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Coble fourfold, $\mathfrak{S}_6$-invariant quartic threefolds, and Wiman–Edge sextics

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To Arnaud Beauville, on the occasion of his 70th birthday

We construct two small resolutions of singularities of the Coble fourfold (the double cover of the four-dimensional projective space branched over the Igusa quartic). We use them to show that all $\mathfrak{S}_6$-invariant three-dimensional quartics are birational to conic bundles over the quintic del Pezzo surface with the discriminant curves from the Wiman–Edge pencil. As an application, we check that $\mathfrak{S}_6$-invariant three-dimensional quartics are unirational, obtain new proofs of rationality of four special quartics among them and irrationality of the others, and describe their Weil divisor class groups as $\mathfrak{S}_6$-representations.

1. Introduction

Consider the projectivization $\mathbb{P}^5$ of the standard permutation representation of the symmetric group $\mathfrak{S}_6$ over an algebraically closed field $\Bbbk$ of characteristic zero, and the invariant hyperplane $\mathbb{P}^4$ given by the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0 \tag{1.1}$$

therein, where $x_1, \dots, x_6$ are homogeneous coordinates in $\mathbb{P}^5$. Consider the classical family of $\mathfrak{S}_6$-invariant quartics X_t, $t \in \Bbbk \cup \{\infty\}$, in this hyperplane defined by the equations

$$(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4) - t(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 = 0; \tag{1.2}$$

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studied in [Beauville 2013]. Every $\mathfrak{S}_6$-invariant quartic in $\mathbb{P}^4$ is one of the quartics X_t; moreover, most of these quartics have automorphism groups isomorphic to $\mathfrak{S}_6$, and every quartic threefold with a faithful $\mathfrak{S}_6$-action is isomorphic to some X_t (see Lemma 3.4). We refer to these quartics as $\mathfrak{S}_6$-*invariant quartics*.

Every quartic X_t is singular along a certain 30-point orbit $\Sigma_{30} \subset \mathbb{P}^4$ of the group $\mathfrak{S}_6$ (see Section 3.1), and Σ_{30} coincides with $\mathrm{Sing}(X_t)$ unless $t = \infty$ or t is in the finite *discriminant set*

$$\mathfrak{D} := \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{7}{10}\right\}. \tag{1.3}$$

For these special values of t the singular locus of X_t is even larger (see Theorem 3.3 for its detailed description).

The quartic $X_{1/4}$ that corresponds to the parameter $t = \frac{1}{4}$ is particularly interesting. Its equation can be written as

$$(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4) - \tfrac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 = 0 \tag{1.4}$$

inside the hyperplane (1.1). It is called the *Igusa quartic*. The Igusa quartic is singular along a union of 15 lines (that itself forms an interesting configuration CR, called the *Cremona–Richmond configuration*). In this sense, $X_{1/4}$ is the most singular of all $\mathfrak{S}_6$-invariant quartics, except for X_∞ (which is a double quadric, i.e., a quadric with an everywhere nonreduced scheme structure).

The quartic $X_{1/2}$ is known as the *Burkhardt quartic*. It has the largest symmetry group among the other quartics in this family (with the exception of X_∞); see [Coble 1906] and Lemma 3.4. It also has many other interesting properties; see for instance [Todd 1936; de Jong et al. 1990; Hunt 1996, Section 5].

The quartics $X_{1/6}$ and $X_{7/10}$ have been studied in [Cheltsov and Shramov 2016b], compare [Todd 1933; 1935; Cheltsov and Shramov 2014].

The double cover of $\mathbb{P}^4$ branched over the Igusa quartic is called the *Coble fourfold*. We denote it by $\mathscr{Y}$ and write

$$\pi : \mathscr{Y} \to \mathbb{P}^4$$

for the double covering morphism. The Coble fourfold can be written as a complete intersection in the weighted projective space $\mathbb{P}(2, 1^6)$ of the hyperplane (1.1) with the hypersurface

$$x_0^2 = (x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4) - \tfrac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2, \tag{1.5}$$

where x_0 is the coordinate of weight 2. The Coble fourfold $\mathscr{Y}$ is singular along the Cremona–Richmond configuration CR, because so is the Igusa quartic. Moreover, it has a big group of symmetries: it carries an action of the symmetric group $\mathfrak{S}_6$ by permutation of coordinates

$$g \cdot (x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6) := (x_0 : x_{g(1)} : x_{g(2)} : x_{g(3)} : x_{g(4)} : x_{g(5)} : x_{g(6)}), \tag{1.6}$$

and also the Galois involution $\sigma : \mathscr{Y} \to \mathscr{Y}$ of the double cover

$$\sigma(x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6) := (-x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6), \tag{1.7}$$

commuting with the symmetric group action. One can check (see Corollary 3.5) that they generate the whole automorphism group

$$\mathrm{Aut}(\mathscr{Y}) \cong \mathfrak{S}_6 \times \boldsymbol{\mu}_2,$$

where $\boldsymbol{\mu}_2$ denotes the group of order 2. Sometimes it is convenient to twist the action of the symmetric group by the Galois involution. The obtained action

$$g \diamond (x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6) := (\epsilon(g)x_0 : x_{g(1)} : x_{g(2)} : x_{g(3)} : x_{g(4)} : x_{g(5)} : x_{g(6)}), \tag{1.8}$$

where $g \in \mathfrak{S}_6$ and $\epsilon(g)$ is the sign of the permutation g, is called *the twisted action*. In contrast, the action (1.6) is called *the natural action*. It is important not to confuse between these two actions, so we strongly recommend the reader to keep an eye on them. Note however, that the actions agree on the alternating group $\mathfrak{A}_6 \subset \mathfrak{S}_6$. Similarly, if G is a subgroup of $\mathfrak{S}_6$, by the natural and the twisted action of G on $\mathscr{Y}$ we mean the restrictions to G of the natural and the twisted actions of $\mathfrak{S}_6$, respectively.

Recall that the group $\mathfrak{S}_6$ has outer automorphisms (in fact, the group $\mathrm{Out}(\mathfrak{S}_6)$ is of order 2; see for instance [Howard et al. 2008]) characterized by the property that they take a transposition in $\mathfrak{S}_6$ to a permutation of cycle type [2, 2, 2]; see Lemma 5.12 for other information about outer automorphisms. If the image of a subgroup $G \subset \mathfrak{S}_6$ under an outer automorphism is not conjugate to G, we call this image a *nonstandard* embedding of G. For instance, we have nonstandard embeddings of $\mathfrak{S}_5$, $\mathfrak{A}_5$, $\mathfrak{S}_4 \times \mathfrak{S}_2$, etc.

The first main result of this paper is a construction of two small resolutions of singularities of the Coble fourfold that are equivariant with respect to maximal proper subgroups of $\mathfrak{S}_6$; note that the rank of the $\mathfrak{S}_6$-invariant Weil divisor class group of $\mathscr{Y}$ (with respect both to the natural and the twisted action of $\mathfrak{S}_6$) equals 1; see Corollary 5.4, hence there are no small resolutions of singularities of $\mathscr{Y}$ equivariant with respect to the entire group $\mathfrak{S}_6$. The varieties $\mathscr{Y}_{4,2}$ and $\mathscr{Y}_{5,1}$ discussed below already appeared in [Farkas and Verra 2016] in a slightly different context. A smooth quintic del Pezzo surface S is unique up to isomorphism, and $\mathrm{Aut}(S) \cong \mathfrak{S}_5$; see for instance [Dolgachev 2012, Section 8.5]; we fix such an isomorphism.

Theorem 1.9. *Consider the twisted* $\mathfrak{S}_6$*-action* (1.8) *on the Coble fourfold* $\mathscr{Y}$:

(i) *For every nonstandard embedding* $\mathfrak{S}_4 \times \mathfrak{S}_2 \hookrightarrow \mathfrak{S}_6$ *there is an* $\mathfrak{S}_4 \times \mathfrak{S}_2$*-equivariant small resolution of singularities*

$$\rho_{4,2} \colon \mathscr{Y}_{4,2} = \mathrm{Bl}_{\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3}(\mathbb{P}^2 \times \mathbb{P}^2) \to \mathscr{Y},$$

where $\mathrm{Bl}_{\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3}(\mathbb{P}^2 \times \mathbb{P}^2)$ *is the blow up of* $\mathbb{P}^2 \times \mathbb{P}^2$ *at a general quadruple of points* $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ *in* $\mathbb{P}^2 \times \mathbb{P}^2$.

(ii) *For every nonstandard embedding* $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ *there is an* $\mathfrak{S}_5$*-equivariant small resolution of singularities*

$$\rho_{5,1} \colon \mathscr{Y}_{5,1} = \mathbb{P}_S(\mathscr{U}_3) \to \mathscr{Y},$$

where S *is the quintic del Pezzo surface and* $\mathscr{U}_3$ *is a vector bundle of rank* 3 *on* S.

(iii) *The maps $\rho_{4,2}$ and $\rho_{5,1}$ are isomorphisms over the complement of the Cremona–Richmond configuration* $\mathrm{CR} \subset \mathscr{Y}$ *and are uniquely defined up to the Galois involution σ of $\mathscr{Y}$ over $\mathbb{P}^4$ by the above properties.*

(iv) *For every nonstandard embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ and every subgroup $\mathfrak{S}_4 \subset \mathfrak{S}_5$ there is a unique $\mathfrak{S}_4$-equivariant small birational map $\theta_1 \colon \mathscr{Y}_{5,1} \dashrightarrow \mathscr{Y}_{4,2}$ such that the diagram*

$$\begin{array}{ccccc} \mathscr{Y}_{5,1} & & \overset{\theta_1}{- - - \to} & & \mathscr{Y}_{4,2} \\ & {\scriptstyle \rho_{5,1}} \searrow & & \swarrow {\scriptstyle \rho_{4,2}} & \\ {\scriptstyle p}\downarrow & & \mathscr{Y} & & \downarrow{\scriptstyle p_1} \\ S & & \xrightarrow{\varphi} & & \mathbb{P}^2 \end{array} \tag{1.10}$$

commutes, where $p \colon \mathscr{Y}_{5,1} = \mathbb{P}_S(\mathscr{U}_3) \to S$ is the natural projection, $p_1 : \mathscr{Y}_{4,2} \to \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the composition of the blow up with the first projection, and φ is the unique $\mathfrak{S}_4$-equivariant birational contraction $S \to \mathbb{P}^2$.

The vector bundle $\mathscr{U}_3$ is described explicitly in Section 2.2.

The Coble fourfold is constructed from the Igusa quartic $X_{1/4}$, but it turns out that it has a very interesting property with respect to all $\mathfrak{S}_6$-invariant quartics. Since the pencil $\{X_t\}$ is generated by $X_{1/4}$ and the double quadric X_∞, we have

$$X_{1/4} \cap X_t = X_\infty \cap X_t \quad \text{for any } t \notin \left\{\tfrac{1}{4}, \infty\right\}.$$

Hence the restriction of $X_{1/4}$ to X_t has multiplicity 2, so that the double cover $\pi : \mathscr{Y} \to \mathbb{P}^4$ splits over X_t. In other words, $\pi^{-1}(X_t)$ is the union of two irreducible components that are isomorphic to X_t and are swapped by the Galois involution (1.7). It is natural here to replace the parameter t in the pencil with the new parameter τ defined by

$$t = \frac{\tau^2 + 1}{4}, \tag{1.11}$$

and define the subvarieties $\mathscr{X}_\tau \subset \mathscr{Y} \subset \mathbb{P}(2, 1^6)$ by (1.1), (1.5), and the formula

$$x_0 + \frac{\tau}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2) = 0. \tag{1.12}$$

Note that $\mathscr{X}_\tau \subset \mathscr{Y}$ is fixed by the natural action of $\mathfrak{S}_6$, but *is not fixed* by the twisted action. This trivial observation leads to various reductions of groups of symmetries.

With this definition of $\mathscr{X}_\tau$ we have an equality (see Lemma 3.12)

$$\pi^{-1}(X_{(\tau^2+1)/4}) = \mathscr{X}_\tau \cup \mathscr{X}_{-\tau}.$$

The map $\sigma \colon \mathscr{X}_\tau \to \mathscr{X}_{-\tau}$ is an isomorphism and the map $\pi \colon \mathscr{X}_\tau \to X_{(\tau^2+1)/4}$ is an isomorphism for all $\tau \neq \infty$. The map $\pi \colon \mathscr{X}_\infty \to (X_\infty)_{\mathrm{red}}$ is the double covering branched over $(X_\infty)_{\mathrm{red}} \cap X_{1/4}$. Thus,

the threefolds $\mathscr{X}_\tau$ have the same singularities as the quartics X_t (except for $\mathscr{X}_\infty$ which becomes smooth away from the $\mathfrak{S}_6$-orbit Σ_{30}; see Remark 3.13).

We consider the preimages of the divisors $\mathscr{X}_\tau$ in the small resolutions $\mathscr{Y}_{5,1}$ and $\mathscr{Y}_{4,2}$:

$$\mathscr{X}_\tau^{5,1} := \rho_{5,1}^{-1}(\mathscr{X}_\tau), \quad \mathscr{X}_\tau^{4,2} := \rho_{4,2}^{-1}(\mathscr{X}_\tau). \tag{1.13}$$

Because of the mixture of the natural and the twisted action, the natural groups of symmetries of the maps $\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ and $\rho_{4,2}\colon \mathscr{X}_\tau^{4,2} \to \mathscr{X}_\tau$ (that is, the groups with respect to which these maps are equivariant) get smaller. In particular, for $\tau \neq 0, \infty$ the first of them reduces to $\mathfrak{A}_5$ and the other to

$$\mathfrak{A}_{4,2} := (\mathfrak{S}_4 \times \mathfrak{S}_2) \cap \mathfrak{A}_6 \cong \mathfrak{S}_4.$$

Our second main result is the following. Recall the discriminant set $\mathfrak{D}$ defined in (1.3).

Theorem 1.14. *The maps*

$$\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau \quad \textit{and} \quad \rho_{4,2}\colon \mathscr{X}_\tau^{4,2} \to \mathscr{X}_\tau$$

are birational contractions for all τ, and are small for $\tau \neq 0$. Similarly, the maps

$$\pi \circ \rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to X_{(\tau^2+1)/4} \quad \textit{and} \quad \pi \circ \rho_{4,2}\colon \mathscr{X}_\tau^{4,2} \to X_{(\tau^2+1)/4}$$

are birational contractions for all $\tau \neq \infty$, and are small for $\tau \neq 0, \infty$. Moreover, $\mathscr{X}_\tau^{5,1}$ is smooth (and thus $\rho_{5,1}$ is a small resolution of singularities of $\mathscr{X}_\tau$) unless

$$t = \frac{\tau^2+1}{4} \in \mathfrak{D}.$$

The above maps are equivariant with respect to the following group actions:

	$\rho_{5,1}$ *or* $\pi \circ \rho_{5,1}$	$\rho_{4,2}$ *or* $\pi \circ \rho_{4,2}$
$\tau \neq 0, \infty$	$\mathfrak{A}_5$	$\mathfrak{A}_{4,2}$
$\tau = 0$ *or* $\tau = \infty$	$\mathfrak{S}_5$	$\mathfrak{S}_{4,2}$

where all subgroups of $\mathfrak{S}_6$ are nonstandard and the action is twisted.

We use the above results to construct an interesting (birational) conic bundle structure on the quartics X_t as follows. The fourfold $\mathscr{Y}_{5,1} = \mathbb{P}_S(\mathscr{U}_3)$ by definition comes with a $\mathbb{P}^2$-fibration $p\colon \mathscr{Y}_{5,1} \to S$ over the quintic del Pezzo surface S. We consider its restriction to the threefolds $\mathscr{X}_\tau^{5,1} \subset \mathscr{Y}_{5,1}$. We show that the maps

$$p\colon \mathscr{X}_\tau^{5,1} \to S$$

are $\mathfrak{A}_5$-equivariant conic bundles (and for $\tau = 0, \infty$ they are $\mathfrak{S}_5$-equivariant). We also discuss their properties, and identify their discriminant curves in S with the Wiman–Edge pencil (see Section 3.2 for its definition and the choice of parametrization) of $\mathfrak{A}_5$-invariant divisors from the linear system $|-2K_S|$.

All this is combined in our third main result. Recall that a flat conic bundle $\mathscr{X} \to S$ is called *standard* if both $\mathscr{X}$ and S are smooth and the relative Picard rank $\rho(\mathscr{X}/S)$ equals 1.

Theorem 1.15. *The map $p\colon \mathscr{X}_\tau^{5,1} \to S$ is a flat conic bundle, equivariant with respect to the group $\mathfrak{A}_5$ (for $\tau = 0, \infty$ it is $\mathfrak{S}_5$-equivariant). It is a standard conic bundle unless*

$$t = \frac{\tau^2+1}{4} \in \mathfrak{D}.$$

Its discriminant locus is the curve $\Delta_{s(\tau)} \subset S$ from the Wiman–Edge pencil, where

$$s(\tau) = \frac{\tau^3-\tau}{5\tau^2+3} \tag{1.16}$$

for an appropriate choice of the resolution $\rho_{5,1}$.

We apply the above results in several ways. First, we prove unirationality of $\mathfrak{S}_6$-invariant quartics X_t (see Corollary 4.2). Further, we give a new and uniform proof of rationality and irrationality of the quartics X_t. For $t \notin \mathfrak{D}$ irrationality follows from the description of the intermediate Jacobian of a resolution of singularities of X_t via the Prym variety arising from the conic bundle; see Theorem 4.4. For $t \in \mathfrak{D}$ we show that the conic bundle can be transformed birationally into the product $S \times \mathbb{P}^1$, hence X_t is rational; see Theorem 4.6. Finally, we describe the class groups $\mathrm{Cl}(X_t)$ of Weil divisors of the quartics X_t as $\mathfrak{S}_6$-representations (see Theorem 5.1), and discuss G-Sarkisov links centered at these quartics for some subgroups $G \subset \mathfrak{S}_6$. We also prove unirationality and irrationality of the threefold $\mathscr{X}_\infty$, and describe its class group as an $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$-representation.

The plan of our paper is the following. In Section 2 we construct the resolutions of the Coble fourfold $\mathscr{Y}$ and prove Theorem 1.9. In Section 3 we discuss the conic bundle structures on the $\mathfrak{S}_6$-invariant quartics induced by the resolutions of the Coble fourfold, and prove Theorems 1.14 and 1.15. In Section 4 we prove rationality and irrationality of the quartics X_t, and in Section 5 we describe the $\mathfrak{S}_6$-action on their class groups. In the Appendix we discuss the Cremona–Richmond configuration $\mathrm{CR} = \mathrm{Sing}(X_{1/4})$ of 15 lines in $\mathbb{P}^4$ and show that such configuration is unique up to a projective transformation of $\mathbb{P}^4$.

Throughout the paper $\Bbbk$ denotes an algebraically closed field of characteristic zero; however, many constructions do not use the assumption that the field is algebraically closed. By $\boldsymbol{\mu}_n$ we denote the cyclic group of order n. Furthermore, we denote by

$$\mathfrak{S}_{n_1,n_2} \cong \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \subset \mathfrak{S}_{n_1+n_2} \quad \text{and} \quad \mathfrak{A}_{n_1,n_2} = \mathfrak{A}_{n_1+n_2} \cap \mathfrak{S}_{n_1,n_2} \subset \mathfrak{A}_{n_1+n_2} \tag{1.17}$$

the subgroup of $\mathfrak{S}_{n_1+n_2}$ that consists of permutations preserving the subsets of the first n_1 and the last n_2 indices, and its intersection with the alternating group $\mathfrak{A}_{n_1+n_2} \subset \mathfrak{S}_{n_1+n_2}$. Note that $\mathfrak{A}_{n-2,2} \cong \mathfrak{S}_{n-2}$.

2. Small resolutions of the Coble fourfold

Recall that the fourfold $\mathscr{Y}$ is defined by (1.5) as the double cover of $\mathbb{P}^4$ (considered as the hyperplane (1.1) in $\mathbb{P}^5$) branched over the Igusa quartic (1.4). It comes with the natural and the twisted actions of the symmetric group $\mathfrak{S}_6$; see (1.6) and (1.8), the double covering $\pi\colon \mathscr{Y} \to \mathbb{P}^4$ and its Galois involution $\sigma\colon \mathscr{Y} \to \mathscr{Y}$; see (1.7), commuting with both actions of $\mathfrak{S}_6$.

The fourfold $\mathscr{Y}$ has been studied by Coble [1915; 1916; 1917]. He showed that $\mathscr{Y}$ is a compactification of the moduli space of ordered sets of 6 points in the projective plane. A modern treatment of $\mathscr{Y}$ has been given in [Dolgachev and Ortland 1988; Matsumoto et al. 1992; Hunt 1996; Howard et al. 2008]; see also [Bauer and Verra 2010]. In particular, Dolgachev and Ortland [1988] proved that $\mathscr{Y}$ can be obtained as the GIT-quotient $(\mathbb{P}^2)^6 /\!\!/ \mathrm{SL}_3(\Bbbk)$ with respect to the diagonal action of $\mathrm{SL}_3(\Bbbk)$. In [Clingher et al. 2019] the variety $\mathscr{Y}$ came up in the study of moduli spaces of K3 surfaces. Hunt [1996] called it the Coble variety (he also denoted it by $\mathscr{Y}$). In the current paper we prefer to call $\mathscr{Y}$ the *Coble fourfold*.

Since the Coble fourfold $\mathscr{Y}$ is singular, it is interesting to construct its resolution of singularities that would be natural from the geometric point of view. One interesting resolution was provided by Naruki [1982]; see also [Hacking et al. 2009; Dolgachev et al. 2005, Section 2]. It has plenty of important properties due to its interpretation as a moduli space of cubic surfaces. However, it is quite big (it has a horde of exceptional divisors). On the other hand, one can observe that the variety $\mathscr{Y}$ has non-$\mathbb{Q}$-factorial singularities, so we can hope to have a nice *small* resolution (i.e., with exceptional locus of codimension 2).

In this section we construct two small resolutions of singularities of $\mathscr{Y}$; one is equivariant with respect to the subgroup $\mathfrak{S}_{4,2} \subset \mathfrak{S}_6$ and another is equivariant with respect to the subgroup $\mathfrak{S}_5 \subset \mathfrak{S}_6$. Note that in both cases a *nonstandard* embedding of the subgroup is used (equivalently, a standard embedding is composed with an outer automorphism of $\mathfrak{S}_6$) and in both cases we consider the *twisted* action of $\mathfrak{S}_6$ on $\mathscr{Y}$.

2.1. ***Blow up of*** $\mathbb{P}^2 \times \mathbb{P}^2$**.** Let W_3 be the irreducible three-dimensional representation of the symmetric group $\mathfrak{S}_4$ with the nontrivial determinant, i.e., a summand of the four-dimensional permutation representation. Explicitly, $\mathrm{W}_3 \cong \mathrm{R}(3, 1)$ in the notation of [Fulton and Harris 1991, Section 4.1]. Choose a $\mathfrak{S}_4$-orbit of length 4

$$\{P_0, P_1, P_2, P_3\} \subset \mathbb{P}(\mathrm{W}_3) \cong \mathbb{P}^2.$$

In appropriate coordinates such quadruple can be written as

$$P_0 = (1:1:1), \quad P_1 = (1:0:0), \quad P_2 = (0:1:0), \quad P_3 = (0:0:1). \tag{2.1}$$

Denote by

$$\overline{P_i P_j} \subset \mathbb{P}(\mathrm{W}_3), \quad 0 \leqslant i < j \leqslant 3,$$

the line passing through the points P_i and P_j.

Consider the diagonal action of $\mathfrak{S}_4$ on $\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$ and the diagonal quadruple

$$\boldsymbol{P} = \{\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3\} \subset \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3), \quad \boldsymbol{P}_i = (P_i, P_i).$$

Note that $\boldsymbol{P}$ is an $\mathfrak{S}_4$-orbit. The vector space $\mathrm{W}_3 \otimes \mathrm{W}_3$ can be regarded as a representation of the group $\mathfrak{S}_{4,2}$; see (1.17), where $\mathfrak{S}_4$ acts diagonally and the nontrivial element of $\mathfrak{S}_2$ interchanges the factors. The linear span of the points $\boldsymbol{P}_i$ in $\mathbb{P}(\mathrm{W}_3 \otimes \mathrm{W}_3)$ induces an embedding of the permutation representation $\Bbbk^4$ of $\mathfrak{S}_4$ (with the trivial action of $\mathfrak{S}_2$) into $\mathrm{W}_3 \otimes \mathrm{W}_3$. We denote by

$$\mathrm{W}_5 := (\mathrm{W}_3 \otimes \mathrm{W}_3)/\Bbbk^4 \tag{2.2}$$

the quotient five-dimensional representation of $\mathfrak{S}_{4,2}$. Note that as a representation of $\mathfrak{S}_4$ it is the direct sum $\mathrm{W}_5|_{\mathfrak{S}_4} \cong \mathrm{R}(2,2) \oplus \mathrm{R}(2,1,1)$; here we again use the (standard) notation oi [Fulton and Harris 1991, Section 4.1].

The linear projection $\mathrm{W}_3 \otimes \mathrm{W}_3 \to \mathrm{W}_5$ induces a rational map

$$\bar{\pi}_{4,2} \colon \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3) \hookrightarrow \mathbb{P}(\mathrm{W}_3 \otimes \mathrm{W}_3) \dashrightarrow \mathbb{P}(\mathrm{W}_5).$$

Note that the center of this projection is the linear span of the orbit $\boldsymbol{P}$ in $\mathbb{P}(\mathrm{W}_3 \otimes \mathrm{W}_3)$, which intersects $\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$ exactly by $\boldsymbol{P}$. Therefore, to regularize the map $\bar{\pi}_{4,2}$ we should consider the blow up $\mathscr{Y}_{4,2}$ of $\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$ in the quadruple $\boldsymbol{P}$

$$\mathscr{Y}_{4,2} := \mathrm{Bl}_{\boldsymbol{P}_0, \boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3}(\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)) \xrightarrow{\beta} \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3) \tag{2.3}$$

with β being the blow up morphism. This induces a commutative diagram:

$$\begin{array}{ccc} & \mathscr{Y}_{4,2} & \\ {\scriptstyle\beta}\swarrow & & \searrow{\scriptstyle\pi_{4,2}} \\ \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3) & \xrightarrow{\bar{\pi}_{4,2}} & \mathbb{P}(\mathrm{W}_5) \end{array} \tag{2.4}$$

By construction the fourfold $\mathscr{Y}_{4,2}$ is smooth and carries a faithful action of $\mathfrak{S}_{4,2}$. The above diagram is $\mathfrak{S}_{4,2}$-equivariant.

We are going to show that the map $\pi_{4,2} \colon \mathscr{Y}_{4,2} \to \mathbb{P}(\mathrm{W}_5)$ defined by the diagram (2.4) factors through the Coble fourfold; more precisely, $\pi_{4,2}$ factors as a composition

$$\mathscr{Y}_{4,2} \xrightarrow{\rho_{4,2}} \mathscr{Y} \xrightarrow{\pi} \mathbb{P}(\mathrm{W}_5),$$

with $\rho_{4,2}$ being a small $\mathfrak{S}_{4,2}$-equivariant resolution of singularities. We accomplish this in two steps.

First, consider the linear projection

$$\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3) \hookrightarrow \mathbb{P}(\mathrm{W}_3 \otimes \mathrm{W}_3) \dashrightarrow \mathbb{P}^5$$

from the linear span of the points $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$; as before, the latter linear span intersects $\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$ exactly by the triple $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$. If $(u_1 : u_2 : u_3)$ and $(v_1 : v_2 : v_3)$ are homogeneous coordinates on the first and the second factors of $\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$ such that (2.1) holds, this map is given by

$$((u_1 : u_2 : u_3), (v_1 : v_2 : v_3)) \mapsto (u_2v_3 : u_3v_1 : u_1v_2 : u_3v_2 : u_1v_3 : u_2v_1), \tag{2.5}$$

and it is easy to describe its structure. We denote by y_1, y_2, y_3, z_1, z_2, and z_3 the homogeneous coordinates on $\mathbb{P}^5$, so that the right-hand side of (2.5) is the point $(y_1 : y_2 : y_3 : z_1 : z_2 : z_3)$.

Lemma 2.6. *The linear projection $\mathbb{P}(W_3)\times\mathbb{P}(W_3)\dashrightarrow\mathbb{P}^5$ with center in the span of the points $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ induces an $\mathfrak{S}_{3,2}$-equivariant commutative diagram*

$$\begin{array}{ccccc} & & \mathrm{Bl}_{\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3}(\mathbb{P}(W_3)\times\mathbb{P}(W_3)) & & \\ & \overset{\beta'}{\swarrow} & & \overset{\rho'_{4,2}}{\searrow} & \\ \mathbb{P}(W_3)\times\mathbb{P}(W_3) & \dashrightarrow & & & \mathscr{Y}'_{4,2} \hookrightarrow \mathbb{P}^5 \end{array}$$

where β' is the blow up, $\mathscr{Y}'_{4,2}\subset\mathbb{P}^5$ is a singular cubic hypersurface given by the equation

$$y_1y_2y_3 = z_1z_2z_3, \tag{2.7}$$

and $\rho'_{4,2}$ is a small birational contraction. The map $\rho'_{4,2}$ contracts

- *the proper transforms of the six planes $\mathbb{P}(W_3)\times P_i$ and $P_i\times\mathbb{P}(W_3)$, $1\leqslant i\leqslant 3$, and*
- *the proper transforms of the three quadrics $\overline{P_iP_j}\times\overline{P_iP_j}$, $1\leqslant i<j\leqslant 3$,*

onto nine lines L_{ij}, $1\leqslant i,\,j\leqslant 3$, given in $\mathbb{P}^5$ by the equations

$$y_k = z_l = 0, \quad k\neq i,\ l\neq j.$$

Moreover, $\rho'_{4,2}$ is an isomorphism over the complement of the lines L_{ij}. Finally, the map $\rho'_{4,2}\circ(\beta')^{-1}$ takes the point $\boldsymbol{P}_0$ to the point $\boldsymbol{P}'_0=(1:1:1:1:1:1)\in\mathscr{Y}'_{4,2}$.

Proof. The map is toric, so everything is easy to describe. We skip the actual computation which is straightforward but tedious. □

The cubic fourfold (2.7) is known as *Perazzo primal*, [Dolgachev 2012, Exercise 9.16; Looijenga 2009, Section 6].

Using the equation (2.7) one can easily check that the union of the nine lines L_{ij} is the singular locus of the cubic $\mathscr{Y}'_{4,2}$.

The second step is to project the cubic $\mathscr{Y}'_{4,2}$ from the point $\boldsymbol{P}'_0$.

Lemma 2.8. *The linear projection $\overline{\pi}'_{4,2}\colon\mathscr{Y}'_{4,2}\dashrightarrow\mathbb{P}(W_5)$ from the point $\boldsymbol{P}'_0$ defines a regular map $\pi''_{4,2}\colon\mathrm{Bl}_{\boldsymbol{P}'_0}(\mathscr{Y}'_{4,2})\to\mathbb{P}(W_5)$ that fits into a commutative diagram*

$$\begin{array}{ccccc} & & \mathrm{Bl}_{\boldsymbol{P}'_0}(\mathscr{Y}'_{4,2}) & \xrightarrow{\rho''_{4,2}} & \mathscr{Y} \\ & \overset{\beta''}{\swarrow} & & \overset{\pi''_{4,2}}{\searrow}\quad\overset{\pi}{\swarrow} & \\ \mathscr{Y}'_{4,2} & & \overset{\overline{\pi}'_{4,2}}{\dashrightarrow} & & \mathbb{P}(W_5) \end{array} \tag{2.9}$$

where $\mathscr{Y}$ is the Coble fourfold, $\pi\colon\mathscr{Y}\to\mathbb{P}(W_5)$ is the double covering, and $\rho''_{4,2}$ is a small birational morphism. Furthermore, the exceptional locus of $\rho''_{4,2}$ is the union of proper transforms of the six planes $\Pi_w\subset\mathscr{Y}'_{4,2}$ given by the equations

$$z_i = y_{w(i)}, \quad 1\leqslant i\leqslant 3,$$

indexed by all bijections $w\colon \{1,2,3\} \to \{1,2,3\}$; the map $\rho''_{4,2}$ contracts them onto six lines in $\mathscr{Y}$ (i.e., rational curves that are isomorphically projected to lines in $\mathbb{P}(\mathrm{W}_5)$), and is an isomorphism over the complement of those.

Proof. Note that the point $\boldsymbol{P}'_0$ is a smooth point of the cubic $\mathscr{Y}'_{4,2}$, so the projection from it factors through a double covering of $\mathbb{P}(\mathrm{W}_5)$; in fact, this is the Stein factorization for the morphism $\pi''_{4,2}$. We have to identify its branch divisor with the Igusa quartic.

Take a point

$$(y_i : z_i) = (y_1 : y_2 : y_3 : z_1 : z_2 : z_3)$$

in $\mathbb{P}^5$ which is different from $\boldsymbol{P}'_0$. The line $M_{(y_i:z_i)}$ in $\mathbb{P}(\mathrm{W}_5)$ passing through the point $(y_i : z_i)$ and the point $\boldsymbol{P}'_0$ can be parametrized as

$$M_{(y_i:z_i)} = \{(\lambda + \mu y_1 : \lambda + \mu y_2 : \lambda + \mu y_3 : \lambda + \mu z_1 : \lambda + \mu z_2 : \lambda + \mu z_3)\}, \tag{2.10}$$

where λ and μ are considered as homogeneous coordinates on this line. Substituting this parametrization into (2.7), we see that the intersection of $M_{(y_i:z_i)}$ with the cubic $\mathscr{Y}'_{4,2}$ is given by the equation

$$(\lambda + \mu y_1)(\lambda + \mu y_2)(\lambda + \mu y_3) = (\lambda + \mu z_1)(\lambda + \mu z_2)(\lambda + \mu z_3).$$

Expanding both sides and canceling the factor μ that corresponds to the intersection point $\boldsymbol{P}'_0$, we can rewrite the above equation as

$$(\boldsymbol{s}_1(y) - \boldsymbol{s}_1(z))\lambda^2 + (\boldsymbol{s}_2(y) - \boldsymbol{s}_2(z))\lambda\mu + (\boldsymbol{s}_3(y) - \boldsymbol{s}_3(z))\mu^2 = 0, \tag{2.11}$$

where $\boldsymbol{s}_d$ denotes the elementary symmetric polynomial of degree d. Restricting (2.11) to the hyperplane

$$y_1 + y_2 + y_3 + z_1 + z_2 + z_3 = 0, \tag{2.12}$$

which is identified by the linear projection $\bar{\pi}'_{4,2}$ from the point $\boldsymbol{P}'_0$ with the space $\mathbb{P}(\mathrm{W}_5)$, we obtain the equation of the double cover over $\mathbb{P}(\mathrm{W}_5)$ we are interested in (embedded into the projectivization of the vector bundle $\mathscr{O}_{\mathbb{P}(\mathrm{W}_5)} \oplus \mathscr{O}_{\mathbb{P}(\mathrm{W}_5)}(-1)$ over $\mathbb{P}(\mathrm{W}_5)$). The branch divisor of $\bar{\pi}'_{4,2}$ is given in the hyperplane (2.12) by the discriminant of the quadratic (2.11)

$$(\boldsymbol{s}_2(y) - \boldsymbol{s}_2(z))^2 - 4(\boldsymbol{s}_1(y) - \boldsymbol{s}_1(z))(\boldsymbol{s}_3(y) - \boldsymbol{s}_3(z)) = 0. \tag{2.13}$$

Let us show that the quartic $X'' \subset \mathbb{P}^4$ defined by equations (2.12) and (2.13) is isomorphic to the Igusa quartic; this will identify the double covering with the Coble fourfold in a way respecting the projection to $\mathbb{P}^4$, that is, ensuring that the upper right triangle in diagram (2.9) is commutative.

To do this we use the following substitutions:

$$\begin{aligned} x_1 &= y_1 - \tfrac{2}{3}\boldsymbol{s}_1(y) + \tfrac{1}{3}\boldsymbol{s}_1(z), \quad & x_4 &= z_1 + \tfrac{1}{3}\boldsymbol{s}_1(y) - \tfrac{2}{3}\boldsymbol{s}_1(z), \\ x_2 &= y_2 - \tfrac{2}{3}\boldsymbol{s}_1(y) + \tfrac{1}{3}\boldsymbol{s}_1(z), \quad & x_5 &= z_2 + \tfrac{1}{3}\boldsymbol{s}_1(y) - \tfrac{2}{3}\boldsymbol{s}_1(z), \\ x_3 &= y_3 - \tfrac{2}{3}\boldsymbol{s}_1(y) + \tfrac{1}{3}\boldsymbol{s}_1(z), \quad & x_6 &= z_3 + \tfrac{1}{3}\boldsymbol{s}_1(y) - \tfrac{2}{3}\boldsymbol{s}_1(z). \end{aligned} \tag{2.14}$$

They express the composition of the projection $\bar{\pi}'_{4,2}$ with a particular identification of its target space $\mathbb{P}(W_5)$ with the hyperplane (1.1) in $\mathbb{P}^5$. A direct verification shows that substituting these expressions into (1.4) of the Igusa quartic we get (2.13). This proves that (2.13) is isomorphic to the cone over the Igusa quartic with the vertex at the point $\boldsymbol{P}'_0$, hence its intersection with (2.12) is isomorphic to the Igusa quartic.

Finally, we describe the exceptional locus of the projection $\pi''_{4,2}$. Clearly, it is the union of those lines $M_{(y_i:z_i)}$ that are contained in the cubic $\mathscr{Y}'_{4,2}$, i.e., the subvariety of those points $(y_i : z_i)$ for which (2.11) is identically zero. This condition can be rewritten as

$$\boldsymbol{s}_1(y) - \boldsymbol{s}_1(z) = \boldsymbol{s}_2(y) - \boldsymbol{s}_2(z) = \boldsymbol{s}_3(y) - \boldsymbol{s}_3(z) = 0$$

Of course, this is equivalent to $(y_i : z_i) \in \Pi_w$ for some permutation w. Thus the exceptional locus is the union of the proper transforms of the planes Π_w. Each of these planes passes through $\boldsymbol{P}'_0$, hence is contracted onto a line in $\mathbb{P}^4 \cong \mathbb{P}(W_5)$. □

Remark 2.15. There is also a computation-free way to identify the branch divisor X'' of the map $\pi''_{4,2}$ with the Igusa quartic. Indeed, note that the singular locus of X'' contains 15 lines (the images of the 9 singular lines L_{ij} of $\mathscr{Y}'_{4,2}$ and the images of the 6 planes Π_w), then check that they form a Cremona–Richmond configuration (e.g., by using Theorem A.8), and then apply Corollary A.14.

Remark 2.16. Using (2.11) it is easy to write the (birational) involution of the double covering $\mathscr{Y}'_{4,2} \dashrightarrow \mathbb{P}^4$ explicitly. Indeed, choose a point $(y_i : z_i) = (y_1 : y_2 : y_3 : z_1 : z_2 : z_3)$ on the cubic $\mathscr{Y}'_{4,2} \subset \mathbb{P}^5$ different from $\boldsymbol{P}'_0$. Using the parametrization (2.10), we see that the point $(y_i : z_i)$ corresponds to $\lambda = 0$. Keeping in mind that $\boldsymbol{s}_3(y) = \boldsymbol{s}_3(z)$ at our point $(y_i : z_i)$, and finding the second root of the (2.11) in λ/μ, we conclude that the involution of the double covering $\mathscr{Y}'_{4,2} \dashrightarrow \mathbb{P}^4$ is given by

$$(y_i : z_i) \mapsto ((\boldsymbol{s}_1(y) - \boldsymbol{s}_1(z))y_i - (\boldsymbol{s}_2(y) - \boldsymbol{s}_2(z)) : (\boldsymbol{s}_1(y) - \boldsymbol{s}_1(z))z_i - (\boldsymbol{s}_2(y) - \boldsymbol{s}_2(z))). \tag{2.17}$$

Furthermore, the induced birational involution of $\mathbb{P}(W_3) \times \mathbb{P}(W_3)$ can be written as

$$\begin{aligned}\bar{\sigma}_{4,2} \colon &((u_1 : u_2 : u_3), (v_1 : v_2 : v_3)) \\ &\mapsto \left(\left(\frac{v_2 - v_3}{\det\left(\begin{smallmatrix} u_2 & u_3 \\ v_2 & v_3 \end{smallmatrix}\right)} : \frac{v_3 - v_1}{\det\left(\begin{smallmatrix} u_3 & u_1 \\ v_3 & v_1 \end{smallmatrix}\right)} : \frac{v_1 - v_2}{\det\left(\begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix}\right)}\right), \left(\frac{u_2 - u_3}{\det\left(\begin{smallmatrix} u_2 & u_3 \\ v_2 & v_3 \end{smallmatrix}\right)} : \frac{u_3 - u_1}{\det\left(\begin{smallmatrix} u_3 & u_1 \\ v_3 & v_1 \end{smallmatrix}\right)} : \frac{u_1 - u_2}{\det\left(\begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix}\right)}\right)\right);\end{aligned} \tag{2.18}$$

to see this one can just compose (2.5) with (2.17) and observe that it gives the same result as a composition of (2.18) with (2.5). Similarly, we deduce from (2.13) that the ramification divisor of the map $\bar{\pi}_{4,2}$ is given by the equation

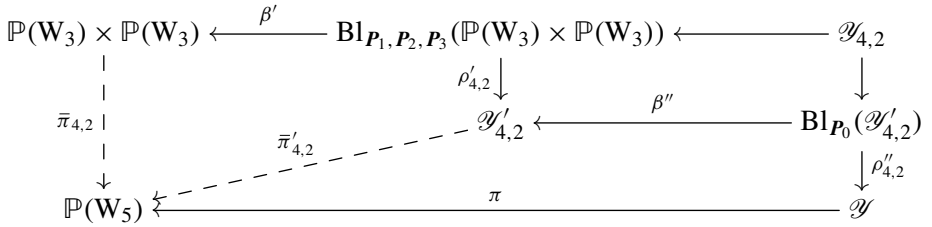
$$\boldsymbol{s}_2(u_2v_3, u_3v_1, u_1v_2) = \boldsymbol{s}_2(u_3v_2, u_1v_3, u_2v_1),$$

that can be compactly rewritten as

$$\det\begin{pmatrix} u_1v_1 & u_2v_2 & u_3v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = 0. \tag{2.19}$$

This gives a determinantal representation of a threefold birational to the Igusa quartic.

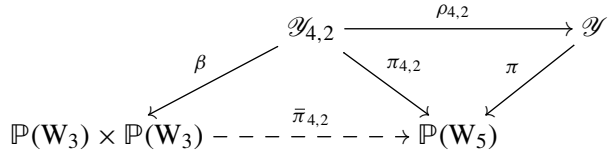
Combining the results of Lemmas 2.6 and 2.8 we obtain a commutative diagram



(2.20)

where the upper right square is Cartesian and the composition $\mathscr{Y}_{4,2}\to\mathbb{P}(W_3)\times\mathbb{P}(W_3)$ of the upper horizontal arrows is the blow up map β.

Proposition 2.21. *The linear projection $\bar{\pi}_{4,2}\colon \mathbb{P}(W_3)\times\mathbb{P}(W_3)\dashrightarrow\mathbb{P}(W_5)$ with center in the span of the points P_0, P_1, P_2, and P_3 gives rise to a commutative diagram*



(2.22)

where $\rho_{4,2}$ is a small resolution of singularities defined uniquely up to a composition with the Galois involution $\sigma\colon\mathscr{Y}\to\mathscr{Y}$. The map $\rho_{4,2}$ contracts

- *the proper transforms of the eight planes $\mathbb{P}(W_3)\times P_i$ and $P_i\times\mathbb{P}(W_3)$, $0\leqslant i\leqslant 3$,*
- *the proper transforms of the six quadrics $\overline{P_iP_j}\times\overline{P_iP_j}$, $0\leqslant i<j\leqslant 3$, and*
- *the proper transform of the diagonal $\mathbb{P}(W_3)\hookrightarrow\mathbb{P}(W_3)\times\mathbb{P}(W_3)$,*

and is an isomorphisms on the complement of those. Moreover, the morphism $\pi_{4,2}$ induces a nonstandard embedding $\mathfrak{S}_{4,2}\to\mathfrak{S}_6$ such that $\rho_{4,2}$ is $\mathfrak{S}_{4,2}$-equivariant with respect to the twisted action of $\mathfrak{S}_{4,2}$ on $\mathscr{Y}$.

Proof. We define the map $\rho_{4,2}$ as the composition of the right vertical arrows in (2.20). Its uniqueness up to σ is evident. We note that the composition

$$\rho_{4,2}\circ\beta^{-1}\colon\mathbb{P}(W_3)\times\mathbb{P}(W_3)\dashrightarrow\mathscr{Y}\subset\mathbb{P}(2,1^6)$$

can be defined by explicit formulas:

$$\begin{aligned}
x_0&=-u_1u_3v_1v_2-u_1u_2v_2v_3-u_2u_3v_1v_3+u_1u_2v_1v_3+u_2u_3v_1v_2+u_1u_3v_2v_3,\\
x_1&=\tfrac{1}{3}(u_2v_3-2u_3v_1-2u_1v_2+u_3v_2+u_1v_3+u_2v_1),\\
x_2&=\tfrac{1}{3}(-2u_2v_3+u_3v_1-2u_1v_2+u_3v_2+u_1v_3+u_2v_1),\\
x_3&=\tfrac{1}{3}(-2u_2v_3-2u_3v_1+u_1v_2+u_3v_2+u_1v_3+u_2v_1),\\
x_4&=\tfrac{1}{3}(u_2v_3+u_3v_1+u_1v_2+u_3v_2-2u_1v_3-2u_2v_1),\\
x_5&=\tfrac{1}{3}(u_2v_3+u_3v_1+u_1v_2-2u_3v_2+u_1v_3-2u_2v_1),\\
x_6&=\tfrac{1}{3}(u_2v_3+u_3v_1+u_1v_2-2u_3v_2-2u_1v_3+u_2v_1).
\end{aligned}\tag{2.23}$$

Indeed, x_0 defines in $\mathscr{Y}$ the ramification divisor of the map π, hence its pullback to $\mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$ coincides (up to a scalar) with the equation (2.19) of the ramification divisor of $\bar{\pi}_{4,2}$. The pullbacks of $x_1, \dots, x_6$ are given by the composition of (2.14) and (2.5), which gives the required formulas. Substituting those into (1.5), we see that the scalar in the formula for x_0 is ± 1. So, (2.23) gives one of the two maps $\rho_{4,2}$, while the other sign choice gives $\sigma \circ \rho_{4,2}$.

For the description of the exceptional locus of $\rho_{4,2}$ we combine the results of Lemmas 2.6 and 2.8 with the simple observation (using (2.5)) that the map $\rho'_{4,2} \circ \beta'^{-1}$ from (2.20) takes the two planes $\mathbb{P}(\mathrm{W}_3) \times P_0$ and $P_0 \times \mathbb{P}(\mathrm{W}_3)$ to the planes Π_w, where w are cycles of length 3; takes the three quadrics $\overline{P_0 P_i} \times \overline{P_0 P_i}$ to the planes Π_w, where w are transpositions; and takes the diagonal to Π_w, where w is the identity permutation.

The space W_5 by definition (2.2) comes with an $\mathfrak{S}_{4,2}$ action, such that the map $\pi_{4,2} \colon \mathscr{Y}_{4,2} \to \mathbb{P}(\mathrm{W}_5)$ obtained by resolving the indeterminacy of the linear projection $\bar{\pi}_{4,2}$ is $\mathfrak{S}_{4,2}$-equivariant. It follows that its branch divisor, which was shown to be the Igusa quartic $X_{1/4}$, is invariant under this action. On the other hand, it is well known that $\mathrm{Aut}(X_{1/4}) \cong \mathfrak{S}_6$ (this follows for instance from [Finkelnberg 1987, Section 3; Hunt 1996, Proposition 3.3.1]; see also Lemma 3.4 below). Thus, we obtain an embedding $\mathfrak{S}_{4,2} \hookrightarrow \mathfrak{S}_6$.

Moreover, for every element $g \in \mathfrak{S}_{4,2}$ the conjugation of the diagram (2.22) by g gives a diagram of the same form. Since $\rho_{4,2}$ is uniquely defined up to σ, we obtain an equality

$$g \circ \rho_{4,2} \circ g^{-1} = \sigma^{k(g)} \circ \rho_{4,2},$$

where $k \colon \mathfrak{S}_{4,2} \to \mathbb{Z}/2\mathbb{Z}$ is a group homomorphism. Using the explicit expression for x_0 provided by (2.19) it is easy to see that transpositions in the group $\mathfrak{S}_{4,2}$ change the sign of x_0. This means that k is the homomorphism of parity $\mathfrak{S}_6 \to \mathbb{Z}/2\mathbb{Z}$ restricted to $\mathfrak{S}_{4,2}$, which means that the map $\rho_{4,2}$ is equivariant with respect to the twisted action (1.8) of $\mathfrak{S}_6$ on $\mathscr{Y}$.

Finally, to show that the embedding $\mathfrak{S}_{4,2} \hookrightarrow \mathfrak{S}_6$ is nonstandard, we use (2.23) to observe that transpositions in $\mathfrak{S}_{4,2}$ go to permutations of cycle type $[2, 2, 2]$ in $\mathfrak{S}_6$. Alternatively, we could notice that the restriction of the representation (1.1) with respect to a standard embedding $\mathfrak{S}_4 \hookrightarrow \mathfrak{S}_6$ decomposes as a direct sum of three irreducible representations of $\mathfrak{S}_4$ (see (5.11) and Lemma 5.12), while (2.2) is the sum of two irreducibles. □

Let us emphasize again that there are exactly two maps $\rho_{4,2}$ that fit into commutative diagram (2.22): the first is given by (2.23) and the second is obtained by its composition with σ, i.e., by the change of sign of x_0. The particular choice (2.23) will lead us to a particular choice of the map $\rho_{5,1}$ in the next subsection.

We write down here a simple consequence of Proposition 2.21 concerning the Weil divisor class group of the Coble fourfold.

Corollary 2.24. *One has* $\mathrm{rk}\,\mathrm{Cl}(\mathscr{Y}) = 6$.

Proof. Since the map $\rho_{4,2}\colon \mathscr{Y}_{4,2} \to \mathscr{Y}$ is a small resolution of singularities, it induces an isomorphism $\mathrm{Cl}(\mathscr{Y}) \cong \mathrm{Pic}(\mathscr{Y}_{4,2})$, and since $\mathscr{Y}_{4,2}$ is the blow up of $\mathbb{P}^2 \times \mathbb{P}^2$ in 4 points, its Picard rank equals 6. □

In Theorem 5.1 we will describe the action of the group $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$ on $\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q}$.

Remark 2.25. For each three-element subset $I \subset \{1, \dots, 6\}$ denote by $\bar{I} \subset \{1, \dots, 6\}$ its complement. Consider the hyperplane $H_I \subset \mathbb{P}^4$ defined in (1.1) by the equation

$$\sum_{i \in I} x_i = 0. \tag{2.26}$$

Note that $H_I = H_{\bar{I}}$. In the terminology of the Appendix these are the ten jail hyperplanes (A.5) of the Cremona–Richmond configuration. The preimage of H_I on $\mathscr{Y}$ splits as the union of two irreducible components. Indeed, consider the subvariety $\mathscr{H}_I \subset \mathscr{Y}$ defined by the (2.26) together with the equation

$$x_0 + \tfrac{1}{2}\left(2\sum_{i \in I} x_i^2 - \sum_{i \in \bar{I}} x_i^2\right) = 0. \tag{2.27}$$

Then it is easy to check that

$$\pi^{-1}(H_I) = \pi^{-1}(H_{\bar{I}}) = \mathscr{H}_I \cup \mathscr{H}_{\bar{I}}.$$

An even easier way to see this splitting is provided by the morphism $\rho_{4,2}$. Indeed, using formulas (2.23) one can check that the preimages on $\mathbb{P}^2 \times \mathbb{P}^2$ of the six hyperplanes H_{124}, H_{125}, H_{134}, H_{136}, H_{235}, and H_{236} are divisors given by equations

$$(u_1 - u_3)v_2 = 0, \quad u_1(v_2 - v_3) = 0, \quad u_3(v_1 - v_2) = 0,$$
$$(u_2 - u_3)v_1 = 0, \quad (u_1 - u_2)v_3 = 0, \quad u_2(v_1 - v_3) = 0,$$

respectively. Each of these divisors is a union of two irreducible components, and each component is the product $\overline{P_i P_j} \times \mathbb{P}^2$ or $\mathbb{P}^2 \times \overline{P_i P_j}$ for appropriate i and j. Note that the action of $\mathfrak{S}_{4,2}$ on the set of all twelve of these irreducible components is transitive. For each I denote

$$\mathscr{H}_I^{4,2} := \rho_{4,2}^{-1}(\mathscr{H}_I).$$

Therefore, if I is one of the above six triples or one of their complements, then $\beta(\mathscr{H}_I^{4,2})$ is one of the above twelve components, hence these divisors $\mathscr{H}_I^{4,2}$ form a single $\mathfrak{S}_{4,2}$-orbit.

Similarly, formulas (2.23) show that the preimages on $\mathbb{P}^2 \times \mathbb{P}^2$ of the remaining four hyperplanes H_{123}, H_{156}, H_{246}, and H_{345} are irreducible divisors singular at the points $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, and $\boldsymbol{P}_3$, respectively. This means that for each of the above four triples I the preimage $\pi_{4,2}^{-1}(H_I)$ of H_I on $\mathscr{Y}_{4,2}$ consists of two irreducible components, one of them being the exceptional divisor of the blow up β over the corresponding point $\boldsymbol{P}_r$. A straightforward computation shows that

$$\mathscr{H}_{123}^{4,2}, \quad \mathscr{H}_{156}^{4,2}, \quad \mathscr{H}_{246}^{4,2}, \quad \mathscr{H}_{345}^{4,2}$$

are the exceptional divisors, while

$$\mathscr{H}^{4,2}_{456}, \quad \mathscr{H}^{4,2}_{234}, \quad \mathscr{H}^{4,2}_{135}, \quad \mathscr{H}^{4,2}_{126}$$

are the proper transforms of irreducible divisors from $\mathbb{P}^2 \times \mathbb{P}^2$.

Using the above observations we can write down the resolution $\rho_{4,2}$ as a blow up. Set

$$\mathscr{H}^{4,2}_{+} = \mathscr{H}^{4,2}_{123} + \mathscr{H}^{4,2}_{156} + \mathscr{H}^{4,2}_{246} + \mathscr{H}^{4,2}_{345}, \quad \text{and} \quad \mathscr{H}^{4,2}_{-} = \mathscr{H}^{4,2}_{456} + \mathscr{H}^{4,2}_{234} + \mathscr{H}^{4,2}_{135} + \mathscr{H}^{4,2}_{126}.$$

Then the divisor $-\mathscr{H}^{4,2}_{+}$ is β-ample. Since $\operatorname{rk}\operatorname{Pic}(\mathscr{Y}_{4,2})^{\mathfrak{S}_{4,2}} = 2$ by definition (2.3) (indeed, the group $\mathfrak{S}_{4,2}$ acts transitively on the set $\{\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \boldsymbol{P}_4\}$ and swaps the factors of $\mathbb{P}(W_3) \times \mathbb{P}(W_3)$) the divisor $\mathscr{H}^{4,2}_{+}$ is $\rho_{4,2}$-ample, so that the divisor $-\mathscr{H}^{4,2}_{-}$ is also $\rho_{4,2}$-ample. We conclude that the small birational morphism $\rho_{4,2}$ is the blow up of the Weil divisor $\mathscr{H}_{456} + \mathscr{H}_{234} + \mathscr{H}_{135} + \mathscr{H}_{126}$ on $\mathscr{Y}$. Note that the other choice of an $\mathfrak{S}_{4,2}$-equivariant small resolution of singularities of $\mathscr{Y}$, that is, the morphism $\sigma \circ \rho_{4,2}$, is the blow up of the Weil divisor $\mathscr{H}_{123} + \mathscr{H}_{156} + \mathscr{H}_{246} + \mathscr{H}_{345}$ on $\mathscr{Y}$.

2.2. ***$\mathbb{P}^2$-bundle over the quintic del Pezzo surface.*** In this section we construct another resolution of the Coble fourfold, using geometry of the quintic del Pezzo surface. Before explaining the construction, we start with recalling this geometry (we refer the reader to [Dolgachev 2012, Section 8.5; Cheltsov and Shramov 2016a, Section 6.2] for more details).

Let S be the (smooth) del Pezzo surface of degree 5. Recall that S can be represented as the blow up of $\mathbb{P}^2$ in four points (in five different ways), and one has $\operatorname{Aut}(S) \cong \mathfrak{S}_5$. The vector space $H^0(S, \omega_S^{-1})$ is the unique irreducible six-dimensional representation of $\mathfrak{S}_5$ (corresponding to the partition (3, 1, 1) in the notation of [Fulton and Harris 1991, Section 4.1]); see [Shepherd-Barron 1989, Lemma 1]; in particular, this representation is invariant under the sign twist. Moreover, the anticanonical line bundle ω_S^{-1} is very ample and defines an $\mathfrak{S}_5$-equivariant embedding

$$S \hookrightarrow \mathbb{P}^5 = \mathbb{P}(H^0(S, \omega_S^{-1})^\vee)$$

such that S is an intersection of five quadrics in $\mathbb{P}^5$. The five-dimensional space of quadrics passing through S in $\mathbb{P}^5$ is an irreducible representation of $\mathfrak{S}_5$; see [Shepherd-Barron 1989, Proposition 2]. We denote by

$$\mathrm{W}_5 := H^0(\mathbb{P}^5, I_S(2))^\vee \tag{2.28}$$

its dual space. Later, we will identify this space with the space defined by (2.2).

Below we consider the Grassmannian $\mathrm{Gr}(2, \mathrm{W}_5^\vee) \cong \mathrm{Gr}(3, \mathrm{W}_5)$ of two-dimensional vector subspaces in $\mathrm{W}_5^\vee$ (respectively, three-dimensional subspaces in W_5) and denote by $\mathscr{U}_2$ and $\mathscr{U}_3$ the tautological rank 2 and rank 3 subbundles in the trivial vector bundles on this Grassmannian with fibers $\mathrm{W}_5^\vee$ and W_5, respectively.

The following result is well known.

Lemma 2.29. *There is an $\mathfrak{S}_5$-equivariant linear embedding $\mathbb{P}^5 \subset \mathbb{P}(\Lambda^3 \mathrm{W}_5)$ such that*

$$S = \mathrm{Gr}(3, \mathrm{W}_5) \cap \mathbb{P}^5 \subset \mathbb{P}(\Lambda^3 \mathrm{W}_5)$$

is a complete intersection of the Grassmannian $\mathrm{Gr}(3, \mathrm{W}_5)$ with $\mathbb{P}^5$.

Proof. We use the technique of excess conormal bundles developed in [Debarre and Kuznetsov 2018, Appendix A]. Since S is an intersection of quadrics, the composition

$$\mathrm{W}_5^\vee \otimes \mathscr{O}_{\mathbb{P}^5} \to I_S(2) \to (I_S/I_S^2)(2)$$

is surjective. The conormal sheaf $\mathcal{N}^\vee_{S/\mathbb{P}^5} \cong I_S/I_S^2$ is locally free of rank 3 on S, hence the above surjection induces an $\mathfrak{S}_5$-equivariant map $S \to \mathrm{Gr}(3, \mathrm{W}_5)$ such that the pullback of the dual tautological bundle $\mathscr{U}_3^\vee$ from $\mathrm{Gr}(3, \mathrm{W}_5)$ to S is isomorphic to $(I_S/I_S^2)(2)$. By adjunction formula we have

$$\det(I_S/I_S^2) \cong \omega_{\mathbb{P}^5|_S} \otimes \omega_S^{-1} \cong \det(\mathrm{W}_5^\vee) \otimes \omega_S^6 \otimes \omega_S^{-1},$$

hence

$$\det((I_S/I_S^2)(2)) \cong \det(\mathrm{W}_5^\vee) \otimes \omega_S^{-1},$$

hence the pullback of $\mathscr{O}_{\mathrm{Gr}(3,\mathrm{W}_5)}(1) \cong \det(\mathscr{U}_3^\vee)$ to S is isomorphic to $\det(\mathrm{W}_5^\vee) \otimes \omega_S^{-1}$. The induced map

$$\Lambda^3 \mathrm{W}_5^\vee \cong H^0(\mathrm{Gr}(3, \mathrm{W}_5), \mathscr{O}_{\mathrm{Gr}(3,\mathrm{W}_5)}(1)) \to H^0(S, \det(\mathrm{W}_5^\vee) \otimes \omega_S^{-1}) \cong \det(\mathrm{W}_5^\vee) \otimes H^0(S, \omega_S^{-1})$$

is $\mathfrak{S}_5$-equivariant and surjective (since the target space is an irreducible $\mathfrak{S}_5$-representation). Moreover, since the $\mathfrak{S}_5$-representation $H^0(S, \omega_S^{-1})$ is invariant under the sign twist, the above composition defines an embedding

$$\mathbb{P}^5 = \mathbb{P}(H^0(S, \omega_S^{-1})^\vee) \hookrightarrow \mathbb{P}(\Lambda^3 \mathrm{W}_5)$$

such that $S \subset \mathrm{Gr}(3, \mathrm{W}_5) \cap \mathbb{P}^5$. It remains to show that this embedding of S is an equality.

Since $\mathrm{Gr}(3, \mathrm{W}_5) \subset \mathbb{P}(\Lambda^3 \mathrm{W}_5)$ is cut out by Plücker quadrics that are parametrized by the space $\mathrm{W}_5^\vee \otimes \det(\mathrm{W}_5^\vee)$, we obtain a map (where the first isomorphism takes place by [Debarre and Kuznetsov 2018, Proposition A.7])

$$\mathrm{W}_5^\vee \otimes \det(\mathrm{W}_5^\vee) \cong H^0(\mathbb{P}(\Lambda^3 \mathrm{W}_5), I_{\mathrm{Gr}(3,\mathrm{W}_5)}(2)) \to H^0(\mathbb{P}^5, I_S(2)) \cong \mathrm{W}_5^\vee \tag{2.30}$$

which by construction commutes with the natural $\mathfrak{S}_5$-action. It is nonzero since $\mathrm{Gr}(3, \mathrm{W}_5)$ does not contain $\mathbb{P}^5$, hence it is an isomorphism by irreducibility of W_5. Since S is an intersection of quadrics, it follows that $S = \mathrm{Gr}(3, \mathrm{W}_5) \cap \mathbb{P}^5$. □

Remark 2.31 (cf. [Shepherd-Barron 1989, Corollary 3]). In (2.30) we obtained an $\mathfrak{S}_5$-equivariant isomorphism $\mathrm{W}_5^\vee \otimes \det(\mathrm{W}_5^\vee) \cong \mathrm{W}_5^\vee$. This allows to identify W_5 as the (unique) irreducible five-dimensional representation of $\mathfrak{S}_5$ with $\det(\mathrm{W}_5)$ being trivial. It corresponds to the Young diagram of the partition $(3, 2)$ in the notation of [Fulton and Harris 1991, Section 4.1].

We denote the restriction of the tautological bundles $\mathscr{U}_2$ and $\mathscr{U}_3$ to S also by $\mathscr{U}_2$ and $\mathscr{U}_3$. The tautological embeddings $\mathscr{U}_2 \hookrightarrow \mathrm{W}_5^\vee \otimes \mathscr{O}_S$ and $\mathscr{U}_3 \hookrightarrow \mathrm{W}_5 \otimes \mathscr{O}_S$ induce $\mathfrak{S}_5$-equivariant maps

$$\mathbb{P}_S(\mathscr{U}_2) \to \mathbb{P}(\mathrm{W}_5^\vee) \quad \text{and} \quad \mathbb{P}_S(\mathscr{U}_3) \to \mathbb{P}(\mathrm{W}_5).$$

Below we describe these maps explicitly. We start with the first of them.

Lemma 2.32. *The image of the map $\varpi : \mathbb{P}_S(\mathscr{U}_2) \to \mathbb{P}(\mathrm{W}_5^\vee)$ is the Segre cubic hypersurface in $\mathbb{P}(\mathrm{W}_5^\vee) \cong \mathbb{P}^4$, and $\mathbb{P}_S(\mathscr{U}_2)$ provides its small $\mathfrak{S}_5$-equivariant resolution of singularities.*

Proof. Let us describe the fiber of ϖ over a point of $\mathbb{P}(\mathrm{W}_5^\vee)$. Thinking of such a point as of a four-dimensional subspace $U_4 \subset \mathrm{W}_5$, we conclude that

$$\varpi^{-1}([U_4]) = \mathrm{Gr}(3, U_4) \cap \mathbb{P}^5 \subset \mathrm{Gr}(3, \mathrm{W}_5) \cap \mathbb{P}^5 = S.$$

Since $\mathrm{Gr}(3, U_4) \cong \mathbb{P}^3$, this intersection is a linear space contained in S, hence either is empty, or is a point, or is a line. Conversely, if $L \subset S$ is a line, then

$$\mathscr{U}_2|_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$$

because $\mathscr{U}_2^\vee$ is globally generated with $\det(\mathscr{U}_2^\vee) \cong \omega_S^{-1}$. Moreover, the section

$$L = \mathbb{P}_L(\mathscr{O}_L) \hookrightarrow \mathbb{P}_L(\mathscr{U}_2|_L) \hookrightarrow \mathbb{P}_S(\mathscr{U}_2)$$

of the projection $\mathbb{P}_L(\mathscr{U}_2|_L) \to L$ is contracted by the map ϖ. This proves that ϖ contracts precisely the exceptional sections over the ten lines of S, hence the image

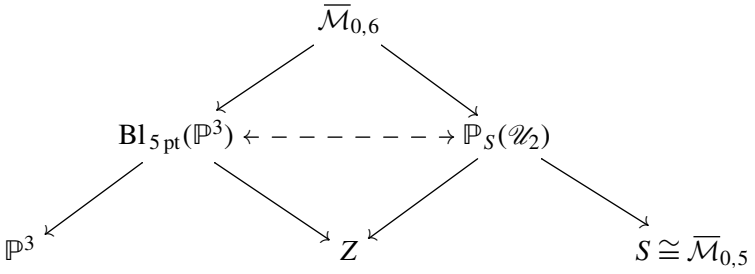
$$Z := \varpi(\mathbb{P}_S(\mathscr{U}_2)) \subset \mathbb{P}(\mathrm{W}_5^\vee)$$

is a hypersurface with ten isolated singular points and the map $\mathbb{P}_S(\mathscr{U}_2) \to Z$ is a small resolution of singularities. On the other hand, since $\det(\mathscr{U}_2) \cong \omega_S$, it follows that

$$\omega_{\mathbb{P}_S(\mathscr{U}_2)} \cong \varpi^* \mathscr{O}_{\mathbb{P}(\mathrm{W}_5^\vee)}(-2).$$

Since the map $\mathbb{P}_S(\mathscr{U}_2) \to Z$ is small, we have $\omega_Z \cong \mathscr{O}_{\mathbb{P}(\mathrm{W}_5^\vee)}(-2)|_Z$, so that Z is a cubic hypersurface. It remains to notice that the only three-dimensional cubic with ten isolated singular points is the Segre cubic; see e.g., [Dolgachev 2016, Proposition 2.1]; alternatively, one can deduce this from the fact that the group $\mathfrak{S}_5$ acting in the irreducible five-dimensional representation $\mathrm{W}_5^\vee$ has a unique cubic invariant, which must thus define the Segre cubic. □

Remark 2.33 [Dolgachev 2016, Section 2; Prokhorov 2010, Proposition 4.6]. The relation of the quintic del Pezzo surface S and the Segre cubic threefold Z extends to an $\mathfrak{S}_5$-equivariant diagram



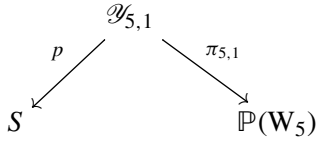
Here $\overline{\mathcal{M}}_{0,n}$ is the moduli spaces of stable rational curves with n marked points, the left outer diagonal arrows provide its Kapranov's representation (the lower left arrow is the blow up of five general points on $\mathbb{P}^3$), the right outer diagonal arrows compose to the forgetful map $\overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,5}$, the inner diagonal arrows contract ten smooth rational curves each (and provide two $\mathfrak{S}_5$-equivariant small resolutions of Z), and the dashed arrow is a flop in these curves.

The above diagram can be thought of as an $\mathfrak{S}_5$-Sarkisov link from the Mori fiber space $\mathbb{P}_S(\mathscr{U}_2) \to S$ to $\mathbb{P}^3$ centered at Z; see Section 5.1 below for explanation of terminology. It is natural to ask what is the $\mathfrak{S}_5$-Sarkisov link starting from $\mathbb{P}_S(\mathscr{U}_3) \to S$. We will see in diagram (2.48) below that it is a symmetric link centered at the Coble fourfold $\mathscr{Y}$.

So, we consider the projectivization $\mathbb{P}_S(\mathscr{U}_3)$ of the rank 3 bundle $\mathscr{U}_3$ and denote it by

$$\mathscr{Y}_{5,1} := \mathbb{P}_S(\mathscr{U}_3).$$

The embedding $\mathscr{U}_3 \hookrightarrow \mathrm{W}_5 \otimes \mathscr{O}_S$ induces an $\mathfrak{S}_5$-equivariant diagram



(2.34)

where p is the natural projection $\mathscr{Y}_{5,1} = \mathbb{P}_S(\mathscr{U}_3) \to S$, and $\pi_{5,1}$ is the composition of the embedding $\mathscr{Y}_{5,1} \hookrightarrow S \times \mathbb{P}(\mathrm{W}_5)$ with the projection to the second factor. In particular, the restriction of the map p to any fiber of $\pi_{5,1}$ is an isomorphism to its image. This allows to consider every fiber

$$S_w := \pi_{5,1}^{-1}(w)$$

of the map $\pi_{5,1}$ as a closed subscheme of S. In the next lemma we describe these subschemes.

For each point $w \in \mathbb{P}(\mathrm{W}_5)$ denote by W_5/w the four-dimensional quotient of the space W_5 by the line in W_5 that corresponds to w. Every two-dimensional subspace in W_5/w gives (by taking preimage) a three-dimensional subspace in W_5 containing w. This allows to consider $\mathrm{Gr}(2, \mathrm{W}_5/w)$ as a subvariety of $\mathrm{Gr}(3, \mathrm{W}_5)$.

Lemma 2.35. *The fiber S_w of the map $\pi_{5,1}$ over a point $w \in \mathbb{P}(\mathrm{W}_5)$ can be described as*

$$S_w = \mathrm{Gr}(2, \mathrm{W}_5/w) \cap \mathbb{P}^5 \subset \mathrm{Gr}(3, \mathrm{W}_5) \cap \mathbb{P}^5 = S.$$

In particular, S_w is either a zero-dimensional scheme of length 2, or a line, or a conic.

Proof. The first equality is obvious. Consequently, S_w is a linear section of the four-dimensional quadric $\mathrm{Gr}(2, \mathrm{W}_5/w)$ of codimension at most 4. So, if S_w is zero-dimensional, it is a scheme of length 2. Furthermore, if S_w is one-dimensional, it is either a line or a conic. It remains to notice that $\dim S_w < \dim S = 2$ since S is irreducible. □

Our goal is to describe the map $\pi_{5,1}$ in (2.34). We start by presenting some surfaces in $\mathscr{Y}_{5,1}$ contracted by it. Recall that S contains 10 lines. Recall also that $\mathscr{U}_3$ is a subbundle in the trivial vector bundle with fiber W_5 over S, so that $\mathscr{Y}_{5,1}$ is a subvariety in $S \times \mathbb{P}(\mathrm{W}_5)$.

Lemma 2.36. *For every line $L \subset S$ there is a unique line $L' \subset \mathbb{P}(\mathrm{W}_5)$ such that for the surface $R_L = L \times L'$ one has*

$$R_L \subset \mathscr{Y}_{5,1} \subset S \times \mathbb{P}(\mathrm{W}_5). \tag{2.37}$$

In particular, the map $\pi_{5,1}$ contracts R_L onto the line L'. Moreover, if $L_1 \neq L_2$ are distinct lines on S then the corresponding lines $L_1', L_2' \subset \mathbb{P}(\mathrm{W}_5)$ are distinct as well.

Proof. Since L is a line on $\mathrm{Gr}(3, \mathrm{W}_5)$, there is a unique two-dimensional subspace $U_2 \subset \mathrm{W}_5$ such that $L \subset \mathbb{P}(\mathrm{W}_5/U_2) \subset \mathrm{Gr}(3, \mathrm{W}_5)$. Then for every point $[U_3]$ of L we have $U_2 \subset U_3$, that is, $U_2 \otimes \mathscr{O}_L \subset \mathscr{U}_3|_L$, hence

$$L \times \mathbb{P}(U_2) = \mathbb{P}_L(U_2 \otimes \mathscr{O}_L) \subset \mathbb{P}_S(\mathscr{U}_3) = \mathscr{Y}_{5,1}.$$

Thus, the line $L' = \mathbb{P}(U_2) \subset \mathbb{P}(\mathrm{W}_5)$ has the required property.

Furthermore, for any two-dimensional subspace $U_2 \subset \mathrm{W}_5$ the intersection

$$\mathbb{P}(\mathrm{W}_5/U_2) \cap S = \mathbb{P}(\mathrm{W}_5/U_2) \cap \mathbb{P}^5$$

is a linear space contained in S, hence is either empty, or a point, or a line. In particular, two distinct lines L_1 and L_2 on S cannot correspond to the same subspace $U_2 \subset \mathrm{W}_5$, hence the corresponding lines L_1' and L_2' in $\mathbb{P}(\mathrm{W}_5)$ are distinct. □

As we already mentioned, a quintic del Pezzo surface is classically represented as the blow up of $\mathbb{P}^2$ in four general points. Let $\varphi\colon S \to \mathbb{P}^2$ be one of such blow up representations with exceptional divisors E_0, E_1, E_2, and E_3. Denote by e_i their classes in $\mathrm{Pic}(S)$, and by ℓ the pullback of the line class from $\mathbb{P}^2$ to S, so that

$$K_S \sim -3\ell + e_0 + e_1 + e_2 + e_3.$$

The line bundle $\mathscr{O}_S(\ell)$ defines the contraction $\varphi\colon S \to \mathbb{P}^2$ and the line bundle $\mathscr{O}_S(2\ell - e_0 - e_1 - e_2 - e_3)$ defines a conic bundle $\bar{\varphi}\colon S \to \mathbb{P}^1$. The combination of φ and $\bar{\varphi}$ defines an embedding

$$\varphi \times \bar{\varphi}\colon S \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1,$$

whose image is a divisor of bidegree $(2, 1)$. Moreover, the composition of $\varphi \times \bar{\varphi}$ with the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ is the anticanonical embedding of S, therefore we have an exact sequence of normal bundles

$$0 \to \mathcal{N}_{S/\mathbb{P}^2\times\mathbb{P}^1} \to \mathcal{N}_{S/\mathbb{P}^5} \to \mathcal{N}_{\mathbb{P}^2\times\mathbb{P}^1/\mathbb{P}^5}|_S \to 0. \tag{2.38}$$

The first of these bundles is isomorphic to

$$\varphi^*\mathscr{O}_{\mathbb{P}^2}(2) \otimes \bar{\varphi}^*\mathscr{O}_{\mathbb{P}^1}(1) \cong \mathscr{O}_S(4\ell - e_0 - e_1 - e_2 - e_3),$$

and the second is isomorphic to $\mathscr{U}_3(6\ell - 2e_0 - 2e_1 - 2e_2 - 2e_3)$ by the proof of Lemma 2.29. The third vector bundle in (2.38) can be computed as follows: We denote by $\mathscr{T}_{\mathbb{P}^n}$ the tangent bundle of $\mathbb{P}^n$.

Lemma 2.39. *For any positive integers m, n we have $\mathcal{N}_{\mathbb{P}^m\times\mathbb{P}^n/\mathbb{P}^{mn+m+n}} \cong \mathscr{T}_{\mathbb{P}^m} \boxtimes \mathscr{T}_{\mathbb{P}^n}$.*

Proof. Let A and B be vector spaces of dimension $m+1$ and $n+1$ respectively. Tensoring pullbacks to $\mathbb{P}(A) \times \mathbb{P}(B)$ of the Euler sequences

$$0 \to \mathscr{O}_{\mathbb{P}(A)} \to A \otimes \mathscr{O}_{\mathbb{P}(A)}(1) \to \mathscr{T}_{\mathbb{P}(A)} \to 0 \quad \text{and} \quad 0 \to \mathscr{O}_{\mathbb{P}(B)} \to B \otimes \mathscr{O}_{\mathbb{P}(B)}(1) \to \mathscr{T}_{\mathbb{P}(B)} \to 0,$$

we obtain an exact sequence

$$\begin{aligned} 0 \to \mathscr{O}_{\mathbb{P}(A)\times\mathbb{P}(B)} \to A \otimes \mathscr{O}_{\mathbb{P}(A)\times\mathbb{P}(B)}(1,0) \oplus B \otimes \mathscr{O}_{\mathbb{P}(A)\times\mathbb{P}(B)}(0,1) \\ \to A \otimes B \otimes \mathscr{O}_{\mathbb{P}(A)\times\mathbb{P}(B)}(1,1) \to \mathscr{T}_{\mathbb{P}(A)} \boxtimes \mathscr{T}_{\mathbb{P}(B)} \to 0. \end{aligned}$$

Comparing it with the restriction to $\mathbb{P}(A) \times \mathbb{P}(B)$ of the Euler sequence

$$0 \to \mathscr{O}_{\mathbb{P}(A)\times\mathbb{P}(B)} \to A \otimes B \otimes \mathscr{O}_{\mathbb{P}(A)\times\mathbb{P}(B)}(1,1) \to \mathscr{T}_{\mathbb{P}(A\otimes B)}|_{\mathbb{P}(A)\times\mathbb{P}(B)} \to 0$$

of $\mathbb{P}(A \otimes B)$ and with the pullbacks of the Euler sequences of $\mathbb{P}(A)$ and $\mathbb{P}(B)$, we obtain an exact sequence

$$0 \to \mathrm{pr}^*_{\mathbb{P}(A)}\,\mathscr{T}_{\mathbb{P}(A)} \oplus \mathrm{pr}^*_{\mathbb{P}(B)}\,\mathscr{T}_{\mathbb{P}(B)} \to \mathscr{T}_{\mathbb{P}(A\otimes B)}|_{\mathbb{P}(A)\times\mathbb{P}(B)} \to \mathscr{T}_{\mathbb{P}(A)} \boxtimes \mathscr{T}_{\mathbb{P}(B)} \to 0,$$

where $\mathrm{pr}_{\mathbb{P}(A)}$ and $\mathrm{pr}_{\mathbb{P}(B)}$ are the projections, which proves the lemma. □

Applying Lemma 2.39 in the case $m = 2$, $n = 1$, we see that the third bundle in (2.38) is isomorphic to

$$\varphi^*(\mathscr{T}_{\mathbb{P}^2}) \otimes \bar{\varphi}^*(\mathscr{T}_{\mathbb{P}^1}) \cong \varphi^*\mathscr{T}_{\mathbb{P}^2} \otimes \mathscr{O}_S(4\ell - 2e_0 - 2e_1 - 2e_2 - 2e_3).$$

So, twisting the normal bundle sequence (2.38) by the line bundle $\mathscr{O}_S(-6\ell + 2e_0 + 2e_1 + 2e_2 + 2e_3)$ we obtain

$$0 \to \mathscr{O}_S(-2\ell + e_0 + e_1 + e_2 + e_3) \to \mathscr{U}_3 \to \varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2)) \to 0 \tag{2.40}$$

Denote by $r_\varphi \colon S \to \mathbb{P}_S(\mathscr{U}_3)$ the section of the projection p induced by the first map in (2.40).

Lemma 2.41. *There is a line $\Gamma_\varphi \subset \mathbb{P}(\mathrm{W}_5)$ and a commutative diagram*

$$\begin{array}{ccc} S & \xrightarrow{r_\varphi} & \mathbb{P}_S(\mathscr{U}_3) \\ {\scriptstyle\bar{\varphi}}\downarrow & & \downarrow{\scriptstyle\pi_{5,1}} \\ \Gamma_\varphi & \hookrightarrow & \mathbb{P}(\mathrm{W}_5) \end{array}$$

that identifies the line Γ_φ with the base of the conic bundle $\bar{\varphi}$. In particular, for any $w \in \Gamma_\varphi$ the fiber $S_w = \pi_{5,1}^{-1}(w)$ is a conic from the pencil $\bar{\varphi}$.

Proof. By definition of r_φ the composition

$$\pi_{5,1} \circ r_\varphi \colon S \to \mathbb{P}(\mathrm{W}_5)$$

is given by the line bundle $\mathscr{O}_S(2\ell - e_0 - e_1 - e_2 - e_3)$ on S, hence factors as the projection $\bar{\varphi}$ followed by a linear embedding. This proves that we have the required diagram. Moreover, it follows that for every $w \in \Gamma$ the fiber $\pi_{5,1}^{-1}(w)$ contains a conic from the pencil $\bar{\varphi}$. By Lemma 2.35 the fiber coincides with this conic. □

For each contraction $\varphi \colon S \to \mathbb{P}^2$ (recall that for a quintic del Pezzo surface S there are five such contractions), define the surface

$$R_\varphi = r_\varphi(S) \subset \mathbb{P}_S(\mathscr{U}_3), \tag{2.42}$$

so that the map $\pi_{5,1}$ contracts it onto the line $\Gamma_\varphi \subset \mathbb{P}(\mathrm{W}_5)$.

Lemma 2.43. *The five lines $\Gamma_\varphi \subset \mathbb{P}(\mathrm{W}_5)$ corresponding to the contractions $\varphi \colon S \to \mathbb{P}^2$ are pairwise disjoint. Moreover, for each φ the line Γ_φ is distinct from the lines $L' \subset \mathbb{P}(\mathrm{W}_5)$ associated with lines L on S in Lemma 2.36.*

Proof. If w is a common point of the curves Γ_φ and $\Gamma_{\varphi'}$, then by Lemma 2.41 the fiber S_w is a conic that belongs to the corresponding pencils $\bar{\varphi}$ and $\bar{\varphi}'$, hence the pencils coincide, hence $\varphi = \varphi'$.

Assume that $\Gamma_\varphi = L'$, where L' is associated with some line $L \subset S$ as in Lemma 2.36. By Lemma 2.36 we have $L \subset S_w$ for each $w \in L' = \Gamma_\varphi$, and by Lemma 2.41 when w runs over Γ_φ the curves S_w run over the corresponding pencil of conics $\bar{\varphi}$. So, the assumption we made implies that every conic in the pencil contains the line L, which is absurd. □

Now we are ready to prove the main result of this subsection.

Proposition 2.44. *The $\mathfrak{S}_5$-equivariant morphism $\pi_{5,1} \colon \mathscr{Y}_{5,1} \to \mathbb{P}(\mathrm{W}_5)$ gives rise to a commutative diagram*

$$\begin{array}{ccccc} & & \mathscr{Y}_{5,1} & \xrightarrow{\rho_{5,1}} & \mathscr{Y} \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle\pi} & \\ S & & \xrightarrow{\pi_{5,1}} & & \mathbb{P}(\mathrm{W}_5) \end{array} \tag{2.45}$$

where $\mathscr{Y}$ is the Coble fourfold, $\pi : \mathscr{Y} \to \mathbb{P}(\mathrm{W}_5)$ is the double covering, and $\rho_{5,1}$ is a small resolution of singularities, defined uniquely up to composition with the Galois involution $\sigma : \mathscr{Y} \to \mathscr{Y}$. Furthermore, the exceptional locus of $\rho_{5,1}$ is the union of 15 *irreducible rational surfaces $\{R_L\}_{L\subset S} \cup \{R_\varphi\}_{\varphi\colon S\to\mathbb{P}^2}$, such that*:

- *$R_L \cong \mathbb{P}^1 \times \mathbb{P}^1$; each of these surfaces is contracted by p onto the line $L \subset S$ and by $\pi_{5,1}$ onto the line $L' \subset \mathbb{P}(\mathrm{W}_5)$.*
- *$R_\varphi \cong S$ with the map $p\colon R_\varphi \to S$ being an isomorphism and with the map $\pi_{5,1}|_{R_\varphi}$ being the conic bundle $\bar{\varphi}\colon R_\varphi \twoheadrightarrow \Gamma_\varphi$ over the line $\Gamma_\varphi \subset \mathbb{P}(\mathrm{W}_5)$.*

Moreover, the morphism $\pi_{5,1}$ induces a nonstandard embedding $\mathfrak{S}_5 \to \mathfrak{S}_6$ such that $\rho_{5,1}$ is $\mathfrak{S}_5$-equivariant with respect either to the natural or to the twisted action of $\mathfrak{S}_5$ on $\mathscr{Y}$.

Using a compatibility result from Proposition 2.50, we will show in Section 2.4 that $\rho_{5,1}$ is $\mathfrak{S}_5$-equivariant with respect to the twisted action of a nonstandard $\mathfrak{S}_5$.

Proof. By Lemma 2.35 the map $\pi_{5,1}$ is generically finite of degree 2. Denote by $R \subset \mathscr{Y}_{5,1}$ the ramification locus of the morphism $\pi_{5,1}\colon \mathscr{Y}_{5,1} \to \mathbb{P}(\mathrm{W}_5)$ and by $B = \pi_{5,1}(R) \subset \mathbb{P}(\mathrm{W}_5)$ its image. Let us show that B is the Igusa quartic. For this we show that B is projectively dual to the Segre cubic $Z = \varpi(\mathbb{P}_S(\mathscr{U}_2))$; see Lemma 2.32.

Indeed, by Lemma 2.35 we know that B is the locus of $w \in \mathbb{P}(\mathrm{W}_5)$ such that S_w is either a double point or a curve. On the other hand, w defines a hyperplane $\mathbb{P}(w^\perp) \subset \mathbb{P}(\mathrm{W}_5^\vee)$ in the dual projective space, and

$$\varpi^{-1}(Z \cap \mathbb{P}(w^\perp)) = \mathbb{P}_S(\mathscr{U}_2) \times_{\mathbb{P}(\mathrm{W}_5^\vee)} \mathbb{P}(w^\perp)$$

is a relative hyperplane in the $\mathbb{P}^1$-bundle $\mathbb{P}_S(\mathscr{U}_2) \to S$. Moreover, the zero locus of the corresponding section of $\mathscr{U}_2^\vee$ is precisely the scheme S_w. If S_w is zero-dimensional then by [Kuznetsov 2016, Lemma 2.1] we have

$$\varpi^{-1}(Z \cap \mathbb{P}(w^\perp)) = \mathrm{Bl}_{S_w}(S),$$

and if it is one-dimensional, then $\varpi^{-1}(Z \cap \mathbb{P}(w^\perp))$ contains the surface $\mathbb{P}_{S_w}(\mathscr{U}_2|_{S_w})$, hence is reducible. Thus, $\varpi^{-1}(Z \cap \mathbb{P}(w^\perp))$ is singular if and only if $w \in B$. Since the singular points of Z are nodes, and ϖ resolves them, it follows that B is the projective dual of Z. Hence $B = X_{1/4}$ is the Igusa quartic (see [Hunt 1996, Proposition 3.3.1]).

It follows from Lemma 2.35 that the map $\pi_{5,1}$ is an étale double cover over $\mathbb{P}(\mathrm{W}_5) \setminus B$, and that the Stein factorization of the map $\pi_{5,1}$ provides a (unique up to σ) decomposition

$$\mathscr{Y}_{5,1} \xrightarrow{\rho_{5,1}} \mathscr{Y} \xrightarrow{\pi} \mathbb{P}(\mathrm{W}_5),$$

where $\rho_{5,1}$ is a birational map.

Let us show that $\rho_{5,1}$ is small. Indeed, since $\det(\mathscr{U}_3) \cong \omega_S$, it follows that

$$\omega_{\mathscr{Y}_{5,1}} \cong \pi_{5,1}^*\mathscr{O}_{\mathbb{P}(\mathrm{W}_5)}(-3) \cong \rho_{5,1}^*\pi^*\mathscr{O}_{\mathbb{P}(\mathrm{W}_5)}(-3). \tag{2.46}$$

On the other hand, π is a double covering branched over a quartic, hence one has $\omega_{\mathscr{Y}} \cong \pi^*\mathscr{O}_{\mathbb{P}(\mathrm{W}_5)}(-3)$. Thus $\omega_{\mathscr{Y}_{5,1}} \cong \rho_{5,1}^*\omega_{\mathscr{Y}}$, i.e., the map $\rho_{5,1}$ is crepant. Since $\mathscr{Y}_{5,1}$ is smooth it follows that the map $\rho_{5,1}$ is an isomorphism over the smooth locus of $\mathscr{Y}$, hence the exceptional locus of $\rho_{5,1}$ is contained in

$$\rho_{5,1}^{-1}(\mathrm{Sing}(\mathscr{Y})) = \pi_{5,1}^{-1}(\mathrm{Sing}(X_{\frac{1}{4}})) = \pi_{5,1}^{-1}(\mathrm{CR}),$$

i.e., in the preimage of the Cremona–Richmond configuration of 15 lines. But by Lemma 2.35 the fibers of $\pi_{5,1}$ are at most one-dimensional, hence $\dim(\pi_{5,1}^{-1}(\mathrm{CR})) \leqslant 2$. This proves that $\rho_{5,1}$ is small.

Next, let us show that

$$\pi_{5,1}^{-1}(\mathrm{CR}) = \left(\bigcup_{\varphi} R_\varphi\right) \cup \left(\bigcup_{L} R_L\right). \tag{2.47}$$

By Lemmas 2.36 and 2.41 the surfaces R_L and R_φ are contracted onto the union of ten lines L' and five lines Γ_φ in $\mathbb{P}(\mathrm{W}_5)$, which are pairwise distinct by Lemmas 2.36 and 2.43. Therefore

$$\mathrm{CR} = \left(\bigcup_{\varphi} \Gamma_\varphi\right) \cup \left(2\bigcup_{L} L'\right).$$

It remains to show that for any $w \in \Gamma_\varphi$ or $w \in L'$ the fiber $S_w = \pi_{5,1}^{-1}(w)$ is contained either in R_φ or in R_L. If $w \in \Gamma_\varphi$, this is proved in Lemma 2.41. Now take $w \in L'$. By Lemma 2.36 we have $L \subset S_w$, hence by Lemma 2.35 the curve S_w is either the line L (hence $S_w \subset R_L$) or a conic (hence $S_w \subset R_\varphi$ for appropriate φ). This proves (2.47).

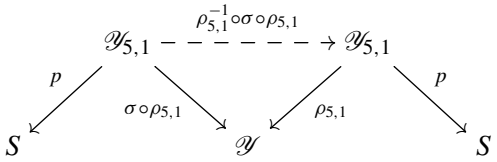
The vector space W_5 by definition (2.28) comes with a natural $\mathfrak{S}_5$-action, and, moreover, the map $\pi_{5,1}\colon \mathscr{Y}_{5,1} \to \mathbb{P}(\mathrm{W}_5)$ is $\mathfrak{S}_5$-equivariant. It follows that its branch divisor $B = X_{1/4}$ is invariant under this action. This gives an embedding $\mathfrak{S}_5 \hookrightarrow \mathrm{Aut}(X_{1/4}) \cong \mathfrak{S}_6 \subset \mathrm{Aut}(\mathscr{Y})$, such that for every element $g \in \mathfrak{S}_5$ the conjugation of the diagram (2.45) by g gives a diagram of the same form. Therefore, one has

$$g \circ \rho_{5,1} \circ g^{-1} = \sigma^{k(g)} \circ \rho_{5,1},$$

where $k\colon \mathfrak{S}_5 \to \mathbb{Z}/2\mathbb{Z}$ is a group homomorphism. If it is trivial, then $\rho_{5,1}$ is equivariant with respect to the natural action, and if k is the homomorphism of parity, then $\rho_{5,1}$ is equivariant with respect to the twisted action (as we mentioned above, we will show in Section 2.4 that k is indeed the homomorphism of parity).

To show that the embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ is nonstandard we use the same argument as in the proof of Proposition 2.21. The restriction of the five-dimensional representation (1.1) to the image of a standard embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ decomposes as a direct sum of two irreducible representations (see Lemma 5.12), while the $\mathfrak{S}_5$-representation W_5 is irreducible by (2.28) and [Shepherd-Barron 1989, Proposition 2]. □

Similarly to the case of $\rho_{4,2}$, the morphism $\rho_{5,1}$ is not uniquely defined even when the corresponding nonstandard subgroup $\mathfrak{S}_5$ is fixed. Moreover, there is a commutative diagram



(2.48)

Here $\rho_{5,1}^{-1}\circ\sigma\circ\rho_{5,1}$ is a small birational map. In fact, we know that $\operatorname{rk}\operatorname{Pic}(S)^{\mathfrak{S}_5}=1$; see for instance [Cheltsov and Shramov 2016a, Lemma 6.2.2(i)]; this means that

$$\operatorname{rk}\operatorname{Cl}(\mathscr{Y}_{5,1})^{\mathfrak{S}_5}=\operatorname{rk}\operatorname{Pic}(\mathscr{Y}_{5,1})^{\mathfrak{S}_5}=2,$$

and therefore $\operatorname{rk}\operatorname{Pic}(\mathscr{Y})^{\mathfrak{S}_5}=1$. The latter implies that $\rho_{5,1}$ and $\sigma\circ\rho_{5,1}$ are the only $\mathfrak{S}_5$-equivariant small resolutions of singularities of $\mathscr{Y}$ and that $\rho_{5,1}^{-1}\circ\sigma\circ\rho_{5,1}$ is an $\mathfrak{S}_5$-flop. Consequently, the diagram (2.48) is an $\mathfrak{S}_5$-Sarkisov link between two copies of the Mori fiber space $\mathscr{Y}_{5,1}\to S$ centered at $\mathscr{Y}$ (see Section 5.1).

Remark 2.49. Recall the notation of Remark 2.25. Denote

$$\mathscr{H}_I^{5,1}:=\rho_{5,1}^{-1}(\mathscr{H}_I),$$

so that one has $\pi_{5,1}^{-1}(H_I)=\pi_{5,1}^{-1}(H_{\bar{I}})=\mathscr{H}_I^{5,1}\cup\mathscr{H}_{\bar{I}}^{5,1}$. One can check that ten out of twenty divisors $\mathscr{H}_I^{5,1}\subset\mathscr{Y}_{5,1}$ are the preimages of lines on S via the map p, and the other ten are relative hyperplane sections for p (this decomposition is the orbit decomposition for the action of $\mathfrak{S}_5$). We denote by $\mathscr{H}_+^{5,1}$ the sum of the divisors of the first type, and by $\mathscr{H}_-^{5,1}$ the sum of the divisors of the second type. The divisor $\mathscr{H}_+^{5,1}$ is the p-pullback of an ample divisor on S, hence it is $\rho_{5,1}$-ample. Consequently, $-\mathscr{H}_-^{5,1}$ is $\rho_{5,1}$-ample, hence the small birational morphism $\rho_{5,1}$ is the blow up of the Weil divisor $\rho_{5,1}(\mathscr{H}_-^{5,1})$ on $\mathscr{Y}$. See Remark 2.57 below for an explicit description of this blow up.

2.3. ***Compatibility of resolutions.*** In this section we relate the resolutions $\mathscr{Y}_{4,2}$ and $\mathscr{Y}_{5,1}$ of the Coble fourfold. Recall that the first of them is associated with a nonstandard embedding $\mathfrak{S}_{4,2}\hookrightarrow\mathfrak{S}_6$, and the second is associated with a nonstandard embedding $\mathfrak{S}_5\hookrightarrow\mathfrak{S}_6$. Note that each (standard or nonstandard) subgroup $\mathfrak{S}_4\subset\mathfrak{S}_6$ can be extended to a subgroup $\mathfrak{S}_{4,2}\subset\mathfrak{S}_6$ and such extension is unique. Indeed, the second factor $\mathfrak{S}_2$ in $\mathfrak{S}_{4,2}\cong\mathfrak{S}_4\times\mathfrak{S}_2$ is just the centralizer of $\mathfrak{S}_4$ in $\mathfrak{S}_6$. Recall also that for each $\mathfrak{S}_4\subset\mathfrak{S}_5=\operatorname{Aut}(S)$ there is a unique $\mathfrak{S}_4$-equivariant contraction $\varphi\colon S\to\mathbb{P}^2$ of the quintic del Pezzo surface S onto the plane.

Proposition 2.50. *Let $\mathfrak{S}_5\hookrightarrow\mathfrak{S}_6$ be a nonstandard embedding. Choose a subgroup $\mathfrak{S}_4\subset\mathfrak{S}_5$ and let $\mathfrak{S}_{4,2}\subset\mathfrak{S}_6$ be its unique extension. Let $\rho_{4,2}\colon\mathscr{Y}_{4,2}\to\mathscr{Y}$ be the $\mathfrak{S}_{4,2}$-equivariant resolution of singularities constructed in Proposition 2.21 and let $\varphi\colon S\to\mathbb{P}^2$ be the unique $\mathfrak{S}_4$-equivariant contraction of the quintic del Pezzo surface. Then there is a unique $\mathfrak{S}_5$-equivariant resolution $\rho_{5,1}\colon\mathscr{Y}_{5,1}\to\mathscr{Y}$ as in Proposition 2.44 and a unique $\mathfrak{S}_4$-equivariant small birational map $\theta_1\colon\mathscr{Y}_{5,1}\dashrightarrow\mathscr{Y}_{4,2}$ such that the diagram* (1.10) *is commutative.*

Of course, if $\rho_{5,1}$ is fixed, there is only one θ_1 such that the inner triangle in the diagram (1.10) commutes, namely, $\theta_1 = \rho_{4,2}^{-1} \circ \rho_{5,1}$. But it is a priori not clear why the outer square commutes. So, to prove Proposition 2.50 we move in the opposite direction: we first construct θ_1 such that the outer square commutes, and after that check that the inner triangle commutes for this θ_1 and for an appropriate choice of $\rho_{5,1}$.

We start with some notation and a lemma. Let $\varphi\colon S \to \mathbb{P}^2$ be the $\mathfrak{S}_4$-equivariant contraction, and, as before, denote by E_0, E_1, E_2, and E_3 the exceptional divisors of the blow up φ, by e_i their classes in $\mathrm{Pic}(S)$ and by ℓ the pullback of the line class of $\mathbb{P}^2$. Recall also the rank 3 bundle $\mathscr{U}_3$ on S.

Since $\mathscr{U}_3^\vee$ is globally generated and $\det(\mathscr{U}_3^\vee)|_{E_i} \cong \omega_S^{-1}|_{E_i} \cong \mathscr{O}_{E_i}(1)$, we have

$$\mathscr{U}_3|_{E_i} \cong \mathscr{O}_{E_i} \oplus \mathscr{O}_{E_i} \oplus \mathscr{O}_{E_i}(-1).$$

Therefore, we have a canonical surjective morphism $\mathscr{U}_3 \to \mathscr{O}_{E_i}(-1)$ of sheaves on S. The sum of these morphisms gives an exact $\mathfrak{S}_4$-equivariant sequence

$$0 \to \mathscr{E} \to \mathscr{U}_3 \to \bigoplus_{i=0}^{3} \mathscr{O}_{E_i}(-1) \to 0 \tag{2.51}$$

and defines a rank 3 vector bundle $\mathscr{E}$ on S equivariant with respect to $\mathfrak{S}_4$.

Lemma 2.52. *One has $\mathscr{E} \cong \mathscr{O}_S(-\ell)^{\oplus 3}$.*

Proof. Consider the composition of the embedding

$$\mathscr{O}_S(-2\ell + e_0 + e_1 + e_2 + e_3) \hookrightarrow \mathscr{U}_3$$

from (2.40) with the projection $\mathscr{U}_3 \to \mathscr{O}_{E_i}(-1)$. If it is equal to zero, then the map $\mathscr{U}_3 \to \mathscr{O}_{E_i}(-1)$ factors through a map $\varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2)) \to \mathscr{O}_{E_i}(-1)$. But the sheaf $\varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2))$ restricts to E_i trivially, hence no such map exists. This contradiction shows that the composition is nontrivial. But since

$$\mathscr{O}_S(-2\ell + e_0 + e_1 + e_2 + e_3)|_{E_i} \cong \mathscr{O}_{E_i}(-1),$$

any nontrivial morphism $\mathscr{O}_S(-2\ell + e_0 + e_1 + e_2 + e_3) \to \mathscr{O}_{E_i}(-1)$ is surjective. Therefore, the sum of these morphisms $\mathscr{O}_S(-2\ell + e_0 + e_1 + e_2 + e_3) \to \bigoplus_{i=0}^3 \mathscr{O}_{E_i}(-1)$ is surjective, hence its kernel is $\mathscr{O}_S(-2\ell)$ and we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_S(-2\ell) & \longrightarrow & \mathscr{O}_S(-2\ell + e_0 + e_1 + e_2 + e_3) & \longrightarrow & \bigoplus_{i=0}^3 \mathscr{O}_{E_i}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{U}_3 & \longrightarrow & \bigoplus_{i=0}^3 \mathscr{O}_{E_i}(-1) & \longrightarrow & 0
\end{array}$$

Taking into account (2.40), we see that the first column extends to an exact sequence

$$0 \to \mathscr{O}_S(-2\ell) \to \mathscr{E} \to \varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2)) \to 0. \tag{2.53}$$

It remains to show that it coincides with the pullback of a twist of the Euler sequence on $\mathbb{P}^2$. Since the pullback functor φ^* is fully faithful, and the Euler sequence is the unique nonsplit extension of $\mathscr{T}_{\mathbb{P}^2}$ by $\mathscr{O}_{\mathbb{P}^2}$, it is enough to show that (2.53) is nonsplit.

Assume on the contrary that there is a splitting $\varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2)) \to \mathscr{E}$. Composing it with the embedding $\mathscr{E} \hookrightarrow \mathscr{U}_3$, we obtain a splitting $\varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2)) \to \mathscr{U}_3$ of (2.40). It induces an embedding

$$S \times_{\mathbb{P}^2} \mathrm{Fl}(1, 2; 3) \cong \mathbb{P}_S(\varphi^*(\mathscr{T}_{\mathbb{P}^2}(-2))) \hookrightarrow \mathbb{P}_S(\mathscr{U}_3) = \mathscr{Y}_{5,1},$$

such that its composition with $\pi_{5,1}$ coincides with the projection

$$S \times_{\mathbb{P}^2} \mathrm{Fl}(1, 2; 3) \to \mathrm{Fl}(1, 2; 3) \to (\mathbb{P}^2)^\vee.$$

But this contradicts the fact that $\rho_{5,1}$ is a small contraction. □

Proof of Proposition 2.50. Let us construct the map θ_1. Let V_1 be a three-dimensional vector space such that the target plane of φ is $\mathbb{P}(V_1)$. We can choose an isomorphism

$$\alpha_1 \colon \mathbb{P}(V_1) \xrightarrow{\sim} \mathbb{P}(\mathrm{W}_3)$$

such that the points of $\mathbb{P}(V_1)$ to which the divisors E_i are contracted by φ go to the points P_i of $\mathbb{P}(\mathrm{W}_3)$ defined by (2.1). Note that such an isomorphism is unique and $\mathfrak{S}_4$-equivariant.

Next, let V_2 be the three-dimensional vector space such that $\mathscr{E} \cong V_2 \otimes \mathscr{O}_S(-\ell)$. Note that the space $V_2 \cong H^0(S, \mathscr{E}(\ell))$ has a natural structure of an $\mathfrak{S}_4$-representation, and the above isomorphism $\mathscr{E} \cong V_2 \otimes \mathscr{O}_S(-\ell)$ is $\mathfrak{S}_4$-equivariant. Under this identification the first map in (2.51) becomes an $\mathfrak{S}_4$-equivariant embedding of sheaves

$$V_2 \otimes \mathscr{O}_S(-\ell) \xrightarrow{\xi} \mathscr{U}_3, \tag{2.54}$$

which is an isomorphism away from the union of E_i. Its dual map extends to an exact $\mathfrak{S}_4$-equivariant sequence

$$0 \to \mathscr{U}_3^\vee \xrightarrow{\xi^\vee} V_2^\vee \otimes \mathscr{O}_S(\ell) \to \bigoplus_{i=0}^{3} \mathscr{O}_{E_i} \to 0. \tag{2.55}$$

The second map defines four linear functions on $V_2^\vee$, i.e., four points on $\mathbb{P}(V_2)$. We can choose an isomorphism

$$\alpha_2 \colon \mathbb{P}(V_2) \xrightarrow{\sim} \mathbb{P}(\mathrm{W}_3)$$

such that these points go to the points P_i of $\mathbb{P}(\mathrm{W}_3)$ defined by (2.1). Again, such an isomorphism is unique and $\mathfrak{S}_4$-equivariant.

Now we put all the above constructions together. The morphism ξ defined by (2.54) induces a birational map

$$S \times \mathbb{P}(V_2) \cong \mathbb{P}_S(V_2 \otimes \mathscr{O}_S(-\ell)) \overset{\xi}{\dashrightarrow} \mathbb{P}_S(\mathscr{U}_3) = \mathscr{Y}_{5,1}.$$

We define θ_1 as the composition

$$\mathscr{Y}_{5,1} \overset{\xi^{-1}}{\dashrightarrow} S \times \mathbb{P}(V_2) \xrightarrow{\varphi\times\mathrm{id}} \mathbb{P}(V_1) \times \mathbb{P}(V_2) \xrightarrow{\alpha_1\times\alpha_2} \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3) \overset{\beta^{-1}}{\dashrightarrow} \mathscr{Y}_{4,2},$$

where the last map is the inverse of the blow up (2.3). Clearly, θ_1 is birational and $\mathfrak{S}_4$-equivariant, since all the maps used in its definition are. Finally, its composition with p_1 equals $\varphi \circ p$ by construction, hence the outer square in (1.10) commutes.

Next, let us show an equality of the maps

$$\pi_{4,2} \circ \theta_1 = \pi_{5,1} \tag{2.56}$$

from $\mathscr{Y}_{5,1}$ to $\mathbb{P}(\mathrm{W}_5)$. For this, consider the diagram

$$\begin{array}{ccccc}
\mathrm{W}_5^\vee \otimes \mathscr{O}_S & \longrightarrow & \mathrm{W}_3^\vee \otimes \mathrm{W}_3^\vee \otimes \mathscr{O}_S & \xrightarrow{(\boldsymbol{P}_0,\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3)} & \bigoplus_{i=0}^3 \mathscr{O}_S \\
\| & & \cong \downarrow (\alpha_2^\vee \otimes \alpha_1^\vee) & & \| \\
\mathrm{W}_5^\vee \otimes \mathscr{O}_S & \xrightarrow{H^0(S,\xi^\vee)} & V_2^\vee \otimes V_1^\vee \otimes \mathscr{O}_S & \longrightarrow & \bigoplus_{i=0}^3 \mathscr{O}_S \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{U}_3^\vee & \xrightarrow{\xi^\vee} & V_2^\vee \otimes \mathscr{O}_S(\ell) & \longrightarrow & \bigoplus_{i=0}^3 \mathscr{O}_{E_i}
\end{array}$$

Here the bottom line is (2.55), the middle line is obtained from it by passing to global sections and tensoring with $\mathscr{O}_S$, and the maps between these lines are induced by evaluation of sections (hence the lower squares commute). The top line is obtained by identification (2.2), the upper-right square commutes by definition of α_1 and α_2. Therefore, there is a unique identification of the spaces $\mathrm{W}_5^\vee$ in this diagram (note that the one in the top line is defined by (2.2), while the other is defined by (2.28)) such that the upper-left square commutes. From now on we use implicitly the induced identification of the spaces W_5.

As a result of this commutativity two morphisms $\mathrm{W}_5^\vee \otimes \mathscr{O}_S \to V_2^\vee \otimes \mathscr{O}_S(\ell)$ in the diagram coincide. One of them induces the rational map

$$S \times \mathbb{P}(V_2) \overset{\xi}{\dashrightarrow} \mathscr{Y}_{5,1} \xrightarrow{\pi_{5,1}} \mathbb{P}(\mathrm{W}_5),$$

and the other induces the rational map

$$S \times \mathbb{P}(V_2) \xrightarrow{\varphi\times\mathrm{id}} \mathbb{P}(V_1) \times \mathbb{P}(V_2) \xrightarrow{\alpha_1\times\alpha_2} \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3) \overset{\overline{\pi}_{4,2}}{\dashrightarrow} \mathbb{P}(\mathrm{W}_5);$$

the map φ appears here because all the global sections of $\mathscr{O}_S(\ell)$ are pullbacks via φ. So, we have an equality of rational maps

$$\overline{\pi}_{4,2} \circ (\alpha_1 \times \alpha_2) \circ (\varphi \times \mathrm{id}) = \pi_{5,1} \circ \xi$$

from $S \times \mathbb{P}(V_2)$ to $\mathbb{P}(\mathrm{W}_5)$. Composing it with the map ξ^{-1} on the right and using (2.4) and the definition of θ_1, we deduce the required equality (2.56).

From (2.56) we further deduce an equality

$$\pi \circ (\rho_{4,2} \circ \theta_1) = \pi_{4,2} \circ \theta_1 = \pi_{5,1}.$$

Hence, the composition $\rho_{4,2} \circ \theta_1$ provides one of the two possible factorizations $\rho_{5,1}$ of the morphism $\pi_{5,1}$. This shows that for one of the two choices of $\rho_{5,1}$, the inner triangle in (1.10) is commutative. □

It is worth noting that if we want to replace the map p_1 in the diagram (1.10) by another projection p_2 and preserve its commutativity, we will have to replace the subgroup $\mathfrak{S}_5$ containing $\mathfrak{S}_4$ by the unique other such subgroup (more precisely, we will have to replace the embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ with the one obtained from it by a conjugation with the factor $\mathfrak{S}_2$ in $\mathfrak{S}_{4,2}$).

Remark 2.57. Recall the notation of Remarks 2.25 and 2.49, and assume that we are in the situation of Proposition 2.50: the resolution $\rho_{4,2}$ is defined by (2.23) and the resolution $\rho_{5,1}$ is such that the diagram (1.10) commutes. Then we have

$$\mathscr{H}^{5,1}_{+} = \mathscr{H}^{5,1}_{123} + \mathscr{H}^{5,1}_{156} + \mathscr{H}^{5,1}_{246} + \mathscr{H}^{5,1}_{345} + \mathscr{H}^{5,1}_{124} + \mathscr{H}^{5,1}_{136} + \mathscr{H}^{5,1}_{235} + \mathscr{H}^{5,1}_{145} + \mathscr{H}^{5,1}_{256} + \mathscr{H}^{5,1}_{346},$$
$$\mathscr{H}^{5,1}_{-} = \mathscr{H}^{5,1}_{456} + \mathscr{H}^{5,1}_{234} + \mathscr{H}^{5,1}_{135} + \mathscr{H}^{5,1}_{126} + \mathscr{H}^{5,1}_{356} + \mathscr{H}^{5,1}_{245} + \mathscr{H}^{5,1}_{146} + \mathscr{H}^{5,1}_{236} + \mathscr{H}^{5,1}_{134} + \mathscr{H}^{5,1}_{125}.$$

Consequently, $\rho_{5,1}$ is the blow up of the Weil divisor

$$\mathscr{H}_{456} + \mathscr{H}_{234} + \mathscr{H}_{135} + \mathscr{H}_{126} + \mathscr{H}_{356} + \mathscr{H}_{245} + \mathscr{H}_{146} + \mathscr{H}_{236} + \mathscr{H}_{134} + \mathscr{H}_{125}$$

on $\mathscr{Y}$.

2.4. ***Proof of Theorem 1.9.*** In Proposition 2.21 we constructed the morphism $\rho_{4,2}$ for some nonstandard subgroup $\mathfrak{S}_{4,2} \subset \mathfrak{S}_6$, and checked that it is $\mathfrak{S}_{4,2}$-equivariant for the twisted action and small. To construct $\rho_{4,2}$ for any other nonstandard embedding, we may use a conjugation by an appropriate element of $\mathfrak{S}_6$. This proves assertion (i).

Similarly to the above, in Proposition 2.44 we constructed a morphism $\rho_{5,1}$ for some nonstandard embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ (and the same trick as above then gives $\rho_{5,1}$ for any other nonstandard $\mathfrak{S}_5 \subset \mathfrak{S}_6$) and checked that it is small. Moreover, the compatibility isomorphism θ_1 was constructed in Proposition 2.50; by the way it proves assertion (iv).

Furthermore, we checked that the morphism $\rho_{5,1}$ is $\mathfrak{S}_5$-equivariant with respect either to the natural or to the twisted action of $\mathfrak{S}_5$ on $\mathscr{Y}$. To show that the action is twisted, we use Proposition 2.50. Choose a subgroup $\mathfrak{S}_4 \subset \mathfrak{S}_5$, a transposition $g \in \mathfrak{S}_4$, and consider the commutative diagram (1.10). Since θ_1 is $\mathfrak{S}_4$-equivariant and $g \circ \rho_{4,2} \circ g^{-1} = \sigma \circ \rho_{4,2}$ (as $\rho_{4,2}$ is equivariant with respect to the twisted action), we have

$$g \circ \rho_{5,1} \circ g^{-1} = g \circ \rho_{4,2} \circ \theta_1 \circ g^{-1} = g \circ \rho_{4,2} \circ g^{-1} \circ \theta_1 = \sigma \circ \rho_{4,2} \circ \theta_1 = \sigma \circ \rho_{5,1},$$

hence $\rho_{5,1}$ is equivariant with respect to the twisted action as well. This completes the proof of assertion (ii).

Finally, recall that we checked in Propositions 2.21 and 2.44 that $\rho_{4,2}$ and $\rho_{5,1}$ are isomorphisms over the complement of the Cremona–Richmond configuration $\mathrm{CR} = \mathrm{Sing}(X_{1/4}) \subset \mathbb{P}^4$. This gives the proof of assertion (iii) and completes the proof of Theorem 1.9. □

3. Conic bundle structures on $\mathfrak{S}_6$-invariant quartics

Recall the pencil $\{X_t\}$ of $\mathfrak{S}_6$-invariant quartics defined by the (1.2) inside the hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$ given by (1.1). In this section we discuss the conic bundle structures on the quartics X_t induced by the resolutions of the Coble fourfold.

3.1. ***$\mathfrak{S}_6$-invariant quartics revisited.*** We start by collecting some facts about automorphism groups of X_t, their singularities and class groups.

Let CR be the Cremona–Richmond configuration of 15 lines with 15 intersection points; see the Appendix. The intersection points of the lines of CR form the orbit

$$\Upsilon_{15} = \{g \cdot (2:2:-1:-1:-1:-1) \mid g \in \mathfrak{S}_6\}.$$

Besides this, we consider also the orbits

$$\begin{aligned}
\Sigma_6 &= \{g \cdot (5:-1:-1:-1:-1:-1) \mid g \in \mathfrak{S}_6\},\\
\Sigma_{10} &= \{g \cdot (1:1:1:-1:-1:-1) \mid g \in \mathfrak{S}_6\},\\
\Sigma_{15} &= \{g \cdot (1:-1:0:0:0:0) \mid g \in \mathfrak{S}_6\},\\
\Sigma_{30} &= \{g \cdot (1:1:\omega:\omega:\omega^2:\omega^2) \mid g \in \mathfrak{S}_6\},
\end{aligned}$$

where ω is a primitive cubic root of unity and the lower index on the left-hand side stands for cardinality of the orbit. We note that

$$\Upsilon_{15} \subset \mathrm{CR}, \quad \Sigma_{30} \subset \mathrm{CR}, \quad (\Sigma_6 \cup \Sigma_{10} \cup \Sigma_{15}) \cap \mathrm{CR} = \varnothing.$$

Remark 3.1. The quartic X_∞ defined by (1.2) with $t = \infty$ is the quadric Q_∞ given by the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0 \tag{3.2}$$

taken with multiplicity 2. Note that

$$Q_\infty \cap \Upsilon_{15} = \varnothing, \quad Q_\infty \cap \mathrm{CR} = \Sigma_{30},$$

and the intersection is transversal.

The singularities of the quartics X_t have been described by van der Geer [1982] in terms of these orbits. Recall the discriminant set $\mathfrak{D}$ defined by (1.3).

Theorem 3.3 [van der Geer 1982, Theorem 4.1]. *One has*

t	$t \notin \mathfrak{D} \cup \{\infty\}$	$t = \frac{1}{4}$	$t = \frac{1}{2}$	$t = \frac{1}{6}$	$t = \frac{7}{10}$
$\mathrm{Sing}(X_t)$	Σ_{30}	CR	$\Sigma_{30} \cup \Sigma_{15}$	$\Sigma_{30} \cup \Sigma_{10}$	$\Sigma_{30} \cup \Sigma_6$

In particular, X_t is normal if $t \neq \infty$.

Moreover, all singular points of the quartics X_t are nodes provided that $t \neq \frac{1}{4}, \infty$.

One can describe automorphism groups of the quartics X_t.

Lemma 3.4. *The following assertions hold*:

(i) *One has* $\mathrm{Aut}(X_{1/2}) \cong \mathrm{PSp}_4(\mathbf{F}_3)$, *where* $\mathbf{F}_3$ *is the field of three elements.*

(ii) *One has* $\mathrm{Aut}(X_t) \cong \mathfrak{S}_6$ *provided that* $t \notin \left\{\frac{1}{2}, \infty\right\}$.

(iii) *If* X *is a normal quartic hypersurface with a faithful action of the group* $\mathfrak{S}_6$, *then* X *is isomorphic to one of the quartics* X_t.

Proof. Assertion (i) is well known; see e.g., [Coble 1906].

Take any $t \neq \infty$. Since the quartic X_t is normal by Theorem 3.3, its hyperplane section is the anticanonical class, hence the group $\mathrm{Aut}(X_t)$ is naturally embedded into $\mathrm{PGL}_5(\Bbbk)$. Moreover, one has $\mathfrak{S}_6 \subset \mathrm{Aut}(X_t)$ by the definition of X_t. It follows from the classification of finite subgroups of $\mathrm{PGL}_5(\Bbbk)$ that either $\mathrm{Aut}(X_t) \cong \mathfrak{S}_6$ or $\mathrm{Aut}(X_t) \cong \mathrm{PSp}_4(\mathbf{F}_3)$; see [Feit 1971, Section 8.5]. But the group $\mathrm{PSp}_4(\mathbf{F}_3)$ has a unique invariant quartic hypersurface in $\mathbb{P}^4$, which is the Burkhardt quartic $X_{1/2}$. This proves assertion (ii).

Finally, assume that X is a normal quartic hypersurface invariant under some faithful action of the group $\mathfrak{S}_6$ on $\mathbb{P}^4$. Using the classification of projective representations of the group $\mathfrak{S}_6$ we deduce that this action comes from an irreducible five-dimensional representation of $\mathfrak{S}_6$; in fact, it is enough to look at the classification of projective representations of the smaller group $\mathfrak{A}_6$, which can be found for instance in [Conway et al. 1985, page 5]. The latter $\mathfrak{S}_6$-representation is unique up to an outer automorphism and a sign twist (cf. Lemma 5.12). This implies assertion (iii). □

Corollary 3.5. *We have* $\mathrm{Aut}(\mathscr{Y}) \cong \mathfrak{S}_6 \times \boldsymbol{\mu}_2$.

Proof. The group on the right-hand side acts on $\mathscr{Y}$ by (1.6) and (1.7), and the action is clearly faithful. It remains to show that any automorphism of $\mathscr{Y}$ belongs to this group. For this we note that the morphism $\pi : \mathscr{Y} \to \mathbb{P}^4$ is defined by the ample generator of $\mathrm{Pic}(\mathscr{Y})$. Indeed, $\mathrm{rk}\,\mathrm{Pic}(\mathscr{Y}) = 1$ by Lefschetz hyperplane section theorem (see [Dolgachev 1982, Theorem 4.2.2]), because $\mathscr{Y}$ is a hypersurface in the weighted projective space $\mathbb{P}(2, 1^5)$. The pullback of the hyperplane in $\mathbb{P}^4$ via π is not divisible in $\mathrm{Pic}(\mathscr{Y})$ by degree reasons, and thus generates $\mathrm{Pic}(\mathscr{Y})$. Hence π is equivariant with respect to any automorphism of $\mathscr{Y}$. This induces a homomorphism $\mathrm{Aut}(\mathscr{Y}) \to \mathrm{PGL}_5(\Bbbk)$ whose kernel is generated by the Galois involution σ. The image of the homomorphism is the subgroup of $\mathrm{PGL}_5(\Bbbk)$ that fixes the branch divisor $X_{1/4}$ of π. Moreover, the latter subgroup acts faithfully on $X_{1/4}$, hence is contained in $\mathrm{Aut}(X_{1/4}) \cong \mathfrak{S}_6$. □

For further reference we state here a description of the class groups of X_t.

Lemma 3.6. *The following table lists the ranks of the class groups of the quartics* X_t:

t	$t \notin \mathfrak{D} \cup \{\infty\}$	$t = \frac{1}{4}$	$t = \frac{1}{2}$	$t = \frac{1}{6}$	$t = \frac{7}{10}$
$\mathrm{rk}\,\mathrm{Cl}(X_t)$	6	1	16	11	7

Proof. First, assume $t \notin \mathfrak{D} \cup \{\infty\}$. Let $\tilde{X}_t$ be the blow up of X_t at its singular points. Then $\tilde{X}_t$ is smooth by Theorem 3.3. Now the assertion follows from [Cynk 2001, Theorem 2] and [Beauville 2013, Lemma 2].

The cases $t = \frac{1}{2}$, $t = \frac{1}{6}$, and $t = \frac{7}{10}$, are discussed in [Kaloghiros 2011, Theorem 1.1(iii); Cheltsov and Shramov 2016b, Sections 5–6].

Finally, consider the case $t = \frac{1}{4}$. As it was already mentioned, the Igusa quartic $X_{1/4}$ is projectively dual to the Segre cubic threefold $Z \subset \mathbb{P}^4$. In fact, projective duality gives an $\mathfrak{S}_6$-equivariant birational map $Z \dashrightarrow X_{1/4}$ that blows up 10 ordinary double points of Z and blows down the proper transforms of 15 planes on Z; see e.g., the proof of [Prokhorov 2010, Lemma 3.10]. In particular, one has

$$\operatorname{rk} \operatorname{Cl}(X_{1/4}) = \operatorname{rk} \operatorname{Cl}(Z) + 10 - 15,$$

and since the class group of the Segre cubic Z has rank 6 (see e.g., [Prokhorov 2013, Theorem 7.1]), we obtain $\operatorname{rk} \operatorname{Cl}(X_{1/4}) = 1$. □

In Theorem 5.1 we will describe the action of the group $\mathfrak{S}_6$ on $\operatorname{Cl}(X_t) \otimes \mathbb{Q}$.

3.2. ***Wiman–Edge pencil.*** Consider the projective plane $\mathbb{P}^2$ with homogeneous coordinates w_1, w_2, and w_3 and the following two polynomials of degree six

$$\begin{aligned} \overline{\Phi}_0(w_1, w_2, w_3) &= (w_2^2 - w_3^2)(w_3^2 - w_1^2)(w_1^2 - w_2^2), \\ \overline{\Phi}_\infty(w_1, w_2, w_3) &= w_1^6 + w_2^6 + w_3^6 + (w_1^2 + w_2^2 + w_3^2)(w_1^4 + w_2^4 + w_3^4) - 12 w_1^2 w_2^2 w_3^2. \end{aligned} \tag{3.7}$$

It is easy to see that the sextic curves on $\mathbb{P}^2$ defined by these polynomials are singular at the following four points

$$(1:1:1), \quad (1:-1:-1), \quad (-1:1:-1), \quad (-1:-1:1), \tag{3.8}$$

hence they induce a pair of global sections

$$\Phi_0, \Phi_\infty \in H^0(S, \omega_S^{-2})$$

of the double anticanonical line bundle on the blow up S of $\mathbb{P}^2$ at the points (3.8), i.e., on the quintic del Pezzo surface. By [Edge 1981] the section Φ_∞ is invariant with respect to the action of $\operatorname{Aut}(S) \cong \mathfrak{S}_5$, while the Φ_0 is acted on by $\mathfrak{S}_5$ via the sign character. Therefore, there is an $\mathfrak{S}_5$-invariant pencil of $\mathfrak{A}_5$-invariant curves $\Delta_s \subset S$ given by the equation

$$\Phi_0 + s\Phi_\infty = 0, \quad s \in \Bbbk \cup \{\infty\}. \tag{3.9}$$

As we already mentioned, the curves Δ_s are double anticanonical divisors on S. We refer to the pencil (3.9) as the *Wiman–Edge pencil*. It was studied in various contexts in [Wiman 1896b; Edge 1981; Inoue and Kato 2005; Cheltsov and Shramov 2016a, Section 6.2; Dolgachev et al. 2018; Zamora 2018].

Theorem 3.10 [Edge 1981; Cheltsov and Shramov 2016a, Theorem 6.2.9]. *The Wiman–Edge pencil contains exactly five singular curves*: Δ_0, $\Delta_{\pm 1/\sqrt{125}}$, *and* $\Delta_{\pm 1/\sqrt{-3}}$. *They can be described as follows*:

- Δ_0 *is the union of* 10 *lines on* S; *it has* 15 *singular points.*
- $\Delta_{\pm 1/\sqrt{-3}}$ *are unions of* 5 *smooth conics*; *each of these curves has* 10 *singular points.*
- $\Delta_{\pm 1/\sqrt{125}}$ *are irreducible rational curves*; *each of these curves has* 6 *singular points.*

Every singular point of any of these curves is a node. The group $\mathfrak{A}_5$ *acts transitively on the set of singular points and on the set of irreducible components of each of these curves.*

Remark 3.11. The curves Δ_0 and Δ_∞ in the Wiman–Edge pencil are not just $\mathfrak{A}_5$-invariant, but also $\mathfrak{S}_5$-invariant. The first of them, as we already mentioned, is the union of 10 lines. The other one is a smooth curve of genus 6 known as the Wiman's sextic curve; see [Wiman 1896b; Edge 1981]; it should not be confused with a smooth plane sextic curve studied by Wiman [1896a]. By construction, Δ_∞ admits a faithful action of the group $\mathfrak{S}_5$, and one can show that its full automorphism group is also $\mathfrak{S}_5$.

3.3. ***Preimages of*** $\mathfrak{S}_6$***-invariant quartics in the Coble fourfold.*** Recall that the Coble fourfold $\mathscr{Y}$ is defined as a complete intersection in the weighted projective space $\mathbb{P}(2, 1^6)$ of the hyperplane (1.1) with the hypersurface (1.5). It comes with a double covering $\pi : \mathscr{Y} \to \mathbb{P}^4$ over the projective space in which the pencil $\{X_t\}$ of $\mathfrak{S}_6$-invariant quartics sits, and with the Galois involution $\sigma : \mathscr{Y} \to \mathscr{Y}$ of the double covering.

As in Section 1, we define a pencil of hypersurfaces $\mathscr{X}_\tau \subset \mathscr{Y}$ by (1.12). By definition each of the varieties $\mathscr{X}_\tau$ is $\mathfrak{S}_6$-invariant with respect to the natural $\mathfrak{S}_6$-action. Moreover, $\mathscr{X}_0$ and $\mathscr{X}_\infty$ are invariant under the whole group $\mathrm{Aut}(\mathscr{Y}) = \mathfrak{S}_6 \times \boldsymbol{\mu}_2$.

Lemma 3.12. *For every* $\tau \neq \infty$ *we have*

$$\pi^{-1}(X_{(\tau^2+1)/4}) = \mathscr{X}_\tau \cup \mathscr{X}_{-\tau},$$

and the involution σ *induces an* $\mathfrak{S}_6$*-equivariant isomorphism* $\sigma : \mathscr{X}_\tau \to \mathscr{X}_{-\tau}$ *for the natural action of* $\mathfrak{S}_6$. *The map* $\pi : \mathscr{X}_\tau \to X_{(\tau^2+1)/4}$ *is an isomorphism for all* $\tau \neq \infty$, *and the map* $\pi : \mathscr{X}_\infty \to X_\infty$ *factors through the double covering over the quadric* $Q_\infty = (X_\infty)_{\mathrm{red}}$ *defined by* (3.2) *that is branched over* $X_{1/4} \cap Q_\infty$. *The map* π *is* $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$*-equivariant for* $\tau = 0, \infty$ *and* $\mathfrak{S}_6$*-equivariant otherwise.*

Proof. The hypersurface $\pi^{-1}(X_{(\tau^2+1)/4}) \subset \mathscr{Y}$ is defined by the equation

$$(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4) - \tfrac{1}{4}(\tau^2+1)(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 = 0,$$

which in view of the equation (1.5) of $\mathscr{Y}$ can be rewritten as

$$\begin{aligned} 0 &= x_0^2 - \tfrac{\tau^2}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2 \\ &= \big(x_0 + \tfrac{\tau}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2\big)\big(x_0 - \tfrac{\tau}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2\big). \end{aligned}$$

Hence $\pi^{-1}(X_{(\tau^2+1)/4})$ is the union of $\mathscr{X}_\tau$ and $\mathscr{X}_{-\tau}$. The Galois involution σ acts by $x_0 \mapsto -x_0$, hence defines an isomorphism between $\mathscr{X}_\tau$ and $\mathscr{X}_{-\tau}$. To check that the map $\pi : \mathscr{X}_\tau \to X_{(\tau^2+1)/4}$ is an isomorphism, just use (1.12) to express x_0 in terms of other x_i; plugging it into the equation of the Coble

fourfold $\mathscr{Y}$, we deduce the equation of the quartic X_t. For $\tau = \infty$ this of course does not work, but the equations of $\mathscr{X}_\infty$ just give

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = x_0^2 - (x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4) = 0$$

which defines a double covering of Q_∞ whose branch locus is $X_0 \cap Q_\infty = X_{1/4} \cap Q_\infty$.

The equivariance of the maps σ and π is obvious. □

Remark 3.13. The singular locus of $\mathscr{X}_\infty$ consists of the unique $\mathfrak{S}_6$-orbit of length 30 that is projected by π to the $\mathfrak{S}_6$-orbit Σ_{30}; see e.g., [Przyjalkowski and Shramov 2016, Section 6].

Now we say a couple of words about the Weil divisor class groups of the threefolds $\mathscr{X}_\tau$. Consider the set

$$\hat{\mathfrak{D}} := \left\{0, \pm 1, \pm\frac{1}{\sqrt{-3}}, \pm\frac{3}{\sqrt{5}}\right\}, \tag{3.14}$$

that is, the preimage of the discriminant set $\mathfrak{D}$ defined in (1.3) under the map (1.11).

Lemma 3.15. *The following table lists the ranks of the class groups of the threefolds $\mathscr{X}_\tau$:*

τ	$\tau \notin \hat{\mathfrak{D}}$	$\tau = 0$	$\tau = \pm 1$	$\tau = \pm\frac{1}{\sqrt{-3}}$	$\tau = \pm\frac{3}{\sqrt{5}}$
$\operatorname{rk} \mathrm{Cl}(\mathscr{X}_\tau)$	6	1	16	11	7

Proof. If we assume that $\tau \neq \infty$, then the assertion follows from Lemma 3.6 in view of Lemma 3.12. For $\tau = \infty$ we argue similarly to the proof of Lemma 3.6 (see the proof of [Przyjalkowski and Shramov 2016, Proposition 6.3]). Let $\tilde{\mathscr{X}}_\infty$ be the blow up of $\mathscr{X}_\infty$ along its singular locus, i.e., the preimage of the $\mathfrak{S}_6$-orbit Σ_{30}; see Remark 3.13. Then $\tilde{\mathscr{X}}_\infty$ is smooth, and one proceeds as in [Cynk 2001, Theorem 2], using the computation of [Beauville 2013, Lemma 2]. □

3.4. ***Pencil of Verra threefolds.*** We consider the pullbacks $\mathscr{X}_\tau^{5,1}$ and $\mathscr{X}_\tau^{4,2}$ of the threefolds $\mathscr{X}_\tau$ to the resolutions $\mathscr{Y}_{5,1}$ and $\mathscr{Y}_{4,2}$ of singularities of the Coble fourfold, so that $\mathscr{X}_\tau^{5,1} \subset \mathscr{Y}_{5,1}$ and $\mathscr{X}_\tau^{4,2} \subset \mathscr{Y}_{4,2}$ are defined by (1.13). In the next section we will study the first of them, but now let us consider the second one. We assume that the map $\rho_{4,2}$ is defined by (2.23).

To simplify the situation, we consider the images of the threefolds $\mathscr{X}_\tau^{4,2}$ with respect to the contraction $\beta\colon \mathscr{Y}_{4,2} \to \mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}(\mathrm{W}_3) \times \mathbb{P}(\mathrm{W}_3)$, see Section 2.1. Define

$$\overline{\mathscr{X}}_\tau^{4,2} = \beta(\mathscr{X}_\tau^{4,2}) \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

As in Section 2.1 we use $(u_1 : u_2 : u_3)$ and $(v_1 : v_2 : v_3)$ for coordinates on the factors of $\mathbb{P}^2 \times \mathbb{P}^2$, and let $\boldsymbol{P}_i = (P_i, P_i)$ with P_i defined by (2.1).

Below we consider divisors of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ (and call them *Verra threefolds*) as conic bundles over the first factor. We write their equations as symmetric 3×3-matrices with coefficients being quadratic polynomials in u_1, u_2, u_3. So, if $q(u) = (q_{ij}(u))$ is such a matrix, the corresponding equation is $q(u)(v) := \sum q_{ij}(u) v_i v_j = 0$.

Proposition 3.16. *The subvariety $\overline{\mathscr{X}}^{4,2}_{\tau} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a Verra threefold given by the equation*

$$q_0(u)(v) + \tau q_\infty(u)(v) = 0, \tag{3.17}$$

where

$$q_0(u) = \frac{1}{2}\begin{pmatrix} 0 & u_3(u_2-u_1) & u_2(u_1-u_3) \\ u_3(u_2-u_1) & 0 & u_1(u_3-u_2) \\ u_2(u_1-u_3) & u_1(u_3-u_2) & 0 \end{pmatrix}, \quad \textit{and} \tag{3.18}$$

$$q_\infty(u) = \frac{1}{6}\begin{pmatrix} 4(u_2^2-u_2u_3+u_3^2) & u_3(u_1+u_2)-2u_1u_2-2u_3^2 & u_2(u_1+u_3)-2u_1u_3-2u_2^2 \\ u_3(u_1+u_2)-2u_1u_2-2u_3^2 & 4(u_1^2-u_1u_3+u_3^2) & u_1(u_2+u_3)-2u_2u_3-2u_1^2 \\ u_2(u_1+u_3)-2u_1u_3-2u_2^2 & u_1(u_2+u_3)-2u_2u_3-2u_1^2 & 4(u_1^2-u_1u_2+u_2^2) \end{pmatrix}. \tag{3.19}$$

Proof. By (1.12), the variety $\overline{\mathscr{X}}^{4,2}_{0}$ is given by the equation $x_0 = 0$. Writing the formula for x_0 from (2.23) in the matrix form, we get (3.18). Similarly, $\overline{\mathscr{X}}^{4,2}_{\infty}$ is given by the equation

$$\tfrac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2) = 0.$$

Substituting expressions for x_i from (2.23) and rewriting everything in the matrix form, we get (3.19). Therefore, (3.17) is the same as (1.12). □

Remark 3.20. Of course, one can cancel the common factor $\frac{1}{2}$ in (3.18) and (3.19). However, we prefer to keep it so that $q_0(u)(v)$ and $q_\infty(u)(v)$ are the same as the two summands in (1.12).

Since the maps $\beta\colon \mathscr{X}^{4,2}_{\tau} \to \overline{\mathscr{X}}^{4,2}_{\tau}$ and $\pi_{4,2} = \pi \circ \rho_{4,2}\colon \mathscr{X}^{4,2}_{\tau} \to X_{(\tau^2+1)/4}$ are birational for all $\tau \neq \infty$, the projection $p_1\colon \overline{\mathscr{X}}^{4,2}_{\tau} \to \mathbb{P}^2$ provides every (reduced) $\mathfrak{S}_6$-invariant quartic with a birational structure of a conic bundle. Similarly, the map $p_1\colon \overline{\mathscr{X}}^{4,2}_{\infty} \to \mathbb{P}^2$ provides a birational structure of a conic bundle on the threefold $\mathscr{X}_\infty$. The explicit formulas of Proposition 3.16 allow to compute their discriminant loci.

Lemma 3.21. *The discriminant curve of the conic bundle $p_1\colon \overline{\mathscr{X}}^{4,2}_{\tau} \to \mathbb{P}^2$ is the curve $\overline{\Delta}_\tau \subset \mathbb{P}^2$ defined by the equation*

$$(5\tau^2+3)\overline{\Phi}_0 + (\tau^3-\tau)\overline{\Phi}_\infty = 0, \tag{3.22}$$

where $\overline{\Phi}_0$ and $\overline{\Phi}_\infty$ are the sextic polynomials (3.7)*, and the coordinates $(w_1 : w_2 : w_3)$ are related to $(u_1 : u_2 : u_3)$ by the formula*

$$u_1 = w_2 + w_3, \quad u_2 = w_1 + w_3, \quad u_3 = w_1 + w_2.$$

Proof. A straightforward computation shows that

$$12\det(q_0(u) + \tau q_\infty(u)) = (5\tau^2+3)\overline{\Phi}_0 + (\tau^3-\tau)\overline{\Phi}_\infty.$$ □

The drawback of this conic bundle model is the lack of flatness. Indeed, it is easy to see that over each of the points P_i (see (2.1)) the matrix $q_0(u)$ is identically zero, so the fiber of $\overline{\mathscr{X}}^{4,2}_{0}$ over P_i is the whole $\mathbb{P}^2$. In the next subsection we check that using the resolution $\mathscr{Y}_{5,1}$ of the Coble fourfold, we obtain flat conic bundles.

3.5. ***Pencil of conic bundles over the quintic del Pezzo surface.*** Recall that $\mathscr{X}_\tau^{5,1} \subset \mathscr{Y}_{5,1}$ is defined in (1.13) as the preimage of the threefold $\mathscr{X}_\tau \subset \mathscr{Y}$ under the resolution $\rho_{5,1}\colon \mathscr{Y}_{5,1} \to \mathscr{Y}$. For its investigation it will be very convenient to use explicit formulas of Section 3.4. So, to benefit from those we assume that we are in the situation of Proposition 2.50, i.e., a subgroup $\mathfrak{S}_4 \subset \mathfrak{S}_5$ and a nonstandard embedding $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ are chosen, the choice of $\rho_{4,2}$ is fixed as in (2.23), the map $\theta_1\colon \mathscr{Y}_{5,1} \dashrightarrow \mathscr{Y}_{4,2}$ is a birational isomorphism for which the outer square of diagram (1.10) commutes, and $\rho_{5,1} = \rho_{4,2} \circ \theta_1$.

Remark 3.23. As we already discussed, for $\tau \neq 0, \infty$ the subvariety $\mathscr{X}_\tau$ is invariant with respect to the *natural* action of $\mathfrak{S}_6$, while the map $\rho_{5,1}\colon \mathscr{Y}_{5,1} \to \mathscr{Y}$ is equivariant with respect to the *twisted* action of $\mathfrak{S}_5 \subset \mathfrak{S}_6$. As a result, the subvariety $\mathscr{X}_\tau^{5,1} \subset \mathscr{Y}^{5,1}$ is only invariant under the action of the subgroup $\mathfrak{A}_6 \cap \mathfrak{S}_5 = \mathfrak{A}_5$, on which the two actions agree. Similarly, the projection $\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ is only $\mathfrak{A}_5$-equivariant. On the other hand, for $\tau = 0$ or $\tau = \infty$, the subvariety $\mathscr{X}_\tau^{5,1} \subset \mathscr{Y}$ is $\mathfrak{S}_5$-invariant and the map $\rho_{5,1}$ is $\mathfrak{S}_5$-equivariant.

Lemma 3.24. *The map $p\colon \mathscr{X}_\tau^{5,1} \to S$ is a flat conic bundle with the discriminant curve $\Delta_{s(\tau)} \subset S$ defined by* (3.9), *where*

$$s(\tau) = \frac{\tau^3 - \tau}{5\tau^2 + 3}. \tag{3.25}$$

The map p is $\mathfrak{A}_5$-equivariant for $\tau \neq 0, \infty$ and $\mathfrak{S}_5$-equivariant for $\tau = 0, \infty$.

Proof. Equivariance of the maps $p\colon \mathscr{X}_\tau^{5,1} \to S$ follows from invariance of $\mathscr{X}_\tau^{5,1}$ discussed in Remark 3.23 and $\mathfrak{S}_5$-equivariance of the $\mathbb{P}^2$-bundle $p\colon \mathscr{Y}_{5,1} \to S$. The restriction of (1.10) gives a commutative diagram

$$\begin{array}{ccccc}
\mathscr{X}_\tau^{5,1} & & \xrightarrow{\quad\theta_1\quad} & & \mathscr{X}_\tau^{4,2} \\
& \searrow^{\rho_{5,1}} & & \swarrow_{\rho_{4,2}} & \downarrow{\scriptstyle \beta} \\
\downarrow{\scriptstyle p} & & \mathscr{X}_\tau & & \overline{\mathscr{X}}_\tau^{4,2} \\
& & & & \downarrow{\scriptstyle p_1} \\
S & & \xrightarrow{\quad\varphi\quad} & & \mathbb{P}^2
\end{array} \tag{3.26}$$

The divisor $\mathscr{X}_\infty^{5,1} \subset \mathscr{Y}$ is the preimage of the quadric threefold $Q_\infty \subset \mathbb{P}(\mathrm{W}_5)$ with respect to the morphism $\pi_{5,1}\colon \mathscr{Y}_{5,1} \to \mathbb{P}(\mathrm{W}_5)$, hence it is the zero locus of a section of the line bundle $\mathscr{O}_{\mathbb{P}_S(\mathscr{U}_3)}(2)$. Since $\mathscr{X}_\tau^{5,1}$ form a pencil, all of them are the zero loci of sections of the same line bundle, hence correspond to symmetric morphisms $\mathscr{U}_3 \to \mathscr{U}_3^\vee$ on S (in particular, $p\colon \mathscr{X}_\tau^{5,1} \to S$ is a conic bundle). Therefore, the discriminant curve of $\mathscr{X}_\tau^{5,1}$ is the zero locus of a morphism

$$\omega_S \cong \det(\mathscr{U}_3) \to \det(\mathscr{U}_3^\vee) \cong \omega_S^{-1},$$

i.e., a double anticanonical divisor.

On the other hand, the above diagram shows that the discriminant locus of $\mathscr{X}_\tau^{5,1}$ contains the proper transform of the discriminant curve $\overline{\Delta}_\tau$ of $\overline{\mathscr{X}}_\tau^{4,2}$ whose equation is (3.22). If $\tau^3 - \tau \neq 0$ it is a sextic

curve passing with multiplicity 2 through each of the points P_i, hence its proper transform to S is a curve on S with equation

$$(5\tau^2+3)\Phi_0+(\tau^3-\tau)\Phi_\infty=0, \tag{3.27}$$

i.e., the curve $\Delta_{s(\tau)}$. In the case when $\tau^3-\tau=0$, the curve $\bar{\Delta}_\tau$ is the union of six lines on $\mathbb{P}^2$, and its proper transform on S is the union of six lines on S. But the conic bundle $p\colon \mathscr{X}_\tau^{5,1}\to S$ is $\mathfrak{A}_5$-equivariant. Thus its discriminant curve is $\mathfrak{A}_5$-invariant, and hence it should also contain the other four lines of S. We conclude this case by noting that the sum of the ten lines Δ_0 on S is a double anticanonical divisor, and it is indeed given by the equation (3.27) with $\tau^3-\tau=0$.

It remains to show that the conic bundle is flat. For this we note that a nonflat point of a conic bundle is a point of multiplicity at least 3 on its discriminant curve. But by Theorem 3.10 all singular points of these curves are nodes. □

Before going further, we discuss some properties of the map $s\colon \mathbb{P}^1\to\mathbb{P}^1$ defined by (3.25).

Lemma 3.28. *The map $s\colon \mathbb{P}^1\to\mathbb{P}^1$ is a triple covering with simple ramification at four points $\tau=\pm\sqrt{-3}$ and $\tau=\pm1/\sqrt{5}$.*

Proof. A direct computation. □

In the next table we list some special values of τ together with the values of the functions $s(\tau)$ and $t(\tau)=(\tau^2+1)/4$ at these points.

τ	0	±1	$\pm\frac{1}{\sqrt{-3}}$	$\boxed{\pm\sqrt{-3}}$	$\mp\frac{3}{\sqrt{5}}$	$\boxed{\pm\frac{1}{\sqrt{5}}}$	∞	$\pm\sqrt{-\frac{3}{5}}$
$s(\tau)$	0		$\mp\frac{1}{\sqrt{-3}}$		$\mp\frac{1}{5\sqrt{5}}$		∞	
$t(\tau)$	$\boxed{\frac{1}{4}}$	$\boxed{\frac{1}{2}}$	$\boxed{\frac{1}{6}}$	$-\frac{1}{2}$	$\boxed{\frac{7}{10}}$	$\frac{3}{10}$	$\boxed{\infty}$	$\frac{1}{10}$

The second row contains the values of the parameter s that correspond to singular members of the Wiman–Edge pencil (see Theorem 3.10) and infinity. The first row contains their preimages; boxed cells mark ramification points of the map $s(\tau)$; see Lemma 3.28. The third row contains the values of the map $t(\tau)$ at these points; boxed cells mark the points of the discriminant set $\mathfrak{D}$ and infinity.

Since the degree of the map s is 3, the same singular curves in the Wiman–Edge pencil may appear as the discriminant loci of the preimages $\mathscr{X}_\tau^{5,1}$ of different quartics X_t. For instance, the Igusa and the Burkhardt quartics both correspond to the union Δ_0 of the ten lines on S. Note also that the quartics $X_{1/6}$ and $X_{7/10}$ share their discriminant curves with nonspecial quartics $X_{-1/2}$ and $X_{3/10}$ respectively. As we will see in Proposition 3.30, these two are characterized by the fact that the corresponding curves in the Wiman–Edge pencil are singular, while the total spaces of the threefolds $\mathscr{X}_\tau^{5,1}$ are smooth. In Section 4 we will see that this subtle difference has a drastic effect on rationality properties.

To proceed we will need the following general result. Its proof can be found in [Beauville 1977, Proposition 1.2] or [Sarkisov 1982, Proposition 1.8], except for the fact that the singularity of $\mathscr{X}_P$ is a node, but this can also be extracted from the arguments in either of these two papers.

Lemma 3.29 [Beauville 1977, Proposition 1.2; Sarkisov 1982, Proposition 1.8]. *Let $p\colon \mathscr{X} \to S$ be a flat conic bundle over a smooth surface S. Assume that its discriminant locus $\Delta \subset S$ has a node at a point $P \in S$. Then $\mathscr{X}$ has a singular point over P if and only if the fiber $\mathscr{X}_P = p^{-1}(P)$ is a conic of corank* 1 (*that is, a union of two distinct lines*), *and in this case the singularity of $\mathscr{X}$ over P is a node at the* (*unique*) *singular point of $\mathscr{X}_P$.*

The next assertion describes the singular loci of the threefolds $\mathscr{X}_\tau^{5,1}$. Recall the morphism $\pi_{5,1}$ defined in (2.34) and the discriminant set $\hat{\mathfrak{D}}$ from (3.14).

Proposition 3.30. *The threefold $\mathscr{X}_\tau^{5,1}$ is smooth for all $\tau \notin \hat{\mathfrak{D}}$ (including $\tau = \infty$). For $\tau \in \hat{\mathfrak{D}}$ the singular locus of $\mathscr{X}_\tau^{5,1}$ is mapped by $\pi_{5,1}$ isomorphically to a subset of $\mathbb{P}^4$ as follows*:

τ	0	± 1	$\pm\frac{1}{\sqrt{-3}}$	$\pm\frac{3}{\sqrt{5}}$
$\pi_{5,1}(\mathrm{Sing}(\mathscr{X}_\tau^{5,1}))$	Υ_{15}	Σ_{15}	Σ_{10}	Σ_6

For $\tau \in \hat{\mathfrak{D}}$ the singularities of $\mathscr{X}_\tau^{5,1}$ form a single $\mathfrak{A}_5$-orbit, every singular point Q of $\mathscr{X}_\tau^{5,1}$ is a node, and the fiber $p^{-1}(p(Q))$ of the conic bundle $p\colon \mathscr{X}_\tau^{5,1} \to S$ passing through Q is the union of two distinct lines intersecting at Q.

Proof. To start with, let us show that for $\tau \neq 0$ the threefold $\mathscr{X}_\tau^{5,1}$ is smooth along the exceptional locus of the morphism $\rho_{5,1}$, which by Proposition 2.44 is the reducible surface

$$\left(\bigcup_L R_L\right) \cup \left(\bigcup_\varphi R_\varphi\right) = \pi_{5,1}^{-1}(\mathrm{CR}) \subset \mathscr{Y}_{5,1}. \tag{3.31}$$

Recall that each of its irreducible components is a smooth surface in $\mathscr{Y}_{5,1}$ (see Lemmas 2.36 and 2.41). Note that a Cartier divisor in a smooth fourfold is smooth along its intersection with a smooth surface provided that their scheme intersection is a smooth curve. So, it is enough to check that the intersections $\mathscr{X}_\tau^{5,1} \cap R_L$ and $\mathscr{X}_\tau^{5,1} \cap R_\varphi$ are smooth curves for all $\tau \neq 0$. But the divisors $\mathscr{X}_\tau^{5,1}$ form a pencil, and $\mathscr{X}_0^{5,1}$ (which by definition is equal to the ramification divisor of $\pi_{5,1}$) contains all these surfaces. Therefore,

$$\mathscr{X}_\tau^{5,1} \cap R_L = \mathscr{X}_\infty^{5,1} \cap R_L \quad \text{and} \quad \mathscr{X}_\tau^{5,1} \cap R_\varphi = \mathscr{X}_\infty^{5,1} \cap R_\varphi.$$

So, it is enough to show that $\mathscr{X}_\infty^{5,1} \cap R_\varphi$ and $\mathscr{X}_\infty^{5,1} \cap R_L$ are smooth curves. But $\mathscr{X}_\infty^{5,1} = \pi_{5,1}^{-1}(Q_\infty)$, while R_φ and R_L are the preimages of the 15 lines of the Cremona–Richmond configuration CR. The quadric Q_∞ intersects all these lines transversally at two points away from the intersection points of the lines by Remark 3.1, and taking into account Lemma 2.35 and Proposition 2.44 we conclude that $\mathscr{X}_\infty^{5,1} \cap R_L$ is the union of two disjoint lines, and $\mathscr{X}_\infty^{5,1} \cap R_\varphi$ is the union of two disjoint smooth conics.

Since the map $\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ is an isomorphism over $\mathbb{P}^4 \setminus \mathrm{CR}$ (because so is the map $\mathscr{Y}_{5,1} \to \mathscr{Y}$), it follows that for all $\tau \neq 0$ we have

$$\mathrm{Sing}(\mathscr{X}_\tau^{5,1}) = \mathrm{Sing}(\mathscr{X}_\tau) \setminus \mathrm{CR},$$

and in view of Lemma 3.12, Theorem 3.3, Remark 3.13, and Lemma 3.29, we obtain the required description of singularities of $\mathscr{X}_\tau^{5,1}$ for $\tau \neq 0$.

Next, consider the case $\tau = 0$. The map $p\colon \mathscr{X}_0^{5,1} \to S$ is a flat conic bundle with the discriminant locus being the curve Δ_0, i.e., the union of the 10 lines on S. It follows that $\mathscr{X}_0^{5,1}$ is smooth over the complement of the 15 intersection points of the lines on S. Since all these points are nodes of Δ_0, Lemma 3.29 shows that the threefold $\mathscr{X}_0^{5,1}$ has a singularity over such a point P if and only if the conic $(\mathscr{X}_0^{5,1})_P = p^{-1}(P)$ is the union of two distinct lines (and then the singular point is a node located at the intersection point of these lines). Since the 15 intersection points of the lines on S form a single $\mathfrak{A}_5$-orbit (see Theorem 3.10), it is enough to check everything over one of them.

Take the intersection point $P \in S$ such that $\varphi(P) = (0:1:1)$. We know from diagram (3.26) that the conic $(\mathscr{X}_0^{5,1})_P$ is isomorphic to the conic $(\overline{\mathscr{X}}_0^{4,2})_{\varphi(P)}$, hence by Proposition 3.16 it is given by the matrix

$$q_0(0:1:1) = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{3.32}$$

Its rank equals 2, hence $(\overline{\mathscr{X}}_0^{4,2})_{\varphi(P)}$, and thus also $(\mathscr{X}_0^{5,1})_P$, is a union of two lines. Moreover, the intersection point of the irreducible components of $(\overline{\mathscr{X}}_0^{4,2})_{\varphi(P)}$ is the point $(0:1:1)$, and using (2.23) we compute that

$$\bar{\pi}_{4,2}((0:1:1),(0:1:1)) = (2:-1:-1:2:-1:-1) \in \Upsilon_{15}.$$

By $\mathfrak{A}_5$-equivariance of the map $\pi_{5,1}$ and transitivity of $\mathfrak{A}_5$-action on Υ_{15} (see Corollary A.4) we conclude that $\pi_{5,1}(\mathrm{Sing}(\mathscr{X}_0^{5,1})) = \Upsilon_{15}$. □

Corollary 3.33. *For all $\tau \neq 0, \infty$ the morphism $\pi_{5,1}\colon \mathscr{X}_\tau^{5,1} \to X_{(\tau^2+1)/4}$ is birational and small. Also, the morphism $\rho_{5,1}\colon \mathscr{X}_\infty^{5,1} \to \mathscr{X}_\infty$ is birational and small.*

Proof. Indeed, as we have seen in the proof of Proposition 3.30, for $\tau \neq 0$ the nontrivial fibers of $\mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ are 30 rational curves, one over each of the 30 intersection points of $\Sigma_{30} = \mathrm{CR} \cap Q_\infty$. Since the map $\pi\colon \mathscr{X}_\tau \to X_{(\tau^2+1)/4}$ is an isomorphism for $\tau \neq \infty$ by Lemma 3.12, the assertion follows. □

Remark 3.34. For $\tau = 0$ the surface (3.31) is equal to the exceptional locus of $\pi_{5,1}\colon \mathscr{X}_0^{5,1} \to X_{1/4}$, hence this morphism is not small, but is still birational.

3.6. ***Proofs of Theorems 1.14 and 1.15.*** For $\tau \neq 0$ the map $\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ is small and birational by Corollary 3.33. The same argument works for $\rho_{4,2}\colon \mathscr{X}_\tau^{4,2} \to \mathscr{X}_\tau$ without changes. Finally, smoothness of $\mathscr{X}_\tau^{5,1}$ for nonspecial τ is proved in Proposition 3.30. The maps $\rho_{5,1}$ and $\pi \circ \rho_{5,1}$ have required

equivariance by Remark 3.23 and Lemma 3.12. The same arguments prove equivariance of the maps $\rho_{4,2}$ and $\pi \circ \rho_{4,2}$. This completes the proof of Theorem 1.14.

Now let us prove Theorem 1.15. By Proposition 3.30 the total spaces of the conic bundles $p\colon \mathscr{X}_\tau^{5,1} \to S$ are smooth for $\tau \notin \hat{\mathfrak{D}}$, so since $\operatorname{rk}\operatorname{Pic}(S) = 5$, to show that p is a standard conic bundle for $\tau \notin \hat{\mathfrak{D}}$ it is enough to check that $\operatorname{rk}\operatorname{Pic}(\mathscr{X}_\tau^{5,1}) = 6$ for these τ. But since the map $\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ is small, we have

$$\operatorname{Pic}(\mathscr{X}_\tau^{5,1}) \cong \operatorname{Cl}(\mathscr{X}_\tau).$$

Thus the assertion of the theorem follows from Lemma 3.15. □

Remark 3.35. Assume the notation of Remark 2.25, and suppose that $t \notin \left\{\frac{1}{4}, \frac{1}{2}, \infty\right\}$. One can check that the restrictions of each hyperplane $H_{ijk} \subset \mathbb{P}^4$ to X_t splits as the union of two smooth quadric surfaces in $H_{ijk} \cong \mathbb{P}^3$. For $t = \frac{1}{4}$ these two quadric surfaces collide into a smooth quadric with a nonreduced structure, and for $t = \frac{1}{2}$ they degenerate into unions of pairs of planes. Considering the preimages of these surfaces on $\mathscr{X}_\tau$, where as usual $t = (\tau^2 + 1)/4$, and using Remarks 2.25 and 2.49, one can describe the small resolutions $\rho_{4,2}$ and $\rho_{5,1}$ of singularities of $\mathscr{X}_\tau$ as blow ups of certain Weil divisors on $\mathscr{X}_\tau$.

4. Rationality

In this section we provide some applications of the results obtained earlier. Namely, we check that all quartics X_t are unirational, give a new and uniform proof of irrationality of $\mathfrak{S}_6$-invariant quartics X_t for $t \notin \mathfrak{D} \cup \{\infty\}$ (and also of the threefold $\mathscr{X}_\infty$), and rationality of X_t for $t \in \mathfrak{D}$.

4.1. ***Unirationality of $\mathfrak{S}_6$-invariant quartics.*** We start with a short proof of unirationality of the $\mathfrak{S}_6$-invariant quartics X_t and the threefold $\mathscr{X}_\infty$. The next fact is well known.

Lemma 4.1. *Let V be an irreducible Verra threefold, i.e., an irreducible hypersurface of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Then V is unirational.*

Proof. Let $p_i\colon V \to \mathbb{P}^2$, $i = 1, 2$, be the natural projections. Both p_i are (possibly nonflat) conic bundles. Let $L \subset \mathbb{P}^2$ be a general line, and put $T = p_2^{-1}(L)$. Since V is irreducible and L is general, the surface T is irreducible by Bertini's theorem. Also, the map p_2 provides the surface T with a conic bundle structure over $L \cong \mathbb{P}^1$, hence T is rational. Note also that $T = V \cap (\mathbb{P}^2 \times L)$ is a divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$, hence the projection $p_1\colon T \to \mathbb{P}^2$ is dominant (actually, T is a rational 2-section of p_1). Since $p_1\colon V \to \mathbb{P}^2$ is a conic bundle, the standard base change argument implies unirationality of V. □

Combining Lemma 4.1 with Proposition 3.16, we obtain

Corollary 4.2. *The quartics X_t, $t \neq \infty$, and the threefold $\mathscr{X}_\infty$, are unirational.*

Remark 4.3. One can use the same approach to prove rationality of the Burkhardt quartic $X_{1/2}$ (this is a classical fact going back to [Todd 1936]; see also Theorem 4.6 below). For this consider the corresponding Verra threefold $\overline{\mathscr{X}}_1^{4,2} \subset \mathbb{P}^2 \times \mathbb{P}^2$ and let $T = p_2^{-1}(\overline{P_1 P_2}) \subset \overline{\mathscr{X}}_1^{4,2}$ be the preimage of the line passing through

two of the points (2.1), that is, the line $v_3 = 0$. As before, T is a divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^1$. Using (3.18) and (3.19) we can rewrite explicitly its equation $q_0(u)(v_1, v_2, 0) + q_\infty(u)(v_1, v_2, 0) = 0$ as

$$(q_0(u) + q_\infty(u))(v_1, v_2, 0) = \tfrac{2}{3}(u_1v_2 + \omega u_2v_1 + \omega^2 u_3v_1 + \omega u_3v_2)(u_1v_2 + \omega^2 u_2v_1 + \omega u_3v_1 + \omega^2 u_3v_2),$$

where ω is a primitive cubic root of unity. Thus we see that $T = T_1 \cup T_2$, where T_i is a divisor of bidegree $(1, 1)$. In particular, each T_i provides a rational section of the conic bundle $p_1 \colon \overline{\mathscr{X}}_1^{4,2} \to \mathbb{P}^2$ and rationality of $\overline{\mathscr{X}}_1^{4,2}$ follows. Since the threefold $\overline{\mathscr{X}}_1^{4,2}$ is birational to the quartic $X_{1/2}$, the rationality of the latter follows as well.

4.2. ***Irrationality of nonspecial $\mathfrak{S}_6$-invariant quartics.*** Beauville [2013] proved that the quartic X_t is irrational provided that $t \notin \mathfrak{D} \cup \{\infty\}$ by using the $\mathfrak{S}_6$-action on the intermediate Jacobian of a suitable resolution of singularities of X_t. By [Beauville 2013], the intermediate Jacobian J_t of the blow up of the 30 singular points of X_t is five-dimensional, and the action of $\mathfrak{S}_6$ on J_t is faithful; on the other hand, if it is a product of Jacobians of curves, it cannot have a faithful $\mathfrak{S}_6$-action. Irrationality of the threefold $\mathscr{X}_\infty$ was proved using the same approach in [Przyjalkowski and Shramov 2016, Proposition 6.3]. With the help of the conic bundle structure on these varieties constructed in Theorem 1.15, we can give another proof of their irrationality.

Theorem 4.4. *If $t \notin \mathfrak{D} \cup \{\infty\}$, then X_t is irrational. Also, the variety $\mathscr{X}_\infty$ is irrational.*

Proof. By Theorem 1.14 it is enough to show that the threefold $\mathscr{X}_\tau^{5,1}$ is irrational for $\tau \notin \hat{\mathfrak{D}}$. By Theorem 1.15 the map $p \colon \mathscr{X}_\tau^{5,1} \to S$ is a standard conic bundle with the nodal discriminant curve Δ_s contained in the linear system $|-2K_S|$. Here $s = s(\tau)$ is given by the formula (1.16). The conic bundle p induces a double cover $\hat{\Delta}_s \to \Delta_s$ that by Lemma 3.29 is branched only over the nodes of the curve Δ_s. Applying [Beauville 1977, Proposition 2.8], we see that the intermediate Jacobian of the threefold $\mathscr{X}_\tau^{5,1}$ is isomorphic as a principally polarized abelian variety to the Prym variety $\mathrm{Prym}(\hat{\Delta}_s, \Delta_s)$. Now [Shokurov 1983, Main Theorem] implies that $\mathrm{Prym}(\hat{\Delta}_s, \Delta_s)$ is not a product of Jacobians of curves, hence $\mathscr{X}_\tau^{5,1}$ is irrational. □

Remark 4.5. The intermediate Jacobian of $\mathscr{X}_\tau^{5,1}$ can be described fairly explicitly. For instance, it was observed by Dimitri Markushevich that it is isogenous to the fifth power of an elliptic curve (whose j-invariant depends on τ). Here is a sketch of his argument. Let $\hat{\mathscr{X}}_\tau \to \mathscr{X}_\tau$ be a minimal $\mathfrak{S}_6$-equivariant resolution of singularities, so that $\mathrm{Jac}(\hat{\mathscr{X}}_\tau) \cong \mathrm{Jac}(\mathscr{X}_\tau^{5,1})$. The action of the group $\mathfrak{S}_6$ on $\mathrm{Jac}(\hat{\mathscr{X}}_\tau)$ can be lifted to an action of the semidirect product $\mathfrak{S}_6 \ltimes H^3(\hat{\mathscr{X}}_\tau, \mathbb{Z})$ on $H^3(\hat{\mathscr{X}}_\tau, \mathbb{C})$. Now [Bernstein and Schwarzman 2006, Theorem 3.1] proves that there is an $\mathfrak{S}_6$-equivariant isomorphism

$$H^3(\hat{\mathscr{X}}_\tau, \mathbb{Q}) \cong \mathbb{Q}(\mathfrak{S}_6) \oplus \lambda\mathbb{Q}(\mathfrak{S}_6),$$

where $\mathbb{Q}(\mathfrak{S}_6)$ is the root lattice associated with the group $\mathfrak{S}_6$ considered as a Weyl group of Dynkin type A_5, and $\lambda = \lambda(\tau)$ is a complex number with positive imaginary part. Therefore, $\mathrm{Jac}(\hat{\mathscr{X}}_\tau)$ is isogenous to $E(\lambda)^5$, where $E(\lambda) = \mathbb{C}/(\mathbb{Z} \oplus \lambda\mathbb{Z})$.

Note by the way, that there is another popular family of threefolds with five-dimensional intermediate Jacobians, namely, smooth cubic threefolds. However, it was pointed out by Beauville that the quartics X_t are not birational to smooth cubics. Indeed, if a quartic X_t is birational to a smooth cubic threefold Y, then the intermediate Jacobian $J(Y)$ is isomorphic to J_t, and thus there is a faithful $\mathfrak{S}_6$-action on $J(Y)$ (note that J_t must coincide with its Griffiths component in this case). Torelli's theorem for smooth cubic threefolds (see [Beauville 1982, Proposition 6]) implies that there is a faithful $\mathfrak{S}_6$-action on Y itself, which is impossible, because the only cubic threefold with a faithful $\mathfrak{S}_6$-action is the Segre cubic that has ten singular points.

It would be interesting to find out if the quartics X_t with $t \notin \mathfrak{D}$ are stably rational or not.

4.3. ***Rationality of special $\mathfrak{S}_6$-invariant quartics.*** The result of Theorem 4.4 is sharp: the threefolds $X_{1/2}$, $X_{1/4}$, $X_{1/6}$, and $X_{7/10}$ are rational. In fact, rationality of the Burkhardt quartic $X_{1/2}$ was proved by Todd [1936] (see also Remark 4.3), rationality of the Igusa quartic $X_{1/4}$ follows from rationality of its projectively dual variety (which is the Segre cubic), and rationality of the quartics $X_{1/6}$ and $X_{7/10}$ is also known; see [Todd 1933; 1935; Cheltsov and Shramov 2016b]. However, using our results one can give a uniform proof of rationality of all these threefolds; this proof does not use explicit rationality constructions.

Theorem 4.6. *The quartics $X_{1/2}$, $X_{1/4}$, $X_{1/6}$, and $X_{7/10}$ are rational.*

Proof. Suppose that $\tau \in \hat{\mathfrak{D}}$, so that $t \in \mathfrak{D}$ and $s \in \{0, \pm 1/\sqrt{125}, \pm 1/\sqrt{-3}\}$, where as usual $t = (\tau^2+1)/4$ and $s = s(\tau)$; see (3.25). By Theorem 1.14 it is enough to show that $\mathscr{X}_\tau^{5,1}$ is rational.

Consider the conic bundle $p\colon \mathscr{X}_\tau^{5,1} \to S$. The singular locus of its discriminant Δ_s is a finite set of nodes; see Theorem 3.10. Actually, by Lemma 3.24 the set $\mathrm{Sing}(\Delta_s)$ consists of 15 points when $t = \frac{1}{4}$ or $t = \frac{1}{2}$, of 10 points when $t = \frac{1}{6}$, and of 6 points when $t = \frac{7}{10}$. We also know from Proposition 3.30 that all singularities of $\mathscr{X}_\tau^{5,1}$ are nodes, and for every singular point Q of $\mathscr{X}_\tau^{5,1}$ the fiber $p^{-1}(p(Q))$ is the union of two lines, with Q being their intersection point.

The conic bundle p is not standard because the threefold $\mathscr{X}_\tau^{5,1}$ is singular, so we start by transforming it to a standard one. Let $\nu\colon \tilde{S} \to S$ be the blow up of the quintic del Pezzo surface S at $\mathrm{Sing}(\Delta_s)$, and consider the base change $p'\colon \mathscr{X}_\tau^{5,1} \times_S \tilde{S} \to \tilde{S}$ of the conic bundle p. Its discriminant curve is the preimage on $\tilde{S}$ of the discriminant curve of p. In particular, it contains all exceptional curves of the blow up ν as irreducible components of multiplicity 2, and the corank of the fibers of p' over the points of each of these curves equals 1. Modifying the conic bundle along these lines as in [Sarkisov 1982, Lemma 1.14] (see also [Debarre and Kuznetsov 2020, Section 2.5]), we can get rid of the corresponding components of the discriminant. In other words, we obtain a small birational map

$$\mathscr{X}_\tau^{5,1} \times_S \tilde{S} \dashrightarrow \tilde{\mathscr{X}}_\tau^{5,1} \tag{4.7}$$

over $\tilde{S}$, such that the threefold $\tilde{\mathscr{X}}_\tau^{5,1}$ comes with a flat conic bundle $\tilde{p}\colon \tilde{\mathscr{X}}_\tau^{5,1} \to \tilde{S}$ whose discriminant curve is the proper transform $\tilde{\Delta}_s \subset \tilde{S}$ of Δ_s with respect to ν. In particular, the curve $\tilde{\Delta}_s$ is smooth (hence also $\tilde{\mathscr{X}}_\tau^{5,1}$ is smooth), and by Theorem 3.10 has ten connected components when $t = \frac{1}{4}$ or $t = \frac{1}{2}$, five

components when $t = \frac{1}{6}$, and just one component when $t = \frac{7}{10}$. Moreover, every connected component of $\tilde{\Delta}_s$ is rational.

Since $\tilde{\Delta}_s$ is smooth, the conic bundle $\tilde{p}$ has only simple degenerations. In particular, it induces an étale double covering over $\tilde{\Delta}_s$. Since every connected component $\tilde{\Delta}_s^{(i)} \subset \tilde{\Delta}_s$ is smooth and rational, the double covering is trivial, hence the preimage $\tilde{p}^{-1}(\tilde{\Delta}_s^{(i)})$ consists of two irreducible components

$$\tilde{p}^{-1}(\tilde{\Delta}_s^{(i)}) = \Theta_i' \cup \Theta_i'',$$

each being a $\mathbb{P}^1$-bundle over $\tilde{\Delta}_s^{(i)}$. Choosing for each i one of them and contracting all chosen components simultaneously over $\tilde{S}$ (see [Sarkisov 1982, 1.17]), we obtain a commutative diagram

$$\begin{array}{ccc} \tilde{\mathscr{X}}_\tau^{5,1} & \longrightarrow & \overline{\mathscr{X}}_\tau^{5,1} \\ & {\scriptstyle\tilde{p}}\searrow \quad \swarrow{\scriptstyle\bar{p}} & \\ & \tilde{S} & \end{array}$$

Here the horizontal arrow is a birational morphism, and $\bar{p}$ is an everywhere nondegenerate conic bundle. Since $\tilde{S}$ is a rational surface, its Brauer group is trivial, hence this $\mathbb{P}^1$-bundle is a projectivization of a vector bundle, hence birational to $\tilde{S} \times \mathbb{P}^1$, hence rational. This means that $\mathscr{X}_\tau^{5,1}$ is also rational. □

Remark 4.8. The birational transformation $\mathscr{X}_\tau^{5,1} \dashrightarrow \mathscr{X}_\tau^{5,1} \times_S \tilde{S} \dashrightarrow \tilde{\mathscr{X}}_\tau^{5,1}$ can be described very explicitly; see Construction I in the proof of [Cheltsov et al. 2019b, Theorem 4.2]. It is a composition of the blow ups of all singular points $Q \in \mathscr{X}_\tau^{5,1}$ followed by the Atiyah flops in the union of proper transforms of the two irreducible components of the conic $p^{-1}(p(Q))$; see Proposition 3.30.

The construction that we used in the proof of Theorem 4.6 has the following consequence, which we will need in Section 5. Recall the notation of (1.17).

Corollary 4.9. *For $\tau \in \hat{\mathfrak{D}}$ the relative divisor class group $\mathrm{Cl}(\mathscr{X}_\tau^{5,1}/S) \otimes \mathbb{Q}$ has the following structure as a representation of the group $\mathfrak{A}_5$:*

τ	0	± 1	$\pm\frac{1}{\sqrt{-3}}$	$\pm\frac{3}{\sqrt{5}}$
$\mathrm{Cl}(\mathscr{X}_\tau^{5,1}/S) \otimes \mathbb{Q}$	$\mathbf{1} \oplus \mathrm{Ind}_{\mathfrak{A}_{3,2}}^{\mathfrak{A}_5}(\mathbf{1})$	$\mathbf{1} \oplus \mathrm{Ind}_{\mathfrak{A}_{3,2}}^{\mathfrak{A}_5}(-\mathbf{1})$	$\mathbf{1} \oplus \mathrm{Ind}_{\mathfrak{A}_4}^{\mathfrak{A}_5}(\mathbf{1})$	$\mathbf{1} \oplus \mathbf{1}$

Here $\mathrm{Ind}_G^{\mathfrak{A}_5}$ stands for the induction functor from the subgroup $G = \mathfrak{A}_4$ or $G = \mathfrak{A}_{3,2} \cong \mathfrak{S}_3$ in $\mathfrak{A}_5$, while $\mathbf{1}$ stands for the trivial representation, and $-\mathbf{1}$ stands for the sign representation of $\mathfrak{S}_3$. The first summand $\mathbf{1}$ in each cell is generated by the canonical class of $\mathscr{X}_\tau^{5,1}$.

Proof. The canonical class $K_{\mathscr{X}_\tau^{5,1}}$ is invariant with respect to the group action, hence generates a trivial subrepresentation in $\mathrm{Cl}(\mathscr{X}_\tau^{5,1}/S) \otimes \mathbb{Q}$. Consider the quotient

$$\mathrm{Cl}_0(\mathscr{X}_\tau^{5,1}/S) \otimes \mathbb{Q} := (\mathrm{Cl}(\mathscr{X}_\tau^{5,1}/S) \otimes \mathbb{Q})/\mathbb{Q}K_{\mathscr{X}_\tau^{5,1}}.$$

To describe it we use the notation introduced in the proof of Theorem 4.6. First, we have

$$\mathrm{Cl}_0(\mathscr{X}_\tau^{5,1}/S) \cong \mathrm{Cl}_0((\mathscr{X}_\tau^{5,1} \times_S \tilde{S})/\tilde{S}).$$

Furthermore, since (4.7) is a small birational map, we have

$$\mathrm{Cl}_0((\mathscr{X}_\tau^{5,1} \times_S \tilde{S})/\tilde{S}) \cong \mathrm{Cl}_0(\tilde{\mathscr{X}}_\tau^{5,1}/\tilde{S}).$$

Finally, it is clear that $\mathrm{Cl}_0(\tilde{\mathscr{X}}_\tau^{5,1}/\tilde{S}) \otimes \mathbb{Q}$ is contained in an $\mathfrak{A}_5$-equivariant exact sequence

$$0 \to \bigoplus \mathbb{Q}[\tilde{\Delta}_{s(\tau)}^{(i)}] \to \bigoplus (\mathbb{Q}[\Theta_i'] \oplus \mathbb{Q}[\Theta_i'']) \to \mathrm{Cl}_0(\tilde{\mathscr{X}}_\tau^{5,1}/\tilde{S}) \otimes \mathbb{Q} \to 0,$$

where we sum up over the set of irreducible components of $\tilde{\Delta}_{s(\tau)}$, and the first map takes the class $[\tilde{\Delta}_{s(\tau)}^{(i)}] \in \mathrm{Pic}(\tilde{S})$ to $[\Theta_i'] + [\Theta_i''] \in \mathrm{Pic}(\tilde{\mathscr{X}}_\tau^{5,1})$. It follows that $\mathrm{Cl}_0(\tilde{\mathscr{X}}_\tau^{5,1}/\tilde{S}) \otimes \mathbb{Q}$ has the basis $[\Theta_i'] - [\Theta_i'']$, and the group $\mathfrak{A}_5$ permutes the basis vectors, possibly changing their signs.

Recall that the group $\mathfrak{A}_5$ acts transitively on the set of irreducible components of $\tilde{\Delta}_{s(\tau)}$ by Theorem 3.10. Let $G \subset \mathfrak{A}_5$ be the stabilizer of some irreducible component of $\tilde{\Delta}_{s(\tau)}$, say, of $\tilde{\Delta}_{s(\tau)}^{(0)}$. The action of G on the set $\{\Theta_0', \Theta_0''\}$ defines a homomorphism $\upsilon\colon G \to \{\pm 1\}$, i.e., a one-dimensional representation of G, and we conclude that

$$\mathrm{Cl}_0(\tilde{\mathscr{X}}_\tau^{5,1}/\tilde{S}) \otimes \mathbb{Q} \cong \mathrm{Ind}_G^{\mathfrak{A}_5}(\upsilon).$$

So, it remains to identify the possible stabilizers G for various τ, and the homomorphisms υ.

When $\tau = \pm 3/\sqrt{5}$, the curve $\tilde{\Delta}_{s(\tau)}$ is irreducible, hence G is the whole group $\mathfrak{A}_5$, and since it has no nontrivial one-dimensional representations, we conclude that

$$\mathrm{Cl}_0(\tilde{\mathscr{X}}_{\pm 3/\sqrt{5}}^{5,1}/\tilde{S}) \otimes \mathbb{Q} \cong \mathrm{Ind}_{\mathfrak{A}_5}^{\mathfrak{A}_5}(\mathbf{1}) \cong \mathbf{1}.$$

When $\tau = \pm 1/\sqrt{-3}$, the curve $\tilde{\Delta}_{s(\tau)}$ has five components, G is the subgroup $\mathfrak{A}_4$ of $\mathfrak{A}_5$, and since again it has no nontrivial one-dimensional representations, we conclude that

$$\mathrm{Cl}_0(\tilde{\mathscr{X}}_{\pm 1/\sqrt{-3}}^{5,1}/\tilde{S}) \otimes \mathbb{Q} \cong \mathrm{Ind}_{\mathfrak{A}_4}^{\mathfrak{A}_5}(\mathbf{1}).$$

When $\tau = 0$ or $\tau = \pm 1$, the curve $\tilde{\Delta}_{s(\tau)}$ has ten components (corresponding to the lines on S) and G is the subgroup $\mathfrak{A}_{3,2} \cong \mathfrak{S}_3$ of $\mathfrak{A}_5$. It remains to show that it fixes the components Θ_0' and Θ_0'' when $\tau = 0$, and swaps them when $\tau = \pm 1$.

The stabilizer $\mathfrak{A}_{3,2}$ of a line $L \subset S$ permutes three points of its intersection with other lines on S. Each of these points, in its turn, is stabilized by a transposition in $\mathfrak{A}_{3,2} \cong \mathfrak{S}_3$. So, it is enough to check how these transpositions act on Θ_0' and Θ_0''.

Consider the point $P = (0 : 1 : 1)$ as in the proof of Proposition 3.30. Then it is easy to see that the subgroup of $\mathfrak{A}_5$ that preserves both lines passing through P is generated by the automorphism

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

of order two of the plane, while the fiber $p^{-1}(P)$ is given by (3.32) in the case $\tau = 0$, and by

$$q_0(0:1:1) + q_\infty(0:1:1) = \tfrac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$$

in the case $\tau = 1$. Now verifying that g fixes the components of the conic $p^{-1}(P)$ if $\tau = 0$ and swaps the components if $\tau = 1$ is straightforward.

The computation in the case $\tau = -1$ is similar to that in the case $\tau = 1$. □

The part of the above argument that identifies the relative class group of a conic bundle in terms of the induced representation is completely general and can be proved for any conic bundle with only simple degenerations, and for an arbitrary group acting on it.

5. Representation structure of the class groups

The main result of this section is the description of the $\mathfrak{S}_6$-action on the class groups of the Coble fourfold and of the quartics X_t, and its applications to the equivariant birational geometry of these varieties. We will be mostly interested in the quartics X_t with $t \neq \frac{1}{4}, \infty$, because the quartic $X_{1/4}$ has nonisolated singularities, and at the same time its class group is not very intriguing by Lemma 3.6 (see Remark 5.24 below), while the quartic X_∞ is nonreduced; however, we will also perform the same computations for the threefold $\mathscr{X}_\infty$.

5.1. ***The result and its applications.*** We start by stating our main result and its consequences. We will use the following notation for representations of the symmetric groups. For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of an integer n (i.e., a nonincreasing sequence of positive integers summing up to n) we denote by

$$\mathrm{R}(\lambda) = \mathrm{R}(\lambda_1, \lambda_2, \dots, \lambda_r)$$

the irreducible $\mathbb{Q}$-representation of the group $\mathfrak{S}_n$ as described in [Fulton and Harris 1991, Section 4.1]. For instance, $\mathrm{R}(n)$ is the trivial representation, while $\mathrm{R}(1^n)$ is the sign representation. Note that the standard permutation representation is the direct sum $\mathrm{R}(n) \oplus \mathrm{R}(n-1, 1)$.

We denote by $\mathrm{R}(\lambda)\boxtimes 1$ and $\mathrm{R}(\lambda)\boxtimes(-1)$ the representations of the group $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$, which are isomorphic to $\mathrm{R}(\lambda)$ when restricted to $\mathfrak{S}_6$ and on which the nontrivial element of $\boldsymbol{\mu}_2$ acts by 1 or -1, respectively.

Theorem 5.1. *The group* $\mathrm{Cl}(\mathscr{Y})$ *is torsion free and there are the following isomorphisms of* $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$*-representations*:

$$\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q} \cong \mathrm{Cl}(\mathscr{X}_\infty) \otimes \mathbb{Q} \cong (\mathrm{R}(6) \boxtimes 1) \oplus (\mathrm{R}(3,3) \boxtimes (-1)).$$

In particular, for the natural action of $\mathfrak{S}_6$ *there are isomorphisms of* $\mathfrak{S}_6$*-representations*

$$\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q} \cong \mathrm{Cl}(\mathscr{X}_\infty) \otimes \mathbb{Q} \cong \mathrm{R}(6) \oplus \mathrm{R}(3,3),$$

while for the twisted action of $\mathfrak{S}_6$ *there are isomorphisms of* $\mathfrak{S}_6$*-representations*

$$\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q} \cong \mathrm{Cl}(\mathscr{X}_\infty) \otimes \mathbb{Q} \cong \mathrm{R}(6) \oplus \mathrm{R}(2,2,2).$$

Finally, there are the following isomorphisms of $\mathfrak{S}_6$*-representations*:

$$\begin{aligned} \mathrm{Cl}(X_t) \otimes \mathbb{Q} &\cong \mathrm{R}(6) \oplus \mathrm{R}(3,3) && \text{for } t \notin \mathfrak{D} \cup \{\infty\} \\ \mathrm{Cl}(X_{1/2}) \otimes \mathbb{Q} &\cong \mathrm{R}(6) \oplus \mathrm{R}(3,3) \oplus \mathrm{R}(3,1^3), \\ \mathrm{Cl}(X_{1/6}) \otimes \mathbb{Q} &\cong \mathrm{R}(6) \oplus \mathrm{R}(3,3) \oplus \mathrm{R}(2,2,2), \\ \mathrm{Cl}(X_{7/10}) \otimes \mathbb{Q} &\cong \mathrm{R}(6) \oplus \mathrm{R}(3,3) \oplus \mathrm{R}(1^6). \end{aligned}$$

The proof of Theorem 5.1 takes the next subsection, and now we discuss its applications to equivariant birational geometry.

Recall that an n-dimensional variety X with an action of a group G is *G-rational* if there exists a G-equivariant birational map between X and $\mathbb{P}^n$ for some action of G on $\mathbb{P}^n$. Also recall that a G-equivariant morphism $\phi\colon X \to S$ of normal varieties acted on by a finite group G is called a *G-Mori fiber space*, if X has terminal singularities, one has $\operatorname{rk}\mathrm{Pic}(X)^G = \operatorname{rk}\mathrm{Cl}(X)^G$, the fibers of ϕ are connected and of positive dimension, the anticanonical divisor $-K_X$ is ϕ-ample, and the relative G-invariant Picard rank $\operatorname{rk}\mathrm{Pic}(X/S)^G$ equals 1.

The first application of Theorem 5.1 is due to the following expectation, which is proved in several particular cases, see [Mella 2004; Shramov 2008; Cheltsov et al. 2019a, Proof of Theorem 1.1].

Conjecture 5.2. *Let X be either a nodal quartic threefold, or a nodal double covering of a smooth three-dimensional quadric branched over its intersection with a quartic. Let G be a finite subgroup in* $\mathrm{Aut}(X)$ *such that*

$$\operatorname{rk}\mathrm{Cl}(X)^G = 1.$$

If there is a G-equivariant birational map $X \dashrightarrow X'$, where $X' \to S'$ is a G-Mori fiber space, then $X \cong X'$. In particular, X is not G-rational.

Of course, this applies to each of the $\mathfrak{S}_6$-invariant quartics X_t with $t \neq \frac{1}{4}, \infty$, and to the threefold $\mathscr{X}_\infty$ as well. For each subgroup $G \subset \mathfrak{S}_6$ the rank of the invariant class group $\mathrm{Cl}(X_t)^G$ can be easily computed from the result of Theorem 5.1 by restricting the representation and computing the multiplicity of the trivial summand. We used GAP [2017] to perform this computation; see http://www.mi-ras.ru/~akuznet/GAP-code/rk-code.txt for the source code. To state our result in a precise form we first introduce our notation for the (conjugacy classes of) subgroups of $\mathfrak{S}_6$ that will be used until the end of Section 5.1. We will also use notation (1.17).

Notation 5.3. Given a subgroup $G \subset \mathfrak{S}_6$ we denote by $\overline{G} \subset \mathfrak{S}_6$ the image of G under an outer automorphism of $\mathfrak{S}_6$ (it is well-defined up to conjugation). Furthermore, if $G_1 \subset \mathfrak{S}_{n_1}, \dots, G_r \subset \mathfrak{S}_{n_r}$ are subgroups and $n_1 + \cdots + n_r \leqslant 6$, then by $G_1 \times \cdots \times G_r$ we denote the corresponding subgroup in

$$\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r} \cong \mathfrak{S}_{n_1,\dots,n_r} \subset \mathfrak{S}_{n_1+\cdots+n_r} \subset \mathfrak{S}_6.$$

Next, we use the notation $\boldsymbol{\mu}_d[c_1, \dots, c_r]$ for a cyclic subgroup of order d generated by a permutation of cycle type $[c_1, \dots, c_r]$. We abbreviate $\boldsymbol{\mu}_5[5]$ to just $\boldsymbol{\mu}_5$.

By V_4 we denote the Klein four-group, i.e., the unique subgroup of order 4 in $\mathfrak{A}_4 \subset \mathfrak{S}_4$. By $\mathrm{V}_{4,2}$ we denote a subgroup of $\mathfrak{S}_{4,2} \subset \mathfrak{S}_6$ whose projection to the first factor $\mathfrak{S}_4$ gives an isomorphism with V_4, while the projection to the second factor $\mathfrak{S}_2$ is surjective.

By D_{2n} we denote the dihedral group of order $2n$. It is naturally embedded into the group $\mathfrak{S}_n$, so for $n \leqslant 6$ it is a subgroup of $\mathfrak{S}_6$; note that $\mathrm{D}_{12} = \overline{\mathfrak{S}}_{3,2}$.

There are four conjugacy classes of subgroups isomorphic to D_8 in $\mathfrak{S}_6$. They can be described as follows. The first class contains subgroups of (the standard) $\mathfrak{S}_4$ in $\mathfrak{S}_6$; according to the above conventions, we will refer to subgroups from this conjugacy class simply as D_8. There are three nontrivial homomorphisms

$$\upsilon_\circ \colon \mathrm{D}_8 \to \boldsymbol{\mu}_2, \quad \upsilon_+ \colon \mathrm{D}_8 \to \boldsymbol{\mu}_2, \quad \upsilon_\times \colon \mathrm{D}_8 \to \boldsymbol{\mu}_2,$$

determined by their kernels

$$\mathrm{Ker}(\upsilon_\circ) = \boldsymbol{\mu}_4[4], \quad \mathrm{Ker}(\upsilon_+) = \mathrm{V}_4, \quad \mathrm{Ker}(\upsilon_\times) = \mathfrak{S}_{2,2}.$$

Thinking of these as of subgroups of symmetries of a square, the first is generated by rotations, the second by reflections with respect to the lines passing through the middle points of its opposite sides, and the third by reflections with respect to the diagonals; this is the mnemonics for the notation $\circ$, $+$, and $\times$. We denote by D_8°, D_8^+, and $\mathrm{D}_8^\times$ the images of the map

$$\mathrm{D}_8 \xrightarrow{(\mathrm{id},\upsilon)} \mathfrak{S}_4 \times \boldsymbol{\mu}_2 \cong \mathfrak{S}_{4,2} \subset \mathfrak{S}_6$$

for $\upsilon = \upsilon_\circ$, υ_+, and $\upsilon_\times$, respectively. Note that $\mathrm{D}_8^\circ = \overline{\mathrm{D}}_8$.

The intersection $\mathfrak{S}_5 \cap \overline{\mathfrak{S}}_5$ of a standard and a nonstandard subgroups $\mathfrak{S}_5$ is a subgroup of order 20 isomorphic to $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, and such groups form a unique conjugacy class of subgroups of order 20 in $\mathfrak{S}_6$. Also, the subgroups $\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$, $\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$, D_{10}, $\mathrm{D}_8 \times \mathfrak{S}_2$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_2$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, and $\mathfrak{S}_{3,3} \rtimes \boldsymbol{\mu}_2$ of $\mathfrak{S}_6$ are unique up to conjugation.

Finally, recall the definitions (1.6) of the natural and (1.8) of the twisted actions of $\mathfrak{S}_6$ on the Coble fourfold $\mathscr{Y}$ and on the threefold $\mathscr{X}_\infty \subset \mathscr{Y}$. Theorem 5.1 implies:

Corollary 5.4. Figure 1 *contains a complete list* (*ordered by cardinality*) *of subgroups* $G \subset \mathfrak{S}_6$ *such that* $\mathrm{rk}\,\mathrm{Cl}(X)^G = 1$, *where* X *is either* X_t, *or* $\mathscr{X}_\infty$, *or* $\mathscr{Y}$. *If* X *is either* $\mathscr{X}_\infty$ *or* $\mathscr{Y}$, *and* G *is any subgroup of* $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$ *that contains the second factor*, *then one also has* $\mathrm{rk}\,\mathrm{Cl}(X)^G = 1$.

In particular, Conjecture 5.2 suggests that the varieties listed in Corollary 5.4 are not G-rational with respect to the corresponding groups.

Another interesting case of G-equivariant behavior arises when $\mathrm{rk}\,\mathrm{Cl}(X)^G = 2$. The following result is well known to experts.

X, *action of* $\mathfrak{S}_6$	G
X_t, $t \notin \mathfrak{D} \cup \{\infty\}$; $\mathscr{X}_\infty$, *natural action*; $\mathscr{Y}$, *natural action*	$\mathfrak{S}_6$, $\mathfrak{A}_6$, $\mathfrak{S}_5$, $\overline{\mathfrak{S}}_5$, $\mathfrak{S}_{3,3} \rtimes \boldsymbol{\mu}_2$, $\mathfrak{A}_5$, $\mathfrak{S}_{4,2}$, $\overline{\mathfrak{S}}_{4,2}$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, $\overline{\mathfrak{S}}_{3,3}$, $\mathfrak{S}_4$, $\overline{\mathfrak{S}}_4$, $\mathfrak{A}_{4,2}$, $\mathfrak{A}_4 \times \mathfrak{S}_2$, $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, $\overline{\mathfrak{S}_3 \times \boldsymbol{\mu}_3}$, $\mathrm{D}_8 \times \mathfrak{S}_2$, $\mathfrak{A}_4$, $\overline{\mathfrak{S}}_{3,2}$, $\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$, $\mathrm{V}_4 \times \boldsymbol{\mu}_2$, $\overline{\mathrm{D}}_8$, $\mathrm{D}_8^\times$, $\overline{\mathfrak{S}}_3$, $\mathrm{V}_{4,2}$
$\mathscr{X}_\infty$, *twisted action*; $\mathscr{Y}$, *twisted action*	$\mathfrak{S}_6$, $\mathfrak{A}_6$, $\mathfrak{S}_5$, $\mathfrak{S}_{3,3} \rtimes \boldsymbol{\mu}_2$, $\mathfrak{A}_5$, $\mathfrak{S}_{4,2}$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, $\mathfrak{S}_{3,3}$, $\mathfrak{S}_4$, $\mathfrak{A}_{4,2}$, $\mathfrak{A}_4 \times \mathfrak{S}_2$, $\mathfrak{S}_3 \times \boldsymbol{\mu}_3$, $\mathfrak{S}_{3,2}$, $\mathfrak{A}_4$, $\mathfrak{S}_3$, $\boldsymbol{\mu}_6[3,2]$
$X_{1/2}$	$\mathfrak{S}_6$, $\mathfrak{A}_6$, $\mathfrak{S}_5$, $\overline{\mathfrak{S}}_5$, $\mathfrak{S}_{3,3} \rtimes \boldsymbol{\mu}_2$, $\mathfrak{A}_5$, $\mathfrak{S}_{4,2}$, $\overline{\mathfrak{S}}_{4,2}$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, $\overline{\mathfrak{S}}_{3,3}$, $\mathfrak{S}_4$, $\mathfrak{A}_{4,2}$, $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, $\mathrm{D}_8 \times \mathfrak{S}_2$
$X_{1/6}$	$\mathfrak{S}_6$, $\mathfrak{A}_6$, $\mathfrak{S}_5$, $\mathfrak{S}_{3,3} \rtimes \boldsymbol{\mu}_2$, $\mathfrak{A}_5$, $\mathfrak{S}_{4,2}$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, $\mathfrak{S}_4$, $\mathfrak{A}_{4,2}$, $\mathfrak{A}_4 \times \mathfrak{S}_2$, $\mathfrak{A}_4$
$X_{7/10}$	$\mathfrak{S}_6$, $\mathfrak{S}_5$, $\overline{\mathfrak{S}}_5$, $\mathfrak{S}_{3,3} \rtimes \boldsymbol{\mu}_2$, $\mathfrak{S}_{4,2}$, $\overline{\mathfrak{S}}_{4,2}$, $\overline{\mathfrak{S}}_{3,3}$, $\mathfrak{S}_4$, $\overline{\mathfrak{S}}_4$, $\mathfrak{A}_4 \times \mathfrak{S}_2$, $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, $\overline{\mathfrak{S}_3 \times \boldsymbol{\mu}_3}$, $\mathrm{D}_8 \times \mathfrak{S}_2$, $\overline{\mathfrak{S}}_{3,2}$, $\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$, $\overline{\mathrm{D}}_8$, $\mathrm{D}_8^\times$, $\mathrm{V}_4 \times \boldsymbol{\mu}_2$, $\overline{\mathfrak{S}}_3$, $\mathrm{V}_{4,2}$

Figure 1. Subgroups $G \subset \mathfrak{S}_6$ such that $\operatorname{rk}\operatorname{Cl}(X)^G = 1$, where X is either X_t, or $\mathscr{X}_\infty$, or $\mathscr{Y}$.

Proposition 5.5 (cf. [Corti 1995; Hacon and McKernan 2013]). *Let X be a terminal Fano variety* (*so that, in particular, the canonical class K_X is a $\mathbb{Q}$-Cartier divisor*). *Let G be a finite subgroup in* $\operatorname{Aut}(X)$ *such that* $\operatorname{rk}\operatorname{Cl}(X)^G = 2$ *and* $\operatorname{rk}\operatorname{Pic}(X)^G = 1$. *Then there exists a unique G-equivariant diagram*:

$$\begin{array}{ccccccccc} X'_+ & \overset{\psi_+}{\dashleftarrow} & & X_+ & \overset{\iota}{\dashrightarrow} & & X_- & \overset{\psi_-}{\dashrightarrow} & X'_- \\ & {\scriptstyle p_+}\swarrow & & \searrow{\scriptstyle f_+} & & \swarrow{\scriptstyle f_-} & & & \searrow{\scriptstyle p_-} \\ Z_+ & & & & X & & & & Z_- \end{array} \tag{5.6}$$

Here $X_\pm$ are varieties with terminal singularities such that

$$\operatorname{rk}\operatorname{Pic}(X_\pm)^G = \operatorname{rk}\operatorname{Cl}(X_\pm)^G = 2, \quad \operatorname{rk}\operatorname{Pic}(X_\pm/X)^G = 1,$$

the maps $f_\pm$ are small birational morphisms, the map ι is a nontrivial G-flop, the maps $\psi_\pm$ are small and birational (*and possibly are just isomorphisms*), *the varieties $X'_\pm$ have terminal singularities*,

$$\operatorname{rk}\operatorname{Pic}(X'_\pm)^G = \operatorname{rk}\operatorname{Cl}(X'_\pm)^G = 2,$$

and each of the maps $p_\pm$, is either a $K_{X'_\pm}$-negative divisorial contraction onto a terminal Fano variety $Z_\pm$ with $\operatorname{rk}\operatorname{Cl}(Z_\pm)^G = 1$, *or a G-Mori fibration.*

The diagram (5.6) is a special case of a so-called *G-Sarkisov link* (that is a G-equivariant version of a usual Sarkisov link; see e.g., [Corti 1995, Definition 3.4] or [Cheltsov 2005, Theorem 1.6.14] for notation). One sometimes says that the link (5.6) is *centered at X*.

Theorem 5.1 allows us to write down a complete list of subgroups $G \subset \mathfrak{S}_6$ for which Proposition 5.5 can be used (as before, we obtained it with the help of GAP [2017]; see http://www.mi-ras.ru/~akuznet/GAP-code/rk-code.txt for the source code).

X, *action of* $\mathfrak{S}_6$	G
X_t, $t \notin \mathfrak{D} \cup \{\infty\}$; $\mathscr{X}_\infty$, *natural action*; $\mathscr{Y}$, *natural action*	$\overline{\mathfrak{A}}_5$, $\mathfrak{S}_{3,3}$, $\overline{\mathfrak{A}}_{4,2}$, $\overline{\mathfrak{A}_4 \times \mathfrak{S}_2}$, $\mathfrak{S}_3 \times \boldsymbol{\mu}_3$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_2$, $\mathfrak{S}_{3,2}$, D_{10}, $\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$, D_8, D_8^+, $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, $\mathfrak{S}_3$, $\mathfrak{A}_{3,2}$, $\boldsymbol{\mu}_6[6]$, $\boldsymbol{\mu}_6[3,2]$, $\boldsymbol{\mu}_5$, $\boldsymbol{\mu}_4[4]$, $\boldsymbol{\mu}_4[4,2]$, $\boldsymbol{\mu}_2[2,2] \times \boldsymbol{\mu}_2[2]$, $\boldsymbol{\mu}_3[3]$, $\boldsymbol{\mu}_2[2,2,2]$
$\mathscr{X}_\infty$, *twisted action*; $\mathscr{Y}$, *twisted action*	$\overline{\mathfrak{S}}_5$, $\overline{\mathfrak{A}}_5$, $\overline{\mathfrak{S}}_{4,2}$, $\overline{\mathfrak{S}}_{3,3}$, $\overline{\mathfrak{A}}_{4,2}$, $\overline{\mathfrak{A}_4 \times \mathfrak{S}_2}$, $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, $\overline{\mathfrak{S}}_3 \times \boldsymbol{\mu}_3$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_2$, $\mathrm{D}_8 \times \mathfrak{S}_2$, D_{10}, $\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$, D_8, $\mathrm{D}_8^\times$, D_8^+, $\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, $\mathfrak{A}_{3,2}$, $\boldsymbol{\mu}_5$, $\boldsymbol{\mu}_4[4,2]$, $\boldsymbol{\mu}_2[2] \times \boldsymbol{\mu}_2[2]$, $\boldsymbol{\mu}_3[3]$
$X_{1/2}$	$\overline{\mathfrak{A}}_5$, $\mathfrak{S}_{3,3}$, $\overline{\mathfrak{S}}_4$, $\overline{\mathfrak{A}}_{4,2}$, $\mathfrak{A}_4 \times \mathfrak{S}_2$, $\overline{\mathfrak{A}_4 \times \mathfrak{S}_2}$, $\overline{\mathfrak{S}}_3 \times \boldsymbol{\mu}_3$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_2$, $\mathfrak{S}_{3,2}$, $\overline{\mathfrak{S}}_{3,2}$, $\mathfrak{A}_4$, D_{10}, D_8, $\overline{\mathrm{D}}_8$, $\mathrm{D}_8^\times$, D_8^+, $\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$, $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, $\mathrm{V}_4 \times \boldsymbol{\mu}_2$
$X_{1/6}$	$\overline{\mathfrak{S}}_5$, $\overline{\mathfrak{S}}_{4,2}$, $\mathfrak{S}_{3,3}$, $\overline{\mathfrak{S}}_{3,3}$, $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$, $\mathfrak{S}_3 \times \boldsymbol{\mu}_3$, $\overline{\mathfrak{S}}_3 \times \boldsymbol{\mu}_3$, $\mathrm{D}_8 \times \boldsymbol{\mu}_2$, $\mathfrak{S}_{3,2}$, $\mathrm{D}_8^\times$, $\boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2$, $\mathfrak{S}_3$, $\boldsymbol{\mu}_6[3,2]$
$X_{7/10}$	$\mathfrak{A}_6$, $\mathfrak{A}_5$, $\mathfrak{S}_{3,3}$, $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, $\mathfrak{A}_{4,2}$, $\overline{\mathfrak{A}_4 \times \mathfrak{S}_2}$, $\mathfrak{S}_3 \times \boldsymbol{\mu}_3$, $\mathfrak{S}_{3,2}$, $\mathfrak{A}_4$, D_8, $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$, $\mathfrak{S}_3$, $\boldsymbol{\mu}_6[6]$, $\boldsymbol{\mu}_6[3,2]$, $\boldsymbol{\mu}_4[4]$, $\boldsymbol{\mu}_2[2,2] \times \boldsymbol{\mu}_2[2]$, $\boldsymbol{\mu}_2[2,2,2]$

Figure 2. Subgroups $G \subset \mathfrak{S}_6$ such that $\operatorname{rk}\operatorname{Cl}(X)^G = 2$, where X is either X_t, or $\mathscr{X}_\infty$, or $\mathscr{Y}$.

Corollary 5.7. Figure 2 *contains a complete list* (*ordered by cardinality*) *of subgroups* $G \subset \mathfrak{S}_6$ *such that* $\operatorname{rk}\operatorname{Cl}(X)^G = 2$, *where* X *is either* X_t, *or* $\mathscr{X}_\infty$, *or* $\mathscr{Y}$. *In particular*, *for each of these varieties there is a* G*-Sarkisov link* (5.6) *centered at* X *with respect to the corresponding groups.*

Example 5.8. If $t \notin \mathfrak{D} \cup \{\infty\}$ and $G = \overline{\mathfrak{A}}_5$, the G-Sarkisov link (5.6) is obtained by restricting the diagram (2.48):

$$\begin{array}{ccccc} & \mathscr{X}_\tau^{5,1} & \overset{\iota}{\dashrightarrow} & \mathscr{X}_{-\tau}^{5,1} & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \pi\circ\rho_{5,1}} \quad {\scriptstyle \pi\circ\rho_{5,1}}\swarrow & & \searrow{\scriptstyle p} \\ S & & X_t & & S \end{array}$$

Here $t = (\tau^2+1)/4$, ι is the restriction of the map $\rho_{5,1}^{-1} \circ \sigma \circ \rho_{5,1}$ to $\mathscr{X}_\tau^{5,1}$ (it is a composition of 30 Atiyah flops), and $\psi_\pm$ are the identity maps. The map ι can be also defined as the map induced by an action of an odd permutation in the subgroup $\overline{\mathfrak{S}}_5 \subset \mathfrak{S}_6$ containing $\overline{\mathfrak{A}}_5$.

Example 5.9. If $G = \overline{\mathfrak{S}}_5$, then the G-Sarkisov link (5.6) for $\mathscr{X}_\infty$ comes from a restriction of the commutative diagram (2.48) to $\mathscr{X}_\infty$ (recall that $\mathscr{X}_\infty$ is $\operatorname{Aut}(\mathscr{Y})$-invariant).

5.2. ***Class group computation.*** In this section we prove Theorem 5.1. We start with a description of the $\mathfrak{S}_5$-action on the Picard group of the quintic del Pezzo surface.

Lemma 5.10. *There is an isomorphism of* $\mathfrak{S}_5$*-representations*

$$\operatorname{Pic}(S) \otimes \mathbb{Q} \cong \mathrm{R}(5) \oplus \mathrm{R}(4,1).$$

Proof. The surface S can be obtained as a blow up of $\mathbb{P}^2$ in four points, and this blow up is $\mathfrak{S}_4$-invariant. Therefore, one has

$$(\mathrm{Pic}(S) \otimes \mathbb{Q})|_{\mathfrak{S}_4} \cong \mathrm{R}(4) \oplus \mathrm{R}(4) \oplus \mathrm{R}(3,1).$$

Here the first summand is the pullback of the line class, and the last two form the permutation representation spanned by the classes of the exceptional divisors of the blow up. Now the assertion easily follows, since

$$\mathrm{R}(5)|_{\mathfrak{S}_4} \cong \mathrm{R}(4), \quad \mathrm{R}(4,1)|_{\mathfrak{S}_4} \cong \mathrm{R}(4) \oplus \mathrm{R}(3,1), \tag{5.11}$$

and moreover, by Pieri's rule [Fulton and Harris 1991, Exercise 4.44] the irreducible $\mathfrak{S}_5$-representations $\mathrm{R}(5)$ and $\mathrm{R}(4,1)$ are the only ones that restrict to $\mathfrak{S}_4$ as sums of $\mathrm{R}(4)$'s and $\mathrm{R}(3,1)$'s. □

Further on we will use a similar argument to describe an $\mathfrak{S}_6$-representation from its restriction to a nonstandard subgroup $\mathfrak{S}_5$. For this the following calculation is quite useful.

Lemma 5.12. *The following table contains all irreducible representations V of $\mathfrak{S}_6$, their images $\overline{V}$ under an outer automorphism of $\mathfrak{S}_6$, and the restrictions of V and $\overline{V}$ to a standard subgroup $\mathfrak{S}_5$.*

$\dim V$	V	$\overline{V}$	$V\|_{\mathfrak{S}_5}$	$\overline{V}\|_{\mathfrak{S}_5}$
1	$\mathrm{R}(6)$		$\mathrm{R}(5)$	
1	$\mathrm{R}(1^6)$		$\mathrm{R}(1^5)$	
5	$\mathrm{R}(5,1)$	$\mathrm{R}(2^3)$	$\mathrm{R}(5) \oplus \mathrm{R}(4,1)$	$\mathrm{R}(2^2,1)$
5	$\mathrm{R}(2,1^4)$	$\mathrm{R}(3^2)$	$\mathrm{R}(2,1^3) \oplus \mathrm{R}(1^5)$	$\mathrm{R}(3,2)$
9	$\mathrm{R}(4,2)$		$\mathrm{R}(4,1) \oplus \mathrm{R}(3,2)$	
9	$\mathrm{R}(2^2,1^2)$		$\mathrm{R}(2^2,1) \oplus \mathrm{R}(2,1^3)$	
10	$\mathrm{R}(4,1^2)$	$\mathrm{R}(3,1^3)$	$\mathrm{R}(4,1) \oplus \mathrm{R}(3,1^2)$	$\mathrm{R}(3,1^2) \oplus \mathrm{R}(2,1^3)$
16	$\mathrm{R}(3,2,1)$		$\mathrm{R}(3,2) \oplus \mathrm{R}(3,1^2) \oplus \mathrm{R}(2^2,1)$	

Proof. The restrictions to $\mathfrak{S}_5$ are computed by Pieri's rule, so we only need to explain the action of an outer automorphism. For this note that an outer automorphism acts on the conjugacy classes of $\mathfrak{S}_6$ by swapping the following cycle types

$$[2] \leftrightarrow [2,2,2], \quad [3] \leftrightarrow [3,3], \quad [6] \leftrightarrow [3,2],$$

and fixing the other types. By using the character table of $\mathfrak{S}_6$ (see for instance [James and Liebeck 1993, Example 19.17]) it is then straightforward to check that an outer automorphism swaps

$$\mathrm{R}(5,1) \leftrightarrow \mathrm{R}(2,2,2), \quad \mathrm{R}(2,1^4) \leftrightarrow \mathrm{R}(3,3), \quad \mathrm{R}(4,1^2) \leftrightarrow \mathrm{R}(3,1^3),$$

and fixes the other irreducible representations. □

Now we are ready to prove the part of Theorem 5.1 concerning the Coble fourfold.

Proposition 5.13. *The group* $\mathrm{Cl}(\mathscr{Y})$ *is torsion free, and there is an isomorphism*

$$\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q} \cong (\mathrm{R}(6) \boxtimes 1) \oplus (\mathrm{R}(3,3) \boxtimes (-1))$$

of representations of the group $\mathrm{Aut}(\mathscr{Y}) \cong \mathfrak{S}_6 \times \boldsymbol{\mu}_2$.

Proof. Since $\mathscr{Y}_{5,1} \to \mathscr{Y}$ is a small $\mathfrak{S}_5$-equivariant resolution, we have an $\mathfrak{S}_5$-equivariant isomorphism $\mathrm{Cl}(\mathscr{Y}) \cong \mathrm{Pic}(\mathscr{Y}_{5,1})$ with respect to the twisted action of a nonstandard subgroup $\mathfrak{S}_5$. Since $\mathscr{Y}_{5,1}$ is a $\mathbb{P}^2$-bundle over the quintic del Pezzo surface S, we have an $\mathfrak{S}_5$-equivariant direct sum decomposition

$$\mathrm{Pic}(\mathscr{Y}_{5,1}) = \mathbb{Z}H \oplus p^*(\mathrm{Pic}(S)).$$

Here the first summand is generated by the pullback of the hyperplane class of $\mathbb{P}^4$ under the map $\pi \circ \rho_{5,1}$, and so is $\mathfrak{S}_5$-invariant. This proves that $\mathrm{Cl}(\mathscr{Y})$ is torsion free.

Furthermore, it follows from Lemma 5.10 that there is an isomorphism of $\mathfrak{S}_5$-representations

$$(\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q})|_{\mathfrak{S}_5} \cong \mathrm{R}(5) \oplus \mathrm{R}(5) \oplus \mathrm{R}(4,1).$$

Since the embedding of $\mathfrak{S}_5 \hookrightarrow \mathfrak{S}_6$ is nonstandard, it follows from Lemma 5.12 that

$$(\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q})|_{\mathfrak{S}_6} \cong \mathrm{R}(6) \oplus \mathrm{R}(2,2,2);$$

we emphasize the fact that this isomorphism holds for the *twisted* action of $\mathfrak{S}_6$ on $\mathscr{Y}$. The first summand $\mathrm{R}(6)$ is generated by the class H, hence lifts to $\mathrm{R}(6) \boxtimes 1$ as a representation of $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$. Since the quotient of $\mathscr{Y}$ by the Galois involution σ is $\mathbb{P}^4$ and its class group is of rank 1, it follows that the action of $\boldsymbol{\mu}_2$ on the second summand $\mathrm{R}(2,2,2)$ is nontrivial. Hence the *natural* action of $\mathfrak{S}_6$ on the second summand is obtained from $\mathrm{R}(2,2,2)$ by the sign twist, i.e., the corresponding representation is $\mathrm{R}(3,3)$ (recall that the sign twist modifies an irreducible representation by a transposition of its partition), and the assertion of the proposition follows. □

Below we will also need to describe certain $\mathfrak{S}_5$-representations from their restrictions to $\mathfrak{A}_5$. For this the following calculation is useful. Denote by R_1, R_3', R_3'', R_4, and R_5 the irreducible representations of the group $\mathfrak{A}_5$ of dimensions 1, 3, 3, 4, and 5, respectively; see for instance [Fulton and Harris 1991, Exercise 3.5].

Lemma 5.14. *The following table contains all irreducible representation of* $\mathfrak{S}_5$ *and their restrictions to* $\mathfrak{A}_5$.

$\mathrm{R}(\lambda)$	$\mathrm{R}(5)$ $\mathrm{R}(1^5)$	$\mathrm{R}(4,1)$ $\mathrm{R}(2,1^3)$	$\mathrm{R}(3,2)$ $\mathrm{R}(2^2,1)$	$\mathrm{R}(3,1^2)$
$\mathrm{R}(\lambda)\|_{\mathfrak{A}_5}$	R_1	R_4	R_5	$R_3' \oplus R_3''$

Proof. It is enough to know that a restriction of an $\mathfrak{S}_5$-representation $\mathrm{R}(\lambda)$ to $\mathfrak{A}_5$ contains the trivial subrepresentation R_1 if and only if $\mathrm{R}(\lambda)$ is trivial or is the sign representation, i.e., if $\lambda = (5)$ or $\lambda = (1^5)$.

This follows from Frobenius reciprocity, because

$$\mathrm{Ind}_{\mathfrak{A}_5}^{\mathfrak{S}_5}(R_1) \cong \mathrm{R}(5) \oplus \mathrm{R}(1^5).$$

With this in mind, there is only one way to represent the dimensions of $\mathrm{R}(\lambda)$ as sums of dimensions of irreducible $\mathfrak{A}_5$-representations. It remains to notice that the $\mathfrak{S}_5$-representation $\mathrm{R}(3, 1^2)$ is defined over $\mathbb{Q}$, while both three-dimensional $\mathfrak{A}_5$-representations R_3' and R_3'' are not, so the restriction of $\mathrm{R}(3, 1^2)$ to $\mathfrak{A}_5$ splits as $R_3' \oplus R_3''$. □

Now we are almost ready to attack the class groups of the quartics X_t. For each τ we have a natural composition

$$\mathrm{Cl}(\mathscr{Y}) \cong \mathrm{Cl}(\mathscr{Y} \setminus \mathrm{CR}) \cong \mathrm{Pic}(\mathscr{Y} \setminus \mathrm{CR}) \xrightarrow{\mathsf{res}} \mathrm{Pic}(\mathscr{X}_\tau \setminus \mathrm{CR}) \hookrightarrow \mathrm{Cl}(\mathscr{X}_\tau \setminus \mathrm{CR}) \cong \mathrm{Cl}(\mathscr{X}_\tau). \tag{5.15}$$

Here res denotes the restriction map. The first and the last isomorphisms take place since the Cremona–Richmond configuration $\mathrm{CR} = \mathrm{Sing}(\mathscr{Y})$ has codimension greater than 1 both in $\mathscr{Y}$ and $\mathscr{X}_\tau$, and the second isomorphism follows from smoothness of $\mathscr{Y} \setminus \mathrm{CR}$.

Lemma 5.16. *For all $\tau \neq 0$ the composition $\mathrm{Cl}(\mathscr{Y}) \to \mathrm{Cl}(\mathscr{X}_\tau)$ of the maps in* (5.15) *is an $\mathfrak{S}_6$-equivariant embedding with respect to the natural action of $\mathfrak{S}_6$. For $\tau = \infty$ it is an $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$-equivariant embedding. Moreover, for $\tau \notin \hat{\mathfrak{D}}$ it is an isomorphism.*

Proof. All the maps in (5.15) are equivariant with respect to the natural action of $\mathfrak{S}_6$ (or of the whole group $\mathfrak{S}_6 \times \boldsymbol{\mu}_2$ in case $\tau = \infty$), hence so is the composition, and it remains to prove injectivity. For this we forget about the $\mathfrak{S}_6$-action and consider the diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathscr{Y}_{5,1}) & \xrightarrow{\mathsf{res}} & \mathrm{Pic}(\mathscr{X}_\tau^{5,1}) \\ {\scriptstyle (\rho_{5,1})_*}\downarrow & & \downarrow{\scriptstyle (\rho_{5,1})_*} \\ \mathrm{Cl}(\mathscr{Y}) & \longrightarrow & \mathrm{Cl}(\mathscr{X}_\tau) \end{array} \tag{5.17}$$

which is easily seen to be commutative. The vertical arrows are isomorphisms, since the birational maps $\rho_{5,1}\colon \mathscr{Y}_{5,1} \to \mathscr{Y}$ and $\rho_{5,1}\colon \mathscr{X}_\tau^{5,1} \to \mathscr{X}_\tau$ for $\tau \neq 0$ are small by Theorems 1.9 and 1.14. So, it is enough to check that the morphism res is injective, which is obvious, since $\mathscr{Y}_{5,1}$ is a $\mathbb{P}^2$-bundle over S and $\mathscr{X}_\tau^{5,1}$ is a (flat) conic bundle inside $\mathscr{Y}_{5,1}$.

Moreover, for $\tau \notin \hat{\mathfrak{D}}$ the conic bundle is standard, hence the image of the top arrow is a sublattice of index 2 or 1, depending on whether the conic bundle has a rational section or not. Since we also know from [Beauville 2013] or Theorem 4.4 that for $\tau \notin \hat{\mathfrak{D}}$ the threefold $\mathscr{X}_\tau^{5,1}$ is not rational, we conclude that the conic bundle $p\colon \mathscr{X}_\tau^{5,1} \to S$ has no rational sections, and thus res is an isomorphism. □

Remark 5.18. Recall that by Lemma 3.12 for $\tau \neq 0, \infty$ one has an isomorphism $\mathscr{X}_\tau \cong X_t$ for $t = (\tau^2+1)/4$. Thus Proposition 5.13 and Lemma 5.16 provide a description of $\mathrm{Cl}(X_t)$ for all $t \notin \mathfrak{D} \cup \{\infty\}$.

It remains to analyze the class groups of the special quartics X_t.

We can think of the map (5.15) as of a map $\mathrm{Cl}(\mathscr{Y}) \to \mathrm{Cl}(X_t)$; this map is $\mathfrak{S}_6$-equivariant, where the action of $\mathfrak{S}_6$ on $\mathscr{Y}$ is natural. We denote the cokernel of this map by

$$\mathrm{ExCl}(X_t) := \mathrm{Cl}(X_t)/\mathrm{Cl}(\mathscr{Y}),$$

and refer to this group as the *excess class group* of X_t. To prove Theorem 5.1 we need to compute the latter group for $t = \frac{1}{2}$, $\frac{1}{6}$, and $\frac{7}{10}$ as an $\mathfrak{S}_6$-representation. For this we need a couple of observations.

Lemma 5.19. *For a standard subgroup $\mathfrak{S}_4 \subset \mathfrak{S}_6$ we have $\mathrm{rk}\,\mathrm{Cl}(X_t)^{\mathfrak{S}_4} = 1$ for any $t \neq \infty$. In particular, we have $\mathrm{rk}\,\mathrm{ExCl}(X_t)^{\mathfrak{S}_4} = 0$ for any $t \neq \infty$.*

Proof. We may assume that $\mathfrak{S}_4$ preserves the homogeneous coordinates x_5 and x_6 on $\mathbb{P}^5$. Denote $\boldsymbol{p}_i := x_1^i + \cdots + x_6^i$. Consider the quotients $\mathbb{P}^5/\mathfrak{S}_4$ and $X_t/\mathfrak{S}_4$. Then

$$\mathbb{P}^5/\mathfrak{S}_4 \cong \mathbb{P}(1, 1, 1, 2, 3, 4),$$

where the weighted homogeneous coordinates of weights 1, 1, 1, 2, 3, and 4 correspond to the $\mathfrak{S}_4$-invariants x_5, x_6, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, and $\boldsymbol{p}_4$, respectively. The quotient variety $X_t/\mathfrak{S}_4$ is given in $\mathbb{P}(1, 1, 1, 2, 3, 4)$ by the equations

$$\boldsymbol{p}_1 = \boldsymbol{p}_4 - t\,\boldsymbol{p}_2^2 = 0,$$

so that $X_t/\mathfrak{S}_4 \cong \mathbb{P}(1, 1, 2, 3)$. Therefore, we have $\mathrm{rk}\,\mathrm{Cl}(X_t)^{\mathfrak{S}_4} = \mathrm{rk}\,\mathrm{Cl}(X_t/\mathfrak{S}_4) = 1$; see for instance [Fulton 1984, 1.7.5]. Since also $\mathrm{rk}\,\mathrm{Cl}(\mathscr{Y})^{\mathfrak{S}_4} = 1$ (see Corollary 5.4), it follows that $\mathrm{rk}\,\mathrm{ExCl}(X_t)^{\mathfrak{S}_4} = 0$. □

Remark 5.20 [Cheltsov et al. 2019a, Remark 2.11]. An argument similar to the proof of Lemma 5.19 was (incorrectly!) used in the proof of [Cheltsov and Shramov 2014, Theorem 1.20] for the standard subgroup $\mathfrak{A}_{4,2} \cong \mathfrak{S}_4$ in $\mathfrak{S}_6$ to deduce that $\mathrm{rk}\,\mathrm{Cl}(X_{1/2})^{\mathfrak{A}_6} = 1$. However, the assertion is correct: it was later obtained in [Cheltsov et al. 2019a, Corollary 2.10] by a different method. Using Theorem 5.1 we can find this rank as well: indeed, one has $\mathrm{rk}\,\mathrm{Cl}(X_{1/2})^{\mathfrak{A}_6} = \mathrm{rk}\,\mathrm{Cl}(X_{1/2})^{\mathfrak{A}_{4,2}} = 1$ by Corollary 5.4.

Lemma 5.21. *For a nonstandard subgroup $\mathfrak{S}_5 \subset \mathfrak{S}_6$ we have $\mathrm{rk}\,\mathrm{Cl}(X_{1/6})^{\mathfrak{S}_5} = 2$. In particular, we have $\mathrm{rk}\,\mathrm{ExCl}(X_{1/6})^{\mathfrak{S}_5} = 1$.*

Proof. By [Cheltsov and Shramov 2016b, Section 6] the quartic $X_{1/6}$ is $\mathfrak{S}_5$-equivariantly isomorphic away from codimension 2 to the blow up $\hat{X}_{1/6}$ of ten lines in $\mathbb{P}^3$, that form a so-called double-five configuration. Therefore we have $\mathrm{Cl}(X_{1/6}) \cong \mathrm{Cl}(\hat{X}_{1/6})$ as $\mathfrak{S}_5$-representations. Furthermore, the group $\mathfrak{S}_5$ acts transitively on this configuration of lines, hence $\mathrm{rk}\,\mathrm{Cl}(\hat{X}_{1/6})^{\mathfrak{S}_5} = 2$. Since also $\mathrm{rk}\,\mathrm{Cl}(\mathscr{Y})^{\mathfrak{S}_5} = 1$ (see Corollary 5.4, and keep in mind that according to Notation 5.3 the nonstandard subgroup $\mathfrak{S}_5 \subset \mathfrak{S}_6$ is denoted by $\overline{\mathfrak{S}}_5$), it follows that $\mathrm{rk}\,\mathrm{ExCl}(X_{1/6})^{\mathfrak{S}_5} = 1$. □

Now we are ready to describe the excess class groups for the special quartics.

Proposition 5.22. *There are the following isomorphisms of $\mathfrak{S}_6$-representations*:

$$\mathrm{ExCl}(X_{1/2}) \otimes \mathbb{Q} \cong \mathrm{R}(3, 1^3), \quad \mathrm{ExCl}(X_{1/6}) \otimes \mathbb{Q} \cong \mathrm{R}(2, 2, 2), \quad \mathrm{ExCl}(X_{7/10}) \otimes \mathbb{Q} \cong \mathrm{R}(1^6).$$

Proof. We replace the quartics $X_{1/2}$, $X_{1/6}$, and $X_{7/10}$ by their partial resolutions of singularities $\mathscr{X}_1^{5,1}$, $\mathscr{X}_{1/\sqrt{-3}}^{5,1}$, and $\mathscr{X}_{3/\sqrt{5}}^{5,1}$, respectively. Similarly to the proof of Lemma 5.16, we obtain isomorphisms of $\mathfrak{A}_5$-representations

$$\mathrm{Cl}(\mathscr{X}_\tau^{5,1}/S) \otimes \mathbb{Q} \cong (\mathrm{Cl}(\mathscr{Y}^{5,1}/S) \oplus (\mathrm{Cl}(\mathscr{X}_\tau^{5,1})/\mathrm{Cl}(\mathscr{Y}^{5,1}))) \otimes \mathbb{Q} \cong R_1 \oplus (\mathrm{ExCl}(X_t) \otimes \mathbb{Q})|_{\mathfrak{A}_5}, \quad (5.23)$$

with the summand R_1 on the right generated by the canonical class. Next we use the computation of Corollary 4.9 to describe the left-hand side of (5.23). Namely, by Corollary 4.9 the left-hand side is isomorphic to $R_1 \oplus \mathrm{Ind}_G^{\mathfrak{A}_5}(\upsilon)$ for a certain subgroup $G \subset \mathfrak{A}_5$ and its one-dimensional representation υ. Canceling the R_1 summands, we obtain an isomorphism

$$(\mathrm{ExCl}(X_t) \otimes \mathbb{Q})|_{\mathfrak{A}_5} \cong \mathrm{Ind}_G^{\mathfrak{A}_5}(\upsilon).$$

It only remains to use the description of the subgroup G and its representation υ also provided by Corollary 4.9.

In the case $t = \frac{1}{2}$, so that $\tau = 1$, it gives

$$(\mathrm{ExCl}(X_{1/2}) \otimes \mathbb{Q})|_{\mathfrak{A}_5} \cong \mathrm{Ind}_{\mathfrak{A}_{3,2}}^{\mathfrak{A}_5}(-\mathbf{1}) \cong R_3' \oplus R_3'' \oplus R_4.$$

Therefore, by Lemma 5.14 we deduce that $(\mathrm{ExCl}(X_{1/2}) \otimes \mathbb{Q})|_{\mathfrak{S}_5}$ is isomorphic either to $\mathrm{R}(3, 1^2) \oplus \mathrm{R}(4, 1)$ or to $\mathrm{R}(3, 1^2) \oplus \mathrm{R}(2, 1^3)$, hence by Lemma 5.12 we have either $\mathrm{ExCl}(X_{1/2}) \otimes \mathbb{Q} \cong \mathrm{R}(4, 1^2)$ or $\mathrm{ExCl}(X_{1/2}) \otimes \mathbb{Q} \cong \mathrm{R}(3, 1^3)$. The first case is impossible by Lemma 5.19, because by Pieri's rule the restriction of the $\mathfrak{S}_6$-representation $\mathrm{R}(4, 1^2)$ to a standard subgroup $\mathfrak{S}_4$ contains a trivial subrepresentation, hence the required result.

Similarly, in the case $t = \frac{1}{6}$, so that $\tau = 1/\sqrt{-3}$, we have

$$(\mathrm{ExCl}(X_{1/6}) \otimes \mathbb{Q})|_{\mathfrak{A}_5} \cong \mathrm{Ind}_{\mathfrak{A}_4}^{\mathfrak{A}_5}(\mathbf{1}) \cong R_1 \oplus R_4.$$

Therefore, by Lemma 5.14 we deduce that $(\mathrm{ExCl}(X_{1/6}) \otimes \mathbb{Q})|_{\mathfrak{S}_5}$ is isomorphic to the sum of one of the representations $\mathrm{R}(5)$ and $\mathrm{R}(1^5)$, and one of the representations $\mathrm{R}(4, 1)$ and $\mathrm{R}(2, 1^3)$. On the other hand, $(\mathrm{ExCl}(X_{1/6}) \otimes \mathbb{Q})|_{\mathfrak{S}_5}$ should contain $\mathrm{R}(5)$ by Lemma 5.21, so it follows that $(\mathrm{ExCl}(X_{1/6}) \otimes \mathbb{Q})|_{\mathfrak{S}_5}$ is either $\mathrm{R}(5) \oplus \mathrm{R}(4, 1)$, or $\mathrm{R}(5) \oplus \mathrm{R}(2, 1^3)$. By Lemma 5.12 only the first of them can be obtained as a restriction of a representation of $\mathfrak{S}_6$ with respect to a nonstandard embedding of $\mathfrak{S}_5$, and the corresponding representation of $\mathfrak{S}_6$ is $\mathrm{R}(2, 2, 2)$. Thus, we have $\mathrm{ExCl}(X_{1/6}) \otimes \mathbb{Q} \cong \mathrm{R}(2, 2, 2)$.

Finally, in the case $t = \frac{7}{10}$, so that $\tau = 3/\sqrt{5}$, we have

$$(\mathrm{ExCl}(X_{7/10}) \otimes \mathbb{Q})|_{\mathfrak{A}_5} \cong R_1,$$

hence $\mathrm{ExCl}(X_{7/10}) \otimes \mathbb{Q}$ is either $\mathrm{R}(6)$ or $\mathrm{R}(1^6)$. Again, the first case is impossible by Lemma 5.19, hence the required result. □

Now we are ready to prove the main result of this section.

Proof of Theorem 5.1. The description of $\mathrm{Cl}(\mathscr{Y}) \otimes \mathbb{Q}$ is given by Proposition 5.13, and the descriptions of $\mathrm{Cl}(\mathscr{X}_\infty) \otimes \mathbb{Q}$ and $\mathrm{Cl}(X_t) \otimes \mathbb{Q}$ for $t \notin \mathfrak{D} \cup \{\infty\}$ follow from a combination of Proposition 5.13 with Lemma 5.16. The last three isomorphisms follow from Proposition 5.22 in view of the definition of the excess class group. □

Remark 5.24. To study G-equivariant birational maps of the remaining $\mathfrak{S}_6$-invariant quartic $X_{1/4}$ to G-Mori fiber spaces, one can replace $X_{1/4}$ by its projective dual, which is the Segre cubic Z. This may be simpler because Z has terminal singularities. The corresponding problem for Z was partially solved in [Avilov 2016, Theorem 1.3]. In particular, if G is a standard subgroup $\mathfrak{A}_5$ in $\mathfrak{S}_6$, then $\mathrm{rk}\,\mathrm{Cl}(Z)^G = 1$ by [Avilov 2016, Proposition 3.1], and we expect that Z, and thus also $X_{1/4}$, is not G-rational. In this case the induced action of G on Z is also given by a standard embedding $\mathfrak{A}_5 \cong G \hookrightarrow \mathrm{Aut}(Z) \cong \mathfrak{S}_6$; see e.g., [Howard et al. 2008, Section 2.2]. On the contrary, if G is a nonstandard subgroup $\mathfrak{A}_5$ in $\mathfrak{S}_6$, then Z is known to be G-rational; see [Prokhorov 2010, 3.16].

Remark 5.25. One of the geometric interpretations of the nontrivial summands of $\mathrm{Cl}(X_t) \otimes \mathbb{Q}$ that appear in Theorem 5.1 is as follows. Suppose that $t \neq \frac{1}{4}, \infty$, so that the singularities of X_t are nodes by Theorem 3.3. Let $\nu\colon \tilde{X}_t \to X_t$ be the blow up of all singular points of X_t, and let $D_1, \dots, D_r$ be the exceptional divisors of ν. Then $\tilde{X}_t$ is smooth, and $D_i \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let M_i^+ and M_i^- be the rulings from two different families on D_i. One can check that there is a natural perfect pairing between the vector subspace in $H^4(\tilde{X}_t, \mathbb{C})$ spanned by the one-cycles $M_i^+ - M_i^-$ and the space $(\mathrm{Cl}(X_t)/\mathrm{Pic}(X_t)) \otimes \mathbb{C}$. Note also that the structure of this subspace of $H^4(\tilde{X}_t, \mathbb{C})$ as an $\mathfrak{S}_6$-representation can be independently deduced from [Schoen 1985, Proposition 1.3] and [Beauville 2013, Lemma 1].

Appendix: Cremona–Richmond configuration

The *Cremona–Richmond configuration* is the configuration CR of 15 lines with 15 triple intersection points in $\mathbb{P}^4$ formed by the singular locus of the Igusa quartic. By a small abuse of terminology, we will sometimes say that the singular locus is the configuration CR itself. We refer the reader to [Cremona 1877; Richmond 1900; Dolgachev 2004, Section 9] for basic properties.

Explicitly, the configuration CR can be described as follows. Consider $\mathbb{P}^4$ as the hyperplane given by (1.1) in $\mathbb{P}^5$ with the usual $\mathfrak{S}_6$-action. For each pairs-splitting

$$\{1, \dots, 6\} = I_1 \sqcup I_2 \sqcup I_3,$$

where $|I_1| = |I_2| = |I_3| = 2$, let $\mathrm{L}_{(I_1|I_2|I_3)}$ be the line in $\mathbb{P}^4$ given by equations

$$x_i = x_j \text{ if } \{i, j\} = I_p \text{ for some } p \in \{1, 2, 3\}.$$

This gives 15 lines in $\mathbb{P}^4$; for instance, $\mathrm{L}_{(1,2|3,4|5,6)}$ is the line given by equations

$$x_1 = x_2, \quad x_3 = x_4, \quad x_5 = x_6, \tag{A.1}$$

and the other lines are obtained from this by the $\mathfrak{S}_6$-action.

Similarly, for every two-element subset $I \subset \{1, \dots, 6\}$ let P_I be the point in $\mathbb{P}^4$ given by equations

$$x_i = x_j \text{ if either } i, j \in I \text{ or } i, j \in \bar{I},$$

where $\bar{I}$ is the complement of I in $\{1, \dots, 6\}$. This gives 15 points in $\mathbb{P}^4$; for instance,

$$\mathrm{P}_{1,2} = (2 : 2 : -1 : -1 : -1 : -1), \tag{A.2}$$

and the other points are obtained from this by the $\mathfrak{S}_6$-action (so, this is the set Υ_{15} defined in Section 3.1).

It is easy to see that P_I lies on $\mathrm{L}_{(I_1|I_2|I_3)}$ if and only if $I = I_p$ for some $p \in \{1, 2, 3\}$, i.e., if I is one of the pairs in the pairs-splitting, or, equivalently, the pairs-splitting extends the pair I. In particular, there are three lines through each of the points (corresponding to three pairs-splittings of $\bar{I}$), and there are three points on each line (corresponding to three pairs in a pairs-splitting). Moreover, the points P_I are the only intersection points of the lines $\mathrm{L}_{(I_1|I_2|I_3)}$. Because of this CR is often referred to as a (15_3)-*configuration*.

In this section we discuss some properties of CR. In particular, in Theorem A.8 we show that CR is determined uniquely up to a projective transformation of $\mathbb{P}^4$ by its combinatorial structure (under a mild nondegeneracy assumption), and that the Igusa quartic is the only quartic whose singular locus contains CR.

We start by a discussion of combinatorics of CR.

Lemma A.3. *The configuration* CR *is combinatorially self-dual*: *an outer automorphism of* $\mathfrak{S}_6$ *induces a bijection between the set of points* P_I *and the set of lines* $\mathrm{L}_{(I_1|I_2|I_3)}$ *that preserves the incidence correspondence.*

Proof. There is a natural bijection between subsets of cardinality two in the set $\{1, \dots, 6\}$, and transpositions in the group $\mathfrak{S}_6$. Similarly, there is a natural bijection between pairs-splittings of the set $\{1, \dots, 6\}$, and elements of cycle type $[2, 2, 2]$ in $\mathfrak{S}_6$. Let us denote the transposition corresponding to a subset $I \subset \{1, \dots, 6\}$ by $w(I)$, and the element of cycle type $[2, 2, 2]$ corresponding to a pairs-splitting (I_1, I_2, I_3) of $\{1, \dots, 6\}$ by $w(I_1, I_2, I_3)$. The incidence relation of lines and points of CR can be reformulated in group-theoretic terms: the line $\mathrm{L}_{(I_1|I_2|I_3)}$ is incident to the point P_I if and only if the permutations $w(I)$ and $w(I_1, I_2, I_3)$ commute (or, which is the same, the composition $w(I) \circ w(I_1, I_2, I_3)$ has cycle type $[2, 2]$).

Choose an outer automorphism α of the group $\mathfrak{S}_6$. The automorphism α interchanges transpositions with elements of cycle type $[2, 2, 2]$. Thus α defines a map from the set of points of CR to the set of lines of CR, and a map from the set of lines of CR to the set of points of CR. Moreover, this map preserves the incidence relation. □

Lemma A.3 implies the following result that we used in the main part of the paper.

Corollary A.4. *Every standard subgroup* $\mathfrak{A}_5 \subset \mathfrak{S}_6$ *acts transitively on the set of lines of* CR, *and every nonstandard subgroup* $\mathfrak{A}_5 \subset \mathfrak{S}_6$ *acts transitively on the set of points of* CR.

Proof. The first assertion is evident from combinatorics, and the second assertion follows from the first one in view of the bijection of Lemma A.3. □

The following description of CR is very useful. Choose a triples-splitting

$$\{1,\dots,6\} = K_0 \sqcup K_1, \quad |K_0| = |K_1| = 3.$$

For each bijection $g\colon K_0 \xrightarrow{\sim} K_1$ let $\Gamma(g)$ be the pairs-splitting formed by all pairs $\{k_0, g(k_0)\}$, where k_0 runs through K_0 (and hence $g(k_0)$ runs through K_1). The 6 lines and 9 points

$$\{\mathrm{L}_{\Gamma(g)}\}_{g\in \mathrm{Iso}(K_0,K_1)} \quad \text{and} \quad \{\mathrm{P}_{k_0,k_1}\}_{(k_0,k_1)\in K_0\times K_1}$$

form a subconfiguration $\mathrm{CR}'_{K_0,K_1} \subset \mathrm{CR}$ of the Cremona–Richmond configuration; see Figure 3. Because of its characteristic shape we call it a *jail configuration*. Note that CR'_{K_0,K_1} is contained in the hyperplane

$$H_{K_0} := \left\{ \sum_{k\in K_0} x_k = 0 \right\} = \left\{ \sum_{k\in K_1} x_k = 0 \right\} =: H_{K_1}. \tag{A.5}$$

We call it the *jail hyperplane*.

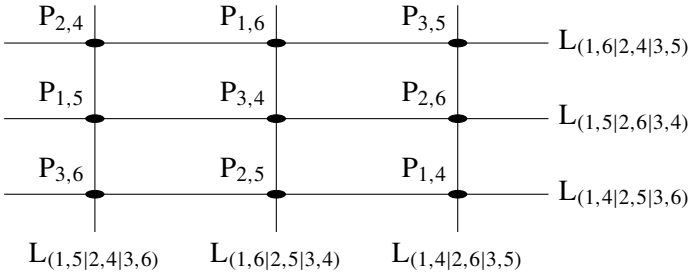


Figure 3. The jail subconfiguration $\mathrm{CR}'_{\{1,2,3\},\{4,5,6\}}$ in the Cremona–Richmond configuration CR.

The remaining 9 lines and 6 points

$$\{\mathrm{L}_{(k_0,k_1|K_0\setminus k_0|K_1\setminus k_1)}\}_{(k_0,k_1)\in K_0\times K_1} \quad \text{and} \quad \{\mathrm{P}_I\}_{I\subset K_0 \text{ or } I\subset K_1}$$

form a complete bipartite graph; see Figure 4; we call it a *bipartite configuration*.

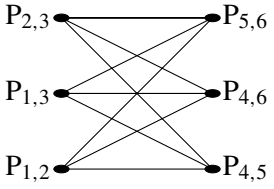


Figure 4. The bipartite subconfiguration $\mathrm{CR}''_{\{1,2,3\},\{4,5,6\}}$ in the Cremona–Richmond configuration CR.

For any decomposition

$$\mathrm{CR} = \mathrm{CR}'_{K_0,K_1} \cup \mathrm{CR}''_{K_0,K_1}$$

into a jail and bipartite subconfiguration its components interact quite weakly: every line $\mathrm{L}_{(k_0,k_1|K_0\setminus k_0|K_1\setminus k_1)}$ from the bipartite component passes through a single point P_{k_0,k_1} in the jail component. This gives a bijection between bipartite lines and jail points (compatible with the natural bijection of both sets with $K_0 \times K_1$).

Lemma A.6. *Let* C *be a configuration of* 15 *lines with* 15 *intersection points in* $\mathbb{P}^4$ *which is not contained in* $\mathbb{P}^3$ *and is combinatorially isomorphic to the Cremona–Richmond configuration. If* $\mathrm{C} = \mathrm{C}' \cup \mathrm{C}''$ *is a jail–bipartite decomposition then the jail component* C′ *spans a hyperplane, and the bipartite component* C″ *spans* $\mathbb{P}^4$.

Proof. The jail component C′ has the shape shown in Figure 3. Two vertical lines do not intersect, hence they span a hyperplane $H' \subset \mathbb{P}^4$. Three horizontal lines intersect each of them, hence they are contained in H'. The last vertical line intersects the horizontal lines, hence it is also contained in H'.

The bipartite component C″ has the shape shown in Figure 4. Assume it is contained in a hyperplane $H'' \subset \mathbb{P}^4$. Then every line of the bipartite component is contained in H''. Since every point of the jail component lies on a line of the bipartite component, it follows that the jail component is also contained in H''. Thus $\mathrm{C} \subset H''$, which contradicts the assumptions of the lemma. □

Remark A.7. The set $\{1, 2, 3, 4, 5, 6\}$ has 10 distinct triples-splittings, giving rise to 10 distinct jail-bipartite decompositions of the Cremona–Richmond configuration. The 10 hyperplanes supporting the jail components of CR appeared in Remark 2.25.

Theorem A.8. *Let* C *be a configuration of* 15 *lines with* 15 *intersection points in* $\mathbb{P}^4$ *which is not contained in* $\mathbb{P}^3$ *and is combinatorially isomorphic to the Cremona–Richmond configuration. Then it is projectively isomorphic to the Cremona–Richmond configuration.*

Proof. Choose a jail-bipartite decomposition $\mathrm{C} = \mathrm{C}' \cup \mathrm{C}''$. Choose five points $P_1, \dots, P_5$ in the bipartite component C″ that are not contained in a hyperplane (this is possible by Lemma A.6), and let H' be the hyperplane containing the jail component C′. Note that $P_i \notin H'$ for all i. Indeed, if $P_i \in H'$ then every line of the bipartite component passing through P_i would be contained in H' (since it also contains a point of the jail component), hence the three points of C″ that are connected to P_i by lines in C″ will be also contained in H'. Applying the same argument to one of these points, we would deduce that the whole bipartite component is contained in H', hence $\mathrm{C} \subset H'$, which contradicts our assumptions.

Assume that the points P_1, P_3, and P_5 are not connected to each other by lines in C″; that is, they are contained in one part of the bipartite component, and P_2, P_4 are contained in the other. Since the points P_i do not lie on a hyperplane, they can be taken to points

$$\begin{aligned} &P_1 = (1:0:0:0:0), \quad P_3 = (0:0:1:0:0), \quad P_5 = (0:0:0:0:1), \\ &P_2 = (0:1:0:0:0), \quad P_4 = (0:0:0:1:0), \end{aligned} \tag{A.9}$$

of $\mathbb{P}^4$ by a projective transformation. Since the hyperplane H' does not pass through the points P_i, it can be simultaneously taken to the hyperplane defined by the equation

$$x_1 - x_2 + x_3 - x_4 + x_5 = 0.$$

Now for each odd i and even j consider the line passing through P_i and P_j. By assumption it belongs to the bipartite component C''. The intersection points of these lines with H' are the following six points

$$\begin{aligned} &P_{12} = (1:1:0:0:0), \quad P_{32} = (0:1:1:0:0), \quad P_{52} = (0:1:0:0:1), \\ &P_{14} = (1:0:0:1:0), \quad P_{34} = (0:0:1:1:0), \quad P_{54} = (0:0:0:1:1). \end{aligned} \tag{A.10}$$

It follows that P_{ij} are points of the jail component C'. Consequently, the following six lines belong to the jail component C':

$$\begin{aligned} &\langle P_{12}, P_{34}\rangle = \{x_1 - x_2 = x_3 - x_4 = x_5 = 0\}, \quad \langle P_{12}, P_{54}\rangle = \{x_1 - x_2 = x_5 - x_4 = x_3 = 0\}, \\ &\langle P_{32}, P_{14}\rangle = \{x_3 - x_2 = x_1 - x_4 = x_5 = 0\}, \quad \langle P_{32}, P_{54}\rangle = \{x_3 - x_2 = x_5 - x_4 = x_1 = 0\}, \\ &\langle P_{52}, P_{14}\rangle = \{x_5 - x_2 = x_1 - x_4 = x_3 = 0\}, \quad \langle P_{52}, P_{34}\rangle = \{x_5 - x_2 = x_3 - x_4 = x_1 = 0\}, \end{aligned}$$

and their three extra intersection points

$$P_{1234} = (1:1:1:1:0), \quad P_{1245} = (1:1:0:1:1), \quad P_{2345} = (0:1:1:1:1) \tag{A.11}$$

also belong to C'. Finally, the last point P_0 of the bipartite component is the point

$$P_0 = \langle P_1, P_{2345}\rangle \cap \langle P_3, P_{1245}\rangle \cap \langle P_5, P_{1234}\rangle = (1:1:1:1:1). \tag{A.12}$$

This proves that such configuration is unique up to a projective transformation. The explicit transformation from $\mathbb{P}^4$ to $\mathbb{P}^5$ that takes the points (A.9), (A.10), (A.11), and (A.12) to the points $\mathrm{P}_{i,j}$ that were defined in (A.2) is given by the matrix

$$\begin{pmatrix} 1 & 1 & -2 & 1 & -2 \\ -2 & 1 & 1 & 1 & -2 \\ -2 & 1 & -2 & 1 & 1 \\ 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 & 1 \end{pmatrix};$$

in particular, the point P_5 is mapped to the point $\mathrm{P}_{1,2}$ in (A.2). This completes the proof of Theorem A.8. □

Remark A.13. Let C be a configuration combinatorially isomorphic to CR. Then one can always project C isomorphically to $\mathbb{P}^3$. In particular, the assumption of Theorem A.8 requiring that the configuration is not contained in $\mathbb{P}^3$ is necessary.

Corollary A.14. *Let* C *be a configuration of* 15 *lines with* 15 *intersection points in* $\mathbb{P}^4$ *which is not contained in* $\mathbb{P}^3$ *and is combinatorially isomorphic to the Cremona–Richmond configuration. Suppose that* X *is a quartic threefold that contains* C *in its singular locus. Then it is projectively isomorphic to the Igusa quartic.*

Proof. By Theorem A.8 it is enough to show that the Igusa quartic X is the unique quartic singular along C. Suppose that X' is another quartic with this property. Since X is irreducible, the intersection $Z = X \cap X'$ is

two-dimensional, and $\deg Z = 16$. Let C' be one of the jail subconfigurations of C. Then C' is contained in a unique two-dimensional smooth quadric T; this quadric is swept out by lines that meet three of the lines in C'. The lines of C' are singular both on X and X', so we conclude that T is contained in Z. It remains to notice that C contains 10 jail subconfigurations, all of them giving rise to different two-dimensional quadrics contained in Z. The degree of the union of these quadrics is 20; this is greater than $\deg Z$, which gives a contradiction. □

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