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We give an equivalence of categories between certain subcategories of modules of pro- p Iwahori–Hecke algebras and modulo p representations.

1. Introduction

Let G be a connected reductive p -adic group and K a compact open subgroup of G . Then one can attach the Hecke algebra \mathcal{H} to this pair (G, K) and we have a functor $\pi \mapsto \pi^K = \{v \in \pi \mid \pi(k)v = v \ (k \in K)\}$ from the category of smooth representations of G to the category of \mathcal{H} -modules. These algebras and functors are powerful tools to study the representation theory of G . In a classical case, namely for smooth representations over the field of complex numbers, this functor gives a bijection between the set of isomorphism classes of irreducible smooth representations of G such that $\pi^K \neq 0$ and the set of isomorphism classes of simple \mathcal{H} -modules. Moreover, the famous theorem of Borel [1976] says that the functor gives an equivalence of categories between the category of smooth representations π of G which is generated by π^K and the category of \mathcal{H} -modules when K is an Iwahori subgroup.

In this paper, we study modulo p representation theory of G . In this case, it is natural to consider a pro- p Iwahori subgroup $I(1)$ which is the pro- p radical of an Iwahori subgroup since any nonzero modulo p representation has a nonzero vector fixed by the pro- p Iwahori subgroup. The corresponding Hecke algebra is called a pro- p Iwahori–Hecke algebra. The aim of this paper is to give a relation between \mathcal{H} -modules and modulo p representations.

Such a relation was first discovered by Vignéras [2007] when $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Based on a classification result due to Barthel and Livné [1995; 1994] and Breuil [2003], she proved that the functor $\pi \mapsto \pi^{I(1)}$ gives a bijection between simple objects. This was enhanced to the level of categories by Ollivier [2009]. Namely she proved that the category of \mathcal{H} -modules is equivalent to the category of modulo p representations of G which are generated by $\pi^{I(1)}$. The quasiinverse of this equivalence is given by $M \mapsto M \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ where $\mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ is the compact induction from the trivial representation of $I(1)$.

However, Ollivier also showed that we cannot expect such correspondence in general. When $G = \mathrm{GL}_2(F)$ where F is a p -adic field such that the number of the residue field is greater than p , for a *supersingular* simple module M (we do not recall the definition of supersingular modules since we do not

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use it in this paper), Ollivier showed that $(M \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$ is not finite-dimensional. Since simple modules of \mathcal{H} are finite-dimensional, it says that we have no equivalence of categories in this case.

Still we can expect that there is such a correspondence if we *avoid* supersingular representations/modules. It was proved by Ollivier and Schneider [2018, Theorem 3.33] that this expectation is true when $G = \mathrm{SL}_2(F)$ when $p \neq 2$ or $F \neq \mathbb{Q}_2$. The aim of this paper is to extend this for any G . We remark that our result is not a generalization of their result since we assume that modules have finite-length which they do not assume.

Let G be a (general) connected reductive p -adic group. In this case, as a consequence of classification theorems [Abe et al. 2017; Abe 2019a] and the calculation of the invariant part of irreducible representations [Abe et al. 2018a], the functor $\pi \mapsto \pi^{I(1)}$ gives a bijection between irreducible modulo p representations of G and simple \mathcal{H} -modules which are *far from supersingular representations/modules*. The aim of this paper is to generalize this correspondence to the level of categories. More precisely, we prove the equivalence of the following two categories:

- The category of \mathcal{H} -modules M such that $\dim(M) < \infty$ and a certain element of the center of \mathcal{H} is invertible on M (see Definition 3.1).
- The category of modulo p representations π of G such that:
 - π is generated by $\pi^{I(1)}$.
 - π has a finite length.
 - Any irreducible subquotient of π is isomorphic to a subquotient of $\mathrm{Ind}_B^G \sigma$ where B is a minimal parabolic subgroup and σ is an irreducible representation of the Levi quotient of B .

Note that an \mathcal{H} -module M is supersingular if and only if certain elements in the center of \mathcal{H} act by zero and a modulo p irreducible admissible representation π of G is supersingular if and only if it is supercuspidal, namely it does not appear as a subquotient of a parabolically induced representation from an irreducible admissible representation of a proper Levi subgroup. Therefore some conditions as above says that M (resp. π) is *far from supersingular modules (resp. representations)*.

We give an outline of the proof. Since the correspondence is true for irreducible representations, by induction on the length, it is sufficient to prove the following (Theorem 3.5): Let M be an \mathcal{H} -module which we are considering. Then $M \rightarrow M \otimes_{\mathcal{H}} \mathrm{Ind}_{I(1)}^G \mathbf{1}$ is injective. This theorem is proved in Section 3. In fact, we prove the injectivity for any $M \in \mathcal{C}$ where the category \mathcal{C} is introduced in Section 3. Here are some reductions:

- Let \mathcal{A} be the Bernstein subalgebra introduced in [Vignéras 2016]. Since we have an embedding $M \hookrightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M)$, it is sufficient to prove the theorem for $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M)$. Note that we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) \in \mathcal{C}$ if $M \in \mathcal{C}$.
- We have a decomposition of $M|_{\mathcal{A}}$ along the *support* (Definition 3.8). We may assume that the support of $M|_{\mathcal{A}}$ is contained in a Weyl chamber.

- Using a result in [Abe 2019a], parabolic inductions and a result of Ollivier and Vignéras [2018], we may assume that the support is the dominant Weyl chamber.
- We prove there exists an \mathcal{A} -module M' such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) \simeq M' \otimes_{\mathcal{A}} \mathcal{H}$. Hence it is sufficient to prove that $M' \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M' \otimes_{\mathcal{A}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ is injective.

By a result in [Abe 2017], both $M \otimes_{\mathcal{A}} \mathcal{H}$ and $M \otimes_{\mathcal{A}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ relate to $\mathrm{c}\text{-Ind}_K^G V$ where K is a special parahoric subgroup and V a certain representation of K . The structure of this representations is studied in [Abe et al. 2017] and using such result we prove the injectivity.

It is almost immediate to prove our main theorem from the above injectivity. This is done in Section 4.

2. Notation and preliminaries

Let F be a nonarchimedean local field of residue characteristic p and G a connected reductive group over F . Let C be an algebraically closed field of characteristic p . This is the coefficient field of representations in this paper. *All representations in this paper are smooth representations over C .*

In general, for any algebraic group H over F , we denote the group of valued points $H(F)$ by the same letter H . Fix a maximal split torus S of G and minimal parabolic subgroup B containing S . The centralizer Z of S in G is a Levi subgroup of B . We denote the unipotent radical of B by U and the opposite of B containing Z by \bar{B} . The unipotent radical of \bar{B} is denoted by \bar{U} .

Consider the reduced apartment corresponding to S and take an alcove A_0 and a special point \mathbf{x}_0 from the closure of A_0 . Let K be the special parahoric subgroup corresponding to \mathbf{x}_0 and I the Iwahori subgroup determined by A_0 . Let $I(1)$ be the pro- p Iwahori subgroup attached to A_0 , namely the pro- p radical of I . The space of C -valued compactly supported $I(1)$ -biinvariant functions \mathcal{H} has a structure of a C -algebra via the convolution product. The algebra \mathcal{H} is called pro- p Iwahori–Hecke algebra. The structure of this algebra is studied by Vignéras [2016].

Let $N_G(S)$ be the normalizer of S in G and put $W_0 = N_G(S)/Z$, $W = N_G(S)/(Z \cap K)$ and $W(1) = N_G(S)/(Z \cap I(1))$. Let G' be the subgroup of G generated by U and \bar{U} . Note that this is not a group of the valued points of an algebraic group in general. Let W_{aff} be the image of $G' \cap N_G(S)$ in W . The action of W_{aff} on the apartment is faithful and therefore it is a subgroup of the group of affine transformations of the apartment. Let S_{aff} be the set of reflections along the walls of A_0 . Then $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system. Denote its length function by ℓ . Let $N_W(A_0)$ be the stabilizer of A_0 in W . Then the group W is the semidirect product of W_{aff} and $N_W(A_0)$. The function ℓ is extended to W , trivially on $N_W(A_0)$. We also inflate ℓ to $W(1)$ via $W(1) \rightarrow W$. We have the Bruhat order on $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ and we extend it to W by $w_1\omega_1 < w_2\omega_2$ if and only if $w_1 < w_2$ and $\omega_1 = \omega_2$ where $w_1, w_2 \in W_{\mathrm{aff}}$ and $\omega_1, \omega_2 \in N_W(A_0)$. For $w_1, w_2 \in W(1)$, we say $w_1 < w_2$ if $\bar{w}_1 < \bar{w}_2$ where \bar{w}_i is the image of w_i in W ($i = 1, 2$). As usual we say $w_1 \leq w_2$ if and only if $w_1 < w_2$ or $w_1 = w_2$.

We give some of structure theorems of \mathcal{H} . For $w \in W(1)$, let T_w be the characteristic function on $I(1)\tilde{w}I(1)$ where $\tilde{w} \in N_G(S)$ is a lift of w . Then T_w does not depend on the choice of a lift and, since we have the bijection $I(1) \backslash G / I(1) \simeq W(1)$, $\{T_w \mid w \in W(1)\}$ is a basis of \mathcal{H} . This basis is called

Iwahori–Matsumoto basis. This basis satisfies the following braid relations:

$$T_{w_1} T_{w_2} = T_{w_1 w_2} \quad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$$

where $w_1, w_2 \in W(1)$. Let $Z_\kappa = (Z \cap K)/(Z \cap I(1))$. Then this is a subgroup of $W(1)$. Since any elements in Z_κ has the length 0 (since it is in the kernel of $W(1) \rightarrow W$), from the braid relations, we have $T_{t_1} T_{t_2} = T_{t_1 t_2}$ for $t_1, t_2 \in Z_\kappa$. In other words, the embedding $C[Z_\kappa] \hookrightarrow \mathcal{H}$ defined by $\sum_{t \in Z_\kappa} c_t t \mapsto \sum_{t \in Z_\kappa} c_t T_t$ is an algebra homomorphism where $C[Z_\kappa]$ is the group ring of Z_κ . Using this embedding, we regard $C[Z_\kappa]$ as a subalgebra of \mathcal{H} .

Let $S_{\text{aff}}(1)$ be the inverse image of S_{aff} in $W(1)$. Then for $s \in S_{\text{aff}}(1)$, we have

$$T_s^2 = c_s T_s$$

for some $c_s \in C[Z_\kappa]$. An element c_s is given in [Vignéras 2016, 4.2].

Define T_w^* as in [loc. cit., 4.3] for $w \in W(1)$. This is also a basis of \mathcal{H} and it satisfies the following: $T_w^* \in T_w + \sum_{v < w} C T_v$ and $T_{w_1}^* T_{w_2}^* = T_{w_1 w_2}^*$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

Let o be a spherical orientation [loc. cit., 5.2]. Note that the set of spherical orientations are canonically bijective with the set of Weyl chambers. For each o , we have another basis $\{E_o(w) \mid w \in W(1)\}$ defined in [loc. cit., 5.3]. The orientations correspond to the Weyl chambers. Let o_- be the orientation corresponding to the antidominant Weyl chamber and set $E(w) = E_{o_-}(w)$.

Set $\Lambda(1) = Z/(Z \cap I(1))$. This is a subgroup of $W(1)$. For $\lambda_1, \lambda_2 \in \Lambda(1)$, the multiplication $E(\lambda_1)E(\lambda_2)$ is simple. Before giving it, we introduce some notation. The pair (G, S) gives a root datum $(X^*(S), \Sigma, X_*(S), \Sigma^\vee)$ and since we have fixed a Borel subgroup we also have a positive system $\Sigma^+ \subset \Sigma$ and the set of simple roots $\Delta \subset \Sigma^+$. An element $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is called dominant if and only if $\langle v, \alpha \rangle \geq 0$ for any $\alpha \in \Sigma^+$. A W_0 -orbit of the set of dominant elements is called a closed Weyl chamber. We also say that $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is regular if $\langle v, \alpha \rangle \neq 0$ for any $\alpha \in \Sigma$. We have a homomorphism $\nu: Z \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R} = \text{Hom}_{\mathbb{Z}}(X^*(S), \mathbb{R})$ characterized by $\nu(z)(\chi) = -\text{val}(\chi(z))$ where $z \in S$, $\chi \in X^*(S)$ and $\text{val}: F^\times \rightarrow \mathbb{Z}$ is the normalized valuation. This homomorphism factors through $Z \rightarrow \Lambda(1)$ and the induced homomorphism $\Lambda(1) \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is denoted by the same letter ν . We let $\Lambda^+(1)$ the set of $\lambda \in \Lambda(1)$ such that $\nu(\lambda)$ is dominant. For $w \in W_0$, let $w(\Lambda^+(1))$ be the set of $\lambda \in \Lambda(1)$ such that $w^{-1}(\nu(\lambda))$ is dominant.

The multiplication $E(\lambda_1)E(\lambda_2)$ is $E(\lambda_1 \lambda_2)$ if $\nu(\lambda_1)$ and $\nu(\lambda_2)$ are in the same closed Weyl chamber (in other words, $\lambda_1, \lambda_2 \in w(\Lambda^+(1))$ for some $w \in W_0$) and otherwise it is zero. In particular, $\mathcal{A} = \bigoplus_{\lambda \in \Lambda(1)} C E(\lambda)$ is a subalgebra of \mathcal{H} . If we fix a closed Weyl chamber \mathcal{C} , then $\bigoplus_{\nu(\lambda) \in \mathcal{C}} C E(\lambda)$ is a subalgebra of \mathcal{A} and the linear map

$$\bigoplus_{\nu(\lambda) \in \mathcal{C}} C E(\lambda) \rightarrow C[\Lambda(1)]$$

defined by $E(\lambda) \mapsto \tau_\lambda$ is an algebra embedding. Here $C[\Lambda(1)]$ is the group ring of $\Lambda(1)$ and we denote the element in $C[\Lambda(1)]$ corresponding to $\lambda \in \Lambda(1)$ by τ_λ , namely $C[\Lambda(1)] = \bigoplus_{\lambda \in \Lambda(1)} C \tau_\lambda$.

Remark 2.1. (1) If $\langle \nu(\lambda), \alpha \rangle = 0$ for any $\alpha \in \Sigma$, then $\nu(\lambda)$ and $\nu(\lambda^{-1})$ are in the same closed Weyl chamber. (In fact, $\nu(\lambda)$ and $\nu(\lambda^{-1})$ are in any closed Weyl chamber.) Hence $E(\lambda)E(\lambda^{-1}) = 1$. In particular, $E(\lambda)$ is invertible.

(2) If $\lambda \in \Lambda(1)$ is in the center of $\Lambda(1)$, then $E(\lambda)$ is also in the center of \mathcal{A} . This follows from the above description of the multiplication.

Let J be a subset of Δ and denote the corresponding standard parabolic subgroup by P_J . Let L_J be the Levi part of P_J containing Z . Then $K \cap L_J$ is a special parahoric subgroup and $I(1)_J = I(1) \cap L_J$ a pro- p Iwahori subgroup. Attached to these, we have many objects. For such objects we add a suffix J , for example, the pro- p Iwahori–Hecke algebra attached to $(L_J, I(1)_J)$ is denoted by \mathcal{H}_J . There are two exceptions: base T_w and $E(w)$ for \mathcal{H}_J is denoted by T_w^J and $E^J(w)$, respectively. For each $J \subset \Delta$, we have two subalgebras \mathcal{H}_J^+ , \mathcal{H}_J^- of \mathcal{H}_J and four algebra homomorphisms j_J^+ , $j_J^{+*}: \mathcal{H}_J^+ \rightarrow \mathcal{H}$ and j_J^- , $j_J^{-*}: \mathcal{H}_J^- \rightarrow \mathcal{H}$. See [Abe 2019b, 2.8] for the definitions. (Here \mathcal{H}_J^+ is denoted by $\mathcal{H}_{P_J}^+$ in [Abe 2019b].)

3. The category \mathcal{C} and a proof of the injectivity

3A. The category \mathcal{C} . The modules in this paper are right modules unless otherwise stated. In this paper, we focus on the full subcategory \mathcal{C} of the category of \mathcal{H} -modules defined using the center \mathcal{Z} of \mathcal{H} . The center \mathcal{Z} is described using the basis $\{E(w)\}$. Since $\Lambda(1)$ is normal in $W(1)$, the group $W(1)$ acts on $\Lambda(1)$ by the conjugate action. For $\lambda \in \Lambda(1)$ denote the orbit through λ by \mathcal{O}_λ . For $\lambda \in \Lambda(1)$, put $z_\lambda = \sum_{\lambda' \in \mathcal{O}_\lambda} E(\lambda')$. Then $\{z_\lambda \mid \lambda \in \Lambda(1)/W(1)\}$ gives a basis of \mathcal{Z} [Vignéras 2014, Theorem 1.2]. Fix a uniformizer ϖ of F and let $\Lambda_S(1)$ be the image of $\{\xi(\varpi) \mid \xi \in X_*(S)\}$.

Definition 3.1. An \mathcal{H} -module M is in \mathcal{C} if and only if z_λ is invertible on M for any $\lambda \in \Lambda_S(1)$.

Lemma 3.2. Let $\lambda \in \Lambda_S(1)$. Then we have the following:

- (1) For $w \in W(1)$, w stabilizes λ if and only if the image of w in W_0 stabilizes $\nu(\lambda)$.
- (2) Let $\{w_1, \dots, w_r\} \subset W(1)$ be a subset of $W(1)$ such that the image in W_0 gives a set of complete representatives of $W_0 / \text{Stab}_{W_0}(\nu(\lambda))$. Then we have $z_\lambda = \sum_{i=1}^r E(w_i \lambda w_i^{-1})$. (Note that $w_i \lambda w_i^{-1}$ depends only on the image of w_i in $W_0 / \text{Stab}_{W_0}(\lambda)$ by (1).)

Proof. Take $\xi \in X_*(S)$ such that $\lambda = \xi(\varpi)^{-1}$. We have $\nu(\lambda) = \xi$. Let $w \in W(1)$ and denote the image of w in W_0 by w_0 . Then we have $w \lambda w^{-1} = (w_0 \xi)(\varpi)^{-1}$. Hence if w_0 stabilizes $\xi = \nu(\lambda)$, then w stabilizes λ . Obviously if w stabilizes λ then w_0 stabilizes $\nu(\lambda)$.

By (1), $\text{Stab}_{W(1)}(\lambda)$ is the inverse image of $\text{Stab}_{W_0}(\lambda)$. Therefore we have $W(1) / \text{Stab}_{W(1)}(\lambda) \simeq W_0 / \text{Stab}_{W_0}(\lambda)$. By the definition, we have $z_\lambda = \sum_{w \in W(1) / \text{Stab}_{W(1)}(\lambda)} E(w \lambda w^{-1})$. Hence we get (2). \square

Lemma 3.3. Let $\lambda, \mu \in \Lambda_S(1)$ and assume that $\nu(\lambda)$ and $\nu(\mu)$ are in the same closed Weyl chamber. We also assume that $\nu(\lambda)$ is regular. Then we have $z_\lambda z_\mu = z_{\lambda\mu}$.

Proof. Take $w_1, \dots, w_r \in W(1)$ such that the images of them in W_0 gives a set of complete representatives of $W_0/\text{Stab}_{W_0}(v(\mu))$. Then we have $z_\mu = \sum_i E(w_i \mu w_i^{-1})$ by the above lemma. Let v_1, \dots, v_s be a set of complete representatives of $W_0 = W(1)/\Lambda(1)$. Then we have $z_\lambda = \sum_j E(v_j \lambda v_j^{-1})$. (Note that $v(\lambda)$ is assumed to be regular.) Since $v(\lambda)$ is regular, for each i , there exists only one $j_i = 1, \dots, r$ such that $v_i(v(\lambda))$ and $w_{j_i}(v(\mu))$ is in the same closed Weyl chamber. Hence we get

$$E(v_i \lambda v_i^{-1})E(w_j \mu w_j^{-1}) = \begin{cases} 0 & j \neq j_i, \\ E(v_i \lambda v_i^{-1} w_j \mu w_j^{-1}) & j = j_i. \end{cases}$$

Moreover, $v(\lambda)$ and $v_i^{-1} w_{j_i}(v(\mu))$ is in the same closed Weyl chamber. Since $v(\lambda)$ and $v(\mu)$ are in the same closed Weyl chamber by the assumption, we get $v_i^{-1} w_{j_i}(v(\mu)) = v(\mu)$. Therefore $v_i^{-1} w_{j_i}$ stabilizes $v(\mu)$. As in the previous lemma, $v_i^{-1} w_{j_i}$ also stabilizes μ . Hence $w_{j_i} \mu w_{j_i}^{-1} = v_i \mu v_i^{-1}$. We get

$$E(v_i \lambda v_i^{-1})E(w_j \mu w_j^{-1}) = \begin{cases} 0 & j \neq j_i, \\ E(v_i \lambda \mu v_i^{-1}) & j = j_i. \end{cases}$$

Now we get

$$z_\lambda z_\mu = \sum_i \sum_j E(v_i \lambda v_i^{-1})E(w_j \mu w_j^{-1}) = \sum_i E(v_i \lambda \mu v_i^{-1}).$$

By the assumption, $v(\lambda \mu)$ is regular and $\lambda \mu \in \Lambda_S(1)$. Hence the last term is $z_{\lambda \mu}$ by the above lemma. \square

Lemma 3.4. *An \mathcal{H} -module M is in \mathcal{C} if and only if for some $\lambda \in \Lambda_S(1)$ such that $v(\lambda)$ is regular, the element z_λ is invertible on M .*

Proof. Assume that there exists $\lambda_0 \in \Lambda_S(1)$ such that $v(\lambda_0)$ is regular and z_{λ_0} is invertible on M . Let $\lambda \in \Lambda_S(1)$ and we prove that λ is also invertible on M . Replacing λ with an element in the orbit through λ , we may assume that $v(\lambda)$ and $v(\lambda_0)$ are in the same closed Weyl chamber. Take a sufficiently large $n \in \mathbb{Z}_{>0}$ such that $v(\lambda_0^n \lambda^{-1})$ is also in the same closed Weyl chamber as $v(\lambda_0)$. Set $\mu = \lambda_0^n \lambda^{-1}$. Then by the above lemma, we have $z_\mu z_\lambda = z_{\lambda_0^n} = z_{\lambda_0}^n$. By the assumption, $z_{\lambda_0}^n$ is invertible on M . Hence z_λ is invertible, namely we have $M \in \mathcal{C}$. \square

3B. Theorem. In the rest of this section, we prove the following:

Theorem 3.5. *If $M \in \mathcal{C}$, then $M \rightarrow M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

3C. Reductions. Define a subalgebra \mathcal{A} of \mathcal{H} by $\mathcal{A} = \bigoplus_{\lambda \in \Lambda(1)} CE(\lambda)$. Let $M \in \mathcal{C}$ and set $M' = \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$.

Remark 3.6. The element z_λ is in the center of \mathcal{H} and $z_\lambda \in \mathcal{A}$. Therefore the action of z_λ on $M' = \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ is induced by that on M . Since $M \in \mathcal{C}$, z_λ is invertible on M for any $\lambda \in \Lambda_S(1)$. Hence the action of z_λ on M' is also invertible. Namely $M' \in \mathcal{C}$.

Defining the action of $X \in \mathcal{H}$ on M' by $(\varphi X)(Y) = \varphi(XY)$ for $\varphi \in \text{Hom}_{\mathcal{A}}(\mathcal{H}, M)$ and $Y \in \mathcal{H}$, M' is a right \mathcal{H} -module. The map $m \mapsto (X \mapsto mX)$ gives an \mathcal{H} -module embedding $M \hookrightarrow M'$ and we have the

following commutative diagram:

$$\begin{array}{ccc} M & \longrightarrow & M \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M' \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1} \end{array}$$

Therefore, to prove [Theorem 3.5](#), it is sufficient to prove that the map $M' \rightarrow M' \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ is injective.

Lemma 3.7. *Any module $M \in \mathcal{C}$ has a functorial decomposition $M = \bigoplus_{w \in W_0} M_w$ as an \mathcal{A} -module such that $E(\mu)$ acts on M_w by:*

- Zero if $w^{-1}\nu(\mu)$ is not dominant.
- Invertible if $w^{-1}\nu(\mu)$ is dominant.

Proof. Fix $\lambda_0 \in \Lambda_S(1)$ such that $\nu(\lambda_0)$ is regular dominant. Put $\lambda_w = n_w \lambda_0 n_w^{-1}$ and set $M_w = ME(\lambda_w)$. Since $\lambda_w \in \Lambda_S(1)$ is central, $E(\lambda_w)$ is also central in \mathcal{A} . Hence M_w is an \mathcal{A} -submodule.

We prove that λ_w is invertible on M_w . Since $\nu(\lambda_0)$ is regular, $\nu(\lambda_v)$ and $\nu(\lambda_w)$ are not in the same closed Weyl chamber if $v \neq w$. Therefore $E(\lambda_v)E(\lambda_w) = 0$. Hence $M_w E(\lambda_v) = 0$ if $v \neq w$. Therefore for $m \in M_w$, we have $mz_{\lambda_0} = \sum_{v \in W_0} mE(\lambda_v) = mE(\lambda_w)$. Hence if $mE(\lambda_w) = 0$ then $mz_{\lambda_0} = 0$, hence $m = 0$ since z_{λ_0} is invertible. Therefore $E(\lambda_w)$ is injective on M_w . We also have that $mz_{\lambda_0}^2 = mE(\lambda_w)z_{\lambda_0} = mz_{\lambda_0}E(\lambda_w) = mE(\lambda_w)^2$ since z_{λ_0} commutes with $E(\lambda_w)$. (Recall that z_{λ_0} is in the center of \mathcal{H} .) Hence $m = m_0E(\lambda_w)$ where $m_0 = mz_{\lambda_0}^{-2}E(\lambda_w) \in M_w$. Therefore $E(\lambda_w)$ is surjective on M_w .

For $\mu \in \Lambda(1)$ such that $w^{-1}(\nu(\mu))$ is not dominant, $\nu(\mu)$ and $\nu(\lambda_w)$ are not in the same closed Weyl chamber. Hence $E(\mu)E(\lambda_w) = 0$. Therefore $E(\mu) = 0$ on M_w . On the other hand, assume that $w^{-1}(\nu(\mu))$ is dominant. Then $\nu(\mu)$ and $\nu(\lambda_w)$ are in the same closed Weyl chamber. Take sufficiently large $n \in \mathbb{Z}_{\geq 0}$ such that $\nu(\lambda_w^n \mu^{-1})$ is also in the same closed Weyl chamber as $\nu(\mu)$. Then we have $E(\lambda_w)^n = E(\lambda_w^n) = E(\lambda_w^n \mu^{-1})E(\mu)$. Since $E(\lambda_w)$ is invertible on M_w , $E(\mu)$ is also invertible on M_w .

We prove $M = \bigoplus_{w \in W_0} M_w$. Since z_{λ_0} is invertible, any element in M can be written mz_{λ_0} for some $m \in M$. We have $mz_{\lambda_0} = \sum_{w \in W_0} mE(\lambda_w) \in \sum_{w \in W_0} M_w$. Hence $M = \sum_{w \in W_0} M_w$. Let $m_w \in M_w$ and assume that $\sum_{w \in W_0} m_w = 0$. Then for each $v \in W_0$ we have $\sum_{w \in W_0} m_w E(\lambda_v) = 0$. Since $m_w E(\lambda_v) = 0$ for $v \neq w$, we have $m_v E(\lambda_v) = 0$. Since the action of $E(\lambda_v)$ on M_v is invertible, $m_v = 0$. \square

Since $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M) = \bigoplus_{w \in W_0} \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M_w)$, to prove $M' \rightarrow M' \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ is injective, it is sufficient to prove that the homomorphism $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M_w) \rightarrow \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, M_w) \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}$ is injective.

Definition 3.8. Let M be an \mathcal{A} -module. We say that $\mathrm{supp} M = w(\Lambda^+(1))$ if and only if $E(\lambda)$ is:

- Zero if $w^{-1}(\nu(\mu))$ is not dominant.
- Invertible if $w^{-1}(\nu(\mu))$ is dominant.

for any $\lambda \in \Lambda(1)$. (Note that we do not define $\mathrm{supp} M$ itself.)

From the above discussions, to prove [Theorem 3.5](#), it is sufficient to prove the following lemma.

Lemma 3.9. *Let M be an \mathcal{A} -module such that $\text{supp } M = w(\Lambda^+(1))$ where $w \in W_0$. Then $\text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

We take a lift n_w of each $w \in W_0$ in $W(1)$ such that $n_{w_1 w_2} = n_{w_1} n_{w_2}$ if $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Let M be an \mathcal{A} -module and $w \in W_0$. We define a new \mathcal{A} -module $n_w M$ as follows. As a vector space, $n_w M = M$ and the action of $E(\lambda) \in \mathcal{A}$ on $n_w M$ is the action of $E(n_w^{-1} \lambda n_w)$ on M . This defines an auto-equivalence of the category of \mathcal{A} -modules. If $\text{supp } M = v(\Lambda^+(1))$, then $\text{supp } n_w M = wv(\Lambda^+(1))$. With this notation, Lemma 3.9 is equivalent to the following.

Lemma 3.10. *Let M be an \mathcal{A} -module such that $\text{supp } M = \Lambda^+(1)$. Then the map $\text{Hom}_{\mathcal{A}}(\mathcal{H}, n_w M) \rightarrow \text{Hom}_{\mathcal{A}}(\mathcal{H}, n_w M) \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

3D. Reduction to $w = w_J$ for some $J \subset \Delta$. For a subset $J \subset \Delta$, let w_J be the longest element in $W_{0,J}$. We prove that we may assume $w = w_J$ for some J in Lemma 3.10.

We relate our M with modules studied in [Abe 2019a]. Consider the homomorphism $\mathcal{A} \rightarrow C[\Lambda(1)]$ defined by

$$E(\lambda) \mapsto \begin{cases} \tau_\lambda & \lambda \in \Lambda^+(1), \\ 0 & \text{otherwise.} \end{cases} \quad (3-1)$$

We regard $C[\Lambda(1)]$ as a right \mathcal{A} -module via this homomorphism. For $w \in W_0$, we also have the \mathcal{A} -module $n_w C[\Lambda(1)]$. Then we consider the module

$$n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

This is a $(C[\Lambda(1)], \mathcal{H})$ -bimodule.

Let M be an \mathcal{A} -module such that $\text{supp } M = \Lambda^+(1)$. Then we define a structure of a right $C[\Lambda(1)]$ -module on M by

$$m \tau_{\lambda_1 \lambda_2^{-1}} = m E(\lambda_1) E(\lambda_2)^{-1}$$

where $\lambda_1, \lambda_2 \in \Lambda^+(1)$ and $m \in M$. (Since $\text{supp } M = \Lambda^+(1)$, $E(\lambda_2)$ is invertible on M .) It is easy to see that this definition is well-defined and define a structure of $C[\Lambda(1)]$ -module. Then we have

$$M \otimes_{C[\Lambda(1)]} n_w C[\Lambda(1)] \simeq n_w M.$$

The isomorphisms are given by $m \otimes f \mapsto mf$ from the left-hand side to the right-hand side and $m \mapsto m \otimes 1$ in the opposite direction. Therefore we have

$$n_w M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]} n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

For each $w \in W_0$, set $\Delta_w = \{\alpha \in \Delta \mid w(\alpha) > 0\}$. Then by [Abe 2019a, Theorem 3.13], if $\Delta_{w_1} = \Delta_{w_2}$, we have

$$n_{w_1} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}.$$

Therefore we get (1) of the next lemma.

Lemma 3.11. *Let M be as in Lemma 3.10. If $w_1, w_2 \in W_0$ satisfies $\Delta_{w_1} = \Delta_{w_2}$, then we have:*

- (1) $n_{w_1}M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2}M \otimes_{\mathcal{A}} \mathcal{H}$.
- (2) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M)$.

Proof. We have proved (1). We prove (2).

Let ι be an automorphism of \mathcal{H} defined in [Vignéras 2016, Proposition 4.23] and $\zeta: \mathcal{H} \rightarrow \mathcal{H}$ an antiautomorphism defined by $\zeta(T_w) = T_{w^{-1}}$. (The linear map ζ is an antihomomorphism by [Abe 2019a, 4.1].) Set $f = \iota \circ \zeta$. Since $\zeta(E(\lambda)) = E_{o_+}(\lambda^{-1})$ [Abe 2019a, Lemma 4.3] and $\iota(E_{o_+}(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda)$ [Vignéras 2016, Lemma 5.31], we have $f(E(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda^{-1})$. In particular, f preserves \mathcal{A} . It is easy to see $f^2(T_w) = T_w$ for any $w \in W(1)$. Hence f^2 is identity.

For a left \mathcal{H} -module N , we define a right \mathcal{H} -module N^f by $N^f = N$ as a vector space and the action of $X \in \mathcal{H}$ on N^f is the action of $f(X)$ on N . Then $m \otimes X \mapsto f(X) \otimes m$ gives an isomorphism $(N^f \otimes_{\mathcal{A}} \mathcal{H})^f \simeq \mathcal{H} \otimes_{\mathcal{A}} N$.

For a right \mathcal{H} -module or \mathcal{A} -module L , set $L^* = \mathrm{Hom}_C(L, C)$. Then this is a left \mathcal{H} -module or \mathcal{A} -module, respectively. Let M be as in the lemma. Since $f(E(\lambda)) = (-1)^{\ell(\lambda)} E(\lambda^{-1})$, we have $\mathrm{supp}(n_{w_1}M^*)^f = w_1(\Lambda^+(1)^{-1}) = w_1w_{\Delta}(\Lambda^+(1))$. Hence $(n_{w_1}M^*)^f = n_{w_1w_{\Delta}}M'$ for some \mathcal{A} -module M' such that $\mathrm{supp} M' = \Lambda^+(1)$. Since $\Delta_{w_1w_{\Delta}} = \Delta \setminus (-w_{\Delta}(\Delta_{w_1}))$, we also have $\Delta_{w_1w_{\Delta}} = \Delta_{w_2w_{\Delta}}$. Hence by (1), we get $n_{w_1w_{\Delta}}M' \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_2w_{\Delta}}M' \otimes_{\mathcal{A}} \mathcal{H}$. Therefore we get $(n_{w_1}M^*)^f \otimes_{\mathcal{A}} \mathcal{H} \simeq (n_{w_2}M^*)^f \otimes_{\mathcal{A}} \mathcal{H}$. Applying $(\cdot)^f$ to the both sides and using $(N^f \otimes_{\mathcal{A}} \mathcal{H})^f \simeq \mathcal{H} \otimes_{\mathcal{A}} N$, we get $\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^* \simeq \mathcal{H} \otimes_{\mathcal{A}} n_{w_2}M^*$. Hence we have $(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^*)^* \simeq (\mathcal{H} \otimes_{\mathcal{A}} n_{w_2}M^*)^*$.

Now we have

$$(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^*)^* = \mathrm{Hom}_C(\mathcal{H} \otimes_{\mathcal{A}} n_{w_1}M^*, C) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M^{**}).$$

Hence we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M^{**}) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M^{**})$. We have an embedding $M \hookrightarrow M^{**}$. Let L be the cokernel. Then $\mathrm{supp} L = \Lambda^+(1)$ and we have an embedding $L \hookrightarrow L^{**}$. Therefore we have an exact sequence $0 \rightarrow M \rightarrow M^{**} \rightarrow L^{**}$ and it gives $0 \rightarrow n_{w_i}M \rightarrow n_{w_i}M^{**} \rightarrow n_{w_i}L^{**}$ for $i = 1, 2$. Hence we get the following commutative diagram with exact rows:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M) & & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M^{**}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M^{**}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}L^{**}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}L^{**}). \end{array}$$

We have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_1}M) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_2}M)$. □

For given $w \in W$, set $J = \Delta \setminus \Delta_w$. Then we have $\Delta_{w_J} = \Delta \setminus J = \Delta_w$. Therefore, to prove [Lemma 3.10](#), we may assume that $w = w_J$ for some $J \subset \Delta$.

3E. Reduction to $w = w_\Delta$. Set

$$\mathcal{A}_w = \bigoplus_{\lambda \in w(\Lambda^+(1))} C E(\lambda) \subset \mathcal{A}. \quad (3-2)$$

Lemma 3.12. *Let M be an \mathcal{A} -module such that $\text{supp } M = w(\Lambda^+(1))$. Then we have $\text{Hom}_{\mathcal{A}}(\mathcal{H}, M) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_w}(\mathcal{H}, M)$.*

Proof. Let $\varphi: \mathcal{H} \rightarrow M$ be an \mathcal{A}_w -module homomorphism and we prove that φ is \mathcal{A} -equivariant. Fix $\lambda_0 \in \Lambda(1)$ such that $w^{-1}(\nu(\lambda_0))$ is dominant and regular. Since $\text{supp } M = w(\Lambda^+(1))$, $E(\lambda_0)$ is invertible on M . For $\mu \in \Lambda(1)$ such that $w^{-1}(\nu(\mu))$ is not dominant, we have $E(\mu)E(\lambda_0) = 0$. Hence for $X \in \mathcal{H}$, we have $\varphi(XE(\mu)) = E(\lambda_0)^{-1}\varphi(XE(\mu)E(\lambda_0)) = 0$. Since $E(\mu) = 0$ on $n_w M$, $E(\mu)\varphi(X) = 0$. Hence we get $\varphi(XE(\mu)) = 0 = E(\mu)\varphi(X)$. Therefore φ is \mathcal{A} -equivariant. \square

For later use, we also prove the following.

Lemma 3.13. *Let M be an \mathcal{A} -module such that $\text{supp } M = w(\Lambda^+(1))$. Then $M \otimes_{\mathcal{A}_w} \mathcal{H} \xrightarrow{\sim} M \otimes_{\mathcal{A}} \mathcal{H}$.*

Proof. Let $m \in M$ and $X \in \mathcal{H}$. We prove $mE(\lambda) \otimes X = m \otimes E(\lambda)X$ in $M \otimes_{\mathcal{A}_w} \mathcal{H}$ for any $\lambda \in \Lambda(1)$. This is true if $w^{-1}(\nu(\lambda))$ is dominant.

Assume that $w^{-1}(\nu(\lambda))$ is not dominant and take $\lambda_0 \in \Lambda(1)$ such that $w^{-1}(\nu(\lambda_0))$ is dominant and $\nu(\lambda), \nu(\lambda_0)$ are not in the same chamber. Then we have $E(\lambda_0)E(\lambda) = 0$. Note that $E(\lambda_0)$ is invertible on M since $\text{supp } M = w(\Lambda^+(1))$. Hence $m \otimes E(\lambda)X = mE(\lambda_0)^{-1} \otimes E(\lambda_0)E(\lambda)X = 0$. On the other hand, $E(\lambda) = 0$ on M , again by $\text{supp } M = w(\Lambda^+(1))$. Hence $mE(\lambda) \otimes X = 0$. We get the lemma. \square

An element $E(\lambda)$ belongs to:

- \mathcal{A}_w if $\langle \nu(\lambda), w(\alpha) \rangle \geq 0$ for any $\alpha \in \Sigma^+$.
- $j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$ if $\langle \nu(\lambda), \alpha \rangle \geq 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_J^+$.

(The second one follows from the following fact: a basis of $\mathcal{H}_J^- \cap \mathcal{A}_J$ is given by $\{E^J(\lambda)\}$ where λ runs through as above [[Abe 2019a](#), Lemma 4.2] and $j_J^{-*}(E^J(\lambda)) = E(\lambda)$ for such λ [[Abe 2019b](#), Lemma 2.6].) Since $w_J(\Sigma^+) = \Sigma_J^- \cup (\Sigma^+ \setminus \Sigma_J^+) \supset \Sigma^+ \setminus \Sigma_J^+$, we have $\mathcal{A}_{w_J} \subset j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$.

Let M be an \mathcal{A} -module. From the above argument, we have

$$\text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}, n_{w_J} M) \simeq \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H} \otimes_{j_J^{-*}(\mathcal{H}_J^-)} j_J^{-*}(\mathcal{H}_J^-), n_{w_J} M) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J} M)).$$

Since $j_J^{-*}(\mathcal{H}_J^- \cap \mathcal{A}_J)$ contains \mathcal{A}_{w_J} , we have $\mathcal{A}_{w_J} \hookrightarrow \mathcal{H}_J^- \cap \mathcal{A}_J \hookrightarrow \mathcal{A}_J$. More precisely, $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J, w_J}$ via $E(\lambda) \mapsto E^J(\lambda)$. (If $E(\lambda) \in \mathcal{A}_{w_J}$, then $w_J^{-1}(\nu(\lambda))$ is dominant with respect to Δ , hence it is also dominant with respect to J . Therefore $E^J(\lambda) \in \mathcal{A}_{J, w_J}$.)

Lemma 3.14. *We regard \mathcal{A}_{w_J} as a subalgebra of \mathcal{A}_J via the above embedding. Let M be an \mathcal{A} -module such that $\text{supp } M = \Lambda^+(1)$. Then $n_{w_J} M$ is uniquely extended to \mathcal{A}_J , namely there exists a unique \mathcal{A}_J -module M_J such that $\text{supp } M_J = \Lambda^+(1)_J$ and $n_{w_J} M_J|_{\mathcal{A}_{w_J}} = n_{w_J} M|_{\mathcal{A}_{w_J}}$.*

Proof. First we prove that $n_{w_J}M$ is uniquely extended to \mathcal{A}_{J,w_J} . Take $\lambda_0 \in \Lambda_S(1)$ such that:

- $\langle v(\lambda_0), \alpha \rangle = 0$ for all $\alpha \in \Sigma_J^+$.
- $\langle v(\lambda_0), \alpha \rangle > 0$ for all $\alpha \in \Sigma^+ \setminus \Sigma_J^+$.

Note that $w_J(\Sigma_J^+) = \Sigma_J^-$ and $w_J(\Sigma^+ \setminus \Sigma_J^+) = \Sigma^+ \setminus \Sigma_J^+$. Hence we have $\lambda_0 \in w_J(\Lambda^+(1))$, $E^J(\lambda_0)$ is central in \mathcal{A}_{J,w_J} (since $\lambda_0 \in \Lambda_S(1)$ is central in $\Lambda(1)$) and $E^J(\lambda_0)$ is invertible by the first condition and [Remark 2.1](#). The embedding $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$ induces $\mathcal{A}_{w_J}[E(\lambda_0)^{-1}] \hookrightarrow \mathcal{A}_{J,w_J}$. We prove that this is surjective. Let $E^J(\mu) \in \mathcal{A}_{J,w_J}$. Then we have $\langle w_J(v(\mu)), \alpha \rangle \geq 0$ for any $\alpha \in \Sigma_J^+$. Therefore, for sufficiently large $n \in \mathbb{Z}_{>0}$, we have $\lambda_0^n \mu \in w_J(\Lambda^+(1))$. The elements $v(\lambda_0)$ and $v(\mu)$ are in the same closed Weyl chamber $w_J v(\Lambda^+(1)_J)$ with respect to J . Hence $E^J(\lambda_0^n)E^J(\mu) = E^J(\lambda_0^n \mu)$ which is in the image of $\mathcal{A}_{w_J} \hookrightarrow \mathcal{A}_{J,w_J}$. Therefore $\mathcal{A}_{w_J}[E(\lambda_0)^{-1}] \hookrightarrow \mathcal{A}_{J,w_J}$ is surjective. Now we get the lemma since $E(\lambda_0)$ is invertible on $n_{w_J}M$. (Recall that $\text{supp } n_{w_J}M = w_J(\Lambda^+(1))$ and $\lambda_0 \in w_J(\Lambda^+(1))$.)

So we have the extension N_J of $n_{w_J}M$ to \mathcal{A}_{J,w_J} . Define the action of $E^J(\lambda)$ on N_J by zero for $\lambda \in \Lambda(1) \setminus w_J(\Lambda^+(1)_J)$. Then N_J is an \mathcal{A}_J -module such that $\text{supp } N_J = w_J(\Lambda^+(1)_J)$ which is desired. From the definition of the support, this is the only way to extend the module N_J to \mathcal{A}_J . We get the lemma. \square

Take M_J as in the lemma. We have

$$\text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}, n_{w_J}M) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M)) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M_J)).$$

Lemma 3.15. *The homomorphisms*

$$\text{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J) \rightarrow \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J, n_{w_J}M_J) \rightarrow \text{Hom}_{\mathcal{A}_{w_J}}(\mathcal{H}_J^-, n_{w_J}M_J)$$

are both isomorphisms.

Proof. The first is an isomorphism by an argument similar to the proof of [Lemma 3.12](#).

Take $\lambda_0 \in \Lambda(1)$ such that:

- $\lambda_0 \in Z(W_J(1))$.
- $\langle v(\lambda_0), \alpha \rangle > 0$ for any $\alpha \in \Sigma^+ \setminus \Sigma_J^+$.

Then $\mathcal{H}_J = \mathcal{H}_J^-[E^J(\lambda_0)^{-1}]$ [[Abe 2019b](#), Proposition 2.5]. Since $E^J(\lambda_0)$ is invertible in \mathcal{A}_J , it is also invertible on $n_{w_J}M_J$. (Note that $n_{w_J}M_J$ is an \mathcal{A}_J -module.) Hence the second homomorphism is an isomorphism. \square

Therefore we get

$$\text{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_J}M) \simeq \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, \text{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J}M_J)).$$

Lemma 3.16. *Let X be an \mathcal{H}_J -module and assume that $X \rightarrow X \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)}^{L_J} \mathbf{1}$ is injective. Then $Y \rightarrow Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is also injective for $Y = \text{Hom}_{(\mathcal{H}_J^-, j_J^{-*})}(\mathcal{H}, X)$.*

Therefore for the proof of [Lemma 3.10](#), it is sufficient to prove that

$$\mathrm{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J} M_J) \rightarrow \mathrm{Hom}_{\mathcal{A}_J}(\mathcal{H}_J, n_{w_J} M_J) \otimes_{\mathcal{H}_J} \mathrm{c}\text{-Ind}_{I(1)_J}^{L_J} \mathbf{1}$$

is injective, namely we may assume that $w = w_\Delta$.

Proof. Set $J' = -w_\Delta(J)$ and put $n = n_{w_\Delta} n_{w_J}$. Then $l \mapsto nln^{-1}$ gives an isomorphism $L_J \rightarrow L_{J'}$ and sends $I(1)_J$ to $I(1)_{J'}$. Therefore it induces an isomorphism $\mathcal{H}_J \rightarrow \mathcal{H}_{J'}$. Define an $\mathcal{H}_{J'}$ -module X' as the pull-back of X by this isomorphism (see [\[Abe 2019a\]](#)). Then $X \rightarrow X \otimes_{\mathcal{H}_J} \mathrm{c}\text{-Ind}_{I(1)_J}^{L_J} \mathbf{1}$ induces $X' \rightarrow X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}$ and the latter map is also injective. By [\[Abe 2019a, Proposition 4.15\]](#), we have $Y \simeq X' \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$. By [\[Vignéras 2015, Proposition 4.1\]](#), the functor $(\cdot) \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$ is exact. Hence, using the assumption in the lemma, the map

$$Y \simeq X' \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H} \rightarrow (X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H}$$

is injective. By [\[Ollivier and Vignéras 2018, Proposition 4.4\]](#)

$$(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H} \simeq (\mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}))^{I(1)}.$$

In particular,

$$(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})^{I(1)_{J'}} \otimes_{(\mathcal{H}_{J'}, j_{J'}^+)} \mathcal{H} \rightarrow \mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1})$$

is injective. Finally, by [\[Ollivier and Vignéras 2018, Corollary 4.7\]](#),

$$\mathrm{Ind}_{P_{J'}}(X' \otimes_{\mathcal{H}_{J'}} \mathrm{c}\text{-Ind}_{I(1)_{J'}}^{L_{J'}} \mathbf{1}) \simeq Y \otimes_{\mathcal{H}} \mathrm{c}\text{-Ind}_{I(1)}^G \mathbf{1}.$$

Combining all of these, we conclude the lemma. □

3E. Some more reductions. By the definition of \mathcal{H}_\emptyset^+ , \mathcal{H}_\emptyset^- and [\[Abe 2019b, Lemma 2.6\]](#), we have:

$$j_\emptyset^+(\mathcal{H}_\emptyset^+) = \mathcal{A}_{w_\Delta} \quad \text{and} \quad j_\emptyset^{-*}(\mathcal{H}_\emptyset^-) = \mathcal{A}_1.$$

See the argument in [Section 3E](#). By these identities, we regard \mathcal{A}_1 and \mathcal{A}_{w_Δ} as a subalgebra of $\mathcal{H}_\emptyset = \mathcal{A}_\emptyset$.

Let M be an \mathcal{A} -module such that $\mathrm{supp} M = \Lambda^+(1)$. By [Lemma 3.12](#), we have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta} M) \simeq \mathrm{Hom}_{\mathcal{A}_{w_\Delta}}(\mathcal{H}, n_{w_\Delta} M)$. By [Lemma 3.14](#), there exists an \mathcal{A}_\emptyset -module M_\emptyset such that $M|_{\mathcal{A}_1} \simeq M_\emptyset|_{\mathcal{A}_1}$. It is easy to see that $n_{w_\Delta} M|_{\mathcal{A}_{w_\Delta}} \simeq n_{w_\Delta} M_\emptyset|_{\mathcal{A}_{w_\Delta}}$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta} M) &\simeq \mathrm{Hom}_{(\mathcal{H}_\emptyset^+, j_\emptyset^+)}(\mathcal{H}, n_{w_\Delta} M_\emptyset) \\ &\simeq \mathrm{Hom}_{(\mathcal{H}_\emptyset^-, j_\emptyset^-)}(\mathcal{H}, M_\emptyset) && \text{[Abe 2019b, Proposition 4.13]} \\ &\simeq M_\emptyset \otimes_{(\mathcal{H}_\emptyset^-, j_\emptyset^{-*})} \mathcal{H} && \text{[Abe 2019b, Corollary 4.19]} \\ &= M \otimes_{\mathcal{A}_1} \mathcal{H}. && (j_\emptyset^{-*}(\mathcal{H}_\emptyset^-) = \mathcal{A}_1) \end{aligned}$$

By [Lemma 3.13](#), we have $M \otimes_{\mathcal{A}_1} \mathcal{H} \simeq M \otimes_{\mathcal{A}} \mathcal{H}$. Hence we get the following lemma:

Lemma 3.17. *We have $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}, n_{w_\Delta} M) \simeq M \otimes_{\mathcal{A}} \mathcal{H}$ for any \mathcal{A} -module M such that $\mathrm{supp} M = \Lambda^+(1)$*

Therefore, to prove [Lemma 3.10](#), hence [Theorem 3.5](#), it is sufficient to prove the following.

Lemma 3.18. *Let M be an \mathcal{A} -module such that $\text{supp } M = \Lambda^+(1)$. Then $M \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

The group algebra $C[Z_K]$ is a subalgebra of \mathcal{A} via the map $t \mapsto T_t = E(t)$ for $t \in Z_K$. Let \hat{Z}_K denote the set of characters of Z_K . Since the order of Z_K is prime to p , M is semisimple as a $C[Z_K]$ -module. Let $\psi \in Z_K$ and set $M_\psi = \{m \in M \mid mT_t = \psi(t)m \ (t \in Z_K)\}$. Since Z_K is normal in $\Lambda(1)$, the conjugate action of $\Lambda(1)$ on Z_K induces the action on \hat{Z}_K . The formula $E(\lambda)T_t = T_{\lambda t \lambda^{-1}} E(\lambda)$ implies that $M_\psi E(\lambda) \subset M_{\lambda^{-1}(\psi)}$. For an orbit ω of this action in \hat{Z}_K , we put $M_\omega = \bigoplus_{\psi \in \omega} M_\psi$. Then M_ω is stable under the \mathcal{A} -action and we have $M = \bigoplus_{\omega} M_\omega$. Therefore we may assume that $M = M_\omega$ for some ω to prove [Lemma 3.18](#).

Let $\alpha \in \Delta$ and consider the image of $Z \cap L'_{\{\alpha\}}$ in $\Lambda(1)$. We denote this subgroup by $\Lambda'_\alpha(1)$. Consider the following condition: ψ is trivial on $Z_K \cap \Lambda'_\alpha(1)$. Since $Z_K \cap \Lambda'_\alpha(1)$ is normal in $\Lambda(1)$, for $t \in Z_K \cap \Lambda'_\alpha(1)$ and $\lambda \in \Lambda(1)$, we have $(\lambda\psi)(t) = \psi(\lambda^{-1}t\lambda) = 1$ if ψ satisfies this condition. Hence this condition only depends on $\Lambda(1)$ -orbit.

We start to prove [Lemma 3.19](#) by induction on $\dim(G)$. Assume that ω is a $\Lambda(1)$ -orbit in \hat{Z}_K . First we assume that there exists $\alpha \in \Delta$ such that ψ is not trivial on $Z_K \cap \Lambda'_\alpha(1)$ for some (equivalently any) $\psi \in \omega$. Then by [\[Abe 2019a, Theorem 3.13\]](#), we have $M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{s_\alpha} M \otimes_{\mathcal{A}} \mathcal{H}$.

We prove that in this case the lemma follows from that for a Levi subgroup. The argument is similar to that in [Section 3E](#). Set $J = \Delta \setminus \{\alpha\}$. Then we have $n_{s_\alpha} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}} \mathcal{H}$ by [Lemma 3.11](#). By [Lemma 3.13](#), we have $n_{w_\Delta w_J} M \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}$. As in the argument in [Section 3E](#) using [\[Abe 2019b, Lemma 2.6\]](#), we have $j_J^+(\mathcal{H}_J^+) \supset \mathcal{A}_{w_\Delta w_J}$. Therefore we have

$$n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H} \simeq (n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J^+) \otimes_{(\mathcal{H}_J^+, j_J^+)} \mathcal{H}.$$

By the same argument of the proof of [Lemma 3.14](#), there exists an \mathcal{A}_J -module M_J such that

$$n_{w_\Delta w_J} M_J|_{\mathcal{A}_{w_\Delta w_J}} = n_{w_\Delta w_J} M|_{\mathcal{A}_{w_\Delta w_J}} \quad \text{and} \quad \text{supp } M_J = \Lambda^+(1)_J.$$

By a similar argument of the proof of [Lemma 3.15](#), the homomorphisms

$$n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J^+ \rightarrow n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}_J \rightarrow n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J$$

are isomorphisms. Now by inductive hypothesis, the homomorphism

$$n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J \rightarrow (n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J) \otimes_{\mathcal{H}_J} \text{c-Ind}_{I(1)_J}^{L_J} \mathbf{1}$$

is injective. By the argument in the proof of [Lemma 3.16](#), this implies that for $Y = (n_{w_\Delta w_J} M_J \otimes_{\mathcal{A}_J} \mathcal{H}_J) \otimes_{(\mathcal{H}_J^+, j_J^+)} \mathcal{H} \simeq n_{w_\Delta w_J} M \otimes_{\mathcal{A}_{w_\Delta w_J}} \mathcal{H}$, the homomorphism

$$Y \rightarrow Y \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$$

is injective. Hence we get the lemma for M .

Therefore we may assume that there is no such α . Hence it is sufficient to prove the following to prove [Lemma 3.18](#).

Lemma 3.19. *Let M be an \mathcal{A} -module such that $\text{supp}(M) = \Lambda^+(1)$ and $Z_\kappa \cap \Lambda'_\alpha(1)$ acts trivially on M for all $\alpha \in \Delta$. Then $M \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective.*

We prove this lemma in [Section 3J](#).

3G. Hecke modules. As discussed in [3D](#), we have the following

$$M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]} (C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}),$$

We decompose this module along the action of Z_κ .

Set $C[\Lambda(1)]_\psi = \{f \in C[\Lambda(1)] \mid \tau_t f = \psi(t) f \ (t \in Z_\kappa)\}$ and for a $\Lambda(1)$ -stable subset $\omega \subset \hat{Z}_\kappa$ we put $C[\Lambda(1)]_\omega = \bigoplus_{\psi \in \omega} C[\Lambda(1)]_\psi$. From the definition, it is obvious that $C[\Lambda(1)]_\omega$ is invariant under the right action of $C[\Lambda(1)]$.

Lemma 3.20. *We have $C[\Lambda(1)]_\omega = \bigoplus_{\psi \in \omega} \{f \in C[\Lambda(1)] \mid f \tau_t = \psi(t) f \ (t \in Z_\kappa)\}$.*

Proof. Let $\psi \in \omega$, $f \in C[\Lambda(1)]_\psi$ and we write $f = \sum_{\lambda \in \Lambda(1)} c_\lambda \tau_\lambda$ where $c_\lambda \in C$. Set

$$e = \#Z_\kappa^{-1} \sum_{t \in Z_\kappa} \psi(t)^{-1} \tau_t \in C[Z_\kappa].$$

Then $ef = f$ and $e\tau_t = \psi(t)e$ for each $t \in Z_\kappa$. We have $e\tau_\lambda \tau_t = e\tau_{\lambda t \lambda^{-1}} \tau_\lambda = (\lambda^{-1}\psi)(t)e\tau_\lambda$. Since $\lambda^{-1}\psi \in \omega$, we get the lemma. \square

Therefore $C[\Lambda(1)]_\omega$ is a two-sided ideal of $C[\Lambda(1)]$. Using Z_κ -action, some objects appearing here are decomposed. Here is a list:

- $C[\Lambda(1)] = C[\Lambda(1)]_\omega \times C[\Lambda(1)]_{\hat{Z}_\kappa \setminus \omega}$ as C -algebras.
- $\mathcal{A} = \mathcal{A}_\omega \times \mathcal{A}_{\hat{Z}_\kappa \setminus \omega}$ as C -algebras with the obvious notation.
- The homomorphism (3-1) induces $\mathcal{A}_\omega \rightarrow C[\Lambda(1)]_\omega$ and $\mathcal{A}_{\hat{Z}_\kappa \setminus \omega} \rightarrow C[\Lambda(1)]_{\hat{Z}_\kappa \setminus \omega}$.

Let M be an \mathcal{A} -module such that $\text{supp } M = \Lambda^+(1)$ and $M = M_\omega$ (see [Section 3F](#)). Then as in [Section 3D](#), M is a $C[\Lambda(1)]$ -module and this action factors through $C[\Lambda(1)] \rightarrow C[\Lambda(1)]_\omega$. Hence we have

$$M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]_\omega} (C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}) \quad (3-3)$$

In [\[Abe 2019a, Section 3\]](#), it is proved that, for any $w \in W_0$, $1 \otimes 1 \mapsto 1 \otimes T_{n_{w_\Delta w^{-1}}}^*$ gives a $(C[\Lambda(1)], \mathcal{H})$ -bimodule homomorphism

$$n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \rightarrow n_{w_\Delta} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}$$

which is injective [[loc. cit.](#), Proposition 3.12]. The homomorphism is compatible with the decomposition $n_w C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H} \simeq n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \oplus n_w C[\Lambda(1)]_{\hat{Z}_\kappa \setminus \omega} \otimes_{\mathcal{A}} \mathcal{H}$. Hence we get the $(C[\Lambda(1)], \mathcal{H})$ -bimodule homomorphism

$$n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \rightarrow n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \quad (3-4)$$

which is again injective. By [loc. cit., Theorem 3.13], the image of this homomorphism only depends on Δ_w . Let X_J be the image of this homomorphism where $J = \Delta_w$. This is a $(C[\Lambda(1)]_\omega, \mathcal{A})$ -module. We have $M \otimes_{\mathcal{A}} \mathcal{H} \simeq M \otimes_{C[\Lambda(1)]_\omega} X_\Delta$ by (3-3).

Lemma 3.21. *If $J' \supset J$, then $X_{J'} \subset X_J$.*

Proof. Note that $\Delta_{w_\Delta w_J} = J$. Hence, by definition, X_J is a $(C[\Lambda(1)], \mathcal{H})$ -submodule in $n_{w_\Delta} C[\Lambda(1)] \otimes_{\mathcal{A}} \mathcal{H}$ generated by $1 \otimes T_{n_{w_\Delta w_J w_\Delta}}^*$. If $J' \supset J$, then

$$\ell(w_\Delta w_J w_{J'} w_\Delta) = \ell(w_J w_{J'}) = \ell(w_{J'}) - \ell(w_J) = \ell(w_\Delta w_{J'} w_\Delta) - \ell(w_\Delta w_J w_\Delta).$$

Hence $T_{n_{w_\Delta w_{J'} w_\Delta}}^* = T_{n_{w_\Delta w_J w_\Delta}}^* T_{n_{w_\Delta w_J w_{J'} w_\Delta}}^*$. Therefore $1 \otimes T_{n_{w_\Delta w_{J'} w_\Delta}}^* \in X_J$. Since $X_{J'}$ is generated by $1 \otimes T_{n_{w_\Delta w_{J'} w_\Delta}}^*$, we have $X_{J'} \subset X_J$. \square

Lemma 3.22. $X_J \in \mathcal{C}$.

Proof. Take $\lambda \in \Lambda_S(1)$ such that $\nu(\lambda)$ is regular dominant. Then we have $z_\lambda = \sum_{v \in W_0} E(n_v \lambda n_v^{-1})$ by Lemma 3.2. Let $f \otimes X \in X_J$. Then, since z_λ is in the center, we have $(f \otimes X)z_\lambda = f \otimes z_\lambda X = f \otimes \sum_{v \in W_0} E(n_v \lambda n_v^{-1})X = f \tau_\lambda \otimes X$ in $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}$. Since $f \mapsto f \tau_\lambda$ is invertible, z_λ is invertible on X_w . \square

Note that $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \simeq n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}_{w_\Delta}} \mathcal{H}$ [Abe 2019a, Proposition 3.12]. Hence $X_\emptyset = n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{(\mathcal{H}_\emptyset^+, j_\emptyset^+)} \mathcal{H}$. This is a parabolically induced module [Vignéras 2015]. By [loc. cit., Example 3.2, Lemma 3.6], we have $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}$. Since $T_{n_w}^* \in T_{n_w} + \sum_{v < w} C[Z_\kappa] T_{n_v}$, we have $n_{w_\Delta} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}^*$.

Set $Y_w = n_{w_\Delta} C[\Lambda(1)]_\omega \otimes T_{n_w}^* \subset X_\emptyset$. Then the subspace Y_w is the image of $n_w C[\Lambda(1)]_\omega \otimes 1$ by the injective homomorphism (3-4). In particular, Y_w is \mathcal{A} -stable and isomorphic to $n_w C[\Lambda(1)]_\omega$. We have $X_\emptyset = \bigoplus_{w \in W_0} Y_w$. This is the decomposition in Lemma 3.7. By the functoriality of the decomposition, we have $X_J = \bigoplus_{w \in W_0} (X_J \cap Y_w)$.

3H. Representations of G . Let ω be a $\Lambda(1)$ -orbit in \hat{Z}_κ such that for any $\alpha \in \Delta$, ψ is trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$ for some (equivalently any) $\psi \in \omega$. Recall that we have fixed a special parahoric subgroup K . Irreducible representations V of K are parametrized by a pair (ψ, J) where ψ is a character of Z_κ and J a certain subset of Δ . Here for V , ψ and J are given by the following: $\psi \simeq V^{I(1)}$ and $W_{0,J} = \text{Stab}_{W_0}(V^{I(1)})$. Note that by the assumption on ω , (ψ, J) gives a parameter for any $\psi \in \omega$ and $J \subset \Delta$ [Abe et al. 2017, III.8]. Let $V_{\psi,J}$ be the irreducible representation of K which corresponds to (ψ, J) and put $V_J = \bigoplus_{\psi \in \omega} V_{\psi^{-1},J}$. In the rest of this paper, we fix a basis of $V_{\psi^{-1},J}^{I(1)}$ for each ψ and J .

Lemma 3.23. (1) *The Hecke algebra $\text{End}_Z(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)})$ is isomorphic to $C[\Lambda(1)]_\omega$.*

(2) *We have the Satake homomorphism*

$$\text{End}_G(\text{c-Ind}_K^G V_J) \hookrightarrow \text{End}_Z(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)}) \simeq C[\Lambda(1)]_\omega$$

and its image is $C[\Lambda^+(1)]_\omega$.

Proof. Let $\mathcal{H}(\psi_1^{-1}, \psi_2^{-1})$ is the space of functions $\varphi: Z \rightarrow C$ such that $\text{supp } \varphi$ is compact and $\varphi(t_1 z t_2) = \psi_1^{-1}(t_1) \varphi(z) \psi_2^{-1}(t_2)$ for any $z \in Z$ and $t_1, t_2 \in Z \cap K$. Since $V_J^{I(1)} \simeq \bigoplus_{\psi \in \omega} \psi^{-1}$, a standard argument for Hecke algebras implies

$$\text{End}_Z(\text{Ind}_{Z \cap K}^Z V_J^{I(1)}) \simeq \bigoplus_{\psi_1, \psi_2 \in \omega} \text{Hom}_Z(\text{c-Ind}_{Z \cap K}^Z \psi_1^{-1}, \text{c-Ind}_{Z \cap K}^Z \psi_2^{-1}) \simeq \bigoplus_{\psi_1, \psi_2 \in \omega} \mathcal{H}(\psi_1^{-1}, \psi_2^{-1}).$$

This space is a subalgebra of \mathcal{H}_Z where \mathcal{H}_Z is the functions φ on Z which is invariant under the left (and equivalently right) multiplication by $Z \cap I(1)$ and whose support is compact. The homomorphism $\varphi \mapsto \sum_{z \in Z/(Z \cap K)} \varphi(z) \tau_z$ gives an isomorphism $\mathcal{H}_Z \simeq C[\Lambda(1)]$. As a subspace of both sides, it is easy to see that we get the desired isomorphism.

The Satake transform

$$\text{Hom}_G(\text{c-Ind}_K^G V_{\psi_1, J}, \text{c-Ind}_K^G V_{\psi_2, J}) \rightarrow \text{Hom}_Z(\text{c-Ind}_{Z \cap K}^Z \psi_1^{-1}, \text{c-Ind}_{Z \cap K}^Z \psi_2^{-1})$$

is defined in [Henniart and Vignéras 2012, 2] and the image is described in [Abe et al. 2018b, Theorem 1.1]. \square

Remark 3.24. In the identification (1) in the lemma, we need to fix an isomorphism $V_J^{I(1)} \simeq \bigoplus_{\psi \in \omega} \psi^{-1}$. We use our fixed basis of $V_{\psi^{-1}, J}^{I(1)}$ for this isomorphism.

By the lemma, $C[\Lambda^+(1)]_\omega$ acts on $\text{c-Ind}_K^G V_J$. Define a representation π_J of G by

$$\pi_J = C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} \text{c-Ind}_K^G V_J.$$

We prove $\pi_J^{I(1)} \simeq X_J$.

Recall that the \mathcal{H} -module $(\text{c-Ind}_K^G V_J)^{I(1)}$ is described as follows. Let \mathcal{H}_f be the Hecke algebra attached to the pair $(K, I(1))$. Then $V_J^{I(1)}$ is naturally a right \mathcal{H}_f -module and the algebra \mathcal{H}_f is a subalgebra of \mathcal{H} with a basis $\{T_w \mid w \in W_0(1)\}$ where $W_0(1)$ is the inverse image of $W_0 \subset W$ in $W(1)$. Then we have $(\text{c-Ind}_K^G V_J)^{I(1)} \simeq V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ [Vignéras 2017, Proposition 7.2].

Remark 3.25. In the argument below, we will use results in [Abe 2017]. In [loc. cit.], we study an \mathcal{H}_f -module denoted by $\eta^J = \bigoplus_{\psi \in \hat{Z}_K} V_{\psi, J}^{I(1)}$. Using a similar argument in [loc. cit.] (or taking a direct summand of results), results are also true for an \mathcal{H}_f -module $V_J^{I(1)}$.

We have an action of $C[\Lambda^+(1)]_\omega$ on $V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ [loc. cit., Proposition 3.4] and the above isomorphism $(\text{c-Ind}_K^G V_J)^{I(1)} \simeq V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ is $C[\Lambda^+(1)]_\omega$ -equivariant. (This can be proved by the same argument in the proof of [loc. cit., Proposition 5.1].)

Lemma 3.26. *Let A be a ring and $S \subset A$ be a multiplicative subset of the center of A . Then for a smooth $A[G]$ -module π , we have $(S^{-1}\pi)^{I(1)} \simeq S^{-1}\pi^{I(1)}$.*

Proof. Both sides can be regarded as a subspace of $S^{-1}\pi$. Any element in $S^{-1}\pi^{I(1)}$ is $I(1)$ -invariant, hence $S^{-1}\pi^{I(1)} \subset (S^{-1}\pi)^{I(1)}$. Let $v/s \in (S^{-1}\pi)^{I(1)}$ where $v \in \pi$ and $s \in S$. Let g_1, \dots, g_n be a representatives of $I(1)/\text{Stab}_{I(1)}(v)$. Since v/s is g_i -invariant, there exists $s_i \in S$ such that $s_i(g_i v - v) = 0$. Therefore $s_1 \cdots s_n(g_i v - v) = 0$. Set $v' = s_1 \cdots s_n v$. Then for any $g \in I(1)$ there exists i and $g' \in \text{Stab}_{I(1)}(v)$

such that $g = g_i g'$. Hence $gv' = s_1 \cdots s_n(g_i g'v) = s_1 \cdots s_n v = v'$. Therefore $v' \in \pi^{I(1)}$. Hence $v/s = v'/(s s_1 \cdots s_n) \in S^{-1} \pi^{I(1)}$. \square

Therefore we have

$$\pi_J^{I(1)} \simeq C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}.$$

By [Abe 2017, Proposition 3.9], we have

$$V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \simeq \text{Im}(n_{w_\Delta w_J} C[\Lambda^+(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H} \rightarrow n_{w_\Delta w_J} C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}).$$

Hence we have an isomorphism $C[\Lambda(1)]_\omega \otimes_{C[\Lambda^+(1)]_\omega} V_J^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \simeq X_J$. Therefore $\pi_J^{I(1)} \simeq X_J$.

We get an embedding $X_J \simeq \pi_J^{I(1)} \hookrightarrow \pi_J$. Hence there exists a homomorphism $X_J \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow \pi_J$. Let $J = \Delta$ and applying $M \otimes_{C[\Lambda(1)]}$ to

$$X_\Delta \rightarrow X_\Delta \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow \pi_\Delta$$

and using $M \otimes_{C[\Lambda(1)]} X_\Delta \simeq M \otimes_{\mathcal{A}} \mathcal{H}$ (3-3), we get

$$M \otimes_{C[\Lambda(1)]} X_\Delta \simeq M \otimes_{\mathcal{A}} \mathcal{H} \rightarrow M \otimes_{\mathcal{A}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow M \otimes_{C[\Lambda(1)]} \pi_\Delta.$$

Hence for Lemma 3.19, it is sufficient to prove that $M \otimes_{C[\Lambda(1)]} X_\Delta \rightarrow M \otimes_{C[\Lambda(1)]} \pi_\Delta$ is injective.

We have an isomorphism $\pi_\emptyset \simeq \text{Ind}_{\bar{B}}^G(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)})$ [Henniart and Vignéras 2012, Theorem 1.2]. (To be precisely, the direct sum of a result in [loc. cit., Theorem 1.2].) An injective embedding $\pi_J \rightarrow \text{Ind}_{\bar{B}}^G(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)}) \simeq \pi_\emptyset$ was given in [loc. cit., Definition 2.1]. Hence we have a diagram of $(C[\Lambda(1)], \mathcal{H})$ -bimodules

$$\begin{array}{ccc} X_J & \longrightarrow & X_\emptyset \\ \downarrow & & \downarrow \\ \pi_J & \longrightarrow & \pi_\emptyset. \end{array}$$

When $J = \emptyset$, $X_J \rightarrow X_\emptyset$ and $\pi_J \rightarrow \pi_\emptyset$ are both identities. Hence this diagram is commutative.

Lemma 3.27. *This diagram is commutative for any J .*

Proof. Fix $\psi^{-1} \in \omega$. It is sufficient to prove that the following diagram is commutative:

$$\begin{array}{ccc} V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} & \longrightarrow & n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \\ \downarrow & & \downarrow \\ (\text{c-Ind}_K^G V_{\psi, J})^{I(1)} & \longrightarrow & \text{Ind}_{\bar{B}}^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}. \end{array} \quad (3-5)$$

Note that this diagram is commutative when $J = \emptyset$.

Let $v_0 \in V_{\psi, J}^{I(1)}$ be our fixed basis. Define $\varphi_J \in (\text{c-Ind}_K^G V_{\psi, J})^{I(1)}$ by $\text{supp } \varphi_J = K$ and $\varphi_J(1) = v_0$. Then the \mathcal{H} -module map $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow \text{c-Ind}_K^G(V_{\psi, J})$ is given by $v_0 \otimes 1 \mapsto \varphi_J$. Define $f_0 \in \text{Ind}_{\bar{B}}^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$ by $\text{supp } f_0 = \bar{B} n_{w_\Delta} I(1)$, $\text{supp } f_0(n_{w_\Delta}^{-1}) = Z \cap K$ and $f_0(n_{w_\Delta}^{-1})(1) = v_0$. Then

the function corresponding to φ_\emptyset under $\text{c-Ind}_K^G V_\emptyset \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_\emptyset^{I(1)})$ is $f_0 T_{n_{w_\Delta}}$ [Abe et al. 2017, IV.9 Proposition].

Set $w = w_\Delta w_J$. Then $X_J = n_w C[\Lambda(1)]_\omega \otimes_{\mathcal{A}} \mathcal{H}$. The homomorphism $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow n_w C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$ is given by $v_0 \otimes 1 \mapsto 1 \otimes T_{n_w}$ [Abe 2017, Lemmas 3.8 and 3.10].

Consider the case of $J = \emptyset$. Then the image of $v_0 \otimes 1 \in V_{\psi, \emptyset}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ under

$$V_{\psi, \emptyset}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow (\text{c-Ind}_K^G V_{\psi, \emptyset})^{I(1)} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$$

is

$$v_0 \otimes 1 \mapsto \varphi_\emptyset \mapsto f_0 T_{n_{w_\Delta}}.$$

On the other hand, the image of $v_0 \otimes 1 \in V_{\psi, \emptyset}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ in $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$ is $1 \otimes T_{n_{w_\Delta}}$. As remarked before the lemma, (3-5) is commutative when $J = \emptyset$. Hence the homomorphism $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$ sends $1 \otimes T_{n_{w_\Delta}}$ to $f_0 T_{n_{w_\Delta}}$. Take λ from the center of $\Lambda(1)$ such that $\langle \alpha, \nu(\lambda) \rangle < 0$ for any $\alpha \in \Sigma^+$. Then by [Abe 2019b, Lemma 2.17], $\ell(n_{w_\Delta} \lambda) = \ell(\lambda) - \ell(n_{w_\Delta})$. Hence by [Vignéras 2016, Theorem 5.25, Example 5.32], $T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1} \lambda) = E(\lambda)$. Therefore $1 \otimes T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1} \lambda) = 1 \otimes E(\lambda) = \tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} \otimes 1$. On the other hand, $f_0 T_{n_{w_\Delta}} E(n_{w_\Delta}^{-1} \lambda) = f_0 E(\lambda) = \tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} f_0$ by [Abe et al. 2017, IV.10 Proposition]. Hence the homomorphism $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$ sends $\tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} \otimes 1$ to $\tau_{n_{w_\Delta}^{-1} \lambda n_{w_\Delta}} f_0$. Therefore $1 \otimes 1$ sends to f_0 .

Let $a = v_0 \otimes 1 \in V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ and we consider the image of a in $\text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$ in the two ways. The image of a in $n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H}$ is $1 \otimes T_{n_{w_\Delta} w^{-1}}^* T_{n_w}$ by [Abe 2017, Proposition 3.11] and the definition of $X_J \rightarrow X_\emptyset$. Therefore the image of a under $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H} \rightarrow n_{w_\Delta} C[\Lambda(1)]_\psi \otimes_{\mathcal{A}} \mathcal{H} \rightarrow \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})^{I(1)}$ is $f_0 T_{n_{w_\Delta} w^{-1}}^* T_{n_w}$.

By [Abe et al. 2017, IV.9 Proposition] (for $J = \Delta$), we have $f_0 T_{n_{w_\Delta} w^{-1}}^* = \sum_{v \leq w_\Delta w^{-1}} f_0 T_{n_v}$. Since $w_\Delta w^{-1} = w_\Delta w_J w_\Delta$, $\{v \in W_0 \mid v \leq w_\Delta w^{-1}\} = w_\Delta W_{0, J} w_\Delta$. Hence

$$f_0 T_{n_{w_\Delta} w^{-1}}^* T_{n_w} = \sum_{v \in W_{J, 0}} f_0 T_{n_{w_\Delta v w_\Delta}} T_{n_{w_\Delta w_J}}.$$

We have

$$\ell(w_\Delta v w_\Delta \cdot w_\Delta w_J) = \ell(w_\Delta v w_J) = \ell(w_\Delta) - \ell(v w_J) = \ell(w_\Delta) - \ell(w_J) + \ell(v) = \ell(w_\Delta w_J) + \ell(w_\Delta v w_\Delta).$$

Hence $T_{n_{w_\Delta v w_\Delta}} T_{n_{w_\Delta w_J}} = T_{n_{w_\Delta v w_J}}$. Therefore, replacing v with $v w_J$, we get $f_0 T_{n_{w_\Delta} w^{-1}}^* T_{n_w} = \sum_{v \in W_{J, 0}} f_0 T_{n_{w_\Delta v}}$. This is the image of φ_J in $\text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_{\psi, \emptyset}^{I(1)})$ by [Abe et al. 2017, IV.7 Corollary]. Hence the diagram (3-5) is commutative if we start with a . Since the element a generates $V_{\psi, J}^{I(1)} \otimes_{\mathcal{H}_f} \mathcal{H}$ as an \mathcal{H} -module, the diagram (3-5) is commutative. \square

Therefore we may regard π_J and X_J as a subspace of π_\emptyset . We have $\pi_\emptyset \simeq \text{Ind}_B^G(\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)})$. By the same argument in the proof of Lemma 3.23, we have $\text{c-Ind}_{Z \cap K}^Z V_J^{I(1)} \simeq C[\Lambda(1)]_\omega$. Here again we use our fixed basis. Hence we have $\pi_\emptyset \simeq \text{Ind}_B^G C[\Lambda(1)]_\omega$. We identify π_J with the image in $\text{Ind}_B^G C[\Lambda(1)]_\omega$.

Remark 3.28. By [Abe et al. 2017, IV.7 Proposition] and the decomposition $G = \bigcup_{w \in W_0} \bar{B}n_w I(1)$ implies that $(\text{Ind}_B^G C[\Lambda(1)]_\omega)^{I(1)} = \bigoplus_{w \in W_0} C[\Lambda(1)]_\omega f_0 T_{n_w}$. Since $X_\emptyset = \bigoplus_{w \in W_0} C[\Lambda(1)]_\omega \otimes T_{n_w}$ (see after the proof of Lemma 3.22) and $X_\emptyset \rightarrow \pi_\emptyset$ sends $1 \otimes 1$ to f_0 (see the proof of the previous lemma), we have $X_\emptyset \simeq \pi_\emptyset^{I(1)}$. Note that $\text{supp } f_0 T_{n_w} = \bar{B}n_{w_\Delta w} I(1)$ [Abe et al. 2017, IV.7 Proposition].

3I. Filtrations. As in the previous subsection, let ω be a $\Lambda(1)$ -orbit in \hat{Z}_κ such that, for some (equivalently any) $\psi \in \omega$, ψ is trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$ for any $\alpha \in \Delta$. In this subsection, we use the following notation: for $A \subset W_0$, $\bar{B}A\bar{B} = \bigcup_{v \in A} \bar{B}n_v \bar{B}$.

For a subset $A \subset W_0$ which is open (namely, if $v_1 \in W_0$, $v_2 \in A$ and $v_1 \geq v_2$ then $v_1 \in A$), we put

$$\pi_{\emptyset, A} = \{f \in \text{Ind}_B^G C[\Lambda(1)]_\psi \mid \text{supp } f \subset \bar{B}A\bar{B}\}.$$

We also put

$$X_{\emptyset, A} = \bigoplus_{v \in A} n_{w_\Delta} C[\Lambda(1)] \otimes T_{n_{w_\Delta v}}.$$

Lemma 3.29. *Let $h \in X_\emptyset$. Then $h \in X_{\emptyset, A}$ if and only if its image in π_\emptyset is in $\pi_{\emptyset, A}$. Namely we have $X_{\emptyset, A} = X_\emptyset \cap \pi_{\emptyset, A}$.*

Proof. Let $H \in \pi_\emptyset$ be the image of h . By the description of $X_\emptyset \rightarrow \pi_\emptyset$ (see Remark 3.28), $h \in X_{\emptyset, A}$ if and only if $\text{supp } H \subset \bar{B}A\bar{B}$. For each $v \in A$, we have

$$\bar{B}vI(1) = \bar{B}v(I(1) \cap v^{-1}\bar{B}v)(I(1) \cap v^{-1}Bv) = \bar{B}v(I(1) \cap v^{-1}Bv) \subset \bar{B}Bv \subset \bigcup_{v' \geq v} \bar{B}v'\bar{B} \subset \bar{B}A\bar{B}.$$

Here we use [Abe 2012, Lemma 2.4]. Hence if $h \in X_{\emptyset, A}$ then $H \in \pi_{\emptyset, A}$.

Assume that $H \in \pi_{\emptyset, A}$ and $\text{supp}(H) \cap \bar{B}vI(1) \neq \emptyset$ for $v \in W_0$. Since H is $I(1)$ -invariant, we have $H(v) \neq 0$. Therefore $v \in A$. Hence $\text{supp}(H) \subset \bigcup_{v \in A} \bar{B}vI(1)$. We get $h \in X_{\emptyset, A}$. \square

Set $X_{J, A} = X_J \cap X_{\emptyset, A}$ and $\pi_{J, A} = \pi_J \cap \pi_{\emptyset, A}$. Let $w \in A$ be a minimal element and put $A' = A \setminus \{w\}$. Then we have an embedding

$$X_{\Delta, A}/X_{\Delta, A'} \hookrightarrow \pi_{\Delta, A}/\pi_{\Delta, A'}.$$

For each $\alpha \in \Delta$, take a lift $a_\alpha \in \Lambda'_\alpha(1)$ of a generator of $\Lambda'_\alpha(1)/(Z_\kappa \cap \Lambda'_\alpha(1))$ such that $\langle v(a_\alpha), \alpha \rangle > 0$ [Abe et al. 2017, III.4].

The element $\#Z_\kappa^{-1} \sum_{\psi \in \omega} \sum_{t \in Z_\kappa} \psi(t)^{-1} \tau_{a_\alpha t}$ is in $C[\Lambda(1)]_\omega$ and does not depend on a choice of a lift (recall that ψ is trivial on $Z_\kappa \cap \Lambda'_\alpha(1)$). We denote it by τ_α . Set $c_w = \prod_{w^{-1}(\alpha) > 0} (1 - \tau_\alpha)$. Then as in [Abe et al. 2017, V.8 Proposition], we have

$$\pi_{\Delta, A}/\pi_{\Delta, A'} = c_w(\pi_{\emptyset, A}/\pi_{\emptyset, A'}). \quad (3-6)$$

The space $\pi_{\emptyset, A}/\pi_{\emptyset, A'}$ can be identified with the space of compactly supported functions on $\bar{B} \setminus \bar{B}w\bar{B}$ with values in $C[\Lambda(1)]_\omega$, which is isomorphic to $C_c^\infty(\bar{B} \setminus \bar{B}w\bar{B}) \otimes_C C[\Lambda(1)]_\omega$ where $C_c^\infty(\bar{B} \setminus \bar{B}w\bar{B})$ is the space of locally constant compact support functions on $\bar{B} \setminus \bar{B}w\bar{B}$ with values in C . Hence it is free as $C[\Lambda(1)]_\omega$ -module. By the following lemma and (3-6), $\pi_{\Delta, A}/\pi_{\Delta, A'}$ is also free.

Lemma 3.30. *The element $c_w \in C[\Lambda(1)]_\omega$ is not a zero divisor.*

Proof. The same proof in [Abe 2019a, Lemma 3.10] can apply. \square

Lemma 3.31. *We have $X_{\Delta,A}/X_{\Delta,A'} = c_w(X_{\emptyset,A}/X_{\emptyset,A'})$.*

Proof. Since $X_{\Delta,A} = \pi_{\Delta,A} \cap X_{\emptyset,A}$, we have

$$X_{\Delta,A}/X_{\Delta,A'} = \pi_{\Delta,A}/\pi_{\Delta,A'} \cap X_{\emptyset,A}/X_{\emptyset,A'}$$

and the right-hand side is

$$c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) \cap X_{\emptyset,A}/X_{\emptyset,A'}.$$

Let H be in this set. Since $\pi_{\emptyset,A}/\pi_{\emptyset,A'}$ is a free $C[\Lambda(1)]_\omega$ -module, the exact sequence $0 \rightarrow \pi_{\emptyset,A'} \rightarrow \pi_{\emptyset,A} \rightarrow \pi_{\emptyset,A}/\pi_{\emptyset,A'} \rightarrow 0$ splits. Hence $\pi_{\emptyset} \simeq \pi_{\emptyset,A'} \oplus (\pi_{\emptyset,A}/\pi_{\emptyset,A'})$. Therefore $c_w\pi_{\emptyset,A} \simeq c_w\pi_{\emptyset,A'} \oplus c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'})$. Hence $c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) \simeq (c_w\pi_{\emptyset,A})/(c_w\pi_{\emptyset,A'})$. Hence there exists $H' \in \pi_{\emptyset,A}$ such that H is the image of $c_w H'$. Since $H \in X_{\Delta,A}/X_{\Delta,A'}$, there exists $h \in X_{\Delta,A}$ such that $c_w H' - h$ is zero in $X_{\Delta,A}/X_{\Delta,A'}$. In particular it is zero in $c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) = (c_w\pi_{\emptyset,A})/(c_w\pi_{\emptyset,A'})$. Therefore there exists $H'' \in \pi_{\emptyset,A'}$ such that $c_w H' - h = c_w H''$. Replacing H' with $H' - H''$, we may assume $c_w H' \in X_{\emptyset,A}$. Recall that H' is a function with values in $C[\Lambda(1)]_\omega$. Since the element c_w is not a zero divisor in $C[\Lambda(1)]_\omega$, $c_w H' \in \pi_{\emptyset,A}$ implies $H' \in \pi_{\emptyset,A}$. Since $c_w H' \in X_{\emptyset}$, $c_w H'$ is $I(1)$ -invariant. Hence H' is also $I(1)$ -invariant, again since c_w is not a zero divisor. Therefore $H' \in \pi_{\emptyset}^{I(1)} = X_{\emptyset}$. Hence $H' \in X_{\emptyset} \cap \pi_{\emptyset,A} = X_{\emptyset,A}$. Therefore $H \in c_w(X_{\emptyset,A}/X_{\emptyset,A'})$. The reverse inclusion $c_w(\pi_{\emptyset,A}/\pi_{\emptyset,A'}) \cap X_{\emptyset,A}/X_{\emptyset,A'} \supset c_w(X_{\emptyset,A}/X_{\emptyset,A'})$ is obvious. We get the lemma. \square

3J. Proof of Lemma 3.19. Let A, A', w be as in the previous subsection.

Lemma 3.32. *The exact sequences of $C[\Lambda(1)]_\omega$ -modules*

$$0 \rightarrow \pi_{\Delta,A'} \rightarrow \pi_{\Delta,A} \rightarrow \pi_{\Delta,A}/\pi_{\Delta,A'} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow X_{\Delta,A'} \rightarrow X_{\Delta,A} \rightarrow X_{\Delta,A}/X_{\Delta,A'} \rightarrow 0$$

split.

Proof. By (3-6) and from the fact that $\pi_{\emptyset,A}/\pi_{\emptyset,A'}$ is free, $\pi_{\Delta,A}/\pi_{\Delta,A'}$ is also free. Hence the first exact sequence splits. Using Lemma 3.31, the same argument can apply for the second sequence. \square

Lemma 3.33. *The inclusion $X_{\Delta,A}/X_{\Delta,A'} \hookrightarrow \pi_{\Delta,A}/\pi_{\Delta,A'}$ has a section as $C[\Lambda(1)]_\omega$ -modules.*

Proof. First we construct a section of $X_{\emptyset,A}/X_{\emptyset,A'} \rightarrow \pi_{\emptyset,A}/\pi_{\emptyset,A'}$. Recall that $X_{\emptyset,A} = \pi_{\emptyset,A}^{I(1)}$. Note that $X_{\emptyset,A}/X_{\emptyset,A'} \simeq C[\Lambda(1)]_\omega$ and the section is given by $f \mapsto f(w)$. For $H \in \pi_{\emptyset,A}$, consider $H' \in \pi_{\emptyset,A}$ which is $I(1)$ -invariant, $\text{supp}(H') = \bar{B}vI(1)$ and $H'(v) = H(v)$. Then $H \mapsto H'$ gives a section of $X_{\emptyset,A}/X_{\emptyset,A'} \rightarrow \pi_{\emptyset,A}/\pi_{\emptyset,A'}$. Multiplying c_w and using (3-6), Lemma 3.31, we get a section of the $C[\Lambda(1)]_\omega$ -module homomorphism $X_{\Delta,A}/X_{\Delta,A'} \rightarrow \pi_{\Delta,A}/\pi_{\Delta,A'}$. \square

Proof of Lemma 3.19. Set $\pi_A^M = M \otimes_{C[\Lambda(1)]_\omega} \pi_{\Delta,A}$ and $X_A^M = M \otimes_{C[\Lambda(1)]_\omega} X_{\Delta,A}$. Then by Lemma 3.32, π_A^M and X_A^M are subspaces of $\pi_{\Delta,A}^M$ and $X_{\Delta,A}^M$, respectively. By Lemma 3.33, $X_A^M/X_{A'}^M \rightarrow \pi_A^M/\pi_{A'}^M$ is injective.

We prove that $X_A^M \rightarrow \pi_A^M$ is injective by induction on $\#A$. We have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_{A'}^M & \longrightarrow & X_A^M & \longrightarrow & X_A^M/X_{A'}^M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_{A'}^M & \longrightarrow & \pi_A^M & \longrightarrow & \pi_A^M/\pi_{A'}^M \longrightarrow 0. \end{array}$$

The homomorphism $X_{A'}^M \rightarrow \pi_{A'}^M$ is injective by inductive hypothesis and $X_A^M/X_{A'}^M \rightarrow \pi_A^M/\pi_{A'}^M$ is injective as we have seen. Hence $X_A^M \rightarrow \pi_A^M$ is injective. Setting $A = W_0$, we get the lemma. \square

4. Theorem

Let \mathcal{C}_f be the full subcategory of \mathcal{C} consisting of finite-dimensional modules. Note that this category is closed under submodules, quotients and extensions.

Theorem 4.1. *Let $M \in \mathcal{C}_f$. Then $(M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \simeq M$.*

Proof. The theorem is true for simple M by [Abe 2019a, main theorem; Abe et al. 2018a, Theorem 4.17 and Theorem 5.11]. We prove the theorem by induction on $\dim(M)$.

Assume that M is not simple and let M' be a proper nonzero submodule of M . Let

$$\pi = \text{Ker}(M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}).$$

By Theorem 3.5, $M \rightarrow (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$ is injective. Then we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^{I(1)} & \longrightarrow & (M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \longrightarrow & (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} \\ & & & & \uparrow \wr & & \uparrow \\ & & & & M' & \hookrightarrow & M. \end{array}$$

Hence $\pi^{I(1)} = 0$. Since $I(1)$ is a pro- p group, $\pi = 0$. Hence $M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1} \rightarrow M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1}$ is injective. Set $M'' = M/M'$. Then we have a commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ (M' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M' \\ \downarrow & & \downarrow \\ (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M \\ \downarrow & & \downarrow \\ (M'' \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)} & \xleftarrow{\sim} & M'' \\ & & \downarrow \\ & & 0 \end{array}$$

with exact columns. Therefore $M \rightarrow (M \otimes_{\mathcal{H}} \text{c-Ind}_{I(1)}^G \mathbf{1})^{I(1)}$ is isomorphic. \square

Corollary 4.2. *Let $\mathcal{C}_{G,f}$ be the category of representations of G consisting of the following objects:*

- *Has a finite length.*
- *Any irreducible subquotient is a subquotient of $\mathrm{Ind}_B^G \sigma$ for a irreducible representation σ of Z .*
- *Is generated by $I(1)$ -invariants.*

Then $\mathcal{C}_f \simeq \mathcal{C}_{G,f}$. The equivalence is given by $\pi \rightarrow \pi^{I(1)}$ and $M \mapsto M \otimes_{\mathcal{H}} \mathrm{c}\text{-}\mathrm{Ind}_{I(1)}^G M$.

Proof. By the classification theorem in [Abe et al. 2017] and [Abe et al. 2018a, Theorem 5.11], if $\pi \in \mathcal{C}_{G,f}$ is irreducible, then $\pi^{I(1)} \in \mathcal{C}_f$. Hence, by induction on the length, if $\pi \in \mathcal{C}_{G,f}$ then $\pi^{I(1)} \in \mathcal{C}_f$.

Let $\pi \in \mathcal{C}_{G,f}$ and we prove that $\pi^{I(1)} \otimes_{\mathcal{H}} \mathrm{c}\text{-}\mathrm{Ind}_{I(1)}^G \mathbf{1} \rightarrow \pi$ is an isomorphism. The homomorphism is surjective since π is generated by $\pi^{I(1)}$. Let π' be the kernel. Then we have an exact sequence

$$0 \rightarrow (\pi')^{I(1)} \rightarrow (\pi^{I(1)} \otimes_{\mathcal{H}} \mathrm{c}\text{-}\mathrm{Ind}_{I(1)}^G \mathbf{1})^{I(1)} \rightarrow \pi^{I(1)}$$

and the last map is isomorphism by the theorem. Hence $(\pi')^{I(1)} = 0$ and it implies $\pi' = 0$. Therefore the homomorphism is also injective. Combining with the previous theorem, we have proved the desired equivalence of categories. \square

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Volume 13 No. 8 2019

Moduli of stable maps in genus one and logarithmic geometry, II	1765
DHRUV RANGANATHAN, KELI SANTOS-PARKER and JONATHAN WISE	
Multiplicity one for wildly ramified representations	1807
DANIEL LE	
Theta operators on unitary Shimura varieties	1829
EHUD DE SHALIT and EYAL Z. GOREN	
Infinitely generated symbolic Rees algebras over finite fields	1879
AKIYOSHI SANNAL and HIROMU TANAKA	
Manin's b -constant in families	1893
AKASH KUMAR SENGUPTA	
Equidimensional adic eigenvarieties for groups with discrete series	1907
DANIEL R. GULOTTA	
Cohomological and numerical dynamical degrees on abelian varieties	1941
FEI HU	
A comparison between pro- p Iwahori–Hecke modules and mod p representations	1959
NORIYUKI ABE	