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On the p-typical de Rham–Witt complex over $W(k)$

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Hesselholt and Madsen (2004) define and study the (absolute, p-typical) de Rham–Witt complex in mixed characteristic, where p is an odd prime. They give as an example an elementary algebraic description of the de Rham–Witt complex over $\mathbb{Z}_{(p)}$, $W_\cdot\Omega^\bullet_{\mathbb{Z}_{(p)}}$. The main goal of this paper is to construct, for k a perfect ring of characteristic $p > 2$, a Witt complex over $A = W(k)$ with an algebraic description which is completely analogous to Hesselholt and Madsen's description for $\mathbb{Z}_{(p)}$. Our Witt complex is not isomorphic to the de Rham–Witt complex; instead we prove that, in each level, the de Rham–Witt complex over $W(k)$ surjects onto our Witt complex, and that the kernel consists of all elements which are divisible by arbitrarily high powers of p. We deduce an explicit description of $W_n\Omega^\bullet_A$ for each $n \geq 1$. We also deduce results concerning the de Rham–Witt complex over certain p-torsion-free perfectoid rings.

Introduction

Fix an odd prime p and a $\mathbb{Z}_{(p)}$-algebra R. Hesselholt and Madsen [2004] define the (absolute, p-typical) de Rham–Witt complex over R to be the initial object in the category of Witt complexes over R. Their definition generalizes the de Rham–Witt complex of Bloch, Deligne and Illusie, which was defined for $\mathbb{F}_p$-algebras. The goal of this paper is to define a Witt complex $E^\bullet_\cdot$ over $A = W(k)$, where k is a perfect ring of characteristic p, and to use this Witt complex to describe the de Rham–Witt complex over $W(k)$ and also to study the de Rham–Witt complex over certain perfectoid rings B.

Among many other conditions, the de Rham–Witt complex $W_\cdot\Omega^\bullet_R$ is a prosystem of differential graded rings. There is an isomorphism $W_n(R) \to W_n\Omega^0_R$, so the degree zero piece of the de Rham–Witt complex is well-understood. For each positive integer n and for every degree d, there is a surjective morphism of differential graded rings

$$\Omega^d_{W_n(R)} \twoheadrightarrow W_n\Omega^d_R,$$

and so it is easy to write down elements of $W_n\Omega^d_R$. On the other hand, especially in the degree one case $d = 1$, it is often difficult to determine which of these elements in $W_n\Omega^1_A$ are nonzero. The author is not aware of a complete algebraic description of the (absolute, p-typical) de Rham–Witt complex in mixed characteristic for any examples other than $\mathbb{Z}_{(p)}$ and polynomial algebras over this ring. One of the goals of the current paper is to give a complete algebraic description of the de Rham–Witt complex over $A = W(k)$, where k is a perfect ring of odd characteristic p. For example, we prove that in the de Rham–Witt complex over $W(k)$, the element $dV^n(1)$ is a nontrivial p^n-torsion element for every integer $n \geq 1$. It is easy to

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see, using the relation $pdV = Vd$, that this element is indeed p^n-torsion, but showing that this element is nonzero takes much more work.

To better analyze relations within the de Rham–Witt complex, we first define in Section 3 a Witt complex $E^\bullet_\cdot$ over $A = W(k)$ which has a simple algebraic description as a $W(k)$-module. The proof that $E^\bullet_\cdot$ is indeed a Witt complex over $W(k)$ is one of the major parts of this paper. It is not isomorphic to the de Rham–Witt complex over $W(k)$; see Remark 3.11. Instead, in each level n and in each positive degree $d \geq 1$, our Witt complex $E^\bullet_\cdot$ is the quotient of the de Rham–Witt complex by the $W(k)$-submodule consisting of all elements which are divisible by arbitrarily large powers of p. In the language of [Hesselholt 2015, Remark 4.8], our Witt complex $E^\bullet_\cdot$ is the p-typical de Rham–Witt complex over $W(k)$ relative to the p-typical λ-ring $(W(k), s_\varphi)$, where s_φ is the ring homomorphism $W(k) \to W(W(k))$ recalled in Proposition 2.1 below.

Our description of $E^\bullet_\cdot$, which we define for each $W(k)$ with k a perfect ring of odd characteristic p, is completely modeled after Hesselholt and Madsen's description of $W_n\Omega^1_{\mathbb{Z}_{(p)}}$ [2004, Example 1.2.4]. They show that for all $n \geq 1$, there is an isomorphism of $\mathbb{Z}_{(p)}$-modules

$$W_n\Omega^1_{\mathbb{Z}_{(p)}} \cong \prod_{i=0}^{n-1} \mathbb{Z}/p^i\mathbb{Z} \cdot dV^i(1). \tag{0.1}$$

This shows that $W_n\Omega^1_{\mathbb{Z}_{(p)}}$ is nonzero if $n \geq 2$. The proof in [Hesselholt and Madsen 2004] involves the topological Hochschild spectrum $T(\mathbb{Z}_{(p)})$. The results below provide an alternative (and elementary) proof that $W_n\Omega^1_{\mathbb{Z}_{(p)}}$ is nonzero if $n \geq 2$.

Of course an elementary algebraic proof of the isomorphism in (0.1) could be given by directly verifying that the stated groups satisfy all the necessary relations to form a Witt complex. It is this approach we follow in the current paper for the case $A = W(k)$, where k is a perfect ring of odd characteristic p. Moreover, we prove that, for such A and for every $n \geq 1$, there is a surjective map

$$W_n\Omega^1_A \twoheadrightarrow \prod_{i=0}^{n-1} A/p^iA \cdot dV^i(1) =: E^1_n, \tag{0.2}$$

and we prove that the kernel of this map consists of all elements of $W_n\Omega^1_A$ which are divisible by arbitrarily large powers of p.

The groups $E^\bullet_n$ in a Witt complex over A are in particular $W_n(A)$-modules, and the $W_n(A)$-module structure we define is also analogous to the description for $\mathbb{Z}_{(p)}$. In the de Rham–Witt complex over $\mathbb{Z}_{(p)}$, and in fact in any Witt complex, for integers $i, j \geq 1$, one has

$$V^j(1)\, dV^i(1) = p^j\, dV^i(1). \tag{0.3}$$

This alone does not completely determine the $W_n(A)$-module structure, but for our specific case $A = W(k)$, there is a ring homomorphism $s_\varphi : A \to W(A)$, and we require that for all $a \in A$ and $x \in E^1_n$, we have $s_\varphi(a)x = a \cdot x$. Here the product $s_\varphi(a)x$ on the left side refers to the $W_n(A)$-module structure we wish to

define, and the product $a \cdot x$ on the right side refers to the A-module structure on E_n^1 that is apparent from the description in (0.2). This requirement completely determines our $W_n(A)$-module structure.

With these prerequisites in mind, the verification that our complex is a Witt complex is largely straightforward. The most difficult step is proving that our complex satisfies

$$Fd[a] = [a]^{p-1}\, d[a] \in E_n^1$$

for every $a \in A$ and for every integer $n \geq 1$. The difficulty, which arises repeatedly in what follows, lies in the fact that the multiplicative Teichmüller lift $[\cdot] : A \to W(A)$ is not related in a simple way to our ring homomorphism lift $s_\varphi : A \to W(A)$.

Once we know that our complex $E_\cdot^\bullet$ is a Witt complex over A, we attain relatively easily a complete algebraic description of the de Rham–Witt complex $W_\cdot\Omega_A^\bullet$. See Section 4 for the proofs of the following results, as well as for a more complete (but longer) description of $W_n\Omega_A^1$ (Corollary 4.10).

Theorem A. *Let k denote a perfect ring of odd characteristic p and let $A = W(k)$.*

(1) *Fix an integer $n \geq 1$. Let $S_n \subseteq W_n\Omega_A^1$ denote $\bigcap_{j=1}^{\infty} p^j W_n\Omega_A^1$, the $W_n(A)$-submodule of all elements which are infinitely p-divisible. Then we have an isomorphism of abelian groups*

$$W_n\Omega_A^1/S_n \cong \prod_{i=0}^{n-1} A/p^i A.$$

(2) *Fix integers $n \geq 1$ and $d \geq 2$. Then we have an isomorphism of abelian groups*

$$W_n\Omega_A^d \cong \prod_{i=0}^{n-1} \Omega_A^d.$$

In Section 5, we turn to describing the de Rham–Witt complex over the quotient ring A/xA, for an element $x \in A$; this is done with the purpose of applying it in the case that A/xA is a perfectoid ring B, and $A = W(B^\flat)$ is the ring of Witt vectors of the tilt of B. Our complete algebraic description of $W_n\Omega_A^1$ makes extensive use of the ring homomorphisms $s_\varphi : A \to W_n(A)$, and in general we have no such ring homomorphisms $B \to W_n(B)$, so our algebraic description of $W_n\Omega_B^1$ is less complete. However, for a certain class of perfectoid rings, we are able to completely describe the kernel of the restriction map $W_{n+1}\Omega_B^1 \to W_n\Omega_B^1$. We phrase the following theorem in slightly more generality, to include also the case $W(k)$ which is proved earlier.

Theorem B. *Let p denote an odd prime. Let S denote either $W(k)$ for k a perfect ring of characteristic p, or else let S denote a p-torsion-free perfectoid ring for which there exists some nonzero p-power torsion element $\omega \in \Omega_S^1$. In either of these cases, the following is a short exact sequence of $W_{n+1}(S)$-modules*:

$$0 \to S \xrightarrow{(-d,p^n)} \Omega_S^1 \oplus S \xrightarrow{V^n + dV^n} W_{n+1}\Omega_S^1 \xrightarrow{R} W_n\Omega_S^1 \to 0.$$

See Propositions 4.7 and 6.12 for the proofs, and also for a description of the module structures. The existence of an element ω as described in the statement is guaranteed, for example, whenever $\zeta_p \in S$ and $d\zeta_p \neq 0$.

One motivation for studying the de Rham–Witt complexes we consider in this paper is our hope to adapt results from [Hesselholt 2006]. That paper concerns the de Rham–Witt complex over the ring of integers in an algebraic closure of a mixed characteristic local field, and we hope to perform a similar analysis in the context of perfectoid rings. Our proofs for perfectoid rings will be modeled after Hesselholt's proof for $\mathcal{O}_{\overline{\mathbb{Q}}_p}$, and our proofs will use an induction argument that requires a precise description of the kernel of restriction $W_{n+1}\Omega^1_B \to W_n\Omega^1_B$. We will pursue this direction in joint work with Irakli Patchkoria.

A second, but indirect, motivation for the current paper is the recent remarkable work of Bhatt, Morrow and Scholze [2016], which makes use of the de Rham–Witt complex in mixed characteristic. Currently this is only a philosophical motivation, however, because they study the *relative* de Rham–Witt complex of Langer and Zink [2004], whereas we study the absolute de Rham–Witt complex of [Hesselholt and Madsen 2003; 2004; Hesselholt 2005]. Our work is not directly relevant to the work of Bhatt, Morrow and Scholze, but it could potentially be relevant to generalizations of their work which involved the *absolute* de Rham–Witt complex.

0.1. ***Notation.*** Throughout this paper, $p > 2$ denotes an odd prime, k is a perfect ring of characteristic p, W denotes p-typical Witt vectors, and $A = W(k)$. To distinguish between the Witt vector Frobenius on $A = W(k)$ and on $W(A)$, we write φ for the Witt vector Frobenius on A and we write F for the Witt vector Frobenius on $W(A)$ and on $W.\Omega^\bullet_A$. Rings in this paper are assumed to be commutative and to have unity, and ring homomorphisms are assumed to map unity to unity. We write Ω^1_R for the R-module of absolute Kähler differentials, i.e., $\Omega^1_R = \Omega_{R/\mathbb{Z}}$ in the notation of [Matsumura 1989, Section 25]. The de Rham–Witt complex we consider is the absolute, p-typical de Rham–Witt complex defined in [Hesselholt and Madsen 2004, Introduction].

1. Background on Witt complexes and the de Rham–Witt complex

Fix k, a perfect ring of odd characteristic p and let $A = W(k)$. The main goal of this paper is to construct a certain Witt complex over A, and to use this Witt complex to deduce properties of the de Rham–Witt complex over A. Similar properties are proven in the work of Hesselholt [2005; 2006] and Hesselholt and Madsen [2003; 2004]; the main difference between our results and these earlier results is that our proofs use only algebra. The only aspect of the current paper which is not elementary is our proof that $\Omega^1_{W(k)}$ has no nontrivial p-torsion (Proposition 2.7), which uses the cotangent complex. The current paper does not use any notions from algebraic topology, such as the spectrum $TR^\cdot_\bullet$.

The current paper does, however, use many standard facts about (p-typical) Witt vectors $W(R)$ and the (p-typical, absolute) de Rham–Witt complex $W.\Omega^\bullet_R$, and it is written with the assumption that the reader is familiar with their basic properties, including the case R is not characteristic p. For background on Witt vectors, we refer to [Illusie 1979] or to the brief introduction given in Section 1 of [Hesselholt

and Madsen 2004]. A thorough treatment of Witt vectors is given in Section 1 of [Hesselholt 2015], but those results are framed in the context of big Witt vectors instead of p-typical Witt vectors.

We work in this section over an arbitrary $\mathbb{Z}_{(p)}$-algebra R, where p is an odd prime. We now recall the basic properties of Witt complexes and the de Rham–Witt complex which we will use. Our reference is [Hesselholt and Madsen 2004].

The de Rham–Witt complex over R (or, more generally, any Witt complex over R) is a prosystem of differential graded rings. The index indicating the position in the prosystem is a positive integer $n = 1, 2, \ldots$ which we refer to as the *level*. The index indicating the degree in the differential graded ring is a nonnegative integer $d = 0, 1, \ldots$ which we refer to as the *degree*. We write E^d_n for the level n, degree d component of a Witt complex $E^\bullet_\cdot$.

Definition 1.1 [Hesselholt and Madsen 2004, Introduction]. Fix an odd prime p and a $\mathbb{Z}_{(p)}$-algebra R. A *Witt complex* over R is the following:

(1) A prodifferential graded ring $E^\bullet_\cdot$ and a strict map of prorings

$$\lambda : W_\cdot(R) \to E^\bullet_\cdot.$$

(2) A strict map of prograded rings

$$F : E^\bullet_\cdot \to E^\bullet_{\cdot-1}$$

such that $F\lambda = \lambda F$ and for all $r \in R$, we have

$$Fd\lambda([r]) = \lambda([r]^{p-1})d\lambda([r]).$$

(3) A strict map of graded $E^\bullet_\cdot$-modules

$$V : F_* E^\bullet_{\cdot-1} \to E^\bullet_\cdot.$$

(In other words,

$$V(F(\omega)\eta) = \omega V(\eta) \text{ for all } \omega \in E^\bullet_\cdot,\ \eta \in E^\bullet_{\cdot-1},$$

and similarly for multiplication on the right.) The map V must further satisfy $V\lambda = \lambda V$ and

$$FdV = d, \quad FV = p.$$

Remark 1.2. In this paper we never consider the prime $p = 2$. See [Hesselholt 2015, Definition 4.1] for a definition of Witt complex which can be used for all primes, or [Costeanu 2008] for a careful treatment of the 2-typical de Rham–Witt complex. One subtlety is that for $p = 2$, the differential does not necessarily satisfy $d \circ d = 0$.

The following theorem defines the de Rham–Witt complex over R as the initial object in the category of Witt complexes over R. Its existence is proved in [Hesselholt and Madsen 2004].

Theorem 1.3 [Hesselholt and Madsen 2004, Theorem A]. *Let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime. There is an initial object $W.\Omega^\bullet_R$ in the category of Witt complexes over R. We call this complex the de Rham–Witt complex over R. Moreover, for every $d \geq 0$ and $n \geq 1$, the canonical map*

$$\Omega^d_{W_n(R)} \to W_n\Omega^d_R$$

is surjective.

The following result, like our last result, is proved in [Hesselholt and Madsen 2004]. It describes the degree 0 piece and the level 1 piece of the de Rham–Witt complex, respectively.

Theorem 1.4. *Let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime.*

(1) [loc. cit., Remark 1.2.2] *The canonical map $\lambda : W_n(R) \to W_n\Omega^0_R$ is an isomorphism for all $n \geq 1$.*

(2) [loc. cit., Theorem D and the first sentence of the proof of Proposition 5.1.1] *The canonical map $\Omega^\bullet_R \to W_1\Omega^\bullet_R$ is an isomorphism.*

Two of the main results of this paper are Propositions 4.7 and 6.12. The main content of these propositions describes, for suitable rings R, the intersection

$$V^n(\Omega^1_R) \cap dV^n(R) \subseteq W_{n+1}\Omega^1_R.$$

Our next proposition, which is true for every $\mathbb{Z}_{(p)}$-algebra R, identifies

$$V^n(\Omega^1_R) + dV^n(R) \subseteq W_{n+1}\Omega^1_R$$

as the kernel of restriction.

Proposition 1.5. *Let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime. Fix integers $d \geq 1$ and $n \geq 1$. Then ω is in the kernel of restriction*

$$W_{n+1}\Omega^d_R \to W_n\Omega^d_R$$

if and only if there exist $\alpha \in \Omega^d_R$ and $\beta \in \Omega^{d-1}_R$ such that

$$\omega = V^n(\alpha) + dV^n(\beta).$$

The difficult part is the *only if* direction. See [Hesselholt and Madsen 2003, Lemma 3.2.4] for a proof in terms of the log de Rham–Witt complex. We recall the idea of that proof. (See also the proof of Proposition 5.7 below for similar arguments.) For every n, d, define

$${}'W_n\Omega^d_R := W_{n+1}\Omega^d_R/(V^n(\Omega^d_R) + dV^n(\Omega^{d-1}_R)).$$

One then shows that ${}'W.\Omega^\bullet_R$ is an initial object in the category of Witt complexes over R, and hence in particular that the natural map

$${}'W_n\Omega^d_R \to W_n\Omega^d_R \tag{1.6}$$

is an isomorphism. That natural map is induced by restriction $W_{n+1}\Omega^d_R \to W_n\Omega^d_R$, so our proposition follows from the injectivity of the map in (1.6).

The following results we recall from [Hesselholt and Madsen 2004] have significantly easier proofs than the previous results we have cited; the proofs of the relations in Proposition 1.7 below are just a few lines of computation.

Proposition 1.7 [Hesselholt and Madsen 2004, Lemma 1.2.1]. *Again let R denote a $\mathbb{Z}_{(p)}$-algebra, where p is an odd prime. The following equalities hold in every Witt complex over R:*

$$dF = pFd, \quad Vd = pdV, \quad V(x_0 dx_1 \cdots dx_m) = V(x_0)dV(x_1)\cdots dV(x_m).$$

2. Results on $W(A)$ and Ω^1_A when $A = W(k)$

Let k denote a perfect ring of odd characteristic p and let $A = W(k)$. In this paper, we study the de Rham–Witt complex over A. In this section, we prove several preliminary results about the degree zero case, $W(A)$, and the level one case, Ω^1_A. Special thanks are due to Bhargav Bhatt and Lars Hesselholt for their assistance with the Ω^1_A proofs.

The following result allows us to view the ring $W(A)$ as an A-algebra. This is a key fact. This is also a similarity between the case $A = W(k)$ and the case $A = \mathbb{Z}_{(p)}$, after which our results are modeled: the ring $W(A)$ is an A-algebra and the ring $W(\mathbb{Z}_{(p)})$ is a $\mathbb{Z}_{(p)}$-algebra. This is also the main reason our methods don't easily translate to more general rings such as ramified extensions of $\mathbb{Z}_p$.

Recall that, to avoid confusion, we write the Witt vector Frobenius differently on $A = W(k)$ from how we write it on $W(A) = W(W(k))$: we write $\varphi : A \to A$ and $F : W(A) \to W(A)$ for these Witt vector Frobenius maps. The map φ is a ring isomorphism, but the map F is not an isomorphism.

Proposition 2.1 [Illusie 1979, (0.1.3.16)]. *Let k denote a perfect ring of characteristic p, let $A = W(k)$, and let $\varphi : A \to A$ denote the Witt vector Frobenius. Then there is a unique ring homomorphism*

$$s_\varphi : A \to W(A)$$

satisfying $F \circ s_\varphi = s_\varphi \circ \varphi$ and such that for all $a \in A$, the ghost components of $s_\varphi(a)$ are $(a, \varphi(a), \varphi^2(a), \ldots)$.

Proof. The ring A is p-torsion free, so this result follows from [Illusie 1979, (0.1.3.16)], provided we know that the ring homomorphism $\varphi : A \to A$ satisfies $\varphi(a) \equiv a^p \bmod pA$ for all $a \in A$. This last congruence is in fact true more generally for any ring $W(R)$ of p-typical Witt vectors. We recall the short proof from [Illusie 1979, Section 0.1.4]. For arbitrary $a \in W(R)$, write $a = [r_0] + V(a_+)$, where $r_0 \in R$ and $a_+ \in W(R)$. We then have

$$\varphi(a) = [r_0]^p + pa_+ \equiv [r_0]^p \bmod pW(R) \equiv ([r_0] + V(a_+))^p \bmod pW(R),$$

where the last congruence uses that $V(x)V(y) = pV(xy) \in pW(R)$ for Witt vectors $x, y \in W(R)$. □

Lemma 2.2. *For every $x \in W(A)$, there exist unique elements $a_0, a_1, \ldots \in A$ for which*

$$x = \sum_{i=0}^{\infty} s_\varphi(a_i) V^i(1) \in W(A). \tag{2.3}$$

Proof. We have

$$\sum_{i=0}^{\infty} s_\varphi(a_i)V^i(1) = \sum_{i=0}^{\infty} V^i(F^i(s_\varphi(a_i))) = \sum_{i=0}^{\infty} V^i(s_\varphi(\varphi^i(a_i))),$$

so the result now follows from the fact that $\varphi : A \to A$ is an isomorphism and that the first component of $s_\varphi(a) \in W(A)$ is a. □

Lemma 2.4. *If $x \in W(A)$ is given as in* (2.3), *then*

$$V(x) = \sum_{i=1}^{\infty} s_\varphi(\varphi^{-1}(a_{i-1}))V^i(1) \in W(A).$$

Proof. This follows from the formula $V(F(x)y) = xV(y)$, for $x, y \in W(A)$ and from the fact that $F(s_\varphi(a_i)) = s_\varphi(\varphi(a_i))$. □

The following result gives explicit formulas for the elements $a_i \in A$ appearing in (2.3) in the specific case that x is a Teichmüller lift of some element $a \in A$. The main technical difficulty of this paper involves studying congruences involving these coefficients.

Lemma 2.5. *In the specific case $x = [a] \in W(A)$ is the Teichmüller lift of an element $a \in A$, then the terms a_i from* (2.3) *are given by the formulas $a_0 = a$ and $a_i = \varphi^{-i}((a^{p^i} - (\varphi(a))^{p^{i-1}})/p^i)$ for $i \geq 1$.*

Proof. This follows using induction on i, by comparing the ghost components of the two sides of (2.3). (Notice that the ghost map is injective because A is p-torsion free.) To simplify the proof, notice that a finite sum

$$s_\varphi(a_0) + s_\varphi(a_1)V(1) + \cdots + s_\varphi(a_n)V^n(1),$$

has ghost components which stabilize in the following pattern

$$(w_0, \ldots, w_{n-1}, w_n, \varphi(w_n), \varphi^2(w_n), \ldots).$$ □

When we define our Witt complex $E^\bullet_\cdot$ in Section 3, we will express E^1_n in terms of quotients A/p^iA. The groups E^1_n in a Witt complex over A always possess a $W_n(A)$-module structure, and the following lemma describes the $W_n(A)$-module structure we put on A/p^iA; notice that this module structure is not the one induced by the obvious projection map $W_n(A) \to A$.

Lemma 2.6. *Let $n, i \geq 1$ be integers and consider the map $W_n(A) \to A/p^iA$ given by*

$$\sum_{j=0}^{n-1} s_\varphi(a_j)V^j(1) \mapsto \sum_{j=0}^{n-1} a_j p^j.$$

This is a surjective ring homomorphism with kernel the ideal in $W_n(A)$ generated by

$$\{p^i, V^j(1) - p^j \mid 0 \leq j \leq n-1\}.$$

Proof. If we view $W_n(A)$ as an A-module via s_φ, then it's clear that the map is a surjective A-module homomorphism. To prove it's a ring homomorphism, we use the formula $V^j(1)V^i(1) = p^j V^i(1)$ for $j \le i$.

We now prove the statement about the kernel. Clearly the proposed elements are in the kernel; we now show an arbitrary element in the kernel is generated by the proposed elements. Assume $\sum_{j=0}^{n-1} s_\varphi(a_j)V^j(1)$ is in the kernel. This means that there exists $a \in A$ such that

$$p^i a = \sum_{j=0}^{n-1} a_j p^j \in A.$$

Applying s_φ to both sides, we find

$$p^i s_\varphi(a) = \sum_{j=0}^{n-1} s_\varphi(a_j) p^j \in W_n(A),$$

and thus

$$p^i s_\varphi(a) + \sum_{j=0}^{n-1} s_\varphi(a_j)(V^j(1) - p^j) = \sum_{j=0}^{n-1} s_\varphi(a_j)V^j(1),$$

which completes the proof. □

This concludes our collection of preliminary results on Witt vectors over $A = W(k)$. We now turn our attention to $\Omega^1_{W(k)}$. We thank Bhargav Bhatt and Lars Hesselholt for their help with the remainder of this section. Our first result, Proposition 2.7, is the most important. It says that multiplication by p is bijective on $\Omega^1_{W(k)}$; we will use this result repeatedly. By contrast, the results from Proposition 2.9 to the end of this section are closer to "reality-checks". For example, Corollary 2.10 below shows that $\Omega^1_{W(k)}$ is not the zero-module.

Proposition 2.7. *Let k denote a perfect ring of characteristic p. Then multiplication by p: $\Omega^1_{W(k)} \to \Omega^1_{W(k)}$ is a bijection.*

Remark 2.8. The proof below is due to Bhargav Bhatt. The tools used in the proof (the cotangent complex and, more generally, the language of derived categories) do not appear elsewhere in this paper, so the reader (or author) who is not comfortable with them is advised to treat the proof of Proposition 2.7 as a black box. See also the proof of [Hesselholt and Madsen 2003, Lemma 2.2.4] for a proof of a related result.

Before giving Bhatt's proof, we point out an elementary argument for surjectivity. The Witt vector Frobenius $\varphi : W(k) \to W(k)$ is surjective on one hand, and on the other hand, $\varphi(a) \equiv a^p \bmod pW(k)$ for every $a \in W(k)$. So for every $a \in W(k)$, we can find $a_0, a_1 \in W(k)$ such that $a = a_0^p + pa_1$. Thus every $da \in \Omega^1_{W(k)}$ is divisible by p, and hence multiplication by p on $\Omega^1_{W(k)}$ is surjective. We are not aware of a similarly elementary proof of injectivity.

Proof. Let $L_{W(k)/\mathbb{Z}}$ denote the cotangent complex. Because $\mathbb{Z} \to W(k)$ is flat, we have

$$L_{W(k)/\mathbb{Z}} \otimes^L_{\mathbb{Z}} \mathbb{F}_p \cong L_{k/\mathbb{F}_p}$$

by [Stacks 2005–, Tag 08QQ]. The right-hand side is zero, because the Frobenius automorphism on k induces a map on $L_{k/\mathbb{F}_p}$ which is simultaneously zero and an isomorphism. Thus the left-hand side is also 0. This implies that multiplication by p on $L_{W(k)/\mathbb{Z}}$ is a quasiisomorphism. In particular, multiplication by p is an isomorphism on $H^0(L_{W(k)/\mathbb{Z}}) \cong \Omega^1_{W(k)}$, which completes the proof. □

Throughout this paper, k denotes a perfect *ring* of characteristic p. We prove Corollary 2.10 below for $W(k)$ by deducing it from Proposition 2.9, which concerns the case of $W(k')$, where k' is a perfect *field* of characteristic p.

Proposition 2.9. *Let k' denote a perfect field of characteristic p. Let $\{x_\alpha\}_{\alpha\in A} \subseteq W(k')$ denote elements such that $\{x_\alpha\}_{\alpha\in A}$ is a transcendence basis for $W(k')[1/p]$ over $\mathbb{Q}$. Then $\{dx_\alpha\}_{\alpha\in A}$ is a basis for $\Omega^1_{W(k')}$ as a $W(k')[1/p]$-vector space.*

Proof. By Proposition 2.7, we have

$$\Omega^1_{W(k')} \cong \Omega^1_{W(k')} \otimes_{W(k')} W(k')[1/p] \cong \Omega^1_{W(k')[1/p]} \cong \Omega^1_{W(k')[1/p]/\mathbb{Q}}.$$

Thus it suffices to prove that if $\{x_\alpha\}_{\alpha\in A}$ is a transcendence basis for a field $K/\mathbb{Q}$, then $\{dx_\alpha\}_{\alpha\in A}$ is a K-basis for $\Omega^1_{K/\mathbb{Q}}$. The result now follows by [Matsumura 1989, Theorem 26.5]. □

Corollary 2.10. *Let k denote a perfect ring of characteristic p. Then the $W(k)$-module $\Omega^1_{W(k)}$ is nonzero.*

Proof. Let $\mathfrak{m} \subseteq k$ denote a maximal ideal. Then $k \to k/\mathfrak{m}$ is a surjection from k onto a perfect field of characteristic p; write $k' = k/\mathfrak{m}$. The induced map $W(k) \to W(k')$ is a surjective ring homomorphism, so $\Omega^1_{W(k)} \to \Omega^1_{W(k')}$ is a surjective $W(k)$-module homomorphism. Because $W(k')$ is uncountable, the field $W(k')[1/p]$ is transcendental over $\mathbb{Q}$, so our result follows from Proposition 2.9. □

Corollary 2.11. *For every integer $n \geq 1$, the $W_n(W(k))$-module $W_n\Omega^1_{W(k)}$ is nonzero.*

Proof. Begin with any nonzero element $\alpha \in \Omega^1_{W(k)}$. We then have $p^{n-1}\alpha \neq 0$ by Proposition 2.7, but on the other hand, $p^{n-1}\alpha = F^{n-1}V^{n-1}(\alpha)$, and so $V^{n-1}(\alpha) \in W_n\Omega^1_{W(k)}$ is nonzero. □

3. A p-adically separated Witt complex over $W(k)$

Let k denote a perfect ring of odd characteristic p and let $A = W(k)$. We are going to define a Witt complex over A. Our definition is modeled after [Hesselholt and Madsen 2004, Example 1.2.4], which gives a completely analogous description of the de Rham–Witt complex over $\mathbb{Z}_{(p)}$.

As an abelian group, we define

$$E_n^0 := W_n(A) \text{ for all } n \geq 1,$$
$$E_n^1 := \prod_{i=0}^{n-1} A/p^i A \cdot dV^i(1) \text{ for all } n \geq 1,$$
$$E_n^d := 0 \text{ for all } n \geq 1,\ d \geq 2;$$

here $dV^i(1)$ should be viewed as a formal basis symbol. The ring structure on $E_n^\bullet$ is obvious with the exception of the multiplication $E_n^0 \times E_n^1 \to E_n^1$, and for this we use the ring homomorphisms from Lemma 2.6 to give $A/p^i A$ the structure of a $W_n(A)$-module. (We note again that the module structure does not arise from the restriction map $W_n(A) \to W_1(A) = A$.) Define $\lambda : W_n(A) \to E_n^0$ to be the identity map and equip E_n^0 with the usual ring structure and with the usual maps R, F, V.

Recalling Lemma 2.2, which guarantees that each element in $W_n(A)$ corresponds to a unique expression $\sum_{i=0}^{n-1} s_\varphi(a_i) \cdot V^i(1)$, we define $d : E_n^0 \to E_n^1$ by the formula

$$d\left(\sum_{i=0}^{n-1} s_\varphi(a_i) V^i(1)\right) = \sum_{i=1}^{n-1} a_i \cdot dV^i(1).$$

Define $R : E_{n+1}^1 \to E_n^1$ by the formula

$$R\left(\sum_{i=0}^{n} a_i \cdot dV^i(1)\right) = \sum_{i=0}^{n-1} a_i \cdot dV^i(1).$$

Define $F : E_{n+1}^1 \to E_n^1$ by the formula

$$F\left(\sum_{i=1}^{n} a_i \cdot dV^i(1)\right) = \sum_{i=0}^{n-1} \varphi(a_{i+1}) \cdot dV^i(1).$$

Define $V : E_n^1 \to E_{n+1}^1$ by the formula

$$V\left(\sum_{i=1}^{n-1} a_i \cdot dV^i(1)\right) = \sum_{i=1}^{n-1} p\varphi^{-1}(a_i) \cdot dV^{i+1}(1).$$

We emphasize that this last definition means in particular that $V(dV^i(1)) = p \cdot dV^{i+1}(1)$.

Remark 3.1. We use the dot $\cdot$ in the notation $A/p^i A \cdot dV^i(1)$ to help distinguish between this $A/p^i A$-module structure and the $W_n(A)$-module structure, which we write without the dot. For example, if we let $\pi_{n,i} : W_n(A) \to A/p^i A$ denote the ring homomorphism from Lemma 2.6, then we would write $x dV^i(1) = \pi_{n,i}(x) \cdot dV^i(1)$. This distinction isn't mathematically important, but we find it helps to reinforce whether we are multiplying by elements in $A/p^i A$ or by elements in $W_n(A)$ or $W(A)$.

Before proving that $E^\bullet_\cdot$ is a Witt complex, we make a preliminary calculation that does not involve Witt vectors. This calculation will be used to verify that

$$Fd([a]) = [a]^{p-1}d([a]) \in E^1_n \tag{3.2}$$

holds for all $n \geq 1$, which is the most difficult step in our verification that $E^\bullet_\cdot$ is a Witt complex.

Remark 3.3. In (3.2), we are being less careful with notation than Hesselholt and Madsen [2004]. In their notation, this equation would be written

$$Fd([a]_{n+1}) = ([a]_n)^{p-1}d([a]_n) \in E^1_n,$$

where the subscripts are indicating $[a]_n \in W_n(A)$ and $[a]_{n+1} \in W_{n+1}(A)$.

Lemma 3.4. *Continue to let $A = W(k)$, where k is a perfect ring of odd characteristic p, and let $\varphi : A \to A$ denote the Witt vector Frobenius. Fix $a \in A$. Then for every $i \geq 1$, we have*

$$\tfrac{1}{p^{i+1}}(a^{p^{i+1}} - \varphi(a)^{p^i}) \equiv \tfrac{1}{p^i}(a^{p^i} - \varphi(a)^{p^{i-1}})a^{p^i(p-1)} \bmod p^i A. \tag{3.5}$$

Proof. The only fact we will use about $\varphi : A \to A$ is that for every $a \in A$, there exists $x \in A$ such that $\varphi(a) = a^p + px$. Multiplying both sides of (3.5) by p^{i+1} and applying the binomial theorem to the powers of $\varphi(a) = a^p + px$, we reduce immediately to proving that

$$\sum_{j=1}^{p^i}\binom{p^i}{j}(a^p)^{p^i-j}(px)^j \equiv pa^{p^i(p-1)}\sum_{j=1}^{p^{i-1}}\binom{p^{i-1}}{j}(a^p)^{p^{i-1}-j}(px)^j \bmod p^{2i+1}A.$$

By distributing the $a^{p^i(p-1)}$ term on the right side, this simplifies to proving that

$$\sum_{j=1}^{p^i}\binom{p^i}{j}a^{p^{i+1}-pj}(px)^j \equiv p\sum_{j=1}^{p^{i-1}}\binom{p^{i-1}}{j}a^{p^{i+1}-pj}(px)^j \bmod p^{2i+1}A.$$

By comparing the coefficients of the $a^m x^n$ monomials, it suffices then to prove the following two claims:

- For every j in the range $1 \leq j \leq p^{i-1}$, we have
$$p^j\binom{p^i}{j} \equiv p^{j+1}\binom{p^{i-1}}{j} \bmod p^{2i+1}.$$
- For every j in the range $p^{i-1}+1 \leq j \leq p^i$, we have
$$p^j\binom{p^i}{j} \equiv 0 \bmod p^{2i+1}.$$

To prove the first claim, we rewrite it as

$$p^j\left(\binom{p^i}{j} - p\binom{p^{i-1}}{j}\right) \equiv 0 \bmod p^{2i+1}.$$

The left side equals 0 if $j = 1$, so we may assume $j \geq 2$ and simplify the expression as

$$p^j \tfrac{p^i}{j!}((p^i - 1)\cdots(p^i - j + 1) - (p^{i-1} - 1)\cdots(p^{i-1} - j + 1)) \equiv 0 \bmod p^{2i+1}.$$

The term inside the parentheses is the difference of two terms which are congruent modulo p^{i-1}, hence the term inside the parentheses is divisible by p^{i-1}. Thus it suffices to show that for every $j \geq 2$ we have

$$p^j \tfrac{p^{2i-1}}{j!} \equiv 0 \bmod p^{2i+1}.$$

Thus it suffices to show that for every $j \geq 2$, we have $j - v_p(j!) \geq 2$, where v_p denotes the p-adic valuation. Because $p \geq 3$, the inequality is true if $j = 2$. For the case $j \geq 3$, again using $p \geq 3$, we compute

$$j - v_p(j!) \geq j - \left(\tfrac{j}{p} + \tfrac{j}{p^2} + \cdots\right) = j - j\tfrac{1}{p(p-1)} \geq j - \tfrac{j}{6} = \tfrac{5j}{6} \geq \tfrac{15}{6} \geq 2,$$

which completes the proof of the first claim.

To prove the second claim, we first treat the case $j = p^i$. Then we need to show that $p^i \geq 2i + 1$, which is true because $p \geq 3$ and $i \geq 1$. For the case $p^{i-1} + 1 \leq j < p^i$, we know the binomial coefficient in the expression has p-adic valuation at least one, so it suffices to prove that $j + 1 \geq 2i + 1$. Thus it suffices to prove that $p^{i-1} + 2 \geq 2i + 1$. Again this holds because $p \geq 3$ and $i \geq 1$. □

Remark 3.6. Lemma 3.4 is false in general if $p = 2$. For example, it is already false in the case $A = \mathbb{Z}_2$, $\varphi = \mathrm{id}$, $a = 2$, and $i = 1$.

We can now state our main theorem of this section; all the main results of this paper are dependent on the following result.

Theorem 3.7. *Let k be a perfect ring of characteristic $p > 2$, and let $A = W(k)$. The complex $E^\bullet_\cdot$ defined above is a Witt complex over A.*

Proof. Many of the required properties are obvious; the main difficulty is proving that for all $a \in A$ and all $n \geq 2$, we have

$$Fd([a]) = [a]^{p-1}d([a]) \in E^1_{n-1}. \tag{3.8}$$

We postpone this verification to the end of the proof.

The following properties are clear:

- For each n, $E^\bullet_n$ is a ring.
- The maps R are ring homomorphisms.
- The map λ is a ring homomorphism that commutes with R.
- The maps F, V commute with λ.
- The maps d, F, V are additive.
- The maps d, F, V commute with R.
- The composition FV is equal to multiplication by p.

Next we check that d verifies the Leibniz rule. Because d is additive and because $ds_\varphi(a) = 0$ for all a, it suffices to prove that for all $1 \le j \le i$, we have

$$d(V^i(1)V^j(1)) = V^i(1)dV^j(1) + V^j(1)dV^i(1).$$

Using the definition of our multiplication $E_n^0 \times E_n^1 \to E_n^1$ and using $V(x)V(y) = pV(xy)$, we see that both sides are equal to $(p^j + p^i A) \cdot dV^i(1)$.

Next we check that F is multiplicative. The only part which isn't obvious is to show that if $x \in E_n^0$ and $y \in E_n^1$, then we have

$$F(xy) = F(x)F(y).$$

Because we already know F is additive, it suffices to check this in the special cases $x = x_1 := s_\varphi(a)$, $x = x_2 := V^i(1)$ with $i \ge 1$, and $y = (b + p^j A) \cdot dV^j(1)$, where $j \ge 1$. We have $F(x_1) = s_\varphi(\varphi(a))$, $F(x_2) = pV^{i-1}(1)$, and $F(y) = (\varphi(b) + p^{j-1}A) \cdot dV^{j-1}(1)$. On the other hand, $x_1 y = (ab + p^j A) \cdot dV^j(1)$ and $F(x_1 y) = (\varphi(ab) + p^{j-1}A) \cdot dV^{j-1}(1) = F(x_1)F(y)$. We also have $x_2 y = (p^i b + p^j A) \cdot dV^j(1)$ and $F(x_2 y) = (p^i \varphi(b) + p^{j-1}A) \cdot dV^{j-1}(1) = F(x_2)F(y)$.

We next check that for all $x \in E_{n+1}^\bullet$ and $y \in E_n^\bullet$, we have

$$V(F(x)y) = xV(y). \tag{3.9}$$

This is obvious if x, y are both in degree zero or both in degree one, thus we only need to consider the case that one of them is degree zero and the other is degree one. It suffices to consider the case that the degree one term has the form $dV^j(1)$ and the degree zero term has the form $s_\varphi(a)V^i(1)$. If $x = dV(1)$ and $y = s_\varphi(a)V^i(1)$, then both sides of (3.9) are zero. If $x = dV^j(1)$ with $j \ge 2$ and $y = s_\varphi(a)V^i(1)$, we compute

$$V(F(x)y) = V(p^i a \cdot dV^{j-1}(1)) = p^{i+1}\varphi^{-1}(a) \cdot dV^j(1) = (s_\varphi(\varphi^{-1}(a))V^{i+1}(1))dV^j(1) = xV(y).$$

If $x = s_\varphi(a)$ and $y = dV^j(1)$, then we compute

$$V(F(x)y) = V(\varphi(a) \cdot dV^j(1)) = ps_\varphi(a)dV^{j+1}(1) = xV(y).$$

If $x = s_\varphi(a)V^i(1)$ with $i \ge 1$ and $y = dV^j(1)$, then we compute

$$\begin{aligned} V(F(x)y) &= V(s_\varphi(\varphi(a))pV^{i-1}(1)dV^j(1)) \\ &= V(\varphi(a)p^i \cdot dV^j(1)) \\ &= ap^{i+1} \cdot dV^{j+1}(1) \\ &= pxdV^{j+1}(1) \\ &= xV(y). \end{aligned}$$

To prove $FdV = d$, we begin with a term $x = s_\varphi(a)V^i(1) \in E_n^0$ and compute

$$\begin{aligned} FdV(x) &= Fd(s_\varphi(\varphi^{-1}(a))V^{i+1}(1)) \\ &= F(\varphi^{-1}(a) \cdot dV^{i+1}(1)) \\ &= a \cdot dV^i(1) \\ &= dx, \end{aligned}$$

as required.

To complete the proof, it remains to prove (3.8). For fixed $n \geq 2$, we compute

$$Fd[a] = Fd\left(\sum_{i=0}^{n-1} s_\varphi(a_i)V^i(1)\right),$$

where the a_i are given by the formulas in Lemma 2.5. We then compute further

$$= \sum_{i=1}^{n-1} F(a_i \cdot dV^i(1)) = \sum_{i=2}^{n-1} \varphi(a_i) \cdot dV^{i-1}(1) = \sum_{i=1}^{n-2} \varphi(a_{i+1}) \cdot dV^i(1).$$

For the other side of (3.8), we have

$$[a]^{p-1}d[a] = \left(\sum_{j=0}^{n-2} s_\varphi(a_j)V^j(1)\right)^{p-1} d\left(\sum_{i=0}^{n-2} s_\varphi(a_i)V^i(1)\right) = \sum_{i=1}^{n-2}\left(a_i\left(\sum_{j=0}^{n-2} a_j p^j\right)^{p-1}\right) \cdot dV^i(1).$$

We are finished if we can prove that, for every i in the range $1 \leq i \leq n-2$, we have

$$\varphi(a_{i+1}) \equiv a_i\left(\sum_{j=0}^{n-2} a_j p^j\right)^{p-1} \bmod p^i A,$$

which is clearly equivalent to proving

$$\varphi(a_{i+1}) \equiv a_i\left(\sum_{j=0}^{i} a_j p^j\right)^{p-1} \bmod p^i A.$$

Because φ is an isomorphism, it suffices to prove

$$\varphi^{i+1}(a_{i+1}) \equiv \varphi^i(a_i)\left(\sum_{j=0}^{i} \varphi^i(a_j) p^j\right)^{p-1} \bmod p^i A.$$

Recall our definition of the a_j terms:

$$[a] = \sum_{j=0}^{\infty} s_\varphi(a_j)V^j(1) \in W(A).$$

Comparing the ghost components of the two sides, we have $a^{p^i} = \sum_{j=0}^{i} \varphi^i(a_j)p^j$ for every $i \geq 0$. Thus we are finished if we can prove

$$\varphi^{i+1}(a_{i+1}) \equiv \varphi^i(a_i)a^{p^i(p-1)} \bmod p^i A.$$

By Lemma 2.5, we have reduced to showing

$$\frac{a^{p^{i+1}} - (\varphi(a))^{p^i}}{p^{i+1}} \equiv \frac{a^{p^i} - (\varphi(a))^{p^{i-1}}}{p^i} a^{p^i(p-1)} \bmod p^i A,$$

which was proved in Lemma 3.4. This completes the proof of (3.8), and this also completes the proof that $E^\bullet_\cdot$ is a Witt complex over A. □

Corollary 3.10. *For every integer n, the ring $E^\bullet_n$ is p-adically separated.*

Proof. This follows immediately from our definition of $E^\bullet_n$: in degree zero, $E^0_n = W_n(A)$, which is p-adically separated because A is p-adically separated. In degree one, we have $p^{n-1}E^1_n = 0$, and hence E^1_n is also p-adically separated. □

Remark 3.11. Our Witt complex $E^\bullet_\cdot$ is not isomorphic to the de Rham–Witt complex $W_\cdot\Omega^\bullet_A$. For example, $E^1_1 = 0$, while on the other hand it was shown in Corollary 2.10 that $W_1\Omega^1_A = \Omega^1_A \neq 0$. Nor is our Witt complex isomorphic to the relative de Rham–Witt complex of Langer and Zink [2004]: in their Witt complex, one always has $dV(1) = 0$. Following the language of [Hesselholt 2015, Remark 4.8], our Witt complex $E^\bullet_\cdot$ is the p-typical de Rham–Witt complex over A relative to the p-typical λ-ring (A, s_φ): this follows from the fact that the elements $s_\varphi(\alpha)$ for $\alpha \in \Omega^1_A$ are all zero in E^d_n, and that the differential map $E^\bullet_\cdot \to E^{\bullet+1}_\cdot$ is A-linear.

4. Applications to the de Rham–Witt complex over $A = W(k)$

Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. In this section, we use our p-adically separated Witt complex $E^\bullet_\cdot$ from Section 3 to give an explicit description (as an A-module) of the de Rham–Witt complex over A.

Remark 4.1. In this section we describe the de Rham–Witt complex over $A = W(k)$ as an A-module. The level n piece of the de Rham–Witt complex over A is always a $W_n(A)$-module. We warn that the $W_n(A)$-module structure does not factor through restriction $W_n(A) \to W_1(A) \cong A$. For example, multiplication by $V(1)$ is nonzero.

As $W_\cdot\Omega^\bullet_A$ is by definition the initial object in the category of Witt complexes over A, we get a natural map $W_\cdot\Omega^\bullet_A \to E^\bullet_\cdot$. The following key result identifies the kernel of this map in degree one.

Proposition 4.2. *Fix any integer $n \geq 1$, and let $S_n \subseteq W_n\Omega^1_A$ be the $W_n(A)$-submodule $\bigcap_{j=1}^{\infty} p^j W_n\Omega^1_A$. The natural map $\eta : W_n\Omega^1_A \to E^1_n$ induces an isomorphism $W_n\Omega^1_A/S_n \cong E^1_n$.*

Proof. Because E^1_n is p-adically separated, we see that S_n is contained in the kernel of the map $W_n\Omega^1_A \to E^1_n$. Consider the composition

$$\Omega^1_{W_n(A)} \to W_n\Omega^1_A \to E^1_n.$$

From our explicit description of E^1_n, we see that this composition is surjective. We will now show that the kernel of this composition is generated as a $W_n(A)$-module by elements of the form

- $ds_\varphi(a)$,
- $(V^j(1) - p^j)dV^i(1)$, and
- $p^i dV^i(1)$.

It is clear that these groups of elements are all in the kernel.

Consider now an arbitrary element $\omega \in \Omega^1_{W_n(A)}$ which is in the kernel; we must show that ω can be expressed as a $W_n(A)$-linear combination of the above elements. Viewing $\Omega^1_{W_n(A)}$ as an A-module via s_φ, we have that an arbitrary element in $\Omega^1_{W_n(A)}$ can be expressed as an A-linear combination of the elements $V^i(1)ds_\varphi(a)$ and $V^j(1)dV^i(1)$ with $0 \le j \le i \le n-1$. Thus we may write

$$\omega = \sum_{i=0}^{n-1} s_\varphi(b_i)V^i(1)ds_\varphi(a_i) + \sum_{0\le j\le i\le n-1} s_\varphi(a_{j,i})V^j(1)dV^i(1),$$

for some elements $b_i, a_i, a_{j,i} \in A$. Because the above itemized elements are all also in the kernel, we deduce that the element

$$\omega' := \sum_{0\le j<i\le n-1} p^j s_\varphi(a_{j,i})dV^i(1)$$

must also be in the kernel. From the explicit description of E^1_n, because ω' is in the kernel of the composition, we have that for each fixed i, we have $\sum_j p^j a_{j,i} \in p^i A$. Thus, for each fixed i, we have that $\sum_j p^j s_\varphi(a_{j,i})dV^i(1)$ is a $W_n(A)$-multiple of $p^i dV^i(1)$. This proves that ω', and hence also ω, is in the $W_n(A)$-submodule generated by the above elements.

We are finished, because $\Omega^1_{W_n(A)} \to W_n\Omega^1_A$ is surjective, and because the images of the above elements in $W_n\Omega^1_A$ are all in the submodule S_n. In fact, the images of the second and third groups of elements are equal to 0 in $W_n\Omega^1_A$: this follows from the identities $p^i dV^i = V^i d$ and

$$V(1)dV^i(1) = V(FdV^i(1)) = V(dV^{i-1}(1)) = pdV^i(1),$$

which hold in every Witt complex. □

The following is modeled after [Hesselholt and Madsen 2003, Section 3.2].

Lemma 4.3. *Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. For every $j \ge 1$, the map*

$$h_j : A \to \Omega^1_A \oplus A, \quad a \mapsto (-da, p^j a),$$

is an A-module homomorphism, where the left-hand side has its A-module structure induced by φ^j and where the right-hand side has component-wise addition and A-module multiplication defined by

$$x \cdot (\alpha, a) = \big(\varphi^j(x)\alpha - \tfrac{1}{p^j} ad\varphi^j(x), \varphi^j(x)a\big).$$

Remark 4.4. For any element $z \in \Omega^1_A$, the term $\frac{1}{p^j} z$ makes sense in Ω^1_A, because multiplication by p is a bijection on Ω^1_A.

Proof. We first check that the right-hand side is actually an A-module with respect to the structure we described. It's clear that $(x_1 + x_2) \cdot (\alpha, a) = x_1 \cdot (\alpha, a) + x_2 \cdot (\alpha, a)$ and that $x \cdot ((\alpha_1, a_1) + (\alpha_2, a_2)) = x \cdot (\alpha_1, a_1) + x \cdot (\alpha_2, a_2)$. Next we compute

$$\begin{aligned} x_1 \cdot (x_2 \cdot (\alpha, a)) &= x_1 \cdot \big(\varphi^j(x_2)\alpha - \tfrac{1}{p^j} ad\varphi^j(x_2), \varphi^j(x_2)a\big) \\ &= \big(\varphi^j(x_1)\big(\varphi^j(x_2)\alpha - \tfrac{1}{p^j} ad\varphi^j(x_2)\big) - \tfrac{1}{p^j}\varphi^j(x_2) ad\varphi^j(x_1), \varphi^j(x_1)\varphi^j(x_2)a\big) \\ &= \big(\varphi^j(x_1x_2)\alpha - \tfrac{1}{p^j} a\varphi^j(x_1) d\varphi^j(x_2) - \tfrac{1}{p^j} a\varphi^j(x_2) d\varphi^j(x_1), \varphi^j(x_1x_2)a\big) \\ &= (x_1x_2) \cdot (\alpha, a). \end{aligned}$$

Notice that so far the $1/p^j$ factor has played no role.

Next we check that the proposed map is an A-module homomorphism; this is where the $1/p^j$ factor becomes important. The map is clearly additive. We then check that, on one hand,

$$\varphi^j(x)a \mapsto (-d(\varphi^j(x)a), p^j\varphi^j(x)a) = (-\varphi^j(x)d(a) - ad(\varphi^j(x)), p^j\varphi^j(x)a),$$

and on the other hand,

$$x \cdot (-da, p^j a) = \big(-\varphi^j(x)da - \tfrac{1}{p^j} p^j ad(\varphi^j(x)), \varphi^j(x)p^j a\big). \qquad \square$$

Let M_j denote the cokernel of the A-module homomorphism h_j from Lemma 4.3. (This module is the analogue of what is denoted ${}_h W_n\omega^i_{(R,M)}$ in [Hesselholt and Madsen 2003, Section 3.2].) We are going to describe the de Rham–Witt complex over A in terms of these modules M_j. First we describe an A-module homomorphism $\Omega^1_A \to W_n\Omega^1_A$.

Given any ring homomorphism $R \to S$, there is an induced R-module homomorphism $\Omega^1_R \to \Omega^1_S$. In what follows, we will often use the following special case. Let $s_\varphi : A \to W(A)$ be the ring homomorphism described in Proposition 2.1. For every $n \geq 1$, composing s_φ with the restriction map induces a ring homomorphism $s_\varphi : A \to W_n(A)$ and hence an A-module homomorphism $s_\varphi : \Omega^1_A \to \Omega^1_{W_n(A)} \to W_n\Omega^1_A$. If we want to be explicit about the codomain, we write $s_{\varphi,n}$ instead of s_φ.

Lemma 4.5. *For every integer $n \geq 2$, the two A-module homomorphisms $s_{\varphi,n-1} \circ \varphi \circ \frac{1}{p}$ and $F \circ s_{\varphi,n}$ mapping $\Omega^1_A \to W_{n-1}\Omega^1_A$ are equal.*

Proof. It suffices to prove the images of a term $a_0 da_1$ are equal, and this follows from the relationships $dF = pFd$ and $s_\varphi \circ \varphi = F \circ s_\varphi$. $\square$

Lemma 4.6. *Fix integers $n \geq j \geq 1$ and let M_j be the cokernel of the A-module homomorphism h_j from Lemma 4.3. Consider $W_{n+1}\Omega^1_A$ as an A-module using the map $s_\varphi : A \to W(A)$. The map*

$$\begin{aligned} M_j &\to W_{n+1}\Omega^1_A, \\ (\alpha, a) &\mapsto V^j(s_\varphi(\alpha)) + dV^j(s_\varphi(a)) \end{aligned}$$

is an A-module homomorphism.

Proof. The map is clearly well-defined, because of the relation $p^j dV^j = V^j d$. We have

$$\begin{aligned} x \cdot (\alpha, a) &= \big(\varphi^j(x)\alpha - \tfrac{1}{p^j} a d\varphi^j(x), \varphi^j(x)a\big) \\ &\mapsto V^j \circ s_\varphi\big(\varphi^j(x)\alpha - \tfrac{1}{p^j} a d\varphi^j(x)\big) + dV^j \circ s_\varphi(\varphi^j(x)a) \\ &= V^j(F^j(s_\varphi(x))s_\varphi(\alpha)) - V^j\big(\tfrac{1}{p^j} s_\varphi(a) dF^j(s_\varphi(x))\big) + dV^j(F^j(s_\varphi(x))s_\varphi(a)) \\ &= V^j(F^j(s_\varphi(x))s_\varphi(\alpha)) - V^j(s_\varphi(a)F^j ds_\varphi(x)) + d(s_\varphi(x)V^j(s_\varphi(a))) \\ &= s_\varphi(x)V^j(s_\varphi(\alpha)) - V^j(s_\varphi(a))ds_\varphi(x) + V^j(s_\varphi(a))ds_\varphi(x) + s_\varphi(x)dV^j(s_\varphi(a)) \\ &= s_\varphi(x)(V^j(s_\varphi(\alpha)) + dV^j(s_\varphi(a))). \end{aligned}$$

□

Proposition 4.7. *Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. Fix any integer $n \geq 1$, and let M_n be the cokernel of the A-module homomorphism from Lemma 4.3. Consider $W_n\Omega^1_A$ and $W_{n+1}\Omega^1_A$ as A-modules via the ring homomorphism $s_\varphi : A \to W(A)$. We have a short exact sequence of A-modules*

$$0 \to M_n \to W_{n+1}\Omega^1_A \xrightarrow{R} W_n\Omega^1_A \to 0, \tag{4.8}$$

where the first map is given by

$$(\alpha, a) \mapsto V^n(s_\varphi(\alpha)) + dV^n(s_\varphi(a)).$$

Proof. Using Lemma 4.6, we see that these are maps of A-modules. Then using Proposition 1.5, we reduce to proving that the map $M_n \to W_{n+1}\Omega^1_A$ is injective. Assume $\alpha \in \Omega^1_A$ and $a \in A$ satisfy $V^n(s_\varphi(\alpha)) + dV^n(s_\varphi(a)) = 0 \in W_{n+1}\Omega^1_A$. Then, because α is divisible by arbitrarily large powers of p, we have that $dV^n(s_\varphi(a))$ is divisible by arbitrarily large powers of p. Write $a' = \varphi^{-n}(a)$. We have

$$dV^n(s_\varphi(a)) = d(s_\varphi(a')V^n(1)) = s_\varphi(a')dV^n(1) + V^n(1)ds_\varphi(a').$$

The term $ds_\varphi(a')$ is divisible by arbitrarily large powers of p, so this implies $s_\varphi(a')dV^n(1)$ is divisible by arbitrarily large powers of p. Thus by Corollary 3.10, the image of $s_\varphi(a')dV^n(1)$ is equal to 0 in E^1_{n+1}, but then by our definition of E^1_{n+1}, we have that a' is divisible by p^n, and hence so is $a = \varphi^n(a')$.

Write $a = p^n a_0$. We then have

$$0 = V^n(s_\varphi(\alpha)) + dV^n(s_\varphi(p^n a_0)) = V^n(s_\varphi(\alpha + da_0)).$$

By Proposition 2.7, the map $p^n : \Omega^1_A \to \Omega^1_A$ is injective. Because $p^n = F^n V^n$, we have that V^n is also injective. This shows that $\alpha = -da_0$, as claimed. □

Remark 4.9. Proposition 4.7 is the main result of this section. The exactness claimed is mostly analogous to [Hesselholt and Madsen 2003, Proposition 3.2.6]; the most interesting part of our result is the fact that the map $A \to \Omega^1_A \oplus A$ surjects onto the kernel of the map $\Omega^1_A \oplus A \to W_{n+1}\Omega^1_A$. This result is difficult to prove because in general it is difficult to prove that elements in the de Rham–Witt complex are nonzero. See [Hesselholt 2005, Proposition 2.2.1] for a result proving this same exactness in the context of the log de Rham–Witt complex over the ring of integers in an algebraic closure of a local field. See also [Illusie 1979, Théorème I.3.8] for a version of this result which is valid in characteristic p.

Using induction, we are able to give the following explicit description of $W_n\Omega^1_A$. The key fact used by the construction is that the maps $\Omega^1_A \oplus A \to W_j\Omega^1_A$ given by $(\alpha, a) \mapsto V^{j-1}(\alpha) + dV^{j-1}(a)$ can be extended to maps into $W_n\Omega^1_A$ using $s_\varphi : A \to W(A)$.

Corollary 4.10. *Continue to assume $A = W(k)$ where k is a perfect ring of odd characteristic p. View $W_{n+1}\Omega^1_A$ as an A-module using the ring homomorphism $s_\varphi : A \to W(A)$. Let $M_0 = \Omega^1_A$, and for every $j \geq 1$, let $M_j = (\Omega^1_A \oplus A)/h_j(A)$ be the cokernel of the A-module homomorphism $h_j : a \mapsto (-da, p^j a)$ from Lemma 4.3. For every integer $n \geq 2$, the map*

$$\prod_{j=0}^{n} M_j \to W_{n+1}\Omega^1_A$$

induced by

$$M_0 \to W_{n+1}\Omega^1_A,$$
$$\alpha_0 \mapsto s_\varphi(\alpha_0)$$

and

$$M_j \to W_{n+1}\Omega^1_A \quad \text{for } j \geq 1,$$
$$(\alpha_j, a_j) \mapsto V^j(s_\varphi(\alpha_j)) + dV^j(s_\varphi(a_j))$$

is an isomorphism of A-modules.

Proof. We know that the map is a homomorphism of A-modules by Lemma 4.6. For every integer $n \geq 1$, consider the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_n & \longrightarrow & \prod_{j=0}^{n} M_j & \longrightarrow & \prod_{j=0}^{n-1} M_j & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_n & \longrightarrow & W_{n+1}\Omega^1_A & \longrightarrow & W_n\Omega^1_A & \longrightarrow & 0. \end{array}$$

The top row is clearly exact. The bottom row is exact by (4.8). The right-hand vertical map is an isomorphism by induction. Thus we are finished by the five lemma. □

Similar, but easier, arguments work also for degrees $d \geq 2$. Our applications involve degree $d = 1$, so we indicate the results more briefly.

Proposition 4.11. *For every $d \geq 2$, $n \geq 1$, we have an exact sequence of A-modules*

$$0 \to \Omega_A^d \xrightarrow{V^n} W_{n+1}\Omega_A^d \to W_n\Omega_A^d \to 0,$$

where the A-module structure on Ω_A^d is given by $a \cdot \alpha := F^n(a)\alpha$, and where the A-module structure on the other two pieces is induced by $s_\varphi : A \to W(A)$.

Proof. The map $V^n : \Omega_A^d \to W_{n+1}\Omega_A^d$ is injective because $F^n \circ V^n = p^n$ is injective on Ω_A^d. We must also show that if $\omega \in W_{n+1}\Omega_A^d$ is in the kernel of R, then we can find $\alpha \in \Omega_A^d$ such that $\omega = V^n(\alpha)$. We know that there exist $\alpha \in \Omega_A^d$ and $\beta \in \Omega_A^{d-1}$ such that

$$V^n(\alpha) + dV^n(\beta) = \omega.$$

But now we are finished, because we can write $\beta = p^n\beta_0$ for some $\beta_0 \in \Omega_A^{d-1}$. (This is where we use that $d \geq 2$.) □

We can deduce the following corollary in the same way as we deduced Corollary 4.10.

Corollary 4.12. *For every $d \geq 2$ and every $n \geq 1$, we have an isomorphism of A-modules*

$$\prod_{i=0}^{n-1} \Omega_A^d \cong W_n\Omega_A^d,$$

where the A-module structure on the i-th piece is given by $a \cdot \alpha_i := \varphi^i(a)\alpha_i$.

Remark 4.13. Much of the author's intuition for the de Rham–Witt complex comes from the cases treated in [Illusie 1979], such as the description of the de Rham–Witt complex over $\mathbb{F}_p[t_1, \ldots, t_r]$ given in [loc. cit., Section I.2]. In this case, the de Rham–Witt complex is 0 in degrees $d > r$. We remark that the absolute, mixed characteristic de Rham–Witt complex we are studying is very different. Consider the easiest case of our setup, $A = \mathbb{Z}_p = W(\mathbb{F}_p)$. Then Ω_A^1 is infinite-dimensional as a $\mathbb{Q}_p$-vector space by Proposition 2.9. Thus $\Omega_A^d := \bigwedge^d \Omega_A^1$ is nonzero for all degrees d. Thus in particular $W_n\Omega_A^d$ is nonzero for all integers $d \geq 0$ and $n \geq 1$.

Remark 4.14. Corollaries 4.10 and 4.12 give an explicit description of the A-module structure of the Witt complex $W.\Omega_A^\bullet$. (Notice that for a general ring $B \neq W(k)$, we cannot expect a B-algebra structure on $W.\Omega_B^\bullet$.) It seems worthwhile to describe the entire Witt complex structure, at least for degrees $d = 0, 1$, in terms of the description from Corollary 4.10. Similar descriptions could be given for higher degrees.

- We already know the A-module structure, so to describe the $W_n(A)$-algebra structure on $W_n\Omega_A^1$, it suffices by Lemma 2.2 to describe the effect of multiplication by $V^j(1)$ on $\prod M_i$. It sends all M_i with $i \leq j$ into the M_j component, via the formulas

$$V^j(1) \cdot \alpha = (\varphi^j(\alpha), 0) \in M_j \text{ for } \alpha \in M_0 = \Omega_A^1, \qquad \text{and}$$
$$V^j(1) \cdot (\alpha_i, a_i) = (p^i\varphi^{j-i}(\alpha_i) + \varphi^{j-i}(da_i), 0) \in M_j, \qquad \text{for } (\alpha_i, a_i) \in M_i, \text{ where } i \leq j.$$

  When $i \geq j$, multiplication by $V^j(1)$ acts on the M_i component as multiplication by p^j.

- To describe the differential $d : W_n(A) \to \prod M_i$, it suffices by Lemma 2.2 to note that $d : s_\varphi(a) \mapsto da \in M_0 = \Omega^1_A$ and that $d : s_\varphi(a_j)V^j(1) \mapsto (0, \varphi^j(a_j)) \in M_j$ for $j \geq 1$.
- The restriction map $R : \prod_{i=0}^n M_i \to \prod_{i=0}^{n-1} M_i$ is the obvious projection map.
- To describe the map $V : \prod_{i=0}^n M_i \to \prod_{i=0}^{n+1} M_i$, we note that

$$\begin{aligned} V : \alpha &\mapsto (\alpha, 0) \in M_1, && \text{where } \alpha \in M_0 = \Omega^1_A, \text{ and} \\ V : (\alpha_i, a_i) &\mapsto (\alpha_i, pa_i) \in M_{i+1}, && \text{where } (\alpha_i, a_i) \in M_i. \end{aligned}$$

- To describe the map $F : \prod_{i=0}^{n+1} M_i \to \prod_{i=0}^n M_i$, we note that

$$\begin{aligned} F : \alpha &\mapsto \varphi(\alpha) \in M_0, && \text{for } \alpha \in M_0 = \Omega^1_A \\ F : (\alpha_1, a_1) &\mapsto p\alpha_1 + da_1 \in M_0, && \text{for } (\alpha_1, a_1) \in M_1, \text{ and} \\ F : (\alpha_i, a_i) &\mapsto (p\alpha_i, a_i) \in M_{i-1}, && \text{for } (\alpha_i, a_i) \in M_i. \end{aligned}$$

Corollary 4.15. *For every $n \geq 1$, the p-torsion submodule of $W_n\Omega^1_A$ is isomorphic to the free A/p-module of rank $n-1$ generated by $p^{j-1}dV^j(1)$, for $j = 1, \ldots, n-1$.*

Proof. Using the fact that multiplication by p is a bijection on Ω^1_A, we see that the p-torsion module in $M_j = (\Omega^1_A \oplus A)/h_j(A)$ is a free A/pA-module of rank 1 generated by $(0, p^{j-1})$. Then from Corollary 4.10, we see that these elements together generate the p-torsion submodule of $W_n\Omega^1_A$. In the factor $M_j \cong (\Omega^1_A \oplus A)/h_j(A)$, a representative (α, a) has element a uniquely determined modulo p^jA. This shows that we have a relation

$$\sum dV^j(p^{j-1}\varphi^j(a)) = 0$$

only if each $a \in pA$. This shows that the proposed elements are free generators, which completes the proof. □

5. The de Rham–Witt complex over A/xA

As usual, let p denote an odd prime, let k denote a perfect ring of characteristic p, and let $A = W(k)$. There are two natural ways to lift elements from A to $W(A)$: the first is our ring homomorphism s_φ, and the second is the multiplicative Teichmüller map. So far in this paper, we have made extensive use of the ring homomorphism s_φ. In this section and the next, we make more frequent use of the Teichmüller map. The reason is that we will be studying the kernel of the natural ring homomorphism $W(A) \to W(A/xA)$ for $x \in A$, and $[x]$ is in this kernel whereas $s_\varphi(x)$ in general is not. For example, $[p]$ is in the kernel of $W(\mathbb{Z}_p) \to W(\mathbb{Z}_p/p\mathbb{Z}_p)$, whereas $s_\varphi(p) = p$ is not.

The exactness in (4.8) above is very useful for making induction arguments involving the de Rham–Witt complex. For example, our proof of Corollary 4.10 was dependent on our Witt complex $E^\bullet_\cdot$ only because $E^\bullet_\cdot$ was used to prove exactness in (4.8). The goal of the remainder of the paper is to prove exactness of the corresponding sequence for the de Rham–Witt complex over a certain class of perfectoid rings. See [Hesselholt 2006, Proposition 2.2.1; Hesselholt and Madsen 2003, Theorem 3.3.8] for related results. In

future joint work with Irakli Patchkoria, we hope to use this exact sequence to provide algebraic proofs of results similar to Hesselholt's p-adic Tate module computation [2006, Proposition 2.3.2]. In this section we prove general results concerning $W_n\Omega^1_{(A/xA)}$ that are valid for arbitrary $x \in A$. In Section 6, we specialize to a certain class of perfectoid rings, in which case we can prove stronger results, including the analogue of the exact sequence in (4.8).

Fix an element $x \in A$. For every integer $n \geq 1$, we have a surjective A-module homomorphism $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$, and Corollary 4.10 gives an explicit description of the domain. We will give explicit A-module generators for the kernel. Unfortunately, this kernel is not generated as an A-module by elements which are homogeneous with respect to the direct sum decomposition from Corollary 4.10.

First we consider the case of level $n = 1$, which will be used repeatedly.

Lemma 5.1. *The kernel of the A-module homomorphism $\Omega^1_A \to \Omega^1_{(A/xA)}$ is generated by $x\alpha$ for $\alpha \in \Omega^1_A$ together with the element dx.*

Proof. This follows immediately from the usual right exact sequence of (A/xA)-modules

$$xA/x^2A \to \Omega^1_A \otimes_A (A/xA) \to \Omega^1_{(A/xA)} \to 0 \tag{5.2}$$

[Matsumura 1989, Theorem 25.2], where the left-most map is given by $xa \mapsto d(xa) \otimes 1$. □

Next we identify the kernel in the degree zero case, $W.\Omega^0_A \to W.\Omega^0_{(A/xA)}$.

Lemma 5.3. *Let K^0 denote the kernel of the ring homomorphism $W(A) \to W(A/xA)$ induced by the projection $A \to A/xA$. Then K^0 consists precisely of elements of the form*

$$\sum_{k=0}^{\infty} s_\varphi(a_k)V^k([x]),$$

where $a_k \in A$.

Proof. It's clear that these elements are in the kernel. We now prove that an arbitrary element in the kernel can be written in this way. Working one level at a time, it suffices to show that if $V^k(y_k)$ is in the kernel, then we can find $a_k \in A$ and $y_{k+1} \in W(A)$ such that

$$V^k(y_k) = s_\varphi(a_k)V^k([x]) + V^{k+1}(y_{k+1}).$$

(Note that this also implies that $V^{k+1}(y_{k+1})$ is in the kernel.) Because

$$s_\varphi(a_k)V^k([x]) = V^k(F^k(s_\varphi(a_k))[x]) = V^k(s_\varphi(\varphi^k(a_k))[x])$$

and $\varphi : A \to A$ is surjective, we can find such elements a_k and y_{k+1}. □

We now do the same thing for the degree one case. In this case, the ring $W(A) \cong \varprojlim W_n(A)$ from Lemma 5.3 gets replaced by the $W(A)$-module, $\varprojlim W_n\Omega^1_A$. Corollary 4.10 leads to an explicit description of this inverse limit as an A-module.

More concretely, we give generators for the kernels of the A-module homomorphisms $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$, and we choose these generators so they are compatible under restriction maps for varying

$n \geq 1$. We view these generators as elements in $\varprojlim W_n\Omega^1_A$. The main work involves studying, for particular choices of A and x, the A-submodule of $\varprojlim W_n\Omega^1_A$ generated by these elements in the kernel. Because these elements involve the Teichmüller lift $[x]$, they do not have a simple description in terms of our decomposition of $W_n\Omega^1_A$ given in Corollary 4.10.

Definition 5.4. Let $M_0 = \Omega^1_A$ and for each integer $j \geq 1$, let M_j be the cokernel of the A-module homomorphism in Lemma 4.3. Let M denote the A-module

$$M = \prod_{j=0}^{\infty} M_j.$$

Let $K^1 \subseteq M$ denote the A-submodule consisting of all elements of the form

$$\sum_{k=0}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])),$$

where $\alpha_k \in \Omega^1_A$ and where $a_k \in A$; here, to make sense of such an expression as an element in M, we use the structures described in Remark 4.14.

Remark 5.5. (1) By Corollary 4.10, M is isomorphic as an A-module to $\varprojlim W_n\Omega^1_A$.

(2) The A-module K^1 depends on our choice of element x, but that element is fixed throughout this section, so we write simply K^1 and not more suggestive notation such as K^1_x.

We will use K^1 from Definition 5.4 to describe the kernel of $W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$; namely, we will show that this kernel is the image of K^1 under the restriction map $R_n : M \to W_n\Omega^1_A$.

Lemma 5.6. *For $n \geq 1$, write R_n for the restriction map $W(A) \to W_n(A)$ and also for the restriction map $M \to W_n\Omega^1_A$. The A-submodule of $W_n\Omega^\bullet_A$ generated by $R_n(K^0)$ and $R_n(K^1)$ and all higher degree terms ($W_n\Omega^d_A$ for $d \geq 2$) is an ideal in the ring $W_n\Omega^\bullet_A$.*

Proof. We have to show that the A-module generated by these elements is closed under multiplication by elements in $W_n\Omega^\bullet_A$. Consider an element $V^k([x])m$, where $m \in W_n\Omega^1_A$. This can be rewritten as $V^k([x]m_0)$, where $m_0 = F^k(m)$. The element m_0 can be written (not uniquely) as

$$m_0 = s_\varphi(\alpha_0) + \sum_{i=1}^{n-k-1}(V^i(s_\varphi(\alpha_i)) + dV^i(s_\varphi(a_i))),$$

and so

$$\begin{aligned}[x]m_0 &= [x]s_\varphi(\alpha_0) + \sum_{i=1}^{n-k-1}([x]V^i(s_\varphi(\alpha_i)) + [x]dV^i(s_\varphi(a_i)))\\ &= [x]s_\varphi(\alpha_0) + \sum_{i=1}^{n-k-1}(V^i([x]^{p^i}s_\varphi(\alpha_i)) + dV^i([x]^{p^i}s_\varphi(a_i)) - V^i(s_\varphi(a_i)[x]^{p^i-1}d[x])).\end{aligned}$$

(Here we used the formula $Fd[x] = [x]^{p-1}d[x]$.) And so

$$V^k([x]m_0) \in R_n(K^1).$$

Now we consider degree 1 terms in our A-module. We first consider a term $V^k([x]s_\varphi(\alpha))$ and then below we consider $dV^k([x])$. We can write an arbitrary element $y \in W_n(A)$ as $\sum_{i=0}^{n-1} s_\varphi(y_i)V^i(1)$, thus it suffices to show that

$$V^k([x]s_\varphi(\alpha))V^i(1) \in R_n(K^1).$$

If $i \le k$, we have

$$V^k([x]s_\varphi(\alpha))V^i(1) = V^k([x]s_\varphi(p^i\alpha)) \in R_n(K^1).$$

If $i > k$, we have

$$V^k([x]s_\varphi(\alpha))V^i(1) = V^i(F^{i-k}([x]s_\varphi(\alpha))) = V^i([x]^{p^{i-k}}s_\varphi\big(\tfrac{1}{p^{i-k}}\varphi^{i-k}(\alpha)\big)) \in R_n(K^1).$$

Similarly, we find

$$dV^k([x])V^i(1) = V^i(dV^{k-i}([x]) = p^i dV^k([x]) \quad \text{for } i \le k$$

and

$$dV^k([x])V^i(1) = V^i(F^{i-k}d[x]) = V^i([x]m) \quad \text{for } i > k \text{ and } m \in W_n\Omega^1_A.$$

It was shown in the degree zero portion of our proof that this latter element is in $R_n(K^1)$. □

Proposition 5.7. *Define $G^\bullet_\cdot$ by*

$$G^0_n := W_n(A)/R_n(K^0), \quad G^1_n := W_n\Omega^1_A/R_n(K^1), \quad G^d_n := 0 \text{ for } d \ge 2.$$

Equipped with the structure maps inherited from $W.\Omega^\bullet_A$, this is a Witt complex over A/xA.

Proof. The main thing to verify is that all of the necessary maps are well-defined. All the various relations required of a Witt complex will then hold automatically since they hold in $W_n\Omega^\bullet_A$.

The fact that $G^\bullet_n$ is a ring follows from Lemma 5.6. Define $\lambda : W_n(A/xA) \to G^0_n$ to be the unique map such that the composition $W_n(A) \to W_n(A/xA) \to G^0_n$ is the projection map; this is possible by Lemma 5.3. To define the differential $d : G^0_n \to G^1_n$, we check that $d(s_\varphi(a)V^k([x])) \in R_n(K^1)$, which follows because

$$d(s_\varphi(a)V^k([x])) = s_\varphi(a)dV^k([x]) + V^k([x]F^k ds_\varphi(a)) = s_\varphi(a)dV^k([x]) + V^k([x]s_\varphi\big(\tfrac{1}{p^k}d\varphi^k(a)\big)),$$

where the last equality holds by Lemma 4.5. Because $R \circ R_n = R_{n-1}$, it is clear that the restriction map R is well-defined. The fact that V is well-defined follows from $VdV^k = pdV^{k+1}$ and the fact that K^1 is closed under multiplication by arbitrary elements in $W(A)$.

To check that F is well-defined on G_n^1, we need to show that $F(R_n(K^1)) \subseteq R_{n-1}(K^1)$, which means that we need to evaluate F on elements

$$\sum_{k=0}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])).$$

The result is immediate from the de Rham–Witt relations, but we need to be careful to treat the $k = 0$ case separately from the $k > 0$ case. We have

$$F([x]s_\varphi(\alpha_0)) = [x]^p s_\varphi\big(\tfrac{1}{p}\varphi(\alpha_0)\big) \quad \text{and} \quad F(s_\varphi(a_0)d[x]) = s_\varphi(\varphi(a_0))[x]^{p-1}d[x],$$

and these elements are in $R_{n-1}(K^1)$ by Lemma 5.6. For $k \geq 1$, we have

$$F(V^k([x]s_\varphi(\alpha_k))) \quad \text{and} \quad F(s_\varphi(a_k)dV^k([x])) \in R_{n-1}(K^1),$$

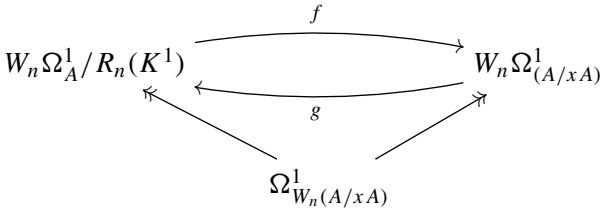
because $FV = p$ and $FdV = d$. □

Proposition 5.8. *We have an isomorphism of A-modules*

$$W_n\Omega_A^1/R_n(K^1) \cong W_n\Omega^1_{(A/xA)}.$$

Proof. Viewing $W_n\Omega^1_{(A/xA)}$ as a Witt complex over A, we have a map of $W_n(A)$-modules $W_n\Omega_A^1 \to W_n\Omega^1_{(A/xA)}$ which induces a map $f : (W_n\Omega_A^1)/R_n(K^1) \to W_n\Omega^1_{(A/xA)}$. Similarly, $G_\cdot^\bullet$ is a Witt complex over A/xA by Proposition 5.7, so we have a map of $W_n(A/xA)$-modules $g : W_n\Omega^1_{(A/xA)} \to (W_n\Omega_A^1)/R_n(K^1)$. We claim that the compositions gf and fg are both the identity map.

Because the maps f and g arise from maps of Witt complexes, the two triangles in the following diagram commute.



Then, because the diagonal maps are both surjective, a diagram chase shows that fg and gf are both the identity map. □

We conclude this section with a technical result about K^1 that will be used in the following section. We include it in this section because it is valid in a more general context than what we consider in Section 6.

Notation 5.9. For every integer $n \geq 1$, let P_n denote the property

- P_n: If $z \in K^1$ and $R_n(z) = 0$, then we can write

$$z = \sum_{k=n}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])).$$

Proposition 5.10. *Assume that $x \notin pA$. If property P_1 holds, then for every integer $n \geq 1$, the property P_n also holds.*

Proof. We prove this using induction on n. Thus assume we know that property P_{n-1} holds for some $n \geq 2$, and assume we have $z \in K^1$ such that $R_n(z) = 0$. By our induction hypothesis, we can assume

$$z = \sum_{k=n-1}^{\infty} (V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])).$$

The terms for $k \geq n$ do not affect the conclusion, so we can in fact assume

$$\begin{aligned} z &= V^{n-1}([x]s_\varphi(\alpha)) + s_\varphi(a)dV^{n-1}([x]) \\ &= V^{n-1}([x]s_\varphi(\alpha)) + dV^{n-1}([x]F^{n-1}(s_\varphi(a))) - V^{n-1}([x]F^{n-1}(ds_\varphi(a))) \\ &= V^{n-1}\big([x]s_\varphi\big(\alpha - \tfrac{1}{p^{n-1}}d(\varphi^{n-1}(a))\big)\big) + dV^{n-1}([x]s_\varphi(\varphi^{n-1}(a))). \end{aligned}$$

Using Proposition 4.7, because we are assuming $R_n(z) = 0$ and that x is not divisible by p, we have that a must be divisible by p^{n-1}, and we find

$$z = V^{n-1}\big([x]s_\varphi\big(\alpha - \tfrac{1}{p^{n-1}}d(\varphi^{n-1}(a))\big) + d([x]s_\varphi(\varphi^{n-1}(a/p^{n-1})))\big).$$

The fact that $R_n(z) = 0$ implies that

$$[x]s_\varphi\big(\alpha - \tfrac{1}{p^{n-1}}d(\varphi^{n-1}(a))\big) + d([x]s_\varphi(\varphi^{n-1}(a/p^{n-1})))$$

satisfies the assumption in property P_1. Hence we have

$$\begin{aligned} z &= V^{n-1}\bigg(\sum_{k=1}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x]))\bigg) \\ &= \sum_{k=1}^{\infty}(V^{k+n-1}([x]s_\varphi(\alpha_k)) + s_\varphi(\varphi^{1-n}(a_k))V^{n-1}(dV^k([x]))) \\ &= \sum_{k=1}^{\infty}(V^{k+n-1}([x]s_\varphi(\alpha_k)) + s_\varphi(\varphi^{1-n}(p^{n-1}a_k))dV^{k+n-1}([x])). \end{aligned}$$

This completes the proof of property P_n. □

Lemma 5.11. *An element $z \in M$ can be written in the form*

$$z = \sum_{k=n}^{\infty}(V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x]))$$

if and only if

$$z \in V^n(K^1) + dV^n(K_0).$$

In particular, property P_n is equivalent to the following:

- *If $z \in K^1$ and $R_n(z) = 0$, then we have*

$$z \in V^n(K^1) + dV^n(K_0).$$

Proof. This follows from the same sorts of manipulations as in the above proofs. The most difficult of these manipulations is showing that

$$s_\varphi(a_n) dV^n([x]) \in V^n(K^1) + dV^n(K_0).$$

Using Lemma 4.5 and the Leibniz rule, one checks that

$$s_\varphi(a_n) dV^n([x]) = V^n\big([x] s_\varphi\big(\varphi^n\big(\tfrac{-1}{p^n} d(a_n)\big)\big)\big) + dV^n(s_\varphi(\varphi^n(a_n))[x]) \in V^n(K^1) + dV^n(K_0). \qquad \square$$

Similar manipulations show the following.

Lemma 5.12. *For every integer $n \geq 1$, we have that $V^n(K^1) + dV^n(K_0) \subseteq M$ is a $W(A)$-submodule.*

Proof. It's clear that the collection of elements of the form

$$z = \sum_{k=n-1}^{\infty} (V^k([x] s_\varphi(\alpha_k)) + s_\varphi(a_k) dV^k([x]))$$

forms an A-module, so we reduce to proving that $V^n(K^1) + dV^n(K_0)$ is closed under multiplication by $V^i(1)$, for $i \geq 1$. Consider first the case $i \geq n$. We have

$$V^i(1) V^n(K^1) = V^i(p^n F^{i-n}(K^1)) \subseteq V^n(K^1), \quad V^i(1) dV^n(K^0) = V^i(F^{i-n} d(K^0)) \subseteq V^n(K^1).$$

Next we consider the case $i < n$. We have

$$\begin{aligned} V^i(1) V^n(K^1) &= V^n(p^i K^1) \subseteq V^n(K^1), \\ V^i(1) dV^n(K^0) &= d(V^i(1) V^n(K^0)) - V^n(K^0) dV^i(1) \subseteq dV^n(K^0). \end{aligned} \qquad \square$$

We cannot expect property P_1 to hold in general, as the following example shows. In the next section we will prove that property P_1 (and hence property P_n for every n) holds when A/xA is a perfectoid ring satisfying Assumption 6.2 below.

Example 5.13. Consider the ring $A = \mathbb{Z}_p$ and the element $x = p \in \mathbb{Z}_p$. Clearly

$$d[p] \in W_2 \Omega^1_{\mathbb{Z}_p}$$

restricts to $dp = 0$ in $\Omega^1_{\mathbb{Z}_p}$. On the other hand, because

$$[p] \equiv p + V(p^{p-1} - 1) \bmod V^2(W(\mathbb{Z}_p)),$$

we have

$$d[p] = -dV(1) \in W_2 \Omega^1_{\mathbb{Z}_p}.$$

The exactness of sequence (4.8) shows this element cannot be written as a $\mathbb{Z}_p$-linear combination of terms in $V(p\Omega^1_{\mathbb{Z}_p}) = V(\Omega^1_{\mathbb{Z}_p})$ and $dV(p) = Vd1 = 0$.

6. Applications to the de Rham–Witt complex over perfectoid rings

As usual, p in this section denotes an odd prime. The term *perfectoid* was originally used in the context of algebras over a field, but we work with the more general notion of *perfectoid ring* which has since been defined; see Definition 6.1 below. Examples of rings satisfying our definition of perfectoid include the p-adic completion of $\mathbb{Z}_p[\zeta_{p^\infty}]$, the p-adic completion of $\mathbb{Z}_p[p^{1/p^\infty}]$, and $\mathbb{O}_{\mathbb{C}_p}$.

Throughout this section, we let B denote a perfectoid ring satisfying Assumption 6.2 below, and we let $A = W(B^\flat)$, where

$$B^\flat := \varprojlim_{x \mapsto x^p} (B/pB)$$

is the tilt of B. The ring $B^\flat$ is a perfect ring of characteristic p. Let $\theta : A = W(B^\flat) \to B$ denote the map θ_1 from [Bhatt et al. 2016, Section 3]. This is the "usual" θ map from p-adic Hodge theory. We will not need the definition of θ; we will only need that it is surjective and its kernel is a principal ideal (by our definition of perfectoid). Throughout this section, $x \in A$ denotes a fixed choice of generator for this principal ideal.

We now explicitly state our definition of perfectoid.

Definition 6.1 [Bhatt et al. 2016, Definition 3.5]. A commutative ring B is called *perfectoid* if it is π-adically complete and separated for some element $\pi \in B$ such that π^p divides p, the Frobenius map $B/pB \to B/pB$ is surjective, and the kernel of $\theta : W(B^\flat) \to B$ is principal.

Assumption 6.2. We further assume that our perfectoid ring B is p-torsion free and that there exists a p-power torsion element $\omega \in \Omega^1_B$ such that the annihilator of ω is contained in $p^n B$ for some integer $n \geq 1$.

Remark 6.3. (1) Assumption 6.2 is satisfied, for example, if the perfectoid ring B is contained in $\mathbb{O}_{\mathbb{C}_p}$ and contains ζ_p. We do not know an elementary argument for this. Fontaine [1981/82, Théorème 1′] gives an elementary argument to show that $d\zeta_p$ is nonzero in Ω^1_R, where $R = \mathbb{O}_{\overline{\mathbb{Q}}_p}$. Bhargav Bhatt has shown us an argument involving the cotangent complex (which was used above in the proof of Proposition 2.7) to deduce that $d\zeta_p \in \Omega^1_{\mathbb{O}_{\mathbb{C}_p}}$ is nonzero. Once one knows that $d\zeta_p \neq 0$, an elementary argument shows that Assumption 6.2 is satisfied. We hope to consider the question, "How restrictive is Assumption 6.2?", in later applications.

(2) Our proofs in this section work for any quotient A/xA satisfying Assumption 6.2, but we do not know any interesting examples where A/xA is not perfectoid. In particular, see the next point.

(3) We have been careful throughout this paper to work with $W(k)$ where k is a perfect ring, instead of restricting our attention to the case where k is a perfect field. That generality is essential for Assumption 6.2 to be reasonable, because when k is a perfect field, the only p-torsion free quotient of $W(k)$ is the zero ring.

The entire goal of this section is to prove Proposition 6.12 below, which identifies the kernel of restriction $W_{n+1}\Omega^1_B \to W_n\Omega^1_B$ in terms of B and Ω^1_B. Using a spectral sequence argument, our result

will follow easily from property P_n described in Notation 5.9. By Proposition 5.10, it will suffice to prove property P_1, which loosely says that if an element in $W_n\Omega^1_A$ is in both $\ker\theta$ and in the kernel of restriction R_1 to Ω^1_A, then the element can be written as $V(\alpha)+dV(a)$, where both α and a are in $\ker\theta$. We now begin the proof that property P_1 holds.

We will apply the following lemma to our fixed $x \in A$ which generates $\ker\theta$, but it also holds for arbitrary $x \in A$.

Lemma 6.4. *Choose $y \in W(A)$ such that $[x] = s_\varphi(x) + V(y)$. Then we have*

$$[x]^p = s_\varphi(\varphi(x)) + py$$

Proof. Apply F to both sides of $[x] = s_\varphi(x) + V(y)$. □

Property P_1 concerns elements which are both in the kernel of $W_n(\theta) : W_n\Omega^1_A \to W_n\Omega^1_{(A/xA)}$ and also in the kernel of restriction $R_1 : W_n\Omega^1_A \to \Omega^1_A$. The following lemma considers the case of a particular element which is obviously in this intersection.

Lemma 6.5. *We have $[x]ds_\varphi(x) - s_\varphi(x)d[x] \in V(K^1) + dV(K^0)$.*

Proof. We use the notation from Lemma 6.4. We compute

$$\begin{aligned}
[x]ds_\varphi(x) - s_\varphi(x)d[x] &= (s_\varphi(x) + V(y))ds_\varphi(x) - s_\varphi(x)d(s_\varphi(x) + V(y)) \\
&= V(y)ds_\varphi(x) - s_\varphi(x)dV(y) \\
&= V(yFds_\varphi(x)) - d(s(x)V(y)) + V(y)ds_\varphi(x) \\
&= V(2yFds_\varphi(x)) - dV(yF(s_\varphi(x))) \\
&= V(2yFds_\varphi(x)) - dV(y[x]^p - py^2).
\end{aligned}$$

Because the term $dV(y[x]^p) \in dV(K^0)$, we reduce to showing the following element is in $V(K^1)$.

$$\begin{aligned}
V(2yFds_\varphi(x)) + dV(py^2) &= V(2yFds_\varphi(x) + 2ydy) \\
&= V(2y(Fds_\varphi(x) + dy)) \\
&= V(2y(Fd([x] - V(y)) + dy)) \\
&= V(2y([x]^{p-1}d[x] - dy + dy)) \in V(K^1).
\end{aligned}$$

This completes the proof. □

Lemma 6.6. *If $x\alpha_1 = 0 \in \Omega^1_A$, then $[x]s_\varphi(\alpha_1) \in V(K^1)$.*

Proof. The key idea is that, because multiplication by p is a bijection on Ω^1_A, we also have that $x\alpha_1/p^N = 0 \in \Omega^1_A$ for every integer $N \geq 1$. Applying Frobenius to both sides, we have $\varphi(x)\varphi(\alpha_1)/p^N = 0 \in \Omega^1_A$. We will apply this observation in the case $N = 2$.

Use the same notation as in Lemma 6.4. We have

$$\begin{aligned}
[x]s_\varphi(\alpha_1) &= s_\varphi(x)s_\varphi(\alpha_1) + V(y)s_\varphi(\alpha_1)\\
&= V(y)s_\varphi(\alpha_1)\\
&= V(yF(s_\varphi(\alpha_1)))\\
&= V\left(ys_\varphi\left(\frac{\varphi(\alpha_1)}{p}\right)\right)\\
&= V\left(pys_\varphi\left(\frac{\varphi(\alpha_1)}{p^2}\right)\right)\\
&= V\left(([x]^p - s_\varphi(\varphi(x)))s_\varphi\left(\frac{\varphi(\alpha_1)}{p^2}\right)\right)\\
&= V\left([x]^p s_\varphi\left(\frac{\varphi(\alpha_1)}{p^2}\right)\right) \in V(K^1).
\end{aligned}$$

□

Using Assumption 6.2, we have a p-power torsion element $\omega \in \Omega^1_B$ with annihilator contained in $p^n B$ for some integer $n \geq 1$. For every integer $r \geq 1$, the following lemma enables us to produce a p-power torsion element $\eta \in \Omega^1_B$ with annihilator contained in $p^{n+r}B$.

Lemma 6.7. *Assume $\omega \in \Omega^1_B$ is such that $\operatorname{Ann}\omega \subseteq p^n B$, where $n \geq 1$ is an integer. If $\eta \in \Omega^1_B$ is an element such that $p^r\eta = \omega$ for some integer $r \geq 1$, then $\operatorname{Ann}\eta \subseteq p^{n+r}B$.*

Proof. It suffices to prove this in the case $r = 1$, so let $\eta \in \Omega^1_B$ be such that $p\eta = \omega$. Let $b \in \operatorname{Ann}\eta$. Then in particular $b \in \operatorname{Ann}\omega$, so we can write $b = p^n b_0$ for some $b_0 \in B$. Then we know

$$0 = b\eta = p^n b_0\eta = p^{n-1}b_0\omega,$$

and hence $p^{n-1}b_0 \in p^n B$. Assumption 6.2 requires that B is p-torsion free, so we deduce that $b_0 \in pB$, and hence $b \in p^{n+1}B$, as required. □

The following is the most important of the preliminary results in this section. If we could prove Proposition 6.8 without using the element ω from Assumption 6.2, then the results of this section would hold for all p-torsion free perfectoid rings.

Proposition 6.8. *If $adx \in x\Omega^1_A$, then $a \in xA$.*

Proof. Our hypothesis implies $a\frac{dx}{p^N} \in \ker\theta$ for every integer $N \geq 0$, and we will show this implies $\theta(a) \in \bigcap p^r B = 0$.

Fix an integer $N \geq 1$. Because $\theta : A \to B$ is surjective, we know the induced map $\Omega^1_A \to \Omega^1_B$ is surjective. Let $\omega_A \in \Omega^1_A$ map to the element $\omega \in \Omega^1_B$ described in Assumption 6.2. Because ω is p-power torsion, we know that $p^m\omega_A \in x\Omega^1_A + Adx$ for some integer $m \geq 1$. Thus, for every integer $N \geq 1$, we can write $(1/p^{N-m})\omega_A = x\alpha_N + a_N\frac{dx}{p^N}$ for some $\alpha_N \in \Omega^1_A$ and $a_N \in A$.

Consider now the element $adx \in x\Omega^1_A$ from the statement of this proposition. We deduce that $a\frac{dx}{p^N} \in x\Omega^1_A$ for every integer $N \geq 1$, so $a\frac{dx}{p^N} \in \ker\theta$ for every integer $N \geq 1$. If we multiply by the

element a_N from the previous paragraph, we know that $aa_N \frac{dx}{p^N}$ is in $\ker\theta$ for every integer $N \geq 1$. If we apply θ to $aa_N \frac{dx}{p^N}$, we see that $\theta(a) \in B$ is in the annihilator of some element η satisfying $p^{N-m}\eta = \omega$. Thus, by Lemma 6.7, we have that $\theta(a) \in B$ is divisible by arbitrarily large powers of p. Thus $a \in xA$, as required. □

Remark 6.9. Proposition 6.8 implies that for our particular rings A and A/xA, the left-most map in the exact sequence (5.2) is injective.

Proposition 6.10. *If* $x\alpha + adx = 0 \in \Omega^1_A$, *then* $[x]s_\varphi(\alpha) + s_\varphi(a)d[x] \in V(K^1) + dV(K^0)$.

Proof. We have $adx = -x\alpha$, so by Proposition 6.8, we know that $a = xa_1$ for some $a_1 \in A$, and thus our assumption means $x(\alpha + a_1 dx) = 0 \in \Omega^1_A$. By Lemma 6.6, we know that $[x](s_\varphi(\alpha) + s_\varphi(a_1)d(s_\varphi(x))) \in V(K^1)$. Thus it suffices to show that

$$[x]s_\varphi(a_1)ds_\varphi(x) - s_\varphi(x)s_\varphi(a_1)d[x] \in V(K^1) + dV(K^0).$$

Thus, by Lemma 5.12, it suffices to show that

$$[x]ds_\varphi(x) - s_\varphi(x)d[x] \in V(K^1) + dV(K^0).$$

So we are done by Lemma 6.5. □

Consider now an arbitrary element $y \in K^1$,

$$y = \sum_{k=0}^{\infty} (V^k([x]s_\varphi(\alpha_k)) + s_\varphi(a_k)dV^k([x])),$$

and assume it restricts to 0 in level one, i.e., assume $R_1(y) = 0 \in \Omega^1_A$. This means that

$$x\alpha_0 + a_0 dx = 0 \in \Omega^1_A.$$

Then Proposition 6.10 shows that property P_1 from Notation 5.9 holds. We immediately deduce the following from Proposition 5.10.

Corollary 6.11. *For every* $n \geq 1$, *property* P_n *from Notation 5.9 holds.*

The following result is the main result of this section. It is modeled after [Hesselholt and Madsen 2003, Proposition 3.2.6]. Compare also Proposition 4.7.

Proposition 6.12. *Let* B *be a perfectoid ring satisfying Assumption 6.2. For every integer* $n \geq 1$, *we have a short exact sequence of* $W_{n+1}(B)$*-modules*

$$0 \to B \to \Omega^1_B \oplus B \to W_{n+1}\Omega^1_B \xrightarrow{R} W_n\Omega^1_B \to 0, \tag{6.13}$$

where the maps and $W_{n+1}(B)$*-module structure are defined as follows. The map* $B \to \Omega^1_B \oplus B$ *is given by* $b \mapsto (-db, p^n b)$. *The map* $\Omega^1_B \oplus B \to W_{n+1}\Omega^1_B$ *is given by* $(\beta, b) \mapsto V^n(\beta) + dV^n(b)$. *The* $W_{n+1}(B)$*-module structure on* B *is given by* F^n. *The* $W_{n+1}(B)$*-module structure on* $\Omega^1_B \oplus B$ *is given by*

$$y \cdot (\omega, b) = (F^n(y)\omega - bF^n(dy), F^n(y)b), \text{ where } y \in W_{n+1}(B).$$

The $W_{n+1}(B)$-module structure on $W_n\Omega^1_B$ is induced by restriction.

Proof. Consider the following short exact sequence of chain complexes (the chain complexes are written horizontally, and the short exact sequences are written vertically):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & B & \longrightarrow & \Omega^1_B \oplus B & \longrightarrow & W_{n+1}\Omega^1_B & \longrightarrow & W_n\Omega^1_B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle W_{n+1}(\theta)} & & \uparrow{\scriptstyle W_n(\theta)} & & \\
0 & \longrightarrow & A & \longrightarrow & \Omega^1_A \oplus A & \longrightarrow & W_{n+1}\Omega^1_A & \longrightarrow & W_n\Omega^1_A & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & xA & \longrightarrow & R_1(K^1) \oplus R_1(K^0) & \longrightarrow & R_{n+1}(K^1) & \longrightarrow & R_n(K^1) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

For convenience, write these chain complexes as $0 \to K_\bullet \to A_\bullet \to B_\bullet \to 0$, where we consider the complexes concentrated in degrees 0 to 3. We must show that $H_n(B_\bullet) \cong 0$ for all n. It's trivial that $H_0(B_\bullet) \cong 0$ and $H_3(B_\bullet) \cong 0$. Using Proposition 1.5, we have also that $H_1(B_\bullet) \cong 0$. This leaves $H_2(B_\bullet)$.

Consider now the long exact sequence in homology [Weibel 1994, Theorem 1.3.1] associated to the above short exact sequence of chain complexes. By Proposition 4.7, we have that $H_n(A_\bullet) \cong 0$ for all n. It follows that $H_2(B_\bullet) \cong H_1(K_\bullet)$. We will finish the proof by showing that $H_1(K_\bullet) \cong 0$.

Consider an element in $R_{n+1}(K^1)$ which restricts to 0 in $W_n\Omega^1_A$. By Corollary 6.11, we know that this element can be written as $V^n([x]s_\varphi(\alpha_n)) + s_\varphi(a_n)dV^n([x])$, for some $\alpha_n \in \Omega^1_A$ and some $a_n \in A$. By Lemma 5.11, such an element lies in $V^n(K^1) + dV^n(K^0)$, and hence is in the image of the map

$$R_1(K^1) \oplus R_1(K^0) \xrightarrow{V^n + dV^n} R_{n+1}(K^1).$$

This shows that $H_1(K_\bullet) \cong 0$, and hence that $H_2(B_\bullet) \cong 0$, as required. □

Example 6.14. As in Example 5.13, the analogue of exactness in (6.13) does not hold for arbitrary quotients of a ring $A = W(k)$. For example, exactness does not hold for $B = \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. In this case, not even the left-most map $B \to \Omega^1_B \oplus B$ is injective. More significantly, we know $W_{n+1}\Omega^1_{(\mathbb{Z}/p\mathbb{Z})}$ is zero for all n, so $dV^n(1) = 0 \in W_{n+1}\Omega^1_{(\mathbb{Z}/p\mathbb{Z})}$ for all $n \geq 1$. By contrast, Proposition 6.12 shows that $dV^n(1) \neq 0$ for all perfectoid rings B satisfying Assumption 6.2.

Remark 6.15. Assume B is a ring for which the sequence in Equation (6.13) is exact. Assume $B_0 \subseteq B$ is a subring satisfying the following two properties:

(1) We have $p^n B \cap B_0 = p^n B_0$.

(2) The B_0-module homomorphism $\Omega^1_{B_0} \to \Omega^1_B$ is injective.

It then follows that the analogue of (6.13) for B_0 is also exact. In foreseeable applications, verifying the first condition will be trivial, but in general it may be difficult to verify the second condition. For example, if B is $\mathbb{O}_{\mathbb{C}_p}$ and B_0 is the valuation ring in an algebraic extension of $\mathbb{Q}_p$, it is not clear whether we should expect the second condition to hold. For this reason, this remark might be more useful in the context of [Hesselholt 2006, Proposition 2.2.1], which shows exactness of a log analogue of (6.13) when $B = \mathbb{O}_{\overline{\mathbb{Q}}_p}$.

Remark 6.16. In this section and the previous section, we have been working with an explicit quotient of the de Rham–Witt complex over $A = W(k)$. Perhaps similar results could be attained by working with an explicit quotient of the de Rham–Witt complex over the polynomial algebra $A[t]$. An explicit description of the de Rham–Witt complex over $A[t]$ is given, in terms of the de Rham–Witt complex over A, in [Hesselholt and Madsen 2004, Theorem B].

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