Algebra & Number Theory

Volume 13
2019
No. 8

msp

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2019 is US $385/year for the electronic version, and $590/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1765

Moduli of stable maps in genus one and logarithmic geometry, II

Dhruv Ranganathan, Keli Santos-Parker and Jonathan Wise

This is the second in a pair of papers developing a framework to apply logarithmic methods in the study of stable maps and singular curves of genus 1. This volume focuses on logarithmic Gromov–Witten theory and tropical geometry. We construct a logarithmically nonsingular and proper moduli space of genus 1 curves mapping to any toric variety. The space is a birational modification of the principal component of the Abramovich–Chen–Gross–Siebert space of logarithmic stable maps and produces logarithmic analogues of Vakil and Zinger's genus one reduced Gromov–Witten theory. We describe the nonarchimedean analytic skeleton of this moduli space and, as a consequence, obtain a full resolution to the tropical realizability problem in genus 1.

1.	Introduction	1765
2.	Preliminaries	1770
3.	Logarithmic maps to toric varieties	1780
4.	Realizability for genus one tropical curves	1793
Acknowledgements		1802
References		1803

1. Introduction

This paper is the second in a pair, exploring the interplay between tropical geometry, logarithmic moduli theory, stable maps, and moduli spaces of genus 1 curves. In the first volume, we used this interplay to construct new nonsingular moduli spaces compactifying the space of elliptic curves in projective space via *radially aligned* stable maps and quasimaps. In this paper, we focus on applications to logarithmic Gromov–Witten theory and tropical geometry.

I. *Realizability of tropical curves.* We give a complete characterization of genus 1 tropical maps that can be realized as tropicalizations of genus 1 curves mapping to tori, completing a study initiated in Speyer's thesis. We show that a combinatorial condition identified by Baker, Payne and Rabinoff is always sufficient. Our proof is independent of these previous results, and is based on the geometry of logarithmic maps.

MSC2010: primary 14N35; secondary 14T05.
Keywords: logarithmic Gromov–Witten theory, tropical realizability, well spacedness condition.

II. *Logarithmic stable maps.* We construct a toroidal moduli space parametrizing maps from pointed genus 1 curves to any toric variety with prescribed contact orders along the toric boundary. This is a desingularization of the principal component of the space of logarithmic stable maps. The boundary complex of this compactification is identified as a space of realizable tropical maps.

1.1. ***Superabundant tropical geometries.*** The realization problem is the crux of the relationship between tropical geometry to algebraic geometry, and is unavoidable in enumerative applications. Given an abstract tropical curve[1] $\sqsubset$ of genus g and a balanced piecewise linear map

$$F : \sqsubset \to \mathbb{R}^r,$$

we ask, *does there exist a nonarchimedean field K extending $\mathbb{C}$, a smooth algebraic curve C over K and a map*

$$\varphi : C \to \mathbb{G}_m^r,$$

such that φ^{trop} coincides with F?

When $\sqsubset$ has genus 0, the only obstruction to lifting is the local balancing condition, and all tropical curves satisfying that condition are realizable. This is reflected in the logarithmic smoothness of the moduli space of genus 0 logarithmic maps [Nishinou and Siebert 2006; Ranganathan 2017a; Tyomkin 2012]. In genus 1, nonlocal obstructions already appear for maps $\sqsubset \to \mathbb{R}$. The obstructions appear when the circuit of $\sqsubset$ is contained in a proper affine subspace of $\mathbb{R}^r$. Speyer [2005; 2014] discovered a sufficient condition for realizability. A weaker necessary condition was identified in [Baker et al. 2016, Section 6]. We provide a characterization of the realizable tropical curves in genus 1 in Theorem A in terms of the geometry of the skeleton of an analytic space of maps.

Let Γ be a marked tropical curve of genus 1 with a unique vertex and n half-edges. Fix a balanced map $\Gamma \to \mathbb{R}^r$. Let $\mathcal{M}_\Gamma(\mathbb{G}_m^r)$ be the moduli space of maps

$$\varphi : C \to \mathbb{G}_m^r,$$

where C is a noncompact smooth algebraic curve of genus 1 with n punctures, and the vanishing orders at infinity of these punctures are specified by the slopes along the edges of Γ in $\mathbb{R}^r$. Let $W_\Gamma(\mathbb{R}^r)$ be the corresponding set of tropical maps

$$\sqsubset \to \mathbb{R}^r$$

whose recession fan is given by $\Gamma \to \mathbb{R}^r$, and satisfy the *well-spacedness condition*, as defined in Section 4.[2] This set can be given the structure of a generalized cone complex.

[1]The first author continues his efforts to popularize Dan Abramovich's convention that algebraic curves be denoted by $C, \mathscr{C}$, while tropical curves be denoted $\sqsubset$, approximating their appearance in nature.

[2]We caution the reader that the meaning attributed by Speyer to well-spacedness is stronger than the one we use here; see Warning 4.4.5.

Given a map $\varphi : \mathscr{C} \to \mathbb{G}_m^r$ over a valued field, one obtains a balanced piecewise linear map from a Berkovich skeleton $\sqsubset$ of the punctured general fiber curve $\mathscr{C}_\eta$ to $\mathbb{R}^r$, i.e., to the skeleton of the torus [Ranganathan 2017b]. This piecewise linear map is the *tropicalization* of φ and is denoted φ^{trop}.

Theorem A. *There exists a continuous and proper tropicalization map*

$$\mathrm{trop} : \mathcal{M}_\Gamma^{\mathrm{an}}(\mathbb{G}_m^r) \to W_\Gamma(\mathbb{R}^r)$$

sending a map $[\varphi]$ *over a valued field to its tropicalization. There is generalized cone complex* $P_\Gamma(\mathbb{R}^r)$ *and a finite morphism*

$$\mathrm{trop}_{\mathfrak{S}} : P_\Gamma(\mathbb{R}^r) \to W_\Gamma(\mathbb{R}^r),$$

which is an isomorphism upon restriction to each cone of the source. The degree of this finite morphism is explicitly computable and the complex $P_\Gamma(\mathbb{R}^r)$ *is a skeleton of the analytic moduli space* $\mathcal{M}_\Gamma^{\mathrm{an}}(\mathbb{G}_m^r)$.

The theorem is proved as Theorem 4.6.2.

The statement that the tropicalization has a finite cover that is a skeleton is a toroidal version of the schön condition, frequently cited in tropical geometry. The skeleton $P_\Gamma(\mathbb{R}^r)$ functions as a parametrizing complex for the tropicalization, as in [Helm and Katz 2012; Tevelev 2007].

1.2. ***Logarithmic stable maps.*** Our tropical investigation leads naturally to an understanding of the geometry of space of logarithmic stable maps to toric varieties in genus 1. The open moduli problem we consider is that of maps

$$(C, p_1, \ldots, p_m) \to Z,$$

where C is a smooth pointed curve of genus 1, the target Z is a toric variety, and the contact orders of the points p_i with the boundary divisors on Z is fixed. There is a natural modular compactification of this space via the theory of logarithmic stable maps [Abramovich and Chen 2014; Chen 2014; Gross and Siebert 2013]. When the genus of the source curve is 0, the resulting moduli space is logarithmically smooth, but in genus 1 can be highly singular and nonequidimensional. We use the insights of Theorem A to construct a logarithmically smooth modular compactification, in parallel with the desingularization of the ordinary stable maps space [Ranganathan et al. 2017; Vakil and Zinger 2008].

Let Z be a proper complex toric variety and $\mathscr{L}_\Gamma(Z)$ the moduli space of genus 1 logarithmic stable maps to Z with discrete data Γ, i.e., Γ records the genus and the contact orders of the marked points with the toric boundary of Z. Let $\mathscr{L}_\Gamma^\circ(Z)$ be the locus of parametrizing maps of positive degree from smooth domains, and let $\overline{\mathscr{L}_\Gamma^\circ}(Z)$ be the closure.

Theorem B. *Consider the following data as a moduli problem on logarithmic schemes*:

(1) *A family of* n*-marked*, *radially aligned logarithmic curves* $C \to S$.

(2) *A logarithmic stable map* $f : C \to Z$ *with contact order* Γ, *such that the map* f *is* well-spaced (*see Definition 3.4.2*).

This moduli problem is represented by a proper and logarithmically smooth stack with logarithmic structure $\mathcal{W}_\Gamma(Z)$ and the natural morphism

$$\mathcal{W}_\Gamma(Z) \to \overline{\mathscr{L}^\circ_\Gamma}(Z)$$

is proper and birational.

See Theorem 4.4.8 for the proof.

The well-spacedness property above is efficiently stated in tropical language, and this is done later in the paper. At a first approximation it may be thought of as forcing a factorization property after composing $C \to Z$ with any rational map $Z \dashrightarrow \mathbb{P}^1$ induced by a character. These logarithmic maps are precisely the ones that have *well-spaced* tropicalizations. A prototype for practical calculations on this space may be found in [Len and Ranganathan 2018].

1.3. ***Motivation for the construction.*** The combinatorics of logarithmic stable maps are essentially part of tropical geometry. Indeed, if the variety Z is a toric variety, taken with its toric boundary, the analytification of the moduli space of logarithmic maps continuously to a polyhedral complex parametrizing tropical curves [Ranganathan 2017b]. The connection is especially transparent in genus 0, see [Ranganathan 2017a]. In genus 1, the tropical realizability problem can be used to predict the desingularization above, as we now explain. The moduli space $\mathscr{L}_\Gamma(Z)$ of genus 1 logarithmic stable maps is highly singular, however, it maps naturally to a logarithmically smooth Artin stack. More precisely, if $\mathscr{A}_Z = [Z/T]$ is the Artin fan of Z obtained by performing a stack quotient on Z by its dense torus, there is a natural map

$$\mathscr{L}_\Gamma(Z) \to \mathscr{L}_\Gamma(\mathscr{A}_Z),$$

where the latter is the space of prestable logarithmic maps to the Artin fan. This space is a logarithmically smooth Artin stack [Abramovich and Wise 2018]. Moreover, the toroidal skeleton of this space is naturally identified with the moduli space of all (not necessarily realizable or even balanced) tropical maps from genus 1 curves to the fan of Z, see [Ranganathan 2017b]. The locus of realizable curves is a sublocus in the moduli space. After subdividing this cone complex, this sublocus is supported on a subcomplex. This subdivision induces a birational modification of $\mathscr{L}_\Gamma(\mathscr{A}_Z)$, and thus a modification of $\mathscr{L}_\Gamma(Z)$. This modification can naturally be identified with the moduli of well-spaced logarithmic maps $\mathcal{W}_\Gamma(Z)$ defined above. The radial alignments developed in [Ranganathan et al. 2017] and recalled in Section 2.9 give rise to the modular interpretation.

The construction of $\mathcal{W}_\Gamma(Z)$ is not a formal lifting of our previous results on ordinary stable maps to the logarithmic category [Ranganathan et al. 2017]. Given an absolute genus 1 stable map $[C \to \mathbb{P}^r]$, if no genus 1 subcurve is contracted, then $[C \to \mathbb{P}^r]$ is a smooth point of the moduli space. However, for a toric variety Z and a genus 1 logarithmic map $[C \to Z]$, the deformations of the map can be obstructed even if no component of C is contracted. This is true even if $Z = \mathbb{P}^r$ with its toric logarithmic structure. This behavior is akin to the genus 1 absolute stable maps theory for semipositive targets. While the tangent bundle of $\mathbb{P}^r$ is ample, the logarithmic tangent bundle of a toric variety is trivial. This allows for a larger space of obstructions to deforming genus 1 logarithmic maps than in the absolute theory. We overcome this by identifying and forcing the stronger factorization property above.

1.4. ***Tropical enumerative geometry and realizability.*** The realizability problem for tropical curves is a combinatorial shadow of the problem of characterizing the closure of the main component in the space of logarithmic maps. The difficulty of the problem has limited tropical enumerative techniques to low target dimensions [Bertrand et al. 2014; Cavalieri et al. 2010; 2016; Mikhalkin 2005] or to genus 0 curves [Gross 2016; 2018; Mandel and Ruddat 2016; Nishinou and Siebert 2006; Ranganathan 2017a].

In the higher genus, higher dimensional situation, there are two directions in which one may generalize the picture above. The first is to develop a systematic method to decompose logarithmic Gromov–Witten invariants, as a sum of virtual invariants over tropical curves [Abramovich et al. 2017a; Parker 2011; Ranganathan 2019]. The second is to analyze the tropical lifting problem and produce a "reduced" curve counting theory that captures the principal component contribution to the virtual count. This paper addresses the second of these in genus one. The realizability theorem in genus 1 allows us to decompose these reduced invariants of any toric variety over tropical curves. The degeneration formula for these invariants is work that we hope to return to. Note that the analogous problem for smooth pairs has recently been treated in [Battistella et al. 2019].

There have been a number of interesting partial results on tropical realizability in the last decade, thanks to the efforts of many [Baker et al. 2016; Cheung et al. 2016; Jensen and Ranganathan 2017; Katz 2012; Mikhalkin 2005; Nishinou 2009; Nishinou and Siebert 2006; Ranganathan 2017b; 2017c; Speyer 2014]. The genus 1 story alone has seen heavy interest. Speyer identified the sufficiency of a strong form of well-spacedness condition for superabundant genus 1 tropical curves using Tate's uniformization theory. Using the group law on the analytification of an elliptic curve, Baker, Payne and Rabinoff showed that a weaker condition was necessary. The existence of genus 1 tropical curves which failed Speyer's condition but were nonetheless realizable was established in [Ranganathan 2017b].

In higher genus, few results are known. Realizability of nonsuperabundant higher genus tropical curves was established by Cheung, Fantini, Park and Ulirsch [2016], and limits of realizable curves can be shown to be realizable [Ranganathan 2017b; 2017c]. Katz [2012] showed that the logarithmic tangent/obstruction complex for degenerate maps gives rise to necessary combinatorial conditions for realizability in higher genus, including a version of well-spacedness. These methods do not prove sufficiency in any cases. A sufficient condition for realizability for some superabundant chain of cycle geometries has recently been shown to hold and used to establish new results in Brill–Noether theory [Jensen and Ranganathan 2017].

1.5. ***User's guide.*** We have written this paper so it may be read independently from the prequel, in which the space of ordinary stable maps to $\mathbb{P}^r$ was considered. In Section 2.9 we recall the preliminary results on radial alignments and their contractions from [Ranganathan et al. 2017]. The moduli space of well-spaced logarithmic maps is constructed in Section 3.4 and the logarithmic unobstructedness appears as Theorem 3.5.1. The tropical well-spacedness condition is discussed and defined precisely in Section 4.4. Finally, tropical realizability results are restated in Theorem 4.4.8 and proved in Section 4.6.

2. Preliminaries

In this section, we recall some preliminaries on singularities of genus 1 and on logarithmic and tropical geometry. There is some overlap between this section and the preliminary material appearing in the prequel to this article [Ranganathan et al. 2017], but we opt to include it for a more self-contained presentation.

2.1. ***Genus* 1 *singularities.*** Let C be a reduced curve over an algebraically closed field k. For an isolated curve singularity (C, p) with normalization $\pi : (\tilde{C}, p_1, \ldots, p_m) \to (C, p)$, recall that m, the cardinality of $\pi^{-1} p$, is called the *number of branches of the singularity*. The δ*-invariant* is defined as

$$\delta := \dim_k(\pi_\star(\mathscr{O}_{\tilde{C}})/\mathscr{O}_C).$$

Based on these two invariants, one defines the genus of (C, p) as

$$g = \delta - m + 1.$$

We will frequently make use of the *seminormalization* of (C, p) in our arguments. The *seminormalization* is a partial resolution of (C, p) to a singularity of genus 0 that is homeomorphic to (C, p). Explicitly, equip the underlying topological space of (C, p) with the subring $\mathscr{A}$ of regular functions on the normalization $\tilde{C}$ that are well defined on the underlying topological space of C. In particular, there are g additional conditions required for a function in $\mathscr{A}$ to descend to (C, p), i.e.,

$$g = \dim_k(\mathscr{A}/\mathscr{O}_C).$$

Let E be a proper Gorenstein curve of genus 1, smooth away from a unique genus 1 singularity. Let $\nu : F \to E$ be the seminormalization and let $\mu : G \to F$ be the normalization. We have inclusions

$$\mathscr{O}_E \subset \nu_\star\mathscr{O}_F \subset \nu_\star\mu_\star\mathscr{O}_G \subset K \quad \text{and} \quad J \supset \omega_E \supset \nu_\star\omega_F \supset \nu_\star\mu_\star\omega_G$$

Here K is the sheaf of meromorphic functions on E and J is the sheaf of meromorphic differentials. For each $X = E, F, G$, the pairs ω_X and $\mathscr{O}_X$ are dual to other another with respect to the residue pairing $K \otimes J \to k$, in the sense that each is the annihilator of the other [Altman and Kleiman 1970, Proposition 1.16(ii)].

Consider the exact sequence

$$0 \to \mathscr{O}_E \to \nu_\star\mathscr{O}_F \to \nu_\star(\mathscr{O}_F)/\mathscr{O}_E \to 0 \tag{1}$$

In the long exact cohomology sequence

$$0 \to H^0(E, \mathscr{O}_E) \to H^0(F, \mathscr{O}_F) \to \nu_\star(\mathscr{O}_F)/\mathscr{O}_E \to H^1(E, \mathscr{O}_E) \to H^1(F, \mathscr{O}_F) \tag{2}$$

the map $H^0(E, \mathscr{O}_E) \to H^0(F, \mathscr{O}_F)$ is an isomorphism because both E and F are proper, connected, and reduced; furthermore $H^1(F, \mathscr{O}_F) = 0$ since F has genus 0. By Serre duality, $H^1(E, \mathscr{O}_E)$ is dual to $H^0(E, \omega_E)$. Since both are 1-dimensional, the choice of a nonzero $\alpha \in H^0(E, \omega_E)$ induces an isomorphism $H^1(E, \mathscr{O}_E) \to k$. The composition

$$\nu_\star(\mathscr{O}_F)/\mathscr{O}_E \to H^1(E, \mathscr{O}_E) \to k$$

may be identified with the residue pairing, sending $f \bmod \mathscr{O}_E$ to $\operatorname{res} f\alpha$. This follows, for example, by the construction of the dualizing sheaf in [Altman and Kleiman 1970, Remark 1.9 and Remark 1.12].

We know that $\omega_F/\mu_\star(\omega_G)$ is spanned by the differentials

$$\frac{dx_i}{x_i} - \frac{dx_j}{x_j} \tag{3}$$

where the x_i are local coordinates of the branches of E at the singular point. As $\omega_E/\nu_\star(\omega_F)$ is 1-dimensional, ω_E is generated relative to $\nu_\star(\omega_F)$ by a differential of the following form:

$$\sum_i \frac{c_i dx_i}{x_i^2} + \frac{c' dx_1}{x_1} \tag{4}$$

If $f \in \mathscr{O}_E$ has the expansion $f(0) + b_i x_i + \cdots$ on the i-th component of F then this differential imposes the constraint

$$c' f(0) + \sum b_i c_i = 0.$$

In order for E to be Gorenstein, ω_E must be a line bundle, so the generators (3) of ω_F must be multiples of the generator (4). This immediately implies $c' = 0$ and that all of the c_i are nonzero. Conversely, if $c' = 0$ and all of the c_i are nonzero, then $c_j x_i - c_i x_j \in \mathscr{O}_E$ and

$$(c_j x_i - c_i x_j) \sum_k \frac{c_k dx_k}{x_k^2} = c_j c_i \frac{dx_i}{x_i} - c_i c_j \frac{dx_j}{x_j}$$

implies that the generators (3) are multiples of (4). This proves the following proposition:

Proposition 2.1.1. *If E is a Gorenstein curve with a genus 1 singularity then ω_E is generated in a neighborhood of its singular point by a meromorphic form* (4), *with $c' = 0$, where the x_i are local parameters for the branches of E at the singular point.*

By consideration of the residue condition imposed by the form (4), we can also obtain a local description of the Gorenstein, genus 1 curve singularities. A more conceptual proof of this result can be found in [Smyth 2011, Proposition A.3].

Proposition 2.1.2. *For each integer $m \geq 0$, there exists a unique Gorenstein singularity (C, p) of genus 1 with m branches. If $m = 1$ then (C, p) can be identified with the cusp $\boldsymbol{V}(y^2 - x^3)$, if $m = 2$ then (C, p) can be identified with the ordinary tacnode $\boldsymbol{V}(y^2 - yx^2)$, and if $m \geq 3$, then (C, p) is the germ at the origin of the union of m general lines through the origin in $\mathbb{A}^{m-1}$.*

2.2. ***Tropical curves.*** We follow standard conventions and definitions for tropical curves and tropical stable maps.

Definition 2.2.1. An *n-marked tropical curve* $\sqsubset$ is a finite graph G with vertex and edge sets V and E, enhanced by

(1) a *marking function* $m : \{1, \dots, n\} \to V$,

(2) a *genus function* $g : V \to \mathbb{N}$,

(3) a *length function* $\ell : E \to \mathbb{R}_+$.

The *genus* of a tropical curve $\sqsubset$ is defined to be

$$g(\sqsubset) = h_1(G) + \sum_{v \in V} g(v)$$

where $h_1(G)$ is the first Betti number of the geometric realization of $\sqsubset$. An n-marked tropical curve is *stable* if (1) every genus 0 vertex has valence at least 3 and (2) every genus 1 vertex has valence at least 1.

More generally, one may permit the length function ℓ above to take values in an arbitrary toric monoid P. This presents us with a natural notion of a family of tropical curves.

Definition 2.2.2. Let σ be a rational polyhedral cone with dual cone S_σ. A *family of n-marked prestable tropical curves over σ* is a tropical curve whose length function takes values in S_σ.

We note that given a tropical curve over σ, each point of σ determines a tropical curve in the usual sense. Indeed, choosing a point of σ is equivalent to choosing a monoid homomorphism

$$\varphi : S_\sigma \to \mathbb{R}_{\geq 0}.$$

Applying this homomorphism to the edge length $\ell(e) \in S_\sigma$ produces a real and positive length for each edge.

2.3. ***Logarithmic geometry: working definitions.*** Let N be a free abelian group of finite rank and X° be a subscheme of a torus $T = \mathbb{G}_m \otimes N$ over a field k equipped with the trivial valuation. Let K be a valued field extending K, with valuation surjective onto $\mathbb{R}$. Then, the tropicalization of X is the image of $X(K)$ under the coordinatewise valuation map

$$T(K) \to \mathbb{R} \otimes N.$$

This set is denoted X^{trop}, and can be given the structure of a fan. This fan distinguishes a partial compactification of T to a toric variety Y. The embedding of the closure $X \hookrightarrow Y$ determines, locally on X, a natural class of *monomial* functions obtained by restricting the monomials on T. These monomials form a sheaf of monoids M_X under multiplication, and a tautological map of monoids

$$\mathcal{O}_X^\star \subset M_X \to \mathcal{O}_X.$$

The quotient is another sheaf of monoids $\overline{M}_X := M_X / \mathcal{O}_X^\star$, and amounts to considering monomial functions up to scalars.

Sections of the groupification $\overline{M}_X^{\mathrm{gp}}$ can be interpreted as piecewise linear functions on X^{trop}. Just as in the toric case, piecewise linear functions on X^{trop} give rise to line bundles on X. Specifically, given a piecewise linear function, the set of algebraic lifts of it in $\overline{M}_X^{\mathrm{gp}}$ form a torsor under the multiplicative group, and therefore a line bundle. This is explained more precisely in Section 2.6 below.

A logarithmic scheme is an object that possesses the main features present above. The requirement that X be embedded in a toric variety can be dropped. Instead, one need only assume that X (locally) admits a morphism to a toric stack. The data of the sheaf $\overline{M}_X$ may be thought of as the sheaf of piecewise linear functions on X.

To be more precise, it is convenient to reverse the logical order and specify the monomials first. Given a scheme S, a logarithmic structure is a sheaf of monoids M_S in its étale topology and sharp homomorphism $\varepsilon : M_S \to \mathcal{O}_S$ (the codomain given its multiplicative monoid structure). Sharpness means that each local section of $\mathcal{O}_S^\star$ has a unique preimage along ε. The quotient $M_X/\varepsilon^{-1}\mathcal{O}_X^\star$ is called the *characteristic monoid* and is denoted $\overline{M}_X$ with its operation denoted *additively*; the image of section α of M_X^{gp} in $\overline{M}_X^{\mathrm{gp}}$ is denoted $\bar{\alpha}$. We assume all logarithmic structures are integral (M_X is contained in its associated group M_X^{gp}) and saturated (if $\alpha \in M_X^{\mathrm{gp}}$ and $n\alpha \in M_X$ for some integer $n \geq 1$ then $\alpha \in M_X$).

Such objects may be assembled into a category. The category of logarithmic schemes has the analogous constructions and notions from scheme theory, keeping track of the tropical data through the sheaves of piecewise linear functions.

For more of the general theory of logarithmic structures, we refer the reader to F. Kato's original article [1989]. A detailed study of the relationship between tropical and logarithmic geometry from a categorical point of view is undertaken in [Cavalieri et al. 2017].

2.4. *Curves and logarithmic structures.* Let (S, M_S) be a logarithmic scheme. A *family of logarithmically smooth curves over* S is a logarithmically smooth, flat, and proper morphism

$$\pi : (C, M_C) \to (S, M_S),$$

with connected and reduced geometric fibers of dimension 1. We recall Kato's structure theorem for logarithmic curves [2000].

Theorem 2.4.1. *Let $C \to S$ be a family of logarithmically smooth curves. If $x \in C$ is a geometric point, then there is an étale neighborhood of C over S, with a strict morphism to an étale-local model $\pi : V \to S$, and $V \to S$ is one of the following*:

- (*The smooth germ*) $V = \mathbb{A}^1_S \to S$, *and the logarithmic structure on V is pulled back from the base.*
- (*The germ of a marked point*) $V = \mathbb{A}^1_S \to S$, *with logarithmic structure pulled back from the toric logarithmic structure on $\mathbb{A}^1$.*
- (*The node*) $V = \mathcal{O}_S[x, y]/(xy = t)$, *for $t \in \mathcal{O}_S$. The logarithmic structure on V is pulled back from the multiplication map $\mathbb{A}^2 \to \mathbb{A}^1$ of toric varieties along a morphism $t : S \to \mathbb{A}^1$ of logarithmic schemes.*

The image of $t \in M_S$ in $\overline{M}_S$ is referred to as the deformation parameter of the node.

Associated to a logarithmic curve $C \to S$ is a family of tropical curves.

Definition 2.4.2. Let $C \to S$ be a family of logarithmically smooth curves and assume that the underlying scheme of S is the spectrum of an algebraically closed field. Then, *the tropicalization* C, denoted C, is obtained as follows: (1) the underlying graph is the marked dual graph of C equipped with the standard genus and marking functions, and (2) given an edge e, the generalized length $\ell(e) = \delta_e \in \overline{M}_S$ is the deformation parameter of the corresponding node of C.

For more about logarithmic curves and their relationship to tropical curves, the reader may consult [Cavalieri et al. 2017].

2.5. *Geometric interpretation of the sections of a logarithmic structure.* Given a logarithmic curve $C \to S$, it will be helpful to interpret sections of the sheaves M_C^{gp}, and $\overline{M}_C^{\mathrm{gp}}$ geometrically.

2.5.1. *The affine and projective lines.* Let $(X, \varepsilon : M_X \to \mathscr{O}_X)$ be a logarithmic scheme. A section of M_X corresponds to a map $X \to \mathbb{A}^1$, the target given its toric logarithmic structure. Let α be such a section and $\bar{\alpha}$ be its image in $\overline{M}_X$. Then $\varepsilon(\alpha)$ is a unit if and only if $\bar{\alpha} = 0$.

With its logarithmic structure, $\mathbb{P}^1$ can be constructed as the quotient of $\mathbb{A}^2 - \{0\}$ by $\mathbb{G}_m$. Any map $X \to \mathbb{P}^1$ lifts locally to $\mathbb{A}^2 - \{0\}$ and can therefore be represented by a pair of sections (ξ, η) of M_X. The ratio $\xi^{-1}\eta$, which is a section of M_X^{gp}, is invariant under the action of $\mathbb{G}_m$, since $\mathbb{G}_m$ acts with the same weight on ξ and η.

Therefore a map $X \to \mathbb{P}^1$ gives a well-defined section α of M_X^{gp}. Not every section of M_X^{gp} arises this way, because the map $(\xi, \eta) : X \to \mathbb{A}^2$ from which α was derived could not meet the origin. This condition implies that, for each geometric point x of X, either $\bar{\xi}_x = 0$ or $\bar{\eta}_x = 0$. In terms of $\bar{\alpha}$, this means that $\bar{\alpha}_x \geq 0$ or $\bar{\alpha}_x \leq 0$. We term this property being *locally comparable to* 0.

Our observations prove the following proposition:

Proposition 2.5.1.1. *Let X be a logarithmic scheme. Maps $X \to \mathbb{P}^1$, the latter given its toric logarithmic structure, may be identified with sections α of M_X^{gp}, whose images $\bar{\alpha}$ in $\overline{M}_X^{\mathrm{gp}}$ are locally comparable to* 0.

Because it has charts, the sheaf $\overline{M}_X^{\mathrm{gp}}$ locally admits a surjection from a constant sheaf, so the condition on $\bar{\alpha}$ in the proposition is open on the base: if X is a family of logarithmic schemes over S and a section α of $\overline{M}_X^{\mathrm{gp}}$ verifies $\bar{\alpha} \geq 0$ or $\bar{\alpha} \leq 0$ for all x in a geometric fiber X_s of X over S then it also verifies that condition for all t in some open neighborhood of s.

This observation is particularly useful for studying infinitesimal deformations of logarithmic maps to $\mathbb{P}^1$, as it is equivalent to deform the section α of M_X^{gp}.

Definition 2.5.1.2. For any logarithmic scheme X, we define $\mathbb{G}_{\log}(X) = \Gamma(X, M_X^{\mathrm{gp}})$. Identifying X with its functor of points, we also write $\mathbb{G}_{\log}(X) = \mathrm{Hom}(X, \mathbb{G}_{\log})$.

Remark 2.5.1.3. The functor $\mathbb{G}_{\log}$ is not representable by a logarithmic scheme; it is analogous to an algebraic space (see [Molcho and Wise 2018], for example). The above considerations may be seen as a demonstration that $\mathbb{P}^1$ is logarithmically étale over $\mathbb{G}_{\log}$. A discussion of the simpler spaces rational curves in $\mathbb{G}_{\log}$ may be found in [Ranganathan and Wise 2019].

We prefer to avoid a discussion of the geometric structure of $\mathbb{G}_{\log}$ in this paper. The reader should feel free to regard maps to $\mathbb{G}_{\log}$ as a convenient shorthand for sections of M_X^{gp} and nothing more. The advantage of treating $\mathbb{G}_{\log}$ as a geometric object, and not merely an abstract sheaf, and working with that object instead of geometric models like $\mathbb{P}^1$, is that the latter approach necessitates an apparently endless process of subdivision and refinement that obscures the geometric essence of our arguments.

2.5.2. *Maps to toric varieties.* The observations above concerning logarithmic maps to $\mathbb{P}^1$ may be extended to all toric varieties. Indeed, if $Z = \operatorname{Spec} k[S_\sigma]$ is an affine toric variety defined by a cone σ and character lattice $N^\vee$, then there is a canonical map

$$S_\sigma \to \Gamma(Z, M_Z),$$

which extends to a map

$$N^\vee \to \Gamma(Z, M_Z^{\text{gp}}).$$

The construction of this map commutes with restriction to open torus invariant subvarieties, and therefore glues to a well-defined map on any toric variety.

Proposition 2.5.2.1. *Let X be a logarithmic scheme and let Z be a toric variety with fan Σ and character lattice $N^\vee$. Morphisms $X \to Z$ may be identified with morphisms $N^\vee \to \Gamma(X, M_X^{\text{gp}})$ such that, for each geometric point x of X, there is a cone $\sigma \in \Sigma$, such that the map*

$$S_\sigma \to \Gamma(X, M_X^{\text{gp}}) \to \Gamma(X, \overline{M}_X^{\text{gp}}) \to \overline{M}_{X,x}^{\text{gp}}$$

factors through $\overline{M}_{X,x}$.

Definition 2.5.2.2. Let N be a finitely generated free abelian group. We write $(N \otimes \mathbb{G}_{\log})(X) = \operatorname{Hom}(N^\vee, \Gamma(X, M_X^{\text{gp}}))$ and use $\operatorname{Hom}(X, N \otimes \mathbb{G}_{\log})$ for the same notion.

Remark 2.5.2.3. The discussion above shows that, if Z is a toric variety with cocharacter lattice N, then there is a canonical logarithmic modification $Z \to N \otimes \mathbb{G}_{\log}$.

2.5.3. *Sections of the characteristic monoid.* Since logarithmic maps $X \to \mathbb{A}^1$ correspond to sections of M_X, maps $X \to [\mathbb{A}^1/\mathbb{G}_m]$ correspond to sections of $M_X/\mathscr{O}_X^\star = \overline{M}_X$. The quotient $[\mathbb{A}^1/\mathbb{G}_m]$ is usually denoted $\mathscr{A}$ and is called the *Artin fan* of $\mathbb{A}^1$.

It is shown in [Cavalieri et al. 2017, Remark 7.3] that, if X is a logarithmic curve over S, and the underlying scheme of S is the spectrum of an algebraically closed field, then sections of $\overline{M}_X$ (which is to say, maps $X \to \mathscr{A}$) may be interpreted as piecewise linear functions on the tropicalization of X that are valued in $\overline{M}_S$ and are linear along the edges with integer slopes.

Similar reasoning, combined with the discussion in Section 2.5.1 shows that maps $X \to [\mathbb{P}^1/\mathbb{G}_m]$ correspond to sections α of $\overline{M}_X^{\text{gp}}$ that are locally comparable to 0. If X is a curve, then these sections are the piecewise linear functions on the tropicalization that are everywhere valued in $\overline{M}_S$ or in $-\overline{M}_S$.

Remark 2.5.3.1. Even though its underlying "space" is an algebraic stack, $[\mathbb{A}^1/\mathbb{G}_m]$ represents a *functor* on logarithmic schemes. This contrasts with the more common situation, where algebraic stacks typically only represent categories fibered in groupoids over schemes.

2.6. ***Line bundles from piecewise linear functions.*** For any logarithmic scheme X, there is a short exact sequence

$$0 \to \mathscr{O}_X^\star \to M_X^{\mathrm{gp}} \to \overline{M}_X^{\mathrm{gp}} \to 0$$

of the sheaves associated to the logarithmic structure. Given a section $\alpha \in \Gamma(X, \overline{M}_X^{\mathrm{gp}})$, the image of α under the coboundary map

$$H^0(X, \overline{M}_X^{\mathrm{gp}}) \to H^1(X, \mathscr{O}_X^\star)$$

is represented by an $\mathscr{O}_X^\star$-torsor $\mathscr{O}_X^\star(-\alpha)$ on X and gives rise to an associated line bundle. Thus, to each piecewise linear function f on $\sqsubset$ that is linear on the edges with integer slopes and takes values in $\overline{M}_S$, we have an associated line bundle $\mathscr{O}(-f)$.

The explicit line bundle obtained by this construction is recorded in [Ranganathan et al. 2017, Section 2].

2.7. ***Tropicalization of morphisms to toric varieties.*** Let Z be a toric variety with dense torus T, equipped with its standard logarithmic structure, and let N and $N^\vee$ be the cocharacter and character lattices of Z.

Let C be a logarithmic curve over S, and assume that the underlying scheme of S is the spectrum of an algebraically closed field. A logarithmic map $\varphi : C \to Z$ induces a map

$$N^\vee \to \Gamma(Z, \overline{M}_Z) \to \Gamma(C, \overline{M}_C) \tag{5}$$

by the discussion in Section 2.5.2.

As remarked in Section 2.5, the sections of $\overline{M}_C$ are piecewise linear functions on the tropicalization $\sqsubset$ of C that are linear with integer slopes along the edges and are valued in $\overline{M}_S^{\mathrm{gp}}$. If we assume in addition that $\overline{M}_S = \mathbb{R}_{\geq 0}$ then we obtain a piecewise linear map

$$\sqsubset \to \mathrm{Hom}(N^\vee, \mathbb{R}) = N_\mathbb{R}$$

that we call the *tropicalization* of $C \to Z$. It will sometimes be convenient to think of this as a map from $\sqsubset \to \Sigma$, where Σ is the fan of Z.

Lemma 2.7.1. *The map $\sqsubset \to N_\mathbb{R}$, constructed above, satisfies the balancing condition.*

Proof. This is proved in [Gross and Siebert 2013, Section 1.4]. □

2.8. *Minimality.*

2.8.1. *Minimal logarithmic structures.* A crucial concept in the theory of logarithmic moduli problems is that of *minimality*. Let LogSch denote the category of fine and saturated logarithmic schemes. Given a moduli stack $\mathfrak{M}$ over LogSch and a logarithmic scheme S, the fiber $\mathfrak{M}(S)$ of the fibered category $\mathfrak{M}$ over S is the groupoid logarithmic geometric objects $[\mathscr{X} \to S]$ defined over S, as specified by the moduli problem.

Logarithmic geometric objects are algebraic schemes or stacks with the additional structure of a sheaf of monoids. The description of $\mathfrak{M}$ as a category fibered in groupoids over LogSch does not furnish such an object: if $\underline{S}$ is a scheme without a chosen logarithmic structure, it does not make mathematical sense to consider the fiber of $\mathfrak{M}$ over $\underline{S}$. Said differently, there is no "underlying scheme, or underlying stack, or underlying category fibered in groupoids over schemes" of $\mathfrak{M}$.

The difficulty that must be overcome is that given an ordinary scheme $\underline{S}$, there are many choices for logarithmic schemes (S, M_S) enhancing $\underline{S}$, and it is unclear which one to pick. The notion of minimality, introduced by Kato and recently clarified and expanded [Abramovich and Chen 2014; Chen 2014; Gillam 2012; Gross and Siebert 2013; Wise 2016a; 2016b] identifies the correct logarithmic structures to allow on $\underline{S}$ as those satisfying a universal property, recalled below.

Assuming that $\mathfrak{M}$ does have an underlying scheme, we arrive at a *necessary* condition for $\mathfrak{M}$ to be representable by a logarithmic scheme. Suppose that $S \to \mathfrak{M}$ is a morphism of logarithmic schemes then the logarithmic structure of $\mathfrak{M}$ pulls back to a logarithmic structure M on the underlying scheme $\underline{S}$ of S. Moreover there is a factorization

$$\begin{array}{ccc} S & & \\ \downarrow & \searrow & \\ & & \mathfrak{M} \\ & \nearrow & \\ (\underline{S}, M) & & \end{array}$$

that is *final* among all such factorizations. This finality condition can be phrased entirely in terms of the moduli problem defining $\mathfrak{M}$, and Gillam shows that if minimal factorizations exist for all $S \to \mathfrak{M}$, and are preserved by base change, then $\mathfrak{M}$ comes from a logarithmic structure on a moduli problem over *schemes* [Gillam 2012; Wise 2016a].

Theorem 2.8.1.1 (Gillam). *When $\mathfrak{M}$ is a category fibered in groupoids over logarithmic schemes that comes from a logarithmic structure on a category fibered in groupoids $\mathfrak{N}$ over schemes, $\mathfrak{N}$ can be recovered from $\mathfrak{M}$ as the subcategory of minimal objects.*

Throughout this paper, we present logarithmic moduli problems and indicate monoidal and tropical (see Section 2.8.2) characterizations of their minimal objects to recover the underlying schematic moduli problems.

2.8.2. *Minimality as tropical representability.* We explain the concept in the case of stable maps for concreteness, where it becomes a tropical concept. This expands on [Gross and Siebert 2013, Remark 1.21].

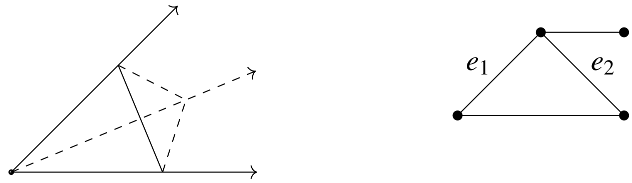


Figure 1. Consider the cone of tropical curves whose underlying graph is shown on the right, such that the edge lengths of e_1 and e_2 are equal. This cone is 3-dimensional. An associated family of logarithmic curves whose minimal monoid is dual to this cone associated family of logarithmic curves is nonminimal, due to the relation that these two edge lengths coincide coincide.

Let $\mathfrak{M}_{g,n}(Z)$ denote stack over LogSch parametrizing logarithmic maps from genus g, n-pointed curves to a toroidal scheme Z. Let Σ be the fan of Z.

Let S be a standard logarithmic point Spec($\mathbb{N} \to \mathbb{C}$) and let $[C \to Z]$ be a logarithmic map over S. As explained in Section 2.7, the morphisms on sheaves of monoids may be dualized to produce a tropical map

$$\sqsubset \to \Sigma.$$

Replacing $\mathbb{N}$ with an arbitrary toric monoid, one obtains a *family* of tropical maps.

From our discussion of minimality, we see that given a logarithmic stable map over Spec($P \to \mathbb{C}$), the monoid be cannot be arbitrary, since by pulling back via a morphism $P \to \mathbb{R}_{\geq 0}$, we must obtain a tropical map. With this observation, there is a clear choice for a universal $P^{\min}$ such that all other enhancements Spec($P \to \mathbb{C}$) of the same underlying map must be pulled back from Spec($P^{\min} \to \mathbb{C}$). That is, we may choose $P^{\min}$ to be the monoid whose dual cone Hom($P^{\min}$, $\mathbb{R}_{\geq 0}$) is the cone of *all* tropical maps of the given combinatorial type. Succinctly, a logarithmic structure is minimal for a given moduli problem if it represents the *tropical deformation space*.

In Figure 1 below, taking $Z = pt$, we depict the duals of the characteristic monoid on the base of a nonminimal family. If one drops the condition that ℓ_1 and ℓ_2 coincide, we obtain the corresponding minimal monoid.

Applying this reasoning at each geometric fiber gives a criterion to check whether any given family of logarithmic maps $C \to Z$ over a logarithmic scheme S is minimal. With the minimal objects identified, we construct a moduli stack as a fibered category over Sch, whose fiber over a scheme $\underline{S}$ is the groupoid of *minimal* logarithmic maps over S.

2.9. ***Preliminaries from the prequel: radial alignments.*** The results of this paper rely on the notion of a radially aligned logarithmic curve and its canonical contraction to a curve with elliptic singularities. These concepts were developed in the companion article [Ranganathan et al. 2017], and we briefly recall the statements that we require.

Let S be a logarithmic scheme enhancing the spectrum of an algebraically closed field and let $C \to S$ be a logarithmic curve over it whose fibers have genus 1 and let $\sqsubset$ be its tropicalization. Given an edge e, we write $\ell(e) \in \overline{M}_S$ for the generalized edge length of this edge. For each vertex $v \in \sqsubset$, there is a unique path from the circuit of $\sqsubset$, namely the smallest subgraph of genus 1, to the chosen vertex v. Write this path as $e_1, \dots, e_n$. Define

$$\lambda(v) = \sum_{i=1}^{n} \ell(e_i).$$

The resulting function λ is a piecewise linear function on $\sqsubset$ with integer slopes, and thus, a global section of $\overline{M}_C$. When S is a general logarithmic scheme and $\pi : C \to S$ a curve, this section glues along specialization morphisms to give rise to a well-defined and canonical global section in $\Gamma(S, \pi_\star \overline{M}_S)$.

Given a logarithmic curve $C \to S$ and a geometric point $s \in S$, we let $\sqsubset_s$ denote the corresponding tropical curve associated to C_s. Recall also that we view a monoid P as being the positive elements in a partially ordered group, with the partial order defined by $a \geq b$ if $a - b \in P \subset P^{\text{gp}}$.

Definition 2.9.1. We say that a logarithmic curve $C \to S$ is *radially aligned* if $\lambda(v)$ and $\lambda(w)$ are comparable for all geometric points s of S and all vertices $v, w \in \sqsubset_s$.

We write $\mathfrak{M}_{1,n}^{\text{rad}}$ for the category fibered in groupoids over logarithmic schemes whose fiber over S is the groupoid of radially aligned logarithmic curves over S having arithmetic genus 1 and n marked points.

The following result is proved in [Ranganathan et al. 2017, Section 3].

Theorem 2.9.2. *The category of radially aligned, prestable, logarithmic curves of genus* 1 *with n marked points is represented by an algebraic stack with logarithmic structure* $\mathfrak{M}_{1,n}^{\text{rad}}$. *The natural map*

$$\mathfrak{M}_{1,n}^{\text{rad}} \to \mathfrak{M}_{1,n},$$

is a logarithmic blowup.

The second major construction in [loc. cit.] is the construction of a contraction to a curve with elliptic singularities, from the data of a radially aligned curve with a chosen "radius of contraction". Let $C \to S$ be a radially aligned logarithmic curve of genus 1. We say that a section $\delta \in \overline{M}_S$ is *comparable to the radii of* C if for each geometric point $s \in S$, the section δ is comparable to $\lambda(v)$ for all vertices $v \in \sqsubset_s$, in the monoid $\overline{M}_S$.

Theorem 2.9.3. *Let* $C \to S$ *be a radially aligned logarithmic curve and* $\delta \in \overline{M}_S$ *a section comparable to the radii of* C. *Then, there exists a partial destabilization*

$$\tilde{C} \to C,$$

and a contraction

$$\tilde{C} \to \overline{C},$$

where $\overline{C} \to S$ *is a family genus* 1 *curves at worst Gorenstein genus* 1 *singularities, such that, for every geometric point s of* S, *if* E *is a component of* C_s *such that* $\lambda(E) < \delta_s$ *then* E *is contracted to a point in* $\overline{C}$.

An intuitive discussion of these concepts are presented in [Ranganathan et al. 2017, Section 3.1]. For working knowledge, reader may visualize the section δ as giving rise to a *circle of radius* δ around the circuit of the tropical curve $\sqsubset$. By subdividing the edges of $\sqsubset$, one may produce a new tropical curve $\tilde{\sqsubset}$ such that every point of $\tilde{\sqsubset}$ at radius δ from the circuit is a vertex. This introduces valency 2 vertices into the tropicalization, and induces the partial destabilization. By contracting the interior of the circle of radius δ in a versal family, one produces a curve with a Gorenstein singularity.

3. Logarithmic maps to toric varieties

We construct the space of radially aligned logarithmic maps to a toric variety. The framework of radial alignments, together with the well-spacedness condition from tropical geometry, will lead to a proof of Theorem B, which is the main result of this section. The symbol Z will denote a proper toric variety with fan Σ.

Recall that a morphism of polyhedral complex $\mathscr{P} \to \mathscr{Q}$ is a continuous map of the underlying topological spaces sending every polyhedron of $\mathscr{P}$ linearly to a polyhedron of $\mathscr{Q}$.

Definition 3.1. A *tropical prestable map* or *tropical map* for short, is a morphism of polyhedral complexes

$$F : \sqsubset \to \Sigma$$

where $\sqsubset$ is an n-marked tropical curve and the following conditions are satisfied:

(1) For each edge $e \in \sqsubset$, the direction of $F(e)$ is an integral vector. When restricted to e, the map has integral slope w_e, taken with respect to this integral direction. This integral slope is referred to as the *expansion factor* of F along e. The expansion factor and primitive edge direction are together referred to as the *contact order* of the edge.

(2) The map f is *balanced*: at all points of $\sqsubset$ the sum of the directional derivatives of F in each tangent direction is zero.

The map is *stable* if it satisfies the following condition: if $p \in \sqsubset$ has valence 2, then the image of $\mathrm{Star}(v)$ is not contained in the relative interior of a single cone of Σ.

Following Section 2.7, given a logarithmic prestable map to a toric variety

$$\begin{array}{ccc} (C, M_C) & \xrightarrow{f} & Z \\ \downarrow & & \\ (S, M_S), & & \end{array}$$

there is an associated family $\sqsubset$ of tropical curves together with a map $[F : \sqsubset \to \Sigma]$, satisfying the axioms of a tropical prestable map.

3.2. ***Radial logarithmic maps.*** We begin with a construction of the stack of radially aligned logarithmic maps.

Proposition 3.2.1. *Let Z be a toric variety. There is an algebraic stack with logarithmic structure, $\mathfrak{W}(Z)$, parametrizing families of radially aligned curves C and logarithmic morphisms $C \to Z$.*

The underlying algebraic stack of $\mathfrak{W}(Z)$ is locally quasifinite over the stack of ordinary prestable maps from radially aligned curves to Z, and its restriction to the open and closed substack of maps with fixed contact orders is quasifinite.

Proof. Let $\mathfrak{M}^{\mathrm{rad}}_{1,n}$ be the stack of radially aligned, n-marked, genus 1 logarithmic curves (Section 2.9) and let C be its universal curve. Then $\mathfrak{W}(Z)$ is the space of logarithmic prestable maps from C to Z, and this is representable by an algebraic stack with a logarithmic structure [Wise 2016a, Corollary 1.1.1]. The local quasifiniteness is a consequence of [Wise 2016b, Theorem 1.1] or [Ranganathan 2017a, Proposition 3.6.3]; under the assumption of fixed contact orders, the combinatorial types of a map $[C \to Z]$ are bounded [Gross and Siebert 2013, Theorem 3.8], and therefore $\mathfrak{W}(Z)$ is quasifinite over the space of maps of underlying schemes. □

Stability in $\mathfrak{W}(Z)$ is defined in terms of the underlying schematic map:

Definition 3.2.2. A radial map $[f : C \to Z]$ in $\mathfrak{W}(Z)$ over $\mathrm{Spec}(\mathbb{C})$ is said to be *stable* if it satisfies the following conditions:

(1) If $D \subset C$ is an irreducible component of genus 0 contracted by f, then D supports at least 3 special points.

(2) If C is a smooth curve of genus 1, then C is not contracted.

A family of ordered logarithmic maps is stable if each geometric fiber is stable.

3.2.3. *Minimal monoids.* We give a tropical description of the logarithmic structure of $\mathfrak{W}(Z)$. We leave it to the reader to verify that this description is correct, either using [Wise 2016a, Appendix C.3] or adapting the arguments from [Chen 2014, Section 3] or [Gross and Siebert 2013, Proposition 1.22].

The minimality condition may be checked on geometric fibers, so we assume that the underlying scheme of S is $\mathrm{Spec}\, C$. Let σ_S be the corresponding dual cone $\mathrm{Hom}(\overline{M}_S, \mathbb{R}_{\geq 0})$ of the characteristic monoid of S. By forgetting the alignment, a radial map $[f]$ above produces a usual logarithmic map with combinatorial type Θ. Letting σ_Θ be the associated cone of tropical maps, we have a morphism of cones

$$\sigma_S \to \sigma_\Theta.$$

In the tropical moduli cone σ_Θ above, the locus of tropical curves whose vertices are ordered in the same manner as C forms a cone $\sigma(f)$.

Definition 3.2.4. Let $f : C \to Z$ be a family of ordered logarithmic maps over a logarithmic base S. The map $[f]$ is a *minimal* ordered logarithmic map if for each geometric point $\bar{s} \in S$, there is an isomorphism of cones

$$\mathrm{Hom}(\overline{M}_{S,\bar{s}}, \mathbb{R}_{\geq 0}) \cong \sigma(f_{\bar{s}}).$$

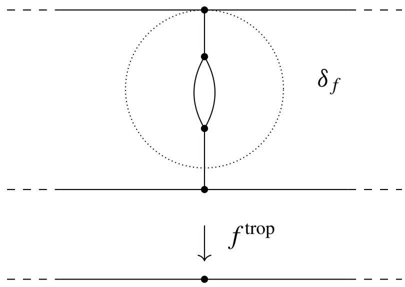


Figure 2. A tropical map from a genus 1 curve to $\Sigma_{\mathbb{P}^1}$ contracting a circuit. The dotted circle corresponds to the circle whose radius is the minimal distance to a vertex supporting a noncontracted flag.

3.3. ***The factorization property.*** To detect the curves that smooth to the main component, we will need to identify certain contractions of the source curve constructed from the tropical maps and use the methods developed in [Ranganathan et al. 2017].

Let $\bar{C}$ be a Gorenstein curve of arithmetic genus 1. We will refer to E, the smallest connected subcurve of C of arithmetic genus 1, as the *circuit component* of C. Given a family $C \to S$, we give the nodes and markings the standard logarithmic structure, and we give C the trivial logarithmic structure near any genus 1 singularities.

Given an aligned logarithmic curve C of genus 1 and a contraction $C \to \bar{C}$, we may equip $\bar{C}$ with the logarithmic structure defined above. This enhances $C \to \bar{C}$ to a logarithmic morphism.

Let $(C, M_C) \to (S, M_S)$ be a radially aligned logarithmic curve and let Z be a toric variety with cocharacter lattice N. We associate a section $\delta_f \in \overline{M}_S$ to a logarithmic map $f : C \to Z$ over S. Let $\sqsubset$ be the tropicalization of the curve C with circuit $\sqsubset_0$. Consider the associated family of tropical maps

$$\varphi : \sqsubset \to N_{\mathbb{R}}$$

If φ does not contract the circuit, then let $\delta_f = 0$. Otherwise, let δ_f be the minimum distance from $\sqsubset_0$ to a 1-valent vertex of $\varphi^{-1}(\varphi(\sqsubset_0))$. That is, the distance to the closest vertex supporting a flag that is not contracted by φ. See Figure 2 for an example.

These data will produce a rational bubbling of the source curve and a contraction thereof. First, subdivide $\sqsubset$ such that every edge of $\varphi^{-1}(\varphi(\sqsubset_0))$ terminates at a vertex; in Figure 2, this amounts to introducing a vertex where the dotted circle crosses the lower vertical edge. This induces a logarithmic modification

$$\tau : \tilde{C} \to C.$$

By the constructions of Section 2.9, we obtain a section comparable to the radii of C, and there is now an induced contraction

$$\gamma : \tilde{C} \to \bar{C},$$

to a curve with a Gorenstein elliptic singularity.

Definition 3.3.1. Keeping the notations above, an ordered logarithmic map $C \to Z$ as above is said to have the *factorization property* if the associated map $\tilde{C} \to C \to Z$ factorizes as

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\quad f \quad} & Z \\ & \searrow \quad \nearrow_{\bar{f}} & \\ & \bar{C}. & \end{array}$$

Remark 3.3.2. Note that it need not be the case that the map $\bar{f}$ is nonconstant on a branch of the elliptic singularity. This is because the map $\bar{C} \to Z$ may have highly degenerate contact order with the boundary of Z; it could be that the entire elliptic component is contracted. However, one may always replace Z with a logarithmic modification $\mathscr{Z}$, i.e., a toric degeneration, such that the genus 1 component maps to the dense torus of one of the components of $\mathscr{Z}$. In such an expansion, there will be at least one branch of the singularity along which the map is nonconstant. For example, if $Z = \mathbb{P}^1$, it could be that the genus 1 subcurve is contracted to one of the relative points 0 or ∞ on $\mathbb{P}^1$. In this case, one must first expand $\mathbb{P}^1$ to a chain $\mathbb{P}^1_{\mathrm{exp}}$, until the curve maps to the dense torus of a component in $\mathbb{P}^1_{\mathrm{exp}}$. The choice of radius forces that the factorization is nonconstant on a branch of the singularity.

Recall from Section 2.5.2 that a map $f : C \to Z$, where Z is a toric variety with character lattice $N^\vee$, induces a homomorphism

$$\alpha : N^\vee \to \Gamma(C, M_C^{\mathrm{gp}}).$$

The factorization property depends only on α and not specifically on the morphism of toric varieties $C \to Z$. For example, if Z were a toric modification of another toric variety Z', then the factorization properties for $C \to Z$ and $C \to Z'$ would coincide. We offer a definition of the factorization property that makes this independence explicit.

Definition 3.3.3. Let N and $N^\vee$ be finitely generated, free abelian groups, dual to each other, and let C be a logarithmic curve over S. Assume given $\alpha : N^\vee \to \Gamma(C, M_C^{\mathrm{gp}})$ and let $\bar{\alpha}$ be the induced morphism valued in $\Gamma(C, \overline{M}_C^{\mathrm{gp}})$. Let $\sqsubset_s$ be the tropicalization of C_s, for each geometric point s of S. Define $\delta_\alpha \in \Gamma(S, \overline{M}_S)$ fiberwise to be the largest $\lambda(v)$, among $v \in \sqsubset$, such that $\bar{\alpha}$ is constant when viewed as a piecewise linear function on $\sqsubset$.

Let $\upsilon : \tilde{C} \to C$ and $\tau : \tilde{C} \to \bar{C}$ be the destabilization and contraction constructed as above. We say that α *satisfies the factorization property* if $\upsilon^\star\alpha$ descends along τ to $N^\vee \to \Gamma(\bar{C}, M_{\bar{C}}^{\mathrm{gp}})$.

Remark 3.3.4. The factorization property is equivalent to requiring that $\tilde{C} \to C \to N \otimes \mathbb{G}_m^{\log}$ factor through the contracted curve $\bar{C}$.

3.4. ***The stack of well-spaced logarithmic maps.*** In this section, we construct a stack that we will later identify as the main component of the moduli space of genus 1 maps to a toric variety.

We begin with some geometric motivation. Let H be a subtorus of the dense torus T of Z. After replacing Z with a toric modification, there is a toric compactification Z_H of the quotient torus T/H and a toric morphism

$$Z \to Z_H,$$

extending the projection $T \to T/H$.

Let $f : C \to Z$ be a radial map over S, let H a subtorus of the dense torus T, and assume that $Z \to Z_H$ exists for some T/H-toric variety Z_H. We say that $[f]$ satisfies the *factorization property for H* if the induced logarithmic map

$$C \to Z \to Z_H$$

satisfies the factorization property.

This definition cannot be applied to an arbitrary toric variety Z and an arbitrary subtorus $H \subset T$, since there may not be a toric map from Z to an equivariant compactification of T/H. For example, consider $Z = \mathbb{P}^2$ and let H be any 1-dimensional subtorus. Since there is no nonconstant map from $\mathbb{P}^2$ to $\mathbb{P}^1$, the assumption fails.

There are two ways in which to overcome the issue. The first is to replace Z with a logarithmic modification, which requires replacing C with a logarithmic modification. This logarithmic modification may not be defined over the base S, until we perform a logarithmic modification of S as well [Abramovich et al. 2017b, Proposition 4.5.2].

It is conceptually simpler to use Definition 3.3.3, which does not require the map $Z \to Z_H$, but only the map of tori $T \to T/H$. Indeed, let $N^\vee$ be the character lattice of T and let $N^\vee_{T/H}$ be the character lattice of T/H. Then the factorization property for $C \to Z \to Z_H$ is equivalent to the factorization property for the composition

$$N^\vee_{T/H} \to N^\vee \to \Gamma(C, M_C^{\mathrm{gp}}).$$

With this as motivation, we arrive at our definition:

Definition 3.4.1. Let $f : C \to Z$ be a map from a radially aligned logarithmic curve to a toric variety Z with dense torus T and character lattice $N^\vee$. Let H be a subtorus of T and let $N^\vee_{T/H}$ be the character lattice of T/H. We say that f *satisfies the factorization property* for H if the map

$$N^\vee_{T/H} \to N^\vee \to \Gamma(C, M_C^{\mathrm{gp}}).$$

satisfies the factorization property of Definition 3.3.3.

Geometrically, the condition is that $C \to N \otimes \mathbb{G}_{\log} \to N_{T/H} \otimes \mathbb{G}_{\log}$ should factor through $\overline{C}_H$, where $\overline{C}_H$ is constructed from $C \to N_{T/H} \otimes \mathbb{G}_{\log}$ as in Section 3.3. The equivalence between the formulations is a tautology: $\mathbb{G}_{\log}$ may simply be defined as the representing objects for global sections of M_X^{gp}.

Definition 3.4.2. Let Z be a toric variety. A radial logarithmic map $f : C \to Z$ is *well-spaced* if f satisfies the factorization property for all subtori H of T.

Let $\mathcal{W}(Z)$ denote the category fibered in groupoids, over logarithmic schemes, of stable, well-spaced, radially aligned logarithmic stable maps to Z.

Given a splitting $N_{T/H}^{\vee} \simeq \mathbb{Z}^r$, the factorization property for H is the conjunction of the factorization properties with respect to the tori dual to the direct summands of $\mathbb{Z}^r$. When $N_{T/H}^{\vee}$ has rank 1, the factorization property asserts that a section of $M_{\tilde{C}}^{\mathrm{gp}}$ descends to a section of $M_{\bar{C}}^{\mathrm{gp}}$. It is in this form that we will verify the algebraicity of the factorization property, below.

Let C be a family of radially aligned logarithmic curves over S, assume that a section $\delta \in \Gamma(S, \overline{M}_S)$ is given, that $\tilde{C} \to C$ and $\tau : \tilde{C} \to \bar{C}$ are the associated semistable model and contraction of genus 1 component, respectively, and that $E \subset \tilde{C}$ is the exceptional locus of τ. We write $\pi : C \to S$ and $\bar{\pi} : \bar{C} \to S$ for the projections. We fix a section α of M_C^{gp}. We will use F for the subfunctor of the functor represented by S on logarithmic schemes consisting of those $f : T \to S$ such that the restriction of α along f has the factorization property — in other words, the pullback of α to $\tilde{C}_T$ descends along $\tilde{C}_T \to \bar{C}_T$.

In the following statements, we will say that the factorization property has a certain trait to mean that the functor F has that trait, relative to the functor represented by S. To unclutter the notation slightly, we will also assume without loss of generality, that $\tilde{C} \to C$ is an isomorphism, since doing so entails no loss of generality.

We begin by showing that the moduli functor of factorized maps is representable.

Theorem 3.4.3. *The factorization property is representable by a* (*not necessarily strict*) *closed embedding of logarithmic schemes.*

The proof proceeds as in Artin's criteria for algebraicity, although we do not need to invoke Artin's criteria directly because the morphism in question will turn out to be a closed embedding.

The following two propositions refine [Ranganathan et al. 2017, Theorem 4.3], with essentially the same proof:

Proposition 3.4.4. *The set of points of S where α satisfies the factorization property is constructible.*

Proof. We wish to show that the locus of points in S where the section α of M_C^{gp} descends to $M_{\bar{C}}^{\mathrm{gp}}$ is constructible in S.

This assertion is local in the constructible topology on S, so we may assume that the dual graph of C is locally constant over S. The assertion is also local in the étale topology, so we can even assume the dual graph is constant. Then there are two obstructions to descending α to a section of $M_{\bar{C}}^{\mathrm{gp}}$. Since $M_{\bar{C}}^{\mathrm{gp}}$ is pulled back from S near $\tau(E)$, the first obstruction is that $\bar{\alpha}$ should be constant on E. But $\pi_\star \overline{M}_C^{\mathrm{gp}}$ is constant over S, so this holds on an open and closed subscheme of S. In particular, this locus is quasicompact and constructible. We may now restrict attention to this locus and assume the first obstruction vanishes.

Now α is a section of $\pi^\star M_S^{\mathrm{gp}} \subset M_C^{\mathrm{gp}}$ near E. Since $\bar{\alpha}$ is constant on E by assumption, we may, after localization in S, divide off a section pulled back from S, to ensure that $\bar{\alpha}(E) = 0$ and α is therefore a section of $\mathcal{O}_C^\star$. We must show that the locus in S where α is pulled back from $\mathcal{O}_{\bar{C}}^\star$ is constructible.

Regarding α instead as a section of $\mathcal{O}_C$, it is equivalent to show that the locus where α is pulled back from $\mathcal{O}_{\bar{C}}$ is constructible. The rest of the proof is now the same as in [Ranganathan et al. 2017, Theorem 4.3]. □

Proposition 3.4.5. *The factorization property satisfies the valuative criterion for properness.*

Proof. We assume that S is the spectrum of a valuation ring, with $j : \eta \to S$ the inclusion of the generic point. We wish to show that if α is a section of M_C^{gp} such that $j^\star\alpha$ is pulled back from a section of $M_{\bar{C}_\eta}$, then α is pulled back from a section of $M_{\bar{C}}$ over $\bar{C}$.

It suffices to assume that S has the maximal logarithmic structure extending the logarithmic structure over η. That is, $M_S = \mathcal{O}_S \times_{j_\star\mathcal{O}_\eta} j_\star M_\eta$. This implies that $M_S^{\mathrm{gp}} = j_\star M_\eta^{\mathrm{gp}}$.

Our task is equivalent to showing that the square (6) is cartesian:

$$\begin{array}{ccc} M_{\bar{C}}^{\mathrm{gp}} & \longrightarrow & j_\star M_{\bar{C}_\eta}^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ \tau_\star M_C^{\mathrm{gp}} & \longrightarrow & j_\star\tau_\star M_{C_\eta}^{\mathrm{gp}} \end{array} \tag{6}$$

Away from $\tau(E)$, we know that φ is an isomorphism, so it suffices to demonstrate the bijectivity of φ near $\tau(E)$, where $M_{\bar{C}} = \bar{\pi}^\star M_S$. Since the map $M_{\bar{C}}^{\mathrm{gp}} = \bar{\pi}^\star M_S^{\mathrm{gp}} \to \tau_\star M_C^{\mathrm{gp}}$ factors through $\tau_\star\pi^\star M_S^{\mathrm{gp}}$, it suffices to show that (7) is cartesian:

$$\begin{array}{ccc} \bar{\pi}^\star M_S^{\mathrm{gp}} & \longrightarrow & j_\star\bar{\pi}^\star M_S^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ \tau_\star\pi^\star M_S^{\mathrm{gp}} & \longrightarrow & j_\star\tau_\star\pi^\star M_S^{\mathrm{gp}} \end{array} \tag{7}$$

This reduces to showing that both of the following two squares are cartesian:

$$\begin{array}{ccc} \bar{\pi}^\star \overline{M}_S^{\mathrm{gp}} & \longrightarrow & j_\star\bar{\pi}^\star \overline{M}_S^{\mathrm{gp}} \\ \downarrow & & \downarrow \\ \tau_\star\pi^\star \overline{M}_S^{\mathrm{gp}} & \longrightarrow & j_\star\tau_\star\pi^\star \overline{M}_S^{\mathrm{gp}} \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{\bar{C}}^\star & \longrightarrow & j_\star\mathcal{O}_{\bar{C}_\eta}^\star \\ \downarrow & & \downarrow \\ \tau_\star\mathcal{O}_C^\star & \longrightarrow & j_\star\tau_\star\mathcal{O}_{C_\eta}^\star \end{array} \tag{8}$$

We can check that the first square is cartesian on fibers, by proper base change for étale pushforward. In that situation it is immediate, because $\pi^\star\overline{M}_S^{\mathrm{gp}}$ is constant on the fibers over S and the fibers of τ are connected. To see that the second square is cartesian, it is sufficient to see that (9) is cartesian:

$$\begin{array}{ccc} \mathcal{O}_{\bar{C}} & \longrightarrow & j_\star\mathcal{O}_{\bar{C}_\eta} \\ \downarrow & & \downarrow \\ \tau_\star\mathcal{O}_C & \longrightarrow & j_\star\tau_\star\mathcal{O}_{C_\eta} \end{array} \tag{9}$$

The rest of the proof is exactly the same as the end of the proof of [Ranganathan et al. 2017, Theorem 4.3]. □

Propositions 3.4.4 and 3.4.5 combine to imply that the locus in S where $\alpha \in M_C^{\mathrm{gp}}$ satisfies the factorization property is a closed subset of S. To give that closed subset a scheme structure, we have two more propositions, the first of which requires the notion of a homogeneous functor, see [SGA 7_{I} 1972, Definition 2.5; Wise 2011, Section 2].

Definition 3.4.6. Let F be a category fibered in groupoids over schemes. We say F is homogeneous if the map

$$F(S') \to F(S) \underset{F(T)}{\times} F(T')$$

is an equivalence whenever $S' = S \amalg_T T'$ is the pushout of an infinitesimal extension $T \subset T'$ and an affine morphism $T \to S$.

Proposition 3.4.7. *The factorization property is homogeneous.*

Proof. Let $\tau : C' \to \bar{C}'$ be a contraction of genus 1 components over S', where $S = T' \amalg_T S$ for a strict infinitesimal extension $T \subset T'$ and a strict affine morphism $\rho : T \to S$. We must show that if $\alpha' \in \Gamma(C', M_{C'}^{\mathrm{gp}})$ and its restrictions β' to T' and α to S satisfy the factorization property, then so does α'. This is in fact immediate in view of (10):

$$M_{S'}^{\mathrm{gp}} = \rho_\star M_{T'}^{\mathrm{gp}} \underset{\rho_\star M_T^{\mathrm{gp}}}{\times} M_S^{\mathrm{gp}} \tag{10}$$

Indeed, this assertion holds trivially on characteristic monoids, since $\overline{M}_{T'}^{\mathrm{gp}} \to \overline{M}_T^{\mathrm{gp}}$ is an isomorphism, so it comes down to the identification (11),

$$\mathcal{O}_{S'} = \rho_\star \mathcal{O}_{T'} \underset{\rho_\star \mathcal{O}_T}{\times} \mathcal{O}_S \tag{11}$$

which is the definition of S'. □

Proposition 3.4.8. *The factorization property holds over a complete noetherian local ring if and only if it holds formally.*

Proof. Here we must show that if S is the spectrum of a complete noetherian local ring with maximal ideal m and S_i is the vanishing locus of m^{i+1} then $\alpha \in \Gamma(C, M_S^{\mathrm{gp}})$ satisfies the factorization property if and only if its restriction to S_i satisfies the factorization property for every i. It is certainly the case that if $\bar{\alpha} \in \Gamma(C, \overline{M}_S^{\mathrm{gp}})$ is pulled back from $\Gamma(\bar{C}, \overline{M}_{\bar{C}}^{\mathrm{gp}})$ modulo every (in fact, *any*) power of m then so does $\bar{\alpha}$. Indeed, this claim amounts to the assertion that if $\bar{\alpha}$ is constant on $E \cap \pi^{-1}(S_0)$ then $\bar{\alpha}$ is constant on E. But $\overline{M}_C$ is an étale sheaf on C, so if $\bar{\alpha}$ vanishes at a point then it vanishes in an open neighborhood of that point. Since the only open subset of E containing $E \cap \pi^{-1}(S_0)$ is E itself, we conclude that $\bar{\alpha}$ descends to $\bar{C}$ if $\bar{\alpha}|_{S_0}$ descends to $\bar{\pi}^{-1}(S_0)$.

Dividing α by a section of M_S^{gp}, we can assume that $\bar{\alpha} = 0$ on E and therefore that α is a section of $\mathcal{O}_C^*$ near E. We can find an open subsets $\overline{U} \subset \bar{C}$ covering $\tau(E)$ such that $\bar{\alpha}$ vanishes on $U = \tau^{-1}\overline{U}$. Then, by assumption, we have $\tau_\star \alpha|_{\overline{U}} \in \mathcal{O}_{\overline{U}}/m^{i+1}\mathcal{O}_{\overline{U}} \subset \tau_\star(\mathcal{O}_U/m^{i+1}\mathcal{O}_U)$. Passing to the limit, it follows that the image of $\tau_\star\alpha$ along $\tau_\star\mathcal{O}_U \to \varprojlim \tau_\star(\mathcal{O}_U/m^{i+1}\mathcal{O}_U)$ lies in $\varprojlim \mathcal{O}_{\overline{U}}/m^{i+1}\mathcal{O}_{\overline{U}} = \mathcal{O}_{\overline{U}}$. But the theorem on formal functions guarantees that $\tau_\star\mathcal{O}_U \to \varprojlim \tau_\star(\mathcal{O}_U/m^{i+1}\mathcal{O}_U)$ is an isomorphism, so α has the factorization property, as required. □

As we will detail below, the propositions proved so far show that F is representable by a closed subscheme of S when restricted to the category of *strict* logarithmic schemes over S. To show that it is representable on the category of all logarithmic schemes over S, we require one more proposition.

Proposition 3.4.9. *The factorization property has minimal monoids and the pullback of a minimal monoid is a minimal monoid.*

Proof. Suppose that α satisfies the factorization property over S_0, where $S_0 \to S$ is a morphism of logarithmic schemes that is an isomorphism on the underlying schemes. Write $C_0 = C \times_S S_0$ and let α_0 be the image of α in $M_{C_0}^{\mathrm{gp}}$, which satisfies the factorization property by assumption. Working locally, we can assume that S is atomic and that the dual graph of C is constant on the closed stratum. Let $\sqsubset$ be that dual graph. Adjusting α by a section of M_S^{gp}, we can assume that $\bar{\alpha}_0$ vanishes on the components contracted by τ. Let Q be the set of all elements of $\overline{M}_S^{\mathrm{gp}}$ that arise as $\bar{\alpha}(v)$ as v ranges over the vertices of $\sqsubset$ contracted by τ. Then Q is contained in the kernel of $\overline{M}_S^{\mathrm{gp}} \to \overline{M}_{S_0}^{\mathrm{gp}}$. The minimal characteristic monoid on which we have the factorization property is therefore the saturation $\overline{N}_S$ of the image of $\overline{M}_S$ in the quotient of $\overline{M}_S^{\mathrm{gp}}$ by the subgroup generated by Q. Since we have a map $\overline{N}_S \to \overline{M}_{S_0}$, we can pull back the logarithmic structure of S_0 to get a logarithmic structure N_S on S on which α has the factorization property. It is immediate from the construction that $\overline{N}_S$ is minimal and that the construction commutes with pullback. □

Proof of Theorem 3.4.3. Let $\tau : C \to \bar{C}$ be a contraction of genus 1 components over S and let α be a section of M_C^{gp}. We wish to show that there is a closed subscheme $S' \subset S$ such that the pullback of α along $f : T \to S$ has the factorization property if and only if f factors through S'. We may assume that S is of finite type, since all of the data in play are locally of finite presentation.

By Propositions 3.4.4 and 3.4.5, the subset of points of S where α satisfies the factorization property is a closed subset S_0 of S. If S_0 is given the reduced subscheme structure then α has the factorization property over S_0. By Proposition 3.4.7, the subscheme structures on S_0 over which α has the factorization property are filtered. Taking the limit of these closed subschemes yields a scheme S', the spectrum of a complete noetherian local ring, such that the factorization property holds formally over S'. But the underlying scheme of S' must be the same as that of S_0, so the ideal of S_0 in S' is nilpotent. Thus S' is actually an infinitesimal neighborhood of S_0 in S and S' is the maximal closed subscheme of S over which the factorization property holds.

Now suppose that $f : T \to S$ is a strict morphism and that α has the factorization property over T. We wish to show f factors through S'. We may assume T is of finite type. Certainly $f(T) \subset S_0$ as a set, so if T_0 is the reduced subscheme structure on T the $f|_{T_0}$ factors through S_0. Then T is an infinitesimal extension of T_0 so the pushout $S_1 = T \amalg_{T_0} S_0$ is an infinitesimal extension of S_0 and α has the factorization property over S_1 by Proposition 3.4.8. It follows that $S_1 \subset S'$ and therefore f factors through S' as required.

By Proposition 3.4.9 the factorization property has minimal monoids, so by Theorem 2.8.1.1, S' with its minimal logarithmic structure represents the factorization property. □

Theorem 3.4.10. *The category $\mathcal{W}(Z)$ is representable by a logarithmic algebraic stack. After fixing the contact orders Γ, the substack $\mathcal{W}_\Gamma(Z)$ of maps with those contact orders is proper.*

Proof. We have just seen that the factorization property is representable by closed (hence proper) morphisms. Since stability is an open condition, this shows $\mathcal{W}(Z)$ is a locally closed substack of $\mathfrak{W}(Z)$. It also shows that $\mathcal{W}(Z)$ is a closed substack of the space $\overline{\mathcal{M}}^{\mathrm{rad}}_{1,n}(Z)$ of stable logarithmic maps from radially aligned curves to Z; this is a logarithmic modification of $\overline{\mathcal{M}}_{1,n}(Z)$, the space of stable logarithmic maps to Z, and is therefore proper. It follows that $\mathcal{W}(Z)$ is proper. □

3.5. *Logarithmic smoothness.* The logarithmic tangent bundle of a toric variety Z is trivial, and is naturally identified with $N \otimes_{\mathbb{Z}} \mathscr{O}_Z$, where $N = N(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ is the cocharacter lattice of the dense torus. Given a radial map $[f : C \to Z]$, the obstructions to deforming the map $[f]$ fixing the deformation of $[C]$ lie in the group

$$\mathrm{Obs}([f]) = H^1(C, f^\star T_Z^{\log}) = H^1(C, \mathscr{O}_C^{\dim Z})$$

with dimension

$$h^1(C, \mathscr{O}_C^{\dim Z}) = g(C) \cdot \dim Z.$$

Consider a torus quotient $T \to T/H$ and choose a compatible equivariant compactification

$$Z \to Z_H,$$

possibly passing from Z to a modification, as in the previous section. The quotient map induces a projection map on logarithmic tangent bundles, extending scalars from

$$N(T) \to N(T/H)$$

Choosing a splitting for the induced map on obstruction groups, we see that if the map $[\bar{f} : C \to \bar{Z}]$ is obstructed, then the map $[f]$ is also obstructed. The well-spacedness condition for radial logarithmic maps removes obstructions arising in this fashion. We now show that these obstructions are the only obstructions that arise.

Theorem 3.5.1. *For any toric variety Z, the stack $\mathcal{W}(Z)$ is logarithmically smooth and unobstructed.*

The proof will require the following lemma.

Lemma 3.5.2. *Let E be a connected Gorenstein curve of genus 1 without genus 0 tails. Let $E_\circ$ be the smooth locus of E. The map $E_\circ \to \mathrm{Pic}^1(E)$ sending x to $\mathcal{O}_E(x)$ is étale.*

Proof. Consider the problem of deforming x, while fixing its image $\mathcal{O}_E(x)$ in the Picard group. The obstructions to these deformations lie in $H^1(E, \mathcal{O}_E(x))$. Since E has no genus 0 tails then ω_E is trivial, and Serre duality yields the requisite vanishing.

To see that the map has relative dimension 0, note that the relative tangent space may be identified with the quotient of $H^0(E, \mathcal{O}_E(x))$ by $H^0(E, \mathcal{O}_E)$. An application of Riemann–Roch shows that this quotient is trivial. □

Corollary 3.5.3. *Let E be a connected Gorenstein curve of genus 1 without genus 0 tails and let $a_1, \dots, a_n \in \mathbb{Z}^m$. Let E_0 be the smooth locus of E. If the a_i span $\mathbb{Q}^m$ then the map $E_0^n \to \operatorname{Pic}(E)^m$ sending $(x_1, \dots, x_n)$ to the tuple of line bundles associated to the divisor with $\mathbb{Z}^m$-coefficients $\sum a_i x_i$.*

Proof. By the elementary divisors theorem, we can assume that the a_i are multiples of the standard basis vectors. Since E_0 is smooth, we may project off the factors of E_0^n where a_i vanishes. Then $E_0^m \to \operatorname{Pic}(E)^m$ is the product of the maps $E_0 \to \operatorname{Pic}^1(E) \xrightarrow{a_i} \operatorname{Pic}(E)$. The maps $E_0 \to \operatorname{Pic}^1(E)$ are étale by Lemma 3.5.2 and multiplication by a_i is étale because we work in characteristic zero. □

Proof of Theorem 3.5.1. We will use the logarithmic infinitesimal criterion for smoothness. We must show that whenever $S \subset S'$ is a strict infinitesimal extension of logarithmic schemes, any morphism $S \to \mathcal{W}(Z)$ can be extended to S', completing diagrams of the following form:

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{W}(Z) \\ \downarrow & \nearrow & \\ S' & & \end{array} \tag{12}$$

This assertion is local in S, so we can restrict to a neighborhood U of a geometric point s, such that the map $\overline{M}_S^{\mathrm{gp}}|_U \to \overline{M}_{S,s}^{\mathrm{gp}}$ is an isomorphism. Let $\sqsubset$ be the tropicalization of C_s.

Filtering the deformations. Let N and $N^\vee$ be the character and cocharacter lattices of Z, respectively. The moduli map $S \to \mathcal{W}(Z)$ gives the data of a curve C and a section $\alpha \in N \otimes \Gamma(C, M_C^{\mathrm{gp}})$. For every torsion free quotient $N \to N'$ we obtain a map $\alpha_{N'} \in N' \otimes \Gamma(C, M_C^{\mathrm{gp}})$ by composition, which we also view as a map $C \to N' \otimes \mathbb{G}_{\log}$. For each such map $\sqsubset \to N' \otimes \mathbb{R}$, there is a largest radius $\delta_{N'} \in \Gamma(S, \overline{M}_S)$ around the minimal circuit in $\sqsubset$ whose interior is contracted by the map.

The radii $\delta_{N'}$ for varying N' are totally ordered and necessarily finite in number. Rename these distinct radii $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_k$.

Since $\bar{\alpha}$ is a piecewise linear function on $\sqsubset$ valued in $\overline{M}_S^{\mathrm{gp}}$ we have $\bar{\alpha}(v) - \bar{\alpha}(w) \in N \otimes \ell \subset N \otimes \overline{M}_S^{\mathrm{gp}}$ whenever v and w are connected by an edge of $\sqsubset$ of length ℓ. We call $(\bar{\alpha}(v) - \bar{\alpha}(w))/\ell$ the *slope* of $\bar{\alpha}$ along that edge.

For each i, we take N_i be the quotient of N by the saturated sublattice spanned of the slopes of $\bar{\alpha}$ along the edges of $\sqsubset$ contained inside the circle of radius δ_i. This gives an sequence of torsion free quotients $N \to N_k \to N_{k-1} \to \cdots \to N_1$.

For each i, let $Z_i = N_i \otimes \mathbb{G}_{\log}$. Then we obtain a sequence of maps:

$$\mathcal{W}(Z) \to \mathcal{W}(Z_k) \to \mathcal{W}(Z_{k-1}) \to \cdots \to \mathcal{W}(Z_1)$$

The first map is logarithmically étale since Z is logarithmically étale over $\operatorname{Hom}(N, \mathbb{G}_{\log})$. It will now suffice to show that $\mathcal{W}(Z_i) \to \mathcal{W}(Z_{i-1})$ is logarithmically smooth for all i. For each i, let α_i be the image of α in $N_i \otimes \Gamma(C, M_C^{\mathrm{gp}})$. We make the following observations:

(1) We have $\delta_i \in \Gamma(S, \overline{M}_S)$ such that $\bar{\alpha}_i \in N_i \otimes \Gamma(C, \overline{M}_C^{\mathrm{gp}})$ is constant on the interior of the circle of radius δ_i around the central vertex of $\sqsubset$.

(2) The slopes of $\bar{\beta}$ on the edges of $\sqsubset$ exiting the circle of radius δ_i span the kernel of $N_{i+1} \to N_i$ as a rational vector space.

The second observation requires a slight argument. By definition, the slopes of $\bar{\alpha}$ within the circle of radius δ_i span the kernel of $N \to N_i$ rationally. But if the edges inside the circle of radius δ_i together with those immediately exiting it spanned a smaller saturated subgroup of than the kernel of $N \to N_{i+1}$ then there would have been another δ_j in between δ_i and δ_{i+1}.

The iterative procedure. The map $\alpha : S \to \mathcal{W}(Z)$ gives families $C \leftarrow \tilde{C} \to \bar{C}$ where $\bar{C}$ is the contraction of the circle of radius δ. We can regard α as an element of $N \otimes \Gamma(\bar{C}, M_{\bar{C}}^{\mathrm{gp}})$. We examine extensions of these data to $C' \leftarrow \tilde{C}' \to \bar{C}'$ and $\alpha' \in N \otimes M_{\bar{C}'}^{\mathrm{gp}}$. The problem is addressed in two steps: first we choose a deformation of C (entailing deformations of $\tilde{C}$ and $\bar{C}$), which is an obstructed problem, and then we try to lift α, which can be obstructed for a fixed choice of C'. We then revise our choice of deformation C' to eliminate the obstruction to lifting α.

The choices of C' form a torsor under $H^1(C, T_{C/S}^{\log})$. We will adjust C' iteratively, lifting α_i to $\alpha_i' \in N_i \otimes \Gamma(\bar{C}', M_{\bar{C}'}^{\mathrm{gp}})$ based on an already selected lift of α_{i-1}. At each step, we will adjust C' by a section of $H^1(C, T_{C/S}^{\log})$ that vanishes on the interior of the circle of radius δ_i, thereby ensuring that our earlier choices are not broken by the later adjustments.

The obstruction group. Let N_i' be the kernel of $N_i \to N_{i-1}$. We indicate how $N_i' \otimes H^1(\bar{C}, \mathscr{O}_{\bar{C}})$) functions as an obstruction group to deforming α_i once α_{i-1} and C' are fixed.

By definition, the lifting problem (12) is equivalent to the problem of extending $\alpha \in N_i \otimes \Gamma(\bar{C}, M_{\bar{C}}^{\mathrm{gp}})$ to $\alpha' \in N_i \otimes \Gamma(\bar{C}', M_{\bar{C}'}^{\mathrm{gp}})$. Recall that α gives an invertible sheaf, $\mathscr{O}_{\bar{C}}(-f(\bar{\alpha}))$ for each $f \in N_i^\vee$, and $f(\alpha)$ is a nowhere vanishing global section of $\mathscr{O}_{\bar{C}}(-f(\bar{\alpha}))$. We will abuse notation slightly and think of $\mathscr{O}_{\bar{C}}(-\bar{\alpha})$ as a family of invertible sheaves indexed by $N_i^\vee$ and α as a trivialization of this family. Let $\bar{\alpha}'$ denote the unique extension of $\bar{\alpha} \in \Gamma(\bar{C}, \overline{M}_{\bar{C}}^{\mathrm{gp}})$ to $\overline{M}_{\bar{C}'}^{\mathrm{gp}}$. Our task is to extend α to a trivialization of $\mathscr{O}_{\bar{C}'}(\bar{\alpha}')$.

If it exists, an extension will necessarily be a trivialization, so the obstruction to the existence of an extension is the isomorphism class of the deformation $\mathscr{O}_{\bar{C}'}(\bar{\alpha}')$, which lies in $N_i \otimes H^1(\bar{C}, \mathscr{O}_{\bar{C}})$. By induction, the image of this obstruction in $N_i' \otimes H^1(\bar{C}, \mathscr{O}_{\bar{C}})$ vanishes, so our obstruction lies in $N_i' \otimes H^1(\bar{C}, \mathscr{O}_{\bar{C}})$.

Deformations of the curve. This obstruction may well be nonzero, but we are still free to choose C'. The choice of C' is a torsor under the deformation group $\mathrm{Def}(C) = H^1(C, T_{C/S}^{\log})$. This gives a homomorphism

$$H^1(C, T_{C/S}^{\log}) = \mathrm{Def}(C) \to \mathrm{Obs}_C(f) = N_i \otimes H^1(\bar{C}, \mathscr{O}_{\bar{C}}) \tag{13}$$

that we wish to show surjects onto $N_i' \otimes H^1(\bar{C}, \mathscr{O}_{\bar{C}})$. Once this is done, we can modify C' to eliminate the obstruction.

Since C is a curve, the formation of $H^1(C, T_{C/S}^{\log})$ and of $H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ commutes with base change. By Nakayama's lemma, we may therefore demonstrate the surjectivity of (13) by checking it on the fibers. We may therefore replace S with a geometric point and assume that S is the spectrum of an algebraically closed field.

We let $\overline{\sqsubset}$ be the dual graph of $\bar{C}$, where the interior of the circle of radius δ_i is treated as a single vertex. For each vertex v_i of $\overline{\sqsubset}$ other than the central vertex v_0, let e_j be the edge of $\overline{\sqsubset}$ that is closest to v_0 (in other words, the edge on which λ has negative slope when it is oriented away from v_j). Let p_j be the node of $\bar{C}$ corresponding to e_j. For $j \neq 0$, the corresponding component of $\bar{C}$ is rational, and therefore $T_{C_j}^{\log}$ has nonnegative degree. It follows that the maps

$$H^0(C_j, T_{C_j}^{\log}) \to H^0(p_j, T_{C_j}^{\log}) \quad \text{and} \quad H^0(C_j, \mathcal{O}_{C_j}) \to H^0(p_j, \mathcal{O}_{C_j})$$

are surjective for all $j \neq 0$. From the normalization sequence, we see that there are decompositions:

$$H^1(C, T_C^{\log}) = \bigoplus_j H^1(C_j, T_{C_j}^{\log}) \quad \text{and} \quad H^1(\bar{C}, \mathcal{O}_C) = \bigoplus_j H^1(C_j, \mathcal{O}_{C_j})$$

Since $H^1(C_j, \mathcal{O}_{C_j}) = 0$ for $j \neq 0$, it follows that we may reduce to the case $C = C_0$ and $\bar{C} = \bar{C}_0$. Note that in this case, $\bar{C}_0$ has no genus 0 tails.

The obstruction class. Now let $p_1, \dots, p_n$ be the external points of C corresponding to the edges $e_1, \dots, e_n$ of $\overline{\sqsubset}$ adjacent to the central vertex. Then $H^1(C, T_{C/S}^{\log})$ contains a copy of $\sum_j T_{p_j/C}$ corresponding to deformation of C as a logarithmic curve by moving the points p_j.

Let $a_j \in N_i'$ be the slope of $\bar{\alpha}$ on e_j and recall that the a_j span N_i' as a rational vector space. Then the obstruction class in $N_i' \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ is given by the following formula:

$$\mathcal{O}_{\bar{C}}(\bar{\alpha}) = \mathcal{O}_{\bar{C}}\Big(\sum a_j p_j\Big) \in N_i' \otimes H^1(\mathcal{O}_{\bar{C}})$$

Thus the obstruction map

$$\sum_{j=1}^{n} T_{p_j/C} \to H^1(C, T_{C/S}^{\log}) \to N_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}) \tag{14}$$

restricts to the tangent map $\sum T_{p_j/C} \to N_i' \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ considered in Corollary 3.5.3. Since the a_i span N_i' rationally and $\bar{C}$ has no genus 0 tails, that corollary implies the desired surjectivity. The points p_j all lie on the boundary of the circle of radius δ_i, so any class in $\sum T_{p_j/C} \subset H^1(C, T_{C/S}^{\log})$ vanishes on the interior of the circle of radius δ_i, as required. □

Remark 3.5.4. The proof shows a stronger smoothness property, because we were able to cancel obstructions using only deformations of the marked points without needing to smooth any of the singularities of $\bar{C}$.

It is a consequence of the proof that we can make following "codimension 1" characterization of the moduli space.

Proposition 3.5.5. *The substack of $\mathfrak{W}(Z)$ parametrizing stable radial maps that satisfy the factorization property for all subtori of codimension* 1 *coincides with the space $\mathcal{W}(Z)$ of well-spaced radial maps.*

Proof. It suffices to treat the case of $Z = \mathbb{G}_{\log}^n$. Assume that a map $C \to \mathbb{G}_{\log}^n$ fails to satisfy the factorization property for some subtorus H in T. In keeping with the notation of the previous proof, we let $N'_H \subset N$ be the associated cocharacter subspace, and let N_H be the quotient of N by N'_H. We demonstrate that the map fails to satisfy the factorization property for a codimension 1 subtorus.

If the map is obstructed, then the cokernel in (14) is nonzero. It follows from Corollary 3.5.3 that for the radius δ associated to the map $\sqsubset \to N_H$, the exiting edge directions a_i of $\sqsubset \to N \otimes \mathbb{R}$ at this radius do not span N'_H rationally. The obstruction class described in the proof above therefore lies in the cokernel

$$N_H / \operatorname{Span}(a_i) \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}).$$

Given any such obstruction class, we may find a projection by a character $N_H / \operatorname{Span}(a_i) \to \mathbb{R}$, such that the class remains nonzero, by projecting onto the 1-dimensional span of the obstruction class. This gives rise to a composition $N \to N_H \to \mathbb{R}$, and an associated map on tori $\mathbb{G}_{\log}^n \to \mathbb{G}_{\log}$. The induced map on logarithmic tangent bundle is given by extending scalars from the projection $N \to \mathbb{R}$. The obstruction to lifting the map to $\mathbb{G}_{\log}$ is the image of the obstruction to lifting the map to $\mathbb{G}_{\log}^n$, under the projection

$$N_H \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}) \to \mathbb{R} \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}).$$

The resulting radial map $C \to \mathbb{G}_{\log}$ therefore does not factorize, as the obstruction class is nonzero by construction. □

4. Realizability for genus one tropical curves

In this section, we use the geometry of the moduli spaces $\mathcal{W}(Z)$ constructed in Section 3 to resolve the tropical realizability problem in genus 1. The results of this section give a precise description of the boundary complex of $\mathcal{W}(Z)$. As a consequence of the smoothness and properness of $\mathcal{W}(Z)$, tropical realizability reduces to a *pointwise* calculation: we examine the unique nontopological condition characterizing the descent of a function from the normalization of a genus 1 singularity, and interpret it tropically as the realizability condition.

4.1. *Moduli of tropical maps.* Fix a pair of dual lattices N and $N^\vee$ of rank r and a complete fan Σ in the vector space $N_{\mathbb{R}}$.

Definition 4.1.1. The *combinatorial type* of a tropical stable map $[\sqsubset \xrightarrow{f} \Sigma]$ consists of:

(1) The finite graph model G underlying $\sqsubset$.

(2) For each vertex $v \in G$, the cone $\sigma_v \in \Sigma$ containing the image of v.

(3) For each edge e, the slope w_e and the primitive vector u_e of $f(e)$.

For tropical maps, the discrete data can be captured by the "least generic" map, defined below.

Definition 4.1.2. The *recession type* of a combinatorial type Θ is obtained from $[\sqsubset \to \Sigma]$ by collapsing all bounded edges of $\sqsubset$ to a single vertex, retaining the contact orders on the unbounded edges.

As explained in [Ranganathan 2017b, Section 2], once one fixes the recession type, there are finitely many combinatorial types of tropical stable maps with this recession type. This boundedness of combinatorial types is the essential content of [Gross and Siebert 2013, Section 3.1].

Given a type Θ, there is a polyhedral cone σ_Θ, whose relative interior parametrizes tropical stable maps with a fixed combinatorial type. This cone serves as a deformation space for maps of type Θ. In [Ranganathan 2017b, Section 2.2], a generalized cone complex $T_\Gamma(\Sigma)$ is constructed, by taking a colimit of the cones above over a natural gluing operation. This is a coarse moduli space for maps of fixed recession type.

Remark 4.1.3 (a moduli stack of tropical maps). It is possible to promote this construction to a fine moduli stack of tropical maps. By replacing the real edge lengths in $\sqsubset$ with monoid-valued edge lengths, one obtains an appropriate notion of a family of tropical stable maps over a cone σ. With this notion of family, the framework in [Cavalieri et al. 2017] produces a cone stack $\mathcal{T}_\Gamma(\Sigma)$, with well-defined evaluation morphisms. The addition of a marked point with trivial contact order functions as a universal curve in this context. We avoid further discussion of this for two reasons. First, we will not need the stacks directly in this work, and can make do with the less conceptually natural, but more concrete generalized cone complex. Second, and more importantly, the precise relationship between the analytification of the moduli space of maps — which coincides with the analytification of the coarse moduli space — remains unclear at present.

4.2. ***Traditional tropicalization and realizability.*** The tropicalization procedure discussed in the early parts of the paper uses the logarithmic structure, and differs from the one involving nonarchimedean geometry. Accounting for the difference is the *tropical realizability problem*, and is the focus of this final section.

Let K be a nonarchimedean field extending $\mathbb{C}$, where the latter is equipped with the trivial valuation. Let Y be a K-scheme or stack, locally of finite type. The *Berkovich analytification* Y^{an} is a locally compact, Hausdorff topological space whose points are naturally identified with equivalence classes of pairs (L, y) where L is a valued field extension of K and y is an L-valued point of Y. The equivalence is the one generated by identifying two such triples $(L, y) \sim (L', y')$ whenever there is an embedding of valued extensions $L \hookrightarrow L'$ sending y to y'. See [Berkovich 1990; Ulirsch 2017b; Yu 2018] for Berkovich spaces and stacks and [Abramovich et al. 2016] for an introduction to analytic spaces in the context of logarithmic geometry.

Given a torus $\mathbb{G}_m^r = \mathrm{Spec}(K[N^\vee])$, the *tropicalization* map is the continuous map

$$\mathrm{trop} : \mathbb{G}^n_{m,\mathrm{an}} \to N \otimes \mathbb{R},$$

that associates to an L-valued point of $\mathbb{G}_m^n$, its coordinatewise valuation. The tropicalization of a subvariety is defined by restriction.

Let $C \to \mathbb{G}_m^n$ be a map to a torus from a smooth curve of genus g. There is a natural factorization of topological spaces

$$\begin{array}{ccc} C^{\mathrm{an}} & \longrightarrow & \mathbb{G}^r_{m,\mathrm{an}} \\ \downarrow & & \downarrow \\ \sqsubset & \longrightarrow & \mathrm{trop}(C^{\mathrm{an}}) \subset \mathbb{R}^r, \end{array}$$

The left vertical map is a deformation retraction onto a *skeleton*; see [Baker et al. 2016] for details. There are at least two natural ways to extract the tropical curve $\sqsubset$ from $[C \to \mathbb{G}_m^r]$.

4.2.1. *Abstract stable reduction.* After choosing coordinates on the target, the map $[\varphi : C \to \mathbb{G}_m^n]$ is given by n invertible functions on C. Let $\hat{C}$ be the smooth projective model for C, and $q_1, \dots, q_n$ the points at which these invertible functions acquire zeros or poles. If the map φ is nonconstant, the pair $(\hat{C}, q_1, \dots, q_n)$ has negative Euler characteristic and thus admits a minimal model $\mathscr{C} \to \operatorname{Spec}(R)$ over the valuation ring of K. Take the underlying graph of $\sqsubset$ to be the dual graph of the special fiber of $\mathscr{C}$. Given an edge e of $\sqsubset$, the corresponding node q_e of $\mathscr{C}$ has a local equation

$$xy = \omega, \quad \omega \in R.$$

Set the length $\ell(e)$ equal to the valuation of the parameter ω.

4.2.2. *Universal property of minimality.* Let $\hat{C} \supset C$ be the projective model of C with boundary $\partial \hat{C} = \{q_1, \dots, q_n\}$ and choose a toric compactification Z of $\mathbb{G}_m^n$ such that the morphism

$$(\hat{C}, \partial \hat{C}) \to (Z, \partial Z).$$

is a logarithmic map. Letting $\mathscr{L}(Z)$ be the space of logarithmic stable maps to Z, this gives rise to a moduli map $\operatorname{Spec}(K) \to \mathscr{L}(Z)$, which, after a base change, extends to a map

$$\operatorname{Spec}(R) \to \mathscr{L}(Z),$$

from the valuation ring. Let k denote the residue field and Γ the value group. Consider the logarithmic map

$$\operatorname{Spec}(\Gamma \to k) \to \mathscr{L}(Z),$$

from the closed point, endowed with the (not necessarily coherent) logarithmic structure from the value group. By the universal property of minimality, this induces a factorization

$$\operatorname{Spec}(\Gamma \to k) \to \operatorname{Spec}(P^{\min} \to k),$$

where $P^{\min}$ is the stalk of the minimal monoid of $\mathscr{L}(Z)$ at the image of the closed point. We obtain a point of the dual cone $\operatorname{Hom}(P^{\min}, \Gamma)$, which, as was previously discussed, is identified with a point in the cone of tropical maps of a fixed combinatorial type. See [Ranganathan 2017b, Section 2] for details.

4.3. ***Expected dimension and superabundance.*** Every tropical stable map $[f]$ of combinatorial type Θ has a deformation space, the moduli cone σ_Θ. Superabundance is the phenomenon wherein this deformation space is larger than expected.

The *overvalence* of a type Θ with underlying graph G is defined as

$$\operatorname{ov}(\Theta) = \sum_{p \in G : \operatorname{val}(p) \geq 4} \operatorname{val}(p) - 3.$$

The overvalence allows us to determine an expected topological dimension of the tropical deformation space as:

$$\mathrm{exp.\,dim}\,\sigma_\Theta = (\dim(\Sigma) - 3)(1 - b_1(G)) + n - \mathrm{ov}(\Theta),$$

where $b_1(\sqsubset)$ is the first Betti number of G. The actual dimension of σ_Θ cannot be less than the expected dimension, but may exceed it. For further details, see [Mikhalkin 2005; Nishinou and Siebert 2006; Ranganathan 2017b].

Definition 4.3.1. A combinatorial type Θ is *superabundant* if the dimension of σ_Θ is strictly larger than the expected dimension.

4.3.2. *Superabundance as tropical obstructedness.* The deformation space of a map $[\varphi : C \to \mathbb{P}^r]$ can be larger than expected because deformations can be obstructed. The dimension of the deformation space can be estimated using Riemann–Roch and the tangent-obstruction complex [Hori et al. 2003, Section 24.4]. One examines the restrictions on the complex structure of the curve that are forced by the map. In some cases, such as when φ multiple covers its image or contracts a component, there are fewer such restrictions than expected.

The situation in tropical geometry is similar. Given a tropical stable map $[f : \sqsubset \to \Sigma]$ and a cycle of edges in $\sqsubset$, the piecewise linearity of f imposes restrictions on the edge lengths of this cycle. In particular, the edge lengths of a cycle are constrained by the condition that the total displacement around each cycle must vanish. If $\dim \Sigma = r$ the map is expected to impose r conditions on the edge lengths of $\sqsubset$ for each cycle, and the conditions imposed by different cycles are expected to be independent. However, if cycles are mapped to linear subspaces, or contracted altogether, there are fewer than the expected number of restrictions.

In genus 1, superabundance can be stated in a simplified form. In the following proposition, and the rest of the section, it will sometimes be convenient to forget the precise fan structure of Σ and consider the map of metric spaces $\sqsubset \to N_\mathbb{R}$.

Proposition 4.3.3. *Let $f : \sqsubset \to \Sigma$ be a tropical map from a tropical curve of genus* 1. *Then, f is superabundant if and only if the image of the circuit $\sqsubset_0$ is contained in a proper affine subspace of Σ. Equivalently, f is superabundant if and only if there exists a character $\chi : N_\mathbb{R} \to \mathbb{R}$ such that the circuit $\sqsubset_0$ is contracted under the composition*

$$\sqsubset \to N_\mathbb{R} \xrightarrow{\chi} \mathbb{R}$$

Proof. The first formulation is well known [Katz 2012; Ranganathan 2017b; Speyer 2014]. For the second, choose a hyperplane containing the circuit and quotient by it. □

4.4. ***Tropical realizability and well-spacedness.*** The tropical realizability problem is as follows.

Question 4.4.1. Given a tropical stable map $f : \sqsubset \to N_\mathbb{R}$, does there exist a smooth curve C over a nonarchimedean field K and a map

$$\varphi : C \to \mathbb{G}_m^r,$$

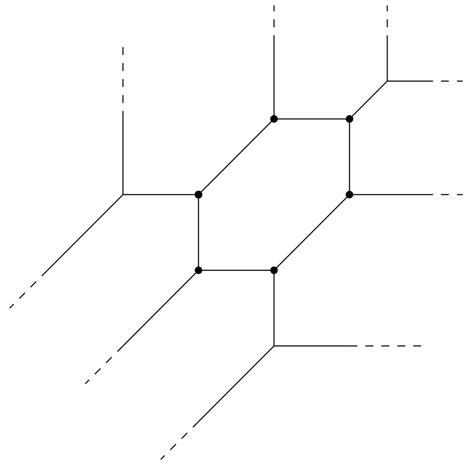


Figure 3. A tropical genus 1 curve in $\Sigma_{\mathbb{P}^2}$ of degree 3 with transverse contact orders. The curve is nonsuperabundant, as the edge directions of the circuit span $\mathbb{R}^2$.

such that $\varphi^{\mathrm{trop}} = f$?

Such a tropical map is said to be *realizable*. Superabundance is intimately related to realizability, as the following result shows. For proofs, see [Cheung et al. 2016; Ranganathan 2017b; Speyer 2014].

Theorem 4.4.2. *Let* $f : \sqsubset \to \Sigma$ *be a tropical stable map of genus* 1 *and combinatorial type* Θ. *If* $\sqsubset$ *has a vertex* v *of genus* 1, *then assume that the local map*

$$\mathrm{Star}(v) \to \Sigma$$

is realizable. If the combinatorial type Θ *is nonsuperabundant*, *then* f *is realizable.*

When a combinatorial type Θ is superabundant, there are additional constraints that are required to characterize the realizable locus.

A *flag* of a tropical curve $\sqsubset$ is a vertex v together with a choice of tangent direction along an edge incident to v. The vertex v will be referred to as the *base* of the flag. Given a piecewise-linear function f on a tropical curve $\sqsubset$, we may speak of the *slope* of f along a flag.

Definition 4.4.3. Let $\sqsubset$ be a tropical curve and let $\sqsubset_0$ be its circuit. Given a flag $t \in \sqsubset$, let $d(t, \sqsubset_0)$ be the distance from the circuit to the base of the flag. A tropical stable map

$$F : \sqsubset \to \mathbb{R}$$

of genus 1 is *well-spaced* if one of the following two conditions are met: either

(1) no open neighborhood of the circuit of $\sqsubset$ is contracted, or

(2) if a neighborhood of the circuit is contracted, let $t_1, \dots, t_k$ be the flags whose base is mapped to $F(\sqsubset_0)$ but along which F has nonzero slope. Then, the minimum of the distances $\{d(t_i, \sqsubset_0)\}_{i=1}^{k}$ occurs at least three times.

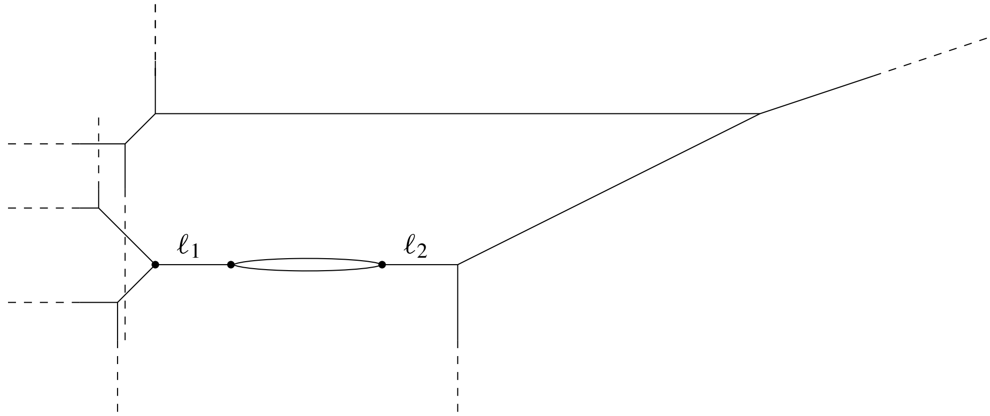


Figure 4. A superabundant tropical stable map to a Hirzebruch surface. The circuit is depicted to be flattened, indicating that its image is a line segment. Projection onto the vertical axis contracts the circuit. The curve is well-spaced if and only if $\ell_1 = \ell_2$.

Well-spacedness when the target is a general fan is formulated by considering projections to $\mathbb{R}$.

Definition 4.4.4. A tropical stable map $\sqsubset \to \Sigma$ of genus 1 is *well-spaced* if for each character

$$\chi : N_{\mathbb{R}} \to \mathbb{R},$$

the induced map $\sqsubset \to \mathbb{R}$ is well-spaced.

Warning 4.4.5. The condition we call well-spacedness is strictly weaker condition than the one given originally by Speyer. In particular, the definition allows that the set of flags with nonzero F-slope $\{t_i\}$ can all be based at the same vertex. In Speyer's definition, there must be distinct vertices achieving this minimum. It has already been shown that Speyer's condition is not a necessary condition in the nontrivalent case [Ranganathan 2017b, Theorem C]. The two definitions coincide when working with trivalent tropical curves whose vertex function is identically zero. To see this, observe that by the balancing condition, if a vertex supports one flag of nonzero F-slope. Thus, if two distinct vertices support flags with nonzero F-slope, then there are at least 4 such flags. We will refer to this stronger condition as *Speyer's condition*; see Figure 5.

Remark 4.4.6. We have chosen to state well-spacedness in terms of projections to 1-dimensional vector spaces, as this is closest to the existing versions of the condition present in the literature. A reader who wishes to see the parallelism with Section 3.3 one could instead impose an appropriate condition on the quotient by any real subspace of $N_{\mathbb{R}}$.

Remark 4.4.7. We make note of a consequence of this condition that is often useful in calculations. Let $\sqsubset \to N_{\mathbb{R}}$ be a tropical map from a genus 1 curve. Let $L \subset N_{\mathbb{R}}$ be the real span of the edge directions of the circuit of $\sqsubset$. Let δ be the minimal radius around the circuit such that the edge directions inside the circle of radius δ span a subspace L' strictly containing L. Let m the difference in dimensions of L

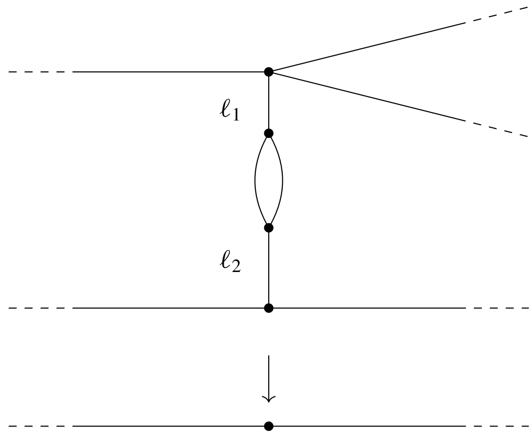


Figure 5. A tropical stable map that is well-spaced, but fails Speyer's condition. This map is well-spaced provided $\ell_1 \leq \ell_2$, as there are three flags with nonzero slope based at the point of minimum distance to the circuit. Speyer's condition forces the equality $\ell_1 = \ell_2$.

and L'. Then if the tropical map is well-spaced, then at the circle of radius δ, the curve $\sqsubset$ exits the circle along least $m+2$ flags.

This brings us to the main result of this section.

Theorem 4.4.8 (realizability of genus 1 tropical curves). *Let* $[\sqsubset \to \Sigma]$ *be a tropical stable map of genus* 1, *and assume there is a minimal logarithmic map* $[C \to Z]$ *whose combinatorial type is that of* $[\sqsubset \to \Sigma]$. *Then* $[\sqsubset \to \Sigma]$ *is realizable if and only if it is well-spaced.*

The proof will be completed in Section 4.6 after we establish some preliminaries in Section 4.5.

4.5. ***Moduli of well-spaced tropical stable maps.*** Let $T_\Gamma(\Sigma)$ be the moduli space of genus 1 tropical stable maps with a fixed recession type $[\Gamma \to \Sigma]$. We abuse notation by understanding that the map to Σ is part of the notation Γ. The well-spacedness condition commutes with automorphisms of tropical curves, and thus descends to a well-defined subset $W_\Gamma(\Sigma)$ of well-spaced tropical stable maps. We specify a subdivision of $T_\Gamma(\Sigma)$ such that $W_\Gamma(\Sigma)$ becomes an equidimensional subcomplex of the expected dimension.

Definition 4.5.1. A *radially aligned* combinatorial type for a genus 1 tropical stable map is a combinatorial type Θ for a tropical stable map, together with a choice of a reflexive and transitive binary relation $\preccurlyeq$ on the vertices such that, if P is a path from the circuit of G to a vertex v that passes a vertex v', then we have the relation

$$v' \preccurlyeq v.$$

Such a *radial combinatorial type* will be denoted $(\Theta, \preccurlyeq)$.

4.5.2. *Constructing the tropical moduli space.* Let $\tilde{W}_\Gamma(\Sigma)$ denote the coarse moduli space of radially aligned tropical stable maps with recession type Γ. Given a radial combinatorial type $(\Theta, \preccurlyeq)$, tropical maps of this type are parametrized by a face of a subdivision of the moduli cone σ_Θ. There is a specialization relation among ordered combinatorial types: a cone $\sigma_{(\Theta', \preccurlyeq')}$ is a face of a cone $\sigma_{(\Theta, \preccurlyeq)}$ if and only if the following conditions hold:

(1) Let G and G' be the underlying graphs of Θ and Θ' respectively. Then, G' is obtained from G by a (possibly trivial) sequence of edge contractions $\alpha : G \to G'$.

(2) The edge contraction $G \to G'$ is order preserving: if $v \preccurlyeq w$ then $\alpha(v) \preccurlyeq \alpha(w)$.

(3) If $v' \in G'$ is a vertex with $\alpha(v') = v \in G$, then the cone $\sigma_{v'}$ is a face of σ_v.

Let $W_\Gamma(\Sigma)$ be the subcomplex of $\tilde{W}_\Gamma(\Sigma)$ parametrizing well-spaced radial tropical maps.

Lemma 4.5.3. *The locus $W_\Gamma(\Sigma)$ is a subcomplex of $\tilde{W}_\Gamma(\Sigma)$, and thus, is itself a generalized cone complex.*

Proof. The well-spacedness condition can be described in terms of the equality of the vertices at minimum distance from the circuit, and thus form a cone of the generalized cone complex. The result follows immediately from this observation. □

Remark 4.5.4. A close relative of the space $W_\Gamma(\Sigma)$ appears in the thesis of Carolin Torchiani, namely the dense open set of $W_\Gamma(\Sigma)$ parametrizing curves with identically zero genus function. In particular, it is proved that this subcomplex is pure-dimensional of the expected dimension [Torchiani 2014, Theorem 3.2.10]. It follows from this that $W_\Gamma(\Sigma)$ is also pure-dimensional. In particular, we consider the following. It would be interesting to examine the fine structure of $W_\Gamma(\Sigma)$ further. What can one say, for instance, about its homotopy type and connectivity properties?

4.6. ***Proof of Theorem 4.4.8.*** We know from Section 3.5 that the moduli space of well-spaced logarithmic stable maps $\mathcal{W}_\Gamma(Z)$ is proper and smooth. By definition, it is the locus of stable maps in $\mathfrak{W}_\Gamma(Z)$ that satisfy the factorization property for every subtorus of the dense torus of Z. Our task is to show that the logarithmic well-spacedness condition is equivalent to the tropical well-spacedness condition. By Proposition 3.5.5, the logarithmic well-spacedness condition is the conjunction of the factorization properties for all 1-dimensional quotients of Z. Since the tropical well-spacedness condition was formulated in terms of 1-dimensional quotients, it suffices to check the equivalence for every subtorus H in Z of codimension 1. Replacing Z with a modification and passing to the quotient, our obligation reduces to checking that *a tropical map $\sqsubset \to \mathbb{R}$, in which all vertices of $\sqsubset$ have genus 0, is well-spaced if and only if it is the tropicalization of a radial map $C \to \mathbb{P}^1$ satisfying the factorization property.*

Let C be a logarithmic curve with tropicalization $\sqsubset$. The map $[\sqsubset \to \mathbb{R}]$ induces a destabilization $\upsilon : \tilde{C} \to C$ and a contraction $\tau : \tilde{C} \to \bar{C}$. The map itself can be regarded as a section $\bar{\alpha}$ of $\overline{M}_C^{\mathrm{gp}}$. This pulls back to $\overline{M}_{\tilde{C}}^{\mathrm{gp}}$ and then descends to $\overline{M}_{\bar{C}}^{\mathrm{gp}}$, since it is constant on the components collapsed by τ. Adding a constant to $\bar{\alpha}$ does not change whether it is well-spaced in either the logarithmic or the tropical sense, so we assume that $\bar{\alpha}$ takes the value 0 on the circuit component of $\bar{C}$.

We must show that $\bar{\alpha}$ lifts to a section α of $M_{\bar{C}}^{\mathrm{gp}}$ if and only if $\sqsubset \to \mathbb{R}$ is well-spaced. Indeed, if α is a section of $M_{\bar{C}}^{\mathrm{gp}}$ then $\upsilon_\star \tau^\star \alpha$ is a section of $\upsilon_\star M_{\tilde{C}}^{\mathrm{gp}} = M_C^{\mathrm{gp}}$ by [Abramovich et al. 2014, Appendix B], and gives a map $C \to \mathbb{P}^1$ with the factorization property.

Let E denote the circuit component of $\bar{C}$ and E° its interior, excluding the nodes where E is joined to the rest of $\bar{C}$ (in other words, the locus in E where the logarithmic structure is pulled back from the base). Since $\bar{\alpha}(E) = 0$, the lift $\alpha|_{E^\circ}$, if it exists, will be in $\mathcal{O}_{E^\circ}^\star \subset M_{E^\circ}$. Regarded as a rational function on E, this lift must have zeroes and poles along the points of attachment between E and the rest of $\bar{C}$ as specified by the outgoing slopes of $\bar{\alpha}$ along the corresponding edges (see Section 2.7). Once $\alpha|_{E^\circ}$ has been found, there is no obstruction to extending it to all of $\bar{C}$, since the rest of the curve is a forest of rational curves and $\bar{\alpha}$ is balanced. The following lemma determines whether $\alpha|_{E^\circ}$ can be found, and completes the proof of the theorem.

Lemma 4.6.1. *Let E be a Gorenstein, genus 1 curve with no nodes and m branches, let $a_1, \dots, a_n$ be nonzero integers, and let P be a partition of $1, \dots, n$ into m parts. Assume that, for each $p \in P$, we have $\sum_{i \in p} a_i = 0$. If $n \geq 3$ then there is a configuration of distinct points $x_1, \dots, x_n$ on E, with each point lying in the component corresponding to its part of the partition, such that $\mathcal{O}_E(\sum a_i x_i)$ is trivial. If $n = 2$ then there is no such configuration.*

Proof. Let $\nu : F \to E$ be the seminormalization and let ω_E be the dualizing sheaf. For any configuration of the x_i, subject to the degree constraint in the statement, there is a rational function f on F with divisor $\sum a_i x_i$, and f is unique up to scaling. We wish to determine whether f descends to E.

Let $y \in F$ be the preimage of the singular point of E and let ϕ be a nonzero global differential on E. Let F_j be the components of F and let $\nu_j : F_j \to E$ be the restrictions of ν and let f_j be the restriction of f to F_j. Let t_j be a local parameter for F_j at y and let b_j be the linear term of the expansion of f_j in terms of t_j. It was shown in Section 2.1 that there are nonzero constants c_j such that f descends to E if and only if

$$\sum_j c_j b_j = 0. \tag{15}$$

We argue that under the hypothesis of the lemma it is possible to configure the x_i on each component F_j to make b_j take any value we like. Indeed, if we decide $f(y)$ should be 1 then f_j has the formula

$$f_j = \prod_i (1 - x_i^{-1} t_j)^{a_i}$$

with the product taken over those i such that x_i lies on F_j. The linear part is

$$b_j = -\sum a_i x_i^{-1}.$$

By adjusting the positions of the x_i, we can arrange for b_j to have any nonzero value we like. If p_j consists of at least 3 points x_i then it is possible to achieve any value for b_j, including 0, but if p_j consists of only two points, x_i and $x_{i'}$ then $a_{i'} = -a_i$ and it is impossible for b_j to take the value 0.

Thus we can solve (15) provided either that there are at least two branches at y or there is one branch containing at least 3 of the x_i. The one remaining case is where there is one branch containing 2 points. In that case, $c_1 \neq 0$ and the remarks in the last paragraph show there is no solution to (15). ☐

The above result determines the dual complex of the space $\mathcal{W}_\Gamma(Z)$, and we obtain the following as a consequence of general structural results about tropicalizations of logarithmic schemes. Let $\mathcal{W}_\Gamma^\circ(Z)$ denote the locus of maps with trivial logarithmic structure.

Theorem 4.6.2. *There is a continuous tropicalization map*

$$\mathrm{trop} : \mathcal{W}_\Gamma^{\circ,\mathrm{an}}(Z) \to W_\Gamma(\Sigma),$$

functorial with respect to evaluation morphisms and forgetful morphisms to the moduli space of curves. Set theoretically, this map sends a family of logarithmic stable maps to its tropicalization. There is a factorization

$$\begin{array}{ccc} \mathcal{W}_\Gamma^{\circ,\mathrm{an}}(Z) & \xrightarrow{\quad\mathrm{trop}\quad} & W_\Gamma(\Sigma) \\ & {\scriptstyle \boldsymbol{p}}\searrow \qquad \nearrow{\scriptstyle \mathrm{trop}_{\mathfrak{S}}} & \\ & \mathrm{P}_\Gamma(\Sigma), & \end{array}$$

where the map $\boldsymbol{p}$ is a deformation retraction onto a generalized cone complex, and admits a canonical continuous section. The map $\mathrm{trop}_{\mathfrak{S}}$ *is finite and is an isomorphism of cones upon restriction to any face.*

Proof. With the identification of the tropical maps that arise as tropicalizations of one-parameter families, the proof of the result is a cosmetic variation on similar results in the literature [Cavalieri et al. 2016; Ranganathan 2017a; 2017b]. By Theorem 4.4.8, the tropicalization of any family of logarithmic stable maps over a valuation ring is well-spaced. Once this is established, the continuity, functoriality, and finiteness of $\mathrm{trop}_{\mathfrak{S}}$ follow from [Ranganathan 2017b, Theorem 2.6.2] and the uniqueness of minimal morphisms of logarithmic schemes up to saturation [Wise 2016b]. The saturation index of a combinatorial type $(\Theta, \preccurlyeq)$ is equal to the cardinality of the fibers of $\mathrm{trop}_{\mathfrak{S}}$, as explained in [Ranganathan 2017a; 2017b]. Since $\mathcal{W}_\Gamma(Z)$ is a toroidal compactification, the existence of a section from the skeleton follows from [Abramovich et al. 2015; Thuillier 2007]. Compatibility with forgetful and evaluation morphisms follows from [Ulirsch 2017a, Theorem 1.1]. ☐

Acknowledgements

Thanks are due to Dan Abramovich, Dori Bejleri, Sebastian Bozlee, Renzo Cavalieri, Dave Jensen, Eric Katz, Diane Maclagan, Davesh Maulik, David Speyer, Jeremy Usatine, and Ravi Vakil for helpful conversations and much encouragement. Special thanks are due to Sam Payne, who first explained the well-spacedness condition to the first author, and pointed out that a deformation theoretic understanding of the phenomenon was likely to be valuable. Luca Battistella and Navid Nabijou kindly pointed out errors in an earlier version of the paper, and we are grateful to them. The referee also provided valuable feedback. Finally, the authors learned that they were working on related ideas after a seminar talk of the first author at Colorado State University when the third author was in the audience; we thank Renzo Cavalieri for creating that opportunity.

Funding. The first author was partially supported by NSF grant DMS-1128155 (Institute for Advanced Study) and the third author was partially supported by NSA Young Investigator's Grants H98230-14-1-0107 and H98230-16-1-0329.

References

[Abramovich and Chen 2014] D. Abramovich and Q. Chen, "Stable logarithmic maps to Deligne–Faltings pairs, II", *Asian J. Math.* **18**:3 (2014), 465–488. MR Zbl

[Abramovich and Wise 2018] D. Abramovich and J. Wise, "Birational invariance in logarithmic Gromov–Witten theory", *Compos. Math.* **154**:3 (2018), 595–620. MR Zbl

[Abramovich et al. 2014] D. Abramovich, S. Marcus, and J. Wise, "Comparison theorems for Gromov–Witten invariants of smooth pairs and of degenerations", *Ann. Inst. Fourier* (*Grenoble*) **64**:4 (2014), 1611–1667. MR Zbl

[Abramovich et al. 2015] D. Abramovich, L. Caporaso, and S. Payne, "The tropicalization of the moduli space of curves", *Ann. Sci. École Norm. Sup.* (4) **48**:4 (2015), 765–809. MR Zbl

[Abramovich et al. 2016] D. Abramovich, Q. Chen, S. Marcus, M. Ulirsch, and J. Wise, "Skeletons and fans of logarithmic structures", pp. 287–336 in *Nonarchimedean and tropical geometry* (St. John, USVI, 2013/Puerto Rico, 2015), edited by M. Baker and S. Payne, Springer, 2016. MR Zbl

[Abramovich et al. 2017a] D. Abramovich, Q. Chen, M. Gross, and B. Siebert, "Decomposition of degenerate Gromov–Witten invariants", preprint, 2017. arXiv

[Abramovich et al. 2017b] D. Abramovich, Q. Chen, S. Marcus, and J. Wise, "Boundedness of the space of stable logarithmic maps", *J. Eur. Math. Soc.* **19**:9 (2017), 2783–2809. MR Zbl

[Altman and Kleiman 1970] A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Math. **146**, Springer, 1970. MR Zbl

[Baker et al. 2016] M. Baker, S. Payne, and J. Rabinoff, "Nonarchimedean geometry, tropicalization, and metrics on curves", *Algebr. Geom.* **3**:1 (2016), 63–105. MR Zbl

[Battistella et al. 2019] L. Battistella, N. Nabijou, and D. Ranganathan, "Curve counting in genus one: elliptic singularities and relative geometry", preprint, 2019. arXiv

[Berkovich 1990] V. G. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields*, Math. Surv. Monogr. **33**, Amer. Math. Soc., Providence, RI, 1990. MR Zbl

[Bertrand et al. 2014] B. Bertrand, E. Brugallé, and G. Mikhalkin, "Genus 0 characteristic numbers of the tropical projective plane", *Compos. Math.* **150**:1 (2014), 46–104. MR Zbl

[Cavalieri et al. 2010] R. Cavalieri, P. Johnson, and H. Markwig, "Tropical Hurwitz numbers", *J. Algebraic Combin.* **32**:2 (2010), 241–265. MR Zbl

[Cavalieri et al. 2016] R. Cavalieri, H. Markwig, and D. Ranganathan, "Tropicalizing the space of admissible covers", *Math. Ann.* **364**:3-4 (2016), 1275–1313. MR Zbl

[Cavalieri et al. 2017] R. Cavalieri, M. Chan, M. Ulirsch, and J. Wise, "A moduli stack of tropical curves", preprint, 2017. arXiv

[Chen 2014] Q. Chen, "Stable logarithmic maps to Deligne–Faltings pairs, I", *Ann. of Math.* (2) **180**:2 (2014), 455–521. MR Zbl

[Cheung et al. 2016] M.-W. Cheung, L. Fantini, J. Park, and M. Ulirsch, "Faithful realizability of tropical curves", *Int. Math. Res. Not.* **2016**:15 (2016), 4706–4727. MR Zbl

[Gillam 2012] W. D. Gillam, "Logarithmic stacks and minimality", *Int. J. Math.* **23**:7 (2012), art. id. 1250069. MR Zbl

[Gross 2016] A. Gross, "Correspondence theorems via tropicalizations of moduli spaces", *Commun. Contemp. Math.* **18**:3 (2016), art. id. 1550043. MR Zbl

[Gross 2018] A. Gross, "Intersection theory on tropicalizations of toroidal embeddings", *Proc. Lond. Math. Soc.* (3) **116**:6 (2018), 1365–1405. MR Zbl

[Gross and Siebert 2013] M. Gross and B. Siebert, "Logarithmic Gromov–Witten invariants", *J. Amer. Math. Soc.* **26**:2 (2013), 451–510. MR Zbl

[Helm and Katz 2012] D. Helm and E. Katz, "Monodromy filtrations and the topology of tropical varieties", *Canad. J. Math.* **64**:4 (2012), 845–868. MR Zbl

[Hori et al. 2003] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay Math. Monogr. **1**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Jensen and Ranganathan 2017] D. Jensen and D. Ranganathan, "Brill–Noether theory for curves of a fixed gonality", preprint, 2017. arXiv

[Kato 1989] K. Kato, "Logarithmic structures of Fontaine–Illusie", pp. 191–224 in *Algebraic analysis, geometry, and number theory* (Baltimore, 1988), edited by J.-I. Igusa, Johns Hopkins Univ. Press, Baltimore, 1989. MR Zbl

[Kato 2000] F. Kato, "Log smooth deformation and moduli of log smooth curves", *Int. J. Math.* **11**:2 (2000), 215–232. MR Zbl

[Katz 2012] E. Katz, "Lifting tropical curves in space and linear systems on graphs", *Adv. Math.* **230**:3 (2012), 853–875. MR Zbl

[Len and Ranganathan 2018] Y. Len and D. Ranganathan, "Enumerative geometry of elliptic curves on toric surfaces", *Israel J. Math.* **226**:1 (2018), 351–385. MR Zbl

[Mandel and Ruddat 2016] T. Mandel and H. Ruddat, "Descendant log Gromov–Witten invariants for toric varieties and tropical curves", 2016. To appear in *Trans. Amer. Math. Soc.* arXiv

[Mikhalkin 2005] G. Mikhalkin, "Enumerative tropical algebraic geometry in $\mathbb{R}^2$", *J. Amer. Math. Soc.* **18**:2 (2005), 313–377. MR Zbl

[Molcho and Wise 2018] S. Molcho and J. Wise, "The logarithmic Picard group and its tropicalization", preprint, 2018. arXiv

[Nishinou 2009] T. Nishinou, "Correspondence theorems for tropical curves, I", preprint, 2009. arXiv

[Nishinou and Siebert 2006] T. Nishinou and B. Siebert, "Toric degenerations of toric varieties and tropical curves", *Duke Math. J.* **135**:1 (2006), 1–51. MR Zbl

[Parker 2011] B. Parker, "Gromov–Witten invariants of exploded manifolds", preprint, 2011. arXiv

[Ranganathan 2017a] D. Ranganathan, "Skeletons of stable maps, I: Rational curves in toric varieties", *J. Lond. Math. Soc.* (2) **95**:3 (2017), 804–832. MR Zbl

[Ranganathan 2017b] D. Ranganathan, "Skeletons of stable maps, II: Superabundant geometries", *Res. Math. Sci.* **4** (2017), art. id. 11. MR Zbl

[Ranganathan 2017c] D. Ranganathan, "Superabundant curves and the Artin fan", *Int. Math. Res. Not.* **2017**:4 (2017), 1103–1115. MR Zbl

[Ranganathan 2019] D. Ranganathan, "Logarithmic Gromov–Witten theory with expansions", preprint, 2019. arXiv

[Ranganathan and Wise 2019] D. Ranganathan and J. Wise, "Rational curves in the logarithmic multiplicative group", 2019. To appear in *Proc. Amer. Math. Soc.* arXiv

[Ranganathan et al. 2017] D. Ranganathan, K. Santos-Parker, and J. Wise, "Moduli of stable maps in genus one and logarithmic geometry, I", 2017. To appear in *Geom. Topol.* arXiv

[SGA 7_{I} 1972] D. S. Rim, "Formal deformation theory", pp. 32–132 in *Groupes de monodromie en géométrie algébrique, I: Exposés I–II, VI–IX* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969), edited by A. Grothendieck, Lecture Notes in Math. **288**, Springer, 1972. MR Zbl

[Smyth 2011] D. I. Smyth, "Modular compactifications of the space of pointed elliptic curves, I", *Compos. Math.* **147**:3 (2011), 877–913. MR Zbl

[Speyer 2005] D. E. Speyer, *Tropical geometry*, Ph.D. thesis, University of California, Berkeley, 2005, Available at https://search.proquest.com/docview/305031446.

[Speyer 2014] D. E. Speyer, "Parameterizing tropical curves, I: Curves of genus zero and one", *Algebra Number Theory* **8**:4 (2014), 963–998. MR Zbl

[Tevelev 2007] J. Tevelev, "Compactifications of subvarieties of tori", *Amer. J. Math.* **129**:4 (2007), 1087–1104. MR Zbl

[Thuillier 2007] A. Thuillier, "Géométrie toroïdale et géométrie analytique non archimédienne: application au type d'homotopie de certains schémas formels", *Manuscripta Math.* **123**:4 (2007), 381–451. MR Zbl

[Torchiani 2014] C. Torchiani, *Enumerative geometry of rational and elliptic tropical curves in* $\mathbb{R}^m$, Ph.D. thesis, Technische Universität Kaiserslautern, 2014, Available at https://tinyurl.com/torchiphd.

[Tyomkin 2012] I. Tyomkin, "Tropical geometry and correspondence theorems via toric stacks", *Math. Ann.* **353**:3 (2012), 945–995. MR Zbl

[Ulirsch 2017a] M. Ulirsch, "Functorial tropicalization of logarithmic schemes: the case of constant coefficients", *Proc. Lond. Math. Soc.* (3) **114**:6 (2017), 1081–1113. MR Zbl

[Ulirsch 2017b] M. Ulirsch, "Tropicalization is a non-archimedean analytic stack quotient", *Math. Res. Lett.* **24**:4 (2017), 1205–1237. MR Zbl

[Vakil and Zinger 2008] R. Vakil and A. Zinger, "A desingularization of the main component of the moduli space of genus-one stable maps into $\mathbb{P}^n$", *Geom. Topol.* **12**:1 (2008), 1–95. MR Zbl

[Wise 2011] J. Wise, "Obstruction theories and virtual fundamental classes", preprint, 2011. arXiv

[Wise 2016a] J. Wise, "Moduli of morphisms of logarithmic schemes", *Algebra Number Theory* **10**:4 (2016), 695–735. MR Zbl

[Wise 2016b] J. Wise, "Uniqueness of minimal morphisms of logarithmic schemes", preprint, 2016. arXiv

[Yu 2018] T. Y. Yu, "Gromov compactness in non-archimedean analytic geometry", *J. Reine Angew. Math.* **741** (2018), 179–210. MR Zbl

Communicated by Brian Conrad
Received 2017-09-01 Revised 2019-05-11 Accepted 2019-07-04

dr508@cam.ac.uk *Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, United Kingdom*

parkerks@med.umich.edu *Medical School, University of Michigan, Ann Arbor, MI, United States*

jonathan.wise@colorado.edu *Department of Mathematics, University of Colorado, Boulder, CO, United States*

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1807

Multiplicity one for wildly ramified representations

Daniel Le

Let F be a totally real field in which p is unramified. Let $\bar{r} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a modular Galois representation which satisfies the Taylor–Wiles hypotheses and is generic at a place v above p. Let $\mathfrak{m}$ be the corresponding Hecke eigensystem. We show that the $\mathfrak{m}$-torsion in the mod p cohomology of Shimura curves with full congruence level at v coincides with the $\mathrm{GL}_2(k_v)$-representation $D_0(\bar{r}|_{G_{F_v}})$ constructed by Breuil and Paškūnas. In particular, it depends only on the local representation $\bar{r}|_{G_{F_v}}$, and its Jordan–Hölder factors appear with multiplicity one. This builds on and extends work of the author with Morra and Schraen and, independently, Hu–Wang, which proved these results when $\bar{r}|_{G_{F_v}}$ was additionally assumed to be tamely ramified. The main new tool is a method for computing Taylor–Wiles patched modules of integral projective envelopes using multitype tamely potentially Barsotti–Tate deformation rings and their intersection theory.

1. Introduction

Let $F/\mathbb{Q}$ be a totally real field which is unramified at a rational prime p. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$. Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ is a Galois representation occurring in the $\mathbb{F}$-cohomology of a Shimura curve $X_{/F}$ with corresponding Hecke eigensystem $\mathfrak{m}$ (see Section 5). Suppose that the corresponding quaternion algebra splits at p. Let v be a place of F dividing p, let K^v be a compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$ and $K_v(n)$ the n-th principal congruence subgroup at v. One expects that the analogues of the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and mod p local-global compatibility for $\mathrm{GL}_2(\mathbb{Q})$ describe the $\mathrm{GL}_2(F_v)$-representation

$$\pi' = \mathrm{Hom}_{G_F}(\bar{r}, \varinjlim_n H^1(X(K^v K_v(n)), \mathbb{F})[\mathfrak{m}_{\bar{r}}])$$

in the completed cohomology of X, at least up to multiplicities, in terms of $\bar{\rho} \overset{\text{def}}{=} \bar{r}|_{G_{F_v}}$. In fact, we study a related representation $\pi = (M^{\min})^*$ (see Section 5), which is minimal with respect to multiplicities. Such analogues are unknown at present, although [Breuil 2014; Emerton et al. 2015] show that if $\bar{r}$ satisfies the usual Taylor–Wiles hypotheses and $\bar{\rho}$ is generic, then π contains one of infinitely many $\mathrm{GL}_2(F_v)$-representations constructed by [Breuil and Paškūnas 2012]. The idea, as explained in [Breuil 2014], behind the constructions of [Breuil and Paškūnas 2012] is that if one can show that the restriction of π to the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_{F_v})$ satisfies certain multiplicity one properties, then π

MSC2010: 11S37.
Keywords: Galois deformations, mod p Langlands program.

must contain a Diamond diagram of the form $D(\bar{\rho}, \iota)$. These multiplicity one properties, which one might view as minimalist conjectures for multiplicities, were established in [Emerton et al. 2015].

That the family of representations containing a diagram $D(\bar{\rho}, \iota)$ is infinite is unfortunate and warrants further investigation of π. One part of a Diamond diagram $D(\bar{\rho}, \iota)$ is a $\mathrm{GL}_2(k_v)$-representation denoted $D_0(\bar{\rho})$, which is a subrepresentation of $\pi|_{\mathrm{GL}_2(\mathcal{O}_{F_v})}$ (see [Breuil 2014, Proposition 9.3]), and thus a subrepresentation of the invariants of π under the first principal congruence subgroup $K_v(1)$ of $\mathrm{GL}_2(\mathcal{O}_{F_v})$. Our main result is the following:

Theorem 1.1 (Corollary 5.2). *If $\bar{r}$ satisfies the Taylor–Wiles hypotheses and $\bar{\rho}$ is generic, see Definition 4.1, then the $\mathrm{GL}_2(k_v)$-representation $\pi^{K_v(1)}$ is isomorphic to $D_0(\bar{\rho})$. In particular, it only depends on $\bar{\rho}$ and is multiplicity free.*

One can view this result as showing that π satisfies a minimality property: $\pi^{K_v(1)}$ is as small as possible. A similar result has been announced by Hu–Wang.

The main tool in the proof of Theorem 1.1 is the Taylor–Wiles patching method. Diamond [1997] and Fujiwara [2006] discovered that the Cohen–Macaulay property of patched modules could be combined with local algebra results of Auslander, Buchsbaum, and Serre to rederive and generalize mod p multiplicity one results of Mazur for modular forms with level away from p. Emerton et al. [2015] proved similar results for modular forms with level at p by introducing two gluing methods to calculate patched modules from smaller ones to which the Diamond–Fujiwara trick applied. The first method is a version of Nakayama's lemma and uses the submodule structure of mod p reductions of Deligne–Lusztig representations. The second method combines the submodule structure above with the intersection theory of special fibers of tamely potentially Barsotti–Tate deformation rings.

When $\bar{\rho}$ is tamely ramified, [Hu and Wang 2018; Le et al. 2016b] show that the patched modules of projective envelopes of irreducible $\mathbb{F}[\mathrm{GL}_2(k_v)]$-modules are cyclic modules by describing the submodule structure of these projective envelopes and using the Nakayama method of [Emerton et al. 2015] (see Proposition 4.6). However, the gluing methods of [loc. cit.] are insufficient when $\bar{\rho}$ is wildly ramified. Indeed, these methods only glue together characteristic p patched modules, but when $\bar{\rho}$ is wildly ramified there is more than one isomorphism class of $\mathbb{F}[\mathrm{GL}_2(k_v)]$-modules satisfying the multiplicity one properties for $\pi^{K_v(1)}$ established in [loc. cit.].

We introduce a variant of the intersection theory method of [loc. cit.], which uses the intersection theory of integral tamely potentially Barsotti–Tate deformation rings. Let $W(\mathbb{F})$ denote the Witt vectors of $\mathbb{F}$. The first step (Proposition 4.6) is to show that the methods of [loc. cit.] still apply to certain quotients of generic $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-projective envelopes (which are projective envelopes in the abelian category of $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-modules generated by lattices in some fixed set of Deligne–Lusztig representations). If such a quotient is reducible rationally, then it can be written as a submodule of the direct sum of two smaller quotients with p-torsion cokernel (see Proposition 2.4). This reflects a kind of transversality: while these subcategories do not give a direct product decomposition of the category of $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-modules, if two subquotients of lattices in two distinct Deligne–Lusztig representations are isomorphic, they must be p-torsion. By exactness of patching and this exact sequence, it turns out that the patched modules of $W(\mathbb{F})[\mathrm{GL}_2(k_v)]$-projective envelopes are then determined by the patched modules of these quotients (this depends crucially on the fact that all such patched modules turn out to be cyclic).

It remains to actually compute these patched modules using intersection theory in a multitype Barsotti–Tate framed deformation space, which we define to be the Zariski closure in the unrestricted framed deformation space of $\bar{\rho}$ of potentially Barsotti–Tate Galois representations with tame inertial type in some fixed set. That the resulting patched module is cyclic comes from the fact that the multitype Barsotti–Tate deformation rings exhibit a similar kind of transversality: two lattices in potentially Barsotti–Tate Galois representations of two distinct generic tame inertial types can be congruent modulo p, but never modulo p^2.

We now give a brief overview of the following sections. In Section 2, we generalize some of the results of [Le et al. 2016b] and prove the key result (Proposition 2.4) gluing integral projective envelopes from their quotients. In Section 3, we define and calculate multitype Barsotti–Tate deformation rings — this is the other key technical input. To compare Kisin modules for varying tame types, it is much more convenient to choose eigenbases for Kisin modules which are not always gauge bases in the sense of [Emerton et al. 2015, Section 7.3]. This requires generalizing [Le et al. 2018, Theorem 4.1]. The main result, Theorem 3.6, of this section computes some multitype Barsotti–Tate framed deformation spaces. In Section 4, we calculate the abstract patched modules of projective envelopes using the Nakayama method and our integral intersection theory method. In Section 5, we apply the results of Section 4 to the cohomology of Shimura curves using the Taylor–Wiles method.

1A. ***Notation.*** If F is any field, we write $\overline{F}$ for a separable closure of F and $G_F := \mathrm{Gal}(\overline{F}/F)$ for the absolute Galois group of F.

Let $f \in \mathbb{N}$ and $q = p^f$. Let $\mathcal{O}_K$ be the Witt vectors $W(\mathbb{F}_q)$ of $\mathbb{F}_q$. Let $K = \mathcal{O}_K[p^{-1}]$ be the unramified extension of $\mathbb{Q}_p$ of degree f. Let E be an extension of K with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbb{F}$. This induces embeddings $\mathcal{O}_K \hookrightarrow \mathcal{O}$ and $\iota_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$. For $i \in \mathbb{Z}/f$, let $\iota_i = \iota_0 \circ \varphi^i$ be the i-th Frobenius twist of ι_0. We fix an embedding $\mathbb{F} \hookrightarrow \overline{\mathbb{F}}_q$. We will denote by $(\cdot)^*$ the $\mathbb{F}$-linear dual, and by $(\cdot)^\vee$ the contragredient of a representation.

Let G (resp. G^{der}) be the algebraic group $\mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p} \mathrm{GL}_2$ (resp. $\mathrm{Res}_{\mathbb{F}_q/\mathbb{F}_p} \mathrm{SL}_2$), and let $T \subset G$ (resp. $T^{\mathrm{der}} \subset G^{\mathrm{der}}$) be the diagonal torus. Let $X^*(T)$ (resp. $X^*(T^{\mathrm{der}})$) denote the group of characters of T (resp. T^{der}). Let $X_*(T)$ and $X_*(T^{\mathrm{der}})$ similarly denote groups of cocharacters. By the embeddings ι_i, $X^*(T)$ is identified with $X^*(T \times_{\mathbb{F}_p} \mathbb{F}) \cong X^*(\prod_{i \in \mathbb{Z}/f} \mathbb{G}_m^2)$, which is identified with $(\mathbb{Z}^2)^{\mathbb{Z}/f}$ in the usual way. A similar identification for $X_*(T)$ is made. For a character $\mu \in X^*(T)$, we write μ_i as the i-th factor of μ so that $\mu = \sum_{i \in \mathbb{Z}/f} \mu_i$.

Let $\eta^{(i)} \in X^*(T)$ (resp. $\alpha^{(i)} \in X_*(T)$) be the dominant fundamental character (resp. the positive coroot) represented by $(1, 0)$ (resp. $(1, -1)$) in the i-th factor and 0 elsewhere. Let $\eta = \sum_{i \in \mathbb{Z}/f} \eta^{(i)}$. Let $\omega^{(i)}$ be the restriction of $\eta^{(i)}$ to T^{der}.

Let W be the Weyl group of G and G^{der}, which is similarly identified with $S_2^{\mathbb{Z}/f}$. Here, S_2 denotes the permutation group on two elements. We denote the trivial element of S_2 by id. Then W acts naturally on $X^*(T)$ and $X^*(T^{\mathrm{der}})$. Let π be the automorphism of $X^*(T)$ and W which acts by a shift so that $\pi(x)_i = x_{i-1}$. Then the action on $X^*(T)$ induced by the relative Frobenius morphism on T is given by $p\pi^{-1}$, while the action of the relative Frobenius on W is given by π.

For a dominant character $\mu \in X^*(T)$ we write $V(\mu)$ for the Weyl module for G defined in [Jantzen 1987, II.2.13(1)]. It has a unique simple G-quotient $L(\mu)$. If $\mu = \sum_i \mu_i$ is p-restricted (i.e., $0 \le \langle \mu, \alpha^{(i)} \rangle \le p$ for all i), then $L(\mu) = \otimes_i L(\mu_i)$ by the Steinberg tensor product theorem as in [Herzig 2009, Theorem 3.9]. Let $F(\mu)$ be the restriction of $L(\mu)$ to $\mathrm{GL}_2(\mathbb{F}_q)$, which remains irreducible by [Herzig 2009, A.1.3]. Every irreducible $\mathrm{GL}_2(\mathbb{F}_q)$-representation is of this form, and we call such a representation a *Serre weight*. Note that $F(\mu) \cong F(\lambda)$ if and only if $\mu \cong \lambda \mod (p-\pi)X^0(T)$, where $X^0(T)$ is the kernel of the restriction map $X^*(T) \to X^*(T^{\mathrm{der}})$.

Recall that to a pair $(s, \lambda) \in W \times X^*(T)$, [Herzig 2009, Lemma 4.2] attaches a (virtual) representation of $\mathrm{GL}_2(\mathbb{F}_q)$, which we denote $R_s(\lambda)$. In each use below, $R_s(\lambda)$ will in fact denote a true representation.

An *inertial type* for a local field L is a continuous E-representation τ of the inertial subgroup I_L, whose action factors through a finite quotient and can be extended to G_L. For our purposes, all inertial types will be two-dimensional. In this case, Henniart [2002, Annexe A] attaches to τ a smooth irreducible finite-dimensional $\mathrm{GL}_2(\mathcal{O}_L)$-representation $\sigma(\tau)$ over E (see also [Emerton et al. 2015, Section 1.9]). We call the association of τ and $\sigma(\tau)$ the inertial local Langlands correspondence. An inertial type τ is called *tame* if τ factors through the tame quotient of I_L. The tame inertial types are exactly those τ such that $\sigma(\tau)$ factors through $\mathrm{GL}_2(k_L)$ where k_L is the residue field of L.

For any characteristic 0 field F, let $\varepsilon : G_F \to \mathbb{Z}_p^\times \subset \mathcal{O}^\times$ denote the p-adic cyclotomic character and $\bar{\varepsilon}$ denote its reduction modulo ϖ. We now let F be K. Let $\mathbb{C}_p(i)$ denote $\varepsilon^i \otimes_E \mathbb{C}_p$, where the tensor product is over any embedding $E \hookrightarrow \mathbb{C}_p$. Let $\rho : G_K \to \mathrm{GL}(V)$ be a continuous representation over E. For each embedding $\kappa : E \hookrightarrow \mathbb{C}_p$, let $\mathrm{HT}_\kappa(V)$ be the multiset of integers such that $-i$ appears with multiplicity $\dim_{\mathbb{C}_p}(V \otimes_\kappa \mathbb{C}_p(i))^{G_K}$. Then in particular $\mathrm{HT}_\kappa(\varepsilon) = \{1\}$ for all embeddings κ. We say that a two-dimensional representation V is (*potentially*) *Barsotti–Tate* if V is (potentially) crystalline with $\mathrm{HT}_\kappa(V) = \{0, 1\}$ for all embeddings κ. If τ is an inertial type, we say that V is potentially Barsotti–Tate of type τ if the action of I_K on the potentially crystalline Dieudonné module of V is isomorphic to τ.

2. Quotients of generic $\mathrm{GL}_2(\mathbb{F}_q)$-projective envelopes

Suppose that $\mu \in X^*(T)$ and that $1 \le \langle \mu - \eta, \alpha^{(i)} \rangle < p - 2$ for all $i \in \mathbb{Z}/f$. Let σ be $F(\mu - \eta)$. Let $\tilde{R}_\mu$ (resp. R_μ) be the projective $\mathcal{O}_K[\mathrm{GL}_2(\mathbb{F}_q)]$-envelope (resp. the projective $\mathbb{F}_q[\mathrm{GL}_2(\mathbb{F}_q)]$-envelope) of σ. Let S be the set $\{\pm\omega^{(i)}\}_i$ and let I be a subset of S. Recall from [Le et al. 2016b, Definition 3.5] that (with respect to μ) we attach to a subset $J \subset S$ a Serre weight σ_J. Let $R_{\mu,I}$ be the universal object among quotients of R_μ that do not contain $\sigma_{\{\omega\}}$ as a Jordan–Hölder factor for all ω in I. Recall from [Le et al. 2016b, Section 3] that there is a filtration $\mathrm{Fil}^{\boldsymbol{k}}$ on R_μ which induces a filtration $\mathrm{Fil}^{\boldsymbol{k}}$ on $R_{\mu,I}$. Similarly, we can construct a filtration $\mathrm{Fil}^k_\otimes = \sum_{|\boldsymbol{k}|=k} \mathrm{Fil}^{\boldsymbol{k}}$ on R_μ and $R_{\mu,I}$. Let $W_{\boldsymbol{k},I}$ be $\mathrm{gr}^{\boldsymbol{k}} R_{\mu,I}$.

Proposition 2.1. *We have an isomorphism* $W_{\boldsymbol{k},I} \cong \bigoplus_{J \subset S, \boldsymbol{k}(J)=\boldsymbol{k}, J \cap I = \varnothing} \sigma_J$.

Proof. This follows from [Le et al. 2016b, Proposition 3.6 and Theorem 3.14]. □

If I is a subset of S such that $I \cap \{\pm\omega^{(i)}\}$ has size at most one for all i, let $T_{\sigma,I}$ be the set of Deligne–Lusztig representations over K of the form $R_w(\mu - w\eta)$ where $w_i = \mathrm{id}$ (resp. $w_i \neq \mathrm{id}$) if $\omega^{(i)} \in I$ (resp. $-\omega^{(i)} \in I$). Fix an embedding $\tilde{R}_\mu \hookrightarrow \bigoplus_{\sigma(\tau)\in T_{\sigma,\varnothing}} \sigma(\tau)$. Let $\tilde{R}_{\mu,I}$ be the quotient of $\tilde{R}_\mu$ isotypic for the set $T_{\sigma,I}$ (which does not depend on the above embedding). Note that $\tilde{R}_{\mu,\varnothing}$ is equal to $\tilde{R}_\mu$.

Proposition 2.2. *The reduction of $\tilde{R}_{\mu,I}$ modulo p is $R_{\mu,I}$.*

Proof. For each $\omega \in I$, $\sigma_{\{\omega\}} \notin \mathrm{JH}(\bar{\sigma}(\tau))$ for all $\sigma(\tau) \in T_{\sigma,I}$. Thus, there is a canonical quotient map $R_{\mu,I} \to \bar{R}_{\mu,I}$, where $\bar{R}_{\mu,I}$ is the reduction of $\tilde{R}_{\mu,I}$. By Proposition 2.1, $R_{\mu,I}$ has length $2^{2f-\#I}$. Since $\bar{R}_{\mu,I}$ is the reduction of a lattice in the direct sum of $2^{f-\#I}$ types, each of whose reduction has length 2^f [Diamond 2007], it also has length $2^{2f-\#I}$. Since both objects have the same length, this surjection must be an isomorphism. □

Again, let $I \subset S$. Let $W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ be $\mathrm{Fil}^{\boldsymbol{k}} R_{\mu,I}/(\mathrm{Fil}^{k+2}_{\otimes} R_{\mu,I} \cap \mathrm{Fil}^{\boldsymbol{k}} R_{\mu,I})$. Note that $W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ is multiplicity free since $W_{\boldsymbol{k},\boldsymbol{k}+1,\varnothing}$ (which is $W_{\boldsymbol{k},\boldsymbol{k}+1}$ in [Le et al. 2016b, Section 3]) is by [loc. cit, Proposition 3.6 and Lemma 3.7].

Proposition 2.3. *Suppose that $J \subset J'$, $\#J' \setminus J = 1$, and $J' \cap I = \varnothing$. Let $\boldsymbol{k}$ and $\boldsymbol{k}'$ be $\boldsymbol{k}(J)$ and $\boldsymbol{k}(J')$, respectively. Then there is a subquotient of $W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ which is the unique up to isomorphism nontrivial extension of σ_J by $\sigma_{J'}$.*

Proof. This follows immediately from Proposition 2.1 and [Le et al. 2016b, Proposition 3.8]. □

Proposition 2.4. *Suppose that the size of $I \cap \{\pm\omega^{(i)}\}$ is at most one for all i and that $I \cap \{\pm\omega^{(j)}\} = \varnothing$ for some j. Then there is an exact sequence*

$$0 \to \tilde{R}_{\mu,I} \to \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}} \to R_{\mu,I\cup\{\pm\omega^{(j)}\}} \to 0, \tag{2-1}$$

where the second (*resp. third*) *map is the sum* (*resp. difference*) *of the natural projections.*

Proof. The second map of (2-1) is clearly injective since it is after inverting p and $\tilde{R}_{\mu,I}$ is $\mathcal{O}_K$-flat. We claim that the cokernel of this map is p-torsion. Let $\sigma_{\{\omega^{(j)}\}} = F(\mu' - \eta)$ and consider a map $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ such that the composition with the projection

$$\tilde{R}_{\mu,I} \twoheadrightarrow R_{\mu,I} \twoheadrightarrow R_{\mu,I}/\mathrm{Fil}^2_{\otimes} R_{\mu,I}$$

is nonzero. The composition of $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ with the natural surjection $\tilde{R}_{\mu,I} \twoheadrightarrow \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}}$ is zero since $\sigma_{\{\omega^{(j)}\}} \notin \mathrm{JH}(R_{\mu,I\cup\{\omega^{(j)}\}})$.

Lemma 2.5. *The image of the composition $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ with the natural surjection $\tilde{R}_{\mu,I} \twoheadrightarrow \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$ contains $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$.*

With Lemma 2.5 and its analogue for $\tilde{R}_{\mu,I\cup\{\omega^{(j)}\}}$, we would see that the image of

$$\tilde{R}_{\mu,I} \to \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$$

contains $p\tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$, establishing our claim.

Proof of Lemma 2.5. Fix a map $\tilde{R}_\mu \to \tilde{R}_{\mu'}$ such that the composition with the projection to $R_{\mu'}/\operatorname{Fil}^2_\otimes R_{\mu'}$ is nonzero. It suffices to show that the image, denoted Q, of the composition of $\tilde{R}_\mu \to \tilde{R}_{\mu'}$ with the above $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I} \twoheadrightarrow \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$ is $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$. On the one hand, we see that Q is in $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$ by reducing modulo p and using Propositions 2.2 and 2.3. Let $\sigma(\tau)$ be a Jordan–Hölder factor of $\tilde{R}_{\mu,I}[p^{-1}]$ and let $\sigma^\circ(\tau) \subset \sigma(\tau)$ be the unique lattice up to homothety with cosocle isomorphic to σ [Emerton et al. 2015, Lemma 4.1.1]. Fix a surjection from $\tilde{R}_{\mu,I}$ to $\sigma^\circ(\tau)$. By reducing $\bmod p$, we see that the image of the composition of $\tilde{R}_{\mu'} \to \tilde{R}_{\mu,I}$ with this surjection is a saturated lattice $\sigma^{\circ\circ}(\tau)$ with cosocle $\sigma_{\{\omega^{(j)}\}}$. Similarly, the image of Q under this surjection is a saturated lattice in $\sigma^{\circ\circ}(\tau)$ with cosocle isomorphic to σ. This lattice is $p\sigma^\circ(\tau)$ by [Emerton et al. 2015, Theorem 5.1.1]. Thus, the composition $Q \subset p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}} \twoheadrightarrow p\sigma^\circ(\tau)$ is an isomorphism upon taking cosocles. We see that Q must be equal to $p\tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}}$. □

Let R be the cokernel of the second map in (2-1), which is p-torsion by our first claim. Then the exact sequence

$$0 \to \tilde{R}_{\mu,I} \to \tilde{R}_{\mu,I\cup\{\omega^{(j)}\}} \oplus \tilde{R}_{\mu,I\cup\{-\omega^{(j)}\}} \to R \to 0 \tag{2-2}$$

induces an exact sequence

$$R_{\mu,I} \to R_{\mu,I\cup\{\omega^{(j)}\}} \oplus R_{\mu,I\cup\{-\omega^{(j)}\}} \to R \to 0 \tag{2-3}$$

by Proposition 2.2. By taking cosocles, (2-3) induces an exact sequence

$$\operatorname{cosoc} R_{\mu,I} \to \operatorname{cosoc} R_{\mu,I\cup\{\omega^{(j)}\}} \oplus \operatorname{cosoc} R_{\mu,I\cup\{-\omega^{(j)}\}} \to \operatorname{cosoc} R \to 0 \tag{2-4}$$

Note that $\operatorname{cosoc} R_{\mu,I}$, $\operatorname{cosoc} R_{\mu,I\cup\{\omega^{(j)}\}}$, and $\operatorname{cosoc} R_{\mu,I\cup\{-\omega^{(j)}\}}$ are all isomorphic to σ and that the composition of first map of (2-4) with either projection is nonzero. Thus $\operatorname{cosoc} R$ is isomorphic to σ and the restriction of the second map of (2-4) to either summand is nonzero. We conclude that the restriction of the second map in (2-3) to either summand is surjective. By definition, the maximal representation which is a quotient of both $R_{\mu,I\cup\{\omega^{(j)}\}}$ and $R_{\mu,I\cup\{-\omega^{(j)}\}}$ is $R_{\mu,I\cup\{\pm\omega^{(j)}\}}$. Thus, there is a surjection $R_{\mu,I\cup\{\pm\omega^{(j)}\}} \twoheadrightarrow R$. On the other hand, it is easy to see that the composition $R_{\mu,I} \to R_{\mu,I\cup\{\omega^{(j)}\}} \oplus R_{\mu,I\cup\{-\omega^{(j)}\}} \to R_{\mu,I\cup\{\pm\omega^{(j)}\}}$ is zero, where the second map is the difference of the natural projections. Thus, there is a surjection $R \twoheadrightarrow R_{\mu,I\cup\{\pm\omega^{(j)}\}}$. Since R and $R_{\mu,I\cup\{\pm\omega^{(j)}\}}$ are finite length objects, they must be isomorphic. □

3. Multitype Barsotti–Tate deformation rings

3A. *Étale φ-modules.* Let K_∞ be the infinite extension obtained by adjoining compatible p-power roots of $-p$ to K. Let $\mathcal{O}_{\mathcal{E},K}$ denote the p-adic completion of $\mathcal{O}_K((v))$, and let $\mathcal{O}_{\mathcal{E}^{\mathrm{un}},K}$ denote the p-adic completion of a maximal connected étale extension of $\mathcal{O}_{\mathcal{E},K}$. For R a complete local Noetherian $\mathcal{O}$-algebra, let $\Phi\text{-}\operatorname{Mod}^{\mathrm{et}}(R)$ be the category of étale φ-modules over $\mathcal{O}_{\mathcal{E},K} \otimes_{\mathbb{Z}_p} R$, and let $\operatorname{Rep}_{G_{K_\infty}}(R)$ be the category of (continuous) representations of G_{K_∞} over R. Fontaine defined an exact antiequivalence of tensor categories

$$\mathbb{V}^* : \Phi\text{-}\operatorname{Mod}^{\mathrm{et}}(R) \to \operatorname{Rep}_{G_{K_\infty}}(R)$$

by $\mathbb{V}^*(\mathcal{M}) = ((\mathcal{M} \otimes \mathcal{O}_{\mathcal{E}^{\mathrm{un}},K})^{\varphi=1})^\vee$.

For a natural number d, let $\varpi_d \in E$ be a root of $u^{p^{df}-1} + p$. Let K_d be the degree d unramified extension of K. We define the fundamental character

$$\begin{aligned} \omega_{df} : G_{K_d} &\to \mathcal{O}^\times \\ g &\mapsto \frac{g(\varpi_d)}{\varpi_d}, \end{aligned}$$

which does not depend on the choice of ϖ_d. For $\alpha \in \mathbb{F}^\times$, denote by nr_α the unramified character of G_K taking a geometric Frobenius element to α.

Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation. If $\bar{\rho}$ is reducible, then it is an extension of

$$\mathrm{nr}_{\alpha'}\, \omega_f^{\sum_{i=0}^{f-1} \mu_{2,i} p^i} \quad \text{by } \mathrm{nr}_\alpha\, \omega_f^{\sum_{i=0}^{f-1} \mu_{1,i} p^i}$$

for some dominant p-restricted character $\mu_{\bar{\rho}} = (\mu_{1,i}, \mu_{2,i})_i \in X^*(T)$ and some α and $\alpha' \in \mathbb{F}^\times$. If $\bar{\rho}$ is irreducible, then $\bar{\rho}$ is

$$\mathrm{Ind}_{G_{K_2}}^{G_K} \mathrm{nr}_{-\alpha}\, \omega_{2f}^{\sum_{i=0}^{f-1} \mu_{1,i} p^i + p^f \sum_{i=0}^{f-1} \mu_{2,i} p^i}$$

where $\mu_{\bar{\rho}}$ again is a dominant p-restricted element of $X^*(T)$ and $\alpha \in \mathbb{F}^\times$. We note that the main result of this paper in the case when $\bar{\rho}$ is irreducible already appears in [Le et al. 2016b; Hu and Wang 2018], and so this case can be ignored if the reader desires. [Buzzard et al. 2010] attaches to $\bar{\rho}$ a set $W(\bar{\rho})$ of Serre weights (see also [Breuil 2014, Section 4, Proposition A.3] with the notation $\mathcal{D}(\bar{\rho})$).

In both the reducible and irreducible cases, we now assume that $\mu_{\bar{\rho}} \in X^*(T)$ with $\mu_i = (\mu_{1,i}, \mu_{2,i}) = (c_i, 1)$ with $3 < c_i < p-2$ for all $i \in \mathbb{Z}/f$. For $i \in \mathbb{Z}/f$, let a_i be an element of $\mathbb{F}$. Let $\mathcal{M} = \prod_i \mathbb{F}((v))\mathfrak{e}^i \oplus \mathbb{F}((v))\mathfrak{f}^i$ be the φ-module defined by

$$\begin{aligned} i \neq 0 &: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}} \mathfrak{e}^i + a_{i-1} v^{c_{f-i}} \mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = v \mathfrak{f}^i, \end{cases} \\ i = 0, \bar{\rho} \text{ reducible} &: \begin{cases} \varphi(\mathfrak{e}^{f-1}) = \alpha v^{c_0} \mathfrak{e}^0 + \alpha a_{f-1} v^{c_0} \mathfrak{f}^0, \\ \varphi(\mathfrak{f}^{f-1}) = \alpha' v \mathfrak{f}^0, \end{cases} \\ i = 0, \bar{\rho} \text{ irreducible} &: \begin{cases} \varphi(\mathfrak{e}^{f-1}) = \alpha v^{c_0} \mathfrak{f}^0, \\ \varphi(\mathfrak{f}^{f-1}) = -v \mathfrak{e}^0, \end{cases} \end{aligned}$$

(here the i-th factor corresponds to the embedding ι_{-i}).

Proposition 3.1. *There are unique values $a_i \in \mathbb{F}$ for $i \in \mathbb{Z}/f$ such that $\mathbb{V}^*(\mathcal{M})$ is isomorphic to the restriction $\bar{\rho}|_{G_{K_\infty}}$.*

Proof. Note that $\bar{\rho}$ is Fontaine–Laffaille by the genericity condition. We use Fontaine–Laffaille theory as in [Breuil 2014, Appendix A]. We address the case when $\bar{\rho}$ is reducible and leave the irreducible case to the reader. Let $M = \bigoplus_{i\in\mathbb{Z}/f} M^{(i)}$ with $M^{(i)} = k_E e^{(i)} \oplus k_E f^{(i)}$ be the Fontaine–Laffaille module with

$$\begin{gathered}\mathrm{Fil}^1 M^{(i)} = M^{(i)}, \quad \mathrm{Fil}^2 M^{(i)} = \mathrm{Fil}^{c_{f-i}} M^{(i)} = k_E f^{(i)}, \quad \mathrm{Fil}^{c_{f-i}+1} M^{(i)} = 0,\\ \varphi(e^{(i)}) = e^{(i+1)}, \quad \varphi_{c_{f-i}}(f^{(i)}) = f^{(i+1)} + a_{i-1}e^{(i+1)} \quad \text{for } i \neq 1,\\ \varphi(e^{(1)}) = \alpha' e^{(2)}, \quad \varphi_{c_{f-1}}(f^{(1)}) = \alpha f^{(2)} + \alpha' a_0 e^{(2)},\end{gathered}$$

for $a_i \in k_E$ such that $\bar\rho \cong \mathrm{Hom}_{\mathrm{Fil}^\bullet,\varphi_\cdot}(M, A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ (see e.g., [Breuil 2014, (16)]).

Let $\mathfrak{M}$ be the $\mathbb{F}_q[\![v]\!] \otimes_{\mathbb{Z}_p} \mathbb{F}$-submodule of $\mathcal{M}$ generated by $(\mathfrak{e}_i)_{i\in\mathbb{Z}/f}$ and $(\mathfrak{f}_i)_{i\in\mathbb{Z}/f}$. Note that φ maps $\mathfrak{M}$ to itself. Then a calculation (see [Emerton et al. 2015, Section 7.4] with $J = \varnothing$) shows that $\Theta_{p-1}(\mathfrak{M}) \cong \mathcal{F}_{p-1}(M)$, where the functors Θ_{p-1} and $\mathcal{F}_{p-1}$ are introduced in [loc. cit, Appendix A]. The result now follows from [loc. cit, Propositions A.3.2 and A.3.3]. □

For the rest of this section, we fix, for each $i \in \mathbb{Z}/f$, $a_i \in \mathbb{F}$, the unique element as in Proposition 3.1. In doing so, we thus fix $\mathcal{M}$. If $\bar\rho$ is irreducible, let $S_{\bar\rho}$ be the set $\{-\omega^{(0)}, \omega^{(1)}, \dots, \omega^{(f-1)}\}$. Otherwise, let $S_{\bar\rho}$ be the set $\{\omega^{(i)} \mid a_{f-1-i} = 0\}$.

Proposition 3.2. *The set $W(\bar\rho)$ equals $\{\sigma_J \mid J \subset S_{\bar\rho}\}$ where σ_J is defined with respect to $\mu_{\bar\rho}$.*

Proof. This follows from a direct calculation using [Breuil 2014, Section 4]. □

3B. *Kisin modules and deformation rings.* To describe tamely potentially Barsotti–Tate deformation rings, we will use the theory of Kisin modules with descent datum. Let τ be the tame principal series type $\eta_1 \oplus \eta_2 : I_K \to \mathrm{GL}_2(\mathbb{F}_q)$ where $\eta_k = \omega_f^{-\boldsymbol{a}_k^{(0)}}$ for $k = 1$ and 2 and

$$\boldsymbol{a}_k^{(j)} = \sum_{i=0}^{f-1} a_{k,-j+i}\, p^i,$$

where $a_{k,i} \in \mathbb{Z}$. We will suppose throughout that $2 \le |a_{1,i} - a_{2,i}| \le p-3$ for all $i \in \mathbb{Z}/f$ and call such a tame principal series type *generic*. We will say a tame inertial type τ' is generic if its restriction to the quadratic unramified extension of K is a generic principal series type.

The *orientation* of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is the element $s \in W$ such that $\boldsymbol{a}_{s_j(1)}^{(j)} > \boldsymbol{a}_{s_j(2)}^{(j)}$. By an abuse of notation, we say that the orientation of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is an orientation for τ if τ can be expressed in terms of $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ as above.

Let R be an $\mathcal{O}$-algebra. For a principal series type τ, we will consider Kisin modules over R with descent datum of type τ (see [Le et al. 2018, Definition 2.4]). We will say that such a Kisin module $\mathfrak{M}_R$ is in $Y^{(0,1),\tau}(R)$ if the cokernels of $\phi_{\mathfrak{M}_R} : \varphi^*(\mathfrak{M}_R) \to \mathfrak{M}_R$ and $\phi_{\det \mathfrak{M}_R} : \varphi^*(\det \mathfrak{M}_R) \to \det \mathfrak{M}_R$ are annihilated by $E(u) = u^{q-1} + p$. Let v be u^{q-1}.

Let s be an orientation for a generic tame principal series type τ and $\mathfrak{M}_R$ be an element of $Y^{(0,1),\tau}(R)$. Then $\mathfrak{M}_R$ can be described by the matrices $\mathrm{Mat}_\beta(\phi^{(i)}_{\mathfrak{M}_R \otimes_R \mathbb{F}, s_{i+1}(2)})$ after choosing an eigenbasis β (see [loc. cit., Definition 2.11]). The following is a generalization of [loc. cit., Theorem 4.1] in the case of GL_2, where β is allowed to have a slightly more general form than a gauge basis.

Theorem 3.3. *Let τ be a tame generic principal series type and let $s = (s_i)_i \in W$ be an orientation for τ. Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$. Let $\mathfrak{M}_R \in Y^{(0,1),\tau}(R)$ with* $\mathrm{Mat}_{\bar\beta}(\phi^{(i)}_{\mathfrak{M}_R\otimes_R\mathbb{F},s_{i+1}(2)})$ *given by*

$$\bar A_1 = \begin{pmatrix} v & \\ a_i v & 1\end{pmatrix}, \quad \bar A_2 = \begin{pmatrix} & 1 \\ v & \end{pmatrix}, \quad \bar A_3 = \begin{pmatrix} & 1 \\ v & a_i\end{pmatrix}, \quad \textit{or} \quad \bar A_4 = \begin{pmatrix} 1 & \\ & v\end{pmatrix}$$

for $i \neq 0$ and $\bar A_j \begin{pmatrix}\alpha & \\ & \alpha'\end{pmatrix}$ for $i = 0$, where $\bar\beta$ is an eigenbasis for $\mathfrak{M}_R \otimes_R \mathbb{F}$. Then there is a unique eigenbasis β of $\mathfrak{M}_R$ up to scaling lifting $\bar\beta$ such that $\mathrm{Mat}_\beta(\phi^{(i)}_{\mathfrak{M}_R,s_{i+1}(2)})$ *is given by*

$$A_1 = \begin{pmatrix} v+p & \\ (X_i + [a_i])v & 1\end{pmatrix}, A_2 = \begin{pmatrix} -Y_i & 1 \\ v & X_i\end{pmatrix}, A_3 = \begin{pmatrix} -p(X_i+[a_i])^{-1} & 1 \\ v & X_i + [a_i]\end{pmatrix}, \textit{ or } A_4 = \begin{pmatrix} 1 & -Y_i \\ & v+p\end{pmatrix},$$

respectively, for $i \neq 0$ and $A_j D(\alpha,\alpha')$ with A_j as above for $i = 0$. Here $[\cdot]$ denotes the Teichmüller lift, $X_i Y_i = p$ for A_2, and

$$D(\alpha,\alpha') = \begin{pmatrix} [\alpha] + X_\alpha & \\ & [\alpha'] + X_{\alpha'}\end{pmatrix}.$$

Proof. The proof is similar to the proofs of [loc. cit., Theorems 4.1 and 4.16] which prove existence and uniqueness of β, respectively. We describe some of the key points. We modify [loc. cit., Definition 4.2], defining $d_R(P) = \min_k 2v_R(r_k) + k$ if $P = \sum_k r_k v^k \in R[\![v]\!]$. Then the analogue of [loc. cit., Proposition 4.3] holds (see [loc. cit., Remark 4.4]). The entry in the middle column of [loc. cit., Table 5] becomes

$$\begin{pmatrix} 1^* & \\ v(\le 0) & 0^*\end{pmatrix}, \quad \begin{pmatrix} \le 0 & 0^* \\ 1^* & \le 0\end{pmatrix}, \quad \begin{pmatrix} \le 0 & 0^* \\ 1^* & \le 0\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0^* & \le 0 \\ & 1^*\end{pmatrix},$$

respectively, and we modify [loc. cit., Definition 4.5] for $E^{(i)}$ appropriately. For $1 \le m, k \le 2$, we define $\delta(A^{(i)}_{mk})$ to be $d_R(E^{(i)}_{mk})$ if $\bar A^{(i)} \neq \bar A_3$. If $\bar A^{(i)} = \bar A_3$, we define $\delta(A^{(i)}_{mk})$ to be $d_R(E^{(i)}_{mk})$ (resp. $d_R(E^{(i)}_{mk}) + 1$) if $k = 1$ (resp. if $k = 2$). Finally, we let

$$\delta(A^{(i)}) = \min_{1\le m,k\le 2}\{\delta(A^{(i)}_{mk})\}.$$

The analogue of [loc. cit., Proposition 4.6] holds, replacing $3 + d_R(x^{(j)})$ with $2 + d_R(x^{(j)})$. We define the notion of pivots for $\bar A^{(i)} \neq \bar A_3$ as in the [loc. cit., Definition 4.8], and define the pivots in the case of $\bar A^{(i)} = \bar A_3$ to be the same as the pivots in the case of $\bar A_2$. The analogue of [loc. cit., Lemma 4.10] holds except that the second equation of [loc. cit.] is changed to $A^{(i)}_{22} = vP_{22} + [a_i] + Q_{22}$ when $\bar A^{(i)} = \bar A_3$. Then the analogues of [loc. cit., Proposition 4.11, Proposition 4.13, and Lemma 4.14] give the eigenbasis β.

We give more details for the algorithm in the case $\bar A^{(i)} = \bar A_3$. We let $\delta > 1$ be an integer. Suppose that $\delta(A^{(i)})$, which is necessarily greater than one, is δ. Then there is an $x \in R[\![v]\!]$ with $d_R(x) \ge \delta - 1$ such that $A'^{,(i)} \stackrel{\text{def}}{=} D_{22}(x)A^{(i)}$ satisfies $\delta(A'^{,(i)}) \ge \delta$ and $\delta(A'^{,(i)}_{21}) > \delta$. Note the crucial role played by the definition of $\delta(A'^{,(i)}_{22})$ as $d_R(E'^{,(i)}_{22}) + 1$ in this case. Moreover, these inequalities still hold after right multiplication by a conjugate of $D_{22}(x)^\varphi$ by a permutation matrix. This is the analogue of [loc. cit., Proposition 4.6], where the notation I^φ is defined.

Suppose next that $\delta(A^{(i)})$ is δ and that $\delta(A^{(i)}_{21}) > \delta$. Then there exists an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A^{\prime,(i)} \overset{\text{def}}{=} U_{12}(x)A^{(i)}$ satisfies $\delta(A^{\prime,(i)}) \geq \delta$ and $\delta(A^{\prime,(i)}_{11}), \delta(A^{\prime,(i)}_{21}) > \delta$ (note that $\delta(A^{\prime,(i)}_{21}) = \delta(A^{(i)}_{21})$). Again, we use that $\delta(A^{\prime,(i)}_{12}) = d_R(E^{\prime,(i)}_{12}) + 1$. Moreover, these inequalities still hold after right multiplication by a conjugate of $U_{12}(x)^{\varphi}$ by a permutation matrix by the genericity assumption.

Suppose next that $\delta(A^{(i)})$ is δ and that $\delta(A^{(i)}_{11}), \delta(A^{(i)}_{21}) > \delta$. Then there is an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A^{\prime,(i)} \overset{\text{def}}{=} D_{11}(x)A^{(i)}$ satisfies $\delta(A^{\prime,(i)}) \geq \delta$ and $\delta(A^{\prime,(i)}_{11}), \delta(A^{\prime,(i)}_{21}), \delta(A^{\prime,(i)}_{12}) > \delta$ using that $A^{(i)}_{11} \in m_R \cdot R[\![v]\!]$. Moreover, these inequalities still hold after right multiplication by a conjugate of $D_{11}(x)^{\varphi}$ by a permutation matrix.

Suppose finally that $\delta(A^{(i)})$ is δ and that $\delta(A^{(i)}_{11}), \delta(A^{(i)}_{21}), \delta(A^{(i)}_{12}) > \delta$. Then there is an $x \in R[\![v]\!]$ with $d_R(x) \geq \delta - 1$ such that $A^{\prime,(i)} \overset{\text{def}}{=} L_{21}(x)A^{(i)}$ satisfies $\delta(A^{\prime,(i)}) \geq \delta + 1$ using again that $A^{(i)}_{11} \in m_R \cdot R[\![v]\!]$. Moreover, these inequalities still hold after right multiplication by a conjugate of $L_{21}(x)^{\varphi}$ by a permutation matrix by the genericity assumption. Repeating these four steps repeatedly gives the analogue of [loc. cit., Proposition 4.13] in this case.

We deduce the forms of A_i from the condition that $v + p$ must divide the determinant. Finally, the analogue of [loc. cit., Theorem 4.16] proves the uniqueness of β up to scaling. In the notation of [loc. cit.], we obtain the equation

$$\tilde{A}^{(i)}_2 + v^2 \tilde{A}^{(i)}_2 M^{(i)} = \tilde{A}^{(i)}_1 + I^{(i+1)} \tilde{A}^{(i)}_1 \tag{3-1}$$

(see [loc. cit., (4.2)]). Suppose that $d_R(I^{(j)}) \geq \delta \geq 1$ for all j. Then one can show that $d_R(I^{(j)}) \geq \delta + 1$ for all j. This implies that $I^{(j)} = 0$ for all j. We again give more details in the case $\bar{A}^{(i)} = \bar{A}_2$ or $\bar{A}_3$. The other cases are treated similarly. Let k be 1 or 2. We first compare the $(k, 1)$-entries of (3-1) to see that $d_R(I^{(i+1)}_{k2}) \geq \delta + 1$. Using this and the $(k, 2)$-entries of (3-1), we see that $d_R(I^{(i+1)}_{k1}) \geq \delta + 1$. □

For the rest of the section, let $\bar{\rho}$ be as in Section 3A and let $\mathcal{M}$ be as in Proposition 3.1 so that $\bar{\rho}|_{G_{K_\infty}}$ is isomorphic to $\mathbb{V}^*(\mathcal{M})$. Moreover, for simplicity, assume that $\bar{\rho}$ is reducible. Recall the definition of $S_{\bar{\rho}}$ from Section 3A.

Let s and s' be in W such that one of the following holds for each $i \in \mathbb{Z}/f$:

(1) s_i and s'_i are both id.

(2) s_i and s'_i are both not id.

(3) s_i is id, but s'_i is not, and $i \in S_{\bar{\rho}}$.

We say that $i \in \mathbb{Z}/f$ is case (1), (2), or (3) if the above relevant condition holds.

Proposition 3.4. *Let s and s' be in W as above. Let τ be the tame generic inertial type with $\sigma(\tau) \cong R_s(\mu_{\bar{\rho}} - s'\eta)$. Let R be the ring $\mathcal{O}[\![(X_i, Y_i)_{i=0}^{f-1}, X_\alpha, X_{\alpha'}]\!]/(h_i)$ where for each $i \in \mathbb{Z}/f$, h_i is Y_i, $X_iY_i - p$, $Y_i - p$, or X_i if $f - 1 - i$ is case* (1), (2) *with $\omega^{(f-i)} \in S_{\bar{\rho}}$,* (2) *with $\omega^{(f-i)} \notin S_{\bar{\rho}}$, or* (3), *respectively.*

Let $\mathcal{M}_R = \prod_i R((v))\mathfrak{e}^i \oplus R((v))\mathfrak{f}^i$ be the φ-module defined by

$$f-i \textit{ is case } (1): \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}-1}(v+p)\mathfrak{e}^i + (X_{i-1}+[a_{i-1}])v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = v\mathfrak{f}^i, \end{cases}$$

$$f-i \textit{ is case } (2),\ \omega^{(f-i)} \in S_{\bar\rho}: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i + X_{i-1}v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f-i \textit{ is case } (2),\ \omega^{(f-i)} \notin S_{\bar\rho}: \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i + (X_{i-1}+[a_{i-1}])v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -p(X_{i-1}+[a_{i-1}])^{-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f-i \textit{ is case } (3): \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}\mathfrak{e}^i + (v+p)\mathfrak{f}^i, \end{cases}$$

with the usual modification for $i=0$. Then $\mathbb{V}^*(\mathcal{M}_R)$ is the restriction to G_{K_∞} of a versal potentially Barsotti–Tate deformation of $\bar\rho$ of type τ.

Proof. Define $w^* \in W$ and $s_\tau \in S_2$ to be the unique elements such that

$$w^*_{f-1} = \mathrm{id} \quad \text{and} \quad (w^*)^{-1}s\pi(w^*) = (s_\tau, \mathrm{id}, \dots, \mathrm{id}).$$

Then the Deligne–Lusztig representations $R_s(\mu_{\bar\rho}-s'\eta)$ and $R_{(s_\tau,\mathrm{id},\dots,\mathrm{id})}((w^*)^{-1}(\mu_{\bar\rho}-s'\eta))$ are isomorphic by [Herzig 2009, Lemma 4.2]. Moreover, (the quadratic base change of) $R_{(s_\tau,\mathrm{id},\dots,\mathrm{id})}((w^*)^{-1}(\mu_{\bar\rho}-s'\eta))$ is a generic principal series. Define $w=(w_i)_i$ by $w_i = (w^*_{f-1-i})^{-1}$ for $i \in \mathbb{Z}/f$. Then one easily checks that w is an orientation for $(w^*)^{-1}(\mu_{\bar\rho}-s'\eta)$. Let $\mathfrak{M}_R$ be the Kisin module (with quadratic unramified descent) of tame inertial type (the quadratic unramified base change of) $\tau(s_\tau, -(w^*)^{-1}(\mu_{\bar\rho}-s'\eta))$ with $A^{(i-1)} = \mathrm{Mat}_\beta(\phi^{(i-1)}_{\mathfrak{M}_R,w_i(2)})$ given by A_1, A_2, A_3, or A_4 if $f-i$ is case (1), $f-i$ is case (2) and $f-i \in S_{\bar\rho}$, $f-i$ is case (2) and $f-i \notin S_{\bar\rho}$, or $f-i$ is case (3), respectively. We claim that $T^*_{dd}(\mathfrak{M}_R \otimes_{\mathcal{O}} \mathbb{F})$ is isomorphic to the restriction to G_{K_∞} of $\bar\rho$. Assuming this, by Theorem 3.3 and the analogue of [Le et al. 2018, Sections 5.2 and 6], $T^*_{dd}(\mathfrak{M}_R)$ is the restriction to G_{K_∞} of a versal potentially Barsotti–Tate deformation of $\bar\rho$ of type τ.

Let L be $K((-p)^{1/e})$ with $e=q-1$ if $s_\tau = \mathrm{id}$ and $K_2((-p)^{1/e})$ with $e=q^2-1$ otherwise. Let Δ be the Galois group $\mathrm{Gal}(L/K)$. We claim that

$$(\mathfrak{M}_R \otimes_{\mathcal{O}_{\mathcal{E},K}} \mathcal{O}_{\mathcal{E},L})^\Delta \cong \mathcal{M}_R.$$

This would finish the proof including the claim in the previous paragraph since the restriction to G_{K_∞} of $\bar\rho$ is isomorphic to $\mathcal{M}$ by Proposition 3.1, and clearly $\mathcal{M}_R \otimes_{\mathcal{O}} \mathbb{F}$ is isomorphic to $\mathcal{M}$.

Let $\mu_{\bar\rho}$ be $(\mu_i)_i$. Let v^λ denote the torus element obtained by applying the coweight λ to $v \stackrel{\mathrm{def}}{=} u^e$. By [Le et al. 2016a, Proposition 3.1.2], we see that a Kisin module (with quadratic unramified descent) of tame inertial type (the quadratic unramified base change of) τ with $\mathrm{Mat}_\beta(\phi^{(i)}_{\mathfrak{M},w_{i+1}(2)})$ given by $A^{(i)}$ (resp. $A^{(i)}s_0^{-1}D(\alpha,\alpha')s_0$) for $i < f-1$ (resp. for $i = f-1$) gives a φ-module $\mathcal{M} = \prod_i \mathbb{F}((v))\mathfrak{e}'^i \oplus \mathbb{F}((v))\mathfrak{f}'^i$

with $\varphi(\mathfrak{e}'^{i-1}, \mathfrak{f}'^{i-1}) = M'_{i-1}(\mathfrak{e}'^{i}, \mathfrak{f}'^{i})$ where

$$\begin{aligned} M'_i &= w_{i+1} A^{(i)} v^{w_{i+1}^{-1}(w^*_{f-1-i})^{-1}(\mu_{f-1-i}-s'_{f-1-i}\eta)} (w_{i+1})^{-1} \\ &= (w^*_{f-2-i})^{-1} A^{(i)} v^{w^*_{f-2-i}(w^*_{f-1-i})^{-1}(\mu_{f-1-i}-s'_{f-1-i}\eta)} w^*_{f-2-i} \\ &= (w^*_{f-2-i})^{-1} A^{(i)} v^{s^{-1}_{f-1-i}(\mu_{f-1-i}-s'_{f-1-i}\eta)} w^*_{f-2-i} \end{aligned}$$

for $i < f-1$ and $M'_{f-1} = A^{(f-1)} s_0^{-1} D(\alpha, \alpha') s_0 s_\tau^{-1} v^{(w^*_0)^{-1}(\mu_0 - s_0\eta)}$. Changing to the bases $(\mathfrak{e}^i, \mathfrak{f}^i) = (\mathfrak{e}'^i, \mathfrak{f}'^i)(w^*_{f-2-i})^{-1}$, we see that $\mathcal{M}$ is given by $(M_i)_i$ where

$$\begin{aligned} M_i &= A^{(i)} v^{s^{-1}_{f-1-i}(\mu_{f-1-i}-s'_{f-1-i}\eta)} w^*_{f-2-i} (w^*_{f-1-i})^{-1} \\ &= A^{(i)} v^{s^{-1}_{f-1-i}(\mu_{f-1-i}-s'_{f-1-i}\eta)} s^{-1}_{f-1-i} \\ &= A^{(i)} s^{-1}_{f-1-i} v^{\mu_{f-1-i}-s'_{f-1-i}\eta} \end{aligned}$$

for $i < f-1$ and

$$\begin{aligned} M'_{f-1} &= A^{(f-1)} s_0^{-1} D(\alpha, \alpha') s_0 s_\tau^{-1} v^{(w^*_0)^{-1}(\mu_0 - s'_0\eta)} (w^*_0)^{-1} \\ &= A^{(f-1)} s_0^{-1} D(\alpha, \alpha') s_0 s_\tau^{-1} (w^*_0)^{-1} v^{\mu_0 - s'_0\eta} \\ &= A^{(f-1)} s_0^{-1} v^{\mu_0 - s'_0\eta} D(\alpha, \alpha'). \end{aligned}$$

The proposition is now deduced by substituting for $A^{(i)}$, s, and $\mu_{\bar\rho}$. □

If τ is an inertial type, let R^τ parametrize potentially Barsotti–Tate (framed) liftings of $\bar\rho$ of type τ. If T is a set of inertial types for K, then we let Spec R^T be the Zariski closure of $\bigcup_{\tau \in T} \operatorname{Spec} R^\tau[p^{-1}]$ in the universal (framed) lifting space Spec $R^\square_{\bar\rho}$ of $\bar\rho$.

For applications to Shimura curves and algebraic modular forms on definite quaternion algebras, it is convenient to consider fixed determinant deformation rings. If $\psi : G_K \to \mathcal{O}^\times$ is a continuous character, let $R^{\psi,\square}_{\bar\rho}$ be the quotient of $R^\square_{\bar\rho}$ parametrizing (framed) liftings of $\bar\rho$ with determinant $\psi\varepsilon$. Let $R^{\psi,\tau}$ be the simultaneous quotient of $R^{\psi,\square}_{\bar\rho}$ and R^τ parametrizing potentially Barsotti–Tate (framed) liftings of $\bar\rho$ of type τ and determinant $\psi\varepsilon$. We can similarly define the quotient $R^{\psi,T}$ of R^T. If $R^{\psi,\tau}$ is nonzero, then R^τ must be nonzero, ψ must lift $\bar\varepsilon^{-1} \det \bar\rho$, and $\psi|_{I_K}$ must be $\det \tau$. For all sets of types T considered below, the determinants of all elements of T coincide.

Now fix a Serre weight σ in $W(\bar\rho)$. Suppose that $\sigma = \sigma_J$ for $J \subset S_{\bar\rho}$ where σ_J is defined with respect to $\mu_{\bar\rho}$. Let I be a subset of S such that $I \cap \{\pm\omega^{(i)}\}$ has size at most one for all $i \in \mathbb{Z}/f$. Let $T_{J,I}$ be the set of inertial types τ such that $\sigma(\tau)$ is of the form $R_s(\mu_{\bar\rho} - s'\eta)$ where s and s' have the restrictions given by the following table:

s_i, s'_i	$i \notin J$	$i \in J$
$\{\pm\omega^{(i)}\} \cap I = \varnothing$	$s_i = s'_i$	$s' \neq \mathrm{id}$
$\omega^{(i)} \in I$	$s_i = s'_i = \mathrm{id}$	$s_i = s'_i \neq \mathrm{id}$
$-\omega^{(i)} \in I$	$s_i = s'_i \neq \mathrm{id}$	$s_i = \mathrm{id}, s'_i \neq \mathrm{id}$

Lemma 3.5. *Define $w_J \in W$ by $w_{J,i-1} = \mathrm{id}$ if and only if $i \notin J$ for all $i \in \mathbb{Z}/f$. Then the set of tame inertial types $T_{J,I}$ corresponds by inertial local Langlands to the set $T_{\sigma, w_J(I)}$ of Deligne–Lusztig representations defined in Section 2.*

Proof. This is a computation using the definitions and [Herzig 2009, Theorem 5.2]. Note that in the notation of [loc. cit.], $\gamma'_{\sigma,\tau}$ in this case is equal to the Kronecker symbol for σ and τ. Another method of proof is to use [Le et al. 2016b, Proposition 2.10] and verify that if $V_\phi(\tau) \cong R_s(\mu)$, then $W^?(\tau) = \mathrm{JH}(\bar{R}_{sw_0}(\mu - sw_0\eta))$. □

Theorem 3.6. *There is an isomorphism to a formal power series ring over $\mathcal{O}[\![(X_i, Y_i)_{i=0}^{f-1}]\!]/(g_i(J, I))_i$ from $R^{T_{J,I}}$, where $g_i(J, I)$ is given by the following table*:

$g_i(J, I)$	$\omega^{(f-1-i)} \notin S_{\bar{\rho}}$	$\omega^{(f-1-i)} \in S_{\bar{\rho}} \setminus J$	$\omega^{(f-1-i)} \in J$
$\{\pm\omega^{(f-1-i)}\} \cap I = \varnothing$	$Y_i(Y_i - p)$	$Y_i(X_iY_i - p)$	$X_i(X_iY_i - p)$
$\omega^{(f-1-i)} \in I$	Y_i	Y_i	$X_iY_i - p$
$-\omega^{(f-1-i)} \in I$	$Y_i - p$	$X_iY_i - p$	X_i

If $I \subset I'$, then $g_i(J, I') \mid g_i(J, I)$ for all $i \in \mathbb{Z}/f$ and $R^{T_{J,I'}}$ is the quotient of $R^{T_{J,I}}$ by the ideal $(g_i(J, I'))_i$. Analogous results hold for $R^{\psi, T_{J,I}}$ provided that ψ is chosen so that $R^{\psi, T_{J,I}}$ is nonzero for any, or equivalently all, choices of I as above.

Remark 3.7. Since twisting by the universal unramified deformation of the trivial character gives an isomorphism $R^T \cong R^{\psi,T}[\![X]\!]$ (assuming $R^{\psi,T}$ is nonzero), the fixed determinant case follows from the first part of Theorem 3.6, and we ignore it below (see [Emerton et al. 2015, Remark 7.2.2]).

Proof. Since $R^{T_{J,I}}$ is naturally a quotient of $R^{\square}_{\bar{\rho}|_{G_{K_\infty}}}$ by [Emerton et al. 2015, Lemma 7.4.3], it suffices to compute the Zariski closure of $\bigcup_{\tau \in T_{J,I}} \operatorname{Spec} R^\tau[p^{-1}]$ in $\operatorname{Spec} R^{\square}_{\bar{\rho}|_{G_{K_\infty}}}$. Let R be the ring

$$\mathcal{O}[\![(X_i, Y_i)_{i=0}^{f-1}, X_\alpha, X_{\alpha'}]\!]/(g_i(J, \varnothing))_i$$

and consider the deformation $\mathcal{M}_R = \prod_i R((v))\mathfrak{e}^i \oplus R((v))\mathfrak{f}^i$ of $\mathcal{M}$ defined by

$$f - i \notin S_{\bar{\rho}} : \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}-1}(v + p - Y_{i-1})\mathfrak{e}^i + v^{c_{f-i}}(X_{i-1} + [a_{i-1}])\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}(X_{i-1} + [a_{i-1}])^{-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f - i \in S_{\bar{\rho}} \setminus J : \begin{cases} \varphi(\mathfrak{e}^{i-1}) &= v^{c_{f-i}-1}(v + p - X_{i-1}Y_{i-1})\mathfrak{e}^i + X_{i-1}v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) &= -Y_{i-1}\mathfrak{e}^i + v\mathfrak{f}^i, \end{cases}$$

$$f - i \in J : \begin{cases} \varphi(\mathfrak{e}^{i-1}) = v^{c_{f-i}}\mathfrak{e}^i + X_{i-1}v^{c_{f-i}}\mathfrak{f}^i, \\ \varphi(\mathfrak{f}^{i-1}) = -Y_{i-1}\mathfrak{e}^i + (v + p - X_{i-1}Y_{i-1})\mathfrak{f}^i, \end{cases}$$

with the usual modification at $i = 0$. Define the deformation functor $D^\square$ by $D^\square(A) = \{(\psi : R \to A, b_A)\}/\cong$ for A a complete local Noetherian $\mathcal{O}$-algebra, where b_A is a basis for the free rank two A-module $\mathbb{V}^*(\psi^*(\mathcal{M}_R))$ whose reduction modulo $\mathfrak{m}_A$ gives $\bar{\rho}$. Then the natural map $D^\square \to \operatorname{Spf} R$ is a $\widehat{\mathrm{GL}}_2$-torsor and is thus formally smooth of dimension 4. Let $D^\square$ be $\operatorname{Spf} R^\square$. One can rescale $\mathfrak{e}^0$ and $\mathfrak{f}^0$ by units, and rescale the other basis vectors appropriately so that the coefficients in the definition of φ which are 1 remain 1. This gives a $\hat{\boldsymbol{G}}_m^2$-action on R, and orbits give isomorphic φ-modules. We claim that the natural map $\operatorname{Spf} R^{\square}/\hat{\boldsymbol{G}}_m^2 \to \operatorname{Spf} R^{\square}_{\bar\rho|_{G_{K_\infty}}}$ is a closed embedding. It suffices to show injectivity on reduced tangent spaces.

Suppose that t is a reduced tangent vector of $\operatorname{Spf} R^{\square}/\hat{\boldsymbol{G}}_m^2$ which maps to zero in $\operatorname{Spf} R^{\square}_{\bar\rho|_{G_{K_\infty}}}$. By formal smoothness, we can extend this to a map $t : R^{\square} \to \mathbb{F}[\varepsilon]/(\varepsilon^2)$. Let $\mathcal{M}_t$ be $\mathcal{M}_R \otimes_{R,t} \mathbb{F}[\varepsilon]/(\varepsilon^2)$ so that $\mathcal{M}_t$ and $\mathcal{M} \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]/(\varepsilon^2)$ are isomorphic. Let M_i (resp. $M_{t,i}$) be the matrices such that $\varphi(\mathfrak{e}^i \otimes_R \mathbb{F}, \mathfrak{f}^i \otimes_R \mathbb{F}) = M_i(\mathfrak{e}^{i+1} \otimes_R \mathbb{F}, \mathfrak{f}^{i+1} \otimes_R \mathbb{F})$ (resp. $\varphi(\mathfrak{e}^i \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2), \mathfrak{f}^i \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2)) = M_{t,i}(\mathfrak{e}^{i+1} \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2), \mathfrak{f}^{i+1} \otimes_R \mathbb{F}[\varepsilon]/(\varepsilon^2))$). Then there are matrices $D_i \in \mathrm{GL}_2(\mathbb{F}((v)))$ such that

$$(\mathrm{id}_3 + \varepsilon D_i) M_i \varphi(\mathrm{id}_3 - \varepsilon D_{i-1}) = M_{t,i}$$

for all $i \in \mathbb{Z}/f$, where id_3 is the 3×3 identity matrix (we can assume without loss of generality that the terms without ε are id_3 by multiplying by their inverses). We first claim that $D_i \in \mathrm{GL}_2(\mathbb{F}[\![v]\!])$ for all $i \in \mathbb{Z}/f$. For each i, let $k_i \in \mathbb{Z}$ be the minimal integer such that $v^{k_i} D_i \in \mathrm{Mat}_3(\mathbb{F}[\![v]\!])$. Then $v^{c_{f-1-i}+k_i} \varphi(\mathrm{id}_3 - \varepsilon D_{i-1}) = v^{c_{f-1-i}+k_i} M_i^{-1}(\mathrm{id}_3 - \varepsilon D_i) M_{t,i} \in \mathrm{Mat}_3(\mathbb{F}[\![v]\!])$, and thus $c_{f-1-i} + k_i \geq p k_{i-1}$. Since $c_{f-1-i} < p-1$, $k_i \geq 2 + p(k_{i-1} - 1)$. If $k_{i-1} \geq n \geq 1$, then $k_i \geq n+1$, from which we derive the contradiction that $k_i \geq n$ for every $n \in \mathbb{N}$. Hence $k_i \leq 0$ for all i.

We next claim that if $f-1-i \notin S_{\bar\rho}$ for some $i \in \mathbb{Z}/f$, then $t(Y_i) = 0$. Suppose for the sake of contradiction that $f-1-i \notin S_{\bar\rho}$ and $t(Y_i) \neq 0$. Let $N_i \in \mathrm{Mat}_3(\mathbb{F}[\![v]\!])$ be such that $\varepsilon N_i = M_{t,i} - M_i$. Then by the formulas for M_i and $M_{t,i}$, the first (resp. second) entry in the top row of N_i is exactly divisible by $v^{c_{f-1-i}-1}$ (resp. v^0). On the other hand, since $D_i M_i - M_i \varphi(D_{i-1}) = N_i$, the first (resp. second) entry in the top row of N_i is divisible by $v^{c_{f-1-i}}$ (resp. v), which is a contradiction. Thus t is a reduced tangent vector of

$$(\operatorname{Spf} R^{\square}/(Y_i : f-1-i \notin S_{\bar\rho}))/\hat{\boldsymbol{G}}_m^2.$$

Let τ be the tame inertial type such that $\sigma(\tau) = R_{w_0}(\mu - w_0\eta)$. Then the natural map from the quotient of

$$\operatorname{Spf} R^{\square}/(\varpi, \{Y_i : f-1-i \notin S_{\bar\rho}\}, \{X_i Y_i : f-1-i \in S_{\bar\rho}\}) \tag{3-2}$$

by $\hat{\boldsymbol{G}}_m^2$ to $\operatorname{Spf} R^{\tau}/\varpi$ is formally smooth by Proposition 3.4. In fact, it is an isomorphism since the domain and codomain are both of dimension $f+4$ over $\mathbb{F}$. Indeed, for the codomain this follows from [Kisin 2008, Theorem 3.3.4] and p-flatness, while for the domain we see directly that (3-2) has dimension $f+6$. Since the map

$$\operatorname{Spf} R^{\square}/(\varpi, \{Y_i : f-1-i \notin S_{\bar\rho}\}, \{X_i Y_i : f-1-i \in S_{\bar\rho}\}) \to \operatorname{Spf} R^{\square}/(\varpi, \{Y_i : f-1-i \notin S_{\bar\rho}\})$$

is an isomorphism on reduced tangent spaces, t is a reduced tangent vector of $\operatorname{Spf} R^{\tau}$. Since $\operatorname{Spf} R^{\tau} \to \operatorname{Spf} R^{\square}_{\bar\rho|_{G_{K_\infty}}}$ is injective on reduced tangent spaces again by [Emerton et al. 2015, Lemma 7.4.3], t is zero.

Finally, since R is p-flat, it suffices to show that if $\#(\{\pm\omega_i\} \cap I) = 1$ for all $i \in \mathbb{Z}/f$, then $\mathbb{V}^*(\mathcal{M}/(g_i(J,I))_i)$ is the restriction to G_{K_∞} of a versal potentially Barsotti–Tate deformation of $\bar\rho$ of the unique type τ in $T_{J,I}$. This follows from Proposition 3.4. □

4. Patching functors and multiplicity one

Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous Galois representation. Again, $\bar{\rho}$ is either an extension of

$$\mathrm{nr}_{\alpha'}\, \omega_f^{\sum_{i=0}^{f-1} \mu_{2,i} p^i} \quad \text{by } \mathrm{nr}_{\alpha}\, \omega_f^{\sum_{i=0}^{f-1} \mu_{1,i} p^i}$$

or is

$$\mathrm{nr}_\alpha \operatorname{Ind}_{G_{K_2}}^{G_K} \omega_{2f}^{\sum_{i=0}^{f-1} \mu_{1,i} p^i + p^f \sum_{i=0}^{f-1} \mu_{2,i} p^i}$$

for some dominant p-restricted character $\mu_{\bar{\rho}} = (\mu_{1,i}, \mu_{2,i})_i \in X^*(T)$ and some α and $\alpha' \in \mathbb{F}^\times$.

Definition 4.1. We say that a dominant p-restricted $\mu \in X^*(T)$ is generic if $2 < \langle \mu, \beta \rangle < p - 3$. We say that $\bar{\rho}$ is generic if $\mu_{\bar{\rho}}$ is generic or if $\bar{\rho}$ is semisimple and 1-generic in the sense of [Le et al. 2016b, Definition 4.1].

Note that if $\bar{\rho}$ is generic, then $\bar{\rho}$ is generic in the sense of [Breuil and Paškūnas 2012, Definition 11.7; Emerton et al. 2015, Definition 2.1.1]. We now assume that $\bar{\rho}$ is not semisimple and is generic. Then a twist of $\bar{\rho}$ is of the form in Section 3A.

We now fix a Serre weight $\sigma \in W(\bar{\rho})$ ($W(\bar{\rho})$ is recalled in Section 3A). Let $\mu \in X^*(T)$ be such that $\sigma \cong F(\mu - \eta)$. If σ is $\sigma_{J(\sigma)}$ with respect to $\mu_{\bar{\rho}}$, define $w_{J(\sigma)} \in W$ by $w_{J(\sigma), i-1} = \mathrm{id}$ if and only if $i \notin J(\sigma)$ for all $i \in \mathbb{Z}/f$ as in Lemma 3.5. Then we set $S_{\bar{\rho}}^{\sigma}$ to be $w(S_{\bar{\rho}})$ with $w = w_{J(\sigma)}^{-1} \pi(w_{J(\sigma)})$.

Lemma 4.2. *The set $W(\bar{\rho})$ is $\{\sigma_J \mid J \subset S_{\bar{\rho}}^{\sigma}\}$ where σ_J is defined in terms of μ.*

Proof. This follows from Proposition 3.2 and [Le et al. 2016b, Proposition 2.4]. □

Let $\psi : G_K \to \mathcal{O}^\times$ be an unramified twist of $\omega_f^{\sum_{i \in \mathbb{Z}/f} (\mu_{1,i} + \mu_{2,i} - 1)}$ lifting $\bar{\varepsilon}^{-1} \det \bar{\rho}$. Suppose that $M_\infty(\cdot)$ is a minimal fixed determinant patching functor over $\mathcal{O}$ for $\bar{\rho}^\vee$ with fixed determinant $\psi^\vee$ (see [Emerton et al. 2015, Definition 6.1.3]). (Note that $\mathcal{D}(\bar{\rho}^\vee)$ in the conventions of [loc. cit., Section 2] is $W(\bar{\rho})$ in ours.) Using contragredients, we identify $R_{\bar{\rho}^\vee}^{\square}$ with $R_{\bar{\rho}}^{\square}$. This identifies R^τ with the (framed) lifting ring of $\bar{\rho}^\vee$ parametrizing lifts $\rho^\vee$ of type $\tau^\vee$ with $\mathrm{HT}_\kappa(\rho^\vee) = \{-1, 0\}$ for all $\kappa : E \hookrightarrow \mathbb{C}_p$. Note that such lifts of $\bar{\rho}^\vee$ are called potentially Barsotti–Tate in [loc. cit., Section 7]. Similar identifications are made for multitype (fixed determinant) potentially Barsotti–Tate deformation rings. For an $\mathcal{O}_K[\mathrm{GL}_2(\mathcal{O}_K)]$-module N, we will denote $M_\infty(N \otimes_{\mathcal{O}_K} \mathcal{O})$ by $M'_\infty(N)$, where tensor product is over the map $\mathcal{O}_K \hookrightarrow \mathcal{O}$ in Section 1A.

Lemma 4.3. *The R_∞-module $M'_\infty(R_\mu / \mathrm{Fil}^2_{\otimes} R_\mu)$ is cyclic.*

Proof. Let τ be the tame type such that $\sigma(\tau) = R_w(\mu - w\eta)$. Then $W(\bar{\rho})$ is exactly $\mathrm{JH}(\bar{\sigma}(\tau))$. Let $\sigma^\circ(\tau) \subset \sigma(\tau)$ be the unique lattice up to homothety with cosocle isomorphic to σ (see [loc. cit., Lemma 4.1.1]). Let $\bar{\sigma}^\circ(\tau)$ be the reduction of $\sigma^\circ(\tau)$. Then the natural map $R_\mu \twoheadrightarrow \bar{\sigma}^\circ(\tau)$ induces a map

$$R_\mu / \mathrm{Fil}^2_{\otimes} R_\mu \twoheadrightarrow \bar{\sigma}^\circ(\tau) / \mathrm{rad}^2 \bar{\sigma}^\circ(\tau). \tag{4-1}$$

By [Le et al. 2016b, Proposition 3.2], the Jordan–Hölder factors of $R_\mu / \mathrm{Fil}^2_{\otimes} R_\mu$ appear without multiplicity. Moreover, those Jordan–Hölder factors which are also in $W(\bar{\rho})$ are in $\mathrm{JH}(\bar{\sigma}^\circ(\tau) / \mathrm{rad}^2 \bar{\sigma}^\circ(\tau))$ by [Emerton et al. 2015, Theorem 5.1.1] (these are exactly the Serre weights σ_J with respect to μ with $J \subset S^{\sigma}_{\bar\rho}$ and $\#J = 1$.). Thus the kernel of the map (4-1) contains no Jordan–Hölder factors in $W(\bar\rho)$. We then see that the induced map

$$M'_\infty(R_\mu/\operatorname{Fil}^2_\otimes R_\mu) \twoheadrightarrow M'_\infty(\bar\sigma^\circ(\tau)/\operatorname{rad}^2\bar\sigma^\circ(\tau))$$

is an isomorphism. As $M'_\infty(\bar\sigma^\circ(\tau))$ is a cyclic R_∞-module by [Emerton et al. 2015, Theorem 10.1.1], so is $M'_\infty(\bar\sigma^\circ(\tau)/\operatorname{rad}^2\bar\sigma^\circ(\tau))$. □

Lemma 4.4. *Suppose that $I \subset S$ such that*

$$\#(I \cap \{\pm\omega^{(i)}\}) + \#(S^{\sigma}_{\bar\rho} \cap \{\pm\omega^{(i)}\}) = 1$$

for all i. Let N be a submodule of $\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I}$, and let $\overline{V}$ be its image in $\operatorname{gr}^k_\otimes R_{\mu,I}$. If $\operatorname{gr}^k_\otimes R_{\mu,I}/\overline{V}$ contains no Serre weights in $W(\bar\rho)$, then

$$(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})/N$$

contains no Jordan–Hölder factors in $W(\bar\rho)$.

Proof. It suffices to show that $\operatorname{gr}^{k+1}_\otimes R_{\mu,I}/\operatorname{gr}^{k+1}_\otimes N$ contains no Jordan–Hölder factors in $W(\bar\rho)$, since by assumption $\operatorname{gr}^k_\otimes R_{\mu,I}/\operatorname{gr}^k_\otimes N$ contains no Jordan–Hölder factors in $W(\bar\rho)$. In fact, it suffices to show that $\operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I}/(N \cap \operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I})$ contains no Jordan–Hölder factors in $W(\bar\rho)$ since $\sum_{|\boldsymbol{k}|=k} \operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I} = \operatorname{gr}^{k+1}_\otimes R_{\mu,I}$.

By Proposition 2.1, a Jordan–Hölder factor of $\operatorname{gr}^{k+1}_\otimes W_{\boldsymbol{k},\boldsymbol{k}+1,I}$ has the form $\sigma_{J'}$ with respect to μ where $J' \cap I = \varnothing$ and there is a $j \in \mathbb{Z}/f$ such that if $\boldsymbol{k}(J') = \boldsymbol{k}'$ then $k'_i = k_i$ for all $i \neq j$ and $k'_j = k_j + 1$. Suppose that $\sigma_{J'} \in W(\bar\rho)$. If $k'_j = 2$, then let $J = J' \setminus \{-w_j\omega^{(j)}\}$ (with w defined in the beginning of the section). Otherwise, $J' \cap \{\pm\omega^{(j)}\} = \{w_j\omega^{(j)}\}$ since we assumed that $\sigma_{J'} \in W(\bar\rho)$. In this case, let $J = J' \setminus \{w_j\omega^{(j)}\}$. Then $\sigma_J \in W(\bar\rho)$ and is thus a Jordan–Hölder factor of $N \cap W_{\boldsymbol{k},\boldsymbol{k}+1,I}$. By Proposition 2.3, $\sigma_{J'}$ is a Jordan–Hölder factor of N. □

The following lemma generalizes [Emerton et al. 2015, Lemma 10.1.13], one of the methods used to compute patched modules.

Lemma 4.5. *Let R be a local ring, and $M'' \subset M' \subset M$ be R-modules such that M'/M'' and M' are minimally generated by the same finite number of elements. Then $M'' \subset \mathfrak{m}M$. If, moreover, M is finitely generated over R, then M/M'' and M are minimally generated by the same number of elements.*

Proof. By Nakayama's lemma, that M'/M'' and M' are minimally generated by the same finite number of elements implies that $M'' \subset \mathfrak{m}M'$ and thus $M'' \subset \mathfrak{m}M$. If M is finitely generated, then another application of Nakayama's lemma implies that M/M'' and M are minimally generated by the same number of elements. □

The following proposition generalizes the results and methods of [Hu and Wang 2018; Le et al. 2016b] by combining Lemmas 4.3, 4.4, and 4.5.

Proposition 4.6. *Suppose that $I \subset S$ such that $\#(I \cap \{\pm\omega^{(i)}\}) + \#(S^{\sigma}_{\bar{\rho}} \cap \{\pm\omega^{(i)}\}) = 1$. Then $M'_\infty(\tilde{R}_{\mu,I})$ is a cyclic R_∞-module.*

Proof. By Nakayama's lemma, it suffices to show that $M'_\infty(R_{\mu,I})$ is a cyclic R_∞-module. We will show that $M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+1}_\otimes R_{\mu,I})$ is a cyclic R_∞-module by induction on k. If $k = 1$, then the result follows from Lemma 4.3.

Now suppose that $M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+1}_\otimes R_{\mu,I})$ is a cyclic R_∞-module. Let $\mathfrak{J}$ be

$$\{J \subset S : k(J) = k,\ J \cap I = \varnothing,\ \sigma_J \in W(\bar{\rho})\}.$$

Recall that for each $J \in \mathfrak{J}$,

$$\overline{V}_J \subset \operatorname{Fil}^k_\otimes R_\mu/\operatorname{Fil}^{k+2}_\otimes R_\mu$$

is defined before [Le et al. 2016b, Proposition 3.9] to be the minimal submodule whose image in $\operatorname{gr}^k_\otimes R_\mu$ contains σ_J. Then we let $\overline{V}_{J,I}$ be the image of $\overline{V}_J$ in $R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I}$. Note that $M'_\infty(\overline{V}_{J,I})$ is a cyclic R_∞-module by Lemma 4.3. Let $\overline{V}$ be $\sum_{J\in\mathfrak{J}} \overline{V}_{J,I} \subset \operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I}$. By Lemma 4.4, the quotient $(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})/\overline{V}$ does not contain any Jordan–Hölder factors in $W(\bar{\rho})$. Thus the natural inclusion $M'_\infty(\overline{V}) \subset M'_\infty(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$ is an equality. In particular,

$$M'_\infty(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$$

is generated by no more than $\#\mathfrak{J}$ elements. On the other hand, $M'_\infty(\operatorname{gr}^k_\otimes R_{\mu,I}) \cong \bigoplus_{J\in\mathfrak{J}} M'_\infty(\sigma_J)$ is generated by (at least) $\#\mathfrak{J}$ elements. By Lemma 4.5 with $M = M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$, $M' = M'_\infty(\operatorname{Fil}^k_\otimes R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$, and $M'' = M'_\infty(\operatorname{gr}^{k+1}_\otimes R_{\mu,I})$, $M'_\infty(R_{\mu,I}/\operatorname{Fil}^{k+2}_\otimes R_{\mu,I})$ is a cyclic R_∞-module. ☐

Proposition 4.7. *The scheme-theoretic support of $M'_\infty(\tilde{R}_{\sigma,I})$ is* $\operatorname{Spec}\big(R_\infty \widehat{\bigotimes}_{R^{\psi,\square}_{\bar{\rho}}} R^{\psi,T_{\sigma,I}}\big)$.

Proof. Since $M'_\infty(\tilde{R}_{\sigma,I})[p^{-1}]$ is isomorphic to $\bigoplus_{\sigma(\tau)\in T_{\sigma,I}} M'_\infty(\sigma(\tau))$, the scheme-theoretic support of $M'_\infty(\tilde{R}_{\sigma,I})[p^{-1}]$ is $\bigcup_{\sigma(\tau)\in T_{\sigma,I}} \operatorname{Spec}\big(R_\infty \widehat{\bigotimes}_{R^{\psi,\square}_{\bar{\rho}}} R^{\psi,\tau}\big)[p^{-1}]$ by the proof of [Emerton et al. 2015, Theorem 9.1.1]. Since $M'_\infty(\tilde{R}_{\sigma,I})$ is $\mathcal{O}$-flat by definition of a patching functor, the scheme-theoretic support of $M'_\infty(\tilde{R}_{\sigma,I})$ is the Zariski closure of that of $M'_\infty(\tilde{R}_{\sigma,I})[p^{-1}]$. The result now follows from the definition of $\operatorname{Spec} R^{\psi,T_{\sigma,I}}$. ☐

In order to weaken the hypotheses on I in Proposition 4.6, we compute an integral scheme intersection, of which the following lemma is the key example.

Lemma 4.8. *There is an exact sequence*

$$0 \to \mathcal{O}[\![Y]\!]/(Y(Y-p)) \to \mathcal{O}[\![Y]\!]/(Y) \oplus \mathcal{O}[\![Y]\!]/(Y-p) \to \mathcal{O}[\![Y]\!]/(Y,p) \to 0,$$

where the second and third maps are the sum and difference, respectively, of the natural projections.

Proof. Given a ring R and ideals I and $J \subset R$, the sequence

$$0 \to R/(I\cap J) \to R/I \oplus R/J \to R/(I+J) \to 0,$$

where the second and third maps are the sum and difference, respectively, of the natural projections, is exact. The lemma follows from this exact sequence and the relations $(Y)\cap(Y-p)=(Y(Y-p))$ and $(Y)+(Y-p)=(Y,p)$ in $\mathcal{O}[\![Y]\!]$. $\square$

The following is our main result in the setting of patching functors. Recall that $\bar\rho$ is generic, but not semisimple.

Theorem 4.9. *Suppose that $I\subset S$ such that $\#(I\cap\{\pm\omega^{(i)}\})+\#(S^\sigma_{\bar\rho}\cap\{\pm\omega^{(i)}\})\le 1$. Then $M'_\infty(\tilde R_{\mu,I})$ is a cyclic R_∞-module.*

Proof. We proceed by induction on $k:=f-\#S^\sigma_{\bar\rho}-\#I$. The case $k=0$ follows from Proposition 4.6. Suppose that $k>0$ and that $(I\cup S^\sigma_{\bar\rho})\cap\{\pm\omega^{(j)}\}=\varnothing$. Then there is an exact sequence

$$0\to\tilde R_{\mu,I}\to\tilde R_{\mu,I\cup\{\omega^{(j)}\}}\oplus\tilde R_{\mu,I\cup\{-\omega^{(j)}\}}\to R_{\mu,I\cup\{\pm\omega^{(j)}\}}\to 0,$$

which induces an exact sequence

$$0\to M'_\infty(\tilde R_{\mu,I})\to M'_\infty(\tilde R_{\mu,I\cup\{\omega^{(j)}\}})\oplus M'_\infty(\tilde R_{\mu,I\cup\{-\omega^{(j)}\}})\to M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})\to 0,$$

where the third map is the sum of two surjections by exactness of $M'_\infty(\cdot)$. By the inductive hypothesis and Proposition 4.7, $M'_\infty(\tilde R_{\mu,I\cup\{\omega^{(j)}\}})$ and $M'_\infty(\tilde R_{\mu,I\cup\{-\omega^{(j)}\}})$ are cyclic R_∞-modules with scheme-theoretic support $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}$ and $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{-\omega^{(j)}\}}}$, respectively. The scheme-theoretic support of $M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ is thus a closed subscheme of the intersections of $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}$ and $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{-\omega^{(j)}\}}}$, which is $\operatorname{Spec} R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p$ by Theorem 3.6 and Lemma 3.5 (we can assume without loss of generality that μ has the form in Section 3 by twisting). Since $M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ is a cyclic R_∞-module, there is a surjection

$$R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p\twoheadrightarrow M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}}).$$

Since $\{\pm\omega^{(j)}\}\cap S^\sigma_{\bar\rho}=\varnothing$, from Proposition 2.1 we see that $M'_\infty(R_{\mu,I\cup\{\omega^{(j)}\}})$ and $M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ have the same Hilbert–Samuel multiplicity. Thus, both sides of the map $R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p\twoheadrightarrow M'_\infty(R_{\mu,I\cup\{\pm\omega^{(j)}\}})$ have the same Hilbert–Samuel multiplicity. Since $R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p$ contains no embedded primes, this map is an isomorphism (see the argument of [Le 2018, Lemma 6.1.1]).

In summary, there is an exact sequence

$$0\to M'_\infty(\tilde R_{\mu,I})\to R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}\oplus R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{-\omega^{(j)}\}}}\to R_\infty\widehat{\bigotimes}_{R^{\psi\square}_{\bar\rho}}R^{\psi,T_{\sigma,I\cup\{\omega^{(j)}\}}}/p\to 0,$$

where the third map is the sum of two surjections. Any lift of a generator under a surjection between two cyclic modules over a local ring is again a generator by Nakayama's lemma. Hence, we can assume that the third map is the difference of the natural projections. Then by Theorem 3.6 and Lemma 3.5, this exact sequence is obtained from taking a completed tensor product with the exact sequence in Lemma 4.8. Hence, we see that $M'_\infty(\tilde R_{\mu,I})\cong R_\infty\widehat{\bigotimes}_{R^{\psi,\square}_{\bar\rho}}R^{\psi,T_{\sigma,I}}$, and in particular that $M'_\infty(\tilde R_{\mu,I})$ is a cyclic R_∞-module. $\square$

5. Global results

Let F be a totally real field in which p is unramified. Let $D_{/F}$ be a quaternion algebra which is unramified at all places dividing p and at most one infinite place, and let $\bar{r}: G_F \to \mathrm{GL}_2(\mathbb{F})$ be a Galois representation. If $D_{/F}$ is indefinite and $K = \prod_w K_w \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$ is an open compact subgroup, then there is a smooth projective curve X_K defined over F and we define $S(K, \mathbb{F})$ to be $H^1((X_K)_{/\bar{F}}, \mathbb{F})$. If $D_{/F}$ is definite, then we let $S(K, \mathbb{F})$ be the space of K-invariant continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^\infty)^\times \to \mathbb{F}.$$

Let S be the union of the set of places in F where $\bar{r}$ is ramified, the set of places in F where D is ramified, and the set of places in F dividing p. Let $\mathbb{T}^{S,\mathrm{univ}}$ be the commutative polynomial algebra over $\mathcal{O}$ generated by the formal variables T_w and S_w for each $w \notin S \cup \{w_1\}$ where w_1 is chosen as in [Emerton et al. 2015, Section 6.2]. Then $\mathbb{T}^{S,\mathrm{univ}}$ acts on $S(K, \mathbb{F})$ with T_w and S_w acting by the usual double coset action of

$$\left[\mathrm{GL}_2(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w & \\ & 1 \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_w})\right] \quad \text{and} \quad \left[\mathrm{GL}_2(\mathcal{O}_{F_w}) \begin{pmatrix} \varpi_w & \\ & \varpi_w \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_{F_w})\right],$$

respectively. Let $\mathbb{T}^{S,\mathrm{univ}} \to \mathbb{F}$ be the map such that the image of $X^2 - T_w X + (\mathbb{N}w) S_w$ in $\mathbb{F}[X]$ is the characteristic polynomial of $\bar{\rho}^\vee(\mathrm{Frob}_w)$, where Frob_w is a geometric Frobenius element at w, and let the kernel be $\mathfrak{m}_{\bar{r}}$.

For the rest of the section, suppose that

(1) $\bar{r}$ is modular, i.e., that there exists K such that $S(K, \mathbb{F})_{\mathfrak{m}_{\bar{r}}}$ is nonzero;

(2) $\bar{r}|_{G_{F(\zeta_p)}}$ is absolutely irreducible;

(3) if $p = 5$ then the image of $\bar{r}(G_{F(\zeta_p)})$ in $\mathrm{PGL}_2(\mathbb{F})$ is not isomorphic to A_5;

(4) $\bar{r}|_{G_{F_w}}$ is generic (Definition 4.1) for all places $w \mid p$; and

(5) $\bar{r}|_{G_{F_w}}$ is nonscalar at all finite places where D ramifies.

Let $v \mid p$ be a place of F, and let $\bar{\rho}$ be $\bar{r}|_{G_{F_v}}$. Let k_v be the residue field of F_v.

We define $S^{\min}$ to be $S(K^v, \otimes_{w \in S, w \neq v} L_w)_{\mathfrak{m}'_{\bar{r}}}$ as in [Emerton et al. 2015, Section 6.5]. We define $M^{\min}$ to be the $\mathbb{F}$-linear dual of $(S^{\min} \otimes_{\mathcal{O}} \mathbb{F})[\mathfrak{m}'_{\bar{r}}]$, factoring out the Galois action in the indefinite case (see [Emerton et al. 2015, Section 6.2]).

Theorem 5.1. *Suppose that $\bar{r}: G_F \to \mathrm{GL}_2(\mathbb{F})$ is a Galois representation satisfying* (1)-(5). *If $\sigma \in W(\bar{\rho})$ and R_σ is the $\mathbb{F}[\mathrm{GL}_2(k_v)]$-projective envelope of σ, then $\mathrm{Hom}_{\mathbb{F}[\mathrm{GL}_2(k_v)]}(R_\sigma, (M^{\min})^*)$ is one-dimensional.*

Proof. The case where $\bar{\rho}$ is semisimple follows from [Le et al. 2016b, Corollary 5.4]. We now assume that $\bar{\rho}$ is not semisimple. Let $\sigma = F(\mu - \eta) \in W(\bar{\rho})$. Identify k_v with a finite field $\mathbb{F}_q$. Then R_σ is $R_\mu \otimes_{\mathbb{F}_q} \mathbb{F}$. Let M_∞ be the minimal fixed determinant patching functor defined in [Emerton et al. 2015, Section 6.5]. By construction, if $\mathfrak{m}_{R_\infty}$ is the maximal ideal of R_∞, then $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(R_\sigma, (M^{\min})^*)$ is the dual of $M_\infty(R_\sigma)/\mathfrak{m}_{R_\infty} = M'_\infty(R_\mu)/\mathfrak{m}_{R_\infty}$, which is one dimensional since $M'_\infty(R_\mu)$ is a cyclic R_∞-module by Theorem 4.9. $\square$

Let $M^{\min}(K_v(1))$ denote the space of coinvariants $(M^{\min})_{K_v(1)}$. Note that $M^{\min}(K_v(1))$ is isomorphic to the dual of $(S(K^v K_v(1), \otimes_{w\in S, w\neq v} L_w) \otimes_{\mathcal{O}} \mathbb{F})[\mathfrak{m}'_{\bar{r}}]$, factoring out the Galois action in the indefinite case, by a standard spectral sequence argument using that $\mathfrak{m}'_{\bar{r}}$ is non-Eisenstein.

Corollary 5.2. *Suppose that $\bar{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ is a Galois representation satisfying* (1)-(5). *Then the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $(M^{\min}(K_v(1)))^*$ is isomorphic to $D_0(\bar{\rho})$. In particular, $(M^{\min}(K_v(1)))^*$ depends only on $\bar{\rho}$ and is multiplicity free.*

Proof. There is an injection $D_0(\bar{\rho}) \hookrightarrow (M^{\min}(K_v(1)))^*$ by [Breuil 2014, Proposition 9.3]. Fix an $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-injective hull $(M^{\min}(K_v(1)))^* \hookrightarrow I$. Since

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(R_\sigma, (M^{\min}(K_v(1)))^*)$$

is one-dimensional for all $\sigma \in W(\bar{\rho})$ by Theorem 5.1, this injective hull factors through $D_0(\bar{\rho})$ by [Breuil and Paškūnas 2012, Theorem 1.1(i)]. Since $D_0(\bar{\rho})$ and $(M^{\min}(K_v(1)))^*$ are finite length $\mathbb{F}[\mathrm{GL}_2(\mathbb{F}_q)]$-modules, they must be isomorphic. Finally, note that $D_0(\bar{\rho})$ is multiplicity free by [Breuil and Paškūnas 2012, Theorem 1.1(ii)]. □

Acknowledgments

Lemma 4.5 originally appeared in [Le et al. ≥ 2019], and we thank Bao Le Hung and Stefano Morra for allowing us to reproduce it here. The idea to use multitype Barsotti–Tate deformation rings grew out of the joint work [loc. cit.]. We thank Bao Le Hung and Stefano Morra for their collaboration and other useful discussions on Kisin modules and étale φ-modules. We thank the anonymous referee for many useful suggestions, comments, and corrections. The debt owed to the work of Christophe Breuil, Matthew Emerton, Toby Gee, Vytautas Paškūnas, and David Savitt will be clear to the reader.

The author was supported by the Simons Foundation under an AMS-Simons travel grant, by the National Science Foundation under the Mathematical Sciences Postdoctoral Research Fellowship No. 1703182, and by the Centre International de Rencontres Mathématiques under the Research in Pairs program No. 1877. We thank CIRM for providing hospitality and excellent working conditions while part of this work was carried out.

References

[Breuil 2014] C. Breuil, "Sur un problème de compatibilité local-global modulo p pour GL_2", *J. Reine Angew. Math.* **692** (2014), 1–76. MR Zbl

[Breuil and Paškūnas 2012] C. Breuil and V. Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc. **1016**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Buzzard et al. 2010] K. Buzzard, F. Diamond, and F. Jarvis, "On Serre's conjecture for mod ℓ Galois representations over totally real fields", *Duke Math. J.* **155**:1 (2010), 105–161. MR Zbl

[Diamond 1997] F. Diamond, "The Taylor–Wiles construction and multiplicity one", *Invent. Math.* **128**:2 (1997), 379–391. MR Zbl

[Diamond 2007] F. Diamond, "A correspondence between representations of local Galois groups and Lie-type groups", pp. 187–206 in *L-functions and Galois representations* (Durham, UK, 2004), edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge Univ. Press, 2007. MR Zbl

[Emerton et al. 2015] M. Emerton, T. Gee, and D. Savitt, "Lattices in the cohomology of Shimura curves", *Invent. Math.* **200**:1 (2015), 1–96. MR Zbl

[Fujiwara 2006] K. Fujiwara, "Deformation rings and Hecke algebras in the totally real case", submitted, 2006. arXiv

[Henniart 2002] G. Henniart, "Sur l'unicité des types pour GL_2", (2002). Appendix to C. Breuil and A. Mézard, "Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $l = p$", *Duke Math. J.* **115**:2 (2002), 205–310. MR Zbl

[Herzig 2009] F. Herzig, "The weight in a Serre-type conjecture for tame n-dimensional Galois representations", *Duke Math. J.* **149**:1 (2009), 37–116. MR Zbl

[Hu and Wang 2018] Y. Hu and H. Wang, "Multiplicity one for the p cohomology of Shimura curves: the tame case", *Math. Res. Lett.* **25**:3 (2018), 843–873. MR Zbl

[Jantzen 1987] J. C. Jantzen, *Representations of algebraic groups*, Pure Appl. Math. **131**, Academic Press, Boston, 1987. MR Zbl

[Kisin 2008] M. Kisin, "Potentially semi-stable deformation rings", *J. Amer. Math. Soc.* **21**:2 (2008), 513–546. MR Zbl

[Le 2018] D. Le, "Lattices in the cohomology of $U(3)$ arithmetic manifolds", *Math. Ann.* **372**:1-2 (2018), 55–89. MR Zbl

[Le et al. 2016a] D. Le, B. V. Le Hung, B. Levin, and S. Morra, "Serre weights and Breuil's lattice conjecture in dimension three", preprint, 2016. arXiv

[Le et al. 2016b] D. Le, S. Morra, and B. Schraen, "Multiplicity one at full congruence level", preprint, 2016. arXiv

[Le et al. 2018] D. Le, B. V. Le Hung, B. Levin, and S. Morra, "Potentially crystalline deformation rings and Serre weight conjectures: shapes and shadows", *Invent. Math.* **212**:1 (2018), 1–107. MR Zbl

[Le et al. $\geq$ 2019] D. Le, B. V. Le Hung, and S. Morra, "Cohomology of $U(3)$ arithmetic manifolds at full congruence level", in preparation.

Communicated by Marie-France Vignéras
Received 2017-10-19 Revised 2019-02-13 Accepted 2019-05-27

le@math.toronto.edu Department of Mathematics, University of Toronto, ON, Canada

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1829

Theta operators on unitary Shimura varieties

Ehud de Shalit and Eyal Z. Goren

We define a theta operator on p-adic vector-valued modular forms on unitary groups of arbitrary signature, over a quadratic imaginary field in which p is inert. We study its effect on Fourier–Jacobi expansions and prove that it extends holomorphically beyond the μ-ordinary locus, when applied to scalar-valued forms.

Introduction	1829
1. Background	1833
2. Differential operators on p-adic modular forms	1843
3. Toroidal compactifications and Fourier–Jacobi expansions	1849
4. Analytic continuation of Θ to the nonordinary locus	1864
5. Theta cycles	1874
Acknowledgments	1875
References	1875

Introduction

Let E be a quadratic imaginary field and p a prime which is inert in E. The purpose of this article is to define a theta operator Θ for p-adic vector-valued modular forms on unitary Shimura varieties of arbitrary signature associated with the extension $E/\mathbb{Q}$, and prove some fundamental results concerning it. Specifically, we prove a formula for the action of Θ in terms of Fourier–Jacobi expansions (Theorem 3.2.5). We also prove that Θ extends to a holomorphic operator outside the μ-ordinary locus, when acting on scalar-valued modular forms in characteristic p (Theorems 4.2.3 and 4.3.2).

When the prime p is *split* in E, general points on the special fiber of the Shimura variety parametrize ordinary abelian varieties. A theta operator, and a whole array of differential operators derived from it, were defined in this context in Eischen's thesis [2012]. Her construction was generalized in [Eischen et al. 2018] to unitary Shimura varieties associated with a general CM field, but still under the ordinariness assumption. In their work, these authors circumvent the study of Θ on Fourier–Jacobi expansions by expressing it in Serre–Tate coordinates at CM points.

"Ordinariness" is a strong assumption. Over the ordinary locus, it provides a unit-root splitting of the Hodge filtration in the cohomology of the universal abelian variety. This allows one to extend Katz's approach to Θ [1977]. The unit-root splitting serves as a p-adic replacement for the Hodge decomposition

MSC2010: primary 11G18; secondary 14G35.
Keywords: Shimura variety, theta operator, modular form.

over the complex numbers, which underlies the construction of similar C^∞-differential operators of Ramanujan and of Maass and Shimura [Shimura 2000, Section III].

In [de Shalit and Goren 2016], we defined a Θ-operator on unitary modular forms of signature (2, 1) and determined its effect on q-expansions, for p *inert* in the quadratic imaginary field E. The main obstacle in this case was that the abelian variety parametrized by a general point of the special fiber of the Shimura variety is not ordinary anymore, but so-called μ-ordinary, and its cohomology does not admit a unit-root splitting. Our approach there, adopted also in the present paper, is to make systematic use of Igusa varieties; we first define the theta operator on them, and show that it descends to the Shimura variety.

Recently, we have learned of the work of Ellen Eischen and Elena Mantovan [2017] in which they construct the same differential operators in the μ-ordinary (p inert) case. Their method is closer to the original idea of Katz, but they replace the unit-root splitting by slope filtration splitting of F-crystals. Their construction is more general than ours, as it applies to unitary Shimura varieties associated with a general CM field. They apply their differential operators to the study of p-adic families of modular forms in the spirit of Serre, Katz and Hida. Their work should have applications to questions of over-convergence, construction of p-adic L-functions and Iwasawa theory. However, the issues addressed in the present paper, the effect of Θ on Fourier–Jacobi expansions and its holomorphic extension beyond the μ-ordinary locus, are not considered there.

We now provide some background and motivation for the study undertaken in this paper. The theta operator for elliptic modular forms is related to an operator already defined by Ramanujan. On q-expansions it is given by

$$f = \sum_n a_n q^n \mapsto \Theta(f) = \sum_n n a_n q^n.$$

Over the complex numbers, this operator does not preserve the space of holomorphic modular forms. However, viewed at the level of q-expansions for p-adic, or mod p, modular forms, it does, at least when one has reasonable demands: in characteristic p one has to multiply $\Theta(f)$ by h, the Hasse invariant, which is a modular form of weight $p-1$ vanishing outside the ordinary locus; p-adically one has to be content with working merely over the ordinary locus.

These aspects were present from the very start in the work of Swinnerton-Dyer [1973] and Serre [1973a; 1973b]. In fact, already in [Serre 1973a], motivated by relation to Galois representations, Serre investigates the notion of *filtration*. The filtration of a q-expansion of a mod p modular form is the minimal weight in which one may find a modular form with that q-expansion; one is interested in its variation under applications of Θ, which at the level of Galois representations corresponds to a cyclotomic twist. Following closely on the heels of these developments, Katz [1977] gave a geometric construction of Θ on (essentially) all modular curves with good reduction at p.

Not much later, Jochnowitz [1982] studied Θ-*cycles*. The basic idea is simple. If $g = \Theta(f)$ has filtration w_0, the series of filtrations w_i of $\Theta^i(g)$, $i = 0, 1, \ldots, p-1$, is a collection of weights that is generally increasing, but not always, because $w_{p-1} = w_0$. The question of the variation of the filtration along the cycles is interesting and has important applications. See [Gross 1990; Jochnowitz 1982]. Further deep uses of the Θ-operator to over-convergence and classicality of p-adic modular forms were given in [Coleman 1996; Coleman et al. 1995].

Katz [1978, Section II] studied such an operator for Hilbert modular forms associated to a totally real field L, and in fact enriched the theory by introducing $g = [L : \mathbb{Q}]$ basic theta operators. These operators were instrumental in his construction of p-adic L-functions for CM fields via the Eisenstein measure. In that work, as in the case of modular curves, strong use is made of the behavior of de Rham cohomology and the unit root splitting over the ordinary locus. The study of these operators was further developed by Andreatta and the second author [Andreatta and Goren 2005], who constructed mod p versions of them by means of the Igusa variety, and provided some results on filtrations, Θ-cycles and relations to cyclotomic twists.

It seemed a natural idea at that point to extend the theory of the theta operator to other Shimura varieties of PEL type. However, two obstacles arise:

(i) The abelian variety classified by a general point of the Shimura variety in positive characteristic may not be ordinary anymore. In particular, its de Rham cohomology may not admit a unit root splitting.

(ii) The natural definition takes modular forms, even if scalar-valued, to vector-valued modular forms.

Bearing in mind the Kodaira–Spencer isomorphism, which is involved in the definition of Θ, the second problem could be anticipated. In the Hilbert modular case, it is the abundance of endomorphisms that allows one to return to scalar-valued modular forms. In spite of these difficulties, progress has been made on other Shimura varieties: As Eischen had already remarked in her thesis, her construction generalizes almost immediately to the symplectic case. Panchishkin and Courtieu discussed similar operators for Siegel modular forms in [Courtieu and Panchishkin 2004, Sections 2 and 3; Panchishkin 2005]. For different aspects in the symplectic case see the papers by Böcherer–Nagaoka [2007] and Ghitza and McAndrew [2016], and additional references therein. For other cases, see the work of Johansson [2013].

Our construction of the theta operator via the Igusa tower was motivated by Gross' construction [1990]. For an application of the Igusa tower to the study of vector-valued p-adic Siegel modular forms see [Ichikawa 2014].

The contents of this paper are as follows. Let E be a quadratic imaginary field, p a rational prime that is inert in E and $\kappa = \mathcal{O}_E/(p)$ its residue field. Let $n \geq m$ be positive integers. Fixing additional data, one obtains a scheme $\mathcal{S}$ over $\mathcal{O}_{E,(p)}$ that parametrizes abelian schemes with $\mathcal{O}_E$-action of signature (n, m), endowed with a principal polarization and level structure. Its complex points are a union of Shimura varieties associated to the unitary group $\mathrm{GU}(n, m)$. Let $S \to \mathrm{Spec}(\kappa)$ denote its special fiber, and let S_s be the base change of $\mathcal{S}$ to $W_s = W_s(\kappa)$.

In Section 1 we collect background material and definitions, and in particular define the type of vector-valued p-adic modular forms that will be considered in this paper. Automorphic vector bundles over $\mathcal{S}$ correspond to representations of the group $\mathrm{GL}_m \times \mathrm{GL}_n$, and there are two "basic" vector bundles, $\mathcal{Q}$ and $\mathcal{P}$, corresponding to the standard representations of the two blocks, from which all others are derived.[1] Characteristic p holds its own idiosyncrasies and there are 3 vector bundles, denoted $\mathcal{Q}$, $\mathcal{P}_0$ and $\mathcal{P}_\mu$, from which all *p-adic* automorphic vector bundles $\mathcal{E}_\rho$ are derived by representation-theoretic constructions; in particular, ρ refers here to a representation of $\mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$. We briefly explain the origin of these vector bundles. The relative cotangent bundle of the universal abelian variety $\mathcal{A} \to \mathcal{S}$ decomposes according to signatures, providing us with vector bundles $\mathcal{P}$, $\mathcal{Q}$ of ranks n, m, respectively. Over the (μ-)ordinary locus S_s^{ord} of S_s, $\mathcal{P}$ admits a filtration $0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0$. The vector bundle $\mathcal{E}_\rho$ lives over S_s^{ord} and is obtained by "twisting" ρ by the triple $(\mathcal{Q}, \mathcal{P}_\mu, \mathcal{P}_0)$ (see page 1841 for details). A mod-p^s modular form of weight ρ is defined to be a section of $\mathcal{E}_\rho$ over S_s^{ord}.

In Section 2 we define the Igusa tower over S_s^{ord} and study its properties. The key fact about the Igusa tower is that the vector bundles $\mathcal{P}_0$, $\mathcal{P}_\mu$ and $\mathcal{Q}$ (unlike $\mathcal{P}$!) are all *canonically trivialized* over it. To be precise, much as in [Katz 1975], the Igusa tower is a double limit of schemes $\{T_{t,s} \mid t, s \geq 1\}$, where $T_{t,s}$ is a scheme over the truncated Witt vectors W_s of length s, and whenever $t \geq s$ a trivialization as above is obtained. Consequently, we are able to propagate, by linear algebra constructions alone, the trivial connection $d : \mathcal{O}_T \to \Omega_{T/W_s}$ for $T = T_{t,s}$, $t \geq s$, to a connection

$$\tilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \Omega_{T/W_s} \cong \mathcal{E}_\rho \otimes \mathcal{P} \otimes \mathcal{Q},$$

the last isomorphism stemming from the Kodaira–Spencer map. When we follow this map by the projection $\mathcal{E}_\rho \otimes \mathcal{P} \otimes \mathcal{Q} \to \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$, and combine it with pull back of modular forms under $T \to S_s^{\mathrm{ord}}$, we obtain an operator

$$\Theta : H^0(S_s^{\mathrm{ord}}, \mathcal{E}_\rho) \to H^0(S_s^{\mathrm{ord}}, \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}).$$

This operator can be iterated and combined with representation-theoretic operations as discussed in the end of Section 2, to produce an array of differential operators $D_\kappa^{\kappa'}$ as in [Eischen et al. 2018; Eischen and Mantovan 2017].

The initial parts of Section 3 are a review of the theory of toroidal compactifications for the case at hand. We follow Faltings and Chai [1990], that relies on the seminal work of Mumford and his school, Skinner and Urban [2014], and the definitive volume by Lan [2013]. In particular, the reader will find a precise explanation of the meaning of the Fourier–Jacobi expansion of a vector-valued modular form

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}), q^{\check{h}}.$$

See page 1857. In this notation our first main theorem states the following.

Theorem (Theorem 3.2.5). *Let ξ be a rank-m cusp. Let f be a global section of $\mathcal{E}_\rho$ and $\sum_{h \in \check{H}^+} a(\check{h}) q^{\check{h}}$ its Fourier–Jacobi expansion at ξ. Then the section $\Theta(f)$ of $\mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$ has the Fourier–Jacobi expansion*

$$\Theta(f) = \sum_{\check{h} \in \check{H}^+} a(\check{h}) \otimes \check{h} \cdot q^{\check{h}}.$$

[1] As the Levi factor of the appropriate parabolic in $\mathrm{GU}(n, m)_{\mathbb{C}}$ is $\mathbb{G}_m \times \mathrm{GL}_m \times \mathrm{GL}_n$ we could, in principle, take also representations that are nontrivial on the first factor. However, we will have no need for this greater generality in this paper and so, here and in the sequel, we will consider automorphic vector bundles associated to representations of $\mathrm{GL}_m \times \mathrm{GL}_n$ only.

The analogous result for the Fourier–Jacobi expansion at a non maximally degenerate cusp (of rank $< m$) should involve also theta operators on lower-rank Shimura varieties acting on the coefficients. For most practical purposes, however, e.g., for a q-expansion principle, rank m cusps suffice.

In Section 4 we consider the extension of the operator Θ to the complement of the μ-ordinary locus. This we are able to do, so far, only for scalar-valued modular forms. The proof requires a partial compactification of a particular Igusa variety as in [de Shalit and Goren 2016], and delicate computations with Dieudonné modules in the spirit of our recent work [de Shalit and Goren 2018]. Let $\mathcal{L} = \det \mathcal{Q}$ and $k \geq 0$.

Theorem (Theorems 4.2.3 and 4.3.2). *Consider the operator*

$$\Theta : H^0(S^{\mathrm{ord}}, \mathcal{L}^k) \to H^0(S^{\mathrm{ord}}, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Then Θ extends holomorphically to an operator

$$\Theta : H^0(S, \mathcal{L}^k) \to H^0(S, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Finally, in Section 5 we introduce the notion of Θ-cycles and recall interesting phenomena observed in [de Shalit and Goren 2016].

Our paper and the work of Eischen and Mantovan suggest several directions in which the theory can be further developed. In addition to those mentioned in [Eischen and Mantovan 2017] we suggest the following problems:

(i) Provide a formula for the Fourier–Jacobi expansion and the theta operator Θ at general cusps.

(ii) Study the extension of Θ to a holomorphic operator for general vector-valued unitary modular forms.

(iii) Develop a theory of mod p operators, such as U and V and characterize the kernel of Θ in terms of V, see [Katz 1977].

(iv) Study Θ-cycles in relation to mod p Galois representations.

1. Background

1.1. *The Shimura variety.*

Linear algebra. We review some background and set up standard notation. Let E be a quadratic imaginary field, embedded in $\mathbb{C}$, $0 \leq m \leq n$ and $\Lambda = \mathcal{O}_E^{n+m}$. Let

$$I_{n,m} = \begin{pmatrix} & & I_m \\ & I_{n-m} & \\ I_m & & \end{pmatrix} \tag{1.1.1}$$

where I_l is the unit matrix of size l, and introduce the perfect hermitian pairing

$$(u, v) = {}^t\bar{u} I_{n,m} v \tag{1.1.2}$$

on Λ. Let

$$\boldsymbol{G} = \mathrm{GU}(\Lambda, (\cdot\,, \cdot))$$

be the group of unitary similitudes of Λ, regarded as a group scheme over $\mathbb{Z}$, and denote by $\nu : \boldsymbol{G} \to \mathbb{G}_m$ the similitude character. For any commutative ring R

$$\boldsymbol{G}(R) = \{g \in \mathrm{GL}_{n+m}(\mathcal{O}_E \otimes R) \mid \forall u, v \in \Lambda \otimes R, \quad (gu, gv) = \nu(g)(u, v)\}.$$

Then $\boldsymbol{G}(\mathbb{R}) = \mathrm{GU}(n, m)$ is the general unitary group of signature (n, m), and $\boldsymbol{G}(\mathbb{C}) \simeq \mathrm{GL}_{n+m}(\mathbb{C}) \times \mathbb{C}^\times$.

Let δ_E be the unique generator of the different $\mathfrak{d}_E$ of E with $\mathrm{Im}(\delta_E) > 0$. The *polarization pairing*

$$\langle u, v\rangle = \mathrm{Tr}_{E/\mathbb{Q}}(\delta_E^{-1}(u, v)) \tag{1.1.3}$$

is then a perfect alternating pairing $\Lambda \times \Lambda \to \mathbb{Z}$ satisfying $\langle au, v\rangle = \langle u, \bar{a}v\rangle$ $(a \in E)$.

Let p be an odd prime which is inert in E, and fix once and for all an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. Let E_p be the completion of E and $\mathcal{O}_p$ its ring of integers. As -1 is a norm from E_p to $\mathbb{Q}_p$, one easily checks that $\boldsymbol{G}_{/\mathcal{O}_p}$ is quasisplit. In fact, over $\mathcal{O}_p$ the lattice $\Lambda_p = \mathbb{Z}_p \otimes \Lambda = \mathcal{O}_p^{n+m}$, equipped with the hermitian form (1.1.2), is isomorphic to the same lattice equipped with the pairing ${}^t\bar{u} J_{n+m} v$, where by J_l we denote the matrix with 1's on the antidiagonal and 0's elsewhere. This will be useful later.

If R is an $\mathcal{O}_{E,(p)}$-algebra then any R-module M endowed with a commuting $\mathcal{O}_E$- action decomposes according to types,

$$M = M(\Sigma) \oplus M(\overline{\Sigma}),$$

where $M(\Sigma)$ is the R-submodule on which $\mathcal{O}_E$ acts via the canonical homomorphism

$$\Sigma : \mathcal{O}_E \hookrightarrow \mathcal{O}_{E,(p)} \to R,$$

while $M(\overline{\Sigma})$ is the part on which it acts via the conjugate homomorphism $\overline{\Sigma}$. Indeed, it is enough to decompose $\mathcal{O}_E \otimes R = R(\Sigma) \times R(\overline{\Sigma})$ as an $\mathcal{O}_E$-algebra. The same notation will be applied to coherent sheaves with $\mathcal{O}_E$-action on schemes defined over $\mathcal{O}_{E,(p)}$.

We denote by κ the field $\mathcal{O}_E/p\mathcal{O}_E$ of p^2 elements.

The Shimura variety and the moduli problem. Fix an integer $N \geq 3$ relatively prime to p. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the adéle ring of $\mathbb{Q}$, where $\mathbb{A}_f = \mathbb{Q} \cdot \hat{\mathbb{Z}}$ are the finite adéles. Let $K_f \subset \boldsymbol{G}(\hat{\mathbb{Z}})$ be an open subgroup of the form $K_f = K^p K_p$, where $K^p \subset \boldsymbol{G}(\mathbb{A}^p)$ is the principal congruence subgroup of level N, and

$$K_p = \boldsymbol{G}(\mathbb{Z}_p) \subset \boldsymbol{G}(\mathbb{Q}_p),$$

which is a hyperspecial maximal compact subgroup at p. Let $K_\infty \subset \boldsymbol{G}(\mathbb{R})$ be the stabilizer of the negative definite subspace spanned by $\{-e_i + e_{n+i} : 1 \leq i \leq m\}$ in $\Lambda_{\mathbb{R}} = \mathbb{C}^{n+m}$, where $\{e_i\}$ stands for the standard basis. This K_∞ is a maximal compact-modulo-center subgroup, isomorphic to $G(U(m) \times U(n))$. By $G(U(m) \times U(n))$ we mean the pairs of matrices $(g_1, g_2) \in \mathrm{GU}(m) \times \mathrm{GU}(n)$ having the same similitude factor. Let $K = K_\infty K_f \subset \boldsymbol{G}(\mathbb{A})$ and $\mathfrak{X} = \boldsymbol{G}(\mathbb{R})/K_\infty$.

To the Shimura datum $(\boldsymbol{G}, \mathfrak{X})$ and the level subgroup K there is associated a Shimura variety Sh_K. It is a quasiprojective nonsingular variety of dimension nm defined over E. If $m = n$ the Shimura variety may even be defined over $\mathbb{Q}$, but we still denote by Sh_K its base-change to E. The complex points of Sh_K are identified, as a complex manifold, with

$$\mathrm{Sh}_K(\mathbb{C}) = \boldsymbol{G}(\mathbb{Q})\backslash\boldsymbol{G}(\mathbb{A})/K.$$

Following Kottwitz [1992] we define a scheme $\mathcal{S}$ over $\mathcal{O}_{E,(p)}$. This $\mathcal{S}$ is a fine moduli space whose R-points, for every $\mathcal{O}_{E,(p)}$-algebra R, classify isomorphism types of tuples $\underline{A} = (A, \iota, \phi, \eta)$ where:

- A is an abelian scheme of dimension $n+m$ over R.
- $\iota : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ has signature (n, m) on the Lie algebra of A.
- $\phi : A \xrightarrow{\sim} A^t$ is a principal polarization whose Rosati involution induces $\iota(a) \mapsto \iota(\bar{a})$ on the image of ι.
- η is an $\mathcal{O}_E$-linear full level-N structure on A compatible with $(\Lambda, \langle\cdot,\cdot\rangle)$ and ϕ [Lan 2013, 1.3.6].

See [Lan 2013, Section 1.4] for the comparison of the various languages used to define the moduli problem.

The generic fiber $\mathcal{S}_E$ of $\mathcal{S}$ is, in general, a union of *several* Shimura varieties, one of which is Sh_K. This is due to the failure of the Hasse principle for $\boldsymbol{G}$, which can happen when $m+n$ is odd [Kottwitz 1992, Section 7]. We also remark that the assumption $N \geq 3$ could be avoided if we were willing to use the language of stacks. As this is not essential to the present paper, we keep the scope slightly limited for the sake of clarity.

As shown by Kottwitz, $\mathcal{S}$ is *smooth* of relative dimension nm over $\mathcal{O}_{E,(p)}$.

The universal abelian variety and its p-divisible group. By virtue of the moduli problem which it represents, $\mathcal{S}$ carries a universal abelian scheme $\mathcal{A}_{/\mathcal{S}}$ equipped with a PEL structure as above. Let

$$S = \mathcal{S} \times_{\mathrm{Spec}(\mathcal{O}_{E,(p)})} \mathrm{Spec}(\kappa)$$

be the special fiber of $\mathcal{S}$. Recall that for any geometric point $x : \mathrm{Spec}(k) \to S$ the p-divisible group of $A = \mathcal{A}_x$ carries a canonical filtration by p-divisible groups

$$\mathrm{Fil}^0 = A[p^\infty] \supset \mathrm{Fil}^1 = A[p^\infty]^0 \supset \mathrm{Fil}^2 = A[p^\infty]^\mu \supset 0, \tag{1.1.4}$$

where $\mathrm{gr}^2 = A[p^\infty]^\mu$ is multiplicative, $\mathrm{gr}^1 = A[p^\infty]^0/A[p^\infty]^\mu$ is local-local and $\mathrm{gr}^0 = A[p^\infty]/A[p^\infty]^0$ is étale. Over $\mathrm{Spec}(k)$ this filtration is even split, i.e., $A[p^\infty]$ is uniquely expressible as a product of multiplicative, local-local and étale p-divisible groups, but this fact is special for algebraically closed (or perfect) fields, while a filtration like (1.1.4) often exists over more general bases.

The special fiber S contains an open dense subset called the μ-ordinary locus, [Wedhorn 1999; Moonen 2004, Theorem 3.2.7], which we denote S^{ord}. It is characterized by the fact that for any geometric point x of S, x lies in S^{ord} if and only if the height of $A[p^\infty]^\mu$ is $2m$, which is as large as it can get. Equivalently, the Newton polygon of $A[p^\infty]$ has slopes 0, $\frac{1}{2}$ and 1 with horizontal lengths $2m$, $2(n-m)$ and $2m$ respectively, which is as low as it can get. In fact, Wedhorn and Moonen show that the isomorphism type of $A[p^\infty]$, as a polarized $\mathcal{O}_E$-group, is the same for all $x \in S^{\mathrm{ord}}(k)$:

$$A[p^\infty] \simeq (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m \times \mathfrak{G}_k^{n-m} \times (\mathcal{O}_E \otimes (\mathbb{Q}_p/\mathbb{Z}_p))^m.$$

Here $\mathfrak{G}_k$ is the p-divisible group denoted by $G_{1/2,1/2}$ in the Dieudonné-Manin classification. It is the unique height-2 one-dimensional connected p-divisible group over k. It is well-known that the ring $\mathcal{O}_p$ acts as endomorphisms of $\mathfrak{G}_k$. We normalize this action so that the induced action of $\mathcal{O}_p$ on the Lie algebra of $\mathfrak{G}_k$ is via $\Sigma : \mathcal{O}_p \twoheadrightarrow \kappa \subset k$, and this pins down $\mathfrak{G}_k$ as an $\mathcal{O}_E$-group up to isomorphism. We polarize it fixing an isomorphism of $\mathfrak{G}_k$ with its Serre dual. The appearance of the inverse different in the first factor is a matter of choice, and is meant to allow a more natural way to write the Weil pairing between the first and last factors, namely

$$\langle a \otimes x, b \otimes y \rangle = \mathrm{Tr}_{E/\mathbb{Q}}(\bar{a}b)\langle x, y\rangle.$$

Over S^{ord}, a filtration like (1.1.4) exists globally, but is far from being split now [de Shalit and Goren 2017, Proposition 2.10]. Nevertheless, its graded pieces are, locally in the proétale topology, isomorphic to the constant p-divisible groups $(\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m$, $\mathfrak{G}_k^{n-m}$ and $(\mathcal{O}_E \otimes (\mathbb{Q}_p/\mathbb{Z}_p))^m$, and the isomorphisms can be taken to respect the endomorphisms and the polarization. This is well-known for gr^0 and gr^2. For gr^1 it follows from the rigidity of isoclinic Barsotti–Tate groups with endomorphisms, namely from the fact that the universal deformation ring of $(\mathfrak{G}_k, \iota)$ where $\iota : \mathcal{O}_p \hookrightarrow \mathrm{End}(\mathfrak{G}_k)$, is $W(k)$ ([Moonen 2004, Corollary 2.1.5], see [loc. cit., Section 3.3.1] for the polarization). This result implies that for any geometric point $x \in S^{\mathrm{ord}}(k)$, $\mathrm{gr}^1 \mathcal{A}[p^\infty]$ becomes isomorphic over $\hat{\mathcal{O}}_{S,x}$ to $\mathfrak{G}_k^{n-m}$, with its additional structures of endomorphisms and polarization. By Artin's approximation theorem [1969] they become isomorphic already over the strict henselization $\mathcal{O}_{S,x}^{\mathrm{sh}}$, which means that they are locally isomorphic in the proétale topology.

The basic vector bundles on S. The Hodge bundle $\omega = \omega_{\mathcal{A}/\mathcal{S}}$ is the pull-back via the zero section $e_{\mathcal{A}} : \mathcal{S} \to \mathcal{A}$ of the relative cotangent sheaf $\Omega_{\mathcal{A}/\mathcal{S}}$ of the universal abelian scheme. It decomposes as

$$\omega = \omega(\Sigma) \oplus \omega(\overline{\Sigma}) = \mathcal{P} \oplus \mathcal{Q}$$

according to types. Thus, $\mathrm{rk}(\mathcal{P}) = n$ and $\mathrm{rk}(\mathcal{Q}) = m$.

Lemma 1.1.1. *The line bundles* $\det(\mathcal{P})$ *and* $\det(\mathcal{Q})$ *are isomorphic over* $\mathcal{S}$.

Proof. The proof is similar to [de Shalit and Goren 2017, Proposition 1.3]. Automorphic vector bundles over the generic fiber $\mathcal{S}_E$ correspond functorially to representations of the group $\mathrm{GL}_m \times \mathrm{GL}_n$, as discussed below on page 1842. The vector bundles $\det(\mathcal{Q})$ and $\det(\mathcal{P})$ correspond to the determinant of GL_m and the *inverse* of the determinant of GL_n. Their ratio therefore corresponds to the determinant of $\mathrm{GL}_m \times \mathrm{GL}_n$. If the level subgroup K is small enough, as we always assume, then the arithmetic group by which we divide the symmetric space to get a complex uniformization of every connected component of $\mathcal{S}_{\mathbb{C}}$ is contained in $\mathrm{SU}(n, m)$. This means that over $\mathbb{C}$, the automorphic line bundle corresponding to det is trivial, hence $\det(\mathcal{P}) \simeq \det(\mathcal{Q})$. From this it is easy to get the claim even over the base $\mathcal{O}_{E,(p)}$. We stress that we do not know a direct moduli-theoretic proof of the claim in the lemma, and we do not know if the particular isomorphism supplied by the complex analytic uniformization is defined over $\overline{\mathbb{Q}}$. See however Corollary 1.1.3 below. □

Over the special fiber S we have the Verschiebung homomorphism $V : \omega \to \omega^{(p)}$ induced by the Verschiebung isogeny $\mathrm{Ver} : \mathcal{A}^{(p)} \to \mathcal{A}$. As V commutes with the endomorphisms it maps $\mathcal{P}$ to $\mathcal{Q}^{(p)}$ and $\mathcal{Q}$ to $\mathcal{P}^{(p)}$. We denote the restriction of V to $\mathcal{P}$ (resp. $\mathcal{Q}$) by $V_{\mathcal{P}}$ (resp. $V_{\mathcal{Q}}$). The homomorphism

$$H = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}^{(p^2)}$$

is called the *Hasse matrix*. We let $\mathcal{L} = \det(\mathcal{Q})$, a line bundle. Then

$$h = \det(H) : \mathcal{L} \to \mathcal{L}^{(p^2)} \simeq \mathcal{L}^{p^2} \tag{1.1.5}$$

is a global section of $\mathcal{L}^{p^2-1}$ called the *μ-ordinary Hasse invariant* [Goldring and Nicole 2017, Appendix B]. Here we used the well-known fact that for a line bundle $\mathcal{L}$ over a scheme in characteristic p, there is a canonical isomorphism between $\mathcal{L}^{(p)}$ and $\mathcal{L}^p$, sending the base-change $s^{(p)} = 1 \otimes s$ of the section s under the absolute Frobenius of S to $s \otimes \cdots \otimes s$. It is an important fact that $h \neq 0$ precisely on S^{ord}. If $n > m$ the zero-divisor of h is even *reduced*, so equals $S^{\mathrm{no}} = S \setminus S^{\mathrm{ord}}$ with its reduced subscheme structure. A proof of this fact may be found in [Wooding 2016, Proposition 7.2.11] but can also be extracted from the Dieudonné module computations in Theorem 4.1.3 below.

If $n = m$ this is not true; h vanishes then on S^{no} to order $p + 1$. There is a variant, though, that will be useful for us in the study of the holomorphicity of the theta operator.

Lemma 1.1.2. *Let $n = m$. Consider the maps of line bundles*

$$h_{\mathcal{Q}} = \det(V_{\mathcal{Q}}) : \det(\mathcal{Q}) \to \det(\mathcal{P})^{(p)} = \det(\mathcal{P})^p \quad \textit{and} \quad h_{\mathcal{P}} = \det(V_{\mathcal{P}}) : \det(\mathcal{P}) \to \det(\mathcal{Q})^{(p)} = \det(\mathcal{Q})^p.$$

Both $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ vanish precisely on S^{no} with multiplicity 1 and the following relation holds:

$$h = h_{\mathcal{P}}^p \circ h_{\mathcal{Q}}.$$

Proof. The claim concerning the vanishing of $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ follows again from [Wooding 2016, Proposition 7.2.11] or from the computations in Theorem 4.3.2 below. The relation $h = h_{\mathcal{P}}^p \circ h_{\mathcal{Q}}$ is a direct consequence of the definition. □

Although the following corollary is weaker than Lemma 1.1.1, it is of interest because its proof is entirely moduli-theoretic.

Corollary 1.1.3. *Let S be of arbitrary signature (n, m). There is an isomorphism*

$$\det(\mathcal{P})^{p+1} \simeq \det(\mathcal{Q})^{p+1}.$$

Proof. Consider first the case of equal signatures (m, m). By comparing divisors of global sections, we obtain from the last lemma an isomorphism of line bundles $\det(\mathcal{P})^p \otimes \det(\mathcal{Q})^{-1} \simeq \det(\mathcal{Q})^p \otimes \det(\mathcal{P})^{-1}$, implying the corollary in this case.

For S of signature (n, m), and a geometric point x of S, we can embed S in a suitable Shimura variety $\mathbb{S}$ of signature $(n+m, n+m)$ by a morphism given on objects by $\underline{A} \mapsto \underline{A} \times \underline{B}_x$, where $\underline{B}_x$ is the abelian variety corresponding to x with the twisted $\mathcal{O}_E$ structure. One easily checks that the pull-back of the relation $\det(\mathcal{P})^{p+1} \simeq \det(\mathcal{Q})^{p+1}$ on $\mathbb{S}$ gives the same relation on S. □

Coming back to the case $m = n$ we have the following lemma.

Lemma 1.1.4. *Over an algebraic closure of κ, we may fix the isomorphism* $\det(\mathcal{P}) \simeq \det(\mathcal{Q}) = \mathcal{L}$ *so that* $h_{\mathcal{P}} = h_{\mathcal{Q}}$, *hence* $h = h_{\mathcal{Q}}^{p+1}$.

Proof. Fix a smooth toroidal compactification $\bar{S}$ of S. As the abelian scheme $\mathcal{A}/S$ extends with the $\mathcal{O}_E$-action to a semiabelian scheme over the toroidal compactification $\bar{S}$ [Lan 2013, Theorem 6.4.1.1] the vector bundles $\mathcal{P}$ and $\mathcal{Q}$, as well as the homomorphisms $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$, extend to $\bar{S}$ as well. In Corollary 3.1.5 below we show that h does not vanish on any irreducible component of the boundary $\bar{S} \setminus S$. The same therefore must be true for $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$. It follows that

$$\operatorname{div}(h_{\mathcal{P}}) = \operatorname{div}(h_{\mathcal{Q}})$$

as divisors on the smooth, complete variety $\bar{S}$. Fix any isomorphism as in Lemma 1.1.1. Having the same divisors, the sections $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ of $\mathcal{L}^{p-1}$ are equal up to a multiplication by a nowhere vanishing function on $\bar{S}$, hence equal up to a scalar on each connected component of $\bar{S}$. By extracting a $p-1$ root from this scalar, we can normalize the isomorphism $\det(\mathcal{P}) \simeq \det(\mathcal{Q})$ so that $h_{\mathcal{P}} = h_{\mathcal{Q}}$. □

We remark that for more general Shimura varieties of PEL type the construction of the Hasse invariant requires substantial work and is due to Goldring and Nicole [2017].

The vector bundles $\mathcal{P}_0$ and $\mathcal{P}_\mu$. The geometric fibers of the subsheaf

$$\mathcal{P}_0 = \ker(V_{\mathcal{P}}) \subset \mathcal{P}$$

have constant rank $n-m$ over an open subset $S_\sharp$ containing the ordinary stratum

$$S^{\text{ord}} \subset S_\sharp \subset S.$$

As the base is nonsingular, this implies that *over* $S_\sharp$ this $\mathcal{P}_0$ is a vector-subbundle of $\mathcal{P}$, hence so is the quotient

$$\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0.$$

In fact, $V_{\mathcal{P}}$ induces there an isomorphism

$$V_{\mathcal{P}} : \mathcal{P}_\mu \simeq \mathcal{Q}^{(p)}, \tag{1.1.6}$$

because as long as its kernel has rank $n-m$, $V_{\mathcal{P}}$ must be surjective. The open subscheme $S_\sharp$ is of much interest, and was analyzed in [de Shalit and Goren 2018]. It is the union of Ekedahl–Oort strata [Oort 2001; Viehmann and Wedhorn 2013] that can be determined precisely. When $m = 1$, for example, its complement in S is zero-dimensional (the superspecial points). When $m < n$ this $S_\sharp$ contains a unique Ekedahl–Oort stratum S^{ao} of dimension $mn-1$. This will be used later on in our work.

The vector bundles $\mathcal{P}_0$, $\mathcal{P}_\mu$ and $\mathcal{Q}$ will turn out to be the building blocks of the mod-p automorphic vector bundles over S^{ord}. See page 1842 for a discussion why we need to substitute the two subquotients $\mathcal{P}_0$ and $\mathcal{P}_\mu$ *in lieu* of the classical automorphic vector bundle $\mathcal{P}$.

It is a remarkable fact that $\mathcal{P}_0$ and $\mathcal{P}_\mu$ can be defined on the ordinary stratum also modulo p^s for any $s \geq 1$, although the Verschiebung isogeny is defined only in characteristic p. One way to see it is as follows. Let $R = \mathcal{O}_{E,(p)}$ and

$$W_s = W_s(\kappa) = W(\kappa)/p^s W(\kappa) = R/p^s R$$

(we identify the Witt vectors $W = W(\kappa)$ with the completion $\mathcal{O}_p$ of R). Denote by S_s^{ord} the open subscheme of $S_s = \mathcal{S} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/p^s R)$ whose underlying topological space is S^{ord}. The filtration of the p-divisible group of $\mathcal{A}$ by its connected and multiplicative parts extends uniquely from S^{ord} to S_s^{ord}. This is well-known for the connected part, and by Cartier duality follows also for the multiplicative part. It is crucial for us that the filtered pieces in (1.1.4) have *constant height* along S_s^{ord}. Moreover, by the same result of Moonen quoted above [2004, Corollary 2.1.5] the graded pieces of $\mathcal{A}[p^\infty]$ with their additional structures of endomorphisms and polarization become isomorphic, locally in the proétale topology on S_s^{ord}, to the constant p-divisible groups $(\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m$, $\mathfrak{G}^{n-m}$ and $(\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m$. (See Section 2.1 below for the p-divisible group $\mathfrak{G}$ over an arbitrary base.) In other words, not only modulo p but modulo p^s as well, we can trivialize $\mathrm{gr}^i \mathcal{A}[p^t]$ with the additional structures after passing to a finite étale covering. This remark will be instrumental in the construction of the big Igusa tower below.

Coming back to the definition of $\mathcal{P}_0$ and $\mathcal{P}_\mu$ over S_s^{ord}, if $t \geq s$ the exact sequence

$$0 \to \mathcal{A}[p^t] \to \mathcal{A} \xrightarrow{p^t} \mathcal{A} \to 0 \tag{1.1.7}$$

shows that $\mathrm{Lie}(\mathcal{A}[p^t]/S_s) \to \mathrm{Lie}(\mathcal{A}/S_s)$ is an isomorphism.[2] The filtration of $\mathcal{A}[p^t]$ induces (over S_s^{ord} only) a filtration of its Lie algebra by $\mathcal{O}_{S_s}$-subbundles, hence a similar filtration of $\mathrm{Lie}(\mathcal{A}/S_s)$. By duality we get (again over S_s^{ord}) a filtration of ω by subbundles, which on its Σ-part yields the exact sequence

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0. \tag{1.1.8}$$

For future reference we record the fact that

$$\mathcal{P}_0 = \omega_{\mathcal{A}[p^\infty]^0/\mathcal{A}[p^\infty]^\mu}, \quad \mathcal{P}_\mu = \omega_{\mathcal{A}[p^\infty]^\mu}(\Sigma), \quad \mathcal{Q} = \omega_{\mathcal{A}[p^\infty]^\mu}(\overline{\Sigma}).$$

We do not know how to extend (1.1.8) in any intelligible way to the s-th infinitesimal thickening of $S_\sharp$, as we did when $s = 1$ using Verschiebung.

[2]For any group scheme G over T we define $\mathrm{Lie}(G/T)$ to be the kernel of the map " mod ε" from $G(T[\varepsilon])$ to $G(T)$, where $\varepsilon^2 = 0$.

1.2. ***p-adic automorphic vector bundles.***

Representations of GL_m. We review some well-known facts from the representation theory of GL_m. Let R be any ring, and $\underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ the category of algebraic representations of GL_m on projective R-modules of finite rank. If $\rho \in \underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$, we denote by $\rho(R)$ the associated projective R-module, endowed with a left $\mathrm{GL}_m(R)$ action. Given an R-scheme S, the functoriality in R allows us to regard $\rho(\mathcal{O}_S) = \mathcal{O}_S \otimes_R \rho(R)$ as a vector bundle with a left $\mathrm{GL}_m(\mathcal{O}_S)$ action on S. The category $\underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ is a rigid tensor category, and if R is a field, it is also abelian. Some special objects of the category are the standard representation st, and the symmetric and exterior powers Sym^r st and $\wedge^r$ st of st, defined as suitable *quotients* of $\otimes^r$ st.

If R is a field of characteristic 0, the category is even semisimple. It is well known that the simple objects are then classified by dominant weights. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$ $(\lambda_i \in \mathbb{Z})$ is a dominant weight of GL_m, the corresponding object is

$$\rho_\lambda = \mathrm{Sym}^{\lambda_1-\lambda_2}(\mathrm{st}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2 \mathrm{st}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_m}(\wedge^m \mathrm{st}). \tag{1.2.1}$$

Note that $\wedge^m$ st is of rank 1, so $\mathrm{Sym}^{\lambda_m}(\wedge^m \mathrm{st}) = \otimes^{\lambda_m}(\wedge^m \mathrm{st})$ makes sense even if λ_m is negative. In Herman Weyl's construction of ρ_λ we assume first that $\lambda_m \geq 0$, view λ as a partition (Young tableau) of size $d = \sum_{i=1}^m \lambda_i$, project $\otimes^d$ st onto a subrepresentation using the Young symmetrizer $c_\lambda = a_\lambda b_\lambda \in \mathbb{Z}[\mathfrak{S}_d]$, and then the resulting quotient is a model for ρ_λ, [Fulton and Harris 1991, Chapter 6]. When λ is not necessarily positive, one reduces to the positive case by a twist by a power of the determinant $\wedge^m$ st.

Recall, however, that over a field of characteristic p the ρ_λ, defined directly by (1.2.1), are in general reducible (e.g., $m = 2$ and $\lambda = (p \geq 0)$), and the category $\underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ is not semisimple. As the Young symmetrizers are only quasiidempotents (i.e., $c_\lambda^2 = n_\lambda c_\lambda$ for some integer n_λ called the hook length of λ, which might be divisible by p) using them to study the representations of GL_m becomes tricky.

A more *geometric* construction of ρ_λ that works *over any ground ring* R, hence produces an element of $\underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ functorially in R, is via the Borel–Weil theorem — see [Fulton and Harris 1991, Claim 23.57] over $\mathbb{C}$. Jantzen [2003, II Section 5], gives this construction of ρ_λ over an arbitrary field, of any characteristic, and not necessarily algebraically closed. It is however clear that the construction is valid over any ring R, and is furthermore functorial in R. Let $\bar{\lambda} = (\lambda_m, \ldots, \lambda_1)$ be the antidominant weight for the standard torus of GL_m, which is the opposite of λ. Let $G = \mathrm{GL}_m$ and let B be the standard upper-triangular Borel subgroup. Let $\bar{\lambda}$ denote also the character of B obtained by first projecting modulo the unipotent radical U to the torus and then applying $\bar{\lambda}$. On the flag variety G/B define the line bundle L_λ by

$$L_\lambda = G \times^B \bar{\lambda}.$$

This is the quotient of $G \times \mathbb{A}^1$ under the equivalence relation $(gb, t) \sim (g, \bar{\lambda}(b)t)$ $(b \in B)$. A global section of L_λ is identified with a map $\sigma : G \to \mathbb{A}^1$ satisfying $\sigma(gb) = \bar{\lambda}(b)^{-1}\sigma(g)$. In particular, letting w be the element of maximal length in the Weyl group (the matrix with 1's on the antidiagonal), we may define such a section on the (open dense) big cell $UwB \subset G$ by

$$\sigma_0(uwb) = \bar{\lambda}(b)^{-1}. \tag{1.2.2}$$

The Borel–Weil theorem says that if λ is dominant, then (a) L_λ is ample and $V_\lambda = H^0(G/B, L_\lambda) \neq 0$, (b) if we let G act on V_λ by left translation, i.e., $(g\sigma)(g') = \sigma(g^{-1}g')$, this becomes a model for ρ_λ, and finally (c) the σ_0 of (1.2.2) extends to a regular section on all of G/B, the group $B \subset G$ acts on it via the character λ, and up to a scalar, σ_0 is the unique highest weight vector in ρ_λ.

This geometric formulation makes it evident that ρ_λ so defined is functorial in R. Moreover, the linear functional

$$\Psi_\lambda : \sigma \mapsto \sigma(w) \in \mathbb{A}^1 \tag{1.2.3}$$

is easily seen to be in $\mathrm{Hom}_{\bar{B}}(\rho_\lambda|_{\bar{B}}, \lambda)$ where $\bar{B}$ is the *lower* triangular Borel. What's more, since $L_{\lambda+\mu} = L_\lambda \otimes L_\mu$ there is a canonical map (multiplication of global sections)

$$m_{\lambda,\mu} : \rho_\lambda \otimes \rho_\mu \to \rho_{\lambda+\mu}, \tag{1.2.4}$$

which is compatible with the functionals Ψ_λ, Ψ_μ and $\Psi_{\lambda+\mu}$. From now on, whenever we write ρ_λ or Ψ_λ we shall have this specific model in mind.

We finally remark that if R is an $\mathbb{F}_p$-algebra, and $\phi : R \to R$ is the absolute Frobenius $\phi(x) = x^p$, then every representation $\rho \in \underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ admits a *Frobenius twist* $\rho^{(p)} = \phi^*(\rho)$. In concrete terms, locally on R we may write ρ in matrices, using a basis of the underlying projective module, and $\rho^{(p)}$ is the representation obtained by raising all the entries of the matrices to power p.

Twisting a representation by a vector bundle. Let S be a scheme over R. For every vector bundle $\mathcal{F}$ of rank m over S we let $\underline{\mathrm{Isom}}(\mathcal{O}_S^m, \mathcal{F})$ be the right GL_m-torsor of isomorphisms between $\mathcal{O}_S^m$ and $\mathcal{F}$, the group scheme $\mathrm{GL}_{m/S}$ acting on the right by precomposition. If $\rho \in \underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ we consider the vector bundle

$$\mathcal{F}_\rho = \underline{\mathrm{Isom}}(\mathcal{O}_S^m, \mathcal{F}) \times^{\mathrm{GL}_m} \rho(\mathcal{O}_S)$$

(contracted product). One should think of $\mathcal{F}_\rho$ as "ρ twisted by $\mathcal{F}$". For example, for a dominant weight λ,

$$\mathcal{F}_{\rho_\lambda} = \mathrm{Sym}^{\lambda_1-\lambda_2}(\mathcal{F}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2\mathcal{F}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_m}(\wedge^m\mathcal{F}).$$

What we have constructed is a *tensor functor* $\rho \rightsquigarrow \mathcal{F}_\rho$ from $\underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$ into the category $\underline{\mathrm{Vec}}_S$ of vector bundles over S. These functors are compatible with base-change of the underlying scheme S, and with isomorphisms $\mathcal{F}_1 \simeq \mathcal{F}_2$ between rank m vector bundles. Thus if over $S' \to S$ the pull-backs of two vector bundles $\mathcal{F}_i$ become isomorphic via an isomorphism ε, this ε induces, over S', functorial isomorphisms $\varepsilon_\rho : \mathcal{F}_{1,\rho} \simeq \mathcal{F}_{2,\rho}$ for every $\rho \in \underline{\mathrm{Rep}}_R(\mathrm{GL}_m)$.

Note that if R is an $\mathbb{F}_p$-algebra, then $\mathcal{F}_{\rho^{(p)}} = \mathcal{F}_\rho^{(p)}$, where for any sheaf $\mathcal{F}$ over S we denote by $\mathcal{F}^{(p)} = \Phi_S^*\mathcal{F}$ its pull-back by the absolute Frobenius of S. By $\mathcal{F}_\rho^{(p)}$ we mean either $(\mathcal{F}_\rho)^{(p)}$ or $(\mathcal{F}^{(p)})_\rho$, the two being canonically identified.

The above generalizes to representations of a product of any number of linear groups, say $M = \prod_{i=1}^{r} \mathrm{GL}_{m_i}$. Given $\rho \in \underline{\mathrm{Rep}}_R(M)$ and vector bundles $\mathcal{F}_i$ of ranks m_i we let

$$\mathcal{E}_\rho = \prod_{i=1}^{r} \underline{\mathrm{Isom}}(\mathcal{O}_S^{m_i}, \mathcal{F}_i) \times^M \rho(\mathcal{O}_S). \tag{1.2.5}$$

We call it the vector bundle obtained by *twisting ρ by the vector bundles $\mathcal{F}_i$*.

p-adic automorphic vector bundles over S_s^{ord}. Classically, automorphic vector bundles on $\mathcal{S}_\mathbb{C}$ are defined in the following way. Every connected component $\mathcal{S}_\mathbb{C}^0$ is of the form $\Gamma \setminus \boldsymbol{G}(\mathbb{R})/K_\infty$ where K_∞ is a maximal compact-modulo-center subgroup, and Γ an arithmetic subgroup of $\boldsymbol{G}(\mathbb{R})$. By a standard procedure due to Harish-Chandra one may embed the symmetric space $\mathfrak{X} = \boldsymbol{G}(\mathbb{R})/K_\infty$ as an open subset of its *compact dual* $\check{\mathfrak{X}}$. In our case the compact dual happens to be the Grassmannian $\mathrm{GL}_{n+m}(\mathbb{C})/P_\mathbb{C}$, where $P_\mathbb{C}$ is the standard maximal parabolic of type (m, n). (The change of variables involved in the Harish-Chandra embedding for $U(n, m)$ is called the Cayley transform, as it generalizes the well-known embedding of the upper half-plane as the open unit disk in $\mathbb{P}_\mathbb{C}^1$ when $n = m = 1$.) The Levi quotient of $P_\mathbb{C}$ is $M_\mathbb{C} = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, and the automorphic vector bundles we consider are attached to representations $\rho \in \underline{\mathrm{Rep}}_\mathbb{C}(M)$.

Let such a representation ρ be given. Let $P_\mathbb{C}$ act on $\rho(\mathbb{C})$ via its quotient $M_\mathbb{C}$, consider the vector bundle

$$\mathrm{GL}_{n+m}(\mathbb{C}) \times^{P_\mathbb{C}} \rho(\mathbb{C})$$

on $\check{\mathfrak{X}} = \mathrm{GL}_{n+m}(\mathbb{C})/P_\mathbb{C}$, and denote by $\tilde{\mathcal{E}}_\rho$ its restriction to $\mathfrak{X}$. Since left multiplication by Γ commutes with right multiplication by $P_\mathbb{C}$, this vector bundle descends to a vector bundle $\mathcal{E}_\rho$ on $\mathcal{S}_\mathbb{C}^0 = \Gamma \setminus \mathfrak{X}$. Using the complex analytic description of the universal abelian variety over $\Gamma \setminus \mathfrak{X}$ one checks that the standard representations of the two blocks in M yield the vector bundles $\mathcal{Q}$ and $\mathcal{P}^\vee$. Easy group theory shows then that this complex analytic construction gives, for any $\rho \in \underline{\mathrm{Rep}}_\mathbb{C}(M)$, a vector bundle which may be canonically identified with the $\mathcal{E}_\rho$ obtained by twisting ρ by the pair of vector bundles $\mathcal{Q}$ and $\mathcal{P}^\vee$, as in the preceding paragraph.

This suggests to adopt the construction outlined on page 1841 as an algebraic construction of automorphic vector bundles that works equally well over the arithmetic scheme $\mathcal{S}$, hence also over its special fiber S.

For the purpose of studying *p-adic vector-valued modular forms* this is however not always sufficient. In the classical complex setting, a great advantage of the construction is that $\tilde{\mathcal{E}}_\rho$ becomes trivial on $\mathfrak{X}$, hence may be described by matrix-valued factors of automorphy. In the mod-p or p-adic theory we need an analogous covering of S^{ord} (or S_s^{ord}), over which our basic building blocks, hence all the $\mathcal{E}_\rho$, will be trivialized. This is crucial both for Katz's theory of p-adic modular forms, and for the construction of Maass–Shimura-like differential operators below. This analogue of $\mathfrak{X}$ is the (*big*) *Igusa tower*, to be described in Section 2.1.

At this point the μ-ordinary case becomes fundamentally different from the ordinary one. If p is split in E, or if p is inert but $m = n$, then both $\mathcal{P}$ and $\mathcal{Q}$ are trivialized over the Igusa tower and everything works well with the usual automorphic vector bundles. However, if p is inert and $m < n$ then $\mathcal{P}$ can not be trivialized over the Igusa tower, nor on any other proétale cover. The best we can do is to trivialize its subquotients $\mathcal{P}_0$ and $\mathcal{P}_\mu$ separately. This explains why we need to start with *three* basic bundles $\mathcal{Q}$, $\mathcal{P}_\mu$ and $\mathcal{P}_0$ over S_s^{ord}, and why our ρ will be an element of $\underline{\mathrm{Rep}}_R(M)$ with

$$M = \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$$

rather than $\mathrm{GL}_m \times \mathrm{GL}_n$ as over $\mathbb{C}$.

After this long discussion, we can finally make the following definition.

Definition 1.2.1. Let $\rho \in \underline{\mathrm{Rep}}_R(M)$ where $M = \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$, and define $\mathcal{E}_\rho$ on S_s^{ord} by (1.2.5) with $(\mathcal{Q}, \mathcal{P}_\mu, \mathcal{P}_0)$ replacing the $\mathcal{F}_i$. We call $\mathcal{E}_\rho$ the *p-adic automorphic vector bundle of weight* $\rho \pmod{p^s}$, and $\lim_{\leftarrow s} H^0(S_s^{\mathrm{ord}}, \mathcal{E}_\rho)$ the space of *p-adic (vector-valued)* modular forms *of weight* ρ.

Remarks. (i) Note that by our convention the standard representations of the second and third factors of M correspond to $\mathcal{P}_\mu$ and $\mathcal{P}_0$, while the complex analytic standard representation of GL_n corresponded to $\mathcal{P}^\vee$.

(ii) A p-adic modular form need not come from a global section over $\mathcal{S}$. It is a rigid analytic object, defined over the affinoid which is the generic fiber of the formal completion of $\mathcal{S}$ along S^{ord}. In fact, if $\mathcal{P}_\mu$ and $\mathcal{P}_0$ are "involved" in $\mathcal{E}_\rho$ (in the precise sense that ρ does not come from a representation of the first simple factor of M) then it does not even make sense to ask whether the modular form extends to a global section over $\mathcal{S}$, because the p-adic automorphic vector bundle does not extend there. In order to compare classical and p-adic modular forms we make the following definition.

Definition 1.2.2. Let $\rho \in \underline{\mathrm{Rep}}_R(M)$. We say that the p-adic automorphic vector bundle $\mathcal{E}_\rho$ is *of classical type* if ρ factors through the first factor of M.

A p-adic automorphic vector bundle of classical type is the restriction to S_s^{ord} of a classical automorphic vector bundle. Note however that $\mathcal{P}$, an honest automorphic vector bundle on S_s, is not a p-adic automorphic vector bundle on S_s^{ord} (if $m < n$), as it can not be reconstructed from its graded pieces $\mathcal{P}_0$ and $\mathcal{P}_\mu$.

2. Differential operators on p-adic modular forms

2.1. *The big Igusa tower.*

The p-divisible group $\mathfrak{G}$. Following a long-standing tradition going back to Katz in the *ordinary* case, we want to describe a certain *tower* of (big) Igusa varieties $T_{t,s}$, for all $t, s \geq 1$. The variety $T_{t,s}$ will be an Igusa variety of level p^t over $\mathcal{O}_{E,(p)}/p^s\mathcal{O}_{E,(p)}$. By "tower" we mean that the reduction of $T_{t,s+1}$ modulo p^s will be identified with $T_{t,s}$, and that for a fixed s there will be compatible morphisms from level $p^{t'}$ to level p^t for all $t' \geq t$. This "big Igusa tower" has been defined and studied, in much greater generality, in Mantovan's work [2005].

To describe it, we shall have to choose a model $\mathfrak{G}$ over $W = W(\kappa) = \mathcal{O}_p$ of the p-divisible group that becomes, over $\bar{\kappa}$, the group $\mathfrak{G}_{\bar{\kappa}}$ introduced on page 1835. This choice results in freedom, which grows with t and s, and prevents the $T_{t,s}$ (unlike the *small* Igusa varieties, see below) from being canonically defined. This problem will nevertheless disappear over $W(\bar{\kappa})$, so the reader interested in the construction over $W(\bar{\kappa})/p^s W(\bar{\kappa})$ only, can happily ignore the issue.

The easiest way to fix our model is to choose an elliptic curve $\mathscr{C}$ defined over W, with complex multiplication by $\mathcal{O}_E$ and CM type Σ. The theory of complex multiplication guarantees that such an elliptic curve exists, and has supersingular reduction. We then let $\mathfrak{G} = \mathscr{C}[p^\infty]$ be its p-divisible group. Its special fiber $\mathfrak{G}_\kappa$ is of local-local type, height 2 and dimension 1. The canonical polarization of the elliptic curve supplies an isomorphism of $\mathfrak{G}$ with its Serre dual, hence a compatible system of perfect alternating Weil pairings (for $t \geq 1$)

$$\langle \cdot , \cdot \rangle : \mathfrak{G}[p^t] \times \mathfrak{G}[p^t] \to \mu_{p^t}.$$

The completion $\mathcal{O}_p$ of $\mathcal{O}_E$ maps isomorphically onto $\mathrm{End}(\mathfrak{G}_{/W}) \subset \mathrm{End}(\mathfrak{G}_{/\kappa})$. Furthermore, for any W-algebra R

$$\mathrm{End}_{\mathcal{O}_E}(\mathfrak{G}[p^t]_{/R}) = \mathcal{O}_p/p^t\mathcal{O}_p.$$

We have $\langle \iota(a)u, v\rangle = \langle u, \iota(\bar{a})v\rangle$ for every $a \in \mathcal{O}_E$.

The Igusa moduli problem. If R is a $W_s(\kappa)$-algebra and $A_{/R}$ is fiber-by-fiber μ-ordinary, then its p-divisible group admits a filtration like (1.1.4) whose graded pieces we label $\mathrm{gr}^i A[p^\infty]$. We choose the indices in such a way that locally in the proétale topology on $\mathrm{Spec}(R)$ there exist isomorphisms

$$\epsilon^0 : (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m_R \simeq \mathrm{gr}^0, \quad \epsilon^1 : \mathfrak{G}^{n-m}_R \simeq \mathrm{gr}^1, \quad \epsilon^2 : (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m_R \simeq \mathrm{gr}^2, \tag{2.1.1}$$

respecting the action of $\mathcal{O}_E$ and the pairings. Note that gr^1 is self-dual, while ϵ^0 and ϵ^2 determine each other. For future reference we want to make the pairings on these "model group schemes" explicit. If

$$\alpha = (x_1, \dots, x_m, y_1, \dots, y_{n-m}, z_1, \dots, z_m) \in (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m_R \times \mathfrak{G}^{n-m}_R \times (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m_R,$$

and similarly $\alpha' = (x'_1, \dots, x'_m, y'_1, \dots, y'_{n-m}, z'_1, \dots, z'_m)$, we define

$$\langle \alpha, \alpha' \rangle = \prod_{i=1}^{m} \langle x_i, z'_{m+1-i}\rangle \prod_{j=1}^{n-m} \langle y_j, y'_{n-m+1-j}\rangle \prod_{i=1}^{m} \langle z_i, x'_{m+1-i}\rangle. \tag{2.1.2}$$

In matrix form, writing μ_{p^∞} additively, we take, ${}^t\alpha J_{n+m}\alpha'$ where J_l is the antidiagonal matrix of size l, and *not* ${}^t\alpha I_{n,m}\alpha'$ where $I_{n,m}$ is the matrix (1.1.1). As remarked on page 1833, these two pairings produce *isomorphic* polarized $\mathcal{O}_E$-groups. Thus, there is no real difference which pairing we take at this point, but for later book-keeping purposes, we prefer the one with J_{n+m}.

We call $\epsilon = (\epsilon^0, \epsilon^1, \epsilon^2)$ a *graded symplectic trivialization* of the p-divisible group. A *graded symplectic trivialization* of $A[p^t]$ is a similar system of isomorphisms of the p^t-torsion in the p-divisible groups, defined over R, which is locally étale liftable to a graded symplectic trivialization of the whole p-divisible group.

Definition 2.1.1. The *big Igusa moduli problem of level* p^t over $W_s(\kappa)$, denoted $T_{t,s}$, classifies tuples

$$(\underline{A}, \epsilon)_{/R/W_s},$$

where $\underline{A} \in S_s^{\mathrm{ord}}(R)$ and ϵ is a graded symplectic trivialization of $A[p^t]$ as in (2.1.1), up to isomorphism.

The representability of this moduli problem by a scheme, denoted also $T_{t,s}$, is standard. One only has to check that it is *relatively representable* over S_s^{ord} [Katz and Mazur 1985, Chapter 4]. The maps between the levels are self-evident. The morphism

$$\tau : T_{t,s} \to S_s^{\mathrm{ord}}$$

is a Galois étale covering of S_s^{ord} [Mantovan 2005, Proposition 4].

The *small Igusa variety* of the same level classifies tuples $(\underline{A}, \epsilon^2)$ of the same nature. There is an obvious morphism from the big tower to the small one: "forget ϵ^1". Since ϵ^0 is determined by ϵ^2 we do not have to forget anything more.

The Galois group. The Galois group Δ_t of the covering $\tau : T_{t,s} \to S_s^{\mathrm{ord}}$ is isomorphic to $\mathrm{GL}_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E)$ under

$$\Delta_t \ni \gamma \mapsto [\gamma] = (\gamma_2, \gamma_1) \in \mathrm{GL}_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E),$$

where

$$\gamma(\underline{A}, \epsilon) = (\underline{A}, \epsilon \circ [\gamma]^{-1}). \tag{2.1.3}$$

Here by $U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E)$ we mean the quasisplit unitary group, consisting of matrices g of size $n-m$ satisfying the relation ${}^t\bar{g}J_{n-m}g = J_{n-m}$. As explained before, it is isomorphic to the group of matrices satisfying ${}^t\bar{g}g = I$. By $\epsilon \circ [\gamma]^{-1}$ we mean that we compose ϵ^1 with γ_1^{-1} and ϵ^2 with γ_2^{-1} (the action on ϵ^0 being determined by the one on ϵ^2). As usual, the group Δ_t acts simply transitively on the geometric fibers of the morphism τ.

Trivializing the three basic vector bundles over the Igusa tower. For simplicity write $T = T_{t,s}$, $\Delta = \Delta_t$, and assume that $t \geq s$. There is enough level structure then to "see" the relative Lie algebra of $\mathcal{A}_{/S_s}$ on $\mathcal{A}[p^t]_{/S_s}$, as explained in the paragraph following (1.1.7).

As the cotangent space at the origin of $\mathfrak{d}_E^{-1} \otimes \mu_{p^t/W_s}$ is canonically identified with $\mathcal{O}_E \otimes W_s = W_s(\Sigma) \oplus W_s(\overline{\Sigma})$, the isomorphism ϵ^2 induces canonical trivializations of $\mathcal{O}_E$-vector bundles over T

$$\varepsilon^2 = ((\epsilon^2)^{-1})^* : \mathcal{O}_E \otimes \mathcal{O}_T^m \simeq \mathcal{Q} \oplus \mathcal{P}_\mu$$

(we write $\mathcal{Q}$ for $\tau^*\mathcal{Q}$ etc. as $\tau^*\mathcal{Q}$ is "the" $\mathcal{Q}$ of $\mathcal{A}/T$), or

$$\varepsilon^2(\overline{\Sigma}) : \mathcal{O}_T^m \simeq \mathcal{Q}, \quad \varepsilon^2(\Sigma) : \mathcal{O}_T^m \simeq \mathcal{P}_\mu.$$

Similarly fix, once and for all, an isomorphism of the cotangent space at the origin of $\mathfrak{G}[p^t]_{/W_s}$ (as an $\mathcal{O}_E$-module) with $W_s(\Sigma)$. The isomorphism ϵ^1 induces then also a canonical trivialization over T

$$\varepsilon^1 : \mathcal{O}_T^{n-m} \simeq \mathcal{P}_0.$$

The action (2.1.3) of $\gamma \in \Delta$ on T induces the following action on the trivializations

$$\gamma(\varepsilon^i) = \varepsilon^i \circ {}^t\gamma_i. \tag{2.1.4}$$

($i = 1, 2$). Let us check the last formula, dropping the index i:

$$\gamma(\varepsilon) = (\gamma(\epsilon)^{-1})^* = (\gamma \circ \epsilon^{-1})^* = (\epsilon^{-1})^* \circ \gamma^* = \varepsilon \circ {}^t\gamma,$$

because the matrix representing $[\gamma]^*$ on the cotangent space is the transpose of the matrix representing $[\gamma]_*$ on the Lie algebra, which is simply $[\gamma]$.

2.2. *The theta operator.*

Pretheta. Let ρ be a representation of $\mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$ over W_s, and let $\mathcal{E}_\rho$ be the automorphic vector bundle on S_s^{ord} defined above. We define a connection

$$\tilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \Omega_{S_s/W_s}$$

over S_s^{ord}.

Let $t \geq s$. Denote by $\mathcal{O}_\rho = \rho(\mathcal{O}_T)$ the vector bundle over $T = T_{t,s}$ obtained by twisting the representation ρ by the trivial vector bundles $\mathcal{O}_T^m$, $\mathcal{O}_T^m$ and $\mathcal{O}_T^{n-m}$ as in Definition 1.2.1. The trivial connection on the structure sheaf $\mathcal{O}_T$ induces, by the usual rules, a connection

$$d_\rho : \mathcal{O}_\rho \to \mathcal{O}_\rho \otimes \Omega_{T/W_s}.$$

For example, if $\rho = \rho_\lambda$ where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a dominant weight depending only on the first GL_m factor, so that $\mathcal{O}_\rho$ is given by (1.2.1), then d_ρ is given by the usual rules of differentiation of symmetric powers, exterior powers and duals.

On the other hand the trivializations ε^1 and ε^2 constructed above yield a trivialization

$$\varepsilon_\rho : \mathcal{O}_\rho \simeq \tau^* \mathcal{E}_\rho$$

over T. To get the action of

$$\gamma = (\gamma_2, \gamma_1) \in \Delta = \mathrm{GL}_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E)$$

on ε_ρ we first map γ to $\mathrm{GL}_m(W_s) \times \mathrm{GL}_m(W_s) \times \mathrm{GL}_{n-m}(W_s)$ via

$$\gamma \mapsto \iota(\gamma) = (\bar{\gamma}_2, \gamma_2, \gamma_1)$$

(well defined because $t \geq s$) and let ${}^t[\gamma]_\rho = \rho({}^t\iota(\gamma))$. Then from (2.1.4) we get

$$\gamma(\varepsilon_\rho) = \varepsilon_\rho \circ {}^t[\gamma]_\rho. \tag{2.2.1}$$

Let $U \subset S_s^{\mathrm{ord}}$ be Zariski open. For $f \in H^0(U, \mathcal{E}_\rho)$ define

$$\tilde{\Theta}(f) = (\varepsilon_\rho \otimes 1) \circ d_\rho \circ \varepsilon_\rho^{-1}(\tau^* f) \in H^0(\tau^{-1}(U), \tau^*\mathcal{E}_\rho \otimes_{\mathcal{O}_T} \Omega_{T/W_s}). \tag{2.2.2}$$

Since τ is étale, $\Omega_{T/W_s} = \mathcal{O}_T \otimes_{\mathcal{O}_{S_s}} \Omega_{S_s/W_s}$, so

$$\tilde{\Theta}(f) \in H^0(\tau^{-1}(U), \tau^*\mathcal{E}_\rho \otimes_{\mathcal{O}_{S_s}} \Omega_{S_s/W_s}).$$

We have to show that $\tilde{\Theta}(f) \in H^0(U, \mathcal{E}_\rho \otimes_{\mathcal{O}_{S_s}} \Omega_{S_s/W_s})$, and for that it would suffice to show that it is invariant under Δ. Let $\gamma \in \Delta$. Then by (2.2.1)

$$\gamma(\tilde{\Theta}(f)) = (\varepsilon_\rho \otimes 1) \circ {}^t[\gamma]_\rho \circ d_\rho \circ {}^t[\gamma]_\rho^{-1} \circ \varepsilon_\rho^{-1}(\tau^* f) = \tilde{\Theta}(f).$$

Here we used that (a) $\tau^* f$ is Galois invariant, (b) d_ρ is Galois invariant since τ is étale, and (c) d_ρ commutes with the scalar matrices ${}^t[\gamma]_\rho$. We summarize our construction in the following theorem.

Theorem 2.2.1. *Let $U \subset S_s^{\mathrm{ord}}$ be an open set and $f \in H^0(U, \mathcal{E}_\rho)$. Then*

$$\tilde{\Theta}(f) = (\varepsilon_\rho \otimes 1) \circ d_\rho \circ \varepsilon_\rho^{-1}(\tau^* f) \in H^0(U, \mathcal{E}_\rho \otimes_{\mathcal{O}_{S_s}} \Omega_{S_s/W_s})$$

yields a well-defined connection on $\mathcal{E}_\rho$. The connection defined on $\mathcal{E} \otimes \mathcal{F}$, $\mathcal{E}^\vee$ etc. is the tensor product, dual etc. of the connections defined on the individual sheaves. If $s = 1$ (i.e., we are in characteristic p), then the connection defined on $\mathcal{E}^{(p)}$ is trivial. Hence, if f and g are sections of $\mathcal{E}$ and $\mathcal{F}$, respectively, then on $\mathcal{E}^{(p)} \otimes \mathcal{F}$ we have $\tilde{\Theta}(f^{(p)} \otimes g) = f^{(p)} \otimes \tilde{\Theta}(g)$.

Proof. The functoriality with respect to linear-algebra operations (including Frobenius twist in characteristic p) is clear. The last remark is a general fact about modules with connection. For any vector bundle $\mathcal{E}$ over a base S in characteristic p there is a canonical connection ∇^{can} on $\mathcal{E}^{(p)}$, characterized by $\nabla^{\mathrm{can}}(f^{(p)}) = 0$ for any section f of $\mathcal{E}$, and if ∇ is *any* connection on $\mathcal{E}$, then its pull-back $\nabla^{(p)}$ to $\mathcal{E}^{(p)}$ is canonically identified with ∇^{can}. □

Theta. Using the inverse of the Kodaira–Spencer isomorphism

$$\mathrm{KS} : \mathcal{P} \otimes \mathcal{Q} \simeq \Omega_{S_s/W_s}$$

we may view $\tilde{\Theta}$ as a map from $\mathcal{E}_\rho$ to $\mathcal{E}_\rho \otimes \mathcal{P} \otimes \mathcal{Q}$. We emphasize that this map is not a sheaf homomorphism, as it is only κ-linear and not $\mathcal{O}_{S_s}$-linear. It is better, however, to consider the operator

$$\Theta = (1 \otimes \mathrm{pr}_\mu \otimes 1) \circ (1 \otimes \mathrm{KS}^{-1}) \circ \tilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}. \tag{2.2.3}$$

Here $\mathrm{pr}_\mu : \mathcal{P} \to \mathcal{P}/\mathcal{P}_0 = \mathcal{P}_\mu$ is the canonical projection.

If $s = 1$, in characteristic p over S, we may replace $\mathcal{P}_\mu$ by $\mathcal{Q}^{(p)}$ and pr_μ by V. From the point of view of connections, dividing $\Omega_{S/\kappa}$ by $\ker(V \otimes 1) = \mathcal{P}_0 \otimes \mathcal{Q}$ means that we restrict the connection to the foliation $TS^+ \subset TS$ which has been introduced and studied in [de Shalit and Goren 2018], i.e., use it to differentiate sections of $\mathcal{E}_\rho$ only in the direction of TS^+. Although this voluntarily gives up information encoded in $\tilde{\Theta}$, when restricted to characteristic p, the operator Θ has *four* advantages over its predecessor:

(1) While $\tilde{\Theta}$ has poles along the complement of S^{ord} in S, we shall see that Θ may be analytically continued everywhere, at least when applied to scalar modular forms.

(2) The effect of Θ on Fourier–Jacobi expansions is particularly nice, while the formulae for $\tilde{\Theta}$ contain unpleasant terms.

(3) Restricting the connection to the foliation TS^+ should also result in a nice expansion of Θ at a μ-ordinary point in terms of Moonen's generalized Serre–Tate coordinates [2004]. This is the approach taken in [Eischen and Mantovan 2017]. For the relation between TS^+ and Moonen's generalized Serre–Tate coordinates, see [de Shalit and Goren 2018, Section 3.3, Theorem 13].

(4) Unlike $\tilde{\Theta}$, the operator Θ lands back in a sheaf which is obtained "by linear algebra operations" from $\mathcal{Q}$, $\mathcal{P}_\mu$ and $\mathcal{P}_0$. This will allow us to *iterate* Θ, something which we were prohibited from doing with $\tilde{\Theta}$ due to the presence of $\mathcal{P}$.

2.3. *Higher order differential operators $D_\kappa^{\kappa'}$.* For the sake of completeness we indicate how one gets, by iterating Θ, a whole array of differential operators $D_\kappa^{\kappa'}$. We follow, with minor modifications, Eischen's thesis [2012]. If $\kappa = (a, b, c)$ is a dominant weight of $M = \mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$ we denote the vector bundle $\mathcal{E}_\rho$ associated with the representation $\rho = \rho_\kappa$ by $\mathcal{E}_\kappa$.

Let st be the standard representation of GL_m over W, let a' be a positive dominant weight $a'_1 \geq \cdots \geq a'_m \geq 0$ and $e = \Sigma_{i=1}^m a'_i$. Then in $\underline{\mathrm{Rep}}_W(\mathrm{GL}_m)$ there exists a distinguished homomorphism, unique up to a $W^\times$-multiple,

$$\pi_{a'} : \mathrm{st}^{\otimes e} \to \rho_{a'}.$$

One simply has to normalize the homomorphism resulting from the Young symmetrizer $c_{a'}$ so that it is integral, but not divisible by p. Whether $\pi_{a'}$ can be further normalized to eliminate the $W^\times$-ambiguity depends on which model we take for $\rho_{a'}$, as two such models are canonically isomorphic only up to multiplication by a scalar. Since we agreed to take the models given by the Borel–Weil theorem over W, we do not know how to normalize $\pi_{a'}$ any further or whether it is surjective before inverting p.

Let $\kappa' = (a', b', 0)$ be a dominant weight with a' and b' positive, such that

$$e = \sum_{i=1}^m a'_i = \sum_{i=1}^m b'_i.$$

In [Eischen et al. 2018] such a κ' is called sum-symmetric.

We twist $\rho_{\kappa'} = \rho_{a'} \otimes \rho_{b'} \otimes 1$ by the vector bundles $\mathcal{Q}$ and $\mathcal{P}_\mu$. Recall that $\mathcal{Q}$ is used to twist $\rho_{a'}$ and $\mathcal{P}_\mu$ is used for $\rho_{b'}$, while twisting by $\mathcal{P}_0$ is not needed, as the representation associated with GL_{n-m} is the trivial one. We get

$$\pi_{\kappa'} = \pi_{b'} \otimes \pi_{a'} : (\mathcal{P}_\mu \otimes \mathcal{Q})^{\otimes e} \to \mathcal{E}_{\kappa'}.$$

Let $\kappa = (a, b, c)$ be a dominant weight of M. Consider the e-th iteration of the derivation Θ. It maps the sheaf $\mathcal{E}_\kappa$ to $\mathcal{E}_\kappa \otimes (\mathcal{P}_\mu \otimes \mathcal{Q})^{\otimes e}$. We may now use $\pi_{\kappa'}$ to map $(\mathcal{P}_\mu \otimes \mathcal{Q})^{\otimes e}$ to $\mathcal{E}_{\kappa'}$ and finally apply the homomorphism $m_{\kappa,\kappa'} : \mathcal{E}_\kappa \otimes \mathcal{E}_{\kappa'} \to \mathcal{E}_{\kappa+\kappa'}$ of (1.2.4) to get the differential operator

$$D_\kappa^{\kappa'} = m_{\kappa,\kappa'} \circ (1 \otimes \pi_{\kappa'}) \circ \Theta^e : \mathcal{E}_\kappa \to \mathcal{E}_{\kappa+\kappa'}. \tag{2.3.1}$$

As $m_{\kappa,\kappa'} \circ (1 \otimes \pi_{\kappa'})$ is a sheaf homomorphism this $D_\kappa^{\kappa'}$ is a differential operator of order e. It is well-defined only up to a scalar from $W^\times$. The operators $D_\kappa^{\kappa'}$ allow us to increase the weight by any κ' as long as

$$\kappa' = (a', b', 0), \quad a'_1 \geq \cdots \geq a'_m \geq 0, \quad b'_1 \geq \cdots \geq b'_m \geq 0, \quad \sum_{i=1}^m a'_i = \sum_{i=1}^m b'_i.$$

Example. *Scalar-valued modular forms.* If $\kappa = (k, \dots, k; 0, \dots, 0; 0, \dots, 0)$ then

$$\mathcal{E}_\kappa = \det(\mathcal{Q})^k = \mathcal{L}^k.$$

In this case, global sections of $\mathcal{E}_\kappa$ are scalar-valued modular forms on $\boldsymbol{G}$ of weight k. If we take $\kappa' = (k', \dots, k'; k', \dots, k'; 0, \dots, 0)$ then $D_\kappa^{\kappa'}$ maps $\mathcal{L}^k$ to $\mathcal{L}^{k+k'} \otimes \det(\mathcal{P}_\mu)^{k'}$. If $s = 1$, in characteristic p, we may identify $\det(\mathcal{P}_\mu)$ with $\mathcal{L}^p$ (1.1.6), so $D_\kappa^{\kappa'}$ maps $\mathcal{L}^k$ to $\mathcal{L}^{k+(p+1)k'}$. In these cases $D_\kappa^{\kappa'}$ is obtained by applying Θ iteratively mk times and projecting. If $m = 1$ then $D_\kappa^{\kappa'}$ is simply $\Theta^{k'}$.

3. Toroidal compactifications and Fourier–Jacobi expansions

3.1. *Toroidal compactifications and logarithmic differentials.*

Generalities. Our goal in this section is to show that the operator Θ, defined so far on S_s^{ord}, extends to a partial compactification $\bar{S}_s^{\mathrm{ord}}$, obtained by fixing a smooth toroidal compactification $\bar{S}_s$ of S_s, and removing from it the closure of $S_s^{\mathrm{no}} = S_s \setminus S_s^{\mathrm{ord}}$. Thus

$$\bar{S}_s^{\mathrm{ord}} = \bar{S}_s \setminus \{\text{Zariski closure of } S_s^{\mathrm{no}}\}$$

is an open subset of $\bar{S}_s$. Note that in general the closure of S_s^{no} may meet the boundary of $\bar{S}_s$, although in some special cases, e.g., whenever $m = 1$, S_s^{no} is proper and does not reach the cusps. For a characterization of $\bar{S}_s^{\mathrm{ord}}$ as the nonvanishing locus of the Hasse invariant see page 1859. Once we extend Θ, we shall calculate its effect on Fourier–Jacobi expansions and show that, as in the classical case of GL_2, it is *morally* given by "$q \cdot d/dq$".

The toroidal compactifications $\bar{\mathcal{S}}$ of $\mathcal{S}$ considered below are smooth over $\mathcal{O}_{E,(p)}$ and their boundary $\partial\mathcal{S} = \bar{\mathcal{S}} \setminus \mathcal{S}$ is a divisor with normal crossing. However, they depend on auxiliary combinatorial data, and are not unique. As such, one can not expect $\bar{\mathcal{S}}$ to solve a moduli problem anymore. The universal abelian scheme $\mathcal{A}$ nevertheless *extends canonically* to a semiabelian scheme $\mathcal{G}$ with $\mathcal{O}_E$-action over $\bar{\mathcal{S}}$. We say that a geometric point x of $\partial\mathcal{S}$ is of rank $1 \leq r \leq m$ if the toric part of $\mathcal{G}_x$ has dimension $2r$, i.e., $\mathcal{O}_E$-rank r. Skinner and Urban [2014] call such a point "a point of genus $n + m - 2r$", referring to the dimension of the abelian part of $\mathcal{G}_x$ instead.

Constructing the toroidal compactifications, even if all proofs are omitted, requires several pages of definitions and notation. Lan's book [2013] is an exhaustive, extremely careful and precise reference.

Unfortunately, some notation introduced there is too long to fit in a single line. Following Faltings and Chai [1990], Skinner and Urban [2014, Section 5.4] gave a very readable account of the compactification, which we will follow closely. It is set for signature (n, n), but the modifications needed to treat an arbitrary signature (n, m) are minor. Yet, this forces us to review everything from scratch, rather than use [Skinner and Urban 2014] blindly.

We shall content ourselves with the arithmetical compactification of $\mathrm{Sh}_{K/W}$ (several copies of which comprise $\mathcal{S}_{/W}$). In Section 3.1 only we will write S for $\mathrm{Sh}_{K/W}$ or for its base-change to W_s (rather than to $\kappa = W_1$ as before). As smaller Shimura varieties will show up in the process, we shall write

$$S = S_G = S_{G,K}$$

whenever we need to emphasize the dependence on G or K.

Let $\{e_i\}$ denote the standard basis of $V = E^{n+m}$ and consider, for $0 \le r \le m$,

$$0 \subset V_r = \mathrm{Span}_E\{e_1, \dots, e_r\} \subset V_r^{\perp} = \mathrm{Span}_E\{e_1, \dots, e_n, e_{n+r+1}, \dots, e_{n+m}\} \subset V.$$

If we regard $V = \mathrm{Res}^E_{\mathbb{Q}} \mathbb{A}^{n+m}$ as a $\mathbb{Q}$-vector group, whose $\mathbb{Q}$-rational points are E^{n+m}, this is a $\mathbb{Q}$-rational filtration. The quotient $V(r) = V_r^{\perp}/V_r$ becomes a hermitian space of signature $(n-r, m-r)$ at infinity, and $\Lambda \cap V_r^{\perp}$ projects to a self-dual lattice $\Lambda(r) \subset V(r)$, defining a smaller general unitary group $\boldsymbol{G}_r$. If $n = m = r$ we understand by $\boldsymbol{G}_r$ the group $\mathbb{G}_m$ (accounting for the similitude factor, which is present even if $V(r) = 0$).

The subgroup

$$P_r = \mathrm{Stab}_{\boldsymbol{G}}(V_r)$$

stabilizes also $V_r^{\perp}$, and is a maximal $\mathbb{Q}$-rational parabolic subgroup of $\boldsymbol{G}$. Its unipotent radical is

$$U_r = \{g \in P_r \mid g \text{ acts trivially on } V_r,\ V(r), \text{ and } V/V_r^{\perp}\}.$$

Its Levi quotient, $L_r = P_r/U_r$, is identified with $\mathrm{Res}^E_{\mathbb{Q}} \mathrm{GL}_r \times \boldsymbol{G}_r$ under the map $g \mapsto (g|_{V_r}, g|_{V(r)})$. The center $Z_r = Z(U_r)$ of U_r turns out to be

$$Z_r = \{g \in U_r \mid (g-1)(V_r^{\perp}) = 0, (g-1)(V) \subset V_r\}.$$

In matrix block form

$$P_r = \left\{ g = \begin{pmatrix} A & C & B \\ & D & C' \\ & & \nu\, {}^t\bar{A}^{-1} \end{pmatrix} \in \boldsymbol{G} \right\}, \tag{3.1.1}$$

where A is a square matrix of size r and D is a square matrix of size $(n+m-2r)$. The group U_r is characterized by $\nu = 1$, $A = 1$, $D = 1$, and Z_r by the additional properties $C = 0$, $C' = 0$. When this is the case, $B = -{}^t\bar{B}$. We regard L_r also as a subgroup of P_r, mapping (g, h) to the matrix which in a diagonal block form is $(g, h, \nu(h)^t\bar{g}^{-1})$. Thus $P_r = L_r U_r$.

Every maximal $\mathbb{Q}$-rational parabolic subgroup of $\boldsymbol{G}$ is conjugate to P_r for some r.

Cusp labels, the minimal compactification and the toroidal compactifications. Let $1 \leq r \leq m$. The set of cusp labels of level K and rank r [Skinner and Urban 2014, Section 5.4.2] is the finite set

$$\mathscr{C}_r = [\mathrm{GL}_r(E) \cdot \boldsymbol{G}_r(\mathbb{A}_f)] \cdot U_r(\mathbb{A}_f) \backslash \boldsymbol{G}(\mathbb{A}_f)/K.$$

As before, the rank r will be the $\mathcal{O}_E$-rank of the toric part of the universal semiabelian variety over the corresponding cuspidal component. If $g \in \boldsymbol{G}(\mathbb{A}_f)$ we denote by $[g] = [g]_r = [g]_{r,K} \in \mathscr{C}_r$ the corresponding double coset. The minimal (Baily–Borel) compactification S^* of S is discussed in [Lan 2013, Section 7.2.4] and, when $n = m$, in [Skinner and Urban 2014, Section 5.4.4]. It is a singular compactification admitting a stratification by finitely many locally closed strata

$$S^* = \bigsqcup_{r=0}^{m} \bigsqcup_{[g]_r \in \mathscr{C}_r} S_{\boldsymbol{G}_r, K_{r,g}},$$

where $K_{r,g} = \boldsymbol{G}_r(\mathbb{A}_f) \cap gKg^{-1}$. Each $S_{\boldsymbol{G}_r, K_{r,g}}$ is an $(n-r)(m-r)$-dimensional Shimura variety, so when r attains its maximal value m, it is 0-dimensional. When $r = 0$ we get one stratum, which is the open dense S. The closure of $S_{\boldsymbol{G}_r, K_{r,g}}$ is the union of $S_{\boldsymbol{G}_{r'}, K_{r',g'}}$ for $r \leq r'$ and g' such that the cusp label $[g']_{r'}$ is a specialization of $[g]_r$ in an appropriate sense [Lan 2013, Definition 5.4.2.13]. We call each $S_{\boldsymbol{G}_r, K_{r,g}}$ a *rank r cuspidal component of S^**.

Any toroidal compactification that we consider will be a smooth scheme $\bar{S}_{/W}$ endowed with a proper morphism

$$\pi : \bar{S} \to S^*.$$

Moreover, it will come equipped with a stratification

$$\bar{S} = \bigsqcup_{r=0}^{m} \bigsqcup_{[g]_r \in \mathscr{C}_r} \bigsqcup_{\sigma \in \Sigma_{H^{++}_{g,\mathbb{R}}} / \Gamma_g} Z([g]_r, \sigma)$$

by finitely many smooth, locally closed W-subschemes $Z([g]_r, \sigma)$. The indexing set $\Sigma_{H^{++}_{g,\mathbb{R}}} / \Gamma_g$ will become clear shortly. The morphism π will respect the stratifications.

Every $Z([g]_r, \sigma)$ is constructed in three steps, related to the structure of the semiabelian scheme $\mathcal{G}$ over it, as follows:

- First, $S_{\boldsymbol{G}_r, K_{r,g}}$ is the moduli space of the abelian part of $\mathcal{G}$ (with the associated PEL structure), which is of signature $(n-r, m-r)$, hence is a smooth Shimura variety of dimension $(n-r)(m-r)$ over W. Let $\mathcal{A}_r$ denote the universal abelian scheme over it. In contrast to the abelian part, the toric part of $\mathcal{G}$ is *fixed* by the cusp label $[g]_r$, and is given by

$$T_X = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m)$$

  where $X = X_g$ is a rank-r projective $\mathcal{O}_E$-module determined by g. Thus $\dim(T_X) = 2r$ and $\dim(\mathcal{A}_r) = n+m-2r$. For example, if $g = 1$ (the "standard cusp of rank r") then $X = \mathrm{Hom}(\Lambda \cap V_r, \mathbb{Z})$.

- The second step in the construction of $Z([g]_r, \sigma)$ is the construction of an abelian scheme C which classifies the extensions of $\mathcal{A}_r$ by T_X. Let $X^* = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{O}_E)$ and

$$C = C([g]_r) := \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{A}_r, T_X).$$

  This can be written also as

$$C = X^* \otimes_{\mathcal{O}_E} \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{A}_r, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m) = X^* \otimes_{\mathcal{O}_E} \mathcal{A}_r^t = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t),$$

  using the fact that $\mathrm{Tr}_{E/\mathbb{Q}} \otimes 1 : \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m \to \mathbb{G}_m$ induces an isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{A}_r, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m) \simeq \mathrm{Ext}^1(\mathcal{A}_r, \mathbb{G}_m) = \mathcal{A}_r^t.$$

  The relative dimension of C over $S_{\mathbf{G}_r, K_{r,g}}$ is $r(n+m-2r)$, so its total dimension is

$$(n-r)(m-r) + r(n+m-2r) = nm - r^2.$$

- In the last and final step one uses auxiliary combinatorial data and the theory of toroidal embeddings [Fulton 1993] to construct the $Z([g]_r, \sigma)$. Each of them is a torus torsor over $C([g]_r)$. For details, see the next subsection.

The stratification by disjoint locally closed strata does not shed any light on the way these strata are glued together, even if the closure relations between them are given. However, each stratum $Z = Z([g]_r, \sigma)$ is actually the underlying reduced scheme (the "support") of a formal scheme $\mathfrak{Z} = \mathfrak{Z}([g]_r, \sigma)$ whose over-all dimension (counting the "formal parameters" too) is mn. The semiabelian scheme together with the PEL structure extend from Z to $\mathfrak{Z}$ "in the infinitesimal directions" to give a structure called *degeneration data*. As described originally in [Mumford 1972] in the totally degenerate setting, and later on in [Ash et al. 1975; Faltings and Chai 1990; Lan 2013], this allows one to use *Mumford's construction* to glue all the pieces together. We do not reproduce this construction, but remark that the key to it is the presence of a polarization, which allows, at a crucial step, to use Grothendieck's algebraization theorem.

The torsor Ξ. As our purpose is to establish just enough notation to be able to study Θ at the cusps, and as this will be done only at the *standard* cusps, we shall explain now the third and final step in the construction of $\mathfrak{Z}([g]_r, \sigma)$ under the assumption that $g = 1$. The general case can be treated in a similar manner, transporting all structures by g. While necessary for applications, it does not add much conceptually.

Assume therefore that the cusp label is $[g]_r = [1]_r$ and drop the g from the notation. Let

$$X = \mathrm{Hom}(\Lambda \cap V_r, \mathbb{Z}), \quad Y = \Lambda/(\Lambda \cap V_r^{\perp}).$$

Let $\phi_X : Y \simeq X$ be the isomorphism given by $\phi_X(u)(v) = \langle u, v \rangle$. It satisfies $\phi_X(au) = \bar{a}\phi_X(u)$. If $c \in C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$ we denote by $c^t \in \mathrm{Hom}_{\mathcal{O}_E}(Y, \mathcal{A}_r)$ the unique homomorphism satisfying $\phi_r \circ c^t = c \circ \phi_X$, where $\phi_r : \mathcal{A}_r \simeq \mathcal{A}_r^t$ is the tautological principal polarization of the abelian scheme $\mathcal{A}_r$ over $S_{\boldsymbol{G}_r, K_{r,g}}$.

$$\begin{array}{ccc} X & \xrightarrow{\;c\;} & \mathcal{A}_r^t \\ {\scriptstyle\phi_X}\uparrow & & \uparrow{\scriptstyle\phi_r} \\ Y & \xrightarrow{\;c^t\;} & \mathcal{A}_r \end{array}$$

We construct a torus T_H and use it to define a T_H-torsor Ξ over C which will be basic for the construction of the local charts below. Let

$$H = Z_r(\mathbb{Q}) \cap K$$

where Z_r, as before, is the center of the unipotent radical of P_r, and K the level subgroup. Let $\check{H} = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ and

$$T_H = H \otimes \mathbb{G}_{m/W} = \mathrm{Spec}(W[\check{H}]),$$

the split torus over the Witt vectors with character group $\check{H}$ and cocharacter group H.[3] There is another useful way to think of H, as a rank-r^2 lattice of hermitian bilinear forms on Y (the lattice shrinking as the level increases) [Skinner and Urban 2014, Section 5.4.1]. Simply attach to $h \in H$ the hermitian form $b_h : Y \times Y \to \mathfrak{d}_E^{-1}$ defined by

$$b_h(y, y') = \delta_E^{-1}((h-1)y, y'). \tag{3.1.2}$$

Here $(\cdot\,,\cdot)$ is the pairing on $V_r \times (V/V_r^{\perp})$ induced from (1.1.2). Using the description of Z_r in (3.1.1) we may regard $h \mapsto b_h$ as assigning to $h \in H$ the matrix $\delta_E^{-1}B$.

We denote by Ξ the T_H-torsor over $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$ constructed in [Skinner and Urban 2014], smooth of total dimension mn. Recall that given such a torsor, every character $\chi \in \check{H}$ of T_H determines, by push-out, a $\mathbb{G}_m$-torsor Ξ_χ over C, and the resulting map

$$\chi \mapsto [\Xi_\chi]$$

from $\check{H}$ to the group of $\mathbb{G}_m$-torsors over C is a homomorphism. Conversely, Ξ is uniquely determined by giving such a homomorphism. We proceed to describe Ξ in this way.

If $y, y' \in Y$ let $\chi = [y \otimes y']$ denote the element of $\check{H}$ which sends

$$H \ni h \mapsto \mathrm{Tr}_{E/\mathbb{Q}}\, b_h(y, y') = \langle (h-1)y, y' \rangle \in \mathbb{Z}.$$

Then we require $\Xi_\chi|_c$, the fiber at $c \in C$ of Ξ_χ, to be

$$\mathcal{P}|^{\times}_{c(\phi_X(y)) \times c^t(y')},$$

where $\mathcal{P}$ is the Poincaré bundle over $\mathcal{A}_r^t \times \mathcal{A}_r$. The superscript $\times$ means "the associated $\mathbb{G}_m$-bundle", obtained by removing the zero section. It can be checked that this extends to a homomorphism from $\check{H}$ to the group of $\mathbb{G}_m$-torsors over C. For any $\chi \in \check{H}$ we let $\mathcal{L}(\chi)$ be the line bundle on C whose associated

[3] Skinner and Urban [2014] denote $\check{H}$ by S.

$\mathbb{G}_m$-bundle is Ξ_χ. Over the complex numbers, sections of $\mathcal{L}(\chi)$ are classical theta functions on the abelian scheme C. We shall often denote elements of $\check{H}$ also by $\check{h}$. We have a canonical identification $\mathcal{L}(\check{h}_1+\check{h}_2)=\mathcal{L}(\check{h}_1)\otimes\mathcal{L}(\check{h}_2)$.

Having constructed Ξ we proceed to study its equivariance properties under the group

$$\Gamma=\mathrm{GL}(V_r)(\mathbb{Q})\cap K.$$

Using (3.1.1), this is the group of rational matrices A that also lie in K. Since the action of P_r on Z_r by conjugation factors through $P_r/U_r=L_r$, the group $\Gamma\subset L_r$ acts on Z_r. Using (3.1.1) again, A sends B to $AB\,{}^t\bar{A}$. In particular Γ acts on H, hence it acts on T_H by automorphisms of the torus.

We also have an action of Γ on $C=\mathrm{Hom}_{\mathcal{O}_E}(X,\mathcal{A}_r^t)$ induced from its action on X. Any $\gamma\in\Gamma$ maps $\mathcal{L}(\check{h})|_c$ to $\mathcal{L}(\gamma(\check{h}))|_{\gamma(c)}$. If $\Gamma(\check{h})$ is the stabilizer of $\check{h}\in\check{H}$ then [Skinner and Urban 2014, Lemma 5.1] $\Gamma(\check{h})$ acts trivially on the global sections of $\mathcal{L}(\check{h})$ over C.

Finally, as the push-out of $\Xi|_{\gamma(c)}$ by $[\gamma(y)\otimes\gamma(y')]$ is identically the same as the push out of $\Xi|_c$ by $[y\otimes y']$, or equivalently

$$\Xi|_{\gamma(c)}=(\Xi\times^{T_H,\gamma}T_H)|_c$$

the isomorphism $1\times\gamma:\Xi=\Xi\times^{T_H}T_H\to\Xi\times^{T_H,\gamma}T_H$ of torsors *over* C, yields an action of Γ on Ξ which *covers* its action on C, and is compatible with the Γ-action on T_H. In short, all the constructions so far are equivariant under Γ.

The local charts. Now comes the choice of the auxiliary data involved in the toroidal compactification. Let

$$H_{\mathbb{R}}^+\subset H_{\mathbb{R}}$$

be the cone of positive semidefinite hermitian bilinear forms on $Y_{\mathbb{R}}$ whose radical is a subspace defined over $\mathbb{Q}$ (i.e., the $\mathbb{R}$-span of a subspace of $Y_{\mathbb{Q}}$). Let $\Sigma=\{\sigma\}$ be a Γ-admissible (infinite) rational polyhedral cone decomposition of $H_{\mathbb{R}}^+$ [Lan 2013, Definition 6.1.1.10]. Admissibility means that the action of Γ on $H_{\mathbb{R}}$ permutes the σ's, and that modulo Γ there are only finitely many cones in Σ. By convention, the cones σ do not contain their proper faces, and every face of a cone in Σ also belongs to Σ. In particular, Σ contains the origin as its unique 0-dimensional cone. When we treat all cusp labels, and not only one at a time, an additional assumption has to be imposed about the compatibility of the polyhedral cone decompositions associated with a cusp ξ and with a higher rank cusp to which ξ specializes. It is a nontrivial fact that such polyhedral cone decompositions exist, see Chapter 2 of [Ash et al. 1975]. Moreover, every two Γ-admissible rational polyhedral cone decompositions of $H_{\mathbb{R}}^+$ have a common refinement of the same sort. One can even find such a polyhedral cone decomposition in which every σ is spanned by a part of a basis of H. The $T_{H,\sigma}$ defined below will then be smooth over W, and from now on we assume that this is the case. Lan [2013] calls such a Σ a Γ-*admissible smooth rational polyhedral cone decomposition of* $H_{\mathbb{R}}^+$. If K is small enough so that Γ is *neat*, refinements exist such that, in addition, the closures of σ and $\gamma(\sigma)$, for $\sigma\in\Sigma$ and $1\neq\gamma\in\Gamma$, meet only at the origin.

Each cone $\sigma \in \Sigma$ defines a torus embedding

$$T_H \hookrightarrow T_{H,\sigma} = \mathrm{Spec}(W[\check{H} \cap \sigma^\vee])$$

where $\sigma^\vee \subset \check{H}_{\mathbb{R}}$ is the dual cone and $W = W(\kappa)$ as before. By definition

$$\sigma^\vee = \{v \in \check{H}_{\mathbb{R}} \mid v(u) \geq 0, \forall u \in \sigma\},$$

so, unlike σ, $\sigma^\vee$ contains its faces. Observe that T_H naturally acts on $T_{H,\sigma}$. Since σ does not contain a line, $\sigma^\vee$ has a nonempty interior.

Let

$$\sigma^\perp = \{v \in \check{H}_{\mathbb{R}} \mid v(u) = 0, \forall u \in \sigma\}.$$

When $d_\sigma = \dim(\sigma) < r^2$, $\sigma^\vee \supset \sigma^\perp \neq 0$. Then $Z_{H,\sigma} = \mathrm{Spec}(W[\check{H} \cap \sigma^\perp])$ is a torus, $\dim Z_{H,\sigma} = r^2 - d_\sigma$. In fact, $Z_{H,\sigma}$ is the unique minimal orbit of T_H in its action on $T_{H,\sigma}$, an orbit which lies in the closure of any other orbit. There is an obvious surjection $T_{H,\sigma} \twoheadrightarrow Z_{H,\sigma}$. This surjection admits a section $Z_{H,\sigma} \hookrightarrow T_{H,\sigma}$, corresponding to $W[\check{H} \cap \sigma^\perp] \simeq W[\check{H} \cap \sigma^\vee]/I_\sigma$, where I_σ is the ideal generated by $\check{H} \cap \sigma^\vee \setminus \check{H} \cap \sigma^\perp$. Another way to think of $Z_{H,\sigma}$ is as

$$Z_{H,\sigma} = T_{H,\sigma} \setminus \bigcup_{\tau < \sigma} T_{H,\tau}$$

where τ runs over all the *proper* faces of σ.

The $T_{H,\sigma}$ glue to form a toric variety (locally of finite type, but not of finite type in general) $T_{H,\Sigma}$, in which each $T_{H,\sigma}$ is open and dense:

$$T_{H,\Sigma} = \bigcup_{\sigma \in \Sigma} T_{H,\sigma}.$$

This $T_{H,\Sigma}$ is stratified by the disjoint union of the $Z_{H,\sigma}$. The actions of Γ on H and Σ induce an action of Γ on T_H and a compatible action on $T_{H,\Sigma}$. By our assumption on Σ, $T_{H,\Sigma}$ is smooth over W.

We "spread" this construction over $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$, twisting it by the torsor Ξ, namely we consider

$$\overline{\Xi}_\Sigma = \Xi \times^{T_H} T_{H,\Sigma}. \tag{3.1.3}$$

The group Γ acts on each of the three symbols on the right in a compatible way, so we get an action of Γ on $\overline{\Xi}_\Sigma$.

Let us bring back the reference to the cusp label $[g]_r$, although in the above we tacitly assumed $[g]_r = 1$ and dropped g from the notation. See [Skinner and Urban 2014, Section 5.4.1] for the precise definition of H_g, Σ_g etc. Denote by $H^{++}_{g,\mathbb{R}}$ the set of *positive-definite* hermitian bilinear forms in $H^+_{g,\mathbb{R}}$. For $\sigma \in \Sigma_g$ such that $\sigma \subset H^{++}_{g,\mathbb{R}}$ we let

$$Z([g]_r, \sigma) = \Xi \times^{T_H} Z_{H,\sigma},$$

and let

$$\mathfrak{Z}([g]_r, \sigma)$$

be the formal completion of $\overline{\Xi}_\Sigma$ or, what amounts to be the same, of its open subset $\Xi \times^{T_H} T_{H,\sigma}$, along $Z([g]_r, \sigma)$. These are the local charts at the cuspidal component labeled by $[g]_r$. There is a smooth morphism

$$\mathfrak{Z}([g]_r, \sigma) \to C([g]_r)$$

whose fibers are isomorphic to the completion of $T_{H,\sigma}$ along $Z_{H.\sigma}$. The $\mathfrak{Z}([g]_r, \sigma)$ are nm-dimensional and smooth over W. Each such local chart has $nm - d_\sigma$ "algebraic dimensions" and d_σ "formal dimensions". Specializing the formal variables to 0, one gets the support $Z([g]_r, \sigma)$ of $\mathfrak{Z}([g]_r, \sigma)$, whose dimension is $nm - d_\sigma$. The action of $\gamma \in \Gamma$ on $\overline{\Xi}_\Sigma$ induces an isomorphism γ_* between $\mathfrak{Z}([g]_r, \sigma)$ and $\mathfrak{Z}([g]_r, \gamma(\sigma))$. For comparison, we remark that in [Lan 2013, Section 6.2.5] the $\mathfrak{Z}([g]_r, \sigma)$ are denoted $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ and $Z([g]_r, \sigma)$ are denoted $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Also, under our assumptions the stabilizers denoted in [Lan 2013] by $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ are trivial.

Once we have described the local charts, it remains to construct on each of them the *degeneration data* which allows one to carry on the *Mumford construction*. This results in gluing the various charts together, and at the same time constructing $\mathcal{G}$ with the accompanying PEL structure over the glued scheme. Care has to be taken not only to glue pieces labeled by the same cusp label $[g]_r$, but also to respect the way cusp labels specialize. In the process of gluing, one has to divide by the action of Γ on the formal completion of (3.1.3) along the complement of $\Xi = \Xi \times^{T_H} T_H$. Note that it does not make sense to divide $\overline{\Xi}_\Sigma$ by Γ, just as it did not make sense to divide Ξ, or the abelian scheme C over which it lies, by the action of Γ. For the gluing of the local charts, that we do not review here, see [Lan 2013, Section 6.3]. The final result is [loc. cit., Theorem 6.4.1.1].

Logarithmic differentials. We construct certain formal differentials on the local chart $\mathfrak{Z}([g]_r, \sigma)$, relative to $C([g]_r)$, with logarithmic poles along $Z([g]_r, \sigma)$. We shall denote the module of these differentials

$$\Omega_{\mathfrak{Z}/C}[d \log \infty].$$

They will play an important role in our formulae for Θ.

Notation as above, consider a cone $\sigma \subset H^{++}_{g,\mathbb{R}}$ and let $h_1, \dots, h_{d_\sigma}$ be positive semidefinite, part of a basis of $H = H_g$, such that

$$\sigma = \mathrm{Cone}(h_1, \dots, h_{d_\sigma}).$$

Complete the h_i to a basis $h_1, \dots, h_{r^2}$ of H, let $\{\check{h}_i\}$ be the dual basis of $\check{H} = \mathrm{Hom}(H, \mathbb{Z})$ and introduce formal variables $q_i = q^{\check{h}_i}$ (to be able to write the group structure on $\check{H}$ multiplicatively rather than additively). Then

$$T_{H,\sigma} = \mathrm{Spec}(W[q_1, \dots, q_{d_\sigma}, q^{\pm 1}_{d_\sigma+1}, \dots, q^{\pm 1}_{r^2}])$$

and

$$Z_{H,\sigma} = \mathrm{Spec}(W[q^{\pm 1}_{d_\sigma+1}, \dots, q^{\pm 1}_{r^2}]).$$

Locally on $\mathfrak{Z}([g]_r, \sigma)$ we use as coordinates the pull-back of any system of $nm - r^2$ local coordinates on the base $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}^t_r)$, together with the "algebraic" coordinates $q_{d_\sigma+1}, \dots, q_{r^2}$, and the "formal" coordinates $q_1, \dots, q_{d_\sigma}$. We emphasize that because of the twist by the torsor Ξ in the construction of the local charts, the q_i are not global coordinates. The correct way to think of them is as local sections of the line bundles $\mathcal{L}(-\check{h}_i)$ on C. If the h_i are positive definite, these line bundles will be antiample, and the q_i will not globalize.

If $\check{h} \in \check{H}$ is of the form $\check{h} = \sum n_i \check{h}_i$ we write $q^{\check{h}} = \prod q_i^{n_i}$ and define

$$\omega(\check{h}) = \frac{dq^{\check{h}}}{q^{\check{h}}} = \sum_{i=1}^{r^2} n_i \frac{dq_i}{q_i} \in \Omega_{\mathfrak{Z}/C}[d \log \infty].$$

This $\omega(\check{h})$ is invariant under the action of T_H, essentially since $d\log(q_0 q) = d \log q$. Hence, despite the fact that the q_i were only *local* coordinates, $\omega(\check{h})$ defines a relative differential on all of $\Xi \times^{T_H} T_{H,\sigma}$, as well as on its completion $\mathfrak{Z}([g]_r, \sigma)$ along $Z([g]_r, \sigma) = \Xi \times^{T_H} Z_{H,\sigma}$, with logarithmic poles along $Z([g]_r, \sigma)$. The following proposition is an immediate by-product of the theory of toroidal compactifications.

Proposition 3.1.1. (i) *The differentials $\omega(\check{h})$ are well-defined formal differentials on $\mathfrak{Z}([g]_r, \sigma)$, relative to $C([g]_r)$, with logarithmic poles along $Z([g]_r, \sigma)$. They are independent of the choice of bases and depend only on $\check{h}$.*

(ii) $\omega(\check{h}_1 + \check{h}_2) = \omega(\check{h}_1) + \omega(\check{h}_2)$.

(iii) *The differentials $\omega(\check{h})$ are compatible with gluing of the local charts. If $\gamma \in \Gamma$ then the induced isomorphism between the local charts $\mathfrak{Z}([g]_r, \sigma)$ and $\mathfrak{Z}([g]_r, \gamma(\sigma))$ carries $\omega(\check{h})$ to $\omega(\gamma(\check{h}))$.*

(iv) *The differentials $\omega(\check{h})$ are compatible with the maps between toroidal compactifications obtained from refinements of the admissible smooth rational polyhedral cone decompositions* [Lan 2013, Section 6.4.2].

Fourier–Jacobi expansions. Let $\bar{S}$ be a fixed smooth toroidal compactification of S over W_s $(1 \le s)$ as a base ring. Let $\mathcal{G}$ be the universal semiabelian scheme over $\bar{S}$ and $e_{\mathcal{G}} : \bar{S} \to \mathcal{G}$ its zero section. Then $\omega = e_{\mathcal{G}}^* \Omega^1_{\mathcal{G}/\bar{S}}$ defines an extension of the Hodge bundle to a rank $n+m$ vector bundle with $\mathcal{O}_E$-action on $\bar{S}$. We continue to denote by $\mathcal{P}$ and $\mathcal{Q}$ its subbundles of type Σ and $\bar{\Sigma}$, of ranks n and m respectively.

Let $\bar{S}^{\text{ord}}$ denote the complement in $\bar{S}$ of the Zariski closure of $S \setminus S^{\text{ord}}$. Over this open subset of $\bar{S}$ the semiabelian variety $\mathcal{G}$ is μ-ordinary in the sense that the connected part of its p-divisible group at every geometric point $x : \text{Spec}(k) \to \bar{S}^{\text{ord}}$ satisfies

$$\mathcal{G}_x[p^\infty]^0 \simeq (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m \times \mathfrak{G}_k^{n-m}.$$

To see this, assume that x lies on a rank r cuspidal component, but that the abelian part $\mathcal{A}_x$ of $\mathcal{G}_x$ is not μ-ordinary, i.e., the multiplicative part of $\mathcal{A}_x[p^\infty]$ has height strictly less than $2(m-r)$. Mumford's construction shows that we may deform $\mathcal{G}$ into an abelian variety $\mathcal{A}_y$ (y signifying a point on the base of the deformation "near" x) so that the multiplicative part of $\mathcal{A}_y[p^\infty]$ has height strictly less than $2m$. But such a point y being not μ-ordinary, we conclude that x lies in the closure of $S \setminus S^{\text{ord}}$, contrary to our assumption.

It follows that the filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0$$

extends to a filtration by a subvector bundle over $\bar{S}^{\mathrm{ord}}$. Thus the automorphic vector bundles $\mathcal{E}_\rho$ defined on page 1842 extend to $\bar{S}^{\mathrm{ord}}$ too. In the following discussion fix the representation ρ.

Let $[g]_r \in \mathscr{C}_r$ be a cusp label of rank $0 < r \leq m$ and let $Z = Z([g]_r)$ be the corresponding cuspidal component of ∂S obtained by "gluing" the $Z([g]_r, \sigma)$ for $\sigma \in \Sigma_g$, $\sigma \subset H^{++}_{g,\mathbb{R}}$, and dividing by Γ. Let $\mathfrak{Z}([g]_r)$ be the formal completion of $\bar{S}$ along $Z([g]_r)$. Let $\xi \in S_{G_r, K_{r,g}}$ be a geometric point, and let Z_ξ be the preimage of ξ in Z. Then Z_ξ is obtained by "gluing" the preimage $Z([g]_r, \sigma)_\xi$ of ξ in $Z([g]_r, \sigma)$ for all σ as above, dividing by the action of Γ. Observe that the toric part T_X and the abelian part $\mathcal{A}_{r,\xi}$ of $\mathcal{G}$ are constant over each $Z([g]_r, \sigma)_\xi$. Thus $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$ are trivialized over the preimage $\mathfrak{Z}([g]_r, \sigma)_\xi$ of ξ in the local chart $\mathfrak{Z}([g]_r, \sigma)$, hence so is $\mathcal{E}_\rho$. In general, however, $\mathcal{E}_\rho$ will not be trivial over $\mathfrak{Z}([g]_r)_\xi$.

Our ξ is a point of the minimal compactification S^* (over W_s). The completed local ring $\hat{\mathcal{O}}_{S^*,\xi}$ is described in [Skinner and Urban 2014, Theorem 5.3; Lan 2013, Proposition 7.2.3.16]. In the following, let $\check{H}^+$ be the set of elements of $\check{H}$ which are nonnegative on $H^+_{\mathbb{R}}$.

Proposition 3.1.2. *There is a canonical isomorphism between $\hat{\mathcal{O}}_{S^*,\xi}$ and the ring $\mathcal{FJ}_\xi$ of all formal power series*

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}) q^{\check{h}}$$

which are invariant under Γ. Here $a(\check{h}) \in H^0(C_\xi, \mathcal{L}(\check{h}))$ where $C_\xi = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}^t_{r,\xi})$ is the abelian variety which is the fiber of C over ξ.

Recall that $\pi : \bar{S} \to S^*$ was the map between the toroidal compactification and the minimal one. There is a similar description of the completion of the stalk of $\pi_* \mathcal{E}_\rho$ at ξ [Skinner and Urban 2014, Proposition 5.5].

Proposition 3.1.3. *The completion of $(\pi_* \mathcal{E}_\rho)_\xi$ is canonically isomorphic to the $\hat{\mathcal{O}}_{S^*,\xi}$-module of formal power series*

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}) q^{\check{h}}$$

which are invariant under Γ. Here $a(\check{h}) \in H^0(C_\xi, \mathcal{L}(\check{h}) \otimes \mathcal{E}_\rho)$.

The action of Γ on $a(\check{h})$ demands an explanation, and for that we must bring back the dependence on $[g]_r \in \mathscr{C}_r$ and even on g itself. Still assuming that we are at the standard cusp, i.e., $[g]_r = [1]_r$, we may replace the representative $g = 1$ by $g = \gamma \in \Gamma = \mathrm{GL}(V_r) \cap K = \mathrm{GL}_r(E) \cap K$. The following changes then take place. The lattice $\Lambda \cap V_r$ is replaced by $\gamma(\Lambda \cap V_r) = \Lambda \cap V_r$, so does not change. The subgroups X and Y therefore remain the same, but γ acts on them nontrivially. This induces an action of γ on the abelian variety $C_\xi = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}^t_{r,\xi})$ classifying the extensions $\mathcal{G}$ of $\mathcal{A}_{r,\xi}$ by T_X, as well as an action on the torus T_X. Thus γ induces an isomorphism

$$\gamma_* : \mathcal{G}_c \simeq \mathcal{G}_{\gamma(c)}$$

($c \in C_\xi$), which on the toric part is the given automorphism of T_X, and on the abelian part induces the identity. This induces isomorphisms $\gamma_* = (\gamma^*)^{-1}$ from the fibers of $\mathcal{P}_0$, $\mathcal{P}_\mu$ and $\mathcal{Q}$ at c to the corresponding fibers at $\gamma(c)$. As $\mathcal{P}_0$ depends only on the abelian part, the action of γ_* on it is trivial. Assume, for simplicity, that $\mathcal{E}_\rho = \mathcal{P}_\mu$. Then $a(\check{h})$ is a section (over C_ξ) of $\mathcal{L}(\check{h}) \otimes \mathcal{P}_\mu$ and $\gamma(a(\check{h}))$ will be the section of $\mathcal{L}(\gamma\check{h}) \otimes \mathcal{P}_\mu$ satisfying

$$\gamma(a(\check{h}))|_{\gamma(c)} = \gamma_*(a(\check{h})|_c).$$

We also remark that in [Skinner and Urban 2014, Proposition 5.5] the automorphic vector bundle is incorrectly assumed to be constant along C_ξ. We thank one of the referees for pointing this out to us. However, in one important case, that will be needed below, this is true. If ξ is a rank m cusp, the three basic automorphic vector bundles $\mathcal{Q}$, $\mathcal{P}_\mu$ and $\mathcal{P}_0$ depend only on the toric part and the abelian part of the universal semiabelian scheme separately, and (unlike $\mathcal{P}$) do not depend on the extension class parametrized by C_ξ. This implies that they are constant along C_ξ and so is every p-adic automorphic vector bundle generated by them.

In the sequel we shall only need the case of the *maximally degenerate* cusps, i.e., $r = m$. Now the Shimura variety $S_{\boldsymbol{G}_m, K_{m,g}}$ is 0-dimensional, and ξ is one of its (schematic) points. The abelian variety C_ξ is $m(n-m)$-dimensional. In this case $\mathcal{P}_\mu$ is the Σ-part of the cotangent space at the origin to

$$T_X = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m)$$

(if $r < m$ it also captures part of the cotangent space at the origin of $\mathcal{A}_{r,\xi}$). In other words, we may identify

$$\mathcal{P}_\mu \simeq \mathcal{O}_{C_\xi} \otimes_{\Sigma, \mathcal{O}_E} X$$

and $\gamma_* : \mathcal{P}_\mu|_c \to \mathcal{P}_\mu|_{\gamma(c)}$ with $\gamma_* \otimes \gamma_*$. Similarly we may identify $\mathcal{Q} \simeq \mathcal{O}_{C_\xi} \otimes_{\bar{\Sigma}, \mathcal{O}_E} X$. As the action of γ on X is via the contragredient $\mathrm{st}^\vee$ of the standard representation, it follows that to obtain the action of $\gamma \in \Gamma$ on $\rho(W)$ in general, we have first to embed γ as $\iota^\vee(\gamma) := ({}^t\bar{\gamma}^{-1}, {}^t\gamma^{-1}, 1)$ in $\mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$. (Recall that these three factors correspond to $\mathcal{Q}$, $\mathcal{P}_\mu$ and $\mathcal{P}_0$ in this order, see Section 1.2.) The action of γ on $a(\check{h}) \in H^0(C_\xi, \mathcal{L}(\check{h})) \otimes_W \rho(W)$ will then be via $\gamma_* \otimes \rho(\iota^\vee(\gamma))$.

We remark that when $n = m$ the Fourier–Jacobi expansions are in fact Fourier expansions in the naive sense, and the $a(\check{h})$ are scalars.

The Fourier–Jacobi expansion of the Hasse invariant. Assume now that $s = 1$, i.e., we are again over the special fiber in characteristic p, and the automorphic vector bundle is the line bundle $\mathcal{L}^{p^2-1}$ where $\mathcal{L} = \det(\mathcal{Q})$. Let $h \in H^0(S, \mathcal{L}^{p^2-1})$ be the Hasse invariant, previously denoted h (1.1.5).

Proposition 3.1.4. *The Fourier–Jacobi expansion of h at a rank-m cusp is* 1.

Proof. Let us check the claim at the standard cusp. Fix a local chart $\mathfrak{Z}([1]_m, \sigma)$ as above. As we have seen, $\mathcal{Q}$, hence also $\mathcal{L}$, are trivialized there. The trivialization is obtained from a similar trivialization of the p-divisible group of the toric part T_X of the semiabelian variety $\mathcal{G}$. As the isogeny Ver acts like the identity on $T_X[p^\infty]$, the Hasse invariant maps a trivializing section ℓ of $\mathcal{L}$ over $\mathfrak{Z}([1]_m, \sigma)$ to $\ell^{(p^2)}$. It follows that in terms of the basis ℓ^{p^2-1} of $\mathcal{L}^{p^2-1}$, its Fourier–Jacobi expansion is simply 1. Note that a choice of another κ-rational section ℓ will result in the same value for h. □

Corollary 3.1.5. *The open set $\bar{S}^{\text{ord}} \subset \bar{S}$ is the nonvanishing locus of h.*

Proof. By definition, $\bar{S}^{\text{ord}}$ is the complement of the Zariski closure of S^{no}, which is the vanishing locus of h in S^{ord}. It is therefore clear that h vanishes on its complement, and to prove the corollary it is enough to check that h does not vanish on any irreducible component of $\partial S = \bar{S} \setminus S$. But by [Lan 2015] any such irreducible component contains a rank m cusp, so the claim follows from the previous proposition. □

3.2. *Analytic continuation of Θ to the boundary and its effect on Fourier–Jacobi expansions.*

The partial toroidal compactification of the Igusa scheme. Fix $s \geq 1$ and work over W_s as a base ring. Since the semiabelian scheme $\mathcal{G}$ over $\bar{S}_s^{\text{ord}}$ is μ-ordinary, the *relative* moduli problem defining the big Igusa scheme of level p^t makes sense over $\bar{S}_s^{\text{ord}}$. More precisely, for an R-valued point of $\bar{S}_s^{\text{ord}}$ denote by $\mathcal{G}_R$ the pull-back of $\mathcal{G}$ to $\text{Spec}(R)$. Then $\mathcal{G}_R[p^\infty]^0$ is still isomorphic, locally in the proétale topology on $\text{Spec}(R)$, to an extension of $\mathfrak{G}^{n-m}$ by $(\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m$. The relative moduli problem $\overline{T}_{t,s}$ classifies Igusa structures (ϵ^1, ϵ^2) on $\mathcal{G}_R$ as in (2.1.1). The compatibility with Weil pairings is imposed on ϵ^1 only, as there is no ϵ^0 to pair with ϵ^2. This makes sense even if $\mathcal{G}_R$ is not an abelian scheme, while when it is, ϵ^0 is determined by ϵ^2. We call the resulting scheme $\overline{T}_{t,s}$. The following proposition is then obvious.

Proposition 3.2.1. (1) *The partially compactified Igusa scheme $\overline{T}_{t,s}$ is a finite étale Galois cover of $\bar{S}_s^{\text{ord}}$ with Galois group Δ_t.*

(2) *If $t \geq s$ then the basic vector bundles $\mathcal{P}_0$, $\mathcal{P}_\mu$ and $\mathcal{Q}$ are canonically trivialized over $\overline{T}_{t,s}$.*

We continue to denote by $\tau : \overline{T}_{t,s} \to \bar{S}_s^{\text{ord}}$ the covering map and by $\varepsilon^1, \varepsilon^2$ the resulting trivializations over $\overline{T}_{t,s}$. The definition of $\tilde{\Theta}$ over $\bar{S}_s^{\text{ord}}$ is then precisely the same as over the open ordinary stratum S_s^{ord}, see (2.2.2).

The extended Θ operator. To extend the definition of Θ we need to recall how the Kodaira–Spencer isomorphism extends to the toroidal compactification. The answer is given by [Lan 2013, Theorem 6.4.1.1, part 4]. See also [Faltings and Chai 1990, Chapter III, Corollary 9.8]. In our case [Lan 2013, Definition 6.3.1] it translates into the following.

Proposition 3.2.2. *The Kodaira Spencer isomorphism extends to an isomorphism*

$$\text{KS} : \mathcal{P} \otimes \mathcal{Q} \simeq \Omega^1_{\bar{S}/W}[d \log \infty]$$

over $\bar{S}$.

The inverse isomorphism KS^{-1} therefore maps $\Omega^1_{\bar{S}/W}$ to sections of $\mathcal{P} \otimes \mathcal{Q}$ *vanishing along the boundary* ∂S. We deduce the following.

Proposition 3.2.3. *The formula*

$$\Theta = (1 \otimes \mathrm{pr}_\mu \otimes 1) \circ (1 \otimes \mathrm{KS}^{-1}) \circ \tilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$$

defines an extension of Θ *over* $\bar{S}^{\mathrm{ord}}$. *For any section* f *of* $\mathcal{E}_\rho$, $\Theta(f)$ *vanishes along* $\partial \bar{S}^{\mathrm{ord}}$.

The isomorphism between $\mathcal{P}_\mu \otimes \mathcal{Q}$ *and* $\check{H} \otimes \mathcal{O}_{\bar{S}}$ *when* $r = m$. We now turn to determining the effect of Θ on Fourier–Jacobi expansions. This will be done at *maximally degenerate* cusps only. We therefore take $r = m$ and denote by $\xi \in S^*$ a cusp of rank m. Note that there are only finitely many such cusps. Nevertheless, there are sufficiently many of them to lie in every irreducible component of $\bar{S}$ [Lan 2015]. This will allow us to apply the q-expansion principle with these cusps only, not having to worry about expansions at lower rank cusps, where the formulae are not as nice.

Lemma 3.2.4. *Let* $x \in \bar{S}$ *be any point lying above* ξ. *Let* g *be a representative of the cusp label* $[g]_m$ *to which* ξ *belongs,* $H = H_g$ *the rank-*m^2 *lattice of hermitian bilinear forms on* $Y = Y_g$ *as on page 1854, and* $\check{H}$ *its* $\mathbb{Z}$*-dual. Then there is a canonical identification of the completed stalk* $(\mathcal{P}_\mu \otimes \mathcal{Q})^\wedge_x$ *with* $\check{H} \otimes \hat{\mathcal{O}}_{\bar{S},x}$,

$$(\mathcal{P}_\mu \otimes \mathcal{Q})^\wedge_x \simeq \check{H} \otimes \hat{\mathcal{O}}_{\bar{S},x}. \tag{3.2.1}$$

This identification is compatible with the natural action of Γ *on both sides.*

Proof. Let $R = \hat{\mathcal{O}}_{\bar{S},x}$. It is enough to deal with the standard cusp. When $r = m$ the stalks of the vector bundles $\mathcal{P}_\mu$ and $\mathcal{Q}$ are the Σ and $\bar{\Sigma}$-parts of ω_{T_X}, the cotangent bundle of the toric part of $\mathcal{G}$. Since $T_X = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m)$, it follows that there are canonical identifications

$$\mathcal{P}^\wedge_{\mu,x} = X \otimes_{\mathcal{O}_E,\Sigma} R, \quad \mathcal{Q}^\wedge_x = X \otimes_{\mathcal{O}_E,\bar{\Sigma}} R.$$

The map $Y \otimes Y \to \check{H}$ described in the course of the construction of the torsor Ξ on page 1854 yields an isomorphism

$$(Y \otimes_{\mathcal{O}_E,\Sigma} R) \otimes_R (Y \otimes_{\mathcal{O}_E,\bar{\Sigma}} R) \simeq \check{H} \otimes R = \mathrm{Hom}(H, R).$$

Explicitly, $(y \otimes 1) \otimes (y' \otimes 1)$ goes to the map sending $h \in H$ to $((h-1)y', y)$. Using the isomorphism $\phi_X : Y \simeq X$ we get the isomorphism (3.2.1).

Let us verify that the isomorphism given in the lemma is compatible with the natural actions of our group Γ on $\mathcal{P}_\mu \otimes \mathcal{Q}$ and $\check{H}$. On page 1857 we computed the action of $\gamma \in \Gamma$ on $(\mathcal{P}_\mu \otimes \mathcal{Q})^\wedge_x$ to be through ${}^t\gamma^{-1} \times {}^t\bar{\gamma}^{-1} \in \mathrm{GL}_m \times \mathrm{GL}_m$. On the other hand, γ acts on $h \in H$ via $h \mapsto \gamma h {}^t\bar{\gamma}$. As $\check{H}$ is the $\mathbb{Z}$-dual of H, these actions match each other. □

The main theorem.

Theorem 3.2.5. *Let* ξ *be a rank-*m *cusp. Let* f *be a section of* $\mathcal{E}_\rho$ *and*

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}) \cdot q^{\check{h}}$$

its Fourier–Jacobi expansion at ξ, as in Proposition 3.1.3. Then the section $\Theta(f)$ of $\mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$ has the Fourier–Jacobi expansion

$$\Theta(f) = \sum_{\check{h} \in \check{H}^+} a(\check{h}) \otimes \check{h} \cdot q^{\check{h}},$$

using the identification from Lemma 3.2.4.

Proof. We may work over W_s as a base ring. Fix any $t \geq s$ and let $\overline{T} = \overline{T}_{t,s}$. We assume that ξ is the standard cusp of rank m (regarded as a k-valued point of S^*, where k is algebraically closed and contains κ), and fix a geometric point x in the toroidal compactification lying above it. Fix a local chart $\mathfrak{Z}([g]_m, \sigma)$ containing x (where $[g]_m = [1]_m$ by our assumptions) and let $\mathfrak{Z}([g]_m, \sigma)_\xi$ be the preimage of ξ in it. As the abelian and toric parts of $\mathcal{G}$ are constant over $\mathfrak{Z}([g]_m, \sigma)_\xi$ we may fix admissible trivializations ϵ^1 and ϵ^2 of the graded pieces gr^1 and gr^2 of $\mathcal{G}[p^t]^0$, over the complete local ring at x. Indeed, the point ξ on the 0-dimensional Shimura variety $S_{\mathbf{G}_m, K_{m,g}}$ corresponds to an $n-m$ dimensional abelian variety $\mathcal{A}_m$ over the algebraically closed field k, with associated PEL structure of signature $(n-m, 0)$. Fix a symplectic trivialization

$$\epsilon^1 : \mathfrak{G}[p^t]^{n-m} \simeq \mathcal{A}_m[p^t] = \mathrm{gr}^1 .$$

Similarly, using the standard basis of $\Lambda \cap V_m$ we get a standard basis on X, which gives us a trivialization

$$\epsilon^2 : (\mathfrak{d}_E^{-1} \otimes \mu_{p^t})^m \simeq T_X[p^t] = \mathrm{gr}^2 .$$

As usual, since $t \geq s$, these trivializations induce trivializations of $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$, hence of $\mathcal{E}_\rho$. They also determine a choice of a point $\bar{x}$ in $\overline{T}$ above x. (If σ is replaced by a Γ-equivalent cone $\gamma(\sigma)$ the trivialization ϵ^2 is twisted by the action of γ on X, and this results in a different $\bar{x}$. The choice of ϵ^1 was also arbitrary, and effects the point $\bar{x}$ in a similar way.)

We use $R = \hat{\mathcal{O}}_{\overline{T},\bar{x}} \simeq \hat{\mathcal{O}}_{\bar{S},x}$ as the ring in which we compute Θ. Recall that the Fourier–Jacobi coefficient $a(\check{h})$ is a section of the vector bundle $\mathcal{L}(\check{h}) \otimes \mathcal{E}_\rho$ over the abelian scheme C of relative dimension $m(n-m)$ and that $\mathcal{E}_\rho$ has already been trivialized by our choices. Trivializing also the pull-back of the line bundles $\mathcal{L}(\check{h})$ to $\mathrm{Spec}(R)$, we may write the ring R as

$$R = W_s(\bar{\kappa})[\![u_1, \ldots, u_{m(n-m)}, q_1, \ldots, q_{m^2}]\!],$$

where the u_i are pull-backs of local coordinates on C at the image of x, and we may assume that the $a(\check{h})$ are (vector-valued) functions of the u_i. We now have

$$d(\tau^* f) = \sum_{\check{h} \in \check{H}^+} da(\check{h}) \cdot q^{\check{h}} + \sum_{h \in \check{H}^+} a(\check{h}) \cdot \frac{dq^{\check{h}}}{q^{\check{h}}} \cdot q^{\check{h}}.$$

Recall that the image of $dq^{\check{h}}/q^{\check{h}}$ modulo Ω_{C/W_s} is $\omega(\check{h}) \in \Omega_{\mathfrak{Z}/C}[d \log \infty]$. To complete the proof of the theorem we shall show the following two claims:

(1) For any $\eta \in \Omega_{C/W}$, we have $\eta \in \mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})$.

(2) The resulting isomorphism $\mathrm{KS}^{-1} : \Omega_{\mathfrak{Z}/C}[d \log \infty] \simeq \mathcal{P}_\mu \otimes \mathcal{Q} \simeq \check{H} \otimes R$ (see Lemma 3.2.4) carries $\omega(\check{h})$ to $\check{h} \otimes 1$.

Indeed, by (1), when we follow the definition of Θ and mod out by $\mathcal{P}_0 \otimes \mathcal{Q}$, the first sum, containing the $da(\check{h})$'s, disappears. The second sum provides the desired formula, by (2).

Proof of (1). This follows from the discussion of the Kodaira–Spencer map for semiabelian schemes in [Lan 2013, Section 4.6.1]. Let C assume the role of the base-scheme denoted there by S, and $\mathcal{G}$ the semiabelian scheme denoted there by $G^\natural$. Then Lan constructs a Kodaira–Spencer map for semiabelian schemes $\mathrm{KS}_{\mathcal{G}/C}$, which in our case is an isomorphism

$$\mathrm{KS}_{\mathcal{G}/C} : \mathcal{P}_0 \otimes \mathcal{Q} \simeq \Omega_{C/W_s}.$$

Note that Lan allows the abelian part to deform as well, but in our case $\mathcal{A} = \mathcal{A}_m$ is constant. This implies that the Kodaira–Spencer map, which is a priori defined on $\omega_{\mathcal{A}} \otimes \omega_{\mathcal{G}}$, factors through its quotient $\omega_{\mathcal{A}} \otimes \omega_T$. In addition, because of the constraints imposed by the endomorphisms, we may restrict it to $\omega_{\mathcal{A}}(\Sigma) \otimes \omega_T(\bar{\Sigma}) = \mathcal{P}_0 \otimes \mathcal{Q}$ without losing any information. Finally, [Lan 2013, Remark 4.6.2.7 and Theorem 4.6.3.16] imply that the diagram

$$\begin{array}{ccc} \mathcal{P}_0 \otimes \mathcal{Q} & \xrightarrow{\mathrm{KS}_{\mathcal{G}/C}} & \Omega_{C/W_s} \\ \cap & & \cap \\ \mathcal{P} \otimes \mathcal{Q} & \xrightarrow{\mathrm{KS}} & \Omega_{\bar{S}/W_s}[d \log \infty] \end{array}$$

is commutative, and this proves (1). □

Proof of (2). The second claim goes to the root of how KS is defined on $\bar{S}$. See [Lan 2013, Section 4.6.2], especially the discussion on page 269, preceding Definition 4.6.2.10. Fix a basis $y_1, \dots, y_m$ of Y and let $\chi_i = \phi_X(y_i)$ be the corresponding basis of X. Then as a basis of $\check{H}$ we may take the elements $\check{h}_{ij} = [y_i \otimes y_j]$ (see the proof of Lemma 3.2.4). The corresponding element of the stalk of $\mathcal{P}_\mu \otimes \mathcal{Q}$ at x is $\chi_i \otimes \chi_j$. The variable $q_{ij} = q^{\check{h}_{ij}}$ is then a generator of the invertible R-module denoted in [loc. cit.] by $I(y_i, \chi_j)$, and the extended Kodaira–Spencer homomorphism is defined in [loc. cit., Definition 4.6.2.12] so that it takes $\chi_i \otimes \chi_j$ to $d \log(q_{ij}) = \omega(\check{h}_{ij})$. The base schemes S and S_1 in [loc. cit.] are in our case $\mathrm{Spec}(R)$ and its generic point. □

Corollary 3.2.6. *Let $f \in H^0(S_s^{\mathrm{ord}}, \mathcal{E}_\rho)$ and let h be the Hasse invariant* (1.1.5). *Then $\Theta(hf) = h\Theta(f)$.*

Proof. Obvious. □

Corollary 3.2.7. (i) *Let $f \in H^0(\mathcal{S}^{\mathrm{ord}}, \mathcal{E}_\rho)$. Then $\Theta(f) = 0$ if and only if the Fourier–Jacobi expansion of f at every rank m cusp is constant.*

(ii) *$f \in \ker(\Theta)$ if and only if its Fourier–Jacobi expansion at every rank m cusp is supported on $\check{h} \in p\check{H}^+$.*

Proof. (i) This follows from our theorem and the FJ-*expansion principle*: a p-adic modular form vanishes if and only if its Fourier–Jacobi expansion at every rank m cusp vanishes. This principle was proved in

[Lan 2013, Proposition 7.1.2.14], under the assumption that every irreducible component of $\bar{S}$ contains at least one rank m cuspidal stratum $Z([g]_m, \sigma)$. This assumption was later verified, for our Shimura variety among others, in Corollary A.2.3 of [Lan 2015].

(ii) Follows by the same argument, noticing that for $a(\check{h}) \otimes \check{h}$ to vanish, it is necessary and sufficient that either $a(\check{h}) = 0$ or $\check{h} \in p\check{H}^+$. □

4. Analytic continuation of Θ to the nonordinary locus

4.1. *The almost ordinary locus.*

The stratum S^{ao}. In this section we assume that $n > m$, as the question we are about to discuss requires different considerations when $n = m$, which will be handled separately. Let S denote, as in the beginning, the special fiber of the Shimura variety $\mathcal{S}$. Thus S is nm-dimensional, smooth over $\kappa = \mathbb{F}_{p^2}$, and is stratified by the Ekedahl–Oort strata [Oort 2001; Moonen 2001; Viehmann and Wedhorn 2013]. The (μ-)ordinary stratum S^{ord} is open and dense, and the operator Θ acts on sections of the automorphic vector bundle $\mathcal{E}_\rho$ over it, sending them to sections of $\mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q} \simeq \mathcal{E}_\rho \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}$,

$$\Theta : H^0(S^{\mathrm{ord}}, \mathcal{E}_\rho) \to H^0(S^{\mathrm{ord}}, \mathcal{E}_\rho \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Here we have used the fact that in characteristic p the vector bundle homomorphism $V_{\mathcal{P}} : \mathcal{P} \to \mathcal{Q}^{(p)}$ is surjective with kernel $\mathcal{P}_0$, so induces an isomorphism $\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0 \simeq \mathcal{Q}^{(p)}$. Our goal in this section is to study the analytic continuation of Θ to all of S. This is reminiscent of the fact that the theta operator on GL_2 (denoted by $A\theta$ in [Katz 1977]) extends holomorphically across the supersingular points of the modular curve.

Proposition 4.1.1. *There exists a unique EO stratum S^{ao} of dimension $nm - 1$. The homomorphism $V_{\mathcal{P}}$ is still surjective in every geometric fiber over S^{ao}, so $\mathcal{P}_0 = \mathcal{P}[V_{\mathcal{P}}]$ extends to a rank $n - m$ vector bundle over $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$. The same applies to $\mathcal{P}_\mu$ and of course to $\mathcal{Q}$, hence every p-adic automorphic vector bundle $\mathcal{E}_\rho$ extends canonically to the open set $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$.*

We call S^{ao} the *almost-ordinary* locus. It is the divisor of the Hasse invariant h on $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$, and, like any other EO stratum, is nonsingular.

Proof. The uniqueness of the EO stratum in codimension 1 is proved in [Wooding 2016, Corollary 3.4.5], where it is deduced from the classification of the EO strata by Weyl group elements and the calculation of their dimensions in [Moonen 2001]. The assertion on $V_{\mathcal{P}}$ being surjective in every geometric fiber follows from the computation of the Dieudonné space at geometric points of S^{no} [Wooding 2016, Proposition 3.5.6], reviewed below. Since the base scheme is a nonsingular variety, constancy of the fibral rank of $V_{\mathcal{P}}$ suffices to conclude that $\mathcal{P}_0$ and $\mathcal{P}_\mu$ are locally free sheaves. Finally, $\mathcal{E}_\rho$ is constructed by twisting the representation ρ of $\mathrm{GL}_m \times \mathrm{GL}_m \times \mathrm{GL}_{n-m}$ (with values in κ) by the vector bundles $\mathcal{Q}$, $\mathcal{P}_\mu$ and $\mathcal{P}_0$ as on page 1841. □

Dieudonné spaces. Let k be a perfect field of characteristic p. For the following see [Oda 1969; Bültel and Wedhorn 2006; Wedhorn 2001, (5.3)]. A polarized Dieudonné space over k is a finite dimensional k-vector space D equipped with a nondegenerate skew-symmetric pairing $\langle \cdot , \cdot \rangle$ and two linear maps $F : D^{(p)} \to D$ and $V : D \to D^{(p)}$ such that $\mathrm{Im}(F) = \ker(V)$ and $\mathrm{Im}(V) = \ker(F)$, and such that $\langle Fx, y\rangle = \langle x, Vy\rangle^{(p)}$ for every $x \in D^{(p)}$ and $y \in D$. It follows immediately from the definition that $\dim D = 2g$ and F and V have rank g. If M is a principally polarized Dieudonné module over $W(k)$ then $D = M/pM$ is a polarized Dieudonné space. If A is a principally polarized abelian variety over k then its de Rham cohomology $D = H^1_{dR}(A/k)$ is equipped with a canonical structure of a Dieudonné space, which may also be identified with the (contravariant) Dieudonné module of $A[p]$. The Hodge filtration is then related to F via

$$\omega = H^0(A, \Omega^1) = (D^{(p)}[F])^{(p^{-1})}.$$

It is essential for this that we work over a perfect base.

A polarized $\mathcal{O}_E$-Dieudonné space is a polarized Dieudonné space admitting, in addition, endomorphisms by $\mathcal{O}_E$, for which F and V are $\mathcal{O}_E$-linear and $\langle ax, y\rangle = \langle x, \bar{a}y\rangle$ $(a \in \mathcal{O}_E)$. Assume that k contains κ. Then $D(\Sigma)$ and $D(\overline{\Sigma})$ are set in duality by the pairing, hence are each of dimension g, V maps $D(\Sigma)$ to $D(\overline{\Sigma})^{(p)}$ and $D(\overline{\Sigma})$ to $D(\Sigma)^{(p)}$ and a similar statement, going backwards, holds for F. The type (n, m) of ω ($n = \dim \omega(\Sigma)$, $m = \dim \omega(\overline{\Sigma})$) is called the *type*, or *signature*, of D.

Over a nonperfect base $\mathrm{Spec}(R)$ (in characteristic p, say, as this is all that we need) one can still associate to a principally polarized abelian scheme A/R, or to its p-divisible group, a Dieudonné *crystal* as in [Grothendieck 1974], and when *evaluated* at $(\mathrm{Spec}(R) \subset \mathrm{Spec}(R))$ it yields a polarized R-module $D(A/R)$ with an F and a V as before, which may be identified with $H^1_{dR}(A/R)$. If R is an equicharacteristic PD-thickening of k then in fact $D(A/R) = R \otimes_k D(A_k/k)$ with the polarization, F and V extended R-linearly. The Hodge filtration can not be read from $D(A/R)$ any more. In fact, Grothendieck's theorem asserts that giving $(D(A/R), \omega)$ is tantamount to giving the deformation of A from $\mathrm{Spec}(k)$ to $\mathrm{Spec}(R)$. We shall apply these remarks later on when k is algebraically closed, $x \in S(k)$ is a geometric point, and $R = \mathcal{O}_{S,x}/\mathfrak{m}^2_{S,x}$ is its first infinitesimal neighborhood.

Let k be an algebraically closed field containing κ. Consider the following polarized $\mathcal{O}_E$-Dieudonné spaces. We use the convention that $\mathcal{O}_E$ acts on the e_i via Σ and on the f_i via $\overline{\Sigma}$. Write $\mathfrak{G}_\Sigma$ for the p-divisible group $\mathfrak{G}_k$ equipped with the $\mathcal{O}_E$-action inducing Σ on the tangent space, and likewise $\mathfrak{G}_{\overline{\Sigma}}$.

(i) $D(\mathfrak{G}_\Sigma[p]) = \mathrm{Span}_k\{e_1, f_1\}$, $\langle e_1, f_1\rangle = 1$, $Ff_1^{(p)} = e_1$, $Fe_1^{(p)} = 0$, $Vf_1 = e_1^{(p)}$, $Ve_1 = 0$. Here $\omega = ke_1$ and the signature is $(1, 0)$.

(i)' $D(\mathfrak{G}_{\overline{\Sigma}}[p]) = \mathrm{Span}_k\{e_2, f_2\}$, $\langle f_2, e_2\rangle = 1$, $Fe_2^{(p)} = f_2$, $Ff_2^{(p)} = 0$, $Ve_2 = f_2^{(p)}$, $Vf_2 = 0$. Note that $D(\mathfrak{G}_{\overline{\Sigma}}[p]) = D(\mathfrak{G}_\Sigma[p])^{(p)}$, $\omega = kf_2$ and the signature is $(0, 1)$.

(ii) $AO(2, 1) = \mathrm{Span}_k\{e_i, f_i \mid 1 \le i \le 3\}$, $\langle e_1, f_3\rangle = \langle f_2, e_2\rangle = \langle f_1, e_3\rangle = 1$ (and $\langle e_i, f_j\rangle = 0$ if $i + j \ne 4$); F and V are given by the following table, where to ease notation the $^{(p)}$ is left out:

	e_1	e_2	e_3	f_1	f_2	f_3
F	0	f_1	0	e_1	0	e_2
V	0	0	f_2	0	e_1	e_3

This is the Dieudonné space denoted by $\bar{B}(3)$ in [Bültel and Wedhorn 2006]. Here $\omega = \mathrm{Span}_k\{e_1, e_3, f_2\}$ and $\mathcal{P}_0 = \omega(\Sigma)[V] = ke_1$.

(iii) $AO(3, 1) = \mathrm{Span}_k\{e_i, f_i \mid 1 \le i \le 4\}$, $\langle e_1, f_4\rangle = \langle e_2, f_3\rangle = \langle f_2, e_3\rangle = \langle f_1, e_4\rangle = 1$ (and $\langle e_i, f_j\rangle = 0$ if $i + j \neq 5$); F and V are given by the following table, where to ease notation the $^{(p)}$ is left out:

	e_1	e_2	e_3	e_4	f_1	f_2	f_3	f_4
F	0	f_1	0	0	e_1	e_2	0	e_3
V	0	0	0	f_3	0	e_1	e_3	e_4

This is the Dieudonné space denoted by $\bar{B}(4)$ in [Bültel and Wedhorn 2006]. Here ω is equal to $\mathrm{Span}_k\{e_1, e_3, e_4, f_3\}$ and $\mathcal{P}_0 = \omega(\Sigma)[V] = \mathrm{Span}_k\{e_1, e_3\}$.

Proposition 4.1.2 [Wooding 2016, Proposition 3.5.6]. *Let $x \in S^{\mathrm{ao}}(k)$ be an almost-ordinary geometric point. Then $D_x = D(\mathcal{A}_x/k)$ is isomorphic to the following*:

(i) $n = m + 1$:

$$D = AO(2, 1) \oplus \bigoplus_{i=1}^{m-1} \mathrm{Span}_k\{e_i^{\mu}, e_i^{et}, f_i^{\mu}, f_i^{et}\}$$

where $\langle e_i^{\mu}, f_i^{et}\rangle = \langle f_i^{\mu}, e_i^{et}\rangle = 1$ (and $\langle e_i^{\mu}, f_i^{\mu}\rangle = \langle e_i^{et}, f_i^{et}\rangle = 0$), and F and V are given by the following table:

	e_i^{μ}	e_i^{et}	f_i^{μ}	f_i^{et}
F	0	f_i^{et}	0	e_i^{et}
V	f_i^{μ}	0	e_i^{μ}	0

(ii) $n \ge m + 2$:

$$D = AO(3, 1) \oplus \bigoplus_{i=1}^{m-1} \mathrm{Span}_k\{e_i^{\mu}, e_i^{et}, f_i^{\mu}, f_i^{et}\} \oplus D(\mathfrak{G}_{\Sigma}[p])^{n-m-2}.$$

The Kodaira–Spencer isomorphism along the almost ordinary stratum. The following result is the key to the analytic continuation of the theta operator, which will be proved in the next section.

Theorem 4.1.3. *Let*

$$\psi = (\mathrm{pr}_{\mu} \otimes 1) \circ \mathrm{KS}^{-1} : \Omega_{S/\kappa} \to \mathcal{P}_{\mu} \otimes \mathcal{Q}$$

be the composition of the inverse of the Kodaira–Spencer isomorphism and the projection from $\mathcal{P}$ to $\mathcal{P}_{\mu} = \mathcal{P}/\mathcal{P}_0$ (well-defined over $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$). Let $u = 0$ be a local equation of the divisor S^{ao} in a Zariski open set $U \subset S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$. Then $\psi(du)$ vanishes along $S^{\mathrm{ao}} \cap U$.

Remark. Compare with [de Shalit and Goren 2016, Proposition 3.11]. In terms of the foliation $\mathcal{T}S^+$ introduced in [de Shalit and Goren 2018] the theorem asserts that at any point $x \in S^{\text{ao}}$ this foliation is tangential to S^{ao}, i.e $\mathcal{T}S^+|_x \subset \mathcal{T}S^{\text{ao}}|_x$. In [de Shalit and Goren 2018] we studied a certain open subset $S_\sharp \subset S$ which was a union of Ekedahl–Oort strata, including S^{ord}, S^{ao} and a unique minimal EO stratum denoted there S^{fol}, of dimension m^2. The subset $S_\sharp$ and the foliation TS^+ are related to the geometry of auxiliary Shimura varieties of parahoric level structure at p, and seem to play an important role. In [loc. cit., Theorem 25], it was proved that $\mathcal{T}S^+$ is tangential to S^{fol}. In view of these two results, claiming tangentiality to S^{ao} and S^{fol}, it is reasonable to expect that $\mathcal{T}S^+$ is tangential to *every* EO strata in $S_\sharp$. The proofs of the known cases, whether in [loc. cit.] or here, invoke delicate computations with Dieudonné modules, and at present we see no conceptual reason justifying our expectation, which could avoid such computations.

Proof. Let k be an algebraically closed field containing κ, $x \in S^{\text{ao}}(k)$ a geometric point and $D_x = D(\mathcal{A}_x/k)$. Let $R = \mathcal{O}_{S,x}/\mathfrak{m}^2_{S,x}$ and $d : R \to \Omega_{S/k}|_x = \mathfrak{m}_{S,x}/\mathfrak{m}^2_{S,x}$ the canonical derivation $df = (f - f(x)) \mod \mathfrak{m}^2_{S,x}$. Let $D = H^1_{dR}(\mathcal{A}/R)$. The Gauss–Manin connection on $H^1_{dR}(\mathcal{A}/S)$ induces a map

$$\nabla : D \to D_x \otimes_k \Omega_{S/k}|_x$$

satisfying $\nabla(r\alpha) = r(x)\nabla(\alpha) + \alpha \otimes dr$, which by abuse of language we call the Gauss–Manin connection on D. It is easy to see that every $\alpha \in D_x$ has a unique extension to a horizontal section $\alpha \in D$, i.e., a section satisfying $\nabla(\alpha) = 0$. Thus, we may identify D with $R \otimes_k D_x$, the horizontal sections being D_x. Since the Gauss–Manin connection commutes with isogenies, $V : D \to D^{(p)}$ and $F : D^{(p)} \to D$ map horizontal sections to horizontal sections. For the same reason, if x, y are horizontal sections of D, their pairing $\langle x, y\rangle$ is horizontal for d, i.e., lies in k.

We now distinguish between two cases:

I. Assume that $n = m + 1$. Then

$$D_x = \text{Span}_k\{\underline{e}_1, e_2, \underline{e}_3, f_1, \underline{f}_2, f_3, \underline{e}^\mu_i, e^{et}_i, \underline{f}^\mu_i, f^{et}_i\}_{1\le i\le m-1}$$

where the first six vectors span $AO(2, 1)$, as in Proposition 4.1.2(i). For the convenience of the reader we have underlined the vectors spanning ω_x. The module D is spanned by the same vectors over R, and the pairings and the tables giving F and V remain the same over R.

We now write the most general deformation of ω_x to a projective submodule of D which is invariant under the endomorphisms and isotropic. An easy check yields that it is given by

$$\omega = \text{Span}_R\{\tilde{e}_1, \tilde{e}_3, \tilde{f}_2, \tilde{e}^\mu_i, \tilde{f}^\mu_i\}_{1\le i\le m-1},$$

where

- $\tilde{e}_1 = e_1 + ue_2 + \sum_{i=1}^{m-1} u_i e^{et}_i$,
- $\tilde{e}_3 = e_3 + ve_2 + \sum_{i=1}^{m-1} v_i e^{et}_i$,
- $\tilde{f}_2 = f_2 - vf_1 + uf_3 + \sum_{i=1}^{m-1} w_i f^{et}_i$,

- $\tilde{e}_i^{\mu} = e_i^{\mu} + w_i e_2 + \sum_{j=1}^{m-1} w_{ij} e_j^{et}$,
- $\tilde{f}_i^{\mu} = f_i^{\mu} - v_i f_1 + u_i f_3 + \sum_{j=1}^{m-1} w_{ji} f_j^{et}$.

The mn parameters $u, u_i, v, v_i, w_i, w_{ij}$ are, according to Grothendieck, the local parameters of R, serving as a basis of $\mathfrak{m}_R$ over k. It follows that $\mathcal{P}_0$ is indeed of rank 1, as claimed before, spanned over R by $\tilde{e}_1$, while $\mathcal{Q}$ is spanned over R by the m vectors $\tilde{f}_2, \tilde{f}_i^{\mu}$. Furthermore, computing the Hasse matrix $H = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}}$ in the bases $(\tilde{f}_2, \tilde{f}_i^{\mu})$ and $(\tilde{f}_2^{(p^2)}, \tilde{f}_i^{\mu(p^2)})$ of $\mathcal{Q}$ and $\mathcal{Q}^{(p^2)}$ we get

$$H = \begin{pmatrix} u & u_1 & u_2 & \cdots & u_{m-1} \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & 1 \end{pmatrix},$$

so the (trivialized) Hasse invariant h is simply u. Since we know that S^{ao} is the zero divisor of h, $S^{\mathrm{ao}} \cap \mathrm{Spec}(R)$ is given by the equation $u = 0$. Note, in passing, that this proves that the zero divisor of the Hasse invariant is reduced and equal to the nonordinary locus.

To compute $\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})$ recall how it is defined. From the Gauss–Manin connection we get a homomorphism of sheaves

$$\overline{\nabla} : \omega \to \Omega_{S/\kappa} \otimes (H^1_{dR}(\mathcal{A}/S)/\omega)$$

which induces a homomorphism $\mathcal{P} \to \Omega_{S/\kappa} \otimes \mathcal{Q}^{\vee}$. This induces the map

$$\mathrm{KS} : \mathcal{P} \otimes \mathcal{Q} \to \Omega_{S/\kappa}$$

which happens to be an isomorphism. We begin by computing the Gauss–Manin connection on $\mathcal{P}_0$:

$$\nabla(\tilde{e}_1) = e_2 \otimes du + \sum_{i=1}^{m-1} e_i^{et} \otimes du_i.$$

Projecting D_x to $D_x/\omega_x = H^1(\mathcal{A}_x, \mathcal{O})$ and noting that e_2, e_i^{et} modulo ω_x are a basis for the dual $\mathcal{Q}^{\vee}$ of $\mathcal{Q}$, equipped with the conjugate action of $\mathcal{O}_E$ via Σ, we find out that

$$\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})|_x = \mathrm{Span}_k\{du, du_i\}.$$

From the definition of ψ it follows that $\psi(du)|_x = 0$. Now assume that u is a *global* generator of the ideal of S^{ao} in a Zariski open set U. Then we conclude that $\psi(du) = 0$ along $S^{\mathrm{ao}} \cap U$ as claimed.

II. The proof of the theorem in the case $n - m \geq 2$ is similar, using Proposition 4.1.2(ii). Here

$$D_x = \mathrm{Span}_k\{\underline{e}_1, e_2, \underline{e}_3, \underline{e}_4, f_1, f_2, \underline{f}_3, f_4, \underline{e}_i^{\mu}, e_i^{et}, \underline{f}_i^{\mu}, f_i^{et}, \underline{e}_j^{\sharp}, f_j^{\sharp}\}_{1 \leq i \leq m-1, 1 \leq j \leq n-m-2}$$

where the first eight vectors span $AO(3,1)$ and for every j the vectors $e_j^{\sharp}, f_j^{\sharp}$ span a copy of $D(\mathfrak{G}_{\Sigma}[p])$. For convenience we have again underlined the vectors spanning ω_x. The most general deformation of ω_x in $D = R \otimes_k D_x$ is spanned by the following vectors:

- $\tilde{e}_1 = e_1 - ue_2 + \sum_{i=1}^{m-1} u_i e_i^{et}$.
- $\tilde{e}_3 = e_3 + ve_2 + \sum_{i=1}^{m-1} v_i e_i^{et}$.
- $\tilde{e}_4 = e_4 + we_2 + \sum_{i=1}^{m-1} w_i e_i^{et}$.
- $\tilde{f}_3 = f_3 + wf_1 + vf_2 + uf_4 - \sum_{l=1}^{m-1} x_l f_l^{et} - \sum_{k=1}^{n-m-2} y_k f_k^{\sharp}$.
- $\tilde{e}_i^{\mu} = e_i^{\mu} + x_i e_2 + \sum_{l=1}^{m-1} x_{il} e_l^{et}$.
- $\tilde{f}_i^{\mu} = f_i^{\mu} - w_i f_1 - v_i f_2 + u_i f_4 + \sum_{l=1}^{m-1} x_{li} f_l^{et} + \sum_{k=1}^{n-m-2} y_{ki} f_k^{\sharp}$.
- $\tilde{e}_j^{\sharp} = e_j^{\sharp} + y_j e_2 + \sum_{l=1}^{m-1} y_{jl} e_l^{et}$.

The nm parameters $u, v, w, u_i, v_i, w_i, x_i, x_{il}, y_k, y_{kj}$ form a basis of $\mathfrak{m}_R$ over k. Calculating the Hasse matrix H yields exactly the same $m \times m$ matrix as above, hence $u = 0$ is again the local infinitesimal equation of S^{ao}. The submodule $\mathcal{P}_0$ is $n - m$ dimensional, and is spanned by $\tilde{e}_1, \tilde{e}_3$ and the $\tilde{e}_j^{\sharp}$. Calculating KS we find that

$$\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})|_x = \mathrm{Span}_k\{du, du_i, dv, dv_i, dy_j, dy_{jl}\}$$

$(1 \le i, l \le m-1,\ 1 \le j \le n-m-2)$ so as before $\psi(du)|_x = 0$. We conclude the proof as in the first case. □

4.2. *Analytic continuation of Θ (m < n).*

Compactification of a certain intermediate Igusa cover. Recall the Igusa tower $T_{t,s}$ over S_s^{ord} that has been constructed in Section 2.1. Let

$$\Delta_t^1 = \mathrm{SL}_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E) \lhd \Delta_t$$

and denote by $T_{t,s}^1$ the intermediate covering of S_s^{ord} fixed by Δ_t^1. It is a Galois étale cover of S_s^{ord} with Galois group $(\mathcal{O}_E/p^t\mathcal{O}_E)^{\times}$. In this section let $T = T_{1,1}^1$, and let $\tau : T \to S^{\mathrm{ord}}$ be the covering map, whose Galois group is identified with $\kappa^{\times}$.

Let $\mathcal{L} = \det(\mathcal{Q})$ and recall that the Hasse invariant $h \in H^0(S, \mathcal{L}^{p^2-1})$ (1.1.5).

Lemma 4.2.1. (i) *The line bundle $\mathcal{L}$ is canonically trivialized over T, i.e., there is a canonical isomorphism $\varepsilon : \mathcal{O}_T \simeq \tau^*\mathcal{L}$.*

(ii) *Denoting by a the global section of $\tau^*\mathcal{L}$ corresponding to the section* “1” *under the trivialization, we have $a^{p^2-1} = \tau^* h$.*

Proof. (i) The canonical trivialization $\varepsilon^2(\overline{\Sigma}) : \mathcal{O}_{T_{1,1}}^m \simeq \tau_{1,1}^*\mathcal{Q}$ over the big Igusa variety $T_{1,1}$ induces a canonical trivialization on the determinants $\varepsilon : \mathcal{O}_{T_{1,1}} \simeq \tau_{1,1}^*\mathcal{L}$. The latter descends to T because it is invariant under Δ_1^1.

(ii) Since Ver is the identity on μ_p, the trivialization ϵ^2 of $\mathrm{gr}^2\,\mathcal{A}[p]$ satisfies

$$\mathrm{Ver} \circ \mathrm{Ver}^{(p)} \circ (\epsilon^2)^{(p^2)} = \epsilon^2.$$

Passing to cohomology (recall $\varepsilon^2 = ((\epsilon^2)^{-1})^*$) yields the relation $(\varepsilon^2(\overline{\Sigma}))^{(p^2)} = H \circ \varepsilon^2(\overline{\Sigma})$ where H, recall, is $V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}}$. Taking determinants we get

$$\varepsilon^{(p^2)} = h \circ \varepsilon$$

and evaluating at "1" gives the desired relation. □

The following Kummer-type result was proved in [de Shalit and Goren 2016, Section 2.4.2] for signature (2,1) and the proof easily generalizes. See also [Goren 2001]. Let

$$S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}.$$

Consider the fiber product

$$T' = \mathcal{L} \times_{\mathcal{L}^{p^2-1}} S' \tag{4.2.1}$$

where the two maps to $\mathcal{L}^{p^2-1}$ are $\lambda \mapsto \lambda^{p^2-1}$ and h. Note that the pull-back of $\mathcal{L}$ from S' to T' admits a tautological p^2-1 root of h extending a, which we still call a. Then $T' \to S'$ is finite flat of degree p^2-1, is Galois étale with Galois group $\kappa^\times$ over S^{ord}, and totally (tamely) ramified along S^{ao}. It satisfies a universal property with respect to extracting a p^2-1 root from the section h; see [loc. cit.]. From part (ii) of the last proposition it follows that there is a canonical map

$$T \to T'.$$

Since both source and target are $\kappa^\times$-torsors over S^{ord} and the map respects the $\kappa^\times$ action, this map is an isomorphism of T with the preimage of S^{ord} in T'. In this way we may identify T' with a (partial) compactification of T. We then have the following.

Proposition 4.2.2. (i) *The morphism $\tau' : T' \to S'$ is finite flat of degree p^2-1, Galois étale with Galois group $\kappa^\times$ over S^{ord}, and totally (tamely) ramified along S^{ao}.*

(ii) *T' is everywhere nonsingular.*

(iii) *Let $x \in S^{\mathrm{ao}}(k)$ be a geometric point, and $y \in T'(k)$ the unique geometric point mapping to it. Then there are formal parameters u, v_i $(1 \le i \le nm-1)$ at x such that $u = 0$ is the infinitesimal equation of S^{ao}, and such that as formal parameters on T' at y we can take w, v_i where $w^{p^2-1} = u$.*

(iv) *T' and $T = T^1_{1,1}$ are irreducible.*

Proof. The proof is the same as in the case of signature (2, 1) [de Shalit and Goren 2016, Section 2.4.3, Proposition 2.16]. □

The main theorem for scalar-valued modular forms. We can now prove the analytic continuation of Θ in characteristic p, when applied to *scalar-valued* p-adic modular forms. Recall that $\mathcal{L} = \det(\mathcal{Q})$.

Theorem 4.2.3. *Assume that $m < n$. Consider the operator*

$$\Theta : H^0(S^{\mathrm{ord}}, \mathcal{L}^k) \to H^0(S^{\mathrm{ord}}, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Then Θ extends holomorphically to an operator

$$\Theta : H^0(S, \mathcal{L}^k) \to H^0(S, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Remark. The analytic continuation of Θ to global modular forms is a characteristic p phenomenon and does not seem to extend to S_s (i.e., modulo p^s) for $s > 1$. Had it extended for all s, we would have obtained, for any algebraic modular form f of weight k, a well-defined rigid analytic "$\Theta(f)$", of weight $k + p + 1$, on the whole rigid analytic space associated to $\mathcal{S}$. By GAGA (and the Köcher principle) this $\Theta(f)$ would have been algebraic. However, the Maass–Shimura operators in characteristic 0 do not preserve the space of classical modular forms.

Proof. Let $f \in H^0(S, \mathcal{L}^k)$. Then $\Theta(f)$ is a section of $\mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}$ over S^{ord} and we have to show that it extends holomorphically to S. Since S is nonsingular, it is enough to show that it extends holomorphically to $S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$, an open set whose complement is of codimension 2. Indeed, Zariski locally we may trivialize the vector bundles, and then any coordinate of $\Theta(f)$ becomes a meromorphic function, whose polar set, if nonempty, should have codimension 1.

Let $\tau' : T' \to S'$ be the intermediate Igusa variety constructed above. Over T (the preimage of S^{ord}) we can write the trivialization ε of $\mathcal{L}$ as $f \mapsto f/a^k$. This introduces a pole of order k, at most, along $T^{\mathrm{ao}} = \tau'^{-1}(S^{\mathrm{ao}})$. Let $y \in T^{\mathrm{ao}}$ be a geometric point and $x = \tau'(y)$. Let u, v_i be formal parameters at x and w, v_i formal parameters at y as in Proposition 4.2.2. Let

$$f/a^k = \sum_{r=-k}^{\infty} g_r(v) w^r$$

be the Taylor expansion of f/a^k in $\hat{\mathcal{O}}_{T',y}$, where the $g_r(v)$ are power series in the v_i. Note that $du = d(w^{p^2-1}) = -w^{p^2-2} dw$. Thus,

$$\begin{aligned} d(f/a^k) &= \sum_{r=-k}^{\infty} w^r dg_r(v) + \sum_{r=-k}^{\infty} r g_r(v) w^{r-1} dw \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^{r-(p^2-1)} du \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^r u^{-1} du. \end{aligned}$$

When we compute

$$\tilde{\Theta}(f) = a^k \left(\sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^r u^{-1} du \right),$$

which we know descends to S', we see that the first sum becomes holomorphic (a vanishes along T^{ao}), while the second sum retains a simple pole along S^{ao}. However, to get $\Theta(f)$ we must still apply the vector-bundle homomorphism ψ. Theorem 4.1.3 says that $\psi(du)$ vanishes along S^{ao}, hence the simple pole disappears and $\Theta(f)$ is holomorphic at x. This being true at every $x \in S^{\mathrm{ao}}$, we conclude that $\Theta(f)$ is everywhere holomorphic. □

4.3. *Analytic continuation of* Θ *(m = n).* We briefly indicate the modifications in the proof which are necessary to deal with the case $m = n$. In this case $\mathrm{rk}(\ker(V_{\mathcal{P}}))$ changes when we move from S^{ord} to S^{ao}, so $\mathcal{P}_0$ and $\mathcal{P}_\mu$ do not extend, with the same definitions, to vector bundles over $S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$. As such, we cannot extend Θ beyond S^{ord} using pr_μ. Instead, we apply $(V_{\mathcal{P}} \otimes 1) \circ \mathrm{KS}^{-1}$ to $\Omega_{S/\kappa}$, a map that gives the same result as $(\mathrm{pr}_\mu \otimes 1) \circ \mathrm{KS}^{-1}$ over S^{ord} in characteristic p, but does not make sense over S_s^{ord} for $s > 1$. Let $\mathcal{L} = \det(\mathcal{Q})$ as before.

Preliminary results on the Igusa variety when m = n. Let $T = T^1_{1,1}$ as before. Let T' be defined by (4.2.1). As before, it is a partial compactification of T. Since the divisor of the Hasse invariant is not reduced when $n = m$ (see page 1836 and Lemma 1.1.4), the proof of the irreducibility of T as in [de Shalit and Goren 2016, Proposition 2.16] breaks down.

Proposition 4.3.1. (i) *The morphism $T' \to S'$ is finite flat of degree $p^2 - 1$, with Galois group $\kappa^\times$.*

(ii) *T' is nonsingular.*

(iii) *T' and the Igusa variety T decompose into $p + 1$ irreducible components T'_ζ (resp. T_ζ) labeled by ζ such that $\zeta^{p+1} = 1$. More canonically,*

$$\pi_0(T) = \pi_0(T') \simeq \kappa^\times/\mathbb{F}_p^\times.$$

(iv) *The map $T'_\zeta \to S'$ is totally (tamely) ramified over S^{ao} of degree $p - 1$.*

Proof. The proof of (i) is the same as when $n > m$. Our T' is still obtained from S' by extracting a $p^2 - 1$ root of h. However, this time $h = h_{\mathcal{Q}}^{p+1}$ where $h_{\mathcal{Q}}$ is in $H^0(S', \mathcal{L}^{p-1})$ so $T' = \bigsqcup T'_\zeta$ where $T'_\zeta = S'[\sqrt[p-1]{\zeta h_{\mathcal{Q}}}]$. As the divisor of $h_{\mathcal{Q}}$ is reduced and equal to S^{ao}, the rest of the proof is similar to the case $n > m$. □

The main theorem when m = n.

Theorem 4.3.2. *Assume that $m = n$. The operator*

$$\Theta = (1 \otimes V_{\mathcal{P}} \otimes 1) \circ (1 \otimes \mathrm{KS}^{-1}) \circ \tilde{\Theta} : H^0(S^{\mathrm{ord}}, \mathcal{L}^k) \to H^0(S^{\mathrm{ord}}, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q})$$

extends holomorphically to an operator

$$\Theta : H^0(S, \mathcal{L}^k) \to H^0(S, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Proof. As before, let

$$\psi = (V_{\mathcal{P}} \otimes 1) \circ \mathrm{KS}^{-1} : \Omega_{S/\kappa} \to \mathcal{Q}^{(p)} \otimes \mathcal{Q}.$$

Let $x \in S^{\mathrm{ao}}(k)$ be a geometric point. The Dieudonné module D_x is now given by

$$D_x = \mathrm{Span}_k\{\underline{e}_i^\mu, \underline{f}_i^\mu, e_i^{et}, f_i^{et}, \underline{e}^\sharp, f^\sharp, e^\flat, \underline{f}^\flat\}_{1 \le i \le m-1}$$

where $\mathrm{Span}_k\{e^\sharp, f^\sharp\}$ is isomorphic to $D(\mathfrak{G}_\Sigma[p])$ and $\mathrm{Span}_k\{e^\flat, f^\flat\}$ to $D(\mathfrak{G}_{\bar\Sigma}[p])$ [Wooding 2016, Proposition 3.5.6]. The underlined vectors span ω_x. As before, we let $R = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^2$ and $D = R \otimes_k D_x$. The most general deformation of ω_x to $\omega \subset D$ compatible with the endomorphisms and the polarization is spanned by

- $\tilde{e}_i^\mu = e_i^\mu + w_i e^\flat + \sum_{j=1}^{m-1} w_{ij} e_j^{et}$,
- $\tilde{e}^\sharp = e^\sharp + u e^\flat + \sum_{j=1}^{m-1} u_j e_j^{et}$,
- $\tilde{f}_i^\mu = f_i^\mu + u_i f^\sharp + \sum_{j=1}^{m-1} w_{ji} f_j^{et}$,
- $\tilde{f}^\flat = f^\flat + u f^\sharp + \sum_{j=1}^{m-1} w_j f_j^{et}$.

The m^2 quantities w_i, w_{ij}, u, u_i then form a system of local (infinitesimal) parameters at x. The matrix of $V_{\mathcal{Q}}$ in the bases $\{\tilde{f}^\flat, \tilde{f}_i^\mu\}$ of $\mathcal{Q}$ and $\{(\tilde{e}^\sharp)^{(p)}, (\tilde{e}_i^\mu)^{(p)}\}$ of $\mathcal{P}^{(p)}$ is

$$\begin{pmatrix} u & u_1 & \cdots & u_{m-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The infinitesimal equation of $S^{\mathrm{ao}} \cap \mathrm{Spec}(R)$ is $u = 0$. As before we compute the Kodaira–Spencer homomorphism and find out that

$$\mathrm{KS}(e^\sharp) = du \wedge e^\flat + \sum_{j=1}^{m-1} du_j \wedge e_j^{et},$$

which means that

$$\mathrm{KS}(e^\sharp \otimes \mathcal{Q}|_x) = \mathrm{Span}_k\{du, du_j\} \subset \Omega_S|_x.$$

This implies that $\mathrm{KS}^{-1}(du) \in e^\sharp \otimes \mathcal{Q}|_x$. However, $V_{\mathcal{P}}$ is expressible in the same bases as above by the matrix

$$\begin{pmatrix} u & w_1 & \cdots & w_{m-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

which means that $\ker(V_{\mathcal{P}})$ is 1-dimensional at x, and spanned by $e^\sharp$. Thus if $u = 0$ is a local equation of S^{ao}, $\psi(du)$ vanishes along S^{ao}. This yields Theorem 4.1.3 when $m = n$.

Let $f \in H^0(S, \mathcal{L}^k)$. Then $\Theta(f)$ is a section of $\mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}$ over S^{ord} and we have to show that it extends holomorphically to S. Since S is nonsingular, as in the case $n > m$, it is enough to show that it extends holomorphically to $S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$.

Let $\tau' : T' \to S'$ be the intermediate Igusa variety constructed above. Let a be, as before, the tautological $p^2 - 1$ root of h over T'; it vanishes to order 1 along $T^{\mathrm{ao}} = \tau'^{-1}(S^{\mathrm{ao}})$. Over T (the preimage of S^{ord}), where a does not vanish, we can write the trivialization ε of $\mathcal{L}$ as $f \mapsto f/a^k$. This introduces a pole of order k, at most, along T^{ao}. Let $y \in T^{\mathrm{ao}}$ be a geometric point and $x = \tau'(y)$. Let ζ be the $p+1$ root of 1 such that $y \in T'_\zeta$. Let u, v_i be formal parameters at x and w, v_i formal parameters at y so that $u = 0$ is a local equation of S^{ao} and

$$u = w^{p-1}.$$

Let

$$f/a^k = \sum_{r=-k}^{\infty} g_r(v) w^r$$

be the Taylor expansion of f/a^k in $\hat{\mathcal{O}}_{T',y}$, where the $g_r(v)$ are power series in the v_i. Note that $du = d(w^{p-1}) = -w^{p-2}dw$. Thus, similarly to the case $n > m$

$$\begin{aligned} d(f/a^k) &= \sum_{r=-k}^{\infty} w^r dg_r(v) + \sum_{r=-k}^{\infty} r g_r(v) w^{r-1} dw \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^{r-(p-1)} du \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^r u^{-1} du. \end{aligned}$$

We conclude the proof as in the case $n > m$. □

5. Theta cycles

For the group GL_2, the application of the theta operator to mod p modular forms was linked to twisting Galois representations by the cyclotomic character (see [Serre 1973a] over $\mathbb{Q}$ and [Andreatta and Goren 2005] over a totally real base field). The variation of the weight filtration upon iteration of Θ was of much interest in this context. While the connection to Galois representations in the unitary case requires further study, our goal here is to present a similar behavior on the level of q-expansions. We consider only signature $(n, 1)$, $n > 1$, as signature $(1, 1)$ is essentially the case of modular curves.

In this section, let S be a *connected component* of the special fiber of a unitary Shimura variety of signature $(n, 1)$, so that $\mathcal{P}_\mu \otimes \mathcal{Q} \simeq \mathcal{Q}^{(p)} \otimes \mathcal{Q} \simeq \mathcal{L}^{p+1}$. The theta operator maps $\mathcal{L}^k$ to $\mathcal{L}^{k+p+1}$ and may be iterated. The index set $\check{H}$ for the Fourier–Jacobi expansions at a given level and a given rank-1 cusp may be identified with $\mathbb{Z}$ so that $\check{H}^+$ is identified with the nonnegative integers. The effect of Θ on Fourier–Jacobi expansions is

$$\Theta\Big(\sum_{n\geq 0} a(n) \cdot q^n\Big) = \sum_{n \geq 0} a(n) n \cdot q^n. \tag{5.0.1}$$

Given the q-expansion principle and the irreducibility of the Igusa variety $T^1_{1,1}$ (see Proposition 4.2.2), the proofs of the following results are verbatim as for signature (2, 1), see [de Shalit and Goren 2016, Sections 3.1–3.3].

Lemma 5.0.1. *Let ξ be a rank 1 cusp on S^*. Let ℓ be the nonzero section of $\mathcal{L}$ used to trivialize $\mathcal{L}$ at a formal neighborhood of ξ as before. Consider the homomorphism*

$$FJ_\xi : \bigoplus_{k=0}^{\infty} H^0(S, \mathcal{L}^k) \to \mathcal{FJ}_\xi,$$

where $\mathcal{FJ}_\xi$ is as in Proposition 3.1.2 and where we have identified formal sections of $\mathcal{L}^k$ near ξ with elements of $\hat{\mathcal{O}}_{S^*,\xi}$ by dividing the sections by ℓ^k. Then the kernel of FJ_ξ is given by the ideal

$$\ker(FJ_\xi) = (h - 1),$$

where h is the Hasse invariant.

Given an element $f = f(q) \in \mathcal{FJ}_\xi$ of the form $FJ_\xi(g)$, $g \in H^0(S, \mathcal{L}^k)$, we denote by $\omega(f)$ the minimal $k \geq 0$ for which there exists such a g. We call $\omega(f)$ the *filtration* of f. By the previous lemma, if f arises from g of weight k then $\omega(f) \equiv k \mod (p^2 - 1)$.

Proposition 5.0.2. *Let $f \in H^0(S, \mathcal{L}^k)$ be in the image of Θ, i.e., $f = \Theta(g)$.*

(i) *We have $\Theta^{p-1}(f) = fh$ where h is the Hasse invariant.*

(ii) *The sequence $\omega(\Theta^i(f))$ $i = 0, 1, 2, \ldots, p-1$ increases by $p+1$ at each step, except for a single $i = i_0(f) < p-1$ for which $\omega(\Theta^{i+1}(f)) = \omega(\Theta^i(f)) - p^2 + p + 2$.*

The combinatorics of weights has some peculiarities not present in the case of elliptic modular forms, see [de Shalit and Goren 2016].

Acknowledgments

We would like to thank Ellen Eischen and Elena Mantovan for discussions relating to the contents of both our papers. It is a pleasure to thank G. Rosso for bringing their work to our attention, and P. Kassaei for valuable comments. We thank the referees for very useful comments.

Much of this paper was written during visits to the Hebrew University and to McGill University and it is our pleasant duty to thank these institutes for their hospitality. This research was supported by NSERC grant 223148 and ISF grant 276/17.

References

[Andreatta and Goren 2005] F. Andreatta and E. Z. Goren, *Hilbert modular forms: mod p and p-adic aspects*, Mem. Amer. Math. Soc. **819**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl

[Artin 1969] M. Artin, "Algebraic approximation of structures over complete local rings", *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), 23–58. MR Zbl

[Ash et al. 1975] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Lie Groups Hist. Frontiers Appl. **4**, Math. Sci. Press, Brookline, MA, 1975. MR Zbl

[Böcherer and Nagaoka 2007] S. Böcherer and S. Nagaoka, "On mod p properties of Siegel modular forms", *Math. Ann.* **338**:2 (2007), 421–433. MR Zbl

[Bültel and Wedhorn 2006] O. Bültel and T. Wedhorn, "Congruence relations for Shimura varieties associated to some unitary groups", *J. Inst. Math. Jussieu* **5**:2 (2006), 229–261. MR Zbl

[Coleman 1996] R. F. Coleman, "Classical and overconvergent modular forms", *Invent. Math.* **124**:1-3 (1996), 215–241. MR Zbl

[Coleman et al. 1995] R. F. Coleman, F. Q. Gouvêa, and N. Jochnowitz, "E_2, Θ, and overconvergence", *Int. Math. Res. Not.* **1995**:1 (1995), 23–41. MR Zbl

[Courtieu and Panchishkin 2004] M. Courtieu and A. Panchishkin, *Non-Archimedean L-functions and arithmetical Siegel modular forms*, 2nd ed., Lecture Notes in Math. **1471**, Springer, 2004. MR Zbl

[Eischen 2012] E. E. Eischen, "p-adic differential operators on automorphic forms on unitary groups", *Ann. Inst. Fourier* (*Grenoble*) **62**:1 (2012), 177–243. MR Zbl

[Eischen and Mantovan 2017] E. Eischen and E. Mantovan, "p-adic families of automorphic forms in the μ-ordinary setting", 2017. To appear in *Amer. J. Math.* arXiv

[Eischen et al. 2018] E. Eischen, J. Fintzen, E. Mantovan, and I. Varma, "Differential operators and families of automorphic forms on unitary groups of arbitrary signature", *Doc. Math.* **23** (2018), 445–495. MR Zbl

[Faltings and Chai 1990] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik (3) **22**, Springer, 1990. MR Zbl

[Fulton 1993] W. Fulton, *Introduction to toric varieties*, Ann. of Math. Stud. **131**, Princeton Univ. Press, 1993. MR Zbl

[Fulton and Harris 1991] W. Fulton and J. Harris, *Representation theory: a first course*, Grad. Texts in Math. **129**, Springer, 1991. MR Zbl

[Ghitza and McAndrew 2016] A. Ghitza and A. McAndrew, "Theta operators on Siegel modular forms and Galois representations", preprint, 2016. arXiv

[Goldring and Nicole 2017] W. Goldring and M.-H. Nicole, "The μ-ordinary Hasse invariant of unitary Shimura varieties", *J. Reine Angew. Math.* **728** (2017), 137–151. MR Zbl

[Goren 2001] E. Z. Goren, "Hilbert modular forms modulo p^m: the unramified case", *J. Number Theory* **90**:2 (2001), 341–375. MR Zbl

[Gross 1990] B. H. Gross, "A tameness criterion for Galois representations associated to modular forms (mod p)", *Duke Math. J.* **61**:2 (1990), 445–517. MR Zbl

[Grothendieck 1974] A. Grothendieck, *Groupes de Barsotti–Tate et cristaux de Dieudonné* (Montréal, 1970), Séminaire Math. Supér. **45**, Presses Univ. Montréal, 1974. MR Zbl

[Ichikawa 2014] T. Ichikawa, "Vector-valued p-adic Siegel modular forms", *J. Reine Angew. Math.* **690** (2014), 35–49. MR Zbl

[Jantzen 2003] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Math. Surv. Monogr. **107**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Jochnowitz 1982] N. Jochnowitz, "A study of the local components of the Hecke algebra mod l", *Trans. Amer. Math. Soc.* **270**:1 (1982), 253–267. MR Zbl

[Johansson 2013] C. Johansson, "Classicality for small slope overconvergent automorphic forms on some compact PEL Shimura varieties of type C", *Math. Ann.* **357**:1 (2013), 51–88. MR Zbl

[Katz 1975] N. M. Katz, "Higher congruences between modular forms", *Ann. of Math.* (2) **101** (1975), 332–367. MR Zbl

[Katz 1977] N. M. Katz, "A result on modular forms in characteristic p", pp. 53–61 in *Modular functions of one variable, V* (Bonn, Germany, 1976), edited by J.-P. Serre and D. B. Zagier, Lecture Notes in Math. **601**, Springer, 1977. MR Zbl

[Katz 1978] N. M. Katz, "p-adic L-functions for CM fields", *Invent. Math.* **49**:3 (1978), 199–297. MR Zbl

[Katz and Mazur 1985] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Ann. of Math. Stud. **108**, Princeton Univ. Press, 1985. MR Zbl

[Kottwitz 1992] R. E. Kottwitz, "Points on some Shimura varieties over finite fields", *J. Amer. Math. Soc.* **5**:2 (1992), 373–444. MR Zbl

[Lan 2013] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Math. Soc. Monogr. Series **36**, Princeton Univ. Press, 2013. MR Zbl

[Lan 2015] K.-W. Lan, "Boundary strata of connected components in positive characteristics", 2015. Appendix to X. Wan, "Families of nearly ordinary Eisenstein series on unitary groups", *Algebra Number Theory* **9**:9 (2015), 1955–2054. MR Zbl

[Mantovan 2005] E. Mantovan, "On the cohomology of certain PEL-type Shimura varieties", *Duke Math. J.* **129**:3 (2005), 573–610. MR Zbl

[Moonen 2001] B. Moonen, "Group schemes with additional structures and Weyl group cosets", pp. 255–298 in *Moduli of abelian varieties* (Texel Island, Netherlands, 1999), edited by C. Faber et al., Progr. Math. **195**, Birkhäuser, Basel, 2001. MR Zbl

[Moonen 2004] B. Moonen, "Serre–Tate theory for moduli spaces of PEL type", *Ann. Sci. École Norm. Sup.* (4) **37**:2 (2004), 223–269. MR Zbl

[Mumford 1972] D. Mumford, "An analytic construction of degenerating abelian varieties over complete rings", *Compos. Math.* **24** (1972), 239–272. MR Zbl

[Oda 1969] T. Oda, "The first de Rham cohomology group and Dieudonné modules", *Ann. Sci. École Norm. Sup.* (4) **2** (1969), 63–135. MR Zbl

[Oort 2001] F. Oort, "A stratification of a moduli space of abelian varieties", pp. 345–416 in *Moduli of abelian varieties* (Texel Island, Netherlands, 1999), edited by C. Faber et al., Progr. Math. **195**, Birkhäuser, Basel, 2001. MR Zbl

[Panchishkin 2005] A. A. Panchishkin, "The Maass–Shimura differential operators and congruences between arithmetical Siegel modular forms", *Mosc. Math. J.* **5**:4 (2005), 883–918. MR Zbl

[Serre 1973a] J.-P. Serre, "Congruences et formes modulaires (d'après H. P. F. Swinnerton-Dyer)", exposé 416, pp. 319–338 in *Séminaire Bourbaki* 1971/1972, Lecture Notes in Math. **317**, Springer, 1973. MR Zbl

[Serre 1973b] J.-P. Serre, "Formes modulaires et fonctions zêta p-adiques", pp. 191–268 in *Modular functions of one variable, III* (Antwerp, 1972), edited by W. Kuyk and J.-P. Serre, Lecture Notes in Math. **350**, Springer, 1973. Correction in *Modular functions of one variable, IV*, Lecture Notes in Math. **476**, Springer, 1975, pp. 149–150. MR Zbl

[de Shalit and Goren 2016] E. de Shalit and E. Z. Goren, "A theta operator on Picard modular forms modulo an inert prime", *Res. Math. Sci.* **3** (2016), art. id. 28. MR Zbl

[de Shalit and Goren 2017] E. de Shalit and E. Z. Goren, "Supersingular curves on Picard modular surfaces modulo an inert prime", *J. Number Theory* **171** (2017), 391–421. MR Zbl

[de Shalit and Goren 2018] E. de Shalit and E. Z. Goren, "Foliations on unitary Shimura varieties in positive characteristic", *Compos. Math.* **154**:11 (2018), 2267–2304. MR Zbl

[Shimura 2000] G. Shimura, *Arithmeticity in the theory of automorphic forms*, Math. Surv. Monogr. **82**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Skinner and Urban 2014] C. Skinner and E. Urban, "The Iwasawa main conjectures for GL_2", *Invent. Math.* **195**:1 (2014), 1–277. MR Zbl

[Swinnerton-Dyer 1973] H. P. F. Swinnerton-Dyer, "On l-adic representations and congruences for coefficients of modular forms", pp. 1–55 in *Modular functions of one variable, III* (Antwerp, 1972), edited by W. Kuyk and J.-P. Serre, Lecture Notes in Math. **350**, Springer, 1973. Correction in *Modular functions of one variable, IV*, Lecture Notes in Math. **476**, Springer, 1975, pp. 149. MR Zbl

[Viehmann and Wedhorn 2013] E. Viehmann and T. Wedhorn, "Ekedahl–Oort and Newton strata for Shimura varieties of PEL type", *Math. Ann.* **356**:4 (2013), 1493–1550. MR Zbl

[Wedhorn 1999] T. Wedhorn, "Ordinariness in good reductions of Shimura varieties of PEL-type", *Ann. Sci. École Norm. Sup.* (4) **32**:5 (1999), 575–618. MR Zbl

[Wedhorn 2001] T. Wedhorn, "The dimension of Oort strata of Shimura varieties of PEL-type", pp. 441–471 in *Moduli of abelian varieties* (Texel Island, Netherlands, 1999), edited by C. Faber et al., Progr. Math. **195**, Birkhäuser, Basel, 2001. MR Zbl

[Wooding 2016] A. W. L. J. Wooding, *The Ekedahl–Oort stratification of unitary Shimura varieties*, Ph.D. thesis, McGill University, 2016.

Communicated by Brian Conrad
Received 2018-01-01 Revised 2019-01-09 Accepted 2019-06-13

ehud.deshalit@mail.huji.ac.il *Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel*
eyal.goren@mcgill.ca *Department of Mathematics and Statistics, McGill University, Montreal, Canada*

mathematical sciences publishers msp

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1879

Infinitely generated symbolic Rees algebras over finite fields

Akiyoshi Sannai and Hiromu Tanaka

For the polynomial ring over an arbitrary field with twelve variables, there exists a prime ideal whose symbolic Rees algebra is not finitely generated.

1. Introduction

Let A be a polynomial ring over a field k with finitely many variables. For a field L satisfying $k \subset L \subset \mathrm{Frac}(A)$, Hilbert's fourteenth problem asks whether or not the ring $L \cap A$ is finitely generated over k. In 1958, Nagata [1960] found the first counterexample to this problem over arbitrary sufficiently large fields. For more examples we refer to [Roberts 1990; Kuroda 2005; Totaro 2008]. On the other hand, this problem is related to the following question raised by Cowsik [1984].

Question 1.1. Let A be a polynomial ring over a field with finitely many variables and let P be a prime ideal of A. Set $P^{(m)} := P^m A_P \cap A$. Then is the symbolic Rees algebra $R_S(P) := \bigoplus_{m=0}^{\infty} P^{(m)}$ a finitely generated k-algebra?

Indeed, Roberts [1985] settled Question 1.1 negatively, using Nagata's counterexample mentioned above. Roberts's construction is valid only over sufficiently large fields of characteristic zero, although Nagata's example is independent of the characteristic of the base field. This is because Roberts's proof requires a theorem of Bertini type that fails in positive characteristic (see [Roberts 1985, line 7 on page 591]). On the other hand, it is known for experts that Roberts's method works, after suitable modifications, for the case where k is not algebraic over a finite field. Roughly speaking, counterexamples over such fields can be found after replacing the theorem of Bertini type and Nagata's counterexample used in [Roberts 1985] by [Diaz and Harbater 1991, Theorem 2.1] and the blowup of $\mathbb{P}^2$ along general nine points, respectively. In this sense, Question 1.1 is still open if k is algebraic over a finite field.

The purpose of this paper is to give the negative answer to Question 1.1 over an arbitrary base field. More specifically, the main theorem is as follows.

Theorem 1.2 (see Theorem 3.7). *Let k be a field. Let A be the polynomial ring over k with twelve variables. Then there exists a prime ideal $\mathfrak{p}$ of A whose symbolic Rees algebra $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring.*

MSC2010: 13H10.
Keywords: symbolic Rees algebras, Mori dream spaces, Cowsik's question.

Sketch of the proof. We overview some of the ideas used in the proof of Theorem 1.2. Let us treat the case where $k = \mathbb{F}_p$. Our method is based on a geometric description of symbolic Rees algebras that was pointed out by Cutkosky [1991] in a certain special case. We start with a projective smooth surface V over $\mathbb{F}_p$, constructed by Totaro, that has a nef divisor M which is not semiample. We embed V into the eleven-dimensional projective space $\mathbb{P}^{11}_{\mathbb{F}_p}$ (see Lemma 3.5). Thanks to a theorem of Bertini type over finite fields, we can find a smooth curve W on V that is linearly equivalent to $stH|_V - tM$ for a hyperplane divisor H of $\mathbb{P}^{11}_{\mathbb{F}_p}$ under the assumption that $t \gg s \gg 0$. Take a homogeneous prime ideal $\mathfrak{p}$ on $A = \mathbb{F}_p[x_0, \dots, x_{11}]$ that defines W. Let $f : X \to \mathbb{P}^{11}_{\mathbb{F}_p}$ be the blowup along W. Set $D := f^*H$ and let E be the f-exceptional prime divisor on X. Then $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring if and only if the Cox ring of X is not a noetherian ring (see Proposition 2.14). In particular it suffices to find a nef divisor on X that is not semiample. By choosing s and t carefully, we can find such a divisor (see Proposition 3.3(3)). For more details, see Section 3.

Related topics. It is worth mentioning that, concerning Question 1.1, many authors have studied the case where P is the prime ideal of $k[x, y, z]$ that defines a space monomial curve (t^a, t^b, t^c) in $\mathbb{A}^3_k$. For instance, Goto, Nishida and Watanabe [1994] proved that for some triples (a, b, c), the associated symbolic Rees algebras are not finitely generated if k is of characteristic zero. It is remarkable that this result is applied to study the compactified moduli space $\overline{\mathcal{M}}_{0,n}$ of pointed rational curves. More specifically, it turns out that $\overline{\mathcal{M}}_{0,n}$ is not a Mori dream space if $n \geq 13$ and the base field is of characteristic zero [Castravet 2009; González and Karu 2016].

Since the case of characteristic zero has such an application, it is natural to consider also the case of positive characteristic. However the situation seems to be subtler. Indeed, if the base field is of positive characteristic, then it is known that the analogous rings of the examples given in [Goto et al. 1994] and [Roberts 1990] are shown to be finitely generated by [Cutkosky 1991; Goto et al. 1994] and [Kurano 1993; 1994], respectively. Then Goto and Watanabe made the following conjecture, which remains to be an open problem.

Conjecture 1.3. Let R be the polynomial ring over a field k with three valuables. Let P be the prime ideal that defines a space monomial curve (t^a, t^b, t^c) in $\mathbb{A}^3_k$. If the characteristic of k is positive, then the symbolic Rees ring $R_S(P) = \bigoplus_{m=0}^{\infty} P^{(m)}$ is finitely generated.

It is known that Conjecture 1.3 is reduced to the case where $k = \overline{\mathbb{F}}_p$. On the other hand, Theorem 1.2 indicates that a symbolic Rees algebra is not necessarily finitely generated in a higher dimensional case, even if the base field is $\overline{\mathbb{F}}_p$. Thus if the Conjecture 1.3 holds true, then its proof depends on some facts that hold only in a lower dimensional situation.

2. Preliminaries

Notation. In this subsection, we summarize notation used in the paper.

We say that X is a *variety* over a field k (or a *k-variety*) if X is an integral scheme which is separated and of finite type over k. We say that X is a *curve* over k or a *k-curve* (resp. a *surface* over k or a *k-surface*) if X is a variety over k with $\dim X = 1$ (resp. $\dim X = 2$).

Given an invertible sheaf L on a proper scheme X over a field k, consider the natural homomorphism

$$H^0(X, L) \otimes_k \mathcal{O}_X \to L. \tag{2.0.1}$$

(1) We say that L is *nef* if $L \cdot C \geq 0$ for any k-curve C on X.

(2) For a k-linear subspace V of $H^0(X, L)$, the *scheme-theoretic base locus* $B(V)$ of V is the closed subscheme of X defined by the image of the composite homomorphism

$$V \otimes_k L^{-1} \hookrightarrow H^0(X, L) \otimes_k L^{-1} \to \mathcal{O}_X,$$

where the latter one is induced by (2.0.1). For the linear system Λ corresponding to V, we set $B(\Lambda) := B(V)$.

(3) We say that L is *globally generated* if (2.0.1) is surjective, i.e., $B(|L|) = \varnothing$.

(4) We say that L is *semiample* if there exists a positive integer n such that $L^{\otimes n}$ is globally generated.

For a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a normal proper variety X over a field, we say that D is *nef* (resp. *semiample*) if there exists a positive integer n such that nD is a Cartier divisor and $\mathcal{O}_X(nD)$ is nef (resp. semiample).

Cox rings. In this subsection, we recall the definition of Cox rings (Definition 2.2) and one of their basic properties (Lemma 2.4).

Definition 2.1. Let k be a field. Let X be a normal variety over k. For a subsemigroup Γ of the group $\mathrm{WDiv}(X)$ of Weil divisors, we set

$$R(X, \Gamma) := \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)),$$

which is called *the multisection ring* of Γ.

Definition 2.2. Let k be a field. Let X be a proper normal variety over k whose divisor class group $\mathrm{Cl}(X)$ is a finitely generated free abelian group. Fix a subgroup Γ of the group $\mathrm{WDiv}(X)$ of Weil divisors such that the induced group homomorphism $\Gamma \to \mathrm{Cl}(X)$ is bijective. We set

$$\mathrm{Cox}(X) := R(X, \Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)),$$

which is called the *Cox ring* of X.

Remark 2.3. If we take another subgroup Γ' satisfying the same property as Γ, then it is known that $R(X, \Gamma)$ and $R(X, \Gamma')$ are isomorphic as k-algebras (see [Gongyo et al. 2015, Remark 2.17]).

Lemma 2.4. *Let k be a field. Let X be a projective normal $\mathbb{Q}$-factorial variety over k whose divisor class group* $\mathrm{Cl}(X)$ *is a finitely generated free abelian group. Assume that*

(a) *X is geometrically integral over k,*

(b) *X is geometrically normal over k,*

(c) $\mathrm{Cox}(X)$ *is a noetherian ring, and*

(d) $\mathbf{Pic}^0_X$ *has dimension zero, where* $\mathbf{Pic}^0_X$ *denotes the identity component of the Picard scheme of X over k* (see [Okawa 2016, Remark 2.4]).

Then, the following assertions hold:

(1) *For any finitely generated subsemigroup Γ_1 of* $\mathrm{WDiv}(X)$, *the multisection ring $R(X, \Gamma_1)$ of Γ_1 is a finitely generated k-algebra.*

(2) *An arbitrary nef Cartier divisor L on X is semiample.*

Proof. By (a) and (b), X is a variety in the sense of [Okawa 2016, the end of Section 1]. Then the conditions (c) and (d) enable us to apply [Okawa 2016, Theorem 2.19], hence X is a Mori dream space in the sense of [Okawa 2016, Definition 2.3]. Then (2) follow from [Okawa 2016, Definition 2.3(2)]. Let us prove (1). By standard arguments (see [Gongyo et al. 2015, discussion in Remark 2.17]), we may assume that Γ_1 is a subgroup of Γ for some subgroup Γ of $\mathrm{WDiv}(X)$. Then the assertion (2) holds by [Okawa 2016, Lemma 2.20]. □

Symbolic Rees algebras. The purpose of this subsection is to prove Proposition 2.14, which gives a relation between symbolic Rees algebras of polynomial rings and Cox rings of blowups of projective spaces. The materials treated in this subsection might be well-known for experts, however we give the details of the proofs for the sake of completeness.

Notation 2.5. (i) Let k be a field and let $A := k[x_0, \ldots, x_n]$ be the polynomial ring equipped with the standard structure of a graded ring. Let M be the homogenous maximal ideal of A. We have $\mathbb{P}^n_k = \mathrm{Proj}\, A$.

(ii) Let W be an integral closed subscheme of $\mathbb{P}^n_k$ and let $f : X \to \mathbb{P}^n_k$ be the blowup along W. For $D := f^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ and the exceptional Cartier divisor E that is the inverse image of W, we set

$$R(X; D, -E) := \bigoplus_{d,e \in \mathbb{Z}_{\geq 0}} H^0(X, dD - eE).$$

(iii) There exists a homogeneous prime ideal $\mathfrak{p}$ of $A := k[x_0, \ldots, x_n]$ that induces the ideal sheaf on $\mathbb{P}^n_k$ corresponding to W. The *symbolic Rees algebra* of $\mathfrak{p}$ is defined as $\bigoplus_{d=0}^{\infty} \mathfrak{p}^{(d)}$, where $\mathfrak{p}^{(d)} := \mathfrak{p}^d A_{\mathfrak{p}} \cap A$.

(iv) Let $\mathcal{I}_W$ be the ideal sheaf on $\mathbb{P}^n_k$ corresponding to W.

Definition 2.6. We use Notation 2.5. For a homogenous ideal I of A, we define the *saturation* I^{sat} of I by

$$I^{\mathrm{sat}} := \bigcup_{\nu=1}^{\infty} \{x \in A \mid M^{\nu} x \subset I\}.$$

Remark 2.7. We use the same notation as in Definition 2.6. By [Hartshorne 1977, Excercise 5.10 in Chapter II], I^{sat} is a homogeneous ideal of A such that both I and I^{sat} define the same closed subscheme on $\mathbb{P}^n_k$ and the equation

$$I^{\mathrm{sat}} = \bigoplus_{d=0}^{\infty} H^0(\mathbb{P}^n_k, \mathcal{I}(d))$$

holds, where $\mathcal{I}$ is the ideal sheaf on $\mathbb{P}^n_k$ associated with I.

Definition 2.8. Let R be a noetherian ring and let J be an ideal of R. We define $\tilde{J}$, called *the Ratliff–Rush ideal associated with* J, by

$$\tilde{J} := \bigcup_{n=0}^{\infty} (J^{n+1} : J^n).$$

The ideal J is said to be *Rattlif–Rush* if $J = \tilde{J}$. It is well-known that $\tilde{J}$ is a Ratliff–Rush ideal (see [Heinzer et al. 1992, Introduction]).

Lemma 2.9. *We use Notation 2.5. Fix a positive integer e and let $\mathfrak{p}^e = \bigcap_{i=0}^{r} \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{p}^e$ such that $\sqrt{\mathfrak{q}_0} = \mathfrak{p}$* (see [Atiyah and Macdonald 1969, Section 4]). *Then the following hold*:

(1) *The equation $\mathfrak{p}^{(e)} = \mathfrak{q}_0$ holds.*

(2) *The equation $(\mathfrak{p}^e)^{\mathrm{sat}} = \bigcap_{i\in L} \mathfrak{q}_i$ holds, where*

$$L := \{i \in \{0, \dots, r\} \mid \sqrt{\mathfrak{q}_i} \neq M\}.$$

Proof. We show (1). Since $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{p}^e$, it follows from [Atiyah and Macdonald 1969, Proposition 4.9] that $\mathfrak{p}^e A_{\mathfrak{p}} = \mathfrak{q}_0 A_{\mathfrak{p}}$. In particular we get equations

$$\mathfrak{p}^{(e)} = \mathfrak{p}^e A_{\mathfrak{p}} \cap A = \mathfrak{q}_0 A_{\mathfrak{p}} \cap A = \mathfrak{q}_0,$$

where the last equation follows from the fact that $\mathfrak{q}_0$ is a $\mathfrak{p}$-primary ideal. Thus (1) holds.

We show (2). First, let us prove $(\mathfrak{p}^e)^{\mathrm{sat}} \subset \bigcap_{i\in L} \mathfrak{q}_i$. Take $x \in (\mathfrak{p}^e)^{\mathrm{sat}}$ and $i \in L$. By definition of the saturation $(\mathfrak{p}^e)^{\mathrm{sat}}$ (see Definition 2.6), there is $\nu \in \mathbb{Z}_{>0}$ such that $M^\nu x \subset \mathfrak{p}^e \subset \mathfrak{q}_i$. As $\sqrt{\mathfrak{q}_i} \neq M$, there is $y \in M \setminus \sqrt{\mathfrak{q}_i}$. Hence $y^\nu x \in \mathfrak{q}_i$. Since $\mathfrak{q}_i$ is a primary ideal, it holds that $x \in \mathfrak{q}_i$. Thus the inclusion $(\mathfrak{p}^e)^{\mathrm{sat}} \subset \bigcap_{i\in L} \mathfrak{q}_i$ holds.

Second we prove the remaining inclusion: $(\mathfrak{p}^e)^{\mathrm{sat}} \supset \bigcap_{i\in L} \mathfrak{q}_i$. If $L = \{0, \dots, r\}$, then there is nothing to show. We may assume that $L \neq \{0, \dots, r\}$. As the primary decomposition $\mathfrak{p}^e = \bigcap_{i=0}^{r} \mathfrak{q}_i$ is minimal, there exists a unique index $i_1 \in \{1, \dots, r\}$ such that $\sqrt{\mathfrak{q}_{i_1}} = M$ (see [Atiyah and Macdonald 1969, Lemma 4.3]). In particular, $L = \{0, \dots, r\} \setminus \{i_1\}$. Since A is a noetherian ring, there exists a positive integer ν such that $M^\nu \subset \mathfrak{q}_{i_1}$. It follows from definition of the saturation $(\mathfrak{p}^e)^{\mathrm{sat}}$ (see Definition 2.6) that $\bigcap_{i\in L} \mathfrak{q}_i = \bigcap_{i\in\{0,\dots,r\}, i\neq i_1} \mathfrak{q}_i \subset (\mathfrak{p}^e)^{\mathrm{sat}}$. □

Lemma 2.10. *Let R be a noetherian ring and let I be an ideal of R generated by a regular sequence $a_1, \dots, a_\mu$ of R. Then the following hold*:

(1) *An* (R/I)*-algebra homomorphism*

$$(R/I)[X_1, \dots, X_\mu] \to \bigoplus_{m=0}^{\infty} I^m/I^{m+1}, \quad X_i \mapsto a_i \mod I^2$$

is an isomorphism, where $I^0 := R$.

(2) *If* I *is a prime ideal of* R *other than* $\{0\}$, *then* I^e *is a Ratliff–Rush ideal for any positive integer* e (*see Definition 2.8*).

(3) *If* I *is a prime ideal of* R, *then for any positive integer* e, *an arbitrary associated prime ideal of* I^e *is equal to* I.

Proof. The assertion (1) holds by the fact that any regular sequence is quasiregular [Matsumura 1989, Theorem 16.2(i)]. The assertion (2) follows from (1) and [Heinzer et al. 1992, (1.2)].

We show (3). By (1), I^m/I^{m+1} is a free (R/I)-module for any $m \in \mathbb{Z}_{\geq 0}$. Consider an exact sequence

$$0 \to I^m/I^{m+1} \to R/I^{m+1} \to R/I^m \to 0.$$

We deduce from induction on e that for any $e \in \mathbb{Z}_{\geq 1}$, an arbitrary associated prime of I^e is equal to I. Thus (3) holds. □

Lemma 2.11. *We use Notation 2.5. Assume that* W *is a local complete intersection scheme. Fix a positive integer* e. *Then the equation* $f_*\mathcal{O}_X(-eE) = \mathcal{I}^e$ *holds as subsheaves of* $\mathcal{O}_{\mathbb{P}^n_k}$.

Proof. Fix a point $z \in \mathbb{P}^n_k$ and set $R := \mathcal{O}_{\mathbb{P}^n_k, z}$. Given a positive integer e, let

$$I := \Gamma(\operatorname{Spec} R, \mathcal{I}|_{\operatorname{Spec} R}), \quad R(I^e) := \bigoplus_{d=0}^{\infty} I^{ed}, \quad g_e : Y_e = \operatorname{Proj} R(I^e) \to \operatorname{Spec} R,$$

where $I^0 := R$ and g_e is the blowup along I^e. We set $Y := Y_1$ and $g := g_1$. Let E_e be the effective Cartier divisor such that $\mathcal{O}_{Y_e}(-E_e) := I^e\mathcal{O}_{Y_e}$. In particular, $E = E_1$. Thanks to [Hartshorne 1977, Exercise 5.13 in Chpater II], we have that $\rho_e : Y \xrightarrow{\sim} Y_e$ and $(\rho_e)_*(eE) = E_e$. We get equations

$$I^e = \widetilde{I^e} = H^0(Y_e, \mathcal{O}_{Y_e}(-E_e)) = H^0(Y, \mathcal{O}_Y(-eE)),$$

where the first equation holds by Lemma 2.10(2), the second one follows from [Heinzer et al. 1992, Fact 2.1] and the third one is obtained by ρ_e. Hence we are done. □

Lemma 2.12. *We use Notation 2.5. Assume that* W *is locally complete intersection. Then* $R(X; D, -E)$ *and* $\bigoplus_{e=0}^{\infty} \mathfrak{p}^{(e)}$ *are isomorphic as* k*-algebras.*

Proof. Fix a nonnegative integer e. We show that $\bigoplus_{d=0}^{\infty} H^0(X, dD - eE)$ is isomorphic to $\mathfrak{p}^{(e)}$. By Lemma 2.11, we have $f_*\mathcal{O}_X(-eE) \simeq \mathcal{I}^e$. By the projection formula, we get

$$f_*\mathcal{O}_X(dD - eE) \simeq \mathcal{I}^e \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(d) = \mathcal{I}^e(d).$$

Thanks to Remark 2.7, we obtain an isomorphism

$$(\mathfrak{p}^e)^{\mathrm{sat}} \simeq \bigoplus_{d=0}^{\infty} H^0(X, dD - eE).$$

Claim 2.13. *Any associated prime ideal of $\mathfrak{p}^e$ is equal to either $\mathfrak{p}$ or M.*

Proof of Claim 2.13. Assume that there exists an associated prime ideal $\mathfrak{q}$ of $\mathfrak{p}^e$ other than $\mathfrak{p}$ or M. Let us derive a contradiction. Since $\mathfrak{q} \neq M = (x_0, \dots, x_n)$, there is x_ℓ that is not contained in $\mathfrak{q}$. Then $\mathfrak{q}A_{x_\ell}$ is an associated prime ideal of $\mathfrak{p}^e A_{x_\ell}$. Take a maximal ideal $\mathfrak{m}$ of A_{x_ℓ} containing $\mathfrak{q}A_{x_\ell}$. Then $\mathfrak{q}A_{\mathfrak{m}}$ is an associated prime ideal of $\mathfrak{p}^e A_{\mathfrak{m}}$ other than $\mathfrak{p}A_{\mathfrak{m}}$. Since W is a local complete intersection scheme, we have that $\mathfrak{p}A_{\mathfrak{m}}$ is a prime ideal generated by a regular sequence, which contradicts Lemma 2.10(3). This completes the proof of Claim 2.13. $\square$

For a minimal primary decomposition $(\mathfrak{p}^e)^{\mathrm{sat}} = \bigcap_{i=0}^{r} \mathfrak{q}_i$ satisfying $\sqrt{\mathfrak{q}_0} = \mathfrak{p}$, we have that

$$\mathfrak{p}^{(e)} = \mathfrak{q}_0 = (\mathfrak{p}^e)^{\mathrm{sat}} \simeq \bigoplus_{d=0}^{\infty} H^0(X, dD - eE),$$

where the first equation holds by Lemma 2.9(1) and the second equation follows from Lemma 2.9(2) and Claim 2.13. This completes the proof of Lemma 2.12 $\square$

Proposition 2.14. *We use Notation 2.5. Assume that W is smooth over k. Then the following are equivalent*:

(1) *$R(X; D, -E)$ is a noetherian ring.*

(2) *$\bigoplus_{e=0}^{\infty} \mathfrak{p}^{(e)}$ is a noetherian ring.*

(3) *The Cox ring* $\mathrm{Cox}(X)$ *of X is a noetherian ring.*

Proof. It follows from Lemma 2.12 that (1) is equivalent to (2). Since X is the blowup of $\mathbb{P}^n_k$ along a smooth scheme W, the assumptions of Lemma 2.4 hold. Then, thanks to Lemma 2.4(1), we have that (3) implies (1). Thus it suffices to show that (1) implies (3). Since it holds that $H^0(X, dD - eE) = 0$ for $d \in \mathbb{Z}_{<0}$ and $e \in \mathbb{Z}$, we get an isomorphism:

$$\bigoplus_{d,e \in \mathbb{Z}, d \geq 0} H^0(X, dD - eE) \xrightarrow{\sim} \bigoplus_{d,e \in \mathbb{Z}} H^0(X, dD - eE).$$

Thus we have a natural inclusion:

$$R(X; D, -E) = \bigoplus_{d,e \in \mathbb{Z}_{\geq 0}} H^0(X, dD - eE) \hookrightarrow \bigoplus_{d,e \in \mathbb{Z}, d \geq 0} H^0(X, dD - eE).$$

The right-hand side is generated by $H^0(X, E)$ as an $R(X; D, -E)$-algebra. Therefore, if $R(X; D, -E)$ is a noetherian ring, then so is $\bigoplus_{d,e \in \mathbb{Z}} H^0(X, dD - eE)$. Hence, also $\mathrm{Cox}(X)$ is a noetherian ring. Thus (1) implies (3). $\square$

3. The main theorem

Construction in a general setting. The purpose of this subsection is to give a sufficient condition under which the blowup of a smooth subvariety in a projective space has a nef Cartier divisor that is not semiample (Notation 3.1, Proposition 3.3).

Notation 3.1. We use notation as follows:

(i) Let k be a field. We work over k unless otherwise specified (e.g., a projective scheme means a scheme that is projective over k).

(ii) Let V be a smooth projective variety. Set $d := \dim V$.

(iii) Let M be a nef Cartier divisor on V which is not semiample.

(iv) Fix a closed immersion: $V \subset \mathbb{P}^n_k$. Let H be a very ample Cartier divisor such that $\mathcal{O}_{\mathbb{P}^n_k}(H) \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. We set H_V to be the pullback of H to V.

(v) Assume that there exists a positive integer r satisfying the following property: if Λ denotes the linear system of $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(r))$ consisting of the effective divisors containing V, then the following conditions hold:

(v-1) The base locus of $|\Lambda|$ is set-theoretically equal to V, i.e., for any point $y \in \mathbb{P}^n_k \setminus V$, there exists a hypersurface S_0 of $\mathbb{P}^n_k$ of degree r such that $V \subset S_0$ and $y \notin S_0$.

(v-2) For any closed point $y \in V$, there exist an open neighborhood U of $y \in \mathbb{P}^n_k$ and hypersurfaces $S_1, \dots, S_{n-\dim V}$ of $\mathbb{P}^n_k$ of degree r such that V is contained in $S_1 \cap \cdots \cap S_{n-\dim V}$ and that two subschemes $V \cap U$ and $S_1 \cap \cdots \cap S_{n-\dim V} \cap U$ of $\mathbb{P}^n_k$ are coincide.

(vi) Assume that there are a smooth prime divisor W on V and positive integers s and t satisfying the following properties:

(vi-1) $st > r$.

(vi-2) $W \sim stH_V - tM$.

(vii) Let $f : X \to \mathbb{P}^n_k$ be the blowup along W. We set $V' := f_*^{-1}V$, $E := \mathrm{Ex}(f)$ and

$$S' := rf^*H - E.$$

Note that E is a smooth prime divisor on X. Let $g : V' \xrightarrow{\simeq} V$ be the induced isomorphism.

(viii) Set

$$L := (st - r)f^*H + S'.$$

Lemma 3.2. *Let k be a field and let $Y := \mathbb{A}^n_k = \mathrm{Spec}\, k[y_1, \dots, y_n]$ be the n-dimensional affine space. For $i \in \{1, \dots, n\}$, set $T_i := V(y_i)$ to be the coordinate hyperplane of $Y = \mathbb{A}^n_k$. Let q be a positive integer satisfying $q \le n - 1$. Set $V := T_1 \cap \cdots \cap T_q$ and $W := T_1 \cap \cdots \cap T_{q+1}$. Let $f : X \to Y$ be the blowup along W and let V' and T_i' be the proper transforms of V and T_i, respectively. Then an equation $V' = T_1' \cap \cdots \cap T_q'$ holds.*

Proof. Since blowups are commutative with flat base changes, we may assume that $q = n-1$. Thus W is the origin and V is a line passing through W. The inclusion $V' \subset T'_1 \cap \cdots \cap T'_{n-1}$ is clear, hence it suffices to prove that $T'_1 \cap \cdots \cap T'_{n-1} \cap E$ is one point, where E denotes the f-exceptional prime divisor. To prove this, we may assume that k is algebraically closed. Then $T'_1 \cap \cdots \cap T'_{n-1} \cap E$ is one point, since there is a canonical bijection between the set $E(k)$ of the closed points of E and the set of the lines on $\mathbb{P}^n_k$ passing through W. □

Proposition 3.3. *We use Notation 3.1. Then the following hold*:

(1) *The base locus of the complete linear system $|S'|$ is contained in V'.*

(2) $L|_{V'} \sim tg^*M$.

(3) *L is a nef Cartier divisor which is not semiample.*

Proof. We show (1). Take a closed point $x \in X \setminus V'$. We set $y := f(x)$. It suffices to show that the base locus $B(|S'|)$ of $|S'|$ does not contain x. We separately treat the following two cases: $y \notin V$ and $y \in V$.

Assume that $y \notin V$. By Notation 3.1(v-1), there exists a hypersurface S_0 of $\mathbb{P}^n_k$ of degree r such that $V \subset S_0$ and $y \notin S_0$. It holds that

$$rf^*H \sim f^*S_0 = S'_0 + aE,$$

where $a \in \mathbb{Z}_{>0}$ and S'_0 is the proper transform of S_0. In particular, we have that

$$B(|S'|) \subset \mathrm{Supp}(S'_0 + E) = f^{-1}(S_0).$$

It follows from $y \notin S_0$ that $x \notin f^{-1}(S_0)$. Hence, $x \notin B(|S'|)$. This completes the proof for the case where $y \notin V$.

Assume that $y \in V$. We have that $x \in E \setminus V'$. By Notation 3.1(v-2), there exist an open neighborhood U of $y \in \mathbb{P}^n_k$ and hypersurfaces $S_1, \ldots, S_{n-\dim V}$ of $\mathbb{P}^n_k$ of degree r such that V is contained in $S_1 \cap \cdots \cap S_{n-\dim V}$ and that two subschemes $V \cap U$ and $S_1 \cap \cdots \cap S_{n-\dim V} \cap U$ of $\mathbb{P}^n_k$ are the same. In particular, $S_1, \ldots, S_{n-\dim V}$ are smooth at y and form a part of a regular system of parameters of $\mathcal{O}_{\mathbb{P}^n_k, y}$ (see [Matsumura 1989, Theorem 17.4]). Therefore, thanks to Cohen's structure theorem, the situation is the same, up to taking the formal completions, as in the statement of Lemma 3.2. It follows from Lemma 3.2 and the faithfully flatness of completions (see [Matsumura 1989, Theorem 7.5(ii)]) that an equation

$$V' \cap f^{-1}(U) = S'_1 \cap \cdots \cap S'_{n-\dim V} \cap f^{-1}(U)$$

holds, where each S'_i denotes the proper transform of S_i. In particular, it holds that $x \notin S'_{i_0}$ for some $i_0 \in \{1, \ldots, n-\dim V\}$. Since S'_{i_0} is smooth at a point y of W, we have that

$$S' = f^*(rH) - E \sim f^*S_{i_0} - E = S'_{i_0}.$$

Thus, in any case, the base locus $B(|S'|)$ does not contain x. Hence, (1) holds.

Assertion (2) holds by the following computation:

$$\begin{aligned} L|_{V'} &= ((st-r)f^*H + S')|_{V'} \\ &\sim g^*((st-r)H_V + (S|_V - W)) \\ &\sim g^*((st-r)H_V + (rH_V - (stH_V - tM))) \\ &\sim tg^*M. \end{aligned}$$

We show (3). Since $L|_{V'}$ is not semiample by (2) and Notation 3.1(iii), neither is L. Thus it suffices to show that $L=(st-r)f^*H+S'$ is nef. Take a curve Γ on X. If $\Gamma \not\subset V'$, then we get $((st-r)f^*H+S')\cdot\Gamma \geq 0$ by (1). If $\Gamma \subset V'$, then (2) implies that $L \cdot \Gamma \geq 0$. In any case, we obtain $L \cdot \Gamma \geq 0$, and hence L is nef. Thus (3) holds. □

Proof of the main theorem. In this subsection, we prove the main theorem of this paper (Theorem 3.7). Theorem 3.7 is a formal consequence of Theorem 3.6 and some results established before. The main part of Theorem 3.6 is to find schemes and divisors satisfying Notation 3.1. To this end, we start with the following lemma.

Lemma 3.4. *Let k be a field. Let V be a smooth projective connected scheme over k such that* $\dim V \geq 2$. *Let W be an ample effective Cartier divisor. Then W is connected.*

Proof. Set $k' := H^0(V, \mathcal{O}_V)$. Note that $k \subset k'$ is a field extension of finite degree. We have natural morphisms:

$$\alpha : V \xrightarrow{\alpha'} \operatorname{Spec} k' \xrightarrow{\beta} \operatorname{Spec} k.$$

We obtain $\alpha'_*\mathcal{O}_V = \mathcal{O}_{\operatorname{Spec} k'}$.

Let us prove that $k \subset k'$ is a separable extension. It suffices to prove that $A := k' \otimes_k \bar{k}$ is reduced for an algebraic closure $\bar{k}$ of k. We have the induced morphism

$$\alpha'' = \alpha' \times_k \bar{k} : V \times_k \bar{k} \to \operatorname{Spec}(k' \otimes_k \bar{k}) = \operatorname{Spec} A.$$

Since $k \to \bar{k}$ is flat, we have that $\alpha''_*\mathcal{O}_{V\times_k \bar{k}} = \mathcal{O}_{\operatorname{Spec} A}$. As $V \times_k \bar{k}$ is reduced, so is A. Therefore, $k \subset k'$ is a separable extension.

We have that α is smooth and β is étale. Then it holds that also α' is smooth by [Fu 2011, Proposition 2.4.1]. Therefore, the problem is reduced to the case where $k = H^0(V, \mathcal{O}_V)$.

We are allowed to replace W by nW for a positive integer n. Hence, by Serre duality and the ampleness of W, we may assume that $H^1(V, \mathcal{O}_V(-W)) = 0$. Then we obtain a surjective k-linear map

$$H^0(V, \mathcal{O}_V) \to H^0(W, \mathcal{O}_W).$$

Since $\dim_k H^0(V, \mathcal{O}_V) = 1$, we get $\dim_k H^0(W, \mathcal{O}_W) = 1$. Therefore, W is connected. □

Lemma 3.5. *The following hold*:

(1) *Let n be an integer such that $n \geq 5$. If k is an algebraically closed field, then there exist a smooth projective surface V over k, a closed immersion $j : V \hookrightarrow \mathbb{P}^n_k$ over k, and a nef Cartier divisor M on V which is not semiample.*

(2) *Let n be an integer such that $n \geq 11$. If k is a field, then there exist a smooth projective surface V over k, a closed immersion $j : V \hookrightarrow \mathbb{P}^n_k$ over k, and a nef Cartier divisor M on V which is not semiample.*

Proof. We show (1). We may assume that $n = 5$. The existence of j is automatic, since any smooth projective surface over k can be embedded in $\mathbb{P}^5_k$. If k is the algebraic closure of a finite field, then the assertion follows from [Totaro 2009, Theorem 6.1]. If k is not algebraic over any finite field, then V can be taken as the direct product of an elliptic curve E and a smooth projective curve. Indeed, there is a Cartier divisor N on E such that $\deg N = 0$ and N is not torsion, i.e., $rN \not\sim 0$ for any positive integer r. This implies that N is a nef Cartier divisor which is not semiample. Hence, its pullback M to V is again a nef Cartier divisor which is not semiample. This completes the proof of (1).

We show (2). We may assume that $n = 11$. First we treat the case where k is a perfect field. By (1), we can find a field extension $k \subset k'$ of finite degree, a connected k'-scheme V of dimension two which is smooth and projective over k', a closed immersion $j' : V \hookrightarrow \mathbb{P}^5_{k'}$ over k' and a nef Cartier divisor M on V which is not semiample. Automatically V is projective over k. Since k is perfect, V is also smooth over k. Thus it suffices to find a closed immersion $j : V \hookrightarrow \mathbb{P}^{11}_k$ over k. Since $k \subset k'$ is a finite separable extension, it is a simple extension. Therefore, there is a closed immersion $i : \operatorname{Spec} k' \hookrightarrow \mathbb{P}^1_k$ over k. We can find a required closed immersion j by using the Segre embedding:

$$j : V \xhookrightarrow{j'} \mathbb{P}^5_{k'} = \mathbb{P}^5_k \times_k k' \xhookrightarrow{\mathrm{id} \times i} \mathbb{P}^5_k \times_k \mathbb{P}^1_k \xhookrightarrow{\mathrm{Segre}} \mathbb{P}^{11}_k.$$

This completes the proof of the case where k is a perfect field.

Second we handle the general case. Let k_0 be the prime field contained in k. Since k_0 is perfect, there exist a smooth projective connected k_0-scheme V_0 of dimension two, a closed immersion $j_0 : V_0 \hookrightarrow \mathbb{P}^{11}_{k_0}$ over k_0, and a nef Cartier divisor M_0 on V_0 which is not semiample. Then $V_0 \times_{k_0} k$ is a scheme which is smooth and projective over k. Since any ring homomorphism between fields is faithfully flat, we can find a connected component V of $V_0 \times_{k_0} k$ such that $M := (\alpha^* M_0)|_V$ is not semiample, where $\alpha : V_0 \times_{k_0} k \to V_0$. Since M_0 is nef, so is M (see [Tanaka 2018, Lemma 2.3]). Clearly, V is a smooth projective surface over k and there is a closed immersion $j : V \hookrightarrow \mathbb{P}^{11}_k$ over k. This completes the proof of (2). □

Theorem 3.6. *The following hold*:

(1) *Let n be an integer such that $n \geq 5$. If k is an algebraically closed field, then there exist a one-dimensional connected closed subscheme W of $\mathbb{P}^n_k$ which is smooth over k and a Cartier divisor L on the blowup X of $\mathbb{P}^n_k$ along W such that L is nef but not semiample.*

(2) *Let n be an integer such that $n \geq 11$. If k is a field, then there exist a one-dimensional connected closed subscheme W of $\mathbb{P}^n_k$ which is smooth over k and a Cartier divisor L on the blowup X of $\mathbb{P}^n_k$ along W such that L is nef but not semiample.*

Proof. We only show (2), as the proof of (1) is easier. Fix a field k. We will find schemes and divisors satisfying the properties of Notation 3.1. Thanks to Lemma 3.5, there exist a smooth projective connected k-scheme V of dimension two, a closed immersion $j : V \hookrightarrow \mathbb{P}^n_k$ over k, and a nef Cartier divisor M on V which is not semiample. Set $d := 2$. Then k, V, M, d, n satisfy properties (i)–(iv) of Notation 3.1.

Since $V = \operatorname{Proj} k[x_0, \dots, x_n]/(h_1, \dots, h_a)$, it holds that the linear system Λ appearing in Notation 3.1(v) satisfies the property (v-1) of Notation 3.1 if $r \geq \max_{1 \leq q \leq a} \deg h_q$. As V is a locally completion intersection scheme, the quasicompactness of V also implies that property (v-2) of Notation 3.1 holds for $r \gg 0$. Therefore, we can find $r \in \mathbb{Z}_{>0}$ satisfying property (v) of Notation 3.1.

We now show that there exist s, t, W satisfying property (vi) of Notation 3.1. If k is an infinite field, then the Bertini theorem enables us to find a positive integer s and a smooth effective divisor W on V such that $W \sim sH_V - M$. Note that W is connected (Lemma 3.4). Thus, s, $t := 1$ and W satisfy property (vi) of Notation 3.1. If k is a finite field, then it follows from [Poonen 2004, Theorem 1.1] that there are positive integers $t \gg s \gg 0$ and a smooth effective divisor W satisfying property (vi) of Notation 3.1. Again by Lemma 3.4, W is connected. In any case, we can find s, t, W satisfying property (vi) of Notation 3.1.

To summarize, we have found V, W, M, d, n, r, s, t over a field k satisfying properties (i)–(viii) of Notation 3.1. By construction, V is a smooth projective surface. In particular, W is a smooth projective curve in $\mathbb{P}^{11}_k$. Thanks to Proposition 3.3, the Cartier divisor

$$L = (st - r) f^* H + S'$$

on X, defined in (viii) of Notation 3.1, is nef but not semiample. □

Theorem 3.7. *The following hold*:

(1) *Let q be an integer such that $q \geq 6$. If k is an algebraically closed field, then there exists a homogeneous prime ideal $\mathfrak{p}$ of the polynomial ring $k[x_1, \dots, x_q]$ with q variables whose symbolic Rees algebra $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring.*

(2) *Let q be an integer such that $q \geq 12$. If k is a field, then there exists a homogeneous prime ideal $\mathfrak{p}$ of the polynomial ring $k[x_1, \dots, x_q]$ with q variables whose symbolic Rees algebra $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring.*

Proof. The assertion follows from Lemma 2.4, Proposition 2.14 and Theorem 3.6. □

Acknowledgements

The authors would like to thank Professors Kazuhiko Kurano and Shinnosuke Okawa for several useful comments and discussion. We are grateful to the referee for valuable comments. The first author would like to thank Professor Shigeru Mukai for his warm encouragement and stimulating discussions. The first author was partially supported by JSPS Grant-in-Aid (S) No 25220701 and JSPS Grant-in-Aid for Young Scientists (B) 16K17581. The second author was funded by EPSRC.

References

[Atiyah and Macdonald 1969] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, Reading, MA, 1969. MR Zbl

[Castravet 2009] A.-M. Castravet, "The Cox ring of $\overline{M}_{0,6}$", *Trans. Amer. Math. Soc.* **361**:7 (2009), 3851–3878. MR Zbl

[Cowsik 1984] R. C. Cowsik, "Symbolic powers and number of defining equations", pp. 13–14 in *Algebra and its applications* (New Delhi, 1981), edited by H. L. Manocha and J. B. Srivastava, Lect. Notes Pure Appl. Math. **91**, Dekker, New York, 1984. MR Zbl

[Cutkosky 1991] S. D. Cutkosky, "Symbolic algebras of monomial primes", *J. Reine Angew. Math.* **416** (1991), 71–89. MR Zbl

[Diaz and Harbater 1991] S. Diaz and D. Harbater, "Strong Bertini theorems", *Trans. Amer. Math. Soc.* **324**:1 (1991), 73–86. MR Zbl

[Fu 2011] L. Fu, *Etale cohomology theory*, Nankai Tracts in Math. **13**, World Sci., Hackensack, NJ, 2011. MR Zbl

[Gongyo et al. 2015] Y. Gongyo, S. Okawa, A. Sannai, and S. Takagi, "Characterization of varieties of Fano type via singularities of Cox rings", *J. Algebraic Geom.* **24**:1 (2015), 159–182. MR Zbl

[González and Karu 2016] J. L. González and K. Karu, "Some non-finitely generated Cox rings", *Compos. Math.* **152**:5 (2016), 984–996. MR Zbl

[Goto et al. 1994] S. Goto, K. Nishida, and K. Watanabe, "Non-Cohen–Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question", *Proc. Amer. Math. Soc.* **120**:2 (1994), 383–392. MR Zbl

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. **52**, Springer, 1977. MR Zbl

[Heinzer et al. 1992] W. Heinzer, D. Lantz, and K. Shah, "The Ratliff–Rush ideals in a Noetherian ring", *Comm. Algebra* **20**:2 (1992), 591–622. MR Zbl

[Kurano 1993] K. Kurano, "Positive characteristic finite generation of symbolic Rees algebras and Roberts' counterexamples to the fourteenth problem of Hilbert", *Tokyo J. Math.* **16**:2 (1993), 473–496. MR Zbl

[Kurano 1994] K. Kurano, "On finite generation of Rees rings defined by filtrations of ideals", *J. Math. Kyoto Univ.* **34**:1 (1994), 73–86. MR Zbl

[Kuroda 2005] S. Kuroda, "A counterexample to the fourteenth problem of Hilbert in dimension three", *Michigan Math. J.* **53**:1 (2005), 123–132. MR Zbl

[Matsumura 1989] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Adv. Math. **8**, Cambridge Univ. Press, 1989. MR Zbl

[Nagata 1960] M. Nagata, "On the fourteenth problem of Hilbert", pp. 459–462 in *Proceedings of the Internatational Congress of Mathematics, VIII* (Edinburgh, 1958), edited by J. A. Todd, Cambridge Univ. Press, 1960. MR Zbl

[Okawa 2016] S. Okawa, "On images of Mori dream spaces", *Math. Ann.* **364**:3-4 (2016), 1315–1342. MR Zbl

[Poonen 2004] B. Poonen, "Bertini theorems over finite fields", *Ann. of Math.* (2) **160**:3 (2004), 1099–1127. MR Zbl

[Roberts 1985] P. C. Roberts, "A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian", *Proc. Amer. Math. Soc.* **94**:4 (1985), 589–592. MR Zbl

[Roberts 1990] P. Roberts, "An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem", *J. Algebra* **132**:2 (1990), 461–473. MR Zbl

[Tanaka 2018] H. Tanaka, "Behavior of canonical divisors under purely inseparable base changes", *J. Reine Angew. Math.* **744** (2018), 237–264. MR Zbl

[Totaro 2008] B. Totaro, "Hilbert's 14th problem over finite fields and a conjecture on the cone of curves", *Compos. Math.* **144**:5 (2008), 1176–1198. MR Zbl

[Totaro 2009] B. Totaro, "Moving codimension-one subvarieties over finite fields", *Amer. J. Math.* **131**:6 (2009), 1815–1833. MR Zbl

Communicated by Keiichi Watanabe
Received 2018-02-15 Revised 2018-06-19 Accepted 2018-07-20

akiyoshi.sannai@riken.jp *Center for Advanced Intelligence Project, RIKEN, Tokyo, Japan*

tanaka@ms.u-tokyo.ac.jp *Graduate School of Mathematical Sciences, The University of Tokyo, Japan*

mathematical sciences publishers msp

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1893

Manin's b-constant in families

Akash Kumar Sengupta

We show that the b-constant (appearing in Manin's conjecture) is constant on very general fibers of a family of algebraic varieties. If the fibers of the family are uniruled, then we show that the b-constant is constant on general fibers.

1. Introduction

Let X be a smooth projective variety over a field of k of characteristic 0 and L a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let $\Lambda_{\text{eff}}(X) \subset \text{NS}(X)_{\mathbb{R}}$ be the cone of pseudoeffective divisors. The Fujita invariant or the a-constant is defined as

$$a(X, L) = \min\{t \in \mathbb{R} \mid [K_X] + t[L] \in \Lambda_{\text{eff}}(X)\}.$$

The invariant $\kappa\epsilon(X, L) = -a(X, L)$ was introduced and studied by Fujita [1987; 1992] under the name Kodaira energy. The a-constant was introduced in the context of Manin's conjecture in [Franke et al. 1989].

The b-constant is defined as follows [Franke et al. 1989; Batyrev and Manin 1990]:

$$b(X, L) = \text{codim of minimal supported face of } \Lambda_{\text{eff}}(X) \text{containing the class of } K_X + a(X, L)L.$$

For a singular variety X, the a- and b-constants of L are defined to be the a- and b-constants of π^*L on a resolution $\pi : \tilde{X} \to X$.

Let $f : X \to T$ be a family of projective varieties and L an f-big and f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. By semicontinuity the a-constant of the fibers $a(X_t, L|_{X_t})$ is constant on very general fiber (see [Lehmann and Tanimoto 2017, Theorem 4.3]). It follows from invariance of log plurigenera that if the fibers are uniruled then the a-constant is constant on general fibers.

In this paper we investigate the behavior of the b-constant in families and answer the questions posed in [Lehmann and Tanimoto 2017]. We prove the following:

Theorem 1.1. *Let $f : X \to T$ be a projective morphism of irreducible varieties over an algebraically closed field k of characteristic* 0, *such that the generic fiber is geometrically integral. Let L be an f-big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then there exists a countable union of proper closed subvarieties $Z = \bigcup_i Z_i \subsetneq T$, such that*

$$b(X_{\bar{t}}, L|_{X_{\bar{t}}}) = b(X_{\bar{\eta}}, L|_{X_{\bar{\eta}}})$$

MSC2010: primary 14E30; secondary 14G05.
Keywords: Manin's conjecture, minimal model program, rational points, Néron–Severi group.

for all $t \in T \setminus Z$, where $\eta \in T$ is the generic point. In particular, the b-constant is constant on very general fibers.

If the fibers of the family are uniruled, then we have the following:

Theorem 1.2. *Let $f : X \to T$ be a projective morphism of irreducible varieties over an algebraically closed field k of characteristic 0, such that the generic fiber is geometrically integral. Let L be an f-big and f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Suppose a general fiber X_t is uniruled. Then there exists a proper closed subscheme $W \subsetneq T$ such that*

$$b(X_{\bar{t}}, L|_{X_{\bar{t}}}) = b(X_{\bar{\eta}}, L|_{X_{\bar{\eta}}})$$

for $t \in T \setminus W$ and $\eta \in T$ is the generic point. In particular, the b-constant is constant on general fibers in a family of uniruled varieties.

One can not replace the very general condition in Theorem 1.1 by just general. For example, in a family of K3-surfaces the b-constant of a fiber is the same as the Picard rank and there exist families where the Picard rank jumps on infinitely many subvarieties. Invariance of the b-constant in general fiber of a family of uniruled varieties was proved in [Lehmann and Tanimoto 2017] under the assumption $\kappa(K_{\tilde{X}_t} + a(X_t, L|_{X_t})\beta^*(L|_{X_t})) = 0$ for some resolution of singularities $\beta : \tilde{X}_t \to X_t$. Theorem 1.2 generalizes their result to get rid of this condition on fibers.

One of the motivations for studying the behavior of a- and b-constants is Manin's conjecture about asymptotic growth of rational points on Fano varieties proposed in [Franke et al. 1989; Batyrev and Manin 1990]. The following version was suggested by Peyre [2003] and later stated in [Le Rudulier 2013; Browning and Loughran 2017].

Manin's conjecture. Let X be a Fano variety defined over a number field F and $\mathcal{L} = (L, \|\cdot\|)$ a big and nef adelically metrized line bundle on X with associated height function $H_{\mathcal{L}}$. Then there exists a thin set $Z \subset X(F)$ such that one has

$$\#\{x \in X(F) \setminus Z \mid H_{\mathcal{L}}(x) \leq B\} \sim c(F, X(F) \setminus Z, \mathcal{L}) B^{a(X,L)} \log B^{b(X,L)-1}$$

as $B \to \infty$.

For the geometric consistency of Manin's conjecture, a necessary condition is that the a- and b-constants achieve a maximum as we vary over subvarieties of X. The behavior of the a- and b-constants in families was used in [Lehmann and Tanimoto 2017] to show this necessary condition. The a- and b-constants also play a role in determining and counting the dominant components of the space $\mathrm{Mor}(\mathbb{P}^1, X)$ of morphisms from $\mathbb{P}^1$ to a smooth Fano variety X (see [Lehmann and Tanimoto 2019] for details).

The ideas in proving our results are as follows. To prove Theorem 1.1, we analyze the behavior of the b-constant under specialization and combine this with the constancy of the Picard rank and the a-constant in very general fibers to obtain the desired conclusion. The key step for Theorem 1.2 is to prove constancy on closed points when $k = \mathbb{C}$. We run a $(K_X + aL)$-MMP over the base T, to obtain a relative minimal model $X \dashrightarrow X'$ where $a = a(X_t, L|_{X_t})$. We pass to a relative canonical model $\phi : X \dashrightarrow Z$ over T and base change to $t \in T$, to obtain $\phi_t : X_t \dashrightarrow Z_t$ as the canonical model for (X_t, aL_{X_t}). Using a version of the global invariant cycles theorem (see Lemma 2.11), we observe that $b(X_t, L_t)$ is same as the rank of the monodromy invariant subspace of $N^1(Y'_z)_{\mathbb{R}}$, where Y'_z is a general fiber of $X'_t \to Z_t$. Then using topological local triviality of algebraic morphisms we conclude that the monodromy invariant subspace has constant rank.

The outline of the paper is as follows. In Section 2 we discuss the preliminaries. In Section 3 and 4 we prove Theorems 1.1 and 1.2 respectively.

2. Preliminaries

In this paper we always work in characteristic 0.

Néron–Severi group. Let X be a smooth proper variety over a field k. The Néron–Severi group $\mathrm{NS}(X)$ is defined as the quotient of the group of Weil divisors, $\mathrm{Cl}(X)$, modulo algebraic equivalence. We denote $N^1(X) = \mathrm{Div}(X)/\equiv$, the quotient of Cartier divisors by numerical equivalence. We denote $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes \mathbb{R}$ and similarly $N^1(X)_{\mathbb{R}}$. By [Néron 1952], $\mathrm{NS}(X)_{\mathbb{R}}$ is a finite-dimensional vector space and its rank $\rho(X)$ is called the Picard rank. If X is a smooth projective variety, then $\mathrm{NS}(X)_{\mathbb{R}} \cong N^1(X)_{\mathbb{R}}$.

Remark 2.1. Let X be a smooth variety over an algebraically closed field k. If $k \subset k'$ is an extension of algebraically closed fields, then the natural homomorphism $\mathrm{NS}(X) \to \mathrm{NS}(X_{k'})$ is an isomorphism. So the Picard rank is unchanged under base extension of algebraically closed fields.

Let $X \to T$ be a smooth proper morphism of irreducible varieties. Suppose $s, t \in T$ such that s is a specialization of t, i.e., s is in the closure of $\{t\}$. Let $X_{\bar{t}}$ denote the base change to the algebraic closure of the residue field $k(t)$.

Proposition 2.2 [Maulik and Poonen 2012, Proposition 3.6]. *In the situation above, it is possible to choose a specialization homomorphism*

$$\mathrm{sp}_{\bar{t},\bar{s}} : \mathrm{NS}(X_{\bar{t}}) \to \mathrm{NS}(X_{\bar{s}})$$

such that:

(a) $\mathrm{sp}_{\bar{t},\bar{s}}$ *is injective. In particular* $\rho(X_{\bar{s}}) \geq \rho(X_{\bar{t}})$.

(b) *If* $\mathrm{sp}_{\bar{t},\bar{s}}$ *maps a class* $[L]$ *to an ample class, then* L *is ample.*

If $\rho(X_{\bar{s}}) = \rho(X_{\bar{t}})$, then the homomorphism $\mathrm{NS}(X_{\bar{t}})_{\mathbb{R}} \to \mathrm{NS}(X_{\bar{s}})_{\mathbb{R}}$ is an isomorphism.

Let $X \to T$ be a smooth projective morphism of irreducible varieties over $\mathbb{C}$. In Section 12 of [Kollár and Mori 1992], the local system $\mathcal{GN}^1(X/T)$ was introduced. This is a sheaf in the analytic topology defined as

$$\mathcal{GN}^1(X/T)(U) = \{\text{sections of } \mathcal{N}^1(X/T) \text{ over } U \text{ with open support}\}$$

for analytic open $U \subset T$, and the functor $\mathcal{N}^1(X/T)$ is defined as $N^1(X \times_T T')$ for any $T' \to T$. It was shown in [Kollár and Mori 1992, 12.2] that $\mathcal{GN}^1(X/T)$ is a local system with finite monodromy and $\mathcal{GN}^1(X/T)|_t = N^1(X_t)$ for very general $t \in T$. We can base change to a finite étale cover of $T' \to T$ so that $\mathcal{GN}^1(X'/T')$ has trivial monodromy. Then we have a natural identification of the fibers of $\mathcal{GN}^1(X'/T')$ and $N^1(X'/T')$. Therefore, for $t' \in T'$ very general, the natural map $N^1(X'/T') \to N^1(X'_{t'})$ is an isomorphism. One can prove the same results over any algebraically closed field of characteristic 0, by using the Lefschetz principle.

Geometric invariants. The pseudoeffective cone $\Lambda_{\mathrm{eff}}(X)$ is the closure of the cone of effective divisor classes in $\mathrm{NS}(X)_{\mathbb{R}}$. The interior of $\Lambda_{\mathrm{eff}}(X)$ is the cone of big divisors $\mathrm{Big}^1(X)_{\mathbb{R}}$.

Definition 2.3. Let L be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$ divisor on X. The a-constant is

$$a(X, L) = \min\{t \in \mathbb{R} \mid K_X + tL \in \Lambda_{\mathrm{eff}}(X)\}.$$

For a singular projective variety we define $a(X, L) := a(\tilde{X}, \pi^*L)$ where $\pi : \tilde{X} \to X$ is a resolution of X. It is invariant under pull-back by a birational morphism of smooth varieties and hence independent of the choice of the resolution. By [Boucksom et al. 2013] we know that $a(X, L) > 0$ if and only if X is uniruled. We note that, by flat base change, the a-constant is independent of base change to another field.

It was shown in [Birkar et al. 2010] that, if X is uniruled with klt singularities and L is ample, then $a(X, L)$ is a rational number. If L is big and not ample, then $a(X, L)$ can be irrational (see [Hassett et al. 2015, Example 6]). For a smooth projective variety X, the function $a(X, \_) : \mathrm{Big}^1(X)_{\mathbb{R}} \to \mathbb{R}$ is a continuous function (see [Lehmann et al. 2018, Lemma 3.2]).

Definition 2.4. A morphism $f : X \to T$ of irreducible varieties is called a family of varieties if the generic fiber is geometrically integral. A family of projective varieties is a projective morphism which is a family of varieties.

We recall the following result about the a-constant in families:

Theorem 2.5 [Lehmann and Tanimoto 2017; Hacon et al. 2013]. *Let $f : X \to T$ be a smooth family of uniruled projective varieties over an algebraically closed field. Let L be an f-big and f-nef $\mathbb{Q}$-Cartier divisor on X. Then there exists a nonempty subset $U \subset T$ such that $a(X_t, L|_{X_t})$ is constant for $t \in U$ and the Iitaka dimension $\kappa(K_{X_t} + a(X_t, L|_{X_t})L|_{X_t})$ is constant for $t \in U$.*

Definition 2.6. Let X be a smooth projective variety over k and L a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The b-constant is defined as

$$b(k, X, L) = \text{codim of minimal supported face of } \Lambda_{\mathrm{eff}}(X) \text{ containing the class of } K_X + a(X, L)L.$$

It is invariant under pullback by a birational morphism of smooth varieties [Hassett et al. 2015]. For a singular variety X we define $b(k, X, L) := b(k, \tilde{X}, \pi^*L)$, by pulling back to a resolution. By Remark 2.1, if we have an extension $k \subset k'$ of algebraically closed fields, the pull back map $\mathrm{NS}(X) \to \mathrm{NS}(X_{k'})$ is an isomorphism and the pseudoeffective cones are isomorphic by flat base change. Also, $K_X + a(X, L)L$ maps to $K_{X_{k'}} + a(X_{k'}, L_{k'})L_{k'}$ under this isomorphism. Therefore the b-constant is unchanged, i.e.,

$b(k', X_{k'}, L_{k'}) = b(k, X, L)$. From now on, when our base field is algebraically closed we write $b(X, L)$ instead of $b(k, X, L)$.

Minimal and canonical models. Let (X, Δ) be a klt pair, with Δ a $\mathbb{R}$-divisor and $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : X \to T$ be a projective morphism. A pair (X', Δ') sitting in a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & T & \end{array}$$

is called a $\mathbb{Q}$-factorial minimal model of (X, Δ) over T if:

(1) X' is $\mathbb{Q}$-factorial.

(2) f' is projective.

(3) ϕ is a birational contraction.

(4) $\Delta' = \phi_* \Delta$.

(5) $K_{X'} + \Delta'$ is f'-nef.

(6) $a(E, X, \Delta) < a(E, X', \Delta')$ for all ϕ-exceptional divisors $E \subset X$. Equivalently, if for a common resolution $p : W \to X$ and $q : W \to X'$, we may write

$$p^*(K_X + \Delta) = q^*(K_{X'} + \Delta') + E$$

where $E \geq 0$ is q-exceptional and the support of E contains the strict transform of the ϕ-exceptional divisors.

A canonical model over T is defined to be a projective morphism $g : Z \to T$ with a surjective morphism $\pi : X' \to Z$ with connected geometric fibers from a minimal model such that $K_{X'} + \Delta' = \pi^* H$ for an $\mathbb{R}$-Cartier divisor H on Z which is ample over T.

Suppose $K_X + \Delta$ is f-pseudoeffective and Δ is f-big, then by [Birkar et al. 2010], we may run a $(K_X + \Delta)$-MMP with scaling to obtain a $\mathbb{Q}$-factorial minimal model (X', Δ') over T. It follows that (X', Δ') is also klt. Then the basepoint freeness theorem implies that $(K_{X'} + \Delta')$ is f'-semiample. Hence there exists a relative canonical model $g : Z \to T$. In particular, if Δ is a $\mathbb{Q}$-divisor, the $\mathcal{O}_T$-algebra

$$\mathfrak{R}(X', \Delta') = \bigoplus_m f'_* \mathcal{O}_{X'}(\lfloor m(K_{X'} + \Delta') \rfloor)$$

is finitely generated. Let $X' \to Z \to \mathrm{Proj}_T(\mathfrak{R}(X', \Delta'))$ be the Stein factorization of the natural morphism. Then Z is the relative canonical model over T.

The following result relates the relative MMP over a base to the MMP of the fibers (see [de Fernex and Hacon 2011, Theorem 4.1; Kollár and Mori 1992, 12.3] for related statements).

Lemma 2.7. *Let $f : X \to T$ be a flat projective morphism of normal varieties. Suppose X is $\mathbb{Q}$-factorial and D be an effective $\mathbb{R}$-divisor such that (X, D) is klt. Let $\psi : X \to Z$ be the contraction of a $K_X + D$-negative extremal ray of $\overline{\mathrm{NE}}(X/T)$. Suppose for $t \in T$ very general, the restriction map $N^1(X/T) \to N^1(X_t)$ is surjective and X_t is $\mathbb{Q}$-factorial.*

Let $t \in T$ be very general. If $\psi_t : X_t \to Z_t$ is not an isomorphism, then it is a contraction of a $K_{X_t} + D_t$-negative extremal ray, and:

(a) *If ψ is of fiber type, so is ψ_t.*

(b) *If ψ is a divisorial contraction of a divisor G, then ψ_t is a divisorial contraction of G_t and $N^1(Z/T) \to N^1(Z_t)$ is surjective.*

(c) *If ψ is a flipping contraction and $\psi^+ : X^+ \to Z$ is the flip, then ψ_t is a flipping contraction and X_t^+ is the flip of $\psi_t : X_t \to Z_t$. Also, $N^1(X^+/T) \to N^1(X_t^+)$ is surjective.*

Proof. Since the natural restriction map $N^1(X/T) \to N^1(X_t)$ is surjective for very general $t \in T$, any curve in X_t that spans a $K_X + D$-negative extremal ray R of $\overline{\mathrm{NE}}(X/T)$, also spans a $K_{X_t} + D_t$ negative extremal ray R_t of $\overline{\mathrm{NE}}(X_t)$. For $t \in T$ general, the base change Z_t is normal and the morphism $X_t \to Z_t$ has connected fibers, hence $\psi_{t*}\mathcal{O}_{X_t} = \mathcal{O}_{Z_t}$. Hence ψ_t is the contraction of the ray R_t for very general $t \in T$.

If ψ is of fiber type, then so is ψ_t for general $t \in T$. Let us assume that ψ is birational.

Suppose ψ is a divisorial contraction of a divisor G. Then all components of G_t are contracted. By the injectivity of $N_1(X_t) \to N_1(X/T)$, we see that ψ_t is an extremal divisorial contraction of G_t (and G_t is irreducible). Since X_t is $\mathbb{Q}$-factorial, we have the surjectivity of $N^1(Z/T) \to N^1(Z_t)$.

Suppose ψ is a flipping contraction and $\phi : X \dashrightarrow X^+$ is the flip. For very general $t \in T$, $X_t \to Z_t$ is a small birational contraction of the ray R_t. Also, $X_t^+ \to Z_t$ is also small birational and $K_{X_t^+} + (\phi_* D)_t$ is ψ^+-ample for $t \in T$ general. Therefore $\phi_t : X_t \dashrightarrow X_t^+$ is the flip. The surjectivity of $N^1(X^+/T) \to N^1(X_t^+)$ follows from ψ_t being an isomorphism in codimension one. ☐

The next proposition allows us to compare minimal and canonical models over a base to those of a general fiber.

Proposition 2.8. *Let $f : X \to T$ be a smooth morphism. Suppose X is smooth and Δ is an f-big and f-nef $\mathbb{R}$-divisor such that (X, Δ) is klt. Suppose the local system $\mathcal{GN}^1(X/T)$ has trivial monodromy. Let $\phi : X \dashrightarrow X'$ be the relative minimal model obtained by running a $(K_X + \Delta)$-MMP over T and $\pi : X' \to Z$ be the morphism to the canonical model over T. Then for a general $t \in T$:*

(1) *The base change $\phi_t : X_t \dashrightarrow X'_t$ is a $\mathbb{Q}$-factorial minimal model of (X_t, Δ_t).*

(2) *Also, $\pi_t : X'_t \to Z_t$ is the canonical model of (X_t, Δ_t).*

Proof. (1) Since $\mathcal{GN}^1(X/T)$ has trivial monodromy, the natural restriction morphism $N^1(X/T) \xrightarrow{\sim} N^1(X_t)$ is an isomorphism for $t \in T$ very general. Then Lemma 2.7 implies that, for very general $t \in T$, the base change $\phi_t : X_t \dashrightarrow X'_t$ is a composition of steps of the $(K_{X_t} + \Delta_t)$-MMP. In particular, X'_t is $\mathbb{Q}$-factorial for a very general $t \in T$. The fibers X'_t have terminal singularities, by [Lehmann et al. 2018, Lemma 2.4]. Hence [Kollár and Mori 1992, 12.1.10] implies that there is a nonempty open $U \subset T$ such that X'_t is $\mathbb{Q}$-factorial for $t \in U$. For a general $t \in T$, the conditions (2)–(6) in the definition of a minimal model follows easily. Therefore, (X'_t, Δ'_t) is a $\mathbb{Q}$-factorial minimal model of (X_t, Δ_t) for general $t \in T$.

(2) Let $g : Z \to T$ be the relative canonical model. Now Z is normal. Therefore, for a general $t \in T$, the base change Z_t is normal and $X'_t \to Z_t$ has geometrically connected fibers. Also, $K_{X'} + \Delta = g^* H$ where H is a π-ample $\mathbb{R}$-Cartier divisor on Z. By adjunction, $K_{X'_t} + \Delta'_t$ is pull-back of an ample $\mathbb{R}$-Cartier divisor on Z_t. Hence, $X'_t \to Z_t$ is the canonical model for general $t \in T$. □

Let X be a smooth uniruled projective variety over an algebraically closed field and L a big and nef $\mathbb{Q}$-divisor on X. The following result (contained in [Lehmann et al. 2018]) gives a geometric interpretation of the b-constant.

Proposition 2.9. *Let $\phi : X \dashrightarrow X'$ be a $K_X + a(X, L)L$-minimal model. Then*:

(1) $b(X, L) = b(X', \phi_* L)$.

(2) *If* $\kappa(K_X + a(X, L)L) = 0$ *then* $b(X, L) = \operatorname{rk} N^1(X')_{\mathbb{R}}$.

(3) *If* $\kappa(K_X + a(X, L)L) > 0$ *and* $\pi : X' \to Z$ *is the morphism to the canonical model and* Y' *is a general fiber of* π. *Then*

$$b(X, L) = \operatorname{rk} N^1(X')_{\mathbb{R}} - \operatorname{rk} N^1_\pi(X')_{\mathbb{R}} = \operatorname{rk}(\operatorname{im}(N^1(X')_{\mathbb{R}} \to N^1(Y')_{\mathbb{R}}))$$

where $N^1_\pi(X')_{\mathbb{R}}$ *is the span of the* π*-vertical divisors and* $N^1(X')_{\mathbb{R}} \to N^1(Y')_{\mathbb{R}}$ *is the restriction map.*

Proof. Part (1) is the statement of Lemma 3.5 in [Lehmann et al. 2018]. Part (2) follows from part (1). By abundance, $K_X + a(X, L)\phi_* L$ is semiample. Then $\kappa(K_X + a(X, L)L) = 0$ implies that $K_X + a(X, L)\phi_* L \equiv 0$. Hence, $b(X, L) = b(X', \phi_* L) = \operatorname{rk} N^1(X')_{\mathbb{R}}$. Part (3) follows from the proof of Theorem 4.5 in [Lehmann et al. 2018]. □

In the case when the fibers are adjoint-rigid, constancy of the b-constant was proved in [Lehmann and Tanimoto 2017].

Proposition 2.10 [Lehmann and Tanimoto 2017, Proposition 4.4]. *Let $f : X \to T$ be a smooth family of projective varieties. Suppose L is an f-big and f-nef Cartier divisor on X. Assume that for a general member X_t, we have $\kappa(K_{X_t} + a(X_t, L_t)L_t) = 0$. Then $b(X_t, L_t)$ is constant for general $t \in T$.*

Global invariant cycles. Let $\pi : X \to Z$ be a morphism of complex algebraic varieties. Then, by Verdier's generalization of Ehresmann's theorem [Verdier 1976, Corolaire 5.1], there exists a Zariski open $U \subset Z$ such that $\pi^{-1}(U) \to U$ is a topologically locally trivial fibration (in the analytic topology), i.e., every point $z \in U$ has a neighborhood $N \subset U$ in the analytic topology, such that there is a fiber preserving homeomorphism

$$\begin{array}{ccc} \pi^{-1}(N) & \xrightarrow{\sim} & N \times F \\ & \searrow \quad \swarrow & \\ & N & \end{array}$$

where $F = \pi^{-1}(z)$. Consequently we have a monodromy action of $\pi_1(U, z)$ on the cohomology of the fiber $H^i(X_z, \mathbb{R})$.

Let $\pi : X \to Z$ be a morphism of normal projective varieties. Note that by generic smoothness and the discussion above, given any resolution of singularities $\mu : \tilde{X} \to X$, we may choose a Zariski open $U \subset Z$ such that $\pi \circ \mu$ is smooth over U and $(\pi \circ \mu)^{-1}(U) \to U$ and $\pi^{-1}(U) \to U$ are topologically locally trivial fibrations.

The following result is an adaptation of Deligne's global invariant cycles theorem [1971] to the case of singular varieties, which helps us to compute the b-constant.

Lemma 2.11. *Let $\pi : X \to Z$ be a morphism of normal projective varieties over $\mathbb{C}$ where X is $\mathbb{Q}$-factorial. Let $\mu : \tilde{X} \to X$ be a resolution of singularities. Let $U \subset Z$ be a Zariski open subset such that $\pi \circ \mu$ is smooth over U and $(\pi \circ \mu)^{-1}(U) \to U$ and $\pi^{-1}(U) \to U$ are topologically locally trivial fibrations (in the analytic topology). Suppose for general $z \in U$, the fiber $X_z := \pi^{-1}(z)$ is rationally connected with rational singularities. Then*

$$\operatorname{im}(N^1(X)_{\mathbb{R}} \to N^1(X_z)_{\mathbb{R}}) \simeq H^2(X_z, \mathbb{R})^{\pi_1(U,z)}$$

for general $z \in U$, where $H^2(X_z, \mathbb{R})^{\pi_1(U,z)}$ is the monodromy invariant subspace.

Proof. Let $\tilde{X}_z$ be the fiber of $\pi \circ \mu$ over z. For $z \in U$ general, $\mu_z : \tilde{X}_z \to X_z$ is a resolution of singularities. Since X_z is rationally connected, $\mathbb{Q}$-linear equivalence and numerical equivalence of $\mathbb{Q}$-Cartier divisors coincide, i.e., $\operatorname{Pic}(X_z)_{\mathbb{Q}} \simeq N^1(X_z)_{\mathbb{Q}}$. We know $h^1(\tilde{X}_z, \mathcal{O}_{\tilde{X}_z}) = h^2(\tilde{X}_z, \mathcal{O}_{\tilde{X}_z}) = 0$ since $\tilde{X}_z$ is smooth rationally connected. We also have $h^1(X_z, \mathcal{O}_{X_z}) = h^2(X_z, \mathcal{O}_{X_z}) = 0$, because X_z has rational singularities. Therefore $H^2(\tilde{X}_z, \mathbb{Q}) \simeq N^1(\tilde{X}_z)_{\mathbb{Q}}$ and $H^2(X_z, \mathbb{Q}) \simeq N^1(X_z)_{\mathbb{Q}}$.

Consider the natural restriction map on cohomology groups $H^2(\tilde{X}, \mathbb{Q}) \to H^2(\tilde{X}_z, \mathbb{Q})$. By Deligne's global invariant cycles theorem [1971] (or [Voisin 2003, 4.3.3]) we know that for $z \in U$,

$$\operatorname{im}(H^2(\tilde{X}, \mathbb{Q}) \to (H^2(\tilde{X}_z, \mathbb{Q})) = H^2(\tilde{X}_z, \mathbb{Q})^{\pi_1(U,z)}.$$

and if $\alpha \in H^2(\tilde{X}_z, \mathbb{Q})^{\pi_1(U,z)}$ is a Hodge class then there is a Hodge class $\tilde{\alpha} \in H^2(\tilde{X}, \mathbb{Q})$ such that $\tilde{\alpha}$ restricts to α. Since $H^2(\tilde{X}_z, \mathbb{Q}) \simeq N^1(\tilde{X}_z)_{\mathbb{Q}}$, we see that

$$\operatorname{im}(H^2(\tilde{X}, \mathbb{Q}) \to H^2(\tilde{X}_z, \mathbb{Q})) \simeq \operatorname{im}(N^1(\tilde{X})_{\mathbb{Q}} \to N^1(\tilde{X}_z)_{\mathbb{Q}})$$

for $z \in U$. In particular

$$\operatorname{im}(N^1(\tilde{X})_{\mathbb{R}} \to N^1(\tilde{X}_z)_{\mathbb{R}}) \simeq H^2(\tilde{X}_z, \mathbb{R})^{\pi_1(U,z)}$$

for $z \in U$.

Now the following diagram of pull-back morphisms commutes

$$\begin{array}{ccc} N^1(X)_{\mathbb{R}} & \xrightarrow{i^*} & N^1(X_z)_{\mathbb{R}} \\ \downarrow{\scriptstyle \mu^*} & & \downarrow{\scriptstyle \mu_z^*} \\ N^1(\tilde{X})_{\mathbb{R}} & \xrightarrow{\tilde{i}^*} & N^1(\tilde{X}_z)_{\mathbb{R}} \end{array}$$

Since $\mu : \tilde{X} \to X$ and $\mu_z : \tilde{X}_z \to X_z$ are resolutions of singularities for general $z \in U$, the vertical morphisms are injective. Therefore

$$\operatorname{im}(i^*) \simeq \operatorname{im}(\mu_z^* \circ i^*) = \operatorname{im}(\tilde{i}^* \circ \mu^*)$$

Since X is $\mathbb{Q}$-factorial, we have $N^1(\tilde{X})_{\mathbb{R}} \simeq \mu^* N^1(X)_{\mathbb{R}} \oplus \bigoplus_j \mathbb{R}E_j$ where E_j are the μ-exceptional divisors. For $z \in U$ general, the restriction of a μ-exceptional divisor E_j to $\tilde{X}_z$ is μ_z-exceptional. In $N^1(\tilde{X}_z)_{\mathbb{R}}$, we have $\operatorname{im}(\mu_z^*) \cap \oplus_j \mathbb{R}E_j^z = 0$ where E_j^z are μ_z-exceptional. Therefore

$$\operatorname{im}(\tilde{i}^* \circ \mu^*) = \operatorname{im}(\tilde{i}^*) \cap \operatorname{im}(\mu_z^*).$$

Recall that we have the isomorphisms given by first Chern class $N^1(\tilde{X}_z)_{\mathbb{R}} \simeq H^2(\tilde{X}_z, \mathbb{R})$ and $N^1(X_z)_{\mathbb{R}} \simeq H^2(X_z, \mathbb{R})$. We know that $\operatorname{im}(\tilde{i}^*) \simeq H^2(\tilde{X}_z, \mathbb{R})^{\pi_1(U,z)}$ and the monodromy actions on $H^2(X_z, \mathbb{R})$ and $H^2(\tilde{X}_z, \mathbb{R})$ commute with the pullback map μ_z^*. Hence

$$\operatorname{im}(\tilde{i}^*) \cap \operatorname{im}(\mu_z^*) \simeq H^2(X_z, \mathbb{R})^{\pi_1(U,z)}.$$

Therefore

$$\operatorname{im}(N^1(X)_{\mathbb{R}} \to N^1(X_z)_{\mathbb{R}}) = \operatorname{im}(\tilde{i}^*) \cap \operatorname{im}(\mu_z^*) \simeq H^2(X_z, \mathbb{R})^{\pi_1(U,z)}$$

for general $z \in U$. □

3. Constancy on very general fibers

Let $f : X \to T$ be a projective morphism and L is an f-big $\mathbb{Q}$-Cartier divisor. We denote $L_{\bar{t}} := L|_{X_{\bar{t}}}$, the restriction to the geometric fiber of t.

Lemma 3.1. *Let $X \to T$ be a smooth projective family of varieties and $s, t \in T$ such that s is a specialization of t:*

(a) *$\Lambda_{\mathrm{eff}}(X_{\bar{t}})$ maps into $\Lambda_{\mathrm{eff}}(X_{\bar{s}})$ under the specialization morphism $\mathrm{sp}_{\bar{t},\bar{s}} : \mathrm{NS}_{\mathbb{R}}(X_{\bar{t}}) \to \mathrm{NS}_{\mathbb{R}}(X_{\bar{s}})$.*

(b) *Suppose $a(X_{\bar{t}}, L_{\bar{t}}) = a(X_{\bar{s}}, L_{\bar{s}})$ and $\rho(X_{\bar{t}}) = \rho(X_{\bar{s}})$. Then $b(X_{\bar{t}}, L_{\bar{t}}) \geq b(X_{\bar{s}}, L_{\bar{s}})$.*

Proof. (a) Let D be an effective divisor in $\mathrm{NS}(X_{\bar{t}})_{\mathbb{R}}$. We may pick a discrete valuation ring R with a morphism $\phi : \operatorname{Spec} R = \{s', t'\} \to T$ where s' and t' map to s and t respectively and t' is the generic point. By Remark 2.1 we have isomorphisms $\mathrm{NS}(X_{\bar{t}}) \xrightarrow{\sim} \mathrm{NS}(X_{\bar{t'}})$ and $\mathrm{NS}(X_{\bar{s}}) \xrightarrow{\sim} \mathrm{NS}(X_{\bar{s'}})$. Therefore we may assume T is the spectrum of a discrete valuation ring R and t is the generic point t'. Now D is defined over a finite extension L of $k(t')$. We can replace R by a discrete valuation ring R_L with quotient field L. Then the image of D under $\operatorname{Pic}(X_{t'}) \xrightarrow{\sim} \operatorname{Pic}(\phi^* X) \to \operatorname{Pic}(X_{s'})$ is effective by semicontinuity.

After passing to the algebraic closure and taking quotient by algebraic equivalence we conclude that, $\mathrm{sp}_{\bar{t},\bar{s}}$ maps D to an effective divisor class.

(b) Since $\rho(X_{\bar{t}}) = \rho(X_{\bar{s}})$, we have an isomorphism $\mathrm{NS}(X_{\bar{t}})_{\mathbb{R}} \to \mathrm{NS}(X_{\bar{s}})_{\mathbb{R}}$. Let $a := a(X_{\bar{s}}, L_{\bar{s}}) = a(X_{\bar{t}}, L_{\bar{t}})$. Note that $\mathrm{sp}_{\bar{t},\bar{s}}$ maps $K_{X_{\bar{t}}} + aL_{\bar{t}}$ to $K_{X_{\bar{s}}} + aL_{\bar{s}}$. Let F be a supporting hyperplane of $\Lambda_{\mathrm{eff}}(X_{\bar{s}})$ corresponding to the minimal supporting face containing $K_{X_{\bar{s}}} + aL_{\bar{s}}$. Since $\Lambda_{\mathrm{eff}}(X_{\bar{t}}) \subset \Lambda_{\mathrm{eff}}(X_{\bar{s}})$, we see that F is a supporting hyperplane of $\Lambda_{\mathrm{eff}}(X_{\bar{t}})$ containing $K_{X_{\bar{t}}} + aL_{\bar{t}}$. Therefore,

$$b(X_{\bar{s}}, L_{\bar{s}}) = \mathrm{codim}(F \cap \Lambda_{\mathrm{eff}}(X_{\bar{s}})) \leq \mathrm{codim}(F \cap \Lambda_{\mathrm{eff}}(X_{\bar{t}})) \leq b(X_{\bar{t}}, L_{\bar{t}}). \qquad \square$$

Lemma 3.2. *Let $X \to T$ a smooth projective family. Let $\eta \in T$ be the generic point. We denote $a = a(X_{\bar{\eta}}, L_{\bar{\eta}})$, $n = \rho(X_{\bar{\eta}})$ and $b = b(X_{\bar{\eta}}, L_{\bar{\eta}})$. For $m \in \mathbb{N}$, define*

$$T_m := \left\{ t \in T \mid a(X_{\bar{t}}, L_{\bar{t}}) \leq a - \tfrac{1}{m} \right\}, \quad T_0 := \{ t \in T \mid \rho(X_{\bar{t}}) > n \}$$

and

$$T_\infty := \{ t \in T \mid a(X_{\bar{t}}, L_{\bar{t}}) = a, \rho(X_{\bar{t}}) = n, b(X_{\bar{t}}, L_{\bar{t}}) < b \}.$$

We let $Z_T := \bigcup_m T_m \cup T_\infty \cup T_0$. Then:

(a) *Z_T is closed under specialization.*

(b) *If we base change by a morphism of schemes $g : T' \to T$, then $Z_{T'} = g^{-1}(Z_T)$.*

Proof. (a) Let $t \in Z_T$ and s a specialization of t in T. If $t \in T_m$ for some $m \in \mathbb{N}$, then Lemma 3.1(a) implies that $K_{X_{\bar{s}}} + a(X_{\bar{t}}, L_{\bar{t}})L_{\bar{s}} \in \Lambda_{\mathrm{eff}}(X_{\bar{s}})$. Therefore, $a(X_{\bar{s}}, L_{\bar{s}}) \leq a(X_{\bar{t}}, L_{\bar{t}})$ and hence $s \in T_m$. If $t \in T_0$, then by Proposition 2.2(a), $\rho(X_{\bar{s}}) \geq \rho(X_{\bar{t}})$ and $s \in T_0$. If $t \notin T_0 \cup \bigcup_m T_m$, then $\rho(X_{\bar{t}}) = n$ and $a(X_{\bar{t}}, L_{\bar{t}}) = a$. Then Lemma 3.1(b) implies $b(X_{\bar{s}}, L_{\bar{s}}) \leq b(X_{\bar{t}}, L_{\bar{t}}) < b$. Therefore $s \in T_\infty$ and Z_T is closed under specialization.

(b) This follows from the fact that the Picard number and a- and b-constants are invariant under algebraically closed base extension. $\square$

Proof of Theorem 1.1. By passing to a resolution of singularities and using generic smoothness, we may exclude a closed subset of T to assume the family $f : X \to T$ is smooth and T is affine. Since our base field k is algebraically closed, we may find a subfield $k' \subset k$ which is the algebraic closure of a field finitely generated over $\mathbb{Q}$, and there exists a finitely generated k'-algebra A such that our family $X \to T$ and L are a base change of a family $X_A \to \mathrm{Spec}\, A$ and a line bundle L_A on X_A. Now $B = \mathrm{Spec}\, A$ is countable and hence $Z_B = \bigcup_{b \in B} \overline{\{b\}}$ is a countable union of closed subsets by Lemma 3.2(a). Now Lemma 3.2(b) implies that Z_T is a countable union of closed subsets. $\square$

4. Family of uniruled varieties

In this section we prove Theorem 1.2 Let $f : X \to T$ be a projective family of uniruled varieties over an algebraically closed field k of characteristic 0 and L an f-nef and f-big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

By a standard argument using the Lefschetz principle, it is enough to prove the statement for $k = \mathbb{C}$. We will henceforth assume that $k = \mathbb{C}$.

We can reduce to the statement for closed points only, as follows. Let us assume that there is an open $U \subset T$ such that $b(X_t, L_t) = b$ is constant for all closed points $t \in U$. Let $s \in U$ and $Z = \overline{\{s\}} \cap U$. By applying Theorem 1.1 to the family over Z, we may find $F = \bigcup_i F_i \subset Z$ a countable union of closed subvarieties such that $b(X_{\bar{t}}, L_{\bar{t}})$ is constant on $Z \setminus F$. Since $\mathbb{C}$ is uncountable, there exists a closed point $t \in Z \setminus F$. Now $s \in Z \setminus F$, since s is the generic point of Z. Therefore, $b(X_{\bar{s}}, L_{\bar{s}}) = b(X_t, L_t) = b$. Since $s \in U$ was arbitrary, we conclude that $b(X_{\bar{t}}, L_{\bar{t}}) = b$ for all $t \in U$. Therefore it is enough to prove the statement for closed points.

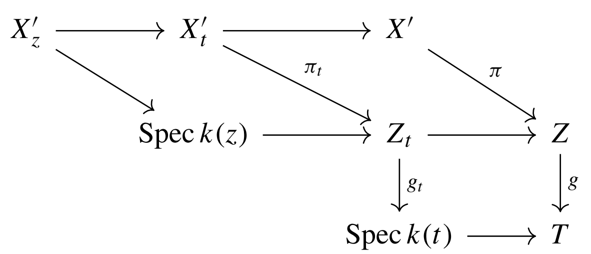
Proof of Theorem 1.2 for closed points when k $= \mathbb{C}$. We may replace X by a resolution, and by generic smoothness, we may exclude a closed subset of the base to assume that $f : X \to T$ is a smooth family. By Theorem 2.5, we can shrink T such that $a(X_t, L_t) = a$ for all $t \in T$ and $\kappa(K_{X_t} + aL_t)$ is independent of t. We may assume that T is affine. Since L is f-big and f-nef, we can replace L by a $\mathbb{Q}$-linearly equivalent divisor to assume that (X, aL) is klt.

Since the local system $\mathcal{GN}^1(X/T)$ has finite monodromy, we can base change to a finite étale cover of T to assume that $\mathcal{GN}^1(X/T)$ has trivial monodromy.

If $\kappa(K_{X_t} + aL_t) = 0$ then we can conclude by Proposition 2.10. Let us assume that $\kappa(K_{X_t} + aL_t) = k > 0$ for all $t \in T$.

Since $K_X + aL$ is f-pseudoeffective and aL is f-big, we may run a $(K_X + aL)$-MMP over T to obtain a relative minimal model $\phi : X \dashrightarrow X'$. Let $\pi : X' \to Z$ be the morphism to the relative canonical model over T. By Proposition 2.8, we may replace T by an open subset to assume that the base change $\phi_t : X_t \dashrightarrow X'_t$ is a $\mathbb{Q}$-factorial minimal model and $\pi_t : X'_t \to Z_t$ is the canonical model for (X_t, aL_t) for all $t \in T$.

For $z \in Z$, we denote the image of z in T by t and let X'_z denote the fiber of $\pi : X' \to Z$ over z.



Let $\mu : \tilde{X} \to X'$ be a resolution of singularities. We may replace T by an open subset to assume that $\tilde{X} \to T$ is smooth. Let $\tilde{X}_z$ be the fiber of $\tilde{\pi} : \tilde{X} \to Z$ over $z \in Z$. By [Verdier 1976, Corrolaire 5.1] we can find a Zariski open $U_Z \subset Z$ such that $\tilde{\pi}$ is smooth over U_Z and $\tilde{\pi}^{-1}(U_Z) \to U_Z$ and $\pi^{-1}(U_Z) \to U_Z$ both are topologically locally trivial fibrations (in the analytic topology). Again we may replace T by a Zariski open $V \subset T$ to assume that $U_Z \to T$ is a topologically locally trivial fibration (in the analytic topology). Let $U_t \subset Z_t$ denote the fiber of U_Z over $t \in T$.

For all $z \in U_Z$, there is a monodromy action of $\pi_1(U_t, z)$ on $H^2(X'_z, \mathbb{Z})$ acting by an integral matrix M_z on the free part. Now for any two points z and z' in U_Z, the fundamental groups $\pi_1(U_t, z)$ and $\pi_1(U_{t'}, z')$ are isomorphic, since $U_Z \to T$ is a locally trivial fibration. Also, the cohomology groups $H^2(X'_z, \mathbb{Z})$ and $H^2(X'_{z'}, \mathbb{Z})$ are isomorphic, because $\pi^{-1}(U_Z) \to U_Z$ is a locally trivial fibration. Since the monodromy actions depend continuously on $z \in U_Z$, we see that M_z is constant. Therefore the monodromy invariant subspaces have constant rank, i.e., $\mathrm{rk}\, H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}$ is constant for all $z \in U_Z$.

By [Hacon and McKernan 2007] we know that a general fiber X'_z is rationally connected and has terminal singularities. Since X'_t is $\mathbb{Q}$-factorial, Lemma 2.11 implies that

$$\mathrm{rk}(\mathrm{im}(N^1(X'_t)_{\mathbb{R}} \to N^1(X'_z)_{\mathbb{R}}) = \mathrm{rk}\, H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}.$$

for general $z \in U_t$. Now using Proposition 2.9(3) we have

$$b(X_t, L_t) = \mathrm{rk}\, H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}$$

for general $z \in U_Z$. Since $\mathrm{rk}\, H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}$ is constant for $z \in U_Z$, we may conclude that $b(X_t, L_t)$ is constant for general $t \in T$.

Acknowledgements

I am very grateful to my advisor Professor János Kollár for fruitful discussions and his constant support and encouragement. I am very thankful to Professor Brian Lehmann and Professor Sho Tanimoto for useful comments on a preliminary version of the paper. I would like to thank Sebastián Olano and Ziquan Zhuang for interesting conversations. I would like to thank the referee for a careful reading of the paper and for useful comments. Partial financial support was provided by the NSF under grant number DMS-1502236.

References

[Batyrev and Manin 1990] V. V. Batyrev and Y. I. Manin, "Sur le nombre des points rationnels de hauteur borné [sic] des variétés algébriques", *Math. Ann.* **286**:1-3 (1990), 27–43. MR Zbl

[Birkar et al. 2010] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, "Existence of minimal models for varieties of log general type", *J. Amer. Math. Soc.* **23**:2 (2010), 405–468. MR Zbl

[Boucksom et al. 2013] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", *J. Algebraic Geom.* **22**:2 (2013), 201–248. MR Zbl

[Browning and Loughran 2017] T. D. Browning and D. Loughran, "Varieties with too many rational points", *Math. Z.* **285**:3-4 (2017), 1249–1267. MR Zbl

[Deligne 1971] P. Deligne, "Théorie de Hodge, II", *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57. MR Zbl

[de Fernex and Hacon 2011] T. de Fernex and C. D. Hacon, "Deformations of canonical pairs and Fano varieties", *J. Reine Angew. Math.* **651** (2011), 97–126. MR Zbl

[Franke et al. 1989] J. Franke, Y. I. Manin, and Y. Tschinkel, "Rational points of bounded height on Fano varieties", *Invent. Math.* **95**:2 (1989), 421–435. MR Zbl

[Fujita 1987] T. Fujita, "On polarized manifolds whose adjoint bundles are not semipositive", pp. 167–178 in *Algebraic geometry* (Sendai, Japan, 1985), edited by T. Oda, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987. MR Zbl

[Fujita 1992] T. Fujita, "On Kodaira energy and adjoint reduction of polarized manifolds", *Manuscripta Math.* **76**:1 (1992), 59–84. MR Zbl

[Hacon and McKernan 2007] C. D. Hacon and J. McKernan, "On Shokurov's rational connectedness conjecture", *Duke Math. J.* **138**:1 (2007), 119–136. MR Zbl

[Hacon et al. 2013] C. D. Hacon, J. McKernan, and C. Xu, "On the birational automorphisms of varieties of general type", *Ann. of Math.* (2) **177**:3 (2013), 1077–1111. MR Zbl

[Hassett et al. 2015] B. Hassett, S. Tanimoto, and Y. Tschinkel, "Balanced line bundles and equivariant compactifications of homogeneous spaces", *Int. Math. Res. Not.* **2015**:15 (2015), 6375–6410. MR Zbl

[Kollár and Mori 1992] J. Kollár and S. Mori, "Classification of three-dimensional flips", *J. Amer. Math. Soc.* **5**:3 (1992), 533–703. MR Zbl

[Le Rudulier 2013] C. Le Rudulier, "Points algébriques de hauteur bornée sur une surface", preprint, 2013, Available at http://cecile.lerudulier.fr/Articles/surfaces.pdf.

[Lehmann and Tanimoto 2017] B. Lehmann and S. Tanimoto, "On the geometry of thin exceptional sets in Manin's conjecture", *Duke Math. J.* **166**:15 (2017), 2815–2869. MR Zbl

[Lehmann and Tanimoto 2019] B. Lehmann and S. Tanimoto, "Geometric Manin's conjecture and rational curves", *Compos. Math.* **155**:5 (2019), 833–862. MR Zbl

[Lehmann et al. 2018] B. Lehmann, S. Tanimoto, and Y. Tschinkel, "Balanced line bundles on Fano varieties", *J. Reine Angew. Math.* **743** (2018), 91–131. MR Zbl

[Maulik and Poonen 2012] D. Maulik and B. Poonen, "Néron–Severi groups under specialization", *Duke Math. J.* **161**:11 (2012), 2167–2206. MR Zbl

[Néron 1952] A. Néron, "Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps", *Bull. Soc. Math. France* **80** (1952), 101–166. MR Zbl

[Peyre 2003] E. Peyre, "Points de hauteur bornée, topologie adélique et mesures de Tamagawa", *J. Théor. Nombres Bordeaux* **15**:1 (2003), 319–349. MR Zbl

[Verdier 1976] J.-L. Verdier, "Stratifications de Whitney et théorème de Bertini–Sard", *Invent. Math.* **36** (1976), 295–312. MR Zbl

[Voisin 2003] C. Voisin, *Hodge theory and complex algebraic geometry, II*, Cambridge Stud. Adv. Math. **77**, Cambridge Univ. Press, 2003. MR Zbl

Communicated by Shigefumi Mori
Received 2018-08-21 Revised 2019-01-20 Accepted 2019-03-10

akashs@math.princeton.edu *Princeton University, Princeton, NJ, United States*

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1907

Equidimensional adic eigenvarieties for groups with discrete series

Daniel R. Gulotta

We extend Urban's construction of eigenvarieties for reductive groups G such that $G(\mathbb{R})$ has discrete series to include characteristic p points at the boundary of weight space. In order to perform this construction, we define a notion of "locally analytic" functions and distributions on a locally $\mathbb{Q}_p$-analytic manifold taking values in a complete Tate $\mathbb{Z}_p$-algebra in which p is not necessarily invertible. Our definition agrees with the definition of locally analytic distributions on p-adic Lie groups given by Johansson and Newton.

1.	Introduction	1907
2.	Modules over complete Tate rings	1910
3.	Locally analytic functions and distributions	1913
4.	Overconvergent cohomology	1921
5.	Eisenstein and cuspidal contributions to characteristic power series	1925
6.	Theory of determinants	1932
7.	Construction of the eigenvariety	1935
	Acknowledgments	1938
	References	1938

1. Introduction

1.1. ***Statement of results.*** The study of p-adic families of automorphic forms began with the work of Hida [1986; 1988; 1994]. Coleman and Mazur [1998] (see also Coleman [1996; 1997]) introduced the eigencurve, which parametrizes overconvergent p-adic modular forms of finite slope. Coleman and Mazur used a geometric definition of p-adic modular forms, based on the original definition of Katz [1973]. It is also possible to define p-adic automorphic forms using a cohomological approach. Several constructions of eigenvarieties are based on overconvergent cohomology, introduced by Stevens [1994] and later generalized by Ash and Stevens [2008]. These include the constructions of Urban [2011] and Hansen [2017]. Emerton [2006b] has also constructed eigenvarieties using a somewhat different cohomological approach.

The eigenvarieties mentioned above are all rigid analytic spaces, so they parametrize forms that have coefficients in $\mathbb{Q}_p$-algebras. Recently, there has been interest in studying forms with coefficients in characteristic p. Liu, Wan, and Xiao [Liu et al. 2017] constructed $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-modules of automorphic forms

MSC2010: primary 11F85; secondary 11S80.
Keywords: eigenvarieties, spectral halo, nonarchimedean functional analysis.

for definite quaternion algebras. By taking quotients of this module, one can obtain both traditional p-adic automorphic forms and forms with coefficients in $\mathbb{F}_p[\![\mathbb{Z}_p^\times]\!]$ whose existence had been conjectured by Coleman. Using these modules, Liu, Wan, and Xiao proved certain cases of a conjecture of Coleman and Mazur and Buzzard and Kilford [2005] concerning the eigenvalues of the U_p operator near the boundary of the weight space. Andreatta, Iovita, and Pilloni [Andreatta et al. 2018] constructed an eigencurve that included characteristic p points by extending Katz's definition of p-adic modular forms.

In this paper, we will show how Urban's eigenvarieties can be extended to include the characteristic p points at the boundary of weight space.

In order to explain our results in more detail, we will first describe the basic idea of overconvergent cohomology. Let G be a connected reductive algebraic group over $\mathbb{Q}$ such that $G_{\mathbb{Q}_p}$ is quasisplit. Let $\mathbb{A}$ be the adeles over $\mathbb{Q}$, let $\mathbb{A}_f^p$ be the finite adeles away from p, let G_∞^+ be the identity component of $G(\mathbb{R})$, and let Z_G be the center of G. Let T_0 be a maximal compact torus of $G(\mathbb{Q}_p)$, and let N_0^- be an open compact subgroup of a maximal unipotent subgroup of $G(\mathbb{Q}_p)$. We may consider the space

$$\mathcal{X} := G(\mathbb{A})/K^p G_\infty^+$$

as a locally $\mathbb{Q}_p$-analytic manifold. Let F be a finite extension of $\mathbb{Q}_p$, and let $\lambda\colon T_0 \to F^\times$ be a continuous homomorphism. Let $\mathcal{D}_{c,\lambda}$ be the space of compactly supported F-valued locally analytic distributions on $\mathcal{X}$, modulo the relations that right translation by N_0^- acts as the identity, right translation by T_0 acts by λ, and translation by $Z_G(\mathbb{Q})$ acts by the identity. One may think of the cohomology groups $H^i(G(\mathbb{Q})/Z_G(\mathbb{Q}), \mathcal{D}_{c,\lambda})$ as spaces of p-adic automorphic forms. One can also study families of p-adic automorphic forms by replacing F with an affinoid $\mathbb{Q}_p$-algebra A.

We are interested in extending overconvergent cohomology to the case where A is a $\mathbb{Z}_p$-algebra. The main challenge is to show that there is a suitable notion of locally analytic A-valued functions and distributions on $\mathcal{X}$. We will define these notions when A is a complete Tate $\mathbb{Z}_p$-algebra.

To see what the definition should be, we recall a fact from p-adic functional analysis: a function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ is locally analytic if and only if it is of the form $f(z) = \sum_{n=0}^\infty a_n \binom{z}{n}$, where $a_n \in A$ and $|a_n|_p$ go to zero exponentially as $n \to \infty$. We will therefore define the space $\mathcal{A}(\mathbb{Z}_p, A)$ of "locally analytic" functions $\mathbb{Z}_p \to A$ to be the set of functions of the form $\sum_{n=0}^\infty a_n \binom{z}{n}$, where $a_n \in A$ and a_n to zero exponentially (i.e., $\alpha^{-n} a_n$ goes to zero for some topologically nilpotent unit α) as $n \to \infty$. If p is invertible in A, then this definition is known to coincide with the usual one. We will make a similar definition for locally analytic functions $\mathbb{Z}_p^k \to A$, and then extend the definition to locally $\mathbb{Q}_p$-analytic manifolds by gluing.

If X is a locally $\mathbb{Q}_p$-analytic manifold, then we will define modules $\mathcal{A}(X, A)$, $\mathcal{D}(X, A)$, $\mathcal{A}_c(X, A)$, $\mathcal{D}_c(X, A)$ of locally analytic functions, distributions, compactly supported functions, and compactly supported distributions, respectively.

Theorem 1.1.1 (Theorem 3.4.2). *The modules $\mathcal{A}(X, A)$, $\mathcal{D}(X, A)$, $\mathcal{A}_c(X, A)$, and $\mathcal{D}_c(X, A)$ satisfy the following properties*:

(1) *$\mathcal{A}(X, A)$ is ring.*

(2) *If $g\colon X \to Y$ is a locally analytic map, then composition with g induces homomorphisms $\mathcal{A}(Y, A) \to \mathcal{A}(X, A)$ and $\mathcal{D}(X, A) \to \mathcal{D}(Y, A)$.*

(3) *The functors $U \mapsto \mathcal{A}(U, A)$ and $U \mapsto \mathcal{D}_c(U, A)$ are sheaves on X.*

(4) *If X has the structure of a finitely generated $\mathbb{Z}_p$-module, then any continuous group homomorphism $X \to A^\times$ is in $\mathcal{A}(X, A)$.*

Remark 1.1.2. Of course, modules of continuous functions and distributions also satisfy the above properties. What makes $\mathcal{A}(X, A)$ and $\mathcal{D}(X, A)$ more like modules of locally analytic functions and distributions is that a map that multiplies all coordinates by p is "completely continuous"; see Proposition 3.3.5 and Lemma 3.3.7 for the precise statement.

Urban [2011] constructed eigenvarieties for reductive groups G such that $G(\mathbb{R})$ has discrete series. We will show how to use the locally analytic distribution modules mentioned above to extend Urban's construction to include characteristic p points.

Theorem 1.1.3 (Theorem 7.4.2). *The reduced eigenvariety* (constructed in [Urban 2011]) *extends to an adic space $\mathcal{E}$ over the weight space $\mathcal{W} = \mathrm{Spa}(\mathbb{Z}_p[[T']], \mathbb{Z}_p[[T']])^{\mathrm{an}}$, where T' is a quotient of a compact subgroup of a maximal torus in $G(\mathbb{Q}_p)$. Furthermore, $\mathcal{E}$ is equidimensional and the projection from $\mathcal{E}$ to the spectral variety $\mathcal{Z}$ is finite and surjective.*

The spectral variety $\mathcal{Z}$ is flat over $\mathcal{W}$, so the existence of characteristic p points of $\mathcal{Z}$ implies the existence of nearby characteristic zero points. It seems to be a difficult problem to prove the existence of boundary points in general; however, in many cases, one can check explicitly that they exist (see for example [Liu et al. 2017; Birkbeck 2019; Johansson and Newton 2018; Ye 2019]).

As this work was being prepared, I became aware that Christian Johansson and James Newton were independently pursuing similar work. In [Johansson and Newton 2019], they adapt Hansen's construction of eigenvarieties to include the boundary of weight space. Their definition of locally analytic distributions on $\mathbb{Z}_p^k$ is essentially the same as ours. To construct distributions on p-adic Lie groups, they use a particular choice of coordinate charts previously studied by Schneider and Teitelbaum. Our definition of locally analytic distributions therefore generalizes theirs.

1.2. ***Summary of Urban's construction and outline of the paper.*** Urban's construction of eigenvarieties is based on the framework of overconvergent cohomology developed by Stevens and Ash and Stevens. In this framework, one first defines a weight space $\mathcal{W}$, as mentioned above. Over any affinoid subspace $\mathcal{U}$ of the weight space $\mathcal{W}$, one defines a complex $C^\bullet$ of projective $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$-modules. The cohomology groups of this complex are the groups $H^i(G(\mathbb{Q})/Z_G(\mathbb{Q}), \mathcal{D}_{c,\lambda})$ mentioned above, where λ is the composition of the quotient $T_0 \to T'$ with the tautological character $T' \to \mathcal{O}_{\mathcal{W}}(\mathcal{U})^\times$.

We consider a weight space that is larger than the one considered by Ash and Stevens and Urban. In particular, our weight space contains opens $\mathcal{U}$ such that the prime p is not invertible on $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$. The main challenge in defining overconvergent cohomology over the larger weight space is to find a suitable notion of “locally analytic” $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$-valued distributions. After recalling the necessary background in Section 2, we define modules of locally analytic functions and distributions and prove some properties of these modules in Section 3. We use these modules to define overconvergent cohomology in Section 4.

Ash and Stevens proposed constructing an eigenvariety whose points correspond to systems of Hecke eigenvalues appearing in the cohomology of the complexes $C^\bullet$; Hansen’s construction uses this approach. Urban took a more K-theoretic approach. Assume that $G(\mathbb{R})$ has discrete series; then cuspidal automorphic forms of regular weight contribute to a single degree q_0 of the cohomology of $C^\bullet$. So the associated systems of Hecke eigenvalues appear a net positive number of times in the formal sum $\sum_i (-1)^{i-q_0} C^i$. Urban showed that after removing the contributions from Eisenstein series, each system of Hecke eigenvalues appears a net nonnegative number of times in the formal sum. The points in Urban’s eigenvariety correspond to those systems of eigenvalues appearing a net positive number of times in the formal sum.

Unfortunately, Urban’s analysis of Eisenstein series contained an error. In order to argue that certain character distributions are uniquely defined, Urban assumed that the region of convergence of an Eisenstein series is (up to translation) a union of Weyl chambers. However, this assumption is not true. In Section 5, we correct this error by giving a new argument for uniqueness.

Urban’s construction of eigenvarieties makes use of the theory of pseudocharacters. We will instead use Chenevier’s theory of determinants [2014], which is equivalent to the theory of pseudocharacters in characteristic zero but better behaved in our setting where the prime p may not be invertible. Section 6 recalls some basic facts about determinants and proves some criteria for establishing that a ratio of two determinants is again a determinant.

Finally, in Section 7, we construct the eigenvariety. We adapt Urban’s construction from the setting of rigid analytic spaces to the setting of adic spaces.

2. Modules over complete Tate rings

2.1. *Definitions.* We begin by recalling the framework necessary for defining modules of locally analytic functions and distributions and for defining eigenvarieties. We will repeat the basic setup of [Buzzard 2007, Section 2; Andreatta et al. 2018, Appendice B]. First, we recall the definition of a Tate ring [Huber 1993, Section 1].

Definition 2.1.1. A *Huber ring* is a topological ring A such that there exists an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ such that A_0 has the I-adic topology. We say that A_0 is a *ring of definition* of A and I is an *ideal of definition* of A_0.

A *Tate ring* is a Huber ring A such that some (equivalently, any) ring of definition A_0 has an ideal of definition that is generated by a topologically nilpotent unit of A.

In Section 7, we will use the framework of adic spaces to construct the eigenvariety. Every analytic adic space can be covered by open subsets of the form $\mathrm{Spa}(A, A^+)$ with A complete Tate, so it is natural to consider this class of rings.

Throughout this section, A will denote a complete Tate ring.

Definition 2.1.2. Let X be a quasicompact topological space, and let M be a topological abelian group. We define $\mathcal{C}(X, M)$ to be the space of continuous functions $X \to M$, with the topology of uniform convergence.

Definition 2.1.3. Let S be a set, and let M be a topological abelian group. We define $c(S, M)$ to be the space of functions $f : S \to M$ such that for any open neighborhood U of the identity in M, the complement of $f^{-1}(U)$ is finite. We give $c(S, M)$ the topology of uniform convergence.

Definition 2.1.4. Let M be a topological A-module. We say that M is *orthonormalizable* if it is isomorphic to $c(S, A)$ for some set S. We say that M is *projective* if it is a direct summand of an orthonormalizable A-module.

Definition 2.1.5. Let M be a topological A-module. We say that a set $B \subset M$ is *bounded* if for all open neighborhoods U of the identity in M, there exists $\alpha \in A^\times$ so that $\alpha B \subseteq U$.

Definition 2.1.6. Let M and N be topological A-modules. We define $\mathcal{L}_b(M, N)$ to be the set of continuous A-module homomorphisms $M \to N$, with the topology of convergence on bounded subsets.

Definition 2.1.7. Let M and N be topological A-modules. We say that an A-module homomorphism $M \to N$ has *finite rank* if its image is a finitely presented A-module. We say that an element of $\mathcal{L}_b(M, N)$ is *completely continuous* if it is in the closure of the subspace of finite rank elements.

2.2. *Spectral theory.*

Definition 2.2.1. We define $A\{\{X\}\}$ to be the set of power series $P(X) = \sum_{n=0}^{\infty} a_n X^n$, $a_n \in A$, such that for any $\alpha \in A^\times$, $\alpha^{-n} a_n \to 0$ as $n \to \infty$.

We say that $P(X) \in A\{\{X\}\}$ is a *Fredholm series* if it has leading coefficient 1.

In Section 7, we will consider the adic space $\mathrm{Spa}(A, A^+)$ for certain complete Tate $\mathbb{Z}_p$-algebras A. If the adic space $\mathrm{Spa}(A, A^+) \times \mathbb{A}^1$ exists, then $A\{\{X\}\}$ is its ring of global sections.

Assume A is Noetherian. If M is a projective A-module and $u : M \to M$ is completely continuous, then we define the Fredholm series $\det(1 - Xu) \in A\{\{X\}\}$ as in [Buzzard 2007, Section 2; Andreatta et al. 2018, Section B.2.4]. To define the series, we express M as a direct summand of an orthonormalizable A-module $c(S, A)$ and extend u to a map $c(S, A) \to c(S, A)$ by having it act as zero on the orthogonal complement of M. The module $c(S, A)$ has a basis consisting of functions sending a single element of S to 1 and the rest to 0. We consider the matrix of u in this basis. The series $\det(1 - Xu)$ is defined to be limit of the characteristic polynomials of finite dimensional submatrices of this matrix. The series does not depend on the choice of embedding.

As in [Urban 2011], we will need to work with complexes. Let $M^\bullet$ be a bounded complex of projective A-modules. We will say that $u^\bullet : M^\bullet \to M^\bullet$ is completely continuous if each u^i is completely continuous. If $u^\bullet$ is completely continuous, then we define

$$\det(1 - Xu^\bullet) := \prod_i \det(1 - Xu^i)^{(-1)^i}.$$

Lemma 2.2.2. *Let $M^\bullet$ be a bounded complex of projective A-modules, and let $u^\bullet, v^\bullet \colon M^\bullet \to M^\bullet$ be completely continuous maps that are homotopy equivalent. Then* $\det(1 - Xu^\bullet) = \det(1 - Xv^\bullet)$.

Proof. For each nonnegative integer k, define the complex $\operatorname{SSym}^k M^\bullet$ so that $(\operatorname{SSym}^k M)^i$ is generated by formal products of k homogeneous elements of $M^\bullet$ of total degree i, subject to the relation a pair of homogeneous elements anticommutes if both have odd degree and commutes otherwise. The differential $d \colon (\operatorname{SSym}^k M)^i \to (\operatorname{SSym}^k M)^{i+1}$ is defined by

$$d(m_1 m_2 \cdots m_k) = (dm_1)m_2 \cdots m_k + (-1)^{\deg m_1} m_1 (dm_2) \cdots m_k + \cdots + (-1)^{i - \deg m_k} m_1 m_2 \cdots (dm_k).$$

The maps $u^\bullet$, $v^\bullet$ induce endomorphisms $\operatorname{SSym}^k u^\bullet$, $\operatorname{SSym}^k v^\bullet$ on $\operatorname{SSym}^k M^\bullet$, and these are completely continuous and homotopy equivalent. We claim that the coefficient of X^k in $\det(1 - Xu^\bullet)^{-1}$ is $\operatorname{tr} \operatorname{SSym}^k u^\bullet$. Indeed, there is a decomposition

$$\begin{aligned}\sum_{k=0}^{\infty} X^k \operatorname{tr} \operatorname{SSym}^k u^\bullet &= \prod_{i \equiv 0(2)} \left(\sum_{k=0}^{\infty} X^k \operatorname{tr} \operatorname{Sym}^k u^i \right) \prod_{i \equiv 1(2)} \left(\sum_{k=0}^{\infty} (-X)^k \operatorname{tr} \wedge^k u^i \right) \\ &= \prod_{i \equiv 0(2)} \det(1 - Xu^i)^{-1} \prod_{i \equiv 1(2)} \det(1 - Xu^i).\end{aligned}$$

Therefore it suffices to show that for each k, $\operatorname{SSym}^k u^\bullet$ and $\operatorname{SSym}^k v^\bullet$ have the same trace. Then we may use the argument of [Urban 2011, Lemma 2.2.8]. □

2.3. *Norms.*

It is often convenient to work with norms on A and on A-modules.

Definition 2.3.1. Let α be a topologically nilpotent unit of A. We define an *α-Banach norm* on A to be a continuous map $|\cdot| \colon A \to \mathbb{R}^{\geq 0}$ satisfying the following conditions:

- $|a + b| \leq \max(|a|, |b|) \quad \forall a, b \in A$.
- $|ab| \leq |a||b| \quad \forall a, b \in A$.
- $|0| = 0$, $|1| = 1$, $|\alpha||\alpha^{-1}| = 1$.
- The norm $|\cdot|$ induces the topology of A.

Definition 2.3.2. Let α be a topologically nilpotent unit of A, let $|\cdot|$ be an α-Banach norm on A, and let M be a topological A-module. We define a *$|\cdot|$-compatible norm* on M to be a continuous map $\|\cdot\| \colon M \to \mathbb{R}^{\geq 0}$ satisfying the following conditions:

- $|m + n| \leq \max(|m|, \|n\|) \quad \forall m, n \in M$.
- $\|am\| \leq |a|\|m\| \quad \forall a \in A, m \in M$.
- $\|0\| = 0$.

If, in addition, $\|\cdot\|$ induces the topology of M, we say that $\|\cdot\|$ is a *Banach norm*.

For any topologically nilpotent unit $\alpha \in A^\times$ and ring of definition A_0 of A containing α, the function $|\cdot|\colon A \to \mathbb{R}^{\geq 0}$ defined by

$$|a| = \inf_{n\in\mathbb{Z}|\alpha^n a\in A_0} p^n$$

is an α-Banach norm.

Furthermore, if M is a topological A-module and M_0 is an open neighborhood of zero in M that is an A_0-module, then the function $\|\cdot\|\colon M \to \mathbb{R}^{\geq 0}$ defined by

$$\|m\| = \inf_{n\in\mathbb{Z}|\alpha^n m\in M_0} p^n$$

is a norm compatible with $|\cdot|$. If the sets of the form $\alpha^n M_0$ are a basis of open neighborhoods of zero, then this norm is Banach.

3. Locally analytic functions and distributions

Now let A be a complete Tate $\mathbb{Z}_p$-algebra, and let X be a locally $\mathbb{Q}_p$-analytic manifold. In this section, we will define modules $\mathcal{A}(X, A)$ and $\mathcal{D}(X, A)$ of "locally analytic" A-valued functions and distributions on X.

The space X can be covered by coordinate patches isomorphic to $\mathbb{Z}_p^k$ for some k. We will first define locally analytic functions on these patches and then show that the construction can be glued.

Naively, one might try to define a function $\mathbb{Z}_p^k \to A$ to be locally analytic if it has a power series expansion in a neighborhood of any point. However, this definition turns out not to be suitable for applications to overconvergent cohomology. In Section 4.3, it will be important that any continuous homomorphism $\mathbb{Z}_p^k \to A^\times$ is in $\mathcal{A}(\mathbb{Z}_p^k, A)$. The homomorphism $\mathbb{Z}_p \to \mathbb{F}_p((T))^\times$ that sends $z \mapsto (1+T)^z$ does not have a power series expansion on any open subset of $\mathbb{Z}_p$. Our criterion for local analyticity will instead be based on Mahler expansions.

3.1. *Preliminaries.*

We will recall some basic facts from p-adic functional analysis.

We will make use of the completed group ring $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}_p^k/p^n\mathbb{Z}_p^k]$.

For $z \in \mathbb{Z}_p^k$, let $[z]$ denote the corresponding group-like element of $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$, and let $\Delta_z = [z] - [0]$. Let I_Δ denote the augmentation ideal of $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$; this is the ideal generated by the Δ_z. The ring $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$ is local with maximal ideal $(p) + I_\Delta$.

We let $\mathbb{Z}_p^k$ act on $\mathcal{C}(\mathbb{Z}_p^k, M)$ by translation: for $g \in \mathcal{C}(\mathbb{Z}_p^k, M)$, $(zg)(y) = g(y+z)$. This action extends to an action of $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$.

We adopt the convention that $\mathbb{N}$ is the set of nonnegative integers. To simplify notation, if $z = (z_1, \dots, z_k) \in \mathbb{Z}_p^k$, and $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, we will write $\binom{z}{n}$ for $\prod_{i=1}^k \binom{z_i}{n_i}$, and we will write $\sum n$ for $\sum_{i=1}^k n_i$.

Lemma 3.1.1 (Mahler's theorem, [Lazard 1965, Théorème II.1.2.4]). *Let M be a complete topological $\mathbb{Z}_p$-module. Suppose that M has a basis of open neighborhoods of zero that are subgroups of M. There is* *an isomorphism* $c(\mathbb{N}^k, M) \xrightarrow{\sim} \mathcal{C}(\mathbb{Z}_p^k, M)$ *that sends* $f \in c(\mathbb{N}^k, M)$ *to a function* $g \in \mathcal{C}(\mathbb{Z}_p^k, M)$ *defined by*

$$g(z) = \sum_{n \in \mathbb{N}^k} f(n)\binom{z}{n}.$$

We say that the right-hand side of the above equation is the Mahler expansion of g.

Lemma 3.1.2 (Amice's theorem). *Let* F *be a closed subfield of* $\mathbb{C}_p$, *and let* $LA_h(\mathbb{Z}_p^k, F)$ *be the space of functions* $\mathbb{Z}_p^k \to F$ *that extend to an analytic function* $\mathbb{Z}_p^k + p^h\mathcal{O}_{\mathbb{C}_p}^k \to \mathbb{C}_p$. *For* $f \in LA_h(\mathbb{Z}_p^k, F)$, *define*

$$|f| := \sup_{z \in \mathbb{Z}_p^k + p^h \mathcal{O}_{\mathbb{C}_p}^k} |f(z)|_p.$$

Then the functions $\lfloor \frac{n_1}{p^h} \rfloor! \cdots \lfloor \frac{n_k}{p^h} \rfloor! \binom{z}{n}$ *form an orthonormal basis for the Banach space* $LA_h(\mathbb{Z}_p^k, F)$. *In other words, every* $f \in LA_h(\mathbb{Z}_p^k, F)$ *can be expressed uniquely in the form*

$$f(z) = \sum_{n \in \mathbb{N}^k} a_n \left\lfloor \frac{n_1}{p^h} \right\rfloor! \cdots \left\lfloor \frac{n_k}{p^h} \right\rfloor! \binom{z}{n},$$

and $|f| = \sup_{n \in \mathbb{N}^k} |a_n|_p$.

Proof. This follows from [Amice 1964, Chapitre 3] (see also [Colmez 2010, Théorème I.4.7]). □

The following formulas concerning the p-adic valuations of $n!$, where n is a nonnegative integer, are well known:

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \quad \text{and} \quad \frac{n}{p-1} - \log_p(n+1) \le v_p(n!) \le \frac{n}{p-1}.$$

Consequently, if F is a closed subfield of $\mathbb{C}_p$, and $f : \mathbb{Z}_p^k \to F$ is a continuous function with the Mahler expansion $f(z) = \sum_{n \in \mathbb{N}^k} a_n \binom{z}{n}$, then f is locally analytic if and only if $|a_n|_p$ go to zero exponentially in $\sum n$.

3.2. *Definitions.* The above facts suggest that we should define a function $\mathbb{Z}_p^k \to A$ to be "locally analytic" if the coefficients of its Mahler expansion decrease to zero exponentially.

We choose a topologically nilpotent $\alpha \in A^\times$.

Definition 3.2.1. Let $r \in \mathbb{R}^+$. We define $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to be the space of functions $f \in \mathcal{C}(\mathbb{Z}_p^k, A)$ such that for any open neighborhood U of zero in A, there exists $N \in \mathbb{N}$ so that for all integers $n > N$ and all $\delta \in I_\Delta^n$, $\alpha^{\lfloor -rn \rfloor}\delta f \in \mathcal{C}(\mathbb{Z}_p^k, U)$.

For any open neighborhood U of zero in A, we define $U_r \subset \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to be the set of all $f \in \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ such that $\alpha^{\lfloor -rn \rfloor}\delta f \in \mathcal{C}(\mathbb{Z}_p^k, U)$ for all $n \in \mathbb{N}$ and all $\delta \in I_\Delta^n$. We define a topology on $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ by making sets of the form U_r a basis of open neighborhoods of zero.

We define $\mathcal{A}(\mathbb{Z}_p^k, A) := \varinjlim_r \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$.

We will not choose a topology on $\mathcal{A}(\mathbb{Z}_p^k, A)$.

The connection between this definition and Mahler expansions will be explained by Lemma 3.2.3.

The definition of $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ is invariant under affine changes of coordinates.

For any topologically nilpotent unit $\alpha' \in A^\times$ and sufficiently small $r' \in \mathbb{R}^+$, $\mathcal{A}^{(\alpha',r')}(\mathbb{Z}_p^k, A)$ injects into $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$. So the directed systems $(\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A))_{r\in\mathbb{R}^+}$ and $(\mathcal{A}^{(\alpha',r)}(\mathbb{Z}_p^k, A))_{r\in\mathbb{R}^+}$ are cofinal, and $\mathcal{A}(\mathbb{Z}_p^k, A)$ does not depend on the choice of α. If F is a closed subfield of $\mathbb{C}_p$, then by Lemma 3.1.2, there are continuous injections with dense image

$$\mathcal{A}^{(p,1/(p-1)p^h)}(\mathbb{Z}_p^k, F) \hookrightarrow LA_h(\mathbb{Z}_p^k, F) \hookrightarrow \mathcal{A}^{(p,r)}(\mathbb{Z}_p^k, F)$$

for any $r < 1/((p-1)p^h)$, so the directed systems $(\mathcal{A}^{(p,r)}(\mathbb{Z}_p^k, F))_{r\in\mathbb{R}^+}$ and $(LA_h(\mathbb{Z}_p^k, F))_{h\in\mathbb{N}}$ are also cofinal.

The module $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ can also be defined (albeit less symmetrically) using α-Banach norms. Choose a ring of definition A_0 of A containing α, and define an α-Banach norm $|\cdot|\colon A \to \mathbb{R}^{\geq 0}$ as in Section 2.3. Define $\|\cdot\|_0\colon \mathcal{C}(\mathbb{Z}_p^k, A) \to \mathbb{R}^{\geq 0}$ by

$$\|f\|_0 = \sup_{z\in\mathbb{Z}_p^k} |f(z)|.$$

The sets $\{f \in A \mid \|f\|_0 \leq s\}$, $s \in \mathbb{R}^{\geq 0}$, form a basis of open neighborhoods of the identity in A. Hence in Definition 3.2.1, we can restrict our attention to neighborhoods of this form. Therefore

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) = \{f \in \mathcal{C}(\mathbb{Z}_p^k, A) \mid \limsup_{n\to\infty} \sup_{\delta\in I_\Delta^n} \|\alpha^{\lfloor -rn\rfloor}\delta f\|_0 = 0\},$$

and the topology on $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ is induced by the norm $\|\cdot\|_r\colon \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to \mathbb{R}^{\geq 0}$ defined by

$$\|f\|_r = \sup_{n\in\mathbb{N}} \sup_{\delta\in I_\Delta^n} \|\alpha^{\lfloor -rn\rfloor}\delta f\|_0.$$

The functions $\|\cdot\|_0$ and $\|\cdot\|_r$ are Banach norms compatible with $|\cdot|$.

Presumably, it would be reasonable to define $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, M)$ and $\mathcal{A}(\mathbb{Z}_p^k, M)$ for any topological A-module M that is locally convex in the sense that for some (equivalently, any) ring of definition A_0 of A, M has a basis of open neighborhoods of the identity that are A_0-modules. (We would just replace A with M in the above definition.) However, we will not need this additional generality.

Definition 3.2.2. Let $r \in \mathbb{R}^+$. We define $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to be the closure of the image of $\mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^k, A), A)$ in $\mathcal{L}_b(\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A), A)$.

We define $\mathcal{D}(\mathbb{Z}_p^k, A) = \varprojlim_r \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$.

The definition of $\mathcal{D}(\mathbb{Z}_p^k, A)$ does not depend on the choice of α.

We chose the definitions of $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ and $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ so that these modules would be orthonormalizable, as we will now show.

Lemma 3.2.3. *There is an isomorphism* $\mathrm{Ser}\colon c(\mathbb{N}^k, A) \xrightarrow{\sim} \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ *that sends* $f \in c(\mathbb{N}^k, A)$ *to a function* $g \in \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ *defined by*

$$g(z) = \sum_{n\in\mathbb{N}^k} \alpha^{\lceil r\Sigma n\rceil} f(n) \binom{z}{n}. \tag{3.2.4}$$

Moreover, if $c(\mathbb{N}^k, A)$ is given the supremum norm and $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ is given the norm $\|\cdot\|_r$, then Ser *is an isometry.*

There is an isomorphism $\mathrm{Ev}\colon \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \xrightarrow{\sim} c(\mathbb{N}^k, A)$ *that sends $\phi \in \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to a function $h \in c(\mathbb{N}^k, A)$ defined by*

$$h(n) = \alpha^{\lceil r\sum n\rceil}\phi\Bigl(\binom{z}{n}\Bigr). \tag{3.2.5}$$

Hence $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ and $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ are orthonormalizable.

Proof. Let $f \in c(\mathbb{N}^k, A)$, and let g be defined by (3.2.4). By Mahler's theorem, $g \in \mathcal{C}(\mathbb{Z}_p^k, A)$. We observe that for any $h \in \mathcal{C}(\mathbb{Z}_p^k, A)$ and $\delta \in \mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$, $\|\delta h\|_0 \le \|h\|_0$. Furthermore, if $\delta \in I_\Delta^m$, then $\delta\binom{z}{n} = 0$ whenever $\sum n < m$. So

$$\|\alpha^{\lfloor -rm\rfloor}\delta g\|_0 \le \sup_{\sum n\ge m} |\alpha^{\lfloor -rm\rfloor + \lceil r\sum n\rceil} f(n)| \le \sup_{\sum n\ge m} |f(n)|.$$

It follows that $g \in \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$, and Ser is has operator norm ≤ 1.

By Mahler's theorem, we can recover f from g:

$$f(n) = \alpha^{\lfloor -r\sum n\rfloor}(\Delta_{e_1}^{n_1}\cdots\Delta_{e_k}^{n_k} g)(0), \tag{3.2.6}$$

where $e_1, \dots, e_k$ are the standard basis for $\mathbb{Z}_p^k$. Since

$$|f(n)| \le \sup_{\delta\in(I_\Delta)^{\sum n}} \|\alpha^{\lfloor -r\sum n\rfloor}\delta g\|_0,$$

the relation (3.2.6) determines a map $\mathrm{Coeff}\colon \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to c(\mathbb{Z}_p^k, A)$ that is a left-inverse of Ser, and Coeff has operator norm ≤ 1. To see that Coeff is also a right-inverse of Ser, observe that $(\mathrm{Ser}\circ\mathrm{Coeff})(g)$ and g agree on $\mathbb{N}^k$, which is dense in $\mathbb{Z}_p^k$. Since Ser and Coeff both have operator norm ≤ 1, they must be isometries.

The map Ser induces an isomorphism $\mathrm{Ser}^*\colon \mathcal{L}_b(\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A), A) \xrightarrow{\sim} \mathcal{L}_b(c(\mathbb{N}^k, A), A)$. The pairing $c(\mathbb{N}^k, A) \times c(\mathbb{N}^k, A) \to A$ defined by $(f, h) \mapsto \sum_{n\in\mathbb{N}^k} f(n)h(n)$ identifies $c(\mathbb{N}^k, A)$ isometrically with a closed submodule of $\mathcal{L}_b(c(\mathbb{N}^k, A), A)$. For any $\phi \in \mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^k, A), A)$, the function $n \mapsto \phi\bigl(\binom{z}{n}\bigr)$ is bounded, so in particular $\alpha^{\lceil r\sum n\rceil}\phi\bigl(\binom{z}{n}\bigr) \to 0$ as $\sum n \to \infty$. Hence the image of Ser^* is contained in $c(\mathbb{N}^k, A)$. Furthermore, the image contains all elements of $c(\mathbb{N}^k, A)$ that are supported on a finite subset of $\mathbb{N}^k$, and these elements are dense in $c(\mathbb{N}^k, A)$. □

Lemma 3.2.3 makes it clear that for $r' < r$, there are natural injections

$$\mathcal{D}^{(\alpha,r')}(\mathbb{Z}_p^k, A) \hookrightarrow \mathcal{L}_b(\mathcal{A}^{(\alpha,r')}(\mathbb{Z}_p^k, A), A) \hookrightarrow \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$$
$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \hookrightarrow \mathcal{L}_b(\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A), A) \hookrightarrow \mathcal{A}^{(\alpha,r')}(\mathbb{Z}_p^k, A).$$

3.3. ***Properties of locally analytic functions and distributions.*** In this section, we check that $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ has some properties that one would expect of locally analytic functions.

Lemma 3.3.1. *Multiplication induces a continuous map $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \times \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$.*

Proof. This follows Lemma 3.2.3 and the fact that for $m, n \in \mathbb{N}$, $\binom{z}{n}\binom{z}{m}$ is of the form $\sum_{i=0}^{m+n} a_i \binom{z}{i}$ with $a_i \in \mathbb{Z}$. □

Lemma 3.3.2. *Let*

$$f : \mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p) \to \mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p)$$

be a $\mathbb{Z}_p$-module homomorphism. For any $r, s \in \mathbb{R}^+$, there is at most one continuous A-linear homomorphism $\tilde{f}$ making the following diagram commute:

$$\begin{array}{ccccc} \mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p) & \longrightarrow & \mathcal{C}(\mathbb{Z}_p^k, A) & \hookleftarrow & \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \\ \downarrow f & & \downarrow & & \downarrow \tilde{f} \\ \mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p) & \longrightarrow & \mathcal{C}(\mathbb{Z}_p^j, A) & \hookleftarrow & \mathcal{A}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \end{array}$$

and there is at most one continuous A-linear homomorphism $\tilde{f}^*$ making the following diagram commute:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p), \mathbb{Z}_p) & \longrightarrow & \mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^j, A), A) & \hookrightarrow & \mathcal{D}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \\ \downarrow f^* & & \downarrow & & \downarrow \tilde{f}^* \\ \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p), \mathbb{Z}_p) & \longrightarrow & \mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^k, A), A) & \hookrightarrow & \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \end{array}$$

If either homomorphism exists, we say that it is induced by f.

Proof. If the maps $\tilde{f}$ and $\tilde{f}^*$ exist, then their matrices in the basis of Lemma 3.2.3 can be deduced from the matrix of f in the basis of Mahler's theorem. More specifically, if we write

$$f\left(\binom{z}{n}\right) = \sum_{m \in \mathbb{N}^j} f_{nm} \binom{z}{m}$$

with $f_{nm} \in \mathbb{Z}_p$, then the matrix coefficients of $\tilde{f}$ must be

$$\tilde{f}_{nm} = \alpha^{\lceil r \sum n \rceil - \lceil s \sum m \rceil} f_{nm},$$

and the matrix coefficients of $\tilde{f}^*$ must be

$$\tilde{f}^*_{nm} = \tilde{f}_{mn} = \alpha^{\lceil r \sum m \rceil - \lceil s \sum n \rceil} f_{mn}.$$ □

Lemma 3.3.3. *There exists $t_0 \in \mathbb{R}^+$ so that for any $r, s \in \mathbb{R}^+$, $j, k \in \mathbb{N}$, and any $\mathbb{Z}_p$-module homomorphism $f : \mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p) \to \mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p)$ that induces a continuous homomorphism*

$$\mathcal{A}^{(p,r)}(\mathbb{Z}_p^k, \mathbb{Q}_p) \to \mathcal{A}^{(p,s)}(\mathbb{Z}_p^j, \mathbb{Q}_p),$$

f also induces continuous homomorphisms

$$\tilde{f} : \mathcal{A}^{(\alpha,rt)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,st)}(\mathbb{Z}_p^j, A) \quad \textit{and} \quad \tilde{f}^* : \mathcal{D}^{(\alpha,st)}(\mathbb{Z}_p^j, A) \to \mathcal{D}^{(\alpha,rt)}(\mathbb{Z}_p^k, A)$$

for all $t \in (0, t_0)$.

Proof. If the map $\tilde{f}\colon \mathcal{A}^{(\alpha,rt)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,st)}(\mathbb{Z}_p^j, A)$ exists, then in the notation of the previous lemma, its matrix coefficients must be given by

$$\tilde{f}_{nm} = \alpha^{\lceil rt\sum n\rceil - \lceil st\sum m\rceil} f_{nm}.$$

Conversely, if there is a continuous map with these matrix coefficients, then it is the desired map $\tilde{f}$.

The $\tilde{f}_{nm}$ are the matrix coefficients of a continuous map if and only if the following two conditions are satisfied:

(1) $\tilde{f}_{nm}$ are bounded.

(2) For any fixed n, $\tilde{f}_{nm} \to 0$ as $\sum m \to \infty$.

The terms with $r\sum n - s\sum m \geq 0$ are certainly bounded, so we only need to worry about terms with $r\sum n - s\sum m < 0$. There exists a positive integer ℓ so that p^ℓ/α is power bounded. If the $p^{\lceil rt\ell\sum n\rceil - \lceil st\ell\sum n\rceil} f_{nm}$ (considered as elements of $\mathbb{Q}_p$) are bounded (resp. go to zero as $\sum m \to \infty$), then the same will be true of the $\alpha^{\lceil rt\sum n\rceil - \lceil st\sum m\rceil} f_{nm}$ (considered as elements of A). So we may take $t_0 = \ell^{-1}$.

Similarly, if the map $\tilde{f}^*\colon \mathcal{D}^{(\alpha,st)}(\mathbb{Z}_p^j, A) \to \mathcal{D}^{(\alpha,rt)}(\mathbb{Z}_p^k, A)$ exists, then its matrix coefficients satisfy $\tilde{f}^*_{mn} = \tilde{f}_{nm}$. The $\tilde{f}^*_{mn}$ are the coefficients of a continuous map if and only if condition (1) above and the following condition are satisfied:

(2′) For any fixed m, $\tilde{f}_{nm} \to 0$ as $\sum n \to \infty$.

Since $f_{nm} \in \mathbb{Z}_p$ and $\alpha^{\lceil rt\sum n\rceil} \to 0$ as $\sum n \to \infty$, condition (2′) will always be satisfied. □

Proposition 3.3.4. *Let $g\colon \mathbb{Z}_p^j \to \mathbb{Z}_p^k$ be a (globally) analytic function. For some $r_0 \in \mathbb{R}^+$ depending on α but not on g, j, k, composition with g induces continuous A-linear homomorphisms*

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \quad \textit{and} \quad \mathcal{D}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \to \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$$

for all $s < r < r_0$.

Proof. There are continuous maps

$$\mathcal{A}^{(p,1/(p-1))}(\mathbb{Z}_p^k, \mathbb{Q}_p) \to LA_0(\mathbb{Z}_p^k, \mathbb{Q}_p) \xrightarrow{g^*} LA_0(\mathbb{Z}_p^j, \mathbb{Q}_p) \to \mathcal{A}^{(p,1/(p-1)-\epsilon)}(\mathbb{Z}_p^j, \mathbb{Q}_p)$$

for any $\epsilon \in (0, 1/(p-1))$. Applying Lemma 3.3.3 then yields the desired result. □

If $j = 1$, then the maps exist even if $r = s$. We do not know if the same is true for $j > 1$. When $j = 1$, one can prove existence by considering the norm on LA_0 defined in Lemma 3.1.2 and using the fact that $v_p(n!) - \sum_{i=1}^k v_p(m_i!) \geq \lfloor (n - \sum m)/p \rfloor$. (The same idea will be used in the proof of Proposition 3.3.5.) However, for $j > 1$, $\sum_{i=1}^j v_p(n_i!) - \sum_{i=1}^k v_p(m_i!)$ can be zero for arbitrarily large values of $\sum n - \sum m$.

Proposition 3.3.5. *Let S be a set of coset representatives of $\mathbb{Z}_p^k/p\mathbb{Z}_p^k$. The homeomorphism $\mathbb{Z}_p^k \times S \xrightarrow{\sim} \mathbb{Z}_p^k$ defined by $(z, s) \mapsto pz + s$ determines an isomorphism*

$$\mathcal{C}(\mathbb{Z}_p^k, A) \cong \mathcal{C}(\mathbb{Z}_p^k, A)^{\oplus p^k}$$

which induces isomorphisms

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \cong \mathcal{A}^{(\alpha,pr)}(\mathbb{Z}_p^k, A)^{\oplus p^k} \quad \textit{and} \quad \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \cong \mathcal{D}^{(\alpha,pr)}(\mathbb{Z}_p^k, A)^{\oplus p^k}$$

for all sufficiently small $r \in \mathbb{R}^+$.

Proof. First, consider the case $k = 1$. Applying Lemma 3.3.3 along with translation invariance, we see that it is then enough to check that composition with the function

$$g(z) = pz$$

defines a continuous homomorphism

$$\mathcal{A}^{(p,1/2p^2)}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathcal{A}^{(p,1/2p)}(\mathbb{Z}_p, \mathbb{Q}_p)$$

and that composition with the function

$$h(z) = \begin{cases} z/p & z \in p\mathbb{Z}_p, \\ 0 & z \in \mathbb{Z}_p^\times, \end{cases}$$

defines a continuous homomorphism

$$\mathcal{A}^{(p,1/2p)}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathcal{A}^{(p,1/2p^2)}(\mathbb{Z}_p, \mathbb{Q}_p).$$

Define g_{nm}, h_{nm} by

$$g\left(\binom{z}{n}\right) = \sum_{m=0}^{\infty} g_{nm}\binom{z}{m}, \quad h\left(\binom{z}{n}\right) = \sum_{m=0}^{\infty} h_{nm}\binom{z}{m}.$$

By the same reasoning as in Lemma 3.3.3, we just need to verify that:

(1) $v_p(g_{nm}) - \frac{m}{2p} + \frac{n}{2p^2}$ is bounded below for $pm \geq n$.

(2) For any n, $v_p(g_{nm}) - \frac{m}{2p} + \frac{n}{2p^2} \to \infty$ as $m \to \infty$.

(3) $v_p(h_{nm}) - \frac{m}{2p^2} + \frac{n}{2p}$ is bounded below for $m \geq pn$.

(4) For any n, $v_p(h_{nm}) - \frac{m}{2p^2} + \frac{n}{2p} \to \infty$ as $m \to \infty$.

Applying Lemma 3.1.2 gives

$$v_p(g_{nm}) - v_p(m!) \geq -v_p(\lfloor n/p \rfloor !).$$

For $n \geq pm$ this implies

$$v_p(g_{nm}) \geq \sum_{i=1}^{\infty} (\lfloor m/p^i \rfloor - \lfloor n/p^{i+1} \rfloor) \geq \lfloor m/p - n/p^2 \rfloor.$$

Similarly, Lemma 3.1.2 implies

$$v_p(h_{nm}) \geq \sum_{i=1}^{\infty} (\lfloor m/p^{i+1} \rfloor - \lfloor n/p^i \rfloor) \geq \lfloor m/p^2 - n/p \rfloor.$$

This proves the case $k = 1$.

We reduce the general case to the case $k = 1$ as follows. Since the above modules are all preserved by translation, if the proposition is true for one choice of S, it is true for any choice of S. In particular, we may assume S is a product of k sets of coset representatives of $\mathbb{Z}/p\mathbb{Z}$. Then, since multiplication by p does not mix coordinates, the argument is essentially the same as in the $k = 1$ case. □

Lemma 3.3.6. *Any continuous homomorphism* $\lambda\colon \mathbb{Z}_p^k \to A^\times$ *is in* $\mathcal{A}(\mathbb{Z}_p^k, A)$.

Proof. Lemma 3.3.1 allows us to reduce to the one-dimensional case, and Proposition 3.3.5 allows us to replace $\mathbb{Z}_p^k$ with an open sublattice. So it suffices to consider the case where $k = 1$ and $(\lambda(1) - 1)/\alpha$ is topologically nilpotent. In that case, since

$$\lambda(z) = \sum_{n=0}^{\infty} \binom{z}{n} (\lambda(1) - 1)^n,$$

$\lambda \in \mathcal{A}^{(\alpha,1)}(\mathbb{Z}_p, A)$. □

Lemma 3.3.7. *For any* $0 < s < r$, *the inclusions*

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \hookrightarrow \mathcal{A}^{(\alpha,s)}(\mathbb{Z}_p^k, A) \quad \text{and} \quad \mathcal{D}^{(\alpha,s)} \hookrightarrow \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$$

are completely continuous.

Proof. In the orthonormal bases of Lemma 3.2.3, these inclusions are represented by diagonal matrices with diagonal entries of the form $\alpha^{\lfloor r \sum n \rfloor - \lfloor s \sum n \rfloor}$. As $\sum n \to \infty$, the entries go to zero. □

3.4. *Gluing.* Propositions 3.3.4 and 3.3.5 show that it makes sense to define locally analytic functions and distributions on arbitrary locally $\mathbb{Q}_p$-analytic manifolds by gluing.

Definition 3.4.1. Let k be a nonnegative integer, and let X be a locally $\mathbb{Q}_p$-analytic manifold of dimension k. Choose a decomposition $X = \bigsqcup_{i \in I} X_i$ for some index set I, and choose an identification of each X_i with $\mathbb{Z}_p^k$. We define

$$\mathcal{A}(X, A) = \prod_{i \in I} \mathcal{A}(X_i, A)$$
$$\mathcal{A}_c(X, A) = \bigoplus_{i \in I} \mathcal{A}(X_i, A)$$
$$\mathcal{D}(X, A) = \bigoplus_{i \in I} \mathcal{D}(X_i, A)$$
$$\mathcal{D}_c(X, A) = \prod_{i \in I} \mathcal{D}(X_i, A).$$

By Propositions 3.3.4 and 3.3.5, the above definitions do not depend on the choice of decomposition.

Theorem 3.4.2. *The modules* $\mathcal{A}(X, A)$, $\mathcal{D}(X, A)$, $\mathcal{A}_c(X, A)$, *and* $\mathcal{D}_c(X, A)$ *satisfy the following properties*:

(1) *$\mathcal{A}(X, A)$ is ring.*

(2) *If $g\colon X \to Y$ is a locally analytic map, then composition with g induces homomorphisms $\mathcal{A}(Y, A) \to \mathcal{A}(X, A)$ and $\mathcal{D}(X, A) \to \mathcal{D}(Y, A)$.*

(3) *The functors $U \mapsto \mathcal{A}(U, A)$ and $U \mapsto \mathcal{D}_c(U, A)$ are sheaves on X.*

(4) *If X has the structure of a finitely generated $\mathbb{Z}_p$-module, then any continuous group homomorphism $X \to A^\times$ is in $\mathcal{A}(X, A)$.*

Proof. The claims all follow immediately from the results of Section 3.3. □

3.5. ***Geometric interpretation of distributions.*** The modules of locally analytic distributions have an alternative interpretation as rings of sections of adic spaces. This interpretation will not be used elsewhere in the paper, but it gives further evidence that our definition of distributions is reasonable. For background on adic spaces, see [Huber 1993; Huber 1994; Huber 1996] or [Scholze and Weinstein 2019, Sections 2–5].

Let $D = \mathrm{Spa}(\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!], \mathbb{Z}_p[\![\mathbb{Z}_p^k]\!])$. Suppose that the Tate algebra $A\langle T_1, \ldots, T_n\rangle$ is sheafy for each nonnegative integer n. Let A^+ be an open and integrally closed subring of A. Let $Y = D \times_{\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)} \mathrm{Spa}(A, A^+)$. We can construct Y as follows. There is an isomorphism $\mathbb{Z}_p[\![T_1, \ldots, T_k]\!] \cong \mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$ that sends $T_i \mapsto \Delta_{e_i}$, where the e_i form a basis of $\mathbb{Z}_p^k$; this isomorphism is known as the multivariable Amice transform. For any positive rational $r = m/n$, let $B_r = A\langle T_1, \ldots, T_k, T_1^n/\alpha^m, \ldots, T_k^n/\alpha^m\rangle$, and let B_r^+ be the normal closure of $A^+\langle T_1, \ldots, T_n, T_1^n/\alpha^m, \ldots, T_n^n/\alpha^m\rangle$ in B_r. Then Y is formed by gluing the affinoids $Y_r := \mathrm{Spa}(B_r, B_r^+)$.

There are canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p), \mathbb{Z}_p) &\cong \mathcal{O}_D(D), \\ \mathcal{D}(\mathbb{Z}_p^k, A) &\cong \mathcal{O}_Y(Y), \\ \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) &\cong \mathcal{O}_Y(Y_r) \quad \forall r \in \mathbb{Q}^+. \end{aligned}$$

4. Overconvergent cohomology

Now we use the modules constructed in Section 3 to define overconvergent cohomology. We mostly repeat the setup of [Urban 2011, Sections 3–4]; see also [Ash and Stevens 2008; Hansen 2017, Sections 2–3].

4.1. ***Locally symmetric spaces.*** Let $\mathbb{A}$ (resp. $\mathbb{A}_f$, $\mathbb{A}_f^p$) be the ring of adeles (resp. finite adeles, finite adeles away from p) of $\mathbb{Q}$.

Let G be a connected reductive algebraic group over $\mathbb{Q}$. We will assume that $G(\mathbb{Q}_p)$ is quasisplit. Let B, T, N, N^- be compatible choices of a Borel subgroup, maximal torus, maximal unipotent subgroup, and opposite unipotent subgroup, respectively, of $G(\mathbb{Q}_p)$.

We will need some results from [Bruhat and Tits 1972]. Note that $G_{\mathbb{Q}_p}$ admits a valued root datum ("donnée radicielle valuée") by [Bruhat and Tits 1984, 4.2.3 Théorème].

Let I be an Iwahori subgroup of $G(\mathbb{Q}_p)$ compatible with B (see for example [Bruhat and Tits 1972, Section 6.5]; note that this reference denotes the Iwahori by B). Then I admits a factorization $I = N_0 T_0 N_0^-$, where $N_0^- = N^- \cap I$, $T_0 = T \cap I$, $N_0 = N \cap I$. Let K^p be an open compact subgroup of $\mathbb{A}_f^p$, and let $K = K^p I$. We assume that K is neat; see Definition 4.1.1 below. Let G_∞^+ be the identity component of $G(\mathbb{R})$, and let K_∞ be a maximal compact modulo center subgroup of G_∞^+. Let Z_G be the center of G.

The space

$$\mathcal{X} := G(\mathbb{A})/K^p G_\infty^+$$

may be considered as a locally $\mathbb{Q}_p$-analytic manifold. Let A be a complete Noetherian Tate $\mathbb{Z}_p$-algebra. In Section 3, we defined the module $\mathcal{D}_c(\mathcal{X}, A)$ of "locally analytic" compactly supported A-valued distributions on $\mathcal{X}$.

Let $\lambda\colon T_0 \to A^\times$ be a continuous homomorphism. By Lemma 3.3.6, $\lambda \in \mathcal{A}(T_0, A)$. We will assume that $\ker\lambda$ contains $(Z_G(\mathbb{Q})K^p G_\infty^+ \cap T_0)$. We define $\mathcal{D}_{c,\lambda}(\mathcal{X}, A)$ to be the quotient of $\mathcal{D}_c(\mathcal{X}, A)$ obtained by constraining right-translation by N_0^- to act by the identity, right-translation by T_0 to act by λ, and translation by $Z_G(\mathbb{Q})$ to act by the identity.

The group $G(\mathbb{Q})/Z_G(\mathbb{Q})$ acts on $\mathcal{D}_{c,\lambda}(\mathcal{X}, A)$ by left-translation. Moreover, $\mathcal{D}_{c,\lambda}(\mathcal{X}, A)$ is a direct sum of modules induced from much smaller subgroups of $G(\mathbb{Q})/Z_G(\mathbb{Q})$. We can write $G(\mathbb{A})$ as a finite union

$$G(\mathbb{A}) = \bigsqcup_i G(\mathbb{Q}) g_i G_\infty^+ K.$$

Let Γ_i be the image of $g_i G_\infty^+ K g_i^{-1} \cap G(\mathbb{Q})$ in $G(\mathbb{Q})/Z_G(\mathbb{Q})$. Then

$$\mathcal{D}_{c,\lambda}(\mathcal{X}, A) \cong \bigoplus_i \operatorname{Ind}_{\Gamma_i}^{G(\mathbb{Q})/Z_G(\mathbb{Q})} \mathcal{D}_\lambda(g_i I, A)$$

where $\mathcal{D}_\lambda(g_i I, A)$ is the quotient of $\mathcal{D}(g_i I, A)$ obtained by constraining right-translation by N_0^- to act as the identity and right-translation by T_0 to act as λ. Here Γ_i acts on $\mathcal{D}_\lambda(g_i I, A)$ by left-translation.

The existence of the Iwahori factorization implies that the map $N_0 \to g_i I$ given by $n \mapsto g_i n$ induces an isomorphism of A-modules

$$\mathcal{D}(N_0, A) \xrightarrow{\sim} \mathcal{D}_\lambda(g_i I, A).$$

This identification induces a Γ_i-action on $\mathcal{D}(N_0, A)$, which can be described as follows. Any $x \in I$ has an Iwahori factorization $x = \boldsymbol{n}(x)\boldsymbol{t}(x)\boldsymbol{n}^-(x)$ with $\boldsymbol{n}(x) \in N_0$, $\boldsymbol{t}(x) \in T_0$, $\boldsymbol{n}^-(x) \in N_0^-$, and the functions $\boldsymbol{n}$, $\boldsymbol{t}$, and $\boldsymbol{n}^-$ are analytic. The action of Γ_i on $\mathcal{D}(N_0, A)$ is given by

$$\gamma \cdot [x] = \lambda(\boldsymbol{t}(g_i^{-1}\gamma g_i x))[\boldsymbol{n}(g_i^{-1}\gamma g_i x)]$$

for $\gamma \in \Gamma_i$, $x \in N_0$. Here $[x]$ denotes the Dirac delta distribution supported at x.

Now consider the locally symmetric space

$$S_G(K) := G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K.$$

Then $S_G(K) \cong \bigsqcup_i \mathcal{Y}_i$ where

$$\mathcal{Y}_i := \Gamma_i \backslash G_\infty^+ / K_\infty.$$

Definition 4.1.1. We say that K is *neat* if all of the Γ_i are torsion-free.

As mentioned above, we assume that K^p has been chosen so that K is neat. Then each $\mathcal{Y}_i$ is a manifold with fundamental group Γ_i.

The manifold $S_G(K)$ has a Borel–Serre compactification $\overline{S_G(K)}$, which is homotopy equivalent to $S_G(K)$. Any finite triangulation of $\overline{S_G(K)}$ determines a resolution

$$0 \to C_d(\Gamma_i) \to \cdots \to C_1(\Gamma_i) \to C_0(\Gamma_i) \to \mathbb{Z} \to 0$$

where the $C_j(\Gamma_i)$ are free $\mathbb{Z}[\Gamma_i]$-modules of finite rank and d is the dimension of $S_G(K)$. We define a complex $C_\lambda^\bullet$ by

$$C_\lambda^j := \bigoplus_i \mathrm{Hom}_{\Gamma_i}(C_j(\Gamma_i), \mathcal{D}_\lambda(g_i I, A)). \tag{4.1.2}$$

Then

$$R\Gamma^\bullet(G(\mathbb{Q})/Z_G(\mathbb{Q}), \mathcal{D}_{c,\lambda}(\mathcal{X}, A)) \cong \bigoplus_i R\Gamma^\bullet(\Gamma_i, \mathcal{D}_\lambda(g_i I, A)) \cong C_\lambda^\bullet$$

in the derived category of A-modules.

4.2. ***Hecke action.*** We choose a projective resolution

$$\cdots \to C_1(G(\mathbb{Q})/Z_G(\mathbb{Q})) \to C_0(G(\mathbb{Q})/Z_G(\mathbb{Q})) \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ as a $G(\mathbb{Q})/Z_G(\mathbb{Q})$-module as well as maps of complexes of Γ_i-modules

$$C_\bullet(\Gamma_i) \to C_\bullet(G(\mathbb{Q})/Z_G(\mathbb{Q})) \quad \text{and} \quad C_\bullet(G(\mathbb{Q})/Z_G(\mathbb{Q})) \to C_\bullet(\Gamma_i)$$

that are homotopy inverses of each other. Then any $f \in \mathrm{End}_{G(\mathbb{Q})/Z_G(\mathbb{Q})}(\mathcal{D}_{c,\lambda}(\mathcal{X}, A))$ defines an operator $[f] \in \mathrm{End}(C_\lambda^\bullet)$ by

$$\begin{aligned} C_\lambda^j &\to \bigoplus_i \mathrm{Hom}_{\Gamma_i}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_\lambda(g_i I, A)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{G(\mathbb{Q})/Z_G(\mathbb{Q})}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_{c,\lambda}(\mathcal{X}, A)) \\ &\xrightarrow{f} \mathrm{Hom}_{G(\mathbb{Q})/Z_G(\mathbb{Q})}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_{c,\lambda}(\mathcal{X}, A)) \\ &\xrightarrow{\sim} \bigoplus_i \mathrm{Hom}_{\Gamma_i}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_\lambda(g_i I, A)) \\ &\to C_\lambda^j. \end{aligned}$$

For any f, g, $[f][g]$ is homotopy equivalent to $[fg]$.

For any $g \in G(\mathbb{A}_f^p)$, the double coset operator $K^p g K^p$ acts on $\mathcal{D}_{c,\lambda}$ and determines a Hecke operator $[K^p g K^p]$ on $C_\lambda^\bullet$.

Let

$$T^- := \{t \in T \mid t^{-1} N_0^- t \subseteq N_0^-\}.$$

For $t \in T^-$, the double coset operator $N_0^- t N_0^-$ acts on $\mathcal{D}_{c,\lambda}$ and determines an operator $[N_0^- t N_0^-]$ on $C_\lambda^\bullet$. We will sometimes denote this operator by u_t.

Remark 4.2.1. Our definition of the Hecke operators at p differs slightly from that of previous references on overconvergent cohomology, which made use of a choice of "right $*$-action". Our definition is instead meant to be analogous to the one used in Emerton's theory of completed cohomology [2006a; 2006b]. The two approaches will yield the same eigenvariety. The only essential difference between the approaches is that, to define a "right $*$-action", one chooses a splitting of $0 \to T_0 \to T \to T/T_0 \to 0$, and then uses this splitting to twist the Hecke operators so that T_0 acts trivially.

Let S be the set of finite places at which K^p is not maximal hyperspecial. Let $\mathbb{A}_f^{p,S}$ be the adeles away from p and S, and let $K^{p,S}$ be the image of K^p in $\mathbb{A}_f^{p,S}$. We define the Hecke algebra

$$\mathcal{H}_G := C_c^\infty(K^{p,S}\backslash G(\mathbb{A}_f^{p,S})/K^{p,S} \times N_0^-\backslash N_0^- T^- N_0^-/N_0^-, \mathbb{Z}_p).$$

4.3. ***Topological properties of Hecke operators.*** In order to apply the spectral theory introduced in Section 2.2, we will need to choose a particular description of $C_\lambda^\bullet$ as a limit of complexes of projective modules. The logarithm induces a bijection between N_0 and a finite free $\mathbb{Z}_p$-module; we use this bijection to define a coordinate chart on N_0. This chart allows us to define the projective modules $\mathcal{D}^{(\alpha,r)}(N_0, A)$ for some arbitrarily chosen topologically nilpotent unit $\alpha \in A$. Define

$$C^i_{\lambda,\alpha,r} := \bigoplus_j \operatorname{Hom}_{\Gamma_j}(C_i(\Gamma_j), \mathcal{D}^{(\alpha,r)}(N_0, A)).$$

Lemma 4.3.1. *For all sufficiently small r and all $\epsilon > 0$, the differential $d\colon C_\lambda^{i+1} \to C_\lambda^i$ extends to a map $C^{i+1}_{\lambda,\alpha,r} \to C^i_{\lambda,\alpha,r+\epsilon}$.*

Proof. It is enough to check that for sufficiently small r and all $\epsilon > 0$, left translation by any $\gamma \in \Gamma_i$ maps $\mathcal{D}^{(\alpha,r)}(N_0, A)$ into $\mathcal{D}^{(\alpha,r+\epsilon)}(N_0, A)$. This follows from the description of the action in Section 4.1 along with Lemmas 3.3.1 and 3.3.6 and Proposition 3.3.4. □

If $\underline{r} = (r_0, \dots, r_d)$ is chosen such that the differentials $C^{i+1}_{\lambda,\alpha,r_{i+1}} \to C^i_{\lambda,\alpha,r_i}$ are defined, then we denote the corresponding complex by $C^\bullet_{\lambda,\alpha,\underline{r}}$.

Choose some $t \in T^-$ such that $t^{-1}N_0 t \subset N_0^p$. Let $\mathcal{H}'_G$ be the ideal of $\mathcal{H}_G$ generated by u_t.

Lemma 4.3.2. *There exists $r_0 \in \mathbb{R}^+$ so that for all $r \in (0, r_0)$, $\epsilon \in \mathbb{R}^+$, and $f \in \mathcal{H}'_G$, f determines a continuous map $C^i_{\lambda,\alpha,r} \to C^i_{\lambda,\alpha,r/p+\epsilon}$, and hence f determines a completely continuous map $C^i_{\lambda,\alpha,r} \to C^i_{\lambda,\alpha,r}$.*

Proof. We can show that Hecke operators away from p map $C^i_{\lambda,\alpha,r}$ into $C^i_{\lambda,\alpha,r+\epsilon}$ using essentially the same argument as in Lemma 4.3.1. It remains to show that u_t maps $C^i_{\lambda,\alpha,r}$ into $C^i_{\lambda,\alpha,r/p+\epsilon}$. The action of u_t can be built from functions of the form

$$[x] \mapsto \lambda(\boldsymbol{t}(\iota(x)))[\boldsymbol{n}(\iota(x))]$$

where $\iota(x)$ takes the form

$$\iota(x) = ht^{-1}\boldsymbol{n}(n^- x)\boldsymbol{t}(n^- x)t$$

for some $n^- \in N_0^-$, $h \in I$. (See for example [Emerton 2006a, Lemma 4.2.19].) In particular, $\boldsymbol{n}(\iota(x))$ belongs to a single right coset of $t^{-1}N_0 t \subset N_0^p$. The argument proceeds as before, except that we also need to use Proposition 3.3.5 and Lemma 3.3.7. □

4.4. ***Characteristic power series.*** For any $f \in \mathcal{H}'_G$, we define the power series

$$\det(1 - Xf \mid C^\bullet_\lambda) := \det(1 - Xf \mid C^\bullet_{\lambda,\alpha,\underline{r}})$$

for any $\alpha, \underline{r}$ for which the complex $C^\bullet_{\lambda,\alpha,\underline{r}}$ is defined and the u_t operator is completely continuous. Choosing a different α and $\underline{r}$ conjugates the matrix of f by a diagonal matrix, so the power series does not depend on them.

Similarly, we define $\det(1 - Xf \mid C^i_\lambda) := \det(1 - Xf \mid C^i_{\lambda,\alpha,\underline{r}})$. Consider the Fredholm series

$$P_+(X) := \prod_{i=0}^{d} \det(1 - Xu_t \mid C^i_\lambda).$$

Suppose that $P_+(X)$ factors as $Q_+(X)S_+(X)$, with $Q_+(X) \in A[X]$, $S_+(X) \in A\{\{X\}\}$, that $Q_+(X)$ and $S_+(X)$ are relatively prime, and that the leading coefficient of $Q_+(X)$ is invertible. Let $Q_+^*(X) = X^{\deg Q_+} Q_+(X^{-1})$. By [Andreatta et al. 2018, Théorème B.2], there is a decomposition $C^\bullet_{\lambda,\alpha,\underline{r}} = N^\bullet_{\alpha,\underline{r}} \oplus F^\bullet_{\alpha,\underline{r}}$, where $Q_+^*(u_t)$ annihilates $N^\bullet_{\alpha,\underline{r}}$ and acts invertibly on $F^\bullet_{\alpha,\underline{r}}$, and the $N^i_{\alpha,\underline{r}}$ are finitely generated and projective.

Lemma 4.4.1. *For any α, α' and $\underline{r}, \underline{r}'$ such that $N^\bullet_{\alpha,\underline{r}}$ and $N^\bullet_{\alpha',\underline{r}'}$ are defined, they are canonically isomorphic.*

Proof. Choose $\underline{r}''$ so that $C^\bullet_{\lambda,\alpha,\underline{r}''}$ injects into $C^\bullet_{\lambda,\alpha,\underline{r}}$ and $C^\bullet_{\lambda,\alpha',\underline{r}'}$. The operator $1 - Q_+^*(u_t)/Q_+^*(0)$ acts as the identity on $N^\bullet_{\alpha,\underline{r}}$, and for sufficiently large n, $(1 - Q_+^*(u_t)/Q_+^*(0))^n$ factors through $N^\bullet_{\alpha,\underline{r}''}$. So we get a canonical isomorphism $N^\bullet_{\alpha,\underline{r}} \cong N^\bullet_{\alpha,\underline{r}''}$, and similarly there is a canonical isomorphism $N^\bullet_{\alpha',\underline{r}'} \cong N^\bullet_{\alpha,\underline{r}''}$. □

Corollary 4.4.2. *There is a decomposition $C^\bullet_\lambda = N^\bullet \oplus F^\bullet$, where $Q_+^*(u_t)$ annihilates $N^\bullet$ and acts invertibly on $F^\bullet$, and the N^i are finitely generated and projective.*

5. Eisenstein and cuspidal contributions to characteristic power series

5.1. ***Preliminaries.*** In this section, we will write $C^\bullet_{G,K^p,\lambda}$ for $C^\bullet_\lambda$ to make it clear which group we are considering. We will also assume that $G(\mathbb{R})$ has discrete series (i.e., $G(\mathbb{R})$ admits representations with essentially square integrable matrix coefficients, or equivalently $G(\mathbb{R})$ has a maximal torus that is compact modulo $Z_G(\mathbb{R})$), since otherwise Urban's eigenvariety will be empty.

In order to construct Urban's eigenvariety, we need the characteristic power series of the Hecke operators to be Fredholm series. However, the power series $\det(1 - Xf \mid C^\bullet_{G,K^p,\lambda})$ includes contributions from both cusp forms and Eisenstein series, and the Eisenstein contribution is generally only a ratio of Fredholm series. We will now define a complex $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$ whose characteristic power series only includes contributions from cusp forms. (This complex will only be useful for defining characteristic power series; we make no attempt to remove the Eisenstein series from the cohomology.)

We will mostly follow [Urban 2011, Section 4.6]. However, there is an error in the handling of the Eisenstein series in [loc. cit.] that we will need to correct. The region of convergence of an Eisenstein series is generally not a union of Weyl chambers. (For example, Sp(6) has two conjugacy classes of parabolic subgroups whose Levis are isomorphic to $\mathrm{GL}(2) \times \mathrm{GL}(1)$. The region of convergence of Eisenstein series coming from these parabolics contains one or two full Weyl chambers and fractions of three others.) Consequently, the set $\mathcal{W}^M_{\mathrm{Eis}}$ defined in [loc. cit.] should depend on the weight of the Eisenstein series. A more careful argument is therefore needed to show that character distribution $I^{cl}_{G,0}(f, \mu)$ has a unique p-adic interpolation. In fact, it appears that the character distribution of Eisenstein series coming from a single parabolic subgroup will generally *not* have a unique interpolation. We will show, however, that the sum of distributions coming from parabolic subgroups that have a common Levi will have a unique interpolation.

Let W_G denote the Weyl group of G. Let Φ_G, $\Phi^\vee_G$ denote the set of roots and coroots, respectively, of the pair $(G_{\mathbb{Q}_p}, T)$, where T is the torus chosen in Section 4. Let Φ^+_G and Φ^-_G denote the subset of roots that are positive and negative, respectively, with respect to B, and we make a similar definition for coroots. Let ρ denote half the sum of the roots in Φ^+_G.

Let F be a finite extension of $\mathbb{Q}_p$. We say that $\mu\colon T_0 \to F^\times$ is an algebraic weight if it can be extended to a homomorphism of algebraic groups $T_F \to (\mathbb{G}_m)_F$. We say that an algebraic weight μ is dominant (resp. regular dominant) if $\langle \alpha^\vee, \mu\rangle \geq 0$ (resp. > 0) for all $\alpha^\vee \in \Phi^{\vee+}_G$.

Suppose that μ is dominant. Then $\mathcal{D}_\mu(g_i I, F)$ has a (nonzero) quotient that is a finite-dimensional F-vector space. We will write L^G_μ for the corresponding local system on either $S_G(K)$ or $\overline{S_G(K)}$.

Lemma 5.1.1. *Let $f = u_t \otimes f^p \in \mathcal{H}'_G$, and let $\mu\colon T_0 \to F^\times$ be an algebraic dominant weight. Then*

$$\det(1 - Xf \mid C^\bullet_{G,K^p,\mu}) \equiv \det(1 - Xf \mid H^\bullet(S_G(K), L^G_\mu)) \pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}$$

where

$$N(\mu, t) := \inf_{w \in W_G \setminus \{\mathrm{id}\}} |t^{(w-1)(\mu+\rho)}|_p.$$

Proof. For the degree 1 term, this is [Urban 2011, Lemma 4.5.2]. The argument used there also works for higher degree terms. ☐

In Section 7, we will consider a family of weights having the property that for any $n \in \mathbb{N}$, the set of points corresponding to regular dominant weights μ satisfying $p^n \mid N(\mu, t)$ is Zariski dense. The characteristic power series for the whole family can then be determined from the $\det(1 - Xf \mid H^\bullet(S_G(K), L^G_\mu))$.

If μ is regular dominant, then the cuspidal subspace of $H^i(S_G(K), L^G_\mu)$ is the interior cohomology $H^i_!(S_G(K), L^G_\mu)$ [Li and Schwermer 2004, Section 5.3], and furthermore (since we assume $G(\mathbb{R})$ has discrete series) the interior cohomology is nonzero only in the middle degree [Borel and Wallach 1980, Theorem III.5.1]. Hence either $\det(1 - Xf \mid H^\bullet_!(S_G(K), L^G_\mu))$ or its reciprocal is a polynomial.

Our goal is to prove a version of Lemma 5.1.1 in which $C^\bullet_{G,K^p,\mu}$ is replaced by a complex $C^\bullet_{G,K^p,\mu,\mathrm{cusp}}$ that we will define, and $H^\bullet(S_G(K), L^G_\mu)$ is replaced by $H^\bullet_!(S_G(K), L^G_\mu)$.

5.2. ***Cohomology of the Borel–Serre boundary.*** Eisenstein series arise from the Borel–Serre boundary $\partial S_G(K) := \overline{S_G(K)} \setminus S_G(K)$ of $S_G(K)$. The boundary has a stratification by locally symmetric spaces of parabolic subgroups of G.

We warn the reader that the Borel–Serre compactification $\overline{S_G(K)}$ is slightly strange. When constructing a locally symmetric space, one usually takes a quotient by the identity component of either $Z_G(\mathbb{R})$ or $A_G(\mathbb{R})$, where A_G is the $\mathbb{Q}$-split part of Z_G. In order to construct Urban's eigenvariety, we need to choose the former option, but the Borel–Serre compactification behaves better with respect to the latter. Consequently, if M is a Levi subgroup of G, then the locally symmetric space for M should be constructed by taking a quotient by the identity component of $Z_G(\mathbb{R})A_M(\mathbb{R})$ rather than that of $Z_M(\mathbb{R})$. However, it will turn out that we only need to consider Levi subgroups for which the two quotients are the same; see Section 5.4 for more details.

Let P be a parabolic subgroup of G, let N be the maximal unipotent subgroup of P, and let $M = P/N$ be its Levi quotient. Let $K^p_P = K^p \cap P(\mathbb{A}^p_f)$, $K_{P,p} = I \cap P(\mathbb{Q}_p)$, $K_P = K^p_P K_{P,p}$. We can define a locally symmetric space $S_P(K_P)$, and there is a locally closed immersion

$$\iota\colon S_P(K_P) \to \overline{S_G(K)}.$$

If P' is another parabolic subgroup of G, then $S_P(K_P)$ and $S_{P'}(K_{P'})$ will have the same image in $\overline{S_G(K)}$ if and only if $P(\mathbb{A}_f)$ and $P'(\mathbb{A}_f)$ are conjugate by an element of $K^p I$.

Let K^p_M, I_M be the images of K^p_P, $K_{P,p}$ in $M(\mathbb{A}^p_f)$, $M(\mathbb{Q}_p)$, respectively. The group I_M is an Iwahori subgroup of M. Let $K_M = I_M K^p_M$. The locally symmetric space $S_P(K_P)$ is a nilmanifold bundle over $S_M(K_M)$. Let

$$\pi : S_P(K_P) \to S_M(K_M)$$

denote the projection.

We can relate $R\pi_*\iota^* L^G_\mu$ to local systems on $S_M(K_M)$ using the Kostant decomposition [Borel and Wallach 1980, Theorem III.3.1]. To define the local systems on $S_M(K_M)$, we first need to choose a quasisplit torus T_M of M. The parabolic subgroup $P_{\mathbb{Q}_p}$ contains a conjugate of B. There is a decomposition $G(\mathbb{Q}_p) = I N_G(S)(\mathbb{Q}_p) B(\mathbb{Q}_p)$, where $N_G(S)$ is the normalizer of the maximal split subtorus S of T; this follows from [Bruhat and Tits 1972, Section 4.2.5, Théorème 5.1.3] as well as from [loc. cit., Proposition 7.3.1]. So $iwBw^{-1}i^{-1} \subseteq P_{\mathbb{Q}_p}$ for some $i \in I$, $w \in N_G(S)(\mathbb{Q}_p)$. We choose i and w to minimize the length of the image of w in the Weyl group W_G. Let T_M be the image of $iwTw^{-1}i^{-1}$ in $M_{\mathbb{Q}_p}$. The obvious isomorphism $T \xrightarrow{\sim} T_M$ determines a length-preserving injection of Weyl groups $W_M \hookrightarrow W_G$. Let W^M denote a set of minimal length coset representatives of $W_M \backslash W_G$.

We have the following isomorphism in the derived category of constructible sheaves on $S_M(K_M)$.

$$R\pi_*\iota^* L^G_\mu \cong \bigoplus_{w' \in W^M} L^M_{w^{-1}(w'(\mu+\rho)-\rho)}[l(w') - \dim N] \tag{5.2.1}$$

Here $l(w')$ denotes the length of w'. To see that the splitting exists in the derived category and not just at the level of cohomology, we observe that the $L^M_{w^{-1}(w'(\mu+\rho)-\rho)}$ have distinct central characters.

5.3. *Hecke action.* We will now define an action of $\mathcal{H}_G$ on the cohomology of $S_M(K_M)$ by constructing a homomorphism $\mathcal{H}_G \to \mathcal{H}_M$. The map $R\pi_*\iota^*$ will be equivariant for this action.

As explained in [Urban 2011, Corollary 4.6.3], for any summand of (5.2.1) with $w \neq w'$, the Hecke eigenvalues of $u_t \in \mathcal{H}'_G$ acting on the cohomology of this summand will be divisible by $N(\mu, t)$. Since our goal is to prove a cuspidal analogue of Lemma 5.1.1, we may ignore these summands and just consider the one with $w = w'$. We are therefore only interested in the local system

$$L^M_{w^{-1}(w(\mu+\rho)-\rho)} = L^M_{\mu+(1-w^{-1})\rho}.$$

Our definition of the homomorphism $\mathcal{H}_G \to \mathcal{H}_M$ will be the same as that of [Urban 2011, 4.1.8], except that our convention for the Hecke operators makes some normalization factors disappear. The Hecke algebra $\mathcal{H}_G$ is generated by operators of the form u_t for $t \in T^-$ and $[K_v g K_v]$ for $v \notin S$, $g \in G(\mathbb{Q}_v)$. Let $u_t \in \mathcal{H}_G$ act as $t^{(1-w^{-1})\rho} u_t \in \mathcal{H}_M$. The double coset $K_v g K_v$ decomposes as a finite union $\bigsqcup_j K_v p_j K_v$ with $p_j \in P(\mathbb{Q}_v)$. Let $[K_v g K_v]$ act as $\sum_j [K_{M,v} m_j K_{M,v}]$, where m_j is the image of p_j in $M(\mathbb{Q}_v)$.

Lemma 5.3.1. *The homomorphism $\mathcal{H}_G \to \mathcal{H}_M$ defined above makes the map*

$$R\pi_*\iota^*\colon H^\bullet(S_G(K), L^G_\mu) \to H^\bullet(S_M(K_M), L^M_{\mu+(1-w^{-1})\rho})[l(w) - \dim N]$$

$\mathcal{H}_G$-equivariant.

Proof. The argument is essentially the same as that of [Urban 2011, 4.1.8, 4.6.1–3]. □

5.4. *Image of the map $R\pi_*\iota^*$.* To simplify some of the analysis that follows, we will observe that some Levis have $H^\bullet(S_M(K_M), L^M_\mu) = 0$ for a Zariski dense subset of weights μ, and hence they cannot contribute to the characteristic power series. The Levi M can have a nonzero contribution only if the following conditions hold (see [Urban 2011, Theorem 4.7.3(ii)′]):

(1) $M(\mathbb{R})$ has discrete series.

(2) The center Z_M of M is generated by its maximal split subgroup, its maximal compact subgroup, and Z_G.

Now assume that M satisfies the above two conditions. We will define an involution

$$\theta\colon X_*(T_M/Z_G) \to X_*(T_M/Z_G).$$

To do this, we first decompose $X_*(T_M/Z_G) \otimes \mathbb{Q}$ into several pieces. We have

$$X_*(T_M/Z_G) \otimes \mathbb{Q} \cong (X_*(Z_M/Z_G) \oplus X_*(T_M \cap M^{\mathrm{der}})) \otimes \mathbb{Q}.$$

There is an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_*(Z_M/Z_G)$. This representation has open kernel, so it becomes semisimple after tensoring with $\mathbb{Q}$. We define θ to be the operator that acts as 1 on the isotypic component of the trivial representation and as -1 on its orthogonal complement and on $X_*(T_M \cap M^{\mathrm{der}})$. Although it is not immediately obvious that θ preserves the lattice $X_*(T_M/Z_G) \subset X_*(T_M/Z_G) \otimes \mathbb{Q}$, the following alternative description of θ will show that it does.

Let T'_M be a maximal torus of $M_{\mathbb{R}}$ that is compact modulo $(Z_M)_{\mathbb{R}}$. Such a torus exists by assumption (1). If C is an algebraically closed field equipped with inclusions $\mathbb{R} \hookrightarrow C$ and $\mathbb{Q}_p \hookrightarrow C$, then the tori $(T_M)_C$ and $(T'_M)_C$ are conjugate in M_C. Each way of expressing $(T'_M)_C$ as a conjugate of $(T_M)_C$ determines an isomorphism $X_*(T_M/Z_G) \simeq X_*(T'_M/Z_G)$. We claim that under any such isomorphism, the action of complex conjugation on $X_*(T'_M/Z_G)$ induces the involution θ on $X_*(T_M/Z_G)$. Indeed, since T'_M is compact modulo center, complex conjugation acts as -1 on $X_*(T'_M \cap M^{\mathrm{der}})$, and assumption (2) guarantees that any element of $X_*(Z_M/Z_G)$ that is fixed by complex conjugation is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

By [Li and Schwermer 2004, Section 3.2], the image of

$$R\pi_*\iota^* \colon H^\bullet(S_G(K), L^G_\mu) \to H^\bullet(S_M(K_M), L^M_{\mu+(1-w^{-1})\rho})[l(w) - \dim N]$$

can have nonzero intersection with the cuspidal part

$$H^\bullet_!(S_M(K_M), L^M_{\mu+(1-w^{-1})\rho})[l(w) - \dim N]$$

only if

$$\langle \alpha^\vee, w(1+\theta)(\mu+\rho))\rangle < 0 \quad \forall \alpha^\vee \in \Phi^{\vee+}_G \setminus \Phi^{\vee+}_M. \tag{5.4.1}$$

If the above equation holds and no Eisenstein series arising from M has a pole at $-w(\mu+\rho)$, then the image contains the cuspidal part. In particular, the image contains the cuspidal part if (5.4.1) is satisfied and

$$|\langle \alpha^\vee, (1+\theta)\mu\rangle| \geq 4|\langle \alpha^\vee, \rho\rangle| \quad \forall \alpha^\vee \in \Phi^\vee_G \setminus \Phi^\vee_M. \tag{5.4.2}$$

The constraint (5.4.1) is archimedean in nature, and therefore appears to provide an obstacle to interpolating Eisenstein series p-adically. To get around this issue, we will combine contributions from parabolic subgroups having common Levis.

We will call a Levi subgroup "relevant" if it satisfies the two conditions listed at the beginning of this section, and we will call a parabolic subgroup relevant if its Levi is relevant. Let $\mathcal{P}_0$ be the set of relevant parabolic subgroups of G, modulo the relation that P_1 and P_2 are considered equivalent if $P_1(\mathbb{A}_f)$ and $P_2(\mathbb{A}_f)$ are conjugate by an element of K^pI. Let $\mathcal{P}$ be the set of relevant parabolic subgroups of G, modulo the relation that P_1 and P_2 are considered equivalent if $P_1(\mathbb{Q}_p)$ and $P_2(\mathbb{Q}_p)$ are conjugate by an element of I. Let $\mathcal{M}$ be the set of relevant Levi subgroups of G, modulo the relation that M_1 and M_2 are considered equivalent if $M_1(\mathbb{Q}_p)$ and $M_2(\mathbb{Q}_p)$ are conjugate by an element of I. There are surjections $\mathcal{P}_0 \to \mathcal{P} \to \mathcal{M}$.

Choose representatives of each element of $\mathcal{P}_0$ and $\mathcal{M}$. If P is the representative of an element of $\mathcal{P}_0$ and M is the representative of its image in $\mathcal{M}$, choose a $g \in G(\mathbb{Q})$ so that $M \subset gPg^{-1}$ and the image of g in $G(\mathbb{Q}_p)$ is in I. This choice determines an identification of M with the Levi quotient of P. We will sometimes identify elements of $\mathcal{P}_0$ and $\mathcal{M}$ with the chosen representatives.

Let $M \in \mathcal{M}$. Assume μ is chosen so that (5.4.2) is satisfied. Then there is exactly one parabolic subgroup P_μ containing M for which (5.4.1) will be satisfied: it is the parabolic determined by the set of coroots $\alpha^\vee$ satisfying

$$\langle \alpha^\vee, (1+\theta)\mu \rangle < 0.$$

So μ determines a section $\mathcal{M} \to \mathcal{P}$ of the projection $\mathcal{P} \to \mathcal{M}$. Let $\mathcal{P}_\mu$ be the image of this section, and let $\mathcal{P}_{0,\mu}$ be the preimage of $\mathcal{P}_\mu$ in $\mathcal{P}_0$. At the end of Section 5.2, we associated each parabolic subgroup of G with an element of W_G; this association determines a map $w\colon \mathcal{P} \to W_G$.

Lemma 5.4.3. *We have*

$$l(w(P_\mu)) = \tfrac{1}{2}|(\Phi_G^- \setminus \Phi_M^-) \cap \theta(\Phi_G^+ \setminus \Phi_M^+)|, \quad (1-\theta)(1-w(P_\mu)^{-1})\rho = \sum_{\alpha \in (\Phi_G^- \setminus \Phi_M^-) \cap \theta(\Phi_G^+ \setminus \Phi_M^+)} \alpha.$$

In particular, $l(w(P_\mu))$ *and* $(1-\theta)(1-w(P_\mu)^{-1})\rho$ *do not depend on* μ.

Proof. By definition,

$$l(w(P_\mu)) = |\{\alpha \in \Phi_G \setminus \Phi_M \mid \langle \alpha^\vee, (1+\theta)\mu \rangle > 0, \langle \alpha^\vee, \mu \rangle < 0\}|$$

Observe that if $\langle \alpha^\vee, \mu \rangle < 0 < \langle \alpha^\vee, \theta\mu \rangle$, then exactly one of the inequalities

$$\langle \alpha^\vee, \mu \rangle < 0 < \langle \alpha^\vee, (1+\theta)\mu \rangle, \quad \langle (-\alpha^\vee\theta), \mu \rangle < 0 < \langle (-\alpha^\vee\theta), (1+\theta)\mu \rangle$$

will be satisfied, and otherwise neither will be satisfied. So

$$|\{\alpha \in \Phi_G \setminus \Phi_M \mid \langle \alpha^\vee, \mu \rangle < 0 < \langle \alpha^\vee, \theta\mu \rangle\}| = 2l(w(P_\mu)).$$

This proves the first item. The same observation also proves the second item. □

We will write $l(M)$ for $l(w(P_\mu))$ and $\rho(M, \mu)$ for $(1-w(P_\mu)^{-1})\rho$.

Now we are almost ready to write down an analogue of Lemma 5.1.1 for cusp forms. The boundary components of $S_G(K)$ whose Eisenstein series contribute to the characteristic power series $\det(1 - Xf \mid H^\bullet(S_G(K), L_\mu^G)) \bmod \mathcal{O}_F[\![N(\mu,t)X]\!]$ are in bijection with elements of $\mathcal{P}_{0,\mu}$. Given $M \in \mathcal{M}$, a choice of a preimage P of M in $\mathcal{P}_{0,\mu}$ determines an open compact subgroup K_M^p of $M(\mathbb{A}_f^p)$, as described in Section 5.2. Let $\mathcal{K}_M^p$ be the collection of all such subgroups.

The analysis of the last few sections gives us the following identity.

Lemma 5.4.4. *For any dominant algebraic weight* $\mu\colon T \to F^\times$ *satisfying* (5.4.2),

$$\frac{\det(1 - Xf \mid H^\bullet(S_G(K), L_\mu^G))}{\det(1 - Xf \mid H_!^\bullet(S_G(K), L_\mu^G))} \equiv \prod_{M \in \mathcal{M}} \prod_{K_M \in \mathcal{K}_{M,\mu}^p} \det(1 - Xf \mid H_!^\bullet(S_M(K_M), L_{\mu+\rho(M,\mu)}^M))^{(-1)^{\dim N - l(M)}} \pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}.$$

In order to interpolate the local systems p-adically, we need to replace $\mathcal{K}_{M,\mu}^p$ and $\rho(M,\mu)$ with something independent of μ.

Proposition 5.4.5. *For any dominant algebraic weights $\mu\colon T\to F^\times$ and $\mu_0\colon T\to F_0^\times$ satisfying* (5.4.2),

$$\frac{\det(1-Xf\mid H^\bullet(S_G(K),L_\mu^G))}{\det(1-Xf\mid H_!^\bullet(S_G(K),L_\mu^G))} \equiv \prod_{M\in\mathcal{M}}\prod_{K_M\in\mathcal{K}_{M,\mu_0}} \det(1-Xf\mid H_!^\bullet(S_M(K_M),L^M_{\mu+\rho(M,\mu_0)}))^{(-1)^{\dim N-l(M)}} \pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}.$$

Proof. We claim that local systems $L^M_{\mu+\rho(M,\mu)}$, $L^M_{\mu+\rho(M,\mu_0)}$ are isomorphic. The isomorphism class of each local system depends only the restriction of the weight to M^{der}. The operator $(1-\theta)/2$ acts as the identity on the character lattice of M^{der}, so the claim follows from Lemma 5.4.3. Furthermore, the isomorphism of local systems induces an $\mathcal{H}_G$-equivariant isomorphism on cohomology. (The isomorphism on cohomology is not $\mathcal{H}_M$-equivariant — the actions of u_t differ by a factor of $t^{\rho(M,\mu)-\rho(M,\mu_0)}$. However, the two homomorphisms $\mathcal{H}_G\to\mathcal{H}_M$ also differ by the same factor, and so the differences cancel each other.)

It remains to explain why can replace $\mathcal{K}_{M,\mu}$ with $\mathcal{K}_{M,\mu_0}$. Essentially, we need to show that if $\pi=\pi_\infty\otimes\pi_p\otimes\pi_f^p$ is an automorphic representation of M, then

$$\sum_{K_M^p\in\mathcal{K}_{M,\mu}^p}\mathrm{tr}(\mathbf{1}_{K_M^p}\mid\pi_f^p)=\mathrm{tr}(\mathbf{1}_{K^p}\mid\mathrm{Ind}_{P_\mu(\mathbb{A}_f^p)}^{G(\mathbb{A}_f^p)}\pi_f^p)$$

is independent of μ. By [Bernstein and Zelevinsky 1977, 2.9–2.10], for any place v, the composition series of the local factor of $\mathrm{Ind}_{P_\mu(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\pi_f^p$ at v is independent of μ. It follows that the trace of $\mathbf{1}_{K^p}$ does not depend on μ. ∎

5.5. *The complex $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$.* Now we fix an algebraic dominant weight μ_0 and let $\lambda\colon T\to A^\times$ be any weight. We define $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$ inductively, assuming that analogous complexes have already been defined for $M\in\mathcal{M}$:

$$C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}:=C^\bullet_{G,K^p,\lambda}\oplus\bigoplus_{M\in\mathcal{M}}\bigoplus_{K_M^p\in\mathcal{K}_{M,\mu_0}^p}C^\bullet_{M,K_M^p,\lambda+\rho(M,\mu_0),\mathrm{cusp}}[l(M)-\dim N-1].$$

Proposition 5.5.1. *Let F be a finite extension of $\mathbb{Q}_p$, let $\mu\colon T\to F^\times$ be an algebraic dominant weight, and let $f=u_t\otimes f^p\in\mathcal{H}'_G$. If μ is sufficiently general, then*

$$\det(1-Xf\mid C^\bullet_{G,K^p,\mu,\mathrm{cusp}})\equiv\det(1-Xf\mid H_!^\bullet(S_G(K),L_\mu^G))\pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}.$$

Proof. By induction, we may assume that the proposition holds for all Levi subgroups of G.

$$\begin{aligned}\det(1-Xf\mid C^\bullet_{G,K^p,\mu,\mathrm{cusp}})&\\ \equiv\det(1-Xf\mid H^\bullet(S_G(K),L_\mu^G))&\prod_{M,K_M}\det(1-Xf\mid H_!^\bullet(S_M(K_M),L^M_{\mu+\rho(M,\mu_0)}))^{(-1)^{l(M)-\dim N+1}}\\ \equiv\det(1-Xf\mid H_!^\bullet(S_G(K),L_\mu^G))&\pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}\end{aligned}$$

where we used the induction hypothesis and Lemma 5.1.1 in the second line and Proposition 5.4.5 in the third line. We also use the fact that $\rho(M, \mu_0)$ is M-dominant, and so $\mathcal{O}_F[\![N(\mu+\rho(M,\mu_0),t)X]\!] \subseteq \mathcal{O}_F[\![N(\mu,t)X]\!]$. □

The analysis of Section 4.4 applies equally well to $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$. For any $f \in \mathcal{H}'_G$, we may define a characteristic power series $\det(1-Xf \mid C^\bullet_{G,K^p,\lambda,\mathrm{cusp}})$. If the Fredholm series $P_+(X) = \prod_i \det(1-Xf \mid C^i_{G,K^p,\lambda,\mathrm{cusp}})$ has a factorization $P_+ = Q_+S_+$ with Q_+ a polynomial with invertible leading coefficient, then this factorization induces a decomposition $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}} = N^\bullet \oplus F^\bullet$.

Remark 5.5.2. One can use Proposition 7.1.2 to show that for $f \in \mathcal{H}'_G$, $\det(1-Xf \mid C^\bullet_{G,K^p,\lambda,\mathrm{cusp}})$ is a Fredholm series. We will not need to prove this fact for arbitrary A and λ, so we leave the details of the argument as an exercise for the reader.

6. Theory of determinants

Urban's eigenvariety construction makes use of pseudocharacters. Chenevier's theory of determinants [2014] is equivalent to the theory of pseudocharacters when the rings involved are $\mathbb{Q}$-algebras [Chenevier 2014, Proposition 1.27], but is better behaved in general. Since we work with rings in which p is not invertible, we will use determinants. (However, it is probably not strictly necessary to use determinants, as we work with rings that are p-torsion-free. See Corollary 7.2.2 and the proof of Lemma 7.4.1.)

We will recall some basic definitions from [Chenevier 2014] and prove a lemma concerning the ratio of two determinants.

Definition 6.1 [Chenevier 2014, Sections 1.1–1.5]. Let A be commutative ring, and let R be an A-module. An *A-valued polynomial law on R* is a rule that assigns to any commutative A-algebra B a map of sets $D_B : R \otimes_A B \to B$ that is functorial in the sense that for any A-algebra homomorphism $f : B \to B'$,

$$D_{B'} \circ (\mathrm{id}_R \otimes f) = f \circ D_B.$$

Let d be a nonnegative integer. We say that a polynomial law D is *homogeneous of degree d* if

$$D_B(br) = b^d D_B(r) \quad \forall B, b \in B, r \in R \otimes_A B.$$

Now assume that R is an A-algebra. We say that a polynomial law D is *multiplicative* if

$$D_B(1) = 1, \quad D_B(rr') = D_B(r)D_B(r') \quad \forall B, r, r' \in R \otimes_A B.$$

We say that a polynomial law D is a *determinant of dimension d* if it is homogeneous of degree d and multiplicative.

Example 6.2. Let M be an R-module that is projective of rank d as an A-module. Then the rule that sends $r \in R \otimes_A B$ to $\det(r \mid M \otimes_A B)$ is a determinant of dimension d.

Lemma 6.3 [Roby 1963, Proposition I.1]. *Let A be a commutative ring, and let R be an A-module. Let D be an A-valued polynomial law on R that is homogeneous of degree d, let n be a positive integer, and* *let $r_1, \dots, r_n \in R$. Then $D_{A[X_1,\dots,X_n]}(X_1r_1 + \cdots + X_nr_n)$ is a homogeneous polynomial of degree d in $X_1, \dots, X_n$.*

Lemma 6.4. *Let A be a commutative ring, let R be an A-algebra, and let D^+, D^- be A-valued determinants on R of dimension d_+, d_-, respectively, with $d_+ \geq d_-$. Let $d = d_+ - d_-$. There is at most one determinant D of dimension d satisfying $D^+_B(r) = D^-_B(r)D_B(r)$ for all A-algebras B and all $r \in R \otimes_A B$.*

The following are equivalent:

(1) *There exists a determinant D satisfying the above condition.*

(2) *For any commutative A-algebra B and $r \in R \otimes_A B$, the quotient*

$$D^+_{B[X]}(1 + Xr)/D^-_{B[X]}(1 + Xr)$$

exists in $B[X]$ and has degree at most d.

(3) *For any positive integer n and $r_1, \dots, r_n \in R$, the quotient*

$$D^+_{A[X_1,\dots,X_n]}(1 + X_1r_1 + \cdots + X_nr_n)/D^-_{A[X_1,\dots,X_n]}(1 + X_1r_1 + \cdots + X_nr_n)$$

exists in $A[X_1, \dots, X_n]$ and has total degree at most d.

Proof. Let B be a commutative A-algebra, and let $r \in R \otimes_A B$. If $D^-_B(r)$ is not a zero divisor and the quotient $D^+_B(r)/D^-_B(r)$ exists, then we will denote this quotient by $F_B(r)$. Note that $D^-_{B[X]}(1 + Xr)$ has constant term 1 by functoriality with respect to $X \mapsto 0$, so it is not a zero divisor. Similarly, $D^-_{A[X_1,\dots,X_n]}(1 + X_1r_1 + \cdots + X_nr_n)$ has constant term 1 and is not a zero divisor.

First, we check that D is uniquely determined if it exists. Suppose D is a determinant satisfying the conditions of the lemma. Let B be a commutative A-algebra, and let $r \in R \otimes_A B$. We claim that the following quantities are equal:

- $D_B(r)$.
- The coefficient of X^dY^0 in $D_{B[X,Y]}(Y + Xr)$.
- The coefficient of X^d in $D_{B[X]}(1 + Xr)$.

To see that the first and second quantities are equal, apply functoriality with respect to $X \mapsto 1$, $Y \mapsto 0$, using Lemma 6.3 to show that $D_{B[X,Y]}(Y + Xr)$ has no X^nY^0 term for $n \neq d$. To see that the second and third quantities are equal, apply functoriality with respect to $Y \mapsto 1$, using Lemma 6.3 to show that $D_{B[X,Y]}$ has no X^dY^n term for $n \neq 0$. Finally, observe that $F_{B[X]}(1 + Xr)$ must exist and must equal $D_{B[X]}(1 + Xr)$. So $D_B(r)$ must be equal to the coefficient of X^d in $F_{B[X]}(1 + Xr)$. Hence D is uniquely determined if it exists.

Lemma 6.3 shows that (1) $\Rightarrow$ (3).

Now we prove (3) $\Rightarrow$ (2). Assume (3) holds. Choose a commutative A-algebra B and $r \in B \otimes_A R$. Condition (3) implies that $F_{A[X_1,\dots,X_n]}(1 + X_1r_1 + \cdots + X_nr_n)$ exists and has total degree at most d. Then by functoriality with respect to $X_i \mapsto Xb_i$, $F_{B[X]}(1 + Xr)$ exists and has degree at most d. This proves (3) $\Rightarrow$ (2).

Now we will show that (2) $\Rightarrow$ (1). Assume that condition (2) holds. Define $D_B(r)$ be the coefficient of X^d in $F_{B[X]}(1+Xr)$. We know that $D^+_{B[X]}(1+Xr)$ (resp. $D^-_{B[X]}(1+Xr)$, $F_{B[X]}(1+Xr)$) has degree at most d_+ (resp. d_-, d), and we have already showed that the coefficient of X^{d_+} (resp. X^{d_-}, X^d) is $D^+_B(r)$ (resp. $D^-_B(r)$, $D_B(r)$). So $D^+_B(r) = D^-_B(r)D_B(r)$.

It remains to show that D is a determinant. Since D^+ and D^- are functorial, D is as well. To show that D is homogeneous of degree d, observe that the map $X \mapsto bX$ multiplies the coefficient of X^d in $F_{B[X]}(1+Xr)$ by b^d.

Finally, we check that D is multiplicative. We have $F_{B[X]}(1+X) = (1+X)^d$, so $D_B(1) = 1$. Observe that $D_B(r_1)D_B(r_2)$ is the coefficient of $(X_1X_2)^d$ in $F_{B[X_1,X_2]}(1+X_1r_1+X_2r_2+X_1X_2r_1r_2)$. This is the same as the coefficient of $X_1^0X_2^0X_3^d$ in $F_{B[X_1,X_2,X_3]}(1+X_1r_1+X_2r_2+X_3r_1r_2)$, since X_3^d is the only monomial of total degree d in $B[X_1, X_2, X_3]$ that maps to $(X_1X_2)^d$ under $X_3 \mapsto X_1X_2$. Then applying $X_1 \mapsto 0$, $X_2 \mapsto 0$, we find that $D_B(r_1)D_B(r_2)$ is the coefficient of X_3^d in $F_{B[X_3]}(1+X_3r_1r_2)$, which is $D_B(r_1r_2)$. This concludes the proof that (2) $\Rightarrow$ (1). □

Corollary 6.5. *Retain the notation of Lemma 6.4. Let $A \hookrightarrow A'$ be an injective map of commutative rings. Suppose that there exists an A'-valued determinant D' on $R \otimes_A A'$ satisfying $D^+_B(r) = D^-_B(r)D'_B(r)$ for any A'-algebra B and $r \in R \otimes_A B$. Then there exists an A-valued determinant D on R satisfying $D^+_B(r) = D^-_B(r)D_B(r)$ for any A-algebra B and $r \in R \otimes_A B$.*

Proof. Apply the equivalence (1) $\Leftrightarrow$ (3) of Lemma 6.4. Observe that $F_{A[X_1,\dots,X_n]}(1+X_1r_1+\dots+X_nr_n)$ exists and has degree total degree $\le d$ if and only if $F_{A[\![X_1,\dots,X_n]\!]}(1+X_1r_1+\dots+X_nr_n)$ is a polynomial of total degree $\le d$. Since $A \hookrightarrow A'$ is injective, if $F_{A'[\![X_1,\dots,X_n]\!]}(1+X_1r_1+\dots+X_nr_n)$ is a polynomial of total degree $\le d$, then $F_{A[\![X_1,\dots,X_n]\!]}(1+X_1r_1+\dots+X_nr_n)$ is as well. □

Definition 6.6 [Chenevier 2014, Section 1.17, Lemma 1.19(i)]. Let D be an A-valued determinant on R. We denote by $\ker(D)$ the set of $r \in R$ such that for all B and all $r' \in B \otimes_A R$, $D_B(1+r'r) = 1$.

Remark 6.7. Let M be projective A-module of rank d, let $\rho\colon R \to \operatorname{End}(M)$ be a homomorphism, and let D be the determinant associated with ρ, as in Example 6.2. Then $\ker\rho \subseteq \ker D$. Conversely, if $r \in \ker D$, then $D_{A[X]}(X-r) = X^d$, so $r^d \in \ker\rho$ by the Cayley–Hamilton theorem.

Remark 6.8. Chenevier also defines the Cayley–Hamilton ideal $\mathrm{CH}(D)$. Assume D comes from a homomorphism $\rho\colon R \to \operatorname{End}(M)$. Then $\mathrm{CH}(D) \subseteq \ker\rho$. So we might think of $\ker D$ as an upper bound for $\ker\rho$ and $\mathrm{CH}(D)$ as a lower bound. If A is Noetherian, then since $\ker\rho \subseteq \ker D$, $R/\ker D$ is a finite A-module. However, $R/\mathrm{CH}(D)$ need not be a finite A-module, making it more difficult to use $\mathrm{CH}(D)$ in the construction of eigenvarieties.

For a concrete example, consider $A = \mathbb{Q}$, $M = \mathbb{Q}^2$, $R = \mathbb{Q}[T_1, T_2, \dots]$, and let ρ be a map that sends each T_i to a nilpotent upper triangular matrix. Then $\ker D = (T_1, T_2, \dots)$ (so $R/\ker D \cong \mathbb{Q}$), $\ker \mathrm{CH}(D) = (T_1, T_2, \dots)^2$ (so $R/\mathrm{CH}(D)$ is not finite type over $\mathbb{Q}$), and $R/\ker\rho$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{Q}[\epsilon]/(\epsilon^2)$.

7. Construction of the eigenvariety

7.1. ***Weight space and Fredholm series.*** Now we are ready to define the eigenvariety following [Urban 2011, Section 5]. We will use Huber's theory of adic spaces [1993; 1994; 1996]; see also [Scholze and Weinstein 2019, Sections 2–5] for a modern introduction. Some aspects of our approach follow [Andreatta et al. 2018, Appendice B].

We return to the setup of sections 4–5. We continue to assume that $G(\mathbb{R})$ has discrete series. Let T' be the quotient of T_0 by the closure of $Z_G(\mathbb{Q})G_\infty^+ K^p \cap T_0$. We define the weight space

$$\mathcal{W} := \operatorname{Spa}(\mathbb{Z}_p[\![T']\!], \mathbb{Z}_p[\![T']\!])^{\mathrm{an}}.$$

Let $\mathcal{U} = \operatorname{Spa}(A, A^+)$ be an open affinoid subset of $\mathcal{W}$ with A a complete Tate $\mathbb{Z}_p$-algebra (which is automatically Noetherian). Let $\lambda\colon T_0 \to A^\times$ be the tautological character induced by the map $T_0 \to T' \to \mathbb{Z}_p[\![T']\!]$.

For any $f \in \mathcal{H}'_G \otimes_{\mathbb{Z}_p} A$, let

$$P_f(X) := \det(1 - Xf \mid C^\bullet_{G,K^p,\lambda,\mathrm{cusp}})^{(-1)^{d/2}}.$$

Note that $d = \dim S_G(K)$ is even since $G(\mathbb{R})$ has discrete series. If $\mathcal{V}$ is an open subspace of $\mathcal{W}$, and $f \in \mathcal{H}'_G \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{W}}(\mathcal{V})$, then we define $P_f(X)$ by gluing.

Definition 7.1.1. Let $\mathcal{V}$ be an open subspace of $\mathcal{W}$. A series $f \in \mathcal{O}_{\mathcal{W}}(\mathcal{V})[\![X]\!]$ is called a Fredholm series if it is the power series expansion of some global section of $\mathcal{V} \times \mathbb{A}^1$ and its leading coefficient is 1.

This definition agrees with Definition 2.2.1 if $\mathcal{V} = \operatorname{Spa}(A, A^+)$ with A a complete Tate $\mathbb{Z}_p$-algebra.

Proposition 7.1.2. *For $f \in \mathcal{H}'_G$, the series $P_f(X) \in \mathcal{O}_{\mathcal{W}}(\mathcal{W})[\![X]\!]$ is a Fredholm series.*

Proof. Observe that $\mathcal{O}_{\mathcal{W}}(\mathcal{W}) = \mathcal{O}^+_{\mathcal{W}}(\mathcal{W}) = \mathbb{Z}_p[\![T']\!]$. Let T'_{tf} be a maximal torsion-free subgroup of T'. The topology on $\mathbb{Z}_p[\![T'_{\mathrm{tf}}]\!]$ is induced by any norm corresponding to a Gauss point of the wide open polydisc $\operatorname{Spa}(\mathbb{Z}_p[\![T'_{\mathrm{tf}}]\!], \mathbb{Z}_p[\![T'_{\mathrm{tf}}]\!]) \times_{\operatorname{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Similarly, the topology on $\mathbb{Z}_p[\![T']\!]$ is induced by a supremum of a finite collection of norms corresponding to Gauss points of $\mathcal{W} \times_{\operatorname{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. So it suffices to check that the restrictions of $P_f(X)$ to Gauss points of $\mathcal{W}$ are Fredholm series. The Gauss points are characteristic zero points, so we may apply the argument of [Urban 2011, Theorem 4.7.3(iii)] along with Proposition 5.5.1. ☐

We will write $P(X)$ for $P_{u_t}(X)$. We define the spectral variety $\mathcal{Z} \subset \mathcal{W} \times \mathbb{A}^1$ to be the zero locus of $P(X)$, and we define $w\colon \mathcal{Z} \to \mathcal{W}$ to be the projection. We also define

$$P_+(X) := \prod_i \det(1 - Xu_t \mid C^i_{G,K^p,\lambda,\mathrm{cusp}}).$$

7.2. ***Weight space and its characteristic zero subspace.*** Before constructing the eigenvariety, we will prove a result that will allow us to deduce information about the behavior of the eigenvariety at the boundary from the characteristic zero part of the eigenvariety.

Lemma 7.2.1. *Let $\mathcal{U} = \mathrm{Spa}(A, A^+)$ by an affinoid adic space. Assume that A is finitely generated over a Noetherian ring of definition. Let $a \in A$ be an element that is not a zero divisor.*

(1) *For any open $\mathcal{V} \subseteq \mathcal{U}$, a is not a zero divisor in $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$.*

(2) *Assume A is Tate. There exists rational subset $\mathcal{V} \subseteq \mathcal{U}$ such that the restriction $A \to \mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is injective and $a \in \mathcal{O}_{\mathcal{U}}(\mathcal{V})^\times$.*

Proof. To prove the first item, it suffices to consider the case where $\mathcal{V}$ is a rational subset. Then $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ is flat over A by [Huber 1993, Corollary 1.7(i)] and $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is flat over $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ by [Huber 1996, Proposition 1.6.7(i), Lemma 1.7.6]. Since the multiplication-by-a map is injective on A, it must be injective on $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ as well.

To prove the second item, choose a Noetherian ring of definition $A_0 \subset A$ and a topologically nilpotent $\alpha \in A^\times \cap A_0$. After multiplying a by a power of α, we may assume $a \in A_0$. By the Artin–Rees lemma, there exists an integer $k \geq 1$ so that $\alpha^n A_0 \cap a A_0 \subseteq \alpha^{n-k} a A_0$ for all $n \geq k$. Let $\mathcal{V} \subset \mathcal{U}$ be the rational subset defined by the inequality $|a| \geq |\alpha^k|$. We claim that $A \to \mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is injective.

The ring $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is the α-adic completion of $A_0[1/a]$, and the completion of $A_0[\alpha^k/a]$ is a ring of definition. Let $b \in A$, and suppose the image of b in $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is zero. Then for each $n \in \mathbb{N}$, $b \in \alpha^n A_0[\alpha^k/a]$. Then there exists $m \in \mathbb{N}$ so that $a^m b \in \alpha^n (a, \alpha^k)^m A_0$. One can then show by induction on m that $b \in \alpha^n A_0$. Since A_0 is α-adically separated, this implies $b = 0$. □

Corollary 7.2.2. *Let $\mathcal{U} = \mathrm{Spa}(A, A^+)$ be an open affinoid subspace of $\mathcal{W}$, with A complete Tate. Then p is not a zero divisor of $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$, and there exists a rational subset $\mathcal{V} \subseteq \mathcal{U}$ such that $A \to \mathcal{O}_{\mathcal{W}}(\mathcal{V})$ is injective and $p \in \mathcal{O}_{\mathcal{W}}(\mathcal{V})^\times$.*

7.3. *Pieces of the eigenvariety.* Now we construct the individual pieces of the eigenvariety. Let $z \in \mathcal{Z}$. By [Andreatta et al. 2018, Corollaire B.1], there exists an open affinoid neighborhood $\mathcal{U} = \mathrm{Spa}(A, A^+)$ of $w(z)$ and a factorization $P_+(X) = Q_+(X) S_+(X)$, with $Q_+(X) \in A[X]$, $S_+(X) \in A\{\{X\}\}$, such that $Q_+(X)$ and $S_+(X)$ are relatively prime, Q_+ vanishes at x, and the leading coefficient of Q_+ is invertible. The factorization of P_+ induces a factorization $P(X) = Q(X)S(X)$ satisfying similar properties. The factorization also determines a subcomplex $N^\bullet$ of $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$, as described in Sections 4.4 and 5.5.

Proposition 7.3.1. *Let D^+ be the determinant associated with the action of $\mathcal{H}_G \otimes_{\mathbb{Z}_p} A$ on $\bigoplus_{i \equiv d/2 (2)} N^i$, and let D^- be the determinant associated with the action of $\mathcal{H}_G \otimes_{\mathbb{Z}_p} A$ on $\bigoplus_{i \equiv d/2+1(2)} N^i$. Then there exists a determinant D so that $D^+ = D^- D$.*

Proof. Let $R = \mathcal{H}_G \otimes_{\mathbb{Z}_p} A$. As in Lemma 6.4, if B is an A-algebra and $r \in R \otimes_A B$ such that $D^-_B(r)$ is not a zero divisor and the ratio $D^+_B(r)/D^-_B(r)$ exists in B, we write $F_B(r)$ for this ratio. Let d_+ and d_- be the dimensions of D^+ and D^-, respectively, and let $d = d_+ - d_-$.

By Corollary 7.2.2, we can find a rational subset $\mathcal{V} \subset \mathcal{U}$ so that the restriction $A \to \mathcal{O}_{\mathcal{W}}(\mathcal{V})$ is injective and p is invertible on $\mathcal{V}$. Since $\mathcal{V}$ is a reduced rigid space, the natural map $\mathcal{O}_{\mathcal{W}}(\mathcal{V}) \to \prod_x k_x$ is injective, where the product runs over rigid analytic points $x \in \mathcal{V}$ and k_x is the residue field of x. For each x, write $D^+|_{k_x}$ (resp. $D^-|_{k_x}$) for the base change of D^+ (resp. D^-) along $A \to k_x$. By [Urban 2011, Lemma 4.1.12

and Theorem 4.7.3iii], the difference of the pseudocharacters corresponding to $D^+|_{k_x}$ and $D^-|_{k_x}$ is again a pseudocharacter. By the equivalence of pseudocharacters and determinants in characteristic zero [Chenevier 2014, Proposition 1.27], there exists a determinant $D|_{k_x}$ satisfying $D^+|_{k_x} = D^-|_{k_x} D|_{k_x}$. Then by Corollary 6.5, there exists a determinant D satisfying $D^+ = D^- D$. □

Let

$$h_{\mathcal{U},Q_+} := (\mathcal{H}_G \otimes_{\mathbb{Z}_p} A)/\ker(D).$$

We will use the extension $A \to h_{\mathcal{U},Q_+}$ to construct an adic space $\mathcal{E}_{\mathcal{U},Q_+}$ over $\mathcal{U}$.

Lemma 7.3.2. *The ring $h_{\mathcal{U},Q_+}$ is a finite A-module.*

Proof. Since $\ker(D)$ contains any operator that annihilates $N^\bullet$, $h_{\mathcal{U},Q_+}$ can be identified with a subquotient of $\bigoplus_i \operatorname{End} N^i$. In particular, $h_{\mathcal{U},Q_+}$ must be finitely generated as an A-module. □

We give $h_{\mathcal{U},Q_+}$ the "A-module topology" defined in [Huber 1994, Section 2].

Lemma 7.3.3. *The ring $h_{\mathcal{U},Q_+}$ is Tate and has a Noetherian ring of definition.*

Proof. Choose A-module generators $a_1, \dots, a_n$ of $h_{\mathcal{U},Q_+}$. Choose $m_{ijk} \in h_{\mathcal{U},Q_+}$ so that for each i, j, $a_i a_j = \sum_{k=1}^n m_{ijk} a_k$. Let A_0 be a ring of definition of A, and let α be a topologically nilpotent unit of A contained in A_0. There exists an integer ℓ so that $\alpha^\ell m_{ijk} \in A_0$ for all i, j, k. Let $h_{\mathcal{U},Q_+,0}$ be the A_0-submodule of $h_{\mathcal{U},Q_+}$ generated by $1, \alpha^\ell a_1, \dots, \alpha^\ell a_n$; then $h_{\mathcal{U},Q_+,0}$ is an open subring of $h_{\mathcal{U},Q_+}$. Then $h_{\mathcal{U},Q_+,0}$ is Noetherian since A_0 is Noetherian, and $h_{\mathcal{U},Q_+,0}$ inherits the α-adic topology from A_0. So $h_{\mathcal{U},Q_+}$ has a Noetherian ring of definition and is Tate. □

Let $h^+_{\mathcal{U},Q_+}$ be the normal closure of A^+ in $h_{\mathcal{U},Q_+}$. Then $(h_{\mathcal{U},Q_+}, h^+_{\mathcal{U},Q_+})$ is a Huber pair. We define $\mathcal{E}_{\mathcal{U},Q_+} := \operatorname{Spa}(h_{\mathcal{U},Q_+}, h^+_{\mathcal{U},Q_+})$.

Since $Q^*(X)$ is the characteristic polynomial of u acting on $N^\bullet$, it follows from [Chenevier 2014, Lemma 1.12(iv)] that $Q^*(u)$ is in $\ker(D)$, and so there is a canonical map $\mathcal{E}_{\mathcal{U},Q_+} \to \mathcal{Z}$.

7.4. *Gluing.* We will glue the $\mathcal{E}_{\mathcal{U},Q_+}$ as in [Buzzard 2007, Section 5]. We need the following lemma to verify that the pieces can be glued.

Lemma 7.4.1. *If $\mathcal{U}' \subset \mathcal{U}$ are affinoid subspaces of $\mathcal{W}$, then there is a canonical isomorphism $\mathcal{E}_{\mathcal{U}',Q_+} \cong \mathcal{E}_{\mathcal{U},Q_+} \times_{\mathcal{U}} \mathcal{U}'$.*

Proof. By Corollary 7.2.2, p is not a zero divisor in $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$ and $\mathcal{O}_{\mathcal{W}}(\mathcal{U}')$. Hence $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$ and $\mathcal{O}_{\mathcal{W}}(\mathcal{U}')$ are torsion-free $\mathbb{Z}$-modules. The map $\mathcal{O}_{\mathcal{W}}(\mathcal{U}) \to \mathcal{O}_{\mathcal{W}}(\mathcal{U}')$ is flat by [Huber 1996, Lemma 1.7.6]. By the argument of [Rydh 2008, Proposition I.2.2.4], if A is a ring that is a torsion-free $\mathbb{Z}$-module, then the kernel of an A-valued determinant is the same as the kernel of the associated pseudocharacter. By [Rydh 2008, Proposition I.2.2.8], the formation of the kernel of a pseudocharacter commutes with flat base change. So the formation of the kernel of a determinant commutes with the base change $\mathcal{O}_{\mathcal{W}}(\mathcal{U}) \to \mathcal{O}_{\mathcal{W}}(\mathcal{U}')$. Then $h_{\mathcal{U}',Q_+} \cong h_{\mathcal{U},Q_+} \otimes_{\mathcal{O}_{\mathcal{W}}(\mathcal{U})} \mathcal{O}_{\mathcal{W}}(\mathcal{U}')$. Since $h_{\mathcal{U},Q_+}$ is finite over $\mathcal{O}_{\mathcal{W}}(\mathcal{U}')$, it follows that $\mathcal{E}_{\mathcal{U}',Q_+} \cong \mathcal{E}_{\mathcal{U},Q_+} \times_{\mathcal{U}} \mathcal{U}'$. □

One can also show, using essentially the same proof as [Urban 2011, Proposition 5.3.5], that if Q_+ and Q'_+ are relatively prime, then there is a canonical isomorphism $\mathcal{E}_{\mathcal{U},Q_+Q'_+} \cong \mathcal{E}_{\mathcal{U},Q_+} \sqcup \mathcal{E}_{\mathcal{U},Q'_+}$.

Theorem 7.4.2. *The $\mathcal{E}_{\mathcal{U},Q_+}$ can be glued to form an adic space $\mathcal{E}$. Furthermore, $\mathcal{E}$ is equidimensional in the sense of [Huber 1996, Definition 1.8.1] and the morphism $\mathcal{E} \to \mathcal{Z}$ is finite and surjective.*

Proof. To show that the morphism $\mathcal{E} \to \mathcal{Z}$ is finite, we observe that $\mathcal{Z}$ can be covered by open sets whose preimage in $\mathcal{E}$ is contained in some $\mathcal{E}_{\mathcal{U},Q_+}$. The finiteness of the morphism $\mathcal{E} \to \mathcal{Z}$ then follows from the finiteness of the maps $\mathcal{E}_{\mathcal{U},Q_+} \to \mathcal{U}$.

Now we check that the morphism is surjective. Let $z \in \mathcal{Z}$, and let k be the residue field of z. Observe that $\operatorname{Spec} \mathcal{H}_G \to \operatorname{Spec} \mathbb{Z}_p[u_t]$ is surjective, so $\mathcal{H}_G \otimes_{\mathbb{Z}_p[u_t]} k$ cannot be the zero ring. The image of $\ker(D)$ in $\mathcal{H}_G \otimes_{\mathbb{Z}_p[u_t]} k$ is contained in the kernel of the base change of D to $\mathcal{H}_G \otimes_{\mathbb{Z}_p[u_t]} k$. Therefore the image of $\ker(D)$ cannot be the unit ideal, and so there must be a point of $\mathcal{E}$ lying above z.

Finally, we show that $\mathcal{E}$ is equidimensional. The weight space $\mathcal{W}$ has the same dimension as its characteristic zero part. By [Urban 2011, Theorem 5.3.7(iii)], the characteristic zero part of $\mathcal{E}$ is equidimensional of dimension $\dim \mathcal{W}$. By Lemma 7.2.1(2), any nonempty open $\mathcal{U} \subseteq \mathcal{E}$ has nonempty characteristic zero part, so it has dimension at least $\dim \mathcal{W}$. Conversely, since $\mathcal{E}$ is locally finite over $\mathcal{W}$, $\mathcal{E}$ has dimension at most $\dim \mathcal{W}$ by [Huber 1996, Example 1.8.9(ii)]. □

Acknowledgments

I would like to thank my advisor, Eric Urban, for suggesting that I consider the problem of adapting cohomological constructions of eigenvarieties to include the boundary of weight space, and for helpful discussions. I would also like to thank David Hansen and Shrenik Shah for helpful discussions. I would like to thank the anonymous referee for pointing out some mistakes and providing many helpful suggestions for improving the exposition. Discussions with Ana Caraiani and Christian Johansson prompted me to make some improvements to the exposition in Section 5. I would like to thank Christian Johansson and James Newton for informing me about their work.

References

[Amice 1964] Y. Amice, “Interpolation p-adique”, *Bull. Soc. Math. France* **92** (1964), 117–180. MR Zbl

[Andreatta et al. 2018] F. Andreatta, A. Iovita, and V. Pilloni, “Le halo spectral”, *Ann. Sci. École Norm. Sup.* (4) **51**:3 (2018), 603–655. MR Zbl

[Ash and Stevens 2008] A. Ash and G. Stevens, “p-adic deformations of arithmetic cohomology”, preprint, 2008, Available at http://math.bu.edu/people/ghs/preprints/Ash-Stevens-02-08.pdf.

[Bernstein and Zelevinsky 1977] I. N. Bernstein and A. V. Zelevinsky, “Induced representations of reductive p-adic groups, I”, *Ann. Sci. École Norm. Sup.* (4) **10**:4 (1977), 441–472. MR Zbl

[Birkbeck 2019] C. Birkbeck, “Slopes of overconvergent Hilbert modular forms”, *Exp. Math.* (online publication January 2019).

[Borel and Wallach 1980] A. Borel and N. R. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Ann. of Math. Studies **94**, Princeton Univ. Press, 1980. MR Zbl

[Bruhat and Tits 1972] F. Bruhat and J. Tits, “Groupes réductifs sur un corps local”, *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5–251. MR Zbl

[Bruhat and Tits 1984] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local, II: Schémas en groupes, existence d'une donnée radicielle valuée", *Inst. Hautes Études Sci. Publ. Math.* **60** (1984), 197–376. MR Zbl

[Buzzard 2007] K. Buzzard, "Eigenvarieties", pp. 59–120 in *L-functions and Galois representations* (Durham, UK, 2004), edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge Univ. Press, 2007. MR Zbl

[Buzzard and Kilford 2005] K. Buzzard and L. J. P. Kilford, "The 2-adic eigencurve at the boundary of weight space", *Compos. Math.* **141**:3 (2005), 605–619. MR Zbl

[Chenevier 2014] G. Chenevier, "The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings", pp. 221–285 in *Automorphic forms and Galois representations, I* (Durham, UK, 2011), edited by F. Diamond et al., London Math. Soc. Lecture Note Ser. **414**, Cambridge Univ. Press, 2014. MR Zbl

[Coleman 1996] R. F. Coleman, "Classical and overconvergent modular forms", *Invent. Math.* **124**:1-3 (1996), 215–241. MR Zbl

[Coleman 1997] R. F. Coleman, "p-adic Banach spaces and families of modular forms", *Invent. Math.* **127**:3 (1997), 417–479. MR Zbl

[Coleman and Mazur 1998] R. Coleman and B. Mazur, "The eigencurve", pp. 1–113 in *Galois representations in arithmetic algebraic geometry* (Durham, UK, 1996), edited by A. J. Scholl and R. L. Taylor, London Math. Soc. Lecture Note Ser. **254**, Cambridge Univ. Press, 1998. MR Zbl

[Colmez 2010] P. Colmez, "Fonctions d'une variable p-adique", pp. 13–59 in *Représentations p-adiques de groupes p-adiques, II: Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (ϕ, Γ)*-modules*, edited by L. Berger et al., Astérisque **330**, Soc. Math. France, Paris, 2010. MR Zbl

[Emerton 2006a] M. Emerton, "Jacquet modules of locally analytic representations of p-adic reductive groups, I: Construction and first properties", *Ann. Sci. École Norm. Sup.* (4) **39**:5 (2006), 775–839. MR Zbl

[Emerton 2006b] M. Emerton, "On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms", *Invent. Math.* **164**:1 (2006), 1–84. MR Zbl

[Hansen 2017] D. Hansen, "Universal eigenvarieties, trianguline Galois representations, and p-adic Langlands functoriality", *J. Reine Angew. Math.* **730** (2017), 1–64. MR Zbl

[Hida 1986] H. Hida, "Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms", *Invent. Math.* **85**:3 (1986), 545–613. MR Zbl

[Hida 1988] H. Hida, "On p-adic Hecke algebras for GL_2 over totally real fields", *Ann. of Math.* (2) **128**:2 (1988), 295–384. MR Zbl

[Hida 1994] H. Hida, "p-adic ordinary Hecke algebras for GL(2)", *Ann. Inst. Fourier* (*Grenoble*) **44**:5 (1994), 1289–1322. MR Zbl

[Huber 1993] R. Huber, "Continuous valuations", *Math. Z.* **212**:3 (1993), 455–477. MR Zbl

[Huber 1994] R. Huber, "A generalization of formal schemes and rigid analytic varieties", *Math. Z.* **217**:4 (1994), 513–551. MR Zbl

[Huber 1996] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Math. **E30**, Vieweg & Sohn, Brunswick, 1996. MR Zbl

[Johansson and Newton 2018] C. Johansson and J. Newton, "Parallel weight 2 points on Hilbert modular eigenvarieties and the parity conjecture", 2018. To appear in *Forum Math. Sigma*. arXiv

[Johansson and Newton 2019] C. Johansson and J. Newton, "Extended eigenvarieties for overconvergent cohomology", *Algebra Number Theory* **13**:1 (2019), 93–158. MR Zbl

[Katz 1973] N. M. Katz, "p-adic properties of modular schemes and modular forms", pp. 69–190 in *Modular functions of one variable, III* (Antwerp, 1972), edited by W. Kuyk and J.-P. Serre, Lecture Notes in Math. **350**, Springer, 1973. MR Zbl

[Lazard 1965] M. Lazard, "Groupes analytiques p-adiques", *Inst. Hautes Études Sci. Publ. Math.* **26** (1965), 389–603. MR Zbl

[Li and Schwermer 2004] J.-S. Li and J. Schwermer, "On the Eisenstein cohomology of arithmetic groups", *Duke Math. J.* **123**:1 (2004), 141–169. MR Zbl

[Liu et al. 2017] R. Liu, D. Wan, and L. Xiao, "The eigencurve over the boundary of weight space", *Duke Math. J.* **166**:9 (2017), 1739–1787. MR Zbl

[Roby 1963] N. Roby, "Lois polynomes et lois formelles en théorie des modules", *Ann. Sci. École Norm. Sup.* (3) **80** (1963), 213–348. MR Zbl

[Rydh 2008] D. Rydh, *Families of cycles and the Chow scheme*, Ph.D. thesis, KTH Royal Institute of Technology, 2008, Available at https://people.kth.se/~dary/thesis.

[Scholze and Weinstein 2019] P. Scholze and J. Weinstein, "Berkeley lectures on p-adic geometry", lecture notes, Univ. California, Berkeley, 2019, Available at https://tinyurl.com/scholwein.

[Stevens 1994] G. Stevens, "Rigid analytic modular symbols", preprint, 1994, Available at https://tinyurl.com/glennrams.

[Urban 2011] E. Urban, "Eigenvarieties for reductive groups", *Ann. of Math.* (2) **174**:3 (2011), 1685–1784. MR Zbl

[Ye 2019] L. Ye, "Slopes in eigenvarieties for definite unitary groups", draft of doctoral thesis, 2019, Available at https://tinyurl.com/yeslopes.

Communicated by Christopher Skinner
Received 2018-08-27 Revised 2019-04-25 Accepted 2019-06-24

daniel.gulotta@maths.ox.ac.uk *Department of Mathematics, Columbia University, New York, USA*
Current address: *Mathematical Institute, University of Oxford, United Kingdom*

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1941

Cohomological and numerical dynamical degrees on abelian varieties

Fei Hu

We show that for a self-morphism of an abelian variety defined over an algebraically closed field of arbitrary characteristic, the second cohomological dynamical degree coincides with the first numerical dynamical degree.

A list of symbols can be found on page 1957.

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field $\boldsymbol{k}$, and f a surjective morphism of X to itself. Inspired by Esnault and Srinivas [2013] and Truong [2016], we associate to this map two dynamical degrees as follows. Let ℓ be a prime different from the characteristic of $\boldsymbol{k}$. As a consequence of Deligne [1974] and Katz and Messing [1974], the characteristic polynomial of f on the ℓ-adic étale cohomology group $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ is independent of ℓ, and has integer coefficients, and algebraic integer roots (see [Esnault and Srinivas 2013, Proposition 2.3]; see also [Kleiman 1968]). The *i-th cohomological dynamical degree* $\chi_i(f)$ of f is then defined as the spectral radius of the pullback action f^* on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$, i.e.,

$$\chi_i(f) = \rho(f^*|_{H^i_{\text{ét}}(X,\mathbb{Q}_\ell)}).$$

Alternatively, one can also define dynamical degrees using algebraic cycles. Indeed, let $N^k(X)$ denote the group of algebraic cycles of codimension k modulo numerical equivalence. Note that $N^k(X)$ is a finitely generated free abelian group [Kleiman 1968, Theorem 3.5], and hence the characteristic polynomial of f on $N^k(X)$ has integer coefficients and algebraic integer roots. We define the *k-th numerical dynamical degree* $\lambda_k(f)$ of f as the spectral radius of the pullback action f^* on $N^k(X)_\mathbb{R} := N^k(X) \otimes_\mathbb{Z} \mathbb{R}$, i.e.,

$$\lambda_k(f) = \rho(f^*|_{N^k(X)_\mathbb{R}}).$$

When $\boldsymbol{k} \subseteq \mathbb{C}$, we may associate to (X, f) a projective (and hence compact Kähler) manifold $X_\mathbb{C}$ and a surjective holomorphic map $f_\mathbb{C}$. Then by the comparison theorem and Hodge theory, it is not hard to show that $\chi_{2k}(f) = \lambda_k(f)$; both of them also agree with the usual dynamical degree defined by the Dolbeault

The author was partially supported by a UBC-PIMS Postdoctoral Fellowship.
MSC2010: primary 14G17; secondary 14K05, 16K20.
Keywords: dynamical degree, abelian variety, endomorphism algebra, étale cohomology, algebraic cycle, positive characteristic.

cohomology group $H^{k,k}(X_{\mathbb{C}}, \mathbb{C})$ in the context of complex dynamics (see e.g., [Dinh and Sibony 2017, Section 4]).

For an arbitrary algebraically closed field $\boldsymbol{k}$ (in particular, of positive characteristic), Esnault and Srinivas [2013] proved that for an automorphism of a smooth projective surface, the second cohomological dynamical degree coincides with the first numerical dynamical degree. Their proof relies on the Enriques–Bombieri–Mumford classification of surfaces in arbitrary characteristic. In general, Truong [2016] raised the following question (among many others):

Question 1.1 (cf. [Truong 2016, Question 2]). Let X be a smooth projective variety defined over an algebraically closed field $\boldsymbol{k}$, and f a surjective morphism of X to itself. Then is $\chi_{2k}(f) = \lambda_k(f)$ for any $1 \le k \le \dim X$?

The above question turns out to be related to Weil's Riemann hypothesis (proved by Deligne in the early 1970s). More precisely, when X_0 is a smooth projective variety defined over a finite field $\boldsymbol{F}_q$, we let X denote the base change of X_0 to the algebraic closure $\overline{\boldsymbol{F}}_q$ of $\boldsymbol{F}_q$ and let F denote the Frobenius endomorphism of X (with respect to $\boldsymbol{F}_q$). Then Deligne's celebrated theorem asserts that all eigenvalues of $F^*|_{H^i_{\text{ét}}(X,\mathbb{Q}_\ell)}$ are algebraic integers of modulus $q^{i/2}$ [Deligne 1974, Théorème 1.6]. In particular, we have $\chi_i(F) = q^{i/2}$. On the other hand, the k-th numerical dynamical degree $\lambda_k(F)$ of F is equal to q^k. See [Truong 2016, Section 4] for more details.

Truong [2016] proved a slightly weaker statement that

$$h_{\text{ét}}(f) := \max_i \log \chi_i(f) = \max_k \log \lambda_k(f) =: h_{\text{alg}}(f),$$

which is enough to conclude that the (étale) entropy $h_{\text{ét}}(f)$ coincides with the algebraic entropy $h_{\text{alg}}(f)$ in the sense of [Esnault and Srinivas 2013, Section 6.3]. As a consequence, the spectral radius of the action f^* on the even degree étale cohomology $H^{2\bullet}_{\text{ét}}(X, \mathbb{Q}_\ell)$ is the same as the spectral radius of f^* on the total cohomology $H^{\bullet}_{\text{ét}}(X, \mathbb{Q}_\ell)$.[1] Note that when $\boldsymbol{k} \subseteq \mathbb{C}$, by the fundamental work of Gromov [2003] and Yomdin [1987], the algebraic entropy is also equal to the topological entropy $h_{\text{top}}(f_{\mathbb{C}})$ of the topological dynamical system $(X_{\mathbb{C}}, f_{\mathbb{C}})$; see [Dinh and Sibony 2017, Section 4] for more details.

In this article, we give an affirmative answer to Question 1.1 in the case that X is an abelian variety and $k = 1$.

Theorem 1.2. *Let X be an abelian variety defined over an algebraically closed field $\boldsymbol{k}$, and f a surjective self-morphism of X. Then $\chi_2(f) = \lambda_1(f)$.*

Remark 1.3. (1) When f is an automorphism of an abelian surface X, the theorem was already known by Esnault and Srinivas [2013, Section 4]. Even in this two-dimensional case, their proof is quite involved. Actually, after a standard specialization argument, they applied the celebrated Tate theorem [1966] (see also [Mumford 1970, Appendix I, Theorem 3]), which asserts that the minimal polynomial of the geometric Frobenius endomorphism is a product of distinct monic irreducible polynomials. Then they

[1] Recently, this was reproved by Shuddhodan [2019] using a number-theoretic method, where the author introduced a zeta function $Z(X, f, t)$ for a dynamical system (X, f) defined over a finite field.

had four cases to analyze according to its irreducibility and degree. Our proof is more explicit in the sense that we will eventually determine all eigenvalues of $f^*|_{N^1(X)_{\mathbb{R}}}$.

(2) Because of the lack of an explicit characterization of higher-codimensional cycles (up to numerical equivalence) like the Néron–Severi group $\mathrm{NS}(X)$ sitting inside the endomorphism algebra $\mathrm{End}^0(X)$, it would be very interesting to consider the case $k \geq 2$ next.

2. Preliminaries on abelian varieties

We refer to [Mumford 1970; Milne 1986] for standard notation and terminologies on abelian varieties and to page 1957 for a list of symbols.

For the convenience of the reader, we include several important structure theorems on the étale cohomology groups, the endomorphism algebras and the Néron–Severi groups of abelian varieties. We refer to [Mumford 1970, Sections 19–21] for more details.

First, the étale cohomology groups of abelian varieties are simple to describe.

Theorem 2.1 [Milne 1986, Theorem 15.1]. *Let X be an abelian variety of dimension g defined over $\boldsymbol{k}$, and let ℓ be a prime different from* $\mathrm{char}\,\boldsymbol{k}$. *Let $T_\ell X := \varprojlim_n X_{\ell^n}(\boldsymbol{k})$ be the Tate module of X, which is a free $\mathbb{Z}_\ell$-module of rank $2g$.*

(a) *There is a canonical isomorphism*

$$H^1_{\text{ét}}(X, \mathbb{Z}_\ell) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell X, \mathbb{Z}_\ell).$$

(b) *The cup-product pairing induces isomorphisms*

$$\bigwedge^i H^1_{\text{ét}}(X, \mathbb{Z}_\ell) \simeq H^i_{\text{ét}}(X, \mathbb{Z}_\ell),$$

for all i. In particular, $H^i_{\text{ét}}(X, \mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-module of rank $\binom{2g}{i}$.

Furthermore, the functor T_ℓ induces an ℓ-adic representation of the endomorphism algebra. In general, we have:

Theorem 2.2 [Mumford 1970, Section 19, Theorem 3]. *For any two abelian varieties X and Y, the group* $\mathrm{Hom}(X, Y)$ *of homomorphisms of X into Y is a finitely generated free abelian group, and the natural homomorphism of $\mathbb{Z}_\ell$-modules*

$$\mathrm{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$$

induced by $T_\ell\colon$ $\mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$ is injective.

For a homomorphism $f\colon X \to Y$ of abelian varieties, its *degree* $\deg f$ is defined to be the order of the kernel $\ker f$, if it is finite, and 0 otherwise. In particular, the degree of an isogeny is always a positive integer.

Theorem 2.3 [Mumford 1970, Section 19, Theorem 4]. *For any $\alpha \in \mathrm{End}(X)$, there is a unique monic polynomial $P_\alpha(t) \in \mathbb{Z}[t]$ of degree $2g$ such that $P_\alpha(n) = \deg(n_X - \alpha)$ for all integers n. Moreover, $P_\alpha(t)$ is the characteristic polynomial of α acting on $T_\ell X$, i.e., $P_\alpha(t) = \det(t - T_\ell \alpha)$, and $P_\alpha(\alpha) = 0$ as an endomorphism of X.*

We call $P_\alpha(t)$ as in Theorem 2.3 the *characteristic polynomial of* α. On the other hand, we can assign to each α the characteristic polynomial $\chi_\alpha(t)$ of α as an element of the semisimple $\mathbb{Q}$-algebra $\mathrm{End}^0(X)$. Namely, we define $\chi_\alpha(t)$ to be the characteristic polynomial of the left multiplication $\alpha_L\colon \beta \mapsto \alpha\beta$ for $\beta \in \mathrm{End}^0(X)$ which is a $\mathbb{Q}$-linear transformation on $\mathrm{End}^0(X)$. Note that the above definition of $\chi_\alpha(t)$ makes no use of the fact that $\mathrm{End}^0(X)$ is semisimple. Actually, for semisimple $\mathbb{Q}$-algebras, it is much more useful to consider the so-called reduced characteristic polynomials.

We recall some basic definitions on semisimple algebras (see [Reiner 2003, Section 9] for more details).

Definition 2.4. Let R be a finite-dimensional semisimple algebra over a field F with $\operatorname{char} F = 0$, and write

$$R = \bigoplus_{i=1}^{k} R_i,$$

where each R_i is a simple F-algebra. For any element $r \in R$, as above, we denote by $\chi_r(t)$ the *characteristic polynomial of* r. Namely, $\chi_r(t)$ is the characteristic polynomial of the left multiplication $r_L\colon r' \mapsto rr'$ for $r' \in R$. Let K_i be the center of R_i. Then there exists a finite field extension E_i/K_i splitting R_i [Reiner 2003, Section 7b], i.e., we have

$$h_i\colon R_i \otimes_{K_i} E_i \xrightarrow{\sim} \mathrm{M}_{d_i}(E_i), \qquad \text{where } [R_i : K_i] = d_i^2.$$

Write $r = r_1 + \cdots + r_k$ with each $r_i \in R_i$. We first define the *reduced characteristic polynomial* $\chi_{r_i}^{\mathrm{red}}(t)$ *of* r_i as follows [Reiner 2003, Definition 9.13]:

$$\chi_{r_i}^{\mathrm{red}}(t) := \mathrm{N}_{K_i/F}(\det(t\boldsymbol{I}_{d_i} - h_i(r_i \otimes_{K_i} 1_{E_i}))) \in F[t].$$

It turns out that $\det(t\boldsymbol{I}_{d_i} - h_i(r_i \otimes_{K_i} 1_{E_i}))$ lies in $K_i[t]$, and is independent of the choice of the splitting field E_i of R_i [Reiner 2003, Theorem 9.3]. The *reduced norm of* r_i is defined by

$$\mathrm{N}_{R_i/F}^{\mathrm{red}}(r_i) := \mathrm{N}_{K_i/F}(\det(h_i(r_i \otimes_{K_i} 1_{E_i}))) \in F.$$

Finally, as one expects, the *reduced characteristic polynomial* $\chi_r^{\mathrm{red}}(t)$ and the *reduced norm* $\mathrm{N}_{R/F}^{\mathrm{red}}(r)$ *of* r are defined by the products

$$\chi_r^{\mathrm{red}}(t) := \prod_{i=1}^{k} \chi_{r_i}^{\mathrm{red}}(t) \quad \text{and} \quad \mathrm{N}_{R/F}^{\mathrm{red}}(r) := \prod_{i=1}^{k} \mathrm{N}_{R_i/F}^{\mathrm{red}}(r_i).$$

Remark 2.5. (1) It follows from [Reiner 2003, Theorem 9.14] that

$$\chi_r(t) = \prod_{i=1}^{k} \chi_{r_i}(t) = \prod_{i=1}^{k} \chi_{r_i}^{\mathrm{red}}(t)^{d_i}. \tag{2-1}$$

(2) Note that reduced characteristic polynomials and norms are not affected by change of ground field [Reiner 2003, Theorem 9.27].

We now apply the above algebraic setting to $R = \mathrm{End}^0(X)$. For any $\alpha \in \mathrm{End}(X)$, let $\chi_\alpha^{\mathrm{red}}(t)$ denote the reduced characteristic polynomial of α as an element of the semisimple $\mathbb{Q}$-algebra $\mathrm{End}^0(X)$. For simplicity, let us first consider the case when $X = A$ is a simple abelian variety and hence $D := \mathrm{End}^0(A)$ is a division ring. Let K denote the center of D which is a field, and K_0 the maximal totally real subfield of K. Set

$$d^2 = [D : K], e = [K : \mathbb{Q}] \quad \text{and} \quad e_0 = [K_0 : \mathbb{Q}].$$

Then the equality (2-1) reads as

$$\chi_\alpha(t) = \chi_\alpha^{\mathrm{red}}(t)^d.$$

The lemma below shows that the two polynomials $P_\alpha(t)$ and $\chi_\alpha(t)$ are closely related. Its proof relies on a characterization of normal forms of D over $\mathbb{Q}$.

For convenience, we include the following definition. Let R be a finite-dimensional associative algebra over an infinite field F. A *norm form* on R over F is a nonzero polynomial function

$$N_{R/F} \colon R \to F$$

(i.e., in terms of a basis of R over F, $N_{R/F}(r)$ can be written as a polynomial over F in the components of r) such that $N_{R/F}(rr') = N_{R/F}(r)N_{R/F}(r')$ for all $r, r' \in R$.

Lemma 2.6. *Using notation as above, for any $\alpha \in \mathrm{End}(A)$, we have*

$$P_\alpha(t) = \chi_\alpha^{\mathrm{red}}(t)^m,$$

where $m = 2g/(ed)$ is a positive integer. In particular, the two polynomials $P_\alpha(t)$ and $\chi_\alpha(t)$ have the same complex roots (apart from multiplicities).[2]

Proof. By the lemma in [Mumford 1970, Section 19] (located between Corollary 3 and Theorem 4, page 179), any norm form of D over $\mathbb{Q}$ is of the following type

$$(\mathrm{N}_{K/\mathbb{Q}} \circ \mathrm{N}_{D/K}^{\mathrm{red}})^k \colon D \to \mathbb{Q}$$

for a suitable nonnegative integer k, where $\mathrm{N}_{D/K}^{\mathrm{red}}$ is the reduced norm (aka canonical norm form in the sense of Mumford) of D over K. Now for each $n \in \mathbb{Z}$, we have

$$\chi_\alpha^{\mathrm{red}}(n) = \mathrm{N}_{K/\mathbb{Q}} \circ \mathrm{N}_{D/K}^{\mathrm{red}}(n_A - \alpha).$$

On the other hand, the action of D on $V_\ell A := T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ defines the determinant map

$$\det \colon D \to \mathbb{Q}_\ell,$$

which actually takes on values in $\mathbb{Q}$ and is a norm form of degree $2g$. Indeed, let $V_\ell\alpha$ denote the induced map of α on $V_\ell A$, then $P_\alpha(n) = \deg(n_A - \alpha) = \det(n_A - \alpha) = \det(n - V_\ell\alpha)$ for all integers n (see

[2] I would like to thank Yuri Zarhin for showing me an argument using the canonical norm form to prove this Lemma 2.6.

Theorem 2.3). Applying the aforementioned lemma in [Mumford 1970, Section 19] to this det, we obtain that for a suitable m,

$$\det(\psi) = (\mathrm{N}_{K/\mathbb{Q}} \circ \mathrm{N}^{\mathrm{red}}_{D/K}(\psi))^m$$

for all $\psi \in D$. It is easy to see that m is $2g/(ed)$. Then by taking $\psi = n_A - \alpha$, we have that $P_\alpha(n) = \chi^{\mathrm{red}}_\alpha(n)^m$ for all integers n. This yields that $P_\alpha(t) = \chi^{\mathrm{red}}_\alpha(t)^m$. □

It is straightforward to generalize Lemma 2.6 to the case that X is the n-th power A^n of a simple abelian variety A since $\mathrm{End}^0(A^n) = \mathrm{M}_n(\mathrm{End}^0(A))$ is still a simple $\mathbb{Q}$-algebra.

Lemma 2.7. *Let A be a simple abelian variety and $X = A^n$. Let $\chi^{\mathrm{red}}_\alpha(t)$ denote the reduced characteristic polynomial of α as an element of the simple $\mathbb{Q}$-algebra $\mathrm{End}^0(X) = \mathrm{M}_n(D)$ with $D = \mathrm{End}^0(A)$. Then*

$$\chi_\alpha(t) = \chi^{\mathrm{red}}_\alpha(t)^{dn} \text{ and } P_\alpha(t) = \chi^{\mathrm{red}}_\alpha(t)^m,$$

where $m = 2g/(edn)$ is a positive integer. In particular, these two polynomials $P_\alpha(t)$ and $\chi_\alpha(t)$ have the same complex roots (apart from multiplicities).

We recall the following useful structure theorems on $\mathrm{NS}^0(X)$ which play a crucial role in the proof of our main theorem.

Theorem 2.8 [Mumford 1970, Section 21, Application III]. *Fix a polarization $\phi\colon X \to \hat{X}$ that is an isogeny from X to its dual $\hat{X}$ induced from some ample line bundle $\mathscr{L}_0$ (we suppress this $\mathscr{L}_0$ since it does not make an appearance here henceforth). Then the natural map*

$$\mathrm{NS}^0(X) \to \mathrm{End}^0(X) \quad \text{via} \quad \mathscr{L} \mapsto \phi^{-1} \circ \phi_{\mathscr{L}}$$

is injective and its image is precisely the subspace $\{\psi \in \mathrm{End}^0(X) \mid \psi^\dagger = \psi\}$ of symmetric elements of $\mathrm{End}^0(X)$ under the Rosati involution † which maps ψ to $\psi^\dagger := \phi^{-1} \circ \hat{\psi} \circ \phi$.

Theorem 2.9 [Mumford 1970, Section 21, Theorems 2 and 6]. *The endomorphism $\mathbb{R}$-algebra $\mathrm{End}(X)_\mathbb{R} := \mathrm{End}^0(X) \otimes_\mathbb{Q} \mathbb{R}$ is isomorphic to a product of copies of $\mathrm{M}_r(\mathbb{R})$, $\mathrm{M}_r(\mathbb{C})$ and $\mathrm{M}_r(\boldsymbol{H})$. Moreover, one can fix an isomorphism so that it carries the Rosati involution into the standard involution $\boldsymbol{A} \mapsto \bar{\boldsymbol{A}}^\mathsf{T}$. In particular, $\mathrm{NS}(X)_\mathbb{R} := \mathrm{NS}^0(X) \otimes_\mathbb{Q} \mathbb{R}$ is isomorphic to a product of Jordan algebras of the following types:*

$$\mathscr{H}_r(\mathbb{R}) = r \times r \text{ symmetric real matrices},$$

$$\mathscr{H}_r(\mathbb{C}) = r \times r \text{ Hermitian complex matrices},$$

$$\mathscr{H}_r(\boldsymbol{H}) = r \times r \text{ Hermitian quaternionic matrices.}$$

3. Proof of Theorem 1.2

3.1. ***Some results on dynamical degrees.*** We first prepare some results used later to prove our main theorem. Recall that in complex dynamics, the dynamical degrees are bimeromorphic invariants of the dynamics system (see e.g., [Dinh and Sibony 2017, Theorem 4.2]). We have also shown the birational invariance of numerical dynamical degrees in arbitrary characteristic [Hu 2019, Lemma 2.8]. Below is a similar consideration which should be of interest in its own right. Note, however, that we have not shown the birational invariance of cohomological dynamical degrees, which is actually one of the questions raised by Truong [2016, Question 5].

Lemma 3.1. *Let $\pi : X \to Y$ be a surjective morphism of smooth projective varieties defined over $\mathbf{k}$. Let f and g be surjective self-morphisms of X and Y, respectively, such that $\pi \circ f = g \circ \pi$. Then $\chi_i(f) \geq \chi_i(g)$ for any $0 \leq i \leq 2 \dim Y$ and $\lambda_k(f) \geq \lambda_k(g)$ for any $0 \leq k \leq \dim Y$.*

Proof. We have the following commutative diagram of $\mathbb{Q}_\ell$-vector spaces:

$$\begin{array}{ccc} H^i_{\text{ét}}(Y, \mathbb{Q}_\ell) & \xrightarrow{\pi^*} & H^i_{\text{ét}}(X, \mathbb{Q}_\ell) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^i_{\text{ét}}(Y, \mathbb{Q}_\ell) & \xrightarrow{\pi^*} & H^i_{\text{ét}}(X, \mathbb{Q}_\ell). \end{array}$$

The first part follows readily from [Kleiman 1968, Proposition 1.2.4] which asserts that the pullback map π^* on ℓ-adic étale cohomology is injective and hence $\pi^* H^i_{\text{ét}}(Y, \mathbb{Q}_\ell)$ is an f^*-invariant subspace of $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$. The second part is similar; see also [Hu 2019, Lemma 2.8] for a stronger version. □

The following useful inequality was already noticed by Truong [2016]. We provide a proof for the sake of completeness.

Lemma 3.2. *Let X be a smooth projective varieties defined over $\mathbf{k}$, and f a surjective self-morphism of X. Then we have $\lambda_k(f) \leq \chi_{2k}(f)$ for any $0 \leq k \leq \dim X$.*

Proof. Note that the ℓ-adic étale cohomology $H^\bullet_{\text{ét}}(X, \mathbb{Q}_\ell)$ is a Weil cohomology after the noncanonical choice of an isomorphism $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$ [Kleiman 1968, Example 1.2.5]. So we have the following cycle map

$$\gamma^k_X : \operatorname{CH}^k(X) \to H^{2k}_{\text{ét}}(X, \mathbb{Q}_\ell),$$

where the k-th Chow group $\operatorname{CH}^k(X)$ of X denotes the group of algebraic cycles of codimension k modulo linear equivalence, i.e., $\operatorname{CH}^k(X) := Z^k(X)/\sim$. Recall that a cycle $Z \in Z^k(X)$ is homologically equivalent to zero if $\gamma^k_X(Z) = 0$. Also, it is well-known that homological equivalence $\sim_{\text{hom}}$ is finer than numerical equivalence $\equiv$ [Kleiman 1968, Proposition 1.2.3]. Hence we have the following diagram of finite-dimensional $\mathbb{Q}_\ell$-vector spaces (respecting the natural pullback action f^* by the functoriality of the cycle map):

$$\begin{array}{ccc} (\operatorname{CH}^k(X)/\sim_{\text{hom}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \hookrightarrow & H^{2k}_{\text{ét}}(X, \mathbb{Q}_\ell) \\ \downarrow\!\!\!\downarrow & & \\ (\operatorname{CH}^k(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = N^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell. & & \end{array} \tag{3-1}$$

Thus Lemma 3.2 follows. □

Remark 3.3. When $k = 1$, by a theorem of Matsusaka [1957], homological equivalence coincides with numerical equivalence (in general, Grothendieck's standard conjecture D predicts that they are equal for all k). Furthermore, after tensoring with $\mathbb{Q}$, both of them are also equivalent to algebraic equivalence $\approx$. Namely, we have

$$\mathrm{NS}(X)_{\mathbb{Q}} = (\mathrm{CH}^1(X)/\approx) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq (\mathrm{CH}^1(X)/\sim_{\mathrm{hom}}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, the cycle map γ_X^1 induces an injection

$$N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H^2_{\text{ét}}(X, \mathbb{Q}_\ell).$$

3.2. ***Extension of the pullback action to endomorphism algebras.*** For an endomorphism α of an abelian variety X, the following easy lemma sheds the light on the connection between the first numerical dynamical degree $\lambda_1(\alpha)$ of α and the induced action α^* on the endomorphism $\mathbb{Q}$-algebra $\mathrm{End}^0(X)$, while the latter is closely related to the matrix representation of α in $\mathrm{End}(X)_{\mathbb{R}}$ or $\mathrm{End}(X)_{\mathbb{C}}$ (see e.g., Lemma 3.5).

Lemma 3.4. *Fix a polarization $\phi\colon X \to \hat{X}$ as in Theorem 2.8. For any endomorphism α of X, we can extend the pullback action α^* on $\mathrm{NS}^0(X)$ to $\mathrm{End}^0(X)$ as follows*:

$$\alpha^*\colon \mathrm{End}^0(X) \to \mathrm{End}^0(X) \quad \textit{via} \quad \psi \mapsto \alpha^*\psi := \alpha^\dagger \circ \psi \circ \alpha.^3$$

Proof. We shall identify $\mathrm{NS}^0(X) \ni \mathscr{L}$ with the subspace of symmetric elements $\phi^{-1} \circ \phi_{\mathscr{L}}$ of the endomorphism $\mathbb{Q}$-algebra $\mathrm{End}^0(X)$ in virtue of Theorem 2.8. Then the natural pullback action α^* on $\mathrm{NS}^0(X)$ could be reinterpreted in the following way:

$$\alpha^*\colon \mathrm{NS}^0(X) \to \mathrm{NS}^0(X)$$
$$\phi^{-1} \circ \phi_{\mathscr{L}} \mapsto \phi^{-1} \circ \phi_{\alpha^*\mathscr{L}}.$$

Note that $\phi^{-1} \circ \phi_{\alpha^*\mathscr{L}} = \phi^{-1} \circ \hat{\alpha} \circ \phi_{\mathscr{L}} \circ \alpha = \alpha^\dagger \circ \phi^{-1} \circ \phi_{\mathscr{L}} \circ \alpha$, where $\hat{\alpha}$ is the induced dual endomorphism of $\hat{X}$ and $\alpha^\dagger = \phi^{-1} \circ \hat{\alpha} \circ \phi$ is the Rosati involution of α; for the first equality, see [Mumford 1970, Section 15, Theorem 1]. This gives rise to an action of α on the whole endomorphism algebra $\mathrm{End}^0(X)$ by sending $\psi \in \mathrm{End}^0(X)$ to $\alpha^\dagger \circ \psi \circ \alpha$. It is easy to see that the restriction of $\alpha^*|_{\mathrm{End}^0(X)}$ to $\mathrm{NS}^0(X)$ is just the natural pullback action α^* on $\mathrm{NS}^0(X)$. ☐

The lemma below plays a crucial role in the proof of our main theorem by giving a characterization of the above induced action α^* on certain endomorphism algebras of abelian varieties. Here we consider a more general version from the aspect of linear algebra.

Lemma 3.5. (1) *If $\boldsymbol{A} \in \mathrm{M}_n(\mathbb{R})$, then the linear transformation*

$$f_{\boldsymbol{A}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R}) \quad \textit{via } \boldsymbol{B} \mapsto \boldsymbol{A}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{A}$$

of n^2-dimensional $\mathbb{R}$-vector space $\mathrm{M}_n(\mathbb{R})$ could be represented by $\boldsymbol{A} \otimes \boldsymbol{A}$, the Kronecker product of $\boldsymbol{A}$ and itself.

[3] Here by abuse of notation, we still denote this action by α^*. We would always write $\alpha^*|_{\mathrm{End}^0(X)}$ to emphasize the acting space in practice.

(2) *If* $\boldsymbol{A} \in \mathrm{M}_n(\mathbb{C})$, *then the following linear transformation*

$$f_{\boldsymbol{A}} \colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}) \quad \textit{via } \boldsymbol{B} \mapsto \overline{\boldsymbol{A}}^{\mathsf{T}} \boldsymbol{B} \boldsymbol{A}$$

of n^2*-dimensional* $\mathbb{C}$*-vector space* $\mathrm{M}_n(\mathbb{C})$ *could be represented by* $\boldsymbol{A} \otimes \overline{\boldsymbol{A}}$, *the Kronecker product of* $\boldsymbol{A}$ *and its complex conjugate* $\overline{\boldsymbol{A}}$.

(3) *If* $\boldsymbol{A} \in \mathrm{M}_n(\mathbb{C})$, *then the following linear transformation*

$$f_{\boldsymbol{A}} \colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C}) \quad \textit{via } \boldsymbol{B} \mapsto \overline{\boldsymbol{A}}^{\mathsf{T}} \boldsymbol{B} \boldsymbol{A}$$

of $2n^2$*-dimensional* $\mathbb{R}$*-vector space* $\mathrm{M}_n(\mathbb{C})$ *could be represented by the block diagonal matrix* $(\boldsymbol{A} \otimes \overline{\boldsymbol{A}}) \oplus (\overline{\boldsymbol{A}} \otimes \boldsymbol{A})$.

Proof. We first prove the assertion (2) since the proof of the first one is essentially the same. Choose the standard $\mathbb{C}$-basis $\{\boldsymbol{e}_{ij}\}$ of $\mathrm{M}_n(\mathbb{C})$, where $\boldsymbol{e}_{ij}$ denotes the $n \times n$ complex matrix whose (i, j)-entry is 1, and 0 elsewhere. We also adopt the standard vectorization

$$\mathrm{vec} \colon \mathrm{M}_n(\mathbb{C}) \xrightarrow{\simeq} \mathbb{C}^{n^2}$$

of $\mathrm{M}_n(\mathbb{C})$, which converts $n \times n$ matrices into column vectors so that

$$\{\mathrm{vec}(\boldsymbol{e}_{11}), \mathrm{vec}(\boldsymbol{e}_{21}), \dots, \mathrm{vec}(\boldsymbol{e}_{n1}), \mathrm{vec}(\boldsymbol{e}_{12}), \dots, \mathrm{vec}(\boldsymbol{e}_{n2}), \dots, \mathrm{vec}(\boldsymbol{e}_{1n}), \dots, \mathrm{vec}(\boldsymbol{e}_{nn})\} \tag{3-2}$$

forms the standard $\mathbb{C}$-basis of $\mathbb{C}^{n^2}$. Write $\boldsymbol{A} = (a_{ij})_{n \times n}$ with $a_{ij} \in \mathbb{C}$. Then we have

$$\overline{\boldsymbol{A}}^{\mathsf{T}} \cdot \boldsymbol{e}_{ij} = \bar{a}_{i1} \boldsymbol{e}_{1j} + \bar{a}_{i2} \boldsymbol{e}_{2j} + \cdots + \bar{a}_{in} \boldsymbol{e}_{nj}.$$

Hence under the basis (3-2), it is easy to verify that the left multiplication by $\overline{\boldsymbol{A}}^{\mathsf{T}}$ on the $\mathbb{C}$-vector space $\mathrm{M}_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ is represented by the block diagonal matrix $\overline{\boldsymbol{A}} \oplus \overline{\boldsymbol{A}} \oplus \cdots \oplus \overline{\boldsymbol{A}} = \boldsymbol{I}_n \otimes \overline{\boldsymbol{A}}$. Similarly, since $\boldsymbol{e}_{ij} \cdot \boldsymbol{A} = a_{j1} \boldsymbol{e}_{i1} + a_{j2} \boldsymbol{e}_{i2} + \cdots + a_{jn} \boldsymbol{e}_{in}$, one can check that under the basis (3-2), the right multiplication by $\boldsymbol{A}$ is represented by $\boldsymbol{A} \otimes \boldsymbol{I}_n$. Therefore, our linear map $f_{\boldsymbol{A}}$ is represented by the matrix product $(\boldsymbol{I}_n \otimes \overline{\boldsymbol{A}}) \cdot (\boldsymbol{A} \otimes \boldsymbol{I}_n) = \boldsymbol{A} \otimes \overline{\boldsymbol{A}}$. Thus the assertion (2) follows.

For the last assertion, we just need to combine the assertion (2) with the following general fact: if $\boldsymbol{M} \in \mathrm{M}_n(\mathbb{C})$, then the associated $2n \times 2n$ real matrix

$$\begin{pmatrix} \mathrm{Re}\, \boldsymbol{M} & -\mathrm{Im}\, \boldsymbol{M} \\ \mathrm{Im}\, \boldsymbol{M} & \mathrm{Re}\, \boldsymbol{M} \end{pmatrix}$$

is similar to the block diagonal matrix $\boldsymbol{M} \oplus \overline{\boldsymbol{M}}$. Indeed, one can easily verify that

$$\begin{pmatrix} \boldsymbol{I}_n & -i \boldsymbol{I}_n \\ -i \boldsymbol{I}_n & \boldsymbol{I}_n \end{pmatrix}^{-1} \cdot \begin{pmatrix} \mathrm{Re}\, \boldsymbol{M} & -\mathrm{Im}\, \boldsymbol{M} \\ \mathrm{Im}\, \boldsymbol{M} & \mathrm{Re}\, \boldsymbol{M} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{I}_n & -i \boldsymbol{I}_n \\ -i \boldsymbol{I}_n & \boldsymbol{I}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{M}} \end{pmatrix}.$$

Applying the above fact to the complex matrix $\boldsymbol{A} \otimes \overline{\boldsymbol{A}}$ coming from the assertion (2), one gets the assertion (3) and hence Lemma 3.5 follows. ☐

3.3. ***Several standard reductions towards the proof.*** Before proving our main Theorem 1.2, we start with some standard reductions. The lemma below reduces the general case to the splitting product case.

Lemma 3.6. *In order to prove Theorem 1.2, it suffices to consider the following case*:

- *the abelian variety* $X = A_1^{n_1} \times \cdots \times A_s^{n_s}$, *where the* A_j *are mutually nonisogenous simple abelian varieties*, *and*
- *the surjective self-morphism* f *of* X *is a surjective endomorphism* α *which can be written as* $\alpha_1 \times \cdots \times \alpha_s$ *with* $\alpha_j \in \mathrm{End}(A_j^{n_j})$.

Proof. We claim that it suffices to consider the case when $f = \alpha$ is a surjective endomorphism. Indeed, any morphism (i.e., regular map) of abelian varieties is a composite of a homomorphism with a translation [Milne 1986, Corollary 2.2]. Hence we can write f as $t_x \circ \alpha$ for a surjective endomorphism $\alpha \in \mathrm{End}(X)$ and $x \in X(\boldsymbol{k})$. Note however that $t_x \in \mathrm{Aut}^0(X) \simeq X$ acts as identity on $H^1_{\text{ét}}(X, \mathbb{Q}_\ell)$ and hence on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ for all i. It follows from the functoriality of the pullback map on ℓ-adic étale cohomology that $\chi_i(f) = \chi_i(\alpha)$. Similarly, we also get $\lambda_k(f) = \lambda_k(\alpha)$ for all k. So the claim follows, and from now on our $f = \alpha$ is an isogeny.

We then make another claim as follows.

Claim 3.7. *Towards the proof of Theorem 1.2*, *we are free to replace our pair* (X, α) *by any of the following pairs*:

(1) (X, α^m), *for any positive integer* m.

(2) $(X, m\alpha)$, *for any positive integer* m.

(3) $(X', \alpha' := g \circ \alpha \circ h)$, *where* $g\colon X \to X'$ *and* $h\colon X' \to X$ *are isogenies such that* $h \circ g = m_X$ *and* $g \circ h = m_{X'}$ *with* $m = \deg g$.

Proof of Claim 3.7. The first part follows from the functoriality of the pullback map. For the second one, we note that $m\alpha = m_X \circ \alpha = \alpha \circ m_X$, where m_X is the multiplication by m map. Using the isomorphism $H^1_{\text{ét}}(X, \mathbb{Z}_\ell) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell X, \mathbb{Z}_\ell)$, one can easily see that the induced pullback map m_X^* on $H^1_{\text{ét}}(X, \mathbb{Q}_\ell)$ is also the multiplication by m map, and hence $m_X^*|_{H^i_{\text{ét}}(X,\mathbb{Q}_\ell)}$ is represented by the diagonal matrix $m^i \cdot \mathrm{id}_{H^i_{\text{ét}}(X,\mathbb{Q}_\ell)}$; see e.g., Theorem 2.1. It follows from the diagram (3-1) in the proof of Lemma 3.2 that the pullback map m_X^* on each $N^k(X)_\mathbb{R}$ is also represented by the diagonal matrix $m^{2k} \cdot \mathrm{id}_{N^k(X)_\mathbb{R}}$. In particular, we have $\chi_i(m\alpha) = m^i \chi_i(\alpha)$ and $\lambda_k(m\alpha) = m^{2k}\lambda_k(\alpha)$, which yields the part (2).

For the last part, it is easy to verify that $\alpha' \circ g = g \circ (m\alpha)$ and $h \circ \alpha' = (m\alpha) \circ h$. By applying Lemma 3.1 to the isogenies g and h, we have $\chi_i(\alpha') = \chi_i(m\alpha)$ and $\lambda_k(\alpha') = \lambda_k(m\alpha)$. Then combining with the second part, the third one follows. So we have proved Claim 3.7. □

Let us go back to the proof of Lemma 3.6. By Poincaré's complete reducibility theorem [Mumford 1970, Section 19, Theorem 1], we know that X is isogenous to the product $A_1^{n_1} \times \cdots \times A_s^{n_s}$, where the A_j

are mutually nonisogenous simple abelian varieties. Then

$$\mathrm{End}^0(X) \simeq \bigoplus_{j=1}^{s} \mathrm{End}^0(A_j^{n_j}),$$

so that we can write α as $\alpha_1 \times \cdots \times \alpha_s$ with $\alpha_j \in \mathrm{End}^0(A_j^{n_j})$. Using the reductions (2) and (3) in Claim 3.7, we only need to consider the case when X itself is the product variety and each α_j belongs to $\mathrm{End}(A_j^{n_j})$, as stated in the lemma. □

Remark 3.8. We are keen to further reduce the situation of Lemma 3.6 to the case when $X = A^n$ is a power of some simple abelian variety A, as Esnault and Srinivas did [2013, proof of Proposition 6.2]. However, to the best of our knowledge, it does not seem to be straightforward. More precisely, let X and α be as in Lemma 3.6. Suppose that Theorem 1.2 holds for every $A_j^{n_j}$ and surjective endomorphism $\alpha_j \in \mathrm{End}(A_j^{n_j})$, i.e., $\lambda_1(\alpha_j) = \chi_2(\alpha_j)$ for all j. We wish to show that Theorem 1.2 also holds for X and α. Note that

$$\mathrm{NS}(X) \simeq \bigoplus_{j=1}^{s} \mathrm{NS}(A_j^{n_j}).^4$$

It follows that

$$\lambda_1(\alpha) = \max_j \{\lambda_1(\alpha_j)\} = \max_j \{\chi_2(\alpha_j)\}. \tag{3-3}$$

On the other hand, by the Künneth formula, we have

$$\begin{aligned} H^1_{\text{ét}}(X, \mathbb{Q}_\ell) &\simeq \bigoplus_j H^1_{\text{ét}}(A_j^{n_j}, \mathbb{Q}_\ell), \quad \text{and} \\ H^2_{\text{ét}}(X, \mathbb{Q}_\ell) &\simeq \bigoplus_j H^2_{\text{ét}}(A_j^{n_j}, \mathbb{Q}_\ell) \oplus \bigoplus_{j<k} (H^1_{\text{ét}}(A_j^{n_j}, \mathbb{Q}_\ell) \otimes H^1_{\text{ét}}(A_k^{n_k}, \mathbb{Q}_\ell)). \end{aligned}$$

However, we are not able to deduce that $\chi_2(\alpha) = \max_j\{\chi_2(\alpha_j)\}$ due to the appearance of the tensor product of the $H^1_{\text{ét}}$.

For the sake of completeness, let us explain this obstruction in a more precise way. We denote by $P_{\alpha_j}(t) \in \mathbb{Z}[t]$ the characteristic polynomial of α_j (or equivalently $T_\ell \alpha_j$, by Theorem 2.3). Set $g_j = \dim A_j^{n_j}$. Denote all complex roots of $P_{\alpha_j}(t)$ by $\omega_{j,1}, \ldots, \omega_{j,2g_j}$. Without loss of generality, we may assume that

$$|\omega_{j,1}| \geq \cdots \geq |\omega_{j,2g_j}| \quad \text{for all } 1 \leq j \leq s, \text{ and } |\omega_{1,1}| \geq \cdots \geq |\omega_{s,1}|. \tag{3-4}$$

It follows from Theorem 2.1 that $\chi_2(\alpha_j) = |\omega_{j,1}| \cdot |\omega_{j,2}|$ for all j. Suppose that

$$\max_j \{\chi_2(\alpha_j)\} = \chi_2(\alpha_{j_0}) = |\omega_{j_0,1}| \cdot |\omega_{j_0,2}| \quad \text{for some } j_0. \tag{3-5}$$

[4] In general, one has $\mathrm{NS}(X \times_{\boldsymbol{k}} Y) \simeq \mathrm{NS}(X) \oplus \mathrm{NS}(Y) \oplus \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Alb}(X), \mathrm{Pic}^0(Y))$; see e.g., [Tate 1966, the proof of Theorem 3]. See also [Bost and Charles 2016, Section 3.2] and references therein for more details about the divisorial correspondences.

Note that j_0 may not be 1. If $|\omega_{2,1}| \le |\omega_{1,2}|$ (in particular, j_0 is 1), then

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{1,2}| = \chi_2(\alpha_1) = \max_j \{\chi_2(\alpha_j)\} = \lambda_1(\alpha).$$

So we are done in this case. However, if $|\omega_{2,1}| > |\omega_{1,2}|$, then

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{2,1}| \ge |\omega_{j_0,1}| \cdot |\omega_{j_0,2}| = \chi_2(\alpha_{j_0}) = \max_j \{\chi_2(\alpha_j)\} = \lambda_1(\alpha).$$

There is no obvious reason to exclude the worst case $j_0 = 1$ which yields that

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{2,1}| > |\omega_{1,1}| \cdot |\omega_{1,2}| = \chi_2(\alpha_1) = \max_j \{\chi_2(\alpha_j)\} = \lambda_1(\alpha).$$

To proceed, we observe that over complex number field $\mathbb{C}$, the above pathology does not happen because each eigenvalue $\omega_{j,2}$ turns out to be the complex conjugate of $\omega_{j,1}$. This fact follows from the Hodge decomposition $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$, which does not seem to exist in étale cohomology as far as we know. But we still believe that $\omega_{j,2} = \overline{\omega_{j,1}}$ for all j. (As a consequence of our main theorem, we will see that this is actually true; see Remark 3.10.) The following lemma makes use of this observation to reduce the splitting product case as in Lemma 3.6 to the case when $X = A^n$ for some simple abelian variety A.

Lemma 3.9. *In order to prove Theorem 1.2, it suffices to show that if A^n is a power of a simple abelian variety A and $\alpha \in \operatorname{End}(A^n)$ is a surjective endomorphism of A^n, then $\lambda_1(\alpha) = |\omega_1|^2$, where ω_1 is one of the complex roots of the characteristic polynomial $P_\alpha(t)$ of α with the maximal absolute value.*

Proof. Thanks to Lemma 3.6, let us consider the case when the abelian variety $X = A_1^{n_1} \times \cdots \times A_s^{n_s}$, where the A_j are mutually nonisogenous simple abelian varieties, and $\alpha = \alpha_1 \times \cdots \times \alpha_s$ is a surjective endomorphism of X with $\alpha_j \in \operatorname{End}(A_j^{n_j})$. We assume that the reader has been familiar with the notation introduced in Remark 3.8, in particular, (3-3)–(3-5). Applying the hypothesis of Lemma 3.9 to each $A_j^{n_j}$ and α_j, we have $\lambda_1(\alpha_j) = |\omega_{j,1}|^2$. It follows from Lemma 3.2 and Theorem 2.1 that $\lambda_1(\alpha_j) \le \chi_2(\alpha_j) = |\omega_{j,1}| \cdot |\omega_{j,2}|$. Hence $\lambda_1(\alpha_j) = \chi_2(\alpha_j)$ and $|\omega_{j,1}| = |\omega_{j,2}|$ for all j which tells us $j_0 = 1$. This yields that

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{1,2}| = \chi_2(\alpha_1) = \max_j \{\chi_2(\alpha_j)\} = \max_j \{\lambda_1(\alpha_j)\} = \lambda_1(\alpha).$$

The first and second equalities follow again from Theorem 2.1, the third one holds because $j_0 = 1$, (3-3) gives the last one. □

3.4. ***Proof of Theorem 1.2.*** We are now ready to prove the main theorem.

Proof of Theorem 1.2. By Lemma 3.9, we can assume that $X = A^n$ for some simple abelian variety A and $\alpha \in \operatorname{End}(X)$ is a surjective endomorphism of X. Let $P_\alpha(t) \in \mathbb{Z}[t]$ be the characteristic polynomial of α (see Theorem 2.3). Set $g = \dim X$. Denote all complex roots of $P_\alpha(t)$ by $\omega_1, \dots, \omega_{2g}$. Without loss of generality, we may assume that

$$|\omega_1| \ge \cdots \ge |\omega_{2g}|.$$

We shall prove that

$$\lambda_1(\alpha) = |\omega_1|^2, \tag{3-6}$$

which will conclude the proof of the theorem by Lemma 3.9.

Under the above assumption, the endomorphism algebra $\mathrm{End}^0(X)$ is the simple $\mathbb{Q}$-algebra $\mathrm{M}_n(D)$ of all $n \times n$ matrices with entries in the division ring $D := \mathrm{End}^0(A)$. Let K denote the center of D, and K_0 the maximal totally real subfield of K. As usual, we set

$$d^2 = [D : K], \quad e = [K : \mathbb{Q}] \quad \text{and} \quad e_0 = [K_0 : \mathbb{Q}].$$

Note that by Lemma 3.4, the natural pullback action α^* on NS^0 can be extended to an action α^* on the whole endomorphism $\mathbb{Q}$-algebra $\mathrm{End}^0(X)$ as follows:

$$\alpha^* \colon \mathrm{End}^0(X) \to \mathrm{End}^0(X) \quad \text{via } \psi \mapsto \alpha^\dagger \circ \psi \circ \alpha.$$

On the other hand, by tensoring with $\mathbb{R}$, we know that

$$\mathrm{End}(X)_\mathbb{R} = \mathrm{End}^0(X) \otimes_\mathbb{Q} \mathbb{R} \simeq \mathrm{M}_n(D) \otimes_\mathbb{Q} \mathbb{R} \simeq \mathrm{M}_n(D \otimes_\mathbb{Q} \mathbb{R})$$

is either a product of $\mathrm{M}_r(\mathbb{R})$, $\mathrm{M}_r(\mathbb{C})$ or $\mathrm{M}_r(\boldsymbol{H})$ with $\mathrm{NS}(X)_\mathbb{R}$ being a product of $\mathscr{H}_r(\mathbb{R})$, $\mathscr{H}_r(\mathbb{C})$ or $\mathscr{H}_r(\boldsymbol{H})$, the corresponding subspace of symmetric/Hermitian matrices (see Theorem 2.9). When there is no risk of confusion, for simplicity, we still denote the induced action $\alpha^* \otimes_\mathbb{Q} 1_\mathbb{R}$ by α^*. In particular, we would write $\alpha^*|_{\mathrm{End}(X)_\mathbb{R}}$ and $\alpha^*|_{\mathrm{NS}(X)_\mathbb{R}}$ to emphasize the acting spaces.

According to Albert's classification of the endomorphism $\mathbb{Q}$-algebra D of a simple abelian variety A [Mumford 1970, Section 21, Theorem 2], we have the following four cases.

Case 1. D is of Type I(e): $d = 1$, $e = e_0$ and $D = K = K_0$ is a totally real algebraic number field and the involution (on D) is the identity. In this case,

$$\mathrm{End}(X)_\mathbb{R} \simeq \bigoplus_{i=1}^{e_0} \mathrm{M}_n(\mathbb{R}) \quad \text{and} \quad \mathrm{NS}(X)_\mathbb{R} \simeq \bigoplus_{i=1}^{e_0} \mathscr{H}_n(\mathbb{R}).$$

For our $\alpha \in \mathrm{End}(X)$, let us denote its image $\alpha \otimes_\mathbb{Z} 1_\mathbb{R}$ in $\mathrm{End}(X)_\mathbb{R}$ by the block diagonal matrix $\boldsymbol{A}_\alpha = \boldsymbol{A}_{\alpha,1} \oplus \cdots \oplus \boldsymbol{A}_{\alpha,e_0}$ with each $\boldsymbol{A}_{\alpha,i} \in \mathrm{M}_n(\mathbb{R})$. Then the Rosati involution $\alpha^\dagger$ of α could be represented by the transpose $\boldsymbol{A}_\alpha^\mathsf{T} = \boldsymbol{A}_{\alpha,1}^\mathsf{T} \oplus \cdots \oplus \boldsymbol{A}_{\alpha,e_0}^\mathsf{T}$ (see Theorem 2.9). Hence we can rewrite the induced action α^* on $\mathrm{End}(X)_\mathbb{R}$ in the following matrix form:

$$\boldsymbol{B} = \boldsymbol{B}_1 \oplus \cdots \oplus \boldsymbol{B}_{e_0} \mapsto \boldsymbol{A}_\alpha^\mathsf{T} \boldsymbol{B} \boldsymbol{A}_\alpha = \boldsymbol{A}_{\alpha,1}^\mathsf{T} \boldsymbol{B}_1 \boldsymbol{A}_{\alpha,1} \oplus \cdots \oplus \boldsymbol{A}_{\alpha,e_0}^\mathsf{T} \boldsymbol{B}_{e_0} \boldsymbol{A}_{\alpha,e_0}.$$

Thanks to Lemma 3.5(1), for each i, the linear transformation defined by the mapping

$$\boldsymbol{B}_i \in \mathrm{M}_n(\mathbb{R}) \mapsto \boldsymbol{A}_{\alpha,i}^\mathsf{T} \boldsymbol{B}_i \boldsymbol{A}_{\alpha,i} \in \mathrm{M}_n(\mathbb{R}),$$

can be represented by the Kronecker product $\boldsymbol{A}_{\alpha,i}\otimes\boldsymbol{A}_{\alpha,i}$. Hence the above linear transformation $\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}}$ on the e_0n^2-dimensional $\mathbb{R}$-vector space $\mathrm{End}(X)_{\mathbb{R}}$ is represented by the block diagonal matrix

$$(\boldsymbol{A}_{\alpha,1}\otimes\boldsymbol{A}_{\alpha,1})\oplus\cdots\oplus(\boldsymbol{A}_{\alpha,e_0}\otimes\boldsymbol{A}_{\alpha,e_0}).$$

For each $1\le i\le e_0$, denote all eigenvalues of $\boldsymbol{A}_{\alpha,i}$ by $\pi_{i,1},\ldots,\pi_{i,n}$. It thus follows from the above discussion that all eigenvalues of the linear transformation $\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}}$ are exactly $\pi_{i,j}\pi_{i,k}$ with $1\le j,k\le n$ and $1\le i\le e_0$. In particular, if $\boldsymbol{v}_{i,j}$ and $\boldsymbol{v}_{i,k}$ denote eigenvectors of $\boldsymbol{A}_{\alpha,i}$ corresponding to $\pi_{i,j}$ and $\pi_{i,k}$, respectively, then

$$\boldsymbol{v}_{i,j}\otimes\boldsymbol{v}_{i,k}=\mathrm{vec}(\boldsymbol{v}_{i,j}^{\mathsf{T}}\otimes\boldsymbol{v}_{i,k})=\mathrm{vec}(\boldsymbol{v}_{i,k}\otimes\boldsymbol{v}_{i,j}^{\mathsf{T}})=\mathrm{vec}(\boldsymbol{v}_{i,k}\cdot\boldsymbol{v}_{i,j}^{\mathsf{T}})$$

is the eigenvector of $\boldsymbol{A}_{\alpha,i}\otimes\boldsymbol{A}_{\alpha,i}$ corresponding to $\pi_{i,j}\pi_{i,k}$.[5] Now, according to Remark 2.5, the reduced characteristic polynomial $\chi_\alpha^{\mathrm{red}}(t)$ of α is independent of the change of the ground field, and hence equal to the reduced characteristic polynomial $\chi^{\mathrm{red}}_{\alpha\otimes_{\mathbb{Z}}1_{\mathbb{R}}}(t)$ of $\alpha\otimes_{\mathbb{Z}}1_{\mathbb{R}}\in\mathrm{End}(X)_{\mathbb{R}}$, while the latter by Definition 2.4 is just the characteristic polynomial $\det(t\boldsymbol{I}_{e_0n}-\boldsymbol{A}_\alpha)$ of $\boldsymbol{A}_\alpha$. Hence, without loss of generality, we may assume that $\omega_1=\pi_{1,1}$ by Lemma 2.7.

We now have two subcases to consider. If $\pi_{1,1}\in\mathbb{R}$ so that $\boldsymbol{v}_{1,1}$ is also a real eigenvector, then $\boldsymbol{v}_{1,1}\otimes\boldsymbol{v}_{1,1}$ is a real eigenvector of $\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}}$ corresponding to the eigenvalue $\pi_{1,1}^2$. This eigenvector is the associated column vector of the real symmetric matrix $\boldsymbol{v}_{1,1}\otimes\boldsymbol{v}_{1,1}^{\mathsf{T}}=\boldsymbol{v}_{1,1}^{\mathsf{T}}\otimes\boldsymbol{v}_{1,1}$. Next, let us assume that $\pi_{1,1}\in\mathbb{C}\setminus\mathbb{R}$. Then $\bar{\pi}_{1,1}$ is another eigenvalue of $\boldsymbol{A}_{\alpha,1}$ with the corresponding eigenvector $\bar{\boldsymbol{v}}_{1,1}$, since $\boldsymbol{A}_{\alpha,1}$ is defined over $\mathbb{R}$. It follows that $\boldsymbol{v}_{1,1}\otimes\bar{\boldsymbol{v}}_{1,1}+\bar{\boldsymbol{v}}_{1,1}\otimes\boldsymbol{v}_{1,1}$ is a real eigenvector of $\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}}$ corresponding to the eigenvalue $\pi_{1,1}\bar{\pi}_{1,1}=|\pi_{1,1}|^2$; moreover, it is the associated column vector of the real symmetric matrix

$$\boldsymbol{v}_{1,1}^{\mathsf{T}}\otimes\bar{\boldsymbol{v}}_{1,1}+\bar{\boldsymbol{v}}_{1,1}^{\mathsf{T}}\otimes\boldsymbol{v}_{1,1}=\bar{\boldsymbol{v}}_{1,1}\otimes\boldsymbol{v}_{1,1}^{\mathsf{T}}+\bar{\boldsymbol{v}}_{1,1}^{\mathsf{T}}\otimes\boldsymbol{v}_{1,1}.$$

In either case, we have shown that the spectral radii of $\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}}$ and $\alpha^*|_{\mathrm{NS}(X)_{\mathbb{R}}}$ coincide, both equal to $|\pi_{1,1}|^2$. In summary, we have

$$|\omega_1|^2=|\pi_{1,1}|^2=\rho(\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}})=\rho(\alpha^*|_{\mathrm{NS}(X)_{\mathbb{R}}})=\lambda_1(\alpha).$$

For the last equality, see Remark 3.3. So we conclude the proof of the equality (3-6) in this case.

Case 2. *D is of Type II(e): $d=2$, $e=e_0$, $K=K_0$ is a totally real algebraic number field and D is an indefinite quaternion division algebra over K.* Hence

$$\mathrm{End}(X)_{\mathbb{R}}\simeq\bigoplus_{i=1}^{e_0}\mathrm{M}_{2n}(\mathbb{R})\quad\text{and}\quad\mathrm{NS}(X)_{\mathbb{R}}\simeq\bigoplus_{i=1}^{e_0}\mathscr{H}_{2n}(\mathbb{R}).$$

The rest is exactly the same as Case 1.

[5]Note that due to multiplicities of eigenvalues, $\boldsymbol{A}_{\alpha,i}$ does not necessarily have n distinct eigenvalues. Thus, $\boldsymbol{v}_{i,j}$ and $\boldsymbol{v}_{i,k}$ may be the same for different j and k. Also, not all eigenvectors of $\boldsymbol{A}_{\alpha,i}\otimes\boldsymbol{A}_{\alpha,i}$ have to arise in this way, namely, being the tensor products $\boldsymbol{v}_{i,j}\otimes\boldsymbol{v}_{i,k}$. For instance, one could consider a Jordan block $J_{\lambda,2}\in\mathrm{M}_2(\mathbb{R})$ with the eigenvalue λ, but $J_{\lambda,2}\otimes J_{\lambda,2}\sim J_{\lambda^2,1}\oplus J_{\lambda^2,3}$.

Case 3. D is of Type III(e): $d = 2$, $e = e_0$, $K = K_0$ is a totally real algebraic number field and D is a definite quaternion division algebra over K. In this case,

$$\mathrm{End}(X)_{\mathbb{R}} \simeq \bigoplus_{i=1}^{e_0} \mathrm{M}_n(\boldsymbol{H}) \quad \text{and} \quad \mathrm{NS}(X)_{\mathbb{R}} \simeq \bigoplus_{i=1}^{e_0} \mathscr{H}_n(\boldsymbol{H}),$$

where $\boldsymbol{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ is the standard quaternion algebra over $\mathbb{R}$. Clearly, $\boldsymbol{H}$ can be embedded, in a standard way (see e.g., [Reiner 2003, Example 9.4]), into $\mathrm{M}_2(\mathbb{C}) \simeq \boldsymbol{H} \otimes_{\mathbb{R}} \mathbb{C}$. This induces a natural embedding of $\mathrm{M}_n(\boldsymbol{H})$ into $\mathrm{M}_{2n}(\mathbb{C}) \simeq \mathrm{M}_n(\boldsymbol{H}) \otimes_{\mathbb{R}} \mathbb{C}$ as follows [Lee 1949, Section 4]:

$$\iota\colon\ \mathrm{M}_n(\boldsymbol{H}) \hookrightarrow \mathrm{M}_{2n}(\mathbb{C}) \quad \text{via } \boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2 \boldsymbol{j} \mapsto \iota(\boldsymbol{A}) := \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ -\overline{\boldsymbol{A}}_2 & \overline{\boldsymbol{A}}_1 \end{pmatrix}.$$

In particular, a quaternionic matrix $\boldsymbol{A}$ is Hermitian if and only if its image $\iota(\boldsymbol{A})$ is a Hermitian complex matrix.

For brevity, we only consider the case $e_0 = 1$ (to deal with the general case, the only cost is to introduce an index i as we have done in Case 1 since the matrices involved are block diagonal matrices). Denote the image $\alpha \otimes_{\mathbb{Z}} 1_{\mathbb{R}}$ of α in $\mathrm{M}_n(\boldsymbol{H})$ by $\boldsymbol{A}_\alpha = \boldsymbol{A}_1 + \boldsymbol{A}_2 \boldsymbol{j}$ with $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathrm{M}_n(\mathbb{C})$. Then the Rosati involution $\alpha^\dagger$ of α could be represented by the quaternionic conjugate transpose $\boldsymbol{A}_\alpha^* = \overline{\boldsymbol{A}}_\alpha^{\mathsf{T}}$ (see Theorem 2.9), whose image under ι is just the complex conjugate transpose $\iota(\boldsymbol{A}_\alpha)^*$ (aka Hermitian transpose) of $\iota(\boldsymbol{A}_\alpha)$. Similar as in Lemma 3.4, the action α^* on $\mathrm{End}(X)_{\mathbb{R}} \simeq \mathrm{M}_n(\boldsymbol{H})$ can be extended to

$$\mathrm{End}(X)_{\mathbb{C}} := \mathrm{End}(X)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{M}_{2n}(\mathbb{C}).$$

By abuse of notation, we still denote this induced action by $\alpha^*\colon\ \mathrm{M}_{2n}(\mathbb{C}) \to \mathrm{M}_{2n}(\mathbb{C})$, which maps $\boldsymbol{B}$ to $\iota(\boldsymbol{A}_\alpha)^* \cdot \boldsymbol{B} \cdot \iota(\boldsymbol{A}_\alpha)$. It follows from Lemma 3.5(2) that $\alpha^*|_{\mathrm{M}_{2n}(\mathbb{C})}$ could be represented by the Kronecker product $\iota(\boldsymbol{A}_\alpha) \otimes \overline{\iota(\boldsymbol{A}_\alpha)}$.

Note that our $\mathrm{End}(X)_{\mathbb{C}} \simeq \mathrm{M}_{2n}(\mathbb{C})$ is a central simple $\mathbb{C}$-algebra. Then by Definition 2.4 and Remark 2.5, the reduced characteristic polynomial $\chi_\alpha^{\mathrm{red}}(t)$ of α is equal to the characteristic polynomial $\det(t\boldsymbol{I}_{2n} - \iota(\boldsymbol{A}_\alpha))$ of the complex matrix $\iota(\boldsymbol{A}_\alpha)$. Thanks to [Lee 1949, Theorem 5], the $2n$ eigenvalues of $\iota(\boldsymbol{A}_\alpha)$ fall into n pairs, each pair consisting of two conjugate complex numbers; denote them by $\pi_1, \ldots, \pi_n, \pi_{n+1} = \bar{\pi}_1, \ldots, \pi_{2n} = \bar{\pi}_n$. In fact, it is easy to verify that if $\pi_i \in \mathbb{C}$ is an eigenvalue of $\iota(\boldsymbol{A}_\alpha)$ so that

$$\iota(\boldsymbol{A}_\alpha) \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{v}_i \end{pmatrix} = \pi_i \begin{pmatrix} \boldsymbol{u}_i \\ \boldsymbol{v}_i \end{pmatrix}, \quad \text{then } \iota(\boldsymbol{A}_\alpha) \begin{pmatrix} -\bar{\boldsymbol{v}}_i \\ \bar{\boldsymbol{u}}_i \end{pmatrix} = \bar{\pi}_i \begin{pmatrix} -\bar{\boldsymbol{v}}_i \\ \bar{\boldsymbol{u}}_i \end{pmatrix},$$

i.e., $\bar{\pi}_i$ is also an eigenvalue of $\iota(\boldsymbol{A}_\alpha)$ corresponding to the eigenvector $(-\bar{\boldsymbol{v}}_i^{\mathsf{T}}, \bar{\boldsymbol{u}}_i^{\mathsf{T}})^{\mathsf{T}}$. Therefore, without loss of generality, we may assume that $\omega_1 = \pi_1$ by Lemma 2.7.

Let $(\boldsymbol{u}_1^{\mathsf{T}}, \boldsymbol{v}_1^{\mathsf{T}})^{\mathsf{T}}$ denote an eigenvector of $\iota(\boldsymbol{A}_\alpha)$ corresponding to the eigenvalue π_1. Then $(-\bar{\boldsymbol{v}}_1^{\mathsf{T}}, \bar{\boldsymbol{u}}_1^{\mathsf{T}})^{\mathsf{T}}$ is an eigenvector of $\iota(\boldsymbol{A}_\alpha)$ corresponding to the eigenvalue $\bar{\pi}_1$. Since the linear transformation $\alpha^*|_{\mathrm{End}(X)_{\mathbb{C}}}$ can be represented by $\iota(\boldsymbol{A}_\alpha) \otimes \overline{\iota(\boldsymbol{A}_\alpha)}$ (see Lemma 3.5(2)), we see that both $(\boldsymbol{u}_1^{\mathsf{T}}, \boldsymbol{v}_1^{\mathsf{T}})^{\mathsf{T}} \otimes (\bar{\boldsymbol{u}}_1^{\mathsf{T}}, \bar{\boldsymbol{v}}_1^{\mathsf{T}})^{\mathsf{T}}$ and $(-\bar{\boldsymbol{v}}_1^{\mathsf{T}}, \bar{\boldsymbol{u}}_1^{\mathsf{T}})^{\mathsf{T}} \otimes (-\boldsymbol{v}_1^{\mathsf{T}}, \boldsymbol{u}_1^{\mathsf{T}})^{\mathsf{T}}$ are eigenvectors of $\alpha^*|_{\mathrm{End}(X)_{\mathbb{C}}}$, corresponding to the same eigenvalue $\pi_1\bar{\pi}_1$.

Recall that these two eigenvectors are the associated column vectors of the Hermitian complex matrices

$$\begin{pmatrix}\bar{\boldsymbol{u}}_1\\ \bar{\boldsymbol{v}}_1\end{pmatrix}\otimes(\boldsymbol{u}_1^{\mathsf{T}},\boldsymbol{v}_1^{\mathsf{T}})=\begin{pmatrix}\bar{\boldsymbol{u}}_1\\ \bar{\boldsymbol{v}}_1\end{pmatrix}\cdot(\boldsymbol{u}_1^{\mathsf{T}},\boldsymbol{v}_1^{\mathsf{T}})\quad\text{and}\quad\begin{pmatrix}-\boldsymbol{v}_1\\ \boldsymbol{u}_1\end{pmatrix}\otimes(-\bar{\boldsymbol{v}}_1^{\mathsf{T}},\bar{\boldsymbol{u}}_1^{\mathsf{T}})=\begin{pmatrix}-\boldsymbol{v}_1\\ \boldsymbol{u}_1\end{pmatrix}\cdot(-\bar{\boldsymbol{v}}_1^{\mathsf{T}},\bar{\boldsymbol{u}}_1^{\mathsf{T}}),$$

respectively. It is then easy to verify that

$$\begin{pmatrix}\bar{\boldsymbol{u}}_1\\ \bar{\boldsymbol{v}}_1\end{pmatrix}\cdot(\boldsymbol{u}_1^{\mathsf{T}},\boldsymbol{v}_1^{\mathsf{T}})+\begin{pmatrix}-\boldsymbol{v}_1\\ \boldsymbol{u}_1\end{pmatrix}\cdot(-\bar{\boldsymbol{v}}_1^{\mathsf{T}},\bar{\boldsymbol{u}}_1^{\mathsf{T}})=\begin{pmatrix}\bar{\boldsymbol{u}}_1\boldsymbol{u}_1^{\mathsf{T}}+\boldsymbol{v}_1\bar{\boldsymbol{v}}_1^{\mathsf{T}} & \bar{\boldsymbol{u}}_1\boldsymbol{v}_1^{\mathsf{T}}-\boldsymbol{v}_1\bar{\boldsymbol{u}}_1^{\mathsf{T}}\\ \bar{\boldsymbol{v}}_1\boldsymbol{u}_1^{\mathsf{T}}-\boldsymbol{u}_1\bar{\boldsymbol{v}}_1^{\mathsf{T}} & \bar{\boldsymbol{v}}_1\boldsymbol{v}_1^{\mathsf{T}}+\boldsymbol{u}_1\bar{\boldsymbol{u}}_1^{\mathsf{T}}\end{pmatrix}$$

is a Hermitian complex matrix lying in the image of ι. In other words, this sum belongs to $\mathrm{NS}(X)_{\mathbb{C}}$. Hence, similar as in Case 1, the spectral radii of $\alpha^*|_{\mathrm{NS}(X)_{\mathbb{C}}}$ and $\alpha^*|_{\mathrm{End}(X)_{\mathbb{C}}}$ coincide, both equal to $|\pi_1|^2$. Overall, we have

$$|\omega_1|^2=|\pi_1|^2=\rho(\alpha^*|_{\mathrm{End}(X)_{\mathbb{C}}})=\rho(\alpha^*|_{\mathrm{NS}(X)_{\mathbb{C}}})=\rho(\alpha^*|_{\mathrm{NS}(X)_{\mathbb{R}}})=\lambda_1(\alpha).$$

We thus conclude the proof of the equality (3-6) in this case.

Case 4. D is of Type IV(e_0,d): $e=2e_0$ and D is a division algebra over the CM-field $K\supsetneq K_0$ (i.e., K is a totally imaginary quadratic extension of a totally real algebraic number field K_0). Then

$$\mathrm{End}(X)_{\mathbb{R}}\simeq\bigoplus_{i=1}^{e_0}\mathrm{M}_{dn}(\mathbb{C})\quad\text{and}\quad\mathrm{NS}(X)_{\mathbb{R}}\simeq\bigoplus_{i=1}^{e_0}\mathscr{H}_{dn}(\mathbb{C}).$$

For simplicity, we just deal with the case $e_0=1$. Denote the image of α in $\mathrm{End}(X)_{\mathbb{R}}$ by the matrix $\boldsymbol{A}_\alpha\in\mathrm{M}_{dn}(\mathbb{C})$. Again, the Rosati involution $\alpha^\dagger$ of α could be represented by the complex conjugate transpose $\boldsymbol{A}_\alpha^*=\overline{\boldsymbol{A}}_\alpha^{\mathsf{T}}$ (see Theorem 2.9). It follows from Lemma 3.5(2) that the induced linear map $\alpha^*|_{\mathrm{M}_{dn}(\mathbb{C})}$ on the d^2n^2-dimensional $\mathbb{C}$-vector space $\mathrm{M}_{dn}(\mathbb{C})$ is represented by the Kronecker product $\boldsymbol{A}_\alpha\otimes\overline{\boldsymbol{A}}_\alpha$; however, the induced linear map $\alpha^*|_{\mathrm{End}(X)_{\mathbb{R}}}$ on the $2d^2n^2$-dimensional $\mathbb{R}$-vector space $\mathrm{End}(X)_{\mathbb{R}}$ is represented by the block diagonal matrix $(\boldsymbol{A}_\alpha\otimes\overline{\boldsymbol{A}}_\alpha)\oplus(\overline{\boldsymbol{A}}_\alpha\otimes\boldsymbol{A}_\alpha)$ by Lemma 3.5(3), though we do not need this fact later.

Note that the center of our $\mathbb{R}$-algebra $\mathrm{End}(X)_{\mathbb{R}}\simeq\mathrm{M}_{dn}(\mathbb{C})$ is $\mathbb{C}$. Then by Definition 2.4 and Remark 2.5, the reduced characteristic polynomial $\chi_\alpha^{\mathrm{red}}(t)$ of α is equal to the product of the characteristic polynomial $\det(t\boldsymbol{I}_{dn}-\boldsymbol{A}_\alpha)$ of $\boldsymbol{A}_\alpha$ and its complex conjugate. We denote all of its complex roots by $\pi_1,\dots,\pi_{dn},\bar{\pi}_1,\dots,\bar{\pi}_{dn}$. Without loss of generality, we may assume that $\omega_1=\pi_1$ by Lemma 2.7. Let $\boldsymbol{v}_1$ be a complex eigenvector of $\boldsymbol{A}_\alpha$ corresponding to the eigenvalue π_1. Then $\boldsymbol{v}_1\otimes\bar{\boldsymbol{v}}_1$ is an eigenvector of $\boldsymbol{A}_\alpha\otimes\overline{\boldsymbol{A}}_\alpha$ corresponding to the eigenvalue $\pi_1\bar{\pi}_1=|\pi_1|^2$. Note that $\boldsymbol{v}_1\otimes\bar{\boldsymbol{v}}_1$ is the associated column vector of the Hermitian complex matrix $\bar{\boldsymbol{v}}_1\otimes\boldsymbol{v}_1^{\mathsf{T}}=\boldsymbol{v}_1^{\mathsf{T}}\otimes\bar{\boldsymbol{v}}_1\in\mathrm{NS}(X)_{\mathbb{R}}$. Hence, in this last case, we also have

$$|\omega_1|^2=|\pi_1|^2=\rho(\alpha^*|_{\mathrm{M}_{dn}(\mathbb{C})})=\rho(\alpha^*|_{\mathrm{NS}(X)_{\mathbb{R}}})=\lambda_1(\alpha).$$

We thus finally complete the proof of Theorem 1.2. □

Remark 3.10. (1) It follows from our proof, in particular from the key equality (3-6), as well as Birkhoff's generalization of the Perron–Frobenius theorem, that either $\omega_2 = \omega_1 \in \mathbb{R}$ or $\omega_2 = \bar{\omega}_1 \neq \omega_1$. This is true for any complex torus X because by the Hodge decomposition we have $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$, where $H^{1,0}(X) = H^0(X, \Omega_X^1)$. A natural question is whether it is true for all ω_i in general, i.e., either $\omega_{2i} = \omega_{2i-1} \in \mathbb{R}$ or $\omega_{2i} = \bar{\omega}_{2i-1} \neq \omega_{2i-1}$ for any $2 \leq i \leq g = \dim X$.
(2) If our self-morphism f is not surjective or α is not an isogeny, one can also proceed by replacing X by the image $\alpha(X)$, which is still an abelian variety of dimension less than $\dim X$.

List of symbols

$\boldsymbol{k}$	an algebraically closed field of arbitrary characteristic
ℓ	a prime different from char $\boldsymbol{k}$
X	an abelian variety of dimension g defined over $\boldsymbol{k}$
$\hat{X}$	the dual abelian variety $\mathrm{Pic}^0(X)$ of X
α, ψ	endomorphisms of X
$\hat{\alpha}, \hat{\psi}$	the induced dual endomorphisms of $\hat{X}$
$\mathrm{End}(X)$	the endomorphism ring of X
$\mathrm{End}^0(X)$	$\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, the endomorphism $\mathbb{Q}$-algebra of X
$\mathrm{End}(X)_{\mathbb{R}}$	$\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{End}^0(X) \otimes_{\mathbb{Q}} \mathbb{R}$, the endomorphism $\mathbb{R}$-algebra of X
$\mathrm{M}_n(R)$	the ring of all $n \times n$ matrices with entries in a ring R
$\phi_{\mathscr{L}}$	the induced homomorphism of a line bundle $\mathscr{L}$ on X: $\phi_{\mathscr{L}}\colon X \to \hat{X}, \quad x \mapsto t_x^*\mathscr{L} \otimes \mathscr{L}^{-1}$
$\phi = \phi_{\mathscr{L}_0}$	a fixed polarization of X induced from some ample line bundle $\mathscr{L}_0$
$\dagger$	the Rosati involution on $\mathrm{End}^0(X)$ defined in the following way: $\psi \mapsto \psi^\dagger := \phi^{-1} \circ \hat{\psi} \circ \phi$, for any $\psi \in \mathrm{End}^0(X)$
$\mathrm{NS}(X)$	$\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, the Néron–Severi group of X
$\mathrm{NS}^0(X)$	$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = N^1(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$ (see Remark 3.3)
$\mathrm{NS}(X)_{\mathbb{R}}$	$\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{NS}^0(X) \otimes_{\mathbb{Q}} \mathbb{R} = N^1(X)_{\mathbb{R}}$
$N^k(X)_{\mathbb{R}}$	$N^k(X) \otimes_{\mathbb{Z}} \mathbb{R}$, the $\mathbb{R}$-vector space of numerical equivalent classes of codimension-k cycles (with $0 \leq k \leq g = \dim X$)
$H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$	$H^i_{\text{ét}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, the ℓ-adic étale cohomology group of degree i
$T_\ell X$	the Tate module $\varprojlim_n X_{\ell^n}(\boldsymbol{k})$ of X, a free $\mathbb{Z}_\ell$-module of rank $2g$
$T_\ell \alpha$	the induced endomorphism on $T_\ell X$
A	a simple abelian variety defined over $\boldsymbol{k}$
D	$\mathrm{End}^0(A)$, the endomorphism $\mathbb{Q}$-algebra of A
K	the center of the division ring $D = \mathrm{End}^0(A)$
K_0	the maximal totally real subfield of K
$\boldsymbol{H}$	the standard quaternion algebra over $\mathbb{R}$

Acknowledgments

I would like to thank Dragos Ghioca and Zinovy Reichstein for their constant support, Yuri Zarhin and Yishu Zeng for helpful discussions, Tuyen Trung Truong for reading an earlier draft of this article and for his inspiring comments. Special thanks go to the referees of my other paper [Hu 2019] since one of their comments motivates this article initially. Finally, I am grateful to the referee for his/her many helpful and invaluable suggestions which significantly improve the exposition of the paper.

References

[Bost and Charles 2016] J.-B. Bost and F. Charles, "Some remarks concerning the Grothendieck period conjecture", *J. Reine Angew. Math.* **714** (2016), 175–208. MR Zbl

[Deligne 1974] P. Deligne, "La conjecture de Weil, I", *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307. MR Zbl

[Dinh and Sibony 2017] T.-C. Dinh and N. Sibony, "Equidistribution problems in complex dynamics of higher dimension", *Int. J. Math.* **28**:7 (2017), art. id. 1750057. MR Zbl

[Esnault and Srinivas 2013] H. Esnault and V. Srinivas, "Algebraic versus topological entropy for surfaces over finite fields", *Osaka J. Math.* **50**:3 (2013), 827–846. MR Zbl

[Gromov 2003] M. Gromov, "On the entropy of holomorphic maps", *Enseign. Math.* (2) **49**:3-4 (2003), 217–235. MR Zbl

[Hu 2019] F. Hu, "A theorem of Tits type for automorphism groups of projective varieties in arbitrary characteristic", *Math. Ann.* (online publication February 2019).

[Katz and Messing 1974] N. M. Katz and W. Messing, "Some consequences of the Riemann hypothesis for varieties over finite fields", *Invent. Math.* **23** (1974), 73–77. MR Zbl

[Kleiman 1968] S. L. Kleiman, "Algebraic cycles and the Weil conjectures", pp. 359–386 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[Lee 1949] H. C. Lee, "Eigenvalues and canonical forms of matrices with quaternion coefficients", *Proc. Roy. Irish Acad. Sect. A.* **52** (1949), 253–260. MR Zbl

[Matsusaka 1957] T. Matsusaka, "The criteria for algebraic equivalence and the torsion group", *Amer. J. Math.* **79** (1957), 53–66. MR Zbl

[Milne 1986] J. S. Milne, "Abelian varieties", pp. 103–150 in *Arithmetic geometry* (Storrs, CT, 1984), edited by G. Cornell and J. H. Silverman, Springer, 1986. MR Zbl

[Mumford 1970] D. Mumford, *Abelian varieties*, Tata Inst. Fundam. Res. Stud. Math. **5**, Oxford Univ. Press, 1970. MR Zbl

[Reiner 2003] I. Reiner, *Maximal orders*, London Mathematical Society Monographs. New Series **28**, Clarendon, Oxford, 2003. MR Zbl

[Shuddhodan 2019] K. V. Shuddhodan, "Constraints on the cohomological correspondence associated to a self map", *Compos. Math.* **155**:6 (2019), 1047–1056. MR Zbl

[Tate 1966] J. Tate, "Endomorphisms of abelian varieties over finite fields", *Invent. Math.* **2** (1966), 134–144. MR Zbl

[Truong 2016] T. T. Truong, "Relations between dynamical degrees, Weil's Riemann hypothesis and the standard conjectures", preprint, 2016. arXiv

[Yomdin 1987] Y. Yomdin, "Volume growth and entropy", *Israel J. Math.* **57**:3 (1987), 285–300. MR Zbl

Communicated by Hélène Esnault
Received 2018-10-23 Revised 2019-04-17 Accepted 2019-07-15

hf@u.nus.edu *Department of Mathematics, University of British Columbia, Vancouver BC, Canada*

ALGEBRA AND NUMBER THEORY 13:8 (2019)
dx.doi.org/10.2140/ant.2019.13.1959

A comparison between pro-p Iwahori–Hecke modules and mod p representations

Noriyuki Abe

We give an equivalence of categories between certain subcategories of modules of pro-p Iwahori–Hecke algebras and modulo p representations.

1. Introduction

Let G be a connected reductive p-adic group and K a compact open subgroup of G. Then one can attach the Hecke algebra $\mathcal{H}$ to this pair (G, K) and we have a functor $\pi \mapsto \pi^K = \{v \in \pi \mid \pi(k)v = v \ (k \in K)\}$ from the category of smooth representations of G to the category of $\mathcal{H}$-modules. These algebras and functors are powerful tools to study the representation theory of G. In a classical case, namely for smooth representations over the field of complex numbers, this functor gives a bijection between the set of isomorphism classes of irreducible smooth representations of G such that $\pi^K \neq 0$ and the set of isomorphism classes of simple $\mathcal{H}$-modules. Moreover, the famous theorem of Borel [1976] says that the functor gives an equivalence of categories between the category of smooth representations π of G which is generated by π^K and the category of $\mathcal{H}$-modules when K is an Iwahori subgroup.

In this paper, we study modulo p representation theory of G. In this case, it is natural to consider a pro-p Iwahori subgroup $I(1)$ which is the pro-p radical of an Iwahori subgroup since any nonzero modulo p representation has a nonzero vector fixed by the pro-p Iwahori subgroup. The corresponding Hecke algebra is called a pro-p Iwahori–Hecke algebra. The aim of this paper is to give a relation between $\mathcal{H}$-modules and modulo p representations.

Such a relation was first discovered by Vignéras [2007] when $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Based on a classification result due to Barthel and Livné [1995; 1994] and Breuil [2003], she proved that the functor $\pi \mapsto \pi^{I(1)}$ gives a bijection between simple objects. This was enhanced to the level of categories by Ollivier [2009]. Namely she proved that the category of $\mathcal{H}$-modules is equivalent to the category of modulo p representations of G which are generated by $\pi^{I(1)}$. The quasiinverse of this equivalence is given by $M \mapsto M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ where $\text{c-Ind}_{I(1)}^G \mathbf{1}$ is the compact induction from the trivial representation of $I(1)$.

However, Ollivier also showed that we cannot expect such correspondence in general. When $G = \mathrm{GL}_2(F)$ where F is a p-adic field such that the number of the residue field is greater than p, for a *supersingular* simple module M (we do not recall the definition of supersingular modules since we do not

MSC2010: primary 22E50; secondary 20C08.
Keywords: pro-p Iwahori–Hecke algebra, mod p representations.

use it in this paper), Ollivier showed that $(M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1})^{I(1)}$ is not finite-dimensional. Since simple modules of $\mathcal{H}$ are finite-dimensional, it says that we have no equivalence of categories in this case.

Still we can expect that there is such a correspondence if we *avoid* supersingular representations/modules. It was proved by Ollivier and Schneider [2018, Theorem 3.33] that this expectation is true when $G = \mathrm{SL}_2(F)$ when $p \neq 2$ or $F \neq \mathbb{Q}_2$. The aim of this paper is to extend this for any G. We remark that our result is not a generalization of their result since we assume that modules have finite-length which they do not assume.

Let G be a (general) connected reductive p-adic group. In this case, as a consequence of classification theorems [Abe et al. 2017; Abe 2019a] and the calculation of the invariant part of irreducible representations [Abe et al. 2018a], the functor $\pi \mapsto \pi^{I(1)}$ gives a bijection between irreducible modulo p representations of G and simple $\mathcal{H}$-modules which are *far from supersingular representations/modules*. The aim of this paper is to generalize this correspondence to the level of categories. More precisely, we prove the equivalence of the following two categories:

- The category of $\mathcal{H}$-modules M such that $\dim(M) < \infty$ and a certain element of the center of $\mathcal{H}$ is invertible on M (see Definition 3.1).
- The category of modulo p representations π of G such that:
  - π is generated by $\pi^{I(1)}$.
  - π has a finite length.
  - Any irreducible subquotient of π is isomorphic to a subquotient of $\mathrm{Ind}_B^G \sigma$ where B is a minimal parabolic subgroup and σ is an irreducible representation of the Levi quotient of B.

Note that an $\mathcal{H}$-module M is supersingular if and only if certain elements in the center of $\mathcal{H}$ act by zero and a modulo p irreducible admissible representation π of G is supersingular if and only if it is supercuspidal, namely it does not appear as a subquotient of a parabolically induced representation from an irreducible admissible representation of a proper Levi subgroup. Therefore some conditions as above says that M (resp. π) is *far from supersingular modules (resp. representations)*.

We give an outline of the proof. Since the correspondence is true for irreducible representations, by induction on the length, it is sufficient to prove the following (Theorem 3.5): Let M be an $\mathcal{H}$-module which we are considering. Then $M \to M \otimes_{\mathcal{H}} \mathrm{Ind}_{I(1)}^{G} \mathbf{1}$ is injective. This theorem is proved in Section 3. In fact, we prove the injectivity for any $M \in \mathcal{C}$ where the category $\mathcal{C}$ is introduced in Section 3. Here are some reductions:

- Let $\mathcal{A}$ be the Bernstein subalgebra introduced in [Vignéras 2016]. Since we have an embedding $M \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M)$, it is sufficient to prove the theorem for $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M)$. Note that we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) \in \mathcal{C}$ if $M \in \mathcal{C}$.
- We have a decomposition of $M|_{\mathcal{A}}$ along the *support* (Definition 3.8). We may assume that the support of $M|_{\mathcal{A}}$ is contained in a Weyl chamber.

- Using a result in [Abe 2019a], parabolic inductions and a result of Ollivier and Vignéras [2018], we may assume that the support is the dominant Weyl chamber.
- We prove there exists an $\mathcal{A}$-module M' such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) \simeq M' \otimes_{\mathcal{A}} \mathcal{H}$. Hence it is sufficient to prove that $M' \otimes_{\mathcal{A}} \mathcal{H} \to M' \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}$ is injective.

By a result in [Abe 2017], both $M \otimes_{\mathcal{A}} \mathcal{H}$ and $M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}$ relate to $\text{c-Ind}_{K}^{G} V$ where K is a special parahoric subgroup and V a certain representation of K. The structure of this representations is studied in [Abe et al. 2017] and using such result we prove the injectivity.

It is almost immediate to prove our main theorem from the above injectivity. This is done in Section 4.

2. Notation and preliminaries

Let F be a nonarchimedean local field of residue characteristic p and G a connected reductive group over F. Let C be an algebraically closed field of characteristic p. This is the coefficient field of representations in this paper. *All representations in this paper are smooth representations over C.*

In general, for any algebraic group H over F, we denote the group of valued points $H(F)$ by the same letter H. Fix a maximal split torus S of G and minimal parabolic subgroup B containing S. The centralizer Z of S in G is a Levi subgroup of B. We denote the unipotent radical of B by U and the opposite of B containing Z by $\overline{B}$. The unipotent radical of $\overline{B}$ is denoted by $\overline{U}$.

Consider the reduced apartment corresponding to S and take an alcove $\boldsymbol{A}_0$ and a special point $\mathbf{x}_0$ from the closure of $\boldsymbol{A}_0$. Let K be the special parahoric subgroup corresponding to $\mathbf{x}_0$ and I the Iwahori subgroup determined by $\boldsymbol{A}_0$. Let $I(1)$ be the pro-p Iwahori subgroup attached to $\boldsymbol{A}_0$, namely the pro-p radical of I. The space of C-valued compactly supported $I(1)$-biinvariant functions $\mathcal{H}$ has a structure of a C-algebra via the convolution product. The algebra $\mathcal{H}$ is called pro-p Iwahori–Hecke algebra. The structure of this algebra is studied by Vignéras [2016].

Let $N_G(S)$ be the normalizer of S in G and put $W_0 = N_G(S)/Z$, $W = N_G(S)/(Z \cap K)$ and $W(1) = N_G(S)/(Z \cap I(1))$. Let G' be the subgroup of G generated by U and $\overline{U}$. Note that this is not a group of the valued points of an algebraic group in general. Let W_{aff} be the image of $G' \cap N_G(S)$ in W. The action of W_{aff} on the apartment is faithful and therefore it is a subgroup of the group of affine transformations of the apartment. Let S_{aff} be the set of reflections along the walls of $\boldsymbol{A}_0$. Then $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system. Denote its length function by ℓ. Let $N_W(\boldsymbol{A}_0)$ be the stabilizer of $\boldsymbol{A}_0$ in W. Then the group W is the semidirect product of W_{aff} and $N_W(\boldsymbol{A}_0)$. The function ℓ is extended to W, trivially on $N_W(\boldsymbol{A}_0)$. We also inflate ℓ to $W(1)$ via $W(1) \to W$. We have the Bruhat order on $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ and we extend it to W by $w_1\omega_1 < w_2\omega_2$ if and only if $w_1 < w_2$ and $\omega_1 = \omega_2$ where $w_1, w_2 \in W_{\mathrm{aff}}$ and $\omega_1, \omega_2 \in N_W(\boldsymbol{A}_0)$. For $w_1, w_2 \in W(1)$, we say $w_1 < w_2$ if $\overline{w}_1 < \overline{w}_2$ where $\overline{w}_i$ is the image of w_i in W $(i = 1, 2)$. As usual we say $w_1 \le w_2$ if and only if $w_1 < w_2$ or $w_1 = w_2$.

We give some of structure theorems of $\mathcal{H}$. For $w \in W(1)$, let T_w be the characteristic function on $I(1)\tilde{w}I(1)$ where $\tilde{w} \in N_G(S)$ is a lift of w. Then T_w does not depend on the choice of a lift and, since we have the bijection $I(1)\backslash G/I(1) \simeq W(1)$, $\{T_w \mid w \in W(1)\}$ is a basis of $\mathcal{H}$. This basis is called

Iwahori–Matsumoto basis. This basis satisfies the following braid relations:

$$T_{w_1}T_{w_2} = T_{w_1w_2} \quad \text{if } \ell(w_1w_2) = \ell(w_1) + \ell(w_2)$$

where $w_1, w_2 \in W(1)$. Let $Z_\kappa = (Z \cap K)/(Z \cap I(1))$. Then this is a subgroup of $W(1)$. Since any elements in Z_κ has the length 0 (since it is in the kernel of $W(1) \to W$), from the braid relations, we have $T_{t_1}T_{t_2} = T_{t_1t_2}$ for $t_1, t_2 \in Z_\kappa$. In other words, the embedding $C[Z_\kappa] \hookrightarrow \mathcal{H}$ defined by $\sum_{t \in Z_\kappa} c_t t \mapsto \sum_{t \in Z_\kappa} c_t T_t$ is an algebra homomorphism where $C[Z_\kappa]$ is the group ring of Z_κ. Using this embedding, we regard $C[Z_\kappa]$ as a subalgebra of $\mathcal{H}$.

Let $S_{\text{aff}}(1)$ be the inverse image of S_{aff} in $W(1)$. Then for $s \in S_{\text{aff}}(1)$, we have

$$T_s^2 = c_s T_s$$

for some $c_s \in C[Z_\kappa]$. An element c_s is given in [Vignéras 2016, 4.2].

Define T_w^* as in [loc. cit., 4.3] for $w \in W(1)$. This is also a basis of $\mathcal{H}$ and it satisfies the following: $T_w^* \in T_w + \sum_{v<w} CT_v$ and $T_{w_1}^* T_{w_2}^* = T_{w_1w_2}^*$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

Let o be a spherical orientation [loc. cit., 5.2]. Note that the set of spherical orientations are canonically bijective with the set of Weyl chambers. For each o, we have another basis $\{E_o(w) \mid w \in W(1)\}$ defined in [loc. cit., 5.3]. The orientations correspond to the Weyl chambers. Let o_- be the orientation corresponding to the antidominant Weyl chamber and set $E(w) = E_{o_-}(w)$.

Set $\Lambda(1) = Z/(Z \cap I(1))$. This is a subgroup of $W(1)$. For $\lambda_1, \lambda_2 \in \Lambda(1)$, the multiplication $E(\lambda_1)E(\lambda_2)$ is simple. Before giving it, we introduce some notation. The pair (G, S) gives a root datum $(X^*(S), \Sigma, X_*(S), \Sigma^\vee)$ and since we have fixed a Borel subgroup we also have a positive system $\Sigma^+ \subset \Sigma$ and the set of simple roots $\Delta \subset \Sigma^+$. An element $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is called dominant if and only if $\langle v, \alpha \rangle \geq 0$ for any $\alpha \in \Sigma^+$. A W_0-orbit of the set of dominant elements is called a closed Weyl chamber. We also say that $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is regular if $\langle v, \alpha \rangle \neq 0$ for any $\alpha \in \Sigma$. We have a homomorphism $\nu \colon Z \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} = \operatorname{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$ characterized by $\nu(z)(\chi) = -\operatorname{val}(\chi(z))$ where $z \in S$, $\chi \in X^*(S)$ and $\operatorname{val} \colon F^\times \to \mathbb{Z}$ is the normalized valuation. This homomorphism factors through $Z \to \Lambda(1)$ and the induced homomorphism $\Lambda(1) \to X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is denoted by the same letter ν. We let $\Lambda^+(1)$ the set of $\lambda \in \Lambda(1)$ such that $\nu(\lambda)$ is dominant. For $w \in W_0$, let $w(\Lambda^+(1))$ be the set of $\lambda \in \Lambda(1)$ such that $w^{-1}(\nu(\lambda))$ is dominant.

The multiplication $E(\lambda_1)E(\lambda_2)$ is $E(\lambda_1\lambda_2)$ if $\nu(\lambda_1)$ and $\nu(\lambda_2)$ are in the same closed Weyl chamber (in other words, $\lambda_1, \lambda_2 \in w(\Lambda^+(1))$ for some $w \in W_0$) and otherwise it is zero. In particular, $\mathcal{A} = \bigoplus_{\lambda \in \Lambda(1)} CE(\lambda)$ is a subalgebra of $\mathcal{H}$. If we fix a closed Weyl chamber $\mathcal{C}$, then $\bigoplus_{\nu(\lambda) \in \mathcal{C}} CE(\lambda)$ is a subalgebra of $\mathcal{A}$ and the linear map

$$\bigoplus_{\nu(\lambda) \in \mathcal{C}} CE(\lambda) \to C[\Lambda(1)]$$

defined by $E(\lambda) \mapsto \tau_\lambda$ is an algebra embedding. Here $C[\Lambda(1)]$ is the group ring of $\Lambda(1)$ and we denote the element in $C[\Lambda(1)]$ corresponding to $\lambda \in \Lambda(1)$ by τ_λ, namely $C[\Lambda(1)] = \bigoplus_{\lambda \in \Lambda(1)} C\tau_\lambda$.

Remark 2.1. (1) If $\langle \nu(\lambda), \alpha\rangle = 0$ for any $\alpha \in \Sigma$, then $\nu(\lambda)$ and $\nu(\lambda^{-1})$ are in the same closed Weyl chamber. (In fact, $\nu(\lambda)$ and $\nu(\lambda^{-1})$ are in any closed Weyl chamber.) Hence $E(\lambda)E(\lambda^{-1}) = 1$. In particular, $E(\lambda)$ is invertible.

(2) If $\lambda \in \Lambda(1)$ is in the center of $\Lambda(1)$, then $E(\lambda)$ is also in the center of $\mathcal{A}$. This follows from the above description of the multiplication.

Let J be a subset of Δ and denote the corresponding standard parabolic subgroup by P_J. Let L_J be the Levi part of P_J containing Z. Then $K \cap L_J$ is a special parahoric subgroup and $I(1)_J = I(1) \cap L_J$ a pro-p Iwahori subgroup. Attached to these, we have many objects. For such objects we add a suffix J, for example, the pro-p Iwahori–Hecke algebra attached to $(L_J, I(1)_J)$ is denoted by $\mathcal{H}_J$. There are two exceptions: base T_w and $E(w)$ for $\mathcal{H}_J$ is denoted by T_w^J and $E^J(w)$, respectively. For each $J \subset \Delta$, we have two subalgebras $\mathcal{H}_J^+, \mathcal{H}_J^-$ of $\mathcal{H}_J$ and four algebra homomorphisms $j_J^+, j_J^{+*}\colon \mathcal{H}_J^+ \to \mathcal{H}$ and $j_J^-, j_J^{-*}\colon \mathcal{H}_J^- \to \mathcal{H}$. See [Abe 2019b, 2.8] for the definitions. (Here $\mathcal{H}_J^+$ is denoted by $\mathcal{H}_{P_J}^+$ in [Abe 2019b].)

3. The category $\mathcal{C}$ and a proof of the injectivity

3A. *The category $\mathcal{C}$.* The modules in this paper are right modules unless otherwise stated. In this paper, we focus on the full subcategory $\mathcal{C}$ of the category of $\mathcal{H}$-modules defined using the center $\mathcal{Z}$ of $\mathcal{H}$. The center $\mathcal{Z}$ is described using the basis $\{E(w)\}$. Since $\Lambda(1)$ is normal in $W(1)$, the group $W(1)$ acts on $\Lambda(1)$ by the conjugate action. For $\lambda \in \Lambda(1)$ denote the orbit through λ by $\mathcal{O}_\lambda$. For $\lambda \in \Lambda(1)$, put $z_\lambda = \sum_{\lambda' \in \mathcal{O}_\lambda} E(\lambda')$. Then $\{z_\lambda \mid z \in \Lambda(1)/W(1)\}$ gives a basis of $\mathcal{Z}$ [Vignéras 2014, Theorem 1.2]. Fix a uniformizer ϖ of F and let $\Lambda_S(1)$ be the image of $\{\xi(\varpi) \mid \xi \in X_*(S)\}$.

Definition 3.1. An $\mathcal{H}$-module M is in $\mathcal{C}$ if and only if z_λ is invertible on M for any $\lambda \in \Lambda_S(1)$.

Lemma 3.2. *Let $\lambda \in \Lambda_S(1)$. Then we have the following*:

(1) *For $w \in W(1)$, w stabilizes λ if and only if the image of w in W_0 stabilizes $\nu(\lambda)$.*

(2) *Let $\{w_1, \dots, w_r\} \subset W(1)$ be a subset of $W(1)$ such that the image in W_0 gives a set of complete representatives of $W_0/\operatorname{Stab}_{W_0}(\nu(\lambda))$. Then we have $z_\lambda = \sum_{i=1}^r E(w_i \lambda w_i^{-1})$. (Note that $w_i\lambda w_i^{-1}$ depends only on the image of w_i in $W_0/\operatorname{Stab}_{W_0}(\lambda)$ by (1).)*

Proof. Take $\xi \in X_*(S)$ such that $\lambda = \xi(\varpi)^{-1}$. We have $\nu(\lambda) = \xi$. Let $w \in W(1)$ and denote the image of w in W_0 by w_0. Then we have $w\lambda w^{-1} = (w_0\xi)(\varpi)^{-1}$. Hence if w_0 stabilizes $\xi = \nu(\lambda)$, then w stabilizes λ. Obviously if w stabilizes λ then w_0 stabilizes $\nu(\lambda)$.

By (1), $\operatorname{Stab}_{W(1)}(\lambda)$ is the inverse image of $\operatorname{Stab}_{W_0}(\lambda)$. Therefore we have $W(1)/\operatorname{Stab}_{W(1)}(\lambda) \simeq W_0/\operatorname{Stab}_{W_0}(\lambda)$. By the definition, we have $z_\lambda = \sum_{w \in W(1)/\operatorname{Stab}_{W(1)}(\lambda)} E(w\lambda w^{-1})$. Hence we get (2). □

Lemma 3.3. *Let $\lambda, \mu \in \Lambda_S(1)$ and assume that $\nu(\lambda)$ and $\nu(\mu)$ are in the same closed Weyl chamber. We also assume that $\nu(\lambda)$ is regular. Then we have $z_\lambda z_\mu = z_{\lambda\mu}$.*

Proof. Take $w_1, \dots, w_r \in W(1)$ such that the images of them in W_0 gives a set of complete representatives of $W_0/\operatorname{Stab}_{W_0}(\nu(\mu))$. Then we have $z_\mu = \sum_i E(w_i \mu w_i^{-1})$ by the above lemma. Let $v_1, \dots, v_s$ be a set of complete representatives of $W_0 = W(1)/\Lambda(1)$. Then we have $z_\lambda = \sum_j E(v_i \lambda v_i^{-1})$. (Note that $\nu(\lambda)$ is assumed to be regular.) Since $\nu(\lambda)$ is regular, for each i, there exists only one $j_i = 1, \dots, r$ such that $v_i(\nu(\lambda))$ and $w_{j_i}(\nu(\mu))$ is in the same closed Weyl chamber. Hence we get

$$E(v_i \lambda v_i^{-1}) E(w_j \mu w_j^{-1}) = \begin{cases} 0 & j \neq j_i, \\ E(v_i \lambda v_i^{-1} w_j \mu w_j^{-1}) & j = j_i. \end{cases}$$

Moreover, $\nu(\lambda)$ and $v_i^{-1} w_{j_i}(\nu(\mu))$ is in the same closed Weyl chamber. Since $\nu(\lambda)$ and $\nu(\mu)$ are in the same closed Weyl chamber by the assumption, we get $v_i^{-1} w_{j_i}(\nu(\mu)) = \nu(\mu)$. Therefore $v_i^{-1} w_{j_i}$ stabilizes $\nu(\mu)$. As in the previous lemma, $v_i^{-1} w_{j_i}$ also stabilizes μ. Hence $w_{j_i} \mu w_{j_i}^{-1} = v_i \mu v_i^{-1}$. We get

$$E(v_i \lambda v_i^{-1}) E(w_j \mu w_j^{-1}) = \begin{cases} 0 & j \neq j_i, \\ E(v_i \lambda \mu v_i^{-1}) & j = j_i. \end{cases}$$

Now we get

$$z_\lambda z_\mu = \sum_i \sum_j E(v_i \lambda v_i^{-1}) E(w_j \mu w_j^{-1}) = \sum_i E(v_i \lambda \mu v_i^{-1}).$$

By the assumption, $\nu(\lambda\mu)$ is regular and $\lambda\mu \in \Lambda_S(1)$. Hence the last term is $z_{\lambda\mu}$ by the above lemma. □

Lemma 3.4. *An $\mathcal{H}$-module M is in $\mathcal{C}$ if and only if for some $\lambda \in \Lambda_S(1)$ such that $\nu(\lambda)$ is regular, the element z_λ is invertible on M.*

Proof. Assume that there exists $\lambda_0 \in \Lambda_S(1)$ such that $\nu(\lambda_0)$ is regular and z_{λ_0} is invertible on M. Let $\lambda \in \Lambda_S(1)$ and we prove that λ is also invertible on M. Replacing λ with an element in the orbit through λ, we may assume that $\nu(\lambda)$ and $\nu(\lambda_0)$ are in the same closed Weyl chamber. Take a sufficiently large $n \in \mathbb{Z}_{>0}$ such that $\nu(\lambda_0^n \lambda^{-1})$ is also in the same closed Weyl chamber as $\nu(\lambda_0)$. Set $\mu = \lambda_0^n \lambda^{-1}$. Then by the above lemma, we have $z_\mu z_\lambda = z_{\lambda_0^n} = z_{\lambda_0}^n$. By the assumption, $z_{\lambda_0}^n$ is invertible on M. Hence z_λ is invertible, namely we have $M \in \mathcal{C}$. □

3B. *Theorem.* In the rest of this section, we prove the following:

Theorem 3.5. *If $M \in \mathcal{C}$, then $M \to M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

3C. *Reductions.* Define a subalgebra $\mathcal{A}$ of $\mathcal{H}$ by $\mathcal{A} = \bigoplus_{\lambda \in \Lambda(1)} C E(\lambda)$. Let $M \in \mathcal{C}$ and set $M' = \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M)$.

Remark 3.6. The element z_λ is in the center of $\mathcal{H}$ and $z_\lambda \in \mathcal{A}$. Therefore the action of z_λ on $M' = \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ is induced by that on M. Since $M \in \mathcal{C}$, z_λ is invertible on M for any $\lambda \in \Lambda_S(1)$. Hence the action of z_λ on M' is also invertible. Namely $M' \in \mathcal{C}$.

Defining the action of $X \in \mathcal{H}$ on M' by $(\varphi X)(Y) = \varphi(XY)$ for $\varphi \in \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ and $Y \in \mathcal{H}$, M' is a right $\mathcal{H}$-module. The map $m \mapsto (X \mapsto mX)$ gives an $\mathcal{H}$-module embedding $M \hookrightarrow M'$ and we have the following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1} \\ \big\downarrow & & \big\downarrow \\ M' & \longrightarrow & M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1} \end{array}$$

Therefore, to prove Theorem 3.5, it is sufficient to prove that the map $M' \to M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}$ is injective.

Lemma 3.7. *Any module $M \in \mathcal{C}$ has a functorial decomposition $M = \bigoplus_{w \in W_0} M_w$ as an $\mathcal{A}$-module such that $E(\mu)$ acts on M_w by:*

- *Zero if $w^{-1}\nu(\mu)$ is not dominant.*
- *Invertible if $w^{-1}\nu(\mu)$ is dominant.*

Proof. Fix $\lambda_0 \in \Lambda_S(1)$ such that $\nu(\lambda_0)$ is regular dominant. Put $\lambda_w = n_w \lambda_0 n_w^{-1}$ and set $M_w = ME(\lambda_w)$. Since $\lambda_w \in \Lambda_S(1)$ is central, $E(\lambda_w)$ is also central in $\mathcal{A}$. Hence M_w is an $\mathcal{A}$-submodule.

We prove that λ_w is invertible on M_w. Since $\nu(\lambda_0)$ is regular, $\nu(\lambda_v)$ and $\nu(\lambda_w)$ are not in the same closed Weyl chamber if $v \neq w$. Therefore $E(\lambda_v)E(\lambda_w) = 0$. Hence $M_w E(\lambda_v) = 0$ if $v \neq w$. Therefore for $m \in M_w$, we have $m z_{\lambda_0} = \sum_{v \in W_0} m E(\lambda_v) = m E(\lambda_w)$. Hence if $mE(\lambda_w) = 0$ then $m z_{\lambda_0} = 0$, hence $m = 0$ since z_{λ_0} is invertible. Therefore $E(\lambda_w)$ is injective on M_w. We also have that $m z_{\lambda_0}^2 = mE(\lambda_w) z_{\lambda_0} = m z_{\lambda_0} E(\lambda_w) = mE(\lambda_w)^2$ since z_{λ_0} commutes with $E(\lambda_w)$. (Recall that z_{λ_0} is in the center of $\mathcal{H}$.) Hence $m = m_0 E(\lambda_w)$ where $m_0 = m z_{\lambda_0}^{-2} E(\lambda_w) \in M_w$. Therefore $E(\lambda_w)$ is surjective on M_w.

For $\mu \in \Lambda(1)$ such that $w^{-1}(\nu(\mu))$ is not dominant, $\nu(\mu)$ and $\nu(\lambda_w)$ are not in the same closed Weyl chamber. Hence $E(\mu)E(\lambda_w) = 0$. Therefore $E(\mu) = 0$ on M_w. On the other hand, assume that $w^{-1}(\nu(\mu))$ is dominant. Then $\nu(\mu)$ and $\nu(\lambda_w)$ are in the same closed Weyl chamber. Take sufficiently large $n \in \mathbb{Z}_{\geq 0}$ such that $\nu(\lambda_w^n \mu^{-1})$ is also in the same closed Weyl chamber as $\nu(\mu)$. Then we have $E(\lambda_w)^n = E(\lambda_w^n) = E(\lambda_w^n \mu^{-1}) E(\mu)$. Since $E(\lambda_w)$ is invertible on M_w, $E(\mu)$ is also invertible on M_w.

We prove $M = \bigoplus_{w \in W_0} M_w$. Since z_{λ_0} is invertible, any element in M can be written $m z_{\lambda_0}$ for some $m \in M$. We have $m z_{\lambda_0} = \sum_{w \in W_0} m E(\lambda_w) \in \sum_{w \in W_0} M_w$. Hence $M = \sum_{w \in W_0} M_w$. Let $m_w \in M_w$ and assume that $\sum_{w \in W_0} m_w = 0$. Then for each $v \in W_0$ we have $\sum_{w \in W_0} m_w E(\lambda_v) = 0$. Since $m_w E(\lambda_v) = 0$ for $v \neq w$, we have $m_v E(\lambda_v) = 0$. Since the action of $E(\lambda_v)$ on M_v is invertible, $m_v = 0$. □

Since $\operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M) = \bigoplus_{w \in W_0} \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M_w)$, to prove $M' \to M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}$ is injective, it is sufficient to prove that the homomorphism $\operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M_w) \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M_w) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}$ is injective.

Definition 3.8. Let M be an $\mathcal{A}$-module. We say that $\operatorname{supp} M = w(\Lambda^+(1))$ if and only if $E(\lambda)$ is:

- Zero if $w^{-1}(\nu(\mu))$ is not dominant.
- Invertible if $w^{-1}(\nu(\mu))$ is dominant.

for any $\lambda \in \Lambda(1)$. (Note that we do not define $\operatorname{supp} M$ itself.)

From the above discussions, to prove Theorem 3.5, it is sufficient to prove the following lemma.

Lemma 3.9. *Let M be an $\mathcal{A}$-module such that* $\operatorname{supp} M = w(\Lambda^+(1))$ *where $w \in W_0$. Then* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M) \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ *is injective.*

We take a lift n_w of each $w \in W_0$ in $W(1)$ such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Let M be an $\mathcal{A}$-module and $w \in W_0$. We define a new $\mathcal{A}$-module $n_w M$ as follows. As a vector space, $n_w M = M$ and the action of $E(\lambda) \in \mathcal{A}$ on $n_w M$ is the action of $E(n_w^{-1} \lambda n_w)$ on M. This defines an auto-equivalence of the category of $\mathcal{A}$-modules. If $\operatorname{supp} M = v(\Lambda^+(1))$, then $\operatorname{supp} n_w M = wv(\Lambda^+(1))$. With this notation, Lemma 3.9 is equivalent to the following.

Lemma 3.10. *Let M be an $\mathcal{A}$-module such that* $\operatorname{supp} M = \Lambda^+(1)$. *Then the map* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, n_w M) \to \operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, n_w M) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ *is injective.*

3D. ***Reduction to $w = w_J$ for some $J \subset \Delta$.*** For a subset $J \subset \Delta$, let w_J be the longest element in $W_{0,J}$. We prove that we may assume $w = w_J$ for some J in Lemma 3.10.

We relate our M with modules studied in [Abe 2019a]. Consider the homomorphism $\mathcal{A} \to C[\Lambda(1)]$ defined by

$$E(\lambda) \mapsto \begin{cases} \tau_\lambda & \lambda \in \Lambda^+(1), \\ 0 & \text{otherwise.} \end{cases} \tag{3-1}$$

We regard $C[\Lambda(1)]$ as a right $\mathcal{A}$-module via this homomorphism. For $w \in W_0$, we also have the $\mathcal{A}$-module $n_w C[\Lambda(1)]$. Then we consider the module

$$n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

This is a $(C[\Lambda(1)], \mathcal{H})$-bimodule.

Let M be an $\mathcal{A}$-module such that $\operatorname{supp} M = \Lambda^+(1)$. Then we define a structure of a right $C[\Lambda(1)]$-module on M by

$$m\tau_{\lambda_1 \lambda_2^{-1}} = mE(\lambda_1)E(\lambda_2)^{-1}$$

where $\lambda_1, \lambda_2 \in \Lambda^+(1)$ and $m \in M$. (Since $\operatorname{supp} M = \Lambda^+(1)$, $E(\lambda_2)$ is invertible on M.) It is easy to see that this definition is well-defined and define a structure of $C[\Lambda(1)]$-module. Then we have

$$M \otimes_{C[\Lambda(1)]} n_w C[\Lambda(1)] \simeq n_w M.$$

The isomorphisms are given by $m \otimes f \mapsto mf$ from the left-hand side to the right-hand side and $m \mapsto m \otimes 1$ in the opposite direction. Therefore we have

$$n_w M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]} \otimes n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

For each $w \in W_0$, set $\Delta_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$. Then by [Abe 2019a, Theorem 3.13], if $\Delta_{w_1} = \Delta_{w_2}$, we have

$$n_{w_1} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

Therefore we get (1) of the next lemma.

Lemma 3.11. *Let M be as in Lemma 3.10. If $w_1, w_2 \in W_0$ satisfies $\Delta_{w_1} = \Delta_{w_2}$, then we have*:

(1) $n_{w_1} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2} M \otimes_{\mathcal{A}} \mathcal{H}$.

(2) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2} M)$.

Proof. We have proved (1). We prove (2).

Let ι be an automorphism of $\mathcal{H}$ defined in [Vignéras 2016, Proposition 4.23] and $\zeta\colon \mathcal{H} \to \mathcal{H}$ an antiautomorphism defined by $\zeta(T_w) = T_{w^{-1}}$. (The linear map ζ is an antihomomorphism by [Abe 2019a, 4.1].) Set $f = \iota \circ \zeta$. Since $\zeta(E(\lambda)) = E_{o_+}(\lambda^{-1})$ [Abe 2019a, Lemma 4.3] and $\iota(E_{o_+}(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda)$ [Vignéras 2016, Lemma 5.31], we have $f(E(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda^{-1})$. In particular, f preserves $\mathcal{A}$. It is easy to see $f^2(T_w) = T_w$ for any $w \in W(1)$. Hence f^2 is identity.

For a left $\mathcal{H}$-module N, we define a right $\mathcal{H}$-module N^f by $N^f = N$ as a vector space and the action of $X \in \mathcal{H}$ on N^f is the action of $f(X)$ on N. Then $m \otimes X \mapsto f(X) \otimes m$ gives an isomorphism $(N^f \otimes_{\mathcal{A}} \mathcal{H})^f \simeq \mathcal{H} \otimes_{\mathcal{A}} N$.

For a right $\mathcal{H}$-module or $\mathcal{A}$-module L, set $L^* = \mathrm{Hom}_C(L, C)$. Then this is a left $\mathcal{H}$-module or $\mathcal{A}$-module, respectively. Let M be as in the lemma. Since $f(E(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda^{-1})$, we have $\mathrm{supp}(n_{w_1} M^*)^f = w_1(\Lambda^+(1)^{-1}) = w_1 w_\Delta(\Lambda^+(1))$. Hence $(n_{w_1} M^*)^f = n_{w_1 w_\Delta} M'$ for some $\mathcal{A}$-module M' such that $\mathrm{supp}\, M' = \Lambda^+(1)$. Since $\Delta_{w_1 w_\Delta} = \Delta \setminus (-w_\Delta(\Delta_{w_1}))$, we also have $\Delta_{w_1 w_\Delta} = \Delta_{w_2 w_\Delta}$. Hence by (1), we get $n_{w_1 w_\Delta} M' \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2 w_\Delta} M' \otimes_{\mathcal{A}} \mathcal{H}$. Therefore we get $(n_{w_1} M^*)^f \otimes_{\mathcal{A}} \mathcal{H} \simeq (n_{w_2} M^*)^f \otimes_{\mathcal{A}} \mathcal{H}$. Applying $(\cdot)^f$ to the both sides and using $(N^f \otimes_{\mathcal{A}} \mathcal{H})^f \simeq \mathcal{H} \otimes_{\mathcal{A}} N$, we get $\mathcal{H} \otimes_{\mathcal{A}} n_{w_1} M^* \simeq \mathcal{H} \otimes_{\mathcal{A}} n_{w_2} M^*$. Hence we have $(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1} M^*)^* \simeq (\mathcal{H} \otimes_{\mathcal{A}} n_{w_2} M^*)^*$.

Now we have

$$(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1} M^*)^* = \mathrm{Hom}_C(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1} M^*, C) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} M^{**}).$$

Hence we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} M^{**}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2} M^{**})$. We have an embedding $M \hookrightarrow M^{**}$. Let L be the cokernel. Then $\mathrm{supp}\, L = \Lambda^+(1)$ and we have an embedding $L \hookrightarrow L^{**}$. Therefore we have an exact sequence $0 \to M \to M^{**} \to L^{**}$ and it gives $0 \to n_{w_i} M \to n_{w_i} M^{**} \to n_{w_i} L^{**}$ for $i = 1, 2$. Hence we get the following commutative diagram with exact rows:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} M) & & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2} M) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} M^{**}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2} M^{**}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} L^{**}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2} L^{**}).
\end{array}$$

We have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1} M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2} M)$. □

For given $w \in W$, set $J = \Delta \setminus \Delta_w$. Then we have $\Delta_{w_J} = \Delta \setminus J = \Delta_w$. Therefore, to prove Lemma 3.10, we may assume that $w = w_J$ for some $J \subset \Delta$.

3E. *Reduction to $w = w_\Delta$.* Set

$$\mathcal{A}_w = \bigoplus_{\lambda \in w(\Lambda^+(1))} CE(\lambda) \subset \mathcal{A}. \tag{3-2}$$

Lemma 3.12. *Let M be an $\mathcal{A}$-module such that* $\operatorname{supp} M = w(\Lambda^+(1))$. *Then we have* $\operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, M) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}_w}(\mathcal{H}, M)$.

Proof. Let $\varphi\colon \mathcal{H} \to M$ be an $\mathcal{A}_w$-module homomorphism and we prove that φ is $\mathcal{A}$-equivariant. Fix $\lambda_0 \in \Lambda(1)$ such that $w^{-1}(\nu(\lambda_0))$ is dominant and regular. Since $\operatorname{supp} M = w(\Lambda^+(1))$, $E(\lambda_0)$ is invertible on M. For $\mu \in \Lambda(1)$ such that $w^{-1}(\nu(\mu))$ is not dominant, we have $E(\mu)E(\lambda_0) = 0$. Hence for $X \in \mathcal{H}$, we have $\varphi(XE(\mu)) = E(\lambda_0)^{-1}\varphi(XE(\mu)E(\lambda_0)) = 0$. Since $E(\mu) = 0$ on $n_w M$, $E(\mu)\varphi(X) = 0$. Hence we get $\varphi(XE(\mu)) = 0 = E(\mu)\varphi(X)$. Therefore φ is $\mathcal{A}$-equivariant. □

For later use, we also prove the following.

Lemma 3.13. *Let M be an $\mathcal{A}$-module such that* $\operatorname{supp} M = w(\Lambda^+(1))$. *Then $M \otimes_{\mathcal{A}_w} \mathcal{H} \xrightarrow{\sim} M \otimes_{\mathcal{A}} \mathcal{H}$.*

Proof. Let $m \in M$ and $X \in \mathcal{H}$. We prove $mE(\lambda) \otimes X = m \otimes E(\lambda)X$ in $M \otimes_{\mathcal{A}_w} \mathcal{H}$ for any $\lambda \in \Lambda(1)$. This is true if $w^{-1}(\nu(\lambda))$ is dominant.

Assume that $w^{-1}(\nu(\lambda))$ is not dominant and take $\lambda_0 \in \Lambda(1)$ such that $w^{-1}(\nu(\lambda_0))$ is dominant and $\nu(\lambda)$, $\nu(\lambda_0)$ are not in the same chamber. Then we have $E(\lambda_0)E(\lambda) = 0$. Note that $E(\lambda_0)$ is invertible on M since $\operatorname{supp} M = w(\Lambda^+(1))$. Hence $m \otimes E(\lambda)X = mE(\lambda_0)^{-1} \otimes E(\lambda_0)E(\lambda)X = 0$. On the other hand, $E(\lambda) = 0$ on M, again by $\operatorname{supp} M = w(\Lambda^+(1))$. Hence $mE(\lambda) \otimes X = 0$. We get the lemma. □

An element $E(\lambda)$ belongs to:

- $\mathcal{A}_w$ if $\langle \nu(\lambda), w(\alpha) \rangle \geq 0$ for any $\alpha \in \Sigma^+$.
- $j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$ if $\langle \nu(\lambda), \alpha \rangle \geq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_J^+$.

(The second one follows from the following fact: a basis of $\mathcal{H}_J^- \cap \mathcal{A}_J$ is given by $\{E^J(\lambda)\}$ where λ runs through as above [Abe 2019a, Lemma 4.2] and $j_J^{-*}(E^J(\lambda)) = E(\lambda)$ for such λ [Abe 2019b, Lemma 2.6].) Since $w_J(\Sigma^+) = \Sigma_J^- \cup (\Sigma^+ \setminus \Sigma_J^+) \supset \Sigma^+ \setminus \Sigma_J^+$, we have $\mathcal{A}_{w_J} \subset j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$.

Let M be an $\mathcal{A}$-module. From the above argument, we have

$$\operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}, n_{w_J} M) \simeq \operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H} \otimes_{j_J^{-*}(\mathcal{H}_J^-)} j_J^{-*}(\mathcal{H}_J^-), n_{w_J} M) \simeq \operatorname{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J} M)).$$

Since $j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$ contains $\mathcal{A}_{w_J}$, we have $\mathcal{A}_{w_J} \hookrightarrow \mathcal{H}_J^- \cap \mathcal{A}_J \hookrightarrow \mathcal{A}_J$. More precisely, $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$ via $E(\lambda) \mapsto E^J(\lambda)$. (If $E(\lambda) \in \mathcal{A}_{w_J}$, then $w_J^{-1}(\nu(\lambda))$ is dominant with respect to Δ, hence it is also dominant with respect to J. Therefore $E^J(\lambda) \in \mathcal{A}_{J,w_J}$.)

Lemma 3.14. *We regard $\mathcal{A}_{w_J}$ as a subalgebra of $\mathcal{A}_J$ via the above embedding. Let M be an $\mathcal{A}$-module such that* $\operatorname{supp} M = \Lambda^+(1)$. *Then $n_{w_J} M$ is uniquely extended to $\mathcal{A}_J$, namely there exists a unique $\mathcal{A}_J$-module M_J such that* $\operatorname{supp} M_J = \Lambda^+(1)_J$ *and* $n_{w_J} M_J|_{\mathcal{A}_{w_J}} = n_{w_J} M|_{\mathcal{A}_{w_J}}$.

Proof. First we prove that $n_{w_J}M$ is uniquely extended to $\mathcal{A}_{J,w_J}$. Take $\lambda_0 \in \Lambda_S(1)$ such that:

- $\langle \nu(\lambda_0), \alpha \rangle = 0$ for all $\alpha \in \Sigma_J^+$.
- $\langle \nu(\lambda_0), \alpha \rangle > 0$ for all $\alpha \in \Sigma^+ \setminus \Sigma_J^+$.

Note that $w_J(\Sigma_J^+) = \Sigma_J^-$ and $w_J(\Sigma^+ \setminus \Sigma_J^+) = \Sigma^+ \setminus \Sigma_J^+$. Hence we have $\lambda_0 \in w_J(\Lambda^+(1))$, $E^J(\lambda_0)$ is central in $\mathcal{A}_{J,w_J}$ (since $\lambda_0 \in \Lambda_S(1)$ is central in $\Lambda(1)$) and $E^J(\lambda_0)$ is invertible by the first condition and Remark 2.1. The embedding $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$ induces $\mathcal{A}_{w_J}[E(\lambda_0)^{-1}] \hookrightarrow \mathcal{A}_{J,w_J}$. We prove that this is surjective. Let $E^J(\mu) \in \mathcal{A}_{J,w_J}$. Then we have $\langle w_J(\nu(\mu)), \alpha \rangle \geq 0$ for any $\alpha \in \Sigma_J^+$. Therefore, for sufficiently large $n \in \mathbb{Z}_{>0}$, we have $\lambda_0^n \mu \in w_J(\Lambda^+(1))$. The elements $\nu(\lambda_0)$ and $\nu(\mu)$ are in the same closed Weyl chamber $w_J \nu(\Lambda^+(1)_J)$ with respect to J. Hence $E^J(\lambda_0^n)E^J(\mu) = E^J(\lambda_0^n \mu)$ which is in the image of $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$. Therefore $\mathcal{A}_{w_J}[E(\lambda_0)^{-1}] \hookrightarrow \mathcal{A}_{J,w_J}$ is surjective. Now we get the lemma since $E(\lambda_0)$ is invertible on $n_{w_J}M$. (Recall that $\operatorname{supp} n_{w_J}M = w_J(\Lambda^+(1))$ and $\lambda_0 \in w_J(\Lambda^+(1))$.)

So we have the extension N_J of $n_{w_J}M$ to $\mathcal{A}_{J,w_J}$. Define the action of $E^J(\lambda)$ on N_J by zero for $\lambda \in \Lambda(1) \setminus w_J(\Lambda^+(1)_J)$. Then N_J is an $\mathcal{A}_J$-module such that $\operatorname{supp} N_J = w_J(\Lambda^+(1)_J)$ which is desired. From the definition of the support, this is the only way to extend the module N_J to $\mathcal{A}_J$. We get the lemma. $\square$

Take M_J as in the lemma. We have

$$\operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}, n_{w_J}M) \simeq \operatorname{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M)) \simeq \operatorname{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M_J)).$$

Lemma 3.15. *The homomorphisms*

$$\operatorname{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J) \to \operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J, n_{w_J}M_J) \to \operatorname{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M_J)$$

are both isomorphisms.

Proof. The first is an isomorphism by an argument similar to the proof of Lemma 3.12.

Take $\lambda_0 \in \Lambda(1)$ such that:

- $\lambda_0 \in Z(W_J(1))$.
- $\langle \nu(\lambda_0), \alpha \rangle > 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_J^+$.

Then $\mathcal{H}_J = \mathcal{H}_J^-[E^J(\lambda_0)^{-1}]$ [Abe 2019b, Proposition 2.5]. Since $E^J(\lambda_0)$ is invertible in $\mathcal{A}_J$, it is also invertible on $n_{w_J}M_J$. (Note that $n_{w_J}M_J$ is an $\mathcal{A}_J$-module.) Hence the second homomorphism is an isomorphism. $\square$

Therefore we get

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_J}M) \simeq \operatorname{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \operatorname{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J)).$$

Lemma 3.16. *Let X be an $\mathcal{H}_J$-module and assume that $X \to X \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$ is injective. Then $Y \to Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}$ is also injective for $Y = \operatorname{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, X)$.*

Therefore for the proof of Lemma 3.10, it is sufficient to prove that

$$\mathrm{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J) \to \mathrm{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J) \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$$

is injective, namely we may assume that $w = w_\Delta$.

Proof. Set $J' = -w_\Delta(J)$ and put $n = n_{w_\Delta} n_{w_J}$. Then $l \mapsto nln^{-1}$ gives an isomorphism $L_J \to L_{J'}$ and sends $I(1)_J$ to $I(1)_{J'}$. Therefore it induces an isomorphism $\mathcal{H}_J \to \mathcal{H}_{J'}$. Define an $\mathcal{H}_{J'}$-module X' as the pull-back of X by this isomorphism (see [Abe 2019a]). Then $X \to X \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$ induces $X' \to X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}$ and the latter map is also injective. By [Abe 2019a, Proposition 4.15], we have $Y \simeq X' \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$. By [Vignéras 2015, Proposition 4.1], the functor $(\cdot) \otimes_{(\mathcal{H}_{J'}^+, j_{J'}^+)} \mathcal{H}$ is exact. Hence, using the assumption in the lemma, the map

$$Y \simeq X' \otimes_{(\mathcal{H}_{J'}^+, j_{J'}^+)} \mathcal{H} \to (X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}^+, j_{J'}^+)} \mathcal{H}$$

is injective. By [Ollivier and Vignéras 2018, Proposition 4.4]

$$(X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}^+, j_{J'}^+)} \mathcal{H} \simeq (\mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}))^{I(1)}.$$

In particular,

$$(X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}^+, j_{J'}^+)} \mathcal{H} \to \mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})$$

is injective. Finally, by [Ollivier and Vignéras 2018, Corollary 4.7],

$$\mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \text{c-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}) \simeq Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^{G} \mathbf{1}.$$

Combining all of these, we conclude the lemma. □

3F. *Some more reductions.* By the definition of $\mathcal{H}_\varnothing^+$, $\mathcal{H}_\varnothing^-$ and [Abe 2019b, Lemma 2.6], we have:

$$j_\varnothing^+(\mathcal{H}_\varnothing^+) = \mathcal{A}_{w_\Delta} \quad \text{and} \quad j_\varnothing^{-*}(\mathcal{H}_\varnothing^-) = \mathcal{A}_1.$$

See the argument in Section 3E. By these identities, we regard $\mathcal{A}_1$ and $\mathcal{A}_{w_\Delta}$ as a subalgebra of $\mathcal{H}_\varnothing = \mathcal{A}_\varnothing$.

Let M be an $\mathcal{A}$-module such that $\operatorname{supp} M = \Lambda^+(1)$. By Lemma 3.12, we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta}M) \simeq \mathrm{Hom}_{\mathcal{A}_{w_\Delta}}(\mathcal{H}, n_{w_\Delta}M)$. By Lemma 3.14, there exists an $\mathcal{A}_\varnothing$-module $M_\varnothing$ such that $M|_{\mathcal{A}_1} \simeq M_\varnothing|_{\mathcal{A}_1}$. It is easy to see that $n_{w_\Delta}M|_{\mathcal{A}_{w_\Delta}} \simeq n_{w_\Delta}M_\varnothing|_{\mathcal{A}_{w_\Delta}}$. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta}M) &\simeq \mathrm{Hom}_{(\mathcal{H}_\varnothing^+, j_\varnothing^+)}(\mathcal{H}, n_{w_\Delta}M_\varnothing) \\
&\simeq \mathrm{Hom}_{(\mathcal{H}_\varnothing^-, j_\varnothing^-)}(\mathcal{H}, M_\varnothing) && \text{[Abe 2019b, Proposition 4.13]} \\
&\simeq M_\varnothing \otimes_{(\mathcal{H}_\varnothing^-, j_\varnothing^{-*})} \mathcal{H} && \text{[Abe 2019b, Corollary 4.19]} \\
&= M \otimes_{\mathcal{A}_1} \mathcal{H}. && (j_\varnothing^{-*}(\mathcal{H}_\varnothing^-) = \mathcal{A}_1)
\end{aligned}$$

By Lemma 3.13, we have $M \otimes_{\mathcal{A}_1} \mathcal{H} \simeq M \otimes_{\mathcal{A}} \mathcal{H}$. Hence we get the following lemma:

Lemma 3.17. *We have* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta}M) \simeq M \otimes_{\mathcal{A}} \mathcal{H}$ *for any* $\mathcal{A}$*-module* M *such that* $\operatorname{supp} M = \Lambda^+(1)$

Therefore, to prove Lemma 3.10, hence Theorem 3.5, it is sufficient to prove the following.

Lemma 3.18. *Let M be an $\mathcal{A}$-module such that* $\operatorname{supp} M = \Lambda^+(1)$. *Then $M \otimes_{\mathcal{A}} \mathcal{H} \to M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

The group algebra $C[Z_\kappa]$ is a subalgebra of $\mathcal{A}$ via the map $t \mapsto T_t = E(t)$ for $t \in Z_\kappa$. Let $\hat{Z}_\kappa$ denote the set of characters of Z_κ. Since the order of Z_κ is prime to p, M is semisimple as a $C[Z_\kappa]$-module. Let $\psi \in Z_\kappa$ and set $M_\psi = \{m \in M \mid mT_t = \psi(t)m\ (t \in Z_\kappa)\}$. Since Z_κ is normal in $\Lambda(1)$, the conjugate action of $\Lambda(1)$ on Z_κ induces the action on $\hat{Z}_\kappa$. The formula $E(\lambda)T_t = T_{\lambda t \lambda^{-1}} E(\lambda)$ implies that $M_\psi E(\lambda) \subset M_{\lambda^{-1}(\psi)}$. For an orbit ω of this action in $\hat{Z}_\kappa$, we put $M_\omega = \bigoplus_{\psi \in \omega} M_\psi$. Then M_ω is stable under the $\mathcal{A}$-action and we have $M = \bigoplus_\omega M_\omega$. Therefore we may assume that $M = M_\omega$ for some ω to prove Lemma 3.18.

Let $\alpha \in \Delta$ and consider the image of $Z \cap L'_{\{\alpha\}}$ in $\Lambda(1)$. We denote this subgroup by $\Lambda'_\alpha(1)$. Consider the following condition: ψ is trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$. Since $Z_\kappa \cap \Lambda'_\alpha(1)$ is normal in $\Lambda(1)$, for $t \in Z_\kappa \cap \Lambda'_\alpha(1)$ and $\lambda \in \Lambda(1)$, we have $(\lambda\psi)(t) = \psi(\lambda^{-1} t \lambda) = 1$ if ψ satisfies this condition. Hence this condition only depends on $\Lambda(1)$-orbit.

We start to prove Lemma 3.19 by induction on $\dim(G)$. Assume that ω is a $\Lambda(1)$-orbit in $\hat{Z}_\kappa$. First we assume that there exists $\alpha \in \Delta$ such that ψ is not trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$ for some (equivalently any) $\psi \in \omega$. Then by [Abe 2019a, Theorem 3.13], we have $M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{s_\alpha} M \otimes_{\mathcal{A}} \mathcal{H}$.

We prove that in this case the lemma follows from that for a Levi subgroup. The argument is similar to that in Section 3E. Set $J = \Delta \setminus \{\alpha\}$. Then we have $n_{s_\alpha} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}} \mathcal{H}$ by Lemma 3.11. By Lemma 3.13, we have $n_{w_\Delta w_J} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}$. As in the argument in Section 3E using [Abe 2019b, Lemma 2.6], we have $j_J^+(\mathcal{H}_J^+) \supset \mathcal{A}_{w_\Delta w_J}$. Therefore we have

$$n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H} \simeq (n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J^+) \otimes_{(\mathcal{H}_J^+, j_J^+)} \mathcal{H}.$$

By the same argument of the proof of Lemma 3.14, there exists an $\mathcal{A}_J$-module M_J such that

$$n_{w_\Delta w_J} M_J|_{\mathcal{A}_{w_\Delta w_J}} = n_{w_\Delta w_J} M|_{\mathcal{A}_{w_\Delta w_J}} \quad \text{and} \quad \operatorname{supp} M_J = \Lambda^+(1)_J.$$

By a similar argument of the proof of Lemma 3.15, the homomorphisms

$$n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J^+ \to n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J \to n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J$$

are isomorphisms. Now by inductive hypothesis, the homomorphism

$$n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J \to (n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J) \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$$

is injective. By the argument in the proof of Lemma 3.16, this implies that for $Y = (n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J) \otimes_{(\mathcal{H}_J^+, j_J^+)} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}$, the homomorphism

$$Y \to Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$$

is injective. Hence we get the lemma for M.

Therefore we may assume that there is no such α. Hence it is sufficient to prove the following to prove Lemma 3.18.

Lemma 3.19. *Let M be an $\mathcal{A}$-module such that* $\operatorname{supp}(M)=\Lambda^+(1)$ *and $Z_\kappa\cap\Lambda'_\alpha(1)$ acts trivially on M for all $\alpha\in\Delta$. Then $M\otimes_{\mathcal{A}}\mathcal{H}\to M\otimes_{\mathcal{A}}\text{c-Ind}_{I(1)}^G\mathbf{1}$ is injective.*

We prove this lemma in Section 3J.

3G. ***Hecke modules.*** As discussed in 3D, we have the following

$$M\otimes_{\mathcal{A}}\mathcal{H}\simeq M\otimes_{C[\Lambda(1)]}(C[\Lambda(1)]\otimes_{\mathcal{A}}\mathcal{H}),$$

We decompose this module along the action of Z_κ.

Set $C[\Lambda(1)]_\psi=\{f\in C[\Lambda(1)]\mid \tau_t f=\psi(t)f\ (t\in Z_\kappa)\}$ and for a $\Lambda(1)$-stable subset $\omega\subset\hat{Z}_\kappa$ we put $C[\Lambda(1)]_\omega=\bigoplus_{\psi\in\omega}C[\Lambda(1)]_\psi$. From the definition, it is obvious that $C[\Lambda(1)]_\omega$ is invariant under the right action of $C[\Lambda(1)]$.

Lemma 3.20. *We have $C[\Lambda(1)]_\omega=\bigoplus_{\psi\in\omega}\{f\in C[\Lambda(1)]\mid f\tau_t=\psi(t)f\ (t\in Z_\kappa)\}$.*

Proof. Let $\psi\in\omega$, $f\in C[\Lambda(1)]_\psi$ and we write $f=\sum_{\lambda\in\Lambda(1)}c_\lambda\tau_\lambda$ where $c_\lambda\in C$. Set

$$e=\#Z_\kappa^{-1}\sum_{t\in Z_\kappa}\psi(t)^{-1}\tau_t\in C[Z_\kappa].$$

Then $ef=f$ and $e\tau_t=\psi(t)e$ for each $t\in Z_\kappa$. We have $e\tau_\lambda\tau_t=e\tau_{\lambda t\lambda^{-1}}\tau_\lambda=(\lambda^{-1}\psi)(t)e\tau_\lambda$. Since $\lambda^{-1}\psi\in\omega$, we get the lemma. □

Therefore $C[\Lambda(1)]_\omega$ is a two-sided ideal of $C[\Lambda(1)]$. Using Z_κ-action, some objects appearing here are decomposed. Here is a list:

- $C[\Lambda(1)]=C[\Lambda(1)]_\omega\times C[\Lambda(1)]_{\hat{Z}_\kappa\setminus\omega}$ as C-algebras.
- $\mathcal{A}=\mathcal{A}_\omega\times\mathcal{A}_{\hat{Z}_\kappa\setminus\omega}$ as C-algebras with the obvious notation.
- The homomorphism (3-1) induces $\mathcal{A}_\omega\to C[\Lambda(1)]_\omega$ and $\mathcal{A}_{\hat{Z}_\kappa\setminus\omega}\to C[\Lambda(1)]_{\hat{Z}_\kappa\setminus\omega}$.

Let M be an $\mathcal{A}$-module such that $\operatorname{supp}M=\Lambda^+(1)$ and $M=M_\omega$ (see Section 3F). Then as in Section 3D, M is a $C[\Lambda(1)]$-module and this action factors through $C[\Lambda(1)]\to C[\Lambda(1)]_\omega$. Hence we have

$$M\otimes_{\mathcal{A}}\mathcal{H}\simeq M\otimes_{C[\Lambda(1)]_\omega}(C[\Lambda(1)]_\omega\otimes_{\mathcal{A}}\mathcal{H})\tag{3-3}$$

In [Abe 2019a, Section 3], it is proved that, for any $w\in W_0$, $1\otimes1\mapsto1\otimes T^*_{n_{w_\Delta w^{-1}}}$ gives a $(C[\Lambda(1)],\mathcal{H})$-bimodule homomorphism

$$n_wC[\Lambda(1)]\otimes_{\mathcal{A}}\mathcal{H}\to n_{w_\Delta}C[\Lambda(1)]\otimes_{\mathcal{A}}\mathcal{H}$$

which is injective [loc. cit., Proposition 3.12]. The homomorphism is compatible with the decomposition $n_wC[\Lambda(1)]\otimes_{\mathcal{A}}\mathcal{H}\simeq n_wC[\Lambda(1)]_\omega\otimes_{\mathcal{A}}\mathcal{H}\oplus n_wC[\Lambda(1)]_{\hat{Z}_\kappa\setminus\omega}\otimes_{\mathcal{A}}\mathcal{H}$. Hence we get the $(C[\Lambda(1)],\mathcal{H})$-bimodule homomorphism

$$n_wC[\Lambda(1)]_\omega\otimes_{\mathcal{A}}\mathcal{H}\to n_{w_\Delta}C[\Lambda(1)]_\omega\otimes_{\mathcal{A}}\mathcal{H}\tag{3-4}$$

which is again injective. By [loc. cit., Theorem 3.13], the image of this homomorphism only depends on Δ_w. Let X_J be the image of this homomorphism where $J = \Delta_w$. This is a $(C[\Lambda(1)]_\omega, \mathcal{A})$-module. We have $M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]_\omega} X_\Delta$ by (3-3).

Lemma 3.21. *If $J' \supset J$, then $X_{J'} \subset X_J$.*

Proof. Note that $\Delta_{w_\Delta w_J} = J$. Hence, by definition, X_J is a $(C[\Lambda(1)], \mathcal{H})$-submodule in $n_{w_\Delta} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}$ generated by $1 \otimes T^*_{n_{w_\Delta w_J w_\Delta}}$. If $J' \supset J$, then

$$\ell(w_\Delta w_J w_{J'} w_\Delta) = \ell(w_J w_{J'}) = \ell(w_{J'}) - \ell(w_J) = \ell(w_\Delta w_{J'} w_\Delta) - \ell(w_\Delta w_J w_\Delta).$$

Hence $T^*_{n_{w_\Delta w_{J'} w_\Delta}} = T^*_{n_{w_\Delta w_J w_\Delta}} T^*_{n_{w_\Delta w_J w_{J'} w_\Delta}}$. Therefore $1 \otimes T^*_{n_{w_\Delta w_{J'} w_\Delta}} \in X_J$. Since $X_{J'}$ is generated by $1 \otimes T^*_{n_{w_\Delta w_{J'} w_\Delta}}$, we have $X_{J'} \subset X_J$. □

Lemma 3.22. *$X_J \in \mathcal{C}$.*

Proof. Take $\lambda \in \Lambda_S(1)$ such that $\nu(\lambda)$ is regular dominant. Then we have $z_\lambda = \sum_{v \in W_0} E(n_v \lambda n_v^{-1})$ by Lemma 3.2. Let $f \otimes X \in X_J$. Then, since z_λ is in the center, we have $(f \otimes X) z_\lambda = f \otimes z_\lambda X = f \otimes \sum_{v \in W_0} E(n_v \lambda n_v^{-1}) X = f \tau_\lambda \otimes X$ in $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}$. Since $f \mapsto f \tau_\lambda$ is invertible, z_λ is invertible on X_w. □

Note that $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}_{w_\Delta}} \mathcal{H}$ [Abe 2019a, Proposition 3.12]. Hence $X_\varnothing = n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{(\mathcal{H}^+_\varnothing, j^+_\varnothing)} \mathcal{H}$. This is a parabolically induced module [Vignéras 2015]. By [loc. cit., Example 3.2, Lemma 3.6], we have $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}$. Since $T^*_{n_w} \in T_{n_w} + \sum_{v < w} C[Z_\kappa] T_{n_v}$, we have $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T^*_{n_w}$.

Set $Y_w = n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T^*_{n_w} \subset X_\varnothing$. Then the subspace Y_w is the image of $n_w C[\Lambda(1)]_\omega \otimes 1$ by the injective homomorphism (3-4). In particular, Y_w is $\mathcal{A}$-stable and isomorphic to $n_w C[\Lambda(1)]_\omega$. We have $X_\varnothing = \bigoplus_{w \in W_0} Y_w$. This is the decomposition in Lemma 3.7. By the functoriality of the decomposition, we have $X_J = \bigoplus_{w \in W_0} (X_J \cap Y_w)$.

3H. *Representations of G.* Let ω be a $\Lambda(1)$-orbit in $\hat{Z}_\kappa$ such that for any $\alpha \in \Delta$, ψ is trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$ for some (equivalently any) $\psi \in \omega$. Recall that we have fixed a special parahoric subgroup K. Irreducible representations V of K are parametrized by a pair (ψ, J) where ψ is a character of Z_κ and J a certain subset of Δ. Here for V, ψ and J are given by the following: $\psi \simeq V^{I(1)}$ and $W_{0,J} = \operatorname{Stab}_{W_0}(V^{I(1)})$. Note that by the assumption on ω, (ψ, J) gives a parameter for any $\psi \in \omega$ and $J \subset \Delta$ [Abe et al. 2017, III.8]. Let $V_{\psi,J}$ be the irreducible representation of K which corresponds to (ψ, J) and put $V_J = \bigoplus_{\psi \in \omega} V_{\psi^{-1},J}$. In the rest of this paper, we fix a basis of $V^{I(1)}_{\psi^{-1},J}$ for each ψ and J.

Lemma 3.23. (1) *The Hecke algebra $\operatorname{End}_Z(\text{c-Ind}^Z_{Z\cap K} V^{I(1)}_J)$ is isomorphic to $C[\Lambda(1)]_\omega$.*

(2) *We have the Satake homomorphism*

$$\operatorname{End}_G(\text{c-Ind}^G_K V_J) \hookrightarrow \operatorname{End}_Z(\text{c-Ind}^Z_{Z\cap K} V^{I(1)}_J) \simeq C[\Lambda(1)]_\omega$$

and its image is $C[\Lambda^+(1)]_\omega$.

Proof. Let $\mathcal{H}(\psi_1^{-1},\psi_2^{-1})$ is the space of functions $\varphi\colon Z\to C$ such that $\operatorname{supp}\varphi$ is compact and $\varphi(t_1zt_2)=\psi_1^{-1}(t_1)\varphi(z)\psi_2^{-1}(t_2)$ for any $z\in Z$ and $t_1,t_2\in Z\cap K$. Since $V_J^{I(1)}\simeq\bigoplus_{\psi\in\omega}\psi^{-1}$, a standard argument for Hecke algebras implies

$$\operatorname{End}_Z(\operatorname{Ind}_{Z\cap K}^Z V_J^{I(1)})\simeq\bigoplus_{\psi_1,\psi_2\in\omega}\operatorname{Hom}_Z(\text{c-Ind}_{Z\cap K}^Z\psi_1^{-1},\text{c-Ind}_{Z\cap K}^Z\psi_2^{-1})\simeq\bigoplus_{\psi_1,\psi_2\in\omega}\mathcal{H}(\psi_1^{-1},\psi_2^{-1}).$$

This space is a subalgebra of $\mathcal{H}_Z$ where $\mathcal{H}_Z$ is the functions φ on Z which is invariant under the left (and equivalently right) multiplication by $Z\cap I(1)$ and whose support is compact. The homomorphism $\varphi\mapsto\sum_{z\in Z/(Z\cap K)}\varphi(z)\tau_z$ gives an isomorphism $\mathcal{H}_Z\simeq C[\Lambda(1)]$. As a subspace of both sides, it is easy to see that we get the desired isomorphism.

The Satake transform

$$\operatorname{Hom}_G(\text{c-Ind}_K^G V_{\psi_1,J},\text{c-Ind}_K^G V_{\psi_2,J})\to\operatorname{Hom}_Z(\text{c-Ind}_{Z\cap K}^Z\psi_1^{-1},\text{c-Ind}_{Z\cap K}^Z\psi_2^{-1})$$

is defined in [Henniart and Vignéras 2012, 2] and the image is described in [Abe et al. 2018b, Theorem 1.1]. □

Remark 3.24. In the identification (1) in the lemma, we need to fix an isomorphism $V_J^{I(1)}\simeq\bigoplus_{\psi\in\omega}\psi^{-1}$. We use our fixed basis of $V_{\psi^{-1},J}^{I(1)}$ for this isomorphism.

By the lemma, $C[\Lambda^+(1)]_\omega$ acts on $\text{c-Ind}_K^G V_J$. Define a representation π_J of G by

$$\pi_J=C[\Lambda(1)]_\omega\otimes_{C[\Lambda^+(1)]_\omega}\text{c-Ind}_K^G V_J.$$

We prove $\pi_J^{I(1)}\simeq X_J$.

Recall that the $\mathcal{H}$-module $(\text{c-Ind}_K^G V_J)^{I(1)}$ is described as follows. Let $\mathcal{H}_{\mathrm{f}}$ be the Hecke algebra attached to the pair $(K,I(1))$. Then $V_J^{I(1)}$ is naturally a right $\mathcal{H}_{\mathrm{f}}$-module and the algebra $\mathcal{H}_{\mathrm{f}}$ is a subalgebra of $\mathcal{H}$ with a basis $\{T_w\mid w\in W_0(1)\}$ where $W_0(1)$ is the inverse image of $W_0\subset W$ in $W(1)$. Then we have $(\text{c-Ind}_K^G V_J)^{I(1)}\simeq V_J^{I(1)}\otimes_{\mathcal{H}_{\mathrm{f}}}\mathcal{H}$ [Vignéras 2017, Proposition 7.2].

Remark 3.25. In the argument below, we will use results in [Abe 2017]. In [loc. cit.], we study an $\mathcal{H}_{\mathrm{f}}$-module denoted by $\eta^J=\bigoplus_{\psi\in\hat{Z}_\kappa}V_{\psi,J}^{I(1)}$. Using a similar argument in [loc. cit.] (or taking a direct summand of results), results are also true for an $\mathcal{H}_{\mathrm{f}}$-module $V_J^{I(1)}$.

We have an action of $C[\Lambda^+(1)]_\omega$ on $V_J^{I(1)}\otimes_{\mathcal{H}_{\mathrm{f}}}\mathcal{H}$ [loc. cit., Proposition 3.4] and the above isomorphism $(\text{c-Ind}_K^G V_J)^{I(1)}\simeq V_J^{I(1)}\otimes_{\mathcal{H}_{\mathrm{f}}}\mathcal{H}$ is $C[\Lambda^+(1)]_\omega$-equivariant. (This can be proved by the same argument in the proof of [loc. cit., Proposition 5.1].)

Lemma 3.26. *Let A be a ring and $S\subset A$ be a multiplicative subset of the center of A. Then for a smooth $A[G]$-module π, we have $(S^{-1}\pi)^{I(1)}\simeq S^{-1}\pi^{I(1)}$.*

Proof. Both sides can be regarded as a subspace of $S^{-1}\pi$. Any element in $S^{-1}\pi^{I(1)}$ is $I(1)$-invariant, hence $S^{-1}\pi^{I(1)}\subset(S^{-1}\pi)^{I(1)}$. Let $v/s\in(S^{-1}\pi)^{I(1)}$ where $v\in\pi$ and $s\in S$. Let $g_1,\dots,g_n$ be a representatives of $I(1)/\operatorname{Stab}_{I(1)}(v)$. Since v/s is g_i-invariant, there exists $s_i\in S$ such that $s_i(g_iv-v)=0$. Therefore $s_1\cdots s_n(g_iv-v)=0$. Set $v'=s_1\cdots s_nv$. Then for any $g\in I(1)$ there exists i and $g'\in\operatorname{Stab}_{I(1)}(v)$

such that $g = g_i g'$. Hence $gv' = s_1 \cdots s_n(g_i g' v) = s_1 \cdots s_n v = v'$. Therefore $v' \in \pi^{I(1)}$. Hence $v/s = v'/(ss_1 \cdots s_n) \in S^{-1}\pi^{I(1)}$. □

Therefore we have

$$\pi_J^{I(1)} \simeq C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} V_J^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}.$$

By [Abe 2017, Proposition 3.9], we have

$$V_J^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H} \simeq \mathrm{Im}(n_{w_\Delta w_J} C[\Lambda^+(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \to n_{w_\Delta w_J} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}).$$

Hence we have an isomorphism $C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} V_J^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H} \simeq X_J$. Therefore $\pi_J^{I(1)} \simeq X_J$.

We get an embedding $X_J \simeq \pi_J^{I(1)} \hookrightarrow \pi_J$. Hence there exists a homomorphism $X_J \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \to \pi_J$. Let $J = \Delta$ and applying $M \otimes_{C[\Lambda(1)]}$ to

$$X_\Delta \to X_\Delta \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \to \pi_\Delta$$

and using $M \otimes_{C[\Lambda(1)]} X_\Delta \simeq M \otimes_{\mathcal{A}} \mathcal{H}$ (3-3), we get

$$M \otimes_{C[\Lambda(1)]} X_\Delta \simeq M \otimes_{\mathcal{A}} \mathcal{H} \to M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1} \to M \otimes_{C[\Lambda(1)]} \pi_\Delta.$$

Hence for Lemma 3.19, it is sufficient to prove that $M \otimes_{C[\Lambda(1)]} X_\Delta \to M \otimes_{C[\Lambda(1)]} \pi_\Delta$ is injective.

We have an isomorphism $\pi_\varnothing \simeq \mathrm{Ind}_{\overline{B}}^G(\text{c-Ind}_{Z\cap K}^Z V_J^{I(1)})$ [Henniart and Vignéras 2012, Theorem 1.2]. (To be precisely, the direct sum of a result in [loc. cit., Theorem 1.2].) An injective embedding $\pi_J \to \mathrm{Ind}_{\overline{B}}^G(\text{c-Ind}_{Z\cap K}^Z V_J^{I(1)}) \simeq \pi_\varnothing$ was given in [loc. cit., Definition 2.1]. Hence we have a diagram of $(C[\Lambda(1)], \mathcal{H})$-bimodules

$$\begin{array}{ccc} X_J & \longrightarrow & X_\varnothing \\ \downarrow & & \downarrow \\ \pi_J & \longrightarrow & \pi_\varnothing. \end{array}$$

When $J = \varnothing$, $X_J \to X_\varnothing$ and $\pi_J \to \pi_\varnothing$ are both identities. Hence this diagram is commutative.

Lemma 3.27. *This diagram is commutative for any J.*

Proof. Fix $\psi^{-1} \in \omega$. It is sufficient to prove that the following diagram is commutative:

$$\begin{array}{ccc} V_{\psi,J}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H} & \longrightarrow & n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \\ \downarrow & & \downarrow \\ (\text{c-Ind}_K^G V_{\psi,J})^{I(1)} & \longrightarrow & \mathrm{Ind}_{\overline{B}}^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})^{I(1)}. \end{array} \tag{3-5}$$

Note that this diagram is commutative when $J = \varnothing$.

Let $v_0 \in V_{\psi,J}^{I(1)}$ be our fixed basis. Define $\varphi_J \in (\text{c-Ind}_K^G V_{\psi,J})^{I(1)}$ by $\mathrm{supp}\,\varphi_J = K$ and $\varphi_J(1) = v_0$. Then the $\mathcal{H}$-module map $V_{\psi,J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \to \text{c-Ind}_K^G(V_{\psi,J})$ is given by $v_0 \otimes 1 \mapsto \varphi_J$. Define $f_0 \in \mathrm{Ind}_{\overline{B}}^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})^{I(1)}$ by $\mathrm{supp}\, f_0 = \overline{B} n_{w_\Delta} I(1)$, $\mathrm{supp}\, f_0(n_{w_\Delta}^{-1}) = Z \cap K$ and $f_0(n_{w_\Delta}^{-1})(1) = v_0$. Then the function corresponding to $\varphi_\varnothing$ under $\text{c-Ind}_K^G V_\varnothing \to \text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_\varnothing^{I(1)})$ is $f_0 T_{n_{w_\Delta}}$ [Abe et al. 2017, IV.9 Proposition].

Set $w = w_\Delta w_J$. Then $X_J = n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}$. The homomorphism $V_{\psi,J}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H} \to n_w C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$ is given by $v_0 \otimes 1 \mapsto 1 \otimes T_{n_w}$ [Abe 2017, Lemmas 3.8 and 3.10].

Consider the case of $J = \varnothing$. Then the image of $v_0 \otimes 1 \in V_{\psi,\varnothing}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}$ under

$$V_{\psi,\varnothing}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H} \to (\text{c-Ind}_K^G V_{\psi,\varnothing})^{I(1)} \to \text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})^{I(1)}$$

is

$$v_0 \otimes 1 \mapsto \varphi_\varnothing \mapsto f_0 T_{n_{w_\Delta}}.$$

On the other hand, the image of $v_0 \otimes 1 \in V_{\psi,\varnothing}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}$ in $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$ is $1 \otimes T_{n_{w_\Delta}}$. As remarked before the lemma, (3-5) is commutative when $J = \varnothing$. Hence the homomorphism $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \to \text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})^{I(1)}$ sends $1 \otimes T_{n_{w_\Delta}}$ to $f_0 T_{n_{w_\Delta}}$. Take λ from the center of $\Lambda(1)$ such that $\langle \alpha, \nu(\lambda)\rangle < 0$ for any $\alpha \in \Sigma^+$. Then by [Abe 2019b, Lemma 2.17], $\ell(n_{w_\Delta}\lambda) = \ell(\lambda) - \ell(n_{w_\Delta})$. Hence by [Vignéras 2016, Theorem 5.25, Example 5.32], $T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1}\lambda) = E(\lambda)$. Therefore $1 \otimes T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1}\lambda) = 1 \otimes E(\lambda) = \tau_{n_{w_\Delta}^{-1}\lambda n_{w_\Delta}} \otimes 1$. On the other hand, $f_0 T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1}\lambda) = f_0 E(\lambda) = \tau_{n_{w_\Delta}^{-1}\lambda n_{w_\Delta}} f_0$ by [Abe et al. 2017, IV.10 Proposition]. Hence the homomorphism $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \to \text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})^{I(1)}$ sends $\tau_{n_{w_\Delta}^{-1}\lambda n_{w_\Delta}} \otimes 1$ to $\tau_{n_{w_\Delta}^{-1}\lambda n_{w_\Delta}} f_0$. Therefore $1 \otimes 1$ sends to f_0.

Let $a = v_0 \otimes 1 \in V_{\psi,J}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}$ and we consider the image of a in $\text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})^{I(1)}$ in the two ways. The image of a in $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$ is $1 \otimes T^*_{n_{w_\Delta w^{-1}}} T_{n_w}$ by [Abe 2017, Proposition 3.11] and the definition of $X_J \to X_\varnothing$. Therefore the image of a under $V_{\psi,J}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H} \to n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \to \text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})$ is $f_0 T^*_{n_{w_\Delta w^{-1}}} T_{n_w}$.

By [Abe et al. 2017, IV.9 Proposition] (for $J = \Delta$), we have $f_0 T^*_{n_{w_\Delta w^{-1}}} = \sum_{v \le w_\Delta w^{-1}} f_0 T_{n_v}$. Since $w_\Delta w^{-1} = w_\Delta w_J w_\Delta$, $\{v \in W_0 \mid v \le w_\Delta w^{-1}\} = w_\Delta W_{0,J} w_\Delta$. Hence

$$f_0 T^*_{n_{w_\Delta w^{-1}}} T_{n_w} = \sum_{v \in W_{J,0}} f_0 T_{n_{w_\Delta v w_\Delta}} T_{n_{w_\Delta w_J}}.$$

We have

$$\ell(w_\Delta v w_\Delta \cdot w_\Delta w_J) = \ell(w_\Delta v w_J) = \ell(w_\Delta) - \ell(v w_J) = \ell(w_\Delta) - \ell(w_J) + \ell(v) = \ell(w_\Delta w_J) + \ell(w_\Delta v w_\Delta).$$

Hence $T_{n_{w_\Delta v w_\Delta}} T_{n_{w_\Delta w_J}} = T_{n_{w_\Delta v w_J}}$. Therefore, replacing v with $v w_J$, we get $f_0 T^*_{n_{w_\Delta w^{-1}}} T_{n_w} = \sum_{v \in W_{J,0}} f_0 T_{n_{w_\Delta v}}$. This is the image of φ_J in $\text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_{\psi,\varnothing}^{I(1)})$ by [Abe et al. 2017, IV.7 Corollary]. Hence the diagram (3-5) is commutative if we start with a. Since the element a generates $V_{\psi,J}^{I(1)} \otimes_{\mathcal{H}_\mathrm{f}} \mathcal{H}$ as an $\mathcal{H}$-module, the diagram (3-5) is commutative. □

Therefore we may regard π_J and X_J as a subspace of $\pi_\varnothing$. We have $\pi_\varnothing \simeq \text{Ind}_B^G(\text{c-Ind}_{Z\cap K}^Z V_J^{I(1)})$. By the same argument in the proof of Lemma 3.23, we have $\text{c-Ind}_{Z\cap K}^Z V_J^{I(1)} \simeq C[\Lambda(1)]_\omega$. Here again we use our fixed basis. Hence we have $\pi_\varnothing \simeq \text{Ind}_B^G C[\Lambda(1)]_\omega$. We identify π_J with the image in $\text{Ind}_B^G C[\Lambda(1)]_\omega$.

Remark 3.28. By [Abe et al. 2017, IV.7 Proposition] and the decomposition $G = \bigcup_{w\in W_0} \overline{B} n_w I(1)$ implies that $(\operatorname{Ind}_{\overline{B}}^G C[\Lambda(1)]_\omega)^{I(1)} = \bigoplus_{w\in W_0} C[\Lambda(1)]_\omega f_0 T_{n_w}$. Since $X_\varnothing = \bigoplus_{w\in W_0} C[\Lambda(1)]_\omega \otimes T_{n_w}$ (see after the proof of Lemma 3.22) and $X_\varnothing \to \pi_\varnothing$ sends $1\otimes 1$ to f_0 (see the proof of the previous lemma), we have $X_\varnothing \simeq \pi_\varnothing^{I(1)}$. Note that $\operatorname{supp} f_0 T_{n_w} = \overline{B} n_{w_\Delta w} I(1)$ [Abe et al. 2017, IV.7 Proposition].

3I. *Filtrations.* As in the previous subsection, let ω be a $\Lambda(1)$-orbit in $\hat{Z}_\kappa$ such that, for some (equivalently any) $\psi\in\omega$, ψ is trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$ for any $\alpha\in\Delta$. In this subsection, we use the following notation: for $A\subset W_0$, $\overline{B} A \overline{B} = \bigcup_{v\in A} \overline{B} n_v \overline{B}$.

For a subset $A\subset W_0$ which is open (namely, if $v_1\in W_0$, $v_2\in A$ and $v_1\geq v_2$ then $v_1\in A$), we put

$$\pi_{\varnothing,A} = \{f\in \operatorname{Ind}_{\overline{B}}^G C[\Lambda(1)]_\psi \mid \operatorname{supp} f\subset \overline{B} A \overline{B}\}.$$

We also put

$$X_{\varnothing,A} = \bigoplus_{v\in A} n_{w_\Delta} C[\Lambda(1)]\otimes T_{n_{w_\Delta v}}.$$

Lemma 3.29. *Let $h\in X_\varnothing$. Then $h\in X_{\varnothing,A}$ if and only if its image in $\pi_\varnothing$ is in $\pi_{\varnothing,A}$. Namely we have $X_{\varnothing,A} = X_\varnothing \cap \pi_{\varnothing,A}$.*

Proof. Let $H\in\pi_\varnothing$ be the image of h. By the description of $X_\varnothing\to\pi_\varnothing$ (see Remark 3.28), $h\in X_{\varnothing,A}$ if and only if $\operatorname{supp} H\subset \overline{B} A I(1)$. For each $v\in A$, we have

$$\overline{B} v I(1) = \overline{B} v (I(1)\cap v^{-1}\overline{B} v)(I(1)\cap v^{-1} B v) = \overline{B} v (I(1)\cap v^{-1} B v) \subset \overline{B} B v \subset \bigcup_{v'\geq v} \overline{B} v' \overline{B} \subset \overline{B} A \overline{B}.$$

Here we use [Abe 2012, Lemma 2.4]. Hence if $h\in X_{\varnothing,A}$ then $H\in\pi_{\varnothing,A}$.

Assume that $H\in\pi_{\varnothing,A}$ and $\operatorname{supp}(H)\cap \overline{B} v I(1)\neq\varnothing$ for $v\in W_0$. Since H is $I(1)$-invariant, we have $H(v)\neq 0$. Therefore $v\in A$. Hence $\operatorname{supp}(H)\subset \bigcup_{v\in A}\overline{B} v I(1)$. We get $h\in X_{\varnothing,A}$. □

Set $X_{J,A} = X_J\cap X_{\varnothing,A}$ and $\pi_{J,A} = \pi_J\cap\pi_{\varnothing,A}$. Let $w\in A$ be a minimal element and put $A' = A\setminus\{w\}$. Then we have an embedding

$$X_{\Delta,A}/X_{\Delta,A'} \hookrightarrow \pi_{\Delta,A}/\pi_{\Delta,A'}.$$

For each $\alpha\in\Delta$, take a lift $a_\alpha\in\Lambda'_\alpha(1)$ of a generator of $\Lambda'_\alpha(1)/(Z_\kappa\cap\Lambda'_\alpha(1))$ such that $\langle\nu(a_\alpha),\alpha\rangle>0$ [Abe et al. 2017, III.4].

The element $\#Z_\kappa^{-1}\sum_{\psi\in\omega}\sum_{t\in Z_\kappa}\psi(t)^{-1}\tau_{a_\alpha t}$ is in $C[\Lambda(1)]_\omega$ and does not depend on a choice of a lift (recall that ψ is trivial on $Z_\kappa\cap\Lambda'_\alpha(1)$). We denote it by τ_α. Set $c_w = \prod_{w^{-1}(\alpha)>0}(1-\tau_\alpha)$. Then as in [Abe et al. 2017, V.8 Proposition], we have

$$\pi_{\Delta,A}/\pi_{\Delta,A'} = c_w(\pi_{\varnothing,A}/\pi_{\varnothing,A'}). \tag{3-6}$$

The space $\pi_{\varnothing,A}/\pi_{\varnothing,A'}$ can be identified with the space of compactly supported functions on $\overline{B}\backslash\overline{B} w\overline{B}$ with values in $C[\Lambda(1)]_\omega$, which is isomorphic to $C_c^\infty(\overline{B}\backslash\overline{B} w\overline{B})\otimes_C C[\Lambda(1)]_\omega$ where $C_c^\infty(\overline{B}\backslash\overline{B} w\overline{B})$ is the space of locally constant compact support functions on $\overline{B}\backslash\overline{B} w\overline{B}$ with values in C. Hence it is free as $C[\Lambda(1)]_\omega$-module. By the following lemma and (3-6), $\pi_{\Delta,A}/\pi_{\Delta,A'}$ is also free.

Lemma 3.30. *The element $c_w \in C[\Lambda(1)]_\omega$ is not a zero divisor.*

Proof. The same proof in [Abe 2019a, Lemma 3.10] can apply. ☐

Lemma 3.31. *We have $X_{\Delta,A}/X_{\Delta,A'} = c_w(X_{\varnothing,A}/X_{\varnothing,A'})$.*

Proof. Since $X_{\Delta,A} = \pi_{\Delta,A} \cap X_{\varnothing,A}$, we have

$$X_{\Delta,A}/X_{\Delta,A'} = \pi_{\Delta,A}/\pi_{\Delta,A'} \cap X_{\varnothing,A}/X_{\varnothing,A'}$$

and the right-hand side is

$$c_w(\pi_{\varnothing,A}/\pi_{\varnothing,A'}) \cap X_{\varnothing,A}/X_{\varnothing,A'}.$$

Let H be in this set. Since $\pi_{\varnothing,A}/\pi_{\varnothing,A'}$ is a free $C[\Lambda(1)]_\omega$-module, the exact sequence $0 \to \pi_{\varnothing,A'} \to \pi_{\varnothing,A} \to \pi_{\varnothing,A}/\pi_{\varnothing,A'} \to 0$ splits. Hence $\pi_\varnothing \simeq \pi_{\varnothing,A'} \oplus (\pi_{\varnothing,A}/\pi_{\varnothing,A'})$. Therefore $c_w\pi_{\varnothing,A} \simeq c_w\pi_{\varnothing,A'} \oplus c_w(\pi_{\varnothing,A}/\pi_{\varnothing,A'})$. Hence $c_w(\pi_{\varnothing,A}/\pi_{\varnothing,A'}) \simeq (c_w\pi_{\varnothing,A})/(c_w\pi_{\varnothing,A'})$. Hence there exists $H' \in \pi_{\varnothing,A}$ such that H is the image of c_wH'. Since $H \in X_{\Delta,A}/X_{\Delta,A'}$, there exists $h \in X_{\Delta,A}$ such that $c_wH' - h$ is zero in $X_{\Delta,A}/X_{\Delta,A'}$. In particular it is zero in $c_w(\pi_{\varnothing,A}/\pi_{\varnothing,A'}) = (c_w\pi_{\varnothing,A})/(c_w\pi_{\varnothing,A'})$. Therefore there exists $H'' \in \pi_{\varnothing,A'}$ such that $c_wH' - h = c_wH''$. Replacing H' with $H' - H''$, we may assume $c_wH' \in X_{\varnothing,A}$. Recall that H' is a function with values in $C[\Lambda(1)]_\omega$. Since the element c_w is not a zero divisor in $C[\Lambda(1)]_\omega$, $c_wH' \in \pi_{\varnothing,A}$ implies $H' \in \pi_{\varnothing,A}$. Since $c_wH' \in X_\varnothing$, c_wH' is $I(1)$-invariant. Hence H' is also $I(1)$-invariant, again since c_w is not a zero divisor. Therefore $H' \in \pi_\varnothing^{I(1)} = X_\varnothing$. Hence $H' \in X_\varnothing \cap \pi_{\varnothing,A} = X_{\varnothing,A}$. Therefore $H \in c_w(X_{\varnothing,A}/X_{\varnothing,A'})$. The reverse inclusion $c_w(\pi_{\varnothing,A}/\pi_{\varnothing,A'}) \cap X_{\varnothing,A}/X_{\varnothing,A'} \supset c_w(X_{\varnothing,A}/X_{\varnothing,A'})$ is obvious. We get the lemma. ☐

3J. *Proof of Lemma 3.19.* Let A, A', w be as in the previous subsection.

Lemma 3.32. *The exact sequences of $C[\Lambda(1)]_\omega$-modules*

$$0 \to \pi_{\Delta,A'} \to \pi_{\Delta,A} \to \pi_{\Delta,A}/\pi_{\Delta,A'} \to 0 \quad \text{and} \quad 0 \to X_{\Delta,A'} \to X_{\Delta,A} \to X_{\Delta,A}/X_{\Delta,A'} \to 0$$

split.

Proof. By (3-6) and from the fact that $\pi_{\varnothing,A}/\pi_{\varnothing,A'}$ is free, $\pi_{\Delta,A}/\pi_{\Delta,A'}$ is also free. Hence the first exact sequence splits. Using Lemma 3.31, the same argument can apply for the second sequence. ☐

Lemma 3.33. *The inclusion $X_{\Delta,A}/X_{\Delta,A'} \hookrightarrow \pi_{\Delta,A}/\pi_{\Delta,A'}$ has a section as $C[\Lambda(1)]_\omega$-modules.*

Proof. First we construct a section of $X_{\varnothing,A}/X_{\varnothing,A'} \to \pi_{\varnothing,A}/\pi_{\varnothing,A'}$. Recall that $X_{\varnothing,A} = \pi_{\varnothing,A}^{I(1)}$. Note that $X_{\varnothing,A}/X_{\varnothing,A'} \simeq C[\Lambda(1)]_\omega$ and the section is given by $f \mapsto f(w)$. For $H \in \pi_{\varnothing,A}$, consider $H' \in \pi_{\varnothing,A}$ which is $I(1)$-invariant, $\operatorname{supp}(H') = \bar{B}vI(1)$ and $H'(v) = H(v)$. Then $H \mapsto H'$ gives a section of $X_{\varnothing,A}/X_{\varnothing,A'} \to \pi_{\varnothing,A}/\pi_{\varnothing,A'}$. Multiplying c_w and using (3-6), Lemma 3.31, we get a section of the $C[\Lambda(1)]_\omega$-module homomorphism $X_{\Delta,A}/X_{\Delta,A'} \to \pi_{\Delta,A}/\pi_{\Delta,A'}$. ☐

Proof of Lemma 3.19. Set $\pi_A^M = M \otimes_{C[\Lambda(1)]_\omega} \pi_{\Delta,A}$ and $X_A^M = M \otimes_{C[\Lambda(1)]_\omega} X_{\Delta,A}$. Then by Lemma 3.32, $\pi_{A'}^M$ and $X_{A'}^M$ are a subspaces of π_A^M and X_A^M, respectively. By Lemma 3.33, $X_A^M/X_{A'}^M \to \pi_A^M/\pi_{A'}^M$ is injective.

We prove that $X_A^M \to \pi_A^M$ is injective by induction on $\#A$. We have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{A'}^M & \longrightarrow & X_A^M & \longrightarrow & X_A^M/X_{A'}^M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_{A'}^M & \longrightarrow & \pi_A^M & \longrightarrow & \pi_A^M/\pi_{A'}^M & \longrightarrow & 0.
\end{array}$$

The homomorphism $X_{A'}^M \to \pi_{A'}^M$ is injective by inductive hypothesis and $X_A^M/X_{A'}^M \to \pi_A^M/\pi_{A'}^M$ is injective as we have seen. Hence $X_A^M \to \pi_A^M$ is injective. Setting $A = W_0$, we get the lemma. □

4. Theorem

Let $\mathcal{C}_{\mathrm{f}}$ be the full subcategory of $\mathcal{C}$ consisting of finite-dimensional modules. Note that this category is closed under submodules, quotients and extensions.

Theorem 4.1. *Let $M \in \mathcal{C}_{\mathrm{f}}$. Then $(M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \simeq M$.*

Proof. The theorem is true for simple M by [Abe 2019a, main theorem; Abe et al. 2018a, Theorem 4.17 and Theorem 5.11]. We prove the theorem by induction on $\dim(M)$.

Assume that M is not simple and let M' be a proper nonzero submodule of M. Let

$$\pi = \operatorname{Ker}(M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \to M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}).$$

By Theorem 3.5, $M \to (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$ is injective. Then we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi^{I(1)} & \longrightarrow & (M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \longrightarrow & (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \\
 & & & & \wr\uparrow & & \uparrow \\
 & & & & M' & \hookrightarrow & M.
\end{array}$$

Hence $\pi^{I(1)} = 0$. Since $I(1)$ is a pro-p group, $\pi = 0$. Hence $M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \to M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective. Set $M'' = M/M'$. Then we have a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
(M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M' \\
\downarrow & & \downarrow \\
(M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \longleftarrow & M \\
\downarrow & & \downarrow \\
(M'' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M'' \\
 & & \downarrow \\
 & & 0
\end{array}$$

with exact columns. Therefore $M \to (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$ is isomorphic. □

Corollary 4.2. *Let $\mathcal{C}_{G,\mathrm{f}}$ be the category of representations of G consisting of the following objects*:

- *Has a finite length.*
- *Any irreducible subquotient is a subquotient of $\mathrm{Ind}_B^G \sigma$ for a irreducible representation σ of Z.*
- *Is generated by $I(1)$-invariants.*

Then $\mathcal{C}_\mathrm{f} \simeq \mathcal{C}_{G,\mathrm{f}}$. The equivalence is given by $\pi \to \pi^{I(1)}$ and $M \mapsto M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G M$.

Proof. By the classification theorem in [Abe et al. 2017] and [Abe et al. 2018a, Theorem 5.11], if $\pi \in \mathcal{C}_{G,\mathrm{f}}$ is irreducible, then $\pi^{I(1)} \in \mathcal{C}_\mathrm{f}$. Hence, by induction on the length, if $\pi \in \mathcal{C}_{G,\mathrm{f}}$ then $\pi^{I(1)} \in \mathcal{C}_\mathrm{f}$.

Let $\pi \in \mathcal{C}_{G,\mathrm{f}}$ and we prove that $\pi^{I(1)} \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \to \pi$ is an isomorphism. The homomorphism is surjective since π is generated by $\pi^{I(1)}$. Let π' be the kernel. Then we have an exact sequence

$$0 \to (\pi')^{I(1)} \to (\pi^{I(1)} \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \to \pi^{I(1)}$$

and the last map is isomorphism by the theorem. Hence $(\pi')^{I(1)} = 0$ and it implies $\pi' = 0$. Therefore the homomorphism is also injective. Combining with the previous theorem, we have proved the desired equivalence of categories. □

Acknowledgment

The author was supported by JSPS KAKENHI Grant Number 18H01107. The author thanks the referee for giving helpful comments.

References

[Abe 2012] N. Abe, "Generalized Jacquet modules of parabolically induced representations", *Publ. Res. Inst. Math. Sci.* **48**:2 (2012), 419–473. MR Zbl

[Abe 2017] N. Abe, "Change of weight theorem for pro-p-Iwahori Hecke algebras", pp. 1–13 in *Around Langlands correspondences*, edited by F. Brumley et al., Contemp. Math. **691**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl

[Abe 2019a] N. Abe, "Modulo p parabolic induction of pro-p-Iwahori Hecke algebra", *J. Reine Angew. Math.* **749** (2019), 1–64. MR Zbl

[Abe 2019b] N. Abe, "Parabolic inductions for pro-p-Iwahori Hecke algebras", *Adv. Math.* **355** (2019).

[Abe et al. 2017] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, "A classification of irreducible admissible p representations of p-adic reductive groups", *J. Amer. Math. Soc.* **30**:2 (2017), 495–559. MR Zbl

[Abe et al. 2018a] N. Abe, G. Henniart, and M.-F. Vignéras, "On pro-p-Iwahori invariants of R-representations of reductive p-adic groups", *Represent. Theory* **22** (2018), 119–159. MR Zbl

[Abe et al. 2018b] N. Abe, F. Herzig, and M.-F. Vignéras, "Inverse Satake isomorphism and change of weight", preprint, 2018. arXiv

[Barthel and Livné 1994] L. Barthel and R. Livné, "Irreducible modular representations of GL_2 of a local field", *Duke Math. J.* **75**:2 (1994), 261–292. MR Zbl

[Barthel and Livné 1995] L. Barthel and R. Livné, "Modular representations of GL_2 of a local field: the ordinary, unramified case", *J. Number Theory* **55**:1 (1995), 1–27. MR Zbl

[Borel 1976] A. Borel, "Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup", *Invent. Math.* **35** (1976), 233–259. MR Zbl

[Breuil 2003] C. Breuil, "Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$, I", *Compos. Math.* **138**:2 (2003), 165–188. MR Zbl

[Henniart and Vignéras 2012] G. Henniart and M.-F. Vignéras, "Comparison of compact induction with parabolic induction", *Pacific J. Math.* **260**:2 (2012), 457–495. MR Zbl

[Ollivier 2009] R. Ollivier, "Le foncteur des invariants sous l'action du pro-p-Iwahori de $\mathrm{GL}_2(F)$", *J. Reine Angew. Math.* **635** (2009), 149–185. MR Zbl

[Ollivier and Schneider 2018] R. Ollivier and P. Schneider, "A canonical torsion theory for pro-p Iwahori–Hecke modules", *Adv. Math.* **327** (2018), 52–127. MR Zbl

[Ollivier and Vignéras 2018] R. Ollivier and M.-F. Vignéras, "Parabolic induction in characteristic p", *Selecta Math.* (*N.S.*) **24**:5 (2018), 3973–4039. MR Zbl

[Vignéras 2007] M.-F. Vignéras, "Représentations irréductibles de $\mathrm{GL}(2, F)$ modulo p", pp. 548–563 in *L-functions and Galois representations* (Durham, UK, 2004), edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge Univ. Press, 2007. MR Zbl

[Vignéras 2014] M.-F. Vignéras, "The pro-p-Iwahori–Hecke algebra of a reductive p-adic group, II", *Münster J. Math.* **7**:1 (2014), 363–379. Correction in **9**:1 (2016), 119. MR Zbl

[Vignéras 2015] M.-F. Vignéras, "The pro-p Iwahori Hecke algebra of a reductive p-adic group, V: Parabolic induction", *Pacific J. Math.* **279**:1-2 (2015), 499–529. MR Zbl

[Vignéras 2016] M.-F. Vignéras, "The pro-p-Iwahori Hecke algebra of a reductive p-adic group, I", *Compos. Math.* **152**:4 (2016), 693–753. MR Zbl

[Vignéras 2017] M.-F. Vignéras, "The pro-p-Iwahori Hecke algebra of a reductive p-adic group, III: Spherical Hecke algebras and supersingular modules", *J. Inst. Math. Jussieu* **16**:3 (2017), 571–608. MR Zbl

Communicated by Marie-France Vignéras
Received 2018-12-20 Revised 2019-03-01 Accepted 2019-05-27

abenori@ms.u-tokyo.ac.jp *Graduate School of Mathematical Sciences, The University of Tokyo, Japan*

Guidelines for Authors

Authors may submit manuscripts in PDF format on-line at the Submission page at the ANT website.

Originality. Submission of a manuscript acknowledges that the manuscript is original and and is not, in whole or in part, published or under consideration for publication elsewhere. It is understood also that the manuscript will not be submitted elsewhere while under consideration for publication in this journal.

Language. Articles in *ANT* are usually in English, but articles written in other languages are welcome.

Length There is no a priori limit on the length of an *ANT* article, but *ANT* considers long articles only if the significance-to-length ratio is appropriate. Very long manuscripts might be more suitable elsewhere as a memoir instead of a journal article.

Required items. A brief abstract of about 150 words or less must be included. It should be self-contained and not make any reference to the bibliography. If the article is not in English, two versions of the abstract must be included, one in the language of the article and one in English. Also required are keywords and subject classifications for the article, and, for each author, postal address, affiliation (if appropriate), and email address.

Format. Authors are encouraged to use LaTeX but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages (Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc.) allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with details about how your graphics were generated.

White space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Proofs. Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

Algebra & Number Theory

Volume 13 No. 8 2019

Moduli of stable maps in genus one and logarithmic geometry, II 1765
DHRUV RANGANATHAN, KELI SANTOS-PARKER and JONATHAN WISE

Multiplicity one for wildly ramified representations 1807
DANIEL LE

Theta operators on unitary Shimura varieties 1829
EHUD DE SHALIT and EYAL Z. GOREN

Infinitely generated symbolic Rees algebras over finite fields 1879
AKIYOSHI SANNAI and HIROMU TANAKA

Manin's *b*-constant in families 1893
AKASH KUMAR SENGUPTA

Equidimensional adic eigenvarieties for groups with discrete series 1907
DANIEL R. GULOTTA

Cohomological and numerical dynamical degrees on abelian varieties 1941
FEI HU

A comparison between pro-p Iwahori–Hecke modules and mod p representations 1959
NORIYUKI ABE

1937-0652(2019)13:8;1-5