \geq 1$. Then the restriction map*

$$\operatorname{Ext}^n_{\mathcal{C}_A}(k, M) \to \operatorname{Ext}^n_{\mathcal{C}_{A(m)}}(k, M)$$

vanishes for $m > \operatorname{amp}(M) + n$.

Proof. First we observe that on the initial page of the spectral sequence

$$E_1^{i,j} = 0 \text{ in the region } 2i + j \geq \mu, \text{ and in } 2i + j < \nu \text{ since } \operatorname{gr} A \text{ is Koszul.}$$

Indeed, by Lemma A.10 we know P_{i+j} is generated by elements e of degree $\geq i + j$. Any degree i map $\phi : P_{i+j} \to \operatorname{gr} M$ takes e to an element $\phi(e)$ of degree $\geq 2i + j$. Assuming $2i + j \geq \mu$ we must have $\phi = 0$. When $\operatorname{gr} A$ is Koszul $|\phi(e)| = 2i + j$ and therefore $\phi = 0$ if $2i + j < \nu$.

Consequently, on the infinite page $E_\infty^{i,j} = \operatorname{gr}^i \operatorname{Ext}^{i+j}_{\mathcal{C}_A}(k, M)$ also vanishes when $2i + j \geq \mu$ or $2i + j < \nu$. In other words, for a fixed n, the filtration $\operatorname{Fil}^i \operatorname{Ext}^n_{\mathcal{C}_A}(k, M)$ becomes stationary and therefore zero for $i \geq \mu - n$, and it equals $\operatorname{Ext}^n_{\mathcal{C}_A}(k, M)$ for $i < \nu - n$ since we know the filtration is exhaustive as observed earlier. This allows us to strengthen the conclusion of Corollary A.9 when $\operatorname{gr} A$ is Koszul. Its proof shows that the restriction map in question is zero as long as $m > \mu - \nu + n$. □

The uniformity of how deep we have to go to get vanishing of the restriction map plays a critical role in this paper.

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