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We extend Urban's construction of eigenvarieties for reductive groups G such that $G(\mathbb{R})$ has discrete series to include characteristic p points at the boundary of weight space. In order to perform this construction, we define a notion of "locally analytic" functions and distributions on a locally $\mathbb{Q}_p$-analytic manifold taking values in a complete Tate $\mathbb{Z}_p$-algebra in which p is not necessarily invertible. Our definition agrees with the definition of locally analytic distributions on p-adic Lie groups given by Johansson and Newton.

1. Introduction

1.1. ***Statement of results.*** The study of p-adic families of automorphic forms began with the work of Hida [1986; 1988; 1994]. Coleman and Mazur [1998] (see also Coleman [1996; 1997]) introduced the eigencurve, which parametrizes overconvergent p-adic modular forms of finite slope. Coleman and Mazur used a geometric definition of p-adic modular forms, based on the original definition of Katz [1973]. It is also possible to define p-adic automorphic forms using a cohomological approach. Several constructions of eigenvarieties are based on overconvergent cohomology, introduced by Stevens [1994] and later generalized by Ash and Stevens [2008]. These include the constructions of Urban [2011] and Hansen [2017]. Emerton [2006b] has also constructed eigenvarieties using a somewhat different cohomological approach.

The eigenvarieties mentioned above are all rigid analytic spaces, so they parametrize forms that have coefficients in $\mathbb{Q}_p$-algebras. Recently, there has been interest in studying forms with coefficients in characteristic p. Liu, Wan, and Xiao [Liu et al. 2017] constructed $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-modules of automorphic forms

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for definite quaternion algebras. By taking quotients of this module, one can obtain both traditional p-adic automorphic forms and forms with coefficients in $\mathbb{F}_p[\![\mathbb{Z}_p^\times]\!]$ whose existence had been conjectured by Coleman. Using these modules, Liu, Wan, and Xiao proved certain cases of a conjecture of Coleman and Mazur and Buzzard and Kilford [2005] concerning the eigenvalues of the U_p operator near the boundary of the weight space. Andreatta, Iovita, and Pilloni [Andreatta et al. 2018] constructed an eigencurve that included characteristic p points by extending Katz's definition of p-adic modular forms.

In this paper, we will show how Urban's eigenvarieties can be extended to include the characteristic p points at the boundary of weight space.

In order to explain our results in more detail, we will first describe the basic idea of overconvergent cohomology. Let G be a connected reductive algebraic group over $\mathbb{Q}$ such that $G_{\mathbb{Q}_p}$ is quasisplit. Let $\mathbb{A}$ be the adeles over $\mathbb{Q}$, let $\mathbb{A}_f^p$ be the finite adeles away from p, let G_∞^+ be the identity component of $G(\mathbb{R})$, and let Z_G be the center of G. Let T_0 be a maximal compact torus of $G(\mathbb{Q}_p)$, and let N_0^- be an open compact subgroup of a maximal unipotent subgroup of $G(\mathbb{Q}_p)$. We may consider the space

$$\mathcal{X} := G(\mathbb{A})/K^p G_\infty^+$$

as a locally $\mathbb{Q}_p$-analytic manifold. Let F be a finite extension of $\mathbb{Q}_p$, and let $\lambda\colon T_0 \to F^\times$ be a continuous homomorphism. Let $\mathcal{D}_{c,\lambda}$ be the space of compactly supported F-valued locally analytic distributions on $\mathcal{X}$, modulo the relations that right translation by N_0^- acts as the identity, right translation by T_0 acts by λ, and translation by $Z_G(\mathbb{Q})$ acts by the identity. One may think of the cohomology groups $H^i(G(\mathbb{Q})/Z_G(\mathbb{Q}), \mathcal{D}_{c,\lambda})$ as spaces of p-adic automorphic forms. One can also study families of p-adic automorphic forms by replacing F with an affinoid $\mathbb{Q}_p$-algebra A.

We are interested in extending overconvergent cohomology to the case where A is a $\mathbb{Z}_p$-algebra. The main challenge is to show that there is a suitable notion of locally analytic A-valued functions and distributions on $\mathcal{X}$. We will define these notions when A is a complete Tate $\mathbb{Z}_p$-algebra.

To see what the definition should be, we recall a fact from p-adic functional analysis: a function $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ is locally analytic if and only if it is of the form $f(z) = \sum_{n=0}^\infty a_n \binom{z}{n}$, where $a_n \in A$ and $|a_n|_p$ go to zero exponentially as $n \to \infty$. We will therefore define the space $\mathcal{A}(\mathbb{Z}_p, A)$ of "locally analytic" functions $\mathbb{Z}_p \to A$ to be the set of functions of the form $\sum_{n=0}^\infty a_n \binom{z}{n}$, where $a_n \in A$ and a_n to zero exponentially (i.e., $\alpha^{-n} a_n$ goes to zero for some topologically nilpotent unit α) as $n \to \infty$. If p is invertible in A, then this definition is known to coincide with the usual one. We will make a similar definition for locally analytic functions $\mathbb{Z}_p^k \to A$, and then extend the definition to locally $\mathbb{Q}_p$-analytic manifolds by gluing.

If X is a locally $\mathbb{Q}_p$-analytic manifold, then we will define modules $\mathcal{A}(X, A)$, $\mathcal{D}(X, A)$, $\mathcal{A}_c(X, A)$, $\mathcal{D}_c(X, A)$ of locally analytic functions, distributions, compactly supported functions, and compactly supported distributions, respectively.

Theorem 1.1.1 (Theorem 3.4.2). *The modules $\mathcal{A}(X, A)$, $\mathcal{D}(X, A)$, $\mathcal{A}_c(X, A)$, and $\mathcal{D}_c(X, A)$ satisfy the following properties*:

(1) $\mathcal{A}(X, A)$ *is ring.*

(2) *If* $g\colon X \to Y$ *is a locally analytic map, then composition with* g *induces homomorphisms* $\mathcal{A}(Y, A) \to \mathcal{A}(X, A)$ *and* $\mathcal{D}(X, A) \to \mathcal{D}(Y, A)$.

(3) *The functors* $U \mapsto \mathcal{A}(U, A)$ *and* $U \mapsto \mathcal{D}_c(U, A)$ *are sheaves on* X.

(4) *If* X *has the structure of a finitely generated* $\mathbb{Z}_p$*-module, then any continuous group homomorphism* $X \to A^\times$ *is in* $\mathcal{A}(X, A)$.

Remark 1.1.2. Of course, modules of continuous functions and distributions also satisfy the above properties. What makes $\mathcal{A}(X, A)$ and $\mathcal{D}(X, A)$ more like modules of locally analytic functions and distributions is that a map that multiplies all coordinates by p is "completely continuous"; see Proposition 3.3.5 and Lemma 3.3.7 for the precise statement.

Urban [2011] constructed eigenvarieties for reductive groups G such that $G(\mathbb{R})$ has discrete series. We will show how to use the locally analytic distribution modules mentioned above to extend Urban's construction to include characteristic p points.

Theorem 1.1.3 (Theorem 7.4.2). *The reduced eigenvariety* (constructed in [Urban 2011]) *extends to an adic space* $\mathcal{E}$ *over the weight space* $\mathcal{W} = \mathrm{Spa}(\mathbb{Z}_p[\![T']\!], \mathbb{Z}_p[\![T']\!])^{\mathrm{an}}$, *where* T' *is a quotient of a compact subgroup of a maximal torus in* $G(\mathbb{Q}_p)$. *Furthermore,* $\mathcal{E}$ *is equidimensional and the projection from* $\mathcal{E}$ *to the spectral variety* $\mathcal{Z}$ *is finite and surjective.*

The spectral variety $\mathcal{Z}$ is flat over $\mathcal{W}$, so the existence of characteristic p points of $\mathcal{Z}$ implies the existence of nearby characteristic zero points. It seems to be a difficult problem to prove the existence of boundary points in general; however, in many cases, one can check explicitly that they exist (see for example [Liu et al. 2017; Birkbeck 2019; Johansson and Newton 2018; Ye 2019]).

As this work was being prepared, I became aware that Christian Johansson and James Newton were independently pursuing similar work. In [Johansson and Newton 2019], they adapt Hansen's construction of eigenvarieties to include the boundary of weight space. Their definition of locally analytic distributions on $\mathbb{Z}_p^k$ is essentially the same as ours. To construct distributions on p-adic Lie groups, they use a particular choice of coordinate charts previously studied by Schneider and Teitelbaum. Our definition of locally analytic distributions therefore generalizes theirs.

1.2. ***Summary of Urban's construction and outline of the paper.*** Urban's construction of eigenvarieties is based on the framework of overconvergent cohomology developed by Stevens and Ash and Stevens. In this framework, one first defines a weight space $\mathcal{W}$, as mentioned above. Over any affinoid subspace $\mathcal{U}$ of the weight space $\mathcal{W}$, one defines a complex $C^\bullet$ of projective $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$-modules. The cohomology groups of this complex are the groups $H^i(G(\mathbb{Q})/Z_G(\mathbb{Q}), \mathcal{D}_{c,\lambda})$ mentioned above, where λ is the composition of the quotient $T_0 \to T'$ with the tautological character $T' \to \mathcal{O}_{\mathcal{W}}(\mathcal{U})^\times$.

We consider a weight space that is larger than the one considered by Ash and Stevens and Urban. In particular, our weight space contains opens $\mathcal{U}$ such that the prime p is not invertible on $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$. The main challenge in defining overconvergent cohomology over the larger weight space is to find a suitable notion of "locally analytic" $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$-valued distributions. After recalling the necessary background in Section 2, we define modules of locally analytic functions and distributions and prove some properties of these modules in Section 3. We use these modules to define overconvergent cohomology in Section 4.

Ash and Stevens proposed constructing an eigenvariety whose points correspond to systems of Hecke eigenvalues appearing in the cohomology of the complexes $C^\bullet$; Hansen's construction uses this approach. Urban took a more K-theoretic approach. Assume that $G(\mathbb{R})$ has discrete series; then cuspidal automorphic forms of regular weight contribute to a single degree q_0 of the cohomology of $C^\bullet$. So the associated systems of Hecke eigenvalues appear a net positive number of times in the formal sum $\sum_i (-1)^{i-q_0} C^i$. Urban showed that after removing the contributions from Eisenstein series, each system of Hecke eigenvalues appears a net nonnegative number of times in the formal sum. The points in Urban's eigenvariety correspond to those systems of eigenvalues appearing a net positive number of times in the formal sum.

Unfortunately, Urban's analysis of Eisenstein series contained an error. In order to argue that certain character distributions are uniquely defined, Urban assumed that the region of convergence of an Eisenstein series is (up to translation) a union of Weyl chambers. However, this assumption is not true. In Section 5, we correct this error by giving a new argument for uniqueness.

Urban's construction of eigenvarieties makes use of the theory of pseudocharacters. We will instead use Chenevier's theory of determinants [2014], which is equivalent to the theory of pseudocharacters in characteristic zero but better behaved in our setting where the prime p may not be invertible. Section 6 recalls some basic facts about determinants and proves some criteria for establishing that a ratio of two determinants is again a determinant.

Finally, in Section 7, we construct the eigenvariety. We adapt Urban's construction from the setting of rigid analytic spaces to the setting of adic spaces.

2. Modules over complete Tate rings

2.1. *Definitions.* We begin by recalling the framework necessary for defining modules of locally analytic functions and distributions and for defining eigenvarieties. We will repeat the basic setup of [Buzzard 2007, Section 2; Andreatta et al. 2018, Appendice B]. First, we recall the definition of a Tate ring [Huber 1993, Section 1].

Definition 2.1.1. A *Huber ring* is a topological ring A such that there exists an open subring $A_0 \subset A$ and a finitely generated ideal $I \subset A_0$ such that A_0 has the I-adic topology. We say that A_0 is a *ring of definition* of A and I is an *ideal of definition* of A_0.

A *Tate ring* is a Huber ring A such that some (equivalently, any) ring of definition A_0 has an ideal of definition that is generated by a topologically nilpotent unit of A.

In Section 7, we will use the framework of adic spaces to construct the eigenvariety. Every analytic adic space can be covered by open subsets of the form $\mathrm{Spa}(A, A^+)$ with A complete Tate, so it is natural to consider this class of rings.

Throughout this section, A will denote a complete Tate ring.

Definition 2.1.2. Let X be a quasicompact topological space, and let M be a topological abelian group. We define $\mathcal{C}(X, M)$ to be the space of continuous functions $X \to M$, with the topology of uniform convergence.

Definition 2.1.3. Let S be a set, and let M be a topological abelian group. We define $c(S, M)$ to be the space of functions $f\colon S \to M$ such that for any open neighborhood U of the identity in M, the complement of $f^{-1}(U)$ is finite. We give $c(S, M)$ the topology of uniform convergence.

Definition 2.1.4. Let M be a topological A-module. We say that M is *orthonormalizable* if it is isomorphic to $c(S, A)$ for some set S. We say that M is *projective* if it is a direct summand of an orthonormalizable A-module.

Definition 2.1.5. Let M be a topological A-module. We say that a set $B \subset M$ is *bounded* if for all open neighborhoods U of the identity in M, there exists $\alpha \in A^\times$ so that $\alpha B \subseteq U$.

Definition 2.1.6. Let M and N be topological A-modules. We define $\mathcal{L}_b(M, N)$ to be the set of continuous A-module homomorphisms $M \to N$, with the topology of convergence on bounded subsets.

Definition 2.1.7. Let M and N be topological A-modules. We say that an A-module homomorphism $M \to N$ has *finite rank* if its image is a finitely presented A-module. We say that an element of $\mathcal{L}_b(M, N)$ is *completely continuous* if it is in the closure of the subspace of finite rank elements.

2.2. *Spectral theory.*

Definition 2.2.1. We define $A\{\{X\}\}$ to be the set of power series $P(X) = \sum_{n=0}^{\infty} a_n X^n$, $a_n \in A$, such that for any $\alpha \in A^\times$, $\alpha^{-n} a_n \to 0$ as $n \to \infty$.

We say that $P(X) \in A\{\{X\}\}$ is a *Fredholm series* if it has leading coefficient 1.

In Section 7, we will consider the adic space $\mathrm{Spa}(A, A^+)$ for certain complete Tate $\mathbb{Z}_p$-algebras A. If the adic space $\mathrm{Spa}(A, A^+) \times \mathbb{A}^1$ exists, then $A\{\{X\}\}$ is its ring of global sections.

Assume A is Noetherian. If M is a projective A-module and $u\colon M \to M$ is completely continuous, then we define the Fredholm series $\det(1 - Xu) \in A\{\{X\}\}$ as in [Buzzard 2007, Section 2; Andreatta et al. 2018, Section B.2.4]. To define the series, we express M as a direct summand of an orthonormalizable A-module $c(S, A)$ and extend u to a map $c(S, A) \to c(S, A)$ by having it act as zero on the orthogonal complement of M. The module $c(S, A)$ has a basis consisting of functions sending a single element of S to 1 and the rest to 0. We consider the matrix of u in this basis. The series $\det(1 - Xu)$ is defined to be limit of the characteristic polynomials of finite dimensional submatrices of this matrix. The series does not depend on the choice of embedding.

As in [Urban 2011], we will need to work with complexes. Let $M^\bullet$ be a bounded complex of projective A-modules. We will say that $u^\bullet\colon M^\bullet \to M^\bullet$ is completely continuous if each u^i is completely continuous. If $u^\bullet$ is completely continuous, then we define

$$\det(1 - Xu^\bullet) := \prod_i \det(1 - Xu^i)^{(-1)^i}.$$

Lemma 2.2.2. *Let $M^\bullet$ be a bounded complex of projective A-modules, and let $u^\bullet, v^\bullet \colon M^\bullet \to M^\bullet$ be completely continuous maps that are homotopy equivalent. Then* $\det(1 - Xu^\bullet) = \det(1 - Xv^\bullet)$.

Proof. For each nonnegative integer k, define the complex $\operatorname{SSym}^k M^\bullet$ so that $(\operatorname{SSym}^k M)^i$ is generated by formal products of k homogeneous elements of $M^\bullet$ of total degree i, subject to the relation a pair of homogeneous elements anticommutes if both have odd degree and commutes otherwise. The differential $d \colon (\operatorname{SSym}^k M)^i \to (\operatorname{SSym}^k M)^{i+1}$ is defined by

$$d(m_1 m_2 \cdots m_k) = (dm_1)m_2 \cdots m_k + (-1)^{\deg m_1} m_1(dm_2) \cdots m_k + \cdots + (-1)^{i - \deg m_k} m_1 m_2 \cdots (dm_k).$$

The maps $u^\bullet$, $v^\bullet$ induce endomorphisms $\operatorname{SSym}^k u^\bullet$, $\operatorname{SSym}^k v^\bullet$ on $\operatorname{SSym}^k M^\bullet$, and these are completely continuous and homotopy equivalent. We claim that the coefficient of X^k in $\det(1 - Xu^\bullet)^{-1}$ is $\operatorname{tr} \operatorname{SSym}^k u^\bullet$. Indeed, there is a decomposition

$$\begin{aligned}\sum_{k=0}^{\infty} X^k \operatorname{tr} \operatorname{SSym}^k u^\bullet &= \prod_{i \equiv 0(2)} \left(\sum_{k=0}^{\infty} X^k \operatorname{tr} \operatorname{Sym}^k u^i \right) \prod_{i \equiv 1(2)} \left(\sum_{k=0}^{\infty} (-X)^k \operatorname{tr} \wedge^k u^i \right) \\ &= \prod_{i \equiv 0(2)} \det(1 - Xu^i)^{-1} \prod_{i \equiv 1(2)} \det(1 - Xu^i).\end{aligned}$$

Therefore it suffices to show that for each k, $\operatorname{SSym}^k u^\bullet$ and $\operatorname{SSym}^k v^\bullet$ have the same trace. Then we may use the argument of [Urban 2011, Lemma 2.2.8]. □

2.3. *Norms.* It is often convenient to work with norms on A and on A-modules.

Definition 2.3.1. Let α be a topologically nilpotent unit of A. We define an *α-Banach norm* on A to be a continuous map $|\cdot| \colon A \to \mathbb{R}^{\geq 0}$ satisfying the following conditions:

- $|a + b| \leq \max(|a|, |b|) \quad \forall a, b \in A$.
- $|ab| \leq |a||b| \quad \forall a, b \in A$.
- $|0| = 0$, $|1| = 1$, $|\alpha||\alpha^{-1}| = 1$.
- The norm $|\cdot|$ induces the topology of A.

Definition 2.3.2. Let α be a topologically nilpotent unit of A, let $|\cdot|$ be an α-Banach norm on A, and let M be a topological A-module. We define a *$|\cdot|$-compatible norm* on M to be a continuous map $\|\cdot\| \colon M \to \mathbb{R}^{\geq 0}$ satisfying the following conditions:

- $|m + n| \leq \max(|m|, \|n\|) \quad \forall m, n \in M$.
- $\|am\| \leq |a|\|m\| \quad \forall a \in A, m \in M$.
- $\|0\| = 0$.

If, in addition, $\|\cdot\|$ induces the topology of M, we say that $\|\cdot\|$ is a *Banach norm*.

For any topologically nilpotent unit $\alpha \in A^\times$ and ring of definition A_0 of A containing α, the function $|\cdot|\colon A \to \mathbb{R}^{\geq 0}$ defined by

$$|a| = \inf_{n\in\mathbb{Z} \mid \alpha^n a \in A_0} p^n$$

is an α-Banach norm.

Furthermore, if M is a topological A-module and M_0 is an open neighborhood of zero in M that is an A_0-module, then the function $\|\cdot\|\colon M \to \mathbb{R}^{\geq 0}$ defined by

$$\|m\| = \inf_{n\in\mathbb{Z} \mid \alpha^n m \in M_0} p^n$$

is a norm compatible with $|\cdot|$. If the sets of the form $\alpha^n M_0$ are a basis of open neighborhoods of zero, then this norm is Banach.

3. Locally analytic functions and distributions

Now let A be a complete Tate $\mathbb{Z}_p$-algebra, and let X be a locally $\mathbb{Q}_p$-analytic manifold. In this section, we will define modules $\mathcal{A}(X, A)$ and $\mathcal{D}(X, A)$ of "locally analytic" A-valued functions and distributions on X.

The space X can be covered by coordinate patches isomorphic to $\mathbb{Z}_p^k$ for some k. We will first define locally analytic functions on these patches and then show that the construction can be glued.

Naively, one might try to define a function $\mathbb{Z}_p^k \to A$ to be locally analytic if it has a power series expansion in a neighborhood of any point. However, this definition turns out not to be suitable for applications to overconvergent cohomology. In Section 4.3, it will be important that any continuous homomorphism $\mathbb{Z}_p^k \to A^\times$ is in $\mathcal{A}(\mathbb{Z}_p^k, A)$. The homomorphism $\mathbb{Z}_p \to \mathbb{F}_p((T))^\times$ that sends $z \mapsto (1+T)^z$ does not have a power series expansion on any open subset of $\mathbb{Z}_p$. Our criterion for local analyticity will instead be based on Mahler expansions.

3.1. ***Preliminaries.*** We will recall some basic facts from p-adic functional analysis.

We will make use of the completed group ring $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}_p^k/p^n\mathbb{Z}_p^k]$.

For $z \in \mathbb{Z}_p^k$, let $[z]$ denote the corresponding group-like element of $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$, and let $\Delta_z = [z] - [0]$. Let I_Δ denote the augmentation ideal of $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$; this is the ideal generated by the Δ_z. The ring $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$ is local with maximal ideal $(p) + I_\Delta$.

We let $\mathbb{Z}_p^k$ act on $\mathcal{C}(\mathbb{Z}_p^k, M)$ by translation: for $g \in \mathcal{C}(\mathbb{Z}_p^k, M)$, $(zg)(y) = g(y+z)$. This action extends to an action of $\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$.

We adopt the convention that $\mathbb{N}$ is the set of nonnegative integers. To simplify notation, if $z = (z_1, \dots, z_k) \in \mathbb{Z}_p^k$, and $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, we will write $\binom{z}{n}$ for $\prod_{i=1}^k \binom{z_i}{n_i}$, and we will write $\sum n$ for $\sum_{i=1}^k n_i$.

Lemma 3.1.1 (Mahler's theorem, [Lazard 1965, Théorème II.1.2.4]). *Let M be a complete topological $\mathbb{Z}_p$-module. Suppose that M has a basis of open neighborhoods of zero that are subgroups of M. There is* *an isomorphism* $c(\mathbb{N}^k, M) \xrightarrow{\sim} \mathcal{C}(\mathbb{Z}_p^k, M)$ *that sends* $f \in c(\mathbb{N}^k, M)$ *to a function* $g \in \mathcal{C}(\mathbb{Z}_p^k, M)$ *defined by*

$$g(z) = \sum_{n \in \mathbb{N}^k} f(n) \binom{z}{n}.$$

We say that the right-hand side of the above equation is the Mahler expansion of g.

Lemma 3.1.2 (Amice's theorem). *Let* F *be a closed subfield of* $\mathbb{C}_p$, *and let* $LA_h(\mathbb{Z}_p^k, F)$ *be the space of functions* $\mathbb{Z}_p^k \to F$ *that extend to an analytic function* $\mathbb{Z}_p^k + p^h \mathcal{O}_{\mathbb{C}_p}^k \to \mathbb{C}_p$. *For* $f \in LA_h(\mathbb{Z}_p^k, F)$, *define*

$$|f| := \sup_{z \in \mathbb{Z}_p^k + p^h \mathcal{O}_{\mathbb{C}_p}^k} |f(z)|_p.$$

Then the functions $\lfloor \frac{n_1}{p^h} \rfloor! \cdots \lfloor \frac{n_k}{p^h} \rfloor! \binom{z}{n}$ *form an orthonormal basis for the Banach space* $LA_h(\mathbb{Z}_p^k, F)$. *In other words, every* $f \in LA_h(\mathbb{Z}_p^k, F)$ *can be expressed uniquely in the form*

$$f(z) = \sum_{n \in \mathbb{N}^k} a_n \left\lfloor \frac{n_1}{p^h} \right\rfloor! \cdots \left\lfloor \frac{n_k}{p^h} \right\rfloor! \binom{z}{n},$$

and $|f| = \sup_{n \in \mathbb{N}^k} |a_n|_p$.

Proof. This follows from [Amice 1964, Chapitre 3] (see also [Colmez 2010, Théorème I.4.7]). □

The following formulas concerning the p-adic valuations of $n!$, where n is a nonnegative integer, are well known:

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor \quad \text{and} \quad \frac{n}{p-1} - \log_p(n+1) \le v_p(n!) \le \frac{n}{p-1}.$$

Consequently, if F is a closed subfield of $\mathbb{C}_p$, and $f : \mathbb{Z}_p^k \to F$ is a continuous function with the Mahler expansion $f(z) = \sum_{n \in \mathbb{N}^k} a_n \binom{z}{n}$, then f is locally analytic if and only if $|a_n|_p$ go to zero exponentially in $\sum n$.

3.2. *Definitions.* The above facts suggest that we should define a function $\mathbb{Z}_p^k \to A$ to be "locally analytic" if the coefficients of its Mahler expansion decrease to zero exponentially.

We choose a topologically nilpotent $\alpha \in A^\times$.

Definition 3.2.1. Let $r \in \mathbb{R}^+$. We define $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to be the space of functions $f \in \mathcal{C}(\mathbb{Z}_p^k, A)$ such that for any open neighborhood U of zero in A, there exists $N \in \mathbb{N}$ so that for all integers $n > N$ and all $\delta \in I_\Delta^n$, $\alpha^{\lfloor -rn \rfloor} \delta f \in \mathcal{C}(\mathbb{Z}_p^k, U)$.

For any open neighborhood U of zero in A, we define $U_r \subset \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to be the set of all $f \in \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ such that $\alpha^{\lfloor -rn \rfloor} \delta f \in \mathcal{C}(\mathbb{Z}_p^k, U)$ for all $n \in \mathbb{N}$ and all $\delta \in I_\Delta^n$. We define a topology on $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ by making sets of the form U_r a basis of open neighborhoods of zero.

We define $\mathcal{A}(\mathbb{Z}_p^k, A) := \varinjlim_r \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$.

We will not choose a topology on $\mathcal{A}(\mathbb{Z}_p^k, A)$.

The connection between this definition and Mahler expansions will be explained by Lemma 3.2.3.

The definition of $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ is invariant under affine changes of coordinates.

For any topologically nilpotent unit $\alpha' \in A^\times$ and sufficiently small $r' \in \mathbb{R}^+$, $\mathcal{A}^{(\alpha',r')}(\mathbb{Z}_p^k, A)$ injects into $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$. So the directed systems $(\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A))_{r\in\mathbb{R}^+}$ and $(\mathcal{A}^{(\alpha',r)}(\mathbb{Z}_p^k, A))_{r\in\mathbb{R}^+}$ are cofinal, and $\mathcal{A}(\mathbb{Z}_p^k, A)$ does not depend on the choice of α. If F is a closed subfield of $\mathbb{C}_p$, then by Lemma 3.1.2, there are continuous injections with dense image

$$\mathcal{A}^{(p,1/(p-1)p^h)}(\mathbb{Z}_p^k, F) \hookrightarrow LA_h(\mathbb{Z}_p^k, F) \hookrightarrow \mathcal{A}^{(p,r)}(\mathbb{Z}_p^k, F)$$

for any $r < 1/((p-1)p^h)$, so the directed systems $(\mathcal{A}^{(p,r)}(\mathbb{Z}_p^k, F))_{r\in\mathbb{R}^+}$ and $(LA_h(\mathbb{Z}_p^k, F))_{h\in\mathbb{N}}$ are also cofinal.

The module $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ can also be defined (albeit less symmetrically) using α-Banach norms. Choose a ring of definition A_0 of A containing α, and define an α-Banach norm $|\cdot|\colon A \to \mathbb{R}^{\geq 0}$ as in Section 2.3. Define $\|\cdot\|_0\colon \mathcal{C}(\mathbb{Z}_p^k, A) \to \mathbb{R}^{\geq 0}$ by

$$\|f\|_0 = \sup_{z\in\mathbb{Z}_p^k} |f(z)|.$$

The sets $\{f \in A \mid \|f\|_0 \leq s\}$, $s \in \mathbb{R}^{\geq 0}$, form a basis of open neighborhoods of the identity in A. Hence in Definition 3.2.1, we can restrict our attention to neighborhoods of this form. Therefore

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) = \{f \in \mathcal{C}(\mathbb{Z}_p^k, A) \mid \limsup_{n\to\infty} \sup_{\delta\in I_\Delta^n} \|\alpha^{\lfloor -rn\rfloor}\delta f\|_0 = 0\},$$

and the topology on $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ is induced by the norm $\|\cdot\|_r\colon \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to \mathbb{R}^{\geq 0}$ defined by

$$\|f\|_r = \sup_{n\in\mathbb{N}} \sup_{\delta\in I_\Delta^n} \|\alpha^{\lfloor -rn\rfloor}\delta f\|_0.$$

The functions $\|\cdot\|_0$ and $\|\cdot\|_r$ are Banach norms compatible with $|\cdot|$.

Presumably, it would be reasonable to define $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, M)$ and $\mathcal{A}(\mathbb{Z}_p^k, M)$ for any topological A-module M that is locally convex in the sense that for some (equivalently, any) ring of definition A_0 of A, M has a basis of open neighborhoods of the identity that are A_0-modules. (We would just replace A with M in the above definition.) However, we will not need this additional generality.

Definition 3.2.2. Let $r \in \mathbb{R}^+$. We define $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to be the closure of the image of $\mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^k, A), A)$ in $\mathcal{L}_b(\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A), A)$.

We define $\mathcal{D}(\mathbb{Z}_p^k, A) = \varprojlim_r \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$.

The definition of $\mathcal{D}(\mathbb{Z}_p^k, A)$ does not depend on the choice of α.

We chose the definitions of $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ and $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ so that these modules would be orthonormalizable, as we will now show.

Lemma 3.2.3. *There is an isomorphism* $\mathrm{Ser}\colon c(\mathbb{N}^k, A) \xrightarrow{\sim} \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ *that sends* $f \in c(\mathbb{N}^k, A)$ *to a function* $g \in \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ *defined by*

$$g(z) = \sum_{n\in\mathbb{N}^k} \alpha^{\lceil r\Sigma n\rceil} f(n)\binom{z}{n}. \tag{3.2.4}$$

Moreover, if $c(\mathbb{N}^k, A)$ is given the supremum norm and $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ is given the norm $\|\cdot\|_r$, then Ser *is an isometry.*

There is an isomorphism Ev: $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \xrightarrow{\sim} c(\mathbb{N}^k, A)$ *that sends $\phi \in \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ to a function $h \in c(\mathbb{N}^k, A)$ defined by*

$$h(n) = \alpha^{\lceil r\sum n\rceil}\phi\left(\binom{z}{n}\right). \tag{3.2.5}$$

Hence $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ and $\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ are orthonormalizable.

Proof. Let $f \in c(\mathbb{N}^k, A)$, and let g be defined by (3.2.4). By Mahler's theorem, $g \in \mathcal{C}(\mathbb{Z}_p^k, A)$. We observe that for any $h \in \mathcal{C}(\mathbb{Z}_p^k, A)$ and $\delta \in \mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$, $\|\delta h\|_0 \le \|h\|_0$. Furthermore, if $\delta \in I_\Delta^m$, then $\delta\binom{z}{n} = 0$ whenever $\sum n < m$. So

$$\|\alpha^{\lfloor -rm\rfloor}\delta g\|_0 \le \sup_{\sum n\ge m} |\alpha^{\lfloor -rm\rfloor + \lceil r\sum n\rceil} f(n)| \le \sup_{\sum n\ge m} |f(n)|.$$

It follows that $g \in \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$, and Ser is has operator norm ≤ 1.

By Mahler's theorem, we can recover f from g:

$$f(n) = \alpha^{\lfloor -r\sum n\rfloor}(\Delta_{e_1}^{n_1}\cdots\Delta_{e_k}^{n_k} g)(0), \tag{3.2.6}$$

where $e_1, \dots, e_k$ are the standard basis for $\mathbb{Z}_p^k$. Since

$$|f(n)| \le \sup_{\delta\in(I_\Delta)^{\sum n}} \|\alpha^{\lfloor -r\sum n\rfloor}\delta g\|_0,$$

the relation (3.2.6) determines a map Coeff: $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to c(\mathbb{Z}_p^k, A)$ that is a left-inverse of Ser, and Coeff has operator norm ≤ 1. To see that Coeff is also a right-inverse of Ser, observe that $(\mathrm{Ser}\circ\mathrm{Coeff})(g)$ and g agree on $\mathbb{N}^k$, which is dense in $\mathbb{Z}_p^k$. Since Ser and Coeff both have operator norm ≤ 1, they must be isometries.

The map Ser induces an isomorphism $\mathrm{Ser}^*\colon \mathcal{L}_b(\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A), A) \xrightarrow{\sim} \mathcal{L}_b(c(\mathbb{N}^k, A), A)$. The pairing $c(\mathbb{N}^k, A)\times c(\mathbb{N}^k, A) \to A$ defined by $(f, h) \mapsto \sum_{n\in\mathbb{N}^k} f(n)h(n)$ identifies $c(\mathbb{N}^k, A)$ isometrically with a closed submodule of $\mathcal{L}_b(c(\mathbb{N}^k, A), A)$. For any $\phi \in \mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^k, A), A)$, the function $n \mapsto \phi\left(\binom{z}{n}\right)$ is bounded, so in particular $\alpha^{\lceil r\sum n\rceil}\phi\left(\binom{z}{n}\right) \to 0$ as $\sum n \to \infty$. Hence the image of Ser^* is contained in $c(\mathbb{N}^k, A)$. Furthermore, the image contains all elements of $c(\mathbb{N}^k, A)$ that are supported on a finite subset of $\mathbb{N}^k$, and these elements are dense in $c(\mathbb{N}^k, A)$. □

Lemma 3.2.3 makes it clear that for $r' < r$, there are natural injections

$$\mathcal{D}^{(\alpha,r')}(\mathbb{Z}_p^k, A) \hookrightarrow \mathcal{L}_b(\mathcal{A}^{(\alpha,r')}(\mathbb{Z}_p^k, A), A) \hookrightarrow \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$$
$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \hookrightarrow \mathcal{L}_b(\mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A), A) \hookrightarrow \mathcal{A}^{(\alpha,r')}(\mathbb{Z}_p^k, A).$$

3.3. ***Properties of locally analytic functions and distributions.*** In this section, we check that $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$ has some properties that one would expect of locally analytic functions.

Lemma 3.3.1. *Multiplication induces a continuous map $\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)\times\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$.*

Proof. This follows Lemma 3.2.3 and the fact that for $m, n \in \mathbb{N}$, $\binom{z}{n}\binom{z}{m}$ is of the form $\sum_{i=0}^{m+n} a_i \binom{z}{i}$ with $a_i \in \mathbb{Z}$. □

Lemma 3.3.2. *Let*

$$f : \mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p) \to \mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p)$$

be a $\mathbb{Z}_p$-module homomorphism. For any $r, s \in \mathbb{R}^+$, there is at most one continuous A-linear homomorphism $\tilde{f}$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p) \arrow[r] \arrow[d, "f"] & \mathcal{C}(\mathbb{Z}_p^k, A) \arrow[d] & \mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \arrow[l, hook'] \arrow[d, dashed, "\tilde{f}"] \\
\mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p) \arrow[r] & \mathcal{C}(\mathbb{Z}_p^j, A) & \mathcal{A}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \arrow[l, hook']
\end{tikzcd}$$

and there is at most one continuous A-linear homomorphism $\tilde{f}^*$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p), \mathbb{Z}_p) \arrow[r] \arrow[d, "f^*"] & \mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^j, A), A) \arrow[r, hook] \arrow[d] & \mathcal{D}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \arrow[d, dashed, "\tilde{f}^*"] \\
\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p), \mathbb{Z}_p) \arrow[r] & \mathcal{L}_b(\mathcal{C}(\mathbb{Z}_p^k, A), A) \arrow[r, hook] & \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)
\end{tikzcd}$$

If either homomorphism exists, we say that it is induced by f.

Proof. If the maps $\tilde{f}$ and $\tilde{f}^*$ exist, then their matrices in the basis of Lemma 3.2.3 can be deduced from the matrix of f in the basis of Mahler's theorem. More specifically, if we write

$$f\left(\binom{z}{n}\right) = \sum_{m \in \mathbb{N}^j} f_{nm} \binom{z}{m}$$

with $f_{nm} \in \mathbb{Z}_p$, then the matrix coefficients of $\tilde{f}$ must be

$$\tilde{f}_{nm} = \alpha^{\lceil r \sum n \rceil - \lceil s \sum m \rceil} f_{nm},$$

and the matrix coefficients of $\tilde{f}^*$ must be

$$\tilde{f}^*_{nm} = \tilde{f}_{mn} = \alpha^{\lceil r \sum m \rceil - \lceil s \sum n \rceil} f_{mn}.$$ □

Lemma 3.3.3. *There exists $t_0 \in \mathbb{R}^+$ so that for any $r, s \in \mathbb{R}^+$, $j, k \in \mathbb{N}$, and any $\mathbb{Z}_p$-module homomorphism $f : \mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p) \to \mathcal{C}(\mathbb{Z}_p^j, \mathbb{Z}_p)$ that induces a continuous homomorphism*

$$\mathcal{A}^{(p,r)}(\mathbb{Z}_p^k, \mathbb{Q}_p) \to \mathcal{A}^{(p,s)}(\mathbb{Z}_p^j, \mathbb{Q}_p),$$

f also induces continuous homomorphisms

$$\tilde{f} : \mathcal{A}^{(\alpha,rt)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,st)}(\mathbb{Z}_p^j, A) \quad \textit{and} \quad \tilde{f}^* : \mathcal{D}^{(\alpha,st)}(\mathbb{Z}_p^j, A) \to \mathcal{D}^{(\alpha,rt)}(\mathbb{Z}_p^k, A)$$

for all $t \in (0, t_0)$.

Proof. If the map $\tilde{f}\colon \mathcal{A}^{(\alpha,rt)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,st)}(\mathbb{Z}_p^j, A)$ exists, then in the notation of the previous lemma, its matrix coefficients must be given by

$$\tilde{f}_{nm} = \alpha^{\lceil rt\sum n\rceil - \lceil st\sum m\rceil} f_{nm}.$$

Conversely, if there is a continuous map with these matrix coefficients, then it is the desired map $\tilde{f}$.

The $\tilde{f}_{nm}$ are the matrix coefficients of a continuous map if and only if the following two conditions are satisfied:

(1) $\tilde{f}_{nm}$ are bounded.

(2) For any fixed n, $\tilde{f}_{nm} \to 0$ as $\sum m \to \infty$.

The terms with $r\sum n - s\sum m \geq 0$ are certainly bounded, so we only need to worry about terms with $r\sum n - s\sum m < 0$. There exists a positive integer ℓ so that p^ℓ/α is power bounded. If the $p^{\lceil rt\ell\sum n\rceil - \lceil st\ell\sum n\rceil} f_{nm}$ (considered as elements of $\mathbb{Q}_p$) are bounded (resp. go to zero as $\sum m \to \infty$), then the same will be true of the $\alpha^{\lceil rt\sum n\rceil - \lceil st\sum m\rceil} f_{nm}$ (considered as elements of A). So we may take $t_0 = \ell^{-1}$.

Similarly, if the map $\tilde{f}^*\colon \mathcal{D}^{(\alpha,st)}(\mathbb{Z}_p^j, A) \to \mathcal{D}^{(\alpha,rt)}(\mathbb{Z}_p^k, A)$ exists, then its matrix coefficients satisfy $\tilde{f}^*_{mn} = \tilde{f}_{nm}$. The $\tilde{f}^*_{mn}$ are the coefficients of a continuous map if and only if condition (1) above and the following condition are satisfied:

(2′) For any fixed m, $\tilde{f}_{nm} \to 0$ as $\sum n \to \infty$.

Since $f_{nm} \in \mathbb{Z}_p$ and $\alpha^{\lceil rt\sum n\rceil} \to 0$ as $\sum n \to \infty$, condition (2′) will always be satisfied. □

Proposition 3.3.4. *Let $g\colon \mathbb{Z}_p^j \to \mathbb{Z}_p^k$ be a (globally) analytic function. For some $r_0 \in \mathbb{R}^+$ depending on α but not on g, j, k, composition with g induces continuous A-linear homomorphisms*

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \to \mathcal{A}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \quad \text{and} \quad \mathcal{D}^{(\alpha,s)}(\mathbb{Z}_p^j, A) \to \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$$

for all $s < r < r_0$.

Proof. There are continuous maps

$$\mathcal{A}^{(p,1/(p-1))}(\mathbb{Z}_p^k, \mathbb{Q}_p) \to LA_0(\mathbb{Z}_p^k, \mathbb{Q}_p) \xrightarrow{g^*} LA_0(\mathbb{Z}_p^j, \mathbb{Q}_p) \to \mathcal{A}^{(p,1/(p-1)-\epsilon)}(\mathbb{Z}_p^j, \mathbb{Q}_p)$$

for any $\epsilon \in (0, 1/(p-1))$. Applying Lemma 3.3.3 then yields the desired result. □

If $j = 1$, then the maps exist even if $r = s$. We do not know if the same is true for $j > 1$. When $j = 1$, one can prove existence by considering the norm on LA_0 defined in Lemma 3.1.2 and using the fact that $v_p(n!) - \sum_{i=1}^k v_p(m_i!) \geq \lfloor (n - \sum m)/p \rfloor$. (The same idea will be used in the proof of Proposition 3.3.5.) However, for $j > 1$, $\sum_{i=1}^j v_p(n_i!) - \sum_{i=1}^k v_p(m_i!)$ can be zero for arbitrarily large values of $\sum n - \sum m$.

Proposition 3.3.5. *Let S be a set of coset representatives of $\mathbb{Z}_p^k/p\mathbb{Z}_p^k$. The homeomorphism $\mathbb{Z}_p^k \times S \xrightarrow{\sim} \mathbb{Z}_p^k$ defined by $(z, s) \mapsto pz + s$ determines an isomorphism*

$$\mathcal{C}(\mathbb{Z}_p^k, A) \cong \mathcal{C}(\mathbb{Z}_p^k, A)^{\oplus p^k}$$

which induces isomorphisms

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \cong \mathcal{A}^{(\alpha,pr)}(\mathbb{Z}_p^k, A)^{\oplus p^k} \quad \textit{and} \quad \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \cong \mathcal{D}^{(\alpha,pr)}(\mathbb{Z}_p^k, A)^{\oplus p^k}$$

for all sufficiently small $r \in \mathbb{R}^+$.

Proof. First, consider the case $k = 1$. Applying Lemma 3.3.3 along with translation invariance, we see that it is then enough to check that composition with the function

$$g(z) = pz$$

defines a continuous homomorphism

$$\mathcal{A}^{(p,1/2p^2)}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathcal{A}^{(p,1/2p)}(\mathbb{Z}_p, \mathbb{Q}_p)$$

and that composition with the function

$$h(z) = \begin{cases} z/p & z \in p\mathbb{Z}_p, \\ 0 & z \in \mathbb{Z}_p^\times, \end{cases}$$

defines a continuous homomorphism

$$\mathcal{A}^{(p,1/2p)}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathcal{A}^{(p,1/2p^2)}(\mathbb{Z}_p, \mathbb{Q}_p).$$

Define g_{nm}, h_{nm} by

$$g\left(\binom{z}{n}\right) = \sum_{m=0}^{\infty} g_{nm}\binom{z}{m}, \quad h\left(\binom{z}{n}\right) = \sum_{m=0}^{\infty} h_{nm}\binom{z}{m}.$$

By the same reasoning as in Lemma 3.3.3, we just need to verify that:

(1) $v_p(g_{nm}) - \frac{m}{2p} + \frac{n}{2p^2}$ is bounded below for $pm \geq n$.
(2) For any n, $v_p(g_{nm}) - \frac{m}{2p} + \frac{n}{2p^2} \to \infty$ as $m \to \infty$.
(3) $v_p(h_{nm}) - \frac{m}{2p^2} + \frac{n}{2p}$ is bounded below for $m \geq pn$.
(4) For any n, $v_p(h_{nm}) - \frac{m}{2p^2} + \frac{n}{2p} \to \infty$ as $m \to \infty$.

Applying Lemma 3.1.2 gives

$$v_p(g_{nm}) - v_p(m!) \geq -v_p(\lfloor n/p \rfloor !).$$

For $n \geq pm$ this implies

$$v_p(g_{nm}) \geq \sum_{i=1}^{\infty} (\lfloor m/p^i \rfloor - \lfloor n/p^{i+1} \rfloor) \geq \lfloor m/p - n/p^2 \rfloor.$$

Similarly, Lemma 3.1.2 implies

$$v_p(h_{nm}) \geq \sum_{i=1}^{\infty} (\lfloor m/p^{i+1} \rfloor - \lfloor n/p^i \rfloor) \geq \lfloor m/p^2 - n/p \rfloor.$$

This proves the case $k = 1$.

We reduce the general case to the case $k = 1$ as follows. Since the above modules are all preserved by translation, if the proposition is true for one choice of S, it is true for any choice of S. In particular, we may assume S is a product of k sets of coset representatives of $\mathbb{Z}/p\mathbb{Z}$. Then, since multiplication by p does not mix coordinates, the argument is essentially the same as in the $k = 1$ case. $\square$

Lemma 3.3.6. *Any continuous homomorphism $\lambda\colon \mathbb{Z}_p^k \to A^\times$ is in $\mathcal{A}(\mathbb{Z}_p^k, A)$.*

Proof. Lemma 3.3.1 allows us to reduce to the one-dimensional case, and Proposition 3.3.5 allows us to replace $\mathbb{Z}_p^k$ with an open sublattice. So it suffices to consider the case where $k = 1$ and $(\lambda(1) - 1)/\alpha$ is topologically nilpotent. In that case, since

$$\lambda(z) = \sum_{n=0}^{\infty} \binom{z}{n} (\lambda(1) - 1)^n,$$

$\lambda \in \mathcal{A}^{(\alpha,1)}(\mathbb{Z}_p, A)$. $\square$

Lemma 3.3.7. *For any $0 < s < r$, the inclusions*

$$\mathcal{A}^{(\alpha,r)}(\mathbb{Z}_p^k, A) \hookrightarrow \mathcal{A}^{(\alpha,s)}(\mathbb{Z}_p^k, A) \quad \textit{and} \quad \mathcal{D}^{(\alpha,s)} \hookrightarrow \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A)$$

are completely continuous.

Proof. In the orthonormal bases of Lemma 3.2.3, these inclusions are represented by diagonal matrices with diagonal entries of the form $\alpha^{\lfloor r \sum n \rfloor - \lfloor s \sum n \rfloor}$. As $\sum n \to \infty$, the entries go to zero. $\square$

3.4. ***Gluing.*** Propositions 3.3.4 and 3.3.5 show that it makes sense to define locally analytic functions and distributions on arbitrary locally $\mathbb{Q}_p$-analytic manifolds by gluing.

Definition 3.4.1. Let k be a nonnegative integer, and let X be a locally $\mathbb{Q}_p$-analytic manifold of dimension k. Choose a decomposition $X = \bigsqcup_{i \in I} X_i$ for some index set I, and choose an identification of each X_i with $\mathbb{Z}_p^k$. We define

$$\mathcal{A}(X, A) = \prod_{i \in I} \mathcal{A}(X_i, A)$$
$$\mathcal{A}_c(X, A) = \bigoplus_{i \in I} \mathcal{A}(X_i, A)$$
$$\mathcal{D}(X, A) = \bigoplus_{i \in I} \mathcal{D}(X_i, A)$$
$$\mathcal{D}_c(X, A) = \prod_{i \in I} \mathcal{D}(X_i, A).$$

By Propositions 3.3.4 and 3.3.5, the above definitions do not depend on the choice of decomposition.

Theorem 3.4.2. *The modules $\mathcal{A}(X, A)$, $\mathcal{D}(X, A)$, $\mathcal{A}_c(X, A)$, and $\mathcal{D}_c(X, A)$ satisfy the following properties:*

(1) *$\mathcal{A}(X, A)$ is ring.*

(2) *If $g\colon X \to Y$ is a locally analytic map, then composition with g induces homomorphisms $\mathcal{A}(Y, A) \to \mathcal{A}(X, A)$ and $\mathcal{D}(X, A) \to \mathcal{D}(Y, A)$.*

(3) *The functors $U \mapsto \mathcal{A}(U, A)$ and $U \mapsto \mathcal{D}_c(U, A)$ are sheaves on X.*

(4) *If X has the structure of a finitely generated $\mathbb{Z}_p$-module, then any continuous group homomorphism $X \to A^\times$ is in $\mathcal{A}(X, A)$.*

Proof. The claims all follow immediately from the results of Section 3.3. $\square$

3.5. ***Geometric interpretation of distributions.*** The modules of locally analytic distributions have an alternative interpretation as rings of sections of adic spaces. This interpretation will not be used elsewhere in the paper, but it gives further evidence that our definition of distributions is reasonable. For background on adic spaces, see [Huber 1993; Huber 1994; Huber 1996] or [Scholze and Weinstein 2019, Sections 2–5].

Let $D = \operatorname{Spa}(\mathbb{Z}_p[\![\mathbb{Z}_p^k]\!], \mathbb{Z}_p[\![\mathbb{Z}_p^k]\!])$. Suppose that the Tate algebra $A\langle T_1, \dots, T_n\rangle$ is sheafy for each nonnegative integer n. Let A^+ be an open and integrally closed subring of A. Let $Y = D \times_{\operatorname{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)} \operatorname{Spa}(A, A^+)$. We can construct Y as follows. There is an isomorphism $\mathbb{Z}_p[\![T_1, \dots, T_k]\!] \cong \mathbb{Z}_p[\![\mathbb{Z}_p^k]\!]$ that sends $T_i \mapsto \Delta_{e_i}$, where the e_i form a basis of $\mathbb{Z}_p^k$; this isomorphism is known as the multivariable Amice transform. For any positive rational $r = m/n$, let $B_r = A\langle T_1, \dots, T_k, T_1^n/\alpha^m, \dots, T_k^n/\alpha^m\rangle$, and let B_r^+ be the normal closure of $A^+\langle T_1, \dots, T_n, T_1^n/\alpha^m, \dots, T_n^n/\alpha^m\rangle$ in B_r. Then Y is formed by gluing the affinoids $Y_r := \operatorname{Spa}(B_r, B_r^+)$.

There are canonical isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathbb{Z}_p}(\mathcal{C}(\mathbb{Z}_p^k, \mathbb{Z}_p), \mathbb{Z}_p) &\cong \mathcal{O}_D(D), \\ \mathcal{D}(\mathbb{Z}_p^k, A) &\cong \mathcal{O}_Y(Y), \\ \mathcal{D}^{(\alpha,r)}(\mathbb{Z}_p^k, A) &\cong \mathcal{O}_Y(Y_r) \quad \forall r \in \mathbb{Q}^+. \end{aligned}$$

4. Overconvergent cohomology

Now we use the modules constructed in Section 3 to define overconvergent cohomology. We mostly repeat the setup of [Urban 2011, Sections 3–4]; see also [Ash and Stevens 2008; Hansen 2017, Sections 2–3].

4.1. ***Locally symmetric spaces.*** Let $\mathbb{A}$ (resp. $\mathbb{A}_f$, $\mathbb{A}_f^p$) be the ring of adeles (resp. finite adeles, finite adeles away from p) of $\mathbb{Q}$.

Let G be a connected reductive algebraic group over $\mathbb{Q}$. We will assume that $G(\mathbb{Q}_p)$ is quasisplit. Let B, T, N, N^- be compatible choices of a Borel subgroup, maximal torus, maximal unipotent subgroup, and opposite unipotent subgroup, respectively, of $G(\mathbb{Q}_p)$.

We will need some results from [Bruhat and Tits 1972]. Note that $G_{\mathbb{Q}_p}$ admits a valued root datum ("donnée radicielle valuée") by [Bruhat and Tits 1984, 4.2.3 Théorème].

Let I be an Iwahori subgroup of $G(\mathbb{Q}_p)$ compatible with B (see for example [Bruhat and Tits 1972, Section 6.5]; note that this reference denotes the Iwahori by B). Then I admits a factorization $I = N_0 T_0 N_0^-$, where $N_0^- = N^- \cap I$, $T_0 = T \cap I$, $N_0 = N \cap I$. Let K^p be an open compact subgroup of $\mathbb{A}_f^p$, and let $K = K^p I$. We assume that K is neat; see Definition 4.1.1 below. Let G_∞^+ be the identity component of $G(\mathbb{R})$, and let K_∞ be a maximal compact modulo center subgroup of G_∞^+. Let Z_G be the center of G.

The space

$$\mathcal{X} := G(\mathbb{A})/K^p G_\infty^+$$

may be considered as a locally $\mathbb{Q}_p$-analytic manifold. Let A be a complete Noetherian Tate $\mathbb{Z}_p$-algebra. In Section 3, we defined the module $\mathcal{D}_c(\mathcal{X}, A)$ of "locally analytic" compactly supported A-valued distributions on $\mathcal{X}$.

Let $\lambda\colon T_0 \to A^\times$ be a continuous homomorphism. By Lemma 3.3.6, $\lambda \in \mathcal{A}(T_0, A)$. We will assume that $\ker \lambda$ contains $(Z_G(\mathbb{Q}) K^p G_\infty^+ \cap T_0)$. We define $\mathcal{D}_{c,\lambda}(\mathcal{X}, A)$ to be the quotient of $\mathcal{D}_c(\mathcal{X}, A)$ obtained by constraining right-translation by N_0^- to act by the identity, right-translation by T_0 to act by λ, and translation by $Z_G(\mathbb{Q})$ to act by the identity.

The group $G(\mathbb{Q})/Z_G(\mathbb{Q})$ acts on $\mathcal{D}_{c,\lambda}(\mathcal{X}, A)$ by left-translation. Moreover, $\mathcal{D}_{c,\lambda}(\mathcal{X}, A)$ is a direct sum of modules induced from much smaller subgroups of $G(\mathbb{Q})/Z_G(\mathbb{Q})$. We can write $G(\mathbb{A})$ as a finite union

$$G(\mathbb{A}) = \bigsqcup_i G(\mathbb{Q}) g_i G_\infty^+ K.$$

Let Γ_i be the image of $g_i G_\infty^+ K g_i^{-1} \cap G(\mathbb{Q})$ in $G(\mathbb{Q})/Z_G(\mathbb{Q})$. Then

$$\mathcal{D}_{c,\lambda}(\mathcal{X}, A) \cong \bigoplus_i \operatorname{Ind}_{\Gamma_i}^{G(\mathbb{Q})/Z_G(\mathbb{Q})} \mathcal{D}_\lambda(g_i I, A)$$

where $\mathcal{D}_\lambda(g_i I, A)$ is the quotient of $\mathcal{D}(g_i I, A)$ obtained by constraining right-translation by N_0^- to act as the identity and right-translation by T_0 to act as λ. Here Γ_i acts on $\mathcal{D}_\lambda(g_i I, A)$ by left-translation.

The existence of the Iwahori factorization implies that the map $N_0 \to g_i I$ given by $n \mapsto g_i n$ induces an isomorphism of A-modules

$$\mathcal{D}(N_0, A) \xrightarrow{\sim} \mathcal{D}_\lambda(g_i I, A).$$

This identification induces a Γ_i-action on $\mathcal{D}(N_0, A)$, which can be described as follows. Any $x \in I$ has an Iwahori factorization $x = \boldsymbol{n}(x)\boldsymbol{t}(x)\boldsymbol{n}^-(x)$ with $\boldsymbol{n}(x) \in N_0$, $\boldsymbol{t}(x) \in T_0$, $\boldsymbol{n}^-(x) \in N_0^-$, and the functions $\boldsymbol{n}$, $\boldsymbol{t}$, and $\boldsymbol{n}^-$ are analytic. The action of Γ_i on $\mathcal{D}(N_0, A)$ is given by

$$\gamma \cdot [x] = \lambda(\boldsymbol{t}(g_i^{-1}\gamma g_i x))[\boldsymbol{n}(g_i^{-1}\gamma g_i x)]$$

for $\gamma \in \Gamma_i$, $x \in N_0$. Here $[x]$ denotes the Dirac delta distribution supported at x.

Now consider the locally symmetric space

$$S_G(K) := G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K.$$

Then $S_G(K) \cong \bigsqcup_i \mathcal{Y}_i$ where

$$\mathcal{Y}_i := \Gamma_i \backslash G_\infty^+ / K_\infty.$$

Definition 4.1.1. We say that K is *neat* if all of the Γ_i are torsion-free.

As mentioned above, we assume that K^p has been chosen so that K is neat. Then each $\mathcal{Y}_i$ is a manifold with fundamental group Γ_i.

The manifold $S_G(K)$ has a Borel–Serre compactification $\overline{S_G(K)}$, which is homotopy equivalent to $S_G(K)$. Any finite triangulation of $\overline{S_G(K)}$ determines a resolution

$$0 \to C_d(\Gamma_i) \to \cdots \to C_1(\Gamma_i) \to C_0(\Gamma_i) \to \mathbb{Z} \to 0$$

where the $C_j(\Gamma_i)$ are free $\mathbb{Z}[\Gamma_i]$-modules of finite rank and d is the dimension of $S_G(K)$. We define a complex $C^\bullet_\lambda$ by

$$C^j_\lambda := \bigoplus_i \operatorname{Hom}_{\Gamma_i}(C_j(\Gamma_i), \mathcal{D}_\lambda(g_i I, A)). \tag{4.1.2}$$

Then

$$R\Gamma^\bullet(G(\mathbb{Q})/Z_G(\mathbb{Q}), \mathcal{D}_{c,\lambda}(\mathcal{X}, A)) \cong \bigoplus_i R\Gamma^\bullet(\Gamma_i, \mathcal{D}_\lambda(g_i I, A)) \cong C^\bullet_\lambda$$

in the derived category of A-modules.

4.2. ***Hecke action.*** We choose a projective resolution

$$\cdots \to C_1(G(\mathbb{Q})/Z_G(\mathbb{Q})) \to C_0(G(\mathbb{Q})/Z_G(\mathbb{Q})) \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ as a $G(\mathbb{Q})/Z_G(\mathbb{Q})$-module as well as maps of complexes of Γ_i-modules

$$C_\bullet(\Gamma_i) \to C_\bullet(G(\mathbb{Q})/Z_G(\mathbb{Q})) \quad \text{and} \quad C_\bullet(G(\mathbb{Q})/Z_G(\mathbb{Q})) \to C_\bullet(\Gamma_i)$$

that are homotopy inverses of each other. Then any $f \in \operatorname{End}_{G(\mathbb{Q})/Z_G(\mathbb{Q})}(\mathcal{D}_{c,\lambda}(\mathcal{X}, A))$ defines an operator $[f] \in \operatorname{End}(C^\bullet_\lambda)$ by

$$\begin{aligned}
C^j_\lambda &\to \bigoplus_i \operatorname{Hom}_{\Gamma_i}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_\lambda(g_i I, A)) \\
&\xrightarrow{\sim} \operatorname{Hom}_{G(\mathbb{Q})/Z_G(\mathbb{Q})}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_{c,\lambda}(\mathcal{X}, A)) \\
&\xrightarrow{f} \operatorname{Hom}_{G(\mathbb{Q})/Z_G(\mathbb{Q})}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_{c,\lambda}(\mathcal{X}, A)) \\
&\xrightarrow{\sim} \bigoplus_i \operatorname{Hom}_{\Gamma_i}(C_j(G(\mathbb{Q})/Z_G(\mathbb{Q})), \mathcal{D}_\lambda(g_i I, A)) \\
&\to C^j_\lambda.
\end{aligned}$$

For any f, g, $[f][g]$ is homotopy equivalent to $[fg]$.

For any $g \in G(\mathbb{A}^p_f)$, the double coset operator $K^p g K^p$ acts on $\mathcal{D}_{c,\lambda}$ and determines a Hecke operator $[K^p g K^p]$ on $C^\bullet_\lambda$.

Let

$$T^- := \{t \in T \mid t^{-1} N^-_0 t \subseteq N^-_0\}.$$

For $t \in T^-$, the double coset operator $N^-_0 t N^-_0$ acts on $\mathcal{D}_{c,\lambda}$ and determines an operator $[N^-_0 t N^-_0]$ on $C^\bullet_\lambda$. We will sometimes denote this operator by u_t.

Remark 4.2.1. Our definition of the Hecke operators at p differs slightly from that of previous references on overconvergent cohomology, which made use of a choice of "right $*$-action". Our definition is instead meant to be analogous to the one used in Emerton's theory of completed cohomology [2006a; 2006b]. The two approaches will yield the same eigenvariety. The only essential difference between the approaches is that, to define a "right $*$-action", one chooses a splitting of $0 \to T_0 \to T \to T/T_0 \to 0$, and then uses this splitting to twist the Hecke operators so that T_0 acts trivially.

Let S be the set of finite places at which K^p is not maximal hyperspecial. Let $\mathbb{A}_f^{p,S}$ be the adeles away from p and S, and let $K^{p,S}$ be the image of K^p in $\mathbb{A}_f^{p,S}$. We define the Hecke algebra

$$\mathcal{H}_G := C_c^\infty(K^{p,S}\backslash G(\mathbb{A}_f^{p,S})/K^{p,S} \times N_0^-\backslash N_0^- T^- N_0^-/N_0^-, \mathbb{Z}_p).$$

4.3. *Topological properties of Hecke operators.* In order to apply the spectral theory introduced in Section 2.2, we will need to choose a particular description of $C_\lambda^\bullet$ as a limit of complexes of projective modules. The logarithm induces a bijection between N_0 and a finite free $\mathbb{Z}_p$-module; we use this bijection to define a coordinate chart on N_0. This chart allows us to define the projective modules $\mathcal{D}^{(\alpha,r)}(N_0, A)$ for some arbitrarily chosen topologically nilpotent unit $\alpha \in A$. Define

$$C^i_{\lambda,\alpha,r} := \bigoplus_j \operatorname{Hom}_{\Gamma_j}(C_i(\Gamma_j), \mathcal{D}^{(\alpha,r)}(N_0, A)).$$

Lemma 4.3.1. *For all sufficiently small r and all $\epsilon > 0$, the differential $d\colon C_\lambda^{i+1} \to C_\lambda^i$ extends to a map $C^{i+1}_{\lambda,\alpha,r} \to C^i_{\lambda,\alpha,r+\epsilon}$.*

Proof. It is enough to check that for sufficiently small r and all $\epsilon > 0$, left translation by any $\gamma \in \Gamma_i$ maps $\mathcal{D}^{(\alpha,r)}(N_0, A)$ into $\mathcal{D}^{(\alpha,r+\epsilon)}(N_0, A)$. This follows from the description of the action in Section 4.1 along with Lemmas 3.3.1 and 3.3.6 and Proposition 3.3.4. □

If $\underline{r} = (r_0, \dots, r_d)$ is chosen such that the differentials $C^{i+1}_{\lambda,\alpha,r_{i+1}} \to C^i_{\lambda,\alpha,r_i}$ are defined, then we denote the corresponding complex by $C^\bullet_{\lambda,\alpha,\underline{r}}$.

Choose some $t \in T^-$ such that $t^{-1}N_0 t \subset N_0^p$. Let $\mathcal{H}'_G$ be the ideal of $\mathcal{H}_G$ generated by u_t.

Lemma 4.3.2. *There exists $r_0 \in \mathbb{R}^+$ so that for all $r \in (0, r_0)$, $\epsilon \in \mathbb{R}^+$, and $f \in \mathcal{H}'_G$, f determines a continuous map $C^i_{\lambda,\alpha,r} \to C^i_{\lambda,\alpha,r/p+\epsilon}$, and hence f determines a completely continuous map $C^i_{\lambda,\alpha,r} \to C^i_{\lambda,\alpha,r}$.*

Proof. We can show that Hecke operators away from p map $C^i_{\lambda,\alpha,r}$ into $C^i_{\lambda,\alpha,r+\epsilon}$ using essentially the same argument as in Lemma 4.3.1. It remains to show that u_t maps $C^i_{\lambda,\alpha,r}$ into $C^i_{\lambda,\alpha,r/p+\epsilon}$. The action of u_t can be built from functions of the form

$$[x] \mapsto \lambda(\boldsymbol{t}(\iota(x)))[\boldsymbol{n}(\iota(x))]$$

where $\iota(x)$ takes the form

$$\iota(x) = ht^{-1}\boldsymbol{n}(n^- x)\boldsymbol{t}(n^- x)t$$

for some $n^- \in N_0^-$, $h \in I$. (See for example [Emerton 2006a, Lemma 4.2.19].) In particular, $\boldsymbol{n}(\iota(x))$ belongs to a single right coset of $t^{-1}N_0 t \subset N_0^p$. The argument proceeds as before, except that we also need to use Proposition 3.3.5 and Lemma 3.3.7. $\square$

4.4. ***Characteristic power series.*** For any $f \in \mathcal{H}'_G$, we define the power series

$$\det(1 - Xf \mid C_\lambda^\bullet) := \det(1 - Xf \mid C_{\lambda,\alpha,\underline{r}}^\bullet)$$

for any $\alpha, \underline{r}$ for which the complex $C_{\lambda,\alpha,\underline{r}}^\bullet$ is defined and the u_t operator is completely continuous. Choosing a different α and $\underline{r}$ conjugates the matrix of f by a diagonal matrix, so the power series does not depend on them.

Similarly, we define $\det(1 - Xf \mid C_\lambda^i) := \det(1 - Xf \mid C_{\lambda,\alpha,\underline{r}}^i)$. Consider the Fredholm series

$$P_+(X) := \prod_{i=0}^{d} \det(1 - Xu_t \mid C_\lambda^i).$$

Suppose that $P_+(X)$ factors as $Q_+(X)S_+(X)$, with $Q_+(X) \in A[X]$, $S_+(X) \in A\{\{X\}\}$, that $Q_+(X)$ and $S_+(X)$ are relatively prime, and that the leading coefficient of $Q_+(X)$ is invertible. Let $Q_+^*(X) = X^{\deg Q_+} Q_+(X^{-1})$. By [Andreatta et al. 2018, Théorème B.2], there is a decomposition $C_{\lambda,\alpha,\underline{r}}^\bullet = N_{\alpha,\underline{r}}^\bullet \oplus F_{\alpha,\underline{r}}^\bullet$, where $Q_+^*(u_t)$ annihilates $N_{\alpha,\underline{r}}^\bullet$ and acts invertibly on $F_{\alpha,\underline{r}}^\bullet$, and the $N_{\alpha,\underline{r}}^i$ are finitely generated and projective.

Lemma 4.4.1. *For any α, α' and $\underline{r}, \underline{r}'$ such that $N_{\alpha,\underline{r}}^\bullet$ and $N_{\alpha',\underline{r}'}^\bullet$ are defined, they are canonically isomorphic.*

Proof. Choose $\underline{r}''$ so that $C_{\lambda,\alpha,\underline{r}''}^\bullet$ injects into $C_{\lambda,\alpha,\underline{r}}^\bullet$ and $C_{\lambda,\alpha',\underline{r}'}^\bullet$. The operator $1 - Q_+^*(u_t)/Q_+^*(0)$ acts as the identity on $N_{\alpha,\underline{r}}^\bullet$, and for sufficiently large n, $(1 - Q_+^*(u_t)/Q_+^*(0))^n$ factors through $N_{\alpha,\underline{r}''}^\bullet$. So we get a canonical isomorphism $N_{\alpha,\underline{r}}^\bullet \cong N_{\alpha,\underline{r}''}^\bullet$, and similarly there is a canonical isomorphism $N_{\alpha',\underline{r}'}^\bullet \cong N_{\alpha,\underline{r}''}^\bullet$. $\square$

Corollary 4.4.2. *There is a decomposition $C_\lambda^\bullet = N^\bullet \oplus F^\bullet$, where $Q_+^*(u_t)$ annihilates $N^\bullet$ and acts invertibly on $F^\bullet$, and the N^i are finitely generated and projective.*

5. Eisenstein and cuspidal contributions to characteristic power series

5.1. ***Preliminaries.*** In this section, we will write $C_{G,K^p,\lambda}^\bullet$ for $C_\lambda^\bullet$ to make it clear which group we are considering. We will also assume that $G(\mathbb{R})$ has discrete series (i.e., $G(\mathbb{R})$ admits representations with essentially square integrable matrix coefficients, or equivalently $G(\mathbb{R})$ has a maximal torus that is compact modulo $Z_G(\mathbb{R})$), since otherwise Urban's eigenvariety will be empty.

In order to construct Urban's eigenvariety, we need the characteristic power series of the Hecke operators to be Fredholm series. However, the power series $\det(1 - Xf \mid C_{G,K^p,\lambda}^\bullet)$ includes contributions from both cusp forms and Eisenstein series, and the Eisenstein contribution is generally only a ratio of Fredholm series. We will now define a complex $C_{G,K^p,\lambda,\mathrm{cusp}}^\bullet$ whose characteristic power series only includes contributions from cusp forms. (This complex will only be useful for defining characteristic power series; we make no attempt to remove the Eisenstein series from the cohomology.)

We will mostly follow [Urban 2011, Section 4.6]. However, there is an error in the handling of the Eisenstein series in [loc. cit.] that we will need to correct. The region of convergence of an Eisenstein series is generally not a union of Weyl chambers. (For example, $\mathrm{Sp}(6)$ has two conjugacy classes of parabolic subgroups whose Levis are isomorphic to $\mathrm{GL}(2)\times\mathrm{GL}(1)$. The region of convergence of Eisenstein series coming from these parabolics contains one or two full Weyl chambers and fractions of three others.) Consequently, the set $\mathcal{W}^M_{\mathrm{Eis}}$ defined in [loc. cit.] should depend on the weight of the Eisenstein series. A more careful argument is therefore needed to show that character distribution $I^{cl}_{G,0}(f,\mu)$ has a unique p-adic interpolation. In fact, it appears that the character distribution of Eisenstein series coming from a single parabolic subgroup will generally *not* have a unique interpolation. We will show, however, that the sum of distributions coming from parabolic subgroups that have a common Levi will have a unique interpolation.

Let W_G denote the Weyl group of G. Let Φ_G, $\Phi_G^\vee$ denote the set of roots and coroots, respectively, of the pair $(G_{\mathbb{Q}_p}, T)$, where T is the torus chosen in Section 4. Let Φ_G^+ and Φ_G^- denote the subset of roots that are positive and negative, respectively, with respect to B, and we make a similar definition for coroots. Let ρ denote half the sum of the roots in Φ_G^+.

Let F be a finite extension of $\mathbb{Q}_p$. We say that $\mu\colon T_0\to F^\times$ is an algebraic weight if it can be extended to a homomorphism of algebraic groups $T_F\to(\mathbb{G}_m)_F$. We say that an algebraic weight μ is dominant (resp. regular dominant) if $\langle\alpha^\vee,\mu\rangle\geq 0$ (resp. >0) for all $\alpha^\vee\in\Phi_G^{\vee+}$.

Suppose that μ is dominant. Then $\mathcal{D}_\mu(g_iI,F)$ has a (nonzero) quotient that is a finite-dimensional F-vector space. We will write L^G_μ for the corresponding local system on either $S_G(K)$ or $\overline{S_G(K)}$.

Lemma 5.1.1. *Let $f=u_t\otimes f^p\in\mathcal{H}'_G$, and let $\mu\colon T_0\to F^\times$ be an algebraic dominant weight. Then*

$$\det(1-Xf\mid C^\bullet_{G,K^p,\mu})\equiv\det(1-Xf\mid H^\bullet(S_G(K),L^G_\mu))\pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}$$

where

$$N(\mu,t):=\inf_{w\in W_G\setminus\{\mathrm{id}\}}|t^{(w-1)(\mu+\rho)}|_p.$$

Proof. For the degree 1 term, this is [Urban 2011, Lemma 4.5.2]. The argument used there also works for higher degree terms. □

In Section 7, we will consider a family of weights having the property that for any $n\in\mathbb{N}$, the set of points corresponding to regular dominant weights μ satisfying $p^n\mid N(\mu,t)$ is Zariski dense. The characteristic power series for the whole family can then be determined from the $\det(1-Xf\mid H^\bullet(S_G(K),L^G_\mu))$.

If μ is regular dominant, then the cuspidal subspace of $H^i(S_G(K),L^G_\mu)$ is the interior cohomology $H^i_!(S_G(K),L^G_\mu)$ [Li and Schwermer 2004, Section 5.3], and furthermore (since we assume $G(\mathbb{R})$ has discrete series) the interior cohomology is nonzero only in the middle degree [Borel and Wallach 1980, Theorem III.5.1]. Hence either $\det(1-Xf\mid H^\bullet_!(S_G(K),L^G_\mu))$ or its reciprocal is a polynomial.

Our goal is to prove a version of Lemma 5.1.1 in which $C^\bullet_{G,K^p,\mu}$ is replaced by a complex $C^\bullet_{G,K^p,\mu,\mathrm{cusp}}$ that we will define, and $H^\bullet(S_G(K), L^G_\mu)$ is replaced by $H^\bullet_!(S_G(K), L^G_\mu)$.

5.2. ***Cohomology of the Borel–Serre boundary.*** Eisenstein series arise from the Borel–Serre boundary $\partial S_G(K) := \overline{S_G(K)} \setminus S_G(K)$ of $S_G(K)$. The boundary has a stratification by locally symmetric spaces of parabolic subgroups of G.

We warn the reader that the Borel–Serre compactification $\overline{S_G(K)}$ is slightly strange. When constructing a locally symmetric space, one usually takes a quotient by the identity component of either $Z_G(\mathbb{R})$ or $A_G(\mathbb{R})$, where A_G is the $\mathbb{Q}$-split part of Z_G. In order to construct Urban's eigenvariety, we need to choose the former option, but the Borel–Serre compactification behaves better with respect to the latter. Consequently, if M is a Levi subgroup of G, then the locally symmetric space for M should be constructed by taking a quotient by the identity component of $Z_G(\mathbb{R})A_M(\mathbb{R})$ rather than that of $Z_M(\mathbb{R})$. However, it will turn out that we only need to consider Levi subgroups for which the two quotients are the same; see Section 5.4 for more details.

Let P be a parabolic subgroup of G, let N be the maximal unipotent subgroup of P, and let $M = P/N$ be its Levi quotient. Let $K^p_P = K^p \cap P(\mathbb{A}^p_f)$, $K_{P,p} = I \cap P(\mathbb{Q}_p)$, $K_P = K^p_P K_{P,p}$. We can define a locally symmetric space $S_P(K_P)$, and there is a locally closed immersion

$$\iota\colon S_P(K_P) \to \overline{S_G(K)}.$$

If P' is another parabolic subgroup of G, then $S_P(K_P)$ and $S_{P'}(K_{P'})$ will have the same image in $\overline{S_G(K)}$ if and only if $P(\mathbb{A}_f)$ and $P'(\mathbb{A}_f)$ are conjugate by an element of K^pI.

Let K^p_M, I_M be the images of K^p_P, $K_{P,p}$ in $M(\mathbb{A}^p_f)$, $M(\mathbb{Q}_p)$, respectively. The group I_M is an Iwahori subgroup of M. Let $K_M = I_M K^p_M$. The locally symmetric space $S_P(K_P)$ is a nilmanifold bundle over $S_M(K_M)$. Let

$$\pi : S_P(K_P) \to S_M(K_M)$$

denote the projection.

We can relate $R\pi_*\iota^* L^G_\mu$ to local systems on $S_M(K_M)$ using the Kostant decomposition [Borel and Wallach 1980, Theorem III.3.1]. To define the local systems on $S_M(K_M)$, we first need to choose a quasisplit torus T_M of M. The parabolic subgroup $P_{\mathbb{Q}_p}$ contains a conjugate of B. There is a decomposition $G(\mathbb{Q}_p) = IN_G(S)(\mathbb{Q}_p)B(\mathbb{Q}_p)$, where $N_G(S)$ is the normalizer of the maximal split subtorus S of T; this follows from [Bruhat and Tits 1972, Section 4.2.5, Théorème 5.1.3] as well as from [loc. cit., Proposition 7.3.1]. So $iwBw^{-1}i^{-1} \subseteq P_{\mathbb{Q}_p}$ for some $i \in I$, $w \in N_G(S)(\mathbb{Q}_p)$. We choose i and w to minimize the length of the image of w in the Weyl group W_G. Let T_M be the image of $iwTw^{-1}i^{-1}$ in $M_{\mathbb{Q}_p}$. The obvious isomorphism $T \xrightarrow{\sim} T_M$ determines a length-preserving injection of Weyl groups $W_M \hookrightarrow W_G$. Let W^M denote a set of minimal length coset representatives of $W_M \backslash W_G$.

We have the following isomorphism in the derived category of constructible sheaves on $S_M(K_M)$.

$$R\pi_*\iota^* L^G_\mu \cong \bigoplus_{w' \in W^M} L^M_{w^{-1}(w'(\mu+\rho)-\rho)}[l(w') - \dim N] \tag{5.2.1}$$

Here $l(w')$ denotes the length of w'. To see that the splitting exists in the derived category and not just at the level of cohomology, we observe that the $L^M_{w^{-1}(w'(\mu+\rho)-\rho)}$ have distinct central characters.

5.3. *Hecke action.* We will now define an action of $\mathcal{H}_G$ on the cohomology of $S_M(K_M)$ by constructing a homomorphism $\mathcal{H}_G \to \mathcal{H}_M$. The map $R\pi_*\iota^*$ will be equivariant for this action.

As explained in [Urban 2011, Corollary 4.6.3], for any summand of (5.2.1) with $w \neq w'$, the Hecke eigenvalues of $u_t \in \mathcal{H}'_G$ acting on the cohomology of this summand will be divisible by $N(\mu, t)$. Since our goal is to prove a cuspidal analogue of Lemma 5.1.1, we may ignore these summands and just consider the one with $w = w'$. We are therefore only interested in the local system

$$L^M_{w^{-1}(w(\mu+\rho)-\rho)} = L^M_{\mu+(1-w^{-1})\rho}.$$

Our definition of the homomorphism $\mathcal{H}_G \to \mathcal{H}_M$ will be the same as that of [Urban 2011, 4.1.8], except that our convention for the Hecke operators makes some normalization factors disappear. The Hecke algebra $\mathcal{H}_G$ is generated by operators of the form u_t for $t \in T^-$ and $[K_v g K_v]$ for $v \notin S$, $g \in G(\mathbb{Q}_v)$. Let $u_t \in \mathcal{H}_G$ act as $t^{(1-w^{-1})\rho} u_t \in \mathcal{H}_M$. The double coset $K_v g K_v$ decomposes as a finite union $\bigsqcup_j K_v p_j K_v$ with $p_j \in P(\mathbb{Q}_v)$. Let $[K_v g K_v]$ act as $\sum_j [K_{M,v} m_j K_{M,v}]$, where m_j is the image of p_j in $M(\mathbb{Q}_v)$.

Lemma 5.3.1. *The homomorphism $\mathcal{H}_G \to \mathcal{H}_M$ defined above makes the map*

$$R\pi_*\iota^* \colon H^\bullet(S_G(K), L^G_\mu) \to H^\bullet(S_M(K_M), L^M_{\mu+(1-w^{-1})\rho})[l(w) - \dim N]$$

$\mathcal{H}_G$-equivariant.

Proof. The argument is essentially the same as that of [Urban 2011, 4.1.8, 4.6.1–3]. ☐

5.4. *Image of the map $R\pi_*\iota^*$.* To simplify some of the analysis that follows, we will observe that some Levis have $H^\bullet(S_M(K_M), L^M_\mu) = 0$ for a Zariski dense subset of weights μ, and hence they cannot contribute to the characteristic power series. The Levi M can have a nonzero contribution only if the following conditions hold (see [Urban 2011, Theorem 4.7.3(ii)′]):

(1) $M(\mathbb{R})$ has discrete series.

(2) The center Z_M of M is generated by its maximal split subgroup, its maximal compact subgroup, and Z_G.

Now assume that M satisfies the above two conditions. We will define an involution

$$\theta \colon X_*(T_M/Z_G) \to X_*(T_M/Z_G).$$

To do this, we first decompose $X_*(T_M/Z_G) \otimes \mathbb{Q}$ into several pieces. We have

$$X_*(T_M/Z_G) \otimes \mathbb{Q} \cong (X_*(Z_M/Z_G) \oplus X_*(T_M \cap M^{\mathrm{der}})) \otimes \mathbb{Q}.$$

There is an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_*(Z_M/Z_G)$. This representation has open kernel, so it becomes semisimple after tensoring with $\mathbb{Q}$. We define θ to be the operator that acts as 1 on the isotypic component of the trivial representation and as -1 on its orthogonal complement and on $X_*(T_M \cap M^{\mathrm{der}})$. Although it is not immediately obvious that θ preserves the lattice $X_*(T_M/Z_G) \subset X_*(T_M/Z_G) \otimes \mathbb{Q}$, the following alternative description of θ will show that it does.

Let T'_M be a maximal torus of $M_{\mathbb{R}}$ that is compact modulo $(Z_M)_{\mathbb{R}}$. Such a torus exists by assumption (1). If C is an algebraically closed field equipped with inclusions $\mathbb{R} \hookrightarrow C$ and $\mathbb{Q}_p \hookrightarrow C$, then the tori $(T_M)_C$ and $(T'_M)_C$ are conjugate in M_C. Each way of expressing $(T'_M)_C$ as a conjugate of $(T_M)_C$ determines an isomorphism $X_*(T_M/Z_G) \simeq X_*(T'_M/Z_G)$. We claim that under any such isomorphism, the action of complex conjugation on $X_*(T'_M/Z_G)$ induces the involution θ on $X_*(T_M/Z_G)$. Indeed, since T'_M is compact modulo center, complex conjugation acts as -1 on $X_*(T'_M \cap M^{\mathrm{der}})$, and assumption (2) guarantees that any element of $X_*(Z_M/Z_G)$ that is fixed by complex conjugation is fixed by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

By [Li and Schwermer 2004, Section 3.2], the image of

$$R\pi_*\iota^*\colon H^\bullet(S_G(K), L^G_\mu) \to H^\bullet(S_M(K_M), L^M_{\mu+(1-w^{-1})\rho})[l(w) - \dim N]$$

can have nonzero intersection with the cuspidal part

$$H^\bullet_!(S_M(K_M), L^M_{\mu+(1-w^{-1})\rho})[l(w) - \dim N]$$

only if

$$\langle \alpha^\vee, w(1+\theta)(\mu+\rho)\rangle < 0 \quad \forall \alpha^\vee \in \Phi^{\vee+}_G \setminus \Phi^{\vee+}_M. \tag{5.4.1}$$

If the above equation holds and no Eisenstein series arising from M has a pole at $-w(\mu+\rho)$, then the image contains the cuspidal part. In particular, the image contains the cuspidal part if (5.4.1) is satisfied and

$$|\langle \alpha^\vee, (1+\theta)\mu\rangle| \geq 4|\langle \alpha^\vee, \rho\rangle| \quad \forall \alpha^\vee \in \Phi^\vee_G \setminus \Phi^\vee_M. \tag{5.4.2}$$

The constraint (5.4.1) is archimedean in nature, and therefore appears to provide an obstacle to interpolating Eisenstein series p-adically. To get around this issue, we will combine contributions from parabolic subgroups having common Levis.

We will call a Levi subgroup "relevant" if it satisfies the two conditions listed at the beginning of this section, and we will call a parabolic subgroup relevant if its Levi is relevant. Let $\mathcal{P}_0$ be the set of relevant parabolic subgroups of G, modulo the relation that P_1 and P_2 are considered equivalent if $P_1(\mathbb{A}_f)$ and $P_2(\mathbb{A}_f)$ are conjugate by an element of K^pI. Let $\mathcal{P}$ be the set of relevant parabolic subgroups of G, modulo the relation that P_1 and P_2 are considered equivalent if $P_1(\mathbb{Q}_p)$ and $P_2(\mathbb{Q}_p)$ are conjugate by an element of I. Let $\mathcal{M}$ be the set of relevant Levi subgroups of G, modulo the relation that M_1 and M_2 are considered equivalent if $M_1(\mathbb{Q}_p)$ and $M_2(\mathbb{Q}_p)$ are conjugate by an element of I. There are surjections $\mathcal{P}_0 \to \mathcal{P} \to \mathcal{M}$.

Choose representatives of each element of $\mathcal{P}_0$ and $\mathcal{M}$. If P is the representative of an element of $\mathcal{P}_0$ and M is the representative of its image in $\mathcal{M}$, choose a $g \in G(\mathbb{Q})$ so that $M \subset gPg^{-1}$ and the image of g in $G(\mathbb{Q}_p)$ is in I. This choice determines an identification of M with the Levi quotient of P. We will sometimes identify elements of $\mathcal{P}_0$ and $\mathcal{M}$ with the chosen representatives.

Let $M \in \mathcal{M}$. Assume μ is chosen so that (5.4.2) is satisfied. Then there is exactly one parabolic subgroup P_μ containing M for which (5.4.1) will be satisfied: it is the parabolic determined by the set of coroots $\alpha^\vee$ satisfying

$$\langle \alpha^\vee, (1+\theta)\mu \rangle < 0.$$

So μ determines a section $\mathcal{M} \to \mathcal{P}$ of the projection $\mathcal{P} \to \mathcal{M}$. Let $\mathcal{P}_\mu$ be the image of this section, and let $\mathcal{P}_{0,\mu}$ be the preimage of $\mathcal{P}_\mu$ in $\mathcal{P}_0$. At the end of Section 5.2, we associated each parabolic subgroup of G with an element of W_G; this association determines a map $w : \mathcal{P} \to W_G$.

Lemma 5.4.3. *We have*

$$l(w(P_\mu)) = \tfrac{1}{2}|(\Phi_G^- \setminus \Phi_M^-) \cap \theta(\Phi_G^+ \setminus \Phi_M^+)|, \quad (1-\theta)(1-w(P_\mu)^{-1})\rho = \sum_{\alpha \in (\Phi_G^- \setminus \Phi_M^-) \cap \theta(\Phi_G^+ \setminus \Phi_M^+)} \alpha.$$

In particular, $l(w(P_\mu))$ and $(1-\theta)(1-w(P_\mu)^{-1})\rho$ do not depend on μ.

Proof. By definition,

$$l(w(P_\mu)) = |\{\alpha \in \Phi_G \setminus \Phi_M \mid \langle \alpha^\vee, (1+\theta)\mu \rangle > 0, \langle \alpha^\vee, \mu \rangle < 0\}|$$

Observe that if $\langle \alpha^\vee, \mu \rangle < 0 < \langle \alpha^\vee, \theta\mu \rangle$, then exactly one of the inequalities

$$\langle \alpha^\vee, \mu \rangle < 0 < \langle \alpha^\vee, (1+\theta)\mu \rangle, \quad \langle (-\alpha^\vee\theta), \mu \rangle < 0 < \langle (-\alpha^\vee\theta), (1+\theta)\mu \rangle$$

will be satisfied, and otherwise neither will be satisfied. So

$$|\{\alpha \in \Phi_G \setminus \Phi_M \mid \langle \alpha^\vee, \mu \rangle < 0 < \langle \alpha^\vee, \theta\mu \rangle\}| = 2l(w(P_\mu)).$$

This proves the first item. The same observation also proves the second item. □

We will write $l(M)$ for $l(w(P_\mu))$ and $\rho(M,\mu)$ for $(1-w(P_\mu)^{-1})\rho$.

Now we are almost ready to write down an analogue of Lemma 5.1.1 for cusp forms. The boundary components of $S_G(K)$ whose Eisenstein series contribute to the characteristic power series $\det(1 - Xf \mid H^\bullet(S_G(K), L_\mu^G)) \bmod \mathcal{O}_F[\![N(\mu,t)X]\!]$ are in bijection with elements of $\mathcal{P}_{0,\mu}$. Given $M \in \mathcal{M}$, a choice of a preimage P of M in $\mathcal{P}_{0,\mu}$ determines an open compact subgroup K_M^p of $M(\mathbb{A}_f^p)$, as described in Section 5.2. Let $\mathcal{K}_M^p$ be the collection of all such subgroups.

The analysis of the last few sections gives us the following identity.

Lemma 5.4.4. *For any dominant algebraic weight $\mu : T \to F^\times$ satisfying (5.4.2),*

$$\frac{\det(1 - Xf \mid H^\bullet(S_G(K), L_\mu^G))}{\det(1 - Xf \mid H_!^\bullet(S_G(K), L_\mu^G))} \equiv \prod_{M \in \mathcal{M}} \prod_{K_M \in \mathcal{K}_{M,\mu}^p} \det(1 - Xf \mid H_!^\bullet(S_M(K_M), L^M_{\mu+\rho(M,\mu)}))^{(-1)^{\dim N - l(M)}} \pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}.$$

In order to interpolate the local systems p-adically, we need to replace $\mathcal{K}_{M,\mu}^p$ and $\rho(M,\mu)$ with something independent of μ.

Proposition 5.4.5. *For any dominant algebraic weights $\mu\colon T \to F^\times$ and $\mu_0\colon T \to F_0^\times$ satisfying (5.4.2),*

$$\frac{\det(1 - Xf \mid H^\bullet(S_G(K), L^G_\mu))}{\det(1 - Xf \mid H^\bullet_!(S_G(K), L^G_\mu))} \equiv \prod_{M\in\mathcal{M}} \prod_{K_M\in\mathcal{K}_{M,\mu_0}} \det(1 - Xf \mid H^\bullet_!(S_M(K_M), L^M_{\mu+\rho(M,\mu_0)}))^{(-1)^{\dim N - l(M)}} \pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}.$$

Proof. We claim that local systems $L^M_{\mu+\rho(M,\mu)}$, $L^M_{\mu+\rho(M,\mu_0)}$ are isomorphic. The isomorphism class of each local system depends only the restriction of the weight to M^{der}. The operator $(1-\theta)/2$ acts as the identity on the character lattice of M^{der}, so the claim follows from Lemma 5.4.3. Furthermore, the isomorphism of local systems induces an $\mathcal{H}_G$-equivariant isomorphism on cohomology. (The isomorphism on cohomology is not $\mathcal{H}_M$-equivariant — the actions of u_t differ by a factor of $t^{\rho(M,\mu)-\rho(M,\mu_0)}$. However, the two homomorphisms $\mathcal{H}_G \to \mathcal{H}_M$ also differ by the same factor, and so the differences cancel each other.)

It remains to explain why can replace $\mathcal{K}_{M,\mu}$ with $\mathcal{K}_{M,\mu_0}$. Essentially, we need to show that if $\pi = \pi_\infty \otimes \pi_p \otimes \pi_f^p$ is an automorphic representation of M, then

$$\sum_{K_M^p\in\mathcal{K}^p_{M,\mu}} \operatorname{tr}(\mathbf{1}_{K_M^p} \mid \pi_f^p) = \operatorname{tr}(\mathbf{1}_{K^p} \mid \operatorname{Ind}^{G(\mathbb{A}_f^p)}_{P_\mu(\mathbb{A}_f^p)} \pi_f^p)$$

is independent of μ. By [Bernstein and Zelevinsky 1977, 2.9–2.10], for any place v, the composition series of the local factor of $\operatorname{Ind}^{G(\mathbb{A}_f)}_{P_\mu(\mathbb{A}_f)} \pi_f^p$ at v is independent of μ. It follows that the trace of $\mathbf{1}_{K^p}$ does not depend on μ. □

5.5. *The complex $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$.* Now we fix an algebraic dominant weight μ_0 and let $\lambda\colon T \to A^\times$ be any weight. We define $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$ inductively, assuming that analogous complexes have already been defined for $M \in \mathcal{M}$:

$$C^\bullet_{G,K^p,\lambda,\mathrm{cusp}} := C^\bullet_{G,K^p,\lambda} \oplus \bigoplus_{M\in\mathcal{M}} \bigoplus_{K_M^p\in\mathcal{K}^p_{M,\mu_0}} C^\bullet_{M,K_M^p,\lambda+\rho(M,\mu_0),\mathrm{cusp}}[l(M) - \dim N - 1].$$

Proposition 5.5.1. *Let F be a finite extension of $\mathbb{Q}_p$, let $\mu\colon T \to F^\times$ be an algebraic dominant weight, and let $f = u_t \otimes f^p \in \mathcal{H}'_G$. If μ is sufficiently general, then*

$$\det(1 - Xf \mid C^\bullet_{G,K^p,\mu,\mathrm{cusp}}) \equiv \det(1 - Xf \mid H^\bullet_!(S_G(K), L^G_\mu)) \pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}.$$

Proof. By induction, we may assume that the proposition holds for all Levi subgroups of G.

$$\begin{aligned}\det(1 - Xf \mid C^\bullet_{G,K^p,\mu,\mathrm{cusp}}) &\\ \equiv \det(1 - Xf \mid H^\bullet(S_G(K), L^G_\mu)) &\prod_{M,K_M} \det(1 - Xf \mid H^\bullet_!(S_M(K_M), L^M_{\mu+\rho(M,\mu_0)}))^{(-1)^{l(M)-\dim N+1}}\\ \equiv \det(1 - Xf \mid H^\bullet_!(S_G(K), L^G_\mu)) &\pmod{\mathcal{O}_F[\![N(\mu,t)X]\!]}\end{aligned}$$

where we used the induction hypothesis and Lemma 5.1.1 in the second line and Proposition 5.4.5 in the third line. We also use the fact that $\rho(M, \mu_0)$ is M-dominant, and so $\mathcal{O}_F[\![N(\mu+\rho(M,\mu_0),t)X]\!] \subseteq \mathcal{O}_F[\![N(\mu,t)X]\!]$. □

The analysis of Section 4.4 applies equally well to $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$. For any $f \in \mathcal{H}'_G$, we may define a characteristic power series $\det(1 - Xf \mid C^\bullet_{G,K^p,\lambda,\mathrm{cusp}})$. If the Fredholm series $P_+(X) = \prod_i \det(1 - Xf \mid C^i_{G,K^p,\lambda,\mathrm{cusp}})$ has a factorization $P_+ = Q_+ S_+$ with Q_+ a polynomial with invertible leading coefficient, then this factorization induces a decomposition $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}} = N^\bullet \oplus F^\bullet$.

Remark 5.5.2. One can use Proposition 7.1.2 to show that for $f \in \mathcal{H}'_G$, $\det(1 - Xf \mid C^\bullet_{G,K^p,\lambda,\mathrm{cusp}})$ is a Fredholm series. We will not need to prove this fact for arbitrary A and λ, so we leave the details of the argument as an exercise for the reader.

6. Theory of determinants

Urban's eigenvariety construction makes use of pseudocharacters. Chenevier's theory of determinants [2014] is equivalent to the theory of pseudocharacters when the rings involved are $\mathbb{Q}$-algebras [Chenevier 2014, Proposition 1.27], but is better behaved in general. Since we work with rings in which p is not invertible, we will use determinants. (However, it is probably not strictly necessary to use determinants, as we work with rings that are p-torsion-free. See Corollary 7.2.2 and the proof of Lemma 7.4.1.)

We will recall some basic definitions from [Chenevier 2014] and prove a lemma concerning the ratio of two determinants.

Definition 6.1 [Chenevier 2014, Sections 1.1–1.5]. Let A be commutative ring, and let R be an A-module. An *A-valued polynomial law on R* is a rule that assigns to any commutative A-algebra B a map of sets $D_B \colon R \otimes_A B \to B$ that is functorial in the sense that for any A-algebra homomorphism $f \colon B \to B'$,

$$D_{B'} \circ (\mathrm{id}_R \otimes f) = f \circ D_B.$$

Let d be a nonnegative integer. We say that a polynomial law D is *homogeneous of degree d* if

$$D_B(br) = b^d D_B(r) \quad \forall B, b \in B, r \in R \otimes_A B.$$

Now assume that R is an A-algebra. We say that a polynomial law D is *multiplicative* if

$$D_B(1) = 1, \quad D_B(rr') = D_B(r) D_B(r') \quad \forall B, r, r' \in R \otimes_A B.$$

We say that a polynomial law D is a *determinant of dimension d* if it is homogeneous of degree d and multiplicative.

Example 6.2. Let M be an R-module that is projective of rank d as an A-module. Then the rule that sends $r \in R \otimes_A B$ to $\det(r \mid M \otimes_A B)$ is a determinant of dimension d.

Lemma 6.3 [Roby 1963, Proposition I.1]. *Let A be a commutative ring, and let R be an A-module. Let D be an A-valued polynomial law on R that is homogeneous of degree d, let n be a positive integer, and* *let $r_1, \dots, r_n \in R$. Then $D_{A[X_1,\dots,X_n]}(X_1 r_1 + \cdots + X_n r_n)$ is a homogeneous polynomial of degree d in $X_1, \dots, X_n$.*

Lemma 6.4. *Let A be a commutative ring, let R be an A-algebra, and let D^+, D^- be A-valued determinants on R of dimension d_+, d_-, respectively, with $d_+ \geq d_-$. Let $d = d_+ - d_-$. There is at most one determinant D of dimension d satisfying $D^+_B(r) = D^-_B(r) D_B(r)$ for all A-algebras B and all $r \in R \otimes_A B$.*

The following are equivalent:

(1) *There exists a determinant D satisfying the above condition.*

(2) *For any commutative A-algebra B and $r \in R \otimes_A B$, the quotient*

$$D^+_{B[X]}(1 + Xr)/D^-_{B[X]}(1 + Xr)$$

exists in $B[X]$ and has degree at most d.

(3) *For any positive integer n and $r_1, \dots, r_n \in R$, the quotient*

$$D^+_{A[X_1,\dots,X_n]}(1 + X_1 r_1 + \cdots + X_n r_n)/D^-_{A[X_1,\dots,X_n]}(1 + X_1 r_1 + \cdots + X_n r_n)$$

exists in $A[X_1, \dots, X_n]$ and has total degree at most d.

Proof. Let B be a commutative A-algebra, and let $r \in R \otimes_A B$. If $D^-_B(r)$ is not a zero divisor and the quotient $D^+_B(r)/D^-_B(r)$ exists, then we will denote this quotient by $F_B(r)$. Note that $D^-_{B[X]}(1 + Xr)$ has constant term 1 by functoriality with respect to $X \mapsto 0$, so it is not a zero divisor. Similarly, $D^-_{A[X_1,\dots,X_n]}(1 + X_1 r_1 + \cdots + X_n r_n)$ has constant term 1 and is not a zero divisor.

First, we check that D is uniquely determined if it exists. Suppose D is a determinant satisfying the conditions of the lemma. Let B be a commutative A-algebra, and let $r \in R \otimes_A B$. We claim that the following quantities are equal:

- $D_B(r)$.
- The coefficient of $X^d Y^0$ in $D_{B[X,Y]}(Y + Xr)$.
- The coefficient of X^d in $D_{B[X]}(1 + Xr)$.

To see that the first and second quantities are equal, apply functoriality with respect to $X \mapsto 1$, $Y \mapsto 0$, using Lemma 6.3 to show that $D_{B[X,Y]}(Y + Xr)$ has no $X^n Y^0$ term for $n \neq d$. To see that the second and third quantities are equal, apply functoriality with respect to $Y \mapsto 1$, using Lemma 6.3 to show that $D_{B[X,Y]}$ has no $X^d Y^n$ term for $n \neq 0$. Finally, observe that $F_{B[X]}(1 + Xr)$ must exist and must equal $D_{B[X]}(1 + Xr)$. So $D_B(r)$ must be equal to the coefficient of X^d in $F_{B[X]}(1 + Xr)$. Hence D is uniquely determined if it exists.

Lemma 6.3 shows that (1) $\Rightarrow$ (3).

Now we prove (3) $\Rightarrow$ (2). Assume (3) holds. Choose a commutative A-algebra B and $r \in B \otimes_A R$. Condition (3) implies that $F_{A[X_1,\dots,X_n]}(1 + X_1 r_1 + \cdots + X_n r_n)$ exists and has total degree at most d. Then by functoriality with respect to $X_i \mapsto X b_i$, $F_{B[X]}(1 + Xr)$ exists and has degree at most d. This proves (3) $\Rightarrow$ (2).

Now we will show that (2) $\Rightarrow$ (1). Assume that condition (2) holds. Define $D_B(r)$ be the coefficient of X^d in $F_{B[X]}(1+Xr)$. We know that $D^+_{B[X]}(1+Xr)$ (resp. $D^-_{B[X]}(1+Xr)$, $F_{B[X]}(1+Xr)$) has degree at most d_+ (resp. d_-, d), and we have already showed that the coefficient of X^{d_+} (resp. X^{d_-}, X^d) is $D^+_B(r)$ (resp. $D^-_B(r)$, $D_B(r)$). So $D^+_B(r) = D^-_B(r)D_B(r)$.

It remains to show that D is a determinant. Since D^+ and D^- are functorial, D is as well. To show that D is homogeneous of degree d, observe that the map $X \mapsto bX$ multiplies the coefficient of X^d in $F_{B[X]}(1+Xr)$ by b^d.

Finally, we check that D is multiplicative. We have $F_{B[X]}(1+X) = (1+X)^d$, so $D_B(1) = 1$. Observe that $D_B(r_1)D_B(r_2)$ is the coefficient of $(X_1X_2)^d$ in $F_{B[X_1,X_2]}(1+X_1r_1+X_2r_2+X_1X_2r_1r_2)$. This is the same as the coefficient of $X_1^0X_2^0X_3^d$ in $F_{B[X_1,X_2,X_3]}(1+X_1r_1+X_2r_2+X_3r_1r_2)$, since X_3^d is the only monomial of total degree d in $B[X_1, X_2, X_3]$ that maps to $(X_1X_2)^d$ under $X_3 \mapsto X_1X_2$. Then applying $X_1 \mapsto 0$, $X_2 \mapsto 0$, we find that $D_B(r_1)D_B(r_2)$ is the coefficient of X_3^d in $F_{B[X_3]}(1+X_3r_1r_2)$, which is $D_B(r_1r_2)$. This concludes the proof that (2) $\Rightarrow$ (1). $\square$

Corollary 6.5. *Retain the notation of Lemma 6.4. Let $A \hookrightarrow A'$ be an injective map of commutative rings. Suppose that there exists an A'-valued determinant D' on $R \otimes_A A'$ satisfying $D^+_B(r) = D^-_B(r)D'_B(r)$ for any A'-algebra B and $r \in R \otimes_A B$. Then there exists an A-valued determinant D on R satisfying $D^+_B(r) = D^-_B(r)D_B(r)$ for any A-algebra B and $r \in R \otimes_A B$.*

Proof. Apply the equivalence (1) $\Leftrightarrow$ (3) of Lemma 6.4. Observe that $F_{A[X_1,\dots,X_n]}(1+X_1r_1+\cdots+X_nr_n)$ exists and has degree total degree $\le d$ if and only if $F_{A[\![X_1,\dots,X_n]\!]}(1+X_1r_1+\cdots+X_nr_n)$ is a polynomial of total degree $\le d$. Since $A \hookrightarrow A'$ is injective, if $F_{A'[\![X_1,\dots,X_n]\!]}(1+X_1r_1+\cdots+X_nr_n)$ is a polynomial of total degree $\le d$, then $F_{A[\![X_1,\dots,X_n]\!]}(1+X_1r_1+\cdots+X_nr_n)$ is as well. $\square$

Definition 6.6 [Chenevier 2014, Section 1.17, Lemma 1.19(i)]. Let D be an A-valued determinant on R. We denote by $\ker(D)$ the set of $r \in R$ such that for all B and all $r' \in B \otimes_A R$, $D_B(1+r'r) = 1$.

Remark 6.7. Let M be projective A-module of rank d, let $\rho\colon R \to \mathrm{End}(M)$ be a homomorphism, and let D be the determinant associated with ρ, as in Example 6.2. Then $\ker\rho \subseteq \ker D$. Conversely, if $r \in \ker D$, then $D_{A[X]}(X-r) = X^d$, so $r^d \in \ker\rho$ by the Cayley–Hamilton theorem.

Remark 6.8. Chenevier also defines the Cayley–Hamilton ideal $\mathrm{CH}(D)$. Assume D comes from a homomorphism $\rho\colon R \to \mathrm{End}(M)$. Then $\mathrm{CH}(D) \subseteq \ker\rho$. So we might think of $\ker D$ as an upper bound for $\ker\rho$ and $\mathrm{CH}(D)$ as a lower bound. If A is Noetherian, then since $\ker\rho \subseteq \ker D$, $R/\ker D$ is a finite A-module. However, $R/\mathrm{CH}(D)$ need not be a finite A-module, making it more difficult to use $\mathrm{CH}(D)$ in the construction of eigenvarieties.

For a concrete example, consider $A = \mathbb{Q}$, $M = \mathbb{Q}^2$, $R = \mathbb{Q}[T_1, T_2, \dots]$, and let ρ be a map that sends each T_i to a nilpotent upper triangular matrix. Then $\ker D = (T_1, T_2, \dots)$ (so $R/\ker D \cong \mathbb{Q}$), $\ker\mathrm{CH}(D) = (T_1, T_2, \dots)^2$ (so $R/\mathrm{CH}(D)$ is not finite type over $\mathbb{Q}$), and $R/\ker\rho$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{Q}[\epsilon]/(\epsilon^2)$.

7. Construction of the eigenvariety

7.1. *Weight space and Fredholm series.* Now we are ready to define the eigenvariety following [Urban 2011, Section 5]. We will use Huber's theory of adic spaces [1993; 1994; 1996]; see also [Scholze and Weinstein 2019, Sections 2–5] for a modern introduction. Some aspects of our approach follow [Andreatta et al. 2018, Appendice B].

We return to the setup of sections 4–5. We continue to assume that $G(\mathbb{R})$ has discrete series. Let T' be the quotient of T_0 by the closure of $Z_G(\mathbb{Q})G_\infty^+K^p \cap T_0$. We define the weight space

$$\mathcal{W} := \mathrm{Spa}(\mathbb{Z}_p[[T']], \mathbb{Z}_p[[T']])^{\mathrm{an}}.$$

Let $\mathcal{U} = \mathrm{Spa}(A, A^+)$ be an open affinoid subset of $\mathcal{W}$ with A a complete Tate $\mathbb{Z}_p$-algebra (which is automatically Noetherian). Let $\lambda\colon T_0 \to A^\times$ be the tautological character induced by the map $T_0 \to T' \to \mathbb{Z}_p[[T']]$.

For any $f \in \mathcal{H}'_G \otimes_{\mathbb{Z}_p} A$, let

$$P_f(X) := \det(1 - Xf \mid C^\bullet_{G,K^p,\lambda,\mathrm{cusp}})^{(-1)^{d/2}}.$$

Note that $d = \dim S_G(K)$ is even since $G(\mathbb{R})$ has discrete series. If $\mathcal{V}$ is an open subspace of $\mathcal{W}$, and $f \in \mathcal{H}'_G \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{W}}(\mathcal{V})$, then we define $P_f(X)$ by gluing.

Definition 7.1.1. Let $\mathcal{V}$ be an open subspace of $\mathcal{W}$. A series $f \in \mathcal{O}_{\mathcal{W}}(\mathcal{V})[[X]]$ is called a Fredholm series if it is the power series expansion of some global section of $\mathcal{V} \times \mathbb{A}^1$ and its leading coefficient is 1.

This definition agrees with Definition 2.2.1 if $\mathcal{V} = \mathrm{Spa}(A, A^+)$ with A a complete Tate $\mathbb{Z}_p$-algebra.

Proposition 7.1.2. *For $f \in \mathcal{H}'_G$, the series $P_f(X) \in \mathcal{O}_{\mathcal{W}}(\mathcal{W})[[X]]$ is a Fredholm series.*

Proof. Observe that $\mathcal{O}_{\mathcal{W}}(\mathcal{W}) = \mathcal{O}^+_{\mathcal{W}}(\mathcal{W}) = \mathbb{Z}_p[[T']]$. Let T'_{tf} be a maximal torsion-free subgroup of T'. The topology on $\mathbb{Z}_p[[T'_{\mathrm{tf}}]]$ is induced by any norm corresponding to a Gauss point of the wide open polydisc $\mathrm{Spa}(\mathbb{Z}_p[[T'_{\mathrm{tf}}]], \mathbb{Z}_p[[T'_{\mathrm{tf}}]]) \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Similarly, the topology on $\mathbb{Z}_p[[T']]$ is induced by a supremum of a finite collection of norms corresponding to Gauss points of $\mathcal{W} \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. So it suffices to check that the restrictions of $P_f(X)$ to Gauss points of $\mathcal{W}$ are Fredholm series. The Gauss points are characteristic zero points, so we may apply the argument of [Urban 2011, Theorem 4.7.3(iii)] along with Proposition 5.5.1. □

We will write $P(X)$ for $P_{u_t}(X)$. We define the spectral variety $\mathcal{Z} \subset \mathcal{W} \times \mathbb{A}^1$ to be the zero locus of $P(X)$, and we define $w\colon \mathcal{Z} \to \mathcal{W}$ to be the projection. We also define

$$P_+(X) := \prod_i \det(1 - Xu_t \mid C^i_{G,K^p,\lambda,\mathrm{cusp}}).$$

7.2. *Weight space and its characteristic zero subspace.* Before constructing the eigenvariety, we will prove a result that will allow us to deduce information about the behavior of the eigenvariety at the boundary from the characteristic zero part of the eigenvariety.

Lemma 7.2.1. *Let $\mathcal{U} = \mathrm{Spa}(A, A^+)$ by an affinoid adic space. Assume that A is finitely generated over a Noetherian ring of definition. Let $a \in A$ be an element that is not a zero divisor.*

1. *For any open $\mathcal{V} \subseteq \mathcal{U}$, a is not a zero divisor in $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$.*
2. *Assume A is Tate. There exists rational subset $\mathcal{V} \subseteq \mathcal{U}$ such that the restriction $A \to \mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is injective and $a \in \mathcal{O}_{\mathcal{U}}(\mathcal{V})^\times$.*

Proof. To prove the first item, it suffices to consider the case where $\mathcal{V}$ is a rational subset. Then $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ is flat over A by [Huber 1993, Corollary 1.7(i)] and $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is flat over $\mathcal{O}_{\mathcal{U}}(\mathcal{U})$ by [Huber 1996, Proposition 1.6.7(i), Lemma 1.7.6]. Since the multiplication-by-a map is injective on A, it must be injective on $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ as well.

To prove the second item, choose a Noetherian ring of definition $A_0 \subset A$ and a topologically nilpotent $\alpha \in A^\times \cap A_0$. After multiplying a by a power of α, we may assume $a \in A_0$. By the Artin–Rees lemma, there exists an integer $k \geq 1$ so that $\alpha^n A_0 \cap a A_0 \subseteq \alpha^{n-k} a A_0$ for all $n \geq k$. Let $\mathcal{V} \subset \mathcal{U}$ be the rational subset defined by the inequality $|a| \geq |\alpha^k|$. We claim that $A \to \mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is injective.

The ring $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is the α-adic completion of $A_0[1/a]$, and the completion of $A_0[\alpha^k/a]$ is a ring of definition. Let $b \in A$, and suppose the image of b in $\mathcal{O}_{\mathcal{U}}(\mathcal{V})$ is zero. Then for each $n \in \mathbb{N}$, $b \in \alpha^n A_0[\alpha^k/a]$. Then there exists $m \in \mathbb{N}$ so that $a^m b \in \alpha^n (a, \alpha^k)^m A_0$. One can then show by induction on m that $b \in \alpha^n A_0$. Since A_0 is α-adically separated, this implies $b = 0$. □

Corollary 7.2.2. *Let $\mathcal{U} = \mathrm{Spa}(A, A^+)$ be an open affinoid subspace of $\mathcal{W}$, with A complete Tate. Then p is not a zero divisor of $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$, and there exists a rational subset $\mathcal{V} \subseteq \mathcal{U}$ such that $A \to \mathcal{O}_{\mathcal{W}}(\mathcal{V})$ is injective and $p \in \mathcal{O}_{\mathcal{W}}(\mathcal{V})^\times$.*

7.3. *Pieces of the eigenvariety.* Now we construct the individual pieces of the eigenvariety. Let $z \in \mathcal{Z}$. By [Andreatta et al. 2018, Corollaire B.1], there exists an open affinoid neighborhood $\mathcal{U} = \mathrm{Spa}(A, A^+)$ of $w(z)$ and a factorization $P_+(X) = Q_+(X)S_+(X)$, with $Q_+(X) \in A[X]$, $S_+(X) \in A\{\{X\}\}$, such that $Q_+(X)$ and $S_+(X)$ are relatively prime, Q_+ vanishes at x, and the leading coefficient of Q_+ is invertible. The factorization of P_+ induces a factorization $P(X) = Q(X)S(X)$ satisfying similar properties. The factorization also determines a subcomplex $N^\bullet$ of $C^\bullet_{G,K^p,\lambda,\mathrm{cusp}}$, as described in Sections 4.4 and 5.5.

Proposition 7.3.1. *Let D^+ be the determinant associated with the action of $\mathcal{H}_G \otimes_{\mathbb{Z}_p} A$ on $\bigoplus_{i \equiv d/2 (2)} N^i$, and let D^- be the determinant associated with the action of $\mathcal{H}_G \otimes_{\mathbb{Z}_p} A$ on $\bigoplus_{i \equiv d/2+1 (2)} N^i$. Then there exists a determinant D so that $D^+ = D^- D$.*

Proof. Let $R = \mathcal{H}_G \otimes_{\mathbb{Z}_p} A$. As in Lemma 6.4, if B is an A-algebra and $r \in R \otimes_A B$ such that $D^-_B(r)$ is not a zero divisor and the ratio $D^+_B(r)/D^-_B(r)$ exists in B, we write $F_B(r)$ for this ratio. Let d_+ and d_- be the dimensions of D^+ and D^-, respectively, and let $d = d_+ - d_-$.

By Corollary 7.2.2, we can find a rational subset $\mathcal{V} \subset \mathcal{U}$ so that the restriction $A \to \mathcal{O}_{\mathcal{W}}(\mathcal{V})$ is injective and p is invertible on $\mathcal{V}$. Since $\mathcal{V}$ is a reduced rigid space, the natural map $\mathcal{O}_{\mathcal{W}}(\mathcal{V}) \to \prod_x k_x$ is injective, where the product runs over rigid analytic points $x \in \mathcal{V}$ and k_x is the residue field of x. For each x, write $D^+|_{k_x}$ (resp. $D^-|_{k_x}$) for the base change of D^+ (resp. D^-) along $A \to k_x$. By [Urban 2011, Lemma 4.1.12

and Theorem 4.7.3iii], the difference of the pseudocharacters corresponding to $D^+|_{k_x}$ and $D^-|_{k_x}$ is again a pseudocharacter. By the equivalence of pseudocharacters and determinants in characteristic zero [Chenevier 2014, Proposition 1.27], there exists a determinant $D|_{k_x}$ satisfying $D^+|_{k_x} = D^-|_{k_x} D|_{k_x}$. Then by Corollary 6.5, there exists a determinant D satisfying $D^+ = D^- D$. □

Let

$$h_{\mathcal{U},Q_+} := (\mathcal{H}_G \otimes_{\mathbb{Z}_p} A)/\ker(D).$$

We will use the extension $A \to h_{\mathcal{U},Q_+}$ to construct an adic space $\mathcal{E}_{\mathcal{U},Q_+}$ over $\mathcal{U}$.

Lemma 7.3.2. *The ring $h_{\mathcal{U},Q_+}$ is a finite A-module.*

Proof. Since $\ker(D)$ contains any operator that annihilates $N^\bullet$, $h_{\mathcal{U},Q_+}$ can be identified with a subquotient of $\bigoplus_i \operatorname{End} N^i$. In particular, $h_{\mathcal{U},Q_+}$ must be finitely generated as an A-module. □

We give $h_{\mathcal{U},Q_+}$ the "A-module topology" defined in [Huber 1994, Section 2].

Lemma 7.3.3. *The ring $h_{\mathcal{U},Q_+}$ is Tate and has a Noetherian ring of definition.*

Proof. Choose A-module generators $a_1, \dots, a_n$ of $h_{\mathcal{U},Q_+}$. Choose $m_{ijk} \in h_{\mathcal{U},Q_+}$ so that for each i, j, $a_i a_j = \sum_{k=1}^n m_{ijk} a_k$. Let A_0 be a ring of definition of A, and let α be a topologically nilpotent unit of A contained in A_0. There exists an integer ℓ so that $\alpha^\ell m_{ijk} \in A_0$ for all i, j, k. Let $h_{\mathcal{U},Q_+,0}$ be the A_0-submodule of $h_{\mathcal{U},Q_+}$ generated by $1, \alpha^\ell a_1, \dots, \alpha^\ell a_n$; then $h_{\mathcal{U},Q_+,0}$ is an open subring of $h_{\mathcal{U},Q_+}$. Then $h_{\mathcal{U},Q_+,0}$ is Noetherian since A_0 is Noetherian, and $h_{\mathcal{U},Q_+,0}$ inherits the α-adic topology from A_0. So $h_{\mathcal{U},Q_+}$ has a Noetherian ring of definition and is Tate. □

Let $h^+_{\mathcal{U},Q_+}$ be the normal closure of A^+ in $h_{\mathcal{U},Q_+}$. Then $(h_{\mathcal{U},Q_+}, h^+_{\mathcal{U},Q_+})$ is a Huber pair. We define $\mathcal{E}_{\mathcal{U},Q_+} := \operatorname{Spa}(h_{\mathcal{U},Q_+}, h^+_{\mathcal{U},Q_+})$.

Since $Q^*(X)$ is the characteristic polynomial of u acting on $N^\bullet$, it follows from [Chenevier 2014, Lemma 1.12(iv)] that $Q^*(u)$ is in $\ker(D)$, and so there is a canonical map $\mathcal{E}_{\mathcal{U},Q_+} \to \mathcal{Z}$.

7.4. *Gluing.* We will glue the $\mathcal{E}_{\mathcal{U},Q_+}$ as in [Buzzard 2007, Section 5]. We need the following lemma to verify that the pieces can be glued.

Lemma 7.4.1. *If $\mathcal{U}' \subset \mathcal{U}$ are affinoid subspaces of $\mathcal{W}$, then there is a canonical isomorphism $\mathcal{E}_{\mathcal{U}',Q_+} \cong \mathcal{E}_{\mathcal{U},Q_+} \times_{\mathcal{U}} \mathcal{U}'$.*

Proof. By Corollary 7.2.2, p is not a zero divisor in $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$ and $\mathcal{O}_{\mathcal{W}}(\mathcal{U}')$. Hence $\mathcal{O}_{\mathcal{W}}(\mathcal{U})$ and $\mathcal{O}_{\mathcal{W}}(\mathcal{U}')$ are torsion-free $\mathbb{Z}$-modules. The map $\mathcal{O}_{\mathcal{W}}(\mathcal{U}) \to \mathcal{O}_{\mathcal{W}}(\mathcal{U}')$ is flat by [Huber 1996, Lemma 1.7.6]. By the argument of [Rydh 2008, Proposition I.2.2.4], if A is a ring that is a torsion-free $\mathbb{Z}$-module, then the kernel of an A-valued determinant is the same as the kernel of the associated pseudocharacter. By [Rydh 2008, Proposition I.2.2.8], the formation of the kernel of a pseudocharacter commutes with flat base change. So the formation of the kernel of a determinant commutes with the base change $\mathcal{O}_{\mathcal{W}}(\mathcal{U}) \to \mathcal{O}_{\mathcal{W}}(\mathcal{U}')$. Then $h_{\mathcal{U}',Q_+} \cong h_{\mathcal{U},Q_+} \otimes_{\mathcal{O}_{\mathcal{W}}(\mathcal{U})} \mathcal{O}_{\mathcal{W}}(\mathcal{U}')$. Since $h_{\mathcal{U},Q_+}$ is finite over $\mathcal{O}_{\mathcal{W}}(\mathcal{U}')$, it follows that $\mathcal{E}_{\mathcal{U}',Q_+} \cong \mathcal{E}_{\mathcal{U},Q_+} \times_{\mathcal{U}} \mathcal{U}'$. □

One can also show, using essentially the same proof as [Urban 2011, Proposition 5.3.5], that if Q_+ and Q'_+ are relatively prime, then there is a canonical isomorphism $\mathcal{E}_{\mathcal{U},Q_+Q'_+} \cong \mathcal{E}_{\mathcal{U},Q_+} \sqcup \mathcal{E}_{\mathcal{U},Q'_+}$.

Theorem 7.4.2. *The $\mathcal{E}_{\mathcal{U},Q_+}$ can be glued to form an adic space $\mathcal{E}$. Furthermore, $\mathcal{E}$ is equidimensional in the sense of [Huber 1996, Definition 1.8.1] and the morphism $\mathcal{E} \to \mathcal{Z}$ is finite and surjective.*

Proof. To show that the morphism $\mathcal{E} \to \mathcal{Z}$ is finite, we observe that $\mathcal{Z}$ can be covered by open sets whose preimage in $\mathcal{E}$ is contained in some $\mathcal{E}_{\mathcal{U},Q_+}$. The finiteness of the morphism $\mathcal{E} \to \mathcal{Z}$ then follows from the finiteness of the maps $\mathcal{E}_{\mathcal{U},Q_+} \to \mathcal{U}$.

Now we check that the morphism is surjective. Let $z \in \mathcal{Z}$, and let k be the residue field of z. Observe that $\operatorname{Spec} \mathcal{H}_G \to \operatorname{Spec} \mathbb{Z}_p[u_t]$ is surjective, so $\mathcal{H}_G \otimes_{\mathbb{Z}_p[u_t]} k$ cannot be the zero ring. The image of $\ker(D)$ in $\mathcal{H}_G \otimes_{\mathbb{Z}_p[u_t]} k$ is contained in the kernel of the base change of D to $\mathcal{H}_G \otimes_{\mathbb{Z}_p[u_t]} k$. Therefore the image of $\ker(D)$ cannot be the unit ideal, and so there must be a point of $\mathcal{E}$ lying above z.

Finally, we show that $\mathcal{E}$ is equidimensional. The weight space $\mathcal{W}$ has the same dimension as its characteristic zero part. By [Urban 2011, Theorem 5.3.7(iii)], the characteristic zero part of $\mathcal{E}$ is equidimensional of dimension $\dim \mathcal{W}$. By Lemma 7.2.1(2), any nonempty open $\mathcal{U} \subseteq \mathcal{E}$ has nonempty characteristic zero part, so it has dimension at least $\dim \mathcal{W}$. Conversely, since $\mathcal{E}$ is locally finite over $\mathcal{W}$, $\mathcal{E}$ has dimension at most $\dim \mathcal{W}$ by [Huber 1996, Example 1.8.9(ii)]. □

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