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**Moduli of stable maps in genus one
and logarithmic geometry, II**

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This is the second in a pair of papers developing a framework to apply logarithmic methods in the study of stable maps and singular curves of genus 1. This volume focuses on logarithmic Gromov–Witten theory and tropical geometry. We construct a logarithmically nonsingular and proper moduli space of genus 1 curves mapping to any toric variety. The space is a birational modification of the principal component of the Abramovich–Chen–Gross–Siebert space of logarithmic stable maps and produces logarithmic analogues of Vakil and Zinger’s genus one reduced Gromov–Witten theory. We describe the nonarchimedean analytic skeleton of this moduli space and, as a consequence, obtain a full resolution to the tropical realizability problem in genus 1.

1. Introduction	1765
2. Preliminaries	1770
3. Logarithmic maps to toric varieties	1780
4. Realizability for genus one tropical curves	1793
Acknowledgements	1802
References	1803

1. Introduction

This paper is the second in a pair, exploring the interplay between tropical geometry, logarithmic moduli theory, stable maps, and moduli spaces of genus 1 curves. In the first volume, we used this interplay to construct new nonsingular moduli spaces compactifying the space of elliptic curves in projective space via *radially aligned* stable maps and quasimaps. In this paper, we focus on applications to logarithmic Gromov–Witten theory and tropical geometry.

I. Realizability of tropical curves. We give a complete characterization of genus 1 tropical maps that can be realized as tropicalizations of genus 1 curves mapping to tori, completing a study initiated in Speyer’s thesis. We show that a combinatorial condition identified by Baker, Payne and Rabinoff is always sufficient. Our proof is independent of these previous results, and is based on the geometry of logarithmic maps.

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II. *Logarithmic stable maps.* We construct a toroidal moduli space parametrizing maps from pointed genus 1 curves to any toric variety with prescribed contact orders along the toric boundary. This is a desingularization of the principal component of the space of logarithmic stable maps. The boundary complex of this compactification is identified as a space of realizable tropical maps.

1.1. Superabundant tropical geometries. The realization problem is the crux of the relationship between tropical geometry to algebraic geometry, and is unavoidable in enumerative applications. Given an abstract tropical curve¹ \square of genus g and a balanced piecewise linear map

$$F : \square \rightarrow \mathbb{R}^r,$$

we ask, *does there exist a nonarchimedean field K extending \mathbb{C} , a smooth algebraic curve C over K and a map*

$$\varphi : C \rightarrow \mathbb{G}_m^r,$$

such that φ^{trop} coincides with F ?

When \square has genus 0, the only obstruction to lifting is the local balancing condition, and all tropical curves satisfying that condition are realizable. This is reflected in the logarithmic smoothness of the moduli space of genus 0 logarithmic maps [Nishinou and Siebert 2006; Ranganathan 2017a; Tyomkin 2012]. In genus 1, nonlocal obstructions already appear for maps $\square \rightarrow \mathbb{R}$. The obstructions appear when the circuit of \square is contained in a proper affine subspace of \mathbb{R}^r . Speyer [2005; 2014] discovered a sufficient condition for realizability. A weaker necessary condition was identified in [Baker et al. 2016, Section 6]. We provide a characterization of the realizable tropical curves in genus 1 in [Theorem A](#) in terms of the geometry of the skeleton of an analytic space of maps.

Let Γ be a marked tropical curve of genus 1 with a unique vertex and n half-edges. Fix a balanced map $\Gamma \rightarrow \mathbb{R}^r$. Let $\mathcal{M}_\Gamma(\mathbb{G}_m^r)$ be the moduli space of maps

$$\varphi : C \rightarrow \mathbb{G}_m^r,$$

where C is a noncompact smooth algebraic curve of genus 1 with n punctures, and the vanishing orders at infinity of these punctures are specified by the slopes along the edges of Γ in \mathbb{R}^r . Let $W_\Gamma(\mathbb{R}^r)$ be the corresponding set of tropical maps

$$\square \rightarrow \mathbb{R}^r$$

whose recession fan is given by $\Gamma \rightarrow \mathbb{R}^r$, and satisfy the *well-spacedness condition*, as defined in [Section 4](#).² This set can be given the structure of a generalized cone complex.

¹The first author continues his efforts to popularize Dan Abramovich's convention that algebraic curves be denoted by C , \mathcal{C} , while tropical curves be denoted \square , approximating their appearance in nature.

²We caution the reader that the meaning attributed by Speyer to well-spacedness is stronger than the one we use here; see [Warning 4.4.5](#).

Given a map $\varphi : \mathcal{C} \rightarrow \mathbb{G}_m^r$ over a valued field, one obtains a balanced piecewise linear map from a Berkovich skeleton \square of the punctured general fiber curve \mathcal{C}_η to \mathbb{R}^r , i.e., to the skeleton of the torus [Ranganathan 2017b]. This piecewise linear map is the *tropicalization* of φ and is denoted φ^{trop} .

Theorem A. *There exists a continuous and proper tropicalization map*

$$\text{trop} : \mathcal{M}_\Gamma^{\text{an}}(\mathbb{G}_m^r) \rightarrow W_\Gamma(\mathbb{R}^r)$$

sending a map $[\varphi]$ over a valued field to its tropicalization. There is generalized cone complex $P_\Gamma(\mathbb{R}^r)$ and a finite morphism

$$\text{trop}_\mathcal{C} : P_\Gamma(\mathbb{R}^r) \rightarrow W_\Gamma(\mathbb{R}^r),$$

which is an isomorphism upon restriction to each cone of the source. The degree of this finite morphism is explicitly computable and the complex $P_\Gamma(\mathbb{R}^r)$ is a skeleton of the analytic moduli space $\mathcal{M}_\Gamma^{\text{an}}(\mathbb{G}_m^r)$.

The theorem is proved as [Theorem 4.6.2](#).

The statement that the tropicalization has a finite cover that is a skeleton is a toroidal version of the schön condition, frequently cited in tropical geometry. The skeleton $P_\Gamma(\mathbb{R}^r)$ functions as a parametrizing complex for the tropicalization, as in [[Helm and Katz 2012](#); [Tevelev 2007](#)].

1.2. Logarithmic stable maps. Our tropical investigation leads naturally to an understanding of the geometry of space of logarithmic stable maps to toric varieties in genus 1. The open moduli problem we consider is that of maps

$$(C, p_1, \dots, p_m) \rightarrow Z,$$

where C is a smooth pointed curve of genus 1, the target Z is a toric variety, and the contact orders of the points p_i with the boundary divisors on Z is fixed. There is a natural modular compactification of this space via the theory of logarithmic stable maps [[Abramovich and Chen 2014](#); [Chen 2014](#); [Gross and Siebert 2013](#)]. When the genus of the source curve is 0, the resulting moduli space is logarithmically smooth, but in genus 1 can be highly singular and nonequidimensional. We use the insights of [Theorem A](#) to construct a logarithmically smooth modular compactification, in parallel with the desingularization of the ordinary stable maps space [[Ranganathan et al. 2017](#); [Vakil and Zinger 2008](#)].

Let Z be a proper complex toric variety and $\mathcal{L}_\Gamma(Z)$ the moduli space of genus 1 logarithmic stable maps to Z with discrete data Γ , i.e., Γ records the genus and the contact orders of the marked points with the toric boundary of Z . Let $\mathcal{L}_\Gamma^\circ(Z)$ be the locus of parametrizing maps of positive degree from smooth domains, and let $\overline{\mathcal{L}}_\Gamma(Z)$ be the closure.

Theorem B. *Consider the following data as a moduli problem on logarithmic schemes:*

- (1) *A family of n -marked, radially aligned logarithmic curves $C \rightarrow S$.*
- (2) *A logarithmic stable map $f : C \rightarrow Z$ with contact order Γ , such that the map f is well-spaced (see [Definition 3.4.2](#)).*

This moduli problem is represented by a proper and logarithmically smooth stack with logarithmic structure $\mathcal{W}_\Gamma(Z)$ and the natural morphism

$$\mathcal{W}_\Gamma(Z) \rightarrow \overline{\mathcal{L}}_\Gamma^\circ(Z)$$

is proper and birational.

See [Theorem 4.4.8](#) for the proof.

The well-spacedness property above is efficiently stated in tropical language, and this is done later in the paper. At a first approximation it may be thought of as forcing a factorization property after composing $C \rightarrow Z$ with any rational map $Z \dashrightarrow \mathbb{P}^1$ induced by a character. These logarithmic maps are precisely the ones that have *well-spaced* tropicalizations. A prototype for practical calculations on this space may be found in [\[Len and Ranganathan 2018\]](#).

1.3. Motivation for the construction. The combinatorics of logarithmic stable maps are essentially part of tropical geometry. Indeed, if the variety Z is a toric variety, taken with its toric boundary, the analytification of the moduli space of logarithmic maps continuously to a polyhedral complex parametrizing tropical curves [\[Ranganathan 2017b\]](#). The connection is especially transparent in genus 0, see [\[Ranganathan 2017a\]](#). In genus 1, the tropical realizability problem can be used to predict the desingularization above, as we now explain. The moduli space $\mathcal{L}_\Gamma(Z)$ of genus 1 logarithmic stable maps is highly singular, however, it maps naturally to a logarithmically smooth Artin stack. More precisely, if $\mathcal{A}_Z = [Z/T]$ is the Artin fan of Z obtained by performing a stack quotient on Z by its dense torus, there is a natural map

$$\mathcal{L}_\Gamma(Z) \rightarrow \mathcal{L}_\Gamma(\mathcal{A}_Z),$$

where the latter is the space of prestable logarithmic maps to the Artin fan. This space is a logarithmically smooth Artin stack [\[Abramovich and Wise 2018\]](#). Moreover, the toroidal skeleton of this space is naturally identified with the moduli space of all (not necessarily realizable or even balanced) tropical maps from genus 1 curves to the fan of Z , see [\[Ranganathan 2017b\]](#). The locus of realizable curves is a sublocus in the moduli space. After subdividing this cone complex, this sublocus is supported on a subcomplex. This subdivision induces a birational modification of $\mathcal{L}_\Gamma(\mathcal{A}_Z)$, and thus a modification of $\mathcal{L}_\Gamma(Z)$. This modification can naturally be identified with the moduli of well-spaced logarithmic maps $\mathcal{W}_\Gamma(Z)$ defined above. The radial alignments developed in [\[Ranganathan et al. 2017\]](#) and recalled in [Section 2.9](#) give rise to the modular interpretation.

The construction of $\mathcal{W}_\Gamma(Z)$ is not a formal lifting of our previous results on ordinary stable maps to the logarithmic category [\[Ranganathan et al. 2017\]](#). Given an absolute genus 1 stable map $[C \rightarrow \mathbb{P}^r]$, if no genus 1 subcurve is contracted, then $[C \rightarrow \mathbb{P}^r]$ is a smooth point of the moduli space. However, for a toric variety Z and a genus 1 logarithmic map $[C \rightarrow Z]$, the deformations of the map can be obstructed even if no component of C is contracted. This is true even if $Z = \mathbb{P}^r$ with its toric logarithmic structure. This behavior is akin to the genus 1 absolute stable maps theory for semipositive targets. While the tangent bundle of \mathbb{P}^r is ample, the logarithmic tangent bundle of a toric variety is trivial. This allows for

a larger space of obstructions to deforming genus 1 logarithmic maps than in the absolute theory. We overcome this by identifying and forcing the stronger factorization property above.

1.4. Tropical enumerative geometry and realizability. The realizability problem for tropical curves is a combinatorial shadow of the problem of characterizing the closure of the main component in the space of logarithmic maps. The difficulty of the problem has limited tropical enumerative techniques to low target dimensions [Bertrand et al. 2014; Cavalieri et al. 2010; 2016; Mikhalkin 2005] or to genus 0 curves [Gross 2016; 2018; Mandel and Ruddat 2016; Nishinou and Siebert 2006; Ranganathan 2017a].

In the higher genus, higher dimensional situation, there are two directions in which one may generalize the picture above. The first is to develop a systematic method to decompose logarithmic Gromov–Witten invariants, as a sum of virtual invariants over tropical curves [Abramovich et al. 2017a; Parker 2011; Ranganathan 2019]. The second is to analyze the tropical lifting problem and produce a “reduced” curve counting theory that captures the principal component contribution to the virtual count. This paper addresses the second of these in genus one. The realizability theorem in genus 1 allows us to decompose these reduced invariants of any toric variety over tropical curves. The degeneration formula for these invariants is work that we hope to return to. Note that the analogous problem for smooth pairs has recently been treated in [Battistella et al. 2019].

There have been a number of interesting partial results on tropical realizability in the last decade, thanks to the efforts of many [Baker et al. 2016; Cheung et al. 2016; Jensen and Ranganathan 2017; Katz 2012; Mikhalkin 2005; Nishinou 2009; Nishinou and Siebert 2006; Ranganathan 2017b; 2017c; Speyer 2014]. The genus 1 story alone has seen heavy interest. Speyer identified the sufficiency of a strong form of well-spacedness condition for superabundant genus 1 tropical curves using Tate’s uniformization theory. Using the group law on the analytification of an elliptic curve, Baker, Payne and Rabinoff showed that a weaker condition was necessary. The existence of genus 1 tropical curves which failed Speyer’s condition but were nonetheless realizable was established in [Ranganathan 2017b].

In higher genus, few results are known. Realizability of nonsuperabundant higher genus tropical curves was established by Cheung, Fantini, Park and Ulirsch [2016], and limits of realizable curves can be shown to be realizable [Ranganathan 2017b; 2017c]. Katz [2012] showed that the logarithmic tangent/obstruction complex for degenerate maps gives rise to necessary combinatorial conditions for realizability in higher genus, including a version of well-spacedness. These methods do not prove sufficiency in any cases. A sufficient condition for realizability for some superabundant chain of cycle geometries has recently been shown to hold and used to establish new results in Brill–Noether theory [Jensen and Ranganathan 2017].

1.5. User’s guide. We have written this paper so it may be read independently from the prequel, in which the space of ordinary stable maps to \mathbb{P}^r was considered. In Section 2.9 we recall the preliminary results on radial alignments and their contractions from [Ranganathan et al. 2017]. The moduli space of well-spaced logarithmic maps is constructed in Section 3.4 and the logarithmic unobstructedness appears as Theorem 3.5.1. The tropical well-spacedness condition is discussed and defined precisely in Section 4.4. Finally, tropical realizability results are restated in Theorem 4.4.8 and proved in Section 4.6.

2. Preliminaries

In this section, we recall some preliminaries on singularities of genus 1 and on logarithmic and tropical geometry. There is some overlap between this section and the preliminary material appearing in the prequel to this article [Ranganathan et al. 2017], but we opt to include it for a more self-contained presentation.

2.1. Genus 1 singularities. Let C be a reduced curve over an algebraically closed field k . For an isolated curve singularity (C, p) with normalization $\pi : (\tilde{C}, p_1, \dots, p_m) \rightarrow (C, p)$, recall that m , the cardinality of $\pi^{-1}p$, is called the *number of branches of the singularity*. The δ -invariant is defined as

$$\delta := \dim_k(\pi_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C).$$

Based on these two invariants, one defines the genus of (C, p) as

$$g = \delta - m + 1.$$

We will frequently make use of the *seminormalization* of (C, p) in our arguments. The *seminormalization* is a partial resolution of (C, p) to a singularity of genus 0 that is homeomorphic to (C, p) . Explicitly, equip the underlying topological space of (C, p) with the subring \mathcal{A} of regular functions on the normalization \tilde{C} that are well defined on the underlying topological space of C . In particular, there are g additional conditions required for a function in \mathcal{A} to descend to (C, p) , i.e.,

$$g = \dim_k(\mathcal{A}/\mathcal{O}_C).$$

Let E be a proper Gorenstein curve of genus 1, smooth away from a unique genus 1 singularity. Let $\nu : F \rightarrow E$ be the seminormalization and let $\mu : G \rightarrow F$ be the normalization. We have inclusions

$$\mathcal{O}_E \subset \nu_*\mathcal{O}_F \subset \nu_*\mu_*\mathcal{O}_G \subset K \quad \text{and} \quad J \supset \omega_E \supset \nu_*\omega_F \supset \nu_*\mu_*\omega_G$$

Here K is the sheaf of meromorphic functions on E and J is the sheaf of meromorphic differentials. For each $X = E, F, G$, the pairs ω_X and \mathcal{O}_X are dual to other another with respect to the residue pairing $K \otimes J \rightarrow k$, in the sense that each is the annihilator of the other [Altman and Kleiman 1970, Proposition 1.16(ii)].

Consider the exact sequence

$$0 \rightarrow \mathcal{O}_E \rightarrow \nu_*\mathcal{O}_F \rightarrow \nu_*(\mathcal{O}_F)/\mathcal{O}_E \rightarrow 0 \tag{1}$$

In the long exact cohomology sequence

$$0 \rightarrow H^0(E, \mathcal{O}_E) \rightarrow H^0(F, \mathcal{O}_F) \rightarrow \nu_*(\mathcal{O}_F)/\mathcal{O}_E \rightarrow H^1(E, \mathcal{O}_E) \rightarrow H^1(F, \mathcal{O}_F) \tag{2}$$

the map $H^0(E, \mathcal{O}_E) \rightarrow H^0(F, \mathcal{O}_F)$ is an isomorphism because both E and F are proper, connected, and reduced; furthermore $H^1(F, \mathcal{O}_F) = 0$ since F has genus 0. By Serre duality, $H^1(E, \mathcal{O}_E)$ is dual

to $H^0(E, \omega_E)$. Since both are 1-dimensional, the choice of a nonzero $\alpha \in H^0(E, \omega_E)$ induces an isomorphism $H^1(E, \mathcal{O}_E) \rightarrow k$. The composition

$$v_*(\mathcal{O}_F)/\mathcal{O}_E \rightarrow H^1(E, \mathcal{O}_E) \rightarrow k$$

may be identified with the residue pairing, sending $f \bmod \mathcal{O}_E$ to $\text{res } f\alpha$. This follows, for example, by the construction of the dualizing sheaf in [Altman and Kleiman 1970, Remark 1.9 and Remark 1.12].

We know that $\omega_F/\mu_*(\omega_G)$ is spanned by the differentials

$$\frac{dx_i}{x_i} - \frac{dx_j}{x_j} \tag{3}$$

where the x_i are local coordinates of the branches of E at the singular point. As $\omega_E/v_*(\omega_F)$ is 1-dimensional, ω_E is generated relative to $v_*(\omega_F)$ by a differential of the following form:

$$\sum_i \frac{c_i dx_i}{x_i^2} + \frac{c' dx_1}{x_1} \tag{4}$$

If $f \in \mathcal{O}_E$ has the expansion $f(0) + b_i x_i + \dots$ on the i -th component of F then this differential imposes the constraint

$$c' f(0) + \sum b_i c_i = 0.$$

In order for E to be Gorenstein, ω_E must be a line bundle, so the generators (3) of ω_F must be multiples of the generator (4). This immediately implies $c' = 0$ and that all of the c_i are nonzero. Conversely, if $c' = 0$ and all of the c_i are nonzero, then $c_j x_i - c_i x_j \in \mathcal{O}_E$ and

$$(c_j x_i - c_i x_j) \sum_k \frac{c_k dx_k}{x_k^2} = c_j c_i \frac{dx_i}{x_i} - c_i c_j \frac{dx_j}{x_j}$$

implies that the generators (3) are multiples of (4). This proves the following proposition:

Proposition 2.1.1. *If E is a Gorenstein curve with a genus 1 singularity then ω_E is generated in a neighborhood of its singular point by a meromorphic form (4), with $c' = 0$, where the x_i are local parameters for the branches of E at the singular point.*

By consideration of the residue condition imposed by the form (4), we can also obtain a local description of the Gorenstein, genus 1 curve singularities. A more conceptual proof of this result can be found in [Smyth 2011, Proposition A.3].

Proposition 2.1.2. *For each integer $m \geq 0$, there exists a unique Gorenstein singularity (C, p) of genus 1 with m branches. If $m = 1$ then (C, p) can be identified with the cusp $V(y^2 - x^3)$, if $m = 2$ then (C, p) can be identified with the ordinary tacnode $V(y^2 - yx^2)$, and if $m \geq 3$, then (C, p) is the germ at the origin of the union of m general lines through the origin in \mathbb{A}^{m-1} .*

2.2. Tropical curves. We follow standard conventions and definitions for tropical curves and tropical stable maps.

Definition 2.2.1. An n -marked tropical curve \square is a finite graph G with vertex and edge sets V and E , enhanced by

- (1) a marking function $m : \{1, \dots, n\} \rightarrow V$,
- (2) a genus function $g : V \rightarrow \mathbb{N}$,
- (3) a length function $\ell : E \rightarrow \mathbb{R}_+$.

The genus of a tropical curve \square is defined to be

$$g(\square) = h_1(G) + \sum_{v \in V} g(v)$$

where $h_1(G)$ is the first Betti number of the geometric realization of \square . An n -marked tropical curve is *stable* if (1) every genus 0 vertex has valence at least 3 and (2) every genus 1 vertex has valence at least 1.

More generally, one may permit the length function ℓ above to take values in an arbitrary toric monoid \mathcal{P} . This presents us with a natural notion of a family of tropical curves.

Definition 2.2.2. Let σ be a rational polyhedral cone with dual cone S_σ . A family of n -marked prestable tropical curves over σ is a tropical curve whose length function takes values in S_σ .

We note that given a tropical curve over σ , each point of σ determines a tropical curve in the usual sense. Indeed, choosing a point of σ is equivalent to choosing a monoid homomorphism

$$\varphi : S_\sigma \rightarrow \mathbb{R}_{\geq 0}.$$

Applying this homomorphism to the edge length $\ell(e) \in S_\sigma$ produces a real and positive length for each edge.

2.3. Logarithmic geometry: working definitions. Let N be a free abelian group of finite rank and X° be a subscheme of a torus $T = \mathbb{G}_m \otimes N$ over a field k equipped with the trivial valuation. Let K be a valued field extending k , with valuation surjective onto \mathbb{R} . Then, the tropicalization of X is the image of $X(K)$ under the coordinatewise valuation map

$$T(K) \rightarrow \mathbb{R} \otimes N.$$

This set is denoted X^{trop} , and can be given the structure of a fan. This fan distinguishes a partial compactification of T to a toric variety Y . The embedding of the closure $X \hookrightarrow Y$ determines, locally on X , a natural class of *monomial* functions obtained by restricting the monomials on T . These monomials form a sheaf of monoids M_X under multiplication, and a tautological map of monoids

$$\mathcal{O}_X^* \subset M_X \rightarrow \mathcal{O}_X.$$

The quotient is another sheaf of monoids $\overline{M}_X := M_X / \mathcal{O}_X^*$, and amounts to considering monomial functions up to scalars.

Sections of the groupification $\overline{M}_X^{\text{gp}}$ can be interpreted as piecewise linear functions on X^{trop} . Just as in the toric case, piecewise linear functions on X^{trop} give rise to line bundles on X . Specifically, given a piecewise linear function, the set of algebraic lifts of it in $\overline{M}_X^{\text{gp}}$ form a torsor under the multiplicative group, and therefore a line bundle. This is explained more precisely in [Section 2.6](#) below.

A logarithmic scheme is an object that possesses the main features present above. The requirement that X be embedded in a toric variety can be dropped. Instead, one need only assume that X (locally) admits a morphism to a toric stack. The data of the sheaf \overline{M}_X may be thought of as the sheaf of piecewise linear functions on X .

To be more precise, it is convenient to reverse the logical order and specify the monomials first. Given a scheme S , a logarithmic structure is a sheaf of monoids M_S in its étale topology and sharp homomorphism $\varepsilon : M_S \rightarrow \mathcal{O}_S$ (the codomain given its multiplicative monoid structure). Sharpness means that each local section of \mathcal{O}_S^* has a unique preimage along ε . The quotient $M_X/\varepsilon^{-1}\mathcal{O}_X^*$ is called the *characteristic monoid* and is denoted \overline{M}_X with its operation denoted *additively*; the image of section α of M_X^{gp} in $\overline{M}_X^{\text{gp}}$ is denoted $\bar{\alpha}$. We assume all logarithmic structures are integral (M_X is contained in its associated group M_X^{gp}) and saturated (if $\alpha \in M_X^{\text{gp}}$ and $n\alpha \in M_X$ for some integer $n \geq 1$ then $\alpha \in M_X$).

Such objects may be assembled into a category. The category of logarithmic schemes has the analogous constructions and notions from scheme theory, keeping track of the tropical data through the sheaves of piecewise linear functions.

For more of the general theory of logarithmic structures, we refer the reader to F. Kato’s original article [\[1989\]](#). A detailed study of the relationship between tropical and logarithmic geometry from a categorical point of view is undertaken in [\[Cavaliere et al. 2017\]](#).

2.4. Curves and logarithmic structures. Let (S, M_S) be a logarithmic scheme. A *family of logarithmically smooth curves over S* is a logarithmically smooth, flat, and proper morphism

$$\pi : (C, M_C) \rightarrow (S, M_S),$$

with connected and reduced geometric fibers of dimension 1. We recall Kato’s structure theorem for logarithmic curves [\[2000\]](#).

Theorem 2.4.1. *Let $C \rightarrow S$ be a family of logarithmically smooth curves. If $x \in C$ is a geometric point, then there is an étale neighborhood of C over S , with a strict morphism to an étale-local model $\pi : V \rightarrow S$, and $V \rightarrow S$ is one of the following:*

- (The smooth germ) $V = \mathbb{A}_S^1 \rightarrow S$, and the logarithmic structure on V is pulled back from the base.
- (The germ of a marked point) $V = \mathbb{A}_S^1 \rightarrow S$, with logarithmic structure pulled back from the toric logarithmic structure on \mathbb{A}^1 .
- (The node) $V = \mathcal{O}_S[x, y]/(xy = t)$, for $t \in \mathcal{O}_S$. The logarithmic structure on V is pulled back from the multiplication map $\mathbb{A}^2 \rightarrow \mathbb{A}^1$ of toric varieties along a morphism $t : S \rightarrow \mathbb{A}^1$ of logarithmic schemes.

The image of $t \in M_S$ in \overline{M}_S is referred to as the deformation parameter of the node.

Associated to a logarithmic curve $C \rightarrow S$ is a family of tropical curves.

Definition 2.4.2. Let $C \rightarrow S$ be a family of logarithmically smooth curves and assume that the underlying scheme of S is the spectrum of an algebraically closed field. Then, *the tropicalization* \bar{C} , denoted \square , is obtained as follows: (1) the underlying graph is the marked dual graph of C equipped with the standard genus and marking functions, and (2) given an edge e , the generalized length $\ell(e) = \delta_e \in \bar{M}_S$ is the deformation parameter of the corresponding node of C .

For more about logarithmic curves and their relationship to tropical curves, the reader may consult [Cavalieri et al. 2017].

2.5. Geometric interpretation of the sections of a logarithmic structure. Given a logarithmic curve $C \rightarrow S$, it will be helpful to interpret sections of the sheaves M_C^{gp} , and \bar{M}_C^{gp} geometrically.

2.5.1. The affine and projective lines. Let $(X, \varepsilon : M_X \rightarrow \mathcal{O}_X)$ be a logarithmic scheme. A section of M_X corresponds to a map $X \rightarrow \mathbb{A}^1$, the target given its toric logarithmic structure. Let α be such a section and $\bar{\alpha}$ be its image in \bar{M}_X . Then $\varepsilon(\alpha)$ is a unit if and only if $\bar{\alpha} = 0$.

With its logarithmic structure, \mathbb{P}^1 can be constructed as the quotient of $\mathbb{A}^2 - \{0\}$ by \mathbb{G}_m . Any map $X \rightarrow \mathbb{P}^1$ lifts locally to $\mathbb{A}^2 - \{0\}$ and can therefore be represented by a pair of sections (ξ, η) of M_X . The ratio $\xi^{-1}\eta$, which is a section of M_X^{gp} , is invariant under the action of \mathbb{G}_m , since \mathbb{G}_m acts with the same weight on ξ and η .

Therefore a map $X \rightarrow \mathbb{P}^1$ gives a well-defined section α of M_X^{gp} . Not every section of M_X^{gp} arises this way, because the map $(\xi, \eta) : X \rightarrow \mathbb{A}^2$ from which α was derived could not meet the origin. This condition implies that, for each geometric point x of X , either $\bar{\xi}_x = 0$ or $\bar{\eta}_x = 0$. In terms of $\bar{\alpha}$, this means that $\bar{\alpha}_x \geq 0$ or $\bar{\alpha}_x \leq 0$. We term this property being *locally comparable to 0*.

Our observations prove the following proposition:

Proposition 2.5.1.1. *Let X be a logarithmic scheme. Maps $X \rightarrow \mathbb{P}^1$, the latter given its toric logarithmic structure, may be identified with sections α of M_X^{gp} , whose images $\bar{\alpha}$ in \bar{M}_X^{gp} are locally comparable to 0.*

Because it has charts, the sheaf \bar{M}_X^{gp} locally admits a surjection from a constant sheaf, so the condition on $\bar{\alpha}$ in the proposition is open on the base: if X is a family of logarithmic schemes over S and a section α of \bar{M}_X^{gp} verifies $\bar{\alpha} \geq 0$ or $\bar{\alpha} \leq 0$ for all x in a geometric fiber X_s of X over S then it also verifies that condition for all t in some open neighborhood of s .

This observation is particularly useful for studying infinitesimal deformations of logarithmic maps to \mathbb{P}^1 , as it is equivalent to deform the section α of M_X^{gp} .

Definition 2.5.1.2. For any logarithmic scheme X , we define $\mathbb{G}_{\log}(X) = \Gamma(X, M_X^{\text{gp}})$. Identifying X with its functor of points, we also write $\mathbb{G}_{\log}(X) = \text{Hom}(X, \mathbb{G}_{\log})$.

Remark 2.5.1.3. The functor \mathbb{G}_{\log} is not representable by a logarithmic scheme; it is analogous to an algebraic space (see [Molcho and Wise 2018], for example). The above considerations may be seen as

a demonstration that \mathbb{P}^1 is logarithmically étale over \mathbb{G}_{\log} . A discussion of the simpler spaces rational curves in \mathbb{G}_{\log} may be found in [Ranganathan and Wise 2019].

We prefer to avoid a discussion of the geometric structure of \mathbb{G}_{\log} in this paper. The reader should feel free to regard maps to \mathbb{G}_{\log} as a convenient shorthand for sections of M_X^{gp} and nothing more. The advantage of treating \mathbb{G}_{\log} as a geometric object, and not merely an abstract sheaf, and working with that object instead of geometric models like \mathbb{P}^1 , is that the latter approach necessitates an apparently endless process of subdivision and refinement that obscures the geometric essence of our arguments.

2.5.2. Maps to toric varieties. The observations above concerning logarithmic maps to \mathbb{P}^1 may be extended to all toric varieties. Indeed, if $Z = \text{Spec } k[S_\sigma]$ is an affine toric variety defined by a cone σ and character lattice N^\vee , then there is a canonical map

$$S_\sigma \rightarrow \Gamma(Z, M_Z),$$

which extends to a map

$$N^\vee \rightarrow \Gamma(Z, M_Z^{\text{gp}}).$$

The construction of this map commutes with restriction to open torus invariant subvarieties, and therefore glues to a well-defined map on any toric variety.

Proposition 2.5.2.1. *Let X be a logarithmic scheme and let Z be a toric variety with fan Σ and character lattice N^\vee . Morphisms $X \rightarrow Z$ may be identified with morphisms $N^\vee \rightarrow \Gamma(X, M_X^{\text{gp}})$ such that, for each geometric point x of X , there is a cone $\sigma \in \Sigma$, such that the map*

$$S_\sigma \rightarrow \Gamma(X, M_X^{\text{gp}}) \rightarrow \Gamma(X, \overline{M}_X^{\text{gp}}) \rightarrow \overline{M}_{X,x}^{\text{gp}}$$

factors through $\overline{M}_{X,x}$.

Definition 2.5.2.2. Let N be a finitely generated free abelian group. We write $(N \otimes \mathbb{G}_{\log})(X) = \text{Hom}(N^\vee, \Gamma(X, M_X^{\text{gp}}))$ and use $\text{Hom}(X, N \otimes \mathbb{G}_{\log})$ for the same notion.

Remark 2.5.2.3. The discussion above shows that, if Z is a toric variety with cocharacter lattice N , then there is a canonical logarithmic modification $Z \rightarrow N \otimes \mathbb{G}_{\log}$.

2.5.3. Sections of the characteristic monoid. Since logarithmic maps $X \rightarrow \mathbb{A}^1$ correspond to sections of M_X , maps $X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ correspond to sections of $M_X/\mathcal{O}_X^* = \overline{M}_X$. The quotient $[\mathbb{A}^1/\mathbb{G}_m]$ is usually denoted \mathcal{A} and is called the *Artin fan* of \mathbb{A}^1 .

It is shown in [Cavaliere et al. 2017, Remark 7.3] that, if X is a logarithmic curve over S , and the underlying scheme of S is the spectrum of an algebraically closed field, then sections of \overline{M}_X (which is to say, maps $X \rightarrow \mathcal{A}$) may be interpreted as piecewise linear functions on the tropicalization of X that are valued in \overline{M}_S and are linear along the edges with integer slopes.

Similar reasoning, combined with the discussion in Section 2.5.1 shows that maps $X \rightarrow [\mathbb{P}^1/\mathbb{G}_m]$ correspond to sections α of $\overline{M}_X^{\text{gp}}$ that are locally comparable to 0. If X is a curve, then these sections are the piecewise linear functions on the tropicalization that are everywhere valued in \overline{M}_S or in $-\overline{M}_S$.

Remark 2.5.3.1. Even though its underlying “space” is an algebraic stack, $[\mathbb{A}^1/\mathbb{G}_m]$ represents a *functor* on logarithmic schemes. This contrasts with the more common situation, where algebraic stacks typically only represent categories fibered in groupoids over schemes.

2.6. Line bundles from piecewise linear functions. For any logarithmic scheme X , there is a short exact sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow M_X^{\text{gp}} \rightarrow \overline{M}_X^{\text{gp}} \rightarrow 0$$

of the sheaves associated to the logarithmic structure. Given a section $\alpha \in \Gamma(X, \overline{M}_X^{\text{gp}})$, the image of α under the coboundary map

$$H^0(X, \overline{M}_X^{\text{gp}}) \rightarrow H^1(X, \mathcal{O}_X^*)$$

is represented by an \mathcal{O}_X^* -torsor $\mathcal{O}_X^*(-\alpha)$ on X and gives rise to an associated line bundle. Thus, to each piecewise linear function f on \square that is linear on the edges with integer slopes and takes values in \overline{M}_S , we have an associated line bundle $\mathcal{O}(-f)$.

The explicit line bundle obtained by this construction is recorded in [Ranganathan et al. 2017, Section 2].

2.7. Tropicalization of morphisms to toric varieties. Let Z be a toric variety with dense torus T , equipped with its standard logarithmic structure, and let N and N^\vee be the cocharacter and character lattices of Z .

Let C be a logarithmic curve over S , and assume that the underlying scheme of S is the spectrum of an algebraically closed field. A logarithmic map $\varphi : C \rightarrow Z$ induces a map

$$N^\vee \rightarrow \Gamma(Z, \overline{M}_Z) \rightarrow \Gamma(C, \overline{M}_C) \tag{5}$$

by the discussion in Section 2.5.2.

As remarked in Section 2.5, the sections of \overline{M}_C are piecewise linear functions on the tropicalization \square of C that are linear with integer slopes along the edges and are valued in $\overline{M}_S^{\text{gp}}$. If we assume in addition that $\overline{M}_S = \mathbb{R}_{\geq 0}$ then we obtain a piecewise linear map

$$\square \rightarrow \text{Hom}(N^\vee, \mathbb{R}) = N_{\mathbb{R}}$$

that we call the *tropicalization* of $C \rightarrow Z$. It will sometimes be convenient to think of this as a map from $\square \rightarrow \Sigma$, where Σ is the fan of Z .

Lemma 2.7.1. *The map $\square \rightarrow N_{\mathbb{R}}$, constructed above, satisfies the balancing condition.*

Proof. This is proved in [Gross and Siebert 2013, Section 1.4]. □

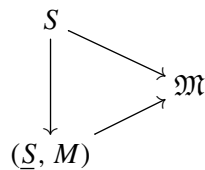
2.8. Minimality.

2.8.1. Minimal logarithmic structures. A crucial concept in the theory of logarithmic moduli problems is that of *minimality*. Let LogSch denote the category of fine and saturated logarithmic schemes. Given a moduli stack \mathfrak{M} over LogSch and a logarithmic scheme S , the fiber $\mathfrak{M}(S)$ of the fibered category \mathfrak{M} over S is the groupoid logarithmic geometric objects $[\mathcal{X} \rightarrow S]$ defined over S , as specified by the moduli problem.

Logarithmic geometric objects are algebraic schemes or stacks with the additional structure of a sheaf of monoids. The description of \mathfrak{M} as a category fibered in groupoids over LogSch does not furnish such an object: if \underline{S} is a scheme without a chosen logarithmic structure, it does not make mathematical sense to consider the fiber of \mathfrak{M} over \underline{S} . Said differently, there is no “underlying scheme, or underlying stack, or underlying category fibered in groupoids over schemes” of \mathfrak{M} .

The difficulty that must be overcome is that given an ordinary scheme \underline{S} , there are many choices for logarithmic schemes (S, M_S) enhancing \underline{S} , and it is unclear which one to pick. The notion of minimality, introduced by Kato and recently clarified and expanded [Abramovich and Chen 2014; Chen 2014; Gillam 2012; Gross and Siebert 2013; Wise 2016a; 2016b] identifies the correct logarithmic structures to allow on \underline{S} as those satisfying a universal property, recalled below.

Assuming that \mathfrak{M} does have an underlying scheme, we arrive at a *necessary* condition for \mathfrak{M} to be representable by a logarithmic scheme. Suppose that $S \rightarrow \mathfrak{M}$ is a morphism of logarithmic schemes then the logarithmic structure of \mathfrak{M} pulls back to a logarithmic structure M on the underlying scheme \underline{S} of S . Moreover there is a factorization



that is *final* among all such factorizations. This finality condition can be phrased entirely in terms of the moduli problem defining \mathfrak{M} , and Gillam shows that if minimal factorizations exist for all $S \rightarrow \mathfrak{M}$, and are preserved by base change, then \mathfrak{M} comes from a logarithmic structure on a moduli problem over schemes [Gillam 2012; Wise 2016a].

Theorem 2.8.1.1 (Gillam). *When \mathfrak{M} is a category fibered in groupoids over logarithmic schemes that comes from a logarithmic structure on a category fibered in groupoids \mathfrak{N} over schemes, \mathfrak{N} can be recovered from \mathfrak{M} as the subcategory of minimal objects.*

Throughout this paper, we present logarithmic moduli problems and indicate monoidal and tropical (see Section 2.8.2) characterizations of their minimal objects to recover the underlying schematic moduli problems.

2.8.2. Minimality as tropical representability. We explain the concept in the case of stable maps for concreteness, where it becomes a tropical concept. This expands on [Gross and Siebert 2013, Remark 1.21].

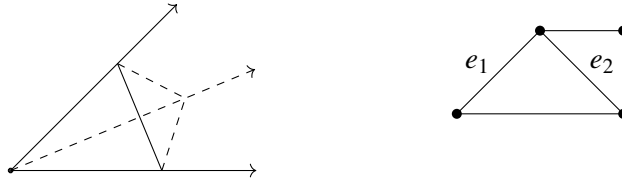


Figure 1. Consider the cone of tropical curves whose underlying graph is shown on the right, such that the edge lengths of e_1 and e_2 are equal. This cone is 3-dimensional. An associated family of logarithmic curves whose minimal monoid is dual to this cone associated family of logarithmic curves is nonminimal, due to the relation that these two edge lengths coincide.

Let $\mathfrak{M}_{g,n}(Z)$ denote stack over LogSch parametrizing logarithmic maps from genus g , n -pointed curves to a toroidal scheme Z . Let Σ be the fan of Z .

Let S be a standard logarithmic point $\text{Spec}(\mathbb{N} \rightarrow \mathbb{C})$ and let $[C \rightarrow Z]$ be a logarithmic map over S . As explained in [Section 2.7](#), the morphisms on sheaves of monoids may be dualized to produce a tropical map

$$\square \rightarrow \Sigma.$$

Replacing \mathbb{N} with an arbitrary toric monoid, one obtains a *family* of tropical maps.

From our discussion of minimality, we see that given a logarithmic stable map over $\text{Spec}(P \rightarrow \mathbb{C})$, the monoid be cannot be arbitrary, since by pulling back via a morphism $P \rightarrow \mathbb{R}_{\geq 0}$, we must obtain a tropical map. With this observation, there is a clear choice for a universal P^{\min} such that all other enhancements $\text{Spec}(P \rightarrow \mathbb{C})$ of the same underlying map must be pulled back from $\text{Spec}(P^{\min} \rightarrow \mathbb{C})$. That is, we may choose P^{\min} to be the monoid whose dual cone $\text{Hom}(P^{\min}, \mathbb{R}_{\geq 0})$ is the cone of *all* tropical maps of the given combinatorial type. Succinctly, a logarithmic structure is minimal for a given moduli problem if it represents the *tropical deformation space*.

In [Figure 1](#) below, taking $Z = pt$, we depict the duals of the characteristic monoid on the base of a nonminimal family. If one drops the condition that ℓ_1 and ℓ_2 coincide, we obtain the corresponding minimal monoid.

Applying this reasoning at each geometric fiber gives a criterion to check whether any given family of logarithmic maps $C \rightarrow Z$ over a logarithmic scheme S is minimal. With the minimal objects identified, we construct a moduli stack as a fibered category over Sch, whose fiber over a scheme \underline{S} is the groupoid of *minimal* logarithmic maps over S .

2.9. Preliminaries from the prequel: radial alignments. The results of this paper rely on the notion of a radially aligned logarithmic curve and its canonical contraction to a curve with elliptic singularities. These concepts were developed in the companion article [[Ranganathan et al. 2017](#)], and we briefly recall the statements that we require.

Let S be a logarithmic scheme enhancing the spectrum of an algebraically closed field and let $C \rightarrow S$ be a logarithmic curve over it whose fibers have genus 1 and let \square be its tropicalization. Given an edge e , we write $\ell(e) \in \overline{M}_S$ for the generalized edge length of this edge. For each vertex $v \in \square$, there is a unique path from the circuit of \square , namely the smallest subgraph of genus 1, to the chosen vertex v . Write this path as e_1, \dots, e_n . Define

$$\lambda(v) = \sum_{i=1}^n \ell(e_i).$$

The resulting function λ is a piecewise linear function on \square with integer slopes, and thus, a global section of \overline{M}_C . When S is a general logarithmic scheme and $\pi : C \rightarrow S$ a curve, this section glues along specialization morphisms to give rise to a well-defined and canonical global section in $\Gamma(S, \pi_* \overline{M}_S)$.

Given a logarithmic curve $C \rightarrow S$ and a geometric point $s \in S$, we let \square_s denote the corresponding tropical curve associated to C_s . Recall also that we view a monoid P as being the positive elements in a partially ordered group, with the partial order defined by $a \geq b$ if $a - b \in P \subset P^{\text{gp}}$.

Definition 2.9.1. We say that a logarithmic curve $C \rightarrow S$ is *radially aligned* if $\lambda(v)$ and $\lambda(w)$ are comparable for all geometric points s of S and all vertices $v, w \in \square_s$.

We write $\mathfrak{M}_{1,n}^{\text{rad}}$ for the category fibered in groupoids over logarithmic schemes whose fiber over S is the groupoid of radially aligned logarithmic curves over S having arithmetic genus 1 and n marked points.

The following result is proved in [Ranganathan et al. 2017, Section 3].

Theorem 2.9.2. *The category of radially aligned, prestable, logarithmic curves of genus 1 with n marked points is represented by an algebraic stack with logarithmic structure $\mathfrak{M}_{1,n}^{\text{rad}}$. The natural map*

$$\mathfrak{M}_{1,n}^{\text{rad}} \rightarrow \mathfrak{M}_{1,n},$$

is a logarithmic blowup.

The second major construction in [loc. cit.] is the construction of a contraction to a curve with elliptic singularities, from the data of a radially aligned curve with a chosen “radius of contraction”. Let $C \rightarrow S$ be a radially aligned logarithmic curve of genus 1. We say that a section $\delta \in \overline{M}_S$ is *comparable to the radii of C* if for each geometric point $s \in S$, the section δ is comparable to $\lambda(v)$ for all vertices $v \in \square_s$, in the monoid \overline{M}_S .

Theorem 2.9.3. *Let $C \rightarrow S$ be a radially aligned logarithmic curve and $\delta \in \overline{M}_S$ a section comparable to the radii of C . Then, there exists a partial destabilization*

$$\tilde{C} \rightarrow C,$$

and a contraction

$$\tilde{C} \rightarrow \bar{C},$$

where $\bar{C} \rightarrow S$ is a family genus 1 curves at worst Gorenstein genus 1 singularities, such that, for every geometric point s of S , if E is a component of C_s such that $\lambda(E) < \delta_s$ then E is contracted to a point in \bar{C} .

An intuitive discussion of these concepts are presented in [Ranganathan et al. 2017, Section 3.1]. For working knowledge, reader may visualize the section δ as giving rise to a *circle of radius δ* around the circuit of the tropical curve \square . By subdividing the edges of \square , one may produce a new tropical curve $\tilde{\square}$ such that every point of $\tilde{\square}$ at radius δ from the circuit is a vertex. This introduces valency 2 vertices into the tropicalization, and induces the partial destabilization. By contracting the interior of the circle of radius δ in a versal family, one produces a curve with a Gorenstein singularity.

3. Logarithmic maps to toric varieties

We construct the space of radially aligned logarithmic maps to a toric variety. The framework of radial alignments, together with the well-spacedness condition from tropical geometry, will lead to a proof of [Theorem B](#), which is the main result of this section. The symbol Z will denote a proper toric variety with fan Σ .

Recall that a morphism of polyhedral complex $\mathcal{P} \rightarrow \mathcal{Q}$ is a continuous map of the underlying topological spaces sending every polyhedron of \mathcal{P} linearly to a polyhedron of \mathcal{Q} .

Definition 3.1. A *tropical prestable map* or *tropical map* for short, is a morphism of polyhedral complexes

$$F : \square \rightarrow \Sigma$$

where \square is an n -marked tropical curve and the following conditions are satisfied:

- (1) For each edge $e \in \square$, the direction of $F(e)$ is an integral vector. When restricted to e , the map has integral slope w_e , taken with respect to this integral direction. This integral slope is referred to as the *expansion factor* of F along e . The expansion factor and primitive edge direction are together referred to as the *contact order* of the edge.
- (2) The map f is *balanced*: at all points of \square the sum of the directional derivatives of F in each tangent direction is zero.

The map is *stable* if it satisfies the following condition: if $p \in \square$ has valence 2, then the image of $\text{Star}(v)$ is not contained in the relative interior of a single cone of Σ .

Following [Section 2.7](#), given a logarithmic prestable map to a toric variety

$$\begin{array}{ccc} (C, M_C) & \xrightarrow{f} & Z \\ \downarrow & & \\ (S, M_S), & & \end{array}$$

there is an associated family \square of tropical curves together with a map $[F : \square \rightarrow \Sigma]$, satisfying the axioms of a tropical prestable map.

3.2. Radial logarithmic maps. We begin with a construction of the stack of radially aligned logarithmic maps.

Proposition 3.2.1. *Let Z be a toric variety. There is an algebraic stack with logarithmic structure, $\mathfrak{W}(Z)$, parametrizing families of radially aligned curves C and logarithmic morphisms $C \rightarrow Z$.*

The underlying algebraic stack of $\mathfrak{W}(Z)$ is locally quasifinite over the stack of ordinary prestable maps from radially aligned curves to Z , and its restriction to the open and closed substack of maps with fixed contact orders is quasifinite.

Proof. Let $\mathfrak{M}_{1,n}^{\text{rad}}$ be the stack of radially aligned, n -marked, genus 1 logarithmic curves (Section 2.9) and let C be its universal curve. Then $\mathfrak{W}(Z)$ is the space of logarithmic prestable maps from C to Z , and this is representable by an algebraic stack with a logarithmic structure [Wise 2016a, Corollary 1.1.1]. The local quasifiniteness is a consequence of [Wise 2016b, Theorem 1.1] or [Ranganathan 2017a, Proposition 3.6.3]; under the assumption of fixed contact orders, the combinatorial types of a map $[C \rightarrow Z]$ are bounded [Gross and Siebert 2013, Theorem 3.8], and therefore $\mathfrak{W}(Z)$ is quasifinite over the space of maps of underlying schemes. \square

Stability in $\mathfrak{W}(Z)$ is defined in terms of the underlying schematic map:

Definition 3.2.2. A radial map $[f : C \rightarrow Z]$ in $\mathfrak{W}(Z)$ over $\text{Spec}(\mathbb{C})$ is said to be *stable* if it satisfies the following conditions:

- (1) If $D \subset C$ is an irreducible component of genus 0 contracted by f , then D supports at least 3 special points.
- (2) If C is a smooth curve of genus 1, then C is not contracted.

A family of ordered logarithmic maps is stable if each geometric fiber is stable.

3.2.3. Minimal monoids. We give a tropical description of the logarithmic structure of $\mathfrak{W}(Z)$. We leave it to the reader to verify that this description is correct, either using [Wise 2016a, Appendix C.3] or adapting the arguments from [Chen 2014, Section 3] or [Gross and Siebert 2013, Proposition 1.22].

The minimality condition may be checked on geometric fibers, so we assume that the underlying scheme of S is $\text{Spec } C$. Let σ_S be the corresponding dual cone $\text{Hom}(\overline{M}_S, \mathbb{R}_{\geq 0})$ of the characteristic monoid of S . By forgetting the alignment, a radial map $[f]$ above produces a usual logarithmic map with combinatorial type Θ . Letting σ_Θ be the associated cone of tropical maps, we have a morphism of cones

$$\sigma_S \rightarrow \sigma_\Theta.$$

In the tropical moduli cone σ_Θ above, the locus of tropical curves whose vertices are ordered in the same manner as C forms a cone $\sigma(f)$.

Definition 3.2.4. Let $f : C \rightarrow Z$ be a family of ordered logarithmic maps over a logarithmic base S . The map $[f]$ is a *minimal* ordered logarithmic map if for each geometric point $\bar{s} \in S$, there is an isomorphism of cones

$$\text{Hom}(\overline{M}_{S,\bar{s}}, \mathbb{R}_{\geq 0}) \cong \sigma(f_{\bar{s}}).$$

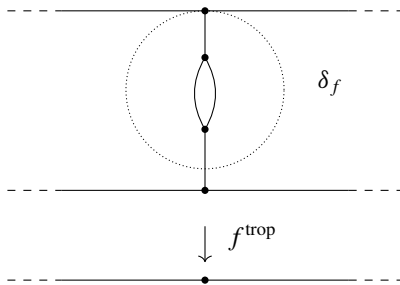


Figure 2. A tropical map from a genus 1 curve to $\Sigma_{\mathbb{P}^1}$ contracting a circuit. The dotted circle corresponds to the circle whose radius is the minimal distance to a vertex supporting a noncontracted flag.

3.3. The factorization property. To detect the curves that smooth to the main component, we will need to identify certain contractions of the source curve constructed from the tropical maps and use the methods developed in [Ranganathan et al. 2017].

Let \bar{C} be a Gorenstein curve of arithmetic genus 1. We will refer to E , the smallest connected subcurve of C of arithmetic genus 1, as the *circuit component* of C . Given a family $C \rightarrow S$, we give the nodes and markings the standard logarithmic structure, and we give C the trivial logarithmic structure near any genus 1 singularities.

Given an aligned logarithmic curve C of genus 1 and a contraction $C \rightarrow \bar{C}$, we may equip \bar{C} with the logarithmic structure defined above. This enhances $C \rightarrow \bar{C}$ to a logarithmic morphism.

Let $(C, M_C) \rightarrow (S, M_S)$ be a radially aligned logarithmic curve and let Z be a toric variety with cocharacter lattice N . We associate a section $\delta_f \in \bar{M}_S$ to a logarithmic map $f : C \rightarrow Z$ over S . Let \square be the tropicalization of the curve C with circuit \square_0 . Consider the associated family of tropical maps

$$\varphi : \square \rightarrow N_{\mathbb{R}}$$

If φ does not contract the circuit, then let $\delta_f = 0$. Otherwise, let δ_f be the minimum distance from \square_0 to a 1-valent vertex of $\varphi^{-1}(\varphi(\square_0))$. That is, the distance to the closest vertex supporting a flag that is not contracted by φ . See Figure 2 for an example.

These data will produce a rational bubbling of the source curve and a contraction thereof. First, subdivide \square such that every edge of $\varphi^{-1}(\varphi(\square_0))$ terminates at a vertex; in Figure 2, this amounts to introducing a vertex where the dotted circle crosses the lower vertical edge. This induces a logarithmic modification

$$\tau : \tilde{C} \rightarrow C.$$

By the constructions of Section 2.9, we obtain a section comparable to the radii of C , and there is now an induced contraction

$$\gamma : \tilde{C} \rightarrow \bar{C},$$

to a curve with a Gorenstein elliptic singularity.

Definition 3.3.1. Keeping the notations above, an ordered logarithmic map $C \rightarrow Z$ as above is said to have the *factorization property* if the associated map $\tilde{C} \rightarrow C \rightarrow Z$ factorizes as

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{f} & Z \\ & \searrow & \nearrow \bar{f} \\ & \bar{C} & \end{array}$$

Remark 3.3.2. Note that it need not be the case that the map \bar{f} is nonconstant on a branch of the elliptic singularity. This is because the map $\bar{C} \rightarrow Z$ may have highly degenerate contact order with the boundary of Z ; it could be that the entire elliptic component is contracted. However, one may always replace Z with a logarithmic modification \mathcal{Z} , i.e., a toric degeneration, such that the genus 1 component maps to the dense torus of one of the components of \mathcal{Z} . In such an expansion, there will be at least one branch of the singularity along which the map is nonconstant. For example, if $Z = \mathbb{P}^1$, it could be that the genus 1 subcurve is contracted to one of the relative points 0 or ∞ on \mathbb{P}^1 . In this case, one must first expand \mathbb{P}^1 to a chain $\mathbb{P}^1_{\text{exp}}$, until the curve maps to the dense torus of a component in $\mathbb{P}^1_{\text{exp}}$. The choice of radius forces that the factorization is nonconstant on a branch of the singularity.

Recall from Section 2.5.2 that a map $f : C \rightarrow Z$, where Z is a toric variety with character lattice N^\vee , induces a homomorphism

$$\alpha : N^\vee \rightarrow \Gamma(C, M_C^{\text{gp}}).$$

The factorization property depends only on α and not specifically on the morphism of toric varieties $C \rightarrow Z$. For example, if Z were a toric modification of another toric variety Z' , then the factorization properties for $C \rightarrow Z$ and $C \rightarrow Z'$ would coincide. We offer a definition of the factorization property that makes this independence explicit.

Definition 3.3.3. Let N and N^\vee be finitely generated, free abelian groups, dual to each other, and let C be a logarithmic curve over S . Assume given $\alpha : N^\vee \rightarrow \Gamma(C, M_C^{\text{gp}})$ and let $\bar{\alpha}$ be the induced morphism valued in $\Gamma(C, \bar{M}_C^{\text{gp}})$. Let \square_s be the tropicalization of C_s , for each geometric point s of S . Define $\delta_\alpha \in \Gamma(S, \bar{M}_S)$ fiberwise to be the largest $\lambda(v)$, among $v \in \square$, such that $\bar{\alpha}$ is constant when viewed as a piecewise linear function on \square .

Let $\nu : \tilde{C} \rightarrow C$ and $\tau : \tilde{C} \rightarrow \bar{C}$ be the destabilization and contraction constructed as above. We say that α satisfies the *factorization property* if $\nu^*\alpha$ descends along τ to $N^\vee \rightarrow \Gamma(\bar{C}, M_{\bar{C}}^{\text{gp}})$.

Remark 3.3.4. The factorization property is equivalent to requiring that $\tilde{C} \rightarrow C \rightarrow N \otimes \mathbb{G}_m^{\text{log}}$ factor through the contracted curve \bar{C} .

3.4. The stack of well-spaced logarithmic maps. In this section, we construct a stack that we will later identify as the main component of the moduli space of genus 1 maps to a toric variety.

We begin with some geometric motivation. Let H be a subtorus of the dense torus T of Z . After replacing Z with a toric modification, there is a toric compactification Z_H of the quotient torus T/H and

a toric morphism

$$Z \rightarrow Z_H,$$

extending the projection $T \rightarrow T/H$.

Let $f : C \rightarrow Z$ be a radial map over S , let H a subtorus of the dense torus T , and assume that $Z \rightarrow Z_H$ exists for some T/H -toric variety Z_H . We say that $[f]$ satisfies the *factorization property for H* if the induced logarithmic map

$$C \rightarrow Z \rightarrow Z_H$$

satisfies the factorization property.

This definition cannot be applied to an arbitrary toric variety Z and an arbitrary subtorus $H \subset T$, since there may not be a toric map from Z to an equivariant compactification of T/H . For example, consider $Z = \mathbb{P}^2$ and let H be any 1-dimensional subtorus. Since there is no nonconstant map from \mathbb{P}^2 to \mathbb{P}^1 , the assumption fails.

There are two ways in which to overcome the issue. The first is to replace Z with a logarithmic modification, which requires replacing C with a logarithmic modification. This logarithmic modification may not be defined over the base S , until we perform a logarithmic modification of S as well [Abramovich et al. 2017b, Proposition 4.5.2].

It is conceptually simpler to use Definition 3.3.3, which does not require the map $Z \rightarrow Z_H$, but only the map of tori $T \rightarrow T/H$. Indeed, let N^\vee be the character lattice of T and let $N_{T/H}^\vee$ be the character lattice of T/H . Then the factorization property for $C \rightarrow Z \rightarrow Z_H$ is equivalent to the factorization property for the composition

$$N_{T/H}^\vee \rightarrow N^\vee \rightarrow \Gamma(C, M_C^{\text{gp}}).$$

With this as motivation, we arrive at our definition:

Definition 3.4.1. Let $f : C \rightarrow Z$ be a map from a radially aligned logarithmic curve to a toric variety Z with dense torus T and character lattice N^\vee . Let H be a subtorus of T and let $N_{T/H}^\vee$ be the character lattice of T/H . We say that f satisfies the *factorization property for H* if the map

$$N_{T/H}^\vee \rightarrow N^\vee \rightarrow \Gamma(C, M_C^{\text{gp}}).$$

satisfies the factorization property of Definition 3.3.3.

Geometrically, the condition is that $C \rightarrow N \otimes \mathbb{G}_{\log} \rightarrow N_{T/H} \otimes \mathbb{G}_{\log}$ should factor through \bar{C}_H , where \bar{C}_H is constructed from $C \rightarrow N_{T/H} \otimes \mathbb{G}_{\log}$ as in Section 3.3. The equivalence between the formulations is a tautology: \mathbb{G}_{\log} may simply be defined as the representing objects for global sections of M_X^{gp} .

Definition 3.4.2. Let Z be a toric variety. A radial logarithmic map $f : C \rightarrow Z$ is *well-spaced* if f satisfies the factorization property for all subtori H of T .

Let $\mathcal{W}(Z)$ denote the category fibered in groupoids, over logarithmic schemes, of stable, well-spaced, radially aligned logarithmic stable maps to Z .

Given a splitting $N_{T/H}^\vee \simeq \mathbb{Z}^r$, the factorization property for H is the conjunction of the factorization properties with respect to the tori dual to the direct summands of \mathbb{Z}^r . When $N_{T/H}^\vee$ has rank 1, the factorization property asserts that a section of $M_{\tilde{C}}^{\text{gp}}$ descends to a section of M_C^{gp} . It is in this form that we will verify the algebraicity of the factorization property, below.

Let C be a family of radially aligned logarithmic curves over S , assume that a section $\delta \in \Gamma(S, \overline{M}_S)$ is given, that $\tilde{C} \rightarrow C$ and $\tau : \tilde{C} \rightarrow \overline{C}$ are the associated semistable model and contraction of genus 1 component, respectively, and that $E \subset \tilde{C}$ is the exceptional locus of τ . We write $\pi : C \rightarrow S$ and $\overline{\pi} : \overline{C} \rightarrow S$ for the projections. We fix a section α of M_C^{gp} . We will use F for the subfunctor of the functor represented by S on logarithmic schemes consisting of those $f : T \rightarrow S$ such that the restriction of α along f has the factorization property — in other words, the pullback of α to \tilde{C}_T descends along $\tilde{C}_T \rightarrow \overline{C}_T$.

In the following statements, we will say that the factorization property has a certain trait to mean that the functor F has that trait, relative to the functor represented by S . To unclutter the notation slightly, we will also assume without loss of generality, that $\tilde{C} \rightarrow C$ is an isomorphism, since doing so entails no loss of generality.

We begin by showing that the moduli functor of factorized maps is representable.

Theorem 3.4.3. *The factorization property is representable by a (not necessarily strict) closed embedding of logarithmic schemes.*

The proof proceeds as in Artin’s criteria for algebraicity, although we do not need to invoke Artin’s criteria directly because the morphism in question will turn out to be a closed embedding.

The following two propositions refine [Ranganathan et al. 2017, Theorem 4.3], with essentially the same proof:

Proposition 3.4.4. *The set of points of S where α satisfies the factorization property is constructible.*

Proof. We wish to show that the locus of points in S where the section α of M_C^{gp} descends to M_C^{gp} is constructible in S .

This assertion is local in the constructible topology on S , so we may assume that the dual graph of C is locally constant over S . The assertion is also local in the étale topology, so we can even assume the dual graph is constant. Then there are two obstructions to descending α to a section of M_C^{gp} . Since M_C^{gp} is pulled back from S near $\tau(E)$, the first obstruction is that $\bar{\alpha}$ should be constant on E . But $\pi_* \overline{M}_C^{\text{gp}}$ is constant over S , so this holds on an open and closed subscheme of S . In particular, this locus is quasicompact and constructible. We may now restrict attention to this locus and assume the first obstruction vanishes.

Now α is a section of $\pi^* M_S^{\text{gp}} \subset M_C^{\text{gp}}$ near E . Since $\bar{\alpha}$ is constant on E by assumption, we may, after localization in S , divide off a section pulled back from S , to ensure that $\bar{\alpha}(E) = 0$ and α is therefore a section of \mathcal{O}_C^* . We must show that the locus in S where α is pulled back from \mathcal{O}_C^* is constructible.

Regarding α instead as a section of \mathcal{O}_C , it is equivalent to show that the locus where α is pulled back from $\mathcal{O}_{\tilde{C}}$ is constructible. The rest of the proof is now the same as in [Ranganathan et al. 2017, Theorem 4.3]. □

Proposition 3.4.5. *The factorization property satisfies the valuative criterion for properness.*

Proof. We assume that S is the spectrum of a valuation ring, with $j : \eta \rightarrow S$ the inclusion of the generic point. We wish to show that if α is a section of M_C^{gp} such that $j^*\alpha$ is pulled back from a section of $M_{\bar{C}_\eta}$, then α is pulled back from a section of $M_{\bar{C}}$ over \bar{C} .

It suffices to assume that S has the maximal logarithmic structure extending the logarithmic structure over η . That is, $M_S = \mathcal{O}_S \times_{j_*\mathcal{O}_\eta} j_*M_\eta$. This implies that $M_S^{\text{gp}} = j_*M_\eta^{\text{gp}}$.

Our task is equivalent to showing that the square (6) is cartesian:

$$\begin{CD} M_C^{\text{gp}} @>>> j_*M_{C_\eta}^{\text{gp}} \\ @VVV @VVV \\ \tau_*M_C^{\text{gp}} @>>> j_*\tau_*M_{C_\eta}^{\text{gp}} \end{CD} \tag{6}$$

Away from $\tau(E)$, we know that φ is an isomorphism, so it suffices to demonstrate the bijectivity of φ near $\tau(E)$, where $M_{\bar{C}} = \bar{\pi}^*M_S$. Since the map $M_C^{\text{gp}} = \bar{\pi}^*M_S^{\text{gp}} \rightarrow \tau_*M_C^{\text{gp}}$ factors through $\tau_*\bar{\pi}^*M_S^{\text{gp}}$, it suffices to show that (7) is cartesian:

$$\begin{CD} \bar{\pi}^*M_S^{\text{gp}} @>>> j_*\bar{\pi}^*M_S^{\text{gp}} \\ @VVV @VVV \\ \tau_*\bar{\pi}^*M_S^{\text{gp}} @>>> j_*\tau_*\bar{\pi}^*M_S^{\text{gp}} \end{CD} \tag{7}$$

This reduces to showing that both of the following two squares are cartesian:

$$\begin{CD} \bar{\pi}^*\bar{M}_S^{\text{gp}} @>>> j_*\bar{\pi}^*\bar{M}_S^{\text{gp}} \\ @VVV @VVV \\ \tau_*\bar{\pi}^*\bar{M}_S^{\text{gp}} @>>> j_*\tau_*\bar{\pi}^*\bar{M}_S^{\text{gp}} \end{CD} \quad \begin{CD} \mathcal{O}_{\bar{C}} @>>> j_*\mathcal{O}_{\bar{C}_\eta} \\ @VVV @VVV \\ \tau_*\mathcal{O}_{\bar{C}} @>>> j_*\tau_*\mathcal{O}_{\bar{C}_\eta} \end{CD} \tag{8}$$

We can check that the first square is cartesian on fibers, by proper base change for étale pushforward. In that situation it is immediate, because $\bar{\pi}^*\bar{M}_S^{\text{gp}}$ is constant on the fibers over S and the fibers of τ are connected. To see that the second square is cartesian, it is sufficient to see that (9) is cartesian:

$$\begin{CD} \mathcal{O}_{\bar{C}} @>>> j_*\mathcal{O}_{\bar{C}_\eta} \\ @VVV @VVV \\ \tau_*\mathcal{O}_{\bar{C}} @>>> j_*\tau_*\mathcal{O}_{\bar{C}_\eta} \end{CD} \tag{9}$$

The rest of the proof is exactly the same as the end of the proof of [Ranganathan et al. 2017, Theorem 4.3]. □

Propositions 3.4.4 and 3.4.5 combine to imply that the locus in S where $\alpha \in M_C^{\text{gp}}$ satisfies the factorization property is a closed subset of S . To give that closed subset a scheme structure, we have two

more propositions, the first of which requires the notion of a homogeneous functor, see [SGA 7_I 1972, Definition 2.5; Wise 2011, Section 2].

Definition 3.4.6. Let F be a category fibered in groupoids over schemes. We say F is homogeneous if the map

$$F(S') \rightarrow F(S) \times_{F(T)} F(T')$$

is an equivalence whenever $S' = S \sqcup_T T'$ is the pushout of an infinitesimal extension $T \subset T'$ and an affine morphism $T \rightarrow S$.

Proposition 3.4.7. *The factorization property is homogeneous.*

Proof. Let $\tau : C' \rightarrow \bar{C}'$ be a contraction of genus 1 components over S' , where $S = T' \sqcup_T S$ for a strict infinitesimal extension $T \subset T'$ and a strict affine morphism $\rho : T \rightarrow S$. We must show that if $\alpha' \in \Gamma(C', M_{C'}^{\text{gp}})$ and its restrictions β' to T' and α to S satisfy the factorization property, then so does α' . This is in fact immediate in view of (10):

$$M_{S'}^{\text{gp}} = \rho_* M_{T'}^{\text{gp}} \times_{\rho_* M_T^{\text{gp}}} M_S^{\text{gp}} \tag{10}$$

Indeed, this assertion holds trivially on characteristic monoids, since $\bar{M}_{T'}^{\text{gp}} \rightarrow \bar{M}_T^{\text{gp}}$ is an isomorphism, so it comes down to the identification (11),

$$\mathcal{O}_{S'} = \rho_* \mathcal{O}_{T'} \times_{\rho_* \mathcal{O}_T} \mathcal{O}_S \tag{11}$$

which is the definition of S' . □

Proposition 3.4.8. *The factorization property holds over a complete noetherian local ring if and only if it holds formally.*

Proof. Here we must show that if S is the spectrum of a complete noetherian local ring with maximal ideal m and S_i is the vanishing locus of m^{i+1} then $\alpha \in \Gamma(C, M_S^{\text{gp}})$ satisfies the factorization property if and only if its restriction to S_i satisfies the factorization property for every i . It is certainly the case that if $\bar{\alpha} \in \Gamma(C, \bar{M}_S^{\text{gp}})$ is pulled back from $\Gamma(\bar{C}, \bar{M}_{\bar{C}}^{\text{gp}})$ modulo every (in fact, any) power of m then so does $\bar{\alpha}$. Indeed, this claim amounts to the assertion that if $\bar{\alpha}$ is constant on $E \cap \pi^{-1}(S_0)$ then $\bar{\alpha}$ is constant on E . But \bar{M}_C is an étale sheaf on C , so if $\bar{\alpha}$ vanishes at a point then it vanishes in an open neighborhood of that point. Since the only open subset of E containing $E \cap \pi^{-1}(S_0)$ is E itself, we conclude that $\bar{\alpha}$ descends to \bar{C} if $\bar{\alpha}|_{S_0}$ descends to $\bar{\pi}^{-1}(S_0)$.

Dividing α by a section of M_S^{gp} , we can assume that $\bar{\alpha} = 0$ on E and therefore that α is a section of \mathcal{O}_C^* near E . We can find an open subsets $\bar{U} \subset \bar{C}$ covering $\tau(E)$ such that $\bar{\alpha}$ vanishes on $U = \tau^{-1}\bar{U}$. Then, by assumption, we have $\tau_* \alpha|_{\bar{U}} \in \mathcal{O}_{\bar{U}}/m^{i+1} \mathcal{O}_{\bar{U}} \subset \tau_*(\mathcal{O}_U/m^{i+1} \mathcal{O}_U)$. Passing to the limit, it follows that the image of $\tau_* \alpha$ along $\tau_* \mathcal{O}_U \rightarrow \varprojlim \tau_*(\mathcal{O}_U/m^{i+1} \mathcal{O}_U)$ lies in $\varprojlim \mathcal{O}_{\bar{U}}/m^{i+1} \mathcal{O}_{\bar{U}} = \mathcal{O}_{\bar{U}}$. But the theorem on formal functions guarantees that $\tau_* \mathcal{O}_U \rightarrow \varprojlim \tau_*(\mathcal{O}_U/m^{i+1} \mathcal{O}_U)$ is an isomorphism, so α has the factorization property, as required. □

As we will detail below, the propositions proved so far show that F is representable by a closed subscheme of S when restricted to the category of *strict* logarithmic schemes over S . To show that it is representable on the category of all logarithmic schemes over S , we require one more proposition.

Proposition 3.4.9. *The factorization property has minimal monoids and the pullback of a minimal monoid is a minimal monoid.*

Proof. Suppose that α satisfies the factorization property over S_0 , where $S_0 \rightarrow S$ is a morphism of logarithmic schemes that is an isomorphism on the underlying schemes. Write $C_0 = C \times_S S_0$ and let α_0 be the image of α in $M_{C_0}^{\text{gp}}$, which satisfies the factorization property by assumption. Working locally, we can assume that S is atomic and that the dual graph of C is constant on the closed stratum. Let \square be that dual graph. Adjusting α by a section of M_S^{gp} , we can assume that $\bar{\alpha}_0$ vanishes on the components contracted by τ . Let Q be the set of all elements of \bar{M}_S^{gp} that arise as $\bar{\alpha}(v)$ as v ranges over the vertices of \square contracted by τ . Then Q is contained in the kernel of $\bar{M}_S^{\text{gp}} \rightarrow \bar{M}_{S_0}^{\text{gp}}$. The minimal characteristic monoid on which we have the factorization property is therefore the saturation \bar{N}_S of the image of \bar{M}_S in the quotient of \bar{M}_S^{gp} by the subgroup generated by Q . Since we have a map $\bar{N}_S \rightarrow \bar{M}_{S_0}$, we can pull back the logarithmic structure of S_0 to get a logarithmic structure N_S on S on which α has the factorization property. It is immediate from the construction that \bar{N}_S is minimal and that the construction commutes with pullback. \square

Proof of Theorem 3.4.3. Let $\tau : C \rightarrow \bar{C}$ be a contraction of genus 1 components over S and let α be a section of M_C^{gp} . We wish to show that there is a closed subscheme $S' \subset S$ such that the pullback of α along $f : T \rightarrow S$ has the factorization property if and only if f factors through S' . We may assume that S is of finite type, since all of the data in play are locally of finite presentation.

By Propositions 3.4.4 and 3.4.5, the subset of points of S where α satisfies the factorization property is a closed subset S_0 of S . If S_0 is given the reduced subscheme structure then α has the factorization property over S_0 . By Proposition 3.4.7, the subscheme structures on S_0 over which α has the factorization property are filtered. Taking the limit of these closed subschemes yields a scheme S' , the spectrum of a complete noetherian local ring, such that the factorization property holds formally over S' . But the underlying scheme of S' must be the same as that of S_0 , so the ideal of S_0 in S' is nilpotent. Thus S' is actually an infinitesimal neighborhood of S_0 in S and S' is the maximal closed subscheme of S over which the factorization property holds.

Now suppose that $f : T \rightarrow S$ is a strict morphism and that α has the factorization property over T . We wish to show f factors through S' . We may assume T is of finite type. Certainly $f(T) \subset S_0$ as a set, so if T_0 is the reduced subscheme structure on T the $f|_{T_0}$ factors through S_0 . Then T is an infinitesimal extension of T_0 so the pushout $S_1 = T \amalg_{T_0} S_0$ is an infinitesimal extension of S_0 and α has the factorization property over S_1 by Proposition 3.4.8. It follows that $S_1 \subset S'$ and therefore f factors through S' as required.

By Proposition 3.4.9 the factorization property has minimal monoids, so by Theorem 2.8.1.1, S' with its minimal logarithmic structure represents the factorization property. \square

Theorem 3.4.10. *The category $\mathcal{W}(Z)$ is representable by a logarithmic algebraic stack. After fixing the contact orders Γ , the substack $\mathcal{W}_\Gamma(Z)$ of maps with those contact orders is proper.*

Proof. We have just seen that the factorization property is representable by closed (hence proper) morphisms. Since stability is an open condition, this shows $\mathcal{W}(Z)$ is a locally closed substack of $\mathfrak{W}(Z)$. It also shows that $\mathcal{W}(Z)$ is a closed substack of the space $\overline{\mathcal{M}}_{1,n}^{\text{rad}}(Z)$ of stable logarithmic maps from radially aligned curves to Z ; this is a logarithmic modification of $\overline{\mathcal{M}}_{1,n}(Z)$, the space of stable logarithmic maps to Z , and is therefore proper. It follows that $\mathcal{W}(Z)$ is proper. \square

3.5. Logarithmic smoothness. The logarithmic tangent bundle of a toric variety Z is trivial, and is naturally identified with $N \otimes_{\mathbb{Z}} \mathcal{O}_Z$, where $N = N(T) = \text{Hom}(\mathbb{G}_m, T)$ is the cocharacter lattice of the dense torus. Given a radial map $[f : C \rightarrow Z]$, the obstructions to deforming the map $[f]$ fixing the deformation of $[C]$ lie in the group

$$\text{Obs}([f]) = H^1(C, f^*T_Z^{\text{log}}) = H^1(C, \mathcal{O}_C^{\dim Z})$$

with dimension

$$h^1(C, \mathcal{O}_C^{\dim Z}) = g(C) \cdot \dim Z.$$

Consider a torus quotient $T \rightarrow T/H$ and choose a compatible equivariant compactification

$$Z \rightarrow Z_H,$$

possibly passing from Z to a modification, as in the previous section. The quotient map induces a projection map on logarithmic tangent bundles, extending scalars from

$$N(T) \rightarrow N(T/H)$$

Choosing a splitting for the induced map on obstruction groups, we see that if the map $[\bar{f} : C \rightarrow \bar{Z}]$ is obstructed, then the map $[f]$ is also obstructed. The well-spacedness condition for radial logarithmic maps removes obstructions arising in this fashion. We now show that these obstructions are the only obstructions that arise.

Theorem 3.5.1. *For any toric variety Z , the stack $\mathcal{W}(Z)$ is logarithmically smooth and unobstructed.*

The proof will require the following lemma.

Lemma 3.5.2. *Let E be a connected Gorenstein curve of genus 1 without genus 0 tails. Let E_\circ be the smooth locus of E . The map $E_\circ \rightarrow \text{Pic}^1(E)$ sending x to $\mathcal{O}_E(x)$ is étale.*

Proof. Consider the problem of deforming x , while fixing its image $\mathcal{O}_E(x)$ in the Picard group. The obstructions to these deformations lie in $H^1(E, \mathcal{O}_E(x))$. Since E has no genus 0 tails then ω_E is trivial, and Serre duality yields the requisite vanishing.

To see that the map has relative dimension 0, note that the relative tangent space may be identified with the quotient of $H^0(E, \mathcal{O}_E(x))$ by $H^0(E, \mathcal{O}_E)$. An application of Riemann–Roch shows that this quotient is trivial. \square

Corollary 3.5.3. *Let E be a connected Gorenstein curve of genus 1 without genus 0 tails and let $a_1, \dots, a_n \in \mathbb{Z}^m$. Let E_0 be the smooth locus of E . If the a_i span \mathbb{Q}^m then the map $E_0^n \rightarrow \text{Pic}(E)^m$ sending (x_1, \dots, x_n) to the tuple of line bundles associated to the divisor with \mathbb{Z}^m -coefficients $\sum a_i x_i$.*

Proof. By the elementary divisors theorem, we can assume that the a_i are multiples of the standard basis vectors. Since E_0 is smooth, we may project off the factors of E_0^n where a_i vanishes. Then $E_0^m \rightarrow \text{Pic}(E)^m$ is the product of the maps $E_0 \rightarrow \text{Pic}^1(E) \xrightarrow{a_i} \text{Pic}(E)$. The maps $E_0 \rightarrow \text{Pic}^1(E)$ are étale by Lemma 3.5.2 and multiplication by a_i is étale because we work in characteristic zero. \square

Proof of Theorem 3.5.1. We will use the logarithmic infinitesimal criterion for smoothness. We must show that whenever $S \subset S'$ is a strict infinitesimal extension of logarithmic schemes, any morphism $S \rightarrow \mathcal{W}(Z)$ can be extended to S' , completing diagrams of the following form:

$$\begin{array}{ccc}
 S & \longrightarrow & \mathcal{W}(Z) \\
 \downarrow & \nearrow & \\
 S' & &
 \end{array}
 \tag{12}$$

This assertion is local in S , so we can restrict to a neighborhood U of a geometric point s , such that the map $\overline{M}_S^{\text{gp}}|_U \rightarrow \overline{M}_{S,s}^{\text{gp}}$ is an isomorphism. Let \square be the tropicalization of C_s .

Filtering the deformations. Let N and N^\vee be the character and cocharacter lattices of Z , respectively. The moduli map $S \rightarrow \mathcal{W}(Z)$ gives the data of a curve C and a section $\alpha \in N \otimes \Gamma(C, M_C^{\text{gp}})$. For every torsion free quotient $N \rightarrow N'$ we obtain a map $\alpha_{N'} \in N' \otimes \Gamma(C, M_C^{\text{gp}})$ by composition, which we also view as a map $C \rightarrow N' \otimes \mathbb{G}_{\log}$. For each such map $\square \rightarrow N' \otimes \mathbb{R}$, there is a largest radius $\delta_{N'} \in \Gamma(S, \overline{M}_S)$ around the minimal circuit in \square whose interior is contracted by the map.

The radii $\delta_{N'}$ for varying N' are totally ordered and necessarily finite in number. Rename these distinct radii $\delta_1 \geq \delta_2 \geq \dots \geq \delta_k$.

Since $\bar{\alpha}$ is a piecewise linear function on \square valued in $\overline{M}_S^{\text{gp}}$ we have $\bar{\alpha}(v) - \bar{\alpha}(w) \in N \otimes \ell \subset N \otimes \overline{M}_S^{\text{gp}}$ whenever v and w are connected by an edge of \square of length ℓ . We call $(\bar{\alpha}(v) - \bar{\alpha}(w))/\ell$ the *slope* of $\bar{\alpha}$ along that edge.

For each i , we take N_i be the quotient of N by the saturated sublattice spanned of the slopes of $\bar{\alpha}$ along the edges of \square contained inside the circle of radius δ_i . This gives an sequence of torsion free quotients $N \rightarrow N_k \rightarrow N_{k-1} \rightarrow \dots \rightarrow N_1$.

For each i , let $Z_i = N_i \otimes \mathbb{G}_{\log}$. Then we obtain a sequence of maps:

$$\mathcal{W}(Z) \rightarrow \mathcal{W}(Z_k) \rightarrow \mathcal{W}(Z_{k-1}) \rightarrow \dots \rightarrow \mathcal{W}(Z_1)$$

The first map is logarithmically étale since Z is logarithmically étale over $\text{Hom}(N, \mathbb{G}_{\log})$. It will now suffice to show that $\mathcal{W}(Z_i) \rightarrow \mathcal{W}(Z_{i-1})$ is logarithmically smooth for all i . For each i , let α_i be the image of α in $N_i \otimes \Gamma(C, M_C^{\text{gp}})$. We make the following observations:

- (1) We have $\delta_i \in \Gamma(S, \overline{M}_S)$ such that $\bar{\alpha}_i \in N_i \otimes \Gamma(C, \overline{M}_C^{\text{gp}})$ is constant on the interior of the circle of radius δ_i around the central vertex of \square .

- (2) The slopes of $\bar{\beta}$ on the edges of \square exiting the circle of radius δ_i span the kernel of $N_{i+1} \rightarrow N_i$ as a rational vector space.

The second observation requires a slight argument. By definition, the slopes of $\bar{\alpha}$ within the circle of radius δ_i span the kernel of $N \rightarrow N_i$ rationally. But if the edges inside the circle of radius δ_i together with those immediately exiting it spanned a smaller saturated subgroup of than the kernel of $N \rightarrow N_{i+1}$ then there would have been another δ_j in between δ_i and δ_{i+1} .

The iterative procedure. The map $\alpha : S \rightarrow \mathcal{W}(Z)$ gives families $C \leftarrow \tilde{C} \rightarrow \bar{C}$ where \bar{C} is the contraction of the circle of radius δ . We can regard α as an element of $N \otimes \Gamma(\bar{C}, M_{\bar{C}}^{\text{gp}})$. We examine extensions of these data to $C' \leftarrow \tilde{C}' \rightarrow \bar{C}'$ and $\alpha' \in N \otimes M_{\tilde{C}'}^{\text{gp}}$. The problem is addressed in two steps: first we choose a deformation of C (entailing deformations of \tilde{C} and \bar{C}), which is an obstructed problem, and then we try to lift α , which can be obstructed for a fixed choice of C' . We then revise our choice of deformation C' to eliminate the obstruction to lifting α .

The choices of C' form a torsor under $H^1(C, T_{C/S}^{\text{log}})$. We will adjust C' iteratively, lifting α_i to $\alpha'_i \in N_i \otimes \Gamma(\bar{C}', M_{\bar{C}'}^{\text{gp}})$ based on an already selected lift of α_{i-1} . At each step, we will adjust C' by a section of $H^1(C, T_{C/S}^{\text{log}})$ that vanishes on the interior of the circle of radius δ_i , thereby ensuring that our earlier choices are not broken by the later adjustments.

The obstruction group. Let N'_i be the kernel of $N_i \rightarrow N_{i-1}$. We indicate how $N'_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ functions as an obstruction group to deforming α_i once α_{i-1} and C' are fixed.

By definition, the lifting problem (12) is equivalent to the problem of extending $\alpha \in N_i \otimes \Gamma(\bar{C}, M_{\bar{C}}^{\text{gp}})$ to $\alpha' \in N_i \otimes \Gamma(\bar{C}', M_{\bar{C}'}^{\text{gp}})$. Recall that α gives an invertible sheaf, $\mathcal{O}_{\bar{C}}(-f(\bar{\alpha}))$ for each $f \in N_i^\vee$, and $f(\alpha)$ is a nowhere vanishing global section of $\mathcal{O}_{\bar{C}}(-f(\bar{\alpha}))$. We will abuse notation slightly and think of $\mathcal{O}_{\bar{C}}(-\bar{\alpha})$ as a family of invertible sheaves indexed by N_i^\vee and α as a trivialization of this family. Let $\bar{\alpha}'$ denote the unique extension of $\bar{\alpha} \in \Gamma(\bar{C}, \bar{M}_{\bar{C}}^{\text{gp}})$ to $\bar{M}_{\bar{C}'}^{\text{gp}}$. Our task is to extend α to a trivialization of $\mathcal{O}_{\bar{C}'}(\bar{\alpha}')$.

If it exists, an extension will necessarily be a trivialization, so the obstruction to the existence of an extension is the isomorphism class of the deformation $\mathcal{O}_{\bar{C}'}(\bar{\alpha}')$, which lies in $N_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$. By induction, the image of this obstruction in $N'_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ vanishes, so our obstruction lies in $N'_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$.

Deformations of the curve. This obstruction may well be nonzero, but we are still free to choose C' . The choice of C' is a torsor under the deformation group $\text{Def}(C) = H^1(C, T_{C/S}^{\text{log}})$. This gives a homomorphism

$$H^1(C, T_{C/S}^{\text{log}}) = \text{Def}(C) \rightarrow \text{Obs}_C(f) = N_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}) \tag{13}$$

that we wish to show surjects onto $N'_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$. Once this is done, we can modify C' to eliminate the obstruction.

Since C is a curve, the formation of $H^1(C, T_{C/S}^{\text{log}})$ and of $H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ commutes with base change. By Nakayama's lemma, we may therefore demonstrate the surjectivity of (13) by checking it on the fibers. We may therefore replace S with a geometric point and assume that S is the spectrum of an algebraically closed field.

We let $\bar{\square}$ be the dual graph of \bar{C} , where the interior of the circle of radius δ_i is treated as a single vertex. For each vertex v_i of $\bar{\square}$ other than the central vertex v_0 , let e_j be the edge of $\bar{\square}$ that is closest to v_0 (in other words, the edge on which λ has negative slope when it is oriented away from v_j). Let p_j be the node of \bar{C} corresponding to e_j . For $j \neq 0$, the corresponding component of \bar{C} is rational, and therefore $T_{C_j}^{\text{log}}$ has nonnegative degree. It follows that the maps

$$H^0(C_j, T_{C_j}^{\text{log}}) \rightarrow H^0(p_j, T_{C_j}^{\text{log}}) \quad \text{and} \quad H^0(C_j, \mathcal{O}_{C_j}) \rightarrow H^0(p_j, \mathcal{O}_{C_j})$$

are surjective for all $j \neq 0$. From the normalization sequence, we see that there are decompositions:

$$H^1(C, T_C^{\text{log}}) = \bigoplus_j H^1(C_j, T_{C_j}^{\text{log}}) \quad \text{and} \quad H^1(\bar{C}, \mathcal{O}_{\bar{C}}) = \bigoplus_j H^1(C_j, \mathcal{O}_{C_j})$$

Since $H^1(C_j, \mathcal{O}_{C_j}) = 0$ for $j \neq 0$, it follows that we may reduce to the case $C = C_0$ and $\bar{C} = \bar{C}_0$. Note that in this case, \bar{C}_0 has no genus 0 tails.

The obstruction class. Now let p_1, \dots, p_n be the external points of C corresponding to the edges e_1, \dots, e_n of $\bar{\square}$ adjacent to the central vertex. Then $H^1(C, T_{C/S}^{\text{log}})$ contains a copy of $\sum_j T_{p_j/C}$ corresponding to deformation of C as a logarithmic curve by moving the points p_j .

Let $a_j \in N'_i$ be the slope of $\bar{\alpha}$ on e_j and recall that the a_j span N'_i as a rational vector space. Then the obstruction class in $N'_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ is given by the following formula:

$$\mathcal{O}_{\bar{C}}(\bar{\alpha}) = \mathcal{O}_{\bar{C}}\left(\sum a_j p_j\right) \in N'_i \otimes H^1(\mathcal{O}_{\bar{C}})$$

Thus the obstruction map

$$\sum_{j=1}^n T_{p_j/C} \rightarrow H^1(C, T_{C/S}^{\text{log}}) \rightarrow N_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}) \tag{14}$$

restricts to the tangent map $\sum T_{p_j/C} \rightarrow N'_i \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}})$ considered in [Corollary 3.5.3](#). Since the a_i span N'_i rationally and \bar{C} has no genus 0 tails, that corollary implies the desired surjectivity. The points p_j all lie on the boundary of the circle of radius δ_i , so any class in $\sum T_{p_j/C} \subset H^1(C, T_{C/S}^{\text{log}})$ vanishes on the interior of the circle of radius δ_i , as required. □

Remark 3.5.4. The proof shows a stronger smoothness property, because we were able to cancel obstructions using only deformations of the marked points without needing to smooth any of the singularities of \bar{C} .

It is a consequence of the proof that we can make following ‘‘codimension 1’’ characterization of the moduli space.

Proposition 3.5.5. *The substack of $\mathfrak{W}(Z)$ parametrizing stable radial maps that satisfy the factorization property for all subtori of codimension 1 coincides with the space $\mathcal{W}(Z)$ of well-spaced radial maps.*

Proof. It suffices to treat the case of $Z = \mathbb{G}_{\log}^n$. Assume that a map $C \rightarrow \mathbb{G}_{\log}^n$ fails to satisfy the factorization property for some subtorus H in T . In keeping with the notation of the previous proof, we let $N'_H \subset N$ be the associated cocharacter subspace, and let N_H be the quotient of N by N'_H . We demonstrate that the map fails to satisfy the factorization property for a codimension 1 subtorus.

If the map is obstructed, then the cokernel in (14) is nonzero. It follows from Corollary 3.5.3 that for the radius δ associated to the map $\square \rightarrow N_H$, the exiting edge directions a_i of $\square \rightarrow N \otimes \mathbb{R}$ at this radius do not span N'_H rationally. The obstruction class described in the proof above therefore lies in the cokernel

$$N_H / \text{Span}(a_i) \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}).$$

Given any such obstruction class, we may find a projection by a character $N_H / \text{Span}(a_i) \rightarrow \mathbb{R}$, such that the class remains nonzero, by projecting onto the 1-dimensional span of the obstruction class. This gives rise to a composition $N \rightarrow N_H \rightarrow \mathbb{R}$, and an associated map on tori $\mathbb{G}_{\log}^n \rightarrow \mathbb{G}_{\log}$. The induced map on logarithmic tangent bundle is given by extending scalars from the projection $N \rightarrow \mathbb{R}$. The obstruction to lifting the map to \mathbb{G}_{\log} is the image of the obstruction to lifting the map to \mathbb{G}_{\log}^n , under the projection

$$N_H \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}) \rightarrow \mathbb{R} \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}).$$

The resulting radial map $C \rightarrow \mathbb{G}_{\log}$ therefore does not factorize, as the obstruction class is nonzero by construction. □

4. Realizability for genus one tropical curves

In this section, we use the geometry of the moduli spaces $\mathcal{W}(Z)$ constructed in Section 3 to resolve the tropical realizability problem in genus 1. The results of this section give a precise description of the boundary complex of $\mathcal{W}(Z)$. As a consequence of the smoothness and properness of $\mathcal{W}(Z)$, tropical realizability reduces to a *pointwise* calculation: we examine the unique nontopological condition characterizing the descent of a function from the normalization of a genus 1 singularity, and interpret it tropically as the realizability condition.

4.1. Moduli of tropical maps. Fix a pair of dual lattices N and N^\vee of rank r and a complete fan Σ in the vector space $N_{\mathbb{R}}$.

Definition 4.1.1. The *combinatorial type* of a tropical stable map $[\square \xrightarrow{f} \Sigma]$ consists of:

- (1) The finite graph model G underlying \square .
- (2) For each vertex $v \in G$, the cone $\sigma_v \in \Sigma$ containing the image of v .
- (3) For each edge e , the slope w_e and the primitive vector u_e of $f(e)$.

For tropical maps, the discrete data can be captured by the “least generic” map, defined below.

Definition 4.1.2. The *recession type* of a combinatorial type Θ is obtained from $[\square \rightarrow \Sigma]$ by collapsing all bounded edges of \square to a single vertex, retaining the contact orders on the unbounded edges.

As explained in [Ranganathan 2017b, Section 2], once one fixes the recession type, there are finitely many combinatorial types of tropical stable maps with this recession type. This boundedness of combinatorial types is the essential content of [Gross and Siebert 2013, Section 3.1].

Given a type Θ , there is a polyhedral cone σ_Θ , whose relative interior parametrizes tropical stable maps with a fixed combinatorial type. This cone serves as a deformation space for maps of type Θ . In [Ranganathan 2017b, Section 2.2], a generalized cone complex $T_\Gamma(\Sigma)$ is constructed, by taking a colimit of the cones above over a natural gluing operation. This is a coarse moduli space for maps of fixed recession type.

Remark 4.1.3 (a moduli stack of tropical maps). It is possible to promote this construction to a fine moduli stack of tropical maps. By replacing the real edge lengths in \square with monoid-valued edge lengths, one obtains an appropriate notion of a family of tropical stable maps over a cone σ . With this notion of family, the framework in [Cavaliere et al. 2017] produces a cone stack $\mathcal{T}_\Gamma(\Sigma)$, with well-defined evaluation morphisms. The addition of a marked point with trivial contact order functions as a universal curve in this context. We avoid further discussion of this for two reasons. First, we will not need the stacks directly in this work, and can make do with the less conceptually natural, but more concrete generalized cone complex. Second, and more importantly, the precise relationship between the analytification of the moduli space of maps — which coincides with the analytification of the coarse moduli space — remains unclear at present.

4.2. Traditional tropicalization and realizability. The tropicalization procedure discussed in the early parts of the paper uses the logarithmic structure, and differs from the one involving nonarchimedean geometry. Accounting for the difference is the *tropical realizability problem*, and is the focus of this final section.

Let K be a nonarchimedean field extending \mathbb{C} , where the latter is equipped with the trivial valuation. Let Y be a K -scheme or stack, locally of finite type. The *Berkovich analytification* Y^{an} is a locally compact, Hausdorff topological space whose points are naturally identified with equivalence classes of pairs (L, y) where L is a valued field extension of K and y is an L -valued point of Y . The equivalence is the one generated by identifying two such triples $(L, y) \sim (L', y')$ whenever there is an embedding of valued extensions $L \hookrightarrow L'$ sending y to y' . See [Berkovich 1990; Ulirsch 2017b; Yu 2018] for Berkovich spaces and stacks and [Abramovich et al. 2016] for an introduction to analytic spaces in the context of logarithmic geometry.

Given a torus $\mathbb{G}_m^r = \text{Spec}(K[N^\vee])$, the *tropicalization* map is the continuous map

$$\text{trop} : \mathbb{G}_{m,\text{an}}^n \rightarrow N \otimes \mathbb{R},$$

that associates to an L -valued point of \mathbb{G}_m^n , its coordinatewise valuation. The tropicalization of a subvariety is defined by restriction.

Let $C \rightarrow \mathbb{G}_m^n$ be a map to a torus from a smooth curve of genus g . There is a natural factorization of topological spaces

$$\begin{array}{ccc} C^{\text{an}} & \longrightarrow & \mathbb{G}_{m,\text{an}}^r \\ \downarrow & & \downarrow \\ \square & \longrightarrow & \text{trop}(C^{\text{an}}) \subset \mathbb{R}^r, \end{array}$$

The left vertical map is a deformation retraction onto a *skeleton*; see [Baker et al. 2016] for details. There are at least two natural ways to extract the tropical curve \square from $[C \rightarrow \mathbb{G}_m^r]$.

4.2.1. Abstract stable reduction. After choosing coordinates on the target, the map $[\varphi : C \rightarrow \mathbb{G}_m^n]$ is given by n invertible functions on C . Let \hat{C} be the smooth projective model for C , and q_1, \dots, q_n the points at which these invertible functions acquire zeros or poles. If the map φ is nonconstant, the pair $(\hat{C}, q_1, \dots, q_n)$ has negative Euler characteristic and thus admits a minimal model $\mathcal{C} \rightarrow \text{Spec}(R)$ over the valuation ring of K . Take the underlying graph of \square to be the dual graph of the special fiber of \mathcal{C} . Given an edge e of \square , the corresponding node q_e of \mathcal{C} has a local equation

$$xy = \omega, \quad \omega \in R.$$

Set the length $\ell(e)$ equal to the valuation of the parameter ω .

4.2.2. Universal property of minimality. Let $\hat{C} \supset C$ be the projective model of C with boundary $\partial\hat{C} = \{q_1, \dots, q_n\}$ and choose a toric compactification Z of \mathbb{G}_m^n such that the morphism

$$(\hat{C}, \partial\hat{C}) \rightarrow (Z, \partial Z).$$

is a logarithmic map. Letting $\mathcal{L}(Z)$ be the space of logarithmic stable maps to Z , this gives rise to a moduli map $\text{Spec}(K) \rightarrow \mathcal{L}(Z)$, which, after a base change, extends to a map

$$\text{Spec}(R) \rightarrow \mathcal{L}(Z),$$

from the valuation ring. Let k denote the residue field and Γ the value group. Consider the logarithmic map

$$\text{Spec}(\Gamma \rightarrow k) \rightarrow \mathcal{L}(Z),$$

from the closed point, endowed with the (not necessarily coherent) logarithmic structure from the value group. By the universal property of minimality, this induces a factorization

$$\text{Spec}(\Gamma \rightarrow k) \rightarrow \text{Spec}(P^{\min} \rightarrow k),$$

where P^{\min} is the stalk of the minimal monoid of $\mathcal{L}(Z)$ at the image of the closed point. We obtain a point of the dual cone $\text{Hom}(P^{\min}, \Gamma)$, which, as was previously discussed, is identified with a point in the cone of tropical maps of a fixed combinatorial type. See [Ranganathan 2017b, Section 2] for details.

4.3. Expected dimension and superabundance. Every tropical stable map $[f]$ of combinatorial type Θ has a deformation space, the moduli cone σ_Θ . Superabundance is the phenomenon wherein this deformation space is larger than expected.

The *overvalence* of a type Θ with underlying graph G is defined as

$$\text{ov}(\Theta) = \sum_{p \in G: \text{val}(p) \geq 4} \text{val}(p) - 3.$$

The overvalence allows us to determine an expected topological dimension of the tropical deformation space as:

$$\text{exp. dim } \sigma_\Theta = (\dim(\Sigma) - 3)(1 - b_1(G)) + n - \text{ov}(\Theta),$$

where $b_1(\square)$ is the first Betti number of G . The actual dimension of σ_Θ cannot be less than the expected dimension, but may exceed it. For further details, see [Mikhalkin 2005; Nishinou and Siebert 2006; Ranganathan 2017b].

Definition 4.3.1. A combinatorial type Θ is *superabundant* if the dimension of σ_Θ is strictly larger than the expected dimension.

4.3.2. Superabundance as tropical obstructedness. The deformation space of a map $[\varphi : C \rightarrow \mathbb{P}^r]$ can be larger than expected because deformations can be obstructed. The dimension of the deformation space can be estimated using Riemann–Roch and the tangent-obstruction complex [Hori et al. 2003, Section 24.4]. One examines the restrictions on the complex structure of the curve that are forced by the map. In some cases, such as when φ multiple covers its image or contracts a component, there are fewer such restrictions than expected.

The situation in tropical geometry is similar. Given a tropical stable map $[f : \square \rightarrow \Sigma]$ and a cycle of edges in \square , the piecewise linearity of f imposes restrictions on the edge lengths of this cycle. In particular, the edge lengths of a cycle are constrained by the condition that the total displacement around each cycle must vanish. If $\dim \Sigma = r$ the map is expected to impose r conditions on the edge lengths of \square for each cycle, and the conditions imposed by different cycles are expected to be independent. However, if cycles are mapped to linear subspaces, or contracted altogether, there are fewer than the expected number of restrictions.

In genus 1, superabundance can be stated in a simplified form. In the following proposition, and the rest of the section, it will sometimes be convenient to forget the precise fan structure of Σ and consider the map of metric spaces $\square \rightarrow N_{\mathbb{R}}$.

Proposition 4.3.3. *Let $f : \square \rightarrow \Sigma$ be a tropical map from a tropical curve of genus 1. Then, f is superabundant if and only if the image of the circuit \square_0 is contained in a proper affine subspace of Σ . Equivalently, f is superabundant if and only if there exists a character $\chi : N_{\mathbb{R}} \rightarrow \mathbb{R}$ such that the circuit \square_0 is contracted under the composition*

$$\square \rightarrow N_{\mathbb{R}} \xrightarrow{\chi} \mathbb{R}$$

Proof. The first formulation is well known [Katz 2012; Ranganathan 2017b; Speyer 2014]. For the second, choose a hyperplane containing the circuit and quotient by it. □

4.4. Tropical realizability and well-spacedness. The tropical realizability problem is as follows.

Question 4.4.1. Given a tropical stable map $f : \square \rightarrow N_{\mathbb{R}}$, does there exist a smooth curve C over a nonarchimedean field K and a map

$$\varphi : C \rightarrow \mathbb{G}_m^r,$$

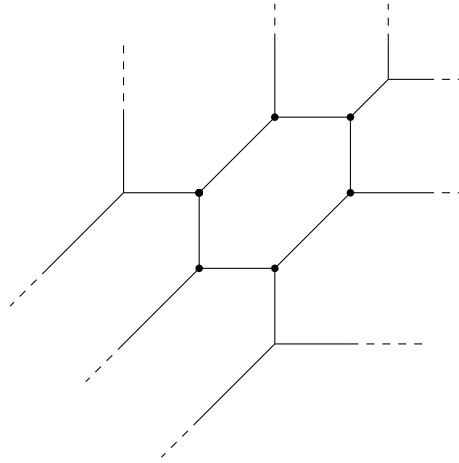


Figure 3. A tropical genus 1 curve in $\Sigma_{\mathbb{P}^2}$ of degree 3 with transverse contact orders. The curve is nonsuperabundant, as the edge directions of the circuit span \mathbb{R}^2 .

such that $\varphi^{\text{trop}} = f$?

Such a tropical map is said to be *realizable*. Superabundance is intimately related to realizability, as the following result shows. For proofs, see [Cheung et al. 2016; Ranganathan 2017b; Speyer 2014].

Theorem 4.4.2. *Let $f : \square \rightarrow \Sigma$ be a tropical stable map of genus 1 and combinatorial type Θ . If \square has a vertex v of genus 1, then assume that the local map*

$$\text{Star}(v) \rightarrow \Sigma$$

is realizable. If the combinatorial type Θ is nonsuperabundant, then f is realizable.

When a combinatorial type Θ is superabundant, there are additional constraints that are required to characterize the realizable locus.

A *flag* of a tropical curve \square is a vertex v together with a choice of tangent direction along an edge incident to v . The vertex v will be referred to as the *base* of the flag. Given a piecewise-linear function f on a tropical curve \square , we may speak of the *slope* of f along a flag.

Definition 4.4.3. Let \square be a tropical curve and let \square_0 be its circuit. Given a flag $t \in \square$, let $d(t, \square_0)$ be the distance from the circuit to the base of the flag. A tropical stable map

$$F : \square \rightarrow \mathbb{R}$$

of genus 1 is *well-spaced* if one of the following two conditions are met: either

- (1) no open neighborhood of the circuit of \square is contracted, or
- (2) if a neighborhood of the circuit is contracted, let t_1, \dots, t_k be the flags whose base is mapped to $F(\square_0)$ but along which F has nonzero slope. Then, the minimum of the distances $\{d(t_i, \square_0)\}_{i=1}^k$ occurs at least three times.

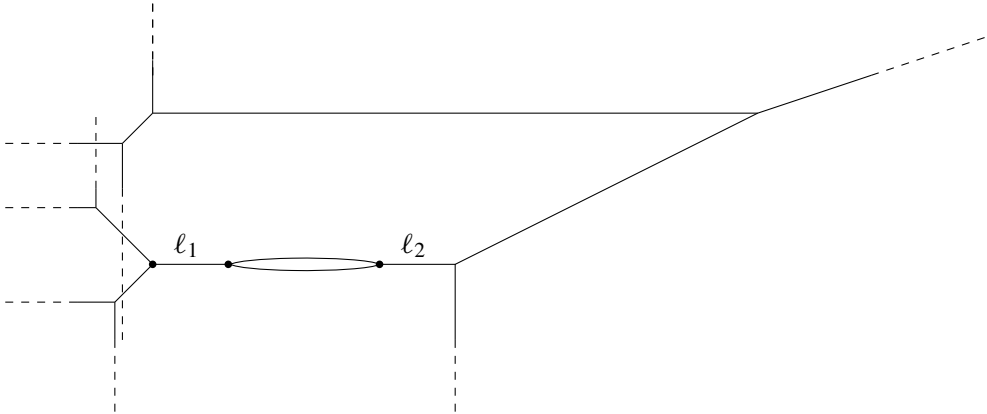


Figure 4. A superabundant tropical stable map to a Hirzebruch surface. The circuit is depicted to be flattened, indicating that its image is a line segment. Projection onto the vertical axis contracts the circuit. The curve is well-spaced if and only if $\ell_1 = \ell_2$.

Well-spacedness when the target is a general fan is formulated by considering projections to \mathbb{R} .

Definition 4.4.4. A tropical stable map $\square \rightarrow \Sigma$ of genus 1 is *well-spaced* if for each character

$$\chi : N_{\mathbb{R}} \rightarrow \mathbb{R},$$

the induced map $\square \rightarrow \mathbb{R}$ is well-spaced.

Warning 4.4.5. The condition we call well-spacedness is strictly weaker condition than the one given originally by Speyer. In particular, the definition allows that the set of flags with nonzero F -slope $\{t_i\}$ can all be based at the same vertex. In Speyer's definition, there must be distinct vertices achieving this minimum. It has already been shown that Speyer's condition is not a necessary condition in the nontrivalent case [Ranganathan 2017b, Theorem C]. The two definitions coincide when working with trivalent tropical curves whose vertex function is identically zero. To see this, observe that by the balancing condition, if a vertex supports one flag of nonzero F -slope. Thus, if two distinct vertices support flags with nonzero F -slope, then there are at least 4 such flags. We will refer to this stronger condition as *Speyer's condition*; see Figure 5.

Remark 4.4.6. We have chosen to state well-spacedness in terms of projections to 1-dimensional vector spaces, as this is closest to the existing versions of the condition present in the literature. A reader who wishes to see the parallelism with Section 3.3 one could instead impose an appropriate condition on the quotient by any real subspace of $N_{\mathbb{R}}$.

Remark 4.4.7. We make note of a consequence of this condition that is often useful in calculations. Let $\square \rightarrow N_{\mathbb{R}}$ be a tropical map from a genus 1 curve. Let $L \subset N_{\mathbb{R}}$ be the real span of the edge directions of the circuit of \square . Let δ be the minimal radius around the circuit such that the edge directions inside the circle of radius δ span a subspace L' strictly containing L . Let m the difference in dimensions of L

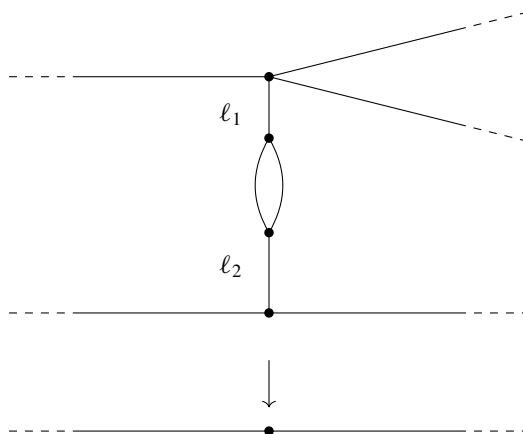


Figure 5. A tropical stable map that is well-spaced, but fails Speyer’s condition. This map is well-spaced provided $l_1 \leq l_2$, as there are three flags with nonzero slope based at the point of minimum distance to the circuit. Speyer’s condition forces the equality $l_1 = l_2$.

and L' . Then if the tropical map is well-spaced, then at the circle of radius δ , the curve \square exits the circle along least $m + 2$ flags.

This brings us to the main result of this section.

Theorem 4.4.8 (realizability of genus 1 tropical curves). *Let $[\square \rightarrow \Sigma]$ be a tropical stable map of genus 1, and assume there is a minimal logarithmic map $[C \rightarrow Z]$ whose combinatorial type is that of $[\square \rightarrow \Sigma]$. Then $[\square \rightarrow \Sigma]$ is realizable if and only if it is well-spaced.*

The proof will be completed in Section 4.6 after we establish some preliminaries in Section 4.5.

4.5. Moduli of well-spaced tropical stable maps. Let $T_\Gamma(\Sigma)$ be the moduli space of genus 1 tropical stable maps with a fixed recession type $[\Gamma \rightarrow \Sigma]$. We abuse notation by understanding that the map to Σ is part of the notation Γ . The well-spacedness condition commutes with automorphisms of tropical curves, and thus descends to a well-defined subset $W_\Gamma(\Sigma)$ of well-spaced tropical stable maps. We specify a subdivision of $T_\Gamma(\Sigma)$ such that $W_\Gamma(\Sigma)$ becomes an equidimensional subcomplex of the expected dimension.

Definition 4.5.1. A *radially aligned* combinatorial type for a genus 1 tropical stable map is a combinatorial type Θ for a tropical stable map, together with a choice of a reflexive and transitive binary relation \preceq on the vertices such that, if P is a path from the circuit of G to a vertex v that passes a vertex v' , then we have the relation

$$v' \preceq v.$$

Such a *radial combinatorial type* will be denoted (Θ, \preceq) .

4.5.2. Constructing the tropical moduli space. Let $\tilde{W}_\Gamma(\Sigma)$ denote the coarse moduli space of radially aligned tropical stable maps with recession type Γ . Given a radial combinatorial type (Θ, \preceq) , tropical maps of this type are parametrized by a face of a subdivision of the moduli cone σ_Θ . There is a specialization relation among ordered combinatorial types: a cone $\sigma_{(\Theta', \preceq')}$ is a face of a cone $\sigma_{(\Theta, \preceq)}$ if and only if the following conditions hold:

- (1) Let G and G' be the underlying graphs of Θ and Θ' respectively. Then, G' is obtained from G by a (possibly trivial) sequence of edge contractions $\alpha : G \rightarrow G'$.
- (2) The edge contraction $G \rightarrow G'$ is order preserving: if $v \preceq w$ then $\alpha(v) \preceq \alpha(w)$.
- (3) If $v' \in G'$ is a vertex with $\alpha(v') = v \in G$, then the cone $\sigma_{v'}$ is a face of σ_v .

Let $W_\Gamma(\Sigma)$ be the subcomplex of $\tilde{W}_\Gamma(\Sigma)$ parametrizing well-spaced radial tropical maps.

Lemma 4.5.3. *The locus $W_\Gamma(\Sigma)$ is a subcomplex of $\tilde{W}_\Gamma(\Sigma)$, and thus, is itself a generalized cone complex.*

Proof. The well-spacedness condition can be described in terms of the equality of the vertices at minimum distance from the circuit, and thus form a cone of the generalized cone complex. The result follows immediately from this observation. □

Remark 4.5.4. A close relative of the space $W_\Gamma(\Sigma)$ appears in the thesis of Carolin Torchiani, namely the dense open set of $W_\Gamma(\Sigma)$ parametrizing curves with identically zero genus function. In particular, it is proved that this subcomplex is pure-dimensional of the expected dimension [Torchiani 2014, Theorem 3.2.10]. It follows from this that $W_\Gamma(\Sigma)$ is also pure-dimensional. In particular, we consider the following. It would be interesting to examine the fine structure of $W_\Gamma(\Sigma)$ further. What can one say, for instance, about its homotopy type and connectivity properties?

4.6. Proof of Theorem 4.4.8. We know from Section 3.5 that the moduli space of well-spaced logarithmic stable maps $\mathcal{W}_\Gamma(Z)$ is proper and smooth. By definition, it is the locus of stable maps in $\mathfrak{W}_\Gamma(Z)$ that satisfy the factorization property for every subtorus of the dense torus of Z . Our task is to show that the logarithmic well-spacedness condition is equivalent to the tropical well-spacedness condition. By Proposition 3.5.5, the logarithmic well-spacedness condition is the conjunction of the factorization properties for all 1-dimensional quotients of Z . Since the tropical well-spacedness condition was formulated in terms of 1-dimensional quotients, it suffices to check the equivalence for every subtorus H in Z of codimension 1. Replacing Z with a modification and passing to the quotient, our obligation reduces to checking that a tropical map $\square \rightarrow \mathbb{R}$, in which all vertices of \square have genus 0, is well-spaced if and only if it is the tropicalization of a radial map $C \rightarrow \mathbb{P}^1$ satisfying the factorization property.

Let C be a logarithmic curve with tropicalization \square . The map $[\square \rightarrow \mathbb{R}]$ induces a destabilization $\nu : \tilde{C} \rightarrow C$ and a contraction $\tau : \tilde{C} \rightarrow \bar{C}$. The map itself can be regarded as a section $\bar{\alpha}$ of \bar{M}_C^{sp} . This pulls back to \bar{M}_C^{sp} and then descends to $\bar{M}_{\bar{C}}^{\text{sp}}$, since it is constant on the components collapsed by τ . Adding a constant to $\bar{\alpha}$ does not change whether it is well-spaced in either the logarithmic or the tropical sense, so we assume that $\bar{\alpha}$ takes the value 0 on the circuit component of \bar{C} .

We must show that $\bar{\alpha}$ lifts to a section α of $M_{\bar{C}}^{\text{gp}}$ if and only if $\square \rightarrow \mathbb{R}$ is well-spaced. Indeed, if α is a section of $M_{\bar{C}}^{\text{gp}}$ then $\nu_{\star}\tau^{\star}\alpha$ is a section of $\nu_{\star}M_{\bar{C}}^{\text{gp}} = M_C^{\text{gp}}$ by [Abramovich et al. 2014, Appendix B], and gives a map $C \rightarrow \mathbb{P}^1$ with the factorization property.

Let E denote the circuit component of \bar{C} and E° its interior, excluding the nodes where E is joined to the rest of \bar{C} (in other words, the locus in E where the logarithmic structure is pulled back from the base). Since $\bar{\alpha}(E) = 0$, the lift $\alpha|_{E^{\circ}}$, if it exists, will be in $\mathcal{O}_{E^{\circ}}^{\star} \subset M_{E^{\circ}}$. Regarded as a rational function on E , this lift must have zeroes and poles along the points of attachment between E and the rest of \bar{C} as specified by the outgoing slopes of $\bar{\alpha}$ along the corresponding edges (see Section 2.7). Once $\alpha|_{E^{\circ}}$ has been found, there is no obstruction to extending it to all of \bar{C} , since the rest of the curve is a forest of rational curves and $\bar{\alpha}$ is balanced. The following lemma determines whether $\alpha|_{E^{\circ}}$ can be found, and completes the proof of the theorem.

Lemma 4.6.1. *Let E be a Gorenstein, genus 1 curve with no nodes and m branches, let a_1, \dots, a_n be nonzero integers, and let P be a partition of $1, \dots, n$ into m parts. Assume that, for each $p \in P$, we have $\sum_{i \in p} a_i = 0$. If $n \geq 3$ then there is a configuration of distinct points x_1, \dots, x_n on E , with each point lying in the component corresponding to its part of the partition, such that $\mathcal{O}_E(\sum a_i x_i)$ is trivial. If $n = 2$ then there is no such configuration.*

Proof. Let $\nu : F \rightarrow E$ be the seminormalization and let ω_E be the dualizing sheaf. For any configuration of the x_i , subject to the degree constraint in the statement, there is a rational function f on F with divisor $\sum a_i x_i$, and f is unique up to scaling. We wish to determine whether f descends to E .

Let $y \in F$ be the preimage of the singular point of E and let ϕ be a nonzero global differential on E . Let F_j be the components of F and let $\nu_j : F_j \rightarrow E$ be the restrictions of ν and let f_j be the restriction of f to F_j . Let t_j be a local parameter for F_j at y and let b_j be the linear term of the expansion of f_j in terms of t_j . It was shown in Section 2.1 that there are nonzero constants c_j such that f descends to E if and only if

$$\sum_j c_j b_j = 0. \tag{15}$$

We argue that under the hypothesis of the lemma it is possible to configure the x_i on each component F_j to make b_j take any value we like. Indeed, if we decide $f(y)$ should be 1 then f_j has the formula

$$f_j = \prod_i (1 - x_i^{-1} t_j)^{a_i}$$

with the product taken over those i such that x_i lies on F_j . The linear part is

$$b_j = - \sum a_i x_i^{-1}.$$

By adjusting the positions of the x_i , we can arrange for b_j to have any nonzero value we like. If p_j consists of at least 3 points x_i then it is possible to achieve any value for b_j , including 0, but if p_j consists of only two points, x_i and $x_{i'}$ then $a_{i'} = -a_i$ and it is impossible for b_j to take the value 0.

Thus we can solve (15) provided either that there are at least two branches at y or there is one branch containing at least 3 of the x_i . The one remaining case is where there is one branch containing 2 points. In that case, $c_1 \neq 0$ and the remarks in the last paragraph show there is no solution to (15). \square

The above result determines the dual complex of the space $\mathcal{W}_\Gamma(Z)$, and we obtain the following as a consequence of general structural results about tropicalizations of logarithmic schemes. Let $\mathcal{W}_\Gamma^\circ(Z)$ denote the locus of maps with trivial logarithmic structure.

Theorem 4.6.2. *There is a continuous tropicalization map*

$$\text{trop} : \mathcal{W}_\Gamma^{\circ, \text{an}}(Z) \rightarrow W_\Gamma(\Sigma),$$

functorial with respect to evaluation morphisms and forgetful morphisms to the moduli space of curves. Set theoretically, this map sends a family of logarithmic stable maps to its tropicalization. There is a factorization

$$\begin{array}{ccc} \mathcal{W}_\Gamma^{\circ, \text{an}}(Z) & \xrightarrow{\text{trop}} & W_\Gamma(\Sigma) \\ & \searrow p & \nearrow \text{trop}_\Theta \\ & P_\Gamma(\Sigma) & \end{array}$$

where the map p is a deformation retraction onto a generalized cone complex, and admits a canonical continuous section. The map trop_Θ is finite and is an isomorphism of cones upon restriction to any face.

Proof. With the identification of the tropical maps that arise as tropicalizations of one-parameter families, the proof of the result is a cosmetic variation on similar results in the literature [Cavalieri et al. 2016; Ranganathan 2017a; 2017b]. By Theorem 4.4.8, the tropicalization of any family of logarithmic stable maps over a valuation ring is well-spaced. Once this is established, the continuity, functoriality, and finiteness of trop_Θ follow from [Ranganathan 2017b, Theorem 2.6.2] and the uniqueness of minimal morphisms of logarithmic schemes up to saturation [Wise 2016b]. The saturation index of a combinatorial type (Θ, \preceq) is equal to the cardinality of the fibers of trop_Θ , as explained in [Ranganathan 2017a; 2017b]. Since $\mathcal{W}_\Gamma(Z)$ is a toroidal compactification, the existence of a section from the skeleton follows from [Abramovich et al. 2015; Thuillier 2007]. Compatibility with forgetful and evaluation morphisms follows from [Ulirsch 2017a, Theorem 1.1]. \square

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References

- [Abramovich and Chen 2014] D. Abramovich and Q. Chen, “Stable logarithmic maps to Deligne–Faltings pairs, II”, *Asian J. Math.* **18**:3 (2014), 465–488. [MR](#) [Zbl](#)
- [Abramovich and Wise 2018] D. Abramovich and J. Wise, “Birational invariance in logarithmic Gromov–Witten theory”, *Compos. Math.* **154**:3 (2018), 595–620. [MR](#) [Zbl](#)
- [Abramovich et al. 2014] D. Abramovich, S. Marcus, and J. Wise, “Comparison theorems for Gromov–Witten invariants of smooth pairs and of degenerations”, *Ann. Inst. Fourier (Grenoble)* **64**:4 (2014), 1611–1667. [MR](#) [Zbl](#)
- [Abramovich et al. 2015] D. Abramovich, L. Caporaso, and S. Payne, “The tropicalization of the moduli space of curves”, *Ann. Sci. École Norm. Sup. (4)* **48**:4 (2015), 765–809. [MR](#) [Zbl](#)
- [Abramovich et al. 2016] D. Abramovich, Q. Chen, S. Marcus, M. Ulirsch, and J. Wise, “Skeletons and fans of logarithmic structures”, pp. 287–336 in *Nonarchimedean and tropical geometry* (St. John, USVI, 2013/Puerto Rico, 2015), edited by M. Baker and S. Payne, Springer, 2016. [MR](#) [Zbl](#)
- [Abramovich et al. 2017a] D. Abramovich, Q. Chen, M. Gross, and B. Siebert, “Decomposition of degenerate Gromov–Witten invariants”, preprint, 2017. [arXiv](#)
- [Abramovich et al. 2017b] D. Abramovich, Q. Chen, S. Marcus, and J. Wise, “Boundedness of the space of stable logarithmic maps”, *J. Eur. Math. Soc.* **19**:9 (2017), 2783–2809. [MR](#) [Zbl](#)
- [Altman and Kleiman 1970] A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*, Lecture Notes in Math. **146**, Springer, 1970. [MR](#) [Zbl](#)
- [Baker et al. 2016] M. Baker, S. Payne, and J. Rabinoff, “Nonarchimedean geometry, tropicalization, and metrics on curves”, *Algebr. Geom.* **3**:1 (2016), 63–105. [MR](#) [Zbl](#)
- [Battistella et al. 2019] L. Battistella, N. Nabijou, and D. Ranganathan, “Curve counting in genus one: elliptic singularities and relative geometry”, preprint, 2019. [arXiv](#)
- [Berkovich 1990] V. G. Berkovich, *Spectral theory and analytic geometry over non-archimedean fields*, Math. Surv. Monogr. **33**, Amer. Math. Soc., Providence, RI, 1990. [MR](#) [Zbl](#)
- [Bertrand et al. 2014] B. Bertrand, E. Brugallé, and G. Mikhalkin, “Genus 0 characteristic numbers of the tropical projective plane”, *Compos. Math.* **150**:1 (2014), 46–104. [MR](#) [Zbl](#)
- [Cavalieri et al. 2010] R. Cavalieri, P. Johnson, and H. Markwig, “Tropical Hurwitz numbers”, *J. Algebraic Combin.* **32**:2 (2010), 241–265. [MR](#) [Zbl](#)
- [Cavalieri et al. 2016] R. Cavalieri, H. Markwig, and D. Ranganathan, “Tropicalizing the space of admissible covers”, *Math. Ann.* **364**:3–4 (2016), 1275–1313. [MR](#) [Zbl](#)
- [Cavalieri et al. 2017] R. Cavalieri, M. Chan, M. Ulirsch, and J. Wise, “A moduli stack of tropical curves”, preprint, 2017. [arXiv](#)
- [Chen 2014] Q. Chen, “Stable logarithmic maps to Deligne–Faltings pairs, I”, *Ann. of Math. (2)* **180**:2 (2014), 455–521. [MR](#) [Zbl](#)
- [Cheung et al. 2016] M.-W. Cheung, L. Fantini, J. Park, and M. Ulirsch, “Faithful realizability of tropical curves”, *Int. Math. Res. Not.* **2016**:15 (2016), 4706–4727. [MR](#) [Zbl](#)
- [Gillam 2012] W. D. Gillam, “Logarithmic stacks and minimality”, *Int. J. Math.* **23**:7 (2012), art. id. 1250069. [MR](#) [Zbl](#)
- [Gross 2016] A. Gross, “Correspondence theorems via tropicalizations of moduli spaces”, *Commun. Contemp. Math.* **18**:3 (2016), art. id. 1550043. [MR](#) [Zbl](#)

- [Gross 2018] A. Gross, “Intersection theory on tropicalizations of toroidal embeddings”, *Proc. Lond. Math. Soc.* (3) **116**:6 (2018), 1365–1405. [MR](#) [Zbl](#)
- [Gross and Siebert 2013] M. Gross and B. Siebert, “Logarithmic Gromov–Witten invariants”, *J. Amer. Math. Soc.* **26**:2 (2013), 451–510. [MR](#) [Zbl](#)
- [Helm and Katz 2012] D. Helm and E. Katz, “Monodromy filtrations and the topology of tropical varieties”, *Canad. J. Math.* **64**:4 (2012), 845–868. [MR](#) [Zbl](#)
- [Hori et al. 2003] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay Math. Monogr. **1**, Amer. Math. Soc., Providence, RI, 2003. [MR](#) [Zbl](#)
- [Jensen and Ranganathan 2017] D. Jensen and D. Ranganathan, “Brill–Noether theory for curves of a fixed gonality”, preprint, 2017. [arXiv](#)
- [Kato 1989] K. Kato, “Logarithmic structures of Fontaine–Illusie”, pp. 191–224 in *Algebraic analysis, geometry, and number theory* (Baltimore, 1988), edited by J.-I. Igusa, Johns Hopkins Univ. Press, Baltimore, 1989. [MR](#) [Zbl](#)
- [Kato 2000] F. Kato, “Log smooth deformation and moduli of log smooth curves”, *Int. J. Math.* **11**:2 (2000), 215–232. [MR](#) [Zbl](#)
- [Katz 2012] E. Katz, “Lifting tropical curves in space and linear systems on graphs”, *Adv. Math.* **230**:3 (2012), 853–875. [MR](#) [Zbl](#)
- [Len and Ranganathan 2018] Y. Len and D. Ranganathan, “Enumerative geometry of elliptic curves on toric surfaces”, *Israel J. Math.* **226**:1 (2018), 351–385. [MR](#) [Zbl](#)
- [Mandel and Ruddat 2016] T. Mandel and H. Ruddat, “Descendant log Gromov–Witten invariants for toric varieties and tropical curves”, 2016. To appear in *Trans. Amer. Math. Soc.* [arXiv](#)
- [Mikhalkin 2005] G. Mikhalkin, “Enumerative tropical algebraic geometry in \mathbb{R}^2 ”, *J. Amer. Math. Soc.* **18**:2 (2005), 313–377. [MR](#) [Zbl](#)
- [Molcho and Wise 2018] S. Molcho and J. Wise, “The logarithmic Picard group and its tropicalization”, preprint, 2018. [arXiv](#)
- [Nishinou 2009] T. Nishinou, “Correspondence theorems for tropical curves, I”, preprint, 2009. [arXiv](#)
- [Nishinou and Siebert 2006] T. Nishinou and B. Siebert, “Toric degenerations of toric varieties and tropical curves”, *Duke Math. J.* **135**:1 (2006), 1–51. [MR](#) [Zbl](#)
- [Parker 2011] B. Parker, “Gromov–Witten invariants of exploded manifolds”, preprint, 2011. [arXiv](#)
- [Ranganathan 2017a] D. Ranganathan, “Skeletons of stable maps, I: Rational curves in toric varieties”, *J. Lond. Math. Soc.* (2) **95**:3 (2017), 804–832. [MR](#) [Zbl](#)
- [Ranganathan 2017b] D. Ranganathan, “Skeletons of stable maps, II: Superabundant geometries”, *Res. Math. Sci.* **4** (2017), art. id. 11. [MR](#) [Zbl](#)
- [Ranganathan 2017c] D. Ranganathan, “Superabundant curves and the Artin fan”, *Int. Math. Res. Not.* **2017**:4 (2017), 1103–1115. [MR](#) [Zbl](#)
- [Ranganathan 2019] D. Ranganathan, “Logarithmic Gromov–Witten theory with expansions”, preprint, 2019. [arXiv](#)
- [Ranganathan and Wise 2019] D. Ranganathan and J. Wise, “Rational curves in the logarithmic multiplicative group”, 2019. To appear in *Proc. Amer. Math. Soc.* [arXiv](#)
- [Ranganathan et al. 2017] D. Ranganathan, K. Santos-Parker, and J. Wise, “Moduli of stable maps in genus one and logarithmic geometry, I”, 2017. To appear in *Geom. Topol.* [arXiv](#)
- [SGA 7_I 1972] D. S. Rim, “Formal deformation theory”, pp. 32–132 in *Groupes de monodromie en géométrie algébrique, I: Exposés I–II, VI–IX* (Séminaire de Géométrie Algébrique du Bois Marie 1967–1969), edited by A. Grothendieck, Lecture Notes in Math. **288**, Springer, 1972. [MR](#) [Zbl](#)
- [Smyth 2011] D. I. Smyth, “Modular compactifications of the space of pointed elliptic curves, I”, *Compos. Math.* **147**:3 (2011), 877–913. [MR](#) [Zbl](#)
- [Speyer 2005] D. E. Speyer, *Tropical geometry*, Ph.D. thesis, University of California, Berkeley, 2005, Available at <https://search.proquest.com/docview/305031446>.
- [Speyer 2014] D. E. Speyer, “Parameterizing tropical curves, I: Curves of genus zero and one”, *Algebra Number Theory* **8**:4 (2014), 963–998. [MR](#) [Zbl](#)

- [Tevelev 2007] J. Tevelev, “Compactifications of subvarieties of tori”, *Amer. J. Math.* **129**:4 (2007), 1087–1104. [MR](#) [Zbl](#)
- [Thuillier 2007] A. Thuillier, “Géométrie toroïdale et géométrie analytique non archimédienne: application au type d’homotopie de certains schémas formels”, *Manuscripta Math.* **123**:4 (2007), 381–451. [MR](#) [Zbl](#)
- [Torchiani 2014] C. Torchiani, *Enumerative geometry of rational and elliptic tropical curves in \mathbb{R}^m* , Ph.D. thesis, Technische Universität Kaiserslautern, 2014, Available at <https://tinyurl.com/torchiphd>.
- [Tyomkin 2012] I. Tyomkin, “Tropical geometry and correspondence theorems via toric stacks”, *Math. Ann.* **353**:3 (2012), 945–995. [MR](#) [Zbl](#)
- [Ulirsch 2017a] M. Ulirsch, “Functorial tropicalization of logarithmic schemes: the case of constant coefficients”, *Proc. Lond. Math. Soc.* (3) **114**:6 (2017), 1081–1113. [MR](#) [Zbl](#)
- [Ulirsch 2017b] M. Ulirsch, “Tropicalization is a non-archimedean analytic stack quotient”, *Math. Res. Lett.* **24**:4 (2017), 1205–1237. [MR](#) [Zbl](#)
- [Vakil and Zinger 2008] R. Vakil and A. Zinger, “A desingularization of the main component of the moduli space of genus-one stable maps into \mathbb{P}^n ”, *Geom. Topol.* **12**:1 (2008), 1–95. [MR](#) [Zbl](#)
- [Wise 2011] J. Wise, “Obstruction theories and virtual fundamental classes”, preprint, 2011. [arXiv](#)
- [Wise 2016a] J. Wise, “Moduli of morphisms of logarithmic schemes”, *Algebra Number Theory* **10**:4 (2016), 695–735. [MR](#) [Zbl](#)
- [Wise 2016b] J. Wise, “Uniqueness of minimal morphisms of logarithmic schemes”, preprint, 2016. [arXiv](#)
- [Yu 2018] T. Y. Yu, “Gromov compactness in non-archimedean analytic geometry”, *J. Reine Angew. Math.* **741** (2018), 179–210. [MR](#) [Zbl](#)

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Volume 13 No. 8 2019

Moduli of stable maps in genus one and logarithmic geometry, II	1765
DHRUV RANGANATHAN, KELI SANTOS-PARKER and JONATHAN WISE	
Multiplicity one for wildly ramified representations	1807
DANIEL LE	
Theta operators on unitary Shimura varieties	1829
EHUD DE SHALIT and EYAL Z. GOREN	
Infinitely generated symbolic Rees algebras over finite fields	1879
AKIYOSHI SANNAI and HIROMU TANAKA	
Manin's b -constant in families	1893
AKASH KUMAR SENGUPTA	
Equidimensional adic eigenvarieties for groups with discrete series	1907
DANIEL R. GULOTTA	
Cohomological and numerical dynamical degrees on abelian varieties	1941
FEI HU	
A comparison between pro- p Iwahori–Hecke modules and mod p representations	1959
NORIYUKI ABE	