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**Cohomological and numerical dynamical degrees  
on abelian varieties**

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# Cohomological and numerical dynamical degrees on abelian varieties

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We show that for a self-morphism of an abelian variety defined over an algebraically closed field of arbitrary characteristic, the second cohomological dynamical degree coincides with the first numerical dynamical degree.

*A list of symbols can be found on page 1957.*

## 1. Introduction

Let  $X$  be a smooth projective variety defined over an algebraically closed field  $k$ , and  $f$  a surjective morphism of  $X$  to itself. Inspired by Esnault and Srinivas [2013] and Truong [2016], we associate to this map two dynamical degrees as follows. Let  $\ell$  be a prime different from the characteristic of  $k$ . As a consequence of Deligne [1974] and Katz and Messing [1974], the characteristic polynomial of  $f$  on the  $\ell$ -adic étale cohomology group  $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$  is independent of  $\ell$ , and has integer coefficients, and algebraic integer roots (see [Esnault and Srinivas 2013, Proposition 2.3]; see also [Kleiman 1968]). The  $i$ -th cohomological dynamical degree  $\chi_i(f)$  of  $f$  is then defined as the spectral radius of the pullback action  $f^*$  on  $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$ , i.e.,

$$\chi_i(f) = \rho(f^*|_{H_{\text{ét}}^i(X, \mathbb{Q}_\ell)}).$$

Alternatively, one can also define dynamical degrees using algebraic cycles. Indeed, let  $N^k(X)$  denote the group of algebraic cycles of codimension  $k$  modulo numerical equivalence. Note that  $N^k(X)$  is a finitely generated free abelian group [Kleiman 1968, Theorem 3.5], and hence the characteristic polynomial of  $f$  on  $N^k(X)$  has integer coefficients and algebraic integer roots. We define the  $k$ -th numerical dynamical degree  $\lambda_k(f)$  of  $f$  as the spectral radius of the pullback action  $f^*$  on  $N^k(X)_{\mathbb{R}} := N^k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ , i.e.,

$$\lambda_k(f) = \rho(f^*|_{N^k(X)_{\mathbb{R}}}).$$

When  $k \subseteq \mathbb{C}$ , we may associate to  $(X, f)$  a projective (and hence compact Kähler) manifold  $X_{\mathbb{C}}$  and a surjective holomorphic map  $f_{\mathbb{C}}$ . Then by the comparison theorem and Hodge theory, it is not hard to show that  $\chi_{2k}(f) = \lambda_k(f)$ ; both of them also agree with the usual dynamical degree defined by the Dolbeault

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cohomology group  $H^{k,k}(X_{\mathbb{C}}, \mathbb{C})$  in the context of complex dynamics (see e.g., [Dinh and Sibony 2017, Section 4]).

For an arbitrary algebraically closed field  $k$  (in particular, of positive characteristic), Esnault and Srinivas [2013] proved that for an automorphism of a smooth projective surface, the second cohomological dynamical degree coincides with the first numerical dynamical degree. Their proof relies on the Enriques–Bombieri–Mumford classification of surfaces in arbitrary characteristic. In general, Truong [2016] raised the following question (among many others):

**Question 1.1** (cf. [Truong 2016, Question 2]). Let  $X$  be a smooth projective variety defined over an algebraically closed field  $k$ , and  $f$  a surjective morphism of  $X$  to itself. Then is  $\chi_{2k}(f) = \lambda_k(f)$  for any  $1 \leq k \leq \dim X$ ?

The above question turns out to be related to Weil’s Riemann hypothesis (proved by Deligne in the early 1970s). More precisely, when  $X_0$  is a smooth projective variety defined over a finite field  $F_q$ , we let  $X$  denote the base change of  $X_0$  to the algebraic closure  $\bar{F}_q$  of  $F_q$  and let  $F$  denote the Frobenius endomorphism of  $X$  (with respect to  $F_q$ ). Then Deligne’s celebrated theorem asserts that all eigenvalues of  $F^*|_{H_{\text{ét}}^i(X, \mathbb{Q}_\ell)}$  are algebraic integers of modulus  $q^{i/2}$  [Deligne 1974, Théorème 1.6]. In particular, we have  $\chi_i(F) = q^{i/2}$ . On the other hand, the  $k$ -th numerical dynamical degree  $\lambda_k(F)$  of  $F$  is equal to  $q^k$ . See [Truong 2016, Section 4] for more details.

Truong [2016] proved a slightly weaker statement that

$$h_{\text{ét}}(f) := \max_i \log \chi_i(f) = \max_k \log \lambda_k(f) =: h_{\text{alg}}(f),$$

which is enough to conclude that the (étale) entropy  $h_{\text{ét}}(f)$  coincides with the algebraic entropy  $h_{\text{alg}}(f)$  in the sense of [Esnault and Srinivas 2013, Section 6.3]. As a consequence, the spectral radius of the action  $f^*$  on the even degree étale cohomology  $H_{\text{ét}}^{2*}(X, \mathbb{Q}_\ell)$  is the same as the spectral radius of  $f^*$  on the total cohomology  $H_{\text{ét}}^*(X, \mathbb{Q}_\ell)$ .<sup>1</sup> Note that when  $k \subseteq \mathbb{C}$ , by the fundamental work of Gromov [2003] and Yomdin [1987], the algebraic entropy is also equal to the topological entropy  $h_{\text{top}}(f_{\mathbb{C}})$  of the topological dynamical system  $(X_{\mathbb{C}}, f_{\mathbb{C}})$ ; see [Dinh and Sibony 2017, Section 4] for more details.

In this article, we give an affirmative answer to Question 1.1 in the case that  $X$  is an abelian variety and  $k = 1$ .

**Theorem 1.2.** *Let  $X$  be an abelian variety defined over an algebraically closed field  $k$ , and  $f$  a surjective self-morphism of  $X$ . Then  $\chi_2(f) = \lambda_1(f)$ .*

**Remark 1.3.** (1) When  $f$  is an automorphism of an abelian surface  $X$ , the theorem was already known by Esnault and Srinivas [2013, Section 4]. Even in this two-dimensional case, their proof is quite involved. Actually, after a standard specialization argument, they applied the celebrated Tate theorem [1966] (see also [Mumford 1970, Appendix I, Theorem 3]), which asserts that the minimal polynomial of the geometric Frobenius endomorphism is a product of distinct monic irreducible polynomials. Then they

<sup>1</sup>Recently, this was reproved by Shuddhodan [2019] using a number-theoretic method, where the author introduced a zeta function  $Z(X, f, t)$  for a dynamical system  $(X, f)$  defined over a finite field.

had four cases to analyze according to its irreducibility and degree. Our proof is more explicit in the sense that we will eventually determine all eigenvalues of  $f^*|_{N^1(X)_\mathbb{R}}$ .

(2) Because of the lack of an explicit characterization of higher-codimensional cycles (up to numerical equivalence) like the Néron–Severi group  $NS(X)$  sitting inside the endomorphism algebra  $\text{End}^0(X)$ , it would be very interesting to consider the case  $k \geq 2$  next.

## 2. Preliminaries on abelian varieties

We refer to [Mumford 1970; Milne 1986] for standard notation and terminologies on abelian varieties and to page 1957 for a list of symbols.

For the convenience of the reader, we include several important structure theorems on the étale cohomology groups, the endomorphism algebras and the Néron–Severi groups of abelian varieties. We refer to [Mumford 1970, Sections 19–21] for more details.

First, the étale cohomology groups of abelian varieties are simple to describe.

**Theorem 2.1** [Milne 1986, Theorem 15.1]. *Let  $X$  be an abelian variety of dimension  $g$  defined over  $\mathbf{k}$ , and let  $\ell$  be a prime different from  $\text{char } \mathbf{k}$ . Let  $T_\ell X := \varprojlim_n X_{\ell^n}(\mathbf{k})$  be the Tate module of  $X$ , which is a free  $\mathbb{Z}_\ell$ -module of rank  $2g$ .*

(a) *There is a canonical isomorphism*

$$H_{\text{ét}}^1(X, \mathbb{Z}_\ell) \simeq \text{Hom}_{\mathbb{Z}_\ell}(T_\ell X, \mathbb{Z}_\ell).$$

(b) *The cup-product pairing induces isomorphisms*

$$\bigwedge^i H_{\text{ét}}^1(X, \mathbb{Z}_\ell) \simeq H_{\text{ét}}^i(X, \mathbb{Z}_\ell),$$

*for all  $i$ . In particular,  $H_{\text{ét}}^i(X, \mathbb{Z}_\ell)$  is a free  $\mathbb{Z}_\ell$ -module of rank  $\binom{2g}{i}$ .*

Furthermore, the functor  $T_\ell$  induces an  $\ell$ -adic representation of the endomorphism algebra. In general, we have:

**Theorem 2.2** [Mumford 1970, Section 19, Theorem 3]. *For any two abelian varieties  $X$  and  $Y$ , the group  $\text{Hom}(X, Y)$  of homomorphisms of  $X$  into  $Y$  is a finitely generated free abelian group, and the natural homomorphism of  $\mathbb{Z}_\ell$ -modules*

$$\text{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow \text{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$$

*induced by  $T_\ell$ :  $\text{Hom}(X, Y) \rightarrow \text{Hom}_{\mathbb{Z}_\ell}(T_\ell X, T_\ell Y)$  is injective.*

For a homomorphism  $f : X \rightarrow Y$  of abelian varieties, its *degree*  $\text{deg } f$  is defined to be the order of the kernel  $\ker f$ , if it is finite, and 0 otherwise. In particular, the degree of an isogeny is always a positive integer.

**Theorem 2.3** [Mumford 1970, Section 19, Theorem 4]. *For any  $\alpha \in \text{End}(X)$ , there is a unique monic polynomial  $P_\alpha(t) \in \mathbb{Z}[t]$  of degree  $2g$  such that  $P_\alpha(n) = \text{deg}(n_X - \alpha)$  for all integers  $n$ . Moreover,  $P_\alpha(t)$  is the characteristic polynomial of  $\alpha$  acting on  $T_\ell X$ , i.e.,  $P_\alpha(t) = \det(t - T_\ell \alpha)$ , and  $P_\alpha(\alpha) = 0$  as an endomorphism of  $X$ .*

We call  $P_\alpha(t)$  as in [Theorem 2.3](#) the *characteristic polynomial of  $\alpha$* . On the other hand, we can assign to each  $\alpha$  the characteristic polynomial  $\chi_\alpha(t)$  of  $\alpha$  as an element of the semisimple  $\mathbb{Q}$ -algebra  $\text{End}^0(X)$ . Namely, we define  $\chi_\alpha(t)$  to be the characteristic polynomial of the left multiplication  $\alpha_L : \beta \mapsto \alpha\beta$  for  $\beta \in \text{End}^0(X)$  which is a  $\mathbb{Q}$ -linear transformation on  $\text{End}^0(X)$ . Note that the above definition of  $\chi_\alpha(t)$  makes no use of the fact that  $\text{End}^0(X)$  is semisimple. Actually, for semisimple  $\mathbb{Q}$ -algebras, it is much more useful to consider the so-called reduced characteristic polynomials.

We recall some basic definitions on semisimple algebras (see [\[Reiner 2003, Section 9\]](#) for more details).

**Definition 2.4.** Let  $R$  be a finite-dimensional semisimple algebra over a field  $F$  with  $\text{char } F = 0$ , and write

$$R = \bigoplus_{i=1}^k R_i,$$

where each  $R_i$  is a simple  $F$ -algebra. For any element  $r \in R$ , as above, we denote by  $\chi_r(t)$  the *characteristic polynomial of  $r$* . Namely,  $\chi_r(t)$  is the characteristic polynomial of the left multiplication  $r_L : r' \mapsto rr'$  for  $r' \in R$ . Let  $K_i$  be the center of  $R_i$ . Then there exists a finite field extension  $E_i/K_i$  splitting  $R_i$  [\[Reiner 2003, Section 7b\]](#), i.e., we have

$$h_i : R_i \otimes_{K_i} E_i \xrightarrow{\sim} M_{d_i}(E_i), \quad \text{where } [R_i : K_i] = d_i^2.$$

Write  $r = r_1 + \dots + r_k$  with each  $r_i \in R_i$ . We first define the *reduced characteristic polynomial  $\chi_{r_i}^{\text{red}}(t)$  of  $r_i$*  as follows [\[Reiner 2003, Definition 9.13\]](#):

$$\chi_{r_i}^{\text{red}}(t) := N_{K_i/F}(\det(t\mathbf{I}_{d_i} - h_i(r_i \otimes_{K_i} 1_{E_i}))) \in F[t].$$

It turns out that  $\det(t\mathbf{I}_{d_i} - h_i(r_i \otimes_{K_i} 1_{E_i}))$  lies in  $K_i[t]$ , and is independent of the choice of the splitting field  $E_i$  of  $R_i$  [\[Reiner 2003, Theorem 9.3\]](#). The *reduced norm of  $r_i$*  is defined by

$$N_{R_i/F}^{\text{red}}(r_i) := N_{K_i/F}(\det(h_i(r_i \otimes_{K_i} 1_{E_i}))) \in F.$$

Finally, as one expects, the *reduced characteristic polynomial  $\chi_r^{\text{red}}(t)$  and the reduced norm  $N_{R/F}^{\text{red}}(r)$  of  $r$*  are defined by the products

$$\chi_r^{\text{red}}(t) := \prod_{i=1}^k \chi_{r_i}^{\text{red}}(t) \quad \text{and} \quad N_{R/F}^{\text{red}}(r) := \prod_{i=1}^k N_{R_i/F}^{\text{red}}(r_i).$$

**Remark 2.5.** (1) It follows from [\[Reiner 2003, Theorem 9.14\]](#) that

$$\chi_r(t) = \prod_{i=1}^k \chi_{r_i}(t) = \prod_{i=1}^k \chi_{r_i}^{\text{red}}(t)^{d_i}. \tag{2-1}$$

(2) Note that reduced characteristic polynomials and norms are not affected by change of ground field [\[Reiner 2003, Theorem 9.27\]](#).

We now apply the above algebraic setting to  $R = \text{End}^0(X)$ . For any  $\alpha \in \text{End}(X)$ , let  $\chi_\alpha^{\text{red}}(t)$  denote the reduced characteristic polynomial of  $\alpha$  as an element of the semisimple  $\mathbb{Q}$ -algebra  $\text{End}^0(X)$ . For simplicity, let us first consider the case when  $X = A$  is a simple abelian variety and hence  $D := \text{End}^0(A)$  is a division ring. Let  $K$  denote the center of  $D$  which is a field, and  $K_0$  the maximal totally real subfield of  $K$ . Set

$$d^2 = [D : K], e = [K : \mathbb{Q}] \quad \text{and} \quad e_0 = [K_0 : \mathbb{Q}].$$

Then the equality (2-1) reads as

$$\chi_\alpha(t) = \chi_\alpha^{\text{red}}(t)^d.$$

The lemma below shows that the two polynomials  $P_\alpha(t)$  and  $\chi_\alpha(t)$  are closely related. Its proof relies on a characterization of normal forms of  $D$  over  $\mathbb{Q}$ .

For convenience, we include the following definition. Let  $R$  be a finite-dimensional associative algebra over an infinite field  $F$ . A *norm form* on  $R$  over  $F$  is a nonzero polynomial function

$$N_{R/F} : R \rightarrow F$$

(i.e., in terms of a basis of  $R$  over  $F$ ,  $N_{R/F}(r)$  can be written as a polynomial over  $F$  in the components of  $r$ ) such that  $N_{R/F}(rr') = N_{R/F}(r)N_{R/F}(r')$  for all  $r, r' \in R$ .

**Lemma 2.6.** *Using notation as above, for any  $\alpha \in \text{End}(A)$ , we have*

$$P_\alpha(t) = \chi_\alpha^{\text{red}}(t)^m,$$

where  $m = 2g/(ed)$  is a positive integer. In particular, the two polynomials  $P_\alpha(t)$  and  $\chi_\alpha(t)$  have the same complex roots (apart from multiplicities).<sup>2</sup>

*Proof.* By the lemma in [Mumford 1970, Section 19] (located between Corollary 3 and Theorem 4, page 179), any norm form of  $D$  over  $\mathbb{Q}$  is of the following type

$$(\mathbb{N}_{K/\mathbb{Q}} \circ \mathbb{N}_{D/K}^{\text{red}})^k : D \rightarrow \mathbb{Q}$$

for a suitable nonnegative integer  $k$ , where  $\mathbb{N}_{D/K}^{\text{red}}$  is the reduced norm (aka canonical norm form in the sense of Mumford) of  $D$  over  $K$ . Now for each  $n \in \mathbb{Z}$ , we have

$$\chi_\alpha^{\text{red}}(n) = \mathbb{N}_{K/\mathbb{Q}} \circ \mathbb{N}_{D/K}^{\text{red}}(n_A - \alpha).$$

On the other hand, the action of  $D$  on  $V_\ell A := T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$  defines the determinant map

$$\det : D \rightarrow \mathbb{Q}_\ell,$$

which actually takes on values in  $\mathbb{Q}$  and is a norm form of degree  $2g$ . Indeed, let  $V_\ell \alpha$  denote the induced map of  $\alpha$  on  $V_\ell A$ , then  $P_\alpha(n) = \deg(n_A - \alpha) = \det(n_A - \alpha) = \det(n - V_\ell \alpha)$  for all integers  $n$  (see

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<sup>2</sup>I would like to thank Yuri Zarhin for showing me an argument using the canonical norm form to prove this Lemma 2.6.

**Theorem 2.3).** Applying the aforementioned lemma in [Mumford 1970, Section 19] to this det, we obtain that for a suitable  $m$ ,

$$\det(\psi) = (\mathbb{N}_{K/\mathbb{Q}} \circ \mathbb{N}_{D/K}^{\text{red}}(\psi))^m$$

for all  $\psi \in D$ . It is easy to see that  $m$  is  $2g/(ed)$ . Then by taking  $\psi = n_A - \alpha$ , we have that  $P_\alpha(n) = \chi_\alpha^{\text{red}}(n)^m$  for all integers  $n$ . This yields that  $P_\alpha(t) = \chi_\alpha^{\text{red}}(t)^m$ .  $\square$

It is straightforward to generalize Lemma 2.6 to the case that  $X$  is the  $n$ -th power  $A^n$  of a simple abelian variety  $A$  since  $\text{End}^0(A^n) = \mathbb{M}_n(\text{End}^0(A))$  is still a simple  $\mathbb{Q}$ -algebra.

**Lemma 2.7.** *Let  $A$  be a simple abelian variety and  $X = A^n$ . Let  $\chi_\alpha^{\text{red}}(t)$  denote the reduced characteristic polynomial of  $\alpha$  as an element of the simple  $\mathbb{Q}$ -algebra  $\text{End}^0(X) = \mathbb{M}_n(D)$  with  $D = \text{End}^0(A)$ . Then*

$$\chi_\alpha(t) = \chi_\alpha^{\text{red}}(t)^{dn} \text{ and } P_\alpha(t) = \chi_\alpha^{\text{red}}(t)^m,$$

where  $m = 2g/(edn)$  is a positive integer. In particular, these two polynomials  $P_\alpha(t)$  and  $\chi_\alpha(t)$  have the same complex roots (apart from multiplicities).

We recall the following useful structure theorems on  $\text{NS}^0(X)$  which play a crucial role in the proof of our main theorem.

**Theorem 2.8** [Mumford 1970, Section 21, Application III]. *Fix a polarization  $\phi : X \rightarrow \hat{X}$  that is an isogeny from  $X$  to its dual  $\hat{X}$  induced from some ample line bundle  $\mathcal{L}_0$  (we suppress this  $\mathcal{L}_0$  since it does not make an appearance here henceforth). Then the natural map*

$$\text{NS}^0(X) \rightarrow \text{End}^0(X) \text{ via } \mathcal{L} \mapsto \phi^{-1} \circ \phi_{\mathcal{L}}$$

is injective and its image is precisely the subspace  $\{\psi \in \text{End}^0(X) \mid \psi^\dagger = \psi\}$  of symmetric elements of  $\text{End}^0(X)$  under the Rosati involution  $^\dagger$  which maps  $\psi$  to  $\psi^\dagger := \phi^{-1} \circ \hat{\psi} \circ \phi$ .

**Theorem 2.9** [Mumford 1970, Section 21, Theorems 2 and 6]. *The endomorphism  $\mathbb{R}$ -algebra  $\text{End}(X)_\mathbb{R} := \text{End}^0(X) \otimes_\mathbb{Q} \mathbb{R}$  is isomorphic to a product of copies of  $\mathbb{M}_r(\mathbb{R})$ ,  $\mathbb{M}_r(\mathbb{C})$  and  $\mathbb{M}_r(\mathbf{H})$ . Moreover, one can fix an isomorphism so that it carries the Rosati involution into the standard involution  $\mathbf{A} \mapsto \overline{\mathbf{A}}^\top$ . In particular,  $\text{NS}(X)_\mathbb{R} := \text{NS}^0(X) \otimes_\mathbb{Q} \mathbb{R}$  is isomorphic to a product of Jordan algebras of the following types:*

$$\mathcal{H}_r(\mathbb{R}) = r \times r \text{ symmetric real matrices,}$$

$$\mathcal{H}_r(\mathbb{C}) = r \times r \text{ Hermitian complex matrices,}$$

$$\mathcal{H}_r(\mathbf{H}) = r \times r \text{ Hermitian quaternionic matrices.}$$

### 3. Proof of Theorem 1.2

**3.1. Some results on dynamical degrees.** We first prepare some results used later to prove our main theorem. Recall that in complex dynamics, the dynamical degrees are bimeromorphic invariants of the dynamics system (see e.g., [Dinh and Sibony 2017, Theorem 4.2]). We have also shown the birational invariance of numerical dynamical degrees in arbitrary characteristic [Hu 2019, Lemma 2.8]. Below is a

similar consideration which should be of interest in its own right. Note, however, that we have not shown the birational invariance of cohomological dynamical degrees, which is actually one of the questions raised by Truong [2016, Question 5].

**Lemma 3.1.** *Let  $\pi : X \rightarrow Y$  be a surjective morphism of smooth projective varieties defined over  $k$ . Let  $f$  and  $g$  be surjective self-morphisms of  $X$  and  $Y$ , respectively, such that  $\pi \circ f = g \circ \pi$ . Then  $\chi_i(f) \geq \chi_i(g)$  for any  $0 \leq i \leq 2 \dim Y$  and  $\lambda_k(f) \geq \lambda_k(g)$  for any  $0 \leq k \leq \dim Y$ .*

*Proof.* We have the following commutative diagram of  $\mathbb{Q}_\ell$ -vector spaces:

$$\begin{array}{ccc} H_{\text{ét}}^i(Y, \mathbb{Q}_\ell) & \xrightarrow{\pi^*} & H_{\text{ét}}^i(X, \mathbb{Q}_\ell) \\ g^* \downarrow & & \downarrow f^* \\ H_{\text{ét}}^i(Y, \mathbb{Q}_\ell) & \xrightarrow{\pi^*} & H_{\text{ét}}^i(X, \mathbb{Q}_\ell). \end{array}$$

The first part follows readily from [Kleiman 1968, Proposition 1.2.4] which asserts that the pullback map  $\pi^*$  on  $\ell$ -adic étale cohomology is injective and hence  $\pi^* H_{\text{ét}}^i(Y, \mathbb{Q}_\ell)$  is an  $f^*$ -invariant subspace of  $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$ . The second part is similar; see also [Hu 2019, Lemma 2.8] for a stronger version.  $\square$

The following useful inequality was already noticed by Truong [2016]. We provide a proof for the sake of completeness.

**Lemma 3.2.** *Let  $X$  be a smooth projective varieties defined over  $k$ , and  $f$  a surjective self-morphism of  $X$ . Then we have  $\lambda_k(f) \leq \chi_{2k}(f)$  for any  $0 \leq k \leq \dim X$ .*

*Proof.* Note that the  $\ell$ -adic étale cohomology  $H_{\text{ét}}^*(X, \mathbb{Q}_\ell)$  is a Weil cohomology after the noncanonical choice of an isomorphism  $\mathbb{Z}_\ell(1) \simeq \mathbb{Z}_\ell$  [Kleiman 1968, Example 1.2.5]. So we have the following cycle map

$$\gamma_X^k : \text{CH}^k(X) \rightarrow H_{\text{ét}}^{2k}(X, \mathbb{Q}_\ell),$$

where the  $k$ -th Chow group  $\text{CH}^k(X)$  of  $X$  denotes the group of algebraic cycles of codimension  $k$  modulo linear equivalence, i.e.,  $\text{CH}^k(X) := Z^k(X)/\sim$ . Recall that a cycle  $Z \in Z^k(X)$  is homologically equivalent to zero if  $\gamma_X^k(Z) = 0$ . Also, it is well-known that homological equivalence  $\sim_{\text{hom}}$  is finer than numerical equivalence  $\equiv$  [Kleiman 1968, Proposition 1.2.3]. Hence we have the following diagram of finite-dimensional  $\mathbb{Q}_\ell$ -vector spaces (respecting the natural pullback action  $f^*$  by the functoriality of the cycle map):

$$\begin{array}{ccc} (\text{CH}^k(X)/\sim_{\text{hom}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \hookrightarrow & H_{\text{ét}}^{2k}(X, \mathbb{Q}_\ell) \\ \downarrow & & \\ (\text{CH}^k(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = N^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell. & & \end{array} \tag{3-1}$$

Thus Lemma 3.2 follows.  $\square$

**Remark 3.3.** When  $k = 1$ , by a theorem of Matsusaka [1957], homological equivalence coincides with numerical equivalence (in general, Grothendieck’s standard conjecture  $D$  predicts that they are equal for



all  $k$ ). Furthermore, after tensoring with  $\mathbb{Q}$ , both of them are also equivalent to algebraic equivalence  $\approx$ . Namely, we have

$$\mathrm{NS}(X)_{\mathbb{Q}} = (\mathrm{CH}^1(X)/\approx) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq (\mathrm{CH}^1(X)/\sim_{\mathrm{hom}}) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In particular, the cycle map  $\gamma_X^1$  induces an injection

$$N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell} \hookrightarrow H_{\mathrm{ét}}^2(X, \mathbb{Q}_{\ell}).$$

**3.2. Extension of the pullback action to endomorphism algebras.** For an endomorphism  $\alpha$  of an abelian variety  $X$ , the following easy lemma sheds the light on the connection between the first numerical dynamical degree  $\lambda_1(\alpha)$  of  $\alpha$  and the induced action  $\alpha^*$  on the endomorphism  $\mathbb{Q}$ -algebra  $\mathrm{End}^0(X)$ , while the latter is closely related to the matrix representation of  $\alpha$  in  $\mathrm{End}(X)_{\mathbb{R}}$  or  $\mathrm{End}(X)_{\mathbb{C}}$  (see e.g., [Lemma 3.5](#)).

**Lemma 3.4.** *Fix a polarization  $\phi: X \rightarrow \hat{X}$  as in [Theorem 2.8](#). For any endomorphism  $\alpha$  of  $X$ , we can extend the pullback action  $\alpha^*$  on  $\mathrm{NS}^0(X)$  to  $\mathrm{End}^0(X)$  as follows:*

$$\alpha^*: \mathrm{End}^0(X) \rightarrow \mathrm{End}^0(X) \quad \text{via} \quad \psi \mapsto \alpha^* \psi := \alpha^{\dagger} \circ \psi \circ \alpha.^3$$

*Proof.* We shall identify  $\mathrm{NS}^0(X) \ni \mathcal{L}$  with the subspace of symmetric elements  $\phi^{-1} \circ \phi_{\mathcal{L}}$  of the endomorphism  $\mathbb{Q}$ -algebra  $\mathrm{End}^0(X)$  in virtue of [Theorem 2.8](#). Then the natural pullback action  $\alpha^*$  on  $\mathrm{NS}^0(X)$  could be reinterpreted in the following way:

$$\begin{aligned} \alpha^*: \mathrm{NS}^0(X) &\rightarrow \mathrm{NS}^0(X) \\ \phi^{-1} \circ \phi_{\mathcal{L}} &\mapsto \phi^{-1} \circ \phi_{\alpha^* \mathcal{L}}. \end{aligned}$$

Note that  $\phi^{-1} \circ \phi_{\alpha^* \mathcal{L}} = \phi^{-1} \circ \hat{\alpha} \circ \phi_{\mathcal{L}} \circ \alpha = \alpha^{\dagger} \circ \phi^{-1} \circ \phi_{\mathcal{L}} \circ \alpha$ , where  $\hat{\alpha}$  is the induced dual endomorphism of  $\hat{X}$  and  $\alpha^{\dagger} = \phi^{-1} \circ \hat{\alpha} \circ \phi$  is the Rosati involution of  $\alpha$ ; for the first equality, see [\[Mumford 1970, Section 15, Theorem 1\]](#). This gives rise to an action of  $\alpha$  on the whole endomorphism algebra  $\mathrm{End}^0(X)$  by sending  $\psi \in \mathrm{End}^0(X)$  to  $\alpha^{\dagger} \circ \psi \circ \alpha$ . It is easy to see that the restriction of  $\alpha^*|_{\mathrm{End}^0(X)}$  to  $\mathrm{NS}^0(X)$  is just the natural pullback action  $\alpha^*$  on  $\mathrm{NS}^0(X)$ . □

The lemma below plays a crucial role in the proof of our main theorem by giving a characterization of the above induced action  $\alpha^*$  on certain endomorphism algebras of abelian varieties. Here we consider a more general version from the aspect of linear algebra.

**Lemma 3.5.** (1) *If  $A \in M_n(\mathbb{R})$ , then the linear transformation*

$$f_A: M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R}) \quad \text{via} \quad B \mapsto A^{\top} B A$$

*of  $n^2$ -dimensional  $\mathbb{R}$ -vector space  $M_n(\mathbb{R})$  could be represented by  $A \otimes A$ , the Kronecker product of  $A$  and itself.*

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<sup>3</sup>Here by abuse of notation, we still denote this action by  $\alpha^*$ . We would always write  $\alpha^*|_{\mathrm{End}^0(X)}$  to emphasize the acting space in practice.

(2) If  $A \in M_n(\mathbb{C})$ , then the following linear transformation

$$f_A: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \quad \text{via } B \mapsto \bar{A}^T B A$$

of  $n^2$ -dimensional  $\mathbb{C}$ -vector space  $M_n(\mathbb{C})$  could be represented by  $A \otimes \bar{A}$ , the Kronecker product of  $A$  and its complex conjugate  $\bar{A}$ .

(3) If  $A \in M_n(\mathbb{C})$ , then the following linear transformation

$$f_A: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \quad \text{via } B \mapsto \bar{A}^T B A$$

of  $2n^2$ -dimensional  $\mathbb{R}$ -vector space  $M_n(\mathbb{C})$  could be represented by the block diagonal matrix  $(A \otimes \bar{A}) \oplus (\bar{A} \otimes A)$ .

*Proof.* We first prove the assertion (2) since the proof of the first one is essentially the same. Choose the standard  $\mathbb{C}$ -basis  $\{e_{ij}\}$  of  $M_n(\mathbb{C})$ , where  $e_{ij}$  denotes the  $n \times n$  complex matrix whose  $(i, j)$ -entry is 1, and 0 elsewhere. We also adopt the standard vectorization

$$\text{vec}: M_n(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^{n^2}$$

of  $M_n(\mathbb{C})$ , which converts  $n \times n$  matrices into column vectors so that

$$\{\text{vec}(e_{11}), \text{vec}(e_{21}), \dots, \text{vec}(e_{n1}), \text{vec}(e_{12}), \dots, \text{vec}(e_{n2}), \dots, \text{vec}(e_{1n}), \dots, \text{vec}(e_{nn})\} \quad (3-2)$$

forms the standard  $\mathbb{C}$ -basis of  $\mathbb{C}^{n^2}$ . Write  $A = (a_{ij})_{n \times n}$  with  $a_{ij} \in \mathbb{C}$ . Then we have

$$\bar{A}^T \cdot e_{ij} = \bar{a}_{i1}e_{1j} + \bar{a}_{i2}e_{2j} + \dots + \bar{a}_{in}e_{nj}.$$

Hence under the basis (3-2), it is easy to verify that the left multiplication by  $\bar{A}^T$  on the  $\mathbb{C}$ -vector space  $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$  is represented by the block diagonal matrix  $\bar{A} \oplus \bar{A} \oplus \dots \oplus \bar{A} = I_n \otimes \bar{A}$ . Similarly, since  $e_{ij} \cdot A = a_{j1}e_{i1} + a_{j2}e_{i2} + \dots + a_{jn}e_{in}$ , one can check that under the basis (3-2), the right multiplication by  $A$  is represented by  $A \otimes I_n$ . Therefore, our linear map  $f_A$  is represented by the matrix product  $(I_n \otimes \bar{A}) \cdot (A \otimes I_n) = A \otimes \bar{A}$ . Thus the assertion (2) follows.

For the last assertion, we just need to combine the assertion (2) with the following general fact: if  $M \in M_n(\mathbb{C})$ , then the associated  $2n \times 2n$  real matrix

$$\begin{pmatrix} \text{Re } M & -\text{Im } M \\ \text{Im } M & \text{Re } M \end{pmatrix}$$

is similar to the block diagonal matrix  $M \oplus \bar{M}$ . Indeed, one can easily verify that

$$\begin{pmatrix} I_n & -iI_n \\ -iI_n & I_n \end{pmatrix}^{-1} \cdot \begin{pmatrix} \text{Re } M & -\text{Im } M \\ \text{Im } M & \text{Re } M \end{pmatrix} \cdot \begin{pmatrix} I_n & -iI_n \\ -iI_n & I_n \end{pmatrix} = \begin{pmatrix} M & \mathbf{0} \\ \mathbf{0} & \bar{M} \end{pmatrix}.$$

Applying the above fact to the complex matrix  $A \otimes \bar{A}$  coming from the assertion (2), one gets the assertion (3) and hence Lemma 3.5 follows. □

**3.3. Several standard reductions towards the proof.** Before proving our main [Theorem 1.2](#), we start with some standard reductions. The lemma below reduces the general case to the splitting product case.

**Lemma 3.6.** *In order to prove [Theorem 1.2](#), it suffices to consider the following case:*

- the abelian variety  $X = A_1^{n_1} \times \cdots \times A_s^{n_s}$ , where the  $A_j$  are mutually nonisogenous simple abelian varieties, and
- the surjective self-morphism  $f$  of  $X$  is a surjective endomorphism  $\alpha$  which can be written as  $\alpha_1 \times \cdots \times \alpha_s$  with  $\alpha_j \in \text{End}(A_j^{n_j})$ .

*Proof.* We claim that it suffices to consider the case when  $f = \alpha$  is a surjective endomorphism. Indeed, any morphism (i.e., regular map) of abelian varieties is a composite of a homomorphism with a translation [[Milne 1986](#), Corollary 2.2]. Hence we can write  $f$  as  $t_x \circ \alpha$  for a surjective endomorphism  $\alpha \in \text{End}(X)$  and  $x \in X(\mathbf{k})$ . Note however that  $t_x \in \text{Aut}^0(X) \simeq X$  acts as identity on  $H_{\text{ét}}^1(X, \mathbb{Q}_\ell)$  and hence on  $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$  for all  $i$ . It follows from the functoriality of the pullback map on  $\ell$ -adic étale cohomology that  $\chi_i(f) = \chi_i(\alpha)$ . Similarly, we also get  $\lambda_k(f) = \lambda_k(\alpha)$  for all  $k$ . So the claim follows, and from now on our  $f = \alpha$  is an isogeny.

We then make another claim as follows.

**Claim 3.7.** *Towards the proof of [Theorem 1.2](#), we are free to replace our pair  $(X, \alpha)$  by any of the following pairs:*

- (1)  $(X, \alpha^m)$ , for any positive integer  $m$ .
- (2)  $(X, m\alpha)$ , for any positive integer  $m$ .
- (3)  $(X', \alpha' := g \circ \alpha \circ h)$ , where  $g: X \rightarrow X'$  and  $h: X' \rightarrow X$  are isogenies such that  $h \circ g = m_X$  and  $g \circ h = m_{X'}$  with  $m = \deg g$ .

*Proof of Claim 3.7.* The first part follows from the functoriality of the pullback map. For the second one, we note that  $m\alpha = m_X \circ \alpha = \alpha \circ m_X$ , where  $m_X$  is the multiplication by  $m$  map. Using the isomorphism  $H_{\text{ét}}^1(X, \mathbb{Z}_\ell) \simeq \text{Hom}_{\mathbb{Z}_\ell}(T_\ell X, \mathbb{Z}_\ell)$ , one can easily see that the induced pullback map  $m_X^*$  on  $H_{\text{ét}}^1(X, \mathbb{Q}_\ell)$  is also the multiplication by  $m$  map, and hence  $m_X^*|_{H_{\text{ét}}^i(X, \mathbb{Q}_\ell)}$  is represented by the diagonal matrix  $m^i \cdot \text{id}_{H_{\text{ét}}^i(X, \mathbb{Q}_\ell)}$ ; see e.g., [Theorem 2.1](#). It follows from the diagram (3-1) in the proof of [Lemma 3.2](#) that the pullback map  $m_X^*$  on each  $N^k(X)_{\mathbb{R}}$  is also represented by the diagonal matrix  $m^{2k} \cdot \text{id}_{N^k(X)_{\mathbb{R}}}$ . In particular, we have  $\chi_i(m\alpha) = m^i \chi_i(\alpha)$  and  $\lambda_k(m\alpha) = m^{2k} \lambda_k(\alpha)$ , which yields the part (2).

For the last part, it is easy to verify that  $\alpha' \circ g = g \circ (m\alpha)$  and  $h \circ \alpha' = (m\alpha) \circ h$ . By applying [Lemma 3.1](#) to the isogenies  $g$  and  $h$ , we have  $\chi_i(\alpha') = \chi_i(m\alpha)$  and  $\lambda_k(\alpha') = \lambda_k(m\alpha)$ . Then combining with the second part, the third one follows. So we have proved [Claim 3.7](#).  $\square$

Let us go back to the proof of [Lemma 3.6](#). By Poincaré's complete reducibility theorem [[Mumford 1970](#), Section 19, Theorem 1], we know that  $X$  is isogenous to the product  $A_1^{n_1} \times \cdots \times A_s^{n_s}$ , where the  $A_j$

are mutually nonisogenous simple abelian varieties. Then

$$\text{End}^0(X) \simeq \bigoplus_{j=1}^s \text{End}^0(A_j^{n_j}),$$

so that we can write  $\alpha$  as  $\alpha_1 \times \cdots \times \alpha_s$  with  $\alpha_j \in \text{End}^0(A_j^{n_j})$ . Using the reductions (2) and (3) in Claim 3.7, we only need to consider the case when  $X$  itself is the product variety and each  $\alpha_j$  belongs to  $\text{End}(A_j^{n_j})$ , as stated in the lemma.  $\square$

**Remark 3.8.** We are keen to further reduce the situation of Lemma 3.6 to the case when  $X = A^n$  is a power of some simple abelian variety  $A$ , as Esnault and Srinivas did [2013, proof of Proposition 6.2]. However, to the best of our knowledge, it does not seem to be straightforward. More precisely, let  $X$  and  $\alpha$  be as in Lemma 3.6. Suppose that Theorem 1.2 holds for every  $A_j^{n_j}$  and surjective endomorphism  $\alpha_j \in \text{End}(A_j^{n_j})$ , i.e.,  $\lambda_1(\alpha_j) = \chi_2(\alpha_j)$  for all  $j$ . We wish to show that Theorem 1.2 also holds for  $X$  and  $\alpha$ . Note that

$$\text{NS}(X) \simeq \bigoplus_{j=1}^s \text{NS}(A_j^{n_j}).^4$$

It follows that

$$\lambda_1(\alpha) = \max_j \{\lambda_1(\alpha_j)\} = \max_j \{\chi_2(\alpha_j)\}. \tag{3-3}$$

On the other hand, by the Künneth formula, we have

$$\begin{aligned} H_{\text{ét}}^1(X, \mathbb{Q}_\ell) &\simeq \bigoplus_j H_{\text{ét}}^1(A_j^{n_j}, \mathbb{Q}_\ell), \quad \text{and} \\ H_{\text{ét}}^2(X, \mathbb{Q}_\ell) &\simeq \bigoplus_j H_{\text{ét}}^2(A_j^{n_j}, \mathbb{Q}_\ell) \oplus \bigoplus_{j < k} (H_{\text{ét}}^1(A_j^{n_j}, \mathbb{Q}_\ell) \otimes H_{\text{ét}}^1(A_k^{n_k}, \mathbb{Q}_\ell)). \end{aligned}$$

However, we are not able to deduce that  $\chi_2(\alpha) = \max_j \{\chi_2(\alpha_j)\}$  due to the appearance of the tensor product of the  $H_{\text{ét}}^1$ .

For the sake of completeness, let us explain this obstruction in a more precise way. We denote by  $P_{\alpha_j}(t) \in \mathbb{Z}[t]$  the characteristic polynomial of  $\alpha_j$  (or equivalently  $T_\ell \alpha_j$ , by Theorem 2.3). Set  $g_j = \dim A_j^{n_j}$ . Denote all complex roots of  $P_{\alpha_j}(t)$  by  $\omega_{j,1}, \dots, \omega_{j,2g_j}$ . Without loss of generality, we may assume that

$$|\omega_{j,1}| \geq \cdots \geq |\omega_{j,2g_j}| \quad \text{for all } 1 \leq j \leq s, \text{ and } |\omega_{1,1}| \geq \cdots \geq |\omega_{s,1}|. \tag{3-4}$$

It follows from Theorem 2.1 that  $\chi_2(\alpha_j) = |\omega_{j,1}| \cdot |\omega_{j,2}|$  for all  $j$ . Suppose that

$$\max_j \{\chi_2(\alpha_j)\} = \chi_2(\alpha_{j_0}) = |\omega_{j_0,1}| \cdot |\omega_{j_0,2}| \quad \text{for some } j_0. \tag{3-5}$$

<sup>4</sup>In general, one has  $\text{NS}(X \times_k Y) \simeq \text{NS}(X) \oplus \text{NS}(Y) \oplus \text{Hom}_k(\text{Alb}(X), \text{Pic}^0(Y))$ ; see e.g., [Tate 1966, the proof of Theorem 3]. See also [Bost and Charles 2016, Section 3.2] and references therein for more details about the divisorial correspondences.

Note that  $j_0$  may not be 1. If  $|\omega_{2,1}| \leq |\omega_{1,2}|$  (in particular,  $j_0$  is 1), then

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{1,2}| = \chi_2(\alpha_1) = \max_j \{\chi_2(\alpha_j)\} = \lambda_1(\alpha).$$

So we are done in this case. However, if  $|\omega_{2,1}| > |\omega_{1,2}|$ , then

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{2,1}| \geq |\omega_{j_0,1}| \cdot |\omega_{j_0,2}| = \chi_2(\alpha_{j_0}) = \max_j \{\chi_2(\alpha_j)\} = \lambda_1(\alpha).$$

There is no obvious reason to exclude the worst case  $j_0 = 1$  which yields that

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{2,1}| > |\omega_{1,1}| \cdot |\omega_{1,2}| = \chi_2(\alpha_1) = \max_j \{\chi_2(\alpha_j)\} = \lambda_1(\alpha).$$

To proceed, we observe that over complex number field  $\mathbb{C}$ , the above pathology does not happen because each eigenvalue  $\omega_{j,2}$  turns out to be the complex conjugate of  $\omega_{j,1}$ . This fact follows from the Hodge decomposition  $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$ , which does not seem to exist in étale cohomology as far as we know. But we still believe that  $\omega_{j,2} = \overline{\omega_{j,1}}$  for all  $j$ . (As a consequence of our main theorem, we will see that this is actually true; see Remark 3.10.) The following lemma makes use of this observation to reduce the splitting product case as in Lemma 3.6 to the case when  $X = A^n$  for some simple abelian variety  $A$ .

**Lemma 3.9.** *In order to prove Theorem 1.2, it suffices to show that if  $A^n$  is a power of a simple abelian variety  $A$  and  $\alpha \in \text{End}(A^n)$  is a surjective endomorphism of  $A^n$ , then  $\lambda_1(\alpha) = |\omega_1|^2$ , where  $\omega_1$  is one of the complex roots of the characteristic polynomial  $P_\alpha(t)$  of  $\alpha$  with the maximal absolute value.*

*Proof.* Thanks to Lemma 3.6, let us consider the case when the abelian variety  $X = A_1^{n_1} \times \dots \times A_s^{n_s}$ , where the  $A_j$  are mutually nonisogenous simple abelian varieties, and  $\alpha = \alpha_1 \times \dots \times \alpha_s$  is a surjective endomorphism of  $X$  with  $\alpha_j \in \text{End}(A_j^{n_j})$ . We assume that the reader has been familiar with the notation introduced in Remark 3.8, in particular, (3-3)–(3-5). Applying the hypothesis of Lemma 3.9 to each  $A_j^{n_j}$  and  $\alpha_j$ , we have  $\lambda_1(\alpha_j) = |\omega_{j,1}|^2$ . It follows from Lemma 3.2 and Theorem 2.1 that  $\lambda_1(\alpha_j) \leq \chi_2(\alpha_j) = |\omega_{j,1}| \cdot |\omega_{j,2}|$ . Hence  $\lambda_1(\alpha_j) = \chi_2(\alpha_j)$  and  $|\omega_{j,1}| = |\omega_{j,2}|$  for all  $j$  which tells us  $j_0 = 1$ . This yields that

$$\chi_2(\alpha) = |\omega_{1,1}| \cdot |\omega_{1,2}| = \chi_2(\alpha_1) = \max_j \{\chi_2(\alpha_j)\} = \max_j \{\lambda_1(\alpha_j)\} = \lambda_1(\alpha).$$

The first and second equalities follow again from Theorem 2.1, the third one holds because  $j_0 = 1$ , (3-3) gives the last one. □

**3.4. Proof of Theorem 1.2.** We are now ready to prove the main theorem.

*Proof of Theorem 1.2.* By Lemma 3.9, we can assume that  $X = A^n$  for some simple abelian variety  $A$  and  $\alpha \in \text{End}(X)$  is a surjective endomorphism of  $X$ . Let  $P_\alpha(t) \in \mathbb{Z}[t]$  be the characteristic polynomial of  $\alpha$  (see Theorem 2.3). Set  $g = \dim X$ . Denote all complex roots of  $P_\alpha(t)$  by  $\omega_1, \dots, \omega_{2g}$ . Without loss of generality, we may assume that

$$|\omega_1| \geq \dots \geq |\omega_{2g}|.$$

We shall prove that

$$\lambda_1(\alpha) = |\omega_1|^2, \tag{3-6}$$

which will conclude the proof of the theorem by [Lemma 3.9](#).

Under the above assumption, the endomorphism algebra  $\text{End}^0(X)$  is the simple  $\mathbb{Q}$ -algebra  $M_n(D)$  of all  $n \times n$  matrices with entries in the division ring  $D := \text{End}^0(A)$ . Let  $K$  denote the center of  $D$ , and  $K_0$  the maximal totally real subfield of  $K$ . As usual, we set

$$d^2 = [D : K], \quad e = [K : \mathbb{Q}] \quad \text{and} \quad e_0 = [K_0 : \mathbb{Q}].$$

Note that by [Lemma 3.4](#), the natural pullback action  $\alpha^*$  on  $\text{NS}^0$  can be extended to an action  $\alpha^*$  on the whole endomorphism  $\mathbb{Q}$ -algebra  $\text{End}^0(X)$  as follows:

$$\alpha^* : \text{End}^0(X) \rightarrow \text{End}^0(X) \quad \text{via } \psi \mapsto \alpha^\dagger \circ \psi \circ \alpha.$$

On the other hand, by tensoring with  $\mathbb{R}$ , we know that

$$\text{End}(X)_\mathbb{R} = \text{End}^0(X) \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_n(D) \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_n(D \otimes_{\mathbb{Q}} \mathbb{R})$$

is either a product of  $M_r(\mathbb{R})$ ,  $M_r(\mathbb{C})$  or  $M_r(\mathbf{H})$  with  $\text{NS}(X)_\mathbb{R}$  being a product of  $\mathcal{H}_r(\mathbb{R})$ ,  $\mathcal{H}_r(\mathbb{C})$  or  $\mathcal{H}_r(\mathbf{H})$ , the corresponding subspace of symmetric/Hermitian matrices (see [Theorem 2.9](#)). When there is no risk of confusion, for simplicity, we still denote the induced action  $\alpha^* \otimes_{\mathbb{Q}} 1_\mathbb{R}$  by  $\alpha^*$ . In particular, we would write  $\alpha^*|_{\text{End}(X)_\mathbb{R}}$  and  $\alpha^*|_{\text{NS}(X)_\mathbb{R}}$  to emphasize the acting spaces.

According to Albert’s classification of the endomorphism  $\mathbb{Q}$ -algebra  $D$  of a simple abelian variety  $A$  [[Mumford 1970](#), Section 21, Theorem 2], we have the following four cases.

**Case 1.**  $D$  is of Type I( $e$ ):  $d = 1$ ,  $e = e_0$  and  $D = K = K_0$  is a totally real algebraic number field and the involution (on  $D$ ) is the identity. In this case,

$$\text{End}(X)_\mathbb{R} \simeq \bigoplus_{i=1}^{e_0} M_n(\mathbb{R}) \quad \text{and} \quad \text{NS}(X)_\mathbb{R} \simeq \bigoplus_{i=1}^{e_0} \mathcal{H}_n(\mathbb{R}).$$

For our  $\alpha \in \text{End}(X)$ , let us denote its image  $\alpha \otimes_{\mathbb{Z}} 1_\mathbb{R}$  in  $\text{End}(X)_\mathbb{R}$  by the block diagonal matrix  $A_\alpha = A_{\alpha,1} \oplus \cdots \oplus A_{\alpha,e_0}$  with each  $A_{\alpha,i} \in M_n(\mathbb{R})$ . Then the Rosati involution  $\alpha^\dagger$  of  $\alpha$  could be represented by the transpose  $A_\alpha^\top = A_{\alpha,1}^\top \oplus \cdots \oplus A_{\alpha,e_0}^\top$  (see [Theorem 2.9](#)). Hence we can rewrite the induced action  $\alpha^*$  on  $\text{End}(X)_\mathbb{R}$  in the following matrix form:

$$B = B_1 \oplus \cdots \oplus B_{e_0} \mapsto A_\alpha^\top B A_\alpha = A_{\alpha,1}^\top B_1 A_{\alpha,1} \oplus \cdots \oplus A_{\alpha,e_0}^\top B_{e_0} A_{\alpha,e_0}.$$

Thanks to [Lemma 3.5\(1\)](#), for each  $i$ , the linear transformation defined by the mapping

$$B_i \in M_n(\mathbb{R}) \mapsto A_{\alpha,i}^\top B_i A_{\alpha,i} \in M_n(\mathbb{R}),$$

can be represented by the Kronecker product  $A_{\alpha,i} \otimes A_{\alpha,i}$ . Hence the above linear transformation  $\alpha^*|_{\text{End}(X)_{\mathbb{R}}}$  on the  $e_0n^2$ -dimensional  $\mathbb{R}$ -vector space  $\text{End}(X)_{\mathbb{R}}$  is represented by the block diagonal matrix

$$(A_{\alpha,1} \otimes A_{\alpha,1}) \oplus \cdots \oplus (A_{\alpha,e_0} \otimes A_{\alpha,e_0}).$$

For each  $1 \leq i \leq e_0$ , denote all eigenvalues of  $A_{\alpha,i}$  by  $\pi_{i,1}, \dots, \pi_{i,n}$ . It thus follows from the above discussion that all eigenvalues of the linear transformation  $\alpha^*|_{\text{End}(X)_{\mathbb{R}}}$  are exactly  $\pi_{i,j}\pi_{i,k}$  with  $1 \leq j, k \leq n$  and  $1 \leq i \leq e_0$ . In particular, if  $v_{i,j}$  and  $v_{i,k}$  denote eigenvectors of  $A_{\alpha,i}$  corresponding to  $\pi_{i,j}$  and  $\pi_{i,k}$ , respectively, then

$$v_{i,j} \otimes v_{i,k} = \text{vec}(v_{i,j}^T \otimes v_{i,k}) = \text{vec}(v_{i,k} \otimes v_{i,j}^T) = \text{vec}(v_{i,k} \cdot v_{i,j}^T)$$

is the eigenvector of  $A_{\alpha,i} \otimes A_{\alpha,i}$  corresponding to  $\pi_{i,j}\pi_{i,k}$ .<sup>5</sup> Now, according to Remark 2.5, the reduced characteristic polynomial  $\chi_{\alpha}^{\text{red}}(t)$  of  $\alpha$  is independent of the change of the ground field, and hence equal to the reduced characteristic polynomial  $\chi_{\alpha \otimes_{\mathbb{Z}} 1_{\mathbb{R}}}^{\text{red}}(t)$  of  $\alpha \otimes_{\mathbb{Z}} 1_{\mathbb{R}} \in \text{End}(X)_{\mathbb{R}}$ , while the latter by Definition 2.4 is just the characteristic polynomial  $\det(tI_{e_0n} - A_{\alpha})$  of  $A_{\alpha}$ . Hence, without loss of generality, we may assume that  $\omega_1 = \pi_{1,1}$  by Lemma 2.7.

We now have two subcases to consider. If  $\pi_{1,1} \in \mathbb{R}$  so that  $v_{1,1}$  is also a real eigenvector, then  $v_{1,1} \otimes v_{1,1}$  is a real eigenvector of  $\alpha^*|_{\text{End}(X)_{\mathbb{R}}}$  corresponding to the eigenvalue  $\pi_{1,1}^2$ . This eigenvector is the associated column vector of the real symmetric matrix  $v_{1,1} \otimes v_{1,1}^T = v_{1,1}^T \otimes v_{1,1}$ . Next, let us assume that  $\pi_{1,1} \in \mathbb{C} \setminus \mathbb{R}$ . Then  $\bar{\pi}_{1,1}$  is another eigenvalue of  $A_{\alpha,1}$  with the corresponding eigenvector  $\bar{v}_{1,1}$ , since  $A_{\alpha,1}$  is defined over  $\mathbb{R}$ . It follows that  $v_{1,1} \otimes \bar{v}_{1,1} + \bar{v}_{1,1} \otimes v_{1,1}$  is a real eigenvector of  $\alpha^*|_{\text{End}(X)_{\mathbb{R}}}$  corresponding to the eigenvalue  $\pi_{1,1}\bar{\pi}_{1,1} = |\pi_{1,1}|^2$ ; moreover, it is the associated column vector of the real symmetric matrix

$$v_{1,1}^T \otimes \bar{v}_{1,1} + \bar{v}_{1,1}^T \otimes v_{1,1} = \bar{v}_{1,1} \otimes v_{1,1}^T + \bar{v}_{1,1}^T \otimes v_{1,1}.$$

In either case, we have shown that the spectral radii of  $\alpha^*|_{\text{End}(X)_{\mathbb{R}}}$  and  $\alpha^*|_{\text{NS}(X)_{\mathbb{R}}}$  coincide, both equal to  $|\pi_{1,1}|^2$ . In summary, we have

$$|\omega_1|^2 = |\pi_{1,1}|^2 = \rho(\alpha^*|_{\text{End}(X)_{\mathbb{R}}}) = \rho(\alpha^*|_{\text{NS}(X)_{\mathbb{R}}}) = \lambda_1(\alpha).$$

For the last equality, see Remark 3.3. So we conclude the proof of the equality (3-6) in this case.

**Case 2.**  $D$  is of Type II(e):  $d = 2, e = e_0, K = K_0$  is a totally real algebraic number field and  $D$  is an indefinite quaternion division algebra over  $K$ . Hence

$$\text{End}(X)_{\mathbb{R}} \simeq \bigoplus_{i=1}^{e_0} M_{2n}(\mathbb{R}) \quad \text{and} \quad \text{NS}(X)_{\mathbb{R}} \simeq \bigoplus_{i=1}^{e_0} \mathcal{H}_{2n}(\mathbb{R}).$$

The rest is exactly the same as Case 1.

<sup>5</sup>Note that due to multiplicities of eigenvalues,  $A_{\alpha,i}$  does not necessarily have  $n$  distinct eigenvalues. Thus,  $v_{i,j}$  and  $v_{i,k}$  may be the same for different  $j$  and  $k$ . Also, not all eigenvectors of  $A_{\alpha,i} \otimes A_{\alpha,i}$  have to arise in this way, namely, being the tensor products  $v_{i,j} \otimes v_{i,k}$ . For instance, one could consider a Jordan block  $J_{\lambda,2} \in M_2(\mathbb{R})$  with the eigenvalue  $\lambda$ , but  $J_{\lambda,2} \otimes J_{\lambda,2} \sim J_{\lambda^2,1} \oplus J_{\lambda^2,3}$ .

**Case 3.**  $D$  is of Type III( $e$ ):  $d = 2$ ,  $e = e_0$ ,  $K = K_0$  is a totally real algebraic number field and  $D$  is a definite quaternion division algebra over  $K$ . In this case,

$$\text{End}(X)_{\mathbb{R}} \simeq \bigoplus_{i=1}^{e_0} M_n(\mathbf{H}) \quad \text{and} \quad \text{NS}(X)_{\mathbb{R}} \simeq \bigoplus_{i=1}^{e_0} \mathcal{H}_n(\mathbf{H}),$$

where  $\mathbf{H} = \left(\frac{-1, -1}{\mathbb{R}}\right)$  is the standard quaternion algebra over  $\mathbb{R}$ . Clearly,  $\mathbf{H}$  can be embedded, in a standard way (see e.g., [Reiner 2003, Example 9.4]), into  $M_2(\mathbb{C}) \simeq \mathbf{H} \otimes_{\mathbb{R}} \mathbb{C}$ . This induces a natural embedding of  $M_n(\mathbf{H})$  into  $M_{2n}(\mathbb{C}) \simeq M_n(\mathbf{H}) \otimes_{\mathbb{R}} \mathbb{C}$  as follows [Lee 1949, Section 4]:

$$\iota: M_n(\mathbf{H}) \hookrightarrow M_{2n}(\mathbb{C}) \quad \text{via } \mathbf{A} = A_1 + A_2\mathbf{j} \mapsto \iota(\mathbf{A}) := \begin{pmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{pmatrix}.$$

In particular, a quaternionic matrix  $\mathbf{A}$  is Hermitian if and only if its image  $\iota(\mathbf{A})$  is a Hermitian complex matrix.

For brevity, we only consider the case  $e_0 = 1$  (to deal with the general case, the only cost is to introduce an index  $i$  as we have done in Case 1 since the matrices involved are block diagonal matrices). Denote the image  $\alpha \otimes_{\mathbb{Z}} 1_{\mathbb{R}}$  of  $\alpha$  in  $M_n(\mathbf{H})$  by  $\mathbf{A}_{\alpha} = A_1 + A_2\mathbf{j}$  with  $A_1, A_2 \in M_n(\mathbb{C})$ . Then the Rosati involution  $\alpha^{\dagger}$  of  $\alpha$  could be represented by the quaternionic conjugate transpose  $\mathbf{A}_{\alpha}^* = \bar{\mathbf{A}}_{\alpha}^{\top}$  (see Theorem 2.9), whose image under  $\iota$  is just the complex conjugate transpose  $\iota(\mathbf{A}_{\alpha})^*$  (aka Hermitian transpose) of  $\iota(\mathbf{A}_{\alpha})$ . Similar as in Lemma 3.4, the action  $\alpha^*$  on  $\text{End}(X)_{\mathbb{R}} \simeq M_n(\mathbf{H})$  can be extended to

$$\text{End}(X)_{\mathbb{C}} := \text{End}(X)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2n}(\mathbb{C}).$$

By abuse of notation, we still denote this induced action by  $\alpha^*: M_{2n}(\mathbb{C}) \rightarrow M_{2n}(\mathbb{C})$ , which maps  $\mathbf{B}$  to  $\iota(\mathbf{A}_{\alpha})^* \cdot \mathbf{B} \cdot \iota(\mathbf{A}_{\alpha})$ . It follows from Lemma 3.5(2) that  $\alpha^*|_{M_{2n}(\mathbb{C})}$  could be represented by the Kronecker product  $\iota(\mathbf{A}_{\alpha}) \otimes \overline{\iota(\mathbf{A}_{\alpha})}$ .

Note that our  $\text{End}(X)_{\mathbb{C}} \simeq M_{2n}(\mathbb{C})$  is a central simple  $\mathbb{C}$ -algebra. Then by Definition 2.4 and Remark 2.5, the reduced characteristic polynomial  $\chi_{\alpha}^{\text{red}}(t)$  of  $\alpha$  is equal to the characteristic polynomial  $\det(tI_{2n} - \iota(\mathbf{A}_{\alpha}))$  of the complex matrix  $\iota(\mathbf{A}_{\alpha})$ . Thanks to [Lee 1949, Theorem 5], the  $2n$  eigenvalues of  $\iota(\mathbf{A}_{\alpha})$  fall into  $n$  pairs, each pair consisting of two conjugate complex numbers; denote them by  $\pi_1, \dots, \pi_n, \pi_{n+1} = \bar{\pi}_1, \dots, \pi_{2n} = \bar{\pi}_n$ . In fact, it is easy to verify that if  $\pi_i \in \mathbb{C}$  is an eigenvalue of  $\iota(\mathbf{A}_{\alpha})$  so that

$$\iota(\mathbf{A}_{\alpha}) \begin{pmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{pmatrix} = \pi_i \begin{pmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{pmatrix}, \quad \text{then } \iota(\mathbf{A}_{\alpha}) \begin{pmatrix} -\bar{\mathbf{v}}_i \\ \bar{\mathbf{u}}_i \end{pmatrix} = \bar{\pi}_i \begin{pmatrix} -\bar{\mathbf{v}}_i \\ \bar{\mathbf{u}}_i \end{pmatrix},$$

i.e.,  $\bar{\pi}_i$  is also an eigenvalue of  $\iota(\mathbf{A}_{\alpha})$  corresponding to the eigenvector  $(-\bar{\mathbf{v}}_i^{\top}, \bar{\mathbf{u}}_i^{\top})^{\top}$ . Therefore, without loss of generality, we may assume that  $\omega_1 = \pi_1$  by Lemma 2.7.

Let  $(\mathbf{u}_1^{\top}, \mathbf{v}_1^{\top})^{\top}$  denote an eigenvector of  $\iota(\mathbf{A}_{\alpha})$  corresponding to the eigenvalue  $\pi_1$ . Then  $(-\bar{\mathbf{v}}_1^{\top}, \bar{\mathbf{u}}_1^{\top})^{\top}$  is an eigenvector of  $\iota(\mathbf{A}_{\alpha})$  corresponding to the eigenvalue  $\bar{\pi}_1$ . Since the linear transformation  $\alpha^*|_{\text{End}(X)_{\mathbb{C}}}$  can be represented by  $\iota(\mathbf{A}_{\alpha}) \otimes \overline{\iota(\mathbf{A}_{\alpha})}$  (see Lemma 3.5(2)), we see that both  $(\mathbf{u}_1^{\top}, \mathbf{v}_1^{\top})^{\top} \otimes (\bar{\mathbf{u}}_1^{\top}, \bar{\mathbf{v}}_1^{\top})^{\top}$  and  $(-\bar{\mathbf{v}}_1^{\top}, \bar{\mathbf{u}}_1^{\top})^{\top} \otimes (-\mathbf{v}_1^{\top}, \mathbf{u}_1^{\top})^{\top}$  are eigenvectors of  $\alpha^*|_{\text{End}(X)_{\mathbb{C}}}$ , corresponding to the same eigenvalue  $\pi_1 \bar{\pi}_1$ .



Recall that these two eigenvectors are the associated column vectors of the Hermitian complex matrices

$$\begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} \otimes (\mathbf{u}_1^\top, \mathbf{v}_1^\top) = \begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} \cdot (\mathbf{u}_1^\top, \mathbf{v}_1^\top) \quad \text{and} \quad \begin{pmatrix} -\mathbf{v}_1 \\ \mathbf{u}_1 \end{pmatrix} \otimes (-\bar{v}_1^\top, \bar{u}_1^\top) = \begin{pmatrix} -\mathbf{v}_1 \\ \mathbf{u}_1 \end{pmatrix} \cdot (-\bar{v}_1^\top, \bar{u}_1^\top),$$

respectively. It is then easy to verify that

$$\begin{pmatrix} \bar{u}_1 \\ \bar{v}_1 \end{pmatrix} \cdot (\mathbf{u}_1^\top, \mathbf{v}_1^\top) + \begin{pmatrix} -\mathbf{v}_1 \\ \mathbf{u}_1 \end{pmatrix} \cdot (-\bar{v}_1^\top, \bar{u}_1^\top) = \begin{pmatrix} \bar{u}_1 \mathbf{u}_1^\top + \mathbf{v}_1 \bar{v}_1^\top & \bar{u}_1 \mathbf{v}_1^\top - \mathbf{v}_1 \bar{u}_1^\top \\ \bar{v}_1 \mathbf{u}_1^\top - \mathbf{u}_1 \bar{v}_1^\top & \bar{v}_1 \mathbf{v}_1^\top + \mathbf{u}_1 \bar{u}_1^\top \end{pmatrix}$$

is a Hermitian complex matrix lying in the image of  $\iota$ . In other words, this sum belongs to  $\text{NS}(X)_\mathbb{C}$ . Hence, similar as in [Case 1](#), the spectral radii of  $\alpha^*|_{\text{NS}(X)_\mathbb{C}}$  and  $\alpha^*|_{\text{End}(X)_\mathbb{C}}$  coincide, both equal to  $|\pi_1|^2$ . Overall, we have

$$|\omega_1|^2 = |\pi_1|^2 = \rho(\alpha^*|_{\text{End}(X)_\mathbb{C}}) = \rho(\alpha^*|_{\text{NS}(X)_\mathbb{C}}) = \rho(\alpha^*|_{\text{NS}(X)_\mathbb{R}}) = \lambda_1(\alpha).$$

We thus conclude the proof of the equality [\(3-6\)](#) in this case.

**Case 4.**  $D$  is of Type IV( $e_0, d$ ):  $e = 2e_0$  and  $D$  is a division algebra over the CM-field  $K \supseteq K_0$  (i.e.,  $K$  is a totally imaginary quadratic extension of a totally real algebraic number field  $K_0$ ). Then

$$\text{End}(X)_\mathbb{R} \simeq \bigoplus_{i=1}^{e_0} M_{dn}(\mathbb{C}) \quad \text{and} \quad \text{NS}(X)_\mathbb{R} \simeq \bigoplus_{i=1}^{e_0} \mathcal{H}_{dn}(\mathbb{C}).$$

For simplicity, we just deal with the case  $e_0 = 1$ . Denote the image of  $\alpha$  in  $\text{End}(X)_\mathbb{R}$  by the matrix  $A_\alpha \in M_{dn}(\mathbb{C})$ . Again, the Rosati involution  $\alpha^\dagger$  of  $\alpha$  could be represented by the complex conjugate transpose  $A_\alpha^* = \bar{A}_\alpha^\top$  (see [Theorem 2.9](#)). It follows from [Lemma 3.5\(2\)](#) that the induced linear map  $\alpha^*|_{M_{dn}(\mathbb{C})}$  on the  $d^2n^2$ -dimensional  $\mathbb{C}$ -vector space  $M_{dn}(\mathbb{C})$  is represented by the Kronecker product  $A_\alpha \otimes \bar{A}_\alpha$ ; however, the induced linear map  $\alpha^*|_{\text{End}(X)_\mathbb{R}}$  on the  $2d^2n^2$ -dimensional  $\mathbb{R}$ -vector space  $\text{End}(X)_\mathbb{R}$  is represented by the block diagonal matrix  $(A_\alpha \otimes \bar{A}_\alpha) \oplus (\bar{A}_\alpha \otimes A_\alpha)$  by [Lemma 3.5\(3\)](#), though we do not need this fact later.

Note that the center of our  $\mathbb{R}$ -algebra  $\text{End}(X)_\mathbb{R} \simeq M_{dn}(\mathbb{C})$  is  $\mathbb{C}$ . Then by [Definition 2.4](#) and [Remark 2.5](#), the reduced characteristic polynomial  $\chi_\alpha^{\text{red}}(t)$  of  $\alpha$  is equal to the product of the characteristic polynomial  $\det(tI_{dn} - A_\alpha)$  of  $A_\alpha$  and its complex conjugate. We denote all of its complex roots by  $\pi_1, \dots, \pi_{dn}, \bar{\pi}_1, \dots, \bar{\pi}_{dn}$ . Without loss of generality, we may assume that  $\omega_1 = \pi_1$  by [Lemma 2.7](#). Let  $\mathbf{v}_1$  be a complex eigenvector of  $A_\alpha$  corresponding to the eigenvalue  $\pi_1$ . Then  $\mathbf{v}_1 \otimes \bar{\mathbf{v}}_1$  is an eigenvector of  $A_\alpha \otimes \bar{A}_\alpha$  corresponding to the eigenvalue  $\pi_1 \bar{\pi}_1 = |\pi_1|^2$ . Note that  $\mathbf{v}_1 \otimes \bar{\mathbf{v}}_1$  is the associated column vector of the Hermitian complex matrix  $\bar{\mathbf{v}}_1 \otimes \mathbf{v}_1^\top = \mathbf{v}_1^\top \otimes \bar{\mathbf{v}}_1 \in \text{NS}(X)_\mathbb{R}$ . Hence, in this last case, we also have

$$|\omega_1|^2 = |\pi_1|^2 = \rho(\alpha^*|_{M_{dn}(\mathbb{C})}) = \rho(\alpha^*|_{\text{NS}(X)_\mathbb{R}}) = \lambda_1(\alpha).$$

We thus finally complete the proof of [Theorem 1.2](#). □

**Remark 3.10.** (1) It follows from our proof, in particular from the key equality (3-6), as well as Birkhoff’s generalization of the Perron–Frobenius theorem, that either  $\omega_2 = \omega_1 \in \mathbb{R}$  or  $\omega_2 = \bar{\omega}_1 \neq \omega_1$ . This is true for any complex torus  $X$  because by the Hodge decomposition we have  $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$ , where  $H^{1,0}(X) = H^0(X, \Omega_X^1)$ . A natural question is whether it is true for all  $\omega_i$  in general, i.e., either  $\omega_{2i} = \omega_{2i-1} \in \mathbb{R}$  or  $\omega_{2i} = \bar{\omega}_{2i-1} \neq \omega_{2i-1}$  for any  $2 \leq i \leq g = \dim X$ .

(2) If our self-morphism  $f$  is not surjective or  $\alpha$  is not an isogeny, one can also proceed by replacing  $X$  by the image  $\alpha(X)$ , which is still an abelian variety of dimension less than  $\dim X$ .

**List of symbols**

$k$	an algebraically closed field of arbitrary characteristic
$\ell$	a prime different from $\text{char } k$
$X$	an abelian variety of dimension $g$ defined over $k$
$\hat{X}$	the dual abelian variety $\text{Pic}^0(X)$ of $X$
$\alpha, \psi$	endomorphisms of $X$
$\hat{\alpha}, \hat{\psi}$	the induced dual endomorphisms of $\hat{X}$
$\text{End}(X)$	the endomorphism ring of $X$
$\text{End}^0(X)$	$\text{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ , the endomorphism $\mathbb{Q}$ -algebra of $X$
$\text{End}(X)_{\mathbb{R}}$	$\text{End}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \text{End}^0(X) \otimes_{\mathbb{Q}} \mathbb{R}$ , the endomorphism $\mathbb{R}$ -algebra of $X$
$M_n(R)$	the ring of all $n \times n$ matrices with entries in a ring $R$
$\phi_{\mathcal{L}}$	the induced homomorphism of a line bundle $\mathcal{L}$ on $X$ : $\phi_{\mathcal{L}}: X \rightarrow \hat{X}, \quad x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$
$\phi = \phi_{\mathcal{L}_0}$	a fixed polarization of $X$ induced from some ample line bundle $\mathcal{L}_0$
$\dagger$	the Rosati involution on $\text{End}^0(X)$ defined in the following way: $\psi \mapsto \psi^{\dagger} := \phi^{-1} \circ \hat{\psi} \circ \phi, \text{ for any } \psi \in \text{End}^0(X)$
$\text{NS}(X)$	$\text{Pic}(X) / \text{Pic}^0(X)$ , the Néron–Severi group of $X$
$\text{NS}^0(X)$	$\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = N^1(X)_{\mathbb{Q}} = \text{NS}(X)_{\mathbb{Q}}$ (see Remark 3.3)
$\text{NS}(X)_{\mathbb{R}}$	$\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \text{NS}^0(X) \otimes_{\mathbb{Q}} \mathbb{R} = N^1(X)_{\mathbb{R}}$
$N^k(X)_{\mathbb{R}}$	$N^k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ , the $\mathbb{R}$ -vector space of numerical equivalent classes of codimension- $k$ cycles (with $0 \leq k \leq g = \dim X$ )
$H_{\text{ét}}^i(X, \mathbb{Q}_{\ell})$	$H_{\text{ét}}^i(X, \mathbb{Z}_{\ell}) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}$ , the $\ell$ -adic étale cohomology group of degree $i$
$T_{\ell} X$	the Tate module $\varprojlim_n X_{\ell^n}(k)$ of $X$ , a free $\mathbb{Z}_{\ell}$ -module of rank $2g$
$T_{\ell} \alpha$	the induced endomorphism on $T_{\ell} X$
$A$	a simple abelian variety defined over $k$
$D$	$\text{End}^0(A)$ , the endomorphism $\mathbb{Q}$ -algebra of $A$
$K$	the center of the division ring $D = \text{End}^0(A)$
$K_0$	the maximal totally real subfield of $K$
$H$	the standard quaternion algebra over $\mathbb{R}$

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## References

- [Bost and Charles 2016] J.-B. Bost and F. Charles, “Some remarks concerning the Grothendieck period conjecture”, *J. Reine Angew. Math.* **714** (2016), 175–208. [MR](#) [Zbl](#)
- [Deligne 1974] P. Deligne, “La conjecture de Weil, I”, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307. [MR](#) [Zbl](#)
- [Dinh and Sibony 2017] T.-C. Dinh and N. Sibony, “Equidistribution problems in complex dynamics of higher dimension”, *Int. J. Math.* **28**:7 (2017), art. id. 1750057. [MR](#) [Zbl](#)
- [Esnault and Srinivas 2013] H. Esnault and V. Srinivas, “Algebraic versus topological entropy for surfaces over finite fields”, *Osaka J. Math.* **50**:3 (2013), 827–846. [MR](#) [Zbl](#)
- [Gromov 2003] M. Gromov, “On the entropy of holomorphic maps”, *Enseign. Math. (2)* **49**:3-4 (2003), 217–235. [MR](#) [Zbl](#)
- [Hu 2019] F. Hu, “A theorem of Tits type for automorphism groups of projective varieties in arbitrary characteristic”, *Math. Ann.* (online publication February 2019).
- [Katz and Messing 1974] N. M. Katz and W. Messing, “Some consequences of the Riemann hypothesis for varieties over finite fields”, *Invent. Math.* **23** (1974), 73–77. [MR](#) [Zbl](#)
- [Kleiman 1968] S. L. Kleiman, “Algebraic cycles and the Weil conjectures”, pp. 359–386 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. [MR](#) [Zbl](#)
- [Lee 1949] H. C. Lee, “Eigenvalues and canonical forms of matrices with quaternion coefficients”, *Proc. Roy. Irish Acad. Sect. A.* **52** (1949), 253–260. [MR](#) [Zbl](#)
- [Matsusaka 1957] T. Matsusaka, “The criteria for algebraic equivalence and the torsion group”, *Amer. J. Math.* **79** (1957), 53–66. [MR](#) [Zbl](#)
- [Milne 1986] J. S. Milne, “Abelian varieties”, pp. 103–150 in *Arithmetic geometry* (Storrs, CT, 1984), edited by G. Cornell and J. H. Silverman, Springer, 1986. [MR](#) [Zbl](#)
- [Mumford 1970] D. Mumford, *Abelian varieties*, Tata Inst. Fundam. Res. Stud. Math. **5**, Oxford Univ. Press, 1970. [MR](#) [Zbl](#)
- [Reiner 2003] I. Reiner, *Maximal orders*, London Mathematical Society Monographs. New Series **28**, Clarendon, Oxford, 2003. [MR](#) [Zbl](#)
- [Shuddhodan 2019] K. V. Shuddhodan, “Constraints on the cohomological correspondence associated to a self map”, *Compos. Math.* **155**:6 (2019), 1047–1056. [MR](#) [Zbl](#)
- [Tate 1966] J. Tate, “Endomorphisms of abelian varieties over finite fields”, *Invent. Math.* **2** (1966), 134–144. [MR](#) [Zbl](#)
- [Truong 2016] T. T. Truong, “Relations between dynamical degrees, Weil’s Riemann hypothesis and the standard conjectures”, preprint, 2016. [arXiv](#)
- [Yomdin 1987] Y. Yomdin, “Volume growth and entropy”, *Israel J. Math.* **57**:3 (1987), 285–300. [MR](#) [Zbl](#)

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