Algebra & Number Theory

Volume 14
2020
No. 1

Birationally superrigid Fano 3-folds of codimension 4

Takuzo Okada

msp

ALGEBRA AND NUMBER THEORY 14:1 (2020)
dx.doi.org/10.2140/ant.2020.14.191

Birationally superrigid Fano 3-folds of codimension 4

Takuzo Okada

We determine birational superrigidity for a quasismooth prime Fano 3-fold of codimension 4 with no projection centers. In particular we prove birational superrigidity for Fano 3-folds of codimension 4 with no projection centers which were recently constructed by Coughlan and Ducat. We also pose some questions and a conjecture regarding the classification of birationally superrigid Fano 3-folds.

1. Introduction

A *prime Fano* 3*-fold* is a normal projective $\mathbb{Q}$-factorial 3-fold X with only terminal singularities such that $-K_X$ is ample and the class group $\mathrm{Cl}(X) \cong \mathbb{Z}$ is generated by $-K_X$. To such X there corresponds the anticanonical graded ring

$$R(X, -K_X) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, -mK_X),$$

and by choosing minimal generators we can embed X into a weighted projective space. By the *codimension* of X we mean the codimension of X in the weighted projective space. Based on analysis by Altınok, Brown, Iano-Fletcher, Kasprzyk, Prokhorov, Reid, and others (see for example [Altınok et al. 2002]) there is a database [Brown and Kasprzyk 2009] of numerical data (such as Hilbert series) coming from graded rings that can be the anticanonical graded ring of a prime Fano 3-fold. Currently it is not a classification, but it serves as a list, meaning that the anticanonical graded ring of a prime Fano 3-fold appears in the database.

The database contains a huge number of candidates, which suggests difficulty in the biregular classification of Fano 3-folds. The aim of this paper is to shed light on the classification of birationally superrigid Fano 3-folds. Here, a Fano 3-fold of Picard number 1 is said to be *birationally superrigid* if any birational map to a Mori fiber space is biregular. We remark that in [Ahmadinezhad and Okada 2018] a possible approach to achieving birational classification of Fano 3-folds is suggested by introducing notion of *solid Fano* 3*-folds*, which are Fano 3-folds not birational to either a conic bundles or a del Pezzo fibration.

Up to codimension 3, we have satisfactory results on the classification of quasismooth prime Fano 3-folds: the classification is complete in codimensions 1 and 2 [Iano-Fletcher 2000; Chen et al. 2011; Altınok 1998] and in codimension 3 the existence is known for all 70 numerical data in the database. Moreover birational superrigidity of quasismooth prime Fano 3-folds of codimension at most 3 has been well studied (see [Iskovskikh and Manin 1971; Corti et al. 2000; Cheltsov and Park 2017; Okada 2014a;

MSC2010: primary 14J45; secondary 14E07, 14E08.
Keywords: Fano variety, Birational rigidity.

Ahmadinezhad and Zucconi 2016; Ahmadinezhad and Okada 2018] and [Okada 2014b, 2018] for solid cases in codimension 2).

For quasismooth prime Fano 3-folds of codimension 4, there are 145 candidates of numerical data in [Brown and Kasprzyk 2009]. In [Brown et al. 2012], existence for 116 data is proved, where the construction is given by birationally modifying a known variety. This process is called unprojection and, as a consequence, a constructed Fano 3-fold corresponding to each of the 116 families admits a Sarkisov link to a Mori fiber space, hence it is not birationally superrigid. The 116 families of Fano 3-folds are characterized as those that possess a singular point which is so called a type I projection center (see [Brown et al. 2012] for details). There are other types of projection centers (such as types $\mathrm{II}_1, \dots, \mathrm{II}_7$, IV according to the database [Brown and Kasprzyk 2009]). Through the known results in codimensions 1, 2 and 3, we can expect that the existence of a projection center violates birational superrigidity. Therefore it is natural to consider prime Fano 3-folds without projection centers for the classification of birational superrigid Fano 3-folds (see also the discussion in Section 5).

According to the database [Brown and Kasprzyk 2009], there are 5 candidates of quasismooth prime Fano 3-folds of codimension 4 with no projection centers. Those are identified by database numbers #25, #166, #282, #308 and #29374. Among them, #29374 corresponds to smooth prime Fano 3-folds of degree 10 embedded in $\mathbb{P}^7$, and it is proved in [Debarre et al. 2012] that they are not birationally superrigid (not even birationally rigid, a weaker notion than superrigidity). Recently Coughlan and Ducat [2018] constructed many prime Fano 3-folds including those corresponding to #25 and #282 and we sometimes refer to these varieties as *cluster Fano* 3*-folds*. There are two constructions, $\mathsf{G}_2^{(4)}$ and C_2 formats (see [Coughlan and Ducat 2018, Section 5.6] for details and see page 205 for concrete descriptions) for #282 and they are likely to sit in different components of the Hilbert scheme.

Theorem 1.1. *Let X be a quasismooth prime Fano* 3*-fold of codimension* 4 *and of numerical type* #282 *which is constructed in either $\mathsf{G}_2^{(4)}$ format or C_2 format. If X is constructed in C_2 format*, *then we assume that X is general. Then X is birationally superrigid.*

For the remaining three candidates #25, #166 and #308, we can prove birational superrigidity in a stronger manner; we are able to prove birational superrigidity for these 3 candidates by utilizing only numerical data. Here, by numerical data for a candidate Fano 3-fold X, we mean the weights of the weighted projective space, degrees of the defining equations, the anticanonical degree $(-K_X)^3$ and the basket of singularities of X (see Section 3). Note that we do not know the existence of Fano 3-folds for #166 and #308.

Theorem 1.2. *Let X be a well-formed quasismooth prime Fano* 3*-fold of codimension* 4 *and of numerical type* #25, #166 *or* #308. *Then X is birationally superrigid.*

2. Birational superrigidity

Basic properties. Throughout this subsection we assume that X is a Fano 3-fold of Picard number 1, that is, X is a normal projective $\mathbb{Q}$-factorial 3-fold such that X has only terminal singularities, $-K_X$ is ample and $\operatorname{rank}\operatorname{Pic}(X) = 1$.

Definition 2.1. We say that X is *birationally superrigid* if any birational map $\sigma: X \dashrightarrow Y$ to a Mori fiber space $Y \to T$ is biregular.

By an *extremal divisorial extraction* $\varphi\colon (E \subset Y) \to (\Gamma \subset X)$, we mean an extremal divisorial contraction $\varphi\colon Y \to X$ from a normal projective $\mathbb{Q}$-factorial variety Y with only terminal singularities such that E is the φ-exceptional divisor and $\Gamma = \varphi(E)$.

Definition 2.2. Let $\mathcal{H} \sim_{\mathbb{Q}} -nK_X$ be a movable linear system, where n is a positive integer. A *maximal singularity* of $\mathcal{H}$ is an extremal extraction $\varphi\colon (E \subset Y) \to (\Gamma \subset X)$ such that

$$c(X, \mathcal{H}) = \frac{a_E(K_X)}{m_E(\mathcal{H})} < \frac{1}{n},$$

where

- $c(X, \mathcal{H}) := \max\{\lambda \mid K_X + \lambda\mathcal{H} \text{ is canonical}\}$ is the *canonical threshold* of $(X, \mathcal{H})$,
- $a_E(K_X)$ is the discrepancy of K_X along E, and
- $m_E(\mathcal{H})$ is the multiplicity along E of the proper transform of $\mathcal{H}$.

We say that an extremal divisorial extraction is a *maximal singularity* if there exists a movable linear system $\mathcal{H}$ such that the extraction is a maximal singularity of $\mathcal{H}$. A subvariety $\Gamma \subset X$ is called a *maximal center* if there is a maximal singularity $Y \to X$ whose center is Γ.

The following is the fundamental theorem in the study of birational superrigidity, which emerged in [Iskovskikh and Manin 1971] and has been simplified and generalized in [Pukhlikov 1998; Corti 1995].

Theorem 2.3 [Corti 1995, Theorem 4.10 and Proposition 2.10]. *If X admits no maximal center, then X is birationally superrigid.*

For the proof of birational superrigidity of a given Fano 3-fold X of Picard number 1 we need to exclude each subvariety of X as a maximal center. In the next subsection we will explain several methods of exclusion under a relatively concrete setting. Here we discuss methods of excluding terminal quotient singular points in a general setting.

For a terminal quotient singular point $\mathsf{p} \in X$ of type $\frac{1}{r}(1, a, r-a)$, where r is coprime to a and $0 < a < r$, there is a unique extremal divisorial extraction $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$, which is the weighted blowup with weight $\frac{1}{r}(1, a, r-a)$, and we call it the *Kawamata blowup* (see [Kawamata 1996] for details). The integer $r > 1$ is called the *index* of $\mathsf{p} \in X$. For the Kawamata blowup $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$, we have $K_Y = \varphi^* K_X + \frac{1}{r} E$ and

$$(E^3) = \frac{r^2}{a(r-a)}.$$

For a divisor D on X, the *order* of D along E, denoted by $\operatorname{ord}_E(D)$, is defined to be the coefficient of E in $\varphi^* D$.

We first explain the most basic method.

Lemma 2.4 [Corti et al. 2000, Lemma 5.2.1]. *Let* $\mathsf{p} \in X$ *a terminal quotient singular point and* $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ *the Kawamata blowup. If* $(-K_Y)^2 \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$, *then* p *is not a maximal center.*

For the application of the above lemma, we need to find a nef divisor on Y. The following result, which is a slight generalization of [Okada 2018, Lemma 6.6], is useful.

Lemma 2.5. *Let* $\mathsf{p} \in X$ *be a terminal quotient singular point and* $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ *the Kawamata blowup. Assume that there are effective Weil divisors* $D_1, \dots, D_k$ *such that the intersection* $\operatorname{Supp}(D_1) \cap \cdots \cap \operatorname{Supp}(D_k)$ *does not contain a curve through* p. *We set*

$$e := \min\left\{ \frac{\operatorname{ord}_E(D_i)}{n_i} \mid 1 \le i \le k \right\},$$

where n_i *is the positive rational number such that* $D_i \sim_{\mathbb{Q}} -n_i K_X$. *Then* $-\varphi^* K_X - \lambda E$ *is a nef divisor for* $0 \le \lambda \le e$.

Proof. We may assume $e > 0$, that is, $\operatorname{Supp}(D_i)$ passes through p for any i. For an effective divisor $D \sim_{\mathbb{Q}} -nK_X$, we call $\operatorname{ord}_E(D)/n$ the vanishing ratio of D along E. For $1 \le i \le k$, we choose a component of D_i, denoted D'_i, which has maximal vanishing ratio along E among the components of D_i. Clearly $D'_1 \cap \cdots \cap D'_k$ does not contain a curve through p and we have

$$e' := \min\left\{ \frac{\operatorname{ord}_E(D'_i)}{n'_i} \mid 1 \le i \le k \right\} \ge e,$$

where $n'_i \in \mathbb{Q}$ is such that $D'_i \sim_{\mathbb{Q}} -n'_i K_X$. Since $D'_1, \dots, D'_k$ are prime divisors, we can apply [Okada 2018, Lemma 6.6] and conclude that $-\varphi^* K_X - e' E$ is nef. Then so is $-\varphi^* K_X - \lambda E$ for any $0 \le \lambda \le e'$ since $-\varphi^* K_X$ is nef, and the proof is completed. □

We have another method of exclusion which can sometimes be effective when Lemma 2.4 is not applicable.

Lemma 2.6. *Let* $\mathsf{p} \in X$ *be a terminal quotient singular point and* $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ *the Kawamata blowup. Suppose that there exists an effective divisor* S *on* X *passing through* p *and a linear system* $\mathcal{L}$ *of divisors on* X *passing through* p *with the following properties*:

(1) $\operatorname{Supp}(S) \cap \operatorname{Bs} \mathcal{L}$ *does not contain a curve passing through* p.

(2) *For a general member* $L \in \mathcal{L}$, *we have*

$$(-K_Y \cdot \tilde{S} \cdot \tilde{L}) \le 0,$$

where $\tilde{S}$, $\tilde{L}$ *are the proper transforms of* S, L *on* Y, *respectively.*

Then p *is not a maximal center.*

Proof. According to [Okada 2018, Lemma 2.20], it suffices to show that there exist infinitely many distinct curves on Y which intersect $-K_Y$ nonpositively and E positively. For a curve or a divisor Δ on X, we denote by $\tilde{\Delta}$ its proper transform on Y.

We write $L \sim -nK_X$. Write $S = \sum m_i S_i + T$, where $m_i > 0$, S_i is a prime divisor and T is an effective divisor which does not pass through p. We have

$$(-K_Y \cdot \tilde{T} \cdot \tilde{L}) = nl(-K_X)^3 \geq 0,$$

where $T \sim -lK_X$ for some $l \geq 0$. Since

$$0 \geq (-K_Y \cdot \tilde{S} \cdot \tilde{L}) = \sum m_i(-K_Y \cdot \tilde{S}_i \cdot \tilde{L}) + (-K_Y \cdot \tilde{T} \cdot \tilde{L}),$$

there is a component S_i for which $(-K_Y \cdot \tilde{S}_i \cdot \tilde{L}) \leq 0$. Since $\mathsf{p} \in S_i \cap \operatorname{Bs} \mathcal{L} \subset \operatorname{Supp}(S) \cap \operatorname{Bs} \mathcal{L}$, we may assume that S is a prime divisor by replacing S by S_i.

Write $\mathcal{L} = \{L_\lambda \mid \lambda \in \mathbb{P}^l\}$. For $\lambda \in \mathbb{P}^l$, we write $S \cdot L_\lambda = \sum_i c_i C_{\lambda,i}$, where $c_i \geq 0$ and $C_{\lambda,i}$ is an irreducible and reduced curve on X. Then,

$$\tilde{S} \cdot \tilde{L}_\lambda = \sum_i c_i \tilde{C}_{\lambda,i} + \Xi,$$

where Ξ is an effective 1-cycle supported on E. Since any component of Ξ is contracted by φ and $-K_Y$ is φ-ample, we have $(-K_Y \cdot \Xi) \geq 0$. Thus, for a general $\lambda \in \mathbb{P}^l$, we have

$$0 \geq (-K_Y \cdot \tilde{S} \cdot \tilde{L}_\lambda) \geq \sum_i c_i(-K_Y \cdot \tilde{C}_{\lambda,i}).$$

It follows that $(-K_Y \cdot \tilde{C}_{\lambda,i}) \leq 0$ for some i. We choose such a $\tilde{C}_{\lambda,i}$ and denote it by $\tilde{C}^\circ_\lambda$. By assumption (1) the set

$$\{\tilde{C}^\circ_\lambda \mid \lambda \in \mathbb{P}^l \text{ is general}\}$$

consists of infinitely many distinct curves. We have $(-K_Y \cdot \tilde{C}^\circ_\lambda) \leq 0$ by the construction. We see that $(E \cdot \tilde{C}^\circ_\lambda) > 0$ since $\tilde{C}^\circ_\lambda$ is the proper transform of a curve passing through p. Therefore p is not a maximal center by [Okada 2018, Lemma 2.20]. □

Fano varieties in a weighted projective space. Let X be a prime Fano 3-fold. As in the introduction, we choose minimal generators of the anticanonical ring $R(X, -K_X)$ and let $X \subset \mathbb{P} := \mathbb{P}(a_0, \dots, a_n)$ be the corresponding embedding. We say that $X \subset \mathbb{P}$ is *anticanonically embedded*. We denote by $x_0, \dots, x_n$ the homogeneous coordinates of $\mathbb{P}$ with $\deg x_i = a_i$. Let

$$F_1 = F_2 = \cdots = F_N = 0$$

be defining equations of X inside $\mathbb{P}$, where $F_i \in \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous polynomial of degree d_i with respect to the grading $\deg x_i = a_i$. We assume that $\mathbb{P}$ is *well-formed*, that is,

$$\gcd\{a_i \mid 0 \leq i \leq n, i \neq j\} = 1$$

for $j = 0, 1, \dots, n$. Note that X is not contained in a linear cone (i.e., a smaller weighted projective space in $\mathbb{P}$) by minimality of generators of $R(X, -K_X)$.

Definition 2.7. We say that X is *well-formed* if $\operatorname{codim}_X(X \cap \operatorname{Sing}(\mathbb{P})) \geq 2$. We say that X is *quasismooth* if the affine cone

$$(F_1 = F_2 = \cdots = F_N = 0) \subset \mathbb{A}^{n+1} = \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_n]$$

is smooth outside the origin.

Remark 2.8. The description of $\operatorname{Sing}(\mathbb{P})$ is given in Remark 2.12 below. Under the assumption that $X \subset \mathbb{P}$ is an anticanonically embedded quasismooth prime Fano 3-fold, we believe that well-formedness is a very mild condition (or perhaps it is automatically satisfied). For example, a quasismooth weighted complete intersection $X \subset \mathbb{P}$ which is not contained in a linear cone is well-formed (see [Iano-Fletcher 2000, Theorem 6.17]).

In the following we assume that $X \subset \mathbb{P}$ is well-formed and quasismooth. For $0 \leq i \leq n$, we define $\mathsf{p}_{x_i} = (0 : \cdots : 1 : \cdots : 0) \in \mathbb{P}$, where the unique 1 is in the $(i+1)$-th position, and we define $D_i = (x_i = 0) \cap X$ which is a Weil divisor such that $D_i \sim -a_i K_X$.

Lemma 2.9. *If $(-K_X)^3 \leq 1$, then no curve on X is a maximal center.*

Proof. The same proof of [Ahmadinezhad and Okada 2018, Lemma 2.1] applies in this setting without any change. □

Lemma 2.10. *Assume that $a_0 \leq a_1 \leq \ldots \leq a_n$. If $a_{n-1} a_n (-K_X)^3 \leq 4$, then no nonsingular point of X is a maximal center.*

Proof. The proof is almost identical to that of [Ahmadinezhad and Okada 2018, Lemma 2.6]. □

Definition 2.11. Let $\mathcal{C} \subset \{x_0, \ldots, x_n\}$ be a nonempty set of homogeneous coordinates. We define

$$\Pi(\mathcal{C}) := \bigcap_{z \in \mathcal{C}} (z = 0) \subset \mathbb{P}, \quad \Pi_X(\mathcal{C}) := \Pi(\mathcal{C}) \cap X \subset X.$$

Sometimes we denote

$$\Pi(\mathcal{C}) = \Pi(x_{i_1}, \ldots, x_{i_m}), \quad \Pi_X(\mathcal{C}) = \Pi_X(x_{i_1}, \ldots, x_{i_m}),$$

when $\mathcal{C} = \{x_{i_1}, \ldots, x_{i_m}\}$. We also define

$$\gcd(\mathcal{C}) := \gcd\{\deg(x_i) \mid x_i \in \mathcal{C}\}.$$

Remark 2.12. We explain some consequences of well-formedness and quasismoothness, which will be frequently used. We keep the above notation and assumptions:

(1) For the singular locus, we have $\operatorname{Sing}(X) = X \cap \operatorname{Sing}(\mathbb{P})$. For the proof see [Dimca 1986, Proposition 8]. Note that $X \subset \mathbb{P}$ is additionally assumed to be a (weighted) complete intersection in [loc. cit.] but the same proof applies.

(2) The singular locus of $\mathbb{P}$ can be described as follows:

$$\operatorname{Sing}(\mathbb{P}) = \bigcup_{\substack{\mathcal{C} \subsetneq \{x_0, \dots, x_n\} \\ \gcd(\{x_0,\dots,x_n\} \setminus \mathcal{C}) > 1}} \Pi(\mathcal{C}).$$

By (1), we have

$$\operatorname{Sing}(X) = \bigcup_{\substack{\mathcal{C} \subsetneq \{x_0, \dots, x_n\} \\ \gcd(\{x_0,\dots,x_n\} \setminus \mathcal{C}) > 1}} \Pi_X(\mathcal{C}).$$

(3) For $\mathcal{C} \subset \{x_0, \dots, x_n\}$, we define

$$\Pi_X^*(\mathcal{C}) := \Pi_X(\mathcal{C}) \cap \Bigg(\bigcap_{z \in \{x_0, \dots, x_n\} \setminus \mathcal{C}} (z \neq 0) \Bigg).$$

Let $\mathcal{C} \subset \{x_0, \dots, x_n\}$ be a subset such that $r := \gcd(\{x_0, \dots, x_n\} \setminus \mathcal{C}) > 1$. Then any point $\mathsf{p} \in X$ which is contained in $\Pi_X^*(\mathcal{C})$ is a cyclic quotient singular point of index r and hence any point $\mathsf{p} \in X$ which is contained in $\Pi_X(\mathcal{C})$ is a cyclic quotient singular point of index divisible by r.

Lemma 2.13. *Let $\mathsf{p} \in X$ be a singular point of type $\frac{1}{2}(1, 1, 1)$ and let*

$$b := \max\{a_i \mid 0 \le i \le n, a_i \text{ is odd}\}.$$

If $2b(-K_X)^3 \le 1$, then p is not a maximal center.

Proof. Let $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\}$ be the set of homogeneous coordinates of odd degree. The set $\Pi_X(\mathcal{C}) = D_{i_1} \cap \dots \cap D_{i_m}$ consists of singular points by Remark 2.12. In particular $\Pi_X(\mathcal{C})$ is a finite set of points since X has only terminal singular points. Let $\varphi \colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup. Then $\operatorname{ord}_E(D_{i_j}) \ge \frac{1}{2}$ since $2D_{i_j}$ is a Cartier divisor passing through p and thus $-b\varphi^* K_X - \frac{1}{2}E$ is nef by Lemma 2.5. We have

$$\big(-b\varphi^* K_X - \tfrac{1}{2}E\big) \cdot (-K_Y)^2 = b(-K_X)^3 - \tfrac{1}{2} \le 0.$$

This shows that $(-K_Y)^2 \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$ and p is not a maximal center by Lemma 2.4. □

Definition 2.14. Let $\mathsf{p} = \mathsf{p}_{x_k} \in X$ be a terminal quotient singular point of type $\frac{1}{a_k}(1, c, a_k - c)$ for some c with $1 \le c \le \frac{1}{2}a_k$. For a nonempty subset $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\} \subset \{x_0, \dots, x_n\}$, we define

$$\operatorname{ivr}_{\mathsf{p}}(\mathcal{C}) := \min_{1 \le j \le m} \left\{ \frac{\bar{a}_{i_j}}{a_{i_j} a_k} \right\},$$

where $\bar{a}_{i_j}$ is the integer such that $1 \le \bar{a}_{i_j} \le a_k$ and $\bar{a}_{i_j}$ is congruent to a_{i_j} modulo a_k, and call it the *initial vanishing ratio* of $\mathcal{C}$ at p.

Definition 2.15. For a terminal quotient singularity p of type $\frac{1}{r}(1, a, r - a)$, we define

$$\operatorname{wp}(\mathsf{p}) := a(r - a),$$

and call it the *weight product* of p.

Lemma 2.16. *Let* $\mathsf{p} = \mathsf{p}_{x_k} \in X$ *be a terminal quotient singular point. Suppose that there exists a subset* $\mathcal{C} \subset \{x_0, \dots, x_n\}$ *satisfying the following properties*:

(1) $\mathsf{p} \in \Pi_X(\mathcal{C})$, *or equivalently* $x_k \notin \mathcal{C}$.

(2) $\Pi_X(\mathcal{C} \cup \{x_k\}) = \varnothing$.

(3) $\operatorname{ivr}_{\mathsf{p}}(\mathcal{C}) \geq \operatorname{wp}(\mathsf{p})(-K_X)^3$.

Then p *is not a maximal center.*

Proof. We write $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\}$. We claim that $\Pi_X(\mathcal{C}) = D_{i_1} \cap \dots \cap D_{i_m}$ is a finite set of points. Indeed, if $\Pi_X(\mathcal{C})$ contains a curve, then $\Pi_X(\mathcal{C} \cup \{x_k\}) = \Pi_X(\mathcal{C}) \cap D_k \neq \varnothing$ since D_k is an ample divisor on X. This is impossible by the assumption (2). Note that we have $\operatorname{ord}_E(D_{i_j}) \geq \bar{a}_{i_j}/a_k$ (see [Ahmadinezhad and Okada 2018, Section 3]) so that

$$e := \min\left\{ \frac{\operatorname{ord}_E(D_{i_j})}{a_{i_j}} \mid 1 \leq j \leq m \right\} \geq \operatorname{ivr}_{\mathsf{p}}(\mathcal{C}).$$

By Lemma 2.5, $-\varphi^* K_X - \operatorname{ivr}_{\mathsf{p}}(\mathcal{C}) E$ is nef and we have

$$(-\varphi^* K_X - \operatorname{ivr}_{\mathsf{p}}(\mathcal{C}) E)(-K_Y)^2 = (-K_X)^3 - \frac{\operatorname{ivr}_{\mathsf{p}}(\mathcal{C})}{\operatorname{wp}(\mathsf{p})} \leq 0$$

by the assumption (3). Therefore $(-K_Y)^2 \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$ and p is not a maximal center. ☐

Let $\mathsf{p} \in X$ be a singular point such that it can be transformed to p_{x_k} by a change of coordinates. For simplicity of the description we assume $\mathsf{p} = \mathsf{p}_{x_0}$ and we set $r = a_0 > 1$. Let $\varphi \colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup. We explain a systematic way to estimate $\operatorname{ord}_E(D_i)$ for coordinates x_i and also an explicit description of φ. It is a consequence of the quasismoothness of X that after renumbering the defining equation we can write

$$F_l = \alpha_l x_0^{m_l} x_{i_l} + (\text{other terms}) \quad \text{for } 1 \leq l \leq n-3,$$

where $\alpha_l \in \mathbb{C} \setminus \{0\}$, m_l is a positive integer, $x_0, x_{i_1}, \dots, x_{i_{n-3}}$ are mutually distinct so that by denoting the other 3 coordinates as $x_{j_1}, x_{j_2}, x_{j_3}$ we have

$$\{x_0, x_{i_1}, \dots, x_{i_{n-3}}, x_{j_1}, x_{j_2}, x_{j_3}\} = \{x_0, \dots, x_n\},$$

and we can choose $x_{j_1}, x_{j_2}, x_{j_3}$ as local orbi-coordinates of X at p. In this case the singular point p is of type

$$\tfrac{1}{r}(a_{j_1}, a_{j_2}, a_{j_3}).$$

Definition 2.17 [Ahmadinezhad and Okada 2018, Definitions 3.6 and 3.7]. For an integer a, we denote by $\bar{a}$ the positive integer such that $\bar{a} \equiv a \pmod{r}$ and $0 < \bar{a} \leq r$. We say that

$$\mathbf{w}(x_1, \dots, x_n) = \tfrac{1}{r}(b_1, \dots, b_n)$$

is an *admissible weight* at p if $b_i \equiv a_i \pmod{r}$ for any i.

For an admissible weight $\mathbf{w}$ at p and a polynomial $f = f(x_0, \dots, x_n)$, we denote by $f^{\mathbf{w}}$ the lowest weight part of f, where we assume that $\mathbf{w}(x_0) = 0$.

We say that an admissible weight $\mathbf{w}$ at p satisfies the *KBL condition* if $x_0^{m_l} x_{i_l} \in F_l^{\mathbf{w}}$ for $1 \le l \le n-3$ and

$$(b_{j_1}, b_{j_2}, b_{j_3}) = (\bar{a}_{j_1}, \bar{a}_{j_2}, \bar{a}_{j_3}).$$

Let $\mathbf{w}(x_1, \dots, x_n) = \frac{1}{r}(b_1, \dots, b_n)$ be an admissible weight at p satisfying the KBL condition. We denote by $\Phi_{\mathbf{w}} \colon Q_{\mathbf{w}} \to \mathbb{P}$ the weighted blowup at p with weight $\mathbf{w}$, and by $Y_{\mathbf{w}}$ the proper transform of X via $\Phi_{\mathbf{w}}$. Then the induced morphism $\varphi_{\mathbf{w}} = \Phi_{\mathbf{w}}|_{Y_{\mathbf{w}}} \colon Y_{\mathbf{w}} \to X$ coincides with the Kawamata blowup at p. From this we see that the exceptional divisor E is isomorphic to

$$E_{\mathbf{w}} := (g_1 = \cdots = g_{n-3} = 0) \subset \mathbb{P}(b_1, \dots, b_n),$$

where $g_l = F_l^{\mathbf{w}}(1, x_1, \dots, x_n)$. Note that the KBL condition implies that the equations defining $E_{\mathbf{w}}$ cut out a copy of $\mathbb{P}(b_{j_1}, b_{j_2}, b_{j_3})$. We refer readers to [Ahmadinezhad and Okada 2018, Section 3] for details.

Lemma 2.18 [Ahmadinezhad and Okada 2018, Lemma 3.9]. *Let $\mathbf{w}(x_1, \dots, x_n) = \frac{1}{r}(b_1, \dots, b_n)$ be an admissible weight at $\mathsf{p} \in X$ satisfying the KBL condition. Then the following assertions hold*:

(1) *We have* $\operatorname{ord}_E(D_i) \ge b_i/r$ *for any* i.

(2) *If $F_l^{\mathbf{w}} = \alpha_l x_0^{m_l} x_{i_l}$, where $\alpha_i \in \mathbb{C} \setminus \{0\}$, for some $1 \le l \le n-3$, then the weight*

$$\mathbf{w}'(x_1, \dots, x_n) = \tfrac{1}{r}(b'_1, \dots, b'_n),$$

where $b'_j = b_j$ for $j \ne l$ and $b'_l = b_l + r$, satisfies the KBL condition. In particular, $\operatorname{ord}_E(D_l) \ge (b_l + r)/r$.

We will use the following notation for a polynomial $f = f(x_0, \dots, x_n)$:

- For a monomial $p = x_0^{e_0} \cdots x_n^{e_n}$, we write $p \in f$ if p appears in f with nonzero (constant) coefficient.
- For a subset $\mathcal{C} \subset \{x_0, \dots, x_n\}$ and $\Pi = \Pi(\mathcal{C})$, we denote by $f|_{\Pi}$ the polynomial in variables $\{x_0, \dots, x_n\} \setminus \mathcal{C}$ obtained by putting $x_i = 0$ for $x_i \in \mathcal{C}$ in f.

3. Proof of birational superrigidity by numerical data

We prove birational superrigidity of codimension 4 quasismooth prime Fano 3-folds with no projections by utilizing only numerical data. The numerical data for each Fano 3-fold will be described in the beginning of the corresponding subsection. The Fano 3-folds are embedded in a weighted projective 7-space, denoted by $\mathbb{P}$, and we use the symbol p, q, r, s, t, u, v, w for the homogeneous coordinates of $\mathbb{P}$. We use the following terminologies: Let $X \subset \mathbb{P}$ be a codimension 4 quasismooth prime Fano 3-fold. For a homogeneous coordinate $z \in \{p, q, \dots, w\}$,

- $D_z := (z = 0) \cap X$ is the Weil divisor on X cut out by z, and
- $\mathsf{p}_z \in \mathbb{P}$ is the point at which only the coordinate z does not vanish.

Note that Theorem 1.2 will follow from Theorems 3.1, 3.2 and 3.4.

Fano 3-folds of numerical type #25. Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #25, whose data consist of the following:

- $X \subset \mathbb{P}(2_p, 5_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$.
- $(-K_X)^3 = \frac{1}{70}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (16, 17, 18, 18, 19, 20, 20, 21, 22)$.
- $\mathcal{B}_X = \left\{7 \times \frac{1}{2}(1,1,1), \frac{1}{5}(1,1,4), \frac{1}{7}(1,2,5)\right\}$.

Here the subscripts $p, q, \dots, w$ of the weights means that they are the homogeneous coordinates of the indicated degrees, and $\mathcal{B}_X$ indicates the numbers and the types of singular points of X.

Theorem 3.1. *Let X be a well-formed quasismooth prime codimension 4 Fano 3-fold of numerical type #25. Then X is birationally superrigid.*

Proof. By Lemmas 2.9 and 2.10, no curve and no nonsingular point on X is a maximal center. By Lemma 2.13, singular points of type $\frac{1}{2}(1,1,1)$ are not maximal centers.

Let p be the singular point of type $\frac{1}{5}(1,1,4)$. Replacing the coordinate v if necessary, we may assume $\mathsf{p} = \mathsf{p}_q$. We set $\mathcal{C} = \{p, s, u, v\}$. We have

$$\mathrm{ivr}_{\mathsf{p}}(\mathcal{C}) = \frac{2}{35} = \mathrm{wp}(\mathsf{p})(-K_X)^3.$$

By Lemma 2.16, it remains to show that $\Pi_X := \Pi_X(\mathcal{C} \cup \{q\}) = \emptyset$. We set $\Pi := \Pi(\mathcal{C} \cup \{q\}) \subset \mathbb{P}$ so that $\Pi_X = \Pi \cap X$. Since $\mathsf{p}_t \notin X$, one of the defining polynomials contain a power of t. By looking at the degrees of $F_1, \dots, F_9$, we have $t^2 \in F_1$. Similarly, we have $r^3 \in F_3$ and $w^2 \in F_9$ after possibly interchanging F_3 and F_4. The monomial t^2 (resp. r^3) is the only monomial of degree 16 (resp. 18) consisting of the variables r, t, w. The monomials w^2 and $t^2 r$ are the only monomials of degree 22 consisting of the variables r, t, w. Hence, rescaling r, t, w, we can write

$$F_1|_\Pi = t^2, \quad F_3|_\Pi = r^3, \quad F_9|_\Pi = w^2 + \alpha t^2 r,$$

for some $\alpha \in \mathbb{C}$. The set Π_X is contained in the common zero locus of the above 3 polynomials inside Π. The equations have only trivial solution and this shows that $\Pi_X = \emptyset$. Thus p is not a maximal center.

Let $\mathsf{p} = \mathsf{p}_s$ be the singular point of type $\frac{1}{7}(1,2,5)$ and set $\mathcal{C} = \{p, q, r\}$. We have

$$\mathrm{ivr}_{\mathsf{p}}(\mathcal{C}) = \frac{1}{7} = \mathrm{wp}(\mathsf{p})(-K_X)^3.$$

By Lemma 2.16, it remains to show that $\Pi_X := \Pi_X(\mathcal{C} \cup \{s\}) = \emptyset$. We set $\Pi := \Pi(\mathcal{C} \cup \{s\}) \subset \mathbb{P}$ so that $\Pi_X = \Pi \cap X$. Since $\mathsf{p}_t, \mathsf{p}_u, \mathsf{p}_v, \mathsf{p}_w \notin X$, we may assume $t^2 \in F_1$, $u^2 \in F_3$, $v^2 \in F_6$ and $w^2 \in F_9$ after possibly interchanging defining polynomials of the same degree. Then we can write

$$F_1|_\Pi = t^2, \quad F_3|_\Pi = u^2 + \alpha v t, \quad F_6|_\Pi = v^2 + \beta w u, \quad F_9|_\Pi = w^2 + \gamma t^2 r,$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. This shows that $\Pi_X = \varnothing$ and thus p is not a maximal center. This completes the proof. □

***Fano 3-folds of numerical type* #166.** Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #166, whose data consist of the following:

- $X \subset \mathbb{P}(2_p, 2_q, 3_r, 3_s, 4_t, 4_u, 5_v, 5_w)$.
- $(-K_X)^3 = \frac{1}{6}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (8, 8, 8, 9, 9, 9, 10, 10, 10)$.
- $\mathcal{B}_X = \left\{11 \times \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2)\right\}$.

Theorem 3.2. *Let X be a well-formed quasismooth prime codimension* 4 *Fano* 3*-fold of numerical type* #166*. Then X is birationally superrigid.*

Proof. By Lemma 2.9 no curve is a maximal center.

Let $\mathsf{p} = (\alpha_p : \alpha_q : \cdots : \alpha_w) \in X$ be a nonsingular point where $\alpha_p, \alpha_q, \ldots, \alpha_w \in \mathbb{C}$. By Remark 2.12, we have $\Pi_X(p, q, r, s, t, u) = \varnothing$ since X does not have a singular point of index 5. Then we can take a coordinate $x \in \{p, q, r, s, t, u\}$ such that $\mathsf{p} \in (x \neq 0)$, i.e., $\alpha_x \neq 0$. The common zero locus of the homogeneous polynomials in the set

$$\{\alpha_y^{\deg x} x^{\deg y} - \alpha_x^{\deg y} y^{\deg x} \mid y \in \{p, q, r, s, t, u, v, w\} \setminus \{x\}\}$$

is a finite set of points including p. Any polynomial in the above set is of degree at most 20 since $x \notin \{v, w\}$. It follows that the base locus of $|\mathcal{I}_{\mathsf{p}}^m(-mlK_X)|$ is a finite set of points, that is, $-lK_X$ isolates p (see [Corti et al. 2000, Definition 5.2.4 and Lemma 5.6.4]), where $l \leq 20$. By the argument in [loc. cit., Section 5.3], we conclude that p is not a maximal center since $20 < 4/(-K_X)^3$.

Let p be a singular point of type $\frac{1}{2}(1, 1, 1)$. After a change of coordinates, we may assume $\mathsf{p} = \mathsf{p}_p$. We set $\mathcal{C} = \{q, r, s, t, u\}$. We have

$$\operatorname{ivr}_{\mathsf{p}}(\mathcal{C}) = \tfrac{1}{6} = \operatorname{wp}(\mathsf{p})(-K_X)^3.$$

Moreover we have $\Pi_X(\mathcal{C} \cup \{p\}) = \varnothing$ because X is quasismooth and it does not have a singular point of index 5. Thus, by Lemma 2.16, p is not a maximal center.

Let p be the singular point of type $\frac{1}{3}(1, 1, 2)$. After a change of coordinates, we may assume $\mathsf{p} = \mathsf{p}_s$. We set $\mathcal{C} = \{p, q, r\}$. Then we have

$$\operatorname{ivr}_{\mathsf{p}}(\mathcal{C}) = \tfrac{1}{3} = \operatorname{wp}(\mathsf{p})(-K_X)^3.$$

By Lemma 2.16, it remains to show that $\Pi_X := \Pi_X(\mathcal{C} \cup \{s\}) = \varnothing$. We set $\Pi := \Pi(\mathcal{C} \cup \{s\}) \subset \mathbb{P}$ so that $\Pi_X = \Pi \cap X$. We have

$$\Pi_X = (F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = F_7|_\Pi = F_8|_\Pi = F_9|_\Pi = 0) \cap \Pi.$$

We see that $F_1|_\Pi, F_2|_\Pi, F_3|_\Pi$ consist only of monomials in variables t, u and that $F_7|_\Pi, F_8|_\Pi, F_9|_\Pi$ consist only of monomials in variables v, w. It follows that

$$\Pi_X(p,q,r,s,v,w) = \Pi_X \cap \Pi(v,w) = (F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = 0) \cap \Pi(p,q,r,s,v,w).$$

We have $\Pi_X(p,q,r,s,v,w) = \varnothing$ since X is well-formed, quasismooth and X has no singular point of index 4 (see Remark 2.12). Hence the equations

$$F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = 0$$

imply $t = u = 0$. Similarly, by considering $\Pi_X(p,q,r,s,t,u) = \varnothing$, we see that the equations

$$F_7|_\Pi = F_8|_\Pi = F_9|_\Pi = 0$$

imply $v = w = 0$. It follows that $\Pi_X = \varnothing$ and p is not a maximal center. Therefore X is birationally superrigid. □

***Fano* 3*-folds of numerical type* #282.** Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #282, whose data consist of the following:

- $X \subset \mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$.
- $(-K_X)^3 = \frac{1}{42}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (16, 17, 18, 18, 19, 20, 20, 21, 22)$.
- $\mathcal{B} = \left\{2 \times \frac{1}{2}(1,1,1), 2 \times \frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5), \frac{1}{7}(1,1,6)\right\}$.

Proposition 3.3. *Let X be a well-formed quasismooth prime codimension* 4 *Fano* 3*-fold of numerical type* #282. *Then no curve and no point is a maximal center except possibly for the singular point of type* $\frac{1}{6}(1,1,5)$.

Proof. By Lemmas 2.9, 2.10 and 2.13, it remains to exclude singular points of type $\frac{1}{3}(1,1,2)$ and $\frac{1}{7}(1,1,6)$ as maximal centers.

Let p be a singular point of type $\frac{1}{3}(1,1,2)$ and let $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup. We claim that $\Pi_X(p,s,t,w) = D_p \cap D_s \cap D_t \cap D_w$ is a finite set of points (containing p). Since X does not contain a singular point of index 10, we may assume that $v^2 \in F_6$. Then, by rescaling v, we have

$$F_6(0,q,r,0,0,u,v,0) = v^2$$

and this shows that $\Pi_X(p,s,t,w) = \Pi_X(p,s,t,v,w)$. The latter set consists of singular points $\{2 \times \frac{1}{3}(1,1,2), \frac{1}{6}(1,1,5)\}$ (see Remark 2.12) and thus $\Pi_X(p,s,t,w)$ is a finite set of points. We have

$$\mathrm{ord}_E(D_p), \mathrm{ord}_E(D_s) \geq \tfrac{1}{3}, \quad \mathrm{ord}_E(D_t), \mathrm{ord}_E(D_w) \geq \tfrac{2}{3}.$$

By Lemma 2.5, $N := -\varphi^* K_X - \frac{1}{21} E$ is a nef divisor on Y and we have $(N \cdot (-K_Y)^2) = 0$. Thus p is not a maximal center by Lemma 2.4.

Let $\mathsf{p} = \mathsf{p}_s$ be the singular point of type $\frac{1}{7}(1, 1, 6)$ and set $\mathcal{C} = \{p, q, r\}$. We have

$$\mathrm{ivr}_{\mathsf{p}}(\mathcal{C}) = \tfrac{1}{7} = \mathrm{wp}(\mathsf{p})(-K_X)^3.$$

We set $\Pi := \Pi(\mathcal{C} \cup \{s\})$. We see that $\mathsf{p}_t, \mathsf{p}_u, \mathsf{p}_v, \mathsf{p}_w \notin X$ since X does not have a singular point of index 8, 9, 10, 11. It follows that $t^2 \in F_1$, $w^2 \in F_9$ and we may assume $u^2 \in F_3$, $v^2 \in F_6$. Then, by rescaling t, u, v, w, we can write

$$F_1|_\Pi = t^2, \quad F_3|_\Pi = \alpha vt + u^2, \quad F_6|_\Pi = \beta wu + v^2, \quad F_9|_\Pi = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. This shows that $\Pi_X(\mathcal{C} \cup \{s\}) = \Pi \cap X = \varnothing$. Thus p is not a maximal center by Lemma 2.16 and the proof is completed. □

***Fano 3-folds of numerical type* #308.** Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #308, whose data consist of the following:

- $X \subset \mathbb{P}(1_p, 5_q, 6_r, 6_s, 7_t, 8_u, 9_v, 10_w)$.
- $(-K_X)^3 = \frac{1}{30}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (14, 15, 16, 16, 17, 18, 18, 19, 20)$.
- $\mathcal{B}_X = \left\{\frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2), \frac{1}{5}(1, 2, 3), 2 \times \frac{1}{6}(1, 1, 5)\right\}$.

Theorem 3.4. *Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #308. Then X is birationally superrigid.*

Proof. By Lemmas 2.9, 2.10 and 2.13 no curve and no nonsingular point is a maximal center and the singular point of type $\frac{1}{2}(1, 1, 1)$ is not a maximal center.

Let p be the singular point of type $\frac{1}{3}(1, 1, 2)$, which is necessarily contained in $(p = q = t = u = w = 0)$, and let $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup. We set $\mathcal{C} = \{p, q, u\}$ and $\Pi = \Pi(\mathcal{C}) \subset \mathbb{P}$. Since $\mathsf{p}_t, \mathsf{p}_w \notin X$, we have $t^2 \in F_1$, $w^2 \in F_9$ and we can write

$$F_1|_\Pi = t^2, \quad F_9|_\Pi = w^2 + \alpha t^2 r + \beta t^2 s,$$

where $\alpha, \beta \in \mathbb{C}$. Thus,

$$\Pi_X(\mathcal{C}) = \Pi \cap X = \Pi_X(p, q, t, u, w),$$

and this consists of two $\frac{1}{6}(1, 1, 5)$ points and p. In particular $D_p \cap D_q \cap D_u = \Pi_X(\mathcal{C})$ is a finite set of points. We have

$$\mathrm{ord}_E(D_p) \geq \tfrac{1}{3}, \quad \mathrm{ord}_E(D_q) \geq \tfrac{2}{3}, \quad \mathrm{ord}_E(D_u) \geq \tfrac{2}{3},$$

hence $N := -8\varphi^* K_X - \frac{2}{3}E$ is a nef divisor on Y by Lemma 2.5. We have

$$(N \cdot (-K_Y)^2) = 8(-K_X)^3 - \tfrac{2}{3^3} \cdot \tfrac{3^2}{2} = -\tfrac{1}{15} < 0.$$

By Lemma 2.4, p is not a maximal center.

Let p be a singular point of type $\frac{1}{6}(1, 1, 5)$. After replacing r and s, we may assume $\mathsf{p} = \mathsf{p}_s$. We set $\mathcal{C} = \{p, q, r\}$. We have

$$\mathrm{ivr}_{\mathsf{p}}(\mathcal{C}) = \frac{1}{6} = \mathrm{wp}(\mathsf{p})(-K_X)^3.$$

Since $\mathsf{p}_t, \mathsf{p}_u, \mathsf{p}_v, \mathsf{p}_w \notin X$, we may assume $t^2 \in F_1$, $u^2 \in F_3$, $v^2 \in F_6$, $w^2 \in F_9$ after possibly interchanging F_3 with F_4 and F_6 with F_7. Then, by setting $\Pi = \Pi(\mathcal{C} \cup \{s\})$ and by rescaling t, u, v, w, we have

$$F_1|_\Pi = t^2, \quad F_3|_\Pi = u^2 + \alpha v t, \quad F_6|_\Pi = v^2 + \beta w u, \quad F_9|_\Pi = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. This shows that $\Pi_X(\mathcal{C} \cup \{s\}) = \varnothing$ and p is not a maximal center by Lemma 2.16.

Finally, let p be a singular point of type $\frac{1}{5}(1, 2, 3)$ and let $\varphi \colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup. Replacing the coordinate w, we may assume $\mathsf{p} = \mathsf{p}_q$. We write

$$F_3 = \lambda q^3 p + \mu q^2 r + \nu q^2 s + q f_{11} + f_{16}, \quad F_4 = \lambda' q^3 p + \mu' q^2 r + \nu' q^2 s + q g_{11} + g_{16},$$

where $\lambda, \mu, \nu, \lambda', \mu', \nu' \in \mathbb{C}$ and $f_{11}, f_{16}, g_{11}, g_{16} \in \mathbb{C}[p, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degrees. Since X is quasismooth at $\mathsf{p} = \mathsf{p}_q$ and is of type $\frac{1}{5}(1, 2, 3)$, the matrix

$$\begin{pmatrix} \frac{\partial F_3}{\partial p}(\mathsf{p}) & \frac{\partial F_3}{\partial r}(\mathsf{p}) & \frac{\partial F_3}{\partial s}(\mathsf{p}) \\ \frac{\partial F_4}{\partial p}(\mathsf{p}) & \frac{\partial F_4}{\partial r}(\mathsf{p}) & \frac{\partial F_4}{\partial s}(\mathsf{p}) \end{pmatrix} = \begin{pmatrix} \lambda & \mu & \nu \\ \lambda' & \mu' & \nu' \end{pmatrix}$$

is of rank 2.

We first consider the case where $\mu\nu' - \nu\mu' \ne 0$. By replacing r and s, we may assume that $\mu = \nu' = 1$ and $\lambda = \nu = \lambda' = \mu' = 0$. We consider the weight at p

$$\mathbf{w}(p, r, s, t, u, v, w) = \frac{1}{5}(1, 1, 1, 2, 3, 4, 5),$$

which is an admissible weight satisfying the KBL condition. Then $F_3^{\mathbf{w}} = q^2 r$ and $F_4^{\mathbf{w}} = q^2 s$, and this implies $\mathrm{ord}_E(D_r), \mathrm{ord}_E(D_s) \ge \frac{6}{5}$ by Lemma 2.18. Note that $\mathrm{ord}_E(D_p) \ge \mathbf{w}(p) = \frac{1}{5}$ by Lemma 2.18. We set $\mathcal{C} = \{p, r, s\}$ and $\Pi = \Pi(\mathcal{C} \cup \{q\})$. By rescaling t, u, v, w, we can write

$$F_1|_\Pi = t^2, \quad F_3|_\Pi = u^2 + \alpha v t, \quad F_6|_\Pi = v^2 + \beta w u, \quad F_9|_\Pi = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. Hence $\Pi_X(\mathcal{C} \cup \{q\}) = \varnothing$. Since D_q is an ample divisor, this implies that $D_p \cap D_r \cap D_s$ is a finite set of points (including p). By Lemma 2.5, $N := -\varphi^* K_X - \frac{1}{5} E$ is a nef divisor on Y. We have

$$(N \cdot (-K_Y)^2) = (-K_X)^3 - \frac{1}{5^3}(E^3) = \frac{1}{30} - \frac{1}{30} = 0,$$

and this shows that p is not a maximal center.

Next we consider the case where $\mu\nu' - \nu\mu' = 0$. By replacing r and s suitably and by possibly interchanging F_3 and F_4, we may assume that

$$F_3 = q^3 p + q f_{11} + f_{16}, \quad F_4 = q^2 s + q g_{11} + g_{16}.$$

Let $\mathbf{w}$ be the same weight at p as in the previous case, which is again an admissible weight satisfying the KBL condition. It is straightforward to see that $F_3^{\mathbf{w}} = q^3 p$, so that $\mathrm{ord}_E(D_p) \ge \frac{6}{5}$. Let $\mathcal{L} \subset |-6K_X|$ be the pencil generated by the sections r and s. Since $\mathrm{ord}_E(D_r) = \frac{1}{5}$ and $\mathrm{ord}_E(D_s) \geq \frac{1}{5}$, a general member $L \in \mathcal{L}$ vanishes along E to order $\frac{1}{5}$ so that $\tilde{L} \sim -6\varphi^* K_X - \frac{1}{5}E$. We have

$$(-K_Y \cdot \tilde{D}_p \cdot \tilde{L}) = 6(-K_X)^3 - \frac{\mathrm{ord}_E(D_p)}{5^2} \cdot (E^3) = \frac{1}{5} - \frac{\mathrm{ord}_E(D_p)}{6} \leq 0$$

since $\mathrm{ord}_E(D_p) \geq \frac{6}{5}$. By Lemma 2.6, p is not a maximal center and the proof is complete. □

4. Birational superrigidity of cluster Fano 3-folds

In this section we prove Theorem 1.1 which follows from Theorems 4.2 and 4.4 below.

#282 by $\mathsf{G}_2^{(4)}$ format. Let X be a quasismooth codimension 4 prime Fano 3-fold of numerical type #282 constructed in $\mathsf{G}_2^{(4)}$ format. Then, by [Coughlan and Ducat 2018, Example 5.5], X is defined by the following polynomials in $\mathbb{P} := \mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$:

$$\begin{aligned}
F_1 &= t^2 - qv + sQ_9,\\
F_2 &= ut - qw + s(v + p^2 t),\\
F_3 &= t(v + p^2 t) - uQ_9 + q(qr + p^4 t),\\
F_4 &= (w + p^4 s)s - P_{12} q + u(u + p^2 s),\\
F_5 &= tw - uv + s(qr + p^4 t),\\
F_6 &= (qr + p^4 t)t - Q_9 w + v(v + p^2 t),\\
F_7 &= rs^2 - wu + tP_{12},\\
F_8 &= P_{12} Q_9 - (vw + p^4 qw + p^2 uv + uqr + str - stp^2),\\
F_9 &= rs(u + p^2 s) - vP_{12} + w(w + p^4 s).
\end{aligned}$$

Here $P_{12}, Q_9 \in \mathbb{C}[p, q, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degree. Recall that $(-K_X)^3 = \frac{1}{42}$.

Lemma 4.1. *The following assertions hold*:

(1) $r^2 \in P_{12}$ *and* $u \in Q_9$.

(2) $X \subset \mathbb{P}$ *is well-formed.*

Proof. It is straightforward to check that X is quasismooth at $\mathsf{p}_r \in X$ if and only if $r^2 \in P_{12}$ and $u \in Q_9$, and this proves (1).

We prove (2). We set

$$\Pi_2 := \Pi_X(p, s, u, w), \quad \Pi_3 := \Pi_X(p, s, t, v, w).$$

It is enough to show that neither Π_2 nor Π_3 contain a surface (note here that $P_{12}|_{\Pi_2} \neq 0$ by (1)). We see that Π_2 is isomorphic to the closed subscheme in $\mathbb{P}(6_q, 6_r, 8_t, 10_v)$ defined by the equations

$$t^2 - qv = tv + q^2 r = qP_{12}|_{\Pi_2} = qrt + v^2 = tP_{12}|_{\Pi_2} = vP_{12}|_{\Pi_2} = 0.$$

We leave the readers to check that Π_2 does not contain a surface. We see that Π_3 is isomorphic to the closed subscheme in $\mathbb{P}(6_q, 6_r, 9_u)$ defined by the equations

$$-uQ_9|_{\Pi_3} + q^2 r = -qP_{12}|_{\Pi_3} + u^2 = P_{12}|_{\Pi_3} Q_9|_{\Pi_3} - uqr = 0.$$

Hence Π_3 does not contain a surface since it is clearly a proper closed subset of the surface $\mathbb{P}(6, 6, 9)$. Thus $X \subset \mathbb{P}$ is well-formed. □

Theorem 4.2. *Let X be a codimension 4 Fano 3-fold of numerical type #282 constructed in $\mathsf{G}_2^{(4)}$ format. Then X is birationally superrigid.*

Proof. By Lemma 4.1, $X \subset \mathbb{P}$ is well-formed. We can apply Proposition 3.3 and it remains to exclude the singular point $\mathsf{p} \in X$ of type $\frac{1}{6}(1, 1, 5)$ as a maximal center. We have $\mathsf{p} = \mathsf{p}_r$ since $\mathsf{p}_r \in X$ and X has a unique singular point of index 6. We set $\mathcal{C} = \{p, q\}$, $\Pi = \Pi(\mathcal{C})$ and $\Gamma := \Pi_X(\mathcal{C}) = \Pi \cap X$.

We will show that Γ is an irreducible and reduced curve. By Lemma 4.1, we can write

$$P_{12}|_{\Pi} = \lambda r^2, \quad Q_9|_{\Pi} = \mu u,$$

where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Then we have

$$\begin{aligned}
F_1|_{\Pi} &= t^2 + \mu s u, & F_4|_{\Pi} &= ws + u^2, & F_7|_{\Pi} &= rs^2 - wu + \lambda t r^2,\\
F_2|_{\Pi} &= ut + sv, & F_5|_{\Pi} &= tw - uv, & F_8|_{\Pi} &= \lambda\mu r^2 u - (vw + str),\\
F_3|_{\Pi} &= tv - \mu u^2, & F_6|_{\Pi} &= -\mu u w + v^2, & F_9|_{\Pi} &= rsu - \lambda v r^2 + w^2.
\end{aligned}$$

We work on the open subset U on which $w \neq 0$. Then $\Gamma \cap U$ is isomorphic to the $\mathbb{Z}/11\mathbb{Z}$-quotient of the affine curve

$$(\lambda r^2 v + \mu^3 r v^6 - 1 = 0) \subset \mathbb{A}^2_{r,v}.$$

It is straightforward to check that the polynomial $\lambda r^2 v + \mu^3 r v^6 - 1$ is irreducible. Thus $\Gamma \cap U$ is an irreducible and reduced affine curve. It is also straightforward to check that

$$\Gamma \cap (w = 0) = X \cap (p = q = w = 0) = \{\mathsf{p}_r, \mathsf{p}_s\}.$$

This shows that Γ is an irreducible and reduced curve.

Let $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup and let $\tilde{\Delta}$ be the proper transform via φ of a divisor or curve Δ on X. We show that $\tilde{D}_p \cap \tilde{D}_q \cap E$ does not contain a curve. Consider the weight

$$\mathbf{w}(p, q, s, t, u, v, w) = \tfrac{1}{6}(1, 6, 1, 2, 3, 4, 5),$$

which is clearly an admissible weight satisfying the KBL condition. We set $g_i = F_i^{\mathbf{w}}(p, q, 1, s, t, u, v, w)$. We have

$$\begin{aligned} g_4 &= (w + p^4)s - \lambda q + u(u + p^2 s), \\ g_7 &= s^2 + \lambda t, \\ g_8 &= \lambda\mu u - st, \\ g_9 &= s(u + p^2 s) - \lambda v. \end{aligned}$$

Since E is isomorphic to the subvariety

$$(g_4 = g_7 = g_8 = g_9 = 0) \subset \mathbb{P}(1_p, 6_q, 1_s, 2_t, 3_u, 4_v, 5_w),$$

it is straightforward to check that $\tilde{D}_p \cap \tilde{D}_q \cap E$ consists of a finite set of points (in fact, 2 points). Thus we have $\tilde{D}_p \cdot \tilde{D}_q = \tilde{\Gamma}$ since $D_p \cdot D_q = \Gamma$.

We have

$$\tilde{D}_p \sim -\varphi^* K_X - \tfrac{1}{6} E, \quad \tilde{D}_q \sim -6\varphi^* K_X - \tfrac{e}{6} E,$$

for some integer $e \geq 6$ and hence

$$(\tilde{D}_p \cdot \tilde{\Gamma}) = (\tilde{D}_p^2 \cdot \tilde{D}_q) = \tfrac{1}{7} - \tfrac{e}{30} < 0.$$

By [Okada 2018, Lemma 2.18], p is not a maximal center. □

#282 *by* C_2 *format.* Let X be a quasismooth codimension 4 prime Fano 3-fold of numerical type #282 constructed in C_2 format. Then, by [Coughlan and Ducat 2018, Example 5.5], X is defined by the following polynomials in $\mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$:

$$\begin{aligned} F_1 &= tR_8 - S_6 Q_{10} + su, \\ F_2 &= tu - wS_6 + sv, \\ F_3 &= rS_6^2 - vR_8 + u^2, \\ F_4 &= tQ_{10} - S_6 P_{12} + sw, \\ F_5 &= rsS_6 - wR_8 + uQ_{10}, \\ F_6 &= rs^2 - P_{12}R_8 + Q_{10}^2, \\ F_7 &= rtS_6 - vQ_{10} + uw, \\ F_8 &= rst - wQ_{10} + uP_{12}, \\ F_9 &= rt^2 - vP_{12} + w^2. \end{aligned}$$

Here $P_{12}, Q_{10}, R_8, S_6 \in \mathbb{C}[p, q, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degree. In the following we assume that $q \in S_6$, and then, we assume that $S_6 = q$ by a change of coordinates.

Lemma 4.3. *Under the above setting, the following assertions hold*:

(1) $r^2 \in P_{12}$, $v \in Q_{10}$ *and* $t \in R_8$.

(2) $X \subset \mathbb{P}$ *is well-formed.*

Proof. We have $\mathsf{p}_r \in X$ and X is quasismooth at p_r if and only if $r^2 \in P_{12}$. Similarly, it is easy to check that if $v \notin Q_{10}$ (resp. $t \notin R_8$), then X is not quasismooth at p_v (resp. p_r). This proves (1). We leave the readers to check that neither Π_2 nor Π_3 contain a surface, where Π_2, Π_3 are those given in the proof of Lemma 4.1, and this proves (2). $\square$

Theorem 4.4. *Let X be a quasismooth prime codimension 4 Fano 3-fold of numerical type #282 constructed by C_2 format. We assume that $q \in S_6$. Then X is birationally superrigid.*

Proof. By Lemma 4.3, we can apply Proposition 3.3 and it remains to exclude the singular point p of type $\frac{1}{6}(1, 1, 5)$ as a maximal center.

The singular point p corresponds to the solution of the equation

$$p = s = t = u = v = w = S_6 = 0,$$

and thus $\mathsf{p} = \mathsf{p}_r$ since $S_6 = q$ by our setting. We set $\mathcal{C} = \{p, q\}$ and $\Pi = \Pi(\mathcal{C})$.

We will show that $\Gamma := \Pi \cap X$ is an irreducible and reduced curve. We have $\Pi_X(\{p, q, r, s\}) = \varnothing$ (see the proof of Proposition 3.3). Hence $\Gamma \cap (s = 0) = \Pi_X(\{p, q, s\})$ does not contain a curve and it remains to show that $\Gamma \cap U_s$ is irreducible and reduced, where $U_s := (s \neq 0) \subset \mathbb{P}$ is the open subset. By Lemma 4.3 we can write

$$P_{12}|_\Pi = \lambda r^2, \quad Q_{10}|_\Pi = \mu v, \quad R_8|_\Pi = \nu t,$$

for some $\lambda, \mu, \nu \in \mathbb{C} \setminus \{0\}$, and we have $S_6|_\Pi = 0$. Note that $F_i|_\Pi = F_i|_\Pi(r, s, t, u, v, w)$ is a polynomial in variables r, s, t, u, v, w and we set $f_i = F_i|_\Pi(r, 1, t, u, v, w)$. Let $C \subset \mathbb{A}^5_{r,t,u,v,w}$ be the affine scheme defined by the equations

$$f_1 = f_2 = \cdots = f_9 = 0.$$

Then $\Gamma \cap U_s$ is isomorphic to the quotient of C by the natural $\mathbb{Z}/7\mathbb{Z}$-action. We have

$$\begin{aligned} f_1 &= \nu t^2 + u, & f_4 &= \mu t v + w, & f_7 &= -\mu v^2 + uw, \\ f_2 &= tu + v, & f_5 &= -\nu t w + \mu u v, & f_8 &= rt - \mu v w + \lambda r^2 u, \\ f_3 &= -\nu t v + u^2, & f_6 &= r - \lambda \nu r^2 t + \mu^2 v^2, & f_9 &= rt^2 - \lambda r^2 v + w^2. \end{aligned}$$

By the equations $f_1 = 0$, $f_2 = 0$ and $f_4 = 0$, we have

$$u = -\nu t^2, \quad v = -tu = \nu t^3, \quad w = -\mu t v = -\mu \nu t^4.$$

By eliminating the variables u, v, w and cleaning up the equations, C is isomorphic to the hypersurface in $\mathbb{A}^2_{r,t}$ defined by

$$r - \lambda \nu r^2 t + \mu^2 \nu^2 t^6 = 0,$$

which is an irreducible and reduced curve since $\mu\nu \neq 0$, and so is $\Gamma \cap U_s$. Thus Γ is an irreducible and reduced curve.

Let $\varphi\colon (E \subset Y) \to (\mathsf{p} \in X)$ be the Kawamata blowup. We have $e := \operatorname{ord}_E(D_q) \geq \frac{6}{6}$ and $\operatorname{ord}_E(D_p) = \frac{1}{6}$ so that we have

$$\tilde{D}_q \sim -6\varphi^* K_X - \tfrac{e}{6} E = -6K_Y + \tfrac{6-e}{6} E, \quad \tilde{D}_p \sim -\varphi^* K_X - \tfrac{1}{6} E = -K_Y.$$

We show that $\tilde{D}_q \cap \tilde{D}_p \cap E$ does not contain a curve. The Kawamata blowup φ is realized as the weighted blowup at p with the weight

$$\mathbf{w}(p, q, s, t, u, v, w) = \tfrac{1}{6}(1, 6, 1, 2, 3, 4, 5),$$

which is an admissible weight satisfying the KBL condition. We have

$$\begin{aligned} F_4^{\mathbf{w}} &= -\lambda q r^2 + t(\mu v + h) + sw, \\ F_6^{\mathbf{w}} &= -\lambda \mu t r^2 + r s^2, \\ F_8^{\mathbf{w}} &= \lambda u r^2 + rst, \\ F_9^{\mathbf{w}} &= -\lambda v r^2 + r t^2, \end{aligned}$$

where we define $h := Q_{10}^{\mathbf{w}} - \mu v$. Note that h is a linear combination of up, tp^2, sp^3, rp^4 and thus h is divisible by p. It follows that E is isomorphic to the subscheme in $\mathbb{P}(1_p, 6_q, 1_s, 2_t, 3_u, 4_v, 5_w)$ defined by the equations

$$\lambda q - t(\mu v + h) - sw = \lambda \mu t - s^2 = \lambda u + st = -\lambda v + t^2 = 0.$$

It is now straightforward to check that $\tilde{D}_q \cap \tilde{D}_p \cap E = (p = q = 0) \cap E$ is a finite set of points (in fact, it consists of 2 points). This shows that $\tilde{D}_q \cdot \tilde{D}_p = \tilde{\Gamma}$ since $D_q \cdot D_p = \Gamma$. We have

$$(\tilde{D}_p \cdot \tilde{\Gamma}) = (\tilde{D}_p^2 \cdot \tilde{D}_q) = 6(-K_X)^3 - \tfrac{e}{6^3}(E^3) = \tfrac{1}{7} - \tfrac{e}{30} < 0$$

since $e \geq 6$. By [Okada 2018, Lemma 2.18] p is not a maximal center. □

5. On further problems

Prime Fano 3-folds with no projection centers. We further investigate birational superrigidity of prime Fano 3-folds of codimension c with no projection centers for $5 \leq c \leq 9$. There are only a few such candidates, which can be summarized as follows.

- In codimension $c \in \{5, 7\}$ there is a unique candidate and it corresponds to smooth prime Fano 3-folds of degree $2c + 2$. All of these Fano 3-folds are rational (see [Iskovskikh and Prokhorov 1999, Corollary 4.3.5 or Section 12.2]) and are not birationally superrigid.
- In codimension 6 there are 2 candidates; one candidate corresponds to smooth prime Fano 3-folds of degree 14 which are birational to smooth cubic 3-folds (see [Takeuchi 1989; Iskovskikh 1979]) and are not birationally superrigid, and the existence is not known for the other candidate which is #78 in the database.

- In codimension 8 there are 2 candidates; one corresponds to smooth prime Fano 3-folds of degree 18 which are rational (see [Iskovskikh and Prokhorov 1999, Corollary 4.3.5 or Section 12.2]), and the existence is not known for the other candidate which is #33 in the database.
- In codimension 9 there is a unique candidate of smooth prime Fano 3-folds of degree 20. However, according to the classification of smooth Fano 3-folds there is no such Fano 3-fold (see e.g., [Takeuchi 1989, Theorem 0.1]).

It follows that, in codimension up to 9, #33 and #78 are the only remaining unknown cases for birational superrigidity (of general members).

Question 5.1. Do there exist prime Fano 3-folds which correspond to #33 or #78? If yes, then is a (general) such Fano 3-fold birationally superrigid?

In codimension 10 and higher there are a lot of candidates of Fano 3-folds with no projection centers. We expect that many of them are nonexistence cases and that there are only a few birationally superrigid Fano 3-folds in higher codimensions.

Question 5.2. Is there a numerical type (in other words, graded ring database ID) #i in codimension greater than 9 such that a (general) quasismooth prime Fano 3-fold of numerical type #i is birationally superrigid?

Classification of birationally superrigid Fano 3-folds. There are many difficulties in the complete classification of birationally superrigid Fano 3-folds. For example, we need to consider Fano 3-folds which are not necessarily quasismooth or not necessarily prime. We also need to understand subtle behaviors of birational superrigidity in a family.

Question 5.3. Is there a birationally superrigid Fano 3-fold which is either of Fano index greater than 1 or has a nonquotient singularity?

Remark 5.4. By recent developments [Pukhlikov 2019; Suzuki 2017; Liu and Zhuang 2019], it is known that there exist birationally superrigid Fano varieties which have nonquotient singularities at least in very high dimensions. On the other hand, only a little is known for Fano varieties of index greater than 1 (see [Pukhlikov 2016]) and there is no example of birationally superrigid Fano varieties of index greater than 1.

We concentrate on quasismooth prime Fano 3-folds. Even in that case, it is necessary to consider those with a projection center, which are not treated in this paper. Let X be a general quasismooth prime Fano 3-fold of codimension c. Then the following are known:

- When $c = 1$, X is birationally superrigid if and only if X does not admit a type I projection center (see [Iskovskikh and Manin 1971; Corti et al. 2000; Cheltsov and Park 2017]).
- When $c = 2, 3$, X is birationally superrigid if and only if X is singular and admits no projection center (see [Iskovskikh and Pukhlikov 1996; Okada 2014a; Ahmadinezhad and Zucconi 2016; Ahmadinezhad and Okada 2018]).

With this evidence we expect the following.

Conjecture 5.5. Let X be a general quasismooth prime Fano 3-fold of codimension at least 2. Then X is birationally superrigid if and only if X is singular and admits no projection centers.

Acknowledgements

Okada would like to thank Stephen Coughlan for fruitful information on cluster Fano 3-folds. He also would like to thank the referees for valuable suggestions. He is partially supported by JSPS KAKENHI Grant Number JP18K03216.

References

[Ahmadinezhad and Okada 2018] H. Ahmadinezhad and T. Okada, "Birationally rigid Pfaffian Fano 3-folds", *Algebr. Geom.* **5**:2 (2018), 160–199. MR Zbl

[Ahmadinezhad and Zucconi 2016] H. Ahmadinezhad and F. Zucconi, "Mori dream spaces and birational rigidity of Fano 3-folds", *Adv. Math.* **292** (2016), 410–445. MR Zbl

[Altınok 1998] S. Altınok, *Graded rings corresponding to polarised K3 surfaces and* $\mathbb{Q}$*-Fano 3-folds*, Ph.D. thesis, University of Warwick, 1998.

[Altınok et al. 2002] S. Altınok, G. Brown, and M. Reid, "Fano 3-folds, $K3$ surfaces and graded rings", pp. 25–53 in *Topology and geometry: commemorating SISTAG* (Singapore, 2001), edited by A. J. Berrick et al., Contemp. Math. **314**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[Brown and Kasprzyk 2009] G. Brown and A. Kasprzyk, "The graded ring database", online database, 2009, Available at www.grdb.co.uk.

[Brown et al. 2012] G. Brown, M. Kerber, and M. Reid, "Fano 3-folds in codimension 4, Tom and Jerry, I", *Compos. Math.* **148**:4 (2012), 1171–1194. MR Zbl

[Cheltsov and Park 2017] I. Cheltsov and J. Park, *Birationally rigid Fano threefold hypersurfaces*, Mem. Amer. Math. Soc. **1167**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl

[Chen et al. 2011] J.-J. Chen, J. A. Chen, and M. Chen, "On quasismooth weighted complete intersections", *J. Algebraic Geom.* **20**:2 (2011), 239–262. MR Zbl

[Corti 1995] A. Corti, "Factoring birational maps of threefolds after Sarkisov", *J. Algebraic Geom.* **4**:2 (1995), 223–254. MR Zbl

[Corti et al. 2000] A. Corti, A. Pukhlikov, and M. Reid, "Fano 3-fold hypersurfaces", pp. 175–258 in *Explicit birational geometry of 3-folds*, edited by A. Corti and M. Reid, Lond. Math. Soc. Lecture Note Ser. **281**, Cambridge Univ. Press, 2000. MR Zbl

[Coughlan and Ducat 2018] S. Coughlan and T. Ducat, "Constructing Fano 3-folds from cluster varieties of rank 2", preprint, 2018. arXiv

[Debarre et al. 2012] O. Debarre, A. Iliev, and L. Manivel, "On the period map for prime Fano threefolds of degree 10", *J. Algebraic Geom.* **21**:1 (2012), 21–59. MR Zbl

[Dimca 1986] A. Dimca, "Singularities and coverings of weighted complete intersections", *J. Reine Angew. Math.* **366** (1986), 184–193. MR Zbl

[Iano-Fletcher 2000] A. R. Iano-Fletcher, "Working with weighted complete intersections", pp. 101–174 in *Explicit birational geometry of 3-folds*, edited by A. Corti, Lond. Math. Soc. Lecture Note Ser. **281**, Cambridge Univ. Press, 2000. MR Zbl

[Iskovskikh 1979] V. A. Iskovskikh, "Birational automorphisms of three-dimensional algebraic varieties", pp. 159–236 in *Current problems in mathematics, XII*, edited by R. V. Gamkrelidze, VINITI, Moscow, 1979. In Russian; translated in *J. Soviet Math.* **13**:6 (1980), 815–868. MR Zbl

[Iskovskikh and Manin 1971] V. A. Iskovskikh and Y. I. Manin, "Three-dimensional quartics and counterexamples to the Lüroth problem", *Mat. Sb.* (*N.S.*) **86(128)**:1(9) (1971), 140–166. In Russian; translated in *Math. USSR-Sb.* **15**:1 (1971), 141–166. MR Zbl

[Iskovskikh and Prokhorov 1999] V. A. Iskovskikh and Y. G. Prokhorov, *Algebraic geometry, V: Fano varieties*, Encycl. Math. Sci. **47**, Springer, 1999. MR Zbl

[Iskovskikh and Pukhlikov 1996] V. A. Iskovskikh and A. V. Pukhlikov, "Birational automorphisms of multidimensional algebraic manifolds", *J. Math. Sci.* **82**:4 (1996), 3528–3613. MR Zbl

[Kawamata 1996] Y. Kawamata, "Divisorial contractions to 3-dimensional terminal quotient singularities", pp. 241–246 in *Higher-dimensional complex varieties* (Trento, Italy, 1994), edited by M. Andreatta and T. Peternell, de Gruyter, Berlin, 1996. MR Zbl

[Liu and Zhuang 2019] Y. Liu and Z. Zhuang, "Birational superrigidity and K-stability of singular Fano complete intersections", *Int. Math. Res. Not.* (online publication August 2019).

[Okada 2014a] T. Okada, "Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections", *Proc. Lond. Math. Soc.* (3) **109**:6 (2014), 1549–1600. MR Zbl

[Okada 2014b] T. Okada, "Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, III", 2014. To appear in *Kyoto J. Math.* arXiv

[Okada 2018] T. Okada, "Birational Mori fiber structures of $\mathbb{Q}$-Fano 3-fold weighted complete intersections, II", *J. Reine Angew. Math.* **738** (2018), 73–129. MR Zbl

[Pukhlikov 1998] A. V. Pukhlikov, "Birational automorphisms of Fano hypersurfaces", *Invent. Math.* **134**:2 (1998), 401–426. MR Zbl

[Pukhlikov 2016] A. V. Pukhlikov, "Birational geometry of Fano hypersurfaces of index two", *Math. Ann.* **366**:1-2 (2016), 721–782. MR Zbl

[Pukhlikov 2019] A. V. Pukhlikov, "Birationally rigid complete intersections with a singular point of high multiplicity", *Proc. Edinb. Math. Soc.* (2) **62**:1 (2019), 221–239. MR Zbl

[Suzuki 2017] F. Suzuki, "Birational rigidity of complete intersections", *Math. Z.* **285**:1-2 (2017), 479–492. MR Zbl

[Takeuchi 1989] K. Takeuchi, "Some birational maps of Fano 3-folds", *Compos. Math.* **71**:3 (1989), 265–283. MR Zbl

Communicated by János Kollár
Received 2019-01-31 Revised 2019-06-08 Accepted 2019-08-26

okada@cc.saga-u.ac.jp *Department of Mathematics, Saga University, Saga, Japan*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2020 is US $415/year for the electronic version, and $620/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 14 No. 1 2020

1937-0652(2020)14:1;1-0