

Algebra & Number Theory

Volume 14
2020
No. 1

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Let F be a local field of mixed characteristic $(0, p)$, let k be a finite extension of its residue field, let \mathcal{H} be the pro- p -Iwahori Hecke k -algebra attached to $\mathrm{GL}_{d+1}(F)$ for some $d \geq 1$. We construct an exact and fully faithful functor from the category of supersingular \mathcal{H} -modules to the category of $\mathrm{Gal}(\bar{F}/F)$ -representations over k . More generally, for a certain k -algebra \mathcal{H}^\sharp surjecting onto \mathcal{H} we define the notion of \sharp -supersingular modules and construct an exact and fully faithful functor from the category of \sharp -supersingular \mathcal{H}^\sharp -modules to the category of $\mathrm{Gal}(\bar{F}/F)$ -representations over k .

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Introduction

Let F be a local field of mixed characteristic $(0, p)$, let $\pi \in \mathcal{O}_F$ be a uniformizer, let k be a finite extension of the residue field \mathbb{F}_q of F . Let $d \in \mathbb{N}$. An important line of current research in number theory is concerned with relating smooth representations of $G = \mathrm{GL}_{d+1}(F)$ over k with finite dimensional representations of $\mathrm{Gal}(\bar{F}/F)$ over k .

At present, the smooth representation theory of G is understood only up to identifying, constructing and describing the still elusive *supercuspidal* representations of G , or equivalently, the *supersingular* representations of G . An important role in better understanding this theory is played by the module theory of the pro- p -Iwahori Hecke k -algebra \mathcal{H} attached to G and a pro- p -Iwahori subgroup I_0 in G . There is a notion of supersingularity for \mathcal{H} -modules which, in contrast to that of supersingularity for G -representations, is transparent and concrete. The notions are compatible in the following sense: at least after replacing k by an algebraically closed extension field, a smooth admissible irreducible G -representation V is supersingular if and only if its space of I_0 -invariants V^{I_0} (which carries a natural

MSC2010: 11F85.

Keywords: pro- p Iwahori Hecke algebra, supersingular module, Galois representation, (φ, Γ) -module.

action by \mathcal{H}) is supersingular if and only if V^{I_0} admits a supersingular subquotient; see [Ollivier and Vignéras 2018]. It is true that the functor $V \mapsto V^{I_0}$ from G -representations to \mathcal{H} -modules often loses information. But the potential of taking into account also its higher derived functors, which again yield (complexes of) \mathcal{H} -modules, has been barely explored so far.

The purpose of the present paper is to explain a method for converting (supersingular) \mathcal{H} -modules into $\text{Gal}(\bar{F}/F)$ -representations over k .

For $F = \mathbb{Q}_p$ we had constructed in [Grosse-Klönne 2016] an exact functor from finite dimensional \mathcal{H} -modules to $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ -representations over k . The construction was inspired by Colmez's functor from $\text{GL}_2(\mathbb{Q}_p)$ -representations to $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ -representations. It was geometric-combinatorial in that it invoked coefficient systems on the Bruhat Tits building of $\text{GL}_n(\mathbb{Q}_p)$. Unfortunately, we see no way to generalize this geometric-combinatorial method to arbitrary finite extensions of F of \mathbb{Q}_p . However, when trying to extract its “algebraic essence”, we found that the functor indeed admits a generalization to any F , albeit now taking on an entirely algebraic and concrete shape. But in fact, it is this concreteness which allows us to not only investigate its behavior on irreducible objects, but also to prove that it accurately preserves extension structures. In this way, even for $F = \mathbb{Q}_p$ we significantly improve on our previous work [Grosse-Klönne 2016].

Let $\text{Rep}(\text{Gal}(\bar{F}/F))$ denote the category whose objects are projective limits of finite dimensional $\text{Gal}(\bar{F}/F)$ -representations over k . Let $\text{Mod}_{ss}(\mathcal{H})$ denote the category of supersingular \mathcal{H} -modules which are inductive limits of their finite dimensional submodules.

Theorem A. *There is an exact and fully faithful functor*

$$\text{Mod}_{ss}(\mathcal{H}) \rightarrow \text{Rep}(\text{Gal}(\bar{F}/F)), \quad M \mapsto V(M).$$

We have $\dim_k(M) = \dim_k(V(M))$ for any $M \in \text{Mod}_{ss}(\mathcal{H})$.

The radical elimination of the group G (and its building) from our approach allows us to improve Theorem A further as follows. We construct k -algebras $\mathcal{H}^{\sharp\sharp}$ and \mathcal{H}^{\sharp} by looking at a certain small set of distinguished generators of \mathcal{H} and by relaxing resp. omitting some of the usual (braid) relations between them. In this way we get a chain of surjective k -algebra morphisms $\mathcal{H}^{\sharp\sharp} \rightarrow \mathcal{H}^{\sharp} \rightarrow \mathcal{H}$. There is again a notion of supersingularity for $\mathcal{H}^{\sharp\sharp}$ -modules and for \mathcal{H}^{\sharp} -modules (which are inductive limits of their finite dimensional submodules; we assume this for all $\mathcal{H}^{\sharp\sharp}$ -, resp. \mathcal{H}^{\sharp} -, resp. \mathcal{H} -modules appearing in this paper). The simple supersingular modules are the same for $\mathcal{H}^{\sharp\sharp}$, for \mathcal{H}^{\sharp} and for \mathcal{H} , but there are more extensions between them in the category of $\mathcal{H}^{\sharp\sharp}$ -modules, resp. of \mathcal{H}^{\sharp} -modules, than in the category of \mathcal{H} -modules. A particular useful category $\text{Mod}_{ss}(\mathcal{H}^{\sharp})$ is formed by what we call \sharp -supersingular \mathcal{H}^{\sharp} -modules. It contains the category of supersingular \mathcal{H} -modules as a full subcategory (but is larger). Now it turns out that the above functor is actually defined on the category of supersingular $\mathcal{H}^{\sharp\sharp}$ -modules, and again with $\dim_k(M) = \dim_k(V(M))$ for any M . When restricting to $\text{Mod}_{ss}(\mathcal{H}^{\sharp})$ we furthermore get:

Theorem A[#]. *There is an exact and fully faithful functor*

$$\text{Mod}_{ss}(\mathcal{H}^{\sharp}) \rightarrow \text{Rep}(\text{Gal}(\bar{F}/F)), \quad M \mapsto V(M).$$

We do not know if the k -algebra \mathcal{H}^\sharp admits a group theoretic interpretation, as does the double coset algebra $\mathcal{H} \cong k[I_0 \backslash G / I_0]$. However, already from the Galois representation theoretic point of view we think that the additional effort taken in proving Theorem A $^\sharp$ (rather than just Theorem A) is justified, since in this way we identify an even larger abelian subcategory of $\text{Rep}(\text{Gal}(\bar{F}/F))$ as a (supersingular) module category of a very concretely given k -algebra. In fact, the additional effort is mostly notational.

We define a standard supersingular \mathcal{H} -module to be an \mathcal{H} -module induced from a supersingular character of a certain subalgebra \mathcal{H}_{aff} of \mathcal{H} with $[\mathcal{H} : \mathcal{H}_{\text{aff}}] = d + 1$. Each simple supersingular \mathcal{H} -module is a subquotient of a standard supersingular \mathcal{H} -module. We also define the notion of a $(d + 1)$ -dimensional standard cyclic $\text{Gal}(\bar{F}/F)$ -representation; in particular, each irreducible $\text{Gal}(\bar{F}/F)$ -representation of dimension $d + 1$ is a $(d + 1)$ -dimensional standard cyclic $\text{Gal}(\bar{F}/F)$ -representation.

Theorem B. *The functor $M \mapsto V(M)$ induces a bijection between standard supersingular \mathcal{H} -modules and $(d + 1)$ -dimensional standard cyclic $\text{Gal}(\bar{F}/F)$ -representations. M is irreducible if and only if $V(M)$ is irreducible.*¹

However, we emphasize that it is rather the much deeper Theorem A (and A $^\sharp$) which proves that supersingular modules are of a strong inherent arithmetic nature.

In Section 5E we gather some generic statements which come close to describing the image of the functor $M \mapsto V(M)$.

Let us now indicate the main features of the construction of the functor. We fix once and for all a Lubin–Tate group for F . More precisely, as this simplifies many formulae, we work with the Lubin–Tate group associated with the Frobenius power series $\Phi(t) = t^q + \pi t$. On the k -algebra $k[[t]][\varphi]$ with commutation relation $\varphi \cdot t = t^q \cdot \varphi$ we let $\Gamma = \mathcal{O}_F^\times$ act by $\gamma \cdot \varphi = \gamma' \varphi$ and $\gamma \cdot t = [\gamma]_\Phi(t)$, where $[\gamma]_\Phi(t) \in k[[t]]$ describes multiplication with γ with respect to Φ and where $\gamma' \in k^\times$ means the image of $\gamma \in \Gamma$ in k^\times . We view a supersingular $\mathcal{H}^{\sharp\sharp}$ -module (or \mathcal{H}^\sharp -module, or \mathcal{H} -module) M as a $k[[t]]$ -module by means of $t|_M = 0$. In $k[[t]][\varphi] \otimes_{k[[t]]} M$ we then use the \mathcal{H} -action on M to define a certain submodule $\nabla(M)$ by giving very explicitly a certain number of generators of it. This is done in such a way that $\Delta(M) = k[[t]][\varphi] \otimes_{k[[t]]} M / \nabla(M)$ naturally receives an action by Γ and is a torsion $k[[t]]$ -module. A very general construction then allows us to endow $\Delta(M)^* \otimes_{k[[t]]} k((t))$ with the structure of a (φ, Γ) -module over $k((t))$. The notion of a (φ, Γ) -module over $k((t))$ with respect to the chosen Lubin–Tate group Φ is explained in full detail in the book [Schneider 2017], where it is also explained that this category is equivalent with the category of representations of $\text{Gal}(\bar{F}/F)$ over k .

It was pointed out by Cédric Pépin that the syntax of the functor $M \mapsto V(M)$ bears strong resemblance with that of Fontaine’s various functors (using “big rings”).

One may wonder which of our results remain valid if the coefficient field k is allowed to be a more general field k containing \mathbb{F}_q , i.e., not necessarily finite. First, this finiteness is invoked for the equivalence of categories between Galois representations and (φ, Γ) -modules. But it is also invoked in the proofs of

¹A numerical version (i.e., comparing cardinalities) of Theorem B was known for quite some time, due to work of Ollivier and Vignéras [2005].

Proposition 3.3 (our main result in Section 3 on recovering a supersingular \mathcal{H}^\sharp -module from subquotients) and of Theorem 5.10 (on recovering M from $\Delta(M)$).

In Section 2B we list some automorphisms of \mathcal{H} (and of \mathcal{H}^\sharp and $\mathcal{H}^{\sharp\sharp}$). They induce autoequivalences of the category of supersingular \mathcal{H} -modules;² thus, precomposing them to $M \mapsto V(M)$ we get more functors satisfying Theorems A, A^\sharp and B.

We end this paper somewhat speculatively by discussing assignments of $\text{Gal}(\bar{F}/F)$ -representations to supersingular G -representations. The functor $M \mapsto V(M)$ invites us to search for meaningful assignments of (complexes of) supersingular \mathcal{H} -modules to supersingular G -representations Y . First we suggest studying the left derived functor of the functor taking Y to the maximal supersingular \mathcal{H} -submodule of Y^{I_0} . This entails working in derived categories and appears to be the most natural approach. Nevertheless, as a variation of this theme we then suggest an *exact* functor from (suitably filtered) G -representations to supersingular \mathcal{H} -modules. It builds on a general procedure of turning complexes of \mathcal{H} -modules into new \mathcal{H} -modules, applied here to complexes arising from E_1 -spectral sequences attached to the said left derived functor.

Apparently, the constructions and results of the present paper call for generalizations into various directions. We mention here just the obvious question of what happens if the pro- p -Iwahori Hecke algebra \mathcal{H} attached to $G = \text{GL}_{d+1}(F)$ is replaced by pro- p -Iwahori Hecke algebras \mathcal{H} attached to other p -adic reductive groups G . In extrapolation of what we did here, the general Langlands philosophy suggests searching for a functor from \mathcal{H} -modules to Galois representations such that in some way the algebraic k -group with root datum dual to that of G shows up on the Galois side—just as it does here in Theorem B. In a subsequent paper we will propose such a functor. However, in its formal shape it will *not* precisely specialize to the functor discussed here if $G = \text{GL}_{d+1}(F)$,³ and Theorem A will *not* be a special case of what we will then prove for general G .

Notations. Let F/\mathbb{Q}_p be a finite field extension. Let \mathbb{F}_q be the residue field of F (with q elements). Let π be a uniformizer in \mathcal{O}_F . Let k be a finite field extension of \mathbb{F}_q .

As explained in [Schneider 2017, Proposition 1.3.4], attached to the Frobenius (or Lubin–Tate) formal power series $\Phi(t) = \pi t + t^q$ is associated a commutative formal group law (the associated Lubin–Tate (formal) group law) $F_\Phi(X, Y)$ over \mathcal{O}_F such that $\Phi(t) \in \text{End}_{\mathcal{O}_F}(F_\Phi(X, Y))$. There is a unique injective homomorphism of rings

$$\mathcal{O}_F \rightarrow \text{End}_{\mathcal{O}_F}(F_\Phi(X, Y)), \quad a \mapsto [a]_\Phi(t)$$

such that $\Phi(t) = [\pi]_\Phi(t)$, see [Schneider 2017, Proposition 1.3.6], where we recall that, by definition,

$$\text{End}_{\mathcal{O}_F}(F_\Phi(X, Y)) = \{h \in \mathcal{O}_F[[t]]; h(0) = 0 \text{ and } h(F_\Phi(X, Y)) = F_\Phi(h(X), h(Y))\}.$$

Lemma 0.1. *Assume that $F \neq \mathbb{Q}_2$. Writing $[a]_\Phi(t) = at + \sum_{i \geq 2} a_i t^i$ (with $a_i \in \mathcal{O}_F$), we have $a_i = 0$ whenever $i - 1 \notin (q - 1)\mathbb{N}$. If $a^{q-1} = 1$ we even have $a_i = 0$ for all $i \geq 2$.*

²But this is not so evident, if true at all, for the category of \sharp -supersingular \mathcal{H}^\sharp -modules

³But of course, it will be closely related

Proof. As $\Phi(t) = \pi t + t^q$, the power series $[a]_\Phi(t) = at + \sum_{i \geq 2} a_i t^i$ is characterized by the formula

$$\pi[a]_\Phi(t) + ([a]_\Phi(t))^q = [a]_\Phi(\pi t + t^q).$$

If $a^{q-1} = 1$ we see that $[a]_\Phi(t) = at$ satisfies this formula. Given a general a , consider the equalities $[a]_\Phi([b]_\Phi(t)) = [b]_\Phi([a]_\Phi(t))$ for all $b \in \mathcal{O}_F$ with $b^{q-1} = 1$. Since we know $[b]_\Phi(t) = bt$, and since $F \neq \mathbb{Q}_2$ implies the existence of primitive such b 's different from 1, we indeed obtain $a_i = 0$ whenever $i - 1 \notin (q - 1)\mathbb{N}$. \square

1. Lubin–Tate (φ, Γ) -modules

In the first two subsections we transpose some constructions and results from the theory of cyclotomic (φ, Γ) -modules over k (i.e., where $F = \mathbb{Q}_p$ and where the underlying Lubin–Tate group is \mathbb{G}_m) to the context of (φ, Γ) -modules over k with respect to the Lubin–Tate group attached to $\Phi(t) = \pi t + t^q$ (with arbitrary F). Namely, we define an exact functor from admissible (torsion) $k[[t]]$ -modules with commuting semilinear actions by $\Gamma = \mathcal{O}_F^\times$ and φ to étale (φ, Γ) -modules over k . The former category is closely related to that of ψ -stable lattices in étale (φ, Γ) -modules \mathbf{D} , and we are lead to transpose some of Colmez's constructions [2010] involving the ψ -stable lattices \mathbf{D}^\natural and \mathbf{D}^\sharp to our context. One difference is that in our context the ψ -operator on $k((t))$ does not satisfy $\psi(1) = 1$, but this necessitates only minor modifications.

We then identify a category of admissible (torsion) $k[[t]]$ -modules with actions by Γ and φ on which the above functor is fully faithful.

1A. (φ, Γ) -modules and torsion $k[[t]]$ -modules. Put $\Phi(t) = \pi t + t^q$. Put $\Gamma = \mathcal{O}_F^\times$. The formula $\gamma \cdot t = [\gamma]_\Phi(t)$ with $\gamma \in \Gamma$ defines an action of Γ by k -algebra automorphisms on $k[[t]]$ and on $k((t))$. Consider the k -algebra

$$\mathfrak{O} = k[[t]][\varphi, \Gamma]$$

with commutation rules given by

$$\gamma\varphi = \varphi\gamma, \quad \gamma t = [\gamma]_\Phi(t)\gamma, \quad \varphi t = t^q\varphi$$

for $\gamma \in \Gamma$. (Here we read $[\gamma]_\Phi(t)\gamma = ([\gamma]_\Phi(t))\gamma$).⁴

Definition. A ψ -operator on $k[[t]]$ is a k -linear map $\psi : k[[t]] \rightarrow k[[t]]$ such that $\psi(\gamma \cdot t) = \gamma \cdot (\psi(t))$ for all $\gamma \in \Gamma$ and such that the following holds true:⁵ if we view φ as acting on $k[[t]]$, then

$$\psi(\varphi(a)x) = a\psi(x) \quad \text{for } a, x \in k[[t]]. \tag{1}$$

Lemma 1.1. *There is a surjective ψ -operator on $k[[t]]$ which extends to a surjective k -linear operator $\psi = \psi_{k((t))}$ on $k((t))$ satisfying formula (1) analogously.*

⁴As $t^q = \Phi(t) = [\pi]_\Phi(t)$ in $k[[t]]$ one may also think of \mathfrak{O} as $\mathfrak{O} = k[[t]][\mathcal{O}_F - \{0\}]$ with commutation rules $at = [a]_\Phi(t)a$ for all $a \in \mathcal{O}_F - \{0\}$.

⁵We do not require $\psi(1) = 1$.

We may choose $\psi_{k((t))}$ on $k((t))$ such that for $m \in \mathbb{Z}$ and $0 \leq i \leq q-1$ we have⁶

$$\psi_{k((t))}(t^{mq+i}) = \begin{cases} \frac{q}{\pi} t^m & i = 0, \\ 0 & 1 \leq i \leq q-2, \\ t^m & i = q-1. \end{cases} \quad (2)$$

Proof. This is explained in [Grosse-Klönne 2019]; it relies on [Schneider and Venjakob 2016, Section 3]. \square

In the following we fix the surjective ψ -operator ψ on $k[[t]]$ satisfying formula (2). We extend it to a k -linear operator $\psi = \psi_{k((t))}$ on $k((t))$ as in Lemma 1.1.

Definition. An étale (φ, Γ) -module over $k((t))$ is an $\mathfrak{O} \otimes_{k[[t]]} k((t))$ -module \mathbf{D} which is finite dimensional over $k((t))$ such that the $k((t))$ -linearized structure map

$$\text{id} \otimes \varphi : k((t)) \otimes_{\varphi, k((t))} \mathbf{D} \xrightarrow{\cong} \mathbf{D}$$

is bijective. We define $\text{Mod}^{\text{et}}(k((t)))$ to be the category of étale (φ, Γ) -module over $k((t))$.

Theorem 1.2 (Fontaine, Kisin–Ren, Schneider). *There is an equivalence between $\text{Mod}^{\text{et}}(k((t)))$ and the category of continuous representations of $\text{Gal}(\bar{F}/F)$ on finite dimensional k -vector spaces.*

Proof. For $F = \mathbb{Q}_p$ and the Frobenius power series $(1+t)^p - 1$ (instead of $\Phi(t) = \pi t + t^q$) this is a theorem of Fontaine, see paragraph 1.2 in [Fontaine 1990]. The analog of the theorem (for an arbitrary Frobenius power series) for a coefficient field of characteristic 0 (hence not k) is due to Kisin and Ren [2009]. A detailed proof of the theorem stated here can be found in Schneider's book [2017]. \square

Definition. A torsion $k[[t]]$ -module Δ is called admissible if

$$\Delta[t] = \{x \in \Delta; tx = 0\}$$

is a finite dimensional k -vector space.

We remark that admissible $k[[t]]$ -modules on which t acts surjectively are precisely the Pontrjagin duals of finitely generated torsion free, and hence free $k[[t]]$ -modules.

Definition. $\text{Mod}^{\text{ad}}(\mathfrak{O})$ is the category of \mathfrak{O} -modules which are finitely generated over $k[[t]][\varphi]$ and admissible (in particular: torsion) over $k[[t]]$.

Lemma 1.3. *The categories $\text{Mod}^{\text{et}}(k((t)))$ and $\text{Mod}^{\text{ad}}(\mathfrak{O})$ are abelian.*

Proof. An $\mathfrak{O} \otimes_{k[[t]]} k((t))$ -module subquotient of an étale (φ, Γ) -module is again an étale (φ, Γ) -module: to see that the étaleness condition (the bijectivity of $\text{id} \otimes \varphi$) is preserved under passing to such subquotients, just notice that it is equivalent with saying that the matrix of φ in an arbitrary $k((t))$ -basis is invertible. Thus, $\text{Mod}^{\text{et}}(k((t)))$ is abelian. (Of course, one could also point to Theorem 1.2.)

An \mathfrak{O} -module subquotient of an object in $\text{Mod}^{\text{ad}}(\mathfrak{O})$ is again an object in $\text{Mod}^{\text{ad}}(\mathfrak{O})$: this is shown in [Emerton 2008, Proposition 3.3]. Thus, $\text{Mod}^{\text{ad}}(\mathfrak{O})$ is abelian. \square

⁶Notice that $\frac{q}{\pi} = 0$ (in k) if $F \neq \mathbb{Q}_p$.

Definition. For a k -vector space Δ we write $\Delta^* = \text{Hom}_k(\Delta, k)$ (algebraic dual). For a $k[[t]]$ -module Δ we endow Δ^* with a $k[[t]]$ -action by putting

$$(S \cdot f)(\delta) = f(S\delta)$$

for $S \in k[[t]]$, $f \in \Delta^*$, $\delta \in \Delta$. If Δ even carries a $k[[t]][\Gamma]$ -module structure then also Δ^* receives one, with $\gamma \in \Gamma$ acting as

$$(\gamma \cdot f)(\delta) = f(\gamma^{-1}\delta)$$

for $\gamma \in \Gamma$, $f \in \Delta^*$, $\delta \in \Delta$.

Proposition 1.4. *For $\Delta \in \text{Mod}^{\text{ad}}(\mathcal{O})$ there is a natural structure of an étale (φ, Γ) -module on $\Delta^* \otimes_{k[[t]]} k((t))$. The contravariant functor*

$$\text{Mod}^{\text{ad}}(\mathcal{O}) \rightarrow \text{Mod}^{\text{et}}(k((t))), \quad \Delta \mapsto \Delta^* \otimes_{k[[t]]} k((t)) \quad (3)$$

is exact.

Proof. The map $\text{id} \otimes \varphi : k[[t]] \otimes_{\varphi, k[[t]]} \Delta \rightarrow \Delta$ gives rise to the $k[[t]]$ -linear map

$$\Delta^* \xrightarrow{(\text{id} \otimes \varphi)^*} (k[[t]] \otimes_{\varphi, k[[t]]} \Delta)^*. \quad (4)$$

On the other hand, we have the $k[[t]]$ -linear map

$$\begin{aligned} k[[t]] \otimes_{\varphi, k[[t]]} (\Delta^*) &\rightarrow (k[[t]] \otimes_{\varphi, k[[t]]} \Delta)^* \\ a \otimes \ell &\mapsto [b \otimes x \mapsto \ell(\psi(ab)x)]. \end{aligned} \quad (5)$$

It is shown in [Grosse-Klönne 2019] that the respective base extended maps (4) $\otimes_{k[[t]]} k((t))$ and (5) $\otimes_{k[[t]]} k((t))$ are bijective. Composing (5) $\otimes_{k[[t]]} k((t))$ with the inverse of (4) $\otimes_{k[[t]]} k((t))$ thus yields a $k((t))$ -linear isomorphism

$$k((t)) \otimes_{\varphi, k((t))} (\Delta^* \otimes_{k[[t]]} k((t))) = k((t)) \otimes_{\varphi, k[[t]]} (\Delta^*) \rightarrow \Delta^* \otimes_{k[[t]]} k((t))$$

and hence the desired φ -operator on $\Delta^* \otimes_{k[[t]]} k((t))$. The exactness of $\Delta \mapsto \Delta^* \otimes_{k[[t]]} k((t))$ follows from the exactness of taking duals and of applying $(\cdot) \otimes_{k[[t]]} k((t))$. \square

1B. ψ -stable lattices in (φ, Γ) -modules.

Lemma 1.5. *Let $\mathbf{D} \in \text{Mod}^{\text{et}}(k((t)))$. There is a natural additive operator $\psi : \mathbf{D} \rightarrow \mathbf{D}$ satisfying*

$$\psi(a\varphi(x)) = \psi(a)x \quad \text{and} \quad \psi(\varphi(a)x) = a\psi(x)$$

for all $a \in k((t))$ and all $x \in \mathbf{D}$, and commuting with the action of Γ .

Proof. We define the composed map

$$\psi : \mathbf{D} \rightarrow k((t)) \otimes_{\varphi, k((t))} \mathbf{D} \rightarrow \mathbf{D}$$

where the first arrow is the inverse of the structure isomorphism $\text{id} \otimes \varphi$ and where the second arrow is given by $a \otimes x \mapsto \psi(a)x$. By construction, it satisfies $\psi(a\varphi(x)) = \psi(a)x$. To see $\psi(\varphi(a)x) = a\psi(x)$ observe that by assumption we may write $x = \sum_i a_i \varphi(d_i)$ with $d_i \in \mathbf{D}$ and $a_i \in k((t))$. We then compute

$$\psi(\varphi(a)x) = \sum_i \psi(\varphi(a)a_i \varphi(d_i)) = \sum_i \psi(\varphi(a)a_i)d_i = a \sum_i \psi(a_i)d_i = a \sum_i \psi(a_i \varphi(d_i)) = a\psi(x).$$

Finally, let $\gamma \in \Gamma$. As γ and φ commute on $k[[t]]$, and as Γ acts semilinearly on \mathbf{D} , the additive map

$$k((t)) \otimes_{\varphi, k((t))} \mathbf{D} \rightarrow k((t)) \otimes_{\varphi, k((t))} \mathbf{D}, \quad a \otimes d \mapsto \gamma(a) \otimes \gamma(d)$$

is the map corresponding to γ on \mathbf{D} under the isomorphism $\text{id} \otimes \varphi$, and under $a \otimes x \mapsto \psi(a)x$ it commutes with γ on \mathbf{D} since γ and ψ commute on $k((t))$. \square

In the following, by a lattice in a $k((t))$ -vector space \mathbf{D} we mean a free $k[[t]]$ -submodule containing a $k((t))$ -basis of \mathbf{D} .

Lemma 1.6. *Let $\mathbf{D} \in \text{Mod}^{\text{et}}(k((t)))$ and let D be a lattice in (the $k((t))$ -vector space underlying) \mathbf{D} . Let $\psi : \mathbf{D} \rightarrow \mathbf{D}$ be the operator constructed in Lemma 1.5:*

- (a) *$\psi(D)$ is a $k[[t]]$ -module.*
- (b) *If $\varphi(D) \subset D$ then $D \subset \psi(D)$.*
- (c) *If $D \subset k[[t]] \cdot \varphi(D)$ then $\psi(D) \subset D$.*
- (d) *If $\psi(D) \subset D$ then $\psi(t^{-1}D) \subset t^{-1}D$, and for each $x \in \mathbf{D}$ there is some $n(x) \in \mathbb{N}$ such that for all $n \geq n(x)$ we have $\psi^n(x) \in t^{-1}D$.*

Proof. (a) Use $\psi(\varphi(a)x) = a\psi(x)$ for $a \in k((t))$ and $x \in \mathbf{D}$.

(b) Choose $a \in k[[t]]$ with $\psi(a) = 1$. For $d \in D$ we have $d = \psi(a\varphi(d))$ which belongs to $\psi(D)$ since $\varphi(D) \subset D$.

(c) Let $d \in D$. By assumption there are $e_i \in D$ and $a_i \in k[[t]]$ with $d = \sum_i a_i \varphi(e_i)$, hence $\psi(d) = \sum_i \psi(a_i)e_i \in D$.

(d) For $i \geq 1$ we have

$$\psi(\varphi^i(t^{-1})D) \subset \varphi^{i-1}(t^{-1})\psi(D) \subset \varphi^{i-1}(t^{-1})D \tag{6}$$

where the second inclusion uses the assumption. From $\varphi(t^{-1}) = t^{-q}$ we get

$$\psi(t^{-1}D) \subset \psi(\varphi(t^{-1})D) \subset t^{-1}D.$$

Moreover, if $n(x) \in \mathbb{N}$ is such that $x \in \varphi^n(t^{-1})D$ for $n \geq n(x)$, then iterated application of formula (6) shows

$$\psi^n(x) \in \psi^n(\varphi^n(t^{-1})D) \subset \psi^{n-1}(\varphi^{n-1}(t^{-1})D) \subset \cdots \subset t^{-1}D$$

for $n \geq n(x)$. \square

Lemma 1.7. (a) *There are lattices D_0, D_1 in \mathbf{D} with*

$$\varphi(D_0) \subset tD_0 \subset D_0 \subset D_1 \subset k[[t]] \cdot \varphi(D_1).$$

(b) *For D_0, D_1 as in (a) and for $n \geq 0$ we have $\psi^n(D_0) \subset \psi^{n+1}(D_0) \subset D_1$.*

Proof. (a) This is a (simplified) subclaim in the proof of Lemma 2.2.10 in [Schneider 2017] (which follows [Colmez 2010, Lemme II 2.3]). One proceeds as follows. Let d_1, \dots, d_r be a $k((t))$ -basis of \mathbf{D} . Then also $\varphi(d_1), \dots, \varphi(d_r)$ is $k((t))$ -basis of \mathbf{D} . We therefore find $\tilde{f}_{ij}, \tilde{g}_{ij} \in k((t))$ with $\varphi(d_j) = \sum_{i=1}^r \tilde{f}_{ij} d_i$ and $d_j = \sum_{i=1}^r \tilde{g}_{ij} \varphi(d_i)$, for any $1 \leq j \leq r$. Choose some $n \geq 0$ with $t^{n(q-1)} \tilde{f}_{ij} \in tk[[t]]$ and $t^{n(q-1)} \tilde{g}_{ij} \in tk[[t]]$ for all i, j . Then $D_0 = \sum_{i=1}^r t^n k[[t]] d_i$ and $D_1 = \sum_{i=1}^r t^{-n} k[[t]] d_i$ work as desired.

(b) Choose $a \in k[[t]]$ with $\psi(a) = 1$. For $x \in D_0$ we have $\psi^n(x) = \psi^{n+1}(a\varphi(x)) \in \psi^{n+1}(D_0)$ since $\varphi(D_0) \subset tD_0$ implies $\varphi(x) \in D_0$ and hence $a\varphi(x) \in D_0$. This shows $\psi^n(D_0) \subset \psi^{n+1}(D_0)$. As $D_0 \subset D_1 \subset k[[t]] \cdot \varphi(D_1)$, an induction using Lemma 1.6(c) shows $\psi^{n+1}(D_0) \subset D_1$. \square

Proposition 1.8. *There exists a unique lattice \mathbf{D}^\sharp in \mathbf{D} with $\psi(\mathbf{D}^\sharp) = \mathbf{D}^\sharp$ and such that for each $x \in \mathbf{D}$ there is some $n \in \mathbb{N}$ with $\psi^n(x) \in \mathbf{D}^\sharp$.*

For any lattice D in \mathbf{D} we have $\psi^n(D) \subset \mathbf{D}^\sharp$ for all $n \gg 0$.

For any lattice D in \mathbf{D} with $\psi(D) = D$ we have $t\mathbf{D}^\sharp \subset D \subset \mathbf{D}^\sharp$.

Proof. Using the previous lemmata, the proof is the same as the one given in [Colmez 2010, Proposition II.4.2]. \square

Proposition 1.9. (a) *For any lattice D in \mathbf{D} contained in \mathbf{D}^\sharp and stable under ψ we have $\psi(D) = D$.*

(b) *The intersection \mathbf{D}^\natural of all lattices in \mathbf{D} contained in \mathbf{D}^\sharp and stable under ψ is itself a lattice, and it satisfies $\psi(\mathbf{D}^\natural) = \mathbf{D}^\natural$.*

Proof. (See [Colmez 2010, Proposition II.5.11 and Corollaire II.5.12].)

(a) Since \mathbf{D}^\sharp as well as D and $\psi(D)$ are lattices in \mathbf{D}^\sharp , both \mathbf{D}^\sharp/D and $\mathbf{D}^\sharp/\psi(D)$ are finite dimensional k -vector spaces. ψ induces an isomorphism $\psi(\mathbf{D}^\sharp)/D = \mathbf{D}^\sharp/\psi(D)$ (as $\psi(D) \subset D$), hence $\psi(D) = D$.

(b) For any D as in (a) we have $t\mathbf{D}^\sharp \subset D$ by what we saw in (a) together with Proposition 1.8. This shows $t\mathbf{D}^\sharp \subset \mathbf{D}^\natural$, hence \mathbf{D}^\natural is indeed a lattice, and $\psi(\mathbf{D}^\natural) = \mathbf{D}^\natural$ follows by applying (a) once more. \square

Lemma 1.10. \mathbf{D}^\natural and \mathbf{D}^\sharp are stable under the action of Γ .

Proof. If D is a lattice in \mathbf{D} , then so is $\gamma \cdot D$ for any $\gamma \in \Gamma$. If in addition $\psi(D) \subset D$, resp. $\psi(D) = D$, then also $\psi(\gamma \cdot D) \subset \gamma \cdot D$, resp. $\psi(\gamma \cdot D) = \gamma \cdot D$. From these observations we immediately get $\gamma \cdot \mathbf{D}^\natural = \mathbf{D}^\natural$ and $\gamma \cdot \mathbf{D}^\sharp = \mathbf{D}^\sharp$. \square

Proposition 1.11. *The functor $\text{Mod}^{\text{ad}}(\mathfrak{O}) \rightarrow \text{Mod}^{\text{et}}(k((t)))$ in Proposition 1.4 sends simple objects to simple objects.*

Proof. Let $\Delta \in \text{Mod}^{\text{ad}}(\mathfrak{O})$ be simple. By construction, ψ on $\Delta^* \otimes_{k[[t]]} k((t))$, when restricted to Δ^* , is the adjoint of φ on Δ . Therefore the simplicity of Δ as an \mathfrak{O} -module means that Δ^* admits no nontrivial $k[[t]]$ -submodule stable under Γ and ψ . If \mathbf{D} is a nonzero (φ, Γ) -submodule of $\Delta^* \otimes_{k[[t]]} k((t))$ then also \mathbf{D}^\sharp is

nonzero and stable under Γ and ψ , see Proposition 1.9 and Lemma 1.10. As $\mathbf{D}^\natural \subset (\Delta^* \otimes_{k[[t]]} k((t)))^\natural \subset \Delta^*$ we get $\mathbf{D}^\natural = \Delta^*$ (since Δ^* is stable under ψ), as desired. \square

Lemma 1.12. *Let $f : \mathbf{D}_1 \rightarrow \mathbf{D}_2$ be a morphism in $\text{Mod}^{\text{et}}(k((t)))$:*

- (a) $f(\mathbf{D}_1^\natural) \subset \mathbf{D}_2^\natural$ and $f(\mathbf{D}_1^\natural) \subset \mathbf{D}_2^\natural$.
- (b) *If $f : \mathbf{D}_1 \rightarrow \mathbf{D}_2$ is injective (resp. surjective), then so is $f : \mathbf{D}_1^\natural \rightarrow \mathbf{D}_2^\natural$.*
- (c) *If $f : \mathbf{D}_1 \rightarrow \mathbf{D}_2$ is injective (resp. surjective), then so is $f : \mathbf{D}_1^\natural \rightarrow \mathbf{D}_2^\natural$.*

Proof. (a) $f(\mathbf{D}_1^\natural)$ is a free $k[[t]]$ -submodule of \mathbf{D}_2 on which ψ acts surjectively. Thus $f(\mathbf{D}_1^\natural) + \mathbf{D}_2^\natural$ is a lattice satisfying the defining condition for \mathbf{D}_2^\natural given in Proposition 1.8, hence $f(\mathbf{D}_1^\natural) + \mathbf{D}_2^\natural = \mathbf{D}_2^\natural$, hence $f(\mathbf{D}_1^\natural) \subset \mathbf{D}_2^\natural$. Next, let $D = \{x \in \mathbf{D}_1^\natural; f(x) \in \mathbf{D}_2^\natural\}$. It is a lattice in \mathbf{D}_1 since \mathbf{D}_1^\natural is a lattice, $f(\mathbf{D}_1^\natural) \subset f(\mathbf{D}_1^\natural) \subset \mathbf{D}_2^\natural$ and $\mathbf{D}_2^\natural/\mathbf{D}_2^\natural$ is a finite dimensional k -vector space. It is also stable under ψ , hence contains \mathbf{D}_1^\natural , hence $f(\mathbf{D}_1^\natural) \subset \mathbf{D}_2^\natural$.

(b) and (c) If $f : \mathbf{D}_1 \rightarrow \mathbf{D}_2$ is injective then obviously so are $f : \mathbf{D}_1^\natural \rightarrow \mathbf{D}_2^\natural$ and $f : \mathbf{D}_1^\natural \rightarrow \mathbf{D}_2^\natural$. If $f : \mathbf{D}_1 \rightarrow \mathbf{D}_2$ is surjective then $f(\mathbf{D}_1^\natural)$ is a lattice in \mathbf{D}_2 stable under ψ , hence contains \mathbf{D}_2^\natural . To see $f(\mathbf{D}_1^\natural) = \mathbf{D}_2^\natural$ we proceed as in [Colmez 2010, Proposition II.4.6(iii)] Namely, choose a lattice D' in \mathbf{D}_1 with $f(D') = \mathbf{D}_2^\natural$. Put $D = \sum_{n \geq 0} \psi^n(D')$. By construction we have $\psi(D) \subset D$ as well as $f(D) = \mathbf{D}_2^\natural$ (since $\psi(\mathbf{D}_2^\natural) = \mathbf{D}_2^\natural$). Proposition 1.8 shows that D is again a lattice. Let $x \in \mathbf{D}_2^\natural$. By Proposition 1.8 we find some $n \in \mathbb{N}$ such that $\psi^n(D) \subset \mathbf{D}_1^\natural$. For such an n , choose $x_n \in \mathbf{D}_2^\natural$ and $\tilde{x}_n \in D$ with $\psi^n(x_n) = x$ and $f(\tilde{x}_n) = x_n$. Put $u_n = \psi^n(\tilde{x}_n) \in \mathbf{D}_1^\natural$. By their construction in Lemma 1.5, the operators ψ on \mathbf{D}_1 and \mathbf{D}_2 commute with f , thus we may compute

$$f(u_n) = f(\psi^n(\tilde{x}_n)) = \psi^n(f(\tilde{x}_n)) = \psi^n(x_n) = x. \quad \square$$

Lemma 1.13. *Let $0 \rightarrow \mathbf{D}_1 \rightarrow \mathbf{D}_2 \rightarrow \mathbf{D}_3 \rightarrow 0$ be an exact sequence in $\text{Mod}^{\text{et}}(k((t)))$. For each i let $D_i \subset \mathbf{D}_i$ be a lattice with $\psi(D_i) = D_i$, and suppose that the above sequence restricts to an exact sequence*

$$0 \rightarrow D_1 \rightarrow D_2 \rightarrow D_3 \rightarrow 0. \quad (7)$$

If we have $D_1 = \mathbf{D}_1^\natural$ and $D_3 = \mathbf{D}_3^\natural$, then we also have $D_2 = \mathbf{D}_2^\natural$. If we have $D_1 = \mathbf{D}_1^\natural$ and $D_3 = \mathbf{D}_3^\natural$ then we also have $D_2 = \mathbf{D}_2^\natural$.

Proof. By Lemma 1.12 the sequence $0 \rightarrow \mathbf{D}_1^\natural \rightarrow \mathbf{D}_2^\natural \rightarrow \mathbf{D}_3^\natural \rightarrow 0$ is exact on the left and on the right. Comparing it with the sequence (7) via $\mathbf{D}_1^\natural = D_1$, $\mathbf{D}_2^\natural \subset D_2$ and $\mathbf{D}_3^\natural = D_3$, we immediately get $\mathbf{D}_2^\natural = D_2$. Next, by Lemma 1.12 the sequence $0 \rightarrow \mathbf{D}_1^\natural \rightarrow \mathbf{D}_2^\natural \rightarrow \mathbf{D}_3^\natural \rightarrow 0$ is exact on the left and on the right. We compare it with the sequence (7) via $D_1 = \mathbf{D}_1^\natural$, $D_2 \subset \mathbf{D}_2^\natural$ and $D_3 = \mathbf{D}_3^\natural$. We claim

$$\psi(\mathbf{D}_1 \cap \mathbf{D}_2^\natural) = D_1 \cap \mathbf{D}_2^\natural.$$

Of course, $\psi(\mathbf{D}_1 \cap \mathbf{D}_2^\natural) \subset D_1 \cap \mathbf{D}_2^\natural$ is clear. To see $D_1 \cap \mathbf{D}_2^\natural \subset \psi(\mathbf{D}_1 \cap \mathbf{D}_2^\natural)$ take $x \in D_1 \cap \mathbf{D}_2^\natural$. Choose $y \in \mathbf{D}_2^\natural$ with $\psi(y) = x$. Choose $y' \in D_2$ mapping to the same element in $\mathbf{D}_3^\natural = D_3$ as y . We then have

$\psi(y') \in D_2 \cap \mathbf{D}_1 = D_1$ and $\psi(y - y') - x \in D_1$, hence there is some $z \in D_1$ with $\psi(z) = \psi(y - y') - x$, hence $x = \psi(y - y' - z) \in \psi(\mathbf{D}_1 \cap \mathbf{D}_2^\sharp)$ since $y - y' \in \mathbf{D}_1 \cap \mathbf{D}_2^\sharp$ and $z \in \mathbf{D}_1 \cap \mathbf{D}_2^\sharp$.

The claim is proven. By the definition of \mathbf{D}_1^\sharp it implies $\mathbf{D}_1 \cap \mathbf{D}_2^\sharp = \mathbf{D}_1^\sharp$, hence $\mathbf{D}_1 \cap \mathbf{D}_2^\sharp = D_1$ since $D_1 = \mathbf{D}_1^\sharp$. Thus, $D_2 = \mathbf{D}_2^\sharp$. \square

Remark. An étale φ -module over $k((t))$ is a $k[[t]][\varphi] \otimes_{k[[t]]} k((t))$ -module \mathbf{D} which is finite dimensional over $k((t))$ such that the $k((t))$ -linearized structure map $\text{id} \otimes \varphi$ is bijective. The above theory of the operator ψ and the lattices \mathbf{D}^\sharp and \mathbf{D}^\natural works analogously for étale φ -modules \mathbf{D} over $k((t))$, i.e., the Γ -action is not really needed.

1C. Partial full faithfulness of $\Delta \mapsto \Delta^* \otimes_{k[[t]]} k((t))$.

Lemma 1.14. *Let N be a k -vector space, and suppose that we are given a k -linear automorphism τ of N , a basis \mathcal{N} of N , integers $0 \leq k_v \leq q - 1$ and units $\alpha_v \in k^\times$ for $v \in \mathcal{N}$. View N as a $k[[t]]$ -module with $tN = 0$ and let Δ denote the quotient of $k[[t]][\varphi] \otimes_{k[[t]]} N$ by the $k[[t]][\varphi]$ -submodule ∇ generated by the elements*

$$1 \otimes v + \alpha_v t^{k_v} \varphi \otimes \tau(v)$$

with $v \in \mathcal{N}$. We then have:

- (a) $k[[t]][\varphi] \otimes_{k[[t]]} N$ is a torsion $k[[t]]$ -module.
- (b) The map $N \rightarrow \Delta[t]$ sending $n \in N$ to the class of $1 \otimes n$ is an isomorphism. In particular, Δ is admissible if N is a finite dimensional k -vector space.
- (c) The action of φ on Δ is injective.

Proof. (a) As $\varphi t = t^q \varphi$ in $k[[t]][\varphi]$ we may write any element in $k[[t]][\varphi] \otimes_{k[[t]]} N$ as a finite sum of elements of the form $a\varphi^n \otimes x$ with $a \in k[[t]]$, $n \geq 0$ and $x \in N$. It is therefore enough to show

$$a\varphi^n \otimes x = 0 \quad \text{for each } a \in t^{q^n} k[[t]] \tag{8}$$

where $n \geq 0$ and $x \in N$. We may write $a = a_0 t^{q^n}$ with $a_0 \in k[[t]]$ and compute

$$a\varphi^n \otimes x = a_0 t^{q^n} \varphi^n \otimes x = a_0 \varphi^n t \otimes x = 0.$$

(b) and (c) We may write

$$k[[t]][\varphi] \otimes_{k[[t]]} N \cong \bigoplus_{v \in \mathcal{N}} \bigoplus_{i \geq 0} \bigoplus_{0 \leq \theta \leq q^i - 1} k \cdot t^\theta \varphi^i \otimes \tau(v).$$

Indeed, that $k[[t]][\varphi] \otimes_{k[[t]]} N$ is a quotient of the right hand side follows from formula (8). It is in fact an isomorphic quotient since all relations between φ and t in $k[[t]][\varphi]$ can be generated from $\varphi t = t^q \varphi$.

Consider the three k -subvector spaces

$$\begin{aligned} 1 \otimes N &= \bigoplus_{v \in \mathcal{N}} k \otimes \tau(v) \\ &= \bigoplus_{v \in \mathcal{N}} k \otimes v, \\ C &= \bigoplus_{v \in \mathcal{N}} \bigoplus_{i > 0} \bigoplus_{0 \leq \theta < q^{i-1}k_v} k \cdot t^\theta \varphi^i \otimes \tau(v), \end{aligned} \tag{9}$$

$$\nabla = \bigoplus_{v \in \mathcal{N}} \bigoplus_{i > 0} \bigoplus_{\epsilon \geq 0} k \cdot t^\epsilon \varphi^{i-1} (1 \otimes v + \alpha_v t^{k_v} \varphi \otimes \tau(v)). \tag{10}$$

Using the formula $\varphi t = t^q \varphi$ we see

$$t^\epsilon \varphi^{i-1} (1 \otimes v + \alpha_v t^{k_v} \varphi \otimes \tau(v)) \in k^\times \cdot t^{\epsilon+q^{i-1}k_v} \varphi^i \otimes \tau(v) + k[[t]] \varphi^{i-1} \otimes v.$$

We also see that in the sum (10) all summands with $\epsilon \geq (q-1)q^{i-1}k_v - 1$ vanish. Equivalently, in the sum (10) only those summands are nonzero for which $\theta = \epsilon + q^{i-1}k_v$ satisfies $q^{i-1}k_v \leq \theta \leq q^i - 1$. Thus we find

$$k[[t]][\varphi] \otimes_{k[[t]]} N \cong 1 \otimes N \bigoplus \nabla \bigoplus C. \tag{11}$$

Let C' , resp. C'' , denote the k -subspace of C spanned by all $t^\theta \varphi^i \otimes \tau(v)$ with $v \in \mathcal{N}$, $i > 1$ and $0 \leq \theta < q^{i-1}k_v$, resp. by all $t^\theta \varphi \otimes \tau(v)$ with $v \in \mathcal{N}$ and $0 \leq \theta < k_v$. Then $\varphi(C) \subset C'$ and $\varphi: C \rightarrow C'$ is injective. On the other hand, $\varphi(1 \otimes N) \subset C''$ and $\varphi: 1 \otimes N \rightarrow C''$ is injective. Since $C' \cap C'' = 0$ we conclude that φ acts injectively on Δ . Now consider the composed map

$$C \rightarrow k[[t]][\varphi] \otimes_{k[[t]]} N \xrightarrow{t(\cdot)} k[[t]][\varphi] \otimes_{k[[t]]} N \rightarrow 1 \otimes N \bigoplus C$$

where the first arrow is the inclusion, the last arrow is the projection. This map is bijective, the critical point being the computation

$$t(k \cdot t^{q^{i-1}k_v-1} \varphi^i \otimes \tau(v)) = k \cdot t^{q^{i-1}k_v} \varphi^i \otimes \tau(v) = k \cdot \varphi^{i-1} t^{k_v} \varphi \otimes \tau(v) \equiv k \cdot \varphi^{i-1} \otimes v$$

modulo ∇ (for $i > 0$). It follows that indeed the image of $1 \otimes N$ in Δ is the kernel of t acting on Δ . \square

Definition. An object $\Delta \in \text{Mod}^{\text{ad}}(\mathcal{O})$ is called standard cyclic if it is generated over $k[[t]][\varphi]$ by $\ker(t|_\Delta) = \Delta[t]$ and if there is a basis of $\Delta[t]$ consisting of Γ -eigenvectors e_0, \dots, e_d such that

$$t^{k_i} \varphi e_{i-1} = \rho_i e_i \quad \text{for all } 0 \leq i \leq d$$

(reading $e_{-1} = e_d$), for certain $0 \leq k_i \leq q-1$ and $\rho_i \in k^\times$ such that $k_i > 0$ for at least one i , as well as $k_i < q-1$ for at least one i .

In the following, we extend any indexing by $0, \dots, d$ to an indexing by \mathbb{Z} in the obvious way (i.e., $k_i = k_{i+d+1}$, $e_i = e_{i+d+1}$, $\rho_i = \rho_{i+d+1}$, $\eta_i = \eta_{i+d+1}$ for all $i \in \mathbb{Z}$). Let ∇ denote the $k[[t]][\varphi]$ -submodule

of $k[[t]][\varphi] \otimes_{k[[t]]} \Delta[t]$ generated by the elements $t^{k_i} \varphi \otimes e_{i-1} - 1 \otimes \rho_i e_i$. The inclusion $\Delta[t] \rightarrow \Delta$ extends to a natural $k[[t]][\varphi]$ -linear map

$$k[[t]][\varphi] \otimes_{k[[t]]} \Delta[t]/\nabla \rightarrow \Delta. \quad (12)$$

Proposition 1.15. *Let $\Delta \in \text{Mod}^{\text{ad}}(\mathfrak{O})$ be standard cyclic, with e_i, k_i, ρ_i, ρ_i as above:*

(a) *t acts surjectively on Δ , and there is a distinguished isomorphism of free $k[[t]]$ -modules of rank $d+1$*

$$\Delta^* \cong k[[t]] \otimes_k (\Delta[t]^*). \quad (13)$$

The map (12) is a $k[[t]][\varphi]$ -linear isomorphism.

- (b) *If for any $1 \leq j \leq d$ there is some $0 \leq i \leq d$ with $k_i \neq k_{i+j}$, then Δ is irreducible as a $k[[t]][\varphi]$ -module.*
- (c) *For $0 \leq i \leq d$ let $\eta_i : \Gamma \rightarrow k^\times$ be the character with $\gamma \cdot e_i = \eta_i(\gamma) e_i$ for all $\gamma \in \Gamma$. Suppose that for any $1 \leq j \leq d$ which satisfies $k_i = k_{i+j}$ for all $0 \leq i \leq d$ there is some $0 \leq i \leq d$ with $\eta_i \neq \eta_{i+j}$. Then Δ is irreducible as an \mathfrak{O} -module.*
- (d) *At least after a finite extension of k we have: Δ admits a filtration such that each associated graded piece is an irreducible standard cyclic object in $\text{Mod}^{\text{ad}}(\mathfrak{O})$. If p does not divide $d+1$ then Δ is even the direct sum of irreducible standard cyclic objects in $\text{Mod}^{\text{ad}}(\mathfrak{O})$.*

Proof. (This is very similar to [Grosse-Klönne 2016, Proposition 6.2].)

(a) For $0 \leq j \leq d$ consider

$$w_j = k_j + qk_{j-1} + \cdots + q^j k_0 + q^{j+1} k_d + \cdots + q^d k_{j+1}.$$

Repeated substitution of $\varphi t = t^q \varphi$ (recall $\Phi(t) = t^q$ modulo π) shows that $t^{w_j} \varphi^{d+1} e_j \in k^\times e_j$. As $k_i > 0$ for at least one i we have $w_j > 0$, and hence $e_j \in t\Delta$. As $\Delta[t]$ is generated over k by all e_j it follows that $\Delta[t] \subset t\Delta$. As Δ is generated over $k[[t]][\varphi]$ by $\Delta[t]$, the equation $\varphi t = t^q \varphi$ therefore shows $\Delta \subset t\Delta$, i.e., t acts surjectively on Δ . We deduce that Δ^* is a torsion free, and hence free $k[[t]]$ -module of rank $d+1$. As Δ is generated over $k[[t]][\varphi]$ by $\Delta[t]$ the map (12) is surjective. But it is also injective, because Lemma 1.14 tells us that it induces an isomorphisms between the respective kernels of t . We view the bijective map (12) as an identification. The proof of Lemma 1.14 yielded a canonical k -vector space decomposition $\Delta = C \oplus \Delta[t]$ where the k -subvector space C of Δ is generated by the image elements of the elements $t^\theta \varphi^r \otimes e \in k[[t]][\varphi] \otimes_{k[[t]]} \Delta[t]$ which do not belong to $1 \otimes \Delta[t]$ (for some $e \in \Delta[t]$, and some $\theta, r \geq 0$). We may thus identify $\Delta[t]^* = \text{Hom}_k(\Delta[t], k)$ with the subspace of $\Delta^* = \text{Hom}_k(\Delta, k)$ consisting of all $f \in \Delta^*$ with $f|_C = 0$. The composition of this k -linear embedding $\Delta[t]^* \rightarrow \Delta^*$ with the projection $\Delta^* \rightarrow (\Delta^*)/t(\Delta^*)$ is a k -linear isomorphism. Therefore, and as Δ^* is free and finitely generated over $k[[t]]$, the $k[[t]]$ -linear map $k[[t]] \otimes_k (\Delta[t]^*) \rightarrow \Delta^*$ extending the k -linear embedding $\Delta[t]^* \rightarrow \Delta^*$ is an isomorphism as stated in formula (13).

(b) Let Z be a nonzero $k[[t]][\varphi]$ -submodule of Δ . With Δ also Z is a torsion $k[[t]]$ -module, hence $\ker(t|_Z) = Z[t]$ is nonzero. For nonzero elements $z = \sum_{0 \leq i \leq d} x_i e_i$ of $Z[t]$ (with $x_i \in k$) put

$$\begin{aligned}\mathcal{D}(z) &= \{0 \leq i \leq d \mid x_i \neq 0\}, \quad \nu(z) = |\mathcal{D}(z)|, \\ \eta(z) &= \max\{k_i \mid i \in \mathcal{D}(z)\}, \quad \Lambda(z) = t^{\eta(z)} \varphi z.\end{aligned}$$

Then $\Lambda(z)$ is again a nonzero element of $Z[t]$. We have

$$\mathcal{D}(\Lambda(z)) = \{i + 1 \mid \eta(z) = k_i \text{ and } i \in \mathcal{D}(z)\}$$

(we read elements in $\{0 \leq i \leq d\}$ modulo $(d + 1)$), in particular $\nu(\Lambda(z)) \leq \nu(z)$. If $\nu(\Lambda(z)) = \nu(z)$ then $\mathcal{D}(\Lambda(z)) = \{i + 1 \mid i \in \mathcal{D}(z)\}$ and $k_i = k_{i+j}$ whenever $i, i + j \in \mathcal{D}(z)$. This implies that if we had $\nu(\Lambda^n(z)) = \nu(z) > 1$ for all $n \geq 0$ then there was some $1 \leq j \leq d$ with $k_i = k_{i+j}$ for all $0 \leq i \leq d$. But this would contradict our hypothesis. Thus, for sufficiently large $n \geq 0$ we have $\nu(\Lambda^n(z)) = 1$, i.e., $\Lambda^n(z) \in k^\times e_i$ for some $0 \leq i \leq d$. For such n we then even have $\Lambda^{n+j}(z) \in k^\times e_{i+j}$ for all $j \geq 0$. It follows that Z contains all e_i , hence $Z = \Delta$.

(c) We use the functions ν, Λ already employed in the proof of (b). Let $0 \neq Z \subset \Delta$ be a nonzero \mathfrak{O} -submodule. Choose a nonzero $z \in Z[t]$ for which $\nu(z)$ is minimal (for all nonzero $z \in Z[t]$). If $\nu(z) = 1$ then we obtain $Z = \Delta$ as in the proof of (b). Now assume $\nu(z) > 1$. For all $n \geq 0$ we have $\nu(\Lambda^n(z)) \leq \nu(z)$, hence $\nu(\Lambda^n(z)) = \nu(z)$ by the choice of z . Thus, writing $z = \sum_{0 \leq i \leq d} x_i e_i$ with $x_i \in k$, we have $x_i \neq 0$ and $x_{i+j} \neq 0$ for some i, j , with j violating the hypothesis in (b). By the hypothesis in (c), replacing i by $i + n$ and z by $\Lambda^n(z)$ we may assume that $\eta_i \neq \eta_{i+j}$. Pick $\gamma \in \Gamma$ with $\eta_i(\gamma) \neq \eta_{i+j}(\gamma)$, and pick $a \in k^\times$ with $ae_i = \gamma \cdot e_i$. Then $az - \gamma \cdot z$ is a nonzero element in $Z[t]$ with $\nu(az - \gamma \cdot z) < \nu(z)$: a contradiction.

(d) Passing to a finite extension of k if necessary we may assume that there is a $(d + 1)$ -st root of $\prod_{i=0}^d \rho_i$ in k . Thus, rescaling the e_i if necessary we may assume $\rho_i = \rho_j$ for all i, j . We argue by induction on d . If Δ itself is not irreducible then there is, by (c), some $1 \leq j \leq d$ which satisfies $k_i = k_{i+j}$ and $\eta_i = \eta_{i+j}$ for all $0 \leq i \leq d$. The minimal such j is a divisor of $d + 1$. Consider the k -subvector space V of $\Delta[t]$ spanned by the vectors $\epsilon_i = e_{ij}$ for $0 \leq i < (d + 1)/j$. Then

$$\left(\prod_{i=1}^j \rho_i^{-1} \right) t^{k_j} \varphi \cdots t^{k_1} \varphi$$

induces the automorphism f of V with $f(\epsilon_i) = \epsilon_{i+1}$ (where we understand $\epsilon_{(d+1)/j} = \epsilon_0$). Choose (after passing to a finite extension of k if necessary) an f -stable filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_{(d+1)/j} = V$ such that each V_i/V_{i-1} is one dimensional. Then define for $0 \leq s \leq (d + 1)/j$ the \mathfrak{O} -submodule $\Delta_s = \mathfrak{O}V_0 + \cdots + \mathfrak{O}V_s$ of Δ . It induces on $\Delta[t]$ the filtration

$$\Delta_s[t] = \Delta_{s-1}[t] + V_s + t^{k_1} \varphi V_s + \cdots + t^{k_{j-1}} \varphi \cdots t^{k_1} \varphi V_s.$$

By construction, each Δ_{i+1}/Δ_i is standard cyclic, and the induction hypothesis applies. If p does not divide $(d + 1)/j$ then there is even an f -stable direct sum decomposition $V = \bigoplus_s V_{[s]}$ with one

dimensional $V_{[s]}$. Then $\Delta = \bigoplus_s \Delta_{[s]}$ with $\Delta_{[s]} = \mathfrak{O} V_{[s]}$ is a direct sum decomposition of Δ , and each $\Delta_{[s]}$ is standard cyclic, and the induction hypothesis applies. \square

Lemma 1.16. *Let $\Delta \in \text{Mod}^{\text{ad}}(\mathfrak{O})$ be standard cyclic and put $\mathbf{D} = \Delta^* \otimes_{k[[t]]} k((t)) \in \text{Mod}^{\text{et}}(k((t)))$, see Proposition 1.4. We have $\mathbf{D}^\natural = \Delta^* = \mathbf{D}^\sharp$.*

Proof. In Proposition 1.15 we saw that Δ^* is a free $k[[t]]$ -module, hence the natural map $\Delta^* \rightarrow \mathbf{D} = \Delta^* \otimes_{k[[t]]} k((t))$ is injective; we view it as an inclusion.

The φ -operator on Δ is the adjoint of the ψ -operator on \mathbf{D} , in such a way that $\psi(\Delta^*) = \Delta^*$ since φ acts injectively on Δ . Therefore the definitions of \mathbf{D}^\natural and \mathbf{D}^\sharp yield $\mathbf{D}^\natural \subset \Delta^* \subset \mathbf{D}^\sharp$. Since \mathbf{D}^\natural is a lattice with $\psi(\mathbf{D}^\natural) = \mathbf{D}^\natural$ we get $t\Delta^\sharp \subset \mathbf{D}^\natural$, together

$$\mathbf{D}^\natural \subset \Delta^* \subset \mathbf{D}^\sharp \quad \text{and} \quad t\mathbf{D}^\sharp \subset \mathbf{D}^\natural. \quad (14)$$

Let e_i and k_i be as in the definition of Δ being standard cyclic.

Formula (14) implies $t\Delta^* \subset \mathbf{D}^\natural$, hence $t(\Delta^*/\mathbf{D}^\natural) = 0$, hence $\Delta^*/\mathbf{D}^\natural$ is dual to a subspace W of $\Delta[t]$ stable under φ . To prove $\mathbf{D}^\natural = \Delta^*$ it is therefore enough to prove that $\Delta[t]$ does not contain a nonzero subspace W stable under φ . Assume that such a W does exist. A nonzero element $\beta \in W$ may be written as $\beta = \sum_{i=0}^d \alpha_i e_i$ with $\alpha_i \in k$. Let $k = \max\{k_{i+1} \mid \alpha_i \neq 0\}$. Since by assumption $k_i > 0$ for at least one i , replacing β by $\varphi^r \beta$ for some $r \in \mathbb{N}$ if necessary, we may assume $k > 0$. But then $t^k \varphi \beta$ is a nonzero linear combination of the e_i , whereas we also have $t\varphi \beta = 0$ since $\varphi \beta \in W \subset \Delta[t]$: a contradiction.

Formula (14) implies $t\mathbf{D}^\sharp \subset \Delta^*$, i.e., $t(\mathbf{D}^\sharp/\Delta^*) = 0$. We endow \mathbf{D}^\sharp and all its submodules with the t -adic topology. By Pontrjagin duality (as recalled e.g., in [Schneider and Venjakob 2016]) we in particular have $\text{Hom}_k^{\text{cont}}(\Delta^*, k) = \Delta$. Now $t(\mathbf{D}^\sharp/\Delta^*) = 0$ means that the kernel W of the natural projection $\text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k) \rightarrow \text{Hom}_k^{\text{cont}}(\Delta^*, k) = \Delta$ is contained in $\text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)[t]$. As t acts injectively on \mathbf{D}^\sharp , it acts surjectively on $\text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)$. Hence, if $\Delta^* \neq \mathbf{D}^\sharp$ then $W \neq 0$ and there is some $\beta \in \text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)$ with $0 \neq t\beta \in W$. Now $t\beta \in W$ means that β maps to an element in $\Delta[t]$. Since on the other hand $tW = 0$ (as $W \subset \text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)[t]$) we may write $\beta = \sum_{i=0}^d \alpha_i \tilde{e}_i$ with $\alpha_i \in k$, where $\tilde{e}_i \in \text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)$ lifts e_i . We then also have $0 \neq t\tilde{e}_{i_0} \in W$ for some i_0 . As φ is injective on W (which follows from the surjectivity of ψ on \mathbf{D}^\sharp and hence on $W^* = \mathbf{D}^\sharp/\Delta^*$) this gives $t^q \varphi \tilde{e}_{i_0} = \varphi t \tilde{e}_{i_0} \neq 0$ in $\text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)$. Together with $W \subset \text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)[t]$ we get $t^{q-1} \varphi e_{i_0} \neq 0$ in Δ . Applying the same argument with $t^{q-1} \varphi \tilde{e}_{i_0}$ instead of \tilde{e}_i (again using that $t^q \varphi \tilde{e}_{i_0} \neq 0$) we see $t^{q-1} \varphi t^{q-1} \varphi e_{i_0} \neq 0$. Next we get $t^{q-1} \varphi t^{q-1} \varphi t^{q-1} \varphi e_{i_0} \neq 0$ etc.. But this means $q-1 = k_i$ for each i , contradicting the hypothesis. We obtain $\Delta^* = \mathbf{D}^\sharp$. \square

Definition. Let $\text{Mod}^\natural(\mathfrak{O})$ denote the subcategory of $\text{Mod}^{\text{ad}}(\mathfrak{O})$ whose objects admit a filtration such that each associated graded piece becomes a standard cyclic object in $\text{Mod}^{\text{ad}}(\mathfrak{O})$ after a suitable field extension of k .

Remark. Proposition 1.15(d) implies that each subquotient in $\text{Mod}^{\text{ad}}(\mathfrak{O})$ of an object in $\text{Mod}^\natural(\mathfrak{O})$ again is an object in $\text{Mod}^\natural(\mathfrak{O})$.

Proposition 1.17. *The restriction of the functor (3) to the category $\text{Mod}^\natural(\mathfrak{O})$ is exact and fully faithful.*

Proof. We already know that the functor is exact. Next, we claim

$$\mathbf{D}^\natural = \Delta^* = \mathbf{D}^\sharp \quad \text{with} \quad \mathbf{D} = \Delta^* \otimes_{k[[t]]} k((t)) \quad (15)$$

for $\Delta \in \text{Mod}^\bullet(\mathfrak{O})$. Indeed, for standard cyclic Δ this is shown in Lemma 1.16. For Δ which become standard cyclic after a field extension k'/k it then follows since the definitions of $(\cdot)^\natural$ and $(\cdot)^\sharp$ in terms of the k -linear operator ψ imply $\mathbf{D}^\natural \otimes_k k' = (\mathbf{D} \otimes_k k')^\natural$ and $\mathbf{D}^\sharp \otimes_k k' = (\mathbf{D} \otimes_k k')^\sharp$. For general $\Delta \in \text{Mod}^\bullet(\mathfrak{O})$ it then follows from Lemma 1.13. We now claim that the reverse functor (on the essential image of the functor under discussion) is given by sending \mathbf{D} to the topological dual $(\mathbf{D}^\natural)'$ of \mathbf{D}^\natural (where we endow \mathbf{D}^\natural with its t -adic topology). Indeed, for \mathbf{D} in this essential image and for $\Delta \in \text{Mod}^\bullet(\mathfrak{O})$ we have natural isomorphisms

$$((\mathbf{D}^\natural)')^* \otimes_{k[[t]]} k((t)) \xrightarrow{\text{(i)}} \mathbf{D}^\natural \otimes_{k[[t]]} k((t)) \cong \mathbf{D}, \quad ((\Delta^* \otimes_{k[[t]]} k((t)))^\natural)' \xrightarrow{\text{(ii)}} (\Delta^*)' \xrightarrow{\text{(iii)}} \Delta,$$

where (i) and (iii) follow from Pontrjagin duality, see e.g., Proposition 5.4 in [Schneider and Venjakob 2016], and where (ii) follows from formula (15). \square

1D. Standard cyclic étale (φ, Γ) -modules.

Proposition 1.18. *Let $\Delta \in \text{Mod}^{\text{ad}}(\mathfrak{O})$ be a standard cyclic object, with $d, e_i, k_i, \rho_i, \eta_i$ as in the definition resp. as in Proposition 1.15. The étale (φ, Γ) -module $\Delta^* \otimes_{k[[t]]} k((t))$ over $k((t))$ admits a $k((t))$ -basis f_0, \dots, f_d such that for all $0 \leq j \leq d$ we have*

$$\varphi(f_{j-1}) = \rho_{j-1}^{-1} t^{1+k_j-q} f_j \quad (16)$$

(reading $f_{-1} = f_d$), and moreover

$$\gamma \cdot f_j - \eta_j^{-1}(\gamma) f_j \in t k[[t]] f_j \quad \text{for all } \gamma \in \Gamma. \quad (17)$$

Proof. We use formula (2).

First we assume $F \neq \mathbb{Q}_p$. Put $N = \bigoplus_{i=0}^d k.e_i$. As explained in the proof of Proposition 1.15, we have a bijective map (12) which we view as an identification. In particular, Lemma 1.14 and its proof apply. In the context of that proof we identify e_i with the class of $1 \otimes e_i$ in Δ . By formula (11) we have a k -linear isomorphism $(1 \otimes N) \oplus C \cong \Delta$ with C as in formula (9). For $0 \leq j \leq d$ we may therefore define $f_j \in \Delta^*$ by asking $f_j(C) = 0$ and $f_j(e_i) = \delta_{ij}$ for $0 \leq i \leq d$. Proposition 1.15 tells us that f_0, \dots, f_d is a $k[[t]]$ -basis of Δ^* . For $\theta, r \geq 0$ and any i, j we have $f_j(t^\theta \varphi^r \otimes e_i) \neq 0$ if and only if $r \equiv j - i$ modulo $(d+1)\mathbb{Z}$ and $\theta = k_j + qk_{j-1} + \dots + qr^{-1}k_{j-r+1}$. As before, $\psi \in \text{End}_k(\Delta^*)$ is defined by $(\psi(f))(x) = f(\varphi(x))$ for $x \in \Delta$, $f \in \Delta^*$. We claim

$$\psi(t^{m+k_j+1} f_j) = \rho_{j-1} \psi_{k((t))}(t^m) t f_{j-1} \quad (18)$$

for all j , all $m \geq -k_j - 1$. Indeed, for $0 \leq i \leq d$ and $\theta, r \geq 0$ we have

$$(\psi(t^{m+k_j+1} f_j))(t^\theta \varphi^r \otimes e_i) = f_j(t^{m+k_j+1} \varphi t^\theta \varphi^r \otimes e_i).$$

If $m + 1 \notin \mathbb{Z}q$ then this shows $(\psi(t^{m+k_j+1}f_j))(t^\theta\varphi^r \otimes e_i) = 0$ by what we pointed out above. But $m + 1 \notin \mathbb{Z}q$ also implies $\psi_{k((t))}(t^m) = 0$. In the case $m + 1 = qn$ (some $n \in \mathbb{Z}$) we compute

$$\begin{aligned} (\psi(t^{m+k_j+1}f_j))(t^\theta\varphi^r \otimes e_i) &= f_j(t^{k_j+qn}\varphi t^\theta\varphi^r \otimes e_i) \\ &= f_j(t^{k_j}\varphi t^{n+\theta}\varphi^r \otimes e_i) \\ &= \rho_{j-1}f_{j-1}(t^{n+\theta}\varphi^r \otimes e_i) \\ &= (\rho_{j-1}\psi_{k((t))}(t^m)tf_{j-1})(t^\theta\varphi^r \otimes e_i) \end{aligned}$$

where we used $\psi_{k((t))}(t^m) = t^{n-1}$. We have proven formula (18).

On the other hand, by tracing the construction in Proposition 1.4 we see that $\varphi(tf_{j-1})$ is characterized by satisfying

$$\psi(t^m\varphi(tf_{j-1})) = \psi_{k((t))}(t^m)tf_{j-1} \quad (19)$$

for all m . Comparing formulae (18) and (19) we find $\varphi(tf_{j-1}) = \rho_{j-1}^{-1}t^{k_j+1}f_j$ which is equivalent with formula (16). Next, for $\gamma \in \Gamma$ we compute

$$(\gamma \cdot f_j)(e_i) = f_j(\gamma^{-1} \cdot e_i) = f_j(\eta_i(\gamma^{-1})e_i) = (\eta_i(\gamma^{-1})f_j)(e_i) = (\eta_j(\gamma^{-1})f_j)(e_i).$$

Here the last equation is trivial if $i = j$, whereas if $i \neq j$ then both sides vanish. This shows $(\gamma \cdot f_j - \eta_j(\gamma^{-1})f_j)|_N = 0$, and hence $\gamma \cdot f_j - \eta_j(\gamma^{-1})f_j \in t\Delta^* = tk[\![t]\!]\{f_0, \dots, f_d\}$. On the other hand, by what we pointed out above, $(\gamma \cdot f_j)(t^\theta\varphi^r \otimes e_i) = f_j([\gamma]_\Phi(t)^\theta\varphi^r \otimes e_i)$ vanishes if $r + i - j \notin (d + 1)\mathbb{Z}$, and this shows $\gamma \cdot f_j \in k[\![t]\!]f_j$. We trivially have $\eta_j(\gamma^{-1})f_j \in k[\![t]\!]f_j$, and hence altogether $\gamma \cdot f_j - \eta_j(\gamma^{-1})f_j \in tk[\![t]\!]\{f_0, \dots, f_d\} \cap k[\![t]\!]f_j = tk[\![t]\!]f_j$, formula (17).

Now we assume $F = \mathbb{Q}_p$. Let us suppose for simplicity that $\pi = q$. For $0 \leq j \leq d$ we may define $f_j \in \Delta^*$ as follows. For $\theta, r \geq 0$ (and any i, j) we require $f_j(t^\theta\varphi^r \otimes e_i) \neq 0$ if and only if $r \equiv j - i$ modulo $(d + 1)\mathbb{Z}$ and there are $a_1, \dots, a_{r-1} \in \{0, 1\}$ such that

$$\theta = k_j + qk_{j-1} + \dots + q^{r-1}k_{j-r+1} + \sum_{i=1}^{r-1} a_i q^{i-1}(1-q);$$

if this is the case we put

$$f_j(t^\theta\varphi^r \otimes e_i) = \rho_{j-1}\rho_{j-2} \cdots \rho_{j-r}.$$

(As usual, the subindices of the ρ_i are read modulo $(d + 1)\mathbb{Z}$.) Again f_0, \dots, f_d is a $k[\![t]\!]$ -basis of Δ^* . Again we claim formula (18). As before we see that both sides vanish if $m \notin \mathbb{Z}q - 1 \cup \mathbb{Z}q$, and coincide if $m \in \mathbb{Z}q - 1$. But the same computation also shows their coincidence if $m = qn$ for some $n \in \mathbb{N}$, as follows:

$$\begin{aligned} (\psi(t^{m+k_j+1}f_j))(t^\theta\varphi^r \otimes e_i) &= f_j(t^{k_j+1}\varphi t^{n+\theta}\varphi^r \otimes e_i) \\ &= \rho_{j-1}f_{j-1}(t^{n+\theta+1}\varphi^r \otimes e_i) \\ &= (\rho_{j-1}\psi_{k((t))}(t^m)tf_{j-1})(t^\theta\varphi^r \otimes e_i) \end{aligned}$$

where we used $\psi_{k((t))}(t^m) = t^n$. With formula (18) being established, the remaining arguments are exactly as before. \square

Definition. We say that an object $\mathbf{D} \in \text{Mod}^{\text{et}}(k((t)))$ of dimension $d+1$ is standard cyclic if it admits a $k((t))$ -basis f_0, \dots, f_d such that there are $\sigma_j \in k^\times$, characters $\alpha_j : \Gamma \rightarrow k^\times$ and $m_j \in \{1-q, \dots, -1, 0\}$ for $0 \leq j \leq d$ satisfying the following conditions:

- $(m_0, \dots, m_d) \notin \{(0, \dots, 0), (1-q, \dots, 1-q)\}$.
- $\varphi(f_{j-1}) = \sigma_j t^{m_j} f_j$ for all j (reading $f_{-1} = f_d$).
- $\gamma \cdot f_j - \alpha_j(\gamma) f_j \in tk[[t]]\{f_0, \dots, f_d\}$ for all $\gamma \in \Gamma$.

Lemma 1.19. (a) *The constant $\prod_{j=0}^d \sigma_j \in k^\times$ as well as, up to cyclic permutation, the ordered tuple $((\alpha_0, m_0), \dots, (\alpha_d, m_d))$, are uniquely determined by the isomorphism class of the (φ, Γ) -module \mathbf{D} .*

(b) *$\alpha_1, \dots, \alpha_d$ are uniquely determined by α_0 and m_0, \dots, m_d .*

Proof. (a) In the following, for elements of $\text{GL}_{d+1}(k((t)))$ we read the (two) respective indices of their entries always modulo $(d+1)\mathbb{Z}$.

The effect of φ on the basis f_0, \dots, f_d is described by $T = (T_{ij})_{0 \leq i, j \leq d} \in \text{GL}_{d+1}(k((t)))$ with $T_{i,i+1} = \sigma_i t^{m_i}$ for $0 \leq i \leq d$, but $T_{i,j} = 0$ for $j \neq i+1$.

Let $\sigma'_j \in k^\times$ and $((\alpha'_0, m'_0), \dots, (\alpha'_d, m'_d))$ be another datum as above, let \mathbf{D}' be an étale (φ, Γ) -module admitting a $k((t))$ -basis f'_0, \dots, f'_d with $\varphi(f'_{j-1}) = \sigma'_j t^{m'_j} f'_j$ and $\gamma \cdot f'_j - \alpha'_j(\gamma) f'_j \in tk[[t]]\{f'_0, \dots, f'_d\}$ for $\gamma \in \Gamma$. Define $T' = (T'_{ij})_{0 \leq i, j \leq d} \in \text{GL}_{d+1}(k((t)))$ similarly as above.

Suppose that there is an isomorphism of (φ, Γ) -modules $\mathbf{D}' \cong \mathbf{D}$. With respect to the bases f_\bullet and f'_\bullet it is described by some $A(t) = (a_{i,j}(t))_{0 \leq i, j \leq d} \in \text{GL}_{d+1}(k((t)))$. In view of $\varphi t = \Phi(t)\varphi$, the compatibility of the isomorphism with the respective φ -actions comes down to the matrix equation

$$T \cdot A(t) = A(\Phi(t)) \cdot T'.$$

For the individual entries this is equivalent with

$$a_{i,j}(t) = \sigma'_j \sigma_i^{-1} t^{m'_j - m_i} a_{i-1,j-1}(\Phi(t))$$

for all i, j . Iteration of this equation yields

$$a_{i,j}(t) = \left(\prod_{\ell=0}^d \sigma'_{j-\ell} \sigma_{i-\ell}^{-1} (\Phi^\ell(t))^{m'_{j-\ell} - m_{i-\ell}} \right) a_{i,j}(\Phi^{d+1}(t))$$

for all i, j . (Here $\Phi^\ell(t)$ resp. $\Phi^{d+1}(t)$ means $\Phi(\Phi(\dots \Phi(t) \dots))$.) From this we deduce that for fixed i, j either $a_{i,j}$ is a nonzero constant and $\prod_{\ell=0}^d \sigma'_{j-\ell} \sigma_{i-\ell}^{-1} = 1$ and $m'_{j-\ell} = m_{i-\ell}$ for all ℓ , or $a_{i,j} = 0$. But since $A(t)$ is invertible we do find i, j with $a_{i,j} \neq 0$. It already follows that $\prod_{j=0}^d \sigma_j = \prod_{j=0}^d \sigma'_j$ and that (m'_0, \dots, m'_d) coincides with (m_0, \dots, m_d) up to cyclic permutation. But since in addition we just saw that A is a constant matrix, with $a_{i,j} = 0$ if and only if $a_{i-1,j-1} = 0$, we see that the same index permutation takes α'_j to α_j .

(b) This follows from the fact that, in view of the defining formulae, D is generated by f_0 as a φ -module over $k((t))$. \square

Proposition 1.20. *The functor $\Delta \mapsto \Delta^* \otimes_{k[[t]]} k((t))$ induces a bijection between the set of standard cyclic objects in $\text{Mod}^{\text{ad}}(\mathfrak{O})$ and the set of standard cyclic objects in $\text{Mod}^{\text{et}}(k((t)))$.*

Proof. $\Delta^* \otimes_{k[[t]]} k((t))$ for a standard cyclic object $\Delta \in \text{Mod}^{\text{ad}}(\mathfrak{O})$ is a standard cyclic object in $\text{Mod}^{\text{et}}(k((t)))$ by Proposition 1.18. With Lemma 1.19(a) we see that the assignment $\Delta \mapsto \Delta^* \otimes_{k[[t]]} k((t))$ is injective on standard cyclic objects in $\text{Mod}^{\text{ad}}(\mathfrak{O})$. It is also surjective: Proposition 1.18 (together with Lemma 1.19(b)) explicitly says how to convert the parameter data describing a standard cyclic object in $\text{Mod}^{\text{et}}(k((t)))$ into the parameter data describing a standard cyclic object in $\text{Mod}^{\text{ad}}(\mathfrak{O})$. \square

Definition. A $(d+1)$ -dimensional standard cyclic $\text{Gal}(\bar{F}/F)$ -representation is a $\text{Gal}(\bar{F}/F)$ -representation over k which corresponds, under the equivalence of categories in Theorem 1.2, to an object in $\text{Mod}^{\text{et}}(k((t)))$ of dimension $d+1$ which is standard cyclic.

2. Hecke algebras and supersingular modules

2A. The pro- p -Iwahori Hecke algebra \mathcal{H} . We introduce the pro- p -Iwahori Hecke algebra \mathcal{H} of $\text{GL}_{d+1}(F)$ with coefficients in k in a slightly unorthodox way, which however is well suited for our later constructions.

Let \bar{T} be a free $\mathbb{Z}/(q-1)$ -module of rank $d+1$. Then $\text{Hom}(\Gamma, \bar{T})$ (with $\Gamma = \mathcal{O}_F^\times$) is also free of rank $d+1$ over $\mathbb{Z}/(q-1)$. We write the group law of \bar{T} multiplicatively, but that of $\text{Hom}(\Gamma, \bar{T})$ we write additively. Let $e^*, \alpha_1^\vee, \dots, \alpha_d^\vee$ be a $\mathbb{Z}/(q-1)$ -basis of $\text{Hom}(\Gamma, \bar{T})$. Put $\alpha_0^\vee = -\sum_{i=1}^d \alpha_i^\vee$. We let the symmetric group \mathfrak{S}_{d+1} act on $\text{Hom}(\Gamma, \bar{T})$ as follows. We think of \mathfrak{S}_{d+1} as the permutation group of $\{0, 1, \dots, d\}$, generated by the transposition $s = (01) \in \mathfrak{S}_{d+1}$ and the cycle $\omega \in \mathfrak{S}_{d+1}$ with $\omega(i) = i+1$ for all $0 \leq i \leq d-1$. We then put

$$\omega \cdot e^* = e^* + \alpha_0^\vee, \quad \omega \cdot \alpha_0^\vee = \alpha_d^\vee \quad \text{and} \quad \omega \cdot \alpha_i^\vee = \alpha_{i-1}^\vee \quad \text{for } 1 \leq i \leq d.$$

If $d = 1$ we put

$$s \cdot e^* = e^* - \alpha_1^\vee, \quad s \cdot \alpha_i^\vee = -\alpha_i^\vee \quad \text{for } i = 0, 1,$$

but if $d \geq 2$ we put

$$s \cdot e^* = e^* - \alpha_1^\vee, \quad s \cdot \alpha_0^\vee = \alpha_0^\vee + \alpha_1^\vee, \quad s \cdot \alpha_1^\vee = -\alpha_1^\vee, \quad s \cdot \alpha_2^\vee = \alpha_1^\vee + \alpha_2^\vee, \quad s \cdot \alpha_i^\vee = \alpha_i^\vee \quad \text{for } 3 \leq i \leq d.$$

One easily checks that there is a unique action of \mathfrak{S}_{d+1} on \bar{T} such that for $\gamma \in \Gamma$ and $f \in \text{Hom}(\Gamma, \bar{T})$ we have

$$\omega \cdot (f(\gamma)) = (\omega \cdot f)(\gamma) \quad \text{and} \quad s \cdot (f(\gamma)) = (s \cdot f)(\gamma).$$

Define $\alpha_1^\vee(\mathbb{F}_q^\times)$ to be the image of the composition $\mathbb{F}_q^\times \rightarrow \Gamma \xrightarrow{\alpha_1^\vee} \bar{T}$ where the first map is the Teichmüller homomorphism.

Definition. (a) The k -algebra \mathcal{H} is generated by elements $T_\omega^{\pm 1}$, T_s and T_t for $t \in \bar{T}$, subject to the following relations (with $t, t' \in \bar{T}$):

$$T_s T_\omega T_s T_\omega^{-1} T_s T_\omega = T_\omega T_s T_\omega^{-1} T_s T_\omega T_s \quad \text{if } d > 1, \quad (20)$$

$$T_s T_\omega^{-m} T_s T_\omega^m = T_\omega^{-m} T_s T_\omega^m T_s \quad \text{for all } 1 < m < d, \quad (21)$$

$$T_s^2 = T_s \tau_s = \tau_s T_s \quad \text{with } \tau_s = \sum_{t \in \alpha_1^\vee(\mathbb{F}_q^\times)} T_t, \quad (22)$$

$$T_\omega T_\omega^{-1} = 1 = T_\omega^{-1} T_\omega, , \quad (23)$$

$$T_\omega^{d+1} T_s = T_s T_\omega^{d+1}, \quad (24)$$

$$T_t T_{t'} = T_{t't}, \quad T_{1_{\bar{T}}} = 1, \quad (25)$$

$$T_t T_\omega = T_\omega T_{\omega \cdot t}, \quad (26)$$

$$T_t T_s = T_s T_{s \cdot t}. \quad (27)$$

Notice that T_ω^{d+1} is central in \mathcal{H} .

(b) \mathcal{H}_{aff} is the k -subalgebra of \mathcal{H} generated by all T_t for $t \in \bar{T}$, by T_ω^{d+1} , T_ω^{-d-1} and by all $T_\omega^m T_s T_\omega^{-m}$ for $m \in \mathbb{Z}$.

(c) \mathcal{H}^\flat is the quotient of \mathcal{H} by the two sided ideal spanned by all elements $T_t - 1$ with $t \in \bar{T}$.

Caution. \mathcal{H}_{aff} differs from the similarly denoted algebra in [Vignéras 2005]. (The difference is that here we include $(T_\omega^{d+1})^\mathbb{Z}$.)

Remark. Let \bar{T} denote the subgroup of $G = \text{GL}_{d+1}(F)$ consisting of diagonal matrices with entries in the image of the Teichmüller homomorphism $\mathbb{F}_q^\times \rightarrow \mathcal{O}_F^\times$. For $\gamma \in \Gamma$ let $\bar{\gamma}$ be its image in \mathbb{F}_q^\times . In \bar{T} define the elements $e^*(\gamma) = \text{diag}(\bar{\gamma}, 1_d)$ and $\alpha_i^\vee(\gamma) = \text{diag}(1_{i-1}, \bar{\gamma}, \bar{\gamma}^{-1}, 1_{d-i})$ for $1 \leq i \leq d$. Define the elements $\omega = (\omega_{ij})_{0 \leq i, j \leq d}$ and $s = (s_{ij})_{0 \leq i, j \leq d}$ of G by $\omega_{d0} = \pi$ and $\omega_{i,i+1} = 1$ (for $0 \leq i \leq d-1$) and $\omega_{ij} = 0$ for all other pairs (i, j) , resp. by $s_{10} = s_{01} = s_{ii} = 1$ for $i \geq 2$, and $s_{ij} = 0$ for all other pairs (i, j) .

Let I_0 denote the pro- p -Iwahori subgroup of G for which $g = (g_{ij})_{0 \leq i, j \leq d} \in G$ belongs to I_0 if and only if all the following conditions are satisfied: $g_{ij} \in \pi \mathcal{O}_F$ for $i > j$, and $g_{ij} \in \mathcal{O}_F$ for $i < j$, and $g_{ii} \in 1 + \pi \mathcal{O}_F$.

Claim. The corresponding pro- p -Iwahori Hecke algebra $k[I_0 \backslash G / I_0]^{\text{op}} \cong \text{End}_{k[G]}(\text{ind}_{I_0}^G k)^{\text{op}}$ is isomorphic with \mathcal{H} , in such a way that the double coset $I_0 g I_0$ for $g \in \bar{T} \cup \{s, \omega\}$ corresponds to the element $T_g \in \mathcal{H}$.

To prove this claim we use the description of $k[I_0 \backslash G / I_0]^{\text{op}}$ worked out by Vignéras [2005] (or rather we use the description of $k[I_0 \backslash G / I_0]^{\text{op}}$ which results from the description of $k[I_0 \backslash G / I_0]$ given in [loc. cit.]).

Let T denote the maximal torus of diagonal matrices in G , let $N(T)$ be its normalizer in G . Let T_1 (resp. T_0) denote the subgroup of T consisting of diagonal matrices with entries in the kernel of $\mathcal{O}_F^\times \rightarrow \mathbb{F}_q^\times$ (resp. in \mathcal{O}_F^\times); thus $T_0 / T_1 \cong \bar{T}$. For $0 \leq i \leq d$ define $s_i = \omega^{1-i} s \omega^{i-1}$. The (classes of) s_0, s_1, \dots, s_d are the Coxeter generators of a Coxeter subgroup W_{aff} of $N(T) / T_0$, and $N(T) / T_0$ is generated by W_{aff} together with the element ω . The length function $\ell : W_{\text{aff}} \rightarrow \mathbb{Z}_{\geq 0}$ can be extended to a function $\ell : N(T) / T_0 \rightarrow \mathbb{Z}_{\geq 0}$

in such a way that $\ell(\omega) = 0$. We again denote by ℓ the induced function $W^{(1)} = N(T)/T_1 \rightarrow \mathbb{Z}_{\geq 0}$. For $w \in W^{(1)}$ and $w' \in N(T)$ lifting w , the double coset $I_0 w' I_0$ only depends on w ; we denote it by T_w . For $0 \leq i \leq d$ let \bar{T}_i be the image of one of the two cocharacters $\mathbb{F}_q^\times \rightarrow \bar{T}$ associated with s_i . (Here we identify \bar{T} with the maximal torus of diagonal matrices in $\mathrm{GL}_{d+1}(\mathbb{F}_q)$. If $1 \leq i \leq d$ then s_i is the simple reflection associated with the coroot α_i^\vee , and $\alpha_i^\vee(\mathbb{F}_q^\times) = \bar{T}_i$.) Now, according to [Vignéras 2005], a k -basis of $k[I_0 \backslash G / I_0]^{\mathrm{op}}$ is given by the set of all T_w for $w \in W^{(1)}$, and the multiplication is uniquely determined by the relations

$$T_w T_{w'} = T_{w'w} \quad \text{for } w, w' \in W^{(1)} \text{ with } \ell(w) + \ell(w') = \ell(ww'), \quad (28)$$

$$T_{s_i}^2 = T_{s_i} \tau_i \quad \text{where } \tau_i = \sum_{t \in \bar{T}_i} T_t \text{ for } 0 \leq i \leq d. \quad (29)$$

In the following we repeatedly use that conjugating these relations by powers of T_ω leads to similar relations (since $\ell(\omega) = 0$). From formula (28) we first deduce $T_{s_i} = T_\omega^{i-1} T_s T_\omega^{1-i}$ and then that $T_\omega^{\pm 1}$ and $T_s = T_{s_1}$ together with the elements T_t for $t \in \bar{T}$ generate $k[I_0 \backslash G / I_0]^{\mathrm{op}}$ as a k -algebra. Next, from $s_i s_{i-1} s_i = s_{i-1} s_i s_{i-1}$ in W_{aff} (for $0 \leq i \leq d$; if $i = 0$ read $i-1 = d$) we get $T_{s_i} T_{s_{i-1}} T_{s_i} = T_{s_{i-1}} T_{s_i} T_{s_{i-1}}$ by applying formula (28) twice, but this comes down to formula (20) (up to conjugation by a power of T_ω). Similarly from $s_i s_j = s_j s_i$ in W_{aff} for $0 \leq i < j-1 \leq d-1$ with $i+d > j$ we get $T_{s_i} T_{s_j} = T_{s_j} T_{s_i}$ by applying formula (28) twice, but this comes down to formula (21) (up to conjugation by a power of T_ω). Formula (29) for any i is a T_ω -power conjugate of formula (22). Finally, formulae (23), (24), (25), (26) and (27) are special instances of formula (28). Conversely, it is not hard to see that these, together with formulae (20), (21) and (22) suffice to generate all relations in $k[I_0 \backslash G / I_0]^{\mathrm{op}}$. The claim is proven.

We add if I denotes the Iwahori subgroup of G containing I_0 , then \mathcal{H}^\flat becomes isomorphic with the Iwahori Hecke algebra $k[I \backslash G / I]^{\mathrm{op}}$.

Definition. A character $\chi : \mathcal{H}_{\mathrm{aff}} \rightarrow k$ is called *supersingular* if the following two conditions are both satisfied:

- (a) There is an $m \in \mathbb{Z}$ with $\chi(T_\omega^m T_s T_\omega^{-m}) = 0$.
- (b) There is an $m \in \mathbb{Z}$ with either $\chi(T_\omega^m T_s T_\omega^{-m}) = -1$ or $\chi(T_\omega^m \tau_s T_\omega^{-m}) = 0$.⁷

Definition. (a) An \mathcal{H} -module M is called *standard supersingular* if it is isomorphic with $\mathcal{H} \otimes_{\mathcal{H}_{\mathrm{aff}}, \chi} k.e$, where $\mathcal{H}_{\mathrm{aff}}$ acts on the one dimensional k -vector space $k.e$ through a supersingular character χ .

Equivalently, M is standard supersingular if and only if $M = \bigoplus_{0 \leq m \leq d} T_\omega^m (M_1)$ with an $\mathcal{H}_{\mathrm{aff}}$ -module M_1 of k -dimension 1 on which $\mathcal{H}_{\mathrm{aff}}$ acts through a supersingular character.⁸

(b) An irreducible \mathcal{H} -module is called *supersingular* if it is a subquotient of a standard supersingular \mathcal{H} -module.

⁷We have $\chi(T_\omega^m \tau_s T_\omega^{-m}) = 0$ if and only if $\chi(T_\omega^m T_t T_\omega^{-m}) \neq 1$ for some $t \in \alpha_1^\vee(\mathbb{F}_q^\times)$, if and only if $\chi(\alpha_{m+1}^\vee(\gamma)) \neq 1$ for some $\gamma \in \Gamma$.

⁸Then $\mathcal{H}_{\mathrm{aff}}$ acts on each $T_\omega^m (M_1)$ through a supersingular character.

A finite dimensional \mathcal{H} -module is called *supersingular* if each of its irreducible subquotients is supersingular.

More generally, an \mathcal{H} -module is called supersingular if it is the inductive limit of its finite dimensional \mathcal{H} -submodules and if each finite dimensional \mathcal{H} -submodule is supersingular.⁹

Remark. For nonzero finite dimensional \mathcal{H} -modules, the above definition of supersingularity is equivalent with the one given by Vignéras. This follows from the discussion in Section 6 of [Vignéras 2017]. There is also a notion of supersingularity for \mathcal{H} -modules which are not necessarily inductive limits of their finite dimensional submodules. In the present paper however, without further mentioning *all \mathcal{H} -modules will be assumed to be inductive limits of their finite dimensional submodules*.

Remark. In the literature on modules over Hecke algebras, the term *standard* module is occasionally used, but this usage is unrelated to our terminology.

2B. The coverings $\mathcal{H}^{\sharp\sharp}$ and \mathcal{H}^{\sharp} of \mathcal{H} .

Definition. (a) Let \mathcal{H}^{\sharp} denote the k -algebra generated by elements $T_{\omega}^{\pm 1}$, T_s and T_t for $t \in \bar{T}$, subject to

- the relations (22), (23), (25), (26),
- the relations (27) for $t = \alpha_i^{\vee}(\gamma)$ (all $0 \leq i \leq d$, $\gamma \in \Gamma$),
- the relation

$$T_{\omega}^{d+1} T_s^2 = T_s^2 T_{\omega}^{d+1}, \quad (30)$$

- the relations

$$T_t T_s^2 = T_s^2 T_t \quad \text{for all } t \in \bar{T}, \quad (31)$$

- the relations

$$T_s^2 T_{\omega} T_s^2 T_{\omega}^{-1} T_s^2 T_{\omega} = T_{\omega} T_s^2 T_{\omega}^{-1} T_s^2 T_{\omega} T_s^2 \quad \text{if } d > 1, \quad (32)$$

$$T_s^2 T_{\omega}^{-m} T_s^2 T_{\omega}^m = T_{\omega}^{-m} T_s^2 T_{\omega}^m T_s^2 \quad \text{for all } 1 < m < d. \quad (33)$$

(b) Let $\mathcal{H}^{\sharp\sharp}$ denote the k -algebra generated by the elements $T_{\omega}^{\pm 1}$, T_s and T_t for $t \in \bar{T}$, subject to

- the relations (22), (23), (25), (26),
- the relations (27) for $t = \alpha_i^{\vee}(\gamma)$ (all $0 \leq i \leq d$, $\gamma \in \Gamma$),
- the relations (31).

Lemma 2.1. *In \mathcal{H} we have the relations (30), (31), (32) and (33).*

⁹It is easy to see that the irreducible subquotients of a supersingular \mathcal{H} -module are the irreducible subquotients of its finite dimensional \mathcal{H} -submodules.

Proof. It is immediate that the relations (27) and (24) imply the relations (31) and (30), respectively. For $1 < m < d$ and $t \in \alpha_1^\vee(\mathbb{F}_q)$ we have $s\omega^m \cdot t = \omega^m \cdot t$, hence $T_s \sum_{t \in \alpha_1^\vee(\mathbb{F}_q)} T_{\omega^m \cdot t} = \sum_{t \in \alpha_1^\vee(\mathbb{F}_q)} T_{\omega^m \cdot t} T_s$. The same applies with $-m$ instead of m , hence

$$T_s T_\omega^{-m} \tau_s T_\omega^m = T_\omega^{-m} \tau_s T_\omega^m T_s \quad \text{and} \quad T_s T_\omega^m \tau_s T_\omega^{-m} = T_\omega^m \tau_s T_\omega^{-m} T_s.$$

This, together with $T_s^2 = \tau_s T_s = T_s \tau_s$ (formula (22)), justifies (i) and (iii) in

$$T_s^2 T_\omega^{-m} T_s^2 T_\omega^m \stackrel{\text{(i)}}{=} \tau_s (T_\omega^{-m} \tau_s T_\omega^m) T_s T_\omega^{-m} T_s T_\omega^m \stackrel{\text{(ii)}}{=} \tau_s (T_\omega^{-m} \tau_s T_\omega^m) T_\omega^{-m} T_s T_\omega^m T_s \stackrel{\text{(iii)}}{=} T_\omega^{-m} T_s^2 T_\omega^m T_s^2,$$

whereas (ii) is justified by (21). We have shown (33). Finally, to see (32) comes down, using (22), (26) and (27), to comparing

$$\begin{aligned} T_\omega T_s^2 T_\omega^{-1} T_s^2 T_\omega T_s^2 &= \left(\sum_{t_1, t_2, t_3 \in \alpha_1^\vee(\mathbb{F}_q)} T_{\omega^{-1} \cdot t_1} T_{\omega^{-1} s \omega \cdot t_2} T_{\omega^{-1} s \omega s \omega^{-1} \cdot t_3} \right) T_\omega T_s T_\omega^{-1} T_s T_\omega T_s, \\ T_s^2 T_\omega^{-1} T_s^2 T_\omega T_s^2 T_\omega &= \left(\sum_{t_1, t_2, t_3 \in \alpha_1^\vee(\mathbb{F}_q)} T_{t_1} T_{s \omega^{-1} \cdot t_2} T_{s \omega^{-1} s \omega \cdot t_3} \right) T_s T_\omega T_s T_\omega^{-1} T_s T_\omega. \end{aligned}$$

That these are equal follows from (20) and equality of the bracketed terms; for the latter observe $\omega s \omega^{-1} s \omega \cdot t = t$ for any $t \in \alpha_1^\vee(\mathbb{F}_q^\times)$. \square

In view of Lemma 2.1 we have natural surjections of k -algebras

$$\mathcal{H}^{\sharp\sharp} \rightarrow \mathcal{H}^\sharp \rightarrow \mathcal{H} \rightarrow \mathcal{H}^\flat.$$

Remark. $\mathcal{H}^{\sharp\sharp}$ (and in particular \mathcal{H}^\sharp and \mathcal{H}) is generated as a k -algebra by $T_\omega^{\pm 1}$, T_s and the $T_{e^*(\gamma)}$ for $\gamma \in \Gamma$.

Lemma 2.2. *There are unique k -algebra involutions ι of \mathcal{H} , \mathcal{H}^\sharp and $\mathcal{H}^{\sharp\sharp}$ with*

$$\iota(T_\omega) = T_\omega, \quad \iota(T_s) = \tau_s - T_s, \quad \iota(T_t) = T_t \quad \text{for } t \in \bar{T}.$$

Proof. This is a slightly tedious but straightforward computation. (For \mathcal{H} see [Vignéras 2005, Corollary 2].) \square

Remark. Besides ι consider the k -algebra involution β of \mathcal{H} , \mathcal{H}^\sharp and $\mathcal{H}^{\sharp\sharp}$ given on generators by

$$\beta(T_\omega) = T_\omega^{-1}, \quad \beta(T_s) = T_s, \quad \beta(T_t) = T_{s \cdot t} \quad \text{for } t \in \bar{T}.$$

Moreover, for any automorphism σ of Γ there is an associated automorphism α_σ of \mathcal{H} , \mathcal{H}^\sharp and $\mathcal{H}^{\sharp\sharp}$ given on generators by

$$\alpha_\sigma(T_\omega) = T_\omega, \quad \alpha_\sigma(T_s) = T_s, \quad \alpha_\sigma(T_{\partial(\sigma(\gamma))}) = T_{\partial(\sigma(\gamma))} \quad \text{for } \gamma \in \Gamma, \partial \in \text{Hom}(\Gamma, \bar{T}).$$

Do ι , β and the α_σ generate the automorphism group of \mathcal{H} (resp. of \mathcal{H}^\sharp , resp. of $\mathcal{H}^{\sharp\sharp}$) modulo inner automorphisms?

Lemma 2.3. *Let M be an $\mathcal{H}^{\sharp\sharp}$ -module. We have a direct sum decomposition*

$$M = M^{T_s = -\text{id}} \bigoplus M^{T_s^2 = 0}.$$

Proof. One computes $\tau_s^2 = (q-1)\tau_s = -\tau_s$ and this shows $T_s = -\text{id}$ on $\text{im}(T_s^2)$ as well as $T_s^2 = 0$ on $\text{im}(T_s^2 - \text{id})$. \square

Let $[0, q-2]^\Phi$ be the set of tuples $\epsilon = (\epsilon_i)_{0 \leq i \leq d}$ with $\epsilon_i \in \{0, \dots, q-2\}$ and $\sum_{0 \leq i \leq d} \epsilon_i \equiv 0$ modulo $(q-1)$. We often read the indices as elements of $\mathbb{Z}/(d+1)$, thus $\epsilon_i = \epsilon_j$ for $i, j \in \mathbb{Z}$ whenever $i-j \in (d+1)\mathbb{Z}$. We let the symmetric group \mathfrak{S}_{d+1} (generated by s, ω as before) act on $[0, q-2]^\Phi$ as follows:

$$(\omega \cdot \epsilon)_0 = \epsilon_d \quad \text{and} \quad (\omega \cdot \epsilon)_i = \epsilon_{i-1} \quad \text{for } 1 \leq i \leq d.$$

If $d = 1$ we put

$$(s \cdot \epsilon)_i = -\epsilon_i \quad \text{for } i = 0, 1,$$

but if $d \geq 2$ we put

$$(s \cdot \epsilon)_1 = -\epsilon_1, \quad (s \cdot \epsilon)_0 = \epsilon_0 + \epsilon_1, \quad (s \cdot \epsilon)_2 = \epsilon_1 + \epsilon_2, \quad (s \cdot \epsilon)_i = \epsilon_i \quad \text{for } 3 \leq i \leq d.^{10}$$

Throughout we assume that all eigenvalues of the T_t for $t \in \bar{T}$ acting on an $\mathcal{H}^{\sharp\sharp}$ -module belong to k . Let M be an $\mathcal{H}^{\sharp\sharp}$ -module. For $a \in [0, q-2]$ and $\epsilon = (\epsilon_i)_{0 \leq i \leq d} \in [0, q-2]^\Phi$ and $j \in \{0, 1\}$ put

$$\begin{aligned} M^\epsilon &= \{x \in M \mid T_{\alpha_i^\vee(\gamma)}^{-1}(x) = \gamma^{\epsilon_i} x \text{ for all } \gamma \in \Gamma, \text{ all } 0 \leq i \leq d\}, \\ M_a^\epsilon &= \{x \in M^\epsilon \mid T_{e^*(\gamma)}(x) = \gamma^a x \text{ for all } \gamma \in \Gamma\}, \\ M_a^\epsilon[j] &= \{x \in M_a^\epsilon \mid T_s^2(x) = jx\}. \end{aligned}$$

The T_t for $t \in \bar{T}$ are of order divisible by $q-1$, hence are diagonalizable on the k -vector space M . Since they commute among each other and with T_s^2 , we may simultaneously diagonalize all these operators (see Lemma 2.3 for T_s^2), hence

$$M = \bigoplus_{\epsilon, a, j} M_a^\epsilon[j]. \tag{34}$$

Lemma 2.4. *For any $\epsilon \in [0, q-2]^\Phi$ and $a \in [0, q-2]$ we have*

$$T_\omega(M_a^\epsilon) = M_{a-\epsilon_0}^{\omega \cdot \epsilon} \quad \text{and} \quad T_s(M^\epsilon) \subset M^{s \cdot \epsilon}.$$

If M is even an \mathcal{H} -module then

$$T_s(M_a^\epsilon) \subset M_{\epsilon_1+a}^{s \cdot \epsilon}. \tag{35}$$

¹⁰Here and below we understand $-\epsilon_i$ to mean the representative in $[0, q-2]$ of the class of $-\epsilon_i$ in $\mathbb{Z}/(q-1)$, and similarly for $\epsilon_0 + \epsilon_1$ and $\epsilon_1 + \epsilon_2$.

Proof. $T_\omega(M^\epsilon) = M^{\omega \cdot \epsilon}$ and $T_s(M^\epsilon) \subset M^{s \cdot \epsilon}$ follow from formulas (26) and (27), respectively, for the $t = \alpha_i^\vee(\gamma)$. For the following computation recall that $\omega \cdot e^* = e^* + \alpha_0^\vee$: For $\gamma \in \Gamma$ and $x \in M_a^\epsilon$ we have

$$T_{e^*(\gamma)} T_\omega(x) = T_\omega T_{(\omega \cdot e^*)(\gamma)}(x) = T_\omega T_{e^*(\gamma)} T_{\alpha_0^\vee(\gamma)}(x) = \gamma^{a - \epsilon_0} T_\omega(x).$$

This shows $T_\omega(M_a^\epsilon) = M_{a - \epsilon_0}^{\omega \cdot \epsilon}$. For formula (35) recall that $s \cdot e^* = e^* - \alpha_1^\vee$ and employ formula (27). \square

Any $x \in M$ can be uniquely written as

$$x = \sum_{a \in [0, q-2]} x_a \quad \text{with } x_a \in \sum_{\epsilon \in [0, q-2]^\Phi} M_a^\epsilon.$$

Given $a \in \mathbb{Z}$ and $x \in M$, we write $x_a = x_{\tilde{a}}$ where $\tilde{a} \in [0, q-2]$ is determined by $a - \tilde{a} \in (q-1)\mathbb{Z}$.

Definition. (a) An \mathcal{H}^\sharp -module M is called *standard supersingular* if the \mathcal{H}^\sharp -action factors through \mathcal{H} , making it a standard supersingular \mathcal{H} -module.

(b) An irreducible \mathcal{H}^\sharp -module is called *supersingular* if it is a subquotient of a standard supersingular \mathcal{H}^\sharp -module. An \mathcal{H}^\sharp -module M is called *supersingular* if it is the inductive limit of finite dimensional \mathcal{H}^\sharp -modules and if each of its irreducible subquotients is supersingular.

(c) An $\mathcal{H}^{\sharp\sharp}$ -module M is called *supersingular* if it satisfies the condition analogous to (b).

(d) A supersingular \mathcal{H}^\sharp -module is called \sharp -*supersingular* if for all $e \in M_a^\epsilon[0]$ with $\epsilon_1 > 0$ we have

$$(T_s e)_{c + \epsilon_1 + a} = 0 \quad \text{for all } q - 1 - \epsilon_1 \leq c \leq q - 2.$$

Lemma 2.5. (a) An \mathcal{H} -module is supersingular if and only if it is supersingular when viewed as an \mathcal{H}^\sharp -module. A supersingular \mathcal{H} -module is \sharp -supersingular when viewed as an \mathcal{H}^\sharp -module.

(b) The category of supersingular \mathcal{H} -modules, the category of supersingular \mathcal{H}^\sharp -modules, the category of supersingular $\mathcal{H}^{\sharp\sharp}$ -modules and the category of \sharp -supersingular \mathcal{H}^\sharp -modules are abelian.

Proof. Statement (a) follows from formula (35). Statement (b) is clear from the definitions. \square

3. Reconstruction of supersingular \mathcal{H}^\sharp -modules

Given an \mathcal{H}^\sharp -module M together with a submodule M_0 such that M/M_0 is supersingular, we address the problem of reconstructing the \mathcal{H}^\sharp -module M from the \mathcal{H}^\sharp -modules M_0 and M/M_0 together with an additional set of data (intended to be sparse). Our proposed solution (Proposition 3.3) critically relies on the braid relations (32) and (33).

Lemma 3.1. Let B_0, \dots, B_n be linear operators on a k -vector space M such that

$$\begin{aligned} B_j^2 &= B_j && \text{for all } 0 \leq j \leq n, \\ B_j B_{j'} B_j &= B_{j'} B_j B_{j'} && \text{for all } 0 \leq j', j \leq n, \\ B_j B_{j'} &= B_{j'} B_j && \text{for all } 0 \leq j' < j \leq n \text{ with } j - j' \geq 2. \end{aligned}$$

Put $\beta = B_n \cdots B_1 B_0$ and let $x \in M$ with $\beta^m x = x$ for some $m \geq 1$. Then we have $B_j x = x$ for each $0 \leq j \leq n$.

Proof. We first claim

$$\beta B_{j+1} = B_j \beta \quad \text{for all } 0 \leq j < n. \quad (36)$$

Indeed,

$$\begin{aligned} \beta B_{j+1} &= B_n \cdots B_{j+2} B_{j+1} B_j B_{j-1} \cdots B_1 B_0 B_{j+1} \\ &= B_n \cdots B_{j+2} B_{j+1} B_j B_{j+1} B_{j-1} \cdots B_1 B_0 \\ &= B_n \cdots B_{j+2} B_j B_{j+1} B_j B_{j-1} \cdots B_1 B_0 \\ &= B_j \beta. \end{aligned}$$

Choose $v \geq 1$ with $mv \geq n$. For $0 \leq j \leq n$ we then compute

$$x \stackrel{(i)}{=} \beta^{mv} x = \beta^{n-j} \beta^{mv-n+j} x \stackrel{(ii)}{=} \beta^{n-j} B_n \beta^{mv-n+j} x \stackrel{(iii)}{=} B_j \beta^{n-j} \beta^{mv-n+j} x = B_j \beta^{mv} x \stackrel{(iv)}{=} B_j x,$$

where (i) and (iv) follow from the hypothesis $\beta^m x = x$, where (ii) follows from $B_n \beta = \beta$ and where (iii) follows from repeated application of formula (36). \square

Proposition 3.2. *Let M be an \mathcal{H}^\sharp -module, let $M_0 \subset M$ be an \mathcal{H}^\sharp -submodule such that M/M_0 is supersingular. Let $\bar{x} \in (M/M_0)^\epsilon$ (some $\epsilon \in [0, q-2]^\Phi$) be such that $\bar{x}\{i\} = T_\omega^{i+1} \bar{x}$ is an eigenvector under T_s , for each $i \in \mathbb{Z}$. For liftings $x \in M$ of \bar{x} put $x\{i\} = T_\omega^{i+1} x$:*

- (a) *If the \mathcal{H}^\sharp -action on M factors through \mathcal{H} then we may choose $x \in M^\epsilon$ such that for each i with $T_s(\bar{x}\{i\}) = 0$ and $(\omega^{i+1} \cdot \epsilon)_1 = 0$ we have $T_s(x\{i\}) = 0$.*
- (b) *If the \mathcal{H}^\sharp -action on M factors through \mathcal{H} then we may choose $x \in M^\epsilon$ such that for each i with $T_s(\bar{x}\{i\}) = -\bar{x}\{i\}$ we have $T_s(x\{i\}) = -x\{i\}$.*
- (c) *We may choose $x \in M^\epsilon$ such that for each i with $T_s^2(\bar{x}\{i\}) = 0$ we have $T_s^2(x\{i\}) = 0$.*
- (d) *We may choose $x \in M^\epsilon$ such that for each i with $T_s^2(\bar{x}\{i\}) = \bar{x}\{i\}$ we have $T_s^2(x\{i\}) = x\{i\}$.*

Proof. (a) Let $i_1 < \cdots < i_r$ be the increasing enumeration of the set of all $0 \leq i \leq d$ with $T_s T_\omega^{i+1}(\bar{x}) = 0$ and $(\omega^{i+1} \cdot \epsilon)_1 = 0$. Replacing M by its \mathcal{H}^\sharp -submodule generated by x and M_0 we may assume that M/M_0 is a subquotient of a standard supersingular \mathcal{H} -module, attached to a supersingular character $\chi : \mathcal{H}_{\text{aff}} \rightarrow k$. If we had $T_s T_\omega^{i+1}(\bar{x}) = 0$ and $(\omega^{i+1} \cdot \epsilon)_1 = 0$ for all $0 \leq i \leq d$ then this would mean $\chi(T_\omega^m T_s T_\omega^{-m}) = 0$ and $\chi(T_\omega^m \tau_s T_\omega^{-m}) \neq 0$ for all $m \in \mathbb{Z}$, in contradiction with the supersingularity of χ . Hence there is some $0 \leq i \leq d$ not occurring among $\{i_1, \dots, i_r\}$. Thus, after a cyclic index shift, we may assume $i_r < d$.

Start with an arbitrary lift $x \in M^\epsilon$ of \bar{x} .

We claim that for any j with $0 \leq j \leq r$, after modifying x if necessary, we can achieve $T_s(x\{i_s\}) = 0$ for all s with $1 \leq s \leq j$. For $j = r$ this is the desired statement.

Let us illustrate the argument in the case $d = 1$ first. (This will logically not be needed for the general case. Notice e.g., that the subarguments (2) and (3) below are required only if $d > 1$.) Then we have $r = 1$ and $i_1 = 0$, and the claim for $j = 1$ states that there is some $\tilde{x} \in M^\epsilon$ lifting \bar{x} with $T_s T_\omega(\tilde{x}) = 0$.

But indeed, $\tilde{x} = x + T_\omega^{-1} T_s T_\omega x$ works: First, \tilde{x} lifts \bar{x} because of $T_s T_\omega \bar{x} = 0$. Next, \tilde{x} belongs to M^ϵ because of $T_\omega^{-1} T_s T_\omega x \in M^\epsilon$ (which follows from $x \in M^\epsilon$ and the assumption $(\omega^{i_1+1} \cdot \epsilon)_1 = 0$). Finally, $T_s T_\omega(\tilde{x}) = 0$, because $T_\omega \tilde{x} \in M^{\omega \cdot \epsilon} = M^{\omega^{i_1+1} \cdot \epsilon}$ and $(\omega^{i_1+1} \cdot \epsilon)_1 = 0$ imply $(T_s + T_s^2) T_\omega \tilde{x} = 0$.

Now let us consider the case of a general d . Induction on j . For $j = 0$ there is nothing to do. Now fix $1 \leq j \leq r$ and assume that x satisfies the condition for $j - 1$, i.e., assume $T_s(x\{i_s\}) = 0$ for all s with $1 \leq s \leq j - 1$. For $-1 \leq i \leq d$ and $0 \leq m < j$ define inductively

$$\begin{aligned} x\{i\}_0 &= x\{i\} = T_\omega^{i+1} x, \\ x\{i\}_{m+1} &= T_\omega^{i-i_{j-m}} T_s(x\{i_{j-m}\}_m). \end{aligned}$$

We establish several subclaims.

(1) $x\{i\}_m \in M^{\omega^{i+1} \cdot \epsilon}$.

For $m = 0$ there is nothing to do. Next, if the claim is true for an arbitrary m , then we have in particular $x\{i_{j-m}\}_m \in M^{\omega^{i_{j-m}+1} \cdot \epsilon}$. By assumption we know $(\omega^{i_{j-m}+1} \cdot \epsilon)_1 = 0$, which implies $T_s(M^{\omega^{i_{j-m}+1} \cdot \epsilon}) \subset M^{\omega^{i_{j-m}+1} \cdot \epsilon}$. Thus, we get $x\{i_{j-m}\}_{m+1} = T_s(x\{i_{j-m}\}_m) \in M^{\omega^{i_{j-m}+1} \cdot \epsilon}$. From this we get $x\{i\}_{m+1} = T_s(x\{i\}_m) \in M^{\omega^{i+1} \cdot \epsilon}$ for general i by applying powers of T_ω to $x\{i_{j-m}\}_{m+1}$.

(2) $T_s(x\{i_s\}_m) = 0$ for all $1 \leq s \leq j$ and all $0 \leq m < j - s$.

We induct on m . For $m = 0$ this is true by induction hypothesis (on j). Now let $0 < m < j - s$ and assume that we know the claim for $m - 1$ instead of m . In particular we then know $T_s(x\{i_s\}_{m-1}) = 0$. We deduce

$$\begin{aligned} T_s(x\{i_s\}_m) &= T_s T_\omega^{i_s-i_{j-m+1}} T_s T_\omega^{i_{j-m+1}-i_s} T_\omega^{i_s-i_{j-m+1}} (x\{i_{j-m+1}\}_{m-1}) \\ &= T_s T_\omega^{i_s-i_{j-m+1}} T_s T_\omega^{i_{j-m+1}-i_s} (x\{i_s\}_{m-1}) \\ &= T_\omega^{i_s-i_{j-m+1}} T_s T_\omega^{i_{j-m+1}-i_s} T_s (x\{i_s\}_{m-1}) \\ &= 0 \end{aligned}$$

where we use the braid relation (21) (which applies since $|i_s - i_{j-m+1}| > 1$ and $i_r < d$). The induction on m is complete.

(3) $T_s(x\{i_s\}_m) = 0$ for all $1 \leq s \leq j$ and all $j - s + 1 < m \leq j$.

We induct on $m + s - j$. The induction begins with $m + s - j = 2$. By (2) we know $T_s(x\{i_{j-m+1}\}_{m-2}) = 0$. Thus, if $i_{j-m+1} + 1 < i_{j-m+2}$, the same argument as in (2) shows $T_s(x\{i_{j-m+1}\}_{m-1}) = 0$ and hence $x\{i\}_m = 0$ for all i , and there is nothing more to do. If however $i_{j-m+1} + 1 = i_{j-m+2}$ we compute

$$\begin{aligned} T_s(x\{i_{j-m+2}\}_m) &= T_s T_\omega T_s T_\omega^{-1} T_s T_\omega (x\{i_{j-m+1}\}_{m-2}) \\ &= T_\omega T_s T_\omega^{-1} T_s T_\omega T_s (x\{i_{j-m+1}\}_{m-2}) \\ &= 0 \end{aligned}$$

where we use the braid relation (20). This settles the case $m + s - j = 2$. For $m + s - j > 2$ we now argue exactly as in (2) again: $T_s(x\{i_s\}_m) = 0$ implies $T_s(x\{i_s\}_{m+1}) = 0$. The induction is complete.

(4) $T_s(x\{i_{j-m}\}_m + x\{i_{j-m}\}_{m+1}) = 0$ for all $0 \leq m < j$.

Indeed, by (1) and our assumption $(\omega^{i_{j-m}+1} \cdot \epsilon)_1 = 0$ we know that $x\{i_{j-m}\}_m$ is fixed under $T_{\alpha_1^\vee(\Gamma)}$ and hence is killed by $T_s^2 + T_s$, as follows from the quadratic relation (22). As $x\{i_{j-m}\}_{m+1} = T_s(x\{i_{j-m}\}_m)$ this gives the claim.

(5) $\tilde{x} = \sum_{0 \leq m \leq j} x\{-1\}_m$ lifts \bar{x} .

Indeed, we have $T_s(x\{i_j\}) \in M_0$ by our defining assumption on i_j . It follows that $x\{-1\}_m \in M_0$ for all $m \geq 1$, hence $x - \tilde{x} \in M_0$.

(6) From (1) we deduce $\tilde{x}\{i\} \in M^{\omega^{i+1} \cdot \epsilon}$. Writing

$$\tilde{x}\{i_s\} = \left(\sum_{0 \leq m < j-s} x\{i_s\}_m \right) + (x\{i_s\}_{j-s} + x\{i_s\}_{j-s+1}) + \left(\sum_{j-s+1 < m \leq j} x\{i_s\}_m \right)$$

we see that (2), (3) and (4) imply $T_s(\tilde{x}\{i_s\}) = 0$ for all s with $1 \leq s \leq j$.

The induction on j is complete; we may substitute \tilde{x} for the old x .

(b) Composing the given \mathcal{H} -module structure on M with the involution ι of Lemma 2.2 we get a new \mathcal{H} -module structure on M . Applying statement (a) to this new \mathcal{H} -module and then translating back via ι , we get statement (b). Notice that here, in contrast to the setting in (a), we *automatically* have $(\omega^{i+1} \cdot \epsilon)_1 = 0$ for each i with $T_s(\bar{x}\{i\}) = -\bar{x}\{i\}$.

(c) Statement (c) is proved in the same way as statement (a), with the following minor modifications: each occurrence of T_s must be replaced by T_s^2 , and in the definition of $x\{i\}_{m+1}$ the alternating sign $(-1)^{m+1}$ must be included, i.e.,

$$x\{i\}_{m+1} = (-1)^{m+1} T_\omega^{i-i_{j-m}} T_s^2(x\{i_{j-m}\}_m) \quad (37)$$

In particular, we then have $x\{i_{j-m}\}_{m+1} = -T_s^2(x\{i_{j-m}\}_m)$. In (2) and (3), the appeal to the braid relations (20), (21) must be replaced by an appeal to the braid relations (32), (33). In (4), the appeal to $T_s^2 + T_s = 0$ on vectors fixed under $T_{\alpha_1^\vee(\Gamma)}$ must be replaced by an appeal to $T_s^4 - T_s^2 = 0$ (it is here where the alternating sign in the defining formula (37) is needed). Notice that here, in contrast to the setting in (a), we do not need to impose $(\omega^{i+1} \cdot \epsilon)_1 = 0$ for each i with $T_s^2(\bar{x}\{i\}) = 0$. (On the one hand, because of $T_s^2(M^\epsilon) \subset M^\epsilon$ for any ϵ the argument analogous to the one in (a)(1) carries over; on the other hand, because of $T_s^4 - T_s^2 = 0$ on all of M the argument analogous to the one in (a)(4) carries over.)

(d) Composing the given \mathcal{H}^\sharp -module structure on M with the involution ι of Lemma 2.2 we get a new \mathcal{H}^\sharp -module structure on M . Applying statement (c) to this new \mathcal{H}^\sharp -module and then translating back via ι , we get statement (d). \square

Proposition 3.3. *Let M be an \mathcal{H}^\sharp -module, let $M_0 \subset M$ be an \mathcal{H}^\sharp -submodule such that M/M_0 is supersingular. The action of \mathcal{H}^\sharp on M is uniquely determined by the following combined data:*

- (a) *The action of \mathcal{H}^\sharp on M_0 and on M/M_0 .*
- (b) *The action of $T_{e^*(\Gamma)}$ and of $T_s T_\omega$ on M .*

(c) *The restriction of T_ω to $(T_s T_\omega)^{-1}(M_0)$, i.e., the map*

$$\{x \in M \mid T_s T_\omega(x) \in M_0\} \xrightarrow{T_\omega} M.$$

(d) *The subspace $\sum_{\substack{\epsilon \in [0, q-2]^\Phi \\ \epsilon_1=0}} M^\epsilon$ of M .*

Proof. The k -algebra \mathcal{H}^\sharp is generated by $T_{e^*(\Gamma)}$, by T_s and by $T_\omega^{\pm 1}$. Therefore we only need to see that the action of T_s and T_ω on M can be reconstructed from the given data (a), (b), (c), (d). Exhausting M/M_0 step by step we may assume that M/M_0 is an irreducible supersingular \mathcal{H}^\sharp -module.

We first show that T_s is uniquely determined. For this we make constant use of Lemma 2.3 (and the decomposition (34)). As $T_s|_{M_0}$ is given to us, it is enough to show that for any nonzero \bar{x} in M/M_0 with either $T_s(\bar{x}) = -\bar{x}$ or $T_s(\bar{x}) = 0$ we find some lifting $x \in M$ such that $T_s(x)$ can be reconstructed. Consider first the case $T_s(\bar{x}) = -\bar{x}$. By the quadratic relation (22) (see Lemma 2.3) we then have $\bar{x} \in \sum_{\substack{\epsilon \in [0, q-2]^\Phi \\ \epsilon_1=0}} (M/M_0)^\epsilon$, and using the datum (d) as well as our knowledge of the subspace $T_s M$ (since $T_s M = T_s T_\omega M$ this is given to us in view of datum (b)), we lift \bar{x} to some $x \in T_s M \cap \sum_{\substack{\epsilon \in [0, q-2]^\Phi \\ \epsilon_1=0}} M^\epsilon$ (use the decomposition (34)). For such x we have $T_s(x) = -x$. Now consider the case $T_s(\bar{x}) = 0$. An arbitrary lifting $x \in M$ of \bar{x} then satisfies $T_s(x) \in M_0$, and $T_s(x)$ is determined by the given data as $T_s(x) = (T_s T_\omega) T_\omega^{-1}(x)$ (notice that the datum (c) is equivalent with the datum $T_s^{-1}(M_0) \xrightarrow{T_\omega^{-1}} M$).

To show that T_ω is uniquely determined, suppose that besides $T_\omega \in \text{Aut}_k(M)$ there is another candidate $\tilde{T}_\omega \in \text{Aut}_k(M)$ extending the data (a), (b), (c), (d) to another \mathcal{H}^\sharp -action on M .

We find and choose some nonzero $\bar{x} \in M/M_0$ such that $T_\omega^j(\bar{x})$ is an eigenvector under T_s , for each $j \in \mathbb{Z}$. For any $x \in M$ lifting \bar{x} we have

$$T_\omega = \tilde{T}_\omega \quad \text{on } M_0 + k \cdot T_\omega^{j-1}(x) \text{ if } T_s T_\omega^j(\bar{x}) = 0 \quad (38)$$

as both \tilde{T}_ω and T_ω respect the datum (c).

Let $i_0 < \dots < i_n$ be the increasing enumeration of the set

$$\{0 \leq i \leq d \mid T_s^2 T_\omega^i \bar{x} = T_\omega^i \bar{x}\}.$$

As M/M_0 is a subquotient of a standard supersingular \mathcal{H} -module, this set is not the full set $\{0 \leq i \leq d\}$. Applying a suitable power of T_ω and reindexing we may assume that 0 does not belong to this set, i.e., that $i_0 > 0$.

Choose a lifting $x \in M$ of \bar{x} such that for each $i \in \{i_0, \dots, i_n\} + \mathbb{Z}(d+1)$ we have $T_s^2 T_\omega^i x = T_\omega^i x$. This is possible by Proposition 3.2. Put $z_0 = x$. For $i \geq 1$ put

$$z_i = \begin{cases} \tilde{T}_\omega z_{i-1} & i \notin \{i_0, \dots, i_n\} + \mathbb{Z}(d+1), \\ T_s^2 \tilde{T}_\omega z_{i-1} & i \in \{i_0, \dots, i_n\} + \mathbb{Z}(d+1). \end{cases}$$

We claim

$$z_i = T_\omega^i x \quad (39)$$

for each $i \geq 0$. Induction on i . The case $i = 0$ is trivial. For $i \geq 1$ with $i \notin \{i_0, \dots, i_n\} + \mathbb{Z}(d+1)$ we compute

$$z_i = \tilde{T}_\omega z_{i-1} \stackrel{(i)}{=} T_\omega z_{i-1} \stackrel{(ii)}{=} T_\omega^i x$$

where in (i) we use statement (38) and in (ii) we use the induction hypothesis. For $i \geq 1$ with $i \in \{i_0, \dots, i_n\} + \mathbb{Z}(d+1)$ we compute

$$z_i = T_s^2 \tilde{T}_\omega z_{i-1} \stackrel{(i)}{=} T_s^2 T_\omega z_{i-1} \stackrel{(ii)}{=} T_\omega^i x$$

where in (i) we use the assumption $T_s T_\omega = T_s \tilde{T}_\omega$, and in (ii) we use the induction hypothesis $T_\omega z_{i-1} = T_\omega^i x$ and the assumption on x . The induction is complete. Put

$$B_{i_j} = \tilde{T}_\omega^{-i_j} T_s^2 \tilde{T}_\omega^{i_j}.$$

The relation (30) implies $B_{i_j} = \tilde{T}_\omega^{-i_j + (d+1)v} T_s^2 \tilde{T}_\omega^{i_j - (d+1)v}$ for each $v \in \mathbb{Z}$. Thus

$$(B_{i_n} \cdots B_{i_1} B_{i_0})^m x \stackrel{(i)}{=} \tilde{T}_\omega^{-m(d+1)} z_{m(d+1)} \stackrel{(ii)}{=} \tilde{T}_\omega^{-m(d+1)} T_\omega^{m(d+1)} x$$

for $m \geq 0$, where (i) follows from the definition of $z_{m(d+1)}$, whereas (ii) follows from formula (39). Choosing m large enough we may assume $T_\omega^{m(d+1)} x = x$ and $\tilde{T}_\omega^{m(d+1)} x = x$ (as T_ω and \tilde{T}_ω are automorphisms of a finite vector space); then

$$(B_{i_n} \cdots B_{i_1} B_{i_0})^m x = x.$$

The braid relations (32), (33) show that the B_{i_j} satisfy the hypotheses of Lemma 3.1 (in particular, the commutation $B_{i_0} B_{i_n} = B_{i_n} B_{i_0}$ if $n > 1$ follows from $i_0 > 0$). This Lemma now tells us $B_{i_j} \cdots B_{i_1} B_{i_0} x = x$ for each $0 \leq j \leq n$. But by the definition of the z_i this means

$$z_i = \tilde{T}_\omega^i x \tag{40}$$

for each $0 \leq i \leq d+1$. When compared with formula (39) this yields $T_\omega = \tilde{T}_\omega$ since M is generated as a k -vector space by M_0 together with the $T_\omega^i x$ (or: the $\tilde{T}_\omega^i x$) for $0 \leq i \leq d$. \square

Remarks. The above proof of Proposition 3.3 shows the following:

- (i) The subspace in (d) could be replaced by the subspace $\{x \in M \mid T_s^2(x) = x\}$.
- (ii) If the \mathcal{H}^\sharp -action factors through an \mathcal{H} -action, then the datum (d) can be entirely left out (T_ω can then be reconstructed without a priori knowledge of T_s).

4. The functor

Here we define a functor $M \mapsto \Delta(M)$ from supersingular $\mathcal{H}^{\sharp\sharp}$ -modules to torsion $k[[t]]$ -modules with φ and Γ actions, as outlined in the introduction. Its entire content is encapsulated in the explicit formula for the elements $h(e)$ introduced below.

Let M be an $\mathcal{H}^{\#}$ -module. View M as a $k[[t]]$ -module with $t = 0$ on M . Let Γ act on M by

$$\gamma \cdot x = T_{e^*(\gamma)}^{-1}(x)$$

for $\gamma \in \Gamma$, making M a $k[[t]][\Gamma]$ -module. We have an isomorphism of $k[[t]][\varphi]$ -modules

$$\mathfrak{O} \otimes_{k[[t]][\Gamma]} M \cong k[[t]][\varphi] \otimes_{k[[t]]} M$$

and hence an action of \mathfrak{O} on $k[[t]][\varphi] \otimes_{k[[t]]} M$.

For $e \in M_a^\epsilon[j]$ (any $\epsilon \in [0, q-2]^\Phi$, any $a \in [0, q-2]$, any $j \in \{0, 1\}$) define the element

$$h(e) = \begin{cases} t^{\epsilon_1} \varphi \otimes T_\omega^{-1}(e) + 1 \otimes e + \sum_{c=0}^{q-2} t^c \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) & j=0, \\ t^{q-1} \varphi \otimes T_\omega^{-1}(e) + 1 \otimes e & j=1 \end{cases}$$

of $k[[t]][\varphi] \otimes_{k[[t]]} M$. Define $\nabla(M)$ to be the $k[[t]][\varphi]$ -submodule of $k[[t]][\varphi] \otimes_{k[[t]]} M$ generated by the elements $h(e)$ for all $e \in M_a^\epsilon[j]$ (all ϵ, a, j). Define

$$\Delta(M) = \frac{k[[t]][\varphi] \otimes_{k[[t]]} M}{\nabla(M)}.$$

Remark. If M is even an \mathcal{H} -module, then in view of formula (35) the definition of $h(e)$ simplifies to become

$$h(e) = \begin{cases} t^{\epsilon_1} \varphi \otimes T_\omega^{-1}(e) + 1 \otimes e + \varphi \otimes T_\omega^{-1}(T_s e) & j=0, \\ t^{q-1} \varphi \otimes T_\omega^{-1}(e) + 1 \otimes e & j=1. \end{cases}$$

In this case it is not necessary to split up M into eigenspaces under the action of $T_{e^*(\Gamma)}$, and the *notation* of many of the subsequent computations simplifies (no underlined subscripts are needed). However, they hardly simplify in mathematical complexity, not even if we restrict to \mathcal{H}^\flat -modules only (in which case always $\epsilon_1 = 0$ and $T_{e^*(\gamma)} = 1$).

Lemma 4.1. *Let $e \in M_a^\epsilon[j]$. The integer*

$$k_e = \begin{cases} \epsilon_1 & j=0, \\ q-1 & j=1, \end{cases}$$

satisfies $k_e \equiv \epsilon_1$ modulo $(q-1)$.

Proof. $j = 1$ means $T_s^2(e) = e$, hence the claim follows from the relation (22). \square

Lemma 4.2. *For $e \in M_a^\epsilon[j]$ we have $\gamma \cdot h(e) = h(T_{e^*(\gamma)}^{-1}(e))$ for all $\gamma \in \Gamma$. In particular, $\nabla(M)$ is stable under the action of Γ , hence is an \mathfrak{O} -submodule of $k[[t]][\varphi] \otimes_{k[[t]]} M$. Hence $\Delta(M)$ is even an \mathfrak{O} -module.*

Proof. First notice that $T_{e^*(\gamma)}^{-1}(e) \in M_a^\epsilon[j]$. In particular, $h(T_{e^*(\gamma)}^{-1}(e))$ is well defined. For $\gamma \in \Gamma$ we find

$$\gamma \cdot (1 \otimes e) = 1 \otimes \gamma \cdot e = 1 \otimes T_{e^*(\gamma)}^{-1}(e). \quad (41)$$

Next, we compute

$$\gamma \cdot (t^{k_e} \varphi \otimes T_\omega^{-1}(e)) \stackrel{(i)}{=} \gamma^{k_e} t^{k_e} \varphi \otimes \gamma \cdot T_\omega^{-1}(e) \stackrel{(ii)}{=} t^{k_e} \varphi \otimes T_\omega^{-1} T_{e^*(\gamma)}^{-1}(e). \quad (42)$$

In (i) we used $\gamma t = [\gamma]_\Phi(t)\gamma$ and $[\gamma]_\Phi(t) \equiv \gamma t$ modulo $t^q k[[t]]$ (Lemma 0.1) and the fact that, since $\pi = 0$ in k , we have $t^q \varphi \otimes M = \Phi(t)\varphi \otimes M = \varphi t \otimes M = 0$. To see (ii) observe

$$\begin{aligned} \gamma \cdot T_\omega^{-1}(e) &= T_{e^*(\gamma)}^{-1} T_\omega^{-1}(e) \\ &= T_\omega^{-1} T_{(\omega^{-1} \cdot e^*)(\gamma)}^{-1}(e) \\ &= T_\omega^{-1} T_{(e^* - \alpha_1^\vee)(\gamma)}^{-1}(e) \\ &= T_\omega^{-1} T_{\alpha_1^\vee(\gamma)}^{-1} T_{e^*(\gamma)}^{-1}(e) \\ &= \gamma^{-k_e} T_\omega^{-1} T_{e^*(\gamma)}^{-1}(e) \end{aligned}$$

where in the last step we use Lemma 4.1. Combining formulae (41) and (42) we are done in the case $j = 1$. In the case $j = 0$ we in addition need the formula

$$\gamma \cdot \sum_{c=0}^{q-2} t^c \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) = \sum_{c=0}^{q-2} t^c \varphi \otimes T_\omega^{-1}((T_s T_{e^*(\gamma)} e)_{\underline{c+\epsilon_1+a}}). \quad (43)$$

Let us prove this (for $e \in M_a^\epsilon[0]$). For $f \in \mathbb{Z}$ and $\gamma \in \Gamma$ we compute

$$\begin{aligned} T_{(\omega^{-1} \cdot e^*)(\gamma)}((T_s e)_{\underline{f}}) &\stackrel{(i)}{=} T_{e^*(\gamma)} T_{\alpha_1^\vee(\gamma^{-1})}((T_s e)_{\underline{f}}) \\ &\stackrel{(ii)}{=} \gamma^{f-\epsilon_1} (T_s e)_{\underline{f}} \\ &= \gamma^{f-\epsilon_1-a} (T_s (\gamma^a e))_{\underline{f}} \\ &= \gamma^{f-\epsilon_1-a} (T_s T_{e^*(\gamma)} e)_{\underline{f}}. \end{aligned} \quad (44)$$

In (i) recall that $\omega^{-1} \cdot e^* = e^* - \alpha_1^\vee$, in (ii) notice that $(T_s e)_{\underline{f}} \in M^{s \cdot \epsilon}$ and $(s \cdot \epsilon)_1 = -\epsilon_1$. For $c \in [0, q-2]$ we deduce

$$\begin{aligned} \gamma \cdot (t^c \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}})) &= \gamma^c t^c \varphi \otimes \gamma \cdot (T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}})) \\ &= \gamma^c t^c \varphi \otimes T_{e^*(\gamma)}^{-1} T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) \\ &= \gamma^c t^c \varphi \otimes T_\omega^{-1} T_{(\omega^{-1} \cdot e^*)(\gamma)}^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) \\ &= t^c \varphi \otimes T_\omega^{-1}((T_s T_{e^*(\gamma)} e)_{\underline{c+\epsilon_1+a}}) \end{aligned}$$

where in the last equality we inserted formula (44). \square

Proposition 4.3. (a) *If M is supersingular and finite dimensional, then we have: $\Delta(M)$ is a torsion $k[[t]]$ -module, generated by M as a $k[[t]][\varphi]$ -module, and φ acts injectively on it. The dual $\Delta(M)^* = \text{Hom}_k(\Delta(M), k)$ is a free $k[[t]]$ -module of rank $\dim_k(M)$. The map $M \rightarrow \Delta(M)$ which sends $m \in M$ to the class of $1 \otimes m$ induces a bijection*

$$M \cong \Delta(M)[t]. \quad (45)$$

(b) $\Delta(M)$ belongs to $\text{Mod}^\bullet(\mathfrak{O})$.

(c) *The assignment $M \mapsto \Delta(M)$ is an exact functor from the category of supersingular $\mathcal{H}^{\sharp\sharp}$ -modules to $\text{Mod}^{ad}(\mathfrak{O})$.*

Proof. (a) Notice first that it is enough to prove these claims after a finite base extensions of k .

Assume first that M is irreducible. It can then be realized as a subquotient of a standard supersingular \mathcal{H} -module N — in fact, it can even be realized as a submodule or as a quotient of such an N . Observing the decomposition (34) for N , we see that there are a k -basis e_0, \dots, e_d of N as well as $0 \leq k_{e_j} \leq q-1$ for $0 \leq j \leq d$, not all of them = 0 and not all of them = $q-1$, such that $\nabla(N)$ is generated by the elements $h(e_j) = t^{k_{e_j}} \varphi \otimes T_\omega^{-1}(e_j) + 1 \otimes e_j$. It follows that $\Delta(N)$ is standard cyclic. Now it is easy to see that $\Delta(M)$ is a subquotient of $\Delta(N)$. Thus, by Proposition 1.15(d), it is standard cyclic as well, at least after a finite extension of k . Therefore all our claims follow from Lemma 1.14 and Proposition 1.15(a).

Now let M be arbitrary (supersingular, finite dimensional). Choose a separated and exhausting descending filtration of M by $\mathcal{H}^{\#\#}$ -submodules $F^\mu M$ with irreducible subquotients $F^{\mu-1}M/F^\mu M$. Since on any standard supersingular \mathcal{H} -module (and hence on any of its subquotients, and hence on any irreducible $\mathcal{H}^{\#\#}$ -module) we have $T_s = -T_s^2$ and hence $\ker(T_s^2) = \ker(T_s)$, the filtration satisfies

$$T_s(F^{\mu-1}M \cap \ker(T_s^2)) \subset F^\mu M \quad (46)$$

for each $\mu \in \mathbb{Z}$. Putting

$$F^\mu = k[[t]][\varphi] \otimes_{k[[t]]} F^\mu M$$

we claim

$$\nabla(F^\mu M) = \nabla(M) \cap F^\mu. \quad (47)$$

Arguing by induction, we may assume that this is known with $\mu-1$ instead of μ . Let \mathcal{E} be a family of elements $e \in (F^{\mu-1}M)_{a_e}^{\epsilon_e}[j_e]$ (for suitable $\epsilon_e \in [0, q-2]^\Phi$ and $a_e \in [0, q-2]$ and $j_e \in \{0, 1\}$ depending on e) which induces a k -basis of $F^{\mu-1}M/F^\mu M$. We consider an expression

$$\sum_{j_1, j_2 \in \mathbb{Z}_{\geq 0}, e \in \mathcal{E}} c_{j_1, j_2, e} t^{j_2} \varphi^{j_1} h(e) \quad (48)$$

with $c_{j_1, j_2, e} \in k$. Assuming that the expression (48) belongs to F^μ we need to see that it even belongs to $\nabla(F^\mu M)$.

Suppose that this is false. We may then define

$$j_1 = \min\{j \geq 0 \mid c_{j, j_2, e} t^{j_2} \varphi^j h(e) \notin \nabla(F^\mu M) \text{ for some } j_2 \geq 0, \text{ some } e \in \mathcal{E}\}.$$

Claim. We find some j_2 and some e with $c_{j_1, j_2, e} t^{j_2} \varphi^{j_1} h(e) \in F^\mu - \nabla(F^\mu M)$.

For $e \in \mathcal{E}$ the expression

$$1 \otimes e + t^{k_e} \varphi \otimes T_\omega^{-1}(e) \quad (49)$$

is congruent to $h(e)$ modulo F^μ , in view of $e \in F^{\mu-1}M$ and formula (46). Therefore, modulo F^μ the expression (48) reads

$$\sum_{j_1, j_2, e} c_{j_1, j_2, e} t^{j_2} \varphi^{j_1} \otimes e + c_{j_1, j_2, e} t^{j_2} \varphi^{j_1} t^{k_e} \varphi \otimes T_\omega^{-1}(e).$$

Notice that $\varphi^{j_1} t^{k_e} \varphi \in k[[t]]\varphi^{j_1+1}$. The claim now follows in view of

$$\frac{\mathbf{F}^{\mu-1}}{\mathbf{F}^\mu} = \bigoplus_{j \geq 0} k[[t]]\varphi^j \otimes_{k[[t]]} \frac{F^{\mu-1}M}{F^\mu M}. \quad (50)$$

The claim proven, we may argue by induction on the number of summands in the expression (48) which do not belong to $\nabla(F^\mu M)$. We may thus assume from the start that the expression (48) consists of a single summand $t^{j_2} \varphi^{j_1} h(e)$, and that moreover $e \notin F^\mu M$ for this e . The aim is then to deduce $t^{j_2} \varphi^{j_1} h(e) \in \nabla(F^\mu M)$, which contradicts our above assumption.

Let us write $\epsilon = \epsilon_e$ and $a = a_e$. The vanishing of $t^{j_2} \varphi^{j_1} h(e)$ modulo \mathbf{F}^μ means, by the decomposition (50) again, that

$$t^{j_2} \varphi^{j_1} \otimes e \stackrel{(i)}{=} 0 \stackrel{(ii)}{=} t^{j_2} \varphi^{j_1} t^{k_e} \varphi \otimes T_\omega^{-1}(e)$$

(i.e., absolute vanishing, not just modulo \mathbf{F}^μ). If $T_s^2(e) = e$ then this shows $t^{j_2} \varphi^{j_1} h(e) = 0$. Now suppose $T_s^2(e) = 0$ (and hence $k_e < q - 1$). The definition of $h(e)$ together with the vanishings (i) and (ii) shows

$$t^{j_2} \varphi^{j_1} h(e) = t^{j_2} \varphi^{j_1} \sum_{c=0}^{q-2} t^c \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}).$$

Since the vanishing (ii) also forces $t^{j_2} \varphi^{j_1} t^{k_e} \varphi \in k[[t]]\varphi^{j_1+1}t$, there is some i and some $j'_2 \geq 0$ with

$$t^{j_2} \varphi^{j_1} = t^{j'_2} \varphi^{j_1} t^i \quad \text{and} \quad i \geq q - k_e.$$

If $k_e = 0$ (and hence $i \geq q$) then again the conclusion is $t^{j_2} \varphi^{j_1} h(e) = 0$. It remains to discuss the case where $0 < k_e < q - 1$. In this case, $(T_s e)_{\underline{c+\epsilon_1+a}} \in M^{s \cdot \epsilon}$ and $(s \cdot \epsilon)_1 = -\epsilon_1$ implies $q - 1 - k_e = k_{(T_s e)_{\underline{c+\epsilon_1+a}}}$ for each c . We thus see

$$\begin{aligned} t^{q-k_e+c} \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) &= t^{1+c} (t^{k_{(T_s e)_{\underline{c+\epsilon_1+a}}}} \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) + 1 \otimes (T_s e)_{\underline{c+\epsilon_1+a}}) \\ &= t^{1+c} h((T_s e)_{\underline{c+\epsilon_1+a}}) - \sum_{c'=0}^{q-2} t^{1+c+c'} \varphi \otimes T_\omega^{-1}((T_s((T_s e)_{\underline{c+\epsilon_1+a}}))_{\underline{c'+c+a}}) \end{aligned}$$

by the definition of $h((T_s e)_{\underline{c+\epsilon_1+a}})$, again since $(T_s e)_{\underline{c+\epsilon_1+a}} \in M^{s \cdot \epsilon}$ and $(s \cdot \epsilon)_1 = -\epsilon_1$. For $0 \leq f \leq q - 2$ we have

$$\sum_{\substack{0 \leq c, c' \leq q-2 \\ c+c'=f}} (T_s((T_s e)_{\underline{c+\epsilon_1+a}}))_{\underline{f+a}} = \sum_{0 \leq c \leq q-2} (T_s((T_s e)_{\underline{c+\epsilon_1+a}}))_{\underline{f+a}} = 0$$

as follows from $T_s^2(e) = 0$. This shows

$$\sum_{c, c'=0}^{q-2} t^{1+c+c'} \varphi \otimes T_\omega^{-1}((T_s((T_s e)_{\underline{c+\epsilon_1+a}}))_{\underline{c'+c+a}}) = 0.$$

Since e belongs to $F^{\mu-1}M$, formula (46) shows $h((T_s e)_{\underline{c+\epsilon_1+a}}) \in \nabla(F^\mu M)$. Together we obtain $t^{q-k_e+c} \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) \in \nabla(F^\mu M)$, hence $t^{i+c} \varphi \otimes T_\omega^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) \in \nabla(F^\mu M)$ for $0 \leq c \leq q - 2$.

This gives

$$t^{j_2} \varphi^{j_1} h(e) = \sum_{c=0}^{q-2} t^{j'_2} \varphi^{j_1} t^{i+c} \varphi \otimes T_{\omega}^{-1}((T_s e)_{\underline{c+\epsilon_1+a}}) \in \nabla(F^{\mu} M),$$

as desired.

Formula (47) is proven. It allows us to deduce all our claims for M from the corresponding claims for the $F^{\mu-1}M/F^{\mu}M$; but for them they have already been established above.

(b) For each irreducible supersingular \mathcal{H} -module M , extending k if necessary, $\Delta(M)$ admits a filtration such that each associated graded piece is a standard cyclic object in $\text{Mod}^{\text{ad}}(\mathfrak{O})$, as pointed out above. Since the functor Δ is exact (see statement (c)) it therefore takes finite dimensional supersingular $\mathcal{H}^{\sharp\sharp}$ -modules to objects in $\text{Mod}^{\bullet\bullet}(\mathfrak{O})$.

(c) It is clear that $M \mapsto \Delta(M)$ is a (covariant) right exact functor. To see left exactness, let $M_1 \rightarrow M_2$ be injective. Since the kernel of $\Delta(M_1) \rightarrow \Delta(M_2)$ is a torsion $k[[t]]$ -module it has, if nonzero, a nonzero vector killed by t . By formula (45) it must belong to (the image of) M_1 , contradicting the injectivity of $M_1 \rightarrow M_2$. \square

5. Standard objects and full faithfulness

5A. The bijection between standard supersingular Hecke modules and standard cyclic Galois representations. Let M be a standard supersingular \mathcal{H} -module, arising from the supersingular character $\chi : \mathcal{H}_{\text{aff}} \rightarrow k$. There is some $e_0 \in M$ such that, putting $e_j = T_{\omega}^{-j} e_0$, we have $M = \bigoplus_{j=0}^d k.e_j$ and \mathcal{H}_{aff} acts on $k.e_0$ by χ . Denote by $\eta_j : \Gamma \rightarrow k^{\times}$ the character through which $T_{e^*(.)}^{-1}$ acts on $k.e_j$, i.e., $T_{e^*(\gamma)}^{-1}(e_j) = \eta_j(\gamma)e_j$ for $\gamma \in \Gamma$.

Lemma 5.1. (a) *There are $0 \leq k_{e_j} \leq q-1$ for $0 \leq j \leq d$, not all of them = 0 and not all of them = $q-1$, such that*

$$t^{k_{e_j}} \varphi \otimes T_{\omega}^{-1}(e_j) = -1 \otimes e_j \tag{51}$$

in $\Delta(M)$ for all $0 \leq j \leq d$.

(b) *If for any $1 \leq j \leq d$ there is some $0 \leq i \leq d$ with $k_{e_i} \neq k_{e_{i+j}}$, then $\Delta(M)$ is irreducible as a $k[[t]][\varphi]$ -module.*

(c) *Suppose that for any $1 \leq j \leq d$ which satisfies $k_{e_i} = k_{e_{i+j}}$ for all $0 \leq i \leq m$ there is some $0 \leq i \leq d$ with $\eta_i \neq \eta_{i+j}$. Then $\Delta(M)$ is irreducible as an \mathfrak{O} -module.*

Proof. For M as above, $\nabla(M)$ is generated by elements of the form $h(e) = t^{k_e} \varphi \otimes T_{\omega}^{-1}(e) + 1 \otimes e$. They give rise to formula (51), hence statement (a). For statements (b) and (c) apply Proposition 1.15; in (c) notice that $\gamma \cdot (1 \otimes e_j) = \eta_j(\gamma) \otimes e_j$ for $\gamma \in \Gamma$. \square

Lemma 5.2. (a) *Conjugating χ by powers of T_ω means cyclically permuting the ordered tuple $((\eta_0, k_{e_0}), \dots, (\eta_d, k_{e_d}))$ associated with χ as above. Knowing the conjugacy class of χ (under powers of T_ω) is equivalent with knowing the tuple $((\eta_0, k_{e_0}), \dots, (\eta_d, k_{e_d}))$ up to cyclic permutations, together with $\chi(T_\omega^{d+1})$.*

(b) (Vignéras) *Two standard supersingular \mathcal{H} -modules are isomorphic if and only if the element $T_\omega^{d+1} \in \mathcal{H}$ acts on them by the same constant in k^\times and if they arise from two supersingular characters $\mathcal{H}_{\text{aff}} \rightarrow k$ which are conjugate under some power of T_ω .*

(c) (Vignéras) *A standard supersingular \mathcal{H} -module M arising from χ is simple if and only if the orbit of χ under conjugation by powers of T_ω has cardinality $d+1$.*

Proof. Statement (a) is clear. For (b) and (c) see [Vignéras 2005, Proposition 3 and Theorem 5]. \square

Proposition 5.3. *The functor $M \mapsto \Delta(M)$ induces a bijection between the set of isomorphism classes of standard supersingular \mathcal{H} -modules and the set of standard cyclic objects in $\text{Mod}^{\text{ad}}(\mathfrak{O})$ of k -dimension $d+1$. If the standard supersingular \mathcal{H} -module M is simple, then $\Delta(M) \in \text{Mod}^{\text{ad}}(\mathfrak{O})$ is simple.*

Proof. This follows from Lemmas 5.1 and 5.2. \square

Theorem 5.4. (1) *The functor $M \mapsto \Delta(M)^* \otimes_{k[[t]]} k((t))$ induces a bijection between the set of isomorphism classes of standard supersingular \mathcal{H} -modules and the set of isomorphism classes of standard cyclic étale (φ, Γ) -modules of dimension $d+1$.*

(2) *The functor $M \mapsto \Delta(M)^* \otimes_{k[[t]]} k((t))$ induces a bijection between the set of isomorphism classes of simple supersingular \mathcal{H} -modules of k -dimension $d+1$ and the set of isomorphism classes of simple étale (φ, Γ) -modules of dimension $d+1$.*

Proof. Statement (a) follows from Propositions 1.20 and 5.3. Statement (b) follows from statement (a) and the full faithfulness of the functor $M \mapsto \Delta(M)^* \otimes_{k[[t]]} k((t))$ on supersingular \mathcal{H} -modules, see Theorem 5.11 below. (To see that if M is simple then so is $\Delta(M)^* \otimes_{k[[t]]} k((t))$ one may alternatively use Proposition 5.3 together with Proposition 1.11.) \square

Remark. We may rewrite (51) as

$$\begin{aligned} t^{k_{e_j}} \varphi \otimes e_{j+1} &= -1 \otimes e_j \quad \text{for } 0 \leq j \leq d-1 \\ t^{k_{e_d}} \varphi \otimes \chi(T_\omega^{-d-1}) e_0 &= -1 \otimes e_d \end{aligned}$$

where we used $T_\omega^{-1}(e_d) = T_\omega^{-d-1}(e_0) = \chi(T_\omega^{-d-1}) e_0$. Thus $(-1)^{d+1} \chi(T_\omega^{-d-1}) \in k^\times$ is the constant referred to in Lemma 1.19.

Corollary 5.5. *The functor $M \mapsto \Delta(M)^* \otimes_{k[[t]]} k((t))$, composed with the functor of Theorem 1.2, induces a bijection between the set of isomorphism classes of standard supersingular \mathcal{H} -modules of k -dimension $d+1$ and the set of isomorphism classes of $(d+1)$ -dimensional standard cyclic $\text{Gal}(\bar{F}/F)$ -representations.*

Proof. Theorem 5.4. \square

Remark. (a) Combining Corollary 5.5 and Theorem 5.4 one can derive the following “numerical Langlands correspondence”: the set of (absolutely) simple $(d+1)$ -dimensional \mathcal{H} -modules with fixed scalar action by T_ω^{d+1} has the same cardinality as the set of (absolutely) irreducible $(d+1)$ -dimensional $\text{Gal}(\bar{F}/F)$ -representations with fixed determinant of Frobenius. This numerical Langlands correspondence was proven already in [Vignéras 2005, Theorem 5].

(b) There is an alternative and arguably more natural definition of supersingularity for \mathcal{H} -modules. Its agreement with the one given in Section 2A, and hence the “numerical Langlands correspondence” with respect to this alternative definition of supersingularity, was proven in [Ollivier 2010].

5B. Reconstruction of an initial segment of \mathbf{M} from $\Delta(\mathbf{M})$. Let $[0, q-1]^\Phi$ be the set of tuples $\mu = (\mu_i)_{0 \leq i \leq d}$ with $\mu_i \in \{0, \dots, q-1\}$ and $\sum_{0 \leq i \leq d} \mu_i \equiv 0 \pmod{q-1}$. We often read the indices as elements of $\mathbb{Z}/(d+1)\mathbb{Z}$, thus $\mu_i = \mu_j$ for $i, j \in \mathbb{Z}$ whenever $i - j \in (d+1)\mathbb{Z}$.

Let Δ be an \mathfrak{O} -module. For $\mu \in [0, q-1]^\Phi$ let $\mathcal{F}\Delta[t]^\mu$ be the k -subvector space of $\Delta[t] = \{x \in \Delta \mid tx = 0\}$ generated by all $x \in \Delta[t]$ satisfying $t^{\mu_i} \varphi \cdots t^{\mu_1} \varphi t^{\mu_0} \varphi x \in \Delta[t]$ for all $0 \leq i \leq d$, as well as $t^{\mu_d} \varphi \cdots t^{\mu_1} \varphi t^{\mu_0} \varphi x \in k^\times x$.

Put $\mathcal{F}\Delta[t] = \bigoplus_{\mu \in [0, q-1]^\Phi} \mathcal{F}\Delta[t]^\mu$ (sum in $\Delta[t]$).

Lemma 5.6. $\mathcal{F}\Delta[t] = \bigoplus_{\mu \in [0, q-1]^\Phi} \mathcal{F}\Delta[t]^\mu$, i.e., the sum is direct.

Proof. Consider the lexicographic enumeration $\mu(1), \mu(2), \mu(3), \dots$ of $[0, q-1]^\Phi$ such that for each pair $r' > r$ there is some $0 \leq i_0 \leq d$ with $\mu_i(r) \geq \mu_i(r')$ for all $i < i_0$, and $\mu_{i_0}(r) > \mu_{i_0}(r')$. Let $\sum_{r \geq 1} x_r = 0$ with $x_r \in \mathcal{F}\Delta[t]^{\mu(r)}$. We prove $x_r = 0$ for all r by induction on r . So, fix r and assume $x_{r'} = 0$ for all $r' < r$, hence $\sum_{r' \geq r} x_{r'} = \sum_{r \geq 1} x_r - \sum_{r' < r} x_{r'} = 0$. For $r' > r$ we have $t^{\mu_d(r)} \varphi \cdots t^{\mu_0(r)} \varphi (x_{r'}) = 0$. Therefore

$$\begin{aligned} 0 &= t^{\mu_d(r)} \varphi \cdots t^{\mu_0(r)} \varphi \left(\sum_{r' \geq r} x_{r'} \right) \\ &= \sum_{r' \geq r} t^{\mu_d(r)} \varphi \cdots t^{\mu_0(r)} \varphi x_{r'} \\ &= t^{\mu_d(r)} \varphi \cdots t^{\mu_0(r)} \varphi x_r \in k^\times x_r \end{aligned}$$

and hence $x_r = 0$. □

We define k -linear endomorphisms T_ω , T_s and $T_{e^*(\gamma)}$ (for $\gamma \in \Gamma$) of $\mathcal{F}\Delta[t]$ as follows. In view of Lemma 5.6 it is enough to define their values on $x \in \mathcal{F}\Delta[t]^\mu$; we put

$$T_\omega(x) = -t^{\mu_0} \varphi x, \quad T_{e^*(\gamma)}(x) = \gamma^{-1} \cdot x, \quad T_s(x) = \begin{cases} -x & \mu_d = q-1 \\ 0 & \mu_d < q-1. \end{cases}$$

Here $\gamma^{-1} \cdot x$ is understood with respect to the Γ -action induced by the \mathfrak{O} -module structure on $\Delta(M)$.

Definition. For an $\mathcal{H}^{\sharp\sharp}$ -module M and $\mu \in [0, q-1]^\Phi$ let $\mathcal{F}M^\mu$ denote the k -subvector space of M consisting of $x \in M$ satisfying the following conditions for all $0 \leq i \leq d$:

$$T_{\alpha_1^\vee(\gamma)}^{-1}(T_\omega^i(x)) = \gamma^{\mu_{i-1}} T_\omega^i(x) \quad \text{for all } \gamma \in \Gamma, \quad (52)$$

$$T_s(T_\omega^i(x)) = \begin{cases} -T_\omega^i(x) & \mu_{i-1} = q-1, \\ 0 & \mu_{i-1} < q-1. \end{cases} \quad (53)$$

Let $\mathcal{F}M$ denote the subspace of M generated by the $\mathcal{F}M^\mu$ for all $\mu \in [0, q-1]^\Phi$.

For $\mu \in [0, q-1]^\Phi$ let $\epsilon_\mu \in [0, q-2]^\Phi$ be the unique element with

$$(\epsilon_\mu)_{-i} \equiv \mu_i \pmod{q-1}. \quad (54)$$

for all i .

Lemma 5.7. (a) We have $\mathcal{F}M^\mu \subset M^{\epsilon_\mu}$.

(b) $\mathcal{F}M$ is an $\mathcal{H}^{\sharp\sharp}$ -submodule of M .

(c) $\mathcal{F}M$ contains each $\mathcal{H}^{\sharp\sharp}$ -submodule of M which is a subquotient of a standard supersingular $\mathcal{H}^{\sharp\sharp}$ -module.

(d) Suppose that M is supersingular. Viewing the isomorphism $\Delta(M)[t] \cong M$ (Proposition 4.3) as an identity, we have $\mathcal{F}M^\mu \subset \mathcal{F}\Delta(M)[t]^\mu$ for each $\mu \in [0, q-1]^\Phi$, and in particular

$$\mathcal{F}M \subset \mathcal{F}\Delta(M)[t]. \quad (55)$$

The operators T_ω , T_s and $T_{e^*(\gamma)}$ acting on $\mathcal{F}\Delta(M)[t]$ as defined above restrict to the operators T_ω , T_s , $T_{e^*(\gamma)} \in \mathcal{H}^{\sharp\sharp}$ acting on $\mathcal{F}M$.

Proof. (a) Let $\mu \in [0, q-1]^\Phi$. For $x \in \mathcal{F}M^\mu$, any $\gamma \in \Gamma$ and any i we compute

$$T_{\alpha_{1-i}^\vee(\gamma)}^{-1}(x) = T_{(\omega^i \cdot \alpha_1^\vee)(\gamma)}^{-1}(x) = T_\omega^{-i} T_{\alpha_1^\vee(\gamma)}^{-1} T_\omega^i(x) = \gamma^{\mu_{i-1}} x = \gamma^{(\epsilon_\mu)_{1-i}} x,$$

i.e., $x \in M^{\epsilon_\mu}$.

(b) Let $\mu \in [0, q-1]^\Phi$ and define $\mu' \in [0, q-1]^\Phi$ by $\mu'_i = \mu_{i+1}$ for all i . For $x \in \mathcal{F}M^\mu$, any $\gamma \in \Gamma$ and any i we compute

$$T_{\alpha_1^\vee(\gamma)}^{-1}(T_\omega^i(T_\omega(x))) = T_{\alpha_1^\vee(\gamma)}^{-1}(T_\omega^{i+1}(x)) = \gamma^{\mu_i} T_\omega^{i+1}(x) = \gamma^{\mu_i} T_\omega^i(T_\omega(x)).$$

We also find $T_s(T_\omega^i(T_\omega(x))) = T_s(T_\omega^{i+1}(x)) = -T_\omega^{i+1}(x) = -T_\omega^i(T_\omega(x))$ if $\mu_i = q-1$, but $T_s(T_\omega^i(T_\omega(x))) = T_s(T_\omega^{i+1}(x)) = 0$ if $\mu_i < q-1$. Together this shows $T_\omega(x) \in \mathcal{F}M^{\mu'}$, i.e., $T_\omega(\mathcal{F}M^\mu) \subset \mathcal{F}M^{\mu'}$. It is immediate from the definitions that $T_s(\mathcal{F}M^\mu) \subset \mathcal{F}M^\mu$. For $x \in \mathcal{F}M^\mu$, any $\gamma, \gamma' \in \Gamma$ and any i we compute

$$T_{\alpha_1^\vee(\gamma)}^{-1} T_\omega^i(T_{e^*(\gamma')}(x)) = T_{\alpha_1^\vee(\gamma)}^{-1} T_{(\omega^{-i} \cdot e^*)(\gamma')} T_\omega^i(x) = T_{(\omega^{-i} \cdot e^*)(\gamma')} \gamma^{\mu_{i-1}} T_\omega^i(x) = \gamma^{\mu_{i-1}} T_\omega^i(T_{e^*(\gamma')}(x)).$$

If $\mu_{i-1} = q - 1$ we also compute

$$\begin{aligned} T_s T_\omega^i (T_{e^*(\gamma')}(x)) &= T_{(s \cdot \omega^{-i} \cdot e^*)(\gamma')} T_s T_\omega^i(x) \\ &= -T_{(s \cdot \omega^{-i} \cdot e^*)(\gamma')} T_\omega^i(x) \\ &= -T_\omega^i (T_{(\omega^i \cdot s \cdot \omega^{-i} \cdot e^*)(\gamma')}(x)) \\ &= -T_\omega^i (T_{e^*(\gamma')}(x)). \end{aligned}$$

Here, in the last equation we use $\omega^i \cdot s \cdot \omega^{-i} \cdot e^* = e^*$ for $2 \leq i \leq d$; for $i = 1$ we use $\omega \cdot s \cdot \omega^{-1} \cdot e^* - e^* = \alpha_0^\vee$ and $T_{\alpha_0^\vee(\gamma')}(x) = T_\omega^{-1} T_{\alpha_1^\vee(\gamma')} T_\omega(x) = \gamma^{-\mu_0} x = x$ (as $\mu_0 = q - 1$); for $i = 0$ we use $s \cdot e^* - e^* = -\alpha_1^\vee$ and $T_{-\alpha_1^\vee(\gamma')}(x) = \gamma^{\mu_1} x = x$ (as $\mu_1 = q - 1$). If however $\mu_{i-1} < q - 1$ then $T_s T_\omega^i (T_{e^*(\gamma')}(x)) = T_{(s \cdot \omega^{-i} \cdot e^*)(\gamma')} T_s T_\omega^i(x) = 0$. Together this shows $T_{e^*(\gamma')}(x) \in \mathcal{F}M^\mu$, i.e., $T_{e^*(\gamma')}(\mathcal{F}M^\mu) \subset \mathcal{F}M^\mu$.

(c) On a standard supersingular $\mathcal{H}^{\sharp\sharp}$ -module, and hence on its subquotients, the actions of T_ω , T_s and $T_{\alpha_1^\vee(\gamma')}$ satisfy formulae (52) and (53), for suitable μ 's.

(d) Let $\mu \in [0, q - 1]^\Phi$ and define $\mu' \in [0, q - 1]^\Phi$ by $\mu'_i = \mu_{i+1}$ for all i . Let $x \in \mathcal{F}M^\mu$. The proof of (b) shows $T_\omega(x) \in \sum_a M_a^{\epsilon_{\mu'}}[0]$ if $\mu'_d = \mu_0 < q - 1$, resp. $T_\omega(x) \in \sum_a M_a^{\epsilon_{\mu'}}[1]$ if $\mu'_d = \mu_0 = q - 1$. In either case, the definition of $\Delta(M)$ then says $T_\omega(x) = -t^{\mu_0} \varphi x$. This shows $\mathcal{F}M^\mu \subset \mathcal{F}\Delta(M)[t]^\mu$ and that the action of T_ω on $\mathcal{F}M$ is indeed as stated. For the actions of T_s and $T_{e^*(\gamma')}$ this is clear anyway. \square

Remark. The inclusion (55) is in fact an equality.

5C. Reconstruction of \sharp -supersingular \mathcal{H}^\sharp -modules M from $\Delta(M)$.

Lemma 5.8. *Let M be an irreducible supersingular \mathcal{H} -module. Let $\mu \in [0, q - 1]^\Phi$, $x \in M$ and $u_{i,c} \in M^{\omega^{-1}s\omega^{i+1}, \epsilon_\mu}$ for $i \geq 0$ and $0 \leq c \leq q - 2$ (with ϵ_μ given by formula (54)). Assume $u_{i,c} = 0$ if*

- (i) $\mu_i = 0$, or
- (ii) $\mu_i = q - 1$ and $c > 0$, or
- (iii) $\mu_i < q - 1$ and $c \geq q - 1 - \mu_i$.

Assume that, if we put $x\{-1\} = x$, then

$$x\{i\} = t^{\mu_i} \varphi(x\{i-1\}) - \sum_{c=0}^{q-2} t^c \varphi u_{i,c}$$

belongs to $M \cong \Delta(M)[t]$ for each $i \geq 0$. Finally, assume that $x\{D\} = x$ for some $D > 0$ with $D + 1 \in \mathbb{Z}(d + 1)$. Then there is some $x' \in M$ with $x - x' \in M^{\epsilon_\mu}$ and such that

$$x'\{i\} = t^{\mu_i} \varphi(\cdots (t^{\mu_1} \varphi(t^{\mu_0} \varphi x')) \cdots)$$

belongs to M for each i , and $x'\{D\} = x'$. Moreover, if x is an eigenvector for $T_{e^*(\Gamma)}$, then x' can be chosen to be an eigenvector for $T_{e^*(\Gamma)}$, with the same eigenvalues.

Proof. It is easy to see that all the irreducible subquotients of a standard supersingular \mathcal{H} -module are isomorphic. In particular, an irreducible supersingular \mathcal{H} -module is isomorphic with a submodule of a standard supersingular \mathcal{H} -module. Therefore we may assume that M itself is a (not necessarily irreducible) standard supersingular \mathcal{H} -module. We then have a direct sum decomposition $M = \bigoplus_{j=0}^d M^{[j]}$ with $\dim_k(M^{[j]}) = 1$ and integers $0 \leq k_j \leq q-1$ such that

$$T_\omega(M^{[j+1]}) = t^{k_j} \varphi(M^{[j+1]}) = M^{[j]} \quad (56)$$

(always reading j modulo $(d+1)$). More precisely, we have $M^{[j]} \subset M^{\epsilon_j}$ for certain $\epsilon_j \in [0, q-2]^\Phi$, and choosing the above k_j minimally, we have $k_j \equiv (\omega \cdot \epsilon_{j+1})_1$ modulo $(q-1)$. It follows that

$$k[t]\varphi M = \bigoplus_{j=0}^d k[t]\varphi M^{[j]} = \bigoplus_{j=0}^d \bigoplus_{c=0}^{k_j} t^c \varphi M^{[j+1]}. \quad (57)$$

For $m \in M$ write $m = \sum_j m^{[j]}$ with $m^{[j]} \in M^{[j]}$. By formulae (56), (57), the defining formula for $x\{i\}$ splits up into the formulae

$$x\{i\}^{[j]} = t^{\mu_i} \varphi(x\{i-1\}^{[j+1]}) - \sum_{c=0}^{q-2} t^c \varphi(u_{i,c}^{[j+1]}) \quad (58)$$

for all j . We use them to show

$$t^c \varphi(u_{i,c}^{[j+1]}) = 0 \quad \text{if } c - \mu_i \notin (q-1)\mathbb{Z}. \quad (59)$$

If $\mu_i \in \{0, q-1\}$ then formula (59) follows from our assumptions on the $u_{i,c}$. Now assume $\mu_i \notin \{0, q-1\}$ and $u_{i,c}^{[j+1]} \neq 0$ for some c . The assumption $u_{i,c} \in M^{\omega^{-1}s\omega^{i+1}\cdot\epsilon_\mu}$ implies $T_\omega(u_{i,c}^{[j+1]}) \in M^{s\omega^{i+1}\cdot\epsilon_\mu}$, and since

$$q-1-\mu_i = q-1-\epsilon_{-i} = (s\omega^{i+1}\cdot\epsilon_\mu)_1 \quad \text{if } \mu_i \notin \{0, q-1\}$$

we get $T_\omega(u_{i,c}^{[j+1]}) = -t^{q-1-\mu_i} \varphi(u_{i,c}^{[j+1]})$, i.e., $k_j = q-1-\mu_i$. Now $\sum_{c=0}^{k_j} t^c \varphi M^{[j+1]}$ is a *direct* sum of one dimensional k -vector spaces, with $x\{i\}^{[j]} \in t^{k_j} \varphi M^{[j+1]}$, $t^{\mu_i} \varphi(x\{i-1\}^{[j+1]}) \in t^{\mu_i} \varphi M^{[j+1]}$ and $t^c \varphi(u_{i,c}^{[j+1]}) \in t^c \varphi M^{[j+1]}$ for all c . Since by assumption $u_{i,c} = 0$ for $c \geq q-1-\mu_i = k_j$, formula (58) shows $t^c \varphi(u_{i,c}^{[j+1]}) = 0$ whenever $c \neq \mu_i$.

Formula (59) is proven. Arguing once more with formulae (56), (57) and (58) shows

$$[t^{\mu_i} \varphi(x\{i-1\}^{[j+1]}) = 0 \text{ or } \varphi(u_{i,0}^{[j+1]}) = 0] \quad \text{if } \mu_i = q-1. \quad (60)$$

In the following, by $u_{i,q-1}$ we mean $u_{i,0}$. If $t^{\mu_i} \varphi(u_{i,\mu_i}^{[j+1]}) \neq 0$ we may write

$$t^{\mu_i} \varphi(x\{i-1\}^{[j+1]}) - t^{\mu_i} \varphi(u_{i,\mu_i}^{[j+1]}) = \rho_{i,j} t^{\mu_i} \varphi(u_{i,\mu_i}^{[j+1]})$$

for some $\rho_{i,j} \in k$, since $t^{\mu_i} \varphi(x\{i-1\}^{[j+1]})$ and $t^{\mu_i} \varphi(u_{i,\mu_i}^{[j+1]})$ belong to the same one-dimensional k -vector space. The upshot of formulae (59) and (60) is then that formula (58) simplifies to become either

$$x\{i\}^{[j]} = t^{\mu_i} \varphi(x\{i-1\}^{[j+1]}) \quad (61)$$

or

$$x\{i\}^{[j]} = \rho_{i,j} t^{\mu_i} \varphi(u_{i,\mu_i}^{[j+1]}) \quad (62)$$

for some $\rho_{i,j} \in k$. Departing from $x^{[j]} = x\{D\}^{[j]}$ we repeatedly substitute formula (61); if this is possible $D+1$ many times we end up with

$$x^{[j]} = x\{D\}^{[j]} = t^{\mu_D} \varphi(\cdots (t^{\mu_1} \varphi(t^{\mu_0} \varphi(x^{[j]}))) \cdots),$$

and in this case we put $n(j) = 0$. Otherwise, after $D+1-n(j)$ many substitutions of formula (61), for some $1 \leq n(j) \leq D+1$, we end the procedure by substituting formula (62) (once) and obtain

$$x^{[j]} = x\{D\}^{[j]} = \rho_j t^{\mu_D} \varphi(\cdots (t^{\mu_{n(j)}} \varphi(t^{\mu_{n(j)-1}} \varphi(u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]}))) \cdots)$$

with $t^{\mu_{n(j)-1}} \varphi(u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]}) \neq 0$, for some $\rho_j \in k$.

We study this second case $n(j) > 0$ further. By construction,

$$w_j\{-1\} = t^{\mu_{n(j)-1}} \varphi(u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]})$$

is nonzero and belongs to M . On the other hand, $u_{n(j)-1,\mu_{n(j)-1}} \in M^{\omega^{-1}s\omega^{n(j)} \cdot \epsilon_\mu}$ implies $T_\omega(u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]}) \in M^{s\omega^{n(j)} \cdot \epsilon_\mu}$ and hence

$$t^{(s\omega^{n(j)} \cdot \epsilon_\mu)_1} \varphi(u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]}) = -T_\omega(u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]}) \in M^{s\omega^{n(j)} \cdot \epsilon_\mu}.$$

Together this means $\mu_{n(j)-1} \equiv (s\omega^{n(j)} \cdot \epsilon_\mu)_1$ modulo $(q-1)$ and $w_j\{-1\} \in M^{s\omega^{n(j)} \cdot \epsilon_\mu}$. But we also have $\mu_{n(j)-1} \equiv (\omega^{n(j)} \cdot \epsilon_\mu)_1$. Combining we see $\mu_{n(j)-1} \equiv -\mu_{n(j)-1}$ modulo $(q-1)$. Hence, we either have $\mu_{n(j)-1} = 0$ or $\mu_{n(j)-1} = (q-1)/2$ or $\mu_{n(j)-1} = q-1$. In view of the assumed vanishings of the $u_{i,c}$ (and of $u_{n(j)-1,\mu_{n(j)-1}}^{[j+1-n(j)]} \neq 0$) this leaves $\mu_{n(j)-1} = q-1$ as the only possibility. It follows that

$$s\omega^{n(j)} \cdot \epsilon_\mu = \omega^{n(j)} \cdot \epsilon_\mu$$

and hence $w_j\{-1\} \in M^{\omega^{n(j)} \cdot \epsilon_\mu}$. Next, again by construction we know that

$$w_j\{s\} = t^{\mu_{n(j)+s}} \varphi(w_j\{s-1\})$$

belongs to M , for $0 \leq s \leq D-n(j)$. By what we learned about $w_j\{-1\}$ this implies $w_j\{s\} = (-1)^{s+1} T_\omega^{s+1} w_j\{-1\} \in M^{\omega^{n(j)+s+1} \cdot \epsilon_\mu}$ by an induction on s (and we also see $\mu_{n(j)+s} \in \{k_0, \dots, k_d\}$ with the k_ℓ from formula (56)). For $s = D-n(j)$ we obtain $x^{[j]} = x\{D\}^{[j]} \in M^{\epsilon_\mu}$.

We now put $x' = \sum_{n(j)=0} x^{[j]}$. □

Lemma 5.9. *Let M be an irreducible supersingular \mathcal{H} -module. Let $\mu \in [0, q-1]^\Phi$ and $x \in M$ such that*

$$x\{i\} = t^{\mu_i} \varphi(\cdots (t^{\mu_1} \varphi(t^{\mu_0} \varphi x)) \cdots)$$

belongs to $M \cong \Delta(M)[t]$ for each $i \geq 0$, and such that $x\{D\} = x$ for some $D > 0$ with $D+1 \in \mathbb{Z}(d+1)$. Then $x \in M^{\epsilon_\mu}$ and $x\{i\} = (-T_\omega)^i x$ for each i .

Proof. This follows from the formulae (56) and (57) in the proof of Lemma 5.8. The argument is very similar to the one given in the proof of Lemma 5.6. \square

Theorem 5.10. *Let M be a \sharp -supersingular \mathcal{H}^\sharp -module. Via the isomorphism $M \cong \Delta(M)[t]$, the action of \mathcal{H}^\sharp on M can be recovered from the action of \mathfrak{O} on $\Delta(M)$.*

Proof. We may assume $\dim_k(M) < \infty$. Define inductively the filtration $(F^i M)_{i \geq 0}$ of M by \mathcal{H}^\sharp -submodules as follows: $F^0 M = 0$, and $F^{i+1} M$ is the preimage of $\mathcal{F}(M/F^i M)$ under the projection $M \rightarrow M/F^i M$. The \mathcal{H}^\sharp -action on the graded pieces can be recovered in view of Lemma 5.7. Exhausting M step by step it is therefore enough to consider the following setting: The action of \mathcal{H}^\sharp has already been recovered on an \mathcal{H}^\sharp -submodule M_0 of M and on the quotient M/M_0 , and the latter is irreducible.

We reconstruct the action of $T_{e^*(\Gamma)}$ on M by means of

$$T_{e^*(\gamma)}(x) = \gamma^{-1} \cdot x \quad \text{for } \gamma \in \Gamma$$

as is tautological from our definitions. Next we are going to reconstruct the decomposition

$$M = \bigoplus_{\substack{\epsilon \in [0, q-2]^\Phi, \\ a \in [0, q-2]}} M_a^\epsilon. \quad (63)$$

Let $D > 0$ be such that $D+1 \in \mathbb{Z}(d+1)$ and $f^{D+1} = \text{id}$ for each k -vector space automorphism f of M . (Such a D does exist. Indeed, M is finite, hence $\text{Aut}_k(M)$ is finite, hence there is some $n \in \mathbb{N}$ with $f^n = \text{id}$ for each $f \in \text{Aut}_k(M)$. Now take $D = (d+1)n - 1$.) For $\epsilon \in [0, q-2]^\Phi$ and $a \in [0, q-2]$ define $M_a^{[\epsilon]}$ to be the k -subspace of M generated by all $x \in M$ with $\gamma \cdot x = \gamma^a x$ (all $\gamma \in \Gamma$) and satisfying the following condition: there is some $\mu \in [0, q-1]^\Phi$ (depending on x) with $\epsilon_\mu = \epsilon$, and there are $u_{i,c} \in M_0^{\omega^{-1}s\omega^{i+1}\cdot\epsilon}$ for $i \geq 0$ and $0 \leq c \leq q-2$ with the following properties: Firstly, $u_{i,c} = 0$ if

- (i) $\mu_i = 0$, or
- (ii) $\mu_i = q-1$ and $c > 0$, or
- (iii) $\mu_i < q-1$ and $c \geq q-1-\mu_i$.

Secondly, putting $x\{-1\} = x$ and

$$x\{i\} = t^{\mu_i} \varphi(x\{i-1\}) - \sum_c t^c \varphi u_{i,c}, \quad (64)$$

we have $x\{i\} \in M \cong \Delta(M)[t]$ for any i , as well as $x\{D\} = x$.

It will be enough to prove $M_a^\epsilon = M_a^{[\epsilon]}$. We first show

$$M_a^\epsilon \subset M_a^{[\epsilon]}. \quad (65)$$

We start with $\bar{x} \in \mathcal{F}(M/M_0)^\mu \cap (M/M_0)_a^\epsilon$ for some μ with $\epsilon_\mu = \epsilon$. By Proposition 3.2 we may lift it to some $x \in M^\epsilon$ such that for each i with $T_s T_\omega^{i+1} \bar{x} = 0$ we have $T_s^2 T_\omega^{i+1} x = 0$. As T_ω maps simultaneous

eigenspaces for the T_t (with $t \in \bar{T}$) again to such simultaneous eigenspaces, and as T_s^2 commutes with the T_t , we may assume $x \in M_a^\epsilon$. Putting

$$x\{i\} = (-T_\omega)^{i+1}x$$

for $-1 \leq i \leq D$, repeated application of Lemma 2.4 shows $x\{i\} \in M_{\underline{a}_{\epsilon,i}}^{\omega^{i+1} \cdot \epsilon}$ with

$$a_{\epsilon,-1} = a, \quad a_{\epsilon,0} = a - \epsilon_0 \quad \text{and} \quad a_{\epsilon,i} = a - \epsilon_0 - \epsilon_{d-i+1} - \cdots - \epsilon_d$$

for $i \leq d$, and then $a_{\epsilon,i} = a_{\epsilon,i'}$ for $i - i' \in \mathbb{Z}(d+1)$.

If $0 \leq \mu_i < q-1$ put

$$u_{i,c} = T_\omega^{-1}((T_s(x\{i\}))_{\underline{c} + \mu_i + a_{\epsilon,i}}).$$

As $\bar{x} \in \mathcal{F}(M/M_0)^\mu$ and $\mu_i < q-1$ we have $u_{i,c} \in M_0$, and as $x\{i\} \in M^{\omega^{i+1} \cdot \epsilon}$ we have $u_{i,c} \in M^{\omega^{-1}s\omega^{i+1} \cdot \epsilon}$, together $u_{i,c} \in M_0^{\omega^{-1}s\omega^{i+1} \cdot \epsilon}$. From $\mu_i < q-1$ we furthermore deduce $k_{x\{i\}} = (\omega^{i+1} \cdot \epsilon)_1 = \mu_i$, and since $T_s^2 x\{i\} = 0$ we then see

$$t^{\mu_i} \varphi(x\{i-1\}) - x\{i\} - \sum_c t^c \varphi u_{i,c} = h(-x\{i\}) = 0. \quad (66)$$

Since furthermore $(T_s(x\{i\}))_{\underline{c} + \mu_i + a_{\epsilon,i}} = 0$ and hence $u_{i,c} = 0$ for $q-1-\mu_i \leq c \leq q-2$ by \sharp -supersingularity (if $0 < \mu_i < q-1$ then $\mu_i = \epsilon_{-i}$), all the conditions on the $u_{i,c}$ in the definition of $x \in M_a^{[\epsilon]}$ are satisfied.

If $\mu_i = q-1$ we have $T_s^2(T_s^2 x\{i\}) = T_s^2 x\{i\}$ and hence $k_{T_s^2 x\{i\}} = q-1$ (independently of the value of μ_i we have $(\omega^{i+1} \cdot \epsilon)_1 \equiv \mu_i$ modulo $(q-1)$), hence

$$t^{q-1} \varphi T_\omega^{-1}(T_s^2 x\{i\}) + T_s^2 x\{i\} = h(T_s^2 x\{i\}) = 0. \quad (67)$$

Similarly we see $k_{(x\{i\} - T_s^2 x\{i\})} = 0$ and hence

$$\varphi T_\omega^{-1}(x\{i\} - T_s^2 x\{i\}) + x\{i\} - T_s^2 x\{i\} = h(x\{i\} - T_s^2 x\{i\}) = 0. \quad (68)$$

We compute

$$\begin{aligned} t^{q-1} \varphi(x\{i-1\}) &= -t^{q-1} \varphi T_\omega^{-1}(x\{i\}) \\ &= -t^{q-1} \varphi T_\omega^{-1} T_s^2(x\{i\}) \\ &= T_s^2(x\{i\}) \\ &= \varphi T_\omega^{-1}(x\{i\} - T_s^2 x\{i\}) + x\{i\} \end{aligned}$$

where the second equality is the result of applying t^{q-1} to formula (68), where the third equality is formula (67) and where the fourth equality is formula (68). Thus, putting $u_{i,0} = T_\omega^{-1}(x\{i\} - T_s^2 x\{i\})$ and $u_{i,c} = 0$ for $c > 0$, we again get formula (66). Moreover, $u_{i,0}$ belongs to M_0 as $\bar{x} \in \mathcal{F}(M/M_0)^\mu$ and $\mu_i = q-1$; but it also belongs to $M^{\omega^{-1}s\omega^{i+1} \cdot \epsilon}$ since $\mu_i = q-1$ implies $\omega^{-1}s\omega^{i+1} \cdot \epsilon = \omega^i \cdot \epsilon$. By construction, $x\{d\} = (-T_\omega)^{d+1}(x)$, hence $x\{D\} = (-T_\omega)^{D+1}x = x$.

It follows that $x \in M_a^{[\epsilon]}$. We have shown that any element in $\mathcal{F}(M/M_0)^\mu \cap (M/M_0)_a^\epsilon$, for μ with $\epsilon_\mu = \epsilon$, lifts to an element in $M_a^\epsilon \cap M_a^{[\epsilon]}$. Since we have $(M/M_0)^\epsilon = \sum_{\mu \in \{0, q-1\}^D} \mathcal{F}(M/M_0)^\mu$ (see Lemma 5.7) and

since this is respected by the action of $T_{e^*(\Gamma)}$, we thus have reduced our problem to showing $(M_0)_{\underline{a}}^\epsilon \subset M_{\underline{a}}^{[\epsilon]}$. But for this we may appeal to an induction on $\dim_k(M)$ (which we may assume to be finite).

We have shown formula (65). Now we show

$$M_{\underline{a}}^{[\epsilon]} \subset M_{\underline{a}}^\epsilon. \quad (69)$$

Let $x \in M_{\underline{a}}^{[\epsilon]}$, $\mu \in [0, q-1]^\Phi$ (with $\epsilon_\mu = \epsilon$) and $u_{i,c}$ be as in the definition of $M_{\underline{a}}^{[\epsilon]}$. Define $x\{i\}$ for $-1 \leq i \leq D$ as in that definition. By Lemma 5.9 and the proof of the inclusion (65) we find $\tilde{x} \in M_{\underline{a}}^\epsilon$ and $\tilde{u}_{i,c} \in M_0^{\omega^{-1}s\omega^{i+1}\cdot\epsilon}$ for $0 \leq i \leq D$ such that, after replacing x by $x - \tilde{x}$ and $u_{i,c}$ by $u_{i,c} - \tilde{u}_{i,c}$, we may assume $x \in M_0$.

Claim. *If $x \in M_0$ and if M_0 is irreducible, then there is some $x' \in (M_0)_{\underline{a}}$ with $x - x' \in (M_0)_{\underline{a}}^\epsilon$ and such that*

$$x'\{i\} = t^{\mu_i} \varphi(\cdots (t^{\mu_1} \varphi(t^{\mu_0} \varphi x')) \cdots)$$

belongs to M_0 for all i , and $x'\{D\} = x'$.

This follows from Lemma 5.8.

If M_0 is not irreducible, choose an \mathcal{H} -submodule M_{00} in M_0 such that M_0/M_{00} is irreducible. By the above claim and again invoking the proof of the inclusion (65), after modifying x by another element of $M_{\underline{a}}^\epsilon$ (now even of $(M_0)_{\underline{a}}^\epsilon$) and suitably modifying the $u_{i,c}$, we may assume $u_{i,c} \in M_{00}$. Thus, it is now enough to solve the problem for the new $x \in (M_0)_{\underline{a}}$ (and the new $u_{i,c} \in M_{00}$). We continue in this way. Since we may assume that $\dim_k(M)$ is finite, an induction on the dimension of M allows us to conclude.

We have reconstructed the decomposition (63) of M .

Now we reconstruct $T_s T_\omega$ acting on M . As we already know the decomposition (63), it is enough to reconstruct $T_s T_\omega(e)$ for $e \in M_{\underline{a}'}^{\epsilon'}$, all ϵ' , a' . Given such e , let \bar{e} be its class in M/M_0 . By Lemma 2.4 there are then ϵ, a such that $T_\omega \bar{e} \in (M/M_0)_{\underline{a}}^\epsilon$.

First assume $\epsilon_1 = 0$. We then reconstruct $T_s T_\omega(e)$ as $T_s T_\omega(e) = t^{q-1} \varphi(e)$. Indeed, to see this we may assume (by Lemma 2.3) that $T_\omega(e)$ is an eigenvector for T_s^2 . If $T_s^2 T_\omega(e) = T_\omega(e)$ and hence $T_s T_\omega(e) = -T_\omega(e)$, the claim follows from the definition of $h(T_\omega(e))$. If $T_s^2 T_\omega(e) = 0$ then in fact $T_s T_\omega(e) = 0$ (since also $\epsilon_1 = 0$), and the definition of $h(T_\omega(e))$ shows $t^{q-1} \varphi(e) = 0$.

Now assume $\epsilon_1 > 0$. This implies $T_s^2 T_\omega(e) = 0$ and $k_{T_\omega(e)} = \epsilon_1$, and by \sharp -supersingularity we get

$$t^{k_{T_\omega(e)}+1} \varphi e = - \sum_{0 \leq c < q-1 - k_{T_\omega(e)}} t^{c+1} \varphi T_\omega^{-1}((T_s T_\omega e)_{c+\epsilon_1+a}).$$

Here $(T_s T_\omega e)_{c+\epsilon_1+a} \in M_0^{s \cdot \epsilon}$ and $q-1 - k_{T_\omega(e)} = (s \cdot \epsilon)_1$. The map

$$\bigoplus_{0 \leq c < q-1 - k_{T_\omega(e)}} M_0^{s \cdot \epsilon} \rightarrow M_0, \quad (y_c)_c \mapsto \sum_{0 \leq c < q-1 - k_{T_\omega(e)}} t^{c+1} \varphi T_\omega^{-1}(y_c)$$

is injective. This is first seen in the case where M_0 is irreducible; it then follows by an obvious devissage argument. We therefore see that the $(T_s T_\omega e)_{c+\epsilon_1+a}$ for $0 \leq c < q-1 - k_{T_\omega(e)}$ can be read off from $t^{k_{T_\omega(e)}+1} \varphi e$, hence also $T_s T_\omega e$ can be read off from $t^{k_{T_\omega(e)}+1} \varphi e$ (by \sharp -supersingularity).

The restriction of T_ω to $\{x \in M \mid T_s T_\omega(x) \in M_0\}$ is reconstructed as follows. Given $\bar{x} \in (M/M_0)_{a-\epsilon_1}^{\omega^{-1} \cdot \epsilon}$ (for some ϵ , some a) with $T_s T_\omega \bar{x} = 0$, we use the decomposition (34) to lift \bar{x} to some $x \in M_{a-\epsilon_1}^{\omega^{-1} \cdot \epsilon}$. Since $(\omega^{-1} \cdot \epsilon)_0 = \epsilon_1$, Lemma 2.4 says $T_\omega x \in M_a^\epsilon$. It then follows from the definitions that

$$T_\omega x = -t^{\epsilon_1} \varphi x - \sum_{c \geq 0} t^c \varphi T_\omega^{-1}((T_s T_\omega x)_{c+\epsilon_1+a}).$$

We have now collected all the data required in Proposition 3.3 for reconstructing M as an \mathcal{H}^\sharp -module. \square

5D. Full faithfulness on \sharp -supersingular \mathcal{H}^\sharp -modules. Let $\text{Rep}(\text{Gal}(\bar{F}/F))$ denote the category of representations of $\text{Gal}(\bar{F}/F)$ on k -vector spaces which are projective limits of finite dimensional continuous $\text{Gal}(\bar{F}/F)$ -representations.

Let $\text{Mod}_{ss}(\mathcal{H}^\sharp)$ denote the category of \sharp -supersingular \mathcal{H}^\sharp -modules. Let $\text{Mod}_{ss}(\mathcal{H})$ and $\text{Mod}_{ss}(\mathcal{H}^{\sharp\sharp})$ denote the categories of supersingular \mathcal{H} -modules and supersingular $\mathcal{H}^{\sharp\sharp}$ -modules, respectively.

Let $M \in \text{Mod}_{ss}(\mathcal{H}^{\sharp\sharp})$ with $\dim_k(M) < \infty$. By Proposition 4.3 we have $\Delta(M) \in \text{Mod}^{\text{ad}}(\mathfrak{O})$, thus $\Delta(M)^* \otimes_{k[[t]]} k((t)) \in \text{Mod}^{\text{et}}(k((t)))$ (see Proposition 1.4). Let $V(M)$ be the object in $\text{Rep}(\text{Gal}(\bar{F}/F))$ assigned to $\Delta(M)^* \otimes_{k[[t]]} k((t))$ by Theorem 1.2. Exhausting an object in $\text{Mod}_{ss}(\mathcal{H}^{\sharp\sharp})$ by its finite dimensional subobjects we see that this construction extends to all of $\text{Mod}_{ss}(\mathcal{H}^{\sharp\sharp})$.

Theorem 5.11. *The assignment*

$$\text{Mod}_{ss}(\mathcal{H}^{\sharp\sharp}) \rightarrow \text{Rep}(\text{Gal}(\bar{F}/F)), \quad M \mapsto V(M) \quad (70)$$

is an exact contravariant functor, with $\dim_k(M) = \dim_k(V(M))$ for any M . Also,

$$\begin{aligned} \text{Mod}_{ss}(\mathcal{H}^\sharp) &\rightarrow \text{Rep}(\text{Gal}(\bar{F}/F)), \quad M \mapsto V(M), \\ \text{Mod}_{ss}(\mathcal{H}) &\rightarrow \text{Rep}(\text{Gal}(\bar{F}/F)), \quad M \mapsto V(M) \end{aligned} \quad (71)$$

are exact and fully faithful contravariant functors.

Proof. Exactness follows from exactness of $M \mapsto \Delta(M)$ (Proposition 4.3), exactness of $\Delta \mapsto \Delta^* \otimes_{k[[t]]} k((t))$ (Proposition 1.4) and exactness of the equivalence functor in Theorem 1.2. From Proposition 4.3 we get $\dim_k(M) = \dim_{k((t))}(\Delta(M)^* \otimes_{k[[t]]} k((t)))$, from Theorem 1.2 we get $\dim_{k((t))}(\Delta(M)^* \otimes_{k[[t]]} k((t))) = \dim_k(V(M))$.

To prove faithfulness on $\text{Mod}_{ss}(\mathcal{H}^\sharp)$, suppose that we are given finite dimensional objects M, M' in $\text{Mod}_{ss}(\mathcal{H}^\sharp)$ and a morphism $\mu : V(M') \rightarrow V(M)$ in $\text{Rep}(\text{Gal}(\bar{F}/F))$. By Theorem 1.2, the latter corresponds to a unique morphism of étale (φ, Γ) -modules

$$\mu : \Delta(M')^* \otimes_{k[[t]]} k((t)) \rightarrow \Delta(M)^* \otimes_{k[[t]]} k((t)).$$

By Proposition 1.17 (which applies since Proposition 4.3 tells us $\Delta(M), \Delta(M') \in \text{Mod}^\sharp(\mathfrak{O})$) it is induced by a unique morphism of \mathfrak{O} -modules $\mu : \Delta(M) \rightarrow \Delta(M')$. Clearly μ takes $\Delta(M)[t]$ to $\Delta(M')[t]$, i.e., it takes M to M' . Applying Theorem 5.10 to both M and M' we see that $\mu : M \rightarrow M'$ is \mathcal{H}^\sharp -equivariant. If $M, M' \in \text{Mod}_{ss}(\mathcal{H}^\sharp)$ are not necessarily finite dimensional, the same conclusion is obtained by exhausting

M, M' by its finite dimensional submodules. We deduce the stated full faithfulness on $\text{Mod}_{ss}(\mathcal{H}^\sharp)$. It implies full faithfulness on $\text{Mod}_{ss}(\mathcal{H})$ (see Lemma 2.5). \square

Example. The analogs of Proposition 3.3 and Theorem 5.11 (on the functor in formula (71)) fail for supersingular $\mathcal{H}^{\sharp\sharp}$ -modules. To see this, take $d = 2$, and endow the 6-dimensional k -vector space M with basis $e_0, e_1, e_2, f_0, f_1, f_2$ with the structure of an $\mathcal{H}^{\sharp\sharp}$ -module as follows. T_t for each $t \in \bar{T}$ acts trivially. Put $T_s(f_0) = T_s(e_1) = T_s(e_2) = 0$ and $T_s(e_0) = -e_0, T_s(f_1) = -f_1, T_s(f_2) = -f_2$. Fix $\alpha \in k$ and put $T_\omega(e_0) = e_1, T_\omega(e_1) = e_2, T_\omega(e_2) = e_0, T_\omega(f_0) = f_1 + \alpha e_1, T_\omega(f_1) = f_2 - \alpha e_2, T_\omega(f_2) = f_0$. This is even an \mathcal{H}^\sharp -module if and only if $\alpha = 0$, if and only if it is decomposable (as an $\mathcal{H}^{\sharp\sharp}$ -module). The corresponding \mathfrak{O} -module $\Delta(M)$ is defined by the relations $\varphi e_0 = -e_1, \varphi e_1 = -e_2, t^{q-1} \varphi e_2 = -e_0, \varphi f_2 = -f_0, t^{q-1} \varphi(f_0 - \alpha e_0) = f_1, t^{q-1} \varphi(f_1 + \alpha e_1) = f_2$. But this \mathfrak{O} -module is in fact independent of α , since $t^{q-1} \varphi e_1 = t^{q-1} \varphi e_0 = 0$. Thus, an $\mathcal{H}^{\sharp\sharp}$ -analog of Theorem 5.11 fails. To see that an $\mathcal{H}^{\sharp\sharp}$ -analog of Proposition 3.3 fails take M_0 to be the k -subvector space of M spanned by e_0, e_1, e_2 ; it is stable under $\mathcal{H}^{\sharp\sharp}$. The action of $\mathcal{H}^{\sharp\sharp}$ on M_0 and on M/M_0 does not depend on α . The actions of $T_\omega^{d+1} = T_\omega^3$, of $T_{e^*(\Gamma)}$ and of $T_s T_\omega$ do not depend on α . We have $(T_s T_\omega)^{-1}(M_0) = M_0 + kf_2$ and hence the restriction of T_ω to $(T_s T_\omega)^{-1}(M_0)$ does not depend on α . We have $M = \sum_\epsilon M^\epsilon$ with $M^\epsilon = 0$ whenever $\epsilon_1 \neq 0$. Thus, an $\mathcal{H}^{\sharp\sharp}$ -analog of Proposition 3.3 would predict that also the action of T_ω (even of \mathcal{H}^\sharp) is independent of α , which however is apparently not the case.

5E. The essential image.

Definition. Let $\text{Hom}(\Gamma, k^\times)^\Phi$ denote the group of $(d+1)$ -tuples $\alpha = (\alpha_0, \dots, \alpha_d)$ of characters $\alpha_j : \Gamma \rightarrow k^\times$. Let \mathfrak{S}_{d+1} act on $\text{Hom}(\Gamma, k^\times)^\Phi$ by the formulae

$$\begin{aligned} (\omega \cdot \alpha)_0 &= \alpha_d \quad \text{and} \quad (\omega \cdot \alpha)_i = \alpha_{i-1} \quad \text{for } 1 \leq i \leq d, \\ (s \cdot \alpha)_0 &= \alpha_1, \quad (s \cdot \alpha)_1 = \alpha_0 \quad \text{and} \quad (s \cdot \alpha)_i = \alpha_i \quad \text{for } 2 \leq i \leq d. \end{aligned}$$

Recall the action of \mathfrak{S}_{d+1} on $[0, q-2]^\Phi$. Combining both (diagonally), we obtain an action of \mathfrak{S}_{d+1} on $\text{Hom}(\Gamma, k^\times)^\Phi \times [0, q-2]^\Phi$.

In Lemma 1.19 we attached to each standard cyclic étale (φ, Γ) -module \mathbf{D} of dimension $d+1$ an ordered tuple $((\alpha_0, m_0), \dots, (\alpha_d, m_d))$ (with integers $m_j \in [1-q, 0]$ and characters $\alpha_j : \Gamma \rightarrow k^\times$), unique up to a cyclic permutation. Sending each m_j to the representative in $[0, q-2]$ of its class in $\mathbb{Z}/(q-1)$, the tuple (m_0, \dots, m_d) gives rise to an element in $[0, q-2]^\Phi$. On the other hand, the tuple $(\alpha_0, \dots, \alpha_d)$ constitutes an element in $\text{Hom}(\Gamma, k^\times)^\Phi$. Taken together we thus attach to \mathbf{D} an element in $\text{Hom}(\Gamma, k^\times)^\Phi \times [0, q-2]^\Phi$, unique up to cyclic permutation. Equivalently, we attach to \mathbf{D} an orbit in $\text{Hom}(\Gamma, k^\times)^\Phi \times [0, q-2]^\Phi$ under the action of the subgroup of \mathfrak{S}_{d+1} generated by ω .

Now let $\mathbf{D}'_1, \mathbf{D}'_2$ be irreducible étale (φ, Γ) -modules over $k((t))$. We say that $\mathbf{D}'_1, \mathbf{D}'_2$ are strongly \mathfrak{S}_{d+1} -linked if they are subquotients of $(d+1)$ -dimensional standard cyclic étale (φ, Γ) -modules $\mathbf{D}_1, \mathbf{D}_2$ respectively, and if $\mathbf{D}_1, \mathbf{D}_2$ give rise to the same \mathfrak{S}_{d+1} -orbit in $\text{Hom}(\Gamma, k^\times)^\Phi \times [0, q-2]^\Phi$. We say that $\mathbf{D}'_1, \mathbf{D}'_2$ are \mathfrak{S}_{d+1} -linked if they are subquotients of $(d+1)$ -dimensional standard cyclic étale

(φ, Γ) -modules $\mathbf{D}_1, \mathbf{D}_2$ respectively, and if $\mathbf{D}_1, \mathbf{D}_2$ give rise to the same \mathfrak{S}_{d+1} -orbit in $[0, q-2]^\Phi$ (or equivalently, if the assigned tuples (up to cyclic permutation) in $[0, q-2]^\Phi$ coincide as *unordered* tuples (with multiplicities)).

Remark. (a) Let \mathbf{D} denote the étale (φ, Γ) -module over $k((t))$ corresponding to $V(M)$, for a finite dimensional supersingular $\mathcal{H}^{\sharp\sharp}$ -module M . Our constructions show $M = \text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k)[t]$ (where \mathbf{D}^\sharp is given the t -adic topology). Moreover:

- (i) Consider the natural map of $k[[t]][\varphi]$ -modules

$$\kappa_{\mathbf{D}} : k[[t]][\varphi] \otimes_{k[[t]]} M \rightarrow \text{Hom}_k^{\text{cont}}(\mathbf{D}^\sharp, k).$$

As a $k[[t]][\varphi]$ -module, $\ker(\kappa_{\mathbf{D}})$ is generated by $\ker(\kappa_{\mathbf{D}}) \cap (k \otimes M + k[[t]]\varphi \otimes M)$.

- (ii) Each irreducible subquotient of \mathbf{D} is a subquotient of a $(d+1)$ -dimensional standard cyclic étale (φ, Γ) -module; more precisely:

- (ii)(1) If \mathbf{D} (or equivalently, M) is indecomposable, then any two irreducible subquotients of \mathbf{D} are \mathfrak{S}_{d+1} -linked.
- (ii)(2) If M is even a supersingular \mathcal{H} -module, and if \mathbf{D} (or equivalently, M) is indecomposable, then any two irreducible subquotients of \mathbf{D} are strongly \mathfrak{S}_{d+1} -linked.
- (ii)(3) If M is even a supersingular \mathcal{H}^\sharp -module, then each irreducible subquotient of \mathbf{D} is a subquotient of a $(d+1)$ -dimensional standard cyclic étale (φ, Γ) -module with parameters $m_j \in \{1-q, 0\}$ and $\alpha_j = 1$ for all j .

- (iii) For any (φ, Γ) -submodule \mathbf{D}_0 of \mathbf{D} the ψ -operator on $\mathbf{D}_0 \cap \mathbf{D}^\sharp$ is surjective.

(b) Does property (i) mean (at least if property (iii) is assumed) that \mathbf{D} is the reduction of a crystalline p -adic $\text{Gal}(\bar{F}/F)$ -representation with Hodge–Tate weights in $[-1, 0]$?

(c) Property (iii) means that the functor $\mathbf{D}_0 \mapsto \mathbf{D}_0^\sharp$ is exact on the category of subquotients \mathbf{D}_0 of \mathbf{D} .

(d) It should not be too hard to show that properties (i), (ii)(1) and (iii) together in fact *characterize* the essential image of the functor (70).

(e) On the other hand, properties (i), (ii)(2) and (iii) together do *not* characterize the essential image of the functor (71). To see this for $d = 1$ consider the following étale (φ, Γ) -module \mathbf{D} (which satisfies (i), (ii)(2), (iii)). It is given by a k -basis $e_0, e_1, f_0, f_1, g_0, g_1$ of $(\mathbf{D}^\sharp)^*[t]$ and the following relations:

$$\varphi e_1 = e_0, \quad \varphi f_1 = f_0, \quad \varphi g_1 = g_0, \quad t^{q-1} \varphi e_0 = e_1, \quad t^{q-1} \varphi f_0 = f_1 + e_1, \quad t^{q-1} \varphi g_0 = g_1 + f_0.$$

Another object not in the essential image is defined by the set of relations

$$\varphi e_1 = e_0, \quad \varphi f_1 = f_0, \quad \varphi g_1 = g_0, \quad t^{q-1} \varphi e_0 = e_1, \quad t^{q-1} \varphi f_0 = f_1 + e_0, \quad t^{q-1} \varphi g_0 = g_1 + f_1.$$

6. From G -representations to \mathcal{H} -modules

6A. Supersingular cohomology. Put $G = \mathrm{GL}_{d+1}(F)$, let I_0 be a pro- p -Iwahori subgroup in G , and fix an isomorphism between \mathcal{H} and the pro- p -Iwahori Hecke algebra $k[I_0 \backslash G / I_0]$ corresponding to $I_0 \subset G$. For a smooth G -representation Y (over k) the subspace Y^{I_0} of I_0 -invariants then receives a natural action by \mathcal{H} . Let us denote by $H_{ss}^0(I_0, Y)$ the maximal supersingular \mathcal{H} -submodule of Y^{I_0} . It is clear that this defines a left exact functor

$$\mathrm{Mod}(G) \rightarrow \mathrm{Mod}_{ss}(\mathcal{H}), \quad Y \mapsto H_{ss}^0(I_0, Y)$$

where $\mathrm{Mod}(G)$ denotes the category of smooth G -representations. The category $\mathrm{Mod}(G)$ is a Grothendieck category [Schneider 2015, Lemma 1] and has enough injective objects [Vignéras 1996, I.5.9]. Let $D^+(G)$ denote the derived category of complexes of smooth G -representations vanishing in negative degrees, let $D_{ss}^+(\mathcal{H})$ denote the derived category of complexes of supersingular \mathcal{H} -modules vanishing in negative degrees. The above functor gives rise to a right derived functor

$$R_{ss}(I_0, .) : D^+(G) \rightarrow D_{ss}^+(\mathcal{H}). \quad (72)$$

Let $D^+(\mathrm{Gal}(\bar{F}/F))$ denote the derived category of complexes in $\mathrm{Rep}(\mathrm{Gal}(\bar{F}/F))$ vanishing in negative degrees. Since the functor V is exact, it induces a functor

$$V : D_{ss}^+(\mathcal{H}) \rightarrow D^+(\mathrm{Gal}(\bar{F}/F)).$$

We may compose them with $R_{ss}(I_0, .)$ to obtain a functor

$$V \circ R_{ss}(I_0, .) : D^+(G) \rightarrow D^+(\mathrm{Gal}(\bar{F}/F)).$$

Remark. The functor $H_{ss}^0(I_0, .)$ is the composite of the left exact functor $\mathrm{Mod}(G) \rightarrow \mathrm{Mod}(\mathcal{H})$, $Y \mapsto Y^{I_0}$ (taking I_0 -invariants) and the left exact functor $\mathrm{Mod}(\mathcal{H}) \rightarrow \mathrm{Mod}_{ss}(\mathcal{H})$, $M \mapsto M_{ss}$ which takes an \mathcal{H} -module to its maximal supersingular \mathcal{H} -submodule. Also $\mathrm{Mod}(\mathcal{H})$ is a Grothendieck category with enough injective objects. Writing $R(I_0, .)$ and $R_{ss}(.)$ for the respective right derived functors, we have a morphism $R_{ss}(I_0, .) \rightarrow R_{ss}(.) \circ R(I_0, .)$.

Remark. Of course, we expect the functor $V \circ R_{ss}(I_0, .)$ to be meaningful only when restricted to (complexes of) supersingular G -representations. The reason is the following theorem of Ollivier and Vignéras [2018]: A smooth admissible irreducible G -representation Y over an algebraic closure \bar{k} of k is supersingular if and only if Y^{I_0} is a supersingular $\mathcal{H} \otimes_k \bar{k}$ -module, if and only if Y^{I_0} admits a supersingular subquotient.

It is known that, beyond the case where $G = \mathrm{GL}_2(\mathbb{Q}_p)$, a smooth admissible irreducible supersingular G -representation Y over k is not uniquely determined by the \mathcal{H} -module Y^{I_0} . Is it perhaps uniquely determined by the derived object $R_{ss}(I_0, Y) \in D_{ss}^+(\mathcal{H})$? It would then also be uniquely determined by the derived object $V(R_{ss}(I_0, Y)) \in D^+(\mathrm{Gal}(\bar{F}/F))$.

Remark. For the universal module $Y = \text{ind}_{I_0}^G k$ we have $H_{ss}^0(I_0, Y) = 0$ since $\mathcal{H} = (\text{ind}_{I_0}^G k)^{I_0}$ does not contain nonzero finite dimensional \mathcal{H} -submodules (let alone supersingular ones).

6B. An exact functor from G -representations to \mathcal{H} -modules. We fix a $(d+1)$ -st root of unity $\xi \in k^\times$ with $\sum_{j=0}^d \xi^j = 0$.

For an \mathcal{H} -module M and $j \in \mathbb{Z}$ let M^{ξ^j} be the \mathcal{H} -module which coincides with M as a module over the k -subalgebra $k[T_s, T_t]_{t \in \bar{T}}$, but with $T_\omega|_{M^{\xi^j}} = \xi^j T_\omega|_M$.

Let $\delta : M_0 \rightarrow M_1$ be a morphism of \mathcal{H} -modules. For $(x_0, x_1) \in M_0 \oplus M_1$ put

$$\begin{aligned} T_\omega((x_0, x_1)) &= (T_\omega(x_0), T_\omega(\delta(x_0)) + \xi T_\omega(x_1)), \\ T_s((x_0, x_1)) &= (T_s(x_0), T_s(x_1)), \\ T_t((x_0, x_1)) &= (T_t(x_0), T_t(x_1)) \quad \text{for } t \in \bar{T}. \end{aligned}$$

Lemma 6.1. *These formulae define an \mathcal{H} -module structure on $M_0 \oplus M_1$; we denote this new \mathcal{H} -module by $M_0 \oplus^\delta M_1$. We have an exact sequence of \mathcal{H} -modules*

$$0 \rightarrow M_1^\xi \rightarrow M_0 \oplus^\delta M_1 \rightarrow M_0 \rightarrow 0. \quad (73)$$

The morphism $\delta : M_0 \rightarrow M_1$ can be recovered from the exact sequence (73).

If there is some $\lambda \in k^\times$ with $T_\omega^{d+1} = \lambda$ on M_0 and on M_1 , then also $T_\omega^{d+1} = \lambda$ on $M_0 \oplus^\delta M_1$,

Proof. By induction on i one shows

$$T_\omega^i((x_0, x_1)) = (T_\omega^i(x_0), \xi^i T_\omega^i(x_1) + \sum_{j=0}^{i-1} \xi^j T_\omega^i(\delta(x_0)))$$

for $i > 0$, and hence $T_\omega^{d+1}((x_0, x_1)) = (T_\omega^{d+1}(x_0), T_\omega^{d+1}(x_1))$. From here, all the required relations are straightforwardly verified, showing that indeed we have defined an \mathcal{H} -module.

Obviously, from the exact sequence (73) both M_0 and M_1 can be recovered. That also δ can be recovered follows from the following more general consideration. Suppose that we are given $\delta : M_0 \rightarrow M_1$ and $\epsilon : N_0 \rightarrow N_1$ and a morphism of \mathcal{H} -modules $f : M_0 \oplus^\delta M_1 \rightarrow N_0 \oplus^\epsilon N_1$ with $f(M_1^\xi) \subset N_1^\xi$. Then there are \mathcal{H} -module homomorphisms $f_0 : M_0 \rightarrow N_0$, $f_1 : M_1^\xi \rightarrow N_1^\xi$ and $\tilde{f} : M_0 \rightarrow N_1^\xi$ with $f((x_0, x_1)) = (f_0(x_0), f_1(x_1) + \tilde{f}(x_0))$. For $x_0 \in M_0$ we compute

$$\begin{aligned} f(T_\omega(x_0, 0)) &= f(T_\omega(x_0), T_\omega(\delta(x_0))) = (T_\omega(f_0(x_0)), T_\omega(f_1(\delta(x_0))) + \xi T_\omega(\tilde{f}(x_0))), \\ T_\omega(f(x_0, 0)) &= T_\omega(f_0(x_0), \tilde{f}(x_0)) = (T_\omega(f_0(x_0)), T_\omega(\epsilon(f_0(x_0))) + \xi T_\omega(\tilde{f}(x_0))). \end{aligned}$$

As $f(T_\omega(x_0, 0)) = T_\omega(f(x_0, 0))$ we deduce $T_\omega(\epsilon(f_0(x_0))) = T_\omega(f_1(\delta(x_0)))$, and since T_ω is an isomorphism even $\epsilon(f_0(x_0)) = f_1(\delta(x_0))$. \square

Let

$$(M_\bullet, \delta_\bullet) = [\dots \xrightarrow{\delta_{-2}} M_{-1} \xrightarrow{\delta_{-1}} M_0 \xrightarrow{\delta_0} M_1 \xrightarrow{\delta_1} M_2 \xrightarrow{\delta_2} \dots]$$

be a complex of \mathcal{H} -modules.

Lemma 6.2. (a) *There is a unique \mathcal{H} -module $\bigoplus_{j \in \mathbb{Z}}^{\delta_{\bullet}} M_j$ with the following properties:*

- *As a k -vector space, $\bigoplus_{j \in \mathbb{Z}}^{\delta_{\bullet}} M_j = \bigoplus_{j \in \mathbb{Z}} M_j$.*
- *For any j we have $\tau(M_j) \subset M_j + M_{j+1}$ for each $\tau \in \mathcal{H}$; in particular, the subspace $M_{\geq j} = \bigoplus_{j' \geq j} M_{j'}$ is an \mathcal{H} -submodule.*
- *The \mathcal{H} -module $M_{\geq j}/M_{\geq j+2}$ is isomorphic with $M_j^{\xi^j} \oplus^{\delta_j} M_{j+1}^{\xi^j}$ as defined in Lemma 6.1.*

(b) *If there is some $\lambda \in k^{\times}$ with $T_{\omega}^{d+1} = \lambda$ on each M_j , then $T_{\omega}^{d+1} = \lambda$ on $\bigoplus_{j \in \mathbb{Z}}^{\delta_{\bullet}} M_j$.*

(c) *The assignment $(M_{\bullet}, \delta_{\bullet}) \mapsto (\bigoplus_{j \in \mathbb{Z}}^{\delta_{\bullet}} M_j, (M_{\geq j})_{j \in \mathbb{Z}})$ is an exact and faithful functor from the category of complexes of \mathcal{H} -modules to the category of filtered \mathcal{H} -modules. The isomorphism class of the complex $(M_{\bullet}, \delta_{\bullet})$ can be recovered from the isomorphism class of the filtered \mathcal{H} -module $(\bigoplus_{j \in \mathbb{Z}}^{\delta_{\bullet}} M_j, (M_{\geq j})_{j \in \mathbb{Z}})$.*

Proof. This is clear from Lemma 6.1. \square

Definition. (a) For a smooth G -representation Y over k and $i \geq 0$ let us denote by $H_{ss}^i(I_0, Y)$ the i -th cohomology group of $R_{ss}(I_0, Y)$, see formula (72).

(b) We say that a smooth G -representation Y over k is *exact* if for each $i \geq 0$ the functor $Y' \mapsto H_{ss}^i(I_0, Y')$ is exact on the category of G -subquotients Y' of Y .

(c) An exhaustive and separated decreasing filtration $(Y^j)_{j \in \mathbb{Z}}$ of a smooth G -representation Y over k is *exact* if Y^j/Y^{j+1} is exact for each j .

Example. A semisimple smooth G -representation is exact.

Let \mathfrak{R}_G denote the following category: objects are smooth G -representations with an exact filtration, morphisms are G -equivariant maps respecting the filtrations (i.e., $f : Y \rightarrow W$ with $f(Y^i) \subset W^i$ for all i). We denote objects $(Y, (Y^i)_{i \in \mathbb{Z}})$ in \mathfrak{R}_G simply by Y^{\bullet} .

Let $\mathfrak{E}(\mathcal{H})$ denote the category of E_1 -spectral sequences in the category of \mathcal{H} -modules.

For $Y^{\bullet} \in \mathfrak{R}_G$ we have the spectral sequence

$$E(Y^{\bullet}) = [E_1^{m,n}(Y^{\bullet}) = H_{ss}^{m+n}(I_0, Y^m/Y^{m+1}) \Rightarrow H_{ss}^{m+n}(I_0, Y)].$$

A morphism $f : Y^{\bullet} \rightarrow W^{\bullet}$ in \mathfrak{R}_G induces morphisms $H_{ss}^m(I_0, Y^i/Y^{i+1}) \rightarrow H_{ss}^m(I_0, W^i/W^{i+1})$ for any m and i , and these induce a morphism of spectral sequences $E(Y^{\bullet}) \rightarrow E(W^{\bullet})$. We thus obtain a functor

$$\mathfrak{R}_G \rightarrow \mathfrak{E}(\mathcal{H}), \quad Y^{\bullet} \mapsto E(Y^{\bullet}).$$

For $r \geq 1$ let \mathcal{Y}_r be the set of equivalence classes of pairs of integers (m, n) , where (m, n) is declared to be equivalent with (m', n') if and only if there is some $j \in \mathbb{Z}$ with $(m, n) = (m' + jr, n' - j(r-1))$. For $y \in \mathcal{Y}_r$ let $E_r^y(Y^{\bullet})$ be the complex of \mathcal{H} -modules whose terms are the $E_r^{m,n}(Y^{\bullet})$ with $(m, n) \in y$, and whose differentials $d_r : E_r^{m,n}(Y^{\bullet}) \rightarrow E_r^{m+r,n-r+1}(Y^{\bullet})$ are given by the spectral sequence. We apply the functor of Lemma 6.2 to $E_r^y(Y^{\bullet})$ to obtain a (filtered) supersingular \mathcal{H} -module $\mathbf{E}_r^y(Y^{\bullet})$.

For a morphism $f : Y^{\bullet} \rightarrow W^{\bullet}$ in \mathfrak{R}_G we have induced \mathcal{H} -linear maps $f_r : \bigoplus_{y \in \mathcal{Y}_r} \mathbf{E}_r^y(Y^{\bullet}) \rightarrow \bigoplus_{y \in \mathcal{Y}_r} \mathbf{E}_r^y(W^{\bullet})$. Notice however that, in general, for a given $y \in \mathcal{Y}_r$ there is no $y' \in \mathcal{Y}_r$ such that $f_r(\mathbf{E}_r^y(Y^{\bullet})) \subset \mathbf{E}_r^{y'}(W^{\bullet})$, even if $r = 1$.

Lemma 6.3. *Let $Y^\bullet \rightarrow W^\bullet \rightarrow X^\bullet$ be a complex in \mathfrak{R}_G such that for each i the induced sequence $0 \rightarrow Y^i/Y^{i+1} \rightarrow W^i/W^{i+1} \rightarrow X^i/X^{i+1} \rightarrow 0$ is exact. We then have an exact sequence of supersingular \mathcal{H} -modules*

$$0 \rightarrow \bigoplus_{y \in \mathcal{Y}_1} E_1^y(Y^\bullet) \rightarrow \bigoplus_{y \in \mathcal{Y}_1} E_1^y(W^\bullet) \rightarrow \bigoplus_{y \in \mathcal{Y}_1} E_1^y(X^\bullet) \rightarrow 0.$$

Proof. This follows from the constructions. \square

Remark. The analog of Lemma 6.3 is false for the maps f_r for $r > 1$.

Remark. For a smooth G -representation Y endowed with an exact filtration, we may apply the functor V of Section 5D to the supersingular \mathcal{H} -module $E_r^y(Y^\bullet)$ (any r). In this way, we assign a $\text{Gal}(\bar{F}/F)$ -representation to Y . We propose this construction as a nonderived alternative to that of Section 6A. Of course, again it will be meaningful only on supersingular G -representations.

We expect that for $G = \text{GL}_2(\mathbb{Q}_p)$, this construction, with $r = 1$, essentially recovers the restriction of Colmez's functor to all supersingular G -representations.¹¹

Acknowledgements

I thank Laurent Berger, Peter Schneider and Gergely Zábrádi for helpful discussions related to this work. I thank Marie-France Vignéras for a very close reading of the text and for detailed suggestions for improvement. I thank the anonymous referees for their careful reading and helpful recommendations. I thank Rachel Ollivier for the invitation to UBC Vancouver in the spring of 2017; some progress on this work was obtained during that visit.

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¹¹i.e., not only to those generated by their I_0 -invariants

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Communicated by Marie-France Vignéras

Received 2018-12-10 Revised 2019-05-06 Accepted 2019-09-01

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The subscription price for 2020 is US \$415/year for the electronic version, and \$620/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

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Algebra & Number Theory

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