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For the polynomial ring over an arbitrary field with twelve variables, there exists a prime ideal whose symbolic Rees algebra is not finitely generated.

1. Introduction

Let A be a polynomial ring over a field k with finitely many variables. For a field L satisfying $k \subset L \subset \mathrm{Frac}(A)$, Hilbert's fourteenth problem asks whether or not the ring $L \cap A$ is finitely generated over k. In 1958, Nagata [1960] found the first counterexample to this problem over arbitrary sufficiently large fields. For more examples we refer to [Roberts 1990; Kuroda 2005; Totaro 2008]. On the other hand, this problem is related to the following question raised by Cowsik [1984].

Question 1.1. Let A be a polynomial ring over a field with finitely many variables and let P be a prime ideal of A. Set $P^{(m)} := P^m A_P \cap A$. Then is the symbolic Rees algebra $R_S(P) := \bigoplus_{m=0}^{\infty} P^{(m)}$ a finitely generated k-algebra?

Indeed, Roberts [1985] settled Question 1.1 negatively, using Nagata's counterexample mentioned above. Roberts's construction is valid only over sufficiently large fields of characteristic zero, although Nagata's example is independent of the characteristic of the base field. This is because Roberts's proof requires a theorem of Bertini type that fails in positive characteristic (see [Roberts 1985, line 7 on page 591]). On the other hand, it is known for experts that Roberts's method works, after suitable modifications, for the case where k is not algebraic over a finite field. Roughly speaking, counterexamples over such fields can be found after replacing the theorem of Bertini type and Nagata's counterexample used in [Roberts 1985] by [Diaz and Harbater 1991, Theorem 2.1] and the blowup of $\mathbb{P}^2$ along general nine points, respectively. In this sense, Question 1.1 is still open if k is algebraic over a finite field.

The purpose of this paper is to give the negative answer to Question 1.1 over an arbitrary base field. More specifically, the main theorem is as follows.

Theorem 1.2 (see Theorem 3.7). *Let k be a field. Let A be the polynomial ring over k with twelve variables. Then there exists a prime ideal $\mathfrak{p}$ of A whose symbolic Rees algebra $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring.*

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Sketch of the proof. We overview some of the ideas used in the proof of Theorem 1.2. Let us treat the case where $k = \mathbb{F}_p$. Our method is based on a geometric description of symbolic Rees algebras that was pointed out by Cutkosky [1991] in a certain special case. We start with a projective smooth surface V over $\mathbb{F}_p$, constructed by Totaro, that has a nef divisor M which is not semiample. We embed V into the eleven-dimensional projective space $\mathbb{P}^{11}_{\mathbb{F}_p}$ (see Lemma 3.5). Thanks to a theorem of Bertini type over finite fields, we can find a smooth curve W on V that is linearly equivalent to $stH|_V - tM$ for a hyperplane divisor H of $\mathbb{P}^{11}_{\mathbb{F}_p}$ under the assumption that $t \gg s \gg 0$. Take a homogeneous prime ideal $\mathfrak{p}$ on $A = \mathbb{F}_p[x_0, \dots, x_{11}]$ that defines W. Let $f : X \to \mathbb{P}^{11}_{\mathbb{F}_p}$ be the blowup along W. Set $D := f^*H$ and let E be the f-exceptional prime divisor on X. Then $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring if and only if the Cox ring of X is not a noetherian ring (see Proposition 2.14). In particular it suffices to find a nef divisor on X that is not semiample. By choosing s and t carefully, we can find such a divisor (see Proposition 3.3(3)). For more details, see Section 3.

Related topics. It is worth mentioning that, concerning Question 1.1, many authors have studied the case where P is the prime ideal of $k[x, y, z]$ that defines a space monomial curve (t^a, t^b, t^c) in $\mathbb{A}^3_k$. For instance, Goto, Nishida and Watanabe [1994] proved that for some triples (a, b, c), the associated symbolic Rees algebras are not finitely generated if k is of characteristic zero. It is remarkable that this result is applied to study the compactified moduli space $\overline{\mathcal{M}}_{0,n}$ of pointed rational curves. More specifically, it turns out that $\overline{\mathcal{M}}_{0,n}$ is not a Mori dream space if $n \geq 13$ and the base field is of characteristic zero [Castravet 2009; González and Karu 2016].

Since the case of characteristic zero has such an application, it is natural to consider also the case of positive characteristic. However the situation seems to be subtler. Indeed, if the base field is of positive characteristic, then it is known that the analogous rings of the examples given in [Goto et al. 1994] and [Roberts 1990] are shown to be finitely generated by [Cutkosky 1991; Goto et al. 1994] and [Kurano 1993; 1994], respectively. Then Goto and Watanabe made the following conjecture, which remains to be an open problem.

Conjecture 1.3. Let R be the polynomial ring over a field k with three valuables. Let P be the prime ideal that defines a space monomial curve (t^a, t^b, t^c) in $\mathbb{A}^3_k$. If the characteristic of k is positive, then the symbolic Rees ring $R_S(P) = \bigoplus_{m=0}^{\infty} P^{(m)}$ is finitely generated.

It is known that Conjecture 1.3 is reduced to the case where $k = \overline{\mathbb{F}}_p$. On the other hand, Theorem 1.2 indicates that a symbolic Rees algebra is not necessarily finitely generated in a higher dimensional case, even if the base field is $\overline{\mathbb{F}}_p$. Thus if the Conjecture 1.3 holds true, then its proof depends on some facts that hold only in a lower dimensional situation.

2. Preliminaries

Notation. In this subsection, we summarize notation used in the paper.

We say that X is a *variety* over a field k (or a *k-variety*) if X is an integral scheme which is separated and of finite type over k. We say that X is a *curve* over k or a *k-curve* (resp. a *surface* over k or a *k-surface*) if X is a variety over k with $\dim X = 1$ (resp. $\dim X = 2$).

Given an invertible sheaf L on a proper scheme X over a field k, consider the natural homomorphism

$$H^0(X, L) \otimes_k \mathcal{O}_X \to L. \tag{2.0.1}$$

(1) We say that L is *nef* if $L \cdot C \geq 0$ for any k-curve C on X.

(2) For a k-linear subspace V of $H^0(X, L)$, the *scheme-theoretic base locus* $B(V)$ of V is the closed subscheme of X defined by the image of the composite homomorphism

$$V \otimes_k L^{-1} \hookrightarrow H^0(X, L) \otimes_k L^{-1} \to \mathcal{O}_X,$$

where the latter one is induced by (2.0.1). For the linear system Λ corresponding to V, we set $B(\Lambda) := B(V)$.

(3) We say that L is *globally generated* if (2.0.1) is surjective, i.e., $B(|L|) = \varnothing$.

(4) We say that L is *semiample* if there exists a positive integer n such that $L^{\otimes n}$ is globally generated.

For a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a normal proper variety X over a field, we say that D is *nef* (resp. *semiample*) if there exists a positive integer n such that nD is a Cartier divisor and $\mathcal{O}_X(nD)$ is nef (resp. semiample).

Cox rings. In this subsection, we recall the definition of Cox rings (Definition 2.2) and one of their basic properties (Lemma 2.4).

Definition 2.1. Let k be a field. Let X be a normal variety over k. For a subsemigroup Γ of the group $\mathrm{WDiv}(X)$ of Weil divisors, we set

$$R(X, \Gamma) := \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)),$$

which is called *the multisection ring* of Γ.

Definition 2.2. Let k be a field. Let X be a proper normal variety over k whose divisor class group $\mathrm{Cl}(X)$ is a finitely generated free abelian group. Fix a subgroup Γ of the group $\mathrm{WDiv}(X)$ of Weil divisors such that the induced group homomorphism $\Gamma \to \mathrm{Cl}(X)$ is bijective. We set

$$\mathrm{Cox}(X) := R(X, \Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)),$$

which is called the *Cox ring* of X.

Remark 2.3. If we take another subgroup Γ' satisfying the same property as Γ, then it is known that $R(X, \Gamma)$ and $R(X, \Gamma')$ are isomorphic as k-algebras (see [Gongyo et al. 2015, Remark 2.17]).

Lemma 2.4. *Let k be a field. Let X be a projective normal $\mathbb{Q}$-factorial variety over k whose divisor class group* $\mathrm{Cl}(X)$ *is a finitely generated free abelian group. Assume that*

(a) *X is geometrically integral over k,*

(b) *X is geometrically normal over k,*

(c) $\mathrm{Cox}(X)$ *is a noetherian ring, and*

(d) $\mathbf{Pic}^0_X$ *has dimension zero, where* $\mathbf{Pic}^0_X$ *denotes the identity component of the Picard scheme of X over k* (see [Okawa 2016, Remark 2.4]).

Then, the following assertions hold:

(1) *For any finitely generated subsemigroup Γ_1 of* $\mathrm{WDiv}(X)$, *the multisection ring $R(X, \Gamma_1)$ of Γ_1 is a finitely generated k-algebra.*

(2) *An arbitrary nef Cartier divisor L on X is semiample.*

Proof. By (a) and (b), X is a variety in the sense of [Okawa 2016, the end of Section 1]. Then the conditions (c) and (d) enable us to apply [Okawa 2016, Theorem 2.19], hence X is a Mori dream space in the sense of [Okawa 2016, Definition 2.3]. Then (2) follow from [Okawa 2016, Definition 2.3(2)]. Let us prove (1). By standard arguments (see [Gongyo et al. 2015, discussion in Remark 2.17]), we may assume that Γ_1 is a subgroup of Γ for some subgroup Γ of $\mathrm{WDiv}(X)$. Then the assertion (2) holds by [Okawa 2016, Lemma 2.20]. $\square$

Symbolic Rees algebras. The purpose of this subsection is to prove Proposition 2.14, which gives a relation between symbolic Rees algebras of polynomial rings and Cox rings of blowups of projective spaces. The materials treated in this subsection might be well-known for experts, however we give the details of the proofs for the sake of completeness.

Notation 2.5. (i) Let k be a field and let $A := k[x_0, \dots, x_n]$ be the polynomial ring equipped with the standard structure of a graded ring. Let M be the homogenous maximal ideal of A. We have $\mathbb{P}^n_k = \mathrm{Proj}\, A$.

(ii) Let W be an integral closed subscheme of $\mathbb{P}^n_k$ and let $f : X \to \mathbb{P}^n_k$ be the blowup along W. For $D := f^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ and the exceptional Cartier divisor E that is the inverse image of W, we set

$$R(X; D, -E) := \bigoplus_{d,e \in \mathbb{Z}_{\geq 0}} H^0(X, dD - eE).$$

(iii) There exists a homogeneous prime ideal $\mathfrak{p}$ of $A := k[x_0, \dots, x_n]$ that induces the ideal sheaf on $\mathbb{P}^n_k$ corresponding to W. The *symbolic Rees algebra* of $\mathfrak{p}$ is defined as $\bigoplus_{d=0}^{\infty} \mathfrak{p}^{(d)}$, where $\mathfrak{p}^{(d)} := \mathfrak{p}^d A_{\mathfrak{p}} \cap A$.

(iv) Let $\mathcal{I}_W$ be the ideal sheaf on $\mathbb{P}^n_k$ corresponding to W.

Definition 2.6. We use Notation 2.5. For a homogenous ideal I of A, we define the *saturation* I^{sat} of I by

$$I^{\mathrm{sat}} := \bigcup_{\nu=1}^{\infty} \{x \in A \mid M^{\nu} x \subset I\}.$$

Remark 2.7. We use the same notation as in Definition 2.6. By [Hartshorne 1977, Excercise 5.10 in Chapter II], I^{sat} is a homogeneous ideal of A such that both I and I^{sat} define the same closed subscheme on $\mathbb{P}^n_k$ and the equation

$$I^{\mathrm{sat}} = \bigoplus_{d=0}^{\infty} H^0(\mathbb{P}^n_k, \mathcal{I}(d))$$

holds, where $\mathcal{I}$ is the ideal sheaf on $\mathbb{P}^n_k$ associated with I.

Definition 2.8. Let R be a noetherian ring and let J be an ideal of R. We define $\tilde{J}$, called *the Ratliff–Rush ideal associated with* J, by

$$\tilde{J} := \bigcup_{n=0}^{\infty} (J^{n+1} : J^n).$$

The ideal J is said to be *Rattlif–Rush* if $J = \tilde{J}$. It is well-known that $\tilde{J}$ is a Ratliff–Rush ideal (see [Heinzer et al. 1992, Introduction]).

Lemma 2.9. *We use Notation 2.5. Fix a positive integer e and let $\mathfrak{p}^e = \bigcap_{i=0}^{r} \mathfrak{q}_i$ be a minimal primary decomposition of $\mathfrak{p}^e$ such that $\sqrt{\mathfrak{q}_0} = \mathfrak{p}$* (see [Atiyah and Macdonald 1969, Section 4]). *Then the following hold*:

(1) *The equation $\mathfrak{p}^{(e)} = \mathfrak{q}_0$ holds.*

(2) *The equation $(\mathfrak{p}^e)^{\mathrm{sat}} = \bigcap_{i \in L} \mathfrak{q}_i$ holds, where*

$$L := \{i \in \{0, \dots, r\} \mid \sqrt{\mathfrak{q}_i} \neq M\}.$$

Proof. We show (1). Since $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{p}^e$, it follows from [Atiyah and Macdonald 1969, Proposition 4.9] that $\mathfrak{p}^e A_{\mathfrak{p}} = \mathfrak{q}_0 A_{\mathfrak{p}}$. In particular we get equations

$$\mathfrak{p}^{(e)} = \mathfrak{p}^e A_{\mathfrak{p}} \cap A = \mathfrak{q}_0 A_{\mathfrak{p}} \cap A = \mathfrak{q}_0,$$

where the last equation follows from the fact that $\mathfrak{q}_0$ is a $\mathfrak{p}$-primary ideal. Thus (1) holds.

We show (2). First, let us prove $(\mathfrak{p}^e)^{\mathrm{sat}} \subset \bigcap_{i \in L} \mathfrak{q}_i$. Take $x \in (\mathfrak{p}^e)^{\mathrm{sat}}$ and $i \in L$. By definition of the saturation $(\mathfrak{p}^e)^{\mathrm{sat}}$ (see Definition 2.6), there is $\nu \in \mathbb{Z}_{>0}$ such that $M^{\nu} x \subset \mathfrak{p}^e \subset \mathfrak{q}_i$. As $\sqrt{\mathfrak{q}_i} \neq M$, there is $y \in M \setminus \sqrt{\mathfrak{q}_i}$. Hence $y^{\nu} x \in \mathfrak{q}_i$. Since $\mathfrak{q}_i$ is a primary ideal, it holds that $x \in \mathfrak{q}_i$. Thus the inclusion $(\mathfrak{p}^e)^{\mathrm{sat}} \subset \bigcap_{i \in L} \mathfrak{q}_i$ holds.

Second we prove the remaining inclusion: $(\mathfrak{p}^e)^{\mathrm{sat}} \supset \bigcap_{i \in L} \mathfrak{q}_i$. If $L = \{0, \dots, r\}$, then there is nothing to show. We may assume that $L \neq \{0, \dots, r\}$. As the primary decomposition $\mathfrak{p}^e = \bigcap_{i=0}^{r} \mathfrak{q}_i$ is minimal, there exists a unique index $i_1 \in \{1, \dots, r\}$ such that $\sqrt{\mathfrak{q}_{i_1}} = M$ (see [Atiyah and Macdonald 1969, Lemma 4.3]). In particular, $L = \{0, \dots, r\} \setminus \{i_1\}$. Since A is a noetherian ring, there exists a positive integer ν such that $M^{\nu} \subset \mathfrak{q}_{i_1}$. It follows from definition of the saturation $(\mathfrak{p}^e)^{\mathrm{sat}}$ (see Definition 2.6) that $\bigcap_{i \in L} \mathfrak{q}_i = \bigcap_{i \in \{0,\dots,r\}, i \neq i_1} \mathfrak{q}_i \subset (\mathfrak{p}^e)^{\mathrm{sat}}$. □

Lemma 2.10. *Let R be a noetherian ring and let I be an ideal of R generated by a regular sequence $a_1, \dots, a_{\mu}$ of R. Then the following hold*:

(1) *An (R/I)-algebra homomorphism*

$$(R/I)[X_1, \dots, X_\mu] \to \bigoplus_{m=0}^{\infty} I^m/I^{m+1}, \quad X_i \mapsto a_i \mod I^2$$

is an isomorphism, where $I^0 := R$.

(2) *If I is a prime ideal of R other than $\{0\}$, then I^e is a Ratliff–Rush ideal for any positive integer e (see Definition 2.8).*

(3) *If I is a prime ideal of R, then for any positive integer e, an arbitrary associated prime ideal of I^e is equal to I.*

Proof. The assertion (1) holds by the fact that any regular sequence is quasiregular [Matsumura 1989, Theorem 16.2(i)]. The assertion (2) follows from (1) and [Heinzer et al. 1992, (1.2)].

We show (3). By (1), I^m/I^{m+1} is a free (R/I)-module for any $m \in \mathbb{Z}_{\geq 0}$. Consider an exact sequence

$$0 \to I^m/I^{m+1} \to R/I^{m+1} \to R/I^m \to 0.$$

We deduce from induction on e that for any $e \in \mathbb{Z}_{\geq 1}$, an arbitrary associated prime of I^e is equal to I. Thus (3) holds. □

Lemma 2.11. *We use Notation 2.5. Assume that W is a local complete intersection scheme. Fix a positive integer e. Then the equation $f_*\mathcal{O}_X(-eE) = \mathcal{I}^e$ holds as subsheaves of $\mathcal{O}_{\mathbb{P}^n_k}$.*

Proof. Fix a point $z \in \mathbb{P}^n_k$ and set $R := \mathcal{O}_{\mathbb{P}^n_k, z}$. Given a positive integer e, let

$$I := \Gamma(\operatorname{Spec} R, \mathcal{I}|_{\operatorname{Spec} R}), \quad R(I^e) := \bigoplus_{d=0}^{\infty} I^{ed}, \quad g_e : Y_e = \operatorname{Proj} R(I^e) \to \operatorname{Spec} R,$$

where $I^0 := R$ and g_e is the blowup along I^e. We set $Y := Y_1$ and $g := g_1$. Let E_e be the effective Cartier divisor such that $\mathcal{O}_{Y_e}(-E_e) := I^e\mathcal{O}_{Y_e}$. In particular, $E = E_1$. Thanks to [Hartshorne 1977, Exercise 5.13 in Chpater II], we have that $\rho_e : Y \xrightarrow{\simeq} Y_e$ and $(\rho_e)_*(eE) = E_e$. We get equations

$$I^e = \widetilde{I^e} = H^0(Y_e, \mathcal{O}_{Y_e}(-E_e)) = H^0(Y, \mathcal{O}_Y(-eE)),$$

where the first equation holds by Lemma 2.10(2), the second one follows from [Heinzer et al. 1992, Fact 2.1] and the third one is obtained by ρ_e. Hence we are done. □

Lemma 2.12. *We use Notation 2.5. Assume that W is locally complete intersection. Then $R(X; D, -E)$ and $\bigoplus_{e=0}^{\infty} \mathfrak{p}^{(e)}$ are isomorphic as k-algebras.*

Proof. Fix a nonnegative integer e. We show that $\bigoplus_{d=0}^{\infty} H^0(X, dD - eE)$ is isomorphic to $\mathfrak{p}^{(e)}$. By Lemma 2.11, we have $f_*\mathcal{O}_X(-eE) \simeq \mathcal{I}^e$. By the projection formula, we get

$$f_*\mathcal{O}_X(dD - eE) \simeq \mathcal{I}^e \otimes_{\mathcal{O}_{\mathbb{P}^n_k}} \mathcal{O}_{\mathbb{P}^n_k}(d) = \mathcal{I}^e(d).$$

Thanks to Remark 2.7, we obtain an isomorphism

$$(\mathfrak{p}^e)^{\mathrm{sat}} \simeq \bigoplus_{d=0}^{\infty} H^0(X, dD - eE).$$

Claim 2.13. *Any associated prime ideal of $\mathfrak{p}^e$ is equal to either $\mathfrak{p}$ or M.*

Proof of Claim 2.13. Assume that there exists an associated prime ideal $\mathfrak{q}$ of $\mathfrak{p}^e$ other than $\mathfrak{p}$ or M. Let us derive a contradiction. Since $\mathfrak{q} \neq M = (x_0, \dots, x_n)$, there is x_ℓ that is not contained in $\mathfrak{q}$. Then $\mathfrak{q}A_{x_\ell}$ is an associated prime ideal of $\mathfrak{p}^e A_{x_\ell}$. Take a maximal ideal $\mathfrak{m}$ of A_{x_ℓ} containing $\mathfrak{q}A_{x_\ell}$. Then $\mathfrak{q}A_{\mathfrak{m}}$ is an associated prime ideal of $\mathfrak{p}^e A_{\mathfrak{m}}$ other than $\mathfrak{p}A_{\mathfrak{m}}$. Since W is a local complete intersection scheme, we have that $\mathfrak{p}A_{\mathfrak{m}}$ is a prime ideal generated by a regular sequence, which contradicts Lemma 2.10(3). This completes the proof of Claim 2.13. □

For a minimal primary decomposition $(\mathfrak{p}^e)^{\mathrm{sat}} = \bigcap_{i=0}^{r} \mathfrak{q}_i$ satisfying $\sqrt{\mathfrak{q}_0} = \mathfrak{p}$, we have that

$$\mathfrak{p}^{(e)} = \mathfrak{q}_0 = (\mathfrak{p}^e)^{\mathrm{sat}} \simeq \bigoplus_{d=0}^{\infty} H^0(X, dD - eE),$$

where the first equation holds by Lemma 2.9(1) and the second equation follows from Lemma 2.9(2) and Claim 2.13. This completes the proof of Lemma 2.12 □

Proposition 2.14. *We use Notation 2.5. Assume that W is smooth over k. Then the following are equivalent*:

(1) *$R(X; D, -E)$ is a noetherian ring.*

(2) *$\bigoplus_{e=0}^{\infty} \mathfrak{p}^{(e)}$ is a noetherian ring.*

(3) *The Cox ring* $\mathrm{Cox}(X)$ *of X is a noetherian ring.*

Proof. It follows from Lemma 2.12 that (1) is equivalent to (2). Since X is the blowup of $\mathbb{P}^n_k$ along a smooth scheme W, the assumptions of Lemma 2.4 hold. Then, thanks to Lemma 2.4(1), we have that (3) implies (1). Thus it suffices to show that (1) implies (3). Since it holds that $H^0(X, dD - eE) = 0$ for $d \in \mathbb{Z}_{<0}$ and $e \in \mathbb{Z}$, we get an isomorphism:

$$\bigoplus_{d,e \in \mathbb{Z}, d \geq 0} H^0(X, dD - eE) \xrightarrow{\sim} \bigoplus_{d,e \in \mathbb{Z}} H^0(X, dD - eE).$$

Thus we have a natural inclusion:

$$R(X; D, -E) = \bigoplus_{d,e \in \mathbb{Z}_{\geq 0}} H^0(X, dD - eE) \hookrightarrow \bigoplus_{d,e \in \mathbb{Z}, d \geq 0} H^0(X, dD - eE).$$

The right-hand side is generated by $H^0(X, E)$ as an $R(X; D, -E)$-algebra. Therefore, if $R(X; D, -E)$ is a noetherian ring, then so is $\bigoplus_{d,e \in \mathbb{Z}} H^0(X, dD - eE)$. Hence, also $\mathrm{Cox}(X)$ is a noetherian ring. Thus (1) implies (3). □

3. The main theorem

Construction in a general setting. The purpose of this subsection is to give a sufficient condition under which the blowup of a smooth subvariety in a projective space has a nef Cartier divisor that is not semiample (Notation 3.1, Proposition 3.3).

Notation 3.1. We use notation as follows:

(i) Let k be a field. We work over k unless otherwise specified (e.g., a projective scheme means a scheme that is projective over k).

(ii) Let V be a smooth projective variety. Set $d := \dim V$.

(iii) Let M be a nef Cartier divisor on V which is not semiample.

(iv) Fix a closed immersion: $V \subset \mathbb{P}^n_k$. Let H be a very ample Cartier divisor such that $\mathcal{O}_{\mathbb{P}^n_k}(H) \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. We set H_V to be the pullback of H to V.

(v) Assume that there exists a positive integer r satisfying the following property: if Λ denotes the linear system of $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(r))$ consisting of the effective divisors containing V, then the following conditions hold:

(v-1) The base locus of $|\Lambda|$ is set-theoretically equal to V, i.e., for any point $y \in \mathbb{P}^n_k \setminus V$, there exists a hypersurface S_0 of $\mathbb{P}^n_k$ of degree r such that $V \subset S_0$ and $y \notin S_0$.

(v-2) For any closed point $y \in V$, there exist an open neighborhood U of $y \in \mathbb{P}^n_k$ and hypersurfaces $S_1, \ldots, S_{n-\dim V}$ of $\mathbb{P}^n_k$ of degree r such that V is contained in $S_1 \cap \cdots \cap S_{n-\dim V}$ and that two subschemes $V \cap U$ and $S_1 \cap \cdots \cap S_{n-\dim V} \cap U$ of $\mathbb{P}^n_k$ are coincide.

(vi) Assume that there are a smooth prime divisor W on V and positive integers s and t satisfying the following properties:

(vi-1) $st > r$.

(vi-2) $W \sim stH_V - tM$.

(vii) Let $f : X \to \mathbb{P}^n_k$ be the blowup along W. We set $V' := f_*^{-1}V$, $E := \mathrm{Ex}(f)$ and

$$S' := rf^*H - E.$$

Note that E is a smooth prime divisor on X. Let $g : V' \xrightarrow{\simeq} V$ be the induced isomorphism.

(viii) Set

$$L := (st - r)f^*H + S'.$$

Lemma 3.2. *Let k be a field and let $Y := \mathbb{A}^n_k = \operatorname{Spec} k[y_1, \ldots, y_n]$ be the n-dimensional affine space. For $i \in \{1, \ldots, n\}$, set $T_i := V(y_i)$ to be the coordinate hyperplane of $Y = \mathbb{A}^n_k$. Let q be a positive integer satisfying $q \leq n - 1$. Set $V := T_1 \cap \cdots \cap T_q$ and $W := T_1 \cap \cdots \cap T_{q+1}$. Let $f : X \to Y$ be the blowup along W and let V' and T_i' be the proper transforms of V and T_i, respectively. Then an equation $V' = T_1' \cap \cdots \cap T_q'$ holds.*

Proof. Since blowups are commutative with flat base changes, we may assume that $q = n-1$. Thus W is the origin and V is a line passing through W. The inclusion $V' \subset T_1' \cap \cdots \cap T_{n-1}'$ is clear, hence it suffices to prove that $T_1' \cap \cdots \cap T_{n-1}' \cap E$ is one point, where E denotes the f-exceptional prime divisor. To prove this, we may assume that k is algebraically closed. Then $T_1' \cap \cdots \cap T_{n-1}' \cap E$ is one point, since there is a canonical bijection between the set $E(k)$ of the closed points of E and the set of the lines on $\mathbb{P}_k^n$ passing through W. □

Proposition 3.3. *We use Notation 3.1. Then the following hold*:

(1) *The base locus of the complete linear system $|S'|$ is contained in V'.*

(2) $L|_{V'} \sim tg^*M$.

(3) *L is a nef Cartier divisor which is not semiample.*

Proof. We show (1). Take a closed point $x \in X \setminus V'$. We set $y := f(x)$. It suffices to show that the base locus $B(|S'|)$ of $|S'|$ does not contain x. We separately treat the following two cases: $y \notin V$ and $y \in V$.

Assume that $y \notin V$. By Notation 3.1(v-1), there exists a hypersurface S_0 of $\mathbb{P}_k^n$ of degree r such that $V \subset S_0$ and $y \notin S_0$. It holds that

$$rf^*H \sim f^*S_0 = S_0' + aE,$$

where $a \in \mathbb{Z}_{>0}$ and S_0' is the proper transform of S_0. In particular, we have that

$$B(|S'|) \subset \operatorname{Supp}(S_0' + E) = f^{-1}(S_0).$$

It follows from $y \notin S_0$ that $x \notin f^{-1}(S_0)$. Hence, $x \notin B(|S'|)$. This completes the proof for the case where $y \notin V$.

Assume that $y \in V$. We have that $x \in E \setminus V'$. By Notation 3.1(v-2), there exist an open neighborhood U of $y \in \mathbb{P}_k^n$ and hypersurfaces $S_1, \ldots, S_{n-\dim V}$ of $\mathbb{P}_k^n$ of degree r such that V is contained in $S_1 \cap \cdots \cap S_{n-\dim V}$ and that two subschemes $V \cap U$ and $S_1 \cap \cdots \cap S_{n-\dim V} \cap U$ of $\mathbb{P}_k^n$ are the same. In particular, $S_1, \ldots, S_{n-\dim V}$ are smooth at y and form a part of a regular system of parameters of $\mathcal{O}_{\mathbb{P}_k^n, y}$ (see [Matsumura 1989, Theorem 17.4]). Therefore, thanks to Cohen's structure theorem, the situation is the same, up to taking the formal completions, as in the statement of Lemma 3.2. It follows from Lemma 3.2 and the faithfully flatness of completions (see [Matsumura 1989, Theorem 7.5(ii)]) that an equation

$$V' \cap f^{-1}(U) = S_1' \cap \cdots \cap S_{n-\dim V}' \cap f^{-1}(U)$$

holds, where each S_i' denotes the proper transform of S_i. In particular, it holds that $x \notin S_{i_0}'$ for some $i_0 \in \{1, \ldots, n - \dim V\}$. Since S_{i_0}' is smooth at a point y of W, we have that

$$S' = f^*(rH) - E \sim f^*S_{i_0} - E = S_{i_0}'.$$

Thus, in any case, the base locus $B(|S'|)$ does not contain x. Hence, (1) holds.

Assertion (2) holds by the following computation:

$$\begin{aligned} L|_{V'} &= ((st-r)f^*H + S')|_{V'} \\ &\sim g^*((st-r)H_V + (S|_V - W)) \\ &\sim g^*((st-r)H_V + (rH_V - (stH_V - tM))) \\ &\sim tg^*M. \end{aligned}$$

We show (3). Since $L|_{V'}$ is not semiample by (2) and Notation 3.1(iii), neither is L. Thus it suffices to show that $L=(st-r)f^*H+S'$ is nef. Take a curve Γ on X. If $\Gamma \not\subset V'$, then we get $((st-r)f^*H+S')\cdot\Gamma \geq 0$ by (1). If $\Gamma \subset V'$, then (2) implies that $L \cdot \Gamma \geq 0$. In any case, we obtain $L\cdot\Gamma \geq 0$, and hence L is nef. Thus (3) holds. □

Proof of the main theorem. In this subsection, we prove the main theorem of this paper (Theorem 3.7). Theorem 3.7 is a formal consequence of Theorem 3.6 and some results established before. The main part of Theorem 3.6 is to find schemes and divisors satisfying Notation 3.1. To this end, we start with the following lemma.

Lemma 3.4. *Let k be a field. Let V be a smooth projective connected scheme over k such that* $\dim V \geq 2$. *Let W be an ample effective Cartier divisor. Then W is connected.*

Proof. Set $k' := H^0(V, \mathcal{O}_V)$. Note that $k \subset k'$ is a field extension of finite degree. We have natural morphisms:

$$\alpha : V \xrightarrow{\alpha'} \operatorname{Spec} k' \xrightarrow{\beta} \operatorname{Spec} k.$$

We obtain $\alpha'_*\mathcal{O}_V = \mathcal{O}_{\operatorname{Spec} k'}$.

Let us prove that $k \subset k'$ is a separable extension. It suffices to prove that $A := k' \otimes_k \bar{k}$ is reduced for an algebraic closure $\bar{k}$ of k. We have the induced morphism

$$\alpha'' = \alpha' \times_k \bar{k} : V \times_k \bar{k} \to \operatorname{Spec}(k' \otimes_k \bar{k}) = \operatorname{Spec} A.$$

Since $k \to \bar{k}$ is flat, we have that $\alpha''_*\mathcal{O}_{V\times_k \bar{k}} = \mathcal{O}_{\operatorname{Spec} A}$. As $V \times_k \bar{k}$ is reduced, so is A. Therefore, $k \subset k'$ is a separable extension.

We have that α is smooth and β is étale. Then it holds that also α' is smooth by [Fu 2011, Proposition 2.4.1]. Therefore, the problem is reduced to the case where $k = H^0(V, \mathcal{O}_V)$.

We are allowed to replace W by nW for a positive integer n. Hence, by Serre duality and the ampleness of W, we may assume that $H^1(V, \mathcal{O}_V(-W)) = 0$. Then we obtain a surjective k-linear map

$$H^0(V, \mathcal{O}_V) \to H^0(W, \mathcal{O}_W).$$

Since $\dim_k H^0(V, \mathcal{O}_V) = 1$, we get $\dim_k H^0(W, \mathcal{O}_W) = 1$. Therefore, W is connected. □

Lemma 3.5. *The following hold*:

(1) *Let n be an integer such that $n \geq 5$. If k is an algebraically closed field, then there exist a smooth projective surface V over k, a closed immersion $j : V \hookrightarrow \mathbb{P}^n_k$ over k, and a nef Cartier divisor M on V which is not semiample.*

(2) *Let n be an integer such that $n \geq 11$. If k is a field, then there exist a smooth projective surface V over k, a closed immersion $j : V \hookrightarrow \mathbb{P}^n_k$ over k, and a nef Cartier divisor M on V which is not semiample.*

Proof. We show (1). We may assume that $n = 5$. The existence of j is automatic, since any smooth projective surface over k can be embedded in $\mathbb{P}^5_k$. If k is the algebraic closure of a finite field, then the assertion follows from [Totaro 2009, Theorem 6.1]. If k is not algebraic over any finite field, then V can be taken as the direct product of an elliptic curve E and a smooth projective curve. Indeed, there is a Cartier divisor N on E such that $\deg N = 0$ and N is not torsion, i.e., $rN \not\sim 0$ for any positive integer r. This implies that N is a nef Cartier divisor which is not semiample. Hence, its pullback M to V is again a nef Cartier divisor which is not semiample. This completes the proof of (1).

We show (2). We may assume that $n = 11$. First we treat the case where k is a perfect field. By (1), we can find a field extension $k \subset k'$ of finite degree, a connected k'-scheme V of dimension two which is smooth and projective over k', a closed immersion $j' : V \hookrightarrow \mathbb{P}^5_{k'}$ over k' and a nef Cartier divisor M on V which is not semiample. Automatically V is projective over k. Since k is perfect, V is also smooth over k. Thus it suffices to find a closed immersion $j : V \hookrightarrow \mathbb{P}^{11}_k$ over k. Since $k \subset k'$ is a finite separable extension, it is a simple extension. Therefore, there is a closed immersion $i : \operatorname{Spec} k' \hookrightarrow \mathbb{P}^1_k$ over k. We can find a required closed immersion j by using the Segre embedding:

$$j : V \xhookrightarrow{j'} \mathbb{P}^5_{k'} = \mathbb{P}^5_k \times_k k' \xhookrightarrow{\mathrm{id} \times i} \mathbb{P}^5_k \times_k \mathbb{P}^1_k \xhookrightarrow{\mathrm{Segre}} \mathbb{P}^{11}_k.$$

This completes the proof of the case where k is a perfect field.

Second we handle the general case. Let k_0 be the prime field contained in k. Since k_0 is perfect, there exist a smooth projective connected k_0-scheme V_0 of dimension two, a closed immersion $j_0 : V_0 \hookrightarrow \mathbb{P}^{11}_{k_0}$ over k_0, and a nef Cartier divisor M_0 on V_0 which is not semiample. Then $V_0 \times_{k_0} k$ is a scheme which is smooth and projective over k. Since any ring homomorphism between fields is faithfully flat, we can find a connected component V of $V_0 \times_{k_0} k$ such that $M := (\alpha^* M_0)|_V$ is not semiample, where $\alpha : V_0 \times_{k_0} k \to V_0$. Since M_0 is nef, so is M (see [Tanaka 2018, Lemma 2.3]). Clearly, V is a smooth projective surface over k and there is a closed immersion $j : V \hookrightarrow \mathbb{P}^{11}_k$ over k. This completes the proof of (2). □

Theorem 3.6. *The following hold*:

(1) *Let n be an integer such that $n \geq 5$. If k is an algebraically closed field, then there exist a one-dimensional connected closed subscheme W of $\mathbb{P}^n_k$ which is smooth over k and a Cartier divisor L on the blowup X of $\mathbb{P}^n_k$ along W such that L is nef but not semiample.*

(2) *Let n be an integer such that $n \geq 11$. If k is a field, then there exist a one-dimensional connected closed subscheme W of $\mathbb{P}^n_k$ which is smooth over k and a Cartier divisor L on the blowup X of $\mathbb{P}^n_k$ along W such that L is nef but not semiample.*

Proof. We only show (2), as the proof of (1) is easier. Fix a field k. We will find schemes and divisors satisfying the properties of Notation 3.1. Thanks to Lemma 3.5, there exist a smooth projective connected k-scheme V of dimension two, a closed immersion $j : V \hookrightarrow \mathbb{P}^n_k$ over k, and a nef Cartier divisor M on V which is not semiample. Set $d := 2$. Then k, V, M, d, n satisfy properties (i)–(iv) of Notation 3.1.

Since $V = \operatorname{Proj} k[x_0, \ldots, x_n]/(h_1, \ldots, h_a)$, it holds that the linear system Λ appearing in Notation 3.1(v) satisfies the property (v-1) of Notation 3.1 if $r \geq \max_{1 \leq q \leq a} \deg h_q$. As V is a locally completion intersection scheme, the quasicompactness of V also implies that property (v-2) of Notation 3.1 holds for $r \gg 0$. Therefore, we can find $r \in \mathbb{Z}_{>0}$ satisfying property (v) of Notation 3.1.

We now show that there exist s, t, W satisfying property (vi) of Notation 3.1. If k is an infinite field, then the Bertini theorem enables us to find a positive integer s and a smooth effective divisor W on V such that $W \sim sH_V - M$. Note that W is connected (Lemma 3.4). Thus, $s, t := 1$ and W satisfy property (vi) of Notation 3.1. If k is a finite field, then it follows from [Poonen 2004, Theorem 1.1] that there are positive integers $t \gg s \gg 0$ and a smooth effective divisor W satisfying property (vi) of Notation 3.1. Again by Lemma 3.4, W is connected. In any case, we can find s, t, W satisfying property (vi) of Notation 3.1.

To summarize, we have found V, W, M, d, n, r, s, t over a field k satisfying properties (i)–(viii) of Notation 3.1. By construction, V is a smooth projective surface. In particular, W is a smooth projective curve in $\mathbb{P}^{11}_k$. Thanks to Proposition 3.3, the Cartier divisor

$$L = (st - r) f^* H + S'$$

on X, defined in (viii) of Notation 3.1, is nef but not semiample. □

Theorem 3.7. *The following hold:*

(1) *Let q be an integer such that $q \geq 6$. If k is an algebraically closed field, then there exists a homogeneous prime ideal $\mathfrak{p}$ of the polynomial ring $k[x_1, \ldots, x_q]$ with q variables whose symbolic Rees algebra $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring.*

(2) *Let q be an integer such that $q \geq 12$. If k is a field, then there exists a homogeneous prime ideal $\mathfrak{p}$ of the polynomial ring $k[x_1, \ldots, x_q]$ with q variables whose symbolic Rees algebra $\bigoplus_{m=0}^{\infty} \mathfrak{p}^{(m)}$ is not a noetherian ring.*

Proof. The assertion follows from Lemma 2.4, Proposition 2.14 and Theorem 3.6. □

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