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We determine birational superrigidity for a quasismooth prime Fano 3-fold of codimension 4 with no projection centers. In particular we prove birational superrigidity for Fano 3-folds of codimension 4 with no projection centers which were recently constructed by Coughlan and Ducat. We also pose some questions and a conjecture regarding the classification of birationally superrigid Fano 3-folds.

1. Introduction

A *prime Fano 3-fold* is a normal projective \mathbb{Q} -factorial 3-fold X with only terminal singularities such that $-K_X$ is ample and the class group $\mathrm{Cl}(X) \cong \mathbb{Z}$ is generated by $-K_X$. To such X there corresponds the anticanonical graded ring

$$R(X, -K_X) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, -mK_X),$$

and by choosing minimal generators we can embed X into a weighted projective space. By the *codimension* of X we mean the codimension of X in the weighted projective space. Based on analysis by Altınok, Brown, Iano-Fletcher, Kasprzyk, Prokhorov, Reid, and others (see for example [Altınok et al. 2002]) there is a database [Brown and Kasprzyk 2009] of numerical data (such as Hilbert series) coming from graded rings that can be the anticanonical graded ring of a prime Fano 3-fold. Currently it is not a classification, but it serves as a list, meaning that the anticanonical graded ring of a prime Fano 3-fold appears in the database.

The database contains a huge number of candidates, which suggests difficulty in the biregular classification of Fano 3-folds. The aim of this paper is to shed light on the classification of birationally superrigid Fano 3-folds. Here, a Fano 3-fold of Picard number 1 is said to be *birationally superrigid* if any birational map to a Mori fiber space is biregular. We remark that in [Ahmadinezhad and Okada 2018] a possible approach to achieving birational classification of Fano 3-folds is suggested by introducing notion of *solid Fano 3-folds*, which are Fano 3-folds not birational to either a conic bundles or a del Pezzo fibration.

Up to codimension 3, we have satisfactory results on the classification of quasismooth prime Fano 3-folds: the classification is complete in codimensions 1 and 2 [Iano-Fletcher 2000; Chen et al. 2011; Altınok 1998] and in codimension 3 the existence is known for all 70 numerical data in the database. Moreover birational superrigidity of quasismooth prime Fano 3-folds of codimension at most 3 has been well studied (see [Iskovskikh and Manin 1971; Corti et al. 2000; Cheltsov and Park 2017; Okada 2014a;

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Ahmadinezhad and Zucconi 2016; Ahmadinezhad and Okada 2018] and [Okada 2014b, 2018] for solid cases in codimension 2).

For quasismooth prime Fano 3-folds of codimension 4, there are 145 candidates of numerical data in [Brown and Kasprzyk 2009]. In [Brown et al. 2012], existence for 116 data is proved, where the construction is given by birationally modifying a known variety. This process is called unprojection and, as a consequence, a constructed Fano 3-fold corresponding to each of the 116 families admits a Sarkisov link to a Mori fiber space, hence it is not birationally superrigid. The 116 families of Fano 3-folds are characterized as those that possess a singular point which is so called a type I projection center (see [Brown et al. 2012] for details). There are other types of projection centers (such as types $\text{II}_1, \dots, \text{II}_7, \text{IV}$ according to the database [Brown and Kasprzyk 2009]). Through the known results in codimensions 1, 2 and 3, we can expect that the existence of a projection center violates birational superrigidity. Therefore it is natural to consider prime Fano 3-folds without projection centers for the classification of birational superrigid Fano 3-folds (see also the discussion in Section 5).

According to the database [Brown and Kasprzyk 2009], there are 5 candidates of quasismooth prime Fano 3-folds of codimension 4 with no projection centers. Those are identified by database numbers #25, #166, #282, #308 and #29374. Among them, #29374 corresponds to smooth prime Fano 3-folds of degree 10 embedded in \mathbb{P}^7 , and it is proved in [Debarre et al. 2012] that they are not birationally superrigid (not even birationally rigid, a weaker notion than superrigidity). Recently Coughlan and Ducat [2018] constructed many prime Fano 3-folds including those corresponding to #25 and #282 and we sometimes refer to these varieties as *cluster Fano 3-folds*. There are two constructions, $G_2^{(4)}$ and C_2 formats (see [Coughlan and Ducat 2018, Section 5.6] for details and see page 205 for concrete descriptions) for #282 and they are likely to sit in different components of the Hilbert scheme.

Theorem 1.1. *Let X be a quasismooth prime Fano 3-fold of codimension 4 and of numerical type #282 which is constructed in either $G_2^{(4)}$ format or C_2 format. If X is constructed in C_2 format, then we assume that X is general. Then X is birationally superrigid.*

For the remaining three candidates #25, #166 and #308, we can prove birational superrigidity in a stronger manner; we are able to prove birational superrigidity for these 3 candidates by utilizing only numerical data. Here, by numerical data for a candidate Fano 3-fold X , we mean the weights of the weighted projective space, degrees of the defining equations, the anticanonical degree $(-K_X)^3$ and the basket of singularities of X (see Section 3). Note that we do not know the existence of Fano 3-folds for #166 and #308.

Theorem 1.2. *Let X be a well-formed quasismooth prime Fano 3-fold of codimension 4 and of numerical type #25, #166 or #308. Then X is birationally superrigid.*

2. Birational superrigidity

Basic properties. Throughout this subsection we assume that X is a Fano 3-fold of Picard number 1, that is, X is a normal projective \mathbb{Q} -factorial 3-fold such that X has only terminal singularities, $-K_X$ is ample and $\text{rank Pic}(X) = 1$.

Definition 2.1. We say that X is *birationally superrigid* if any birational map $\sigma: X \dashrightarrow Y$ to a Mori fiber space $Y \rightarrow T$ is biregular.

By an *extremal divisorial extraction* $\varphi: (E \subset Y) \rightarrow (\Gamma \subset X)$, we mean an extremal divisorial contraction $\varphi: Y \rightarrow X$ from a normal projective \mathbb{Q} -factorial variety Y with only terminal singularities such that E is the φ -exceptional divisor and $\Gamma = \varphi(E)$.

Definition 2.2. Let $\mathcal{H} \sim_{\mathbb{Q}} -nK_X$ be a movable linear system, where n is a positive integer. A *maximal singularity* of \mathcal{H} is an extremal extraction $\varphi: (E \subset Y) \rightarrow (\Gamma \subset X)$ such that

$$c(X, \mathcal{H}) = \frac{a_E(K_X)}{m_E(\mathcal{H})} < \frac{1}{n},$$

where

- $c(X, \mathcal{H}) := \max\{\lambda \mid K_X + \lambda\mathcal{H} \text{ is canonical}\}$ is the *canonical threshold* of (X, \mathcal{H}) ,
- $a_E(K_X)$ is the discrepancy of K_X along E , and
- $m_E(\mathcal{H})$ is the multiplicity along E of the proper transform of \mathcal{H} .

We say that an extremal divisorial extraction is a *maximal singularity* if there exists a movable linear system \mathcal{H} such that the extraction is a maximal singularity of \mathcal{H} . A subvariety $\Gamma \subset X$ is called a *maximal center* if there is a maximal singularity $Y \rightarrow X$ whose center is Γ .

The following is the fundamental theorem in the study of birational superrigidity, which emerged in [Iskovskikh and Manin 1971] and has been simplified and generalized in [Pukhlikov 1998; Corti 1995].

Theorem 2.3 [Corti 1995, Theorem 4.10 and Proposition 2.10]. *If X admits no maximal center, then X is birationally superrigid.*

For the proof of birational superrigidity of a given Fano 3-fold X of Picard number 1 we need to exclude each subvariety of X as a maximal center. In the next subsection we will explain several methods of exclusion under a relatively concrete setting. Here we discuss methods of excluding terminal quotient singular points in a general setting.

For a terminal quotient singular point $p \in X$ of type $\frac{1}{r}(1, a, r-a)$, where r is coprime to a and $0 < a < r$, there is a unique extremal divisorial extraction $\varphi: (E \subset Y) \rightarrow (p \in X)$, which is the weighted blowup with weight $\frac{1}{r}(1, a, r-a)$, and we call it the *Kawamata blowup* (see [Kawamata 1996] for details). The integer $r > 1$ is called the *index* of $p \in X$. For the Kawamata blowup $\varphi: (E \subset Y) \rightarrow (p \in X)$, we have $K_Y = \varphi^*K_X + \frac{1}{r}E$ and

$$(E^3) = \frac{r^2}{a(r-a)}.$$

For a divisor D on X , the *order* of D along E , denoted by $\text{ord}_E(D)$, is defined to be the coefficient of E in φ^*D .

We first explain the most basic method.

Lemma 2.4 [Corti et al. 2000, Lemma 5.2.1]. *Let $p \in X$ a terminal quotient singular point and $\varphi: (E \subset Y) \rightarrow (p \in X)$ the Kawamata blowup. If $(-K_Y)^2 \notin \text{Int } \overline{\text{NE}}(Y)$, then p is not a maximal center.*

For the application of the above lemma, we need to find a nef divisor on Y . The following result, which is a slight generalization of [Okada 2018, Lemma 6.6], is useful.

Lemma 2.5. *Let $p \in X$ be a terminal quotient singular point and $\varphi: (E \subset Y) \rightarrow (p \in X)$ the Kawamata blowup. Assume that there are effective Weil divisors D_1, \dots, D_k such that the intersection $\text{Supp}(D_1) \cap \dots \cap \text{Supp}(D_k)$ does not contain a curve through p . We set*

$$e := \min \left\{ \frac{\text{ord}_E(D_i)}{n_i} \mid 1 \leq i \leq k \right\},$$

where n_i is the positive rational number such that $D_i \sim_{\mathbb{Q}} -n_i K_X$. Then $-\varphi^* K_X - \lambda E$ is a nef divisor for $0 \leq \lambda \leq e$.

Proof. We may assume $e > 0$, that is, $\text{Supp}(D_i)$ passes through p for any i . For an effective divisor $D \sim_{\mathbb{Q}} -n K_X$, we call $\text{ord}_E(D)/n$ the vanishing ratio of D along E . For $1 \leq i \leq k$, we choose a component of D_i , denoted D'_i , which has maximal vanishing ratio along E among the components of D_i . Clearly $D'_1 \cap \dots \cap D'_k$ does not contain a curve through p and we have

$$e' := \min \left\{ \frac{\text{ord}_E(D'_i)}{n'_i} \mid 1 \leq i \leq k \right\} \geq e,$$

where $n'_i \in \mathbb{Q}$ is such that $D'_i \sim_{\mathbb{Q}} -n'_i K_X$. Since D'_1, \dots, D'_k are prime divisors, we can apply [Okada 2018, Lemma 6.6] and conclude that $-\varphi^* K_X - e' E$ is nef. Then so is $-\varphi^* K_X - \lambda E$ for any $0 \leq \lambda \leq e'$ since $-\varphi^* K_X$ is nef, and the proof is completed. \square

We have another method of exclusion which can sometimes be effective when Lemma 2.4 is not applicable.

Lemma 2.6. *Let $p \in X$ be a terminal quotient singular point and $\varphi: (E \subset Y) \rightarrow (p \in X)$ the Kawamata blowup. Suppose that there exists an effective divisor S on X passing through p and a linear system \mathcal{L} of divisors on X passing through p with the following properties:*

- (1) $\text{Supp}(S) \cap \text{Bs } \mathcal{L}$ does not contain a curve passing through p .
- (2) For a general member $L \in \mathcal{L}$, we have

$$(-K_Y \cdot \tilde{S} \cdot \tilde{L}) \leq 0,$$

where \tilde{S}, \tilde{L} are the proper transforms of S, L on Y , respectively.

Then p is not a maximal center.

Proof. According to [Okada 2018, Lemma 2.20], it suffices to show that there exist infinitely many distinct curves on Y which intersect $-K_Y$ nonpositively and E positively. For a curve or a divisor Δ on X , we denote by $\tilde{\Delta}$ its proper transform on Y .

We write $L \sim -nK_X$. Write $S = \sum m_i S_i + T$, where $m_i > 0$, S_i is a prime divisor and T is an effective divisor which does not pass through p . We have

$$(-K_Y \cdot \tilde{T} \cdot \tilde{L}) = nl(-K_X)^3 \geq 0,$$

where $T \sim -lK_X$ for some $l \geq 0$. Since

$$0 \geq (-K_Y \cdot \tilde{S} \cdot \tilde{L}) = \sum m_i (-K_Y \cdot \tilde{S}_i \cdot \tilde{L}) + (-K_Y \cdot \tilde{T} \cdot \tilde{L}),$$

there is a component S_i for which $(-K_Y \cdot \tilde{S}_i \cdot \tilde{L}) \leq 0$. Since $p \in S_i \cap \text{Bs } \mathcal{L} \subset \text{Supp}(S) \cap \text{Bs } \mathcal{L}$, we may assume that S is a prime divisor by replacing S by S_i .

Write $\mathcal{L} = \{L_\lambda \mid \lambda \in \mathbb{P}^l\}$. For $\lambda \in \mathbb{P}^l$, we write $S \cdot L_\lambda = \sum_i c_i C_{\lambda,i}$, where $c_i \geq 0$ and $C_{\lambda,i}$ is an irreducible and reduced curve on X . Then,

$$\tilde{S} \cdot \tilde{L}_\lambda = \sum_i c_i \tilde{C}_{\lambda,i} + \Xi,$$

where Ξ is an effective 1-cycle supported on E . Since any component of Ξ is contracted by φ and $-K_Y$ is φ -ample, we have $(-K_Y \cdot \Xi) \geq 0$. Thus, for a general $\lambda \in \mathbb{P}^l$, we have

$$0 \geq (-K_Y \cdot \tilde{S} \cdot \tilde{L}_\lambda) \geq \sum_i c_i (-K_Y \cdot \tilde{C}_{\lambda,i}).$$

It follows that $(-K_Y \cdot \tilde{C}_{\lambda,i}) \leq 0$ for some i . We choose such a $\tilde{C}_{\lambda,i}$ and denote it by \tilde{C}_λ° . By assumption (1) the set

$$\{\tilde{C}_\lambda^\circ \mid \lambda \in \mathbb{P}^l \text{ is general}\}$$

consists of infinitely many distinct curves. We have $(-K_Y \cdot \tilde{C}_\lambda^\circ) \leq 0$ by the construction. We see that $(E \cdot \tilde{C}_\lambda^\circ) > 0$ since \tilde{C}_λ° is the proper transform of a curve passing through p . Therefore p is not a maximal center by [Okada 2018, Lemma 2.20]. \square

Fano varieties in a weighted projective space. Let X be a prime Fano 3-fold. As in the introduction, we choose minimal generators of the anticanonical ring $R(X, -K_X)$ and let $X \subset \mathbb{P} := \mathbb{P}(a_0, \dots, a_n)$ be the corresponding embedding. We say that $X \subset \mathbb{P}$ is *anticanonically embedded*. We denote by x_0, \dots, x_n the homogeneous coordinates of \mathbb{P} with $\deg x_i = a_i$. Let

$$F_1 = F_2 = \dots = F_N = 0$$

be defining equations of X inside \mathbb{P} , where $F_i \in \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous polynomial of degree d_i with respect to the grading $\deg x_i = a_i$. We assume that \mathbb{P} is *well-formed*, that is,

$$\gcd\{a_i \mid 0 \leq i \leq n, i \neq j\} = 1$$

for $j = 0, 1, \dots, n$. Note that X is not contained in a linear cone (i.e., a smaller weighted projective space in \mathbb{P}) by minimality of generators of $R(X, -K_X)$.

Definition 2.7. We say that X is *well-formed* if $\text{codim}_X(X \cap \text{Sing}(\mathbb{P})) \geq 2$. We say that X is *quasismooth* if the affine cone

$$(F_1 = F_2 = \cdots = F_N = 0) \subset \mathbb{A}^{n+1} = \text{Spec } \mathbb{C}[x_0, \dots, x_n]$$

is smooth outside the origin.

Remark 2.8. The description of $\text{Sing}(\mathbb{P})$ is given in [Remark 2.12](#) below. Under the assumption that $X \subset \mathbb{P}$ is an anticanonically embedded quasismooth prime Fano 3-fold, we believe that well-formedness is a very mild condition (or perhaps it is automatically satisfied). For example, a quasismooth weighted complete intersection $X \subset \mathbb{P}$ which is not contained in a linear cone is well-formed (see [\[Iano-Fletcher 2000, Theorem 6.17\]](#)).

In the following we assume that $X \subset \mathbb{P}$ is well-formed and quasismooth. For $0 \leq i \leq n$, we define $p_{x_i} = (0 : \cdots : 1 : \cdots : 0) \in \mathbb{P}$, where the unique 1 is in the $(i+1)$ -th position, and we define $D_i = (x_i = 0) \cap X$ which is a Weil divisor such that $D_i \sim -a_i K_X$.

Lemma 2.9. *If $(-K_X)^3 \leq 1$, then no curve on X is a maximal center.*

Proof. The same proof of [\[Ahmadinezhad and Okada 2018, Lemma 2.1\]](#) applies in this setting without any change. □

Lemma 2.10. *Assume that $a_0 \leq a_1 \leq \cdots \leq a_n$. If $a_{n-1}a_n(-K_X)^3 \leq 4$, then no nonsingular point of X is a maximal center.*

Proof. The proof is almost identical to that of [\[Ahmadinezhad and Okada 2018, Lemma 2.6\]](#). □

Definition 2.11. Let $\mathcal{C} \subset \{x_0, \dots, x_n\}$ be a nonempty set of homogeneous coordinates. We define

$$\Pi(\mathcal{C}) := \bigcap_{z \in \mathcal{C}} (z = 0) \subset \mathbb{P}, \quad \Pi_X(\mathcal{C}) := \Pi(\mathcal{C}) \cap X \subset X.$$

Sometimes we denote

$$\Pi(\mathcal{C}) = \Pi(x_{i_1}, \dots, x_{i_m}), \quad \Pi_X(\mathcal{C}) = \Pi_X(x_{i_1}, \dots, x_{i_m}),$$

when $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\}$. We also define

$$\gcd(\mathcal{C}) := \gcd\{\deg(x_i) \mid x_i \in \mathcal{C}\}.$$

Remark 2.12. We explain some consequences of well-formedness and quasismoothness, which will be frequently used. We keep the above notation and assumptions:

- (1) For the singular locus, we have $\text{Sing}(X) = X \cap \text{Sing}(\mathbb{P})$. For the proof see [\[Dimca 1986, Proposition 8\]](#). Note that $X \subset \mathbb{P}$ is additionally assumed to be a (weighted) complete intersection in [\[loc. cit.\]](#) but the same proof applies.

(2) The singular locus of \mathbb{P} can be described as follows:

$$\text{Sing}(\mathbb{P}) = \bigcup_{\substack{\mathcal{C} \subseteq \{x_0, \dots, x_n\} \\ \gcd(\{x_0, \dots, x_n\} \setminus \mathcal{C}) > 1}} \Pi(\mathcal{C}).$$

By (1), we have

$$\text{Sing}(X) = \bigcup_{\substack{\mathcal{C} \subseteq \{x_0, \dots, x_n\} \\ \gcd(\{x_0, \dots, x_n\} \setminus \mathcal{C}) > 1}} \Pi_X(\mathcal{C}).$$

(3) For $\mathcal{C} \subset \{x_0, \dots, x_n\}$, we define

$$\Pi_X^*(\mathcal{C}) := \Pi_X(\mathcal{C}) \cap \left(\bigcap_{z \in \{x_0, \dots, x_n\} \setminus \mathcal{C}} (z \neq 0) \right).$$

Let $\mathcal{C} \subset \{x_0, \dots, x_n\}$ be a subset such that $r := \gcd(\{x_0, \dots, x_n\} \setminus \mathcal{C}) > 1$. Then any point $p \in X$ which is contained in $\Pi_X^*(\mathcal{C})$ is a cyclic quotient singular point of index r and hence any point $p \in X$ which is contained in $\Pi_X(\mathcal{C})$ is a cyclic quotient singular point of index divisible by r .

Lemma 2.13. *Let $p \in X$ be a singular point of type $\frac{1}{2}(1, 1, 1)$ and let*

$$b := \max\{a_i \mid 0 \leq i \leq n, a_i \text{ is odd}\}.$$

If $2b(-K_X)^3 \leq 1$, then p is not a maximal center.

Proof. Let $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\}$ be the set of homogeneous coordinates of odd degree. The set $\Pi_X(\mathcal{C}) = D_{i_1} \cap \dots \cap D_{i_m}$ consists of singular points by [Remark 2.12](#). In particular $\Pi_X(\mathcal{C})$ is a finite set of points since X has only terminal singular points. Let $\varphi: (E \subset Y) \rightarrow (p \in X)$ be the Kawamata blowup. Then $\text{ord}_E(D_{i_j}) \geq \frac{1}{2}$ since $2D_{i_j}$ is a Cartier divisor passing through p and thus $-b\varphi^*K_X - \frac{1}{2}E$ is nef by [Lemma 2.5](#). We have

$$(-b\varphi^*K_X - \frac{1}{2}E) \cdot (-K_Y)^2 = b(-K_X)^3 - \frac{1}{2} \leq 0.$$

This shows that $(-K_Y)^2 \notin \text{Int } \overline{\text{NE}}(Y)$ and p is not a maximal center by [Lemma 2.4](#). □

Definition 2.14. Let $p = p_{x_k} \in X$ be a terminal quotient singular point of type $\frac{1}{a_k}(1, c, a_k - c)$ for some c with $1 \leq c \leq \frac{1}{2}a_k$. For a nonempty subset $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\} \subset \{x_0, \dots, x_n\}$, we define

$$\text{ivr}_p(\mathcal{C}) := \min_{1 \leq j \leq m} \left\{ \frac{\bar{a}_{i_j}}{a_{i_j} a_k} \right\},$$

where \bar{a}_{i_j} is the integer such that $1 \leq \bar{a}_{i_j} \leq a_k$ and \bar{a}_{i_j} is congruent to a_{i_j} modulo a_k , and call it the *initial vanishing ratio* of \mathcal{C} at p .

Definition 2.15. For a terminal quotient singularity p of type $\frac{1}{r}(1, a, r - a)$, we define

$$\text{wp}(p) := a(r - a),$$

and call it the *weight product* of p .

Lemma 2.16. *Let $p = p_{x_k} \in X$ be a terminal quotient singular point. Suppose that there exists a subset $\mathcal{C} \subset \{x_0, \dots, x_n\}$ satisfying the following properties:*

- (1) $p \in \Pi_X(\mathcal{C})$, or equivalently $x_k \notin \mathcal{C}$.
- (2) $\Pi_X(\mathcal{C} \cup \{x_k\}) = \emptyset$.
- (3) $\text{ivr}_p(\mathcal{C}) \geq \text{wp}(p)(-K_X)^3$.

Then p is not a maximal center.

Proof. We write $\mathcal{C} = \{x_{i_1}, \dots, x_{i_m}\}$. We claim that $\Pi_X(\mathcal{C}) = D_{i_1} \cap \dots \cap D_{i_m}$ is a finite set of points. Indeed, if $\Pi_X(\mathcal{C})$ contains a curve, then $\Pi_X(\mathcal{C} \cup \{x_k\}) = \Pi_X(\mathcal{C}) \cap D_k \neq \emptyset$ since D_k is an ample divisor on X . This is impossible by the assumption (2). Note that we have $\text{ord}_E(D_{i_j}) \geq \bar{a}_{i_j}/a_k$ (see [Ahmadinezhad and Okada 2018, Section 3]) so that

$$e := \min \left\{ \frac{\text{ord}_E(D_{i_j})}{a_{i_j}} \mid 1 \leq j \leq m \right\} \geq \text{ivr}_p(\mathcal{C}).$$

By Lemma 2.5, $-\varphi^*K_X - \text{ivr}_p(\mathcal{C})E$ is nef and we have

$$(-\varphi^*K_X - \text{ivr}_p(\mathcal{C})E)(-K_Y)^2 = (-K_X)^3 - \frac{\text{ivr}_p(\mathcal{C})}{\text{wp}(p)} \leq 0$$

by the assumption (3). Therefore $(-K_Y)^2 \notin \text{Int } \overline{\text{NE}}(Y)$ and p is not a maximal center. \square

Let $p \in X$ be a singular point such that it can be transformed to p_{x_k} by a change of coordinates. For simplicity of the description we assume $p = p_{x_0}$ and we set $r = a_0 > 1$. Let $\varphi: (E \subset Y) \rightarrow (p \in X)$ be the Kawamata blowup. We explain a systematic way to estimate $\text{ord}_E(D_i)$ for coordinates x_i and also an explicit description of φ . It is a consequence of the quasismoothness of X that after renumbering the defining equation we can write

$$F_l = \alpha_l x_0^{m_l} x_{i_l} + (\text{other terms}) \quad \text{for } 1 \leq l \leq n-3,$$

where $\alpha_l \in \mathbb{C} \setminus \{0\}$, m_l is a positive integer, $x_0, x_{i_1}, \dots, x_{i_{n-3}}$ are mutually distinct so that by denoting the other 3 coordinates as $x_{j_1}, x_{j_2}, x_{j_3}$ we have

$$\{x_0, x_{i_1}, \dots, x_{i_{n-3}}, x_{j_1}, x_{j_2}, x_{j_3}\} = \{x_0, \dots, x_n\},$$

and we can choose $x_{j_1}, x_{j_2}, x_{j_3}$ as local orbi-coordinates of X at p . In this case the singular point p is of type

$$\frac{1}{r}(a_{j_1}, a_{j_2}, a_{j_3}).$$

Definition 2.17 [Ahmadinezhad and Okada 2018, Definitions 3.6 and 3.7]. For an integer a , we denote by \bar{a} the positive integer such that $\bar{a} \equiv a \pmod{r}$ and $0 < \bar{a} \leq r$. We say that

$$\mathbf{w}(x_1, \dots, x_n) = \frac{1}{r}(b_1, \dots, b_n)$$

is an *admissible weight* at p if $b_i \equiv a_i \pmod{r}$ for any i .

For an admissible weight \mathbf{w} at \mathfrak{p} and a polynomial $f = f(x_0, \dots, x_n)$, we denote by $f^{\mathbf{w}}$ the lowest weight part of f , where we assume that $\mathbf{w}(x_0) = 0$.

We say that an admissible weight \mathbf{w} at \mathfrak{p} satisfies the *KBL condition* if $x_0^{m_l} x_{i_l} \in F_l^{\mathbf{w}}$ for $1 \leq l \leq n-3$ and

$$(b_{j_1}, b_{j_2}, b_{j_3}) = (\bar{a}_{j_1}, \bar{a}_{j_2}, \bar{a}_{j_3}).$$

Let $\mathbf{w}(x_1, \dots, x_n) = \frac{1}{r}(b_1, \dots, b_n)$ be an admissible weight at \mathfrak{p} satisfying the KBL condition. We denote by $\Phi_{\mathbf{w}}: Q_{\mathbf{w}} \rightarrow \mathbb{P}$ the weighted blowup at \mathfrak{p} with weight \mathbf{w} , and by $Y_{\mathbf{w}}$ the proper transform of X via $\Phi_{\mathbf{w}}$. Then the induced morphism $\varphi_{\mathbf{w}} = \Phi_{\mathbf{w}}|_{Y_{\mathbf{w}}}: Y_{\mathbf{w}} \rightarrow X$ coincides with the Kawamata blowup at \mathfrak{p} . From this we see that the exceptional divisor E is isomorphic to

$$E_{\mathbf{w}} := (g_1 = \dots = g_{n-3} = 0) \subset \mathbb{P}(b_1, \dots, b_n),$$

where $g_l = F_l^{\mathbf{w}}(1, x_1, \dots, x_n)$. Note that the KBL condition implies that the equations defining $E_{\mathbf{w}}$ cut out a copy of $\mathbb{P}(b_{j_1}, b_{j_2}, b_{j_3})$. We refer readers to [Ahmadinezhad and Okada 2018, Section 3] for details.

Lemma 2.18 [Ahmadinezhad and Okada 2018, Lemma 3.9]. *Let $\mathbf{w}(x_1, \dots, x_n) = \frac{1}{r}(b_1, \dots, b_n)$ be an admissible weight at $\mathfrak{p} \in X$ satisfying the KBL condition. Then the following assertions hold:*

- (1) *We have $\text{ord}_E(D_i) \geq b_i/r$ for any i .*
- (2) *If $F_l^{\mathbf{w}} = \alpha_l x_0^{m_l} x_{i_l}$, where $\alpha_l \in \mathbb{C} \setminus \{0\}$, for some $1 \leq l \leq n-3$, then the weight*

$$\mathbf{w}'(x_1, \dots, x_n) = \frac{1}{r}(b'_1, \dots, b'_n),$$

where $b'_j = b_j$ for $j \neq l$ and $b'_l = b_l + r$, satisfies the KBL condition. In particular, $\text{ord}_E(D_l) \geq (b_l + r)/r$.

We will use the following notation for a polynomial $f = f(x_0, \dots, x_n)$:

- For a monomial $p = x_0^{e_0} \dots x_n^{e_n}$, we write $p \in f$ if p appears in f with nonzero (constant) coefficient.
- For a subset $\mathcal{C} \subset \{x_0, \dots, x_n\}$ and $\Pi = \Pi(\mathcal{C})$, we denote by $f|_{\Pi}$ the polynomial in variables $\{x_0, \dots, x_n\} \setminus \mathcal{C}$ obtained by putting $x_i = 0$ for $x_i \in \mathcal{C}$ in f .

3. Proof of birational superrigidity by numerical data

We prove birational superrigidity of codimension 4 quasismooth prime Fano 3-folds with no projections by utilizing only numerical data. The numerical data for each Fano 3-fold will be described in the beginning of the corresponding subsection. The Fano 3-folds are embedded in a weighted projective 7-space, denoted by \mathbb{P} , and we use the symbol p, q, r, s, t, u, v, w for the homogeneous coordinates of \mathbb{P} . We use the following terminologies: Let $X \subset \mathbb{P}$ be a codimension 4 quasismooth prime Fano 3-fold. For a homogeneous coordinate $z \in \{p, q, \dots, w\}$,

- $D_z := (z = 0) \cap X$ is the Weil divisor on X cut out by z , and
- $\mathfrak{p}_z \in \mathbb{P}$ is the point at which only the coordinate z does not vanish.

Note that [Theorem 1.2](#) will follow from [Theorems 3.1, 3.2 and 3.4](#).

Fano 3-folds of numerical type #25. Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #25, whose data consist of the following:

- $X \subset \mathbb{P}(2_p, 5_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$.
- $(-K_X)^3 = \frac{1}{70}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (16, 17, 18, 18, 19, 20, 20, 21, 22)$.
- $\mathcal{B}_X = \{7 \times \frac{1}{2}(1, 1, 1), \frac{1}{5}(1, 1, 4), \frac{1}{7}(1, 2, 5)\}$.

Here the subscripts p, q, \dots, w of the weights means that they are the homogeneous coordinates of the indicated degrees, and \mathcal{B}_X indicates the numbers and the types of singular points of X .

Theorem 3.1. *Let X be a well-formed quasismooth prime codimension 4 Fano 3-fold of numerical type #25. Then X is birationally superrigid.*

Proof. By [Lemmas 2.9 and 2.10](#), no curve and no nonsingular point on X is a maximal center. By [Lemma 2.13](#), singular points of type $\frac{1}{2}(1, 1, 1)$ are not maximal centers.

Let p be the singular point of type $\frac{1}{5}(1, 1, 4)$. Replacing the coordinate v if necessary, we may assume $p = p_q$. We set $\mathcal{C} = \{p, s, u, v\}$. We have

$$\mathrm{ivr}_p(\mathcal{C}) = \frac{2}{35} = \mathrm{wp}(p)(-K_X)^3.$$

By [Lemma 2.16](#), it remains to show that $\Pi_X := \Pi_X(\mathcal{C} \cup \{q\}) = \emptyset$. We set $\Pi := \Pi(\mathcal{C} \cup \{q\}) \subset \mathbb{P}$ so that $\Pi_X = \Pi \cap X$. Since $p_t \notin X$, one of the defining polynomials contain a power of t . By looking at the degrees of F_1, \dots, F_9 , we have $t^2 \in F_1$. Similarly, we have $r^3 \in F_3$ and $w^2 \in F_9$ after possibly interchanging F_3 and F_4 . The monomial t^2 (resp. r^3) is the only monomial of degree 16 (resp. 18) consisting of the variables r, t, w . The monomials w^2 and t^2r are the only monomials of degree 22 consisting of the variables r, t, w . Hence, rescaling r, t, w , we can write

$$F_1|_{\Pi} = t^2, \quad F_3|_{\Pi} = r^3, \quad F_9|_{\Pi} = w^2 + \alpha t^2 r,$$

for some $\alpha \in \mathbb{C}$. The set Π_X is contained in the common zero locus of the above 3 polynomials inside Π . The equations have only trivial solution and this shows that $\Pi_X = \emptyset$. Thus p is not a maximal center.

Let $p = p_s$ be the singular point of type $\frac{1}{7}(1, 2, 5)$ and set $\mathcal{C} = \{p, q, r\}$. We have

$$\mathrm{ivr}_p(\mathcal{C}) = \frac{1}{7} = \mathrm{wp}(p)(-K_X)^3.$$

By [Lemma 2.16](#), it remains to show that $\Pi_X := \Pi_X(\mathcal{C} \cup \{s\}) = \emptyset$. We set $\Pi := \Pi(\mathcal{C} \cup \{s\}) \subset \mathbb{P}$ so that $\Pi_X = \Pi \cap X$. Since $p_t, p_u, p_v, p_w \notin X$, we may assume $t^2 \in F_1$, $u^2 \in F_3$, $v^2 \in F_6$ and $w^2 \in F_9$ after possibly interchanging defining polynomials of the same degree. Then we can write

$$F_1|_{\Pi} = t^2, \quad F_3|_{\Pi} = u^2 + \alpha vt, \quad F_6|_{\Pi} = v^2 + \beta wu, \quad F_9|_{\Pi} = w^2 + \gamma t^2 r,$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. This shows that $\Pi_X = \emptyset$ and thus \mathfrak{p} is not a maximal center. This completes the proof. \square

Fano 3-folds of numerical type #166. Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #166, whose data consist of the following:

- $X \subset \mathbb{P}(2_p, 2_q, 3_r, 3_s, 4_t, 4_u, 5_v, 5_w)$.
- $(-K_X)^3 = \frac{1}{6}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (8, 8, 8, 9, 9, 9, 10, 10, 10)$.
- $\mathcal{B}_X = \left\{ 11 \times \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2) \right\}$.

Theorem 3.2. *Let X be a well-formed quasismooth prime codimension 4 Fano 3-fold of numerical type #166. Then X is birationally superrigid.*

Proof. By [Lemma 2.9](#) no curve is a maximal center.

Let $\mathfrak{p} = (\alpha_p : \alpha_q : \dots : \alpha_w) \in X$ be a nonsingular point where $\alpha_p, \alpha_q, \dots, \alpha_w \in \mathbb{C}$. By [Remark 2.12](#), we have $\Pi_X(p, q, r, s, t, u) = \emptyset$ since X does not have a singular point of index 5. Then we can take a coordinate $x \in \{p, q, r, s, t, u\}$ such that $\mathfrak{p} \in (x \neq 0)$, i.e., $\alpha_x \neq 0$. The common zero locus of the homogeneous polynomials in the set

$$\{\alpha_y^{\deg x} x^{\deg y} - \alpha_x^{\deg y} y^{\deg x} \mid y \in \{p, q, r, s, t, u, v, w\} \setminus \{x\}\}$$

is a finite set of points including \mathfrak{p} . Any polynomial in the above set is of degree at most 20 since $x \notin \{v, w\}$. It follows that the base locus of $|\mathcal{I}_{\mathfrak{p}}^m(-mlK_X)|$ is a finite set of points, that is, $-lK_X$ isolates \mathfrak{p} (see [\[Corti et al. 2000, Definition 5.2.4 and Lemma 5.6.4\]](#)), where $l \leq 20$. By the argument in [\[loc. cit., Section 5.3\]](#), we conclude that \mathfrak{p} is not a maximal center since $20 < 4/(-K_X)^3$.

Let \mathfrak{p} be a singular point of type $\frac{1}{2}(1, 1, 1)$. After a change of coordinates, we may assume $\mathfrak{p} = \mathfrak{p}_p$. We set $\mathcal{C} = \{q, r, s, t, u\}$. We have

$$\text{ivr}_{\mathfrak{p}}(\mathcal{C}) = \frac{1}{6} = \text{wp}(\mathfrak{p})(-K_X)^3.$$

Moreover we have $\Pi_X(\mathcal{C} \cup \{p\}) = \emptyset$ because X is quasismooth and it does not have a singular point of index 5. Thus, by [Lemma 2.16](#), \mathfrak{p} is not a maximal center.

Let \mathfrak{p} be the singular point of type $\frac{1}{3}(1, 1, 2)$. After a change of coordinates, we may assume $\mathfrak{p} = \mathfrak{p}_s$. We set $\mathcal{C} = \{p, q, r\}$. Then we have

$$\text{ivr}_{\mathfrak{p}}(\mathcal{C}) = \frac{1}{3} = \text{wp}(\mathfrak{p})(-K_X)^3.$$

By [Lemma 2.16](#), it remains to show that $\Pi_X := \Pi_X(\mathcal{C} \cup \{s\}) = \emptyset$. We set $\Pi := \Pi(\mathcal{C} \cup \{s\}) \subset \mathbb{P}$ so that $\Pi_X = \Pi \cap X$. We have

$$\Pi_X = (F_1|_{\Pi} = F_2|_{\Pi} = F_3|_{\Pi} = F_7|_{\Pi} = F_8|_{\Pi} = F_9|_{\Pi} = 0) \cap \Pi.$$

We see that $F_1|_\Pi, F_2|_\Pi, F_3|_\Pi$ consist only of monomials in variables t, u and that $F_7|_\Pi, F_8|_\Pi, F_9|_\Pi$ consist only of monomials in variables v, w . It follows that

$$\Pi_X(p, q, r, s, v, w) = \Pi_X \cap \Pi(v, w) = (F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = 0) \cap \Pi(p, q, r, s, v, w).$$

We have $\Pi_X(p, q, r, s, v, w) = \emptyset$ since X is well-formed, quasismooth and X has no singular point of index 4 (see [Remark 2.12](#)). Hence the equations

$$F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = 0$$

imply $t = u = 0$. Similarly, by considering $\Pi_X(p, q, r, s, t, u) = \emptyset$, we see that the equations

$$F_7|_\Pi = F_8|_\Pi = F_9|_\Pi = 0$$

imply $v = w = 0$. It follows that $\Pi_X = \emptyset$ and \mathfrak{p} is not a maximal center. Therefore X is birationally superrigid. \square

Fano 3-folds of numerical type #282. Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #282, whose data consist of the following:

- $X \subset \mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$.
- $(-K_X)^3 = \frac{1}{42}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (16, 17, 18, 18, 19, 20, 20, 21, 22)$.
- $\mathcal{B} = \{2 \times \frac{1}{2}(1, 1, 1), 2 \times \frac{1}{3}(1, 1, 2), \frac{1}{6}(1, 1, 5), \frac{1}{7}(1, 1, 6)\}$.

Proposition 3.3. *Let X be a well-formed quasismooth prime codimension 4 Fano 3-fold of numerical type #282. Then no curve and no point is a maximal center except possibly for the singular point of type $\frac{1}{6}(1, 1, 5)$.*

Proof. By Lemmas [2.9](#), [2.10](#) and [2.13](#), it remains to exclude singular points of type $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{7}(1, 1, 6)$ as maximal centers.

Let \mathfrak{p} be a singular point of type $\frac{1}{3}(1, 1, 2)$ and let $\varphi: (E \subset Y) \rightarrow (\mathfrak{p} \in X)$ be the Kawamata blowup. We claim that $\Pi_X(p, s, t, w) = D_p \cap D_s \cap D_t \cap D_w$ is a finite set of points (containing \mathfrak{p}). Since X does not contain a singular point of index 10, we may assume that $v^2 \in F_6$. Then, by rescaling v , we have

$$F_6(0, q, r, 0, 0, u, v, 0) = v^2$$

and this shows that $\Pi_X(p, s, t, w) = \Pi_X(p, s, t, v, w)$. The latter set consists of singular points $\{2 \times \frac{1}{3}(1, 1, 2), \frac{1}{6}(1, 1, 5)\}$ (see [Remark 2.12](#)) and thus $\Pi_X(p, s, t, w)$ is a finite set of points. We have

$$\text{ord}_E(D_p), \text{ord}_E(D_s) \geq \frac{1}{3}, \quad \text{ord}_E(D_t), \text{ord}_E(D_w) \geq \frac{2}{3}.$$

By [Lemma 2.5](#), $N := -\varphi^*K_X - \frac{1}{21}E$ is a nef divisor on Y and we have $(N \cdot (-K_Y)^2) = 0$. Thus \mathfrak{p} is not a maximal center by [Lemma 2.4](#).

Let $p = p_s$ be the singular point of type $\frac{1}{7}(1, 1, 6)$ and set $\mathcal{C} = \{p, q, r\}$. We have

$$\mathrm{iv}_p(\mathcal{C}) = \frac{1}{7} = \mathrm{wp}(p)(-K_X)^3.$$

We set $\Pi := \Pi(\mathcal{C} \cup \{s\})$. We see that $p_t, p_u, p_v, p_w \notin X$ since X does not have a singular point of index 8, 9, 10, 11. It follows that $t^2 \in F_1$, $w^2 \in F_9$ and we may assume $u^2 \in F_3$, $v^2 \in F_6$. Then, by rescaling t, u, v, w , we can write

$$F_1|_{\Pi} = t^2, \quad F_3|_{\Pi} = \alpha vt + u^2, \quad F_6|_{\Pi} = \beta wu + v^2, \quad F_9|_{\Pi} = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. This shows that $\Pi_X(\mathcal{C} \cup \{s\}) = \Pi \cap X = \emptyset$. Thus p is not a maximal center by [Lemma 2.16](#) and the proof is completed. \square

Fano 3-folds of numerical type #308. Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #308, whose data consist of the following:

- $X \subset \mathbb{P}(1_p, 5_q, 6_r, 6_s, 7_t, 8_u, 9_v, 10_w)$.
- $(-K_X)^3 = \frac{1}{30}$.
- $\deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (14, 15, 16, 16, 17, 18, 18, 19, 20)$.
- $B_X = \left\{ \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2), \frac{1}{5}(1, 2, 3), 2 \times \frac{1}{6}(1, 1, 5) \right\}$.

Theorem 3.4. *Let X be a well-formed quasismooth prime Fano 3-fold of numerical type #308. Then X is birationally superrigid.*

Proof. By [Lemmas 2.9, 2.10](#) and [2.13](#) no curve and no nonsingular point is a maximal center and the singular point of type $\frac{1}{2}(1, 1, 1)$ is not a maximal center.

Let p be the singular point of type $\frac{1}{3}(1, 1, 2)$, which is necessarily contained in $(p = q = t = u = w = 0)$, and let $\varphi: (E \subset Y) \rightarrow (p \in X)$ be the Kawamata blowup. We set $\mathcal{C} = \{p, q, u\}$ and $\Pi = \Pi(\mathcal{C}) \subset \mathbb{P}$. Since $p_t, p_w \notin X$, we have $t^2 \in F_1$, $w^2 \in F_9$ and we can write

$$F_1|_{\Pi} = t^2, \quad F_9|_{\Pi} = w^2 + \alpha t^2 r + \beta t^2 s,$$

where $\alpha, \beta \in \mathbb{C}$. Thus,

$$\Pi_X(\mathcal{C}) = \Pi \cap X = \Pi_X(p, q, t, u, w),$$

and this consists of two $\frac{1}{6}(1, 1, 5)$ points and p . In particular $D_p \cap D_q \cap D_u = \Pi_X(\mathcal{C})$ is a finite set of points. We have

$$\mathrm{ord}_E(D_p) \geq \frac{1}{3}, \quad \mathrm{ord}_E(D_q) \geq \frac{2}{3}, \quad \mathrm{ord}_E(D_u) \geq \frac{2}{3},$$

hence $N := -8\varphi^*K_X - \frac{2}{3}E$ is a nef divisor on Y by [Lemma 2.5](#). We have

$$(N \cdot (-K_Y)^2) = 8(-K_X)^3 - \frac{2}{3} \cdot \frac{3^2}{2} = -\frac{1}{15} < 0.$$

By [Lemma 2.4](#), p is not a maximal center.

Let p be a singular point of type $\frac{1}{6}(1, 1, 5)$. After replacing r and s , we may assume $p = p_s$. We set $\mathcal{C} = \{p, q, r\}$. We have

$$\mathrm{ivr}_p(\mathcal{C}) = \frac{1}{6} = \mathrm{wp}(p)(-K_X)^3.$$

Since $p_t, p_u, p_v, p_w \notin X$, we may assume $t^2 \in F_1, u^2 \in F_3, v^2 \in F_6, w^2 \in F_9$ after possibly interchanging F_3 with F_4 and F_6 with F_7 . Then, by setting $\Pi = \Pi(\mathcal{C} \cup \{s\})$ and by rescaling t, u, v, w , we have

$$F_1|_{\Pi} = t^2, \quad F_3|_{\Pi} = u^2 + \alpha vt, \quad F_6|_{\Pi} = v^2 + \beta wu, \quad F_9|_{\Pi} = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. This shows that $\Pi_X(\mathcal{C} \cup \{s\}) = \emptyset$ and p is not a maximal center by [Lemma 2.16](#).

Finally, let p be a singular point of type $\frac{1}{5}(1, 2, 3)$ and let $\varphi: (E \subset Y) \rightarrow (p \in X)$ be the Kawamata blowup. Replacing the coordinate w , we may assume $p = p_q$. We write

$$F_3 = \lambda q^3 p + \mu q^2 r + \nu q^2 s + q f_{11} + f_{16}, \quad F_4 = \lambda' q^3 p + \mu' q^2 r + \nu' q^2 s + q g_{11} + g_{16},$$

where $\lambda, \mu, \nu, \lambda', \mu', \nu' \in \mathbb{C}$ and $f_{11}, f_{16}, g_{11}, g_{16} \in \mathbb{C}[p, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degrees. Since X is quasismooth at $p = p_q$ and is of type $\frac{1}{5}(1, 2, 3)$, the matrix

$$\begin{pmatrix} \frac{\partial F_3}{\partial p}(p) & \frac{\partial F_3}{\partial r}(p) & \frac{\partial F_3}{\partial s}(p) \\ \frac{\partial F_4}{\partial p}(p) & \frac{\partial F_4}{\partial r}(p) & \frac{\partial F_4}{\partial s}(p) \end{pmatrix} = \begin{pmatrix} \lambda & \mu & \nu \\ \lambda' & \mu' & \nu' \end{pmatrix}$$

is of rank 2.

We first consider the case where $\mu\nu' - \nu\mu' \neq 0$. By replacing r and s , we may assume that $\mu = \nu' = 1$ and $\lambda = \nu = \lambda' = \mu' = 0$. We consider the weight at p

$$\mathbf{w}(p, r, s, t, u, v, w) = \frac{1}{5}(1, 1, 1, 2, 3, 4, 5),$$

which is an admissible weight satisfying the KBL condition. Then $F_3^{\mathbf{w}} = q^2 r$ and $F_4^{\mathbf{w}} = q^2 s$, and this implies $\mathrm{ord}_E(D_r), \mathrm{ord}_E(D_s) \geq \frac{6}{5}$ by [Lemma 2.18](#). Note that $\mathrm{ord}_E(D_p) \geq \mathbf{w}(p) = \frac{1}{5}$ by [Lemma 2.18](#). We set $\mathcal{C} = \{p, r, s\}$ and $\Pi = \Pi(\mathcal{C} \cup \{q\})$. By rescaling t, u, v, w , we can write

$$F_1|_{\Pi} = t^2, \quad F_3|_{\Pi} = u^2 + \alpha vt, \quad F_6|_{\Pi} = v^2 + \beta wu, \quad F_9|_{\Pi} = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. Hence $\Pi_X(\mathcal{C} \cup \{q\}) = \emptyset$. Since D_q is an ample divisor, this implies that $D_p \cap D_r \cap D_s$ is a finite set of points (including p). By [Lemma 2.5](#), $N := -\varphi^* K_X - \frac{1}{5}E$ is a nef divisor on Y . We have

$$(N \cdot (-K_Y)^2) = (-K_X)^3 - \frac{1}{53}(E^3) = \frac{1}{30} - \frac{1}{30} = 0,$$

and this shows that p is not a maximal center.

Next we consider the case where $\mu\nu' - \nu\mu' = 0$. By replacing r and s suitably and by possibly interchanging F_3 and F_4 , we may assume that

$$F_3 = q^3 p + q f_{11} + f_{16}, \quad F_4 = q^2 s + q g_{11} + g_{16}.$$

Let \mathbf{w} be the same weight at p as in the previous case, which is again an admissible weight satisfying the KBL condition. It is straightforward to see that $F_3^{\mathbf{w}} = q^3 p$, so that $\mathrm{ord}_E(D_p) \geq \frac{6}{5}$. Let $\mathcal{L} \subset |-6K_X|$ be

the pencil generated by the sections r and s . Since $\text{ord}_E(D_r) = \frac{1}{5}$ and $\text{ord}_E(D_s) \geq \frac{1}{5}$, a general member $L \in \mathcal{L}$ vanishes along E to order $\frac{1}{5}$ so that $\tilde{L} \sim -6\varphi^*K_X - \frac{1}{5}E$. We have

$$(-K_Y \cdot \tilde{D}_p \cdot \tilde{L}) = 6(-K_X)^3 - \frac{\text{ord}_E(D_p)}{5^2} \cdot (E^3) = \frac{1}{5} - \frac{\text{ord}_E(D_p)}{6} \leq 0$$

since $\text{ord}_E(D_p) \geq \frac{6}{5}$. By [Lemma 2.6](#), p is not a maximal center and the proof is complete. \square

4. Birational superrigidity of cluster Fano 3-folds

In this section we prove [Theorem 1.1](#) which follows from [Theorems 4.2](#) and [4.4](#) below.

#282 by $G_2^{(4)}$ format. Let X be a quasismooth codimension 4 prime Fano 3-fold of numerical type #282 constructed in $G_2^{(4)}$ format. Then, by [\[Coughlan and Ducat 2018, Example 5.5\]](#), X is defined by the following polynomials in $\mathbb{P} := \mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$:

$$\begin{aligned} F_1 &= t^2 - qv + sQ_9, \\ F_2 &= ut - qw + s(v + p^2t), \\ F_3 &= t(v + p^2t) - uQ_9 + q(qr + p^4t), \\ F_4 &= (w + p^4s)s - P_{12}q + u(u + p^2s), \\ F_5 &= tw - uv + s(qr + p^4t), \\ F_6 &= (qr + p^4t)t - Q_9w + v(v + p^2t), \\ F_7 &= rs^2 - wu + tP_{12}, \\ F_8 &= P_{12}Q_9 - (vw + p^4qw + p^2uv + uqr + str - stp^2), \\ F_9 &= rs(u + p^2s) - vP_{12} + w(w + p^4s). \end{aligned}$$

Here $P_{12}, Q_9 \in \mathbb{C}[p, q, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degree. Recall that $(-K_X)^3 = \frac{1}{42}$.

Lemma 4.1. *The following assertions hold:*

- (1) $r^2 \in P_{12}$ and $u \in Q_9$.
- (2) $X \subset \mathbb{P}$ is well-formed.

Proof. It is straightforward to check that X is quasismooth at $p_r \in X$ if and only if $r^2 \in P_{12}$ and $u \in Q_9$, and this proves (1).

We prove (2). We set

$$\Pi_2 := \Pi_X(p, s, u, w), \quad \Pi_3 := \Pi_X(p, s, t, v, w).$$

It is enough to show that neither Π_2 nor Π_3 contain a surface (note here that $P_{12}|_{\Pi_2} \neq 0$ by (1)). We see that Π_2 is isomorphic to the closed subscheme in $\mathbb{P}(6_q, 6_r, 8_t, 10_v)$ defined by the equations

$$t^2 - qv = tv + q^2r = qP_{12}|_{\Pi_2} = qrt + v^2 = tP_{12}|_{\Pi_2} = vP_{12}|_{\Pi_2} = 0.$$

We leave the readers to check that Π_2 does not contain a surface. We see that Π_3 is isomorphic to the closed subscheme in $\mathbb{P}(6_q, 6_r, 9_u)$ defined by the equations

$$-uQ_9|_{\Pi_3} + q^2r = -qP_{12}|_{\Pi_3} + u^2 = P_{12}|_{\Pi_3}Q_9|_{\Pi_3} - uqr = 0.$$

Hence Π_3 does not contain a surface since it is clearly a proper closed subset of the surface $\mathbb{P}(6, 6, 9)$. Thus $X \subset \mathbb{P}$ is well-formed. \square

Theorem 4.2. *Let X be a codimension 4 Fano 3-fold of numerical type #282 constructed in $G_2^{(4)}$ format. Then X is birationally superrigid.*

Proof. By Lemma 4.1, $X \subset \mathbb{P}$ is well-formed. We can apply Proposition 3.3 and it remains to exclude the singular point $p \in X$ of type $\frac{1}{6}(1, 1, 5)$ as a maximal center. We have $p = p_r$ since $p_r \in X$ and X has a unique singular point of index 6. We set $\mathcal{C} = \{p, q\}$, $\Pi = \Pi(\mathcal{C})$ and $\Gamma := \Pi_X(\mathcal{C}) = \Pi \cap X$.

We will show that Γ is an irreducible and reduced curve. By Lemma 4.1, we can write

$$P_{12}|_{\Pi} = \lambda r^2, \quad Q_9|_{\Pi} = \mu u,$$

where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Then we have

$$\begin{aligned} F_1|_{\Pi} &= t^2 + \mu su, & F_4|_{\Pi} &= ws + u^2, & F_7|_{\Pi} &= rs^2 - wu + \lambda tr^2, \\ F_2|_{\Pi} &= ut + sv, & F_5|_{\Pi} &= tw - uv, & F_8|_{\Pi} &= \lambda \mu r^2 u - (vw + str), \\ F_3|_{\Pi} &= tv - \mu u^2, & F_6|_{\Pi} &= -\mu uw + v^2, & F_9|_{\Pi} &= rsu - \lambda vr^2 + w^2. \end{aligned}$$

We work on the open subset U on which $w \neq 0$. Then $\Gamma \cap U$ is isomorphic to the $\mathbb{Z}/11\mathbb{Z}$ -quotient of the affine curve

$$(\lambda r^2 v + \mu^3 r v^6 - 1 = 0) \subset \mathbb{A}_{r,v}^2.$$

It is straightforward to check that the polynomial $\lambda r^2 v + \mu^3 r v^6 - 1$ is irreducible. Thus $\Gamma \cap U$ is an irreducible and reduced affine curve. It is also straightforward to check that

$$\Gamma \cap (w = 0) = X \cap (p = q = w = 0) = \{p_r, p_s\}.$$

This shows that Γ is an irreducible and reduced curve.

Let $\varphi: (E \subset Y) \rightarrow (p \in X)$ be the Kawamata blowup and let $\tilde{\Delta}$ be the proper transform via φ of a divisor or curve Δ on X . We show that $\tilde{D}_p \cap \tilde{D}_q \cap E$ does not contain a curve. Consider the weight

$$\mathbf{w}(p, q, s, t, u, v, w) = \frac{1}{6}(1, 6, 1, 2, 3, 4, 5),$$

which is clearly an admissible weight satisfying the KBL condition. We set $g_i = F_i^w(p, q, 1, s, t, u, v, w)$. We have

$$\begin{aligned} g_4 &= (w + p^4)s - \lambda q + u(u + p^2s), \\ g_7 &= s^2 + \lambda t, \\ g_8 &= \lambda \mu u - st, \\ g_9 &= s(u + p^2s) - \lambda v. \end{aligned}$$

Since E is isomorphic to the subvariety

$$(g_4 = g_7 = g_8 = g_9 = 0) \subset \mathbb{P}(1_p, 6_q, 1_s, 2_t, 3_u, 4_v, 5_w),$$

it is straightforward to check that $\tilde{D}_p \cap \tilde{D}_q \cap E$ consists of a finite set of points (in fact, 2 points). Thus we have $\tilde{D}_p \cdot \tilde{D}_q = \tilde{\Gamma}$ since $D_p \cdot D_q = \Gamma$.

We have

$$\tilde{D}_p \sim -\varphi^* K_X - \frac{1}{6}E, \quad \tilde{D}_q \sim -6\varphi^* K_X - \frac{e}{6}E,$$

for some integer $e \geq 6$ and hence

$$(\tilde{D}_p \cdot \tilde{\Gamma}) = (\tilde{D}_p^2 \cdot \tilde{D}_q) = \frac{1}{7} - \frac{e}{30} < 0.$$

By [Okada 2018, Lemma 2.18], p is not a maximal center. □

#282 by C_2 format. Let X be a quasismooth codimension 4 prime Fano 3-fold of numerical type #282 constructed in C_2 format. Then, by [Coughlan and Ducat 2018, Example 5.5], X is defined by the following polynomials in $\mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w)$:

$$\begin{aligned} F_1 &= tR_8 - S_6Q_{10} + su, \\ F_2 &= tu - wS_6 + sv, \\ F_3 &= rS_6^2 - vR_8 + u^2, \\ F_4 &= tQ_{10} - S_6P_{12} + sw, \\ F_5 &= rsS_6 - wR_8 + uQ_{10}, \\ F_6 &= rs^2 - P_{12}R_8 + Q_{10}^2, \\ F_7 &= rtS_6 - vQ_{10} + uw, \\ F_8 &= rst - wQ_{10} + uP_{12}, \\ F_9 &= rt^2 - vP_{12} + w^2. \end{aligned}$$

Here $P_{12}, Q_{10}, R_8, S_6 \in \mathbb{C}[p, q, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degree. In the following we assume that $q \in S_6$, and then, we assume that $S_6 = q$ by a change of coordinates.

Lemma 4.3. *Under the above setting, the following assertions hold:*

- (1) $r^2 \in P_{12}, v \in Q_{10}$ and $t \in R_8$.
- (2) $X \subset \mathbb{P}$ is well-formed.

Proof. We have $p_r \in X$ and X is quasismooth at p_r if and only if $r^2 \in P_{12}$. Similarly, it is easy to check that if $v \notin Q_{10}$ (resp. $t \notin R_8$), then X is not quasismooth at p_v (resp. p_r). This proves (1). We leave the readers to check that neither Π_2 nor Π_3 contain a surface, where Π_2, Π_3 are those given in the proof of [Lemma 4.1](#), and this proves (2). \square

Theorem 4.4. *Let X be a quasismooth prime codimension 4 Fano 3-fold of numerical type #282 constructed by C_2 format. We assume that $q \in S_6$. Then X is birationally superrigid.*

Proof. By [Lemma 4.3](#), we can apply [Proposition 3.3](#) and it remains to exclude the singular point p of type $\frac{1}{6}(1, 1, 5)$ as a maximal center.

The singular point p corresponds to the solution of the equation

$$p = s = t = u = v = w = S_6 = 0,$$

and thus $p = p_r$ since $S_6 = q$ by our setting. We set $\mathcal{C} = \{p, q\}$ and $\Pi = \Pi(\mathcal{C})$.

We will show that $\Gamma := \Pi \cap X$ is an irreducible and reduced curve. We have $\Pi_X(\{p, q, r, s\}) = \emptyset$ (see the proof of [Proposition 3.3](#)). Hence $\Gamma \cap (s = 0) = \Pi_X(\{p, q, s\})$ does not contain a curve and it remains to show that $\Gamma \cap U_s$ is irreducible and reduced, where $U_s := (s \neq 0) \subset \mathbb{P}$ is the open subset. By [Lemma 4.3](#) we can write

$$P_{12}|_{\Pi} = \lambda r^2, \quad Q_{10}|_{\Pi} = \mu v, \quad R_8|_{\Pi} = vt,$$

for some $\lambda, \mu, v \in \mathbb{C} \setminus \{0\}$, and we have $S_6|_{\Pi} = 0$. Note that $F_i|_{\Pi} = F_i|_{\Pi}(r, s, t, u, v, w)$ is a polynomial in variables r, s, t, u, v, w and we set $f_i = F_i|_{\Pi}(r, 1, t, u, v, w)$. Let $C \subset \mathbb{A}_{r,t,u,v,w}^5$ be the affine scheme defined by the equations

$$f_1 = f_2 = \cdots = f_9 = 0.$$

Then $\Gamma \cap U_s$ is isomorphic to the quotient of C by the natural $\mathbb{Z}/7\mathbb{Z}$ -action. We have

$$\begin{aligned} f_1 &= vt^2 + u, & f_4 &= \mu tv + w, & f_7 &= -\mu v^2 + uw, \\ f_2 &= tu + v, & f_5 &= -vtw + \mu uv, & f_8 &= rt - \mu vw + \lambda r^2 u, \\ f_3 &= -vtv + u^2, & f_6 &= r - \lambda vr^2 t + \mu^2 v^2, & f_9 &= rt^2 - \lambda r^2 v + w^2. \end{aligned}$$

By the equations $f_1 = 0$, $f_2 = 0$ and $f_4 = 0$, we have

$$u = -vt^2, \quad v = -tu = vt^3, \quad w = -\mu tv = -\mu vt^4.$$

By eliminating the variables u, v, w and cleaning up the equations, C is isomorphic to the hypersurface in $\mathbb{A}_{r,t}^2$ defined by

$$r - \lambda vr^2 t + \mu^2 v^2 t^6 = 0,$$

which is an irreducible and reduced curve since $\mu v \neq 0$, and so is $\Gamma \cap U_s$. Thus Γ is an irreducible and reduced curve.

Let $\varphi: (E \subset Y) \rightarrow (p \in X)$ be the Kawamata blowup. We have $e := \text{ord}_E(D_q) \geq \frac{6}{5}$ and $\text{ord}_E(D_p) = \frac{1}{6}$ so that we have

$$\tilde{D}_q \sim -6\varphi^*K_X - \frac{e}{6}E = -6K_Y + \frac{6-e}{6}E, \quad \tilde{D}_p \sim -\varphi^*K_X - \frac{1}{6}E = -K_Y.$$

We show that $\tilde{D}_q \cap \tilde{D}_p \cap E$ does not contain a curve. The Kawamata blowup φ is realized as the weighted blowup at p with the weight

$$\mathbf{w}(p, q, s, t, u, v, w) = \frac{1}{6}(1, 6, 1, 2, 3, 4, 5),$$

which is an admissible weight satisfying the KBL condition. We have

$$F_4^{\mathbf{w}} = -\lambda q r^2 + t(\mu v + h) + s w,$$

$$F_6^{\mathbf{w}} = -\lambda \mu t r^2 + r s^2,$$

$$F_8^{\mathbf{w}} = \lambda u r^2 + r s t,$$

$$F_9^{\mathbf{w}} = -\lambda v r^2 + r t^2,$$

where we define $h := Q_{10}^{\mathbf{w}} - \mu v$. Note that h is a linear combination of up, tp^2, sp^3, rp^4 and thus h is divisible by p . It follows that E is isomorphic to the subscheme in $\mathbb{P}(1_p, 6_q, 1_s, 2_t, 3_u, 4_v, 5_w)$ defined by the equations

$$\lambda q - t(\mu v + h) - s w = \lambda \mu t - s^2 = \lambda u + s t = -\lambda v + t^2 = 0.$$

It is now straightforward to check that $\tilde{D}_q \cap \tilde{D}_p \cap E = (p = q = 0) \cap E$ is a finite set of points (in fact, it consists of 2 points). This shows that $\tilde{D}_q \cdot \tilde{D}_p = \tilde{\Gamma}$ since $D_q \cdot D_p = \Gamma$. We have

$$(\tilde{D}_p \cdot \tilde{\Gamma}) = (\tilde{D}_p^2 \cdot \tilde{D}_q) = 6(-K_X)^3 - \frac{e}{6^3}(E^3) = \frac{1}{7} - \frac{e}{30} < 0$$

since $e \geq 6$. By [Okada 2018, Lemma 2.18] p is not a maximal center. □

5. On further problems

Prime Fano 3-folds with no projection centers. We further investigate birational superrigidity of prime Fano 3-folds of codimension c with no projection centers for $5 \leq c \leq 9$. There are only a few such candidates, which can be summarized as follows.

- In codimension $c \in \{5, 7\}$ there is a unique candidate and it corresponds to smooth prime Fano 3-folds of degree $2c + 2$. All of these Fano 3-folds are rational (see [Iskovskikh and Prokhorov 1999, Corollary 4.3.5 or Section 12.2]) and are not birationally superrigid.
- In codimension 6 there are 2 candidates; one candidate corresponds to smooth prime Fano 3-folds of degree 14 which are birational to smooth cubic 3-folds (see [Takeuchi 1989; Iskovskikh 1979]) and are not birationally superrigid, and the existence is not known for the other candidate which is #78 in the database.

- In codimension 8 there are 2 candidates; one corresponds to smooth prime Fano 3-folds of degree 18 which are rational (see [Iskovskikh and Prokhorov 1999, Corollary 4.3.5 or Section 12.2]), and the existence is not known for the other candidate which is #33 in the database.
- In codimension 9 there is a unique candidate of smooth prime Fano 3-folds of degree 20. However, according to the classification of smooth Fano 3-folds there is no such Fano 3-fold (see e.g., [Takeuchi 1989, Theorem 0.1]).

It follows that, in codimension up to 9, #33 and #78 are the only remaining unknown cases for birational superrigidity (of general members).

Question 5.1. Do there exist prime Fano 3-folds which correspond to #33 or #78? If yes, then is a (general) such Fano 3-fold birationally superrigid?

In codimension 10 and higher there are a lot of candidates of Fano 3-folds with no projection centers. We expect that many of them are nonexistence cases and that there are only a few birationally superrigid Fano 3-folds in higher codimensions.

Question 5.2. Is there a numerical type (in other words, graded ring database ID) # i in codimension greater than 9 such that a (general) quasismooth prime Fano 3-fold of numerical type # i is birationally superrigid?

Classification of birationally superrigid Fano 3-folds. There are many difficulties in the complete classification of birationally superrigid Fano 3-folds. For example, we need to consider Fano 3-folds which are not necessarily quasismooth or not necessarily prime. We also need to understand subtle behaviors of birational superrigidity in a family.

Question 5.3. Is there a birationally superrigid Fano 3-fold which is either of Fano index greater than 1 or has a nonquotient singularity?

Remark 5.4. By recent developments [Pukhlikov 2019; Suzuki 2017; Liu and Zhuang 2019], it is known that there exist birationally superrigid Fano varieties which have nonquotient singularities at least in very high dimensions. On the other hand, only a little is known for Fano varieties of index greater than 1 (see [Pukhlikov 2016]) and there is no example of birationally superrigid Fano varieties of index greater than 1.

We concentrate on quasismooth prime Fano 3-folds. Even in that case, it is necessary to consider those with a projection center, which are not treated in this paper. Let X be a general quasismooth prime Fano 3-fold of codimension c . Then the following are known:

- When $c = 1$, X is birationally superrigid if and only if X does not admit a type I projection center (see [Iskovskikh and Manin 1971; Corti et al. 2000; Cheltsov and Park 2017]).
- When $c = 2, 3$, X is birationally superrigid if and only if X is singular and admits no projection center (see [Iskovskikh and Pukhlikov 1996; Okada 2014a; Ahmadinezhad and Zucconi 2016; Ahmadinezhad and Okada 2018]).

With this evidence we expect the following.

Conjecture 5.5. Let X be a general quasismooth prime Fano 3-fold of codimension at least 2. Then X is birationally superrigid if and only if X is singular and admits no projection centers.

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