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Manin's b-constant in families

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We show that the b-constant (appearing in Manin's conjecture) is constant on very general fibers of a family of algebraic varieties. If the fibers of the family are uniruled, then we show that the b-constant is constant on general fibers.

1. Introduction

Let X be a smooth projective variety over a field of k of characteristic 0 and L a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let $\Lambda_{\text{eff}}(X) \subset \text{NS}(X)_{\mathbb{R}}$ be the cone of pseudoeffective divisors. The Fujita invariant or the a-constant is defined as

$$a(X, L) = \min\{t \in \mathbb{R} \mid [K_X] + t[L] \in \Lambda_{\text{eff}}(X)\}.$$

The invariant $\kappa\epsilon(X, L) = -a(X, L)$ was introduced and studied by Fujita [1987; 1992] under the name Kodaira energy. The a-constant was introduced in the context of Manin's conjecture in [Franke et al. 1989].

The b-constant is defined as follows [Franke et al. 1989; Batyrev and Manin 1990]:

$$b(X, L) = \text{codim of minimal supported face of } \Lambda_{\text{eff}}(X) \text{containing the class of } K_X + a(X, L)L.$$

For a singular variety X, the a- and b-constants of L are defined to be the a- and b-constants of π^*L on a resolution $\pi : \tilde{X} \to X$.

Let $f : X \to T$ be a family of projective varieties and L an f-big and f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. By semicontinuity the a-constant of the fibers $a(X_t, L|_{X_t})$ is constant on very general fiber (see [Lehmann and Tanimoto 2017, Theorem 4.3]). It follows from invariance of log plurigenera that if the fibers are uniruled then the a-constant is constant on general fibers.

In this paper we investigate the behavior of the b-constant in families and answer the questions posed in [Lehmann and Tanimoto 2017]. We prove the following:

Theorem 1.1. *Let $f : X \to T$ be a projective morphism of irreducible varieties over an algebraically closed field k of characteristic* 0, *such that the generic fiber is geometrically integral. Let L be an f-big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then there exists a countable union of proper closed subvarieties $Z = \bigcup_i Z_i \subsetneq T$, such that*

$$b(X_{\bar{t}}, L|_{X_{\bar{t}}}) = b(X_{\bar{\eta}}, L|_{X_{\bar{\eta}}})$$

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for all $t \in T \setminus Z$, where $\eta \in T$ is the generic point. In particular, the b-constant is constant on very general fibers.

If the fibers of the family are uniruled, then we have the following:

Theorem 1.2. *Let $f : X \to T$ be a projective morphism of irreducible varieties over an algebraically closed field k of characteristic 0, such that the generic fiber is geometrically integral. Let L be an f-big and f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Suppose a general fiber X_t is uniruled. Then there exists a proper closed subscheme $W \subsetneq T$ such that*

$$b(X_{\bar{t}}, L|_{X_{\bar{t}}}) = b(X_{\bar{\eta}}, L|_{X_{\bar{\eta}}})$$

for $t \in T \setminus W$ and $\eta \in T$ is the generic point. In particular, the b-constant is constant on general fibers in a family of uniruled varieties.

One can not replace the very general condition in Theorem 1.1 by just general. For example, in a family of K3-surfaces the b-constant of a fiber is the same as the Picard rank and there exist families where the Picard rank jumps on infinitely many subvarieties. Invariance of the b-constant in general fiber of a family of uniruled varieties was proved in [Lehmann and Tanimoto 2017] under the assumption $\kappa(K_{\tilde{X}_t} + a(X_t, L|_{X_t})\beta^*(L|_{X_t})) = 0$ for some resolution of singularities $\beta : \tilde{X}_t \to X_t$. Theorem 1.2 generalizes their result to get rid of this condition on fibers.

One of the motivations for studying the behavior of a- and b-constants is Manin's conjecture about asymptotic growth of rational points on Fano varieties proposed in [Franke et al. 1989; Batyrev and Manin 1990]. The following version was suggested by Peyre [2003] and later stated in [Le Rudulier 2013; Browning and Loughran 2017].

Manin's conjecture. Let X be a Fano variety defined over a number field F and $\mathcal{L} = (L, \|\cdot\|)$ a big and nef adelically metrized line bundle on X with associated height function $H_{\mathcal{L}}$. Then there exists a thin set $Z \subset X(F)$ such that one has

$$\#\{x \in X(F) \setminus Z \mid H_{\mathcal{L}}(x) \leq B\} \sim c(F, X(F) \setminus Z, \mathcal{L}) B^{a(X,L)} \log B^{b(X,L)-1}$$

as $B \to \infty$.

For the geometric consistency of Manin's conjecture, a necessary condition is that the a- and b-constants achieve a maximum as we vary over subvarieties of X. The behavior of the a- and b-constants in families was used in [Lehmann and Tanimoto 2017] to show this necessary condition. The a- and b-constants also play a role in determining and counting the dominant components of the space $\mathrm{Mor}(\mathbb{P}^1, X)$ of morphisms from $\mathbb{P}^1$ to a smooth Fano variety X (see [Lehmann and Tanimoto 2019] for details).

The ideas in proving our results are as follows. To prove Theorem 1.1, we analyze the behavior of the b-constant under specialization and combine this with the constancy of the Picard rank and the a-constant in very general fibers to obtain the desired conclusion. The key step for Theorem 1.2 is to prove constancy on closed points when $k = \mathbb{C}$. We run a $(K_X + aL)$-MMP over the base T, to obtain a relative minimal model $X \dashrightarrow X'$ where $a = a(X_t, L|_{X_t})$. We pass to a relative canonical model $\phi : X \dashrightarrow Z$ over T and base change to $t \in T$, to obtain $\phi_t : X_t \dashrightarrow Z_t$ as the canonical model for (X_t, aL_{X_t}). Using a version of the global invariant cycles theorem (see Lemma 2.11), we observe that $b(X_t, L_t)$ is same as the rank of the monodromy invariant subspace of $N^1(Y'_z)_{\mathbb{R}}$, where Y'_z is a general fiber of $X'_t \to Z_t$. Then using topological local triviality of algebraic morphisms we conclude that the monodromy invariant subspace has constant rank.

The outline of the paper is as follows. In Section 2 we discuss the preliminaries. In Section 3 and 4 we prove Theorems 1.1 and 1.2 respectively.

2. Preliminaries

In this paper we always work in characteristic 0.

Néron–Severi group. Let X be a smooth proper variety over a field k. The Néron–Severi group $\mathrm{NS}(X)$ is defined as the quotient of the group of Weil divisors, $\mathrm{Cl}(X)$, modulo algebraic equivalence. We denote $N^1(X) = \mathrm{Div}(X)/\equiv$, the quotient of Cartier divisors by numerical equivalence. We denote $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes \mathbb{R}$ and similarly $N^1(X)_{\mathbb{R}}$. By [Néron 1952], $\mathrm{NS}(X)_{\mathbb{R}}$ is a finite-dimensional vector space and its rank $\rho(X)$ is called the Picard rank. If X is a smooth projective variety, then $\mathrm{NS}(X)_{\mathbb{R}} \cong N^1(X)_{\mathbb{R}}$.

Remark 2.1. Let X be a smooth variety over an algebraically closed field k. If $k \subset k'$ is an extension of algebraically closed fields, then the natural homomorphism $\mathrm{NS}(X) \to \mathrm{NS}(X_{k'})$ is an isomorphism. So the Picard rank is unchanged under base extension of algebraically closed fields.

Let $X \to T$ be a smooth proper morphism of irreducible varieties. Suppose $s, t \in T$ such that s is a specialization of t, i.e., s is in the closure of $\{t\}$. Let $X_{\bar{t}}$ denote the base change to the algebraic closure of the residue field $k(t)$.

Proposition 2.2 [Maulik and Poonen 2012, Proposition 3.6]. *In the situation above, it is possible to choose a specialization homomorphism*

$$\mathrm{sp}_{\bar{t},\bar{s}} : \mathrm{NS}(X_{\bar{t}}) \to \mathrm{NS}(X_{\bar{s}})$$

such that:

(a) $\mathrm{sp}_{\bar{t},\bar{s}}$ *is injective. In particular* $\rho(X_{\bar{s}}) \geq \rho(X_{\bar{t}})$.

(b) *If* $\mathrm{sp}_{\bar{t},\bar{s}}$ *maps a class* $[L]$ *to an ample class, then* L *is ample.*

If $\rho(X_{\bar{s}}) = \rho(X_{\bar{t}})$, then the homomorphism $\mathrm{NS}(X_{\bar{t}})_{\mathbb{R}} \to \mathrm{NS}(X_{\bar{s}})_{\mathbb{R}}$ is an isomorphism.

Let $X \to T$ be a smooth projective morphism of irreducible varieties over $\mathbb{C}$. In Section 12 of [Kollár and Mori 1992], the local system $\mathcal{GN}^1(X/T)$ was introduced. This is a sheaf in the analytic topology defined as

$$\mathcal{GN}^1(X/T)(U) = \{\text{sections of } \mathcal{N}^1(X/T) \text{ over } U \text{ with open support}\}$$

for analytic open $U \subset T$, and the functor $\mathcal{N}^1(X/T)$ is defined as $N^1(X \times_T T')$ for any $T' \to T$. It was shown in [Kollár and Mori 1992, 12.2] that $\mathcal{GN}^1(X/T)$ is a local system with finite monodromy and $\mathcal{GN}^1(X/T)|_t = N^1(X_t)$ for very general $t \in T$. We can base change to a finite étale cover of $T' \to T$ so that $\mathcal{GN}^1(X'/T')$ has trivial monodromy. Then we have a natural identification of the fibers of $\mathcal{GN}^1(X'/T')$ and $N^1(X'/T')$. Therefore, for $t' \in T'$ very general, the natural map $N^1(X'/T') \to N^1(X'_{t'})$ is an isomorphism. One can prove the same results over any algebraically closed field of characteristic 0, by using the Lefschetz principle.

Geometric invariants. The pseudoeffective cone $\Lambda_{\text{eff}}(X)$ is the closure of the cone of effective divisor classes in $\text{NS}(X)_{\mathbb{R}}$. The interior of $\Lambda_{\text{eff}}(X)$ is the cone of big divisors $\text{Big}^1(X)_{\mathbb{R}}$.

Definition 2.3. Let L be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$ divisor on X. The a-constant is

$$a(X, L) = \min\{t \in \mathbb{R} \mid K_X + tL \in \Lambda_{\text{eff}}(X)\}.$$

For a singular projective variety we define $a(X, L) := a(\tilde{X}, \pi^* L)$ where $\pi : \tilde{X} \to X$ is a resolution of X. It is invariant under pull-back by a birational morphism of smooth varieties and hence independent of the choice of the resolution. By [Boucksom et al. 2013] we know that $a(X, L) > 0$ if and only if X is uniruled. We note that, by flat base change, the a-constant is independent of base change to another field.

It was shown in [Birkar et al. 2010] that, if X is uniruled with klt singularities and L is ample, then $a(X, L)$ is a rational number. If L is big and not ample, then $a(X, L)$ can be irrational (see [Hassett et al. 2015, Example 6]). For a smooth projective variety X, the function $a(X, \_) : \text{Big}^1(X)_{\mathbb{R}} \to \mathbb{R}$ is a continuous function (see [Lehmann et al. 2018, Lemma 3.2]).

Definition 2.4. A morphism $f : X \to T$ of irreducible varieties is called a family of varieties if the generic fiber is geometrically integral. A family of projective varieties is a projective morphism which is a family of varieties.

We recall the following result about the a-constant in families:

Theorem 2.5 [Lehmann and Tanimoto 2017; Hacon et al. 2013]. *Let $f : X \to T$ be a smooth family of uniruled projective varieties over an algebraically closed field. Let L be an f-big and f-nef $\mathbb{Q}$-Cartier divisor on X. Then there exists a nonempty subset $U \subset T$ such that $a(X_t, L|_{X_t})$ is constant for $t \in U$ and the Iitaka dimension $\kappa(K_{X_t} + a(X_t, L|_{X_t})L|_{X_t})$ is constant for $t \in U$.*

Definition 2.6. Let X be a smooth projective variety over k and L a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The b-constant is defined as

$$b(k, X, L) = \text{codim of minimal supported face of } \Lambda_{\text{eff}}(X) \text{ containing the class of } K_X + a(X, L)L.$$

It is invariant under pullback by a birational morphism of smooth varieties [Hassett et al. 2015]. For a singular variety X we define $b(k, X, L) := b(k, \tilde{X}, \pi^* L)$, by pulling back to a resolution. By Remark 2.1, if we have an extension $k \subset k'$ of algebraically closed fields, the pull back map $\text{NS}(X) \to \text{NS}(X_{k'})$ is an isomorphism and the pseudoeffective cones are isomorphic by flat base change. Also, $K_X + a(X, L)L$ maps to $K_{X_{k'}} + a(X_{k'}, L_{k'})L_{k'}$ under this isomorphism. Therefore the b-constant is unchanged, i.e.,

$b(k', X_{k'}, L_{k'}) = b(k, X, L)$. From now on, when our base field is algebraically closed we write $b(X, L)$ instead of $b(k, X, L)$.

Minimal and canonical models. Let (X, Δ) be a klt pair, with Δ a $\mathbb{R}$-divisor and $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $f : X \to T$ be a projective morphism. A pair (X', Δ') sitting in a diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & T & \end{array}$$

is called a $\mathbb{Q}$-factorial minimal model of (X, Δ) over T if:

(1) X' is $\mathbb{Q}$-factorial.

(2) f' is projective.

(3) ϕ is a birational contraction.

(4) $\Delta' = \phi_* \Delta$.

(5) $K_{X'} + \Delta'$ is f'-nef.

(6) $a(E, X, \Delta) < a(E, X', \Delta')$ for all ϕ-exceptional divisors $E \subset X$. Equivalently, if for a common resolution $p : W \to X$ and $q : W \to X'$, we may write

$$p^*(K_X + \Delta) = q^*(K_{X'} + \Delta') + E$$

where $E \geq 0$ is q-exceptional and the support of E contains the strict transform of the ϕ-exceptional divisors.

A canonical model over T is defined to be a projective morphism $g : Z \to T$ with a surjective morphism $\pi : X' \to Z$ with connected geometric fibers from a minimal model such that $K_{X'} + \Delta' = \pi^* H$ for an $\mathbb{R}$-Cartier divisor H on Z which is ample over T.

Suppose $K_X + \Delta$ is f-pseudoeffective and Δ is f-big, then by [Birkar et al. 2010], we may run a $(K_X + \Delta)$-MMP with scaling to obtain a $\mathbb{Q}$-factorial minimal model (X', Δ') over T. It follows that (X', Δ') is also klt. Then the basepoint freeness theorem implies that $(K_{X'} + \Delta')$ is f'-semiample. Hence there exists a relative canonical model $g : Z \to T$. In particular, if Δ is a $\mathbb{Q}$-divisor, the $\mathcal{O}_T$-algebra

$$\mathfrak{R}(X', \Delta') = \bigoplus_m f'_* \mathcal{O}_{X'}(\lfloor m(K_{X'} + \Delta') \rfloor)$$

is finitely generated. Let $X' \to Z \to \operatorname{Proj}_T(\mathfrak{R}(X', \Delta'))$ be the Stein factorization of the natural morphism. Then Z is the relative canonical model over T.

The following result relates the relative MMP over a base to the MMP of the fibers (see [de Fernex and Hacon 2011, Theorem 4.1; Kollár and Mori 1992, 12.3] for related statements).

Lemma 2.7. *Let $f : X \to T$ be a flat projective morphism of normal varieties. Suppose X is $\mathbb{Q}$-factorial and D be an effective $\mathbb{R}$-divisor such that (X, D) is klt. Let $\psi : X \to Z$ be the contraction of a $K_X + D$-negative extremal ray of $\overline{\mathrm{NE}}(X/T)$. Suppose for $t \in T$ very general, the restriction map $N^1(X/T) \to N^1(X_t)$ is surjective and X_t is $\mathbb{Q}$-factorial.*

Let $t \in T$ be very general. If $\psi_t : X_t \to Z_t$ is not an isomorphism, then it is a contraction of a $K_{X_t} + D_t$-negative extremal ray, and:

(a) *If ψ is of fiber type, so is ψ_t.*

(b) *If ψ is a divisorial contraction of a divisor G, then ψ_t is a divisorial contraction of G_t and $N^1(Z/T) \to N^1(Z_t)$ is surjective.*

(c) *If ψ is a flipping contraction and $\psi^+ : X^+ \to Z$ is the flip, then ψ_t is a flipping contraction and X_t^+ is the flip of $\psi_t : X_t \to Z_t$. Also, $N^1(X^+/T) \to N^1(X_t^+)$ is surjective.*

Proof. Since the natural restriction map $N^1(X/T) \to N^1(X_t)$ is surjective for very general $t \in T$, any curve in X_t that spans a $K_X + D$-negative extremal ray R of $\overline{\mathrm{NE}}(X/T)$, also spans a $K_{X_t} + D_t$ negative extremal ray R_t of $\overline{\mathrm{NE}}(X_t)$. For $t \in T$ general, the base change Z_t is normal and the morphism $X_t \to Z_t$ has connected fibers, hence $\psi_{t*}\mathcal{O}_{X_t} = \mathcal{O}_{Z_t}$. Hence ψ_t is the contraction of the ray R_t for very general $t \in T$.

If ψ is of fiber type, then so is ψ_t for general $t \in T$. Let us assume that ψ is birational.

Suppose ψ is a divisorial contraction of a divisor G. Then all components of G_t are contracted. By the injectivity of $N_1(X_t) \to N_1(X/T)$, we see that ψ_t is an extremal divisorial contraction of G_t (and G_t is irreducible). Since X_t is $\mathbb{Q}$-factorial, we have the surjectivity of $N^1(Z/T) \to N^1(Z_t)$.

Suppose ψ is a flipping contraction and $\phi : X \dashrightarrow X^+$ is the flip. For very general $t \in T$, $X_t \to Z_t$ is a small birational contraction of the ray R_t. Also, $X_t^+ \to Z_t$ is also small birational and $K_{X_t^+} + (\phi_* D)_t$ is ψ^+-ample for $t \in T$ general. Therefore $\phi_t : X_t \dashrightarrow X_t^+$ is the flip. The surjectivity of $N^1(X^+/T) \to N^1(X_t^+)$ follows from ψ_t being an isomorphism in codimension one. □

The next proposition allows us to compare minimal and canonical models over a base to those of a general fiber.

Proposition 2.8. *Let $f : X \to T$ be a smooth morphism. Suppose X is smooth and Δ is an f-big and f-nef $\mathbb{R}$-divisor such that (X, Δ) is klt. Suppose the local system $\mathcal{GN}^1(X/T)$ has trivial monodromy. Let $\phi : X \dashrightarrow X'$ be the relative minimal model obtained by running a $(K_X + \Delta)$-MMP over T and $\pi : X' \to Z$ be the morphism to the canonical model over T. Then for a general $t \in T$:*

(1) *The base change $\phi_t : X_t \dashrightarrow X'_t$ is a $\mathbb{Q}$-factorial minimal model of (X_t, Δ_t).*

(2) *Also, $\pi_t : X'_t \to Z_t$ is the canonical model of (X_t, Δ_t).*

Proof. (1) Since $\mathcal{GN}^1(X/T)$ has trivial monodromy, the natural restriction morphism $N^1(X/T) \xrightarrow{\sim} N^1(X_t)$ is an isomorphism for $t \in T$ very general. Then Lemma 2.7 implies that, for very general $t \in T$, the base change $\phi_t : X_t \dashrightarrow X'_t$ is a composition of steps of the $(K_{X_t} + \Delta_t)$-MMP. In particular, X'_t is $\mathbb{Q}$-factorial for a very general $t \in T$. The fibers X'_t have terminal singularities, by [Lehmann et al. 2018, Lemma 2.4]. Hence [Kollár and Mori 1992, 12.1.10] implies that there is a nonempty open $U \subset T$ such that X'_t is $\mathbb{Q}$-factorial for $t \in U$. For a general $t \in T$, the conditions (2)–(6) in the definition of a minimal model follows easily. Therefore, (X'_t, Δ'_t) is a $\mathbb{Q}$-factorial minimal model of (X_t, Δ_t) for general $t \in T$.

(2) Let $g : Z \to T$ be the relative canonical model. Now Z is normal. Therefore, for a general $t \in T$, the base change Z_t is normal and $X'_t \to Z_t$ has geometrically connected fibers. Also, $K_{X'} + \Delta = g^*H$ where H is a π-ample $\mathbb{R}$-Cartier divisor on Z. By adjunction, $K_{X'_t} + \Delta'_t$ is pull-back of an ample $\mathbb{R}$-Cartier divisor on Z_t. Hence, $X'_t \to Z_t$ is the canonical model for general $t \in T$. □

Let X be a smooth uniruled projective variety over an algebraically closed field and L a big and nef $\mathbb{Q}$-divisor on X. The following result (contained in [Lehmann et al. 2018]) gives a geometric interpretation of the b-constant.

Proposition 2.9. *Let $\phi : X \dashrightarrow X'$ be a $K_X + a(X, L)L$-minimal model. Then*:

(1) $b(X, L) = b(X', \phi_* L)$.

(2) *If $\kappa(K_X + a(X, L)L) = 0$ then $b(X, L) = \operatorname{rk} N^1(X')_{\mathbb{R}}$.*

(3) *If $\kappa(K_X + a(X, L)L) > 0$ and $\pi : X' \to Z$ is the morphism to the canonical model and Y' is a general fiber of π. Then*

$$b(X, L) = \operatorname{rk} N^1(X')_{\mathbb{R}} - \operatorname{rk} N^1_{\pi}(X')_{\mathbb{R}} = \operatorname{rk}(\operatorname{im}(N^1(X')_{\mathbb{R}} \to N^1(Y')_{\mathbb{R}}))$$

where $N^1_{\pi}(X')_{\mathbb{R}}$ is the span of the π-vertical divisors and $N^1(X')_{\mathbb{R}} \to N^1(Y')_{\mathbb{R}}$ is the restriction map.

Proof. Part (1) is the statement of Lemma 3.5 in [Lehmann et al. 2018]. Part (2) follows from part (1). By abundance, $K_X + a(X, L)\phi_* L$ is semiample. Then $\kappa(K_X + a(X, L)L) = 0$ implies that $K_X + a(X, L)\phi_* L \equiv 0$. Hence, $b(X, L) = b(X', \phi_* L) = \operatorname{rk} N^1(X')_{\mathbb{R}}$. Part (3) follows from the proof of Theorem 4.5 in [Lehmann et al. 2018]. □

In the case when the fibers are adjoint-rigid, constancy of the b-constant was proved in [Lehmann and Tanimoto 2017].

Proposition 2.10 [Lehmann and Tanimoto 2017, Proposition 4.4]. *Let $f : X \to T$ be a smooth family of projective varieties. Suppose L is an f-big and f-nef Cartier divisor on X. Assume that for a general member X_t, we have $\kappa(K_{X_t} + a(X_t, L_t)L_t) = 0$. Then $b(X_t, L_t)$ is constant for general $t \in T$.*

Global invariant cycles. Let $\pi : X \to Z$ be a morphism of complex algebraic varieties. Then, by Verdier's generalization of Ehresmann's theorem [Verdier 1976, Corolaire 5.1], there exists a Zariski open $U \subset Z$ such that $\pi^{-1}(U) \to U$ is a topologically locally trivial fibration (in the analytic topology), i.e., every point $z \in U$ has a neighborhood $N \subset U$ in the analytic topology, such that there is a fiber preserving homeomorphism

$$\begin{array}{ccc} \pi^{-1}(N) & \xrightarrow{\sim} & N \times F \\ & \searrow \quad \swarrow & \\ & N & \end{array}$$

where $F = \pi^{-1}(z)$. Consequently we have a monodromy action of $\pi_1(U, z)$ on the cohomology of the fiber $H^i(X_z, \mathbb{R})$.

Let $\pi : X \to Z$ be a morphism of normal projective varieties. Note that by generic smoothness and the discussion above, given any resolution of singularities $\mu : \tilde{X} \to X$, we may choose a Zariski open $U \subset Z$ such that $\pi \circ \mu$ is smooth over U and $(\pi \circ \mu)^{-1}(U) \to U$ and $\pi^{-1}(U) \to U$ are topologically locally trivial fibrations.

The following result is an adaptation of Deligne's global invariant cycles theorem [1971] to the case of singular varieties, which helps us to compute the b-constant.

Lemma 2.11. *Let $\pi : X \to Z$ be a morphism of normal projective varieties over $\mathbb{C}$ where X is $\mathbb{Q}$-factorial. Let $\mu : \tilde{X} \to X$ be a resolution of singularities. Let $U \subset Z$ be a Zariski open subset such that $\pi \circ \mu$ is smooth over U and $(\pi \circ \mu)^{-1}(U) \to U$ and $\pi^{-1}(U) \to U$ are topologically locally trivial fibrations* (*in the analytic topology*). *Suppose for general $z \in U$, the fiber $X_z := \pi^{-1}(z)$ is rationally connected with rational singularities. Then*

$$\operatorname{im}(N^1(X)_{\mathbb{R}} \to N^1(X_z)_{\mathbb{R}}) \simeq H^2(X_z, \mathbb{R})^{\pi_1(U,z)}$$

for general $z \in U$, where $H^2(X_z, \mathbb{R})^{\pi_1(U,z)}$ is the monodromy invariant subspace.

Proof. Let $\tilde{X}_z$ be the fiber of $\pi \circ \mu$ over z. For $z \in U$ general, $\mu_z : \tilde{X}_z \to X_z$ is a resolution of singularities. Since X_z is rationally connected, $\mathbb{Q}$-linear equivalence and numerical equivalence of $\mathbb{Q}$-Cartier divisors coincide, i.e., $\operatorname{Pic}(X_z)_{\mathbb{Q}} \simeq N^1(X_z)_{\mathbb{Q}}$. We know $h^1(\tilde{X}_z, \mathcal{O}_{\tilde{X}_z}) = h^2(\tilde{X}_z, \mathcal{O}_{\tilde{X}_z}) = 0$ since $\tilde{X}_z$ is smooth rationally connected. We also have $h^1(X_z, \mathcal{O}_{X_z}) = h^2(X_z, \mathcal{O}_{X_z}) = 0$, because X_z has rational singularities. Therefore $H^2(\tilde{X}_z, \mathbb{Q}) \simeq N^1(\tilde{X}_z)_{\mathbb{Q}}$ and $H^2(X_z, \mathbb{Q}) \simeq N^1(X_z)_{\mathbb{Q}}$.

Consider the natural restriction map on cohomology groups $H^2(\tilde{X}, \mathbb{Q}) \to H^2(\tilde{X}_z, \mathbb{Q})$. By Deligne's global invariant cycles theorem [1971] (or [Voisin 2003, 4.3.3]) we know that for $z \in U$,

$$\operatorname{im}(H^2(\tilde{X}, \mathbb{Q}) \to (H^2(\tilde{X}_z, \mathbb{Q})) = H^2(\tilde{X}_z, \mathbb{Q})^{\pi_1(U,z)}.$$

and if $\alpha \in H^2(\tilde{X}_z, \mathbb{Q})^{\pi_1(U,z)}$ is a Hodge class then there is a Hodge class $\tilde{\alpha} \in H^2(\tilde{X}, \mathbb{Q})$ such that $\tilde{\alpha}$ restricts to α. Since $H^2(\tilde{X}_z, \mathbb{Q}) \simeq N^1(\tilde{X}_z)_{\mathbb{Q}}$, we see that

$$\operatorname{im}(H^2(\tilde{X}, \mathbb{Q}) \to H^2(\tilde{X}_z, \mathbb{Q})) \simeq \operatorname{im}(N^1(\tilde{X})_{\mathbb{Q}} \to N^1(\tilde{X}_z)_{\mathbb{Q}})$$

for $z \in U$. In particular

$$\operatorname{im}(N^1(\tilde{X})_{\mathbb{R}} \to N^1(\tilde{X}_z)_{\mathbb{R}}) \simeq H^2(\tilde{X}_z, \mathbb{R})^{\pi_1(U,z)}$$

for $z \in U$.

Now the following diagram of pull-back morphisms commutes

$$\begin{array}{ccc} N^1(X)_{\mathbb{R}} & \xrightarrow{i^*} & N^1(X_z)_{\mathbb{R}} \\ \downarrow{\mu^*} & & \downarrow{\mu_z^*} \\ N^1(\tilde{X})_{\mathbb{R}} & \xrightarrow{\tilde{i}^*} & N^1(\tilde{X}_z)_{\mathbb{R}} \end{array}$$

Since $\mu : \tilde{X} \to X$ and $\mu_z : \tilde{X}_z \to X_z$ are resolutions of singularities for general $z \in U$, the vertical morphisms are injective. Therefore

$$\mathrm{im}(i^*) \simeq \mathrm{im}(\mu_z^* \circ i^*) = \mathrm{im}(\tilde{i}^* \circ \mu^*)$$

Since X is $\mathbb{Q}$-factorial, we have $N^1(\tilde{X})_{\mathbb{R}} \simeq \mu^* N^1(X)_{\mathbb{R}} \oplus \bigoplus_j \mathbb{R}E_j$ where E_j are the μ-exceptional divisors. For $z \in U$ general, the restriction of a μ-exceptional divisor E_j to $\tilde{X}_z$ is μ_z-exceptional. In $N^1(\tilde{X}_z)_{\mathbb{R}}$, we have $\mathrm{im}(\mu_z^*) \cap \oplus_j \mathbb{R}E_j^z = 0$ where E_j^z are μ_z-exceptional. Therefore

$$\mathrm{im}(\tilde{i}^* \circ \mu^*) = \mathrm{im}(\tilde{i}^*) \cap \mathrm{im}(\mu_z^*).$$

Recall that we have the isomorphisms given by first Chern class $N^1(\tilde{X}_z)_{\mathbb{R}} \simeq H^2(\tilde{X}_z, \mathbb{R})$ and $N^1(X_z)_{\mathbb{R}} \simeq H^2(X_z, \mathbb{R})$. We know that $\mathrm{im}(\tilde{i}^*) \simeq H^2(\tilde{X}_z, \mathbb{R})^{\pi_1(U,z)}$ and the monodromy actions on $H^2(X_z, \mathbb{R})$ and $H^2(\tilde{X}_z, \mathbb{R})$ commute with the pullback map μ_z^*. Hence

$$\mathrm{im}(\tilde{i}^*) \cap \mathrm{im}(\mu_z^*) \simeq H^2(X_z, \mathbb{R})^{\pi_1(U,z)}.$$

Therefore

$$\mathrm{im}(N^1(X)_{\mathbb{R}} \to N^1(X_z)_{\mathbb{R}}) = \mathrm{im}(\tilde{i}^*) \cap \mathrm{im}(\mu_z^*) \simeq H^2(X_z, \mathbb{R})^{\pi_1(U,z)}$$

for general $z \in U$. □

3. Constancy on very general fibers

Let $f : X \to T$ be a projective morphism and L is an f-big $\mathbb{Q}$-Cartier divisor. We denote $L_{\bar{t}} := L|_{X_{\bar{t}}}$, the restriction to the geometric fiber of t.

Lemma 3.1. *Let $X \to T$ be a smooth projective family of varieties and $s, t \in T$ such that s is a specialization of t:*

(a) *$\Lambda_{\mathrm{eff}}(X_{\bar{t}})$ maps into $\Lambda_{\mathrm{eff}}(X_{\bar{s}})$ under the specialization morphism* $\mathrm{sp}_{\bar{t},\bar{s}} : \mathrm{NS}_{\mathbb{R}}(X_{\bar{t}}) \to \mathrm{NS}_{\mathbb{R}}(X_{\bar{s}})$.

(b) *Suppose $a(X_{\bar{t}}, L_{\bar{t}}) = a(X_{\bar{s}}, L_{\bar{s}})$ and $\rho(X_{\bar{t}}) = \rho(X_{\bar{s}})$. Then $b(X_{\bar{t}}, L_{\bar{t}}) \geq b(X_{\bar{s}}, L_{\bar{s}})$.*

Proof. (a) Let D be an effective divisor in $\mathrm{NS}(X_{\bar{t}})_{\mathbb{R}}$. We may pick a discrete valuation ring R with a morphism $\phi : \mathrm{Spec}\, R = \{s', t'\} \to T$ where s' and t' map to s and t respectively and t' is the generic point. By Remark 2.1 we have isomorphisms $\mathrm{NS}(X_{\bar{t}}) \xrightarrow{\sim} \mathrm{NS}(X_{\bar{t'}})$ and $\mathrm{NS}(X_{\bar{s}}) \xrightarrow{\sim} \mathrm{NS}(X_{\bar{s'}})$. Therefore we may assume T is the spectrum of a discrete valuation ring R and t is the generic point t'. Now D is defined over a finite extension L of $k(t')$. We can replace R by a discrete valuation ring R_L with quotient field L. Then the image of D under $\mathrm{Pic}(X_{t'}) \xrightarrow{\sim} \mathrm{Pic}(\phi^* X) \to \mathrm{Pic}(X_{s'})$ is effective by semicontinuity.

After passing to the algebraic closure and taking quotient by algebraic equivalence we conclude that, $\mathrm{sp}_{\bar{t},\bar{s}}$ maps D to an effective divisor class.

(b) Since $\rho(X_{\bar{t}}) = \rho(X_{\bar{s}})$, we have an isomorphism $\mathrm{NS}(X_{\bar{t}})_{\mathbb{R}} \to \mathrm{NS}(X_{\bar{s}})_{\mathbb{R}}$. Let $a := a(X_{\bar{s}}, L_{\bar{s}}) = a(X_{\bar{t}}, L_{\bar{t}})$. Note that $\mathrm{sp}_{\bar{t},\bar{s}}$ maps $K_{X_{\bar{t}}} + aL_{\bar{t}}$ to $K_{X_{\bar{s}}} + aL_{\bar{s}}$. Let F be a supporting hyperplane of $\Lambda_{\mathrm{eff}}(X_{\bar{s}})$ corresponding to the minimal supporting face containing $K_{X_{\bar{s}}} + aL_{\bar{s}}$. Since $\Lambda_{\mathrm{eff}}(X_{\bar{t}}) \subset \Lambda_{\mathrm{eff}}(X_{\bar{s}})$, we see that F is a supporting hyperplane of $\Lambda_{\mathrm{eff}}(X_{\bar{t}})$ containing $K_{X_{\bar{t}}} + aL_{\bar{t}}$. Therefore,

$$b(X_{\bar{s}}, L_{\bar{s}}) = \mathrm{codim}(F \cap \Lambda_{\mathrm{eff}}(X_{\bar{s}})) \leq \mathrm{codim}(F \cap \Lambda_{\mathrm{eff}}(X_{\bar{t}})) \leq b(X_{\bar{t}}, L_{\bar{t}}). \qquad \square$$

Lemma 3.2. *Let $X \to T$ a smooth projective family. Let $\eta \in T$ be the generic point. We denote $a = a(X_{\bar{\eta}}, L_{\bar{\eta}})$, $n = \rho(X_{\bar{\eta}})$ and $b = b(X_{\bar{\eta}}, L_{\bar{\eta}})$. For $m \in \mathbb{N}$, define*

$$T_m := \left\{t \in T \mid a(X_{\bar{t}}, L_{\bar{t}}) \leq a - \tfrac{1}{m}\right\}, \quad T_0 := \{t \in T \mid \rho(X_{\bar{t}}) > n\}$$

and

$$T_\infty := \{t \in T \mid a(X_{\bar{t}}, L_{\bar{t}}) = a, \rho(X_{\bar{t}}) = n, b(X_{\bar{t}}, L_{\bar{t}}) < b\}.$$

We let $Z_T := \bigcup_m T_m \cup T_\infty \cup T_0$. Then:

(a) *Z_T is closed under specialization.*

(b) *If we base change by a morphism of schemes $g : T' \to T$, then $Z_{T'} = g^{-1}(Z_T)$.*

Proof. (a) Let $t \in Z_T$ and s a specialization of t in T. If $t \in T_m$ for some $m \in \mathbb{N}$, then Lemma 3.1(a) implies that $K_{X_{\bar{s}}} + a(X_{\bar{t}}, L_{\bar{t}})L_{\bar{s}} \in \Lambda_{\mathrm{eff}}(X_{\bar{s}})$. Therefore, $a(X_{\bar{s}}, L_{\bar{s}}) \leq a(X_{\bar{t}}, L_{\bar{t}})$ and hence $s \in T_m$. If $t \in T_0$, then by Proposition 2.2(a), $\rho(X_{\bar{s}}) \geq \rho(X_{\bar{t}})$ and $s \in T_0$. If $t \notin T_0 \cup \bigcup_m T_m$, then $\rho(X_{\bar{t}}) = n$ and $a(X_{\bar{t}}, L_{\bar{t}}) = a$. Then Lemma 3.1(b) implies $b(X_{\bar{s}}, L_{\bar{s}}) \leq b(X_{\bar{t}}, L_{\bar{t}}) < b$. Therefore $s \in T_\infty$ and Z_T is closed under specialization.

(b) This follows from the fact that the Picard number and a- and b-constants are invariant under algebraically closed base extension. $\square$

Proof of Theorem 1.1. By passing to a resolution of singularities and using generic smoothness, we may exclude a closed subset of T to assume the family $f : X \to T$ is smooth and T is affine. Since our base field k is algebraically closed, we may find a subfield $k' \subset k$ which is the algebraic closure of a field finitely generated over $\mathbb{Q}$, and there exists a finitely generated k'-algebra A such that our family $X \to T$ and L are a base change of a family $X_A \to \mathrm{Spec}\, A$ and a line bundle L_A on X_A. Now $B = \mathrm{Spec}\, A$ is countable and hence $Z_B = \bigcup_{b \in B} \overline{\{b\}}$ is a countable union of closed subsets by Lemma 3.2(a). Now Lemma 3.2(b) implies that Z_T is a countable union of closed subsets. $\square$

4. Family of uniruled varieties

In this section we prove Theorem 1.2 Let $f : X \to T$ be a projective family of uniruled varieties over an algebraically closed field k of characteristic 0 and L an f-nef and f-big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

By a standard argument using the Lefschetz principle, it is enough to prove the statement for $k = \mathbb{C}$. We will henceforth assume that $k = \mathbb{C}$.

We can reduce to the statement for closed points only, as follows. Let us assume that there is an open $U \subset T$ such that $b(X_t, L_t) = b$ is constant for all closed points $t \in U$. Let $s \in U$ and $Z = \overline{\{s\}} \cap U$. By applying Theorem 1.1 to the family over Z, we may find $F = \bigcup_i F_i \subset Z$ a countable union of closed subvarieties such that $b(X_{\bar{t}}, L_{\bar{t}})$ is constant on $Z \setminus F$. Since $\mathbb{C}$ is uncountable, there exists a closed point $t \in Z \setminus F$. Now $s \in Z \setminus F$, since s is the generic point of Z. Therefore, $b(X_{\bar{s}}, L_{\bar{s}}) = b(X_t, L_t) = b$. Since $s \in U$ was arbitrary, we conclude that $b(X_{\bar{t}}, L_{\bar{t}}) = b$ for all $t \in U$. Therefore it is enough to prove the statement for closed points.

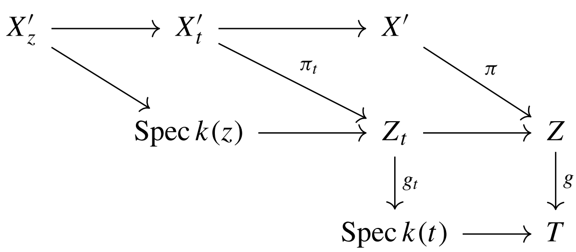
Proof of Theorem 1.2 for closed points when $k = \mathbb{C}$. We may replace X by a resolution, and by generic smoothness, we may exclude a closed subset of the base to assume that $f : X \to T$ is a smooth family. By Theorem 2.5, we can shrink T such that $a(X_t, L_t) = a$ for all $t \in T$ and $\kappa(K_{X_t} + aL_t)$ is independent of t. We may assume that T is affine. Since L is f-big and f-nef, we can replace L by a $\mathbb{Q}$-linearly equivalent divisor to assume that (X, aL) is klt.

Since the local system $\mathcal{GN}^1(X/T)$ has finite monodromy, we can base change to a finite étale cover of T to assume that $\mathcal{GN}^1(X/T)$ has trivial monodromy.

If $\kappa(K_{X_t} + aL_t) = 0$ then we can conclude by Proposition 2.10. Let us assume that $\kappa(K_{X_t} + aL_t) = k > 0$ for all $t \in T$.

Since $K_X + aL$ is f-pseudoeffective and aL is f-big, we may run a $(K_X + aL)$-MMP over T to obtain a relative minimal model $\phi : X \dashrightarrow X'$. Let $\pi : X' \to Z$ be the morphism to the relative canonical model over T. By Proposition 2.8, we may replace T by an open subset to assume that the base change $\phi_t : X_t \dashrightarrow X'_t$ is a $\mathbb{Q}$-factorial minimal model and $\pi_t : X'_t \to Z_t$ is the canonical model for (X_t, aL_t) for all $t \in T$.

For $z \in Z$, we denote the image of z in T by t and let X'_z denote the fiber of $\pi : X' \to Z$ over z.



Let $\mu : \tilde{X} \to X'$ be a resolution of singularities. We may replace T by an open subset to assume that $\tilde{X} \to T$ is smooth. Let $\tilde{X}_z$ be the fiber of $\tilde{\pi} : \tilde{X} \to Z$ over $z \in Z$. By [Verdier 1976, Corrolaire 5.1] we can find a Zariski open $U_Z \subset Z$ such that $\tilde{\pi}$ is smooth over U_Z and $\tilde{\pi}^{-1}(U_Z) \to U_Z$ and $\pi^{-1}(U_Z) \to U_Z$ both are topologically locally trivial fibrations (in the analytic topology). Again we may replace T by a Zariski open $V \subset T$ to assume that $U_Z \to T$ is a topologically locally trivial fibration (in the analytic topology). Let $U_t \subset Z_t$ denote the fiber of U_Z over $t \in T$.

For all $z \in U_Z$, there is a monodromy action of $\pi_1(U_t, z)$ on $H^2(X'_z, \mathbb{Z})$ acting by an integral matrix M_z on the free part. Now for any two points z and z' in U_Z, the fundamental groups $\pi_1(U_t, z)$ and $\pi_1(U_{t'}, z')$ are isomorphic, since $U_Z \to T$ is a locally trivial fibration. Also, the cohomology groups $H^2(X'_z, \mathbb{Z})$ and $H^2(X'_{z'}, \mathbb{Z})$ are isomorphic, because $\pi^{-1}(U_Z) \to U_Z$ is a locally trivial fibration. Since the monodromy actions depend continuously on $z \in U_Z$, we see that M_z is constant. Therefore the monodromy invariant subspaces have constant rank, i.e., $\operatorname{rk} H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}$ is constant for all $z \in U_Z$.

By [Hacon and McKernan 2007] we know that a general fiber X'_z is rationally connected and has terminal singularities. Since X'_t is $\mathbb{Q}$-factorial, Lemma 2.11 implies that

$$\operatorname{rk}(\operatorname{im}(N^1(X'_t)_{\mathbb{R}} \to N^1(X'_z)_{\mathbb{R}}) = \operatorname{rk} H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}.$$

for general $z \in U_t$. Now using Proposition 2.9(3) we have

$$b(X_t, L_t) = \operatorname{rk} H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}$$

for general $z \in U_Z$. Since $\operatorname{rk} H^2(X'_z, \mathbb{R})^{\pi_1(U_t,z)}$ is constant for $z \in U_Z$, we may conclude that $b(X_t, L_t)$ is constant for general $t \in T$.

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References

[Batyrev and Manin 1990] V. V. Batyrev and Y. I. Manin, "Sur le nombre des points rationnels de hauteur borné [sic] des variétés algébriques", *Math. Ann.* **286**:1-3 (1990), 27–43. MR Zbl

[Birkar et al. 2010] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, "Existence of minimal models for varieties of log general type", *J. Amer. Math. Soc.* **23**:2 (2010), 405–468. MR Zbl

[Boucksom et al. 2013] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", *J. Algebraic Geom.* **22**:2 (2013), 201–248. MR Zbl

[Browning and Loughran 2017] T. D. Browning and D. Loughran, "Varieties with too many rational points", *Math. Z.* **285**:3-4 (2017), 1249–1267. MR Zbl

[Deligne 1971] P. Deligne, "Théorie de Hodge, II", *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57. MR Zbl

[de Fernex and Hacon 2011] T. de Fernex and C. D. Hacon, "Deformations of canonical pairs and Fano varieties", *J. Reine Angew. Math.* **651** (2011), 97–126. MR Zbl

[Franke et al. 1989] J. Franke, Y. I. Manin, and Y. Tschinkel, "Rational points of bounded height on Fano varieties", *Invent. Math.* **95**:2 (1989), 421–435. MR Zbl

[Fujita 1987] T. Fujita, "On polarized manifolds whose adjoint bundles are not semipositive", pp. 167–178 in *Algebraic geometry* (Sendai, Japan, 1985), edited by T. Oda, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987. MR Zbl

[Fujita 1992] T. Fujita, "On Kodaira energy and adjoint reduction of polarized manifolds", *Manuscripta Math.* **76**:1 (1992), 59–84. MR Zbl

[Hacon and McKernan 2007] C. D. Hacon and J. McKernan, "On Shokurov's rational connectedness conjecture", *Duke Math. J.* **138**:1 (2007), 119–136. MR Zbl

[Hacon et al. 2013] C. D. Hacon, J. McKernan, and C. Xu, "On the birational automorphisms of varieties of general type", *Ann. of Math.* (2) **177**:3 (2013), 1077–1111. MR Zbl

[Hassett et al. 2015] B. Hassett, S. Tanimoto, and Y. Tschinkel, "Balanced line bundles and equivariant compactifications of homogeneous spaces", *Int. Math. Res. Not.* **2015**:15 (2015), 6375–6410. MR Zbl

[Kollár and Mori 1992] J. Kollár and S. Mori, "Classification of three-dimensional flips", *J. Amer. Math. Soc.* **5**:3 (1992), 533–703. MR Zbl

[Le Rudulier 2013] C. Le Rudulier, "Points algébriques de hauteur bornée sur une surface", preprint, 2013, Available at http://cecile.lerudulier.fr/Articles/surfaces.pdf.

[Lehmann and Tanimoto 2017] B. Lehmann and S. Tanimoto, "On the geometry of thin exceptional sets in Manin's conjecture", *Duke Math. J.* **166**:15 (2017), 2815–2869. MR Zbl

[Lehmann and Tanimoto 2019] B. Lehmann and S. Tanimoto, "Geometric Manin's conjecture and rational curves", *Compos. Math.* **155**:5 (2019), 833–862. MR Zbl

[Lehmann et al. 2018] B. Lehmann, S. Tanimoto, and Y. Tschinkel, "Balanced line bundles on Fano varieties", *J. Reine Angew. Math.* **743** (2018), 91–131. MR Zbl

[Maulik and Poonen 2012] D. Maulik and B. Poonen, "Néron–Severi groups under specialization", *Duke Math. J.* **161**:11 (2012), 2167–2206. MR Zbl

[Néron 1952] A. Néron, "Problèmes arithmétiques et géométriques rattachés à la notion de rang d'une courbe algébrique dans un corps", *Bull. Soc. Math. France* **80** (1952), 101–166. MR Zbl

[Peyre 2003] E. Peyre, "Points de hauteur bornée, topologie adélique et mesures de Tamagawa", *J. Théor. Nombres Bordeaux* **15**:1 (2003), 319–349. MR Zbl

[Verdier 1976] J.-L. Verdier, "Stratifications de Whitney et théorème de Bertini–Sard", *Invent. Math.* **36** (1976), 295–312. MR Zbl

[Voisin 2003] C. Voisin, *Hodge theory and complex algebraic geometry, II*, Cambridge Stud. Adv. Math. **77**, Cambridge Univ. Press, 2003. MR Zbl

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