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Degree of irrationality of very general abelian surfaces

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The degree of irrationality of a projective variety X is defined to be the smallest degree of a rational dominant map to a projective space of the same dimension. For abelian surfaces, Yoshihara computed this invariant in specific cases, while Stapleton gave a sublinear upper bound for very general polarized abelian surfaces (A, L) of degree d. Somewhat surprisingly, we show that the degree of irrationality of a very general polarized abelian surface is uniformly bounded above by 4, independently of the degree of the polarization. This result disproves part of a conjecture of Bastianelli, De Poi, Ein, Lazarsfeld, and Ullery.

1. Introduction

Given a projective variety X of dimension n which is not rational, one can try to quantify how far it is from being rational. When $n = 1$, the natural invariant is the *gonality* of a curve C, defined to be the smallest degree of a branched covering $C' \to \mathbb{P}^1$ (where C' is the normalization of C). One generalization of gonality to higher dimensions is the *degree of irrationality*, defined as:

$$\operatorname{irr}(X) := \min\{\delta > 0 \mid \exists \text{ degree } \delta \text{ rational dominant map } X \dashrightarrow \mathbb{P}^n\}.$$

Recently, there has been significant progress in understanding the case of hypersurfaces of large degree [Bastianelli 2017; Bastianelli et al. 2014; 2017]. The history behind the development of these ideas is described in [Bastianelli et al. 2017]. The results of those works depend on the positivity of the canonical bundles of the varieties in question, so it is interesting to consider what happens in the K_X-trivial case. Our purpose here is to prove the somewhat surprising fact that the degree of irrationality of a very general polarized abelian surface is uniformly bounded above, independently of the degree of the polarization.

To be precise, let $A = A_d$ be an abelian surface carrying a polarization $L = L_d$ of type $(1, d)$ and assume that $\mathrm{NS}(A) \cong \mathbb{Z}[L]$. An argument of Stapleton [2017] showed that there is a positive constant C such that

$$\operatorname{irr}(A) \le C \cdot \sqrt{d}$$

for $d \gg 0$, and it was conjectured in [Bastianelli et al. 2017] that equality holds asymptotically. Our main result shows that this is maximally false:

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Theorem 1.1. *For an abelian surface $A = A_d$ with Picard number $\rho = 1$, one has*

$$\operatorname{irr}(A) \leq 4.$$

As far as we can see, the conjecture of [Bastianelli et al. 2017] for very general polarized K3 surfaces (S_d, B_d) of genus d — namely, that there exist positive constants C_1, C_2 satisfying $C_1 \cdot \sqrt{d} \leq \operatorname{irr}(S_d) \leq C_2 \cdot \sqrt{d}$ for $d \gg 0$ — remains plausible. Here, B_d is an ample line bundle generating $\operatorname{Pic}(S_d)$ with $B_d^2 = 2d - 2$.

For an abelian variety A of dimension n, it has been shown in [Alzati and Pirola 1992] that $\operatorname{irr}(A) \geq n+1$. When A is an abelian surface, we give a geometric proof of the fact that $\operatorname{irr}(A) \geq 3$ in Lemma 3.1. Yoshihara proved that $\operatorname{irr}(A) = 3$ for abelian surfaces A containing a smooth curve of genus 3 [Yoshihara 1996]. On a related note, Voisin [2018] showed that the covering gonality of a very general abelian variety A of dimension n is bounded from below by $f(n)$, where $f(n)$ grows like $\log(n)$, and this lower bound was subsequently improved to $\lceil \frac{1}{2}n + 1 \rceil$ by Martin [2019]. The covering gonality is defined as the minimum integer $c > 0$ such that given a general point $x \in A$, there exists a curve C passing through x with gonality c.

In the proof of our theorem, assuming as we may that L is symmetric, we consider the space $H^0(A, \mathcal{O}_A(2L))^+$ of even sections of $\mathcal{O}_A(2L)$. By imposing suitable multiplicities at the two-torsion points of A, we construct a subspace $V \subset H^0(A, \mathcal{O}_A(2L))^+$ which numerically should define a rational map from A to a surface $S \subset \mathbb{P}^N$. Using bounds on the degree of the map and the degree of S, we construct a rational covering $A \dashrightarrow \mathbb{P}^2$ of degree 4. The main difficulty is to deal with the possibility that $\mathbb{P}_{\text{sub}}(V)$ has a fixed component. Our approach was inspired in part by the work of Bauer [1994; 1998; 1999].

2. Set-up

Let $A = A_d$ be an abelian surface with $\rho(A) = 1$. Assume $\operatorname{NS}(A) \cong \mathbb{Z}[L]$ where L is a polarization of type $(1, d)$ for some fixed $d \geq 1$, so that $L^2 = 2d$ and $h^0(L) = d$. Let

$$\iota : A \to A, \quad x \mapsto -x$$

be the inverse morphism and let $Z = \{p_1, \ldots, p_{16}\}$ be the set of two-torsion points of A (fixed points of ι). We may assume that L is symmetric — that is, $\iota^*\mathcal{O}_A(L) \cong \mathcal{O}_A(L)$ — by replacing L with a suitable translate. In particular, the cyclic group of order two acts on $H^0(A, \mathcal{O}_A(2L))$. The space of *even* sections $H^0(A, \mathcal{O}_A(2L))^+$ of the line bundle $\mathcal{O}_A(2L)$ (sections s with the property that $\iota^* s = s$) has dimension

$$h^0(A, 2L)^+ = 2d + 2$$

(see [Lange and Birkenhake 1992, Corollary 4.6.6]). Since an even section of $\mathcal{O}_A(2L)$ vanishes to even order at any two-torsion point, it is at most

$$1 + 3 + \cdots + (2m - 1) = m^2$$

conditions for an even section to vanish to order $2m$ at a fixed point $p \in Z$ (see [Bauer 1994] and the Appendix to [Bauer 1998] for more details).

Fix any integer solutions $a_1, \dots, a_{16} \geq 0$ to the equation

$$\sum_{i=1}^{16} a_i^2 = 2d - 2,$$

with $a_{15} = 0 = a_{16}$ (this last assumption will be useful in Corollary 3.4). Such a solution always exists by Lagrange's four-square theorem. Let $V \subset H^0(A, \mathcal{O}_A(2L))^+$ be the space of even sections vanishing to order at least $2a_i$ at each point p_i, so that

$$\dim V \geq 2d + 2 - \sum_{i=1}^{16} a_i^2 \geq 4.$$

Projectivizing via subspaces, let $\mathfrak{d} = \mathbb{P}_{\mathrm{sub}}(V) \subseteq |2L|^+$ be the corresponding linear system of divisors, whose dimension is $N := \dim \mathfrak{d} \geq 3$. Write

$$d_i := \mathrm{mult}_{p_i} D$$

for a general divisor $D \in \mathfrak{d}$, so that $d_i \geq 2a_i$.

Remark 2.1. From [Lange and Birkenhake 1992, Section 4.8], it follows that sections of V are pulled back from the singular Kummer surface A/ι, so any divisor $D \in \mathfrak{d}$ is symmetric, i.e., $\iota(D) = D$.

Let $\varphi : A \dashrightarrow \mathbb{P}^N$ be the rational map given by the linear system $\mathfrak{d}$ above (if $\mathfrak{d}$ has a fixed component F, take $\mathfrak{d} - F$), and write $S := \overline{\mathrm{Im}(\varphi)}$ for the image of φ. Regardless of whether or not $\mathfrak{d}$ has a fixed component, we find that:

Proposition 2.2. *$S \subset \mathbb{P}^N$ is an irreducible and nondegenerate surface.*

Proof. Suppose for the sake of contradiction that $\overline{\mathrm{Im}(\varphi)}$ is a nondegenerate curve C. Then $\deg C \geq 3$ since $N \geq 3$, and a hyperplane section of $C \subset \mathbb{P}^N$ pulls back to a divisor with at least three irreducible components. This contradicts the fact that any divisor $D(\sim_{\mathrm{lin}} 2L) \in \mathfrak{d}$ has at most two irreducible components since $\mathrm{NS}(A) \cong \mathbb{Z}[L]$. So the image of φ is a surface. $\square$

The following lemma will also be useful:

Lemma 2.3. *Let $\varphi : X \dashrightarrow \mathbb{P}^n$ be a rational map from a surface X to a projective space of dimension $n \geq 2$, and suppose that its image $S := \overline{\mathrm{Im}(\varphi)} \subset \mathbb{P}^n$ has dimension 2. Let $\mathfrak{d}$ be the linear system corresponding to φ (assuming $\mathfrak{d}$ has no base components). Then for any $D \in \mathfrak{d}$,*

$$\deg \varphi \cdot \deg S \leq D^2.$$

Proof. The indeterminacy locus of φ is a finite set. $\square$

3. Degree bounds

We first begin with an observation, which holds for an arbitrary abelian surface:

Lemma 3.1. *There are no rational dominant maps $A \dashrightarrow \mathbb{P}^2$ of degree* 2.

Proof. Suppose there exists such a map f. We have the following diagram

$$\begin{tikzcd}
 & & A^{[2]} \arrow[r, "s"] & A \\
A \arrow[r, dashed, "f"] & \mathbb{P}^2 \arrow[ur, dashed, "g"] \arrow[r, dashed, "h"'] & K^{[2]}(A) \arrow[u, hook] & =: s^{-1}(0)
\end{tikzcd}$$

where g is the pullback map on 0-cycles, $A^{[2]}$ is the Hilbert scheme of 2 points on A, and s is given by summation composed with the Hilbert–Chow morphism. Since the rational map $s \circ g$ can be extended to a morphism (see [Lange and Birkenhake 1992, Theorem 4.9.4]), it must be constant. So $\overline{\mathrm{Im}(g)}$ is contained in a fiber $s^{-1}(0)$, which is a smooth Kummer K3 surface $K^{[2]}(A)$. Since g is injective, it descends to an injective (and hence birational) map $h : \mathbb{P}^2 \dashrightarrow K^{[2]}(A)$, yielding a contradiction. □

We will now study the numerical properties of the linear series $\mathfrak{d}$ constructed in the previous section. There are two possibilities for $\mathfrak{d}$; either (i) $\mathfrak{d}$ has no fixed component or (ii) $\mathfrak{d}$ has a fixed component, denoted by $F \neq 0$. In fact, with a little more work one can show that the second case does not actually occur; see Remark 3.5.

In the second case, let $\mathfrak{b}$ be the movable component of $\mathfrak{d}$, so that we may write every divisor $D \in \mathfrak{d}$ as

$$D = F + M \quad \text{where } M \in \mathfrak{b}.$$

By definition, $\dim \mathfrak{d} = \dim \mathfrak{b}$. Since $\mathrm{NS}(A) \cong \mathbb{Z}[L]$, $D \sim_{\mathrm{lin}} 2L$ implies $F, M \sim_{\mathrm{alg}} L$ and are irreducible effective divisors for all $M \in \mathfrak{b}$. Choose a general divisor $M \in \mathfrak{b}$ and write

$$m_i := \mathrm{mult}_{p_i} M \quad \text{and} \quad f_i := \mathrm{mult}_{p_i} F,$$

so that $d_i = m_i + f_i \geq 2a_i$ for all i. We claim that F must be symmetric as a divisor. If not, then

$$\iota(M) + \iota(F) = \iota(D) = D = M + F \quad \text{for all } D \in \mathfrak{d}.$$

This implies that $M = \iota(F)$ and $F = \iota(M)$ for all $M \in \mathfrak{b}$, which would mean that M must also be fixed, leading to a contradiction. Hence, F must be symmetric, and likewise for all $M \in \mathfrak{b}$.

We first need an intermediate estimate:

Proposition 3.2. *Assume $\mathfrak{d}$ has a fixed component $F \neq 0$. Keeping the notation as above,*

$$\sum_{i=1}^{16} m_i^2 \geq 2d - 8.$$

Proof. The idea here is to use the Kummer construction to push our fixed curve F onto a K3 surface and apply Riemann–Roch. This is analogous to a proof of Bauer's [1999, Theorem 6.1]. Consider the smooth Kummer K3 surface K associated to A:

$$\begin{array}{ccc} E \subset \hat{A} & \xrightarrow{\gamma} & \hat{A}/\{1,\sigma\} =: K \\ \pi \big\downarrow & & \\ Z \subset A & & \end{array}$$

where π is the blow-up of A along the collection of two-torsion points Z. Since the points in Z are ι-invariant, ι lifts to an involution σ on $\hat{A}$ and the quotient K is a smooth K3 surface. Let E_i denote the exceptional curve over $p_i \in Z$, so that $E = \sum_{i=1}^{16} E_i$ is the exceptional divisor of π. Since F is symmetric, its strict transform

$$\hat{F} = \pi^* F - \sum_{i=1}^{16} f_i E_i,$$

descends to an irreducible curve $\bar{F} \subset K$. We claim that

$$h^0(K, \mathcal{O}_K(\bar{F})) = 1.$$

In fact, if the linear system $|\mathcal{O}_K(\bar{F})|$ were to contain a pencil, then this would give us a pencil of symmetric curves in $|\mathcal{O}_A(F)|$ with the same multiplicities at the two-torsion points, which contradicts F being a fixed component of $\mathfrak{d}$.

On the other hand, it is well-known that an irreducible curve $\bar{F}$ on a K3 surface with $h^0(K, \bar{F}) = 1$ satisfies $(\bar{F})^2 = -2$, so

$$-4 = 2(\bar{F})^2 = (\gamma^* \bar{F})^2 = (\hat{F})^2 = F^2 - \sum_{i=1}^{16} f_i^2 = 2d - \sum_{i=1}^{16} f_i^2 \tag{1}$$

combined with $\sum_{i=1}^{16} f_i m_i \le \sum_{i=1}^{16} \big(\frac{d_i}{2}\big)^2$ yields

$$\sum_{i=1}^{16} d_i^2 = \sum_{i=1}^{16} (f_i^2 + m_i^2 + 2 f_i m_i) \le 2d + 4 + \sum_{i=1}^{16} m_i^2 + \tfrac{1}{2} \sum_{i=1}^{16} d_i^2.$$

After rearranging the terms, we find that

$$\sum_{i=1}^{16} m_i^2 \ge -2d - 4 + \tfrac{1}{2} \sum_{i=1}^{16} d_i^2 \ge -2d - 4 + 2 \sum_{i=1}^{16} a_i^2 = 2d - 8 \tag{2}$$

for a general divisor $D = F + M \in \mathfrak{d}$, which is the desired inequality. □

As an immediate consequence:

Theorem 3.3. *Keeping the notation as before, let $\varphi : A \dashrightarrow \mathbb{P}^N$ be the rational map corresponding to $\mathfrak{d}$ (or $\mathfrak{b}$ if $F \neq 0$), with image S. Then*

$$\deg \varphi \cdot \deg S \le 8. \tag{3}$$

Proof. By applying Proposition 2.2 and blowing-up A along the collection of two-torsion points Z to resolve some of the base points of $\mathfrak{d}$, we arrive at the diagram

$$\begin{tikzcd} \hat{A} := \mathrm{Bl}_Z A \arrow[d, "\pi"'] \arrow[dr, dashed, "\psi"] & \\ A \arrow[r, dashed, "\varphi"'] & S \subset \mathbb{P}^N. \end{tikzcd}$$

(i) If the linear system $\mathfrak{d}$ has no fixed components, the divisors corresponding to ψ are of the form

$$\hat{D} \sim_{\text{lin}} \pi^* D - \sum_{i=1}^{16} d_i E_i,$$

where $\hat{D}$ denotes the strict transform of D. By Lemma 2.3 applied to ψ, we have

$$\deg \varphi \cdot \deg S = \deg \psi \cdot \deg S \le \hat{D}^2 = 4L^2 - \sum_{i=1}^{16} d_i^2 \le 4\left(2d - \sum_{i=1}^{16} a_i^2\right) = 8.$$

(ii) If the linear system $\mathfrak{d}$ has a fixed component $F \ne 0$, replace $\hat{D}$ and d_i in the equation above with $\hat{M}$ and m_i, respectively. Proposition 3.2 then gives an analogous bound. $\square$

Corollary 3.4. *There exists a rational dominant map* $\varphi : A \dashrightarrow \mathbb{P}^2$ *of degree* 4.

Proof. From Remark 2.1, it follows that $\varphi : A \dashrightarrow S \subset \mathbb{P}^N$ factors through the quotient $A \to A/\iota$, so $\deg \varphi$ must be even. In addition, $\deg S \ge 2$ since S is nondegenerate. By Lemma 3.1, it is impossible for S to be rational together with $\deg \varphi = 2$, so $\{\deg \varphi = 2, \deg S = 2, 3\}$ is ruled out by the classification of quadric and cubic surfaces (using the fact that $\rho(A) = 1$).

Together with the upper bound $\deg \varphi \cdot \deg S \le 8$ given by Theorem 3.3, there are two possibilities:

$$\{\deg \varphi = 4,\ \deg S = 2\} \quad \text{and} \quad \{\deg \varphi = 2,\ \deg S = 4\}.$$

Either of these imply the equality in (3), so that we have a morphism $\mathrm{Bl}_Z A \to S$ which fits into the diagram:

$$\begin{tikzcd} E_i \subset \mathrm{Bl}_Z A \arrow[r, "\gamma"] \arrow[d, "\pi"'] \arrow[dr] & K \supset G_i \arrow[d, "\alpha"] \\ A \arrow[r, dashed, "\varphi"'] & S \subset \mathbb{P}^N \end{tikzcd}$$

where K is the smooth Kummer K3 surface, γ is a branched cover of degree 2, and $G_i := \gamma(E_i)$ are (-2)-curves.

In the first case where $\deg \varphi = 4$ and $\deg S = 2$, note that S is rational. In the second case where $\deg \varphi = 2$ and $\deg S = 4$, recall that we chose the multiplicities a_i so that $a_{15} = 0 = a_{16}$. Thus, equality in (3) forces either $d_{15} = 0 = d_{16}$ or $m_{15} = 0 = m_{16}$. This implies that the curves G_{15}, G_{16} are contracted and their images q_{15}, q_{16} under α are double points on S since α is a birational morphism. Projection from a general $(N-3)$-plane containing one but not both of the q_i defines a rational map $A \dashrightarrow \mathbb{P}^2$ of degree 2 (if q_{15} is a cone point of S, pick a general plane passing through q_{16}, and vice versa), which contradicts Lemma 3.1. $\square$

This immediately implies Theorem 1.1.

Remark 3.5. The case when $\mathfrak{d}$ has a fixed component $F \neq 0$ cannot occur. To see this, suppose $F \neq 0$ and note that the two cases given in Corollary 3.4 imply that equality must hold throughout the proof of Proposition 3.2. In particular, $d_i = m_i + f_i$ and $\sum_{i=1}^{16} f_i m_i = \sum_{i=1}^{16} \left(\frac{d_i}{2}\right)^2$ implies $f_i = m_i$ for all i. Combining this with (1) and (2) gives

$$2d + 4 = \sum_{i=1}^{16} f_i^2 = \sum_{i=1}^{16} m_i^2 = 2d - 8,$$

which is a contradiction.

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