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# A comparison between pro- $p$ Iwahori–Hecke modules and mod $p$ representations

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We give an equivalence of categories between certain subcategories of modules of pro- $p$  Iwahori–Hecke algebras and modulo  $p$  representations.

## 1. Introduction

Let  $G$  be a connected reductive  $p$ -adic group and  $K$  a compact open subgroup of  $G$ . Then one can attach the Hecke algebra  $\mathcal{H}$  to this pair  $(G, K)$  and we have a functor  $\pi \mapsto \pi^K = \{v \in \pi \mid \pi(k)v = v \ (k \in K)\}$  from the category of smooth representations of  $G$  to the category of  $\mathcal{H}$ -modules. These algebras and functors are powerful tools to study the representation theory of  $G$ . In a classical case, namely for smooth representations over the field of complex numbers, this functor gives a bijection between the set of isomorphism classes of irreducible smooth representations of  $G$  such that  $\pi^K \neq 0$  and the set of isomorphism classes of simple  $\mathcal{H}$ -modules. Moreover, the famous theorem of Borel [1976] says that the functor gives an equivalence of categories between the category of smooth representations  $\pi$  of  $G$  which is generated by  $\pi^K$  and the category of  $\mathcal{H}$ -modules when  $K$  is an Iwahori subgroup.

In this paper, we study modulo  $p$  representation theory of  $G$ . In this case, it is natural to consider a pro- $p$  Iwahori subgroup  $I(1)$  which is the pro- $p$  radical of an Iwahori subgroup since any nonzero modulo  $p$  representation has a nonzero vector fixed by the pro- $p$  Iwahori subgroup. The corresponding Hecke algebra is called a pro- $p$  Iwahori–Hecke algebra. The aim of this paper is to give a relation between  $\mathcal{H}$ -modules and modulo  $p$  representations.

Such a relation was first discovered by Vignéras [2007] when  $G = \mathrm{GL}_2(\mathbb{Q}_p)$ . Based on a classification result due to Barthel and Livné [1995; 1994] and Breuil [2003], she proved that the functor  $\pi \mapsto \pi^{I(1)}$  gives a bijection between simple objects. This was enhanced to the level of categories by Ollivier [2009]. Namely she proved that the category of  $\mathcal{H}$ -modules is equivalent to the category of modulo  $p$  representations of  $G$  which are generated by  $\pi^{I(1)}$ . The quasiinverse of this equivalence is given by  $M \mapsto M \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  where  $\mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  is the compact induction from the trivial representation of  $I(1)$ .

However, Ollivier also showed that we cannot expect such correspondence in general. When  $G = \mathrm{GL}_2(F)$  where  $F$  is a  $p$ -adic field such that the number of the residue field is greater than  $p$ , for a *supersingular* simple module  $M$  (we do not recall the definition of supersingular modules since we do not

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use it in this paper), Ollivier showed that  $(M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$  is not finite-dimensional. Since simple modules of  $\mathcal{H}$  are finite-dimensional, it says that we have no equivalence of categories in this case.

Still we can expect that there is such a correspondence if we *avoid* supersingular representations/modules. It was proved by Ollivier and Schneider [2018, Theorem 3.33] that this expectation is true when  $G = \text{SL}_2(F)$  when  $p \neq 2$  or  $F \neq \mathbb{Q}_2$ . The aim of this paper is to extend this for any  $G$ . We remark that our result is not a generalization of their result since we assume that modules have finite-length which they do not assume.

Let  $G$  be a (general) connected reductive  $p$ -adic group. In this case, as a consequence of classification theorems [Abe et al. 2017; Abe 2019a] and the calculation of the invariant part of irreducible representations [Abe et al. 2018a], the functor  $\pi \mapsto \pi^{I(1)}$  gives a bijection between irreducible modulo  $p$  representations of  $G$  and simple  $\mathcal{H}$ -modules which are *far from supersingular representations/modules*. The aim of this paper is to generalize this correspondence to the level of categories. More precisely, we prove the equivalence of the following two categories:

- The category of  $\mathcal{H}$ -modules  $M$  such that  $\dim(M) < \infty$  and a certain element of the center of  $\mathcal{H}$  is invertible on  $M$  (see Definition 3.1).
- The category of modulo  $p$  representations  $\pi$  of  $G$  such that:
  - $\pi$  is generated by  $\pi^{I(1)}$ .
  - $\pi$  has a finite length.
  - Any irreducible subquotient of  $\pi$  is isomorphic to a subquotient of  $\text{Ind}_B^G \sigma$  where  $B$  is a minimal parabolic subgroup and  $\sigma$  is an irreducible representation of the Levi quotient of  $B$ .

Note that an  $\mathcal{H}$ -module  $M$  is supersingular if and only if certain elements in the center of  $\mathcal{H}$  act by zero and a modulo  $p$  irreducible admissible representation  $\pi$  of  $G$  is supersingular if and only if it is supercuspidal, namely it does not appear as a subquotient of a parabolically induced representation from an irreducible admissible representation of a proper Levi subgroup. Therefore some conditions as above says that  $M$  (resp.  $\pi$ ) is *far from supersingular modules (resp. representations)*.

We give an outline of the proof. Since the correspondence is true for irreducible representations, by induction on the length, it is sufficient to prove the following (Theorem 3.5): Let  $M$  be an  $\mathcal{H}$ -module which we are considering. Then  $M \rightarrow M \otimes_{\mathcal{H}} \text{Ind}_{I(1)}^G \mathbf{1}$  is injective. This theorem is proved in Section 3. In fact, we prove the injectivity for any  $M \in \mathcal{C}$  where the category  $\mathcal{C}$  is introduced in Section 3. Here are some reductions:

- Let  $\mathcal{A}$  be the Bernstein subalgebra introduced in [Vignéras 2016]. Since we have an embedding  $M \hookrightarrow \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ , it is sufficient to prove the theorem for  $\text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ . Note that we have  $\text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \in \mathcal{C}$  if  $M \in \mathcal{C}$ .
- We have a decomposition of  $M|_{\mathcal{A}}$  along the *support* (Definition 3.8). We may assume that the support of  $M|_{\mathcal{A}}$  is contained in a Weyl chamber.

- Using a result in [Abe 2019a], parabolic inductions and a result of Ollivier and Vignéras [2018], we may assume that the support is the dominant Weyl chamber.
- We prove there exists an  $\mathcal{A}$ -module  $M'$  such that  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) \simeq M' \otimes_{\mathcal{A}} \mathcal{H}$ . Hence it is sufficient to prove that  $M' \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M' \otimes_{\mathcal{A}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  is injective.

By a result in [Abe 2017], both  $M \otimes_{\mathcal{A}} \mathcal{H}$  and  $M \otimes_{\mathcal{A}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  relate to  $\mathrm{c}\text{-Ind}_K^G V$  where  $K$  is a special parahoric subgroup and  $V$  a certain representation of  $K$ . The structure of this representations is studied in [Abe et al. 2017] and using such result we prove the injectivity.

It is almost immediate to prove our main theorem from the above injectivity. This is done in Section 4.

## 2. Notation and preliminaries

Let  $F$  be a nonarchimedean local field of residue characteristic  $p$  and  $G$  a connected reductive group over  $F$ . Let  $C$  be an algebraically closed field of characteristic  $p$ . This is the coefficient field of representations in this paper. *All representations in this paper are smooth representations over  $C$ .*

In general, for any algebraic group  $H$  over  $F$ , we denote the group of valued points  $H(F)$  by the same letter  $H$ . Fix a maximal split torus  $S$  of  $G$  and minimal parabolic subgroup  $B$  containing  $S$ . The centralizer  $Z$  of  $S$  in  $G$  is a Levi subgroup of  $B$ . We denote the unipotent radical of  $B$  by  $U$  and the opposite of  $B$  containing  $Z$  by  $\bar{B}$ . The unipotent radical of  $\bar{B}$  is denoted by  $\bar{U}$ .

Consider the reduced apartment corresponding to  $S$  and take an alcove  $A_0$  and a special point  $\mathbf{x}_0$  from the closure of  $A_0$ . Let  $K$  be the special parahoric subgroup corresponding to  $\mathbf{x}_0$  and  $I$  the Iwahori subgroup determined by  $A_0$ . Let  $I(1)$  be the pro- $p$  Iwahori subgroup attached to  $A_0$ , namely the pro- $p$  radical of  $I$ . The space of  $C$ -valued compactly supported  $I(1)$ -biinvariant functions  $\mathcal{H}$  has a structure of a  $C$ -algebra via the convolution product. The algebra  $\mathcal{H}$  is called pro- $p$  Iwahori–Hecke algebra. The structure of this algebra is studied by Vignéras [2016].

Let  $N_G(S)$  be the normalizer of  $S$  in  $G$  and put  $W_0 = N_G(S)/Z$ ,  $W = N_G(S)/(Z \cap K)$  and  $W(1) = N_G(S)/(Z \cap I(1))$ . Let  $G'$  be the subgroup of  $G$  generated by  $U$  and  $\bar{U}$ . Note that this is not a group of the valued points of an algebraic group in general. Let  $W_{\mathrm{aff}}$  be the image of  $G' \cap N_G(S)$  in  $W$ . The action of  $W_{\mathrm{aff}}$  on the apartment is faithful and therefore it is a subgroup of the group of affine transformations of the apartment. Let  $S_{\mathrm{aff}}$  be the set of reflections along the walls of  $A_0$ . Then  $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$  is a Coxeter system. Denote its length function by  $\ell$ . Let  $N_W(A_0)$  be the stabilizer of  $A_0$  in  $W$ . Then the group  $W$  is the semidirect product of  $W_{\mathrm{aff}}$  and  $N_W(A_0)$ . The function  $\ell$  is extended to  $W$ , trivially on  $N_W(A_0)$ . We also inflate  $\ell$  to  $W(1)$  via  $W(1) \rightarrow W$ . We have the Bruhat order on  $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$  and we extend it to  $W$  by  $w_1 \omega_1 < w_2 \omega_2$  if and only if  $w_1 < w_2$  and  $\omega_1 = \omega_2$  where  $w_1, w_2 \in W_{\mathrm{aff}}$  and  $\omega_1, \omega_2 \in N_W(A_0)$ . For  $w_1, w_2 \in W(1)$ , we say  $w_1 < w_2$  if  $\bar{w}_1 < \bar{w}_2$  where  $\bar{w}_i$  is the image of  $w_i$  in  $W$  ( $i = 1, 2$ ). As usual we say  $w_1 \leq w_2$  if and only if  $w_1 < w_2$  or  $w_1 = w_2$ .

We give some of structure theorems of  $\mathcal{H}$ . For  $w \in W(1)$ , let  $T_w$  be the characteristic function on  $I(1)\tilde{w}I(1)$  where  $\tilde{w} \in N_G(S)$  is a lift of  $w$ . Then  $T_w$  does not depend on the choice of a lift and, since we have the bijection  $I(1) \backslash G / I(1) \simeq W(1)$ ,  $\{T_w \mid w \in W(1)\}$  is a basis of  $\mathcal{H}$ . This basis is called

Iwahori–Matsumoto basis. This basis satisfies the following braid relations:

$$T_{w_1} T_{w_2} = T_{w_1 w_2} \quad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$$

where  $w_1, w_2 \in W(1)$ . Let  $Z_\kappa = (Z \cap K)/(Z \cap I(1))$ . Then this is a subgroup of  $W(1)$ . Since any elements in  $Z_\kappa$  has the length 0 (since it is in the kernel of  $W(1) \rightarrow W$ ), from the braid relations, we have  $T_{t_1} T_{t_2} = T_{t_1 t_2}$  for  $t_1, t_2 \in Z_\kappa$ . In other words, the embedding  $C[Z_\kappa] \hookrightarrow \mathcal{H}$  defined by  $\sum_{t \in Z_\kappa} c_t t \mapsto \sum_{t \in Z_\kappa} c_t T_t$  is an algebra homomorphism where  $C[Z_\kappa]$  is the group ring of  $Z_\kappa$ . Using this embedding, we regard  $C[Z_\kappa]$  as a subalgebra of  $\mathcal{H}$ .

Let  $S_{\text{aff}}(1)$  be the inverse image of  $S_{\text{aff}}$  in  $W(1)$ . Then for  $s \in S_{\text{aff}}(1)$ , we have

$$T_s^2 = c_s T_s$$

for some  $c_s \in C[Z_\kappa]$ . An element  $c_s$  is given in [Vignéras 2016, 4.2].

Define  $T_w^*$  as in [loc. cit., 4.3] for  $w \in W(1)$ . This is also a basis of  $\mathcal{H}$  and it satisfies the following:  $T_w^* \in T_w + \sum_{v < w} C T_v$  and  $T_{w_1}^* T_{w_2}^* = T_{w_1 w_2}^*$  if  $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ .

Let  $o$  be a spherical orientation [loc. cit., 5.2]. Note that the set of spherical orientations are canonically bijective with the set of Weyl chambers. For each  $o$ , we have another basis  $\{E_o(w) \mid w \in W(1)\}$  defined in [loc. cit., 5.3]. The orientations correspond to the Weyl chambers. Let  $o_-$  be the orientation corresponding to the antidominant Weyl chamber and set  $E(w) = E_{o_-}(w)$ .

Set  $\Lambda(1) = Z/(Z \cap I(1))$ . This is a subgroup of  $W(1)$ . For  $\lambda_1, \lambda_2 \in \Lambda(1)$ , the multiplication  $E(\lambda_1)E(\lambda_2)$  is simple. Before giving it, we introduce some notation. The pair  $(G, S)$  gives a root datum  $(X^*(S), \Sigma, X_*(S), \Sigma^\vee)$  and since we have fixed a Borel subgroup we also have a positive system  $\Sigma^+ \subset \Sigma$  and the set of simple roots  $\Delta \subset \Sigma^+$ . An element  $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$  is called dominant if and only if  $\langle v, \alpha \rangle \geq 0$  for any  $\alpha \in \Sigma^+$ . A  $W_0$ -orbit of the set of dominant elements is called a closed Weyl chamber. We also say that  $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$  is regular if  $\langle v, \alpha \rangle \neq 0$  for any  $\alpha \in \Sigma$ . We have a homomorphism  $\nu: Z \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} = \text{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$  characterized by  $\nu(z)(\chi) = -\text{val}(\chi(z))$  where  $z \in S$ ,  $\chi \in X^*(S)$  and  $\text{val}: F^\times \rightarrow \mathbb{Z}$  is the normalized valuation. This homomorphism factors through  $Z \rightarrow \Lambda(1)$  and the induced homomorphism  $\Lambda(1) \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$  is denoted by the same letter  $\nu$ . We let  $\Lambda^+(1)$  the set of  $\lambda \in \Lambda(1)$  such that  $\nu(\lambda)$  is dominant. For  $w \in W_0$ , let  $w(\Lambda^+(1))$  be the set of  $\lambda \in \Lambda(1)$  such that  $w^{-1}(\nu(\lambda))$  is dominant.

The multiplication  $E(\lambda_1)E(\lambda_2)$  is  $E(\lambda_1 \lambda_2)$  if  $\nu(\lambda_1)$  and  $\nu(\lambda_2)$  are in the same closed Weyl chamber (in other words,  $\lambda_1, \lambda_2 \in w(\Lambda^+(1))$  for some  $w \in W_0$ ) and otherwise it is zero. In particular,  $\mathcal{A} = \bigoplus_{\lambda \in \Lambda(1)} C E(\lambda)$  is a subalgebra of  $\mathcal{H}$ . If we fix a closed Weyl chamber  $\mathcal{C}$ , then  $\bigoplus_{\nu(\lambda) \in \mathcal{C}} C E(\lambda)$  is a subalgebra of  $\mathcal{A}$  and the linear map

$$\bigoplus_{\nu(\lambda) \in \mathcal{C}} C E(\lambda) \rightarrow C[\Lambda(1)]$$

defined by  $E(\lambda) \mapsto \tau_\lambda$  is an algebra embedding. Here  $C[\Lambda(1)]$  is the group ring of  $\Lambda(1)$  and we denote the element in  $C[\Lambda(1)]$  corresponding to  $\lambda \in \Lambda(1)$  by  $\tau_\lambda$ , namely  $C[\Lambda(1)] = \bigoplus_{\lambda \in \Lambda(1)} C \tau_\lambda$ .

- Remark 2.1.** (1) If  $\langle \nu(\lambda), \alpha \rangle = 0$  for any  $\alpha \in \Sigma$ , then  $\nu(\lambda)$  and  $\nu(\lambda^{-1})$  are in the same closed Weyl chamber. (In fact,  $\nu(\lambda)$  and  $\nu(\lambda^{-1})$  are in any closed Weyl chamber.) Hence  $E(\lambda)E(\lambda^{-1}) = 1$ . In particular,  $E(\lambda)$  is invertible.
- (2) If  $\lambda \in \Lambda(1)$  is in the center of  $\Lambda(1)$ , then  $E(\lambda)$  is also in the center of  $\mathcal{A}$ . This follows from the above description of the multiplication.

Let  $J$  be a subset of  $\Delta$  and denote the corresponding standard parabolic subgroup by  $P_J$ . Let  $L_J$  be the Levi part of  $P_J$  containing  $Z$ . Then  $K \cap L_J$  is a special parahoric subgroup and  $I(1)_J = I(1) \cap L_J$  a pro- $p$  Iwahori subgroup. Attached to these, we have many objects. For such objects we add a suffix  $J$ , for example, the pro- $p$  Iwahori–Hecke algebra attached to  $(L_J, I(1)_J)$  is denoted by  $\mathcal{H}_J$ . There are two exceptions: base  $T_w$  and  $E(w)$  for  $\mathcal{H}_J$  is denoted by  $T_w^J$  and  $E^J(w)$ , respectively. For each  $J \subset \Delta$ , we have two subalgebras  $\mathcal{H}_J^+$ ,  $\mathcal{H}_J^-$  of  $\mathcal{H}_J$  and four algebra homomorphisms  $j_J^+$ ,  $j_J^{+*}: \mathcal{H}_J^+ \rightarrow \mathcal{H}$  and  $j_J^-$ ,  $j_J^{-*}: \mathcal{H}_J^- \rightarrow \mathcal{H}$ . See [Abe 2019b, 2.8] for the definitions. (Here  $\mathcal{H}_J^+$  is denoted by  $\mathcal{H}_{P_J}^+$  in [Abe 2019b].)

### 3. The category $\mathcal{C}$ and a proof of the injectivity

**3A. The category  $\mathcal{C}$ .** The modules in this paper are right modules unless otherwise stated. In this paper, we focus on the full subcategory  $\mathcal{C}$  of the category of  $\mathcal{H}$ -modules defined using the center  $\mathcal{Z}$  of  $\mathcal{H}$ . The center  $\mathcal{Z}$  is described using the basis  $\{E(w)\}$ . Since  $\Lambda(1)$  is normal in  $W(1)$ , the group  $W(1)$  acts on  $\Lambda(1)$  by the conjugate action. For  $\lambda \in \Lambda(1)$  denote the orbit through  $\lambda$  by  $\mathcal{O}_\lambda$ . For  $\lambda \in \Lambda(1)$ , put  $z_\lambda = \sum_{\lambda' \in \mathcal{O}_\lambda} E(\lambda')$ . Then  $\{z_\lambda \mid \lambda \in \Lambda(1)/W(1)\}$  gives a basis of  $\mathcal{Z}$  [Vignéras 2014, Theorem 1.2]. Fix a uniformizer  $\varpi$  of  $F$  and let  $\Lambda_S(1)$  be the image of  $\{\xi(\varpi) \mid \xi \in X_*(S)\}$ .

**Definition 3.1.** An  $\mathcal{H}$ -module  $M$  is in  $\mathcal{C}$  if and only if  $z_\lambda$  is invertible on  $M$  for any  $\lambda \in \Lambda_S(1)$ .

**Lemma 3.2.** Let  $\lambda \in \Lambda_S(1)$ . Then we have the following:

- (1) For  $w \in W(1)$ ,  $w$  stabilizes  $\lambda$  if and only if the image of  $w$  in  $W_0$  stabilizes  $\nu(\lambda)$ .
- (2) Let  $\{w_1, \dots, w_r\} \subset W(1)$  be a subset of  $W(1)$  such that the image in  $W_0$  gives a set of complete representatives of  $W_0/\text{Stab}_{W_0}(\nu(\lambda))$ . Then we have  $z_\lambda = \sum_{i=1}^r E(w_i \lambda w_i^{-1})$ . (Note that  $w_i \lambda w_i^{-1}$  depends only on the image of  $w_i$  in  $W_0/\text{Stab}_{W_0}(\lambda)$  by (1).)

*Proof.* Take  $\xi \in X_*(S)$  such that  $\lambda = \xi(\varpi)^{-1}$ . We have  $\nu(\lambda) = \xi$ . Let  $w \in W(1)$  and denote the image of  $w$  in  $W_0$  by  $w_0$ . Then we have  $w \lambda w^{-1} = (w_0 \xi)(\varpi)^{-1}$ . Hence if  $w_0$  stabilizes  $\xi = \nu(\lambda)$ , then  $w$  stabilizes  $\lambda$ . Obviously if  $w$  stabilizes  $\lambda$  then  $w_0$  stabilizes  $\nu(\lambda)$ .

By (1),  $\text{Stab}_{W(1)}(\lambda)$  is the inverse image of  $\text{Stab}_{W_0}(\lambda)$ . Therefore we have  $W(1)/\text{Stab}_{W(1)}(\lambda) \simeq W_0/\text{Stab}_{W_0}(\lambda)$ . By the definition, we have  $z_\lambda = \sum_{w \in W(1)/\text{Stab}_{W(1)}(\lambda)} E(w \lambda w^{-1})$ . Hence we get (2).  $\square$

**Lemma 3.3.** Let  $\lambda, \mu \in \Lambda_S(1)$  and assume that  $\nu(\lambda)$  and  $\nu(\mu)$  are in the same closed Weyl chamber. We also assume that  $\nu(\lambda)$  is regular. Then we have  $z_\lambda z_\mu = z_{\lambda\mu}$ .

*Proof.* Take  $w_1, \dots, w_r \in W(1)$  such that the images of them in  $W_0$  gives a set of complete representatives of  $W_0/\text{Stab}_{W_0}(v(\mu))$ . Then we have  $z_\mu = \sum_i E(w_i \mu w_i^{-1})$  by the above lemma. Let  $v_1, \dots, v_s$  be a set of complete representatives of  $W_0 = W(1)/\Lambda(1)$ . Then we have  $z_\lambda = \sum_j E(v_j \lambda v_j^{-1})$ . (Note that  $v(\lambda)$  is assumed to be regular.) Since  $v(\lambda)$  is regular, for each  $i$ , there exists only one  $j_i = 1, \dots, r$  such that  $v_i(v(\lambda))$  and  $w_{j_i}(v(\mu))$  is in the same closed Weyl chamber. Hence we get

$$E(v_i \lambda v_i^{-1})E(w_j \mu w_j^{-1}) = \begin{cases} 0 & j \neq j_i, \\ E(v_i \lambda v_i^{-1} w_{j_i} \mu w_{j_i}^{-1}) & j = j_i. \end{cases}$$

Moreover,  $v(\lambda)$  and  $v_i^{-1} w_{j_i}(v(\mu))$  is in the same closed Weyl chamber. Since  $v(\lambda)$  and  $v(\mu)$  are in the same closed Weyl chamber by the assumption, we get  $v_i^{-1} w_{j_i}(v(\mu)) = v(\mu)$ . Therefore  $v_i^{-1} w_{j_i}$  stabilizes  $v(\mu)$ . As in the previous lemma,  $v_i^{-1} w_{j_i}$  also stabilizes  $\mu$ . Hence  $w_{j_i} \mu w_{j_i}^{-1} = v_i \mu v_i^{-1}$ . We get

$$E(v_i \lambda v_i^{-1})E(w_j \mu w_j^{-1}) = \begin{cases} 0 & j \neq j_i, \\ E(v_i \lambda \mu v_i^{-1}) & j = j_i. \end{cases}$$

Now we get

$$z_\lambda z_\mu = \sum_i \sum_j E(v_i \lambda v_i^{-1})E(w_j \mu w_j^{-1}) = \sum_i E(v_i \lambda \mu v_i^{-1}).$$

By the assumption,  $v(\lambda \mu)$  is regular and  $\lambda \mu \in \Lambda_S(1)$ . Hence the last term is  $z_{\lambda \mu}$  by the above lemma.  $\square$

**Lemma 3.4.** *An  $\mathcal{H}$ -module  $M$  is in  $\mathcal{C}$  if and only if for some  $\lambda \in \Lambda_S(1)$  such that  $v(\lambda)$  is regular, the element  $z_\lambda$  is invertible on  $M$ .*

*Proof.* Assume that there exists  $\lambda_0 \in \Lambda_S(1)$  such that  $v(\lambda_0)$  is regular and  $z_{\lambda_0}$  is invertible on  $M$ . Let  $\lambda \in \Lambda_S(1)$  and we prove that  $\lambda$  is also invertible on  $M$ . Replacing  $\lambda$  with an element in the orbit through  $\lambda$ , we may assume that  $v(\lambda)$  and  $v(\lambda_0)$  are in the same closed Weyl chamber. Take a sufficiently large  $n \in \mathbb{Z}_{>0}$  such that  $v(\lambda_0^n \lambda^{-1})$  is also in the same closed Weyl chamber as  $v(\lambda_0)$ . Set  $\mu = \lambda_0^n \lambda^{-1}$ . Then by the above lemma, we have  $z_\mu z_\lambda = z_{\lambda_0^n} = z_{\lambda_0}^n$ . By the assumption,  $z_{\lambda_0}^n$  is invertible on  $M$ . Hence  $z_\lambda$  is invertible, namely we have  $M \in \mathcal{C}$ .  $\square$

**3B. Theorem.** In the rest of this section, we prove the following:

**Theorem 3.5.** *If  $M \in \mathcal{C}$ , then  $M \rightarrow M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is injective.*

**3C. Reductions.** Define a subalgebra  $\mathcal{A}$  of  $\mathcal{H}$  by  $\mathcal{A} = \bigoplus_{\lambda \in \Lambda(1)} CE(\lambda)$ . Let  $M \in \mathcal{C}$  and set  $M' = \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ .

**Remark 3.6.** The element  $z_\lambda$  is in the center of  $\mathcal{H}$  and  $z_\lambda \in \mathcal{A}$ . Therefore the action of  $z_\lambda$  on  $M' = \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$  is induced by that on  $M$ . Since  $M \in \mathcal{C}$ ,  $z_\lambda$  is invertible on  $M$  for any  $\lambda \in \Lambda_S(1)$ . Hence the action of  $z_\lambda$  on  $M'$  is also invertible. Namely  $M' \in \mathcal{C}$ .

Defining the action of  $X \in \mathcal{H}$  on  $M'$  by  $(\varphi X)(Y) = \varphi(XY)$  for  $\varphi \in \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$  and  $Y \in \mathcal{H}$ ,  $M'$  is a right  $\mathcal{H}$ -module. The map  $m \mapsto (X \mapsto mX)$  gives an  $\mathcal{H}$ -module embedding  $M \hookrightarrow M'$  and we have the



following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & M \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M' \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1} \end{array}$$

Therefore, to prove Theorem 3.5, it is sufficient to prove that the map  $M' \rightarrow M' \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  is injective.

**Lemma 3.7.** *Any module  $M \in \mathcal{C}$  has a functorial decomposition  $M = \bigoplus_{w \in W_0} M_w$  as an  $\mathcal{A}$ -module such that  $E(\mu)$  acts on  $M_w$  by:*

- Zero if  $w^{-1}v(\mu)$  is not dominant.
- Invertible if  $w^{-1}v(\mu)$  is dominant.

*Proof.* Fix  $\lambda_0 \in \Lambda_S(1)$  such that  $v(\lambda_0)$  is regular dominant. Put  $\lambda_w = n_w \lambda_0 n_w^{-1}$  and set  $M_w = ME(\lambda_w)$ . Since  $\lambda_w \in \Lambda_S(1)$  is central,  $E(\lambda_w)$  is also central in  $\mathcal{A}$ . Hence  $M_w$  is an  $\mathcal{A}$ -submodule.

We prove that  $\lambda_w$  is invertible on  $M_w$ . Since  $v(\lambda_0)$  is regular,  $v(\lambda_v)$  and  $v(\lambda_w)$  are not in the same closed Weyl chamber if  $v \neq w$ . Therefore  $E(\lambda_v)E(\lambda_w) = 0$ . Hence  $M_w E(\lambda_v) = 0$  if  $v \neq w$ . Therefore for  $m \in M_w$ , we have  $mz_{\lambda_0} = \sum_{v \in W_0} mE(\lambda_v) = mE(\lambda_w)$ . Hence if  $mE(\lambda_w) = 0$  then  $mz_{\lambda_0} = 0$ , hence  $m = 0$  since  $z_{\lambda_0}$  is invertible. Therefore  $E(\lambda_w)$  is injective on  $M_w$ . We also have that  $mz_{\lambda_0}^2 = mE(\lambda_w)z_{\lambda_0} = mz_{\lambda_0}E(\lambda_w) = mE(\lambda_w)^2$  since  $z_{\lambda_0}$  commutes with  $E(\lambda_w)$ . (Recall that  $z_{\lambda_0}$  is in the center of  $\mathcal{H}$ .) Hence  $m = m_0E(\lambda_w)$  where  $m_0 = mz_{\lambda_0}^{-2}E(\lambda_w) \in M_w$ . Therefore  $E(\lambda_w)$  is surjective on  $M_w$ .

For  $\mu \in \Lambda(1)$  such that  $w^{-1}(v(\mu))$  is not dominant,  $v(\mu)$  and  $v(\lambda_w)$  are not in the same closed Weyl chamber. Hence  $E(\mu)E(\lambda_w) = 0$ . Therefore  $E(\mu) = 0$  on  $M_w$ . On the other hand, assume that  $w^{-1}(v(\mu))$  is dominant. Then  $v(\mu)$  and  $v(\lambda_w)$  are in the same closed Weyl chamber. Take sufficiently large  $n \in \mathbb{Z}_{\geq 0}$  such that  $v(\lambda_w^n \mu^{-1})$  is also in the same closed Weyl chamber as  $v(\mu)$ . Then we have  $E(\lambda_w)^n = E(\lambda_w^n) = E(\lambda_w^n \mu^{-1})E(\mu)$ . Since  $E(\lambda_w)$  is invertible on  $M_w$ ,  $E(\mu)$  is also invertible on  $M_w$ .

We prove  $M = \bigoplus_{w \in W_0} M_w$ . Since  $z_{\lambda_0}$  is invertible, any element in  $M$  can be written  $mz_{\lambda_0}$  for some  $m \in M$ . We have  $mz_{\lambda_0} = \sum_{w \in W_0} mE(\lambda_w) \in \sum_{w \in W_0} M_w$ . Hence  $M = \sum_{w \in W_0} M_w$ . Let  $m_w \in M_w$  and assume that  $\sum_{w \in W_0} m_w = 0$ . Then for each  $v \in W_0$  we have  $\sum_{w \in W_0} m_w E(\lambda_v) = 0$ . Since  $m_w E(\lambda_v) = 0$  for  $v \neq w$ , we have  $m_v E(\lambda_v) = 0$ . Since the action of  $E(\lambda_v)$  on  $M_v$  is invertible,  $m_v = 0$ .  $\square$

Since  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) = \bigoplus_{w \in W_0} \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M_w)$ , to prove  $M' \rightarrow M' \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  is injective, it is sufficient to prove that the homomorphism  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M_w) \rightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M_w) \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$  is injective.

**Definition 3.8.** Let  $M$  be an  $\mathcal{A}$ -module. We say that  $\mathrm{supp} M = w(\Lambda^+(1))$  if and only if  $E(\lambda)$  is:

- Zero if  $w^{-1}v(\mu)$  is not dominant.
- Invertible if  $w^{-1}v(\mu)$  is dominant.

for any  $\lambda \in \Lambda(1)$ . (Note that we do not define  $\mathrm{supp} M$  itself.)

From the above discussions, to prove Theorem 3.5, it is sufficient to prove the following lemma.

**Lemma 3.9.** *Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = w(\Lambda^+(1))$  where  $w \in W_0$ . Then  $\text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is injective.*

We take a lift  $n_w$  of each  $w \in W_0$  in  $W(1)$  such that  $n_{w_1 w_2} = n_{w_1} n_{w_2}$  if  $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$ . Let  $M$  be an  $\mathcal{A}$ -module and  $w \in W_0$ . We define a new  $\mathcal{A}$ -module  $n_w M$  as follows. As a vector space,  $n_w M = M$  and the action of  $E(\lambda) \in \mathcal{A}$  on  $n_w M$  is the action of  $E(n_w^{-1} \lambda n_w)$  on  $M$ . This defines an auto-equivalence of the category of  $\mathcal{A}$ -modules. If  $\text{supp } M = v(\Lambda^+(1))$ , then  $\text{supp } n_w M = wv(\Lambda^+(1))$ . With this notation, Lemma 3.9 is equivalent to the following.

**Lemma 3.10.** *Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = \Lambda^+(1)$ . Then the map  $\text{Hom}_{\mathcal{A}}(\mathcal{H}, n_w M) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{H}, n_w M) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is injective.*

**3D. Reduction to  $w = w_J$  for some  $J \subset \Delta$ .** For a subset  $J \subset \Delta$ , let  $w_J$  be the longest element in  $W_{0,J}$ . We prove that we may assume  $w = w_J$  for some  $J$  in Lemma 3.10.

We relate our  $M$  with modules studied in [Abe 2019a]. Consider the homomorphism  $\mathcal{A} \rightarrow C[\Lambda(1)]$  defined by

$$E(\lambda) \mapsto \begin{cases} \tau_\lambda & \lambda \in \Lambda^+(1), \\ 0 & \text{otherwise.} \end{cases} \quad (3-1)$$

We regard  $C[\Lambda(1)]$  as a right  $\mathcal{A}$ -module via this homomorphism. For  $w \in W_0$ , we also have the  $\mathcal{A}$ -module  $n_w C[\Lambda(1)]$ . Then we consider the module

$$n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

This is a  $(C[\Lambda(1)], \mathcal{H})$ -bimodule.

Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = \Lambda^+(1)$ . Then we define a structure of a right  $C[\Lambda(1)]$ -module on  $M$  by

$$m \tau_{\lambda_1 \lambda_2^{-1}} = m E(\lambda_1) E(\lambda_2)^{-1}$$

where  $\lambda_1, \lambda_2 \in \Lambda^+(1)$  and  $m \in M$ . (Since  $\text{supp } M = \Lambda^+(1)$ ,  $E(\lambda_2)$  is invertible on  $M$ .) It is easy to see that this definition is well-defined and define a structure of  $C[\Lambda(1)]$ -module. Then we have

$$M \otimes_{C[\Lambda(1)]} n_w C[\Lambda(1)] \simeq n_w M.$$

The isomorphisms are given by  $m \otimes f \mapsto mf$  from the left-hand side to the right-hand side and  $m \mapsto m \otimes 1$  in the opposite direction. Therefore we have

$$n_w M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]} n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

For each  $w \in W_0$ , set  $\Delta_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$ . Then by [Abe 2019a, Theorem 3.13], if  $\Delta_{w_1} = \Delta_{w_2}$ , we have

$$n_{w_1} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

Therefore we get (1) of the next lemma.

**Lemma 3.11.** *Let  $M$  be as in Lemma 3.10. If  $w_1, w_2 \in W_0$  satisfies  $\Delta_{w_1} = \Delta_{w_2}$ , then we have:*

- (1)  $n_{w_1}M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2}M \otimes_{\mathcal{A}} \mathcal{H}$ .
- (2)  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M)$ .

*Proof.* We have proved (1). We prove (2).

Let  $\iota$  be an automorphism of  $\mathcal{H}$  defined in [Vignéras 2016, Proposition 4.23] and  $\zeta: \mathcal{H} \rightarrow \mathcal{H}$  an antiautomorphism defined by  $\zeta(T_w) = T_{w^{-1}}$ . (The linear map  $\zeta$  is an antihomomorphism by [Abe 2019a, 4.1].) Set  $f = \iota \circ \zeta$ . Since  $\zeta(E(\lambda)) = E_{o_+}(\lambda^{-1})$  [Abe 2019a, Lemma 4.3] and  $\iota(E_{o_+}(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda)$  [Vignéras 2016, Lemma 5.31], we have  $f(E(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda^{-1})$ . In particular,  $f$  preserves  $\mathcal{A}$ . It is easy to see  $f^2(T_w) = T_w$  for any  $w \in W(1)$ . Hence  $f^2$  is identity.

For a left  $\mathcal{H}$ -module  $N$ , we define a right  $\mathcal{H}$ -module  $N^f$  by  $N^f = N$  as a vector space and the action of  $X \in \mathcal{H}$  on  $N^f$  is the action of  $f(X)$  on  $N$ . Then  $m \otimes X \mapsto f(X) \otimes m$  gives an isomorphism  $(N^f \otimes_{\mathcal{A}} \mathcal{H})^f \simeq \mathcal{H} \otimes_{\mathcal{A}} N$ .

For a right  $\mathcal{H}$ -module or  $\mathcal{A}$ -module  $L$ , set  $L^* = \mathrm{Hom}_C(L, C)$ . Then this is a left  $\mathcal{H}$ -module or  $\mathcal{A}$ -module, respectively. Let  $M$  be as in the lemma. Since  $f(E(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda^{-1})$ , we have  $\mathrm{supp}(n_{w_1}M^*)^f = w_1(\Lambda^+(1)^{-1}) = w_1w_{\Delta}(\Lambda^+(1))$ . Hence  $(n_{w_1}M^*)^f = n_{w_1w_{\Delta}}M'$  for some  $\mathcal{A}$ -module  $M'$  such that  $\mathrm{supp} M' = \Lambda^+(1)$ . Since  $\Delta_{w_1w_{\Delta}} = \Delta \setminus (-w_{\Delta}(\Delta_{w_1}))$ , we also have  $\Delta_{w_1w_{\Delta}} = \Delta_{w_2w_{\Delta}}$ . Hence by (1), we get  $n_{w_1w_{\Delta}}M' \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2w_{\Delta}}M' \otimes_{\mathcal{A}} \mathcal{H}$ . Therefore we get  $(n_{w_1}M^*)^f \otimes_{\mathcal{A}} \mathcal{H} \simeq (n_{w_2}M^*)^f \otimes_{\mathcal{A}} \mathcal{H}$ . Applying  $(\cdot)^f$  to the both sides and using  $(N^f \otimes_{\mathcal{A}} \mathcal{H})^f \simeq \mathcal{H} \otimes_{\mathcal{A}} N$ , we get  $\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^* \simeq \mathcal{H} \otimes_{\mathcal{A}} n_{w_2}M^*$ . Hence we have  $(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^*)^* \simeq (\mathcal{H} \otimes_{\mathcal{A}} n_{w_2}M^*)^*$ .

Now we have

$$(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^*)^* = \mathrm{Hom}_C(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^*, C) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M^{**}).$$

Hence we have  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M^{**}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M^{**})$ . We have an embedding  $M \hookrightarrow M^{**}$ . Let  $L$  be the cokernel. Then  $\mathrm{supp} L = \Lambda^+(1)$  and we have an embedding  $L \hookrightarrow L^{**}$ . Therefore we have an exact sequence  $0 \rightarrow M \rightarrow M^{**} \rightarrow L^{**}$  and it gives  $0 \rightarrow n_{w_i}M \rightarrow n_{w_i}M^{**} \rightarrow n_{w_i}L^{**}$  for  $i = 1, 2$ . Hence we get the following commutative diagram with exact rows:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M) & & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M^{**}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M^{**}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}L^{**}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}L^{**}). \end{array}$$

We have  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M)$ . □

For given  $w \in W$ , set  $J = \Delta \setminus \Delta_w$ . Then we have  $\Delta_{w_J} = \Delta \setminus J = \Delta_w$ . Therefore, to prove Lemma 3.10, we may assume that  $w = w_J$  for some  $J \subset \Delta$ .

**3E. Reduction to  $w = w_\Delta$ .** Set

$$\mathcal{A}_w = \bigoplus_{\lambda \in w(\Lambda^+(1))} CE(\lambda) \subset \mathcal{A}. \quad (3-2)$$

**Lemma 3.12.** *Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = w(\Lambda^+(1))$ . Then we have  $\text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_w}(\mathcal{H}, M)$ .*

*Proof.* Let  $\varphi: \mathcal{H} \rightarrow M$  be an  $\mathcal{A}_w$ -module homomorphism and we prove that  $\varphi$  is  $\mathcal{A}$ -equivariant. Fix  $\lambda_0 \in \Lambda(1)$  such that  $w^{-1}(\nu(\lambda_0))$  is dominant and regular. Since  $\text{supp } M = w(\Lambda^+(1))$ ,  $E(\lambda_0)$  is invertible on  $M$ . For  $\mu \in \Lambda(1)$  such that  $w^{-1}(\nu(\mu))$  is not dominant, we have  $E(\mu)E(\lambda_0) = 0$ . Hence for  $X \in \mathcal{H}$ , we have  $\varphi(XE(\mu)) = E(\lambda_0)^{-1}\varphi(XE(\mu)E(\lambda_0)) = 0$ . Since  $E(\mu) = 0$  on  $n_w M$ ,  $E(\mu)\varphi(X) = 0$ . Hence we get  $\varphi(XE(\mu)) = 0 = E(\mu)\varphi(X)$ . Therefore  $\varphi$  is  $\mathcal{A}$ -equivariant.  $\square$

For later use, we also prove the following.

**Lemma 3.13.** *Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = w(\Lambda^+(1))$ . Then  $M \otimes_{\mathcal{A}_w} \mathcal{H} \xrightarrow{\sim} M \otimes_{\mathcal{A}} \mathcal{H}$ .*

*Proof.* Let  $m \in M$  and  $X \in \mathcal{H}$ . We prove  $mE(\lambda) \otimes X = m \otimes E(\lambda)X$  in  $M \otimes_{\mathcal{A}_w} \mathcal{H}$  for any  $\lambda \in \Lambda(1)$ . This is true if  $w^{-1}(\nu(\lambda))$  is dominant.

Assume that  $w^{-1}(\nu(\lambda))$  is not dominant and take  $\lambda_0 \in \Lambda(1)$  such that  $w^{-1}(\nu(\lambda_0))$  is dominant and  $\nu(\lambda), \nu(\lambda_0)$  are not in the same chamber. Then we have  $E(\lambda_0)E(\lambda) = 0$ . Note that  $E(\lambda_0)$  is invertible on  $M$  since  $\text{supp } M = w(\Lambda^+(1))$ . Hence  $m \otimes E(\lambda)X = mE(\lambda_0)^{-1} \otimes E(\lambda_0)E(\lambda)X = 0$ . On the other hand,  $E(\lambda) = 0$  on  $M$ , again by  $\text{supp } M = w(\Lambda^+(1))$ . Hence  $mE(\lambda) \otimes X = 0$ . We get the lemma.  $\square$

An element  $E(\lambda)$  belongs to:

- $\mathcal{A}_w$  if  $\langle \nu(\lambda), w(\alpha) \rangle \geq 0$  for any  $\alpha \in \Sigma^+$ .
- $j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$  if  $\langle \nu(\lambda), \alpha \rangle \geq 0$  for any  $\alpha \in \Sigma^+ \setminus \Sigma_J^+$ .

(The second one follows from the following fact: a basis of  $\mathcal{H}_J^- \cap \mathcal{A}_J$  is given by  $\{E^J(\lambda)\}$  where  $\lambda$  runs through as above [Abe 2019a, Lemma 4.2] and  $j_J^{-*}(E^J(\lambda)) = E(\lambda)$  for such  $\lambda$  [Abe 2019b, Lemma 2.6].) Since  $w_J(\Sigma^+) = \Sigma_J^- \cup (\Sigma^+ \setminus \Sigma_J^+) \supset \Sigma^+ \setminus \Sigma_J^+$ , we have  $\mathcal{A}_{w_J} \subset j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$ .

Let  $M$  be an  $\mathcal{A}$ -module. From the above argument, we have

$$\text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}, n_{w_J} M) \simeq \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H} \otimes_{j_J^{-*}(\mathcal{H}_J^-)} j_J^{-*}(\mathcal{H}_J^-), n_{w_J} M) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J} M)).$$

Since  $j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$  contains  $\mathcal{A}_{w_J}$ , we have  $\mathcal{A}_{w_J} \hookrightarrow \mathcal{H}_J^- \cap \mathcal{A}_J \hookrightarrow \mathcal{A}_J$ . More precisely,  $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J, w_J}$  via  $E(\lambda) \mapsto E^J(\lambda)$ . (If  $E(\lambda) \in \mathcal{A}_{w_J}$ , then  $w_J^{-1}(\nu(\lambda))$  is dominant with respect to  $\Delta$ , hence it is also dominant with respect to  $J$ . Therefore  $E^J(\lambda) \in \mathcal{A}_{J, w_J}$ .)

**Lemma 3.14.** *We regard  $\mathcal{A}_{w_J}$  as a subalgebra of  $\mathcal{A}_J$  via the above embedding. Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = \Lambda^+(1)$ . Then  $n_{w_J} M$  is uniquely extended to  $\mathcal{A}_J$ , namely there exists a unique  $\mathcal{A}_J$ -module  $M_J$  such that  $\text{supp } M_J = \Lambda^+(1)_J$  and  $n_{w_J} M_J|_{\mathcal{A}_{w_J}} = n_{w_J} M|_{\mathcal{A}_{w_J}}$ .*

*Proof.* First we prove that  $n_{w_J}M$  is uniquely extended to  $\mathcal{A}_{J,w_J}$ . Take  $\lambda_0 \in \Lambda_S(1)$  such that:

- $\langle v(\lambda_0), \alpha \rangle = 0$  for all  $\alpha \in \Sigma_J^+$ .
- $\langle v(\lambda_0), \alpha \rangle > 0$  for all  $\alpha \in \Sigma^+ \setminus \Sigma_J^+$ .

Note that  $w_J(\Sigma_J^+) = \Sigma_J^-$  and  $w_J(\Sigma^+ \setminus \Sigma_J^+) = \Sigma^+ \setminus \Sigma_J^+$ . Hence we have  $\lambda_0 \in w_J(\Lambda^+(1))$ ,  $E^J(\lambda_0)$  is central in  $\mathcal{A}_{J,w_J}$  (since  $\lambda_0 \in \Lambda_S(1)$  is central in  $\Lambda(1)$ ) and  $E^J(\lambda_0)$  is invertible by the first condition and Remark 2.1. The embedding  $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$  induces  $\mathcal{A}_{w_J}[E(\lambda_0)^{-1}] \hookrightarrow \mathcal{A}_{J,w_J}$ . We prove that this is surjective. Let  $E^J(\mu) \in \mathcal{A}_{J,w_J}$ . Then we have  $\langle w_J(v(\mu)), \alpha \rangle \geq 0$  for any  $\alpha \in \Sigma_J^+$ . Therefore, for sufficiently large  $n \in \mathbb{Z}_{>0}$ , we have  $\lambda_0^n \mu \in w_J(\Lambda^+(1))$ . The elements  $v(\lambda_0)$  and  $v(\mu)$  are in the same closed Weyl chamber  $w_J v(\Lambda^+(1)_J)$  with respect to  $J$ . Hence  $E^J(\lambda_0^n)E^J(\mu) = E^J(\lambda_0^n \mu)$  which is in the image of  $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$ . Therefore  $\mathcal{A}_{w_J}[E(\lambda_0)^{-1}] \hookrightarrow \mathcal{A}_{J,w_J}$  is surjective. Now we get the lemma since  $E(\lambda_0)$  is invertible on  $n_{w_J}M$ . (Recall that  $\text{supp } n_{w_J}M = w_J(\Lambda^+(1))$  and  $\lambda_0 \in w_J(\Lambda^+(1))$ .)

So we have the extension  $N_J$  of  $n_{w_J}M$  to  $\mathcal{A}_{J,w_J}$ . Define the action of  $E^J(\lambda)$  on  $N_J$  by zero for  $\lambda \in \Lambda(1) \setminus w_J(\Lambda^+(1)_J)$ . Then  $N_J$  is an  $\mathcal{A}_J$ -module such that  $\text{supp } N_J = w_J(\Lambda^+(1)_J)$  which is desired. From the definition of the support, this is the only way to extend the module  $N_J$  to  $\mathcal{A}_J$ . We get the lemma.  $\square$

Take  $M_J$  as in the lemma. We have

$$\text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}, n_{w_J}M) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M)) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M_J)).$$

**Lemma 3.15.** *The homomorphisms*

$$\text{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J) \rightarrow \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J, n_{w_J}M_J) \rightarrow \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M_J)$$

*are both isomorphisms.*

*Proof.* The first is an isomorphism by an argument similar to the proof of Lemma 3.12.

Take  $\lambda_0 \in \Lambda(1)$  such that:

- $\lambda_0 \in Z(W_J(1))$ .
- $\langle v(\lambda_0), \alpha \rangle > 0$  for any  $\alpha \in \Sigma^+ \setminus \Sigma_J^+$ .

Then  $\mathcal{H}_J = \mathcal{H}_J^-[E^J(\lambda_0)^{-1}]$  [Abe 2019b, Proposition 2.5]. Since  $E^J(\lambda_0)$  is invertible in  $\mathcal{A}_J$ , it is also invertible on  $n_{w_J}M_J$ . (Note that  $n_{w_J}M_J$  is an  $\mathcal{A}_J$ -module.) Hence the second homomorphism is an isomorphism.  $\square$

Therefore we get

$$\text{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_J}M) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J)).$$

**Lemma 3.16.** *Let  $X$  be an  $\mathcal{H}_J$ -module and assume that  $X \rightarrow X \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$  is injective. Then  $Y \rightarrow Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is also injective for  $Y = \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, X)$ .*

Therefore for the proof of Lemma 3.10, it is sufficient to prove that

$$\mathrm{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J} M_J) \rightarrow \mathrm{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J} M_J) \otimes_{\mathcal{H}_J} \mathrm{c}\text{-Ind}_{I(1)_J}^{L_J} \mathbf{1}$$

is injective, namely we may assume that  $w = w_\Delta$ .

*Proof.* Set  $J' = -w_\Delta(J)$  and put  $n = n_{w_\Delta} n_{w_J}$ . Then  $l \mapsto nln^{-1}$  gives an isomorphism  $L_J \rightarrow L_{J'}$  and sends  $I(1)_J$  to  $I(1)_{J'}$ . Therefore it induces an isomorphism  $\mathcal{H}_J \rightarrow \mathcal{H}_{J'}$ . Define an  $\mathcal{H}_{J'}$ -module  $X'$  as the pull-back of  $X$  by this isomorphism (see [Abe 2019a]). Then  $X \rightarrow X \otimes_{\mathcal{H}_J} \mathrm{c}\text{-Ind}_{I(1)_J}^{L_J} \mathbf{1}$  induces  $X' \rightarrow X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}$  and the latter map is also injective. By [Abe 2019a, Proposition 4.15], we have  $Y \simeq X' \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$ . By [Vignéras 2015, Proposition 4.1], the functor  $(\cdot) \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$  is exact. Hence, using the assumption in the lemma, the map

$$Y \simeq X' \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H} \rightarrow (X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$$

is injective. By [Ollivier and Vignéras 2018, Proposition 4.4]

$$(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H} \simeq (\mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}))^{I(1)}.$$

In particular,

$$(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H} \rightarrow \mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})$$

is injective. Finally, by [Ollivier and Vignéras 2018, Corollary 4.7],

$$\mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}) \simeq Y \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}.$$

Combining all of these, we conclude the lemma.  $\square$

**3F. Some more reductions.** By the definition of  $\mathcal{H}_\varnothing^+$ ,  $\mathcal{H}_\varnothing^-$  and [Abe 2019b, Lemma 2.6], we have:

$$j_\varnothing^+(\mathcal{H}_\varnothing^+) = \mathcal{A}_{w_\Delta} \quad \text{and} \quad j_\varnothing^-(\mathcal{H}_\varnothing^-) = \mathcal{A}_1.$$

See the argument in Section 3E. By these identities, we regard  $\mathcal{A}_1$  and  $\mathcal{A}_{w_\Delta}$  as a subalgebra of  $\mathcal{H}_\varnothing = \mathcal{A}_\varnothing$ .

Let  $M$  be an  $\mathcal{A}$ -module such that  $\mathrm{supp} M = \Lambda^+(1)$ . By Lemma 3.12, we have  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta} M) \simeq \mathrm{Hom}_{\mathcal{A}_{w_\Delta}}(\mathcal{H}, n_{w_\Delta} M)$ . By Lemma 3.14, there exists an  $\mathcal{A}_\varnothing$ -module  $M_\varnothing$  such that  $M|_{\mathcal{A}_1} \simeq M_\varnothing|_{\mathcal{A}_1}$ . It is easy to see that  $n_{w_\Delta} M|_{\mathcal{A}_{w_\Delta}} \simeq n_{w_\Delta} M_\varnothing|_{\mathcal{A}_{w_\Delta}}$ . We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta} M) &\simeq \mathrm{Hom}_{(\mathcal{H}_\varnothing^+, j_\varnothing^+)}(\mathcal{H}, n_{w_\Delta} M_\varnothing) \\ &\simeq \mathrm{Hom}_{(\mathcal{H}_\varnothing^-, j_\varnothing^-)}(\mathcal{H}, M_\varnothing) && [\text{Abe 2019b, Proposition 4.13}] \\ &\simeq M_\varnothing \otimes_{(\mathcal{H}_\varnothing^-, j_\varnothing^-)} \mathcal{H} && [\text{Abe 2019b, Corollary 4.19}] \\ &= M \otimes_{\mathcal{A}_1} \mathcal{H}. && (j_\varnothing^-(\mathcal{H}_\varnothing^-) = \mathcal{A}_1) \end{aligned}$$

By Lemma 3.13, we have  $M \otimes_{\mathcal{A}_1} \mathcal{H} \simeq M \otimes_{\mathcal{A}} \mathcal{H}$ . Hence we get the following lemma:

**Lemma 3.17.** *We have  $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta} M) \simeq M \otimes_{\mathcal{A}} \mathcal{H}$  for any  $\mathcal{A}$ -module  $M$  such that  $\mathrm{supp} M = \Lambda^+(1)$*

Therefore, to prove Lemma 3.10, hence Theorem 3.5, it is sufficient to prove the following.

**Lemma 3.18.** *Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = \Lambda^+(1)$ . Then  $M \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is injective.*

The group algebra  $C[Z_\kappa]$  is a subalgebra of  $\mathcal{A}$  via the map  $t \mapsto T_t = E(t)$  for  $t \in Z_\kappa$ . Let  $\hat{Z}_\kappa$  denote the set of characters of  $Z_\kappa$ . Since the order of  $Z_\kappa$  is prime to  $p$ ,  $M$  is semisimple as a  $C[Z_\kappa]$ -module. Let  $\psi \in Z_\kappa$  and set  $M_\psi = \{m \in M \mid mT_t = \psi(t)m \ (t \in Z_\kappa)\}$ . Since  $Z_\kappa$  is normal in  $\Lambda(1)$ , the conjugate action of  $\Lambda(1)$  on  $Z_\kappa$  induces the action on  $\hat{Z}_\kappa$ . The formula  $E(\lambda)T_t = T_{\lambda t \lambda^{-1}}E(\lambda)$  implies that  $M_\psi E(\lambda) \subset M_{\lambda^{-1}(\psi)}$ . For an orbit  $\omega$  of this action in  $\hat{Z}_\kappa$ , we put  $M_\omega = \bigoplus_{\psi \in \omega} M_\psi$ . Then  $M_\omega$  is stable under the  $\mathcal{A}$ -action and we have  $M = \bigoplus_{\omega} M_\omega$ . Therefore we may assume that  $M = M_\omega$  for some  $\omega$  to prove Lemma 3.18.

Let  $\alpha \in \Delta$  and consider the image of  $Z \cap L'_{\{\alpha\}}$  in  $\Lambda(1)$ . We denote this subgroup by  $\Lambda'_\alpha(1)$ . Consider the following condition:  $\psi$  is trivial on  $Z_\kappa \cap \Lambda'_\alpha(1)$ . Since  $Z_\kappa \cap \Lambda'_\alpha(1)$  is normal in  $\Lambda(1)$ , for  $t \in Z_\kappa \cap \Lambda'_\alpha(1)$  and  $\lambda \in \Lambda(1)$ , we have  $(\lambda\psi)(t) = \psi(\lambda^{-1}t\lambda) = 1$  if  $\psi$  satisfies this condition. Hence this condition only depends on  $\Lambda(1)$ -orbit.

We start to prove Lemma 3.19 by induction on  $\dim(G)$ . Assume that  $\omega$  is a  $\Lambda(1)$ -orbit in  $\hat{Z}_\kappa$ . First we assume that there exists  $\alpha \in \Delta$  such that  $\psi$  is not trivial on  $Z_\kappa \cap \Lambda'_\alpha(1)$  for some (equivalently any)  $\psi \in \omega$ . Then by [Abe 2019a, Theorem 3.13], we have  $M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{s_\alpha} M \otimes_{\mathcal{A}} \mathcal{H}$ .

We prove that in this case the lemma follows from that for a Levi subgroup. The argument is similar to that in Section 3E. Set  $J = \Delta \setminus \{\alpha\}$ . Then we have  $n_{s_\alpha} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}} \mathcal{H}$  by Lemma 3.11. By Lemma 3.13, we have  $n_{w_\Delta w_J} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}$ . As in the argument in Section 3E using [Abe 2019b, Lemma 2.6], we have  $j_J^+(\mathcal{H}_J^+) \supset \mathcal{A}_{w_\Delta w_J}$ . Therefore we have

$$n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H} \simeq (n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J^+) \otimes_{(\mathcal{H}_J^+, j_J^+)} \mathcal{H}.$$

By the same argument of the proof of Lemma 3.14, there exists an  $\mathcal{A}_J$ -module  $M_J$  such that

$$n_{w_\Delta w_J} M_J|_{\mathcal{A}_{w_\Delta w_J}} = n_{w_\Delta w_J} M|_{\mathcal{A}_{w_\Delta w_J}} \quad \text{and} \quad \text{supp } M_J = \Lambda^+(1)_J.$$

By a similar argument of the proof of Lemma 3.15, the homomorphisms

$$n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J^+ \rightarrow n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J \rightarrow n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J$$

are isomorphisms. Now by inductive hypothesis, the homomorphism

$$n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J \rightarrow (n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J) \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$$

is injective. By the argument in the proof of Lemma 3.16, this implies that for  $Y = (n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J) \otimes_{(\mathcal{H}_J^+, j_J^+)} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}$ , the homomorphism

$$Y \rightarrow Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$$

is injective. Hence we get the lemma for  $M$ .

Therefore we may assume that there is no such  $\alpha$ . Hence it is sufficient to prove the following to prove Lemma 3.18.

**Lemma 3.19.** *Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp}(M) = \Lambda^+(1)$  and  $Z_\kappa \cap \Lambda'_\alpha(1)$  acts trivially on  $M$  for all  $\alpha \in \Delta$ . Then  $M \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is injective.*

We prove this lemma in Section 3J.

**3G. Hecke modules.** As discussed in 3D, we have the following

$$M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]} (C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}),$$

We decompose this module along the action of  $Z_\kappa$ .

Set  $C[\Lambda(1)]_\psi = \{f \in C[\Lambda(1)] \mid \tau_t f = \psi(t) f \ (t \in Z_\kappa)\}$  and for a  $\Lambda(1)$ -stable subset  $\omega \subset \hat{Z}_\kappa$  we put  $C[\Lambda(1)]_\omega = \bigoplus_{\psi \in \omega} C[\Lambda(1)]_\psi$ . From the definition, it is obvious that  $C[\Lambda(1)]_\omega$  is invariant under the right action of  $C[\Lambda(1)]$ .

**Lemma 3.20.** *We have  $C[\Lambda(1)]_\omega = \bigoplus_{\psi \in \omega} \{f \in C[\Lambda(1)] \mid f \tau_t = \psi(t) f \ (t \in Z_\kappa)\}$ .*

*Proof.* Let  $\psi \in \omega$ ,  $f \in C[\Lambda(1)]_\psi$  and we write  $f = \sum_{\lambda \in \Lambda(1)} c_\lambda \tau_\lambda$  where  $c_\lambda \in C$ . Set

$$e = \#Z_\kappa^{-1} \sum_{t \in Z_\kappa} \psi(t)^{-1} \tau_t \in C[Z_\kappa].$$

Then  $ef = f$  and  $e\tau_t = \psi(t)e$  for each  $t \in Z_\kappa$ . We have  $e\tau_\lambda \tau_t = e\tau_{\lambda t \lambda^{-1}} \tau_\lambda = (\lambda^{-1}\psi)(t)e\tau_\lambda$ . Since  $\lambda^{-1}\psi \in \omega$ , we get the lemma.  $\square$

Therefore  $C[\Lambda(1)]_\omega$  is a two-sided ideal of  $C[\Lambda(1)]$ . Using  $Z_\kappa$ -action, some objects appearing here are decomposed. Here is a list:

- $C[\Lambda(1)] = C[\Lambda(1)]_\omega \times C[\Lambda(1)]_{\hat{Z}_\kappa \setminus \omega}$  as  $C$ -algebras.
- $\mathcal{A} = \mathcal{A}_\omega \times \mathcal{A}_{\hat{Z}_\kappa \setminus \omega}$  as  $C$ -algebras with the obvious notation.
- The homomorphism (3-1) induces  $\mathcal{A}_\omega \rightarrow C[\Lambda(1)]_\omega$  and  $\mathcal{A}_{\hat{Z}_\kappa \setminus \omega} \rightarrow C[\Lambda(1)]_{\hat{Z}_\kappa \setminus \omega}$ .

Let  $M$  be an  $\mathcal{A}$ -module such that  $\text{supp } M = \Lambda^+(1)$  and  $M = M_\omega$  (see Section 3F). Then as in Section 3D,  $M$  is a  $C[\Lambda(1)]$ -module and this action factors through  $C[\Lambda(1)] \rightarrow C[\Lambda(1)]_\omega$ . Hence we have

$$M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]_\omega} (C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}) \quad (3-3)$$

In [Abe 2019a, Section 3], it is proved that, for any  $w \in W_0$ ,  $1 \otimes 1 \mapsto 1 \otimes T_{n_w \Delta w^{-1}}^*$  gives a  $(C[\Lambda(1)], \mathcal{H})$ -bimodule homomorphism

$$n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \rightarrow n_{w\Delta} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}$$

which is injective [loc. cit., Proposition 3.12]. The homomorphism is compatible with the decomposition  $n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \simeq n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \oplus n_w C[\Lambda(1)]_{\hat{Z}_\kappa \setminus \omega} \otimes_{\mathcal{A}} \mathcal{H}$ . Hence we get the  $(C[\Lambda(1)], \mathcal{H})$ -bimodule homomorphism

$$n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \rightarrow n_{w\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \quad (3-4)$$



which is again injective. By [loc. cit., Theorem 3.13], the image of this homomorphism only depends on  $\Delta_w$ . Let  $X_J$  be the image of this homomorphism where  $J = \Delta_w$ . This is a  $(C[\Lambda(1)]_\omega, \mathcal{A})$ -module. We have  $M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]_\omega} X_\Delta$  by (3-3).

**Lemma 3.21.** *If  $J' \supset J$ , then  $X_{J'} \subset X_J$ .*

*Proof.* Note that  $\Delta_{w_\Delta w_J} = J$ . Hence, by definition,  $X_J$  is a  $(C[\Lambda(1)], \mathcal{H})$ -submodule in  $n_{w_\Delta} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}$  generated by  $1 \otimes T_{n_{w_\Delta w_J w_\Delta}}^*$ . If  $J' \supset J$ , then

$$\ell(w_\Delta w_J w_{J'} w_\Delta) = \ell(w_J w_{J'}) = \ell(w_{J'}) - \ell(w_J) = \ell(w_\Delta w_{J'} w_\Delta) - \ell(w_\Delta w_J w_\Delta).$$

Hence  $T_{n_{w_\Delta w_{J'} w_\Delta}}^* = T_{n_{w_\Delta w_J w_\Delta}}^* T_{n_{w_\Delta w_J w_{J'} w_\Delta}}^*$ . Therefore  $1 \otimes T_{n_{w_\Delta w_{J'} w_\Delta}}^* \in X_J$ . Since  $X_{J'}$  is generated by  $1 \otimes T_{n_{w_\Delta w_{J'} w_\Delta}}^*$ , we have  $X_{J'} \subset X_J$ .  $\square$

**Lemma 3.22.**  $X_J \in \mathcal{C}$ .

*Proof.* Take  $\lambda \in \Lambda_S(1)$  such that  $\nu(\lambda)$  is regular dominant. Then we have  $z_\lambda = \sum_{v \in W_0} E(n_v \lambda n_v^{-1})$  by Lemma 3.2. Let  $f \otimes X \in X_J$ . Then, since  $z_\lambda$  is in the center, we have  $(f \otimes X)z_\lambda = f \otimes z_\lambda X = f \otimes \sum_{v \in W_0} E(n_v \lambda n_v^{-1}) X = f \tau_\lambda \otimes X$  in  $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}$ . Since  $f \mapsto f \tau_\lambda$  is invertible,  $z_\lambda$  is invertible on  $X_w$ .  $\square$

Note that  $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}_{w_\Delta}} \mathcal{H}$  [Abe 2019a, Proposition 3.12]. Hence  $X_\emptyset = n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{(\mathcal{H}_\emptyset^+, j_\emptyset^+)} \mathcal{H}$ . This is a parabolically induced module [Vignéras 2015]. By [loc. cit., Example 3.2, Lemma 3.6], we have  $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}$ . Since  $T_{n_w}^* \in T_{n_w} + \sum_{v < w} C[Z_\kappa] T_{n_v}$ , we have  $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}^*$ .

Set  $Y_w = n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}^* \subset X_\emptyset$ . Then the subspace  $Y_w$  is the image of  $n_w C[\Lambda(1)]_\omega \otimes 1$  by the injective homomorphism (3-4). In particular,  $Y_w$  is  $\mathcal{A}$ -stable and isomorphic to  $n_w C[\Lambda(1)]_\omega$ . We have  $X_\emptyset = \bigoplus_{w \in W_0} Y_w$ . This is the decomposition in Lemma 3.7. By the functoriality of the decomposition, we have  $X_J = \bigoplus_{w \in W_0} (X_J \cap Y_w)$ .

**3H. Representations of  $G$ .** Let  $\omega$  be a  $\Lambda(1)$ -orbit in  $\hat{Z}_\kappa$  such that for any  $\alpha \in \Delta$ ,  $\psi$  is trivial on  $Z_\kappa \cap \Lambda'_\alpha(1)$  for some (equivalently any)  $\psi \in \omega$ . Recall that we have fixed a special parahoric subgroup  $K$ . Irreducible representations  $V$  of  $K$  are parametrized by a pair  $(\psi, J)$  where  $\psi$  is a character of  $Z_\kappa$  and  $J$  a certain subset of  $\Delta$ . Here for  $V$ ,  $\psi$  and  $J$  are given by the following:  $\psi \simeq V^{I(1)}$  and  $W_{0,J} = \text{Stab}_{W_0}(V^{I(1)})$ . Note that by the assumption on  $\omega$ ,  $(\psi, J)$  gives a parameter for any  $\psi \in \omega$  and  $J \subset \Delta$  [Abe et al. 2017, III.8]. Let  $V_{\psi,J}$  be the irreducible representation of  $K$  which corresponds to  $(\psi, J)$  and put  $V_J = \bigoplus_{\psi \in \omega} V_{\psi^{-1},J}$ . In the rest of this paper, we fix a basis of  $V_{\psi^{-1},J}^{I(1)}$  for each  $\psi$  and  $J$ .

**Lemma 3.23.** (1) *The Hecke algebra  $\text{End}_Z(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)})$  is isomorphic to  $C[\Lambda(1)]_\omega$ .*

(2) *We have the Satake homomorphism*

$$\text{End}_G(\text{c-Ind}_K^G V_J) \hookrightarrow \text{End}_Z(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)}) \simeq C[\Lambda(1)]_\omega$$

*and its image is  $C[\Lambda^+(1)]_\omega$ .*

*Proof.* Let  $\mathcal{H}(\psi_1^{-1}, \psi_2^{-1})$  is the space of functions  $\varphi: Z \rightarrow C$  such that  $\text{supp } \varphi$  is compact and  $\varphi(t_1 z t_2) = \psi_1^{-1}(t_1) \varphi(z) \psi_2^{-1}(t_2)$  for any  $z \in Z$  and  $t_1, t_2 \in Z \cap K$ . Since  $V_J^{I(1)} \simeq \bigoplus_{\psi \in \omega} \psi^{-1}$ , a standard argument for Hecke algebras implies

$$\text{End}_Z(\text{Ind}_{Z \cap K}^Z V_J^{I(1)}) \simeq \bigoplus_{\psi_1, \psi_2 \in \omega} \text{Hom}_Z(\text{c-Ind}_{Z \cap K}^Z \psi_1^{-1}, \text{c-Ind}_{Z \cap K}^Z \psi_2^{-1}) \simeq \bigoplus_{\psi_1, \psi_2 \in \omega} \mathcal{H}(\psi_1^{-1}, \psi_2^{-1}).$$

This space is a subalgebra of  $\mathcal{H}_Z$  where  $\mathcal{H}_Z$  is the functions  $\varphi$  on  $Z$  which is invariant under the left (and equivalently right) multiplication by  $Z \cap I(1)$  and whose support is compact. The homomorphism  $\varphi \mapsto \sum_{z \in Z/(Z \cap K)} \varphi(z) \tau_z$  gives an isomorphism  $\mathcal{H}_Z \simeq C[\Lambda(1)]$ . As a subspace of both sides, it is easy to see that we get the desired isomorphism.

The Satake transform

$$\text{Hom}_G(\text{c-Ind}_K^G V_{\psi_1, J}, \text{c-Ind}_K^G V_{\psi_2, J}) \rightarrow \text{Hom}_Z(\text{c-Ind}_{Z \cap K}^Z \psi_1^{-1}, \text{c-Ind}_{Z \cap K}^Z \psi_2^{-1})$$

is defined in [Henniart and Vignéras 2012, 2] and the image is described in [Abe et al. 2018b, Theorem 1.1].  $\square$

**Remark 3.24.** In the identification (1) in the lemma, we need to fix an isomorphism  $V_J^{I(1)} \simeq \bigoplus_{\psi \in \omega} \psi^{-1}$ . We use our fixed basis of  $V_{\psi^{-1}, J}^{I(1)}$  for this isomorphism.

By the lemma,  $C[\Lambda^+(1)]_\omega$  acts on  $\text{c-Ind}_K^G V_J$ . Define a representation  $\pi_J$  of  $G$  by

$$\pi_J = C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} \text{c-Ind}_K^G V_J.$$

We prove  $\pi_J^{I(1)} \simeq X_J$ .

Recall that the  $\mathcal{H}$ -module  $(\text{c-Ind}_K^G V_J)^{I(1)}$  is described as follows. Let  $\mathcal{H}_f$  be the Hecke algebra attached to the pair  $(K, I(1))$ . Then  $V_J^{I(1)}$  is naturally a right  $\mathcal{H}_f$ -module and the algebra  $\mathcal{H}_f$  is a subalgebra of  $\mathcal{H}$  with a basis  $\{T_w \mid w \in W_0(1)\}$  where  $W_0(1)$  is the inverse image of  $W_0 \subset W$  in  $W(1)$ . Then we have  $(\text{c-Ind}_K^G V_J)^{I(1)} \simeq V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  [Vignéras 2017, Proposition 7.2].

**Remark 3.25.** In the argument below, we will use results in [Abe 2017]. In [loc. cit.], we study an  $\mathcal{H}_f$ -module denoted by  $\eta^J = \bigoplus_{\psi \in \hat{Z}_K} V_{\psi, J}^{I(1)}$ . Using a similar argument in [loc. cit.] (or taking a direct summand of results), results are also true for an  $\mathcal{H}_f$ -module  $V_J^{I(1)}$ .

We have an action of  $C[\Lambda^+(1)]_\omega$  on  $V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  [loc. cit., Proposition 3.4] and the above isomorphism  $(\text{c-Ind}_K^G V_J)^{I(1)} \simeq V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  is  $C[\Lambda^+(1)]_\omega$ -equivariant. (This can be proved by the same argument in the proof of [loc. cit., Proposition 5.1].)

**Lemma 3.26.** *Let  $A$  be a ring and  $S \subset A$  be a multiplicative subset of the center of  $A$ . Then for a smooth  $A[G]$ -module  $\pi$ , we have  $(S^{-1}\pi)^{I(1)} \simeq S^{-1}\pi^{I(1)}$ .*

*Proof.* Both sides can be regarded as a subspace of  $S^{-1}\pi$ . Any element in  $S^{-1}\pi^{I(1)}$  is  $I(1)$ -invariant, hence  $S^{-1}\pi^{I(1)} \subset (S^{-1}\pi)^{I(1)}$ . Let  $v/s \in (S^{-1}\pi)^{I(1)}$  where  $v \in \pi$  and  $s \in S$ . Let  $g_1, \dots, g_n$  be a representatives of  $I(1)/\text{Stab}_{I(1)}(v)$ . Since  $v/s$  is  $g_i$ -invariant, there exists  $s_i \in S$  such that  $s_i(g_i v - v) = 0$ . Therefore  $s_1 \cdots s_n(g_i v - v) = 0$ . Set  $v' = s_1 \cdots s_n v$ . Then for any  $g \in I(1)$  there exists  $i$  and  $g' \in \text{Stab}_{I(1)}(v)$

such that  $g = g_i g'$ . Hence  $g v' = s_1 \cdots s_n (g_i g' v) = s_1 \cdots s_n v = v'$ . Therefore  $v' \in \pi^{I(1)}$ . Hence  $v/s = v'/(s s_1 \cdots s_n) \in S^{-1} \pi^{I(1)}$ .  $\square$

Therefore we have

$$\pi_J^{I(1)} \simeq C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}.$$

By [Abe 2017, Proposition 3.9], we have

$$V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \simeq \text{Im}(n_{w_\Delta w_J} C[\Lambda^+(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \rightarrow n_{w_\Delta w_J} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}).$$

Hence we have an isomorphism  $C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \simeq X_J$ . Therefore  $\pi_J^{I(1)} \simeq X_J$ .

We get an embedding  $X_J \simeq \pi_J^{I(1)} \hookrightarrow \pi_J$ . Hence there exists a homomorphism  $X_J \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow \pi_J$ . Let  $J = \Delta$  and applying  $M \otimes_{C[\Lambda(1)]}$  to

$$X_\Delta \rightarrow X_\Delta \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow \pi_\Delta$$

and using  $M \otimes_{C[\Lambda(1)]} X_\Delta \simeq M \otimes_{\mathcal{A}} \mathcal{H}$  (3-3), we get

$$M \otimes_{C[\Lambda(1)]} X_\Delta \simeq M \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow M \otimes_{C[\Lambda(1)]} \pi_\Delta.$$

Hence for Lemma 3.19, it is sufficient to prove that  $M \otimes_{C[\Lambda(1)]} X_\Delta \rightarrow M \otimes_{C[\Lambda(1)]} \pi_\Delta$  is injective.

We have an isomorphism  $\pi_\emptyset \simeq \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)})$  [Henniart and Vignéras 2012, Theorem 1.2]. (To be precisely, the direct sum of a result in [loc. cit., Theorem 1.2].) An injective embedding  $\pi_J \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)}) \simeq \pi_\emptyset$  was given in [loc. cit., Definition 2.1]. Hence we have a diagram of  $(C[\Lambda(1)], \mathcal{H})$ -bimodules

$$\begin{array}{ccc} X_J & \longrightarrow & X_\emptyset \\ \downarrow & & \downarrow \\ \pi_J & \longrightarrow & \pi_\emptyset. \end{array}$$

When  $J = \emptyset$ ,  $X_J \rightarrow X_\emptyset$  and  $\pi_J \rightarrow \pi_\emptyset$  are both identities. Hence this diagram is commutative.

**Lemma 3.27.** *This diagram is commutative for any  $J$ .*

*Proof.* Fix  $\psi^{-1} \in \omega$ . It is sufficient to prove that the following diagram is commutative:

$$\begin{array}{ccc} V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} & \longrightarrow & n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \\ \downarrow & & \downarrow \\ (\text{c-Ind}_K^G V_{\psi, J})^{I(1)} & \longrightarrow & \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}. \end{array} \quad (3-5)$$

Note that this diagram is commutative when  $J = \emptyset$ .

Let  $v_0 \in V_{\psi, J}^{I(1)}$  be our fixed basis. Define  $\varphi_J \in (\text{c-Ind}_K^G V_{\psi, J})^{I(1)}$  by  $\text{supp } \varphi_J = K$  and  $\varphi_J(1) = v_0$ . Then the  $\mathcal{H}$ -module map  $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow \text{c-Ind}_K^G(V_{\psi, J})$  is given by  $v_0 \otimes 1 \mapsto \varphi_J$ . Define  $f_0 \in \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$  by  $\text{supp } f_0 = \bar{B} n_{w_\Delta} I(1)$ ,  $\text{supp } f_0(n_{w_\Delta}^{-1}) = Z \cap K$  and  $f_0(n_{w_\Delta}^{-1})(1) = v_0$ . Then

the function corresponding to  $\varphi_\emptyset$  under  $\text{c-Ind}_K^G V_\emptyset \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_\emptyset^{I(1)})$  is  $f_0 T_{n_{w_\Delta}}$  [Abe et al. 2017, IV.9 Proposition].

Set  $w = w_\Delta w_J$ . Then  $X_J = n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}$ . The homomorphism  $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow n_w C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$  is given by  $v_0 \otimes 1 \mapsto 1 \otimes T_{n_w}$  [Abe 2017, Lemmas 3.8 and 3.10].

Consider the case of  $J = \emptyset$ . Then the image of  $v_0 \otimes 1 \in V_{\psi, \emptyset}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  under

$$V_{\psi, \emptyset}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow (\text{c-Ind}_K^G V_{\psi, \emptyset})^{I(1)} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$$

is

$$v_0 \otimes 1 \mapsto \varphi_\emptyset \mapsto f_0 T_{n_{w_\Delta}}.$$

On the other hand, the image of  $v_0 \otimes 1 \in V_{\psi, \emptyset}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  in  $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$  is  $1 \otimes T_{n_{w_\Delta}}$ . As remarked before the lemma, (3-5) is commutative when  $J = \emptyset$ . Hence the homomorphism  $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$  sends  $1 \otimes T_{n_{w_\Delta}}$  to  $f_0 T_{n_{w_\Delta}}$ . Take  $\lambda$  from the center of  $\Lambda(1)$  such that  $\langle \alpha, \nu(\lambda) \rangle < 0$  for any  $\alpha \in \Sigma^+$ . Then by [Abe 2019b, Lemma 2.17],  $\ell(n_{w_\Delta} \lambda) = \ell(\lambda) - \ell(n_{w_\Delta})$ . Hence by [Vignéras 2016, Theorem 5.25, Example 5.32],  $T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1} \lambda) = E(\lambda)$ . Therefore  $1 \otimes T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1} \lambda) = 1 \otimes E(\lambda) = \tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} \otimes 1$ . On the other hand,  $f_0 T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1} \lambda) = f_0 E(\lambda) = \tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} f_0$  by [Abe et al. 2017, IV.10 Proposition]. Hence the homomorphism  $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$  sends  $\tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} \otimes 1$  to  $\tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} f_0$ . Therefore  $1 \otimes 1$  sends to  $f_0$ .

Let  $a = v_0 \otimes 1 \in V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  and we consider the image of  $a$  in  $\text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$  in the two ways. The image of  $a$  in  $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$  is  $1 \otimes T_{n_{w_\Delta} w^{-1}}^* T_{n_w}$  by [Abe 2017, Proposition 3.11] and the definition of  $X_J \rightarrow X_\emptyset$ . Therefore the image of  $a$  under  $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$  is  $f_0 T_{n_{w_\Delta} w^{-1}}^* T_{n_w}$ .

By [Abe et al. 2017, IV.9 Proposition] (for  $J = \Delta$ ), we have  $f_0 T_{n_{w_\Delta} w^{-1}}^* = \sum_{v \leq w_\Delta w^{-1}} f_0 T_{n_v}$ . Since  $w_\Delta w^{-1} = w_\Delta w_J w_\Delta$ ,  $\{v \in W_0 \mid v \leq w_\Delta w^{-1}\} = w_\Delta W_{0, J} w_\Delta$ . Hence

$$f_0 T_{n_{w_\Delta} w^{-1}}^* T_{n_w} = \sum_{v \in W_{J, 0}} f_0 T_{n_{w_\Delta v w_\Delta}} T_{n_{w_\Delta w_J}}.$$

We have

$$\ell(w_\Delta v w_\Delta \cdot w_\Delta w_J) = \ell(w_\Delta v w_J) = \ell(w_\Delta) - \ell(v w_J) = \ell(w_\Delta) - \ell(w_J) + \ell(v) = \ell(w_\Delta w_J) + \ell(w_\Delta v w_\Delta).$$

Hence  $T_{n_{w_\Delta v w_\Delta}} T_{n_{w_\Delta w_J}} = T_{n_{w_\Delta v w_J}}$ . Therefore, replacing  $v$  with  $v w_J$ , we get  $f_0 T_{n_{w_\Delta} w^{-1}}^* T_{n_w} = \sum_{v \in W_{J, 0}} f_0 T_{n_{w_\Delta v}}$ . This is the image of  $\varphi_J$  in  $\text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})$  by [Abe et al. 2017, IV.7 Corollary]. Hence the diagram (3-5) is commutative if we start with  $a$ . Since the element  $a$  generates  $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$  as an  $\mathcal{H}$ -module, the diagram (3-5) is commutative.  $\square$

Therefore we may regard  $\pi_J$  and  $X_J$  as a subspace of  $\pi_\emptyset$ . We have  $\pi_\emptyset \simeq \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)})$ . By the same argument in the proof of Lemma 3.23, we have  $\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)} \simeq C[\Lambda(1)]_\omega$ . Here again we use our fixed basis. Hence we have  $\pi_\emptyset \simeq \text{Ind}_B^G C[\Lambda(1)]_\omega$ . We identify  $\pi_J$  with the image in  $\text{Ind}_B^G C[\Lambda(1)]_\omega$ .

**Remark 3.28.** By [Abe et al. 2017, IV.7 Proposition] and the decomposition  $G = \bigcup_{w \in W_0} \bar{B}n_w I(1)$  implies that  $(\text{Ind}_B^G C[\Lambda(1)]_\omega)^{I(1)} = \bigoplus_{w \in W_0} C[\Lambda(1)]_\omega f_0 T_{n_w}$ . Since  $X_\emptyset = \bigoplus_{w \in W_0} C[\Lambda(1)]_\omega \otimes T_{n_w}$  (see after the proof of Lemma 3.22) and  $X_\emptyset \rightarrow \pi_\emptyset$  sends  $1 \otimes 1$  to  $f_0$  (see the proof of the previous lemma), we have  $X_\emptyset \simeq \pi_\emptyset^{I(1)}$ . Note that  $\text{supp } f_0 T_{n_w} = \bar{B}n_{w_\Delta} I(1)$  [Abe et al. 2017, IV.7 Proposition].

**3I. Filtrations.** As in the previous subsection, let  $\omega$  be a  $\Lambda(1)$ -orbit in  $\hat{Z}_\kappa$  such that, for some (equivalently any)  $\psi \in \omega$ ,  $\psi$  is trivial on  $Z_\kappa \cap \Lambda'_\alpha(1)$  for any  $\alpha \in \Delta$ . In this subsection, we use the following notation: for  $A \subset W_0$ ,  $\bar{B}A\bar{B} = \bigcup_{v \in A} \bar{B}v\bar{B}$ .

For a subset  $A \subset W_0$  which is open (namely, if  $v_1 \in W_0$ ,  $v_2 \in A$  and  $v_1 \geq v_2$  then  $v_1 \in A$ ), we put

$$\pi_{\emptyset, A} = \{f \in \text{Ind}_B^G C[\Lambda(1)]_\psi \mid \text{supp } f \subset \bar{B}A\bar{B}\}.$$

We also put

$$X_{\emptyset, A} = \bigoplus_{v \in A} n_{w_\Delta} C[\Lambda(1)] \otimes T_{n_{w_\Delta} v}.$$

**Lemma 3.29.** *Let  $h \in X_\emptyset$ . Then  $h \in X_{\emptyset, A}$  if and only if its image in  $\pi_\emptyset$  is in  $\pi_{\emptyset, A}$ . Namely we have  $X_{\emptyset, A} = X_\emptyset \cap \pi_{\emptyset, A}$ .*

*Proof.* Let  $H \in \pi_\emptyset$  be the image of  $h$ . By the description of  $X_\emptyset \rightarrow \pi_\emptyset$  (see Remark 3.28),  $h \in X_{\emptyset, A}$  if and only if  $\text{supp } H \subset \bar{B}A\bar{B}$ . For each  $v \in A$ , we have

$$\bar{B}vI(1) = \bar{B}v(I(1) \cap v^{-1}\bar{B}v)(I(1) \cap v^{-1}Bv) = \bar{B}v(I(1) \cap v^{-1}Bv) \subset \bar{B}Bv \subset \bigcup_{v' \geq v} \bar{B}v'\bar{B} \subset \bar{B}A\bar{B}.$$

Here we use [Abe 2012, Lemma 2.4]. Hence if  $h \in X_{\emptyset, A}$  then  $H \in \pi_{\emptyset, A}$ .

Assume that  $H \in \pi_{\emptyset, A}$  and  $\text{supp}(H) \cap \bar{B}vI(1) \neq \emptyset$  for  $v \in W_0$ . Since  $H$  is  $I(1)$ -invariant, we have  $H(v) \neq 0$ . Therefore  $v \in A$ . Hence  $\text{supp}(H) \subset \bigcup_{v \in A} \bar{B}vI(1)$ . We get  $h \in X_{\emptyset, A}$ .  $\square$

Set  $X_{J, A} = X_J \cap X_{\emptyset, A}$  and  $\pi_{J, A} = \pi_J \cap \pi_{\emptyset, A}$ . Let  $w \in A$  be a minimal element and put  $A' = A \setminus \{w\}$ . Then we have an embedding

$$X_{\Delta, A}/X_{\Delta, A'} \hookrightarrow \pi_{\Delta, A}/\pi_{\Delta, A'}.$$

For each  $\alpha \in \Delta$ , take a lift  $a_\alpha \in \Lambda'_\alpha(1)$  of a generator of  $\Lambda'_\alpha(1)/(Z_\kappa \cap \Lambda'_\alpha(1))$  such that  $\langle v(a_\alpha), \alpha \rangle > 0$  [Abe et al. 2017, III.4].

The element  $\#Z_\kappa^{-1} \sum_{\psi \in \omega} \sum_{t \in Z_\kappa} \psi(t)^{-1} \tau_{a_\alpha t}$  is in  $C[\Lambda(1)]_\omega$  and does not depend on a choice of a lift (recall that  $\psi$  is trivial on  $Z_\kappa \cap \Lambda'_\alpha(1)$ ). We denote it by  $\tau_\alpha$ . Set  $c_w = \prod_{w^{-1}(\alpha) > 0} (1 - \tau_\alpha)$ . Then as in [Abe et al. 2017, V.8 Proposition], we have

$$\pi_{\Delta, A}/\pi_{\Delta, A'} = c_w(\pi_{\emptyset, A}/\pi_{\emptyset, A'}). \quad (3-6)$$

The space  $\pi_{\emptyset, A}/\pi_{\emptyset, A'}$  can be identified with the space of compactly supported functions on  $\bar{B} \setminus \bar{B}w\bar{B}$  with values in  $C[\Lambda(1)]_\omega$ , which is isomorphic to  $C_c^\infty(\bar{B} \setminus \bar{B}w\bar{B}) \otimes_C C[\Lambda(1)]_\omega$  where  $C_c^\infty(\bar{B} \setminus \bar{B}w\bar{B})$  is the space of locally constant compact support functions on  $\bar{B} \setminus \bar{B}w\bar{B}$  with values in  $C$ . Hence it is free as  $C[\Lambda(1)]_\omega$ -module. By the following lemma and (3-6),  $\pi_{\Delta, A}/\pi_{\Delta, A'}$  is also free.

**Lemma 3.30.** *The element  $c_w \in C[\Lambda(1)]_\omega$  is not a zero divisor.*

*Proof.* The same proof in [Abe 2019a, Lemma 3.10] can apply.  $\square$

**Lemma 3.31.** *We have  $X_{\Delta,A}/X_{\Delta,A'} = c_w(X_{\emptyset,A}/X_{\emptyset,A'})$ .*

*Proof.* Since  $X_{\Delta,A} = \pi_{\Delta,A} \cap X_{\emptyset,A}$ , we have

$$X_{\Delta,A}/X_{\Delta,A'} = \pi_{\Delta,A}/\pi_{\Delta,A'} \cap X_{\emptyset,A}/X_{\emptyset,A'}$$

and the right-hand side is

$$c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) \cap X_{\emptyset,A}/X_{\emptyset,A'}.$$

Let  $H$  be in this set. Since  $\pi_{\emptyset,A}/\pi_{\emptyset,A'}$  is a free  $C[\Lambda(1)]_\omega$ -module, the exact sequence  $0 \rightarrow \pi_{\emptyset,A'} \rightarrow \pi_{\emptyset,A} \rightarrow \pi_{\emptyset,A}/\pi_{\emptyset,A'} \rightarrow 0$  splits. Hence  $\pi_{\emptyset} \simeq \pi_{\emptyset,A'} \oplus (\pi_{\emptyset,A}/\pi_{\emptyset,A'})$ . Therefore  $c_w\pi_{\emptyset,A} \simeq c_w\pi_{\emptyset,A'} \oplus c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'})$ . Hence  $c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) \simeq (c_w\pi_{\emptyset,A})/(c_w\pi_{\emptyset,A'})$ . Hence there exists  $H' \in \pi_{\emptyset,A}$  such that  $H$  is the image of  $c_w H'$ . Since  $H \in X_{\Delta,A}/X_{\Delta,A'}$ , there exists  $h \in X_{\Delta,A}$  such that  $c_w H' - h$  is zero in  $X_{\Delta,A}/X_{\Delta,A'}$ . In particular it is zero in  $c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) = (c_w\pi_{\emptyset,A})/(c_w\pi_{\emptyset,A'})$ . Therefore there exists  $H'' \in \pi_{\emptyset,A'}$  such that  $c_w H' - h = c_w H''$ . Replacing  $H'$  with  $H' - H''$ , we may assume  $c_w H' \in X_{\emptyset,A}$ . Recall that  $H'$  is a function with values in  $C[\Lambda(1)]_\omega$ . Since the element  $c_w$  is not a zero divisor in  $C[\Lambda(1)]_\omega$ ,  $c_w H' \in \pi_{\emptyset,A}$  implies  $H' \in \pi_{\emptyset,A}$ . Since  $c_w H' \in X_{\emptyset}$ ,  $c_w H'$  is  $I(1)$ -invariant. Hence  $H'$  is also  $I(1)$ -invariant, again since  $c_w$  is not a zero divisor. Therefore  $H' \in \pi_{\emptyset}^{I(1)} = X_{\emptyset}$ . Hence  $H' \in X_{\emptyset} \cap \pi_{\emptyset,A} = X_{\emptyset,A}$ . Therefore  $H \in c_w(X_{\emptyset,A}/X_{\emptyset,A'})$ . The reverse inclusion  $c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) \cap X_{\emptyset,A}/X_{\emptyset,A'} \supset c_w(X_{\emptyset,A}/X_{\emptyset,A'})$  is obvious. We get the lemma.  $\square$

**3J. Proof of Lemma 3.19.** Let  $A, A', w$  be as in the previous subsection.

**Lemma 3.32.** *The exact sequences of  $C[\Lambda(1)]_\omega$ -modules*

$$0 \rightarrow \pi_{\Delta,A'} \rightarrow \pi_{\Delta,A} \rightarrow \pi_{\Delta,A}/\pi_{\Delta,A'} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow X_{\Delta,A'} \rightarrow X_{\Delta,A} \rightarrow X_{\Delta,A}/X_{\Delta,A'} \rightarrow 0$$

*split.*

*Proof.* By (3-6) and from the fact that  $\pi_{\emptyset,A}/\pi_{\emptyset,A'}$  is free,  $\pi_{\Delta,A}/\pi_{\Delta,A'}$  is also free. Hence the first exact sequence splits. Using Lemma 3.31, the same argument can apply for the second sequence.  $\square$

**Lemma 3.33.** *The inclusion  $X_{\Delta,A}/X_{\Delta,A'} \hookrightarrow \pi_{\Delta,A}/\pi_{\Delta,A'}$  has a section as  $C[\Lambda(1)]_\omega$ -modules.*

*Proof.* First we construct a section of  $X_{\emptyset,A}/X_{\emptyset,A'} \rightarrow \pi_{\emptyset,A}/\pi_{\emptyset,A'}$ . Recall that  $X_{\emptyset,A} = \pi_{\emptyset,A}^{I(1)}$ . Note that  $X_{\emptyset,A}/X_{\emptyset,A'} \simeq C[\Lambda(1)]_\omega$  and the section is given by  $f \mapsto f(w)$ . For  $H \in \pi_{\emptyset,A}$ , consider  $H' \in \pi_{\emptyset,A}$  which is  $I(1)$ -invariant,  $\text{supp}(H') = \bar{B}vI(1)$  and  $H'(v) = H(v)$ . Then  $H \mapsto H'$  gives a section of  $X_{\emptyset,A}/X_{\emptyset,A'} \rightarrow \pi_{\emptyset,A}/\pi_{\emptyset,A'}$ . Multiplying  $c_w$  and using (3-6), Lemma 3.31, we get a section of the  $C[\Lambda(1)]_\omega$ -module homomorphism  $X_{\Delta,A}/X_{\Delta,A'} \rightarrow \pi_{\Delta,A}/\pi_{\Delta,A'}$ .  $\square$

*Proof of Lemma 3.19.* Set  $\pi_A^M = M \otimes_{C[\Lambda(1)]_\omega} \pi_{\Delta,A}$  and  $X_A^M = M \otimes_{C[\Lambda(1)]_\omega} X_{\Delta,A}$ . Then by Lemma 3.32,  $\pi_{A'}^M$  and  $X_{A'}^M$  are subspaces of  $\pi_A^M$  and  $X_A^M$ , respectively. By Lemma 3.33,  $X_A^M/X_{A'}^M \rightarrow \pi_A^M/\pi_{A'}^M$  is injective.

We prove that  $X_A^M \rightarrow \pi_A^M$  is injective by induction on  $\#A$ . We have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_{A'}^M & \longrightarrow & X_A^M & \longrightarrow & X_A^M / X_{A'}^M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_{A'}^M & \longrightarrow & \pi_A^M & \longrightarrow & \pi_A^M / \pi_{A'}^M \longrightarrow 0. \end{array}$$

The homomorphism  $X_{A'}^M \rightarrow \pi_{A'}^M$  is injective by inductive hypothesis and  $X_A^M / X_{A'}^M \rightarrow \pi_A^M / \pi_{A'}^M$  is injective as we have seen. Hence  $X_A^M \rightarrow \pi_A^M$  is injective. Setting  $A = W_0$ , we get the lemma.  $\square$

#### 4. Theorem

Let  $\mathcal{C}_f$  be the full subcategory of  $\mathcal{C}$  consisting of finite-dimensional modules. Note that this category is closed under submodules, quotients and extensions.

**Theorem 4.1.** *Let  $M \in \mathcal{C}_f$ . Then  $(M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \simeq M$ .*

*Proof.* The theorem is true for simple  $M$  by [Abe 2019a, main theorem; Abe et al. 2018a, Theorem 4.17 and Theorem 5.11]. We prove the theorem by induction on  $\dim(M)$ .

Assume that  $M$  is not simple and let  $M'$  be a proper nonzero submodule of  $M$ . Let

$$\pi = \text{Ker}(M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}).$$

By Theorem 3.5,  $M \rightarrow (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$  is injective. Then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^{I(1)} & \longrightarrow & (M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \longrightarrow & (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \\ & & & & \uparrow & & \uparrow \\ & & & & M' & \hookrightarrow & M. \end{array}$$

Hence  $\pi^{I(1)} = 0$ . Since  $I(1)$  is a pro- $p$  group,  $\pi = 0$ . Hence  $M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$  is injective. Set  $M'' = M/M'$ . Then we have a commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ (M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M' \\ \downarrow & & \downarrow \\ (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\quad} & M \\ \downarrow & & \downarrow \\ (M'' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M'' \\ & & \downarrow \\ & & 0 \end{array}$$

with exact columns. Therefore  $M \rightarrow (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$  is isomorphic.  $\square$

**Corollary 4.2.** *Let  $\mathcal{C}_{G,f}$  be the category of representations of  $G$  consisting of the following objects:*

- *Has a finite length.*
- *Any irreducible subquotient is a subquotient of  $\text{Ind}_B^G \sigma$  for a irreducible representation  $\sigma$  of  $Z$ .*
- *Is generated by  $I(1)$ -invariants.*

*Then  $\mathcal{C}_f \simeq \mathcal{C}_{G,f}$ . The equivalence is given by  $\pi \rightarrow \pi^{I(1)}$  and  $M \mapsto M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G M$ .*

*Proof.* By the classification theorem in [Abe et al. 2017] and [Abe et al. 2018a, Theorem 5.11], if  $\pi \in \mathcal{C}_{G,f}$  is irreducible, then  $\pi^{I(1)} \in \mathcal{C}_f$ . Hence, by induction on the length, if  $\pi \in \mathcal{C}_{G,f}$  then  $\pi^{I(1)} \in \mathcal{C}_f$ .

Let  $\pi \in \mathcal{C}_{G,f}$  and we prove that  $\pi^{I(1)} \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow \pi$  is an isomorphism. The homomorphism is surjective since  $\pi$  is generated by  $\pi^{I(1)}$ . Let  $\pi'$  be the kernel. Then we have an exact sequence

$$0 \rightarrow (\pi')^{I(1)} \rightarrow (\pi^{I(1)} \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \rightarrow \pi^{I(1)}$$

and the last map is isomorphism by the theorem. Hence  $(\pi')^{I(1)} = 0$  and it implies  $\pi' = 0$ . Therefore the homomorphism is also injective. Combining with the previous theorem, we have proved the desired equivalence of categories.  $\square$

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