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Positivity of anticanonical divisors and F-purity of fibers

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We prove that given a flat generically smooth morphism between smooth projective varieties with F-pure closed fibers, if the source space is Fano, weak Fano or a variety with nef anticanonical divisor, respectively, then so is the target space. We also show that, in arbitrary characteristic, a generically smooth surjective morphism between smooth projective varieties cannot have nef and big relative anticanonical divisor, if the target space has positive dimension.

1. Introduction

Let X be a smooth projective variety over an algebraically closed field. The positivity of the anticanonical divisor $-K_X$ on X is an important notion that helps us know certain geometric properties of X. Let $f : X \to Y$ be a surjective morphism from X to another smooth projective variety Y. Kollár, Miyaoka and Mori [Kollár et al. 1992, Corollary 2.9] proved that, under the assumption that f is smooth, if X is a Fano variety, that is $-K_X$ is ample, then so is Y. It follows from an analogous argument that, under the same assumption, if $-K_X$ is nef, then so is $-K_Y$ [Miyaoka 1993; Fujino and Gongyo 2014, Theorem 1.1; Debarre 2001, Corollary 3.15(a)]. Based on these results, Yasutake asked "what positivity condition is passed from $-K_X$ to $-K_Y$?" Some answers to this question are known in characteristic 0. Fujino and Gongyo [2012, Theorem 1.1] proved that, under the assumption that f is smooth, if X is a weak Fano variety, that is $-K_X$ is nef and big, then so is Y. Birkar and Chen [2016, Theorem 1.1] showed that, under the same assumption, if $-K_X$ is semiample, then so is $-K_Y$. Furthermore, similar but weaker results hold even if f is not smooth (but the characteristic of k is still 0). For example, a result of Prokhorov and Shokurov [2009, Lemma 2.8] (see also [Fujino and Gongyo 2012, Corollary 3.3]) implies that if $-K_X$ is nef and big, then $-K_Y$ is big. Chen and Zhang [2013, Main Theorem] also proved that if $-K_X$ is nef, then $-K_Y$ is pseudoeffective.

In contrast, little was known about the positive characteristic case. In this paper, assuming that the geometric generic fiber has only F-pure or strongly F-regular singularities, we prove that (generalizations of) the statements above hold in positive characteristic, except the one concerning semiampleness. F-purity and strong F-regularity are mild singularities defined in terms of Frobenius splitting properties

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(Definition 2.4), which have a close connection to log canonical and Kawamata log terminal singularities, respectively.

Suppose that the base field k is an algebraically closed field of characteristic $p > 0$. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties, Δ an effective $\mathbb{Q}$-divisor on X with index $\operatorname{ind}\Delta$, and D a $\mathbb{Q}$-divisor on Y. Let $X_{\bar{\eta}}$ denote the geometric generic fiber of f. Then our main theorem is stated as follows:

Theorem 1.1 (Theorem 4.1). *Let y be a scheme-theoretic point in Y such that the following conditions hold*:

(i) $\dim f^{-1}(y) = \dim X - \dim Y$.

(ii) *The support of Δ does not contain any irreducible component of $f^{-1}(y)$.*

(iii) *$(X_{\bar{y}}, \Delta|_{X_{\bar{y}}})$ is F-pure, where $X_{\bar{y}}$ is the geometric fiber over y.*

Suppose that $p \nmid \operatorname{ind}(\Delta)$ and $-(K_X+\Delta+f^*D)$ is nef. Then y is not in the Zariski closure of $\boldsymbol{B}_-(-(K_Y+D))$.

Here, $\boldsymbol{B}_-$ denotes the restricted base locus (Definition 2.8). This locus is empty (resp. has nonempty complement) if and only if the divisor is nef (resp. pseudoeffective). Theorem 1.1 implies, in the case of $\Delta = 0$, that every closed fiber over $\boldsymbol{B}_-(-(K_Y + D))$ is "bad", where "bad" means the fiber is not F-pure or has dimension larger than that of the general fiber.

The following two theorems are corollaries of Theorem 1.1.

Theorem 1.2 (Corollary 4.5). *Suppose that conditions* (i)–(iii) *in Theorem 1.1 hold for every closed point in Y*:

(1) *Assume $p \nmid \operatorname{ind}(\Delta)$. If $-(K_X + \Delta + f^*D)$ is nef, then so is $-(K_Y + D)$.*

(2) *If $-(K_X + \Delta + f^*D)$ is ample, then so is $-(K_Y + D)$.*

Theorem 1.3 (Corollary 4.6). *Suppose that $(X_{\bar{\eta}}, \Delta|_{X_{\bar{\eta}}})$ is F-pure*:

(1) *If $p \nmid \operatorname{ind}(\Delta)$ and $-(K_X + \Delta + f^*D)$ is nef, then $-(K_Y + D)$ is pseudoeffective.*

(2) *If $-(K_X + \Delta + f^*D)$ is ample, then $-(K_Y + D)$ is big.*

(3) *If $(X_{\bar{\eta}}, \Delta|_{X_{\bar{\eta}}})$ is strongly F-regular and $-(K_X + \Delta + f^*D)$ is nef and big, then $-(K_Y + D)$ is big.*

Theorem 1.2 is a generalization of [Kollár et al. 1992, Corollary 2.9] and [Debarre 2001, Corollary 3.15] in positive characteristic. One can also recover [Kollár et al. 1992, Corollary 2.9] in characteristic zero from Theorem 1.2, using reduction to characteristic p. Our proof relies on a study of the positivity of direct image sheaves in terms of the Grothendieck trace of the relative Frobenius morphism. This is completely different from the proof [Kollár et al. 1992, Corollary 2.9] that is an application of their great study regarding rational curves on varieties. Theorem 1.3 should be compared with [Prokhorov and Shokurov 2009, Lemma 2.8] and [Chen and Zhang 2013, Main Theorem].

The following two theorems are direct consequences of Theorems 1.2 and 1.3.

Theorem 1.4 (Corollary 4.7). *Suppose that $(X_{\bar{\eta}}, \Delta|_{X_{\bar{\eta}}})$ is F-pure. If $p \nmid \mathrm{ind}(\Delta)$ and $K_X+\Delta$ is numerically equivalent to $f^*(K_Y+L)$ for some $\mathbb{Q}$-divisor L on Y, then L is pseudoeffective.*

Theorem 1.5 (Corollary 4.9). *Suppose that f is a flat morphism such that every closed fiber is F-pure and the geometric generic fiber is strongly F-regular. If X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y.*

Theorem 1.5 is a positive characteristic counterpart of [Fujino and Gongyo 2012, Theorem 1.1].

For another application of Theorem 1.1, we return to the situation where k is of arbitrary characteristic. Suppose that $f : X \to Y$ is a generically smooth surjective morphism between smooth projective varieties of positive dimension.

Theorem 1.6 (Corollary 4.10 and Theorem 5.4). *The relative anticanonical divisor $-K_{X/Y}$ cannot be both nef and big.*

Theorem 1.7 (Corollary 4.11 and Theorem 5.5). *Suppose that $\omega_{X_{\bar{\eta}}}^{-m}$ is globally generated for an integer $m>0$. Then $f_*\omega_{X/Y}^{-m}$ is not big in the sense of Definition 2.6.*

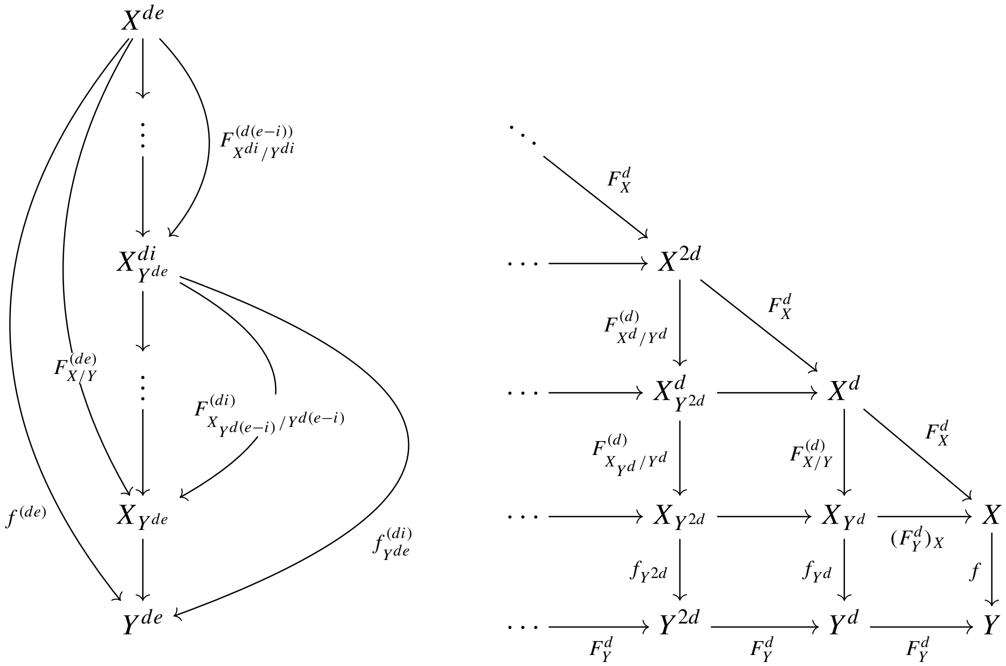
In both the theorems, the characteristic zero case is proved by reduction to positive characteristic. Theorem 1.6 includes [Kollár et al. 1992, Corollary 2.8] which states that $-K_{X/Y}$ is not ample. Theorem 1.7 generalizes [Miyaoka 1993, Corollary 2'] which says that if $\omega_{X/Y}^{-1}$ is f-ample and $\omega_{X/Y}^{-m}$ is f-free for some $m \in \mathbb{Z}_{>0}$, then $f_*\omega_{X/Y}^{-m}$ is not an ample vector bundle.

Notation. Let k be a field. A *k-scheme* is a separated scheme of finite type over k. A *variety* is an integral k-scheme. Let $\varphi : S \to T$ be a morphism of k-schemes and T' a T-scheme. Then, $S_{T'}$ and $\varphi_{T'} : S_{T'} \to T'$ denote the fiber product $S \times_T T'$ and its second projection, respectively. For a Cartier or $\mathbb{Q}$-Cartier divisor D on S (resp. an $\mathcal{O}_S$-module $\mathcal{G}$), the pullback of D (resp. $\mathcal{G}$) to $S_{T'}$ is written as $D_{T'}$ (resp. $\mathcal{G}_{T'}$) if it is well defined. Similarly, for a homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_S$-modules, $\alpha_{T'} : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$ is the pullback of α to $S_{T'}$. Assume that k is of characteristic $p>0$. We say that k is *F-finite* if the field extension k/k^p is finite. Let X be a k-scheme. Then, $F_X : X \to X$ denotes the absolute Frobenius morphism of X. We often write the source of F_X^e as X^e. Let $f : X \to Y$ be a morphism between k-schemes. The same morphism is denoted by $f^{(e)} : X^e \to Y^e$ when we regard X (resp. Y) as X^e (resp. Y^e). We define the e-th relative Frobenius morphism of f to be the morphism $F_{X/Y}^{(e)} := (F_X^e, f^{(e)}) : X^e \to X \times_Y Y^e =: X_{Y^e}$. We write the localization of $\mathbb{Z}$ at $(p) = p\mathbb{Z}$ as $\mathbb{Z}_{(p)}$.

2. Preliminaries

2A. ***Relative Frobenius morphisms and trace maps.*** In this subsection, given a morphism between varieties, we consider the relative Frobenius morphism and its trace map. Let k be an F-finite field and $f : X \to Y$ a morphism from a pure dimensional Gorenstein k-scheme X to a regular variety Y. For each

$d, e \in \mathbb{Z}_{>0}$ we use the notation defined by the following commutative diagram:



Since F_Y is flat, every horizontal morphism in the diagram is a Gorenstein morphism, so every object is a pure dimensional Gorenstein k-scheme. Let ω_X be the dualizing sheaf on X. Let $\mathrm{Tr}_{F^{(1)}_{X/Y}} : F^{(1)}_{X/Y*}\omega_{X^1} \to \omega_{X_{Y^1}}$ denote the morphism obtained by applying the functor $\mathscr{H}om_{\mathcal{O}_{X_{Y^1}}}((\ ), \omega_{X_{Y^1}})$ to the natural morphism ${F^{(1)}_{X/Y}}^{\#} : \mathcal{O}_{X_{Y^1}} \to F^{(1)}_{X/Y*}\mathcal{O}_{X^1}$. Take a Cartier divisor K_X satisfying $\mathcal{O}_X(K_X) \cong \omega_X$. Set $K_{X/Y} := K_X - f^*K_Y$. For each $e \in \mathbb{Z}_{>0}$ we define

$$\phi^{(1)}_{X/Y} := \mathrm{Tr}_{F^{(1)}_{X/Y}} \otimes \mathcal{O}_{X_{Y^1}}(-K_{X_{Y^1}}) : F^{(1)}_{X/Y*}\mathcal{O}_{X^1}((1-p)K_{X^1/Y^1}) \to \mathcal{O}_{X_{Y^1}}, \quad \text{and}$$
$$\phi^{(e+1)}_{X/Y} := (\phi^{(e)}_{X/Y})_{Y^{e+1}} \circ F^{(e)}_{X_{Y^1}/Y^1 *}(\phi^{(1)}_{X^e/Y^e} \otimes \mathcal{O}_{X^e_{Y^{e+1}}}((1-p^e)K_{X^e_{Y^{e+1}}/Y^{e+1}}))$$
$$: {F^{(e+1)}_{X/Y}}_* \mathcal{O}_X((1-p^{e+1})K_{X^{e+1}/Y^{e+1}}) \to \mathcal{O}_{X_{Y^{e+1}}}.$$

Let E be an effective Cartier divisor on X, let $a > 0$ be an integer not divisible by p, and let $d > 0$ be the minimum integer satisfying $a|(p^d - 1)$. Note that an integer $e > 0$ satisfies $a|(p^e - 1)$ if and only if $d|e$. Set $\Delta := E \otimes \frac{1}{a} \in \mathrm{Car}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. For each $e \in d\mathbb{Z}_{>0}$ we define

$$\mathcal{L}^{(e)}_{(X/Y,\Delta)} := \mathcal{O}_{X^e}((1-p^e)(K_{X^e/Y^e} + \Delta^e)) \subseteq \mathcal{O}_{X^e}((1-p^e)K_{X^e/Y^e}),$$
$$\phi^{(d)}_{(X/Y,\Delta)} : F^{(d)}_{X/Y*}\mathcal{L}^{(d)}_{(X/Y,\Delta)} \to F^{(d)}_{X/Y*}\mathcal{O}_{X^d}((1-p^d)K_{X^d/Y^d}) \xrightarrow{\phi^{(d)}_{X/Y}} \mathcal{O}_{X_{Y^d}}, \quad \text{and}$$
$$\phi^{(e+d)}_{(X/Y,\Delta)} := (\phi^{(e)}_{(X/Y,\Delta)})_{Y^e} \circ F^{(e)}_{X_{Y^d}/Y^d *}(\phi^{(d)}_{(X^e/Y^e,\Delta^e)} \otimes (\mathcal{L}^{(e)}_{(X/Y,\Delta)})_{Y^{e+d}}) : {F^{(e+d)}_{X/Y}}_* \mathcal{L}^{(e+d)}_{(X/Y,\Delta)} \to \mathcal{O}_{X_{Y^{e+d}}}.$$

In order to generalize the definitions above, we recall the notion of generalized divisors on a k-scheme. Let X be a k-scheme of pure dimension satisfying S_2 and G_1. An *AC divisor* (or *almost Cartier divisor*) on

X is a reflexive coherent subsheaf of the sheaf of total quotient rings on X that is invertible in codimension one (see [Hartshorne 1994; Miller and Schwede 2012]). For an AC divisor D, the coherent sheaf defining D is denoted by $\mathcal{O}_X(D)$. The set of AC divisors $\mathrm{WSh}(X)$ has a structure of additive group [Hartshorne 1994, Corollary 2.6]. A $\mathbb{Z}_{(p)}$-*AC divisor* (resp. $\mathbb{Q}$-*AC divisor*) is an element of $\mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ (resp. $\mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$). An AC divisor D is said to be *effective* if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$, and a $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Q}$-AC) divisor Δ is said to be *effective* if $\Delta = D \otimes r$ for an effective AC divisor D and an $r \geq 0$. For two AC divisors D and E, the notation $D \leq E$ means that $E - D$ is effective. We use the same notation for $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Q}$-AC) divisors.

Remark 2.1. Each of the natural morphisms

$$\mathrm{WSh}(X) \xrightarrow{(\ )\otimes 1} \mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \to \mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is not necessarily injective (see Example 2.2). Let D and E be AC divisors. Then, D and E are equal as $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Q}$-AC) divisors if and only if $mD = mE$ for some nonzero $m \in \mathbb{Z} \setminus p\mathbb{Z}$ (resp. $m \in \mathbb{Z}$). Furthermore, D and E can be equal as $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Q}$-AC) divisors even when D is effective but E is not (see Example 2.2).

Example 2.2 [Corti 1992, (16.1.2)]. Set $X := \operatorname{Spec} k[x, y, z, z^{-1}]/(x^n - zy^n)$ for an $n \geq 2$. Note that X is integral and Gorenstein but not normal. Let D and E be AC divisors on X defined by $x^{-1}\mathcal{O}_X$ and $y^{-1}\mathcal{O}_X$, respectively. For an $m \geq 1$, one has

$$mD = mE \iff x^{-m}\mathcal{O}_X = y^{-m}\mathcal{O}_X \iff n \mid m.$$

Hence, we see that

- $D \neq E$ as AC divisors,
- $D \otimes 1 = E \otimes 1$ as $\mathbb{Z}_{(p)}$-AC divisors if and only if $p \nmid n$, and
- $D \otimes 1 = E \otimes 1$ as $\mathbb{Q}$-AC divisors.

Furthermore, $D - E$ is not effective but $n(D - E) = 0$ is effective.

Remark 2.3. Let E and K be two AC divisors, take an $\varepsilon \in \mathbb{Z}_{(p)}$ (resp. $\varepsilon \in \mathbb{Q}$) and set $\Delta := E \otimes \varepsilon$. When we consider the $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Q}$-AC) divisor $K + \Delta$, for each $m \in \mathbb{Z}$ with $\varepsilon m \in \mathbb{Z}$, we let $m(K + \Delta)$ denote the AC divisor $mK + (\varepsilon m)E$.

Let $f : X \to Y$ be a morphism from a pure dimensional k-scheme X to a regular variety Y and assume that X satisfies S_2 and G_1. Let E be an effective AC divisor on X, and fix a Gorenstein open subset $U \subseteq X$ such that $\operatorname{codim}_X(X \setminus U) \geq 2$ and $E|_U$ is Cartier. Let $U \xhookrightarrow{\iota} X$ denote the open immersion. Then, for each $e \in \mathbb{Z}_{>0}$, we have the following commutative diagram:

$$\begin{array}{ccccc} U^e & \xrightarrow{F^{(e)}_{U/Y}} & U_{Y^e} & \longrightarrow & U \\ {\scriptstyle \iota^{(e)}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{Y^e}} & & \big\downarrow{\scriptstyle \iota} \\ X^e & \xrightarrow{F^{(e)}_{X/Y}} & X_{Y^e} & \longrightarrow & X \\ & {\scriptstyle f^{(e)}}\searrow & \big\downarrow{\scriptstyle f_{Y^e}} & & \big\downarrow{\scriptstyle f} \\ & & Y^e & \xrightarrow[F^e_Y]{} & Y \end{array}$$

Take an integer $a > 0$ not divisible by p, set $\Delta := E \otimes \frac{1}{a}$, and let d be the minimum positive integer satisfying $a \mid (p^d - 1)$. For each $e \in d\mathbb{Z}_{>0}$, we define

$$\mathcal{L}^{(e)}_{(X/Y,\Delta)} := \iota^{(e)}{}_*\mathcal{L}^{(e)}_{(U/Y,\Delta|_U)} \quad \text{and} \quad \phi^{(e)}_{(X/Y,\Delta)} := \iota_{Y^e *}(\phi^{(e)}_{(U/Y,\Delta|_U)}) : F^{(e)}_{X/Y *}\mathcal{L}^{(e)}_{(X/Y,\Delta)} \to \mathcal{O}_{X_{Y^e}}.$$

Fix $e \in d\mathbb{Z}_{>0}$. Set $\mathcal{L}^{(e)}_{(X,\Delta)} := \mathcal{O}_{X^e}((1-p^e)(K_{X^e} + \Delta^e))$. We can define the morphism $\phi^{(e)}_{(X,\Delta)} : F^e_{X*}\mathcal{L}^{(e)}_{(X,\Delta)} \to \mathcal{O}_X$ by a procedure similar to the one above, replacing $F^{(e)}_{X/Y}$ by F^e_X. In the case where k is perfect and $Y = \operatorname{Spec} k$, one may identify, respectively, F^e_X, $\mathcal{L}^{(e)}_{(X,\Delta)}$ and $\phi^{(e)}_{(X,\Delta)}$ with $F^{(e)}_{X/Y}$, $\mathcal{L}^{(e)}_{(X/Y,\Delta)}$ and $\phi^{(e)}_{(X/Y,\Delta)}$.

We next introduce singularities of pairs defined by the Grothendieck trace map of the Frobenius morphism.

Definition 2.4. Let X be a k-scheme of pure dimension satisfying S_2 and G_1, and let Δ be an effective $\mathbb{Q}$-AC divisor on X:

(i) We say that (X, Δ) is *F-pure* if for each $e \in \mathbb{Z}_{>0}$ and every effective AC divisor E' with $\Delta' := E' \otimes 1/(p^e - 1) \leq \Delta$, the morphism

$$\phi^{(e)}_{(X,\Delta')} : F^e_{X*}\mathcal{O}_X((1-p^e)(K_X + \Delta')) \to \mathcal{O}_X$$

is surjective.

(ii) [Schwede 2008, Definition 3.1] Assume that X is a normal variety. We say that (X, Δ) is *strongly F-regular* if every effective Cartier divisor D on X, the morphism

$$F^e_{X*}\mathcal{O}_X(\lfloor (1-p^e)(K_X+\Delta) - D \rfloor) = F^e_{X*}\mathcal{O}_X((1-p^e)(K_X+\Delta')) \xrightarrow{\phi^{(e)}_{(X,\Delta')}} \mathcal{O}_X$$

is surjective for some $e \in \mathbb{Z}_{>0}$, where $\Delta' := \frac{1}{p^e-1}\lfloor (p^e-1)\Delta + D \rfloor$. Here, $\lfloor\ \rfloor$ denotes the round down.

We simply say that X is *F-pure* (resp. *strongly F-regular*) if $(X, 0)$ is F-pure (resp. strongly F-regular).

Remarks 2.5. (1) With the notation as above, assume that X is normal and affine. Then the above definition of F-purity is equivalent to that in [Hara and Watanabe 2002]. This follows from the fact that $\lfloor (p^e-1)\Delta \rfloor$ is the greatest element of the set S of all divisors E' such that $E' \leq (p^e-1)\Delta$.

(2) When X is not normal, S in (1) does not necessarily have a greatest element. Let X, D and E be as in Example 2.2 with $p \nmid n$ and $n \nmid l$, where $l := p^e - 1$. Then, $\Delta := D \otimes 1 = E \otimes 1$ as $\mathbb{Z}_{(p)}$-divisors and $lD \neq lE$. If S has a greatest element G, then $G \geq lD$ and $G \otimes 1 = l\Delta$, from which one can get $G = lD$. In the same way, we get $G = lE$, so $lD = lE$, a contradiction.

(3) Let (X, Δ) be a strongly F-regular pair, and Δ' an effective $\mathbb{Q}$-divisor on X. Then there is $\varepsilon \in \mathbb{Q}_{>0}$ such that $(X, \Delta + \varepsilon\Delta')$ is again strongly F-regular.

2B. ***Weak positivity.*** In this subsection, we recall the notion of weak positivity introduced by Viehweg [1983]. The definition employed in this paper is slightly different from Viehweg's original one. We work over a field k in this subsection.

Definition 2.6. Let Y be a quasiprojective normal variety, let $\mathcal{G}$ and $\mathcal{G}'$ be coherent sheaves on Y, and let $\mathcal{H}$ be an ample line bundle on Y. Take a subset S of the underlying topological space of Y such that the stalk of $\mathcal{G}$ at any point in S is free, i.e., there is an open subset $Y_0 \subset Y$ such that $S \subseteq Y_0$ and $\mathcal{G}|_{Y_0}$ is locally free:

(i) We say that a morphism $\mathcal{G} \to \mathcal{G}'$ is *surjective over* S if S and the support of the cokernel do not intersect.

(ii) We say that $\mathcal{G}$ is *globally generated over* S if the natural morphism $H^0(Y, \mathcal{G}) \otimes_k \mathcal{O}_Y \to \mathcal{G}$ is surjective over S.

(iii) We say that $\mathcal{G}$ is *weakly positive over* S if for every $\alpha \in \mathbb{Z}_{>0}$, there is $\beta \in \mathbb{Z}_{>0}$ such that $(S^{\alpha\beta}\mathcal{G})^{**} \otimes \mathcal{H}^{\beta}$ is globally generated over S. Here, $S^{\alpha\beta}(\ )$ and $(\ )^{**}$ denote the $\alpha\beta$-th symmetric product and the double dual, respectively.

(iv) We say that $\mathcal{G}$ is *big over* S if there is $\gamma \in \mathbb{Z}_{>0}$ such that $(S^{\gamma}\mathcal{G}) \otimes \mathcal{H}^{-1}$ is weakly positive over S.

We simply say that $\mathcal{G}$ is *generically globally generated* if $\mathcal{G}$ is globally generated over $\{\eta\}$, where η is the generic point of Y. Furthermore, we simply say that $\mathcal{G}$ is *weakly positive* (resp. *big*) if it is weakly positive (resp. big) over $\{\eta\}$.

Remark 2.7. Let Y, $\mathcal{G}$, S and $\mathcal{H}$ be as above:

(1) The above definitions are independent of the choice of $\mathcal{H}$ [Viehweg 1995, Lemma 2.14].

(2) Suppose that $\mathcal{G}$ is a vector bundle over a smooth projective curve Y. Then $\mathcal{G}$ is weakly positive (resp. big) over Y if and only if $\mathcal{G}$ is nef (resp. ample).

2C. ***Augmented and restricted base loci.*** In this subsection, we recall the definition of the augmented and restricted base locus of a $\mathbb{Q}$-Cartier divisor. In this subsection, we work over a field k.

Definition 2.8 [Ein et al. 2006; Mustaţă 2013]. Let Y be a quasiprojective variety and D a $\mathbb{Q}$-Cartier divisor on Y:

(i) The *stable base locus* $\boldsymbol{B}(D)$ of D is defined as the reduced base locus of mD for sufficiently large and divisible integer m.

(ii) The *augmented base locus* is given by

$$\boldsymbol{B}_+(D) := \bigcap_A \boldsymbol{B}(D-A),$$

where A runs over all the ample $\mathbb{Q}$-Cartier divisors on Y.

(iii) The *restricted base locus* (also called the *nonnef locus* or the *diminished base locus*) is defined by

$$\boldsymbol{B}_-(D) := \bigcup_A \boldsymbol{B}(D+A),$$

where A runs over all the ample $\mathbb{Q}$-Cartier divisors on Y.

Remarks 2.9. (1) In [Ein et al. 2006], the variety Y is assumed to be projective.

(2) Assume that Y is projective. Then the following hold:

- $\boldsymbol{B}_+(D) = \varnothing$ if and only if D is ample [loc. cit., Example 1.7].
- $\boldsymbol{B}_+(D) \neq Y$ if and only if D is big [loc. cit., Example 1.7].
- $\boldsymbol{B}_-(D) = \varnothing$ if and only if D is nef [loc. cit., Example 1.18].
- When k is uncountable, $\boldsymbol{B}_-(D) \neq Y$ if and only if D is pseudoeffective [Mustaţă 2013, Section 2].

(3) Assume that Y is a normal projective variety and D is Cartier. Let S be a subset of the underlying topological space of Y. Then, the weak positivity (resp. bigness) of $\mathcal{O}_Y(D)$ is equivalent to saying that $S \cap \boldsymbol{B}_-(D) = \varnothing$ (resp. $S \cap \boldsymbol{B}_+(D) = \varnothing$).

The next lemma can be proved in the same way as in the proof of [Ein et al. 2006, Proposition 1.19].

Lemma 2.10 [Ein et al. 2006, Propositions 1.19]. *Let the notation be as in Definition 2.8. Let H be an ample $\mathbb{Q}$-Cartier divisor on Y and $\{a_m\}$ a sequence of positive rational numbers that converges to zero. Then $\boldsymbol{B}_-(D) = \bigcup_m \boldsymbol{B}(D + a_m H)$.*

3. Auxiliary lemmas

In this section, we prove several lemmas used in the proofs of the main theorems. Throughout this section, the base field k is assumed to be an F-finite field of characteristic $p > 0$.

Lemma 3.1. *Let W be a normal quasiprojective variety and $W_0 \subseteq W$ an open subset. Let $\mathcal{H}$ be an ample line bundle on W and $\mathcal{G}$ a coherent sheaf on W such that $\mathcal{G}|_{W_0}$ is locally free. Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that, for every $m \geq m_0$, there exists a homomorphism $\theta : \bigoplus^n \mathcal{G} \to \mathcal{H}^m$ of $\mathcal{O}_W$-modules which is surjective over W_0.*

Proof. Take $m \in \mathbb{Z}_{>0}$ such that $\mathcal{G}^* \otimes \mathcal{H}^m$ is generated by its global sections. Since W_0 is Noetherian, we get a surjective morphism $\theta' : \bigoplus^n \mathcal{O}_W \twoheadrightarrow \mathcal{G}^* \otimes \mathcal{H}^m$ for some $n \in \mathbb{Z}_{>0}$. We then obtain

$$\bigoplus^n \mathcal{G} \cong \left(\bigoplus^n \mathcal{O}_W\right) \otimes \mathcal{G} \xrightarrow{\theta' \otimes \mathcal{G}} (\mathcal{G}^* \otimes \mathcal{H}^m) \otimes \mathcal{G} \cong \mathcal{H}om(\mathcal{G}, \mathcal{H}^m) \otimes \mathcal{G} \to \mathcal{H}^m.$$

Here, the last morphism is the evaluation morphism, which is surjective over W_0, since $\mathcal{G}|_{W_0}$ is locally free. Hence, the composite of the above morphisms is the desired morphism. □

Lemma 3.2. *Let W be a normal quasiprojective variety and D a Cartier divisor on W. Let $W_0 \subseteq W$ be an open subset, $\mathcal{E}$ a coherent sheaf that is globally generated over W_0, and $\mathcal{G}$ a coherent sheaf on W such that $\mathcal{G}|_{W_0}$ is free. Suppose that there exists a morphism*

$$\varphi : \mathcal{E} \otimes \left(\bigotimes^{p^e} \mathcal{G} \right) \to \mathcal{O}_W(D) \otimes (F_W^e)^* \mathcal{G}$$

that is surjective over W_0. Then $W_0 \cap \boldsymbol{B}_-(D) = \varnothing$.

Proof. Obviously, we may assume that $\mathcal{E} = \bigoplus^d \mathcal{O}_W$ for some $d \in \mathbb{Z}_{>0}$. Then $\mathcal{E} \otimes \big(\bigotimes^{p^e} \mathcal{G}\big) \cong \bigoplus^d \big(\bigotimes^{p^e} \mathcal{G}\big)$. Take an ample line bundle $\mathcal{H}$ on W such that $\mathcal{G} \otimes \mathcal{H}$ is globally generated. Since we have $\big(\bigotimes^{p^e} \mathcal{G}\big) \otimes \mathcal{H}^{p^e} \cong \bigotimes^{p^e} (\mathcal{G} \otimes \mathcal{H})$ and $((F_Y^e)^* \mathcal{G}) \otimes \mathcal{H}^{p^e} \cong (F_Y^e)^*(\mathcal{G} \otimes \mathcal{H})$, replacing $\mathcal{G}$ (resp. φ) by $\mathcal{G} \otimes \mathcal{H}$ (resp. $\varphi \otimes \mathcal{H}^{p^e}$), we may assume that $\mathcal{G}$ is globally generated. Let $S(\mathcal{G}) \subseteq \mathbb{Q}$ be the set of rational numbers r satisfying the following condition: there is $h \in \mathbb{Z}_{>0}$ such that $p^h r \in \mathbb{Z}$ and that the sheaf

$$\mathcal{O}_W(p^h r D) \otimes (F_W^h)^* \mathcal{G}$$

is globally generated over W_0. We then have $0 \in S(\mathcal{G})$. We prove that $S(\mathcal{G})$ is *not* bounded from above. Choose $r \in S$ and $h \in \mathbb{Z}_{>0}$ so that the above conditions hold. We then have the following sequence of morphisms:

$$\begin{aligned} \bigoplus^d \left(\bigotimes^{p^e} (\mathcal{O}_W(p^h r D) \otimes (F_W^h)^* \mathcal{G}) \right) &\cong \mathcal{O}_W(p^{e+h} r D) \otimes (F_W^h)^* \left(\bigoplus^d \left(\bigotimes^{p^e} \mathcal{G} \right) \right) \\ &\xrightarrow{\psi} \mathcal{O}_W(p^{e+h} r D) \otimes (F_W^h)^* (\mathcal{O}_W(D) \otimes (F_W^e)^* \mathcal{G}) \\ &\cong \mathcal{O}_W((p^h + p^{e+h} r) D) \otimes (F_W^{e+h})^* \mathcal{G} \end{aligned}$$

Here, $\psi := ((F_W^h)^* \varphi) \otimes \mathcal{O}_W(p^h r D)$, so it is surjective over W_0, which implies that the last sheaf is globally generated over W_0. We then see that $1/p^e + r = (p^h + p^{e+h} r)/p^{e+h} \in S(\mathcal{G})$, and hence $S(\mathcal{G})$ can not be bounded from above. Next, we show the assertion. Lemma 3.1 shows that we have an ample Cartier divisor H on W and a morphism $\theta : \bigoplus^n \mathcal{G} \to \mathcal{H} := \mathcal{O}_W(H)$ that is surjective over W_0. One can easily check that $S(\mathcal{G}) \subseteq S(\mathcal{H})$, so $S(\mathcal{H})$ is also not bounded from above. Take $0 < r \in S(\mathcal{H})$. Then for some $h \gg 0$, the sheaf $\mathcal{O}_W(p^h r D) \otimes (F_W^h)^* \mathcal{H} \cong \mathcal{O}_W\big(p^h r\big(D + \frac{1}{r} H\big)\big)$ is globally generated over W_0, and so $\boldsymbol{B}\big(D + \frac{1}{r} H\big) \subseteq W \setminus W_0$. Hence, we conclude from Lemma 2.10 that $\boldsymbol{B}_-(D) \subseteq W \setminus W_0$. □

Before stating the next lemma, we recall Keeler's vanishing theorem, which is a relative version of Fujita's vanishing theorem.

Theorem 3.3 [Keeler 2003, Theorem 1.5]. *Let $f : X \to Y$ be a projective morphism between Noetherian schemes, $\mathcal{F}$ a coherent sheaf on X, and $\mathcal{L}$ an f-ample line bundle on X. Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that*

$$R^i f_*(\mathcal{F} \otimes \mathcal{L}^m \otimes \mathcal{N}) = 0$$

for each $m \geq m_0$ and every f-nef line bundle $\mathcal{N}$ on X.

Note that, in the situation of the theorem, a line bundle on X is said to be f-nef if the restriction to each fiber is nef.

Lemma 3.4. *Let $f : X \to Y$ be a surjective morphism between projective varieties, and $\mathcal{A}$ an ample line bundle on X. Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that $f_*(\mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N})$ is generated by its global sections for each $m \geq m_0$ and every nef line bundle $\mathcal{N}$ on X.*

Proof. Let $\mathcal{H}$ be an ample line bundle on Y that is generated by its global sections. Set $n := \dim Y$. Take $m_1 > 0$ so that $\mathcal{A}^{m_1} \otimes f^*\mathcal{H}^{-n}$ is nef. Applying Theorem 3.3, we get $m_2 > 0$ such that $H^i(X, \mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N}) = 0$ and $R^i f_*(\mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N}) = 0$ for each $m \geq m_2$, $i > 0$ and every nef line bundle $\mathcal{N}$ on X. The Leray spectral sequence then implies that $H^i(Y, f_*(\mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N})) = 0$ for each $i > 0$. Fix $m \geq m_0 := m_1 + m_2$ and a nef line bundle $\mathcal{N}$ on X, and set $\mathcal{M} := \mathcal{A}^m \otimes \mathcal{N}$. We then have $\mathcal{F} \otimes \mathcal{M} \otimes f^*\mathcal{H}^{-i} \cong \mathcal{F} \otimes \mathcal{A}^{m-m_1} \otimes (\mathcal{A}^{m_1} \otimes f^*\mathcal{H}^{-n}) \otimes f^*\mathcal{H}^{n-i}$, so the above argument tells us that the i-th cohomology of $f_*(\mathcal{F} \otimes \mathcal{M} \otimes f^*\mathcal{H}^{-i}) \cong (f_*(\mathcal{F} \otimes \mathcal{M})) \otimes \mathcal{H}^{-i}$ vanishes for $0 < i \leq n$. This means that $f_*(\mathcal{F} \otimes \mathcal{M})$ is 0-regular with respect to $\mathcal{H}$, so this sheaf is generated by its global sections thanks to the Castelnuovo–Mumford regularity (see [Lazarsfeld 2004, Theorem 1.8.5]). □

Lemma 3.5. *Let $g : V \to W$ be a surjective projective morphism from a k-scheme V to a variety W, let $\mathcal{A}$ be a g-ample line bundle on V, and let $\mathcal{F}$ be a coherent sheaf on V that is flat over W. Then, there exists $m_0 \in \mathbb{Z}_{>0}$ such that $g_*(\mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N})$ is locally free for each $m \geq m_0$ and every g-nef line bundle $\mathcal{N}$ on V.*

Proof. By Theorem 3.3, there is $m_0 \in \mathbb{Z}_{>0}$ such that $R^i g_*(\mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N}) = 0$ for each $m \geq m_0$, $i > 0$ and every g-nef line bundle $\mathcal{N}$ on V. Fix $m \geq m_0$ and a g-nef line bundle $\mathcal{N}$ on V, and set $\mathcal{M} := \mathcal{A}^m \otimes \mathcal{N}$. For each $i \geq 0$, define the function $h^i : W \to \mathbb{Z}$ by $h^i(w) := \dim_{k(w)} H^i(V_w, (\mathcal{F} \otimes \mathcal{M})|_{V_w})$. By the choice of m_0 and cohomology and base change (see [Hartshorne 1977, Theorem III 12.11]), we obtain that $h^i = 0$ for each $i > 0$, so $\chi((\mathcal{F} \otimes \mathcal{M})|_{V_w}) = h^0(w)$ for every $w \in W$. Then [Hartshorne 1977, Theorem III 9.9 and its proof] implies that h^0 is constant. Hence, our claim follows from Grauert's theorem (see [Hartshorne 1977, Corollary III 12.9]). □

Lemma 3.6. *Let the notation be as in Lemma 3.5. Let $\mathcal{L}$ be a line bundle on V:*

(1) *If $\mathcal{L}$ is g-free, then there exists $n_0 \in \mathbb{Z}_{>0}$ such that the natural morphism*

$$g_*\mathcal{L}^m \otimes g_*(\mathcal{F} \otimes \mathcal{L}^n \otimes g^*\mathcal{P}) \to g_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes g^*\mathcal{P})$$

is surjective for each $n \geq n_0$, $m > 0$ and every line bundle $\mathcal{P}$ on W.

(2) *If $\mathcal{L}$ is g-ample and g-free, then there exists $n_0 \in \mathbb{Z}_{>0}$ such that the natural morphism*

$$g_*\mathcal{L}^m \otimes g_*(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}) \to g_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes \mathcal{N})$$

is surjective for each $n \geq n_0$, $m > 0$ and every g-nef line bundle $\mathcal{N}$ on V.

Proof. We first show that (2) implies (1). Since $\mathcal{L}$ is g-free, g can be decomposed as

$$g : V \xrightarrow{\sigma} Z \xrightarrow{\tau} W$$

and $\mathcal{L} \cong \sigma^*\mathcal{M}$ for a τ-ample and τ-free line bundle $\mathcal{M}$ on Z. Then we have

$$\begin{array}{ccc}
(\tau_*\mathcal{M}^m) \otimes \tau_*((\sigma_*\mathcal{F}) \otimes \mathcal{M}^n \otimes \tau^*\mathcal{P}) & \longrightarrow & \tau_*((\sigma_*\mathcal{F}) \otimes \mathcal{M}^{m+n} \otimes \tau^*\mathcal{P}) \\
\downarrow & & \\
(\tau_*\sigma_*\sigma^*\mathcal{M}^m) \otimes \tau_*((\sigma_*\mathcal{F}) \otimes \mathcal{M}^n \otimes \tau^*\mathcal{P}) & & \downarrow \cong \\
\downarrow \cong & & \\
\tau_*\sigma_*\mathcal{L}^m \otimes \tau_*\sigma_*(\mathcal{F} \otimes \mathcal{L}^n \otimes \sigma^*\tau^*\mathcal{P}) & \longrightarrow & \tau_*\sigma_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes \sigma^*\tau^*\mathcal{P})
\end{array}$$

Here, the isomorphisms follow from the projection formula. If (2) holds, then the top horizontal arrow is surjective, and hence so is the bottom horizontal arrow, so (1) holds. We show (2). Theorem 3.3 tells us that we have $n_0 \in \mathbb{Z}_{>0}$ such that for each $n \geq n_0$ and every g-nef line bundle $\mathcal{N}$ on V, the sheaf $\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}$ is 0-regular with respect to $\mathcal{L}$ and g. Hence, in the case when $m = 1$, our claim follows from the relative Castelnuovo–Mumford regularity (see [Lazarsfeld 2004, Example 1.8.24]). Using this repeatedly, we see that the natural morphism

$$\left(\bigotimes^m g_*\mathcal{L}\right) \otimes g_*(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}) \to g_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes \mathcal{N})$$

is surjective for each $m \in \mathbb{Z}_{>0}$. This morphism factors through

$$(g_*\mathcal{L}^m) \otimes g_*(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}) \to g_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes \mathcal{N}),$$

which completes the proof. □

Lemma 3.7. *Let V be a k-scheme of pure dimension satisfying S_2 and G_1, let W be a regular variety, and let $g : V \to W$ be a flat projective morphism. Let $E \geq 0$ be an AC divisor on V such that $aK_V + E$ is Cartier for some $a \in \mathbb{Z}_{>0}$ with $p \nmid a$. Set $\Delta := E \otimes \frac{1}{a}$. Let $U \subseteq V$ be a Gorenstein open subset. Suppose that the codimension of $(V \setminus U)|_{V_w}$ (resp. $E|_{V_w}$) in V_w is at least 2 (resp. 1) for every $w \in W$:*

(1) [Patakfalvi et al. 2018, Corollary 3.31] *The set $W_0 := \{w \in W \mid (V_{\overline{w}}, \overline{\Delta|_{U_{\overline{w}}}})$ is F-pure$\}$ is an open subset of W. Here, $V_{\overline{w}}$ is the geometric fiber over w and $\overline{\Delta|_{U_{\overline{w}}}}$ is the $\mathbb{Z}_{(p)}$-AC divisor on $V_{\overline{w}}$ obtained as the unique extension of $\Delta|_{U_{\overline{w}}}$.*

(2) *Assume that W_0 is nonempty. Set $V_0 := g^{-1}(W_0)$. Let $\mathcal{A}$ be a line bundle on V such that $\mathcal{A}|_{V_0}$ is $g|_{V_0}$-ample. Then there exists $m_0 \in \mathbb{Z}_{>0}$ such that*

$$g_{W^e*}(\phi^{(e)}_{(V/W,\Delta)} \otimes \mathcal{A}^m_{W^e} \otimes \mathcal{N}_{W^e}) : g^{(e)}{}_*(\mathcal{L}^{(e)}_{(V/W,\Delta)} \otimes \mathcal{A}^{p^e m} \otimes \mathcal{N}^{p^e}) \to g_{W^e*}(\mathcal{A}^m_{W^e} \otimes \mathcal{N}_{W^e})$$

is surjective over W_0 for each $e \in \mathbb{Z}_{>0}$ with $a \mid (p^e - 1)$, $m \geq m_0$ and every line bundle $\mathcal{N}$ on V whose restriction $\mathcal{N}|_{V_0}$ to V_0 is $g|_{V_0}$-nef

Proof. One can prove (1) by the same argument as that in [Patakfalvi et al. 2018]. We prove (2). Let $d > 0$ be the minimum integer such that $a \mid (p^d - 1)$. For simplicity, let $\phi^{(e)}$ (resp. $\mathcal{L}^{(e)}$) denote $\phi^{(e)}_{(V/W,\Delta)}$ (resp. $\mathcal{L}^{(e)}_{(V/W,\Delta)}$) for each $e \in d\mathbb{Z}_{>0}$. Replacing W by W_0, we may assume that $W_0 = W$. The morphism $\phi^{(e)}|_{V_{\overline{w}}} \cong \phi^{(e)}_{(V_{\overline{w}}/\overline{w},\Delta_{\overline{w}})}$ is then surjective for every $w \in W$ and $e \in d\mathbb{Z}_{>0}$, so $\phi^{(e)}$ is surjective. Applying

Theorem 3.3 to the kernel of $\phi^{(d)}$, we obtain $m_0' \in \mathbb{Z}_{>0}$ such that $g_{W^d*}(\phi^{(d)} \otimes \mathcal{A}^m_{W_d} \otimes \mathcal{N}_{W_d})$ is surjective for every $m \geq m_0'$ and g-nef line bundle $\mathcal{N}$ on V. Take $m_0 \geq m_0'$ so that $m_0 A - (K_{V/W} + \Delta)$ is g-nef, where A is a Cartier divisor on V satisfying $\mathcal{O}_V(A) \cong \mathcal{A}$. We fix $m \geq m_0$ and a g-nef line bundle $\mathcal{N}$ on V. Set $\mathcal{M} := \mathcal{A}^m \otimes \mathcal{N}$. We show that $\psi^{(e)} := g_{W^e*}(\phi^{(e)} \otimes \mathcal{M}_{W^e})$ is surjective for each $e \in d\mathbb{Z}_{>0}$. We have already seen that $\psi^{(d)}$ is surjective. Take $e \in d\mathbb{Z}_{>0}$. Assuming the surjectivity of $\psi^{(e)}$, we show that $\psi^{(e+d)}$ is surjective. By the definition of $\phi^{(e+d)}$, we have

$$\begin{aligned}\psi^{(e+d)} &= g_{W^{e+d}*}(\phi^{(e+d)} \otimes \mathcal{M}_{W^{e+d}}) \\ &\cong (F^d_W)^*(g_{W^e*}(\phi^{(e)} \otimes \mathcal{M}_{W^e})) \circ g^{(e)}_{W^{e+d}*}(\phi^{(d)}_{(V^e/W^e,\Delta^e)} \otimes (\mathcal{L}^{(e)} \otimes \mathcal{M}^{p^e})_{W^{e+d}}) \\ &= (F^d_W)^*(\psi^{(e)}) \circ g^{(e)}_{W^{e+d}*}(\phi^{(d)}_{(V^e/W^e,\Delta^e)} \otimes (\mathcal{L}^{(e)} \otimes \mathcal{M}^{p^e})_{W^{e+d}}).\end{aligned}$$

Note that the surjectivity of $\psi^{(e)}$ induces that of $(F^d_W)^*(\psi^{(e)})$, so we only need to show that

$$g^{(e)}_{W^{e+d}*}(\phi^{(d)}_{(V^e/W^e,\Delta^e)} \otimes (\mathcal{L}^{(e)} \otimes \mathcal{M}^{p^e})_{W^{e+d}})$$

is surjective. We can rewrite this morphism as

$$g_{W^d*}(\phi^{(d)} \otimes (\mathcal{O}_V((1-p^e)(K_{V/W}+\Delta)) \otimes \mathcal{M}^{p^e})_{W^d})$$

identifying $g^{(e)} : V^e \to W^e$ with $g : V \to W$. This morphism is surjective if

$$\mathcal{P} := \mathcal{O}_V((1-p^e)(K_{V/W}+\Delta)) \otimes \mathcal{M}^{p^e} \otimes \mathcal{A}^{-m_0}$$

is g-nef, by the choice of m_0'. This g-nefness follows from the isomorphisms

$$\mathcal{P} \cong \mathcal{O}_V((1-p^e)(K_{V/W}+\Delta)) \otimes \mathcal{A}^{mp^e - m_0} \otimes \mathcal{N}^{p^e} \cong \mathcal{O}_V((p^e-1)(m_0 A - (K_{V/W}+\Delta))) \otimes \mathcal{A}^{(m-m_0)p^e} \otimes \mathcal{N}^{p^e}$$

and the choice of m_0. □

4. Main theorems and corollaries

In this section, we prove the main theorems. After this, we give several corollaries.

4A. ***Main theorems.*** In this subsection, we prove Theorems 4.1 and 4.2. *Throughout this subsection, we use the following notation:*

Fix an F-finite field k. Let $f : X \to Y$ be a surjective projective morphism from a pure dimensional quasiprojective k-scheme X satisfying S_2 and G_1 to a normal quasiprojective variety Y. Let E be an effective AC-divisor on X and $a > 0$ an integer such that $aK_X + E$ is Cartier. Set $\Delta := E \otimes a^{-1}$. Let $U \subset X$ be the Gorenstein locus and $W \subseteq Y$ the maximal regular open subset such that $g := f|_V : V \to W$ is flat, where $V := f^{-1}(W)$. Suppose that there exists a scheme-theoretic point $w \in W$ with the following properties:

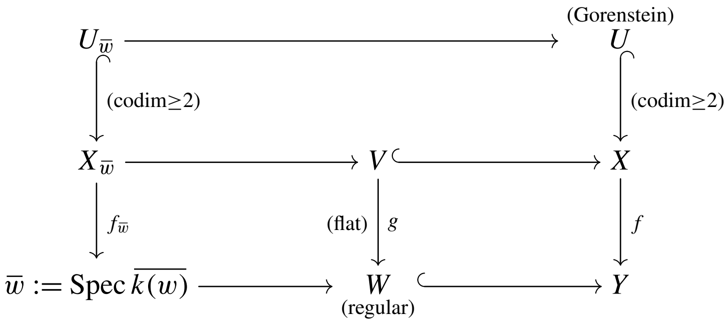
(i) $\operatorname{codim}_{X_w}(X_w \setminus U_w) \geq 2$.

(ii) Supp E does not contain any irreducible component of X_w.

(iii) $(X_{\overline{w}}, \overline{\Delta|_{U_{\overline{w}}}})$ is F-pure.

Here, $X_{\overline{w}}$ (resp. $U_{\overline{w}}$) is the geometric fiber of f (resp. $f|_U$) over w, and $\overline{\Delta|_{U_{\overline{w}}}}$ is the $\mathbb{Z}_{(p)}$-AC divisor on $X_{\overline{w}}$ obtained as the unique extension of $\Delta|_{U_{\overline{w}}}$. Let D be a $\mathbb{Q}$-Cartier divisor on Y.

We now have the following commutative diagram whose squares are cartesian:



Here, "(codim ≥ 2)" means the morphism is an open immersion whose complement is of codimension at least 2.

In this situation, we prove the following two theorems:

Theorem 4.1. *Let the notation be as above. Assume that X is projective and $K_X + \Delta$ is $\mathbb{Q}$-Cartier*:

(1) *If $p \nmid a$ and $-(K_X+\Delta+f^*D)$ is nef, then the closure of $\boldsymbol{B}_-(-(K_W+D|_W))$ in Y does not contain w.*

(2) *If K_X is $\mathbb{Q}$-Cartier and $-(K_X + \Delta + f^*D)$ is ample, then $\boldsymbol{B}_+(-(K_W + D|_W))$ does not contain w.*

Theorem 4.2. *Let the notation be as above. Take $b \in \mathbb{Z}_{>0}$ so that bD is Cartier and set $\mathcal{M} := \mathcal{O}_X(-ab(K_X + \Delta + f^*D))$. Assume that $\mathcal{M}|_U$ is $f|_U$-free for some open subset $U \subseteq X$ containing $f^{-1}(w)$. If $p \nmid a$ and if $f_*\mathcal{M}$ is weakly positive over $\{w\}$, then $w \notin \boldsymbol{B}_-(-(K_W + D|_W))$.*

Remarks 4.3. (1) When X is normal, we may choose a as the Cartier index of $K_X + \Delta$.

(2) When $w \in W$ is the generic point, assumptions (i) and (ii) above hold. However, assumption (iii) does not necessarily hold even if X is smooth and $\Delta = 0$.

(3) If X is a variety, then $\operatorname{codim}_Y(Y \setminus W) \geq 2$. Furthermore, if $\operatorname{codim}_Y(Y \setminus W) \geq 2$ and K_Y is $\mathbb{Q}$-Cartier, then $\boldsymbol{B}_-(-(K_W + D|_W)) = \boldsymbol{B}_-(-(K_Y + D))|_W$.

Remark 4.4. Take $m \in \mathbb{Z}$. In the following proof, $m\Delta$ (resp. mf^*D) denotes $\frac{m}{a}E$ (resp. $f^*(mD)$) if $a \mid m$ (resp. mD is Cartier). Note that there may be two distinct AC divisors on X that are equal as $\mathbb{Z}_{(p)}$-AC divisors.

Proof of Theorem 4.1. We first show that (1) implies (2). By the assumption in (2), Δ is $\mathbb{Q}$-Cartier. Write $a = mp^c$, where $m, c \in \mathbb{Z}_{\geq 0}$ with $p \nmid m$. Take $e \gg 0$. Put $a' := m(p^e+1)$. Then $p \nmid a'$. Set $\Delta' := (p^{e-c}E) \otimes \frac{1}{a'}$. We then have $\Delta' = p^{e-c}a/a'\Delta = p^e/(p^e+1)\Delta \leq \Delta$, so (X, Δ') satisfies assumption (iii). Since $e \gg 0$, we may assume that $-(K_X + \Delta' + f^*D) = -(K_X + \Delta + f^*D) - 1/(p^e+1)\Delta$ is ample. Let H be an ample $\mathbb{Q}$-Cartier divisor on Y such that $-(K_X + \Delta' + f^*(D + H))$ is nef. Then, (1) implies that

$w \notin \boldsymbol{B}_-(-(K_W + D|_W + H|_W))$. Putting $\Gamma := -(K_W + D|_W)$, we obtain that $\boldsymbol{B}_+(\Gamma) \subseteq \boldsymbol{B}\big(\Gamma - \frac{1}{2}H|_W\big) \subseteq \boldsymbol{B}_-(\Gamma - H|_W) \subseteq W \setminus \{w\}$.

We begin the proof of (1). Define W_0 to be the subset of points in W satisfying conditions (i)–(iii). We first claim that $W_0 \subseteq W$ is open. Lemma 3.7 (1) tells us that (iii) is an open condition on W. Set $r := \dim X - \dim Y$ and $Z := (X \setminus U)_{\mathrm{red}}$. Then, condition (i) (resp. (ii)) is equivalent to saying that $\dim Z_w \le r - 2$ (resp. $\dim E_w \le r - 1$). Hence, our claim follows from Chevalley's theorem [EGA IV$_3$ 1966, Corollaire 13.1.5], which says that the function $\delta(w) := \dim Z_w$ (resp. $\delta(w) := \dim E_w$) on W is upper semicontinuous. Next, let us introduce some notation:

(n1) Take $d \in \mathbb{Z}_{>0}$ with $a \mid d$ such that dD and $d(K_X + \Delta)$ are Cartier.

(n2) Let $\mathcal{A}'$ be an ample line bundle on X and put $\mathcal{A}'|_V := \mathcal{A}$.

(n3) Denote $g_*\mathcal{A}$ by $\mathcal{G}$.

Lemmas 3.4–3.7 tell us that, by replacing $\mathcal{A}'$, we may assume that the following conditions hold:

(a1) For every nef line bundle $\mathcal{N}'$ on X and each $0 \le i < d$ with $a \mid i$, the sheaf

$$g_*(\mathcal{O}_V(-i(K_V + \Delta|_V)) \otimes \mathcal{A} \otimes \mathcal{N}) \cong (f_*(\mathcal{O}_X(-i(K_X + \Delta)) \otimes \mathcal{A}' \otimes \mathcal{N}'))\big|_W$$

is a locally free sheaf generated by its global sections, where $\mathcal{N} := \mathcal{N}'|_V$. In particular, $\mathcal{G} = g_*\mathcal{A}$ is locally free (Lemmas 3.4 and 3.5).

(a2) For a g-nef line bundle $\mathcal{N}$ on V, the natural morphism

$$\mathcal{G} \otimes g_*(\mathcal{A}^n \otimes \mathcal{N}) = (g_*\mathcal{A}) \otimes g_*(\mathcal{A}^n \otimes \mathcal{N}) \to g_*(\mathcal{A}^{n+1} \otimes \mathcal{N})$$

is surjective for each $n \in \mathbb{Z}_{>0}$ (Lemma 3.6).

(a3) For each $e \in \mathbb{Z}_{>0}$ with $a \mid (p^e - 1)$, there exists a morphism

$$g_*(\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A}^{p^e}) \to g_{W^e *}\mathcal{A}_{W^e} \cong (F^e_W)^* g_*\mathcal{A} = (F^e_W)^*\mathcal{G}$$

that is surjective over W_0 (Lemma 3.7).

We continue to introduce some notation:

(n4) Take an ample Cartier divisor H on W such that for each $0 \le i < d$, there is a surjective morphism $\bigoplus^t \omega_Y^{-i} \otimes \mathcal{G} \to \mathcal{H} := \mathcal{O}_W(H)$. Such an H exists as shown in Lemma 3.1.

(n5) Fix $e \in \mathbb{Z}_{>0}$ with $a \mid (p^e - 1)$ and write $p^e - 1 = dq + r$ for $q, r \in \mathbb{Z}$ with $0 \le r < d$. Note that $a \mid r$.

(n6) Set $\mathcal{N}'$ to be the nef line bundle $\mathcal{O}_X(-dq(K_X + \Delta + f^*D))$ and

$$\mathcal{N} := \mathcal{N}'|_V \cong \mathcal{O}_V(-dq(K_V + \Delta|_V + g^*(D|_W))).$$

Also, set $\mathcal{P} := \mathcal{O}_W(-dq(K_W + D|_W))$.

(n7) Recall that $\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} := \mathcal{O}_V((1 - p^e)(K_{V/W} + \Delta|_V))$. Here, we identify V^e (resp. W^e) with V (resp. W). Note that $\mathcal{L}_{(V/W,\Delta|_V)}$ is g-nef in this situation.

We prove the assertion. We have

$$\mathcal{N} \otimes g^*\mathcal{P}^{-1} \underset{(n6)}{\cong} \mathcal{O}_V(-dq(K_{V/W} + \Delta|_V)) \underset{(n7)}{\cong} \mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{O}_V(r(K_{V/W} + \Delta|_V)).$$

We then obtain

$$\mathcal{O}_V(-r(K_V + \Delta|_V)) \otimes \mathcal{A} \otimes \mathcal{N} \cong \mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A} \otimes g^*(\mathcal{P} \otimes \omega_W^{-r}),$$

so the projection formula implies that

$$g_*(\mathcal{O}_V(-r(K_V + \Delta|_V)) \otimes \mathcal{A} \otimes \mathcal{N}) \cong g_*(\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A}) \otimes \mathcal{P} \otimes \omega_W^{-r}. \qquad (*)$$

It then follows from (a1) that the right-hand side is globally generated. Hence, we may apply Lemma 3.2 to the composition of the following morphisms which are surjective over W_0:

$$\begin{aligned}\Big(\bigotimes^{p^e} \mathcal{G}\Big) \otimes \bigoplus^{t} ((g_*(\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A})) \otimes \mathcal{P} \otimes \omega_Y^{-r}) &\cong \Big(\bigotimes^{p^e-1} \mathcal{G}\Big) \otimes (g_*(\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A})) \otimes \mathcal{P} \otimes \Big(\bigoplus^{t} \mathcal{G} \otimes \omega_Y^{-r}\Big) \\ &\xrightarrow{(n4)} \Big(\bigotimes^{p^e-1} \mathcal{G}\Big) \otimes (g_*(\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A})) \otimes \mathcal{P} \otimes \mathcal{H} \\ &\xrightarrow{(a2)} (g_*(\mathcal{L}^{(e)}_{(V/W,\Delta|_V)} \otimes \mathcal{A}^{p^e})) \otimes \mathcal{P} \otimes \mathcal{H} \\ &\xrightarrow{(a3)} ((F_Y^e)^*\mathcal{G}) \otimes \mathcal{P} \otimes \mathcal{H}.\end{aligned}$$

Note that $\mathcal{P} \otimes \mathcal{H} \cong \mathcal{O}_W\big(dq\big(-(K_W + D|_W) + \frac{1}{dq}H\big)\big)$. Thus, we obtain

$$\boldsymbol{B}\big(-(K_Y + D) + \tfrac{2}{dq}H\big) \underset{\text{by def}}{\subseteq} \boldsymbol{B}_-\big(-(K_Y + D) + \tfrac{1}{dq}H\big) \underset{\text{Lemma 3.2}}{\subseteq} W \setminus W_0.$$

Since $\dfrac{2}{dq}$ goes to zero as $e \to \infty$, we conclude from Lemma 2.10 that $\boldsymbol{B}_-(-(K_Y + D)) \subseteq W \setminus W_0$. $\square$

Proof of Theorem 4.2. Replacing $f : X \to Y$ by $g : V \to W$, we may assume that $X = V$ and $Y = W$. Then f is flat. Set $Y_0 := Y \setminus (f(X \setminus U))$. Note that $Y_0 \subseteq Y$ is an open subset containing w. We may assume $U = f^{-1}(Y_0)$ by shrinking U. Put $f_0 := f_{Y_0} : U \to Y_0$. Since $\mathcal{M}|_U$ is f_0-free by assumption, f_0 can be decomposed as

$$f_0 : U \xrightarrow{\sigma} T \xrightarrow{\tau} Y_0$$

and $\mathcal{M}|_U \cong \sigma^*\mathcal{R}$ for a τ-ample line bundle $\mathcal{R}$ on T. For each $c \in \mathbb{Z}_{>0}$, the projection formula says that $\sigma_*(\mathcal{M}^c|_U) \cong (\sigma_*\mathcal{O}_U) \otimes \mathcal{R}^c$, so we get

$$(f_*\mathcal{M}^c)|_{Y_0} \cong (f_0)_*(\mathcal{M}^c|_U) \cong \tau_*\sigma_*(\mathcal{M}^c|_U) \cong \tau_*((\sigma_*\mathcal{O}_U) \otimes \mathcal{R}^c).$$

The last sheaf is locally free if $c \gg 0$, as shown in [Hartshorne 1977, Theorem III 9.9 and its proof]. Fix such a c. Replacing b by bc, we may assume that $(f_*\mathcal{M})|_{Y_0}$ is locally free. We then have a closed subset $Z \subset Y$ of codimension at least 2 such that $Y_0 \subseteq Y \setminus Z$ and $(f_*\mathcal{M})|_{Y \setminus Z}$ is locally free. Shrinking Y to $Y \setminus Z$, we may assume that $f_*\mathcal{M}$ is locally free. Take $\alpha \in \mathbb{Z}_{>0}$ and an ample Cartier divisor H on Y. Set

$\mathcal{H} := \mathcal{O}_Y(H)$. Then, there is $\beta \in \mathbb{Z}_{>0}$ such that $(S^{\alpha\beta} f_*\mathcal{M}) \otimes \mathcal{H}^\beta$ is globally generated over $\{w\}$. We may assume that this sheaf is globally generated over Y_0, shrinking Y_0 to a neighborhood of w. We set

(n8) $d := ab\alpha\beta$

and use notation (n2) and (n3) in the proof of Theorem 4.1. Assume that $\mathcal{A}$ satisfies conditions (a2) and (a3). Furthermore, we add the following notation:

(n9) Take $n_0 \in \mathbb{Z}_{>0}$ with $a \mid n_0$ such that for each $n \geq n_0$, $0 \leq i < d$ and every line bundle $\mathcal{Q}$ on Y, the natural morphism

$$(f_*\mathcal{M}) \otimes f_*(\mathcal{M}^n \otimes \mathcal{O}_X(-i(K_X+\Delta)) \otimes \mathcal{A} \otimes f^*\mathcal{Q}) \to f_*(\mathcal{M}^{n+1} \otimes \mathcal{O}_X(-i(K_X+\Delta)) \otimes \mathcal{A} \otimes f^*\mathcal{Q})$$

is surjective over Y_0. We can find such an n_0 by Lemma 3.6.

(n10) Choose $\nu \in \mathbb{Z}_{>0}$ so that

- $\mathcal{H}^\nu \otimes f_*\mathcal{O}_X(-i(K_X+\Delta))$ is generated by its global sections for each $i \in a\mathbb{Z}$ with $abn_0 \leq i < abn_0 + d$, and
- there is a morphism $\bigoplus^t \mathcal{G} \to \mathcal{H}^\nu$ that is surjective over Y_0.

The existence of such a ν is ensured by Lemma 3.1.

(n11) Fix $e \in \mathbb{Z}_{>0}$ with $a \mid (p^e - 1)$ and write $p^e - 1 = dq + r$ for $q, r \in \mathbb{Z}$ with $abn_0 \leq r < abn_0 + d$. Note that $a \mid r$.

We also use notation (n6) and (n7) in the proof of Theorem 4.1. Then,

$$\mathcal{N} = \mathcal{O}_X(-dq(K_X+\Delta+f^*D)) \cong \mathcal{M}^{\alpha\beta q} \quad \text{and}$$
$$\mathcal{O}_X(-r(K_X+\Delta)) \cong \mathcal{M}^{n_0} \otimes \mathcal{O}_X(-(abn_0 - r)(K_X+\Delta)) \otimes g^*\mathcal{O}_Y(abn_0 D).$$

Note that $0 \leq abn_0 - r < d$. Therefore, the morphisms

$$\begin{aligned}
&(S^q(\mathcal{H}^\beta \otimes S^{\alpha\beta} f_*\mathcal{M})) \otimes \mathcal{H}^\nu \otimes f_*(\mathcal{O}_X(-r(K_X+\Delta)) \otimes \mathcal{A}) \\
&\qquad \cong \mathcal{H}^{\beta q+\nu} \otimes (S^q(S^{\alpha\beta} f_*\mathcal{M})) \otimes f_*(\mathcal{O}_X(-r(K_X+\Delta)) \otimes \mathcal{A}) \\
&\qquad \xrightarrow{(n9)} \mathcal{H}^{\beta q+\nu} \otimes f_*(\mathcal{O}_X(-r(K_X+\Delta)) \otimes \mathcal{A} \otimes \mathcal{M}^{\alpha\beta q}) \\
&\qquad \cong \mathcal{H}^{\beta q+\nu} \otimes f_*(\mathcal{O}_X(-r(K_X+\Delta)) \otimes \mathcal{A} \otimes \mathcal{N}) \\
&\qquad \underset{(*)}{\cong} \mathcal{H}^{\beta q+\nu} \otimes f_*(\mathcal{L}^{(e)}_{(X/Y,\Delta)} \otimes \mathcal{A}) \otimes \mathcal{P} \otimes \omega_Y^{-r}
\end{aligned}$$

are surjective over Y_0. Here, the last isomorphism is $(*)$ in the proof of Theorem 4.1. By the choice of β and ν, we see that the first sheaf is globally generated over Y_0, and hence so is the last sheaf. Now, we have the following sequence of morphisms that are surjective over Y_0:

$$\begin{aligned}(\bigotimes^{p^e}\mathcal{G})\otimes\bigoplus^t(\mathcal{H}^{\beta q+\nu}\otimes f_*(\mathcal{L}^{(e)}_{(X/Y,\Delta)}\otimes\mathcal{A})\otimes\mathcal{P}\otimes\omega_Y^{-r})&\\ \cong(\bigotimes^{p^e-1}\mathcal{G})\otimes(f_*(\mathcal{L}^{(e)}_{(X/Y,\Delta)}\otimes\mathcal{A}))\otimes\mathcal{P}\otimes\big(\bigoplus^t\mathcal{G}\otimes\omega_Y^{-r}\big)\otimes\mathcal{H}^{\beta q+\nu}&\\ \xrightarrow{(n10)}(\bigotimes^{p^e-1}\mathcal{G})\otimes(f_*(\mathcal{L}^{(e)}_{(X/Y,\Delta)}\otimes\mathcal{A}))\otimes\mathcal{P}\otimes\mathcal{H}^{\beta q+2\nu}&\\ \xrightarrow{(a2)}(f_*(\mathcal{L}^{(e)}_{(X/Y,\Delta)}\otimes\mathcal{A}^{p^e}))\otimes\mathcal{P}\otimes\mathcal{H}^{\beta q+2\nu}&\\ \xrightarrow{(a3)}((F_Y^e)^*\mathcal{G})\otimes\mathcal{P}\otimes\mathcal{H}^{\beta q+2\nu}.&\end{aligned}$$

Note that $\mathcal{P}\otimes\mathcal{H}^{\beta q+2\nu}\cong\mathcal{O}_X(dq(-(K_Y+D)+(\beta q+2\nu)/(dq)H))$. Replacing e by some larger one if necessary, we may assume that $\beta q>2\nu$. Then,

$$\boldsymbol{B}\Big(-(K_Y+D)+\frac{2\beta q}{dq}H\Big)\underset{\text{by def}}{\subseteq}\boldsymbol{B}_-\Big(-(K_Y+D)+\frac{\beta q+2\nu}{dq}H\Big)\underset{\text{Lemma 3.2}}{\subseteq}Y\setminus Y_0.$$

Since $\dfrac{2\beta q}{dq}=\dfrac{2}{ab\alpha}$ goes to zero as $\alpha\to\infty$, we conclude from Lemma 2.10 that $\boldsymbol{B}_-(-(K_Y+D))\subseteq Y\setminus Y_0$. □

4B. *Corollaries.* In this subsection, we give several corollaries of the main theorems. *Throughout this subsection, we use the following notation:*

Let $f:X\to Y$ be a surjective morphism between regular projective varieties over an F-finite field, Δ an effective $\mathbb{Q}$-divisor on X, and a the Cartier index of Δ. Let D be a $\mathbb{Q}$-divisor on Y. Let $\bar\eta$ denote the geometric generic point of Y.

Corollary 4.5. *Assume that f is flat. Suppose that* Supp Δ *does not contain any component of any fiber, and $(X_{\bar y},\Delta_{\bar y})$ is F-pure for every point $y\in Y$:*

(1) *If $p\nmid a$ and if $-(K_X+\Delta+f^*D)$ is nef, then so is $-(K_Y+D)$.*

(2) *If $-(K_X+\Delta+f^*D)$ is ample, then so is $-(K_Y+D)$.*

Proof. This follows from Theorem 4.1 and Remarks 2.9 immediately. □

The author learned the proof of Corollary 4.6(3) below from professor Yoshinori Gongyo.

Corollary 4.6. *Assume that $(X_{\bar\eta},\Delta_{\bar\eta})$ is F-pure:*

(1) *If $p\nmid a$ and if $-(K_X+\Delta+f^*D)$ is nef, then $-(K_Y+D)$ is pseudoeffective.*

(2) *If $-(K_X+\Delta+f^*D)$ is ample, then $-(K_Y+D)$ is big.*

(3) *If $(X_{\bar\eta},\Delta_{\bar\eta})$ is strongly F-regular and if $-(K_X+\Delta+f^*D)$ is nef and big, then $-(K_Y+D)$ is big.*

Proof. By Remarks 2.9, we see that (1) and (2) follow from (1) and (2) of Theorem 4.1, respectively. We prove (3). By Kodaira's lemma, there is a $\mathbb{Q}$-divisor $\Delta'\geq\Delta$ on X such that $-(K_X+\Delta'+f^*D)$ is ample and $(X_{\bar\eta},\Delta'_{\bar\eta})$ is again strongly F-regular. Hence (3) follows from (2). □

Corollary 4.7. *Assume that $(X_{\bar\eta},\Delta_{\bar\eta})$ is F-pure. If $p\nmid a$ and if $K_X+\Delta$ is numerically equivalent to $f^*(K_Y+L)$ for some $\mathbb{Q}$-divisor L on Y, then L is pseudoeffective.*

Proof. Set $D := -(K_Y + L)$. Then, $K_X + \Delta + f^*D$ is numerically trivial, so it is nef. Hence, by Corollary 4.6(1), we obtain the assertion. □

Remarks 4.8. (1) In the situation of Corollary 4.7, it is known that if $(X_{\bar\eta}, \Delta_{\bar\eta})$ is globally F-split, then $\kappa(L) \geq 0$ (see [Das and Schwede 2017, Theorem B] or [Ejiri 2017, Theorem 3.18]). Of course, $(X_{\bar\eta}, \Delta_{\bar\eta})$ is not necessary globally F-split even if $X_{\bar\eta}$ is a smooth curve and $\Delta = 0$. At the same time, Chen and Zhang proved that the relative canonical divisor of an elliptic fibration has nonnegative Kodaira dimension [Chen and Zhang 2015, 3.2].

(2) In the case when $\dim Y = 1$, Corollary 4.7 follows from a result of Patakfalvi [2014, Theorem 1.6].

Corollary 4.9. *Assume that f is flat and every geometric fiber is F-pure*:

(1) *If X is a Fano variety, that is, $-K_X$ is ample, then so is Y.*

(2) *Suppose that the geometric generic fiber of f is strongly F-regular. If X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y.*

Proof. Putting $\Delta = 0$ and $D = 0$, we see that the assertions follow from Corollaries 4.5(2) and 4.6(3). □

Corollary 4.10. *Assume that Y has positive dimension*:

(1) *If $(X_{\bar\eta}, \Delta_{\bar\eta})$ is F-pure, then $-(K_{X/Y} + \Delta)$ is not ample.*

(2) *If $(X_{\bar\eta}, \Delta_{\bar\eta})$ is strongly F-regular, then $-(K_{X/Y} + \Delta)$ cannot be both nef and big.*

Proof. Set $D := -K_Y$. Then $-(K_X + \Delta + f^*D) = -(K_{X/Y} + \Delta)$. Since $-(K_Y + D) = 0$ is not big, the assertions follows from Corollary 4.6(2) and (3). □

Corollary 4.11. *Assume that Y has positive dimension. Suppose that $\mathcal{O}_X(-ab(K_X + \Delta))|_{X_\eta}$ is globally generated for some $b \in \mathbb{Z}_{>0}$. If $p \nmid a$ and if $(X_{\bar\eta}, \Delta_{\bar\eta})$ is F-pure, then $f_*\mathcal{O}_X(-ab(K_{X/Y} + \Delta))$ is not big.*

Proof. Set $\mathcal{G}_{(-l)} := f_*\mathcal{O}_X(al(K_X + \Delta))$ for each $l \in \mathbb{Z}$. Suppose that $\mathcal{G}_{(b)}$ is big. Then, $\mathcal{H}^{-1} \otimes S^\gamma \mathcal{G}_{(b)}$ is weakly positive for some $\gamma \in \mathbb{Z}_{>0}$ and an ample line bundle $\mathcal{H}$ on Y. Take $n_0 \in \mathbb{Z}_{>0}$ so that the natural morphism $\mathcal{G}_{(b)} \otimes \mathcal{G}_{(n)} \to \mathcal{G}_{(b+n)}$ is generically surjective for each $n \geq n_0$. We can find such an n_0 by Lemma 3.6. Choose $\nu \in \mathbb{Z}_{>0}$ so that $\mathcal{H}^\nu \otimes \mathcal{G}_{(n_0)}$ is globally generated. Fix $l \in \mathbb{Z}$ with $l > \nu$. Using the natural morphism $S^l(S^\gamma \mathcal{G}_{(b)}) \otimes \mathcal{G}_{(n_0)} \to \mathcal{G}_{(bl\gamma+n_0)}$ and [Viehweg 1995, Lemma 2.16], we see that $\mathcal{H}^{\nu-l} \otimes \mathcal{G}_{(bl\gamma+n_0)}$ is weakly positive. Let H be a Cartier divisor on Y satisfying $\mathcal{O}_Y(H) \cong \mathcal{H}$ and set $H' := (l-\nu)/(a(bl\gamma+n_0))H$. The projection formula then shows that

$$\mathcal{H}^{\nu-l} \otimes \mathcal{G}_{(bl\gamma+n_0)} \cong f_*\mathcal{O}_X(-a(bl\gamma+n_0)(K_{X/Y} + \Delta + f^*H')).$$

It then follows from Theorem 4.2 that $\boldsymbol{B}_-(-H') \neq \varnothing$, i.e., $-H'$ is pseudoeffective, a contradiction. □

5. Results in arbitrary characteristic

In this section we generalize several results in Section 4 to arbitrary characteristic. In particular, we prove the characteristic zero counterparts of Corollaries 4.10 and 4.11 (Theorems 5.4 and 5.5). We also deal with a morphism that is special but not necessarily smooth, and show that the image of a Fano variety is again a Fano variety.

To begin with, let us recall the following definition:

Definition 5.1. Let X be a normal variety over a field k of characteristic zero, and Δ an effective $\mathbb{Q}$-Weil divisor on X. Let (X_R, Δ_R) be a model of (X, Δ) over a finitely generated $\mathbb{Z}$-subalgebra R of k. We say that (X, Δ) is of *dense F-pure type* (resp. *strongly F-regular type*) if there exists a dense (resp. dense open) subset $S \subseteq \operatorname{Spec} R$ such that (X_μ, Δ_μ) is *F-pure* (resp. strongly F-regular) for all closed points $\mu \in S$.

Remark 5.2. The above definition can be generalized in an obvious way to the case where X is a finite disjoint union of varieties over k.

Theorem 5.3 [Takagi 2004, Corollary 3.4]. *Let X be a normal variety over a field of characteristic zero, and Δ an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then (X, Δ) is klt if and only if it is of strongly F-regular type.*

Theorem 5.4. *Let k be an algebraically closed field of characteristic zero. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties of positive dimension, and Δ an effective $\mathbb{Q}$-divisor on X. If (X_y, Δ_y) is of dense F-pure type (resp. klt) for every general closed point $y \in Y$, then $-(K_{X/Y} + \Delta)$ cannot be ample (resp. both nef and big).*

Proof. Let X_R, Δ_R, Y_R, y_R and f_R be models of X, Δ, Y, y and f over a finitely generated $\mathbb{Z}$-algebra R, respectively. We may assume that $(X_R)_{y_R}$ is a model of X_y over R. We first suppose that (X_y, Δ_y) is of dense F-pure type for a general closed point $y \in Y$. Then, there is a dense subset $S \subseteq \operatorname{Spec} R$ such that $((X_y)_\mu, \Delta_\mu)$ is F-pure for every $\mu \in S$. Note that $(X_y)_\mu \cong (X_\mu)_{y_\mu}$ and $(\Delta_y)_\mu = (\Delta_\mu)_{y_\mu}$. Corollary 4.10 then implies that $-(K_{X_\mu/Y_\mu} + \Delta_\mu)$ is not ample, which means that $-(K_{X/Y} + \Delta)$ is not ample. We next suppose that (X_y, Δ_y) is klt for every general closed point $y \in Y$. If $-(K_{X/Y} + \Delta)$ is nef and big, then by Kodaira's lemma, there is $\Delta' \geq \Delta$ such that $-(K_{X/Y} + \Delta')$ is ample and (X_y, Δ'_y) is klt for a general closed point $y \in Y$. However, Theorem 5.3 tells us that (X_y, Δ'_y) is of dense F-pure type, which contradicts the above arguments. □

Theorem 5.5. *Let k be an algebraically closed field of characteristic zero. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties of positive dimension, and Δ an effective $\mathbb{Q}$-divisor on X. Assume that (X_y, Δ_y) is of dense F-pure type for a general closed point $y \in Y$. Let $\bar{\eta}$ be a geometric generic point of Y. If $\mathcal{O}_X(-m(K_{X/Y} + \Delta))|_{X_{\bar{\eta}}}$ is globally generated for some $m > 0$ such that $m\Delta$ is integral, then $f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ is not big.*

Proof. Set $\mathcal{G} := f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ and $r := \operatorname{rank} \mathcal{G}$. Since $y \in Y$ is general, f is flat at every point in $f^{-1}(y)$ and $\dim H^0(X_y, -m(K_{X_y} + \Delta_y)) = r$. Let X_R, Δ_R, Y_R, y_R and f_R be models of X, Δ, Y, y and f, respectively. By replacing R if necessary, we may assume that $f_{R*}\mathcal{O}_{X_R}(-m(K_{X_R/Y_R}+\Delta_R))$ and $(X_R)_{y_R}$ are respectively models of $\mathcal{G}$ and X_y. We may further assume that $\dim H^0((X_\mu)_{y_\mu}, -m(K_{X_\mu}+\Delta_\mu)_{y_\mu})=r$ for every $\mu \in \operatorname{Spec} R$. Then, [Hartshorne 1977, Corollary 12.9] implies that the natural morphism

$$\mathcal{G}_\mu = f_{R*}\mathcal{O}_{X_R}(-m(K_{X_R/Y_R}+\Delta_R))|_{Y_\mu} \to f_{\mu*}\mathcal{O}_{X_\mu}(-m(K_{X_\mu/Y_\mu}+\Delta_\mu))$$

is surjective over y_μ. Since $f_{\mu*}\mathcal{O}_{X_\mu}(-m(K_{X_\mu/Y_\mu}+\Delta_\mu))$ is not big as shown in Corollary 4.11, $\mathcal{G}_\mu$ is also not big. Hence, the lemma below completes the proof. □

Lemma 5.6. *Let $\mathcal{G}$ be a torsion-free coherent sheaf on a smooth quasiprojective variety Y over an algebraically closed field of characteristic zero. Let Y_R and $\mathcal{G}_R$ be models of Y and $\mathcal{G}$ respectively over a finitely generated $\mathbb{Z}$-algebra R. If $\mathcal{G}$ is big, then there exists a dense open subset $S \subseteq \operatorname{Spec} R$ such that $\mathcal{G}_\mu$ is big for every $\mu \in S$.*

Proof. Let $Z \subset Y$ be a closed subset of codimension at least 2 such that $\mathcal{G}|_{Y\setminus Z}$ is locally free. Replacing Y by $Y \setminus Z$, we may assume that $\mathcal{G}$ is locally free. By the definition, we have $\gamma \in \mathbb{Z}_{>0}$ such that $\mathcal{H}^{-1}\otimes S^\gamma\mathcal{G}$ is weakly positive for some ample line bundle $\mathcal{H}$ on Y. Then, there is $\beta \in \mathbb{Z}_{>0}$ such that $\mathcal{H}^\beta \otimes S^{2\beta}(\mathcal{H}^{-1}\otimes S^\gamma\mathcal{G}) \cong \mathcal{H}^{-\beta}\otimes S^{2\beta}(S^\gamma\mathcal{G})$ is generically globally generated. Using the natural morphism $S^{2\beta}(S^\gamma\mathcal{G}) \to S^{2\beta+\gamma}\mathcal{G}$, we see that $\mathcal{F} := \mathcal{H}^{-\beta}\otimes S^{2\beta+\gamma}\mathcal{G}$ is generically globally generated, i.e., there is a morphism $\theta : \bigoplus^t \mathcal{O}_Y \to \mathcal{F}$ that is surjective over a dense open subset $V \subseteq Y$, where $t \in \mathbb{Z}_{>0}$. Let θ_R, H_R and V_R be models of θ, H and V over R, respectively. Replacing R if necessary, we may assume that θ_R is surjective over V_R. Thus for every closed point $\mu \in \operatorname{Spec} R$, the morphism $\theta_\mu : \bigoplus^t \mathcal{O}_{X_\mu} \to \mathcal{H}_\mu^{-\beta}\otimes S^{2\beta+\gamma}\mathcal{G}_\mu$ is surjective over V_μ, which means that $\mathcal{G}_\mu$ is big. □

Kollár, Miyaoka and Mori [1992, Corollary 2.9] (compare [Miyaoka 1993, Theorem 3]) proved that images of Fano varieties under smooth morphisms are again Fano varieties. The rest of this paper is devoted to extending this result to toroidal morphisms.

Definition 5.7 [Abramovich and Karu 2000; Kawamata 2002]. Let k be an algebraically closed field of arbitrary characteristic:

(i) Let X be a normal variety and U an open subset of X. We say that the embedding $U \subseteq X$ is *toroidal* if for every closed point $x \in X$, there exists
   - a toric variety V with torus T,
   - a closed point $v \in V$ and
   - an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{V,v}$ of complete local k-algebras such that the ideal of $B := X \setminus U$ maps isomorphically to that of $V \setminus T$.

   Such a pair (V, v) is called a *local model* at $x \in X$. The pair (X, B) is often called a *toroidal variety*.

(ii) Let (X, B) and (Y, C) be toroidal varieties. A *toroidal morphism* $f : (X, B) \to (Y, C)$ is a dominant morphism $f : X \to Y$ with $f(X \setminus B) \subseteq Y \setminus C$ such that for every closed point $x \in X$, there exist
   - local models (V, v) and (W, w) at x and $y := f(x)$, respectively, and

- a toric morphism $g : V \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,x} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{V,v} \\ {\scriptstyle \hat{f}^{\#}}\uparrow & & \uparrow{\scriptstyle \hat{g}^{\#}} \\ \hat{\mathcal{O}}_{Y,y} & \xrightarrow{\cong} & \hat{\mathcal{O}}_{W,w} \end{array}$$

The next theorem is a generalization of [Kollár et al. 1992, Corollary 2.9].

Theorem 5.8. *Let k be an algebraically closed field of any characteristic $p \ge 0$. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties and B a reduced divisor on X. Let Δ be a $\mathbb{Q}$-divisor on X such that $0 \le \Delta \le B$ and that $a\Delta$ is integral for some $0 < a \in \mathbb{Z} \setminus p\mathbb{Z}$. Assume that the following conditions hold*:

(i) *f induces a toroidal morphism $f : (X, B) \to (Y, \varnothing)$.*

(ii) *f is equidimensional.*

(iii) *Every closed fiber of f is reduced.*

(iv) Supp Δ *does not contain any irreducible component of any fiber.*

In this situation, if $-(K_X + \Delta + f^*D)$ is ample for some $\mathbb{Q}$-divisor D on Y, then so is $-(K_Y + D)$.

Proof. Let $x_\lambda \in X$ be a closed point and set $y_\lambda := f(x_\lambda)$. By assumption (i), there is a local model (V_λ, v_λ) (resp. (W_λ, w_λ)) at x_λ (resp. y_λ) and a toric morphism $g_\lambda : V_\lambda \to W_\lambda$. Using Artin's approximation theorem [1969, Corollary 2.6], we obtain a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\rho_\lambda} & T_\lambda & \xrightarrow{\mu_\lambda} & V_\lambda \\ {\scriptstyle f}\downarrow & & {\scriptstyle h_\lambda}\downarrow & & \downarrow{\scriptstyle g_\lambda} \\ Y & \xleftarrow{\sigma_\lambda} & U_\lambda & \xrightarrow{\nu_\lambda} & W_\lambda \end{array} \tag{$*$}$$

such that

- T_λ and U_λ are varieties,
- all the horizontal morphisms are étale, and
- there is a closed point $t_\lambda \in T_\lambda$ such that $\rho_\lambda(t_\lambda) = x_\lambda$ and $\mu_\lambda(t_\lambda) = v_\lambda$.

Let $k[v_1, v_1^{-1}, \dots, v_m, v_m^{-1}]$ (resp. $k[w_1, w_1^{-1}, \dots, w_n, w_n^{-1}]$) be the coordinate ring of the torus of V_λ (resp. W_λ). Set $t_i := \mu_\lambda^* v_i$ and $u_i := \nu_\lambda^* w_i$ for each i. We then see from assumptions (ii) and (iii) that

$$h_\lambda^* u_j = h_\lambda^* \nu_\lambda^* w_j = \mu_\lambda^* g_\lambda^* w_j = \mu_\lambda^* \prod_{l_{j-1} < i \le l_j} v_i = \prod_{l_{j-1} < i \le l_j} t_i$$

for $j = 1, \dots, n$, where $0 = l_0 < l_1 < \cdots < l_n \le m$.

Shrinking T_λ if necessary, we may assume that for any closed point $t \in T_\lambda$, there are $a_1, \dots, a_m \in k$ such that $\mathfrak{m}_t = (t_1 - a_1, \dots, t_m - a_m)$. We may also assume a similar condition for U_λ and $u_1, \dots, u_n$.

Let Λ be a *finite* set of λ such that $X = \bigcup_{\lambda \in \Lambda} \rho_\lambda(T_\lambda)$. When $p = 0$, one can check that diagram $(*)$ can be reduced to characteristic $p \gg 0$ for all $\lambda \in \Lambda$ simultaneously. For this reason, we consider the case of $p > 0$.

We see from [Matsumura 1986, Corollary of Theorem 23.1] that f is flat. Therefore, in order to apply Corollary 4.5, we only need to show that $(Z, \Delta|_Z)$ is F-pure for every closed fiber Z of f. This holds if $(S, (\rho_\lambda^* \Delta)|_S)$ is F-pure for every closed fiber S of h_λ, since ρ_λ is étale. Fix a closed fiber S over $u \in U_\lambda$ and a closed point $t \in T_\lambda$ contained in S. Then, there are $a_1, \dots, a_m, b_1, \dots, b_n \in k$ such that $\mathfrak{m}_t = (t_1 - a_1, \dots, t_m - a_m)$ and $\mathfrak{m}_u = (u_1 - b_1, \dots, u_n - b_n)$. Put $t'_i := t_i - a_i$ and $u'_i := u_i - b_i$ for each i. We then have

$$h_\lambda^* u'_j = \prod_{l_{j-1} < i \le l_j} (t'_i + a_i) - \prod_{l_{j-1} < i \le l_j} a_i$$

for $j = 1, \dots, n$. Set $\delta := \prod_{l_n < i \le m} t_i$. Now, one can easily check that

- the sequence $(h_\lambda^* u'_1, \dots, h_\lambda^* u'_n, \delta)$ is $\mathcal{O}_{T_\lambda, t}$-regular, and
- $(h_\lambda^* u'_1 \cdots h_\lambda^* u'_n)^{q-1} \cdot \delta^{q-1} \notin \mathfrak{m}_t^{[q]}$ for every $q = p^e$.

Then, [Hara and Watanabe 2002, Corollary 2.7] tells us that $(S, \operatorname{div}(\delta)|_S)$ is F-pure around t. Since $\rho_\lambda^* \Delta \le \operatorname{div}(\delta)$, we conclude that $(S, (\rho_\lambda^* \Delta)|_S)$ is F-pure around t. □

Example 5.9. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the canonical basis of $\mathbb{R}^3$. For integers $m, n \ge 0$, we define $\boldsymbol{v}_{m,n} := (1, m, n) \in \mathbb{R}^3$. Let $\Sigma_{m,n}$ be the fan consisting of all the faces of the following cones:

$$\langle \boldsymbol{v}_{m,n}, \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_3 \rangle, \langle \boldsymbol{v}_{m,n}, \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_3 \rangle, \langle \boldsymbol{v}_{m,n}, -\boldsymbol{e}_2, \boldsymbol{e}_3 \rangle, \langle \boldsymbol{v}_{m,n}, \boldsymbol{e}_2, -\boldsymbol{e}_3 \rangle, \langle \boldsymbol{v}_{m,n}, -\boldsymbol{e}_2, -\boldsymbol{e}_3 \rangle,$$
$$\langle -\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_3 \rangle, \langle -\boldsymbol{e}_1, \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_3 \rangle, \langle -\boldsymbol{e}_1, -\boldsymbol{e}_2, \boldsymbol{e}_3 \rangle, \langle -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_3 \rangle, \langle -\boldsymbol{e}_1, -\boldsymbol{e}_2, -\boldsymbol{e}_3 \rangle.$$

Let $X_{m,n}$ be the smooth toric 3-fold corresponding to the fan $\Sigma_{m,n}$ with respect to the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$. Then $X_{m,n}$ is a Fano variety if and only if $m, n \in \{0, 1\}$. The projection $\mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$ induces a toric morphism $f : X_{m,n} \to Y_m$ from $X_{m,n}$ to the Hirzebruch surface $Y_m := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$. Set $\Delta = 0$. Then one can check that f satisfies the assumptions of Theorem 5.8, but it is not smooth. Hence by Theorem 5.8, we see that Y_m is a Del Pezzo surface if $m = 0, 1$. In fact, it is well known that Y_m is a Del Pezzo surface if and only if $m = 0, 1$.

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