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On the definition of quantum Heisenberg category

Jonathan Brundan, Alistair Savage and Ben Webster

We introduce a diagrammatic monoidal category $\mathcal{H}eis_k(z,t)$ which we call the *quantum Heisenberg category*; here, $k \in \mathbb{Z}$ is "central charge" and z and t are invertible parameters. Special cases were known before: for central charge $k = -1$ and parameters $z = q - q^{-1}$ and $t = -z^{-1}$ our quantum Heisenberg category may be obtained from the deformed version of Khovanov's Heisenberg category introduced by Licata and Savage by inverting its polynomial generator, while $\mathcal{H}eis_0(z,t)$ is the affinization of the HOMFLY-PT skein category. We also prove a basis theorem for the morphism spaces in $\mathcal{H}eis_k(z,t)$.

1. Introduction

Fix a commutative ground ring $\mathbb{k}$ and parameters $z, t \in \mathbb{k}^\times$. This paper introduces a family of pivotal monoidal categories $\mathcal{H}eis_k(z,t)$, one for each *central charge* $k \in \mathbb{Z}$. We refer to these categories as *quantum Heisenberg categories*. The terminology is due to a connection to Khovanov's Heisenberg category [2014]: our category for central charge $k = -1$ is a two parameter deformation of the category from [loc. cit.], and is closely related to the one parameter deformation introduced already by Licata and Savage [2013]. The category $\mathcal{H}eis_0(z,t)$ has also already appeared in the literature: it is the *affine HOMFLY-PT skein category* from [Brundan 2017, Section 4]. For more general central charges, our categories are new. They were discovered by mimicking the approach of Brundan [2018], where the definition of the degenerate Heisenberg categories introduced in [Mackaay and Savage 2018] was reformulated.

In fact, we will give three different monoidal presentations of $\mathcal{H}eis_k(z,t)$. They all start from the affine Hecke algebra AH_n associated to the symmetric group $\mathfrak{S}_n$. It is convenient to assemble these algebras for all $n \geq 0$ into a single monoidal category $\mathcal{AH}(z)$. By definition, this is the strict $\mathbb{k}$-linear monoidal category generated by one object $\uparrow$ and two morphisms $x : \uparrow \to \uparrow$ and $\tau : \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$, subject to the

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MSC2010: primary 17B10; secondary 18D10.
Keywords: Heisenberg category, categorification, affine Hecke algebra.

relations

$$\tau \circ (1_\uparrow \otimes x) \circ \tau = x \otimes 1_\uparrow, \tag{1.1}$$

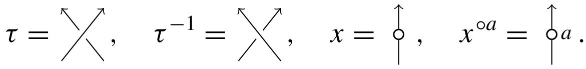
$$\tau \circ \tau = z\tau + 1_{\uparrow\otimes\uparrow}, \tag{1.2}$$

$$(\tau \otimes 1_\uparrow) \circ (1_\uparrow \otimes \tau) \circ (\tau \otimes 1_\uparrow) = (1_\uparrow \otimes \tau) \circ (\tau \otimes 1_\uparrow) \circ (1_\uparrow \otimes \tau). \tag{1.3}$$

The second relation here implies that τ is invertible. We also require that x is invertible, i.e., there is another generator x^{-1} such that

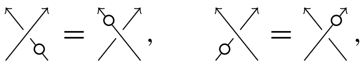
$$x \circ x^{-1} = x^{-1} \circ x = 1_\uparrow. \tag{1.4}$$

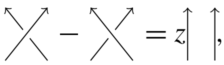
Adopting the usual string calculus for strict monoidal categories, we represent τ, τ^{-1}, x, and more generally $x^{\circ a}$ for any $a \in \mathbb{Z}$, by the diagrams



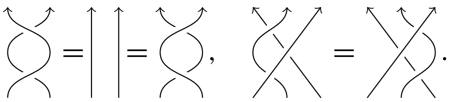
(1.5)

Then the relations (1.1)–(1.3) are equivalent to the following diagrammatic relations:


(1.6)



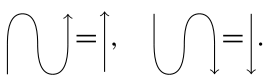
(1.7)


(1.8)

The affine Hecke algebra AH_n itself may be identified with $\mathrm{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$, with its standard generators x_i and τ_j coming from a dot on the i-th string and the positive crossing of the j-th and $(j+1)$-th strings, respectively; our convention for this numbers strings $1, \dots, n$ from right to left. It is often convenient to assume (passing to a quadratic extension if necessary) that $\Bbbk$ contains a root q of the quadratic equation $x^2 - zx - 1 = 0$, so that $z = q - q^{-1}$. The quadratic relation in AH_n may then be written as $(\tau_j - q)(\tau_j + q^{-1}) = 0$. Such a choice of parameter q is not needed in sections 2–4, but is essential for the applications in sections 5–10.

To obtain the quantum Heisenberg category $\mathcal{H}eis_k(z,t)$ from $\mathcal{AH}(z)$, we adjoin a right dual $\downarrow$ to the object $\uparrow$, i.e., we add an additional generating object $\downarrow$ and additional generating morphisms

subject to the relations


(1.9)

Then we add several more generating morphisms subject to relations which ensure that the resulting monoidal category is strictly pivotal, and moreover that there is a distinguished isomorphism

$$\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k} \quad \text{if } k \geq 0$$

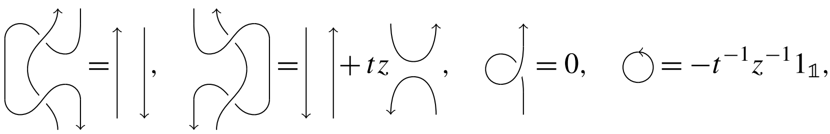
or

$$\uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \cong \downarrow \otimes \uparrow \text{ if } k \leq 0.$$

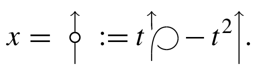
There are various equivalent ways to accomplish this in practice; see Sections 2–4. In these sections, we establish the equivalence of the three approaches and record many other useful relations which follow from the defining ones, including the property already mentioned that $\mathcal{H}eis_k(z,t)$ admits a strictly pivotal structure.

In this paragraph, we explain the approach from Section 4 in the special case $k = -1$. According to Definition 4.1 and (4.14), $\mathcal{H}eis_{-1}(z,t)$ is the strict $\mathbb{k}$-linear monoidal category generated by objects $\uparrow, \downarrow$ and morphisms

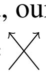
subject to (1.7)–(1.9), the relations



and one more relation, which is equivalent to (1.4). We have *not* included the generating morphism x since, due to a special feature of the $k = -1$ case, it can be recovered from the other generators via the formula



The relations in Definition 4.1 which involve x such as (1.6) are consequences of the other relations with one exception: we must still impose that x is invertible, that is, relation (1.4).

The deformed Heisenberg category $\mathcal{H}(q^2)$ introduced in [Licata and Savage 2013] is (the additive envelope of) the strict $\mathbb{k}$-linear monoidal category defined by the same presentation as in the previous paragraph, with the parameters satisfying $tz = -1$, but *without* the relation (1.4). This follows easily on comparing our presentation with the one in [loc. cit.], using also the fact that our category is strictly pivotal. The generator x denoted by a dot here is not the same as the morphism denoted by a dot in [loc. cit.] (that is simply equal to the right curl); instead, our dot is the "star dot" of [Cautis et al. 2018] (up to renormalization). The Hecke algebra generator $T =$  from [Licata and Savage 2013, Definition 2.1] is related to our τ by $T = q\tau$ (so that the quadratic relation becomes $(T_j - q^2)(T_j + 1) = 0$). Also the generator X appearing just before [loc. cit., Lemma 3.8] is our $-x$. In fact, the category $\mathcal{H}(q^2)$ may be identified with the monoidal subcategory of our category $\mathcal{H}eis_{-1}(z, -z^{-1})$ consisting of all objects and all morphisms which do not involve negative powers of x.

For any $\mathbb{k}$-linear category $\mathcal{C}$, there is an associated strict $\mathbb{k}$-linear monoidal category $\mathcal{E}nd_{\mathbb{k}}(\mathcal{C})$ consisting of $\mathbb{k}$-linear endofunctors and natural transformations. Then one can consider "representations" of $\mathcal{H}eis_k(z,t)$ by considering $\mathbb{k}$-linear monoidal functors into $\mathcal{E}nd_{\mathbb{k}}(\mathcal{C})$ for different choices of $\mathcal{C}$. The motivation for the definition of $\mathcal{H}eis_k(z,t)$ comes from the fact that it acts in this way on other well-known categories appearing in representation theory. If $k=0$ and $t=q^n$ then $\mathcal{H}eis_k(z,t)$ acts on representations of $U_q(\mathfrak{gl}_n)$, with the generating objects $\uparrow$ and $\downarrow$ acting by tensoring with the natural $U_q(\mathfrak{gl}_n)$-module and its dual, respectively; see Section 5. This action is an extension of the monoidal functor from the HOMFLY-PT skein category to the category of finite-dimensional $U_q(\mathfrak{gl}_n)$-modules constructed originally by Turaev [1989]. If $k \neq 0$ then $\mathcal{H}eis_k(z,t)$ acts on representations of the cyclotomic Hecke algebras of level $|k|$ from [Ariki and Koike 1994], with $\uparrow$ and $\downarrow$ acting by induction and restriction functors if $k<0$, or vice versa if $k>0$; see Section 6. When $k=-1$, this specializes to the action of the deformed Heisenberg category on modules over the usual (finite) Hecke algebras associated to the symmetric groups constructed already in [Licata and Savage 2013]. The action of $\mathcal{H}eis_{-l}(z,t)$ on representations of cyclotomic Hecke algebras extends to an action on category $\mathcal{O}$ over the rational Cherednik algebras of type $\mathfrak{S}_n \wr \mathbb{Z}/l$ for all $n \geq 0$, with $\uparrow$ and $\downarrow$ acting by certain Bezrukavnikov–Etingof induction and restriction functors from; [Bezrukavnikov and Etingof 2009]; see Section 7.

We also prove a basis theorem for the morphism spaces in $\mathcal{H}eis_k(z,t)$; see Section 10 for the precise statement. In particular, our basis theorem implies that the *center* $\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})$ of the quantum Heisenberg category is the tensor product $\mathrm{Sym} \otimes \mathrm{Sym}$ of *two* copies of the algebra of symmetric functions. In the degenerate case studied in [Brundan 2018], the basis theorem was proved by treating the cases $k=0$ and $k \neq 0$ separately, appealing to results from [Brundan et al. 2017] and [Mackaay and Savage 2018]; the proofs in [loc. cit.] ultimately exploited analogs of the categorical actions mentioned above, on representations of degenerate cyclotomic Hecke algebras and representations of $\mathfrak{gl}_n(\mathbb{C})$, respectively. In the quantum case, it is still possible to prove the basis theorem when $k=0$ by such an argument, but for nonzero k the approach from [loc. cit.] seems to be unmanageable due to the larger center. Instead, we prove the basis theorem here by following the technique developed in the degenerate case in [Brundan et al. 2018, Theorem 6.4] (and earlier, in the context of Kac–Moody 2-categories, in [Webster 2018]). It depends crucially on the existence of an action of $\mathcal{H}eis_k(z,t)$ on a "sufficiently large" module category, which is obtained by choosing $l \gg 0$ then taking the tensor product of actions of $\mathcal{H}eis_{-l}(z,t)$ and $\mathcal{H}eis_{k+l}(z,1)$ on representations of suitably generic cyclotomic Hecke algebras of levels l and $k+l$, respectively.

The construction of this categorical tensor product involves a remarkable monoidal functor from $\mathcal{H}eis_k(z,t)$ to a certain localization of the symmetric product

$$\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$$

for $k=l+m$ and $t=uv$. This functor is defined in Section 8 and is the quantum analog of the categorical comultiplication from [Brundan et al. 2018, Theorem 5.4]. The particular tensor products exploited to prove the basis theorem are generic examples of *generalized cyclotomic quotients* of $\mathcal{H}eis_k(z,t)$; see Section 9 for the general definition. In fact, these $\Bbbk$-linear categories first appeared in [Webster 2015, Proposition 5.6], but in a rather different form; the precise relationship between the categories of [loc. cit.] and the ones here will be explained in [Brundan et al. 2019].

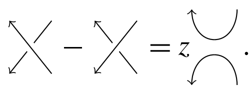
We have stopped short of proving any results about the *decategorification* of $\mathcal{H}eis_k(z,t)$ here, but let us make some remarks about this. There are two complementary points of view:

- One can consider the *Grothendieck ring* $K_0(\mathrm{Kar}(\mathcal{H}eis_k(z,t)))$ of the additive Karoubi envelope of $\mathcal{H}eis_k(z,t)$. For generic z (i.e., when q is not a root of unity), we expect that this is isomorphic to a $\mathbb{Z}$-form for a central reduction of the universal enveloping algebra of the infinite-dimensional Heisenberg Lie algebra, just as was established in the degenerate case in [Brundan et al. 2018, Theorem 1.1]. However, there is a significant obstruction to proving this result in the quantum case: we do not know how to show that the split Grothendieck group $K_0(AH_n)$ of the affine Hecke algebra is isomorphic to that of the finite Hecke algebra.
- Alternatively, one can pass to the *trace* (or zeroth Hochschild homology). In [Cautis et al. 2018], this was computed already for the category $\mathcal{H}(q^2)$ of [Licata and Savage 2013], revealing an interesting connection to the elliptic Hall algebra. Using the basis theorem proved here, we expect it should be possible to extend the calculations made in [loc. cit.] to give a description of the trace of the full category $\mathcal{H}eis_k(z,t)$ for all $k \in \mathbb{Z}$.

In the main body of the article, proofs of all lemmas involving purely diagrammatic manipulations have been omitted. However, we have attempted to give enough details for the reader familiar with the analogous calculations in the degenerate case from [Brundan 2018, Section 2] and [Brundan et al. 2018, Section 5] to be able to reconstruct the proofs. Brundan, Savage and Webster [≥ 2020] are currently preparing a sequel in which we incorporate a (symmetric) Frobenius algebra into the definition of $\mathcal{H}eis_k(z,t)$, in a similar way to the Frobenius Heisenberg categories defined in the degenerate case in [Savage 2019]. We will include full proofs of all of the diagrammatic lemmas in the more general Frobenius setting in this sequel.

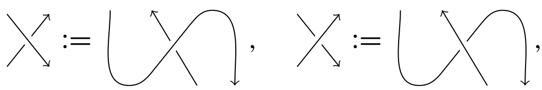
2. First approach

Before formulating our first definition of $\mathcal{H}eis_k(z,t)$, let us make some general remarks. We refer to the relation (1.7) as the *upward skein relation*. Rotating it through ±90° or 180°, one obtains three more skein relations; for example, here is the *leftward skein relation*



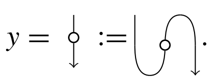
(2.1)

At present, this has no meaning since we have not defined the leftward cups, caps or crossings which it involves! However, already in the monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, we can introduce the rightward crossings



(2.2)

and then we see that the rightward skein relation holds from (1.7). Rotating the two rightward crossings once more by a similar procedure, we obtain positive and negative downward crossings satisfying the downward skein relation. We also define the downward dot



(2.3)

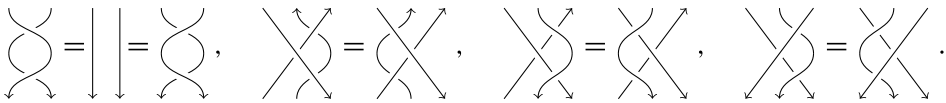
It is immediate from these definitions and (1.9) that dots and crossings slide past rightward cups and caps


(2.4)


(2.5)

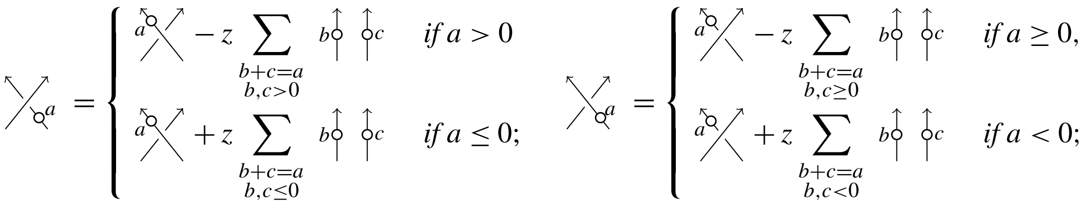

(2.6)

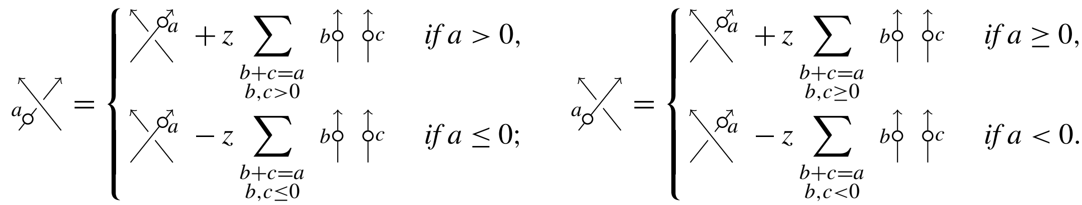
Also, the following relations are easily deduced by attaching rightward cups and caps to the relations in (1.8), then rotating the pictures using the definitions of the rightward/downward crossings:


(2.7)

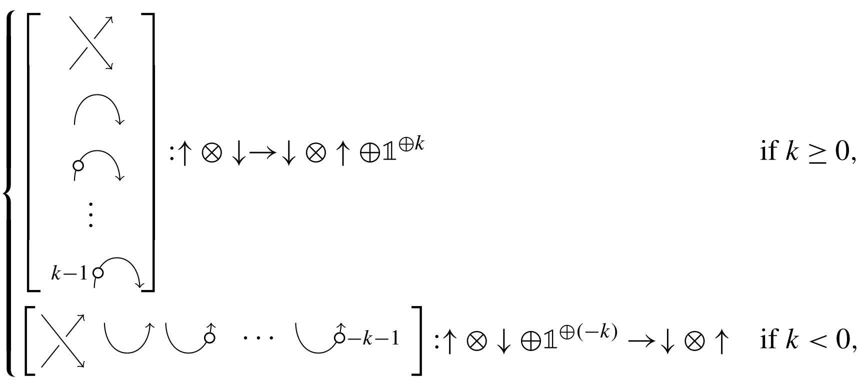
The following lemma will be used repeatedly (often without reference). There are analogous dot slide relations for the rightward and downward crossings (obtained by rotation).

Lemma 2.1. *The following relations hold for* $a \in \mathbb{Z}$:


(2.8)

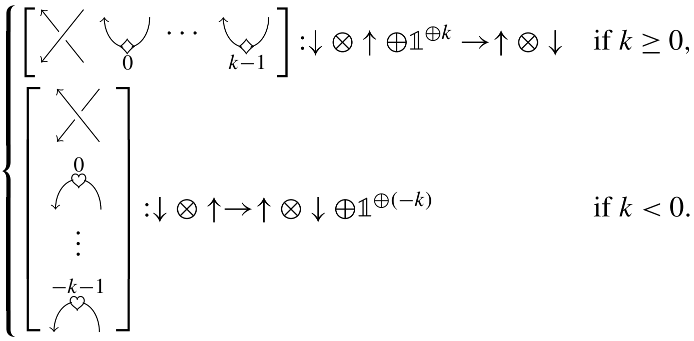

(2.9)

Now we can explain the first way to complete the definition of the quantum Heisenberg category following the scheme outlined in the introduction. The idea is to invert the morphism


(2.10)

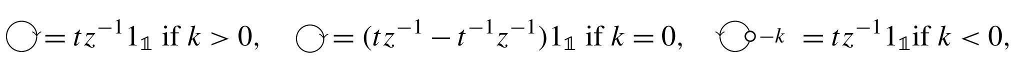
in $\mathrm{Add}(\mathcal{H}eis_k(z,t))$ (where Add denotes the additive envelope).

Definition 2.2. The *quantum Heisenberg category* $\mathcal{H}eis_k(z,t)$ is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, together with the matrix entries of the following morphism which we declare to be a two-sided inverse to the morphism (2.10):



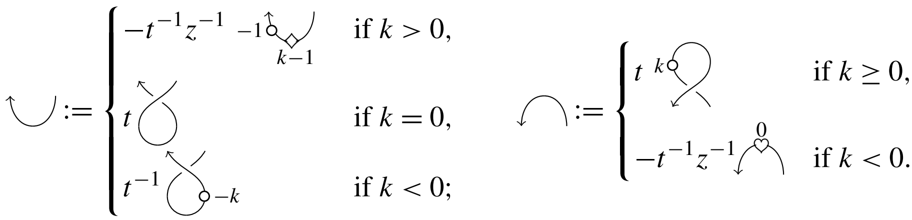
(2.11)

We impose one more essential relation:



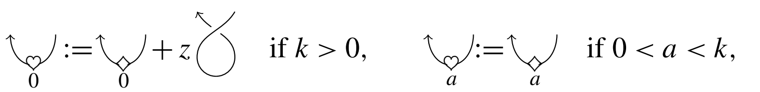
(2.12)

where the leftward cups and caps are defined by the formulas

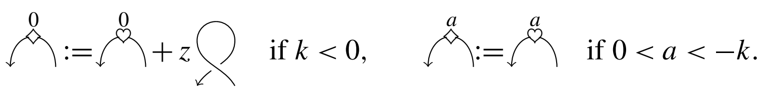


(2.13)

To complete the definition, we introduce a few more shorthands for morphisms. We have already introduced one of the two leftward crossings; define the other one so that the leftward skein relation (2.1) holds. Also set

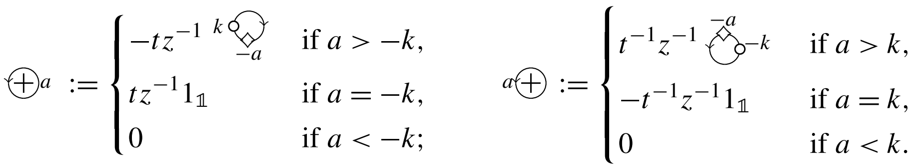


(2.14)



(2.15)

Next, introduce the following $(+)$-*bubbles* assuming $a\leq 0$:



(2.16)

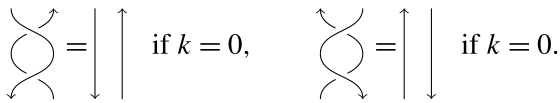
Finally, define the $(+)$-bubbles with label $a>0$ to be the usual bubbles with a dots:

(2.17)

Then define $(-)$-*bubbles* for all $a \in \mathbb{Z}$ by setting

(2.18)

In the case $k = 0$, the assertion that (2.10) and (2.11) are two-sided inverses means that



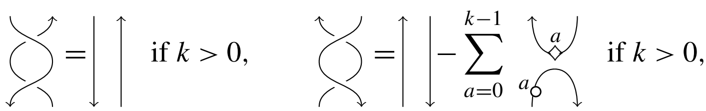
(2.19)

In fact, the defining relations for $\mathcal{H}eis_0(z, t)$ from Definition 2.2 are exactly the same as the ones for the affine HOMFLY-PT skein category $\mathcal{AOS}(z, t)$ from [Brundan 2017, Theorem 1.1 and Section 4]. Thus,

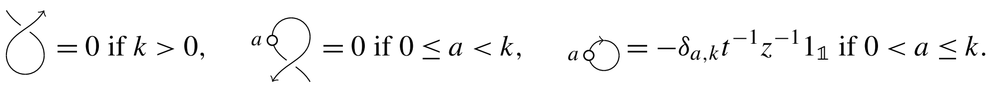
$$\mathcal{H}eis_0(z, t) = \mathcal{AOS}(z, t).$$

In this case, most of the other relations that we need have already been proved in [loc. cit.]. However, the arguments there exploit a theorem of Turaev [1989, Lemma I.3.3] to establish all of the relations that do not involve dots; the approach described below reproves all of these relations in a way that is independent of Turaev's work.

When $k > 0$, the assertion that the morphisms (2.10) and (2.11) are two-sided inverses implies the following relations:

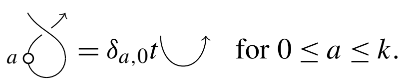


(2.20)



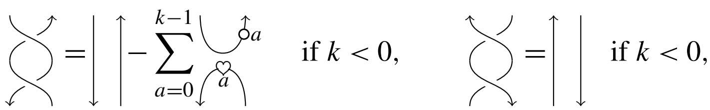
(2.21)

To derive these relations, we multiplied the matrices (2.10) and (2.11) in both orders, then equated the result with the appropriate identity matrix. The following useful relation is an easy exercise at this point; one needs to use (2.8), (2.12), (2.13) and (2.21):

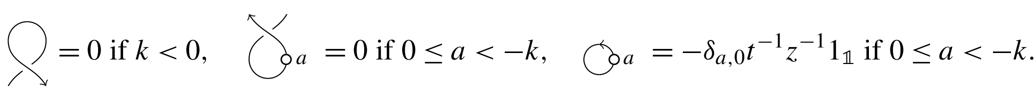


(2.22)

Finally, when $k < 0$, we will need the following relations which are deduced from (2.10) and (2.11) by the same argument as explained in the previous paragraph

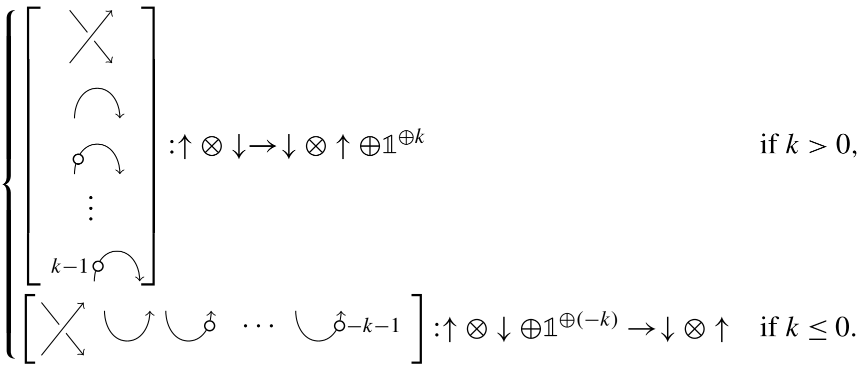


(2.23)



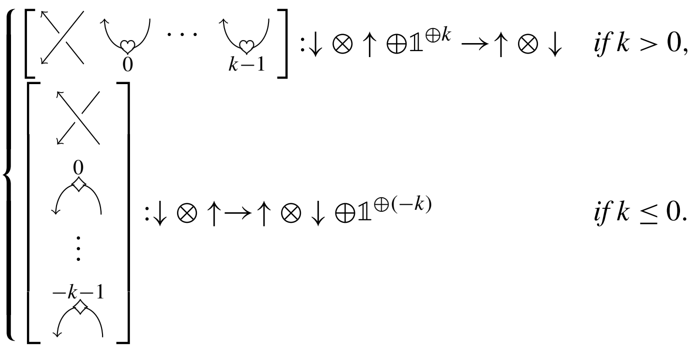
(2.24)

Now we are going to consider the counterpart of the morphism (2.10) defined using the negative instead of positive rightward crossing:



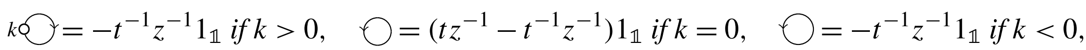
(2.25)

Lemma 2.3. *The morphism (2.25) is invertible with two-sided inverse*

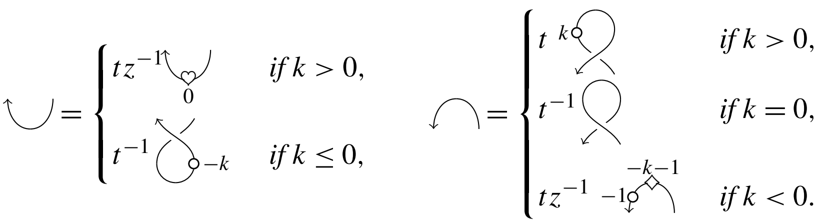


(2.26)

Moreover, we have that



(2.27)



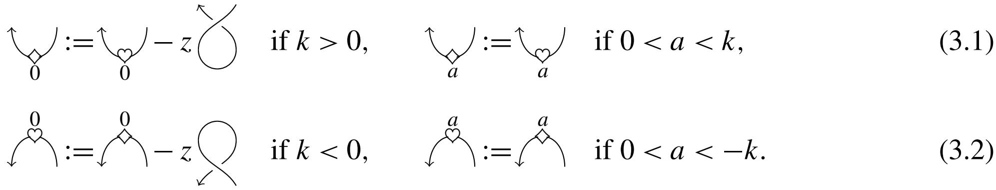
(2.28)

3. Second approach

Our second presentation for $\mathcal{H}eis_k(z,t)$ is very similar to the first presentation, but we invert the morphism (2.25) instead of (2.10).

Definition 3.1. The *quantum Heisenberg category* $\mathcal{H}eis_k(z,t)$ is the strict $\mathbb{k}$-linear monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, together with the matrix entries of the morphism (2.26), which we declare to be a two-sided inverse to (2.25). In addition, we impose the relation (2.27) for the leftward cups and caps which are defined in this approach from

(2.28). Define the other leftward crossing, i.e., the one which does not appear in (2.26), so the leftward skein relation (2.1) holds. Also set

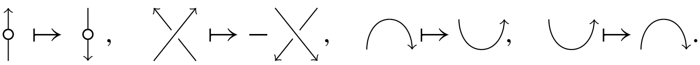


Finally define the (+)- and (−)-bubbles from (2.16)–(2.18) as before.

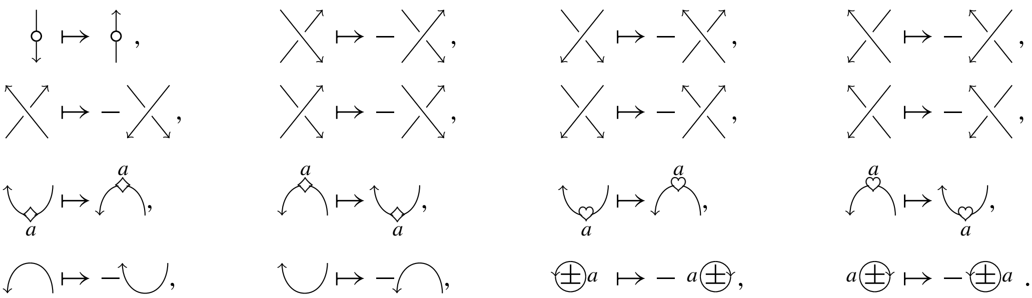
Theorem 3.2. *Definitions 2.2 and 3.1 give two different presentations for the same monoidal category, with all of the named morphisms introduced in the two definitions being the same. Moreover, there is a unique isomorphism of* $\mathbb{k}$*-linear monoidal categories*

$$\Omega_k : \mathcal{H}eis_k(z,t) \to \mathcal{H}eis_{-k}(z,t^{-1})^{\mathrm{op}} \tag{3.3}$$

sending



The effect of Ω_k *on the other morphisms is as follows*:



Proof. To avoid confusion, denote the category $\mathcal{H}eis_k(z,t)$ from Definition 2.2 by $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$ and the one from Definition 3.1 by $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$. The relations and other definitions for the category $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$ in Definition 3.1 and the ones for $\mathcal{H}eis_{-k}^{\mathrm{old}}(z,t^{-1})$ from Definition 2.2 are related by reflecting all diagrams in a horizontal plane and multiplying by $(-1)^{x+y}$, where x is the number of crossings and y is the number of leftward cups and caps (including leftward cups and caps in (+)- and (−)-bubbles but not ones labeled by $\diamond$ or $\heartsuit$). It follows that there are mutually inverse isomorphisms

$$\mathcal{H}eis_{-k}^{\mathrm{old}}(z,t^{-1}) \underset{\Omega_+}{\overset{\Omega_-}{\rightleftarrows}} \mathcal{H}eis_k^{\mathrm{new}}(z,t)^{\mathrm{op}}$$

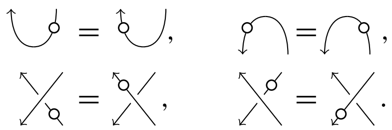
both defined in the same way as the functor Ω_k in the statement of the theorem. Now we apply Lemma 2.3 and Definition 3.1 to construct a strict $\mathbb{k}$-linear monoidal functor

$$\Theta_k : \mathcal{H}eis_k^{\mathrm{new}}(z,t) \to \mathcal{H}eis_k^{\mathrm{old}}(z,t)$$

which is the identity on diagrams. This functor is an isomorphism because it has a two-sided inverse, namely, $\Omega_+ \circ \Theta_{-k} \circ \Omega_-$. Thus, using Θ_k, we may identify $\mathcal{H}eis_k^{\text{new}}(z,t)$ and $\mathcal{H}eis_k^{\text{old}}(z,t)$. Finally, $\Omega_k := \Omega_+$ gives the required symmetry. $\square$

In the remainder of the section, we record some further consequences of the defining relations, thereby showing that $\mathcal{H}eis_k(z,t)$ is strictly pivotal. The first lemma explains how dots slide past leftward cups, caps and crossings. Its generalization to dots with arbitrary multiplicities $n \in \mathbb{Z}$ may also be deduced using induction and the leftward skein relation like in Lemma 2.1.

Lemma 3.3. *The following relations hold*:



(3.4)

(3.5)

Let Sym be the algebra of symmetric functions over $\Bbbk$. This is an infinite rank polynomial algebra with two sets of algebraically independent generators, namely, the *elementary symmetric functions* $e_1, e_2, \ldots$ and the *complete symmetric functions* $h_1, h_2, \ldots$. Adopting the convention that $e_n = h_n := \delta_{n,0}$ for $n \leq 0$, the elementary and complete symmetric functions are related by the following well-known identity [Macdonald 1995, (I.2.6)]:

$$\sum_{r+s=n} (-1)^s e_r h_s = \delta_{n,0}. \tag{3.6}$$

The following lemma, which we may refer to as the *infinite Grassmannian relation* (following Lauda), shows that there is a well-defined homomorphism

$$\beta : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}) \tag{3.7}$$

such that

$$h_n \otimes 1 \mapsto (-1)^{n-1} tz \; {}_{n+k}\!\circlearrowleft\!\oplus, \qquad 1 \otimes h_n \mapsto (-1)^n t^{-1} z \; {}_{-n}\!\circlearrowleft\!\ominus, \tag{3.8}$$

$$e_n \otimes 1 \mapsto t^{-1} z \; \oplus\!\circlearrowleft_{n-k}, \qquad 1 \otimes e_n \mapsto -tz \; \ominus\!\circlearrowleft_{-n}. \tag{3.9}$$

We will prove in Corollary 10.2 that β is actually an *isomorphism*.

Lemma 3.4. *For any $a \in \mathbb{Z}$, we have that*

$$\sum_{\substack{b,c \in \mathbb{Z} \\ b+c=a}} \oplus_b \; {}_c\oplus = \sum_{\substack{b,c \in \mathbb{Z} \\ b+c=a}} \ominus_b \; {}_c\ominus = -\delta_{a,0}\, z^{-2} 1_{\mathbb{1}}. \tag{3.10}$$

Moreover

$$\oplus_a = \delta_{a,-k} t z^{-1} 1_{\mathbb{1}} \quad \textit{if } a \leq -k, \qquad {}_a\oplus = -\delta_{a,k} t^{-1} z^{-1} 1_{\mathbb{1}} \quad \textit{if } a \leq k, \tag{3.11}$$

$${}_a\ominus = \delta_{a,0} t z^{-1} 1_{\mathbb{1}} \quad \textit{if } a \geq 0, \qquad \ominus_a = -\delta_{a,0} t^{-1} z^{-1} 1_{\mathbb{1}} \quad \textit{if } a \geq 0. \tag{3.12}$$

Corollary 3.5. *For an indeterminate w, we have that*

$$\oplus(w)\,\oplus(w) = \ominus(w)\,\ominus(w) = 1_{\mathbb{1}}, \tag{3.13}$$

where

$$\oplus(w) := t^{-1}z\sum_{n\in\mathbb{Z}} \oplus_n\, w^{-n} \in w^k 1_{\mathbb{1}} + w^{k-1}\,\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w^{-1}]\!], \tag{3.14}$$

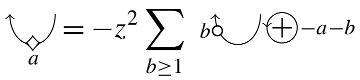
$$\oplus(w) := -tz\sum_{n\in\mathbb{Z}} {}_n\oplus\, w^{-n} \in w^{-k} 1_{\mathbb{1}} + w^{-k-1}\,\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w^{-1}]\!], \tag{3.15}$$

$$\ominus(w) := -tz\sum_{n\in\mathbb{Z}} \ominus_n\, w^{-n} \in 1_{\mathbb{1}} + w\,\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w]\!], \tag{3.16}$$

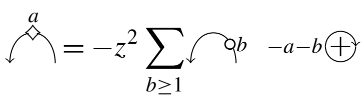
$$\ominus(w) := t^{-1}z\sum_{n\in\mathbb{Z}} {}_n\ominus\, w^{-n} \in 1_{\mathbb{1}} + w\,\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})[\![w]\!]. \tag{3.17}$$

Using the next relations plus (2.14) and (3.2), the leftward cups and caps decorated by $\diamond$ or $\heartsuit$ can be eliminated from any diagram.

Lemma 3.6. *The following relations hold*:



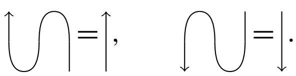
if $0 \le a < k$, (3.18)



if $0 \le a < -k$. (3.19)

The next lemma shows that $\downarrow$ is left dual to $\uparrow$ (as well as being right dual by the original construction). Thus, the monoidal category $\mathcal{H}eis_k(z,t)$ is rigid.

Lemma 3.7. *The following relations hold*:



(3.20)

The final lemma together with (3.4) implies that $\mathcal{H}eis_k(z,t)$ is strictly pivotal, with duality functor

$$* : \mathcal{H}eis_k(z,t) \xrightarrow{\sim} (\mathcal{H}eis_k(z,t)^{\mathrm{op}})^{\mathrm{rev}} \tag{3.21}$$

defined on morphisms by rotating diagrams through 180°.

Lemma 3.8. *The following relations hold*:

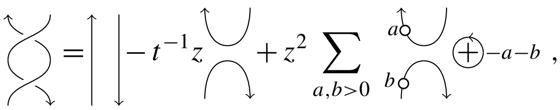
(3.22)

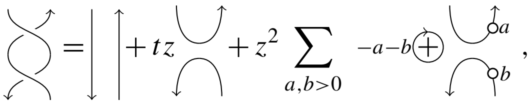
(3.23)

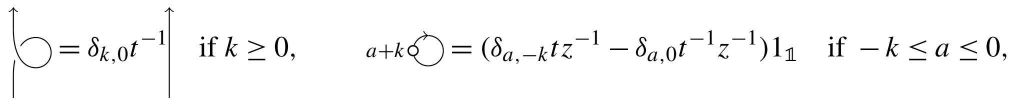
4. Third approach

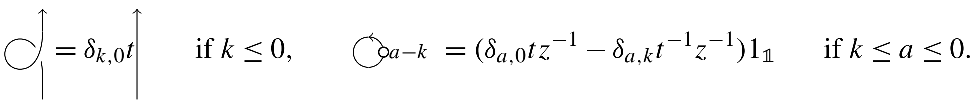
Now we have enough relations in hand to formulate our third presentation for $\mathcal{H}eis_k(z,t)$. This presentation does not involve any leftward cups or caps decorated by $\diamond$ or $\heartsuit$; Lemma 3.6 showed already that these are redundant as generators.

Definition 4.1. The *quantum Heisenberg category* $\mathcal{H}eis_k(z,t)$ is the strict $\Bbbk$-linear monoidal category obtained from $\mathcal{AH}(z)$ by adjoining a right dual $\downarrow$ to $\uparrow$ as explained in the introduction, plus two more generating morphisms and subject to the following additional relations:

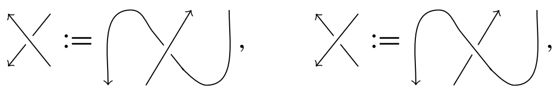

(4.1)


(4.2)

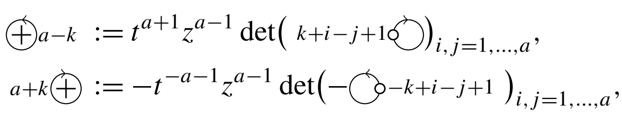

(4.3)


(4.4)

Here, we have used the leftward crossings which are defined in this approach by

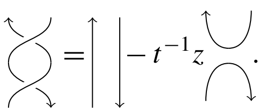

(4.5)

and the (+)-bubbles which are defined for $a \le k$ or $a \le -k$, respectively, by

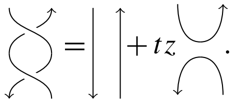

(4.6)
(4.7)

interpreting the determinants as $\delta_{a,0}$ in case $a \le 0$. Finally, define the (+)-bubbles with label $a > 0$ to be the usual bubbles with a dots as in (2.17), then define the (−)-bubbles for all $a \in \mathbb{Z}$ so that (2.18) holds.

Before proving the equivalence of this definition with the earlier ones, we make some remarks about the relations (4.1)–(4.7). If $k \le 1$, the relation (4.1) is equivalent to

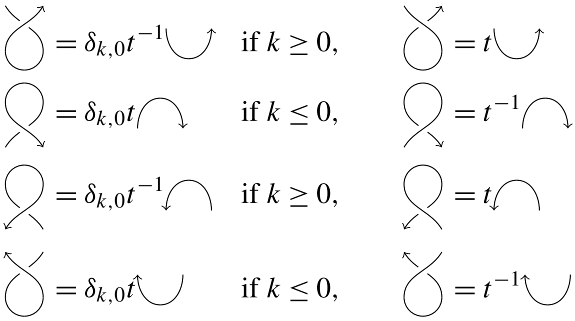

(4.8)

This follows immediately from the definition of the (+)-bubbles from (4.6). Similarly, when $k \geq -1$, the relation (4.2) is equivalent to



(4.9)

Here are some other useful consequences of these relations:



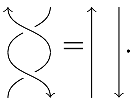
(4.10) if $k \geq 0$, if $k \geq 0$,

(4.11) if $k \leq 0$, if $k \leq 0$,

(4.12) if $k \geq 0$, if $k \geq 0$,

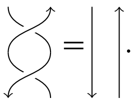
(4.13) if $k \leq 0$, if $k \leq 0$.

These follow from (4.3)–(4.4) on expanding the definitions of the sideways crossings. Then, using (4.13) and the leftward skein relation to convert the negative crossings in (4.8) to positive ones, relation (4.8) can be further simplified in case that $k < 0$: it is equivalent to



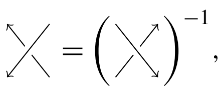
(4.14)

Similarly, (4.9) is equivalent to the following when $k > 0$:



(4.15)

Finally, when $k = 0$, the relations (4.8)–(4.9) together are equivalent to the single assertion



(4.16)

i.e., both of the relations from (2.19).

Theorem 4.2. *The category $\mathcal{H}eis_k(z,t)$ defined by Definition 4.1 is the same as the one from Definitions 2.2 and 3.1, with all morphisms introduced in the third definition being the same as the ones from before.*

Proof. To avoid confusion in the proof, we denote the category from the equivalent Definitions 2.2 and 3.1 by $\mathcal{H}eis_k^{\text{old}}(z,t)$, and the one from Definition 4.1 by $\mathcal{H}eis_k^{\text{new}}(z,t)$. From the evident symmetry in the relations (4.1)–(4.7), it follows that there is an isomorphism

$$\Omega_k : \mathcal{H}eis_k^{\text{new}}(z,t) \to \mathcal{H}eis_{-k}^{\text{new}}(z,t^{-1})^{\text{op}}$$

which reflects diagrams in a horizontal plane and multiplies by $(-1)^{x+y}$ where x is the number of crossings and y is the number of leftward cups and caps. Combining this with (3.3), we are reduced to proving the theorem under the assumption that $k \leq 0$.

We first check that all of the defining relations (4.1)–(4.7) of $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$ are satisfied in $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$, so that there is a strict $\Bbbk$-linear monoidal functor

$$\Theta : \mathcal{H}eis_k^{\mathrm{new}}(z,t) \to \mathcal{H}eis_k^{\mathrm{old}}(z,t)$$

which is the identity on diagrams. For this, note to start with that (4.5) holds in $\mathcal{H}eis_k^{\mathrm{old}}(z,t)$ as we have shown that the latter category is strictly pivotal. The relation (4.6) is almost trivial when $k \leq 0$ and holds thanks to (3.11). For (4.7), the identity holds if $a-k \leq 0$ due again to (3.11), so assume that $a-k > 0$. Then the desired identity is the image under the homomorphism β from (3.7) of the identity

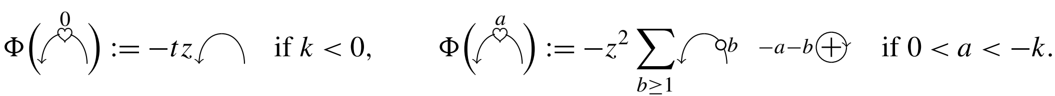
$$(-1)^{a-k-1}t^{-1}z^{-1}h_{a-k} \otimes 1 = -z^{a-k-1}t^{-a+k-1}\det(-tz^{-1}e_{i-j+1} \otimes 1)_{i,j=1,\dots,a-k}$$

in $\mathrm{Sym} \otimes \mathrm{Sym}$. This follows from the well-known identity $h_n = \det(e_{i-j+1})_{i,j=1,\dots,n}$; see [Macdonald 1995, Exercise I.2.8]. It remains to check the relations (4.1)–(4.4). For (4.1)–(4.2) when $k=0$, we just need to check the equivalent form (4.16), which follows by (2.19). For (4.1) when $k<0$, we check the equivalent form (4.14), which holds due to the second relation from (2.23). For (4.2) when $k<0$, we use the first relation from (2.23), expanding the leftward caps decorated by $\heartsuit$ using (2.13) when $a=0$ or (2.15) and (3.19) when $a>0$. Finally, the relations (4.3)–(4.4) follow easily from (2.24), (2.12)–(2.13) and (2.27)–(2.28).

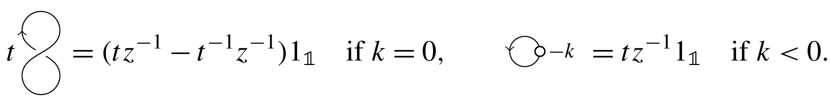
Now we want to show that Θ is an isomorphism. We do this by using the presentation from Definition 2.2 to construct a two-sided inverse

$$\Phi : \mathcal{H}eis_k^{\mathrm{old}}(z,t) \to \mathcal{H}eis_k^{\mathrm{new}}(z,t),$$

still assuming that $k \leq 0$. We define Φ on morphisms by declaring that it takes the rightward cup, the rightward cap, and all dots and crossings (with any orientation) to the corresponding morphisms in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$, and also

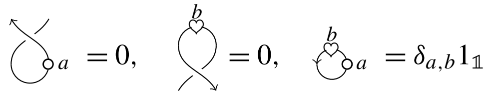


To see that Φ is well defined, we must verify the relations from Definition 2.2. For (2.12), we must check the following in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$:

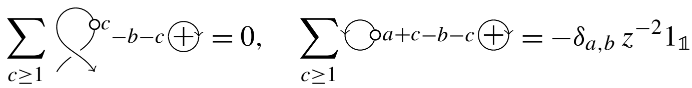


These follow from (4.4) and (4.12). Then the main work is to show that the images under Φ of the morphisms (2.10) and (2.11) are two-sided inverses in $\mathcal{H}eis_k^{\mathrm{new}}(z,t)$. When $k=0$, this is immediate from

(4.16), so suppose that $k < 0$. The images under Φ of the two equations in (2.23) are precisely the known relations (4.2) and (4.14). We are left with checking that the images under Φ of the relations

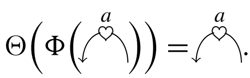


hold in $\mathcal{H}eis_k^{\text{new}}(z,t)$ for $0 \le a, b < -k$. The first of these when $a = 0$ follows by (4.13). To see it for $0 < a < -k$, we first apply the leftward skein relation, then slide the dots past the crossing using the leftward analog of (2.9) which may be deduced from the definition (4.5), and finally appeal to (4.4). The second and third relations follow from (4.11) and (4.4) in the case that $b = 0$. To prove them when $0 < b < -k$, we must show that

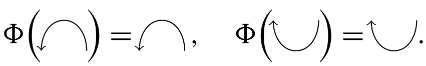


in $\mathcal{H}eis_k^{\text{new}}(z,t)$. For the first identity, it is zero if $b \ge -k$ as the $(+)$-bubble vanishes by (1.3). To see it for $0 < b < -k$, use the skein relation, commute the dots past the crossing, then appeal to (4.4) and (4.11). For the second identity, define a homomorphism $\gamma : \text{Sym} \to \text{End}_{\mathcal{H}eis_k^{\text{new}}(z,t)}(\mathbb{1})$ by sending $e_n \mapsto t^{-1}z$ $n-k$ for $n \ge 0$. Using $h_n = \det(e_{i-j+1})_{i,j=1,\dots,n}$ and (4.7), it follows that γ sends $h_n \mapsto (-1)^{n-1}tz$ $n+k$ for $n \le -k$. Then the identity we are trying to prove follows by applying γ to the identity $\sum_{c\ge1}(-1)^{-k-b-c}e_{k+a+c}h_{-k-b-c} = \delta_{a,b}$, which is (3.6).

To complete the proof, we must show that Θ and Φ are indeed two-sided inverses. To check that $\Theta \circ \Phi = \text{Id}$, the only difficulty is to see that



When $a = 0$, this is immediate from (2.13), while if $0 < a < -k$ it follows from (2.15) and (3.19). To check that $\Phi \circ \Theta = \text{Id}$, the only difficulty is to see that



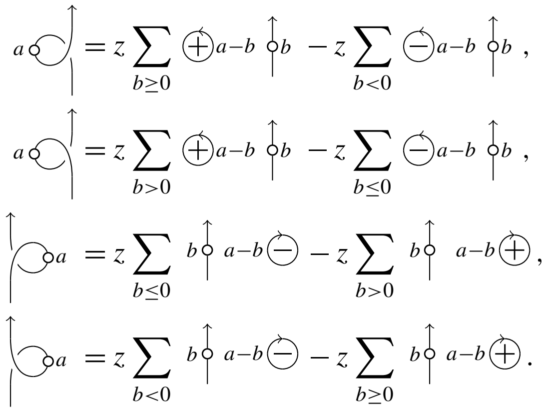
These follow from (2.13) and (4.12)–(4.13). □

Lemma 4.3. *Suppose that $\mathcal{C}$ is a strict $\Bbbk$-linear monoidal category containing objects $\uparrow$ and $\downarrow$ and morphisms , , , and satisfying (1.6)–(1.9). Then $\mathcal{C}$ contains at most one pair of morphisms and which satisfy (4.1)–(4.4) (for the sideways crossings and the $(+)$-bubbles defined via (2.2) and (4.5)–(4.7)).*

Proof. If $k \le 0$, Theorem 4.2 implies that the morphism (2.10) is invertible in $\mathcal{C}$, and is the $(1,1)$-entry of the inverse matrix. This property characterizes uniquely as a morphism in $\mathcal{C}$ when $k \le 0$, independent of the choices of or . Similarly, when $k \ge 0$, the morphism (2.25) is invertible in $\mathcal{C}$, and is the $(1,1)$-entry of the inverse matrix. Thus is characterized uniquely when $k \le 0$. To complete the proof when $k = 0$, it remains to use (4.12)–(4.13), since these show how to express and in terms of and and the two leftward crossings. To complete the proof when $k < 0$, we note instead that the $(2, 1)$-entry of the inverse of (2.10) is $-tz$, so is uniquely determined in $\mathcal{C}$. Then may be recovered uniquely using the relation (2.13) and our knowledge of . Finally when $k > 0$, the $(1, 2)$-entry of the inverse of (2.25) gives $t^{-1}z$ and then may be recovered using (2.28) and our knowledge of . □

To conclude the section, we formulate three more important sets of relations. The first of these explains how to expand *curls*. It is quite surprising that we have never needed to simplify left curls when $k > 0$ (or right curls when $k < 0$) before this point.

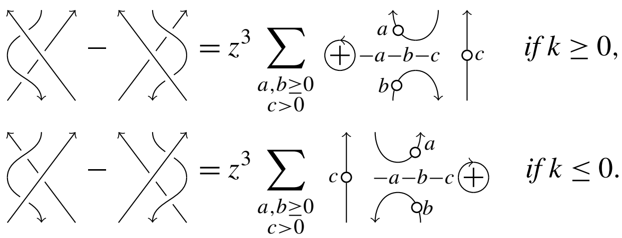
Lemma 4.4. *The following relations hold for any $a \in \mathbb{Z}$:*



(4.17)
(4.18)
(4.19)
(4.20)

The following lemma gives a braid relation for *alternating crossings*. All other variations on the braid relation can be deduced from this plus the original braid relation from (1.8), by arguments similar to the proof of the braid relations in (2.7).

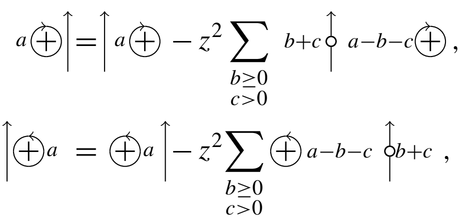
Lemma 4.5. *The following relation holds*:



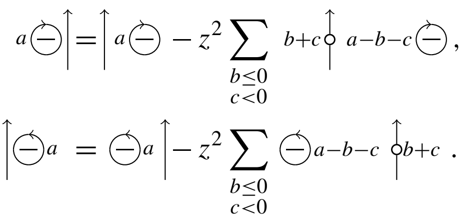
(4.21)
(4.22)

Finally we have the *bubble slides*.

Lemma 4.6. *The following relations hold for any $a \in \mathbb{Z}$:*



(4.23)
(4.24)



(4.25)

(4.26)

5. Action on representations of quantum GL_n

In this section, we construct an action of $\mathcal{H}eis_0(z,t)$ on the category of modules over $U_q(\mathfrak{gl}_n)$ and use this action to produce a family of generators for the center of $U_q(\mathfrak{gl}_n)$. These central elements were introduced originally by Bracken, Gould and Zhang [Gould et al. 1991]. We also determine their images under the Harish-Chandra homomorphism, giving a new approach to some results of Li [2010]. Throughout the section, we work in the generic case, setting

$$\mathbb{k} := \mathbb{Q}(q), \quad z := q - q^{-1}, \quad t := q^n$$

for an indeterminate q. In fact, the formulae which we derive are defined over $\mathbb{Z}[q, q^{-1}]$, hence, they make sense over any ground ring for any invertible q (including roots of unity).

For the precise definition of $U_q(\mathfrak{gl}_n)$, we follow the conventions of [Brundan 2017, Section 3], denoting its standard generators by $\{e_i, f_i, d_j^{\pm 1} \mid i = 1, \dots, n-1,\ j = 1, \dots, n\}$. The usual diagonal generator k_i of the subalgebra $U_q(\mathfrak{sl}_n)$ is $d_i d_{i+1}^{-1}$. The subalgebras of $U_q(\mathfrak{gl}_n)$ generated by the e_i, f_i and $d_j^{\pm}$ are $U_q(\mathfrak{gl}_n)^+$, $U_q(\mathfrak{gl}_n)^-$ and $U_q(\mathfrak{gl}_n)^0$, respectively. We also have the Borel subalgebras $U_q(\mathfrak{gl}_n)^\sharp := U_q(\mathfrak{gl}_n)^0 U_q(\mathfrak{gl}_n)^+$ and $U_q(\mathfrak{gl}_n)^\flat := U_q(\mathfrak{gl}_n)^0 U_q(\mathfrak{gl}_n)^-$. We will often cite Lusztig's book [1993], noting that our q and k_i are Lusztig's v^{-1} and K_i^{-1}.

The natural module V^+ and dual natural module V^- are the left $U_q(\mathfrak{gl}_n)$-modules with bases

$$\{v_i^+ \mid 1 \le i \le n\} \quad \text{and} \quad \{v_i^- \mid 1 \le i \le n\},$$

respectively, on which the generators act by

$$f_i v_j^+ = \delta_{i,j} v_{i+1}^+, \quad e_i v_j^+ = \delta_{i+1,j} v_i^+, \quad d_i v_j^+ = q^{\delta_{i,j}} v_j^+, \tag{5.1}$$

$$f_i v_j^- = \delta_{i+1,j} v_i^-, \quad e_i v_j^- = \delta_{i,j} v_{i+1}^-, \quad d_i v_j^- = q^{-\delta_{i,j}} v_j^-. \tag{5.2}$$

We denote the weight of v_i^+ by ε_i; then v_i^- is of weight $-\varepsilon_i$. Let $\Lambda := \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i$ be the *weight lattice* with inner product $(\cdot,\cdot)$ defined so that $\varepsilon_1, \dots, \varepsilon_n$ are orthonormal. The *positive roots* are $\{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$. By a *weight module* we mean a $U_q(\mathfrak{gl}_n)$-module V that is the sum of its weight spaces $V_\lambda := \{v \in V \mid d_i v = q^{(\lambda,\varepsilon_i)} v\}$ for all $\lambda \in \Lambda$. The *Weyl group* is the symmetric group $\mathfrak{S}_n$. It acts in obvious ways on Λ and on $U_q(\mathfrak{gl}_n)^0 = \mathbb{k}[d_1^{\pm 1}, \dots, d_n^{\pm 1}]$, permuting the generators. Denote the longest element of $\mathfrak{S}_n$ by w_0.

We work with the Hopf algebra structure on $U_q(\mathfrak{gl}_n)$ whose comultiplication Δ satisfies

$$\Delta(e_i) = d_i^{-1} d_{i+1} \otimes e_i + e_i \otimes 1, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes d_i d_{i+1}^{-1}, \quad \Delta(d_j) = d_j \otimes d_j. \tag{5.3}$$

We also need various (anti)automorphisms. First, we have the *bar involution*, which is the antilinear automorphism $- : U_q \to U_q$ defined from $\overline{e_i} := e_i, \overline{f_i} := f_i$ and $\overline{d_i} := d_i^{-1}$. Then there are linear antiautomorphisms T and G defined from

$$T(e_i) := f_i, \qquad T(f_i) := e_i, \qquad T(d_i) := d_i, \tag{5.4}$$

$$G(e_i) := e_{n-i}, \quad G(f_i) := f_{n-i}, \quad G(d_i) := d_{n+1-i}. \tag{5.5}$$

The maps $-$, T and G commute with each other. Finally, we have Lusztig's braid group action, under which the i-th generator of the braid group acts by the automorphism $T_i : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ (which is $T''_{i,-}$ from [Lusztig 1993, Section 37.1.3]) defined for $|j-i| > 1$ and $k \neq i, i+1$ by

$$\begin{aligned} &T_i(e_i) = -f_i d_i d_{i+1}^{-1}, && T_i(e_{i\pm1}) = e_i e_{i\pm1} - q^{-1} e_{i\pm1} e_i, && T_i(e_j) = e_j,\\ &T_i(f_i) = -d_i^{-1} d_{i+1} e_i, && T_i(f_{i\pm1}) = f_{i\pm1} f_i - q f_i f_{i\pm1}, && T_i(f_j) = f_j,\\ &T_i(d_i) = d_{i+1}, && T_i(d_{i+1}) = d_i, && T_i(d_k) = d_k. \end{aligned}$$

A key role is played by the R-matrix. We recall its definition following the approach from [Lusztig 1993, Section 32.1]. Let Θ be the *quasi-R-matrix* from [loc. cit., Section 4.1]. This is an infinite sum of components $\Theta_\alpha \in U_q(\mathfrak{gl}_n)^-_{-\alpha} \otimes U_q(\mathfrak{gl}_n)^+_\alpha$ as α runs over the positive root lattice $\bigoplus_{i=1}^{n-1} \mathbb{N}(\varepsilon_i - \varepsilon_{i+1})$. Let $P : V \otimes W \to W \otimes V$ be the tensor flip. Assuming in addition that V and W are weight modules, let $\Pi : V \otimes W \to V \otimes W$ be the diagonal map defined from

$$\Pi(v \otimes w) := q^{(\lambda,\mu)} v \otimes w$$

for v of weight λ and w of weight μ. Then, for finite-dimensional weight modules V and W, the *R-matrix*

$$R_{V,W} : V \otimes W \xrightarrow{\sim} W \otimes V \tag{5.6}$$

is the $U_q(\mathfrak{gl}_n)$-module isomorphism defined by the composition $\Theta \circ P \circ \Pi$, which makes sense since all but finitely many of the components Θ_α act as zero. The inverse $R_{V,W}^{-1} : W \otimes V \to V \otimes W$ is the map $\Pi^{-1} \circ P^{-1} \circ \overline{\Theta}$, where $\overline{\Theta}$ is obtained from Θ by applying the bar involution to each tensor factor. For finite-dimensional weight modules U, V and W, we have the *hexagon property*

$$R_{U,W} \otimes \mathrm{id}_V \circ \mathrm{id}_U \otimes R_{V,W} = R_{U \otimes V,W}, \quad \mathrm{id}_V \otimes R_{U,W} \circ R_{U,V} \otimes \mathrm{id}_W = R_{U,V \otimes W}. \tag{5.7}$$

This is proved in [Lusztig 1993, Proposition 32.2.2] (our $R_{V,W}$ is Lusztig's ${}_f\mathcal{R}_{W,V}$ taking the function f from [loc. cit., Section 31.1.3] to be $f(\lambda,\mu) := -(\lambda,\mu)$).

In fact, to define the isomorphism $R_{V,W}$, one only needs *one* of the modules V or W to be a finite-dimensional weight module; the other can be an arbitrary $U_q(\mathfrak{gl}_n)$-module. To see this, one just needs to observe that Π extends to a linear map $V \otimes W \to V \otimes W$ when just one of V or W is a weight module on setting

$$\Pi(v \otimes w) := \begin{cases} (d_\lambda \otimes 1)(v \otimes w) & \text{if } w \text{ is a weight vector of weight } \lambda,\\ (1 \otimes d_\lambda)(v \otimes w) & \text{if } v \text{ is a weight vector of weight } \lambda, \end{cases}$$

where $d_\lambda := d_1^{(\lambda,\varepsilon_1)} \cdots d_n^{(\lambda,\varepsilon_n)}$. Then the same formula $R_{V,W} := \Theta \circ P \circ \Pi$ makes sense when only one of V or W is a finite-dimensional weight module, and it still gives an isomorphism of $U_q(\mathfrak{gl}_n)$-modules. Moreover, the hexagon property (5.7) remains true if only two of U, V and W are finite-dimensional weight modules. These assertions follow from the known results in the previous paragraph. For example, to prove that $R_{V,W}$ is an isomorphism assuming that W is a finite-dimensional weight module, let $\rho_W : U_q(\mathfrak{gl}_n) \to \mathrm{End}_\Bbbk(W)$ be the corresponding representation. Then

$$(\rho_W \otimes 1)(\Theta) \in \mathrm{End}_\Bbbk(W) \otimes U_q(\mathfrak{gl}_n) \quad \text{and} \quad (\rho_W \otimes 1)(\overline{\Theta}) \in \mathrm{End}_\Bbbk(W) \otimes U_q(\mathfrak{gl}_n).$$

It suffices to show that these are inverse to each other, since then $R_{V,W} = (\rho_W \otimes 1)(\Theta) \circ P \circ \Pi$ has inverse $\Pi^{-1} \circ P^{-1} \circ (\rho_W \otimes 1)(\overline{\Theta})$ for any module V. We have that

$$(\rho_W \otimes 1)(\Theta) \circ (\rho_W \otimes 1)(\overline{\Theta}) \in \mathrm{End}_\Bbbk(W) \otimes U_q(\mathfrak{gl}_n)$$

and, for any finite-dimensional weight module V with corresponding representation ρ_V, we have

$$(1 \otimes \rho_V)((\rho_W \otimes 1)(\Theta) \circ (\rho_W \otimes 1)(\overline{\Theta})) = 1$$

by the known result. Since the intersection of the annihilators of all finite-dimensional weight modules is zero, this implies that $(\rho_W \otimes 1)(\Theta) \circ (\rho_W \otimes 1)(\overline{\Theta}) = 1$. The proof that $(\rho_W \otimes 1)(\overline{\Theta}) \circ (\rho_W \otimes 1)(\Theta) = 1$ is analogous, as is the proof of the hexagon property when just two of the modules are finite-dimensional weight modules.

The goal now is to derive explicit formulae for $R_{V^\pm,M}$ and $R_{M,V^\pm}$ for any module M. Similar formulae were established already in [Gould et al. 1991, Section III] following the older conventions of Drinfeld and Jimbo. They involve the *higher root elements* defined as follows. Let

$$e_{i,i} = f_{i,i} := z^{-1}, \quad e_{i,i+1} := e_i, \quad f_{i,i+1} := f_i. \tag{5.8}$$

Then when $j - i > 1$ we recursively define

$$e_{i,j} := e_{i,r} e_{r,j} - q^{-1} e_{r,j} e_{i,r}, \quad f_{i,j} := f_{r,j} f_{i,r} - q^{-1} f_{i,r} f_{r,j}, \tag{5.9}$$

where r is any index chosen so that $i < r < j$. It is an induction exercise to see that these elements are well defined independent of the choice of r; see the proof of the following lemma for a more conceptual explanation of this. Alternatively, $e_{i,j}$ and $f_{i,j}$ can be defined using the braid group action: we have that

$$e_{i,j} = T_{j-1} \cdots T_{i+1}(e_i), \quad f_{i,j} = \overline{T_{j-1} \cdots T_{i+1}(f_i)}.$$

Note that

$$T(e_{i,j}) = f_{i,j}, \qquad T(f_{i,j}) = e_{i,j}, \tag{5.10}$$

$$G(e_{i,j}) = e_{n+1-j,n+1-i}, \quad G(f_{i,j}) = f_{n+1-j,n+1-i}. \tag{5.11}$$

However, the bar involution does not fix $e_{i,j}$ or $f_{i,j}$ (except when $j = i + 1$).

Lemma 5.1. *For any $i < j$, the $(\varepsilon_i - \varepsilon_j)$-component $\Theta_{i,j}$ of the quasi-R-matrix Θ satisfies*

$$\Theta_{i,j} = \sum_{\substack{r\geq 1 \\ i=i_0<i_1<\cdots<i_r=j}} z^r f_{i_{r-1},i_r}\cdots f_{i_0,i_1} \otimes \overline{e_{i_{r-1},i_r}\cdots e_{i_0,i_1}} = \sum_{\substack{r\geq 1 \\ i=i_0<i_1<\cdots<i_r=j}} z^r \overline{f_{i_0,i_1}\cdots f_{i_{r-1},i_r}} \otimes e_{i_0,i_1}\cdots e_{i_{r-1},i_r}.$$

Proof. It suffices to derive the first expression. Then the second follows using (5.10) and the identity $(T\otimes T)(\Theta_\alpha) = P(\Theta_\alpha)$, which may easily be deduced from the characterization in [Lusztig 1993, Theorem 4.1.2(a)]. To prove the first expression, we appeal to further results of Lusztig from [loc. cit.]. Let $\boldsymbol{f}$ be Lusztig's "half" quantum group with its standard generators $\theta_1,\ldots,\theta_{n-1}$; see also [Brundan et al. 2014, Section 2.1] which follows the same conventions as here. There are two isomorphisms

$$(-)^+ : \boldsymbol{f} \xrightarrow{\sim} U_q(\mathfrak{gl}_n)^+, \quad \theta_i^+ := e_i, \qquad (-)^- : \boldsymbol{f} \xrightarrow{\sim} U_q(\mathfrak{gl}_n)^-, \quad \theta_i^- := f_i.$$

Consider the convex ordering on the positive roots defined so that $\varepsilon_i - \varepsilon_j < \varepsilon_p - \varepsilon_q$ if either $i < p$ or ($i = p$ and $j < q$); this is the "standard order" as in [loc. cit., Example A.1]. Let $\theta_{i,j}$ be Lusztig's higher root element associated to this ordering, which was denoted $r_{\varepsilon_i-\varepsilon_j}$ in [loc. cit., Section 2.4]. Noting that $(\varepsilon_m - \varepsilon_j, \varepsilon_i - \varepsilon_m)$ is a minimal pair for $\varepsilon_i - \varepsilon_j$, [loc. cit., Theorem 4.2] implies that these satisfy the following recursion: $\theta_{i,i+1} = \theta_i$ and $\theta_{i,j} = \theta_{i,r}\theta_{r,j} - q\theta_{r,j}\theta_{i,r}$ for any $i < r < j$. Comparing with (5.9), it follows that $\theta_{i,j}^+ = \overline{e_{i,j}}$ and $\theta_{i,j}^- = (-q)^{j-i-1} f_{i,j}$; in particular, these equalities justify the independence of r in (5.9). Then we appeal to [loc. cit., Theorem 2.7] (which was extracted from [Lusztig 1993]) to see that $\{\theta_{i_{r-1},i_r}\cdots\theta_{i_0,i_1} \mid r\geq 1, i = i_0 < \cdots < i_r = j\}$ and $\{(1-q^2)^r\theta_{i_{r-1},i_r}\cdots\theta_{i_0,i_1} \mid r \geq 1, i = i_0 < \cdots < i_r = j\}$ are a pair of dual bases for $\boldsymbol{f}_{\varepsilon_i-\varepsilon_j}$ with respect to Lusztig's form. Finally the formula from [Lusztig 1993, Theorem 4.1.2(b)] gives that

$$\Theta_{i,j} = \sum_{\substack{r\geq 1 \\ i=i_0<\cdots<i_r=j}} (-q)^{i-j}(1-q^2)^r \theta^-_{i_{r-1},i_r}\cdots\theta^-_{i_0,i_1} \otimes \theta^+_{i_{r-1},i_r}\cdots\theta^+_{i_0,i_1}.$$

This simplifies to the desired formula. $\square$

For $1 \leq i, j \leq n$ let $e^+_{i,j} \in \mathrm{End}_{\Bbbk}(V^+)$ (resp. $e^-_{i,j} \in \mathrm{End}_{\Bbbk}(V^-)$) be the ij-matrix unit with respect to the basis $v_1^+,\ldots,v_n^+$ (resp. $v_1^-,\ldots,v_n^-$). Then for $i < j$ and $v^\pm \in V^\pm$ we have that

$$e_{i,j}v^+ = e^+_{i,j}v^+, \quad f_{i,j}v^+ = e^+_{j,i}v^+, \quad e_{i,j}v^- = (-q)^{i-j+1}e^-_{j,i}v^-, \quad f_{i,j}v^- = (-q)^{i-j+1}e^-_{i,j}v^-, \tag{5.12}$$

$$\overline{e_{i,j}}v^+ = e^+_{i,j}v^+, \quad \overline{f_{i,j}}v^+ = e^+_{j,i}v^+, \quad \overline{e_{i,j}}v^- = (-q)^{j-i-1}e^-_{j,i}v^-, \quad \overline{f_{i,j}}v^- = (-q)^{j-i-1}e^-_{i,j}v^-. \tag{5.13}$$

These follow easily by induction on $j - i$ using (5.1)–(5.2) and (5.9). Also let

$$x_{i,j} := z^2 \sum_{r=1}^{\min(i,j)} e_{r,i}d_r f_{r,j}d_j, \qquad y_{i,j} := z^2 \sum_{r=\max(i,j)}^{n} d_i f_{i,r} d_r e_{j,r} \tag{5.14}$$

for any $1 \leq i, j \leq n$. Then for $m \geq 0$ we set

$$x^{(m)}_{i,j} := \sum_{i=i_0,i_1,\ldots,i_{m-1},i_m=j} x_{i_0,i_1}\cdots x_{i_{m-1},i_m}, \qquad y^{(m)}_{i,j} := \sum_{i=i_0,i_1,\ldots,i_{m-1},i_m=j} y_{i_0,i_1}\cdots y_{i_{m-1},i_m}. \tag{5.15}$$

In particular, $x_{i,j}^{(0)} = y_{i,j}^{(0)} = \delta_{i,j}$. From (5.11), we get that

$$G(x_{i,j}^{(m)}) = y_{n+1-j,n+1-i}^{(m)}, \quad G(y_{i,j}^{(m)}) = x_{n+1-j,n+1-i}^{(m)}. \tag{5.16}$$

Lemma 5.2. *For any $U_q(\mathfrak{gl}_n)$-module M, the endomorphisms $R_{V^\pm,M}$ and $R_{M,V^\pm}$ and their inverses are given explicitly by the following operators*:

$$\begin{aligned}
R_{V^+,M} &= zP \circ \sum_{i\le j} e_{i,j}^+ \otimes f_{i,j} d_j, & R_{V^+,M}^{-1} &= -zP \otimes \sum_{i\le j} \overline{d_i f_{i,j}} \otimes e_{i,j}^+, \\
R_{M,V^+} &= zP \circ \sum_{i\le j} e_{i,j} d_i \otimes e_{j,i}^+, & R_{M,V^+}^{-1} &= -zP \circ \sum_{i\le j} e_{j,i}^+ \otimes \overline{d_j e_{i,j}}, \\
R_{V^-,M} &= -zP \circ \sum_{i\le j} (-q)^{i-j} e_{j,i}^- \otimes \overline{d_i f_{i,j}}, & R_{V^-,M}^{-1} &= zP \circ \sum_{i\le j} (-q)^{i-j} f_{i,j} d_j \otimes e_{j,i}^-, \\
R_{M,V^-} &= -zP \circ \sum_{i\le j} (-q)^{i-j} \overline{d_j e_{i,j}} \otimes e_{i,j}^-, & R_{M,V^-}^{-1} &= zP \circ \sum_{i\le j} (-q)^{i-j} e_{i,j}^- \otimes e_{i,j} d_i.
\end{aligned}$$

Proof. These are all proved by similar calculations, so we just go through the argument for R_{M,V^-}. Take $v \otimes v_j^- \in M \otimes V^-$. By definition, $R_{M,V^-}(v \otimes v_j^-) = \Theta(v_j^- \otimes d_j^{-1} v)$. To compute the action of Θ, we observe by weight considerations that only its weight components $\Theta_{\varepsilon_i - \varepsilon_j}$ for $i \le j$ are nonzero on $v_j^- \otimes d_j^{-1} v$. Moreover, in the first expression for $\Theta_{i,j}$ from Lemma 5.1, all of the monomials with $r > 1$ act on v_j^- as zero. We deduce that

$$R_{M,V^-}(v \otimes v_j^-) = v_j^- \otimes d_j^{-1} v + z \sum_{i<j} f_{i,j} v_j^- \otimes \overline{e_{i,j} d_j} v.$$

Then we use (5.12) to replace $f_{i,j}$ with $(-q)^{i-j+1} e_{i,j}^-$, the relation $e_{i,j} d_j = q d_j e_{i,j}$, and the definition $\overline{e_{j,j}} = -z^{-1}$ to get

$$R_{M,V^-}(v \otimes v_j^-) = -z e_{j,j}^- v_j^- \otimes \overline{d_j e_{j,j}} v - z \sum_{i<j} (-q)^{i-j} e_{i,j}^- v_j^- \otimes \overline{e_{i,j} d_j} v.$$

Now observe that the expression on the right-hand side of the formula we are trying to prove acts on $v \otimes v_j^-$ in the same way. □

Corollary 5.3. *For any $U_q(\mathfrak{gl}_n)$-module M and $m \in \mathbb{Z}$, we have that*

$$(R_{M,V^+} \circ R_{V^+,M})^m = \begin{cases} \sum_{i,j=1}^n e_{i,j}^+ \otimes x_{i,j}^{(m)} & \text{if } m \ge 0, \\ \sum_{i,j=1}^n e_{i,j}^+ \otimes \overline{y_{i,j}^{(-m)}} & \text{if } m \le 0. \end{cases}$$

Proof. This follows from Lemma 5.2 and Definitions (5.14)–(5.15). □

Now we return to the Heisenberg category $\mathcal{H}eis_0(z,t)$ taking $t := q^n$. Let $\mathcal{OS}(z,t)$ be the *HOMFLY-PT skein category* as defined in the introduction of [Brundan 2017], which is Turaev's Hecke category [1989].

By [Brundan 2017, Theorem 1.1], $\mathcal{OS}(z,t)$ has a presentation by generators and relations which is very similar to the presentation of $\mathcal{H}eis_0(z,t)$ from Definition 2.2 but *without* the morphism x. Consequently, there is a strict $\Bbbk$-linear monoidal functor $\mathcal{OS}(z,t) \to \mathcal{H}eis_0(z,t)$. By [loc. cit., Lemma 4.2], this functor is faithful, so we may use it to *identify* $\mathcal{OS}(z,t)$ with a subcategory of $\mathcal{H}eis_0(z,t)$. Thus, $\mathcal{OS}(z,t)$ is the monoidal subcategory of $\mathcal{H}eis_0(z,t)$ consisting of all objects and all morphisms which do not involve dots (i.e., x or y). In fact, as noted already after Definition 2.2, $\mathcal{H}eis_0(z,t)$ is the *affine* HOMFLY-PT skein category from [loc. cit., Section 4].

Let $U_q(\mathfrak{gl}_n)$-mod be the category of all left $U_q(\mathfrak{gl}_n)$-modules. By [loc. cit., Lemma 3.1] (although the result is much older, e.g., it was exploited already in [Turaev 1989]), there is a monoidal functor

$$\Psi : \mathcal{OS}(z,t) \to U_q(\mathfrak{gl}_n)\text{-mod} \tag{5.17}$$

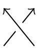
to the category of left $U_q(\mathfrak{gl}_n)$-modules. The functor Ψ sends the generating objects $\uparrow$ and $\downarrow$ to V^+ and V^-, respectively. It maps the various generating morphisms to the following $U_q(\mathfrak{gl}_n)$-module homomorphisms:


$$: v_i^+ \otimes v_j^+ \mapsto \begin{cases} v_j^+ \otimes v_i^+ & \text{if } i<j, \\ q v_j^+ \otimes v_i^+ & \text{if } i=j, \\ v_j^+ \otimes v_i^+ + z v_i^+ \otimes v_j^+ & \text{if } i>j; \end{cases} \tag{5.18}$$


$$: v_i^+ \otimes v_j^- \mapsto \begin{cases} v_j^- \otimes v_i^+ & \text{if } i\neq j, \\ q^{-1} v_j^- \otimes v_i^+ - z \sum_{r=1}^{i-1} (-q)^{-r} v_{j-r}^- \otimes v_{i-r}^+ & \text{if } i=j; \end{cases} \tag{5.19}$$


$$: v_i^- \otimes v_j^- = \begin{cases} v_j^- \otimes v_i^- & \text{if } i>j, \\ q v_j^- \otimes v_i^- & \text{if } i=j, \\ v_j^- \otimes v_i^- + z v_i^- \otimes v_j^- & \text{if } i<j; \end{cases} \tag{5.20}$$


$$: v_i^- \otimes v_j^+ \mapsto \begin{cases} v_j^+ \otimes v_i^- & \text{if } i\neq j, \\ q^{-1} v_j^+ \otimes v_i^- - z \sum_{r=1}^{n-i} (-q)^{-r} v_{j+r}^+ \otimes v_{i+r}^- & \text{if } i=j; \end{cases} \tag{5.21}$$


$$: v_i^+ \otimes v_j^+ \mapsto \begin{cases} v_j^+ \otimes v_i^+ & \text{if } i>j, \\ q^{-1} v_j^+ \otimes v_i^+ & \text{if } i=j, \\ v_j^+ \otimes v_i^+ - z v_i^+ \otimes v_j^+ & \text{if } i<j; \end{cases} \tag{5.22}$$


$$: v_i^+ \otimes v_j^- \mapsto \begin{cases} v_j^- \otimes v_i^+ & \text{if } i\neq j, \\ q v_j^- \otimes v_i^+ + z \sum_{r=1}^{n-i} (-q)^{r} v_{j+r}^- \otimes v_{i+r}^+ & \text{if } i=j; \end{cases} \tag{5.23}$$


$$: v_i^- \otimes v_j^- = \begin{cases} v_j^- \otimes v_i^- & \text{if } i<j, \\ q^{-1} v_j^- \otimes v_i^- & \text{if } i=j, \\ v_j^- \otimes v_i^- - z v_i^- \otimes v_j^- & \text{if } i>j; \end{cases} \tag{5.24}$$


$$: v_i^- \otimes v_j^+ \mapsto \begin{cases} v_j^+ \otimes v_i^- & \text{if } i\neq j, \\ q v_j^+ \otimes v_i^- + z \sum_{r=1}^{i-1} (-q)^{r} v_{j-r}^+ \otimes v_{i-r}^- & \text{if } i=j; \end{cases} \tag{5.25}$$


$$: 1 \mapsto \sum_{j=1}^{n} (-1)^j q^j v_j^- \otimes v_j^+,$$

$$: 1 \mapsto \sum_{j=1}^{n} (-1)^j q^{n+1-j} v_j^+ \otimes v_j^-, \tag{5.26}$$


$$: v_i^+ \otimes v_j^- \mapsto (-1)^i q^{-i} \delta_{i,j},$$

$$: v_i^- \otimes v_j^+ \mapsto (-1)^i q^{i-n-1} \delta_{i,j}. \tag{5.27}$$

These formulae are recorded in many places in the literature going back to the original work [Turaev 1989], but one finds many different choices of normalization. For our choices, (5.18)–(5.21) and (5.22)–(5.25) follow from the formulae for the R-matrix and its inverse from Lemma 5.2, while the formulae (5.26)–(5.27) are derived in [Brundan 2017, Section 3].

Theorem 5.4. *Assuming $t = q^n$ and $z = q - q^{-1}$, there is a strict $\Bbbk$-linear monoidal functor*

$$\widehat{\Psi} : \mathcal{H}eis_0(z,t) \to \mathcal{E}nd_{\Bbbk}(U_q(\mathfrak{gl}_n)\text{-mod})$$

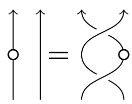
such that $\Psi = \mathrm{Ev} \circ \widehat{\Psi}\big|_{\mathcal{OS}(z,t)}$, where Ev *denotes evaluation on the trivial module. On objects, $\widehat{\Psi}$ takes X to the endofunctor $\Psi(X) \otimes -$, e.g., $\widehat{\Psi}(\uparrow) = V^+ \otimes -$ and $\widehat{\Psi}(\downarrow) = V^- \otimes -$. On morphisms, $\widehat{\Psi}$ sends $f \in \mathrm{Hom}_{\mathcal{OS}(z,t)}(X,Y)$ to the natural transformation $\Psi(f) \otimes 1 : \Psi(X) \otimes - \to \Psi(Y) \otimes -$. Finally, on the additional generating morphism x, it is defined by*

$$\widehat{\Psi}(x)_M := R_{M,V^+} \circ R_{V^+,M} : V^+ \otimes M \to V^+ \otimes M, \qquad v_j^+ \otimes m \mapsto \sum_{i=1}^{n} v_i^+ \otimes x_{i,j} m.$$

Proof. We just need to verify that the relations from Definition 2.2 are satisfied. All of the ones that do not involve x follow immediately since they are already satisfied by the morphisms in the image of the monoidal functor Ψ. Also $R_{V^+,M} \circ R_{M,V^+}$ is invertible since each of these R-matrices is invertible. It just remains to check the relation (1.6). In fact, this is a formal consequence of the hexagon property; see e.g., [Virk 2011, Proposition 3.1.1]. The argument goes as follows. By (5.7), we have for any $U_q(\mathfrak{gl}_n)$-module M that

$$R_{V^+ \otimes M, V^+} \circ R_{V^+, V^+ \otimes M} = R_{V^+,V^+} \otimes \mathrm{id}_M \circ \mathrm{id}_{V^+} \otimes R_{M,V^+} \circ \mathrm{id}_{V^+} \otimes R_{V^+,M} \circ R_{V^+,V^+} \otimes \mathrm{id}_M .$$

This establishes that the image under $\widehat{\Psi}$ of the relation



is satisfied, from which (1.6) easily follows. □

Let $Z_q(\mathfrak{gl}_n)$ be the center of $U_q(\mathfrak{gl}_n)$. It is identified with the endomorphism algebra of the identity functor $\mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}}$; indeed, evaluation on the identity element of the regular representation defines a canonical algebra isomorphism $\mathrm{End}(\mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}}) \xrightarrow{\sim} Z_q(\mathfrak{gl}_n)$. Dotted bubbles are endomorphisms of the unit object of $\mathcal{H}eis_0(z,t)$. Applying the monoidal functor $\widehat{\Psi}$ from Theorem 5.4, we obtain natural transformations

$\widehat{\Psi}\Big($  $\Big) : \mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}} \to \mathrm{Id}_{U_q(\mathfrak{gl}_n)\text{-mod}},$ (5.28)

hence, central elements $z_m \in Z(U_q(\mathfrak{gl}_n))$ for each $m \in \mathbb{Z}$. A calculation using (5.26)–(5.27) and Corollary 5.3 shows that

$$z_m = \begin{cases} \sum_{i=1}^n q^{2i-n-1} x_{i,i}^{(m)} & \text{if } m \geq 0, \\ \sum_{i=1}^n q^{2i-n-1} \overline{y_{i,i}^{(-m)}} & \text{if } m \leq 0. \end{cases} \tag{5.29}$$

We have trivially that $z_0 = [n]_q$. The goal in the remainder of the section is to compute explicit formulae for the images of all the others under the Harish-Chandra homomorphism.

Our argument uses the Harish-Chandra homomorphism in two different forms adapted to the positive and negative Borel subalgebras, respectively. To review the definitions, let $\rho_+ := -\varepsilon_2 - 2\varepsilon_3 - \cdots - (n-1)\varepsilon_n$ and $\rho_- := -(n-1)\varepsilon_1 - \cdots - 2\varepsilon_{n-2} - \varepsilon_{n-1}$, i.e., $\rho_- = w_0(\rho_+)$. For any $\lambda \in \Lambda$, we have the *shift automorphism*

$$S_\lambda : U_q(\mathfrak{gl}_n)^0 \to U_q(\mathfrak{gl}_n)^0, \quad d_i \mapsto q^{(\lambda,\varepsilon_i)} d_i. \tag{5.30}$$

For example, $S_{-\rho_+}(d_i) = q^{i-1} d_i$ and $S_{-\rho_-}(d_i) = q^{n-i} d_i$. Let $U_q(\mathfrak{gl}_n)_0$ be the zero weight space of $U_q(\mathfrak{gl}_n)$, which is a subalgebra containing $U_q(\mathfrak{gl}_n)^0$. Let I_+ (resp. I_-) be the intersection of $U_q(\mathfrak{gl}_n)_0$ with the left ideal of $U_q(\mathfrak{gl}_n)$ generated by $e_1, \dots, e_{n-1}$ (resp. $f_1, \dots, f_{n-1}$). Equivalently, I_+ (resp. I_-) is the intersection of $U_q(\mathfrak{gl}_n)_0$ with the right ideal generated by $f_1, \dots, f_{n-1}$ (resp. $e_1, \dots, e_{n-1}$). It follows that $I_\pm$ is a two-sided ideal of $U_q(\mathfrak{gl}_n)_0$. Let $\mathrm{pr}_\pm : U_q(\mathfrak{gl}_n)_0 \to U_q(\mathfrak{gl}_n)^0$ be the algebra homomorphism defined by projection along the direct sum decomposition $U_q(\mathfrak{gl}_n)_0 = U_q(\mathfrak{gl}_n)^0 \oplus I_\pm$. The two versions of the *Harish-Chandra homomorphism* are

$$HC_\pm := S_{-\rho_\pm} \circ \mathrm{pr}_\pm : U_q(\mathfrak{gl}_n)_0 \to U_q(\mathfrak{gl}_n)^0. \tag{5.31}$$

The following is an extension of the well-known description of $Z_q(\mathfrak{sl}_n)$ from e.g., [Jantzen 1996, 6.25].

Lemma 5.5 [Li 2010, Lemma 2.1]. *The restriction $HC := HC_+|_{Z_q(\mathfrak{gl}_n)}$ defines an algebra isomorphism between $Z_q(\mathfrak{gl}_n)$ and the algebra $\Bbbk[(d_1 \cdots d_n)^{-1}, d_1^2, \dots, d_n^2]^{\mathfrak{S}_n}$.*

The following facts are also well known, but we could not find a suitable reference.

Lemma 5.6. *Each braid group generator $T_i : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ fixes $Z_q(\mathfrak{gl}_n)$ pointwise.*

Proof. Take $c \in Z_q(\mathfrak{gl}_n)$. Let V be an integrable highest weight module. Since V is irreducible, both c and $T_i(c)$ act on V as scalars. These scalars are equal because there is an automorphism $T_i : V \to V$ such that $T_i(cv) = T_i(c)T_i(v)$; see [Lusztig 1993, Section 37.1.1]. This shows that $c - T_i(c)$ acts as zero on every integrable highest weight module. The intersection of the annihilators of all integrable highest weight modules is zero, so this proves that $c = T_i(c)$. □

Lemma 5.7. *The restriction $HC = HC_+|_{Z_q(\mathfrak{gl}_n)}$ is equal also to the restriction $HC_-|_{Z_q(\mathfrak{gl}_n)}$.*

Proof. Let T_{w_0} be the product of simple braid group generators T_i taken in some order corresponding to a reduced expression of w_0. This is an automorphism of $U_q(\mathfrak{gl}_n)$ which switches $U_q(\mathfrak{gl}_n)^\sharp$ and $U_q(\mathfrak{gl}_n)^\flat$, and it sends $d_i \mapsto d_{n+1-i}$. It follows that

$$HC_\mp \circ T_{w_0} = T_{w_0} \circ HC_\pm. \tag{5.32}$$

Clearly, T_{w_0} fixes $\Bbbk[(d_1 \cdots d_n)^{-1}, d_1^2, \dots, d_n^2]^{\mathfrak{S}_n}$ pointwise. It also fixes $Z_q(\mathfrak{gl}_n)$ pointwise by Lemma 5.6. Hence, $HC_-|_{Z_q(\mathfrak{gl}_n)} = HC_- \circ T_{w_0}|_{Z_q(\mathfrak{gl}_n)} = T_{w_0} \circ HC_+|_{Z_q(\mathfrak{gl}_n)} = HC_+|_{Z_q(\mathfrak{gl}_n)}$. □

Lemma 5.8. *The antiautomorphism G fixes $Z_q(\mathfrak{gl}_n)$ pointwise.*

Proof. We have that

$$HC_{\mp} \circ G = G \circ HC_{\pm}. \tag{5.33}$$

Combined with Lemma 5.7, it follows that $HC_+ \circ G\big|_{Z_q(\mathfrak{gl}_n)} = G \circ HC_+\big|_{Z_q(\mathfrak{gl}_n)}$. Also G clearly fixes $\Bbbk[(d_1 \cdots d_n)^{-1}, d_1^2, \dots, d_n^2]^{\mathfrak{S}_n}$ pointwise. Hence, $HC_+ \circ G\big|_{Z_q(\mathfrak{gl}_n)} = HC_+\big|_{Z_q(\mathfrak{gl}_n)}$, which implies the result since HC_+ is injective on $Z_q(\mathfrak{gl}_n)$. □

In particular, this shows that $G(z_m) = z_m$, hence, on applying G to the right-hand side of (5.29) using (5.16), we obtain another formula for z_m:

$$z_m = \begin{cases} \sum_{i=1}^n q^{n+1-2i} y_{i,i}^{(m)} & \text{if } m \geq 0, \\ \sum_{i=1}^n q^{n+1-2i} \overline{x_{i,i}^{(-m)}} & \text{if } m \leq 0. \end{cases} \tag{5.34}$$

Comparing with (5.29), it follows that

$$z_{-m} = \overline{z_m} \tag{5.35}$$

for every $m \in \mathbb{Z}$. From now on, we only consider z_m for $m \geq 1$.

Finally, consider the *modified complete symmetric polynomials*

$$\tilde{h}_m(x_1, \dots, x_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} (q^{-1} z)^{\#\{i_1, \dots, i_m\} - 1} x_{i_1} \cdots x_{i_m}. \tag{5.36}$$

We will use these for all values of $n \geq 0$ (not just the n fixed above for $\mathfrak{gl}_n$). We have that

$$\tilde{h}_m(x_1, \dots, x_n) = q z^{-1} \text{ if } m = 0 \quad \text{and} \quad \tilde{h}_m(x_1, \dots, x_n) = 0 \text{ if } m > 0 \text{ but } n = 0.$$

These elements obviously satisfy the recurrence relation

$$\tilde{h}_m(x_1, \dots, x_n) = \tilde{h}_m(x_1, \dots, x_{n-1}) + q^{-1} z \sum_{r=1}^{m} \tilde{h}_{m-r}(x_1, \dots, x_{n-1}) x_n^r \tag{5.37}$$

for $n > 0$.

Lemma 5.9. $\tilde{h}_m(x_1, \dots, x_n) = \tilde{h}_m(x_1, \dots, x_{n-1}) + \tilde{h}_{m-1}(x_1, \dots, x_n) x_n - q^{-2} \tilde{h}_{m-1}(x_1, \dots, x_{n-1}) x_n.$

Proof. By (5.37) with m replaced by $m-1$, we have that

$$\begin{aligned} \tilde{h}_{m-1}(x_1, \dots, x_n) x_n &= \tilde{h}_{m-1}(x_1, \dots, x_{n-1}) x_n + q^{-1} z \sum_{r=1}^{m-1} \tilde{h}_{m-r-1}(x_1, \dots, x_{n-1}) x_n^{r+1} \\ &= \tilde{h}_{m-1}(x_1, \dots, x_{n-1}) x_n + q^{-1} z \sum_{r=2}^{m} \tilde{h}_{m-r}(x_1, \dots, x_{n-1}) x_n^r \\ &= q^{-2} \tilde{h}_{m-1}(x_1, \dots, x_{n-1}) x_n + q^{-1} z \sum_{r=1}^{m} \tilde{h}_{m-r}(x_1, \dots, x_{n-1}) x_n^r. \end{aligned}$$

Given this, it is easy to see that the right-hand side of the identity we are trying to prove is equal to the right-hand side of (5.37). □

Theorem 5.10. *For any $m \geq 1$ we have that $HC(z_m) = q^{n-1}\tilde{h}_m(d_1^2, \ldots, d_n^2)$.*

Proof. Noting that $q^{1-n} z_m = \sum_{i=1}^n q^{2i-2n} x_{i,i}^{(m)}$ according to (5.29), this follows from the following claim: *for any $m \geq 1$ and $i = 1, \ldots, n$, we have that*

$$HC_+(x_{i,i}^{(m)}) = \tilde{h}_m(d_1^2, \ldots, d_i^2) - q^{-2}\tilde{h}_m(d_1^2, \ldots, d_{i-1}^2). \tag{5.38}$$

To prove (5.38), we proceed by induction on $m+n$. The result is easy to check when $n = 1$. Now assume that $n > 1$. The Harish-Chandra homomorphism HC_+ is compatible with the usual "top left corner" embedding of $U_q(\mathfrak{gl}_{n-1})$ into $U_q(\mathfrak{gl}_n)$. This follows because the restriction of ρ_+ for $\mathfrak{gl}_n$ is the weight ρ_+ for $\mathfrak{gl}_{n-1}$. Also the elements $x_{1,1}^{(m)}, \ldots, x_{n-1,n-1}^{(m)}$ of $U_q(\mathfrak{gl}_{n-1})$ are the same as these elements in $U_q(\mathfrak{gl}_n)$. Thus we get (5.38) for each $i < n$ from the induction hypothesis. It remains to prove (5.38) when $i = n$. We have that

$$q^{1-n} HC_-(z_m) = \sum_{i=1}^n q^{2i-2n} \sum_{j_1,\ldots,j_m} HC_-(z^{2m} e_{j_1,i} d_{j_1} f_{j_1,j_2} d_{j_2} \cdots e_{j_m,j_{m-1}} d_{j_m} f_{j_m,i} d_i).$$

By the definition of HC_-, the terms in this expansion are zero if either $j_1 < i$ or $j_m < i$. Thus, the sum simplifies to give

$$q^{1-n} HC_-(z_m) = \sum_{i=1}^n q^{2i-2n} HC_-(y_{i,i}^{(m-1)}\, d_i^2) = \sum_{i=1}^n HC_-(y_{i,i}^{(m-1)})\, d_i^2.$$

Now we apply G, using Lemma 5.8, (5.33) and (5.11), to see that

$$q^{1-n} HC_+(z_m) = \sum_{i=1}^n HC_+(x_{i,i}^{(m-1)})\, d_i^2.$$

Remembering (5.29), we have now proved that

$$\sum_{i=1}^n q^{2i-2n} HC_+(x_{i,i}^{(m)}) = \sum_{i=1}^n HC_+(x_{i,i}^{(m-1)})\, d_i^2. \tag{5.39}$$

The same identity with n replaced by $(n-1)$ gives

$$\sum_{i=1}^{n-1} q^{2i-2(n-1)} HC_+(x_{i,i}^{(m)}) = \sum_{i=1}^{n-1} HC_+(x_{i,i}^{(m-1)})\, d_i^2. \tag{5.40}$$

By the induction hypothesis, the left-hand side of (5.40) is equal to $\tilde{h}_m(d_1^2, \ldots, d_{n-1}^2)$. Hence, (5.39) can be rewritten to obtain

$$\begin{aligned}
&HC_+(x_{n,n}^{(m)}) + q^{-2}\tilde{h}_m(d_1^2, \ldots, d_{n-1}^2) \\
&\qquad = HC_+(x_{n,n}^{(m-1)}) d_n^2 + \tilde{h}_m(d_1^2, \ldots, d_{n-1}^2) \\
&\qquad = \tilde{h}_m(d_1^2, \ldots, d_{n-1}^2) + \tilde{h}_{m-1}(d_1^2, \ldots, d_n^2) d_n^2 - q^{-2}\tilde{h}_{m-1}(d_1^2, \ldots, d_{n-1}^2) d_n^2,
\end{aligned}$$

where we have used the induction hypothesis again to establish the second equality. This is equal to $\tilde{h}_m(d_1^2, \dots, d_n^2)$ thanks to Lemma 5.9. The conclusion follows. □

Corollary 5.11 [Li 2010, Theorem 4.1]. *$Z_q(\mathfrak{gl}_n)$ is generated by $z_1, \dots, z_n$ and $(d_1 \cdots d_n)^{-1}$.*

Proof. This follows from Lemma 5.5 and Theorem 5.10 since $\mathbb{k}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ is generated by the modified complete symmetric functions $\tilde{h}_1(x_1, \dots, x_n), \dots, \tilde{h}_n(x_1, \dots, x_n)$. □

6. Action on modules over cyclotomic Hecke algebras

Throughout the section, we assume that we are given a polynomial

$$f(w) = f_0 w^l + f_1 w^{l-1} + \cdots + f_l \in \mathbb{k}[w] \tag{6.1}$$

of degree $l \geq 0$ such that $f_0 = 1$ and $f_l = t^2$. Recall from the introduction that the affine Hecke algebra AH_n with its standard generators $x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1}$ is identified with the endomorphism algebra $\mathrm{End}_{\mathcal{AH}(z)}(\uparrow^{\otimes n})$ so that and x_i is the dot on the i-th string and τ_j is the positive crossing of the j-th and $(j+1)$-th strings (numbering strings $1, \dots, n$ from right to left). The *cyclotomic Hecke algebra* H_n^f of level l associated to the polynomial $f(w)$ is the quotient of AH_n by the two-sided ideal generated by $f(x_1)$. We also include the possibility $n = 0$ with the convention that $H_0^f = \mathbb{k}$.

The basis theorem proved in [Ariki and Koike 1994, Theorem 3.10] shows that the following gives a basis for H_n^f as a free $\mathbb{k}$-module:

$$\{x_1^{r_1} \cdots x_n^{r_n} \tau_g \mid 0 \leq r_1, \dots, r_n < l, \, g \in \mathfrak{S}_n\}, \tag{6.2}$$

where τ_g denotes the element of the finite Hecke algebra defined from a reduced expression for the permutation g. By the basis theorem, the obvious homomorphism $H_n^f \to H_{n+1}^f$ sending the generators x_i and τ_j to the elements of H_{n+1}^f with the same names is *injective*. So we may identify H_n^f with a subalgebra of H_{n+1}^f. We denote the induction and restriction functors by

$$\mathrm{ind}_n^{n+1} := H_{n+1}^f \otimes_{H_n^f} - : H_n^f\text{-mod} \to H_{n+1}^f\text{-mod}, \tag{6.3}$$

$$\mathrm{res}_n^{n+1} : H_{n+1}^f\text{-mod} \to H_n^f\text{-mod}\,. \tag{6.4}$$

We are going to make the Abelian category $\bigoplus_{n \geq 0} H_n^f$-mod into a left $\mathcal{H}eis_{-l}(z, f_0^{-1})$-module category, with $\uparrow$ and $\downarrow$ acting as induction and restriction, respectively. In order to do this, we need the *Mackey theorem* for H_n^f: there is an isomorphism of functors

$$\mathrm{ind}_{n-1}^n \circ \mathrm{res}_{n-1}^n \oplus \mathrm{Id}^{\oplus l} \xrightarrow{\sim} \mathrm{res}_n^{n+1} \circ \mathrm{ind}_n^{n+1}\,. \tag{6.5}$$

The standard proof shows that the map

$$H_n^f \otimes_{H_{n-1}^f} H_n^f \oplus \bigoplus_{r=0}^{l-1} H_n^f \to H_{n+1}^f, \quad (u \otimes v, w_0, \dots, w_{l-1}) \mapsto u\tau_n v + \sum_{r=0}^{l-1} w_r x_{n+1}^r \tag{6.6}$$

is an isomorphism of (H_n^f, H_n^f)-bimodules. This implies that there is a unique (H_n^f, H_n^f)-bimodule homomorphism

$$\mathrm{tr}_n^f : H_{n+1}^f \to H_n^f \tag{6.7}$$

such that $\mathrm{tr}_n^f(\tau_n) = 0$ and $\mathrm{tr}_n^f(x_{n+1}^r) = \delta_{r,0}$ for $0 \le r < l$.

Lemma 6.1. *For any $n \ge 0$, we have that $\mathrm{tr}_n^f(f(x_{n+1})) = 0$.*

Proof. For $u, v \in H_{n+1}^f$, write $u \equiv_n v$ as shorthand for $u = v$ in case $n = 0$, or $u - v \in H_n^f \tau_n H_n^f$ in case $n > 0$. We first show by induction on $n = 0, 1, \ldots$ that

$$\tau_n \cdots \tau_1 x_1^a \tau_1 \cdots \tau_n \equiv_n \begin{cases} \sum_{\substack{b+c_1+\cdots+c_n=a \\ b>0, c_1,\ldots,c_n \ge 0}} \left(\prod_{i \text{ with } c_i \neq 0} (-z^2 c_i)\right) x_{n+1}^b x_n^{c_n} \cdots x_1^{c_1} & \text{if } a > 0, \\ \sum_{\substack{b+c_1+\cdots+c_n=a \\ b\le 0, c_1,\ldots,c_n \le 0}} \left(\prod_{i \text{ with } c_i \neq 0} (z^2 c_i)\right) x_{n+1}^b x_n^{c_n} \cdots x_1^{c_1} & \text{if } a \le 0. \end{cases} \tag{6.8}$$

We explain this in detail in the case $a > 0$, since the case $a \le 0$ is similar. The base case is trivial. For the induction step, using the relations depicted in (2.8)–(2.9), we have that

$$\begin{aligned} \tau_n x_n^a \tau_n &= \tau_n \tau_n^{-1} x_{n+1}^a - z \sum_{\substack{b+c=a \\ b,c>0}} \tau_n x_{n+1}^b x_n^c \\ &= x_{n+1}^a - z \sum_{\substack{b+c=a \\ b,c>0}} \tau_n^{-1} x_{n+1}^b x_n^c - z^2 \sum_{\substack{b+c=a \\ b,c>0}} x_{n+1}^b x_n^c \\ &\equiv_n x_{n+1}^a - z^2 \sum_{\substack{b+c+d=a \\ b,c,d>0}} x_{n+1}^b x_n^{c+d} - z^2 \sum_{\substack{b+c=a \\ b,c>0}} x_{n+1}^b x_n^c = x_{n+1}^a - z^2 \sum_{\substack{b+c=a \\ b,c>0}} c x_{n+1}^b x_n^c. \end{aligned}$$

Now take the expression for $\tau_{n-1} \cdots \tau_1 x_1^a \tau_1 \cdots \tau_{n-1}$ given by the induction hypothesis, multiply on left and right by τ_n, and use the above identity plus the observation

$$\tau_n (H_{n-1}^f \tau_{n-1} H_{n-1}^f) \tau_n = H_{n-1}^f \tau_n \tau_{n-1} \tau_n H_{n-1}^f = H_{n-1}^f \tau_{n-1} \tau_n \tau_{n-1} H_{n-1}^f \subseteq H_n^f \tau_n H_n^f.$$

Finally, to deduce the lemma, we multiply (6.8) by f_{l-a} and sum over $a = 0, 1, \ldots, l$ to show

$$\tau_n \cdots \tau_1 f(x_1) \tau_1 \cdots \tau_n \equiv_n f_l + \sum_{a=1}^{l} f_{l-a} \sum_{\substack{b+c_1+\cdots+c_n=a \\ b>0, c_1,\ldots,c_n \ge 0}} \left(\prod_{i \text{ with } c_i \neq 0} (-z^2 c_i) \right) x_{n+1}^b x_n^{c_n} \cdots x_1^{c_1}.$$

The left-hand side is zero by the cyclotomic relation in H_{n+1}^f. The right-hand side is equal to $f(x_{n+1})$ plus terms in the kernel of tr_n^f. ◻

Theorem 6.2. *There is a unique strict $\Bbbk$-linear monoidal functor*

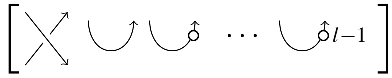
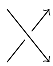
$$\Psi_f : \mathcal{H}eis_{-l}(z, t) \to \mathcal{E}nd_{\Bbbk}\left(\bigoplus_{n \ge 0} H_n^f\text{-mod} \right)$$

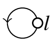
sending the generating object $\uparrow$ (*resp.* $\downarrow$) *to the additive endofunctor that takes an* H_n^f*-module* M *to* $\mathrm{ind}_n^{n+1} M$ (*resp.* $\mathrm{res}_{n-1}^n M$), *and the generating morphisms* x, τ, c *and* d *to the natural transformations defined on the* H_n^f*-module* M *as follows*:

- $\Psi_f(x)_M : H_{n+1}^f \otimes_{H_n^f} M \to H_{n+1}^f \otimes_{H_n^f} M,\ u \otimes v \mapsto u x_{n+1} \otimes v.$
- $\Psi_f(\tau)_M : H_{n+2}^f \otimes_{H_n^f} M \to H_{n+2}^f \otimes_{H_n^f} M,\ u \otimes v \mapsto u\tau_{n+1} \otimes v$ (*where we have identified* $\mathrm{ind}_{n+1}^{n+2} \circ \mathrm{ind}_n^{n+1}$ *with* ind_n^{n+2} *in the obvious way*).
- $\Psi_f(c)_M : M \to \mathrm{res}_n^{n+1}(H_{n+1}^f \otimes_{H_n^f} M),\ v \mapsto 1 \otimes v$, *i.e., it is the unit of the canonical adjunction making* $(\mathrm{ind}_n^{n+1}, \mathrm{res}_n^{n+1})$ *into an adjoint pair of functors.*
- $\Psi_f(d)_M : H_n^f \otimes_{H_{n-1}^f} (\mathrm{res}_{n-1}^n M) \to M,\ u \otimes v \mapsto uv$, *i.e., it is the counit of the canonical adjunction making* $(\mathrm{ind}_{n-1}^n, \mathrm{res}_{n-1}^n)$ *into an adjoint pair of functors.*

Proof. We use the presentation for $\mathcal{H}eis_{-l}(z,t)$ from Definition 2.2. Let us first treat the case $l = 0$. In this case, the polynomial $f(w)$ from (6.1) is 1 and $t^2 = 1$. The category $\bigoplus_{n\geq 0} H_n^f$-mod is simply the category of left $\Bbbk$-modules, and all of the induction and restriction functors are zero. Consequently, almost of the relations are trivially true. The only one that requires any thought is the relation  $= (tz^{-1} - t^{-1}z^{-1})1_{\mathbb{1}}$ from (2.12). This holds because the scalar on the right-hand side is zero as $t^2 = 1$.

Henceforth, we assume that $l > 0$. Then $\mathcal{H}eis_{-l}(z,t)$ is generated by the objects $\uparrow$ and $\downarrow$ and morphisms x, τ, c and d subject to the relations (1.6)–(1.9), plus two more relations:

(1)  is invertible where $\sigma :=$  is defined by (2.2).

(2)  $= tz^{-1}1_{\mathbb{1}}$ where $\gamma :=$  is defined by (2.13), i.e., it is $-t^{-1}z^{-1}$ times the (2, 1)-entry of the inverse of the matrix in (1).

The relations (1.6)–(1.9) are straightforward to check. On H_n^f-modules, $\Psi_f(\sigma)$ comes from the (H_n^f, H_n^f)-bimodule homomorphism $H_n^f \otimes_{H_{n-1}^f} H_n^f \to H_{n+1}^f,\ u \otimes v \mapsto u\tau_n v$. So we get the relation (1) since (6.6) is invertible by the proof of the Mackey theorem. Moreover, we see from (6.6) and the definition that $\Psi_f(\gamma)$ comes from the (H_n^f, H_n^f)-bimodule homomorphisms $-t^{-1}z^{-1}\,\mathrm{tr}_n^f : H_{n+1}^f \to H_n^f$ for all $n \geq 0$. So for (2) we must show that $-t^{-1}z^{-1}\,\mathrm{tr}_n^f(x_{n+1}^l) = tz^{-1}$. This follows from Lemma 6.1 and the definition of tr_n^f, remembering that $t^2 = f_l$. □

If we switch the roles of induction and restriction, we can reformulate Theorem 6.2 in terms of Heisenberg categories of positive central charge. We prefer for this to replace the induction functor ind_n^{n+1} from before (which is the canonical left adjoint to restriction) with the *coinduction functor*

$$\mathrm{coind}_n^{n+1} := \mathrm{Hom}_{H_n^f}(H_{n+1}^f, -) : H_n^f\text{-mod} \to H_{n+1}^f\text{-mod} \tag{6.9}$$

which is its canonical right adjoint.

Theorem 6.3. *There is a unique strict* $\Bbbk$*-linear monoidal functor*

$$\Psi_f^\vee : \mathcal{H}eis_l(z, t^{-1}) \to \mathcal{E}nd_\Bbbk\Big(\bigoplus_{n\geq 0} H_n^f\text{-mod}\Big)$$

sending the generating object $\uparrow$ (*resp.* $\downarrow$) *to the additive endofunctor that takes an* H_n^f*-module* M *to* $\mathrm{res}^n_{n-1} M$ (*resp.* $\mathrm{coind}^{n+1}_n M$), *and the generating morphisms* x, τ, c *and* d *to the natural transformations defined on the* H_n^f*-module* M *as follows*:

- $\Psi_f^\vee(x)_M : \mathrm{res}^n_{n-1} M \to \mathrm{res}^n_{n-1} M$, $v \mapsto x_n v$.
- $\Psi_f^\vee(\tau)_M : \mathrm{res}^n_{n-2} M \to \mathrm{res}^n_{n-2} M$, $v \mapsto -\tau_{n-1}^{-1} v$.
- $\Psi_f^\vee(c)_M : M \to \mathrm{Hom}_{H^f_{n-1}}(H_n^f, \mathrm{res}^n_{n-1} M)$, $v \mapsto (u \mapsto uv)$, *i.e., it is the unit of the canonical adjunction making* $(\mathrm{res}^n_{n-1}, \mathrm{coind}^n_{n-1})$ *into an adjoint pair of functors.*
- $\Psi_f^\vee(d)_M : \mathrm{res}^{n+1}_n(\mathrm{Hom}_{H^f_n}(H^f_{n+1}, M)) \to M$, $\theta \mapsto \theta(1)$, *i.e., it is the counit of the canonical adjunction making* $(\mathrm{res}^{n+1}_n, \mathrm{coind}^{n+1}_n)$ *into an adjoint pair of functors.*

Proof. This may be proved directly in a similar way to the proof of Theorem 6.2. One uses the presentation for $\mathcal{H}eis_l(z, t^{-1})$ from Definition 3.1 instead of the one from Definition 2.2, plus the Mackey isomorphism (6.6) and Lemma 6.1 as before. We leave the details to the reader. □

In fact, we have that $\mathrm{ind}^{n+1}_n \cong \mathrm{coind}^{n+1}_n$. This follows by the uniqueness of adjoints, since Lemma 3.7 and Theorem 6.2 (resp. Theorem 6.3) implies that ind^{n+1}_n is right adjoint to restriction as well as being left adjoint (resp. coind^{n+1}_n is left adjoint to restriction as well as being right adjoint). It follows that all three functors (induction, coinduction and restriction) send finitely generated projective modules to finitely generated projective modules. Hence:

Lemma 6.4. *The restrictions of the functors* Ψ_f *and* $\Psi_f^\vee$ *constructed in Theorems 6.2–6.3 give strict* $\Bbbk$*-linear monoidal functors*

$$\Psi_f : \mathcal{H}eis_{-l}(z, t) \to \mathcal{E}nd_{\Bbbk}\Big(\bigoplus_{n\geq 0} H_n^f\text{-pmod}\Big), \qquad \Psi_f^\vee : \mathcal{H}eis_l(z, t^{-1}) \to \mathcal{E}nd\Big(\bigoplus_{n\geq 0} H_n^f\text{-pmod}\Big),$$

where H_n^f-pmod *denotes the category of finitely generated projective left* H_n^f*-modules.*

7. Action on category $\mathcal{O}$ for rational Cherednik algebras

The Heisenberg action on $\bigoplus_{n\geq 0} H_n^f$-mod from Theorem 6.2 can also be extended to an action on the category $\mathcal{O}$ for rational Cherednik algebras, following an argument of Shan. To explain this in more detail, assume that $\Bbbk = \mathbb{C}$, and consider the complex reflection group $G(l, 1, n) \cong \mathfrak{S}_n \wr \mathbb{Z}/l\mathbb{Z}$ for $l \geq 1$, with reflection representation $\Bbbk^n$ defined as in [Shan 2011, Section 3.1]. Defining a rational Cherednik algebra requires a choice of parameters, for which there are a bewildering number of different parametrizations. We have:

- A single parameter $\kappa \in \Bbbk$, which is the parameter $k_{H,1}$ in [Ginzburg et al. 2003, Remark 3.2] for a reflecting hyperplane H on which the difference of two coordinates vanish.
- An l-tuple $(\kappa_1, \dots, \kappa_l) \in \Bbbk^l$ of parameters, which corresponds to the family $\{k_{H,i}\}_{0\leq i\leq l}$ of parameters in [loc. cit., Remark 3.2] associated to a reflecting hyperplane H on which a single coordinate

vanishes so that $\kappa_i = k_{H,i}$. In [loc. cit.], it is assumed that $k_{H,0} = k_{H,l} = 0$, but adding a constant to all $k_{H,i}$ leaves the algebra unchanged. It is useful for us to incorporate an additional degree of freedom, so we drop the vanishing condition here: our parameter κ_l may be nonzero.

Let H_n be the rational Cherednik algebra attached to these parameters as in [loc. cit., Section 3].

Let $q := \exp(\sqrt{-1}\pi\kappa)$ and $q_i := \exp(\sqrt{-1}\pi(\kappa_i - i/\ell))$ for $i = 1, \dots, l$. One can relate these to the parameters in [Shan 2011] by choosing integers $e \geq 2$ and $(s_1, \dots, s_l)$ then letting $\kappa := 1/e$ and $\kappa_i := \kappa s_i + i/\ell$, so $q_i = q^{s_i}$, for $i = 1, \dots, l$; note that the parameter q in [loc. cit.] is our q^2. Let $\mathcal{O} = \mathcal{O}_{\kappa;\kappa_1,\dots,\kappa_l} := \bigoplus_{n\geq 0} O_n$ where $\mathcal{O}_n$ is the category of H_n-modules introduced in [Ginzburg et al. 2003, Section 3]. Also define

$$f(w) := \prod_{i=1}^{l}(w + q_i^2), \qquad t := q_1 \cdots q_l.$$

By [Ginzburg et al. 2003, Theorem 5.16], there is an exact functor

$$\mathsf{KZ} : \mathcal{O} \to \bigoplus_{n\geq 0} H_n^f\text{-mod}. \tag{7.1}$$

Note that this functor depends on a choice for each n of a basepoint in the subset of $\mathbb{C}^n$ where all entries are distinct and nonzero. Different basepoints give isomorphic functors, but the isomorphism depends on the homotopy class of a path between the basepoints. For simplicity, we assume these basepoints are chosen to lie in the set $\{(b_1, \dots, b_n) \in \mathbb{R}^n \mid 0 < b_1 < \cdots < b_n\}$. Since this is a contractible space, the resulting KZ functors are all canonically isomorphic, and there is no need for us to be more specific.

Matching with the formulae in [Ginzburg et al. 2003; Shan 2011] requires using the isomorphism from the cyclotomic Hecke algebra in [Shan 2011, Section 3.1] to ours that sends the generators $T_0, T_1, \dots, T_{n-1}$ to $-x_1, q\tau_1, \dots, q\tau_{n-1}$. The Hecke algebra generators T_i $(i = 1, \dots, n-1)$ in [Shan 2011] are of the form $-T$ for Hecke algebra generators T from [Ginzburg et al. 2003, Section 5.2.5] associated to reflections in the first type of hyperplane above. Also, T_0 is a scalar multiple (depending on the choice of κ_l) of the Hecke algebra generator T in [loc. cit., Section 5.2.5] associated to a reflection of the second type. The key point in all of this is that the minimal polynomials for x_1 and τ_i $(i = 1, \dots, n-1)$ arising from the key formula in [loc. cit., Section 5.2.5] are $f(w)$ and $(w - q)(w + q^{-1})$ (up to scalars), i.e., we do indeed get defining relations of H_n^f.

The functor KZ is fully faithful on projectives [loc. cit., Theorem 5.16]. Moreover, it intertwines the Bezrukavnikov–Etingof induction and restriction functors denoted $\mathrm{ind}_{b_{n+1}}$ and $\mathrm{res}_{b_{n+1}}$ in [Shan 2011, Section 3.2] with the functors ind_n^{n+1} and res_n^{n+1} thanks to [loc. cit., Theorem 2.1]. These induction and restriction functors also depend on a choice of basepoint with a particular stabilizer, which following Shan we fix to be $(0, 0, \dots, 0, 1)$. (It would be more philosophically consistent with our previous conventions to say that whenever we choose a basepoint for restriction, we choose one of the form $(b_1, \dots, b_n) \in \mathbb{R}^n$ such that $0 \leq b_1 \leq b_2 \leq \cdots \leq b_n$; whether we have equality or strict inequality depends on which stabilizer we wish to have under the action of $G(l, 1, n)$. As before, all such choices give canonically isomorphic functors.)

Theorem 7.1. *There is a strict $\Bbbk$-linear monoidal functor*

$$\widehat{\Psi}_f : \mathcal{H}eis_{-l}(z, t) \to \mathcal{E}nd_{\Bbbk}(\mathcal{O}). \tag{7.2}$$

that makes $\mathcal{O}$ into a module category over $\mathcal{H}eis_{-l}(z, t)$, with $\uparrow$ and $\downarrow$ acting as Bezrukavnikov–Etingof induction and restriction functors, respectively. This can be done in such a way that KZ is a morphism of $\mathcal{H}eis_{-l}(z, t)$-module categories, viewing $\bigoplus_{n\geq 0} H_n^f$-mod as a module category via the functor Ψ_f from Theorem 6.2.

Proof. Our argument is exactly as in the proof of [Shan 2011, Theorem 5.1] using [loc. cit., Lemma 2.4]. We need to show that there are certain natural transformations of functors satisfying specific relations. Theorem 6.2 allows us to define these on the image of the functor KZ via the action of $\mathcal{H}eis_{-l}(z, t)$. The full-faithfulness of KZ allows us to transfer this to an action on the full subcategory of projectives in $\mathcal{O}$. Since $\mathcal{O}$ has enough projectives by [Ginzburg et al. 2003, Corollary 2.8], this action can be extended to an arbitrary object X by presenting X as the cokernel of a map between projectives. The resulting action is well-defined due to the fact that endomorphisms of an object lift to any projective resolution uniquely up to homotopy. □

Remark 7.2. This quantum Heisenberg action is in many ways more convenient for working with category $\mathcal{O}$ over Cherednik algebras than a Kac–Moody 2-category action, since the Heisenberg action requires no special assumptions on parameters. In fact, this action is still well defined if $\Bbbk$ is replaced by a complete local ring, so one can extend the Heisenberg action to deformed category $\mathcal{O}$.

8. Categorical comultiplication

In this section, we construct the quantum analog of the categorical comultiplication from [Brundan et al. 2018, Theorem 5.4]. As discussed in [loc. cit., Theorem 1.3], the name "categorical comultiplication" derives from the relationship of this map to the usual comultiplication on the universal enveloping algebra of the Heisenberg Lie algebra. Since in the quantum case an explicit description of $K_0(\mathrm{Kar}(\mathcal{H}eis_k(z, t)))$ analogous to that of [loc. cit., Theorem 1.1] is not available, we will not make a precise statement along these lines here, but we fully expect an analog of [loc. cit., Theorem 1.3] to hold in all situations where the Grothendieck ring has the expected form. As well as the quantum Heisenberg category $\mathcal{H}eis_k(z, t)$, we will work with $\mathcal{H}eis_l(z, u)$ and $\mathcal{H}eis_m(z, v)$ for $l, m \in \mathbb{Z}$ and $u, v \in \Bbbk^\times$ chosen so that

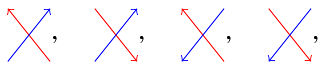
$$k = l + m, \quad t = uv. \tag{8.1}$$

To avoid confusion between these different categories, the reader will want to view the material in this section in color.

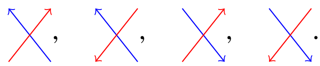
Let $\mathcal{H}eis_l(z, u) \odot \mathcal{H}eis_m(z, v)$ be the symmetric product of $\mathcal{H}eis_l(z, u)$ and $\mathcal{H}eis_m(z, v)$ as defined [Brundan et al. 2018, Section 3]. This is the strict $\Bbbk$-linear monoidal category defined by first taking the free product of $\mathcal{H}eis_l(z,u)$ and $\mathcal{H}eis_m(z,v)$, i.e., the strict $\Bbbk$-linear monoidal category defined by the disjoint union of the given generators and relations of $\mathcal{H}eis_l(z,u)$ and of $\mathcal{H}eis_m(z,v)$, then adjoining isomorphisms $\sigma_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ for each pair of objects $X \in \mathcal{H}eis_l(z,u)$ and $Y \in \mathcal{H}eis_m(z,v)$ subject to the relations

$$\sigma_{X_1 \otimes X_2, Y} = (\sigma_{X_1,Y} \otimes 1_{X_2}) \circ (1_{X_1} \otimes \sigma_{X_2,Y}), \quad \sigma_{X_2,Y} \circ (f \otimes 1_Y) = (1_Y \otimes f) \circ \sigma_{X_1,Y},$$
$$\sigma_{X,Y_1 \otimes Y_2} = (1_{Y_1} \otimes \sigma_{X,Y_2}) \circ (\sigma_{X,Y_1} \otimes 1_{Y_2}), \quad \sigma_{X,Y_2} \circ (1_X \otimes g) = (g \otimes 1_X) \circ \sigma_{X,Y_1}$$

for all $X, X_1, X_2 \in \mathcal{H}eis_l(z,u)$, $Y, Y_1, Y_2 \in \mathcal{H}eis_m(z,v)$ and $f : X_1 \to X_2$, $g : Y_1 \to Y_2$. Morphisms in $\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$ are linear combinations of diagrams colored both blue and red. In these diagrams, as well as the generating morphisms of $\mathcal{H}eis_l(z,u)$ and $\mathcal{H}eis_m(z,v)$, we have the additional two-color crossings



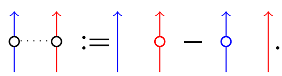
which represent the isomorphisms $\sigma_{X,Y}$ for $X \in \{\uparrow, \downarrow\}$ and $Y \in \{\uparrow, \downarrow\}$, and their inverses



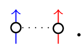
Definition 8.1. Given a diagram D representing a morphism in $\mathcal{H}eis_l(z,u) \odot \mathcal{H}eis_m(z,v)$ and two generic points in this diagram, one on a red string and the other on a blue string, we will denote the morphism represented by

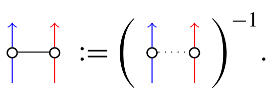
$$(D \text{ with an extra dot at the red point}) - (D \text{ with an extra dot at the blue point})$$

by labeling the points with dots joined by a dotted line. For example



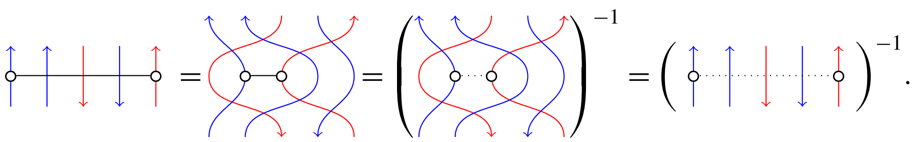
(8.2)

Let $\mathcal{H}eis_l(z,u) \bar{\odot} \mathcal{H}eis_m(z,v)$ be the strict $\Bbbk$-linear monoidal category obtained by localizing at  . This means that we adjoin a two-sided inverse to this morphism, which we denote as a dumbbell

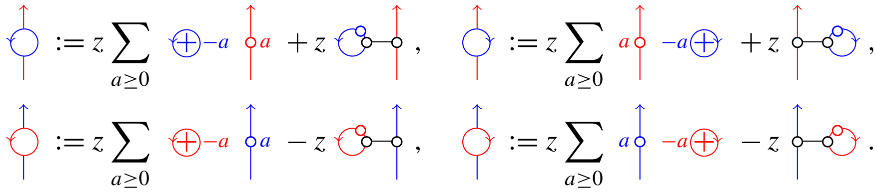


(8.3)

Just as explained in the degenerate case in [Brundan et al. 2018, Sections 4–5], all morphisms whose string diagram is that of an identity morphism with a horizontal dotted line joining two points of different colors are also automatically invertible in the localized category. We also denote the inverses of such morphisms by using a solid dumbbell in place of the dotted one. For instance



We also need the following morphisms, which we refer to as *internal bubbles*:



(8.4)

(8.5)

The category $\mathcal{H}eis_l(z,u)\bar{\odot}\mathcal{H}eis_m(z,v)$ possesses various symmetries which are often useful. Derived from (3.3), we have the strict $\Bbbk$-linear monoidal isomorphism

$$\Omega_{l|m} : \mathcal{H}eis_l(z,u)\bar{\odot}\mathcal{H}eis_m(z,v) \xrightarrow{\sim} (\mathcal{H}eis_{-l}(z,u^{-1})\bar{\odot}\mathcal{H}eis_{-m}(z,v^{-1}))^{\mathrm{op}}, \tag{8.6}$$

which takes a diagram to its mirror image in a horizontal plane multiplied by $(-1)^{x+y}$ where x is the number of one-colored crossings and y is the number of leftward cups and caps (including ones in (+)-, (−)- and internal bubbles). Also, we have

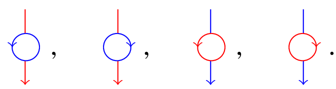
$$\mathrm{flip} : \mathcal{H}eis_l(z,u)\bar{\odot}\mathcal{H}eis_m(z,v) \xrightarrow{\sim} \mathcal{H}eis_m(z,v)\bar{\odot}\mathcal{H}eis_l(z,u) \tag{8.7}$$

defined on diagrams by switching the colors blue and red then multiplying by $(-1)^z$ where z is the total number of dumbbells (both solid and dotted) in the picture. Finally, the category $\mathcal{H}eis_l(z,u)\bar{\odot}\mathcal{H}eis_m(z,v)$ is strictly pivotal, with duality functor

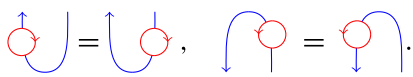
$$* : \mathcal{H}eis_l(z,u)\bar{\odot}\mathcal{H}eis_m(z,v) \xrightarrow{\sim} ((\mathcal{H}eis_l(z,u)\bar{\odot}\mathcal{H}eis_m(z,v))^{\mathrm{op}})^{\mathrm{rev}} \tag{8.8}$$

defined by rotating diagrams through 180° just like in (3.21).

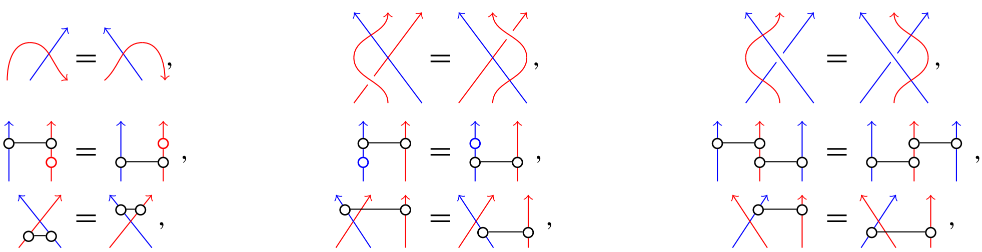
We denote the duals of the internal bubbles (8.4)–(8.5) by

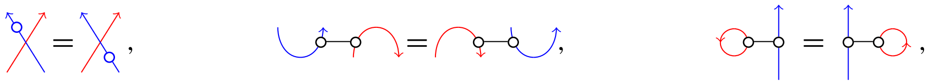


This definition ensures that internal bubbles commute past cups and caps in all possible configurations. For example



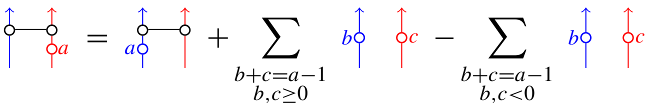
Again as in [Brundan et al. 2018, Sections 4–5], there are many other obvious commuting relations, such as





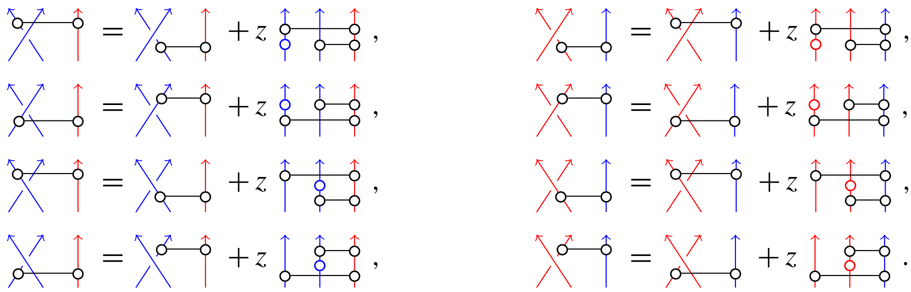
as well as the mirror images of these under the symmetries $\Omega_{l|m}$, flip and $*$. We will appeal to all such relations below without further mention.

Here are some more interesting relations. The first shows how to "teleport" dots across dumbbells (plus a correction term):



(8.9)

for any $a \in \mathbb{Z}$. We also have the following relations to commute dumbbells past one-color crossings:



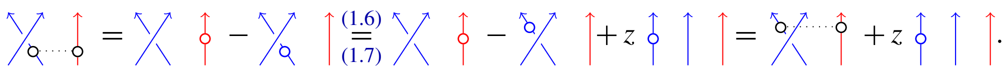
(8.10)

(8.11)

(8.12)

(8.13)

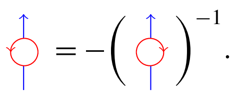
These are all straightforward to prove: one first cancels the solid dumbbells by composing on the top and bottom with their inverses then uses the affine Hecke algebra relations (1.6)–(1.7) to commute dots past crossings in the result. For example, to prove the first relation in (8.10), we have



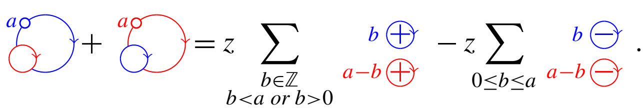
We then compose on the top with a solid dumbbell connecting the red strand and the leftmost blue strand, and compose on the bottom with a solid dumbbell connecting the red strand and the rightmost blue strand.

The following seven lemmas are the quantum analogs of [Brundan et al. 2018, Lemmas 5.6–5.12]. Their proofs are quite similar to the degenerate case.

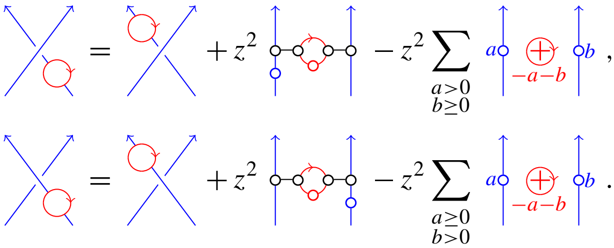
Lemma 8.2. *We have that*



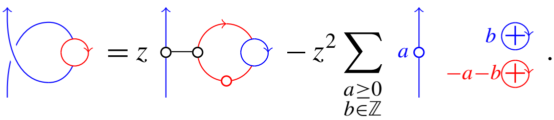
Lemma 8.3. *For any $a \in \mathbb{Z}$, we have that*



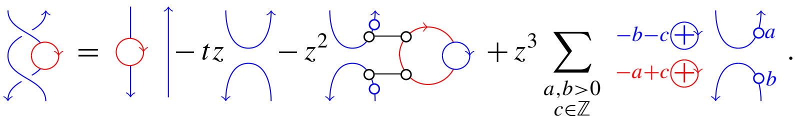
Lemma 8.4. *The following relations hold*:



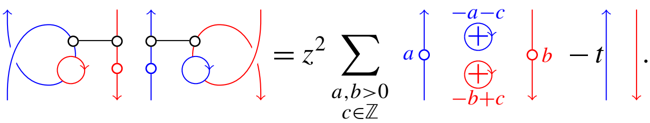
Lemma 8.5. *We have that*



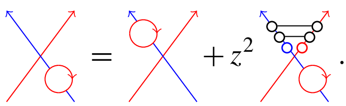
Lemma 8.6. *We have that*



Lemma 8.7. *We have that*



Lemma 8.8. *We have that*

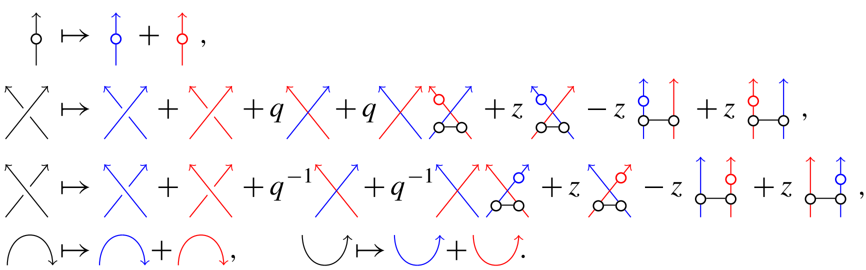


Using these, we can prove the main theorem of the section:

Theorem 8.9. *For $k = l + m$ and $t = uv$, there is a unique strict $\Bbbk$-linear monoidal functor*

$$\Delta_{l|m} : \mathcal{H}eis_k(z,t) \to \mathrm{Add}(\mathcal{H}eis_l(z,u) \bar{\odot} \mathcal{H}eis_m(z,v))$$

such that $\uparrow \mapsto \uparrow \oplus \uparrow$, $\downarrow \mapsto \downarrow \oplus \downarrow$, and on morphisms



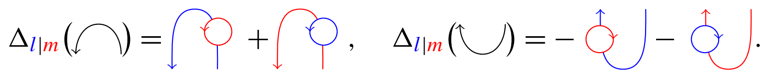
(8.14)

(8.15)

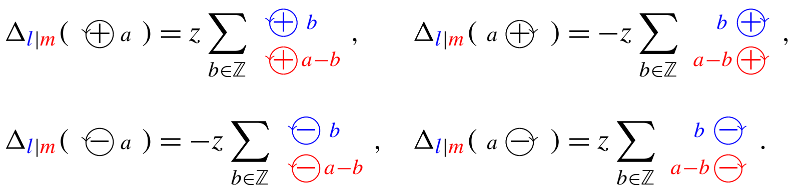
(8.16)

(8.17)

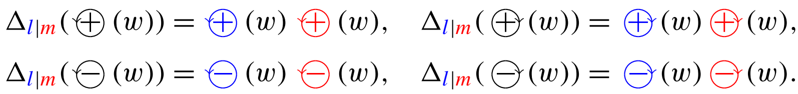
Moreover, we have that


(8.18)

Also, the following hold for all $a \in \mathbb{Z}$:


(8.19)
(8.20)

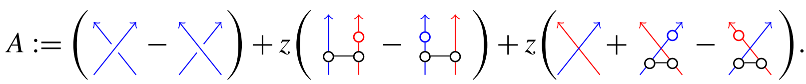
Equivalently, in terms of the generating functions (3.14)–(3.17) *and their analogs in* $\mathcal{H}eis_l(z,u)$ *and* $\mathcal{H}eis_m(z,v)$


(8.21)
(8.22)

Remark 8.10. For the proof, it is helpful to notice that flip $\circ \Delta_{l|m} = \Delta_{m|l}$ (on extending flip to the additive envelopes in the obvious way). However, $\Delta_{l|m}$ does not commute with either of the other symmetries Ω or $*$. In fact, the map $\Omega_{-l|-m} \circ \Delta_{-l|-m} \circ \Omega_k$ would be an equally good alternative choice for the categorical comultiplication map. The only change to the above formulae if one uses this alternative is that one needs to replace q with $-q^{-1}$ in (8.15)–(8.16); this is the "Galois symmetry" in the choice of the root q of the equation $x^2 - zx - 1 = 0$.

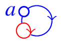
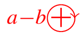
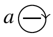
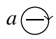
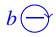
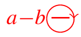
Proof. In view of the uniqueness from Lemma 4.3, we may take (8.14)–(8.18) as the definition of $\Delta_{l|m}$ on generating morphisms, and must check that the images of the relations (1.6)–(1.9) and (4.1)–(4.4) from Definition 4.1 are all satisfied in $\mathrm{Add}(\mathcal{H}eis_l(z,u) \bar{\odot} \mathcal{H}eis_m(z,v))$; we must also check (8.19)–(8.20). The details are sufficiently similar to the degenerate case from the proof of [Brundan et al. 2018, Theorem 5.4] that we only sketch the steps needed below.

First one checks (1.6)–(1.8). For example, to check the skein relation, the image under $\Delta_{l|m}$ of ╳ − ╳ is $A + \mathrm{flip}(A)$ where

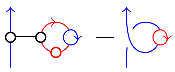
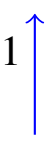


Using the skein relation in $\mathcal{H}eis_l(z,u)$ plus (8.9), A simplifies to $B := z$ ↑ ↑ $+ z$ ↑ ↑. This is what is required since the image under $\Delta_{l|m}$ of z ↑ ↑ is $B + \mathrm{flip}(B)$. The other relations here are checked by similarly explicit calculations. The one for the braid relation is rather long.

The relation (1.9) is easy.

To check (8.19)–(8.20), we assume to start with that $k \geq 0$. Consider the clockwise $(+)$-bubble a⊕. When $a \leq 0$, this is just a scalar (usually zero) due to (3.11) and the assumption $k \geq 0$, and the relation to be checked is trivial. So assume that $a > 0$. Then  $=$ , hence, its image under $\Delta_{l|m}$ is $-$  $-$ , which is indeed equal to $-z\sum_{b\in\mathbb{Z}}$   by Lemma 8.3. This establishes the left-hand identity in (8.19), hence, the left-hand identity in (8.21). The right-hand identity in (8.21) then follows using (3.13), thereby establishing the right-hand identity in (8.19) as well. Next, consider the clockwise $(-)$-bubble . This time the relation to be checked is trivial when $a \geq 0$, so assume that $a < 0$. Then, using the assumption $k \geq 0$ again, we have that  $=$ , hence, its image under $\Delta_{l|m}$ is $-$  $-$ , which is equal to $z\sum_{b\in\mathbb{Z}}$   by Lemma 8.3 (noting when $a < 0 \leq k$ that the term involving $(+)$-bubbles is zero). Then we complete the proof of (8.20) using the equivalent form (8.22) and (3.13) once again. It remains to treat $k \leq 0$. This follows by similar arguments; one starts by considering the counterclockwise $(+)$- and $(-)$-bubbles using the identities obtained by applying $\Omega_{l|m}$ to Lemma 8.3, then gets the clockwise ones using (3.13).

Consider (4.3)–(4.4). The relations involving bubbles follow easily from (8.19)–(8.20). Next consider the right curl relation in (4.3), so $k \geq 0$. Applying $\Delta_{l|m}$ to the relation reveals that we must show that $A + \mathrm{flip}(A) = B + \mathrm{flip}(B)$ where

$A := z$ , $\quad B := \delta_{k,0}t^{-1}$ .

This follows from Lemma 8.5, noting that the only nonzero term in the summation on the right-hand side of that identity is the one with $a = b = 0$ due to the assumption that $k \geq 0$. The argument for the left curl in (4.4) is entirely similar; it uses the identity obtained by applying $* \circ \Omega_{l|m}$ to Lemma 8.5.

Finally, one must check (4.1)–(4.2). This is a calculation just like in the final paragraph of the proof of [Brundan et al. 2018, Theorem 5.4]; ultimately one uses Lemmas 8.6–8.8. □

9. Generalized cyclotomic quotients

In this section, we define some $\Bbbk$-linear categories, namely, the generalized cyclotomic quotients of $\mathcal{H}eis_k(z,t)$. Recall that $x =$  and $y =$ .

Definition 9.1. Suppose we are given monic polynomials

$$f(w) = f_0 w^l + f_1 w^{l-1} + \cdots + f_l \in \Bbbk[w], \tag{9.1}$$

$$g(w) = g_0 w^m + g_1 w^{m-1} + \cdots + g_m \in \Bbbk[w] \tag{9.2}$$

such that $k = m - l$ and $t^2 = f_l/g_m$. Define

$$\mathbb{O}^+(w) = t^{-1}z\sum_{n\in\mathbb{Z}} \mathbb{O}_n^+ w^{-n} := g(w)/f(w) \in w^k + w^{k-1}\Bbbk[\![w^{-1}]\!], \tag{9.3}$$

$$\widetilde{\mathbb{O}}^+(w) = -tz\sum_{n\in\mathbb{Z}} \widetilde{\mathbb{O}}_n^+ w^{-n} := f(w)/g(w) \in w^{-k} + w^{-k-1}\Bbbk[\![w^{-1}]\!], \tag{9.4}$$

$$\mathbb{O}^-(w) = -tz\sum_{n\in\mathbb{Z}} \mathbb{O}_n^- w^{-n} := t^2 g(w)/f(w) \in 1 + w\Bbbk[\![w]\!], \tag{9.5}$$

$$\widetilde{\mathbb{O}}^-(w) = t^{-1}z\sum_{n\in\mathbb{Z}} \widetilde{\mathbb{O}}^-_n w^{-n} := t^{-2} f(w)/g(w) \in 1 + w\Bbbk[\![w]\!]; \tag{9.6}$$

cf. (3.14)–(3.17). Let $\mathcal{I}(f|g)$ be the left tensor ideal generated by the morphisms

$$\{f(x),\ \oplus\, n - \mathbb{O}^+_n \mathbb{1} \mid -k < n < l\}. \tag{9.7}$$

The *generalized cyclotomic quotient* associated to the polynomials $f(w)$ and $g(w)$ is the quotient category

$$\mathcal{H}(f|g) := \mathcal{H}eis_k(z,t)/\mathcal{I}(f|g). \tag{9.8}$$

It is a module category over $\mathcal{H}eis_k(z,t)$.

The following is the quantum analog of [Brundan 2018, Lemma 1.8]; see also [Brundan and Davidson 2017, Lemma 4.14] for the analog in the setting of Kac–Moody 2-categories.

Lemma 9.2. *In the setup of Definition 9.1, $\mathcal{I}(f|g)$ may be defined equivalently as the left tensor ideal generated by*

$$\{g(y),\ n\,\oplus - \widetilde{\mathbb{O}}^+_n \mathbb{1} \mid k < n < m\}. \tag{9.9}$$

Moreover, it contains $\oplus\, n - \mathbb{O}^+_n\mathbb{1}$, $\ominus\, n - \mathbb{O}^-_n\mathbb{1}$, $n\,\oplus - \widetilde{\mathbb{O}}^+_n\mathbb{1}$ and $n\,\ominus - \widetilde{\mathbb{O}}^-_n\mathbb{1}$ for all $n\in\mathbb{Z}$.

Proof. For morphisms $\theta, \phi : X \to Y$, we will write $\theta \equiv \phi$ as shorthand for $\theta - \phi \in \mathcal{I}(f|g)$. By (3.11)–(3.12), we have automatically that $\oplus\, n \equiv \mathbb{O}^+_n\mathbb{1}$ when $n \le -k$, $n\,\oplus \equiv \widetilde{\mathbb{O}}^+_n\mathbb{1}$ when $n \le k$, $\ominus\, n \equiv \mathbb{O}^-_n\mathbb{1}$ when $n \ge 0$, and $n\,\ominus \equiv \widetilde{\mathbb{O}}^-_n\mathbb{1}$ when $n \ge 0$.

In this paragraph, we use ascending induction on n to show that $\oplus\, n \equiv \mathbb{O}^+_n\mathbb{1}$ for all $n\in\mathbb{Z}$. This is immediate from (9.7) if $n < l$, so assume that $n \ge l$. The fact that $f(x) \equiv 0$ implies that

$$\sum_{a=0}^{l} f_a \oplus\, {n-a} + \sum_{a=0}^{l} f_a \ominus\, {n-a} = \sum_{a=0}^{l} f_a \circ\, {n-a} \equiv 0.$$

On the left-hand side of this, the only nonzero $(-)$-bubble arises when $n = a = l$, so it shows that $\sum_{a=0}^{l} f_a \oplus\, {n-a} \equiv \delta_{l,n} f_l t^{-1} z^{-1}\mathbb{1}$. Using the induction hypothesis and $f_l = g_m t^2$, we deduce that $\oplus\, n + \sum_{a=1}^{l} f_a \mathbb{O}^+_{n-a}\mathbb{1} \equiv \delta_{l,n} g_m t z^{-1}\mathbb{1}$. Equating w^{l-n}-coefficients in $f(w)\mathbb{O}^+(w) = g(w)$, we get that $\sum_{a=0}^{l} f_a \mathbb{O}^+_{n-a} = \delta_{l,n} g_m t z^{-1}$. Hence, $\oplus\, n \equiv \mathbb{O}^+_n\mathbb{1}$ as claimed.

Next, we show by descending induction on n that $\ominus\, n \equiv \mathbb{O}^-_n\mathbb{1}$ for all $n\in\mathbb{Z}$. We may assume that $n < 0$. Equating w^{-n}-coefficients in $f(w)\mathbb{O}^+(w) = t^{-2} f(w)\mathbb{O}^-(w)$ gives that

$$\sum_{a=0}^{l} f_{l-a}\mathbb{O}^+_{a+n} = -\sum_{a=0}^{l} f_{l-a}\mathbb{O}^-_{a+n}.$$

Using the induction hypothesis plus the previous paragraph, we deduce that

$$\sum_{a=0}^{l} f_{l-a} \oplus\, {a+n} + f_l \mathbb{O}^-_n + \sum_{a=1}^{l} f_{l-a} \ominus\, {a+n} \equiv 0.$$

But also from $f(x) \equiv 0$ we get that

$$\sum_{a=0}^{l} f_{l-a} \oplus_{a+n} + \sum_{a=0}^{l} f_{l-a} \ominus_{a+n} = \sum_{a=0}^{l} f_{l-a} \bigcirc_{a+n} \equiv 0.$$

Taking the difference of these two identities establishes the induction step.

Using the notation of (3.14)–(3.17), we have now shown that $\oplus(w) \equiv \mathbb{O}^{\pm}(w)\mathbb{1}$. Taking inverses using (3.13), we deduce that $\oplus(w) \equiv \widetilde{\mathbb{O}}^{\pm}(w)\mathbb{1}$. Hence, ${}_n\oplus \equiv \widetilde{\mathbb{O}}_n^{\pm}\mathbb{1}$ for all $n \in \mathbb{Z}$. So we have established the last assertion from the lemma.

Equating w^b-coefficients in $g(w) = f(w)\mathbb{O}^+(w)$ shows that $g_{m-b} = t^{-1}z\sum_{a=0}^{l} f_{l-a}\mathbb{O}^+_{a-b}$. Hence

$$g(y) = \sum_{a=0}^{l} t^{-1} f_{l-a}\Big(z \sum_{b\geq 0} {}^{b}\!\downarrow \oplus_{a-b}\Big) \overset{(4.17)}{=} \sum_{a=0}^{l} t^{-1} f_{l-a} \downarrow\!\bigcirc_{a} \equiv 0.$$

We have now shown that $\mathcal{I}(f|g)$, the left tensor ideal generated by (9.7), contains (9.9). Similarly, the left tensor ideal generated by (9.9) contains (9.7). This completes the proof. □

We assume for the rest of the section that $\Bbbk$ is a field, and that we are given a factorization $t = uv^{-1}$ for $u, v \in \Bbbk^\times$ such that $u^2 = f_l$ and $v^2 = g_m$. Let $\mathcal{V}(f)$ and $\mathcal{V}(g)^\vee$ denote $\bigoplus_{n\geq 0} H_n^f$-pmod and $\bigoplus_{n\geq 0} H_n^g$-pmod viewed as a module categories over $\mathcal{H}eis_{-l}(z,u)$ and $\mathcal{H}eis_m(z,v^{-1})$ via the monoidal functors Ψ_f and $\Psi_g^\vee$ from Lemma 6.4. Let

$$\mathcal{V}(f|g) := \mathcal{V}(f) \boxtimes \mathcal{V}(g)^\vee \tag{9.10}$$

be their linearized Cartesian product, i.e., the $\Bbbk$-linear category with objects that are pairs (X, Y) for $X \in \mathcal{V}(f)$, $Y \in \mathcal{V}(g)^\vee$, and morphisms

$$\mathrm{Hom}_{\mathcal{V}(f|g)}((X,Y),(U,V)) := \mathrm{Hom}_{\mathcal{V}(f)}(X,U) \otimes \mathrm{Hom}_{\mathcal{V}(g)^\vee}(Y,V)$$

with the obvious composition law. There is an equivalence of categories

$$\mathcal{V}(f|g) \to \bigoplus_{r,s\geq 0} (H_r^f \otimes H_s^g)\text{-pmod},$$

hence, $\mathcal{V}(f|g)$ is additive Karoubian. Moreover, $\mathcal{V}(f|g)$ is a module category over the symmetric product $\mathcal{H}eis_{-l}(z,u) \odot \mathcal{H}eis_m(z,v^{-1})$.

Lemma 9.3. *Let V be a finite-dimensional AH_2-module. All eigenvalues of x_2 on V are of the form λ, $q^2\lambda$ or $q^{-2}\lambda$ for eigenvalues λ of x_1 on V.*

Proof. We may assume for the proof that $\Bbbk$ is algebraically closed. Suppose that $v \in V$ is a simultaneous eigenvector for the commuting operators x_1 and x_2 of eigenvalues λ_1 and λ_2, respectively. If $\tau_1 v = qv$ (resp. $\tau_1 v = -q^{-1}v$) then $\lambda_2 = q^2\lambda_1$ (resp. $\lambda_2 = q^{-2}\lambda_1$), as follows easily from the relation $x_2(\tau_1 - z)v = \tau_1 x_1 v$. Otherwise, v and $\tau_1 v$ are linearly independent, in which case the matrix describing the action of x_1 on the subspace with basis $\{v, \tau_1 v\}$ is $\begin{pmatrix} \lambda_1 & -z\lambda_2 \\ 0 & \lambda_2 \end{pmatrix}$. So λ_2 is another eigenvalue of x_1 on V. □

Lemma 9.4. *Assume that $f(w)$ and $g(w)$ split as products of linear factors in $\Bbbk[w]$, and moreover assume that $\lambda\mu^{-1} \notin \{q^{2i} \mid i \in \mathbb{Z}\}$ for all roots λ of $f(w)$ and μ of $g(w)$. Then the categorical action of $\mathcal{H}eis_{-l}(z,u) \odot \mathcal{H}eis_m(z,v^{-1})$ on $\mathcal{V}(f|g)$ defined above extends to an action of the localization $\mathcal{H}eis_{-l}(z,u)\bar{\odot}\mathcal{H}eis_m(z,v^{-1})$ from Definition 8.1.*

Proof. Lemma 9.3 implies that the eigenvalues of $x_1,\dots,x_n$ on any finite-dimensional H_n^f-module are of the form $q^{2i}\lambda$ for $i \in \mathbb{Z}$ and a root λ of $f(w)$. Consequently, the commuting endomorphisms defined by evaluating $\uparrow\ \mathring{\uparrow}$ and $\mathring{\uparrow}\ \uparrow$ on an object of $\mathcal{V}(f|g)$ have eigenvalues contained in the sets $\{q^{2i}\lambda \mid i \in \mathbb{Z}, \lambda \text{ a root of } f(w)\}$ and $\{q^{2j}\mu \mid j \in \mathbb{Z}, \mu \text{ a root of } g(w)\}$, respectively. By the genericity assumption, these sets are disjoint, hence, all eigenvalues of the endomorphism defined by $\mathring{\uparrow}\cdots\mathring{\uparrow} = \uparrow\ \mathring{\uparrow} - \mathring{\uparrow}\ \uparrow$ lie in $\Bbbk^\times$. Consequently, this endomorphism is invertible. □

Lemma 9.4 shows for suitably generic $f(w), g(w)$ that there is a strict $\Bbbk$-linear monoidal functor $\Psi_f\bar{\odot}\Psi_g^\vee : \mathcal{H}eis_{-l}(z,u)\bar{\odot}\mathcal{H}eis_m(z,v^{-1}) \to \mathcal{E}nd_\Bbbk(\mathcal{V}(f|g))$. Composing this functor with the functor $\Delta_{-l|m}$ from Theorem 8.9, we obtain a strict $\Bbbk$-linear monoidal functor

$$\Psi_{f|g} := \Psi_f\bar{\odot}\Psi_g^\vee \circ \Delta_{-l|m} : \mathcal{H}eis_k(z,t) \to \mathcal{E}nd_\Bbbk(\mathcal{V}(f|g)). \tag{9.11}$$

Thus, we have made $\mathcal{V}(f|g)$ into a module category over $\mathcal{H}eis_k(z,t)$.

Theorem 9.5. *Assume that $f(w), g(w)$ satisfy the genericity assumption from Lemma 9.4 so that (9.11) is defined. Let $\mathrm{Ev} : \mathcal{E}nd_\Bbbk(\mathcal{V}(f|g)) \to \mathcal{V}(f|g)$ be the $\Bbbk$-linear functor defined by evaluation on $S := (H_0^f, H_0^g) \in \mathcal{V}(f|g)$. The composition $\mathrm{Ev}\circ\Psi_{f|g}$ factors through the generalized cyclotomic quotient $\mathcal{H}(f|g)$ to induce an equivalence of $\mathcal{H}eis_k(z,t)$-module categories*

$$\psi_{f|g} : \mathrm{Kar}(\mathcal{H}(f|g)) \to \mathcal{V}(f|g).$$

Proof. We first show that $\Psi_{f|g}(\oplus(w))_S \in w^k \operatorname{End}(S)[\![w^{-1}]\!]$ equals $\mathbb{O}^+(w)1_S$. Recalling that $\mathbb{O}^+(w)$ is the expansion at $w=\infty$ of the rational function $g(w)/f(w)$, this follows because

$$\Psi_{f|g}(\oplus(w))_S = \Psi_f(\oplus(w))_{H_0^f} \otimes \Psi_g^\vee(\oplus(w))_{H_0^g}$$

thanks to (8.21), and also $\Psi_f(\oplus(w))_{H_0^f} = 1/f(w)$ and $\Psi_g^\vee(\oplus(w))_{H_0^g} = g(w)$. To see the last two assertions, we first apply Lemma 9.2 to see that $\mathcal{I}(f|1)$, the left tensor ideal of $\mathcal{H}eis_{-l}(z,u)$ generated by $f(x)$, contains all coefficients of the series $\oplus(w) - 1/f(w)\mathbb{1}$; all elements of this ideal act as zero on H_0^f since its generator $f(x)$ acts as zero. Then we apply Lemma 9.2 again to see that $\mathcal{I}(1|g)$, the left tensor ideal of $\mathcal{H}eis_m(z,v^{-1})$ generated by $g(y)$, contains all coefficients of $\oplus(w) - g(w)\mathbb{1}$; all elements of this act as zero on H_0^g.

The previous paragraph shows that $\oplus_n - \mathbb{O}_n^+\mathbb{1}$ acts as zero on S for all $n \in \mathbb{Z}$. Also it is obvious that $f(x)$ acts as zero on S. So the left tensor ideal $\mathcal{I}(f|g)$ acts as zero on S, which proves that $\mathrm{Ev}\circ\Psi_{f|g}$ factors through the quotient $\mathcal{H}(f|g) = \mathcal{H}eis_k(z,t)/\mathcal{I}(f|g)$ to induce a $\Bbbk$-linear functor $\mathcal{H}(f|g) \to \mathcal{V}(f)\boxtimes\mathcal{V}(g)^\vee$. Since $\mathcal{V}(f|g)$ is additive Karoubian, this extends to the Karoubi envelope to induce the functor $\psi_{f|g}$ from the statement of the theorem. Moreover, it is automatic from the definition that $\psi_{f|g}$ is a morphism of $\mathcal{H}eis_k(z,t)$-module categories. It just remains to show that $\psi_{f|g}$ is an equivalence, which we do by showing that it is full, faithful and dense.

First we show that $\psi_{f|g}$ is full and faithful. It suffices to check this on objects $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ that are words in $\uparrow$ and $\downarrow$. We assume moreover that $k \geq 0$; a similar argument with the roles of $\uparrow$ and $\downarrow$ interchanged does the job when $k \leq 0$ too. Let $X^* = X_1^* \otimes \cdots \otimes X_r^*$ be the dual object (here, $\uparrow^* = \downarrow$, $\downarrow^* = \uparrow$). By rigidity, we have a canonical isomorphism $\mathrm{Hom}_{\mathcal{H}(f|g)}(X,Y) \cong \mathrm{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, X^* \otimes Y)$, from which we get a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{H}(f|g)}(X,Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, X^* \otimes Y) \\ \psi_{f|g}\downarrow & & \downarrow\psi_{f|g} \\ \mathrm{Hom}_{\mathcal{V}(f|g)}(X \otimes S, Y \otimes S) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{V}(f|g)}(S, X^* \otimes Y \otimes S). \end{array}$$

The left-hand vertical map in this diagram is an isomorphism if and only if the right-hand vertical map is one. We claim that the left-hand vertical map is an isomorphism when $X = Y = \uparrow^{\otimes n}$. To prove this, the usual straightening algorithm (see the beginning of the proof of Theorem 10.1 for details) shows that $\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\uparrow^{\otimes n})$ is spanned by diagrams in the image of the canonical homomorphism $AH_n \to \mathrm{End}_{\mathcal{H}eis_k(z,t)}(\uparrow^{\otimes n})$, with some number of bubbles added to the right-hand edge. Thus we have an induced homomorphism $H_n^f \to \mathrm{End}_{\mathcal{H}(f|g)}(\uparrow^{\otimes n})$ which is surjective since bubbles on the right-hand edge are scalars in the generalized cyclotomic quotient. On the other hand, $\mathrm{End}_{\mathcal{V}(f|g)}(\uparrow^{\otimes n} \otimes S) = \mathrm{End}_{H_n^f}(H_n^f) = H_n^f$. The claim follows. Hence, the right-hand vertical map is an isomorphism when $X^* \otimes Y = \downarrow^{\otimes n} \otimes \uparrow^{\otimes n}$. Using this, we can show that the right hand vertical map is an isomorphism in general. All of the morphism spaces are zero unless $X^* \otimes Y$ has the same number of $\uparrow$'s as $\downarrow$'s. If all $\downarrow$'s are to the left of all $\uparrow$'s, we are done already, so we may assume that $X^* \otimes Y$ involves $\uparrow \otimes \downarrow$ as a subword. Let U be $X^* \otimes Y$ with the two letters in this subword interchanged and V be $X^* \otimes Y$ with these two letters deleted. Using the isomorphism $\uparrow \otimes \downarrow \cong \downarrow \otimes \uparrow \oplus \mathbb{1}^{\oplus k}$ from (2.10), we get a commuting diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, X^* \otimes Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{H}(f|g)}(\mathbb{1}, U \oplus V^{\oplus k}) \\ \psi_{f|g}\downarrow & & \downarrow\psi_{f|g} \\ \mathrm{Hom}_{\mathcal{V}(f|g)}(S, X^* \otimes Y \otimes S) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{V}(f|g)}(S, U \otimes S \oplus V \otimes S^{\oplus k}) \end{array}$$

By induction, the right-hand vertical map is an isomorphism, hence, so too is the left-hand one.

Finally, we explain why $\psi_{f|g}$ is dense. Let Q be an indecomposable object in $\mathcal{V}(f|g)$. We have that $\downarrow^{\otimes m} \otimes \uparrow^{\otimes n} \otimes S = \downarrow^{\otimes m} \otimes (H_n^f, H_0^g) = (H_n^f, H_m^g) \oplus M$ where M is a direct sum of summands of $(H_{n'}^f, H_{m'}^g)$ with $n' < n$ and $m' < m$. It follows that Q is isomorphic to the image of some idempotent in $\mathrm{End}_{\mathcal{V}(f|g)}(\downarrow^{\otimes m} \otimes \uparrow^{\otimes n} \otimes S)$ for some $m, n \geq 0$. Since we have shown already that $\psi_{f|g}$ is full and faithful, there is a corresponding idempotent in $\mathrm{End}_{\mathcal{H}(f|g)}(\downarrow^{\otimes m} \otimes \uparrow^{\otimes n})$. The latter idempotent defines an object P of $\mathrm{Kar}(\mathcal{H}(f|g))$ such that $\psi_{f|g}(P) \cong Q$. □

Remark 9.6. If $g(w) = 1$ the genericity assumption is vacuous, so Theorem 9.5 gives us an equivalence of categories $\psi_{f|1} : \mathrm{Kar}(\mathcal{H}(f|1)) \to \mathcal{V}(f)$. In other words, the generalized cyclotomic quotient $\mathcal{H}(f|1)$

is Morita equivalent to the "usual" cyclotomic quotient defined by the cyclotomic Hecke algebras H_n^f for all $n \geq 0$. This statement is the quantum analog of [Brundan 2018, Theorem 1.7]; see also [Rouquier 2012, Theorem 4.25] for the analogous result in the setting of Kac–Moody 2-categories.

Remark 9.7. More generally, suppose that there are factorizations $f(w) = f_1(w) f_2(w)$ and $g(w) = g_1(w) g_2(w)$ such that the genericity assumption $\lambda\mu^{-1} \notin \{q^{2i} \mid i \in \mathbb{Z}\}$ holds for λ a root of $f_1(w)$ or $g_1(w)$, and μ a root of $f_2(w)$ or $g_2(w)$. Then a similar argument to the proof of Theorem 9.5 can be used to show that the categories $\mathrm{Kar}(\mathcal{H}(f|g))$ and $\mathrm{Kar}(\mathcal{H}(f_1|g_1) \boxtimes \mathcal{H}(f_2|g_2))$ are equivalent. In particular, applying this to $\mathrm{Kar}(\mathcal{H}(f|1))$ and using the previous remark, it follows that the cyclotomic Hecke algebra H_n^f is Morita equivalent to $\bigoplus_{n_1+n_2=n} H_{n_1}^{f_1} \otimes H_{n_2}^{f_2}$, thereby recovering a result of Dipper and Mathas [2002].

10. Basis theorem

Finally, we prove a basis theorem for the morphism spaces in $\mathcal{H}eis_k(z,t)$. Our proof of this is very similar to the argument in the degenerate case from [Brundan et al. 2018, Theorem 6.4]. Let $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ be objects of $\mathcal{H}eis_k(z,t)$ for $X_i, Y_j \in \{\uparrow, \downarrow\}$. An (X,Y)-*matching* is a bijection between $\{i \mid X_i = \uparrow\} \sqcup \{j \mid Y_j = \downarrow\}$ and $\{i \mid X_i = \downarrow\} \sqcup \{j \mid Y_j = \uparrow\}$. A *reduced lift* of an (X,Y)-matching means a diagram representing a morphism $X \to Y$ such that

- the endpoints of each string are points which correspond under the given matching;
- there are no floating bubbles and no dots on any string;
- there are no self-intersections of strings and no two strings cross each other more than once.

Fix a set $B(X,Y)$ consisting of a choice of reduced lift for each of the (X,Y)-matchings. Let $B_\circ(X,Y)$ be the set of all morphisms that can be obtained from the elements of $B(X,Y)$ by adding dots labeled with integer multiplicities near to the terminus of each string. Also recall the homomorphism $\beta : \mathrm{Sym} \otimes \mathrm{Sym} \to \mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1})$ from (3.7). Using it, we can make the morphism space $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y)$ into a right $\mathrm{Sym} \otimes \mathrm{Sym}$-module: $\phi\theta := \phi \otimes \beta(\theta)$.

Theorem 10.1. *For any ground ring $\Bbbk$, parameters $z, t \in \Bbbk^\times$, and objects $X, Y \in \mathcal{H}eis_k(z,t)$, the morphism space $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y)$ is a free right $\mathrm{Sym} \otimes \mathrm{Sym}$-module with basis $B_\circ(X,Y)$.*

Proof. We just prove this when $k \leq 0$; the result for $k \geq 0$ then follows by applying Ω_k. Let $X = X_r \otimes \cdots \otimes X_1$ and $Y = Y_s \otimes \cdots \otimes Y_1$ be two objects.

We first observe that $B_\circ(X,Y)$ spans $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X,Y)$ as a right $\mathrm{Sym} \otimes \mathrm{Sym}$-module. The defining relations and the additional relations derived in Sections 2, 3 and 4 give Reidemeister-type relations modulo terms with fewer crossings, plus a skein relation and bubble and dot sliding relations. These relations allow diagrams for morphisms in $\mathcal{H}eis_k(z,t)$ to be transformed in a similar way to the way oriented tangles are simplified in skein categories, modulo diagrams with fewer crossings. Hence, there a straightening algorithm to rewrite any diagram representing a morphism $X \to Y$ as a linear combination of the ones in $B_\circ(X,Y)$.

It remains to prove the linear independence. We say $\phi \in B_\circ(X,Y)$ is *positive* if it only involves nonnegative powers of dots. It suffices to show just that the positive morphisms in $B_\circ(X,Y)$ are linearly independent. Indeed, given any linear relation of the form $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i) = 0$ for morphisms $\phi_i \in B_\circ(X,Y)$ and coefficients $\theta_i \in \mathrm{Sym} \otimes \mathrm{Sym}$, we can "clear denominators" by multiplying the termini of the strings by sufficiently large positive powers of dots to reduce to the positive case.

The main step now is to prove the linear independence in the special case that $X = Y = \uparrow^{\otimes n}$. To do this, we need to allow the ground ring $\Bbbk$ to change, so we will add a subscript to our notation, denoting $\mathcal{H}eis_k(z,t)$, $\mathcal{V}(f|g)$, $\mathrm{Sym}\otimes\mathrm{Sym}, \dots$ by ${}_\Bbbk\mathcal{H}eis_k(z,t)$, ${}_\Bbbk\mathcal{V}(f|g)$, ${}_\Bbbk\mathrm{Sym}\otimes_{\Bbbk\Bbbk}\mathrm{Sym}, \dots$ to avoid any confusion. It suffices to prove the linear independence of positive elements of $B_\circ(X,Y)$ in the special case that $\Bbbk = \mathbb{Z}[z^{\pm 1}, t^{\pm 1}]$; one can then use the canonical $\Bbbk$-linear monoidal functor ${}_\Bbbk\mathcal{H}eis_k(z,t) \to \Bbbk \otimes_{\mathbb{Z}[z^{\pm 1},t^{\pm 1}]} {}_{\mathbb{Z}[z^{\pm 1},t^{\pm 1}]}\mathcal{H}eis_k(z,t)$ to deduce the linear independence over an arbitrary ground ring $\Bbbk$ and for arbitrary parameters.

So assume now that $\Bbbk = \mathbb{Z}[z^{\pm 1}, t^{\pm 1}]$ and take a linear relation $\sum_{i=1}^N \phi_i \otimes \beta(\theta_i) = 0$ for positive $\phi_i \in B_\circ(X,Y)$. Choose a so that the multiplicities of dots in all ϕ_i arising in this linear relation are $\leq a$. Also choose $b, c \geq 0$ so that all of the symmetric functions $\theta_i \in {}_\Bbbk\mathrm{Sym}\otimes_{\Bbbk\Bbbk}\mathrm{Sym}$ are polynomials in the elementary symmetric functions $e_1 \otimes 1, \dots, e_b \otimes 1$ and $1 \otimes e_1, \dots, 1 \otimes e_c$. Then choose l, m so that $a < l$, $b + c < m$ and $k = m - l$. Note that $l \geq m$ due to our standing assumption that $k \leq 0$. Let $u_1, \dots, u_b$ and $v_1, \dots, v_c$ be indeterminates and $\mathbb{K}$ be the algebraic closure of the field $\mathbb{Q}(z, t, u_1, \dots, u_b, v_1, \dots, v_c)$. Pick $q \in \mathbb{K}^\times$ so that $z = q - q^{-1}$ and consider the cyclotomic Hecke algebras ${}_\mathbb{K}H_n^f$ and ${}_\mathbb{K}H_n^g$ over $\mathbb{K}$ associated to the polynomials

$$f(w) := w^l + t^2, \quad g(w) = w^m + u_1 w^{m-1} + \dots + u_b w^{m-b} + v_c w^c + \dots + v_1 w + 1.$$

Note the formula for $g(w)$ makes sense because $b + c < m$. Consider the ${}_\mathbb{K}\mathcal{H}eis_k(z,t)$-module category ${}_\mathbb{K}\mathcal{V}(f|g)$ from (9.11) (taking $u := t$ and $v := 1$). Since $\Bbbk \hookrightarrow \mathbb{K}$, there is a canonical $\Bbbk$-linear monoidal functor ${}_\Bbbk\mathcal{H}eis_k(z,t) \to {}_\mathbb{K}\mathcal{H}eis_k(z,t)$, allowing us to view ${}_\mathbb{K}\mathcal{V}(f|g)$ also as a module category over ${}_\Bbbk\mathcal{H}eis_k(z,t)$. Then we evaluate the relation $\sum \phi_i \otimes \beta(\theta_i) = 0$ on ${}_\mathbb{K}S := ({}_\mathbb{K}H_0^f, {}_\mathbb{K}H_0^g)$ to obtain a relation in ${}_\mathbb{K}H_n^f$. By the basis theorem for ${}_\mathbb{K}H_n^f$ from (6.2) and the assumption that $a < l$, the images of $\phi_1, \dots, \phi_N$ in ${}_\mathbb{K}H_n^f$ are linearly independent over $\mathbb{K}$, so we deduce that the image of $\beta(\theta_i)$ in $\mathbb{K}$ is zero for each i. To deduce from this that $\theta_i = 0$, recall that θ_i is a polynomial in $e_1 \otimes 1, \dots, e_b \otimes 1, 1 \otimes e_1, \dots, 1 \otimes e_c$. So we need to show that the images of $\beta(e_1 \otimes 1), \dots, \beta(e_b \otimes 1), \beta(1 \otimes e_1), \dots, \beta(1 \otimes e_c)$ in $\mathbb{K}$ are algebraically independent. In fact, we claim that these images are the indeterminates $u_1, \dots, u_b, v_1, \dots, v_c$, respectively. To prove this, note that the low degree terms of $\mathbb{O}^\pm(w)$ are

$$\mathbb{O}^+(w) = g(w)/f(w) = w^k + u_1 w^{k-1} + \dots + u_b w^{k-b} + \dots \in w^k \mathbb{K}[\![w^{-1}]\!],$$
$$\mathbb{O}^-(w) = t^2 g(w)/f(w) = 1 + v_1 w + \dots + v_c w^c + \dots \in \mathbb{K}[\![w]\!].$$

By (3.9), (9.3)–(9.5) and Lemma 9.2, the images of $\beta(e_n \otimes 1)$ and $\beta(1 \otimes e_n)$ are the w^{k-n}- and w^n-coefficients of $\mathbb{O}^+(u)$ and $\mathbb{O}^-(u)$, respectively, and the claim follows.

We have now proved the linear independence when $X = Y = \uparrow^{\otimes n}$. Returning to the general case, we can use the canonical isomorphism $\mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(X, Y) \cong \mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}, X^* \otimes Y)$ arising from the rigidity to see that the $\mathrm{Sym} \otimes \mathrm{Sym}$-linear independence of the positive morphisms in $B_\circ(X, Y)$ is equivalent to the $\mathrm{Sym} \otimes \mathrm{Sym}$-linear independence of the positive morphisms in $B_\circ(\mathbb{1}, X^* \otimes Y)$. Thus, we are reduced to the case that $X = \mathbb{1}$. Assume this from now on. The set $B_\circ(\mathbb{1}, Y)$ is empty unless Y has the same number n of $\uparrow$'s as $\downarrow$'s. Also we have already proved the linear independence in the case $Y = \downarrow^{\otimes n} \otimes \uparrow^{\otimes n}$. So we may assume that Y has a subword $\uparrow \otimes \downarrow$. Let Z be Y with the two letters in the subword interchanged. By induction, we may assume the linear independence has already been established for $B_\circ(\mathbb{1}, Z)$. Consider a linear relation $\sum_{i=1}^{N} \phi_i \otimes \beta(\theta_i)$ for positive $\phi_i \in B_\circ(\mathbb{1}, Y)$. Recalling the isomorphism $\uparrow \otimes \downarrow \oplus \mathbb{1}^{\oplus(-k)} \xrightarrow{\sim} \downarrow \otimes \uparrow$ from (2.25), multiplying the subword $\uparrow \otimes \downarrow$ on top by the sideways crossing  defines a $\mathrm{Sym} \otimes \mathrm{Sym}$-linear map $s : \mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}, Z)$. Unfortunately, s does not send $B_\circ(\mathbb{1}, Y)$ into $B_\circ(\mathbb{1}, Z)$. However, the image of $B_\circ(\mathbb{1}, Y)$ is related to $B_\circ(\mathbb{1}, Z)$ in a triangular way, which is good enough to complete the argument. The full explanation of this is almost exactly the same as in the degenerate case, so we refer the reader to the last paragraph of the proof of [Brundan et al. 2018, Theorem 6.4] for the details. □

Corollary 10.2. $$\mathrm{End}_{\mathcal{H}eis_k(z,t)}(\mathbb{1}) \cong \mathrm{Sym} \otimes \mathrm{Sym}.$$

References

[Ariki and Koike 1994] S. Ariki and K. Koike, "A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of its irreducible representations", *Adv. Math.* **106**:2 (1994), 216–243. MR Zbl

[Bezrukavnikov and Etingof 2009] R. Bezrukavnikov and P. Etingof, "Parabolic induction and restriction functors for rational Cherednik algebras", *Selecta Math.* (*N.S.*) **14**:3-4 (2009), 397–425. MR Zbl

[Brundan 2017] J. Brundan, "Representations of the oriented skein category", preprint, 2017. arXiv

[Brundan 2018] J. Brundan, "On the definition of Heisenberg category", *Algebr. Comb.* **1**:4 (2018), 523–544. MR Zbl

[Brundan and Davidson 2017] J. Brundan and N. Davidson, "Categorical actions and crystals", pp. 105–147 in *Categorification and higher representation theory*, edited by A. Beliakova and A. D. Lauda, Contemp. Math. **683**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl

[Brundan et al. 2014] J. Brundan, A. Kleshchev, and P. J. McNamara, "Homological properties of finite-type Khovanov–Lauda–Rouquier algebras", *Duke Math. J.* **163**:7 (2014), 1353–1404. MR Zbl

[Brundan et al. 2017] J. Brundan, J. Comes, D. Nash, and A. Reynolds, "A basis theorem for the affine oriented Brauer category and its cyclotomic quotients", *Quantum Topol.* **8**:1 (2017), 75–112. MR Zbl

[Brundan et al. 2018] J. Brundan, A. Savage, and B. Webster, "The degenerate Heisenberg category and its Grothendieck ring", preprint, 2018. arXiv

[Brundan et al. 2019] J. Brundan, A. Savage, and B. Webster, "Heisenberg and Kac–Moody categorification", preprint, 2019. arXiv

[Brundan et al. $\geq$ 2020] J. Brundan, A. Savage, and B. Webster, "Quantum Frobenius Heisenberg categorification", in preparation.

[Cautis et al. 2018] S. Cautis, A. D. Lauda, A. M. Licata, P. Samuelson, and J. Sussan, "The elliptic Hall algebra and the deformed Khovanov Heisenberg category", *Selecta Math.* (*N.S.*) **24**:5 (2018), 4041–4103. MR Zbl

[Dipper and Mathas 2002] R. Dipper and A. Mathas, "Morita equivalences of Ariki–Koike algebras", *Math. Z.* **240**:3 (2002), 579–610. MR Zbl

[Ginzburg et al. 2003] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, "On the category $\mathcal{O}$ for rational Cherednik algebras", *Invent. Math.* **154**:3 (2003), 617–651. MR Zbl

[Gould et al. 1991] M. D. Gould, R. B. Zhang, and A. J. Bracken, "Generalized Gelfand invariants and characteristic identities for quantum groups", *J. Math. Phys.* **32**:9 (1991), 2298–2303. MR Zbl

[Jantzen 1996] J. C. Jantzen, *Lectures on quantum groups*, Grad. Stud. Math. **6**, Amer. Math. Soc., Providence, RI, 1996. MR Zbl

[Khovanov 2014] M. Khovanov, "Heisenberg algebra and a graphical calculus", *Fund. Math.* **225**:1 (2014), 169–210. MR Zbl

[Li 2010] J. Li, "The quantum Casimir operators of $U_q(\mathfrak{gl}_n)$ and their eigenvalues", *J. Phys. A* **43**:34 (2010), art. id. 345202. MR Zbl

[Licata and Savage 2013] A. Licata and A. Savage, "Hecke algebras, finite general linear groups, and Heisenberg categorification", *Quantum Topol.* **4**:2 (2013), 125–185. MR Zbl

[Lusztig 1993] G. Lusztig, *Introduction to quantum groups*, Progr. Math. **110**, Birkhäuser, Boston, 1993. MR Zbl

[Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Math. Monogr., Oxford Univ. Press, 1995. MR Zbl

[Mackaay and Savage 2018] M. Mackaay and A. Savage, "Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification", *J. Algebra* **505** (2018), 150–193. MR Zbl

[Rouquier 2012] R. Rouquier, "Quiver Hecke algebras and 2-Lie algebras", *Algebra Colloq.* **19**:2 (2012), 359–410. MR Zbl

[Savage 2019] A. Savage, "Frobenius Heisenberg categorification", *Algebr. Comb.* **2**:5 (2019), 937–967. Zbl

[Shan 2011] P. Shan, "Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras", *Ann. Sci. Éc. Norm. Sup.* (4) **44**:1 (2011), 147–182. MR Zbl

[Turaev 1989] V. G. Turaev, "Operator invariants of tangles, and R-matrices", *Izv. Akad. Nauk SSSR Ser. Mat.* **53**:5 (1989), 1073–1107. In Russian; translated in *Mat. USSR-Uzv.* **35**:2 (1990), 411–444. Zbl

[Virk 2011] R. Virk, "Derived equivalences and $\mathfrak{sl}_2$-categorifications for $U_q(\mathfrak{gl}_n)$", *J. Algebra* **346** (2011), 82–100. MR Zbl

[Webster 2015] B. Webster, "Canonical bases and higher representation theory", *Compos. Math.* **151**:1 (2015), 121–166. MR Zbl

[Webster 2018] B. Webster, "Unfurling Khovanov–Lauda–Rouquier algebras", preprint, 2018. arXiv

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Characteristic cycles and Gevrey series solutions of A-hypergeometric systems

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We compute the L-characteristic cycle of an A-hypergeometric system and higher Euler–Koszul homology modules of the toric ring. We also prove upper semicontinuity results about the multiplicities in these cycles and apply our results to analyze the behavior of Gevrey solution spaces of the system.

Introduction

Let D denote the Weyl algebra on $X = \mathbb{C}^n$ with coordinates $x = x_1, \dots, x_n$. Let ∂_i denote the variable that acts on $\mathbb{C}[x]$ as $\partial/\partial x_i$ and write $\partial = \partial_1, \dots, \partial_n$. A *weight vector* on D is $L = (L_x, L_\partial) \in \mathbb{Q}^n \times \mathbb{Q}^n$ such that $L_x + L_\partial \geq 0$. Such a vector induces an exhaustive increasing filtration L on D by, for $k \in \mathbb{Q}$,

$$L^k D := \mathbb{C} \cdot \{x^u \partial^v \mid L \cdot (u, v) \leq k\}.$$

Write $L^{<k} D := \bigcup_{\ell < k} L^\ell D$. For any P in $L^k D \setminus L^{<k} D$, set

$$\operatorname{in}_L(P) := P + L^{<k} D \in \operatorname{gr}^{L,k} D := L^k D / L^{<k} D \subseteq \operatorname{gr}^L D \quad \text{and} \quad \deg^L(P) := k.$$

For a left D-ideal I and the D-module $M = D/I$, set

$$\operatorname{gr}^L(I) := \langle \operatorname{in}_L(P) \mid P \in I \rangle \subseteq \operatorname{gr}^L(D) \quad \text{and} \quad \operatorname{gr}^L(M) := \operatorname{gr}^L(D)/\operatorname{gr}^L(I).$$

If $L_x + L_\partial = 0$, the associated graded ring $\operatorname{gr}^L D$ is isomorphic to D and $\operatorname{gr}^L(I)$ can be identified with a left D-ideal, which is also called a *Gröbner deformation* of I in [Saito et al. 2000]. It is suggestive to call $\operatorname{gr}^L(M)$ the Gröbner deformation of M with respect to L. On the other hand, if $L_x + L_\partial > 0$, the associated graded ring $\operatorname{gr}^L D$ is isomorphic to the coordinate ring of $T^*X \cong \mathbb{C}^{2n}$, which is a polynomial ring in $2n$ variables. In this latter case, the *L-characteristic variety* of M is

$$\operatorname{Char}^L(M) := \operatorname{Var}(\operatorname{gr}^L(I)) \subseteq T^*X \cong \mathbb{C}^{2n}. \tag{0-1}$$

The *L-characteristic cycle* of M is the finite formal sum

$$\operatorname{CC}^L(M) := \sum_C \mu^{L,C}(M) \cdot C,$$

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where C runs over the irreducible components of $\mathrm{Char}^L(M)$, and

$$\mu^{L,C}(M) := \ell((\mathrm{gr}^L(M))_{P_C})$$

is the multiplicity of $\mathrm{gr}^L(M)$ along C, where P_C is the defining ideal of C in $\mathrm{gr}^L(D)$ and ℓ denotes the length of a $\mathrm{gr}^L(D)_{P_C}$-module.

The weight vector $F = (\mathbf{0}_n, \mathbf{1}_n) := (0, \dots, 0, 1, \dots, 1) \in \mathbb{Q}^n \times \mathbb{Q}^n$ induces the *order filtration* on D. We notice that $\mathrm{Char}^F(M)$ and $\mathrm{CC}^F(M)$ are called, respectively, the *characteristic variety* and the *characteristic cycle* of M. If M is *holonomic*, that is, the dimension of its characteristic variety is n, then the *rank* of M, defined $\mathrm{rank}(M) := \dim_{\mathbb{C}(x)} \mathbb{C}(x) \otimes_{\mathbb{C}[x]} M$, coincides with the dimension of the space of germs of its holomorphic solutions at any nonsingular point by a result of Kashiwara (see e.g., [Saito et al. 2000, Theorem 1.4.19]). Notice that $\mathrm{rank}(M) = \mu^{F,C}(M)$ for $C = T^*_X X$.

One motivation for the study of L-characteristic cycles comes from the theory of irregularity of holonomic D-modules. For a flavor of this deep and involved theory that fits the goals of this paper, a projective weight vector of the form

$$L = F + (s-1)V$$

where $s \in \mathbb{Q}$ and $V = (-w, w)$, with $w = (0, \dots, 0, 1)$, induces the Kashiwara–Malgrange filtration along the coordinate hyperplane $Y = \{x_n = 0\} \subseteq X = \mathbb{C}^n$. In this case, the L-characteristic variety $\mathrm{Char}^{F+(s-1)V}(M)$ is locally constant with respect to $s \in \mathbb{Q}$, except for at a finite set of values called *algebraic slopes* of M along Y. This is a global version of the algebraic slopes defined and studied by Laurent [1987]. On the other hand, the *analytic slopes* of M along Y were defined as jumps in the *Gevrey* filtration of the *irregularity sheaf* of M along Y by Mebkhout [1990]. The comparison theorem for slopes states that the algebraic and analytic slopes for M along Y coincide, and, even more, the Euler–Poincaré characteristic of the irregularity sheaf can be computed in terms of the L-characteristic cycles of M [Laurent and Mebkhout 1999]. In particular, certain multiplicities in the L-characteristic cycles are closely related to the dimension of the space of Gevrey solutions of M along Y.

Another motivating idea of this article is that the F-characteristic cycle of a Gröbner deformation of a holonomic D-module M is equal to the L-characteristic cycle of M for an appropriate L (see Lemma 3.1 for the precise statement). In particular, the holonomic rank of such a Gröbner deformation is the multiplicity of the component $T^*_X X$ in $\mathrm{CC}^L(M)$.

Our main interest is A-hypergeometric D-modules, also known as GKZ-systems after their introduction and study by Gelfand, Graev, Kapranov, and Zelevinsky [Gelfand et al. 1987; 1989; 1990]. Let

$$A = [a_{ij}] = [a_1 \cdots a_n] \in \mathbb{Z}^{d \times n}$$

be an integral matrix such that the group generated by the columns of A, $\mathbb{Z}A$, is equal to $\mathbb{Z}^d$, and the positive real cone $\mathbb{R}_{\geq 0} A$ over the columns is pointed. Let

$$I_A := \langle \partial^u - \partial^v \mid Au = Av \rangle \subseteq \mathbb{C}[\partial]$$

denote the *toric ideal* of A. For $\beta \in \mathbb{C}^d$, write $E-\beta$ for the sequence of *Euler operators* given by

$$E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i$$

for $i = 1, \dots, d$. The A*-hypergeometric system* of A at $\beta \in \mathbb{C}^d$ is

$$H_A(\beta) := D \cdot (I_A + \langle E-\beta \rangle) \quad \text{with associated module } M_A(\beta) := D/H_A(\beta).$$

A weight vector $L = (L_x, L_\partial) \in \mathbb{Q}^n \times \mathbb{Q}^n$ as above is called *projective* if $L_x + L_\partial = c \cdot \mathbf{1}_n := c \cdot (1, \dots, 1)$ for some constant $c > 0$. Notice that any Euler operator E_i is homogeneous with respect to such a filtration. In [Schulze and Walther 2008], the irreducible components of $\mathrm{CC}^L(M_A(\beta))$ were enumerated, and when β is generic (or not rank-jumping), $\mathrm{CC}^L(M_A(\beta))$ was computed. In this article, we compute $\mathrm{CC}^L(M_A(\beta))$ for any β, along with the characteristic cycles of higher Euler–Koszul homology modules (see Section 1) of the toric ring $\mathbb{C}[\partial]/I_A$. We also provide upper semicontinuity results for some of these multiplicities and apply our results to the Gevrey solution spaces of $M_A(\beta)$.

Outline. In Sections 1–2, we provide background and preliminary results on Euler–Koszul homology and L-characteristic cycles of A-hypergeometric systems. We compute the multiplicities in the characteristic cycles of the Euler–Koszul homology of the toric ring in Section 4, with consequences in Section 5. We provide upper semicontinuity results in Section 6 and study Gevrey solutions of $H_A(\beta)$ in Section 7.

1. Euler–Koszul homology

In this section, we present background related to Euler–Koszul homology, as found in [Matusevich et al. 2005; Schulze and Walther 2009], with some additions needed in the sequel. We use the convention that $0 \in \mathbb{N}$. Recall that a_i denotes the i-th column of the matrix A. Given a subset $\tau \subseteq A$ of the column set of A, the semigroup generated by τ,

$$\mathbb{N}\tau := \left\{ \sum_{a_i \in \tau} j_i a_i \;\middle|\; j_i \in \mathbb{N} \text{ for all } a_i \in \tau \right\},$$

generates the semigroup ring $S_\tau := \mathbb{C}[\mathbb{N}\tau]$. With $\pi_\tau : \mathbb{C}[\partial_\tau] := \mathbb{C}[\partial_j \mid a_j \in \tau] \to S_\tau$ denoting the map induced by τ, we have the isomorphism of rings $S_\tau \cong \mathbb{C}[\partial_\tau]/\ker \pi_\tau$. When convenient, we will abuse notation and also view τ as a matrix.

A subset G of the columns of the matrix A is a *face* of A, denoted $G \preceq A$, if $\mathbb{R}_{\geq 0} G$ is a face of the cone $\mathbb{R}_{\geq 0} A$ and $G = A \cap \mathbb{R}G$. The codimension of a nonempty face G is $\mathrm{codim}(G) := d - \dim(\mathbb{R}G)$, with $\mathrm{codim}(\varnothing) = d$ by convention. Let G^c denote the complement of G in A.

Define a $\mathbb{Z}^d$-grading on D via $\deg(x_i) := -a_i$ and $\deg(\partial_{x_i}) := a_i$. A $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module N is *toric* if it has a filtration

$$0 = N^{(0)} \subseteq N^{(1)} \subseteq \cdots \subseteq N^{(\ell-1)} \subseteq N^{(\ell)} = N$$

such that $N^{(i)}/N^{(i-1)}$, for each i, is a $\mathbb{Z}^d$-graded translate of S_{G_i} for some face $G_i \preceq A$. The *degree set* of a finitely generated $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module N is $\deg(N) := \{\alpha \in \mathbb{Z}^d \mid N_\alpha \neq 0\}$. The *quasidegree set* of N, denoted $\operatorname{qdeg}(N)$, is the Zariski closure of $\deg(N)$ under the natural embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$. A $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module N is *weakly toric* if there is a filtered partially ordered set $(\mathfrak{S}, \leq)$ and a $\mathbb{Z}^d$-graded direct limit

$$\phi_s : N^{(s)} \to \varinjlim_{s \in \mathfrak{S}} N^{(s)} = N,$$

where $N^{(s)}$ is a toric $\mathbb{C}[\partial]$-module for each $s \in \mathfrak{S}$. The *quasidegrees* of N are

$$\operatorname{qdeg}(N) := \bigcup_{s \in \mathfrak{S}} \operatorname{qdeg}(\phi_s(N^{(s)})),$$

where each $\operatorname{qdeg}(\phi_s(N^{(s)}))$ is already defined since $\phi_s(N^{(s)})$ is toric for each s.

Let N be a weakly toric module. Given a homogeneous $y \in D \otimes_{\mathbb{C}[\partial]} N$, define an action of the Euler operators for $1 \leq i \leq d$ by

$$(E_i - \beta_i) \circ y = (E_i - \beta_i + \deg_i(y))y,$$

and extend this action $\mathbb{C}$-linearly to $D \otimes N$. With this sequence of commuting endomorphisms on $D \otimes N$, let $\mathcal{K}^A_\bullet(N, \beta)$ denote the Koszul complex on the left D-module $D \otimes_{\mathbb{C}[\partial]} N$, which we call the *Euler–Koszul complex* of N at β. Its homology is denoted $\mathcal{H}^A_i(N, \beta) := H_i(\mathcal{K}^A_\bullet(N, \beta))$ or simply $\mathcal{H}_i(N, \beta)$ when A is clear from the context. Euler–Koszul homology was first introduced in [Matusevich et al. 2005] for toric modules and extended to weakly toric modules in [Schulze and Walther 2009].

If $b \in \mathbb{Z}^d$, we denote by $N(b)$ a $\mathbb{Z}^d$-graded translated copy of N such that $N(b)_v = N_{v-b}$ for all $v \in \mathbb{Z}^d$. Thus, $\deg(N(b)) = b + \deg(N)$. For example, if $N = S_A = \mathbb{C}[\mathbb{N}A]$ then $N(b) = \mathbb{C}[\mathbb{N}A]t^b$. Euler–Koszul homology is compatible with these graded shifts. Namely, we have

$$\mathcal{H}_q(N(b), \beta) \cong \mathcal{H}_q(N, \beta - b)(b). \tag{1-1}$$

Theorem 1.1 [Schulze and Walther 2009, Theorem 5.4]. *For a weakly toric module N, the following are equivalent*:

(1) $\mathcal{H}_i(N, \beta) = 0$ *for all* $i \geq 0$.

(2) $\mathcal{H}_0(N, \beta) = 0$.

(3) $\beta \notin \operatorname{qdeg}(N)$. □

Theorem 1.2 [Matusevich et al. 2005, Theorem 6.6; Schulze and Walther 2009]. *Let N be a weakly toric module. Then $\mathcal{H}_i(N, \beta) = 0$ for all $i > 0$ and for all $\beta \in \mathbb{C}^d$ if and only if N is a maximal Cohen–Macaulay S_A-module.* □

For a subset $\tau \subseteq A$, given an $\mathbb{N}\tau$-module $\mathbb{S}$, define the S_τ-module $\mathbb{C}\{\mathbb{S}\} := \bigoplus_{s \in \mathbb{S}} \mathbb{C} \cdot t^s$ as a $\mathbb{C}$-vector space with S_τ-action given by $\partial_i \cdot t^s = t^{s+a_i}$. Then $\mathbb{C}\{\mathbb{S}\}$ has a multiplicative structure given by $t^s \cdot t^{s'} = t^{s+s'}$, and $S_\tau \cong \mathbb{C}\{\mathbb{N}\tau\}$ as rings. The *saturation* of τ in $\mathbb{Z}\tau$ is the semigroup $\widetilde{\mathbb{N}\tau} = \mathbb{R}_{\geq 0}\tau \cap \mathbb{Z}\tau$. The *saturation* of S_τ is the semigroup ring of the saturation of τ in $\mathbb{Z}\tau$, which is given by $\tilde{S}_\tau = \mathbb{C}\{\widetilde{\mathbb{N}\tau}\}$ as a $\mathbb{Z}^d$-graded S_τ-module. By [Hochster 1972], $\tilde{S}_\tau$ is a Cohen–Macaulay S_τ-module.

2. Characteristic cycles of A-hypergeometric systems

Let $L = (L_x, L_\partial) \in \mathbb{Q}^{2n}$ be a projective weight vector on D. In this section, we recall from [Schulze and Walther 2008] the description of the L-characteristic variety of an A-hypergeometric system, which includes the computation of the L-characteristic cycle of $H_A(\beta)$ when β is not rank-jumping for A.

Let $h = (h_1, \dots, h_d) \in \mathbb{Q}^d$ be such that $h \cdot a_i > 0$ for $i = 1, \dots, n$. Choose $\varepsilon > 0$ such that $h \cdot a_i + \varepsilon L_{\partial_i} > 0$ for $i = 1, \dots, n$, and denote by H_ε the hyperplane in $\mathbb{P}^d_{\mathbb{Q}}$ given by

$$\{[y_0 : y_1 : \cdots : y_d] \in \mathbb{P}^d_{\mathbb{Q}} \mid \varepsilon y_0 + h_1 y_1 + \cdots + h_d y_d = 0\}.$$

The *L-polyhedron of A* is the convex hull of $\{[1 : \mathbf{0}_d], [L_{\partial_1} : a_1], \dots, [L_{\partial_n} : a_n]\}$ in the affine space $\mathbb{P}^d_{\mathbb{Q}} \setminus H_\varepsilon$. The *$(A, L)$-umbrella*, denoted Φ^L_A, is the set of faces of the L-polyhedron of A that do not contain $[1 : \mathbf{0}_d]$.

We denote by $\Phi^{L,k}_A \subset \Phi^L_A$ the subset of faces τ of dimension k (equivalently, $\dim(\mathbb{C}\tau) = k + 1$). A face τ of Φ^L_A will be identified with $\{j \in \{1, \dots, n\} \mid [L_{\partial_j} : a_j] \in \tau\}$ or with the submatrix of A indexed by this set, when necessary. With this identification, Φ^L_A is an abstract polyhedral complex. For any face $G \preceq A$, set $\Phi^L_G := \{\tau \in \Phi^L_A \mid \tau \subseteq G\}$.

Let (x, ξ) denote the coordinates on $T^*X = T^*\mathbb{C}^n$. For any $\tau \subseteq \{1, \dots, n\}$, let

$$C^\tau_A := \left\{(x, \xi) \in T^*X \;\Big|\; \xi_i = 0 \text{ for } i \notin \tau, \sum_{i \in \tau} a_i x_i \xi_i = 0 \text{ and } \exists t \in (\mathbb{C}^*)^d, \xi_j = t^{a_j}, \forall j \in \tau\right\},$$

and let $\overline{C^\tau_A}$ denote the Zariski closure of C^τ_A in T^*X, with defining ideal $P_\tau \subseteq \mathbb{C}[x, \xi]$. In particular, $\overline{C^\varnothing_A} = T^*_X X$ and $\overline{C^{\{j\}}_A} = T^*_{(x_j=0)} X$.

If N is a $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module and $C = \overline{C^\tau_A}$ for some $\tau \in \Phi^L_A$, we write

$$\mu^{L,\tau}_{A,i}(N, \beta) := \mu^{L,C}(\mathcal{H}_i(N, \beta)) = \ell((\mathrm{gr}^L(\mathcal{H}_i(N, \beta)))_{P_\tau}). \tag{2-1}$$

We will also denote $\mu^{L,\tau}_{A,i}(\beta) := \mu^{L,\tau}_{A,i}(S_A, \beta)$. By [Schulze and Walther 2008, Corollary 4.13],

$$\mu^{L,\tau}_A := \sum_{j=0}^{d} (-1)^j \mu^{L,\tau}_{A,j}(\beta) = \mu^{L,\tau}_{A,0}(\tilde{S}_A, \beta) = \mu^{L,\tau}_{A,0}(S_A[\partial_A^{-1}], \beta) \tag{2-2}$$

is independent of $\beta \in \mathbb{C}^d$.

Note that $\mathrm{rank}(M_A(\beta))$ is equal to $\mu^{F,\varnothing}_{A,0}(\beta)$. Since $M_A(\beta)$ is always holonomic [Gelfand et al. 1987; Adolphson 1994], its rank is always finite. Further, the rank of $M_A(\beta)$ is upper semicontinuous as a function of the parameter β, with a generic value equal to $\mathrm{vol}_{\mathbb{Z}^d}(A)$, the normalized volume in $\mathbb{Z}A = \mathbb{Z}^d$ of the convex hull of the columns of A and the origin [Matusevich et al. 2005; Adolphson 1994; Gelfand et al. 1990]. We recall that the normalized volume function in a lattice Ω, denoted by vol_Ω, is defined so that the volume of the unit simplex in Ω (that is, the convex hull of the origin and a lattice basis of Ω) is one.

A parameter β is said to be *rank-jumping* when $\operatorname{rank}(M_A(\beta)) > \operatorname{vol}_{\mathbb{Z}^d}(A)$. The set of rank-jumping parameters is described in [Matusevich et al. 2005]; namely, with $\varepsilon_A := \sum_{i=1}^n a_i$,

$$\mathcal{E}_A := \{\beta \in \mathbb{C}^d \mid \operatorname{rank}(M_A(\beta)) > \operatorname{vol}_{\mathbb{Z}^d}(A)\} = -\operatorname{qdeg}\Biggl(\bigoplus_{i=0}^{d-1} \operatorname{Ext}^{n-i}_{\mathbb{C}[\partial]}(S_A, \mathbb{C}[\partial])(-\varepsilon_A)\Biggr).$$

Schulze and Walther provided a description of $\mathrm{CC}^L(H_A(\beta))$ when β is not rank-jumping, as summarized through the following two results.

Theorem 2.1 [Schulze and Walther 2008, Theorem 4.21]. *For all $G \preceq A$, if $\tau \in \Phi^L_G$, then*

$$\mu_G^{L,\tau} = \sum_{\tau \subseteq \tau' \in \Phi_G^{L,d'-1}} [\mathbb{Z}G : \mathbb{Z}\tau'] \cdot [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \operatorname{vol}_{\pi(\mathbb{Z}\tau')}(P_{\tau,\tau'} \setminus Q_{\tau,\tau'}),$$

where $d' = \dim(\mathbb{C}G)$, $\pi : \mathbb{Z}\tau' \twoheadrightarrow \mathbb{Z}\tau'/(\mathbb{Z}\tau' \cap \mathbb{Q}\tau)$ is the natural projection and $P_{\tau,\tau'}$ and $Q_{\tau,\tau'}$ denote the convex hull of $\pi(\tau' \cup \{0\})$ and $\pi(\tau' \setminus \tau)$ respectively.

In [Schulze and Walther 2008], Theorem 4.21 is only stated for $G = A$. Theorem 2.1 is a straightforward adaptation that will be useful in the sequel. Note that here we are using (2-1) and (2-2) with A replaced by G, but we still write L for the filtration induced on the Weyl Algebra D_G in the variables $\{x_j \mid j \in G\}$ by the projective weight vector given by the G-coordinates of L_x and L_∂.

Theorem 2.2 [Schulze and Walther 2008, Corollary 4.12]. *The L-characteristic variety of $M_A(\beta)$ is independent of $\beta \in \mathbb{C}^d$ and given by*

$$\operatorname{Char}^L(M_A(\beta)) = \bigcup_{\tau \in \Phi^L_A} \overline{C^\tau_A},$$

where each component $\overline{C^\tau_A}$ is irreducible. Moreover, $\mu^{L,\tau}_{A,0}(\beta) \geq \mu^{L,\tau}_A$, and equality holds if β is not rank-jumping.

Theorem 2.2 implies that when β is not rank-jumping,

$$\mathrm{CC}^L(M_A(\beta)) = \sum_{\tau \in \Phi^L_A} \mu^{L,\tau}_A \cdot \overline{C^\tau_A},$$

and for each $\tau \in \Phi^L_A$, the multiplicity $\mu^{L,\tau}_A$ is computed in Theorem 2.1.

A subset $\tau \subseteq A$ is called *F-homogeneous* if the set of columns of A indexed by τ lie in a common affine hyperplane off the origin. For a subset $\tau \subseteq A$, let $\Delta_\tau = \operatorname{conv}(\tau \cup \{\mathbf{0}\}) \subseteq \mathbb{R}^d$ denote the convex hull of the origin and all the columns of τ.

By [Schulze and Walther 2008, Corollary 4.22 and Remark 4.23],

$$\mu^{L,\varnothing}_A = \operatorname{vol}_{\mathbb{Z}^d}\Biggl(\bigcup_{\tau' \in \Phi^{L,d-1}_A} \Delta_{\tau'} \setminus \operatorname{conv}(\tau')\Biggr). \tag{2-3}$$

Hence if all the facets of the (A, L)-umbrella are F-homogeneous, then

$$\mu_A^{L,\varnothing} = \mathrm{vol}_{\mathbb{Z}^d}\Bigl(\bigcup_{\tau' \in \Phi_A^{L,d-1}} \Delta_{\tau'}\Bigr). \tag{2-4}$$

3. F-characteristic cycles of initial ideals are L-characteristic cycles

Given any real vector $w \in \mathbb{R}^n$ and any left ideal $J \subseteq D$, we can consider the initial ideal $\mathrm{in}_{(-w,w)}(J)$ as defined in [Saito et al. 2000]. We recall that by [loc. cit., Theorem 2.2.1], if $M = D/J$ is a holonomic D-module, then so is $\mathrm{gr}^{(-w,w)}(M) := D/\mathrm{in}_{(-w,w)}(J)$ and, moreover,

$$\mathrm{rank}(\mathrm{gr}^{(-w,w)} M) \le \mathrm{rank}(M). \tag{3-1}$$

On the other hand, by [loc. cit., Lemma 2.1.6], for any weight vector $(u, v) \in \mathbb{R}^{2n}$ and $L = (-w, w) + \epsilon(u, v)$ with $\epsilon > 0$ small enough,

$$\mathrm{gr}^{(u,v)}(\mathrm{gr}^{(-w,w)}(M)) = \mathrm{gr}^L(M). \tag{3-2}$$

Lemma 3.1. *If $M = D/J$ is a holonomic D-module, then for L chosen as in (3-2) with $(u, v) = F$,*

$$\mathrm{CC}^F(\mathrm{gr}^{(-w,w)}(M)) = \mathrm{CC}^L(M).$$

The holonomic rank of $\mathrm{in}_{(-w,w)}(M_A(\beta))$, a central object of study in [loc. cit.], equals the multiplicity $\mu_{A,0}^{L,\varnothing}(\beta)$ for $L = (-w, w) + \epsilon F$ and $\epsilon > 0$ small enough. Notice that, by the form of L, all the facets of Φ_A^L are F-homogeneous. We will see in Section 4 that for any projective weight vector L, the multiplicity $\mu_{A,0}^{L,\varnothing}(\beta)$ equals the rank of a Gröbner deformation of $M_A(\beta)$ (see Corollaries 4.3 and 4.5).

4. Computing multiplicities in L-characteristic cycles

In this section, we use the approach of [Berkesch 2011] to compute the multiplicities in the L-characteristic cycles of Euler–Koszul homology modules of the toric ring S_A. We first recall some definitions from [Berkesch 2011; Berkesch et al. 2018].

For a face $G \preceq A$, consider the union of the lattice translates

$$\mathbb{E}_G^\beta := [\mathbb{Z}^d \cap (\beta + \mathbb{C}G)] \setminus (\mathbb{N}A + \mathbb{Z}G) = \bigsqcup_{b \in B_G^\beta} (b + \mathbb{Z}G), \tag{4-1}$$

where B_G^β is a set of lattice translate representatives. As such, $|B_G^\beta|$ is the number of translates of $\mathbb{Z}G$ appearing in $\mathbb{E}^\beta$, which is by definition equal to the difference between $[\mathbb{Z}^d \cap \mathbb{Q}G : \mathbb{Z}G]$ and the number of translates of $\mathbb{Z}G$ along $\beta + \mathbb{C}G$ that are contained in $\mathbb{N}A + \mathbb{Z}G$.

For a face $\tau \in \Phi_A^L$ of the (A, L)-umbrella, let $\mathbb{E}_\tau^\beta$ denote the union of the ranking lattices $\mathbb{E}_G^\beta$, where $G \preceq A$ contains τ.

Theorem 4.1. *Let L be a projective weight vector and $\tau \in \Phi_A^L$ be a face of the (A, L)-umbrella. For each i and β, the multiplicity $\mu_{A,i}^{L,\tau}(\beta)$, which is the coefficient of $\overline{C}_A^\tau$ in the characteristic cycle $\mathrm{CC}^L(\mathcal{H}_i(S_A, \beta))$*

(see (2-1)), can be computed from the combinatorics of the ranking lattices at β and the (A, L)-umbrella Φ_A^L. More precisely, there is a spectral sequence involving the faces of Φ_A^L that contain τ and the ranking lattices in $\mathbb{E}_\tau^\beta$, from which $\mu_{A,i}^{L,\tau}(\beta)$ can be computed.

Before proving Theorem 4.1, we state some consequences.

Corollary 4.2. *For all $\beta \in \mathbb{C}^d$ and all projective weight vectors L, L',*

$$\mathrm{CC}^L(M_A(\beta)) = \mathrm{CC}^{L'}(M_A(\beta)) \quad \textit{if and only if} \quad \Phi_A^L = \Phi_A^{L'}.$$

Proof. While the only if direction follows from Theorem 2.2, the if direction uses Theorems 2.1, 2.2, and 4.1. □

Corollary 4.3. *For any projective weight vector $L = (u, v)$ on D such that all the facets of Φ_A^L are F-homogeneous,*

$$\mathrm{CC}^F(\mathrm{gr}^{(-v,v)}(M_A(\beta))) = \mathrm{CC}^L(M_A(\beta)).$$

In particular, $\mathrm{rank}(\mathrm{gr}^{(-v,v)}(M_A(\beta))) = \mu_{A,0}^{L,\varnothing}(\beta)$.

Proof. Let $\epsilon > 0$ be as small as necessary in the sequel. Notice first that $\mathrm{CC}^F(\mathrm{gr}^{(-v,v)}(M_A(\beta))) = \mathrm{CC}^{L_\epsilon}(M_A(\beta))$ for $L_\epsilon := (-v, v) + \epsilon F$ by Lemma 3.1. Moreover, by the assumption on the (A, L)-umbrella, we have $\Phi_A^L = \Phi_A^{L+\epsilon F}$. On the other hand, the last n coordinates of $L + \epsilon F$ and L_ϵ are equal to $v + \epsilon \cdot \mathbf{1}_n$, and hence $\Phi_A^L = \Phi_A^{L_\epsilon}$. Thus, the result follows from Corollary 4.2. □

As a particular case of Corollary 4.3, the characteristic cycles, and hence the ranks, of the modules $\mathrm{gr}^{(-\mathbf{1}_n,\mathbf{1}_n)}(M_A(\beta))$ and $M_A(\beta)$ are equal. We next show that [Saito et al. 2000, Corollary 3.2.14] holds with weakened hypotheses.

Corollary 4.4. *For any $\beta \in \mathbb{C}^d$ and any (not necessarily homogeneous) A, the small Gröbner fan of the hypergeometric ideal $H_A(\beta)$ refines the secondary fan of A.*

Proof. It suffices to see that each open cone of the small Gröbner fan of $H_A(\beta)$ is contained in an open cone of the secondary fan of A. Since such an open cone corresponds to a Gröbner deformation with respect to a generic weight vector $w \in \mathbb{R}^n$, it follows that $L = (-w + c \cdot \mathbf{1}_n, w)$ is a projective weight vector for any $c > 0$ and Φ_A^L, which only depends on w, has only F-homogeneous facets. Thus, beginning with generic vectors w, w' with

$$\mathrm{gr}^{(-w,w)}(M_A(\beta)) = \mathrm{gr}^{(-w',w')}(M_A(\beta)),$$

Corollaries 4.2 and 4.3 imply that $\Phi_A^L = \Phi_A^{L'}$ where the last coordinates of L and L' are w and w', respectively. This means that w and w' belong to the same cone of the secondary fan of A. □

Corollary 4.5. *Any projective weight vector $L = (u, v)$ on D has a perturbation L' such that all the facets of the (A, L')-umbrella $\Phi_A^{L'}$ are F-homogeneous and $\mu_{A,0}^{L,\varnothing}(\beta) = \mu_{A,0}^{L',\varnothing}(\beta)$.*

Proof. If $L'(\epsilon) := L + \epsilon(\mathbf{1}_n, -\mathbf{1}_n)$ for $\epsilon > 0$, then there is an $\epsilon_0 > 0$ such that the $L'(\epsilon)$-umbrella is constant for $\epsilon \in (0, \epsilon_0]$. Thus, if we fix $L' = L'(\epsilon_0)$, then all the facets of $\Phi_A^{L'}$ are F-homogeneous. Moreover, by the choice of L', any F-homogeneous facet of Φ_A^L is a facet of $\Phi_A^{L'}$, while each non-F-homogeneous facet τ of Φ_A^L is replaced in $\Phi_A^{L'}$ by the set of facets of $\Phi_\tau^{L''}$, where $L'' := (c\mathbf{1}_n, -\mathbf{1}_n)$ is a projective weight vector for any $c > 1$. This latter set is the set of facets of $\operatorname{conv}(\tau)$ that are not facets of Δ_τ. This proves that $\mu_A^{L,\varnothing} = \mu_A^{L',\varnothing}$ by using (2-3) to compute $\mu_A^{L,\varnothing}$ and (2-4) to compute $\mu_A^{L',\varnothing}$. Analogously, $\mu_G^{L,\varnothing} = \mu_G^{L',\varnothing}$ for any face $G \preceq A$. Finally, the result follows from previous equality and Theorem 4.1. □

Corollary 4.6. *Given any projective weight vector L and $\beta \in \mathbb{C}^d$,*

$$\mu_{A,0}^{L,\varnothing}(\beta) \leq \operatorname{rank}(M_A(\beta)) \leq 4^{(d+1)} \operatorname{vol}(A).$$

Proof. The first inequality is a consequence of (3-1) and Corollaries 4.5 and 4.3. The second is [Berkesch et al. 2018, Corollary 6.2]. □

To prove Theorem 4.1, we will follow the approach used to compute the rank of an A-hypergeometric system from [Berkesch 2011] (see also [Berkesch et al. 2018]). We will use the set

$$\mathcal{C}_A(\beta) := \mathbb{Z}^d \cap (\operatorname{Re}\beta + \mathbb{R}_{\geq 0}A).$$

Note that $\mathcal{C}_A(\beta)$ here is defined differently than in [Berkesch 2011]. However, the quotient between the subsequent modules with the same names, defined using $\mathcal{C}_A(\beta)$ here or as in [loc. cit.], all have quasidegree sets that do not contain β. Hence, by Theorem 1.1, the Euler–Koszul homology modules for modules with the same names here and in [loc. cit.] are isomorphic.

Given a subset

$$J \subseteq \mathcal{J}(\beta) := \{(G, b) \mid G \preceq A,\, b \in B_G^\beta,\, \mathbb{E}_G^\beta \neq \varnothing\}, \tag{4-2}$$

define

$$\mathbb{E}_J^\beta := \bigcup_{(G,b)\in J} (b + \mathbb{Z}G) \quad \text{and} \quad \mathbb{P}_J^\beta := \mathcal{C}_A(\beta) \cap \mathbb{E}_J^\beta.$$

Now define the respective sets and S_A-modules

$$\mathbb{T}^\beta := \mathbb{N}A \cup \Bigg[\bigcup_{b\in\mathbb{P}^\beta_{\mathcal{J}(\beta)}} (b + \widetilde{\mathbb{N}A})\Bigg], \quad T^\beta := \mathbb{C}\{\mathbb{T}^\beta\}, \quad \mathbb{S}_J^\beta := \mathbb{T}^\beta \setminus \mathbb{P}_J^\beta, \quad S_J^\beta := \mathbb{C}\{\mathbb{S}_J^\beta\}, \quad \text{and} \quad P_J^\beta := \frac{T^\beta}{S_J^\beta}.$$

The degree set of P_J^β is $\deg(P_J^\beta) = \mathbb{P}_J^\beta$. If a toric module N is isomorphic to P_J^β for some $J \subseteq \mathcal{J}(\beta)$ and β, then we say that N is a *ranking toric module* determined by J. A *simple ranking toric module* is a module isomorphic to $P_{G,J}^\beta := P_{J(G)}^\beta$, where $G \preceq A$ is a fixed face of A such that $\mathbb{E}_G^\beta \neq \varnothing$ and

$$J(G) := \{(G, b) \in J \mid b \in B_G^\beta\}.$$

When $J = \mathcal{J}(\beta)$, we suppress it from the notation and write P^β and P_G^β in place of P_J^β and $P_{G,J}^\beta$, respectively. If $(G, b) \in J$ and there is not any other pair $(G', b') \in J$ such that $b + \mathbb{Z}G \subsetneq b' + \mathbb{Z}G'$ we say that (G, b) is a maximal pair in J. We denote by $\max(J)$ the set of all maximal pairs in J.

Lemma 4.7. *If $G \preceq A$ and $\tau \in \Phi_A^L$, then the multiplicity $\mu_{A,q}^{L,\tau}(P_G^\beta, \beta)$ of the simple ranking toric module P_G^β is*

$$\mu_{A,q}^{L,\tau}(P_G^\beta, \beta) = |B_G^\beta| \cdot \mu_{A,q}^{L,\tau}(\tilde{S}_G(b), \beta) = \begin{cases} |B_G^\beta| \cdot \binom{\operatorname{codim}(G)}{q} \cdot \mu_G^{L,\tau} & \text{if } \tau \subseteq G, \\ 0 & \text{otherwise}. \end{cases}$$

for any $b \in B_G^\beta$.

Proof. For all $j \notin G$ we have that $\partial_j \cdot \tilde{S}_G = 0$, hence that $\xi_j \cdot \operatorname{gr}^L(\mathcal{H}_0(\tilde{S}_G(b), \beta)) = 0$ where $\xi_j = \operatorname{in}_L(\partial_j) \in \mathbb{C}[x, \xi] \cong \operatorname{gr}^L(D)$. On the other hand, by the definition of P_τ, it is clear that $\xi_j \in P_\tau$ if and only if $j \notin \tau$. Thus, we have that $(\operatorname{gr}^L(\mathcal{H}_0(\tilde{S}_G(b), \beta)))_{P_\tau} = 0$ if $\tau \nsubseteq G$. Now, with $\mu^{L,\overline{C_A^\tau}}$ in place of rank, the arguments in the proof of [Berkesch 2011, Theorem 6.1] yield this result. □

Proof of Theorem 4.1. The argument proving [Berkesch 2011, Theorem 6.6] can be used to obtain this result, when J is chosen to be the right hand side of (4-2) and $\mu^{L,\overline{C_A^\tau}}$ in place of rank. We make note of the necessary modifications below.

To begin, it follows from Theorem 1.2 and (2-2) that

$$\mu_{A,i}^{L,\tau}(\beta) = \begin{cases} \mu_A^{L,\tau} + \mu_{A,1}^{L,\tau}(Q_A, \beta) - \mu_{A,0}^{L,\tau}(Q_A, \beta) & \text{if } i = 0, \\ \mu_{A,i+1}^{L,\tau}(Q_A, \beta) & \text{if } i > 0, \end{cases} \tag{4-3}$$

where Q_A sits in the short exact sequence $0 \to S_A \to S_A[\partial^{-1}] \to Q_A \to 0$. Then [Berkesch 2011, Proposition 5.10] implies that

$$\mu_{A,i}^{L,\tau}(Q_A, \beta) = \mu_{A,i}^{L,\tau}(P_J^\beta, \beta), \tag{4-4}$$

where J is equal to the right hand side of (4-2). Now [loc. cit., Lemmas 6.9, 6.10, 6.11, and 6.14] can be applied verbatim, while Lemma 4.7 replaces the need for [loc. cit., Lemma 6.13]. Finally, as [loc. cit., Lemmas 6.12 and 6.15] hold when rank is replaced with $\mu_A^{L,\overline{C_A^\tau}}$, which is possible since localization at P_τ and $\operatorname{gr}^L(-)$ are exact functors and length is additive, the arguments of the proof of [loc. cit., Theorem 6.6] yield the desired result. In particular, the spectral sequence involved begins with the cellular resolution of P_J^β as constructed in [loc. cit., (6.3)]:

$$0 \to P_J \to I_J^0 \to I_J^1 \to \cdots \to I_J^r \to 0, \tag{4-5}$$

where $I_J^\bullet$ is constructed as follows. Set

$$\begin{aligned} \Delta_J^0 &= \{F \preceq A \mid \exists (F, b) \in \max(J)\}, \\ \Delta_J^p &= \{s \subseteq \Delta_J^0 \mid |s| = p+1\}, \text{ and} \\ F_s &= \bigcap_{G \in s} G \text{ for } s \in \Delta_J^p. \end{aligned}$$

With $r + 1 = |\Delta_J^0|$, let $\Delta = \Delta_J^\beta$ be the standard r-simplex with vertices corresponding to the elements of Δ_J^0. To the p-face of Δ spanned by the vertices corresponding to the elements in $s \in \Delta_J^p$, assign the ranking toric module $P^{\beta}_{F_s,J}$. Choosing the natural maps $P^{\beta}_{F_s,J} \to P^{\beta}_{F_t,J}$ for $s \subseteq t$ induces a cellular complex supported on Δ,

$$I^{\bullet}_J : I^0_J \to I^1_J \to \cdots \to I^r_J \to 0 \quad \text{with } I^p_J = \bigoplus_{s\in\Delta^p_J} P^{\beta}_{F_s,J}. \tag{4-6}$$

Applying Euler–Koszul homology to (4-6) yields a double complex. The desired spectral sequence arises from this double complex after localizing at P_τ and applying $\mathrm{gr}^L(-)$. □

Remark 4.8. If $\beta \in \mathbb{C}^d$ is such that $\max(\mathcal{J}(\beta))$ involves two faces, F_1, F_2, then the proof of Theorem 4.1 shows that

$$\mu^{L,\tau}_{A,0}(\beta) - \mu^{L,\tau}_{A} = \sum_{i=1}^{2}(|B^{\beta}_{F_i}| \cdot [\mathrm{codim}(F_i) - 1] \cdot \mu^{L,\tau}_{F_i}) + |B^{\beta}_{G}| \cdot C^{\beta} \cdot \mu^{L,\tau}_{G}, \tag{4-7}$$

where $G = F_1 \cap F_2$ and the constant C^{β} is given by

$$C^{\beta} = \binom{\mathrm{codim}(G)}{2} - \mathrm{codim}(G) + 1 - \binom{\mathrm{codim}(F_1)}{2} - \binom{\mathrm{codim}(F_2)}{2} + \binom{\mathrm{codim}(\mathbb{C}F_1 + \mathbb{C}F_2)}{2}.$$

Example 4.9. The values of the $\mu^{L,\tau}_{A,0}(\beta)$ for a fixed β are dependent upon the choice of face $\tau \in \Phi^L_A$. For example, consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and the parameter $\beta = (0, 0, -1)^t$, which lies outside the cone $\mathbb{R}_{\geq 0}A$. It turns out that

$$\widetilde{\mathbb{N}A} \setminus \mathbb{N}A = (\beta + \mathbb{N}G_1) \cap \mathbb{R}_{\geq 0}A = (\beta + \mathbb{N}G_1) \setminus \{\beta\},$$

where $G_1 = \{a_3, a_6\}$ and $G_2 = \{a_1, a_2, a_4, a_5, a_7\}$ are facets of A. In particular, $\mathcal{E}_A = \{\beta\}$ and the ranking lattices at β are

$$\mathbb{E}^{\beta} = (\beta + \mathbb{Z}G_1) \cup (\beta + \mathbb{Z}G_2).$$

By Remark 4.8, $\mu^{L,\varnothing}_{A,0}(\beta) - \mu^{L,\varnothing}_{A} = 1$ for any projective weight vector L. On the other hand, $\mu^{L,\tau}_{A,0}(\beta) = \mu^{L,\tau}_{A}$ if $\tau \neq \varnothing$.

Example 4.10. The choice of projective weight vector impacts the resulting stratification via multiplicities of $\mathcal{E}_A$. For example, consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix},$$

which has

$$\widetilde{\mathbb{N}A} \setminus \mathbb{N}A = (\beta + \mathbb{N}G_1) \cup (\beta + \mathbb{N}G_2),$$

where $G_1 = \{a_3, a_4\}$, $G_2 = \{a_5, a_6\} \preceq A$ and $\beta = (1, 0, 0)^t$. Moreover, we also have

$$\mathbb{E}^\beta = (\beta + \mathbb{Z}G_1) \cup (\beta + \mathbb{Z}G_2) \quad \text{and} \quad \mathcal{E}_A = (\beta + \mathbb{C}G_1) \cup (\beta + \mathbb{C}G_2).$$

If $L_\partial = (1, 4, 1, 4, 1, 3, 1)$ and $L_x = 5 \cdot \mathbf{1}_7 - L_\partial$, then $\mu^{L,\varnothing}_{A,0}(\beta') - \mu^{L,\varnothing}_A = 1$ for any $\beta' \in \mathcal{E}_A$. On the other hand, the stratification of $\mathcal{E}_A$ by the rank jump is different:

$$\mu^{F,\varnothing}_{A,0}(\beta') - \mu^{F,\varnothing}_A = \begin{cases} 2 & \text{if } \beta' \in (\beta + \mathbb{C}G_2) \setminus \{\beta\}, \\ 3 & \text{if } \beta' \in (\beta + \mathbb{C}G_1) \setminus \{\beta\}, \\ 4 & \text{if } \beta' = \beta. \end{cases}$$

5. More consequences of the multiplicity computation

For $\tau \in \Phi^L_A$, let

$$j^{L,\tau}_A(\beta) := \mu^{L,\tau}_{A,0}(\beta) - \mu^{L,\tau}_A$$

be the (L, τ)-*multiplicity jump* at β, and let

$$\mathcal{E}^{L,\tau}_A := \{\beta \in \mathbb{C}^d \mid j^{L,\tau}_A(\beta) > 0\}$$

be the (L, τ)-*exceptional set* of A. In this section, we record consequences of Theorem 4.1 and its implications for $\mathcal{E}^{L,\tau}_A$. We also propose a description of $\mathcal{E}^{L,\tau}_A$ and prove it holds in a special case.

Corollary 5.1. *If $\tau \in \Phi^L_A$ is a face of the (A, L)-umbrella such that τ is not contained in any face of A of codimension 2, then $\mathcal{E}^{L,\tau}_A = \varnothing$.*

Proof. Fix $\beta \in \mathbb{C}^d$. By hypothesis, τ is contained in at most one facet of A. Recall that the cellular resolution of P^β_J is made of ranking toric modules P^β_G for faces $G \preceq A$ such that $\mathbb{E}^\beta_G \neq \varnothing$.

If τ is not contained in any proper face of A or it is contained in a unique facet $F \preceq A$ with $\mathbb{E}^\beta_F = \varnothing$, then Lemma 4.7 guarantees that $\mu^{L,\tau}_{A,q}(P^\beta_G, \beta) = 0$ for all $q \geq 0$ for any proper face $G \preceq A$ with $\mathbb{E}^\beta_G \neq \varnothing$. Thus, the formula from Theorem 4.1 computes that $\mu^{L,\tau}_{A,i}(P^\beta, \beta) = 0$ for all $i \geq 0$.

For the remaining case when τ is contained in a unique facet $F \preceq A$ and $\mathbb{E}^\beta_F \neq \varnothing$,

$$\mu^{L,\tau}_{A,i}(P^\beta, \beta) = \mu^{L,\tau}_{A,i}(P^\beta_F, \beta) = |B^\beta_F| \cdot \mu^{L,\tau}_{A,i}(\tilde{S}_F, \beta) = |B^\beta_F| \cdot \binom{1}{i} \cdot \mu^{L,\tau}_F$$

for all $i \geq 0$. Therefore, as in the proof of Theorem 4.1,

$$j^{L,\tau}_A(\beta) = \mu^{L,\tau}_{A,1}(P^\beta, \beta) - \mu^{L,\tau}_{A,0}(P^\beta, \beta) = 0. \qquad \square$$

Remark 5.2. As an immediate consequence of Corollary 5.1, if $\dim(\mathbb{C}\tau) \geq d - 1$, then $\mu^{L,\tau}_{A,0}(\beta)$ is independent of β. Notice that this fact was known when $\dim(\mathbb{C}\tau) = d$ (see [Schulze and Walther 2008, Theorem 3.10]). $\square$

Corollary 5.3. *If $\tau \in \Phi_A^L$ is a face of the (A, L)-umbrella such that τ is contained in a unique face $G \preceq A$ of codimension 2, then*

$$j_A^{L,\tau}(\beta) = \begin{cases} |B_G^\beta| \cdot \mu_G^{L,\tau} & \text{if } (G, b) \in \max(\mathcal{J}(\beta)) \text{ for } b \in B_G^\beta, \text{ see (4-2)}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the proof of Theorem 4.1 and Lemma 4.7,

$$j_A^{L,\tau}(\beta) = \mu_{A,1}^{L,\tau}(P^\beta, \beta) - \mu_{A,0}^{L,\tau}(P^\beta, \beta) = \mu_{A,1}^{L,\tau}(P_{J'}^\beta, \beta) - \mu_{A,0}^{L,\tau}(P_{J'}^\beta, \beta),$$

where $J' = \{(F, b) \in J \mid \tau \subseteq F\}$ for $J = \mathcal{J}(\beta)$. If $(G, b) \in \max(J)$ and $b \in B_G^\beta$, then $J' = J(G)$ and $P_{J'}^\beta = P_G^\beta$. Thus, it is enough to consider the case when $(G, b) \notin \max(J)$ for any b but there exists at least one facet F such that $\tau \subseteq G \preceq F$ and $(F, b) \in \max(J)$. In this case, either $\max(J') = J(F)$ or $J' = J(F) \cup J(F')$ for some other facet F' such that $F \cap F' = G$. Either way, it follows that $\mu_{A,1}^{L,\tau}(P_{J'}^\beta, \beta) - \mu_{A,0}^{L,\tau}(P_{J'}^\beta, \beta) = 0$. □

By Corollaries 5.1 and 5.3, if $\dim(\mathbb{C}\tau) = d - 2$, then $j_A^{L,\tau}(\beta) > 0$ only when there is a (unique) codimension 2 face G of A containing τ and $(G, b) \in \max(\mathcal{J}(\beta))$ for some $b \in B_G^\beta$.

Notation 5.4. For any $\tau \in \Phi_A^L$, let us denote $S_A^\tau := \mathbb{C}[(\mathbb{N}A + \mathbb{Z}\tau) \cap \mathbb{R}_{\geq 0}A]$.

Conjecture 5.5. *There is an equality*

$$\mathcal{E}_A^{L,\tau} = -\operatorname{qdeg}\left(\bigoplus_{q=0}^{d-1} \operatorname{Ext}_{\mathbb{C}[\partial]}^{n-q}(S_A^\tau, \mathbb{C}[\partial])(-\varepsilon_A)\right),$$

where $\varepsilon_A := \sum_{i=1}^n a_i$. In particular, $\mathcal{E}_A^{L,\tau} = \varnothing$ if and only if S_A^τ is Cohen–Macaulay.

As evidence of the truth of Conjecture 5.5, we exhibit a containment between the two sets involved. We then prove the second part of conjecture in the case that $\mathbb{R}_{\geq 0}A$ is a simplicial cone.

Proposition 5.6. *There is a containment*

$$\mathcal{E}_A^{L,\tau} \subseteq -\operatorname{qdeg}\left(\bigoplus_{q=0}^{d-1} \operatorname{Ext}_{\mathbb{C}[\partial]}^{n-q}(S_A^\tau, \mathbb{C}[\partial])(-\varepsilon_A)\right).$$

Proof. By the definition of S_A^τ, it is clear that $S_A^\tau[\partial_\tau^{-1}] = S_A[\partial_\tau^{-1}]$ and thus,

$$\mu_{A,0}^{L,\tau}(S_A, \beta) = \mu_{A,0}^{L,\tau}(S_A[\partial_\tau^{-1}], \beta) = \mu_{A,0}^{L,\tau}(S_A^\tau[\partial_\tau^{-1}], \beta) = \mu_{A,0}^{L,\tau}(S_A^\tau, \beta),$$

where the first and third equalities follows from the definition of $\mu_{A,0}^{L,\tau}$ (see (2-1)) and the fact that $\xi_j = \operatorname{in}_L(\partial_j) \notin P_\tau$ if and only if $j \in \tau$.

If $\beta \notin -\operatorname{qdeg}(\operatorname{Ext}_{\mathbb{C}[\partial]}^{n-q}(S_A^\tau, \mathbb{C}[\partial])(-\varepsilon_A))$ for any $q = 0, \ldots, d-1$, then $\mathcal{H}_i(S_A^\tau, \beta) = 0$ for all $i > 0$ by [Matusevich et al. 2005, Theorem 6.6]. Thus, $\mu_{A,0}^{L,\tau}(S_A^\tau, \beta) = \sum_{j=0}^d (-1)^j \mu_{A,j}^{L,\tau}(S_A^\tau, \beta)$, which is independent of β by [Schulze and Walther 2008, Theorem 4.11] and hence equal to the generic value $\mu_A^{L,\tau}$. In particular, $\beta \notin \mathcal{E}_A^{L,\tau}$. □

Proposition 5.7. *Fix $\beta \in \mathbb{C}^d$ and let J be as in (4-2). If J involves only facets of A satisfying that the intersection of r of them is a face of codimension at most r, then $\mathcal{H}_q(P_J^\beta, \beta) = 0$ for all $q \geq 2$.*

Proof. Consider the cellular resolution of P_J^β as constructed in [Berkesch 2011, (6.3)]:

$$0 \to P_J \to I_J^0 \to I_J^1 \to \cdots \to I_J^r \to 0,$$

where $r+1$ is the cardinality of J. On the other hand, if $K_p := \ker(I_J^p \to I_J^{p+1})$ for $0 \leq p \leq r-1$ and $K_r = I_J^r$, then there are short exact sequences

$$0 \to P_J \to I_J^0 \to K_1 \to 0 \quad \text{and} \quad 0 \to K_p \to I_J^p \to K_{p+1} \to 0 \quad \text{for } 1 \leq p \leq r-1.$$

By the assumption on J, I_J^p is a direct sum of simple ranking toric modules P_G for faces G of codimension at most $p+1$, so by [Berkesch 2011, Proposition 3.2], $\mathcal{H}_q(I_J^p, \beta) = 0$ for all $q \geq p+2$ and $p = 0, \ldots, r$. Therefore

$$\mathcal{H}_q(P_J, \beta) \cong \mathcal{H}_{q+1}(K_1, \beta) \cong \cdots \cong \mathcal{H}_{q+r-1}(K_{r-1}, \beta) \cong \mathcal{H}_{q+r}(I_J^r, \beta) = 0$$

for all $q \geq 2$, as desired. □

Note that if $\mathbb{R}_{\geq 0}A$ is simplicial then any set of facets of A satisfies the property required in Proposition 5.7. To the contrary, Example 4.9 does not satisfy this property.

Theorem 5.8. *Let $\tau \in \Phi_A^L$ and assume that $\mathbb{R}_{\geq 0}A$ is a simplicial cone. Then $\mathcal{E}_A^{L,\tau} = \varnothing$ if and only if S_A^τ is Cohen–Macaulay.*

Proof. The if direction is proven in Proposition 5.6. By the definition of S_A^τ we have that

$$\operatorname{rank}(\mathcal{H}_0(S_A^\tau, \beta)) = \operatorname{vol}(A) + \mu_{A,1}^{F,\varnothing}(P_{J'}) - \mu_{A,0}^{F,\varnothing}(P_{J'}),$$

where $J' := \{(G, b) \in \mathcal{J}(\beta) \mid \tau \subseteq G\}$. If S_A^τ is not Cohen–Macaulay, then by Theorem 1.2, there exists $\beta \in \mathbb{C}^d$ such that $\operatorname{rank}(\mathcal{H}_0(S_A^\tau, \beta)) > \operatorname{vol}(A)$. Since $\mathbb{R}_{\geq 0}A$ is simplicial, by Proposition 5.7 there must be a face G of codimension at least 2 such that $(G, b) \in \max(J')$. Thus, for generic $\beta' \in b + \mathbb{C}G$, we have that $\max(\mathcal{J}(\beta')) = \{(G, b_1), \ldots, (G, b_r)\}$ with $r = |B_G^{\beta'}|$. Now, using Lemma 4.7, we have that

$$\mu_A^{L,\tau}(S_A, \beta') = \mu_A^{L,\tau}(S_A^\tau, \beta') = \mu_A^{L,\tau} + r(\operatorname{codim}(G) - 1) \cdot \mu_G^{L,\tau} > \mu_A^{L,\tau}$$

and thus $\beta' \in \mathcal{E}_A^{L,\tau} \neq \varnothing$. □

6. Upper-semicontinuity and convex filtrations

It was conjectured in [Schulze and Walther 2008] that the multiplicities $\mu_{A,0}^{L,\tau}(\beta)$ are upper semicontinuous in $\beta \in \mathbb{C}^d$ for any projective L and $\tau \in \Phi_A^L$. We prove this conjecture when L and τ satisfy certain conditions with respect to A (see Theorem 6.1 and Corollary 6.6). We also prove Conjecture 5.5 in this setting when $\tau = \varnothing$ (see Corollary 6.5).

Given a submatrix $\sigma \subseteq A$ with rank d, denote by E_i^σ the Euler operator associated with the i-th row of the matrix σ. Let D_σ denote the Weyl algebra associated to the variables $x_\sigma = \{x_i \mid a_i \in \sigma\}$. We have that $\mathbb{Z}A = \mathbb{Z}^d = \bigoplus_{j=1}^r \Lambda_j$, where $r = [\mathbb{Z}^d : \mathbb{Z}\sigma]$ and $\Lambda_j = b_j + \mathbb{Z}\sigma$ for some $b_j \in \mathbb{Z}^d$ with $j = 1, \dots, r$.

If N is a $\mathbb{Z}^d$-graded S_A-module, then $N_j := \bigoplus_{\alpha \in \Lambda_j} N_\alpha$ is an S_σ-module. Let $\mathcal{K}_\bullet^\sigma(N, \beta)$ denote the direct sum over j of the Euler–Koszul complexes on $D_\sigma \otimes_{\mathbb{C}[\partial_\sigma]} N_j(-b_j)$ given by the operators $\{E_i^\sigma - \beta_i + (b_j)_i\}_{i=1}^d$, where each such Euler–Koszul complex is placed in degree b_j. That is,

$$\mathcal{K}^\sigma(N, \beta) := \bigoplus_{j=1}^r \mathcal{K}^\sigma(N_j(-b_j), \beta - b_j)(b_j),$$

where the right-hand side Euler–Koszul complexes where defined before since $N_j(-b_j)$ is a $\mathbb{Z}\sigma$-graded S_σ-module. This definition is independent of the chosen elements $b_1, \dots, b_r \in \mathbb{Z}^d$ by (1-1). With this setup, $D_\sigma \otimes N \cong \bigoplus_{j=1}^r (D_\sigma \otimes N_j)$, and $\mathcal{K}_\bullet^\sigma(N, \beta)$ is a $\mathbb{Z}^d$-graded complex of left D_σ-modules. Set

$$\mathcal{H}_i^\sigma(N, \beta) := H_i(\mathcal{K}_\bullet^\sigma(N, \beta)),$$

and note that these definitions make (1-1) and Theorem 1.1 also valid for the homology modules $\mathcal{H}_i^\sigma(N, \beta)$.

Let L be a projective weight vector, which induces a filtration on D as considered in the introduction. We denote by A^L the submatrix of A whose columns belong to facets of Φ_A^L. We say that L is a *convex* filtration with respect to A if all facets of Φ_A^L are F-homogeneous and

$$\bigcup_{\tau' \in \Phi_A^{L,d-1}} \Delta_{\tau'} \tag{6-1}$$

is a convex polytope, and thus equal to Δ_{A^L}. Notice that, by the inclusion $S_{A^L} \subseteq S_A$, the ring S_A is an S_{A^L}-module.

Theorem 6.1. *If L is a convex filtration with respect to A, then $\mu_{A,0}^{L,\varnothing}(\beta) = \operatorname{rank}(\mathcal{H}_0^{A^L}(S_A, \beta))$. In particular, $\mu_{A,0}^{L,\varnothing}(\beta)$ is upper-semicontinuous in β.*

Before proving Theorem 6.1, we first consider the simple case.

Proposition 6.2. *Let L be a convex filtration of D with respect to A and $G \preceq A$. Then for all $(G, b) \in \mathcal{J}(\beta)$,*

$$\mu_G^{L,\varnothing} = \operatorname{vol}_{\mathbb{Z}G}(G^L) = \operatorname{rank}(\mathcal{H}_0^{G^L}(P_{(G,b)}^\beta, \beta)),$$

where G^L denotes the submatrix of A whose columns belong to facets of Φ_G^L.

Proof. The first equality follows from Theorem 2.1 and (2-4) since L is convex. For the second equality, by definition of the (G, L)-umbrella Φ_G^L, the submatrix G^L of G is such that $\mathbb{R}_{\geq 0}G = \mathbb{R}_{\geq 0}G^L$ and $\operatorname{rank}(G) = \operatorname{rank}(G^L)$. This implies that S_G is a toric S_{G^L}-module. Further,

$$P_{(G,b)}^\beta \subseteq S_G[\partial_G^{-1}](b) \cong \bigoplus_{\alpha \in \Lambda} S_{G^L}[\partial_{G^L}^{-1}](\alpha),$$

where Λ is a finite subset of $b+\mathbb{Z}G$ of cardinality $[\mathbb{Z}G:\mathbb{Z}G^L]$. Since

$$\deg(S_G[\partial_G^{-1}](b)/P^{\beta}_{(G,b)}) = (b+\mathbb{Z}G)\setminus \mathbb{P}^{\beta}_{(G,b)},$$

it follows from the definition of $C_A(\beta)$ that the parameter β does not belong to the quasidegrees set of the weakly toric module $S_G[\partial_G^{-1}](b)/P^{\beta}_{(G,b)}$. Thus, since $S_{G^L}[\partial_{G^L}^{-1}]$ is a Cohen–Macaulay S_{G^L}-module, by Theorem 1.1 and Theorem 1.2, $\mathcal{H}_i^{G^L}(P^{\beta}_{(G,b)},\beta)=0$ for all $i\geq 1$ and

$$\operatorname{rank}(\mathcal{H}_0^{G^L}(P^{\beta}_{(G,b)},\beta)) = [\mathbb{Z}G:\mathbb{Z}G^L]\cdot \operatorname{vol}_{\mathbb{Z}G^L}(G^L) = \operatorname{vol}_{\mathbb{Z}G}(G^L). \qquad \square$$

Remark 6.3. Notice that any weakly toric S_A-module $M\subseteq S_A[\partial_A^{-1}]$ can be viewed as a weakly toric S_{A^L}-module. Indeed, since A^L and A have the same rank, then $\mathbb{Z}A=\bigoplus_{j=1}^r(b_j+\mathbb{Z}A^L)$ for some $b_j\in\mathbb{Z}A$ with $j=1,\dots,r$. Thus $S_A[\partial_A^{-1}]=\bigoplus_{j=1}^r S_{A^L}[\partial_{A^L}^{-1}](b_j)$ as S_{A^L}-modules. Setting $M_j := M\cap S_{A^L}[\partial_{A^L}^{-1}](b_j)$, then M is the direct sum of the weakly toric S_{A^L}-modules M_j. Moreover, for any face $G\preccurlyeq A$,

$$\mathbb{E}_G^{\beta} = \bigsqcup_{b\in B_G^{\beta}}(b+\mathbb{Z}G) = \bigsqcup_{c\in B^{\beta}_{G^L}}(c+\mathbb{Z}G^L), \tag{6-2}$$

where B_G^{β} and $B^{\beta}_{G^L}$ is a set of lattice representatives (see (4-1)).

Lemma 6.4. *The module P^{β} is a direct sum of toric S_{A^L}-modules, and for any face $G\preceq A$ and $q\geq 0$,*

$$\mu^{L,\varnothing}_{A,q}(P_G^{\beta},\beta) = \mu^{F,\varnothing}_{A^L,q}(P_G^{\beta},\beta).$$

Proof. The decomposition of $M=S_A$ as a direct sum of weakly toric S_A^L-modules M_j given in Remark 6.3 induces a decomposition of $S_A[\partial_A^{-1}]/M$ as a direct sum of the weakly toric S_A^L-modules $S_{A^L}[\partial_{A^L}^{-1}](b_j)/M_j$. Then, by the two short exact sequences in the proof of [Berkesch 2011, Proposition 5.10], P^{β} is a direct sum of weakly toric S_{A^L}-modules. Moreover, since $\mathbb{P}^{\beta}=\mathbb{E}^{\beta}\cap\mathcal{C}_A(\beta)$ and $\mathcal{C}_A(\beta)=\mathcal{C}_{A^L}(\beta)$, it follows that P^{β} is a direct sum of toric S_A^L-modules.

On the other hand, if G is a face of A, then by (6-2), $|B_G^{\beta}|[\mathbb{Z}G:\mathbb{Z}G^L]=|B^{\beta}_{G^L}|$. Thus, using Lemma 4.7 and Proposition 6.2,

$$\mu^{L,\varnothing}_{A,q}(P_G^{\beta},\beta) = |B_G^{\beta}|\cdot\binom{\operatorname{codim}(G)}{q}\cdot\operatorname{vol}_{\mathbb{Z}G}(G^L) = |B^{\beta}_{G^L}|\cdot\binom{\operatorname{codim}(G)}{q}\cdot\operatorname{vol}_{\mathbb{Z}G^L}(G^L) = \mu^{F,\varnothing}_{A^L,q}(P_G^{\beta},\beta). \ \square$$

The proof of Theorem 6.1 makes use of the notion of a *holonomic family* from [Matusevich et al. 2005, Definition 2.1], which we now recall. While defined over any algebraic variety B with structure sheaf $\mathcal{O}_B$, we will need only the case when $B=\mathbb{A}^d_{\mathbb{C}}$, affine d-space over $\mathbb{C}$.

If $\beta\in B$, denote by p_β the prime ideal (sheaf) of β and set $\kappa_\beta=\mathcal{O}_{B,\beta}/p_\beta\mathcal{O}_{B,\beta}$, the residue field of the stalk $\mathcal{O}_{B,\beta}$. A *coherent sheaf* of $(D\otimes_{\mathbb{C}}\mathcal{O}_B)$-modules is a quasicoherent sheaf of $\mathcal{O}_B$-modules on B whose sections over each open affine subset $U\subset B$ are finitely generated over the ring of global sections $H^0(B, D\otimes_{\mathbb{C}}\mathcal{O}_U)$. Let $\mathcal{O}_B(x)$ denote the localization at $\langle 0\rangle\in\operatorname{Spec}(\mathbb{C}[x])$ of $\mathcal{O}_B[x]:=\mathbb{C}[x]\otimes_{\mathbb{C}}\mathcal{O}_B\subset D\otimes_{\mathbb{C}}\mathcal{O}_B$. The sheaf-spectrum of $\mathcal{O}_B(x)$ is the base-extended scheme $B(x):=\operatorname{Spec}\mathbb{C}(x)\times_{\operatorname{Spec}\mathbb{C}}B$.

A *holonomic family* over B is a coherent sheaf $\widetilde{\mathcal{M}}$ of left $(D\otimes_{\mathbb{C}}\mathcal{O}_B)$-modules such that

(1) the fibers $\mathcal{M}_\beta = \widetilde{\mathcal{M}} \otimes_{\mathcal{O}_B} \kappa_\beta$ are holonomic D-modules for all $\beta \in B$, and

(2) $\mathcal{O}_B(x) \otimes_{\mathcal{O}_B} \widetilde{\mathcal{M}}$ is coherent on $B(x)$.

Proof of Theorem 6.1. Since $S_A[\partial^{-1}]$ is a maximal Cohen–Macaulay weakly toric S_{A^L}-module,

$$\mathcal{H}_i^{A^L}(S_A[\partial^{-1}], \beta) = 0$$

for all $i > 0$ by Theorem 1.2. Thus, applying Euler–Koszul homology with respect to A^L to the short exact sequence

$$0 \to S_A \to S_A[\partial^{-1}] \to Q \to 0$$

and using that $\mathcal{H}_q^{A^L}(P^\beta, \beta) \simeq \mathcal{H}_q^{A^L}(Q, \beta)$ (see the proof of [Berkesch 2011, Proposition 5.10], which can be adapted to this case), it follows that

$$\operatorname{rank}(\mathcal{H}_0^{A^L}(S_A, \beta)) = \operatorname{rank}(\mathcal{H}_0^{A^L}(S_A[\partial^{-1}], \beta)) + \mu_{A^L,1}^{F,\varnothing}(P^\beta, \beta) - \mu_{A^L,0}^{F,\varnothing}(P^\beta, \beta).$$

The proofs of [loc. cit., Theorem 6.6] and Theorem 4.1 and the induction argument in the proof of [loc. cit., Proposition 6.18] reduces the computation of

$$\mu_{A^L,1}^{F,\varnothing}(P^\beta, \beta) - \mu_{A^L,0}^{F,\varnothing}(P^\beta, \beta) \quad \text{(and respectively } \mu_{A,1}^{L,\varnothing}(P^\beta, \beta) - \mu_{A,0}^{L,\varnothing}(P^\beta, \beta))$$

to that of $\mu_{A^L,q}^{F,\varnothing}(N, \beta)$ (and respectively $\mu_{A,q}^{L,\varnothing}(N, \beta)$) for $q \geq 0$ and simple toric modules $N = P_G^\beta$ with $\mathbb{E}_G^\beta \neq \varnothing$. Thus, by Lemma 6.4,

$$\mu_{A^L,1}^{F,\varnothing}(P^\beta, \beta) - \mu_{A^L,0}^{F,\varnothing}(P^\beta, \beta) = \mu_{A,1}^{L,\varnothing}(P^\beta, \beta) - \mu_{A,0}^{L,\varnothing}(P^\beta, \beta) = \mu_{A,0}^{L,\varnothing}(\beta) - \mu_A^{L,\varnothing}$$

which yields the desired equality.

Finally, since S_A is a toric S_{A^L}-module, $\mathcal{H}_0^{A^L}(S_A, b)$ is a holonomic family by [Matusevich et al. 2005, Theorem 7.5]. Hence [loc. cit., Theorem 2.6] guarantees that

$$\beta \mapsto \operatorname{rank}(\mathcal{H}_0^{A^L}(S_A, \beta))$$

is an upper semicontinuous function. □

Theorem 6.1 provides a way to prove Conjecture 5.5 when L is a convex filtration of D with respect to A and $\tau = \varnothing$.

Corollary 6.5. *If L is a convex filtration of D with respect to A, then*

$$\mathcal{E}_A^{L,\varnothing} = -\operatorname{qdeg}\Big(\bigoplus_{q=0}^{d-1} \operatorname{Ext}_{\mathbb{C}[\partial]}^{n-q}(S_A, \mathbb{C}[\partial])(-\varepsilon_A)\Big).$$

Proof. By the proof of Theorem 6.1, $\mathcal{H}_0^{A^L}(S_A, b)$ is a holonomic family and $\mathcal{E}_A^{L,\varnothing} = \mathcal{E}_{A^L}^{F,\varnothing}$, and thus by [Matusevich et al. 2005, Theorem 9.1],

$$\mathcal{E}_A^{L,\varnothing} = \overline{\deg\Big(2\bigoplus_{i=0}^{d-1} H_{\mathfrak{m}_L}^i(S_A)\Big)^{\text{Zariski}}},$$

where $\mathfrak{m}_L$ denotes the maximal homogeneous ideal in S_{A^L}. However, since $\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}A^L$, the radical of the extended ideal $\mathfrak{m}_L S_A$ in S_A equals $\mathfrak{m}$. Therefore, by applying graded Matlis duality, we obtain the desired result. □

Let L be a filtration on D induced by a projective weight vector. For $\tau \in \Phi_A^L$, we denote by $A^{L,\tau}$ the submatrix of A whose columns belong to facets $\tau' \in \Phi_A^{L,d-1}$ such that $\tau \subseteq \tau'$. We say that L is τ-*convex* if all facets of Φ_A^L containing τ are F-homogeneous and the polytope

$$\bigcup_{\tau \subseteq \tau' \in \Phi_A^{L,d-1}} \Delta_{\tau'} \tag{6-3}$$

is convex, and thus equal to $\Delta_{A^{L,\tau}}$.

We recall that a subset $\eta' \subseteq A$ is said to be a *pyramid* over $\eta \subseteq \eta'$ if

$$\operatorname{rank}_{\mathbb{Z}}(\mathbb{Z}\eta) + |\eta' \setminus \eta| = d,$$

where we denote by $|\lambda|$ the cardinality of a set λ.

Theorem 6.1 can now be generalized as follows.

Corollary 6.6. *If L induces a τ-convex filtration for some $\tau \in \Phi_A^L$ and any $\tau' \in \Phi_A^{L,d-1}$ such that $\tau \subseteq \tau'$ is a pyramid over $\tau' \setminus \tau$, then*

$$\mu_{A,0}^{L,\tau}(\beta) = \operatorname{rank}(\mathcal{H}_0^{A^{L,\tau}}(S_A, \beta)).$$

In particular, $\mu_{A,0}^{L,\tau}(\beta)$ is upper-semicontinuous in β.

Proof. Recall the formula in Theorem 2.1. For any $\tau' \in \Phi_A^L$ containing τ, since τ' is a pyramid over $\tau' \setminus \tau$, it follows that

$$\mathbb{Z}\tau' \cap \mathbb{Q}\tau = \mathbb{Z}\tau, \quad \pi_{\tau,\tau'}(\mathbb{Z}\tau') = \mathbb{Z}(\tau' \setminus \tau), \quad P_{\tau,\tau'} = \Delta_{\tau' \setminus \tau},$$

$Q_{\tau,\tau'}$ is the convex hull of $\tau' \setminus \tau$ (whose volume is zero because τ' is F-homogeneous), and $\operatorname{vol}_{\mathbb{Z}\tau'}(\tau') = \operatorname{vol}_{\mathbb{Z}(\tau' \setminus \tau)}(\tau' \setminus \tau)$. Thus, for any face $G \preceq A$ that contains τ,

$$\mu_G^{L,\tau} = \sum_{\tau \subseteq \tau' \in \Phi_G^{L,d-1}} [\mathbb{Z}G : \mathbb{Z}\tau'] \cdot \operatorname{vol}_{\mathbb{Z}\tau'}(\Delta_{\tau'}) = \operatorname{vol}_{\mathbb{Z}G}\Bigg(\bigcup_{\tau \subseteq \tau' \in \Phi_G^{L,d-1}} \Delta_{\tau'} \Bigg) = \operatorname{vol}_{\mathbb{Z}G}(\Delta_{G^{L,\tau}}).$$

When $(G, b) \in \mathcal{J}(\beta)$, to obtain the equality

$$\operatorname{rank}(\mathcal{H}_0^{G^{L,\tau}}(P_{(G,b)}^\beta, \beta)) = \operatorname{vol}_{\mathbb{Z}G}(\Delta_{G^{L,\tau}}) \tag{6-4}$$

we can proceed as in the proof of Proposition 6.2, but now $\mathbb{R}_{\geq 0}G$ is not equal to $\mathbb{R}_{\geq 0}G^{L,\tau}$, so S_A is only a direct sum of weakly toric $S_{A^{L,\tau}}$-modules (by Remark 6.3) instead of a toric $S_{A^{L,\tau}}$-module. On the other hand, in the proof of Theorem 6.1 we can use P_J^β with $J = \{(G, b) \in \mathcal{J}(\beta) \mid \tau \subseteq G\}$ instead of P^β and consider each P_G^β as a direct sum of weakly toric Cohen–Macaulay $S_{G^{L,\tau}}$-modules.

Finally, by [Schulze and Walther 2009, Remark 5.5.(5)], in the analytic topology, $\mathcal{H}_0^{A^{L,\tau}}(S_A, \beta)$ is locally a holonomic family on $\mathbb{A}^d$. This fact along with [Matusevich et al. 2005, Theorem 2.6] and Theorem 4.1 imply that the function $\beta \mapsto \operatorname{rank}(\mathcal{H}_0^{A^{L,\tau}}(S_A, \beta))$ is upper-semicontinuous. $\square$

7. Gevrey series solutions associated to slopes

Let $\mathcal{D}$ be the sheaf of linear partial differential operators with coefficients in the sheaf $\mathcal{O}_X^{\text{an}}$ of holomorphic functions on $X = \mathbb{C}^n$. The irregularity sheaf of order $s > 1$ of a holonomic $\mathcal{D}$-module $\mathcal{M}$ along a hypersurface Y was introduced and proved to be a perverse sheaf on Y by Mebkhout [1990]. In particular, higher cohomology of the irregularity sheaf vanishes at generic points of Y.

In this section, for a coordinate hyperplane $Y \subset X$, we compute the dimension of the stalk at a generic point $p \in Y$ of the irregularity sheaf of order s of $\mathcal{M}_A(\beta) := \mathcal{D} \otimes_D M_A(\beta)$ along Y for any parameter $\beta \in \mathbb{C}^d$, generalizing results from [Fernández-Fernández 2010]. As a consequence, we provide some formulas for the dimension of the Gevrey solution spaces of $\mathcal{M}_A(\beta)$ in particular cases, and we show that the dimension of the generic stalk of the irregularity sheaf of $\mathcal{M}_A(\beta)$ along Y is upper-semicontinuous in β.

We assume for simplicity that $Y = \operatorname{Var}(x_n)$ and write s instead of $L(s)$ for the filtration given by $L(s) := F + (s-1)V_n$ with $s \geq 1$, where $F = (\mathbf{0}_n, \mathbf{1}_n)$ is the filtration by the order of the differential operators and V_n is the Kashiwara–Malgrange filtration along Y. Recall that this filtration is induced by the weight vector $V_n := (0, \dots, 0, -1, 0, \dots, 0, 1)$, where -1 is the weight for the variable x_n. More precisely, the filtration $L(s)$ is determined by

$$\deg_s \partial_i = \begin{cases} 1 & \text{if } 1 \leq i \leq n-1, \\ s & \text{if } i = n, \end{cases} \quad \text{and} \quad \deg_s(x_i) = 1 - \deg_s(\partial_i).$$

In this section, we call the $(A, L(s))$-umbrella the (A, s)-umbrella, and we denote $\Phi_A^s := \Phi_A^{L(s)}$ for $s \geq 1$.

A global version of Laurent's slope theory [1987] proceeds as follows. Let M be a holonomic D-module. A number $s > 1$ is said to be a *slope* of M along $Y = \operatorname{Var}(x_n)$ if and only if the s-characteristic variety $\operatorname{Char}^s(M)$ of M along Y is not homogeneous with respect to the weight vector $F = (\mathbf{0}_n, \mathbf{1}_n)$.

Remark 7.1. Denote by A' the submatrix of A defined by the first $n-1$ columns and by Δ' the convex hull of the columns of A' and the origin. Note that a_n/s belongs to a hyperplane off the origin that contains a facet of $\Delta_{A'}$ if and only if there exists a facet of the (A, s)-umbrella, in other words an element of $\Phi_A^{s,d-1}$, that is not F-homogeneous. Moreover, by [Schulze and Walther 2008, Corollary 4.18], this condition holds if and only if $s > 1$ is a slope of $M_A(\beta)$ along $\operatorname{Var}(x_n)$.

Let $\mathcal{O}_{\widehat{X|Y}}$ denote the formal completion of $\mathcal{O}_X$ along Y. A *germ* $f \in \mathcal{O}_{\widehat{X|Y},p}$ with $p \in Y$ is a formal series

$$f = \sum_{m=0}^{\infty} f_m(x_1, \dots, x_{n-1}) x_n^m$$

such that there exists some open subset $U \subseteq \mathbb{C}^{n-1}$ so that f_m is a holomorphic function in U for all $m \geq 0$. The formal series $f \in \mathcal{O}_{\widehat{X|Y},p}$ is said to be a *Gevrey series* of order $s \in \mathbb{R}$ along Y at $p \in Y$ if the series

$$\rho_s(f) := \sum_{m=0}^{\infty} \frac{f_m(x_1, \ldots, x_{n-1})}{(m!)^{s-1}} x_n^m$$

is convergent at p. Moreover, if $\rho_{s'}(f)$ is not convergent at p for any $s' < s$, then s is said to be the *Gevrey index* of f along Y at p. Denote by $\mathcal{O}_{X|Y}(s)$ the subsheaf of $\mathcal{O}_{\widehat{X|Y}}$ whose germs are Gevrey series of order s along Y.

The *irregularity sheaf* of a $\mathcal{D}$-module $\mathcal{M}$ along Y of order $s > 1$ is

$$\mathrm{Irr}_Y^{(s)}(\mathcal{M}) := \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}(s)/\mathcal{O}_{X|Y}).$$

For $s = \infty$, the sheaf $\mathrm{Irr}_Y^{\infty}(\mathcal{M})$ is simply called the *irregularity sheaf* of $\mathcal{M}$ along Y. If M is a D-module, we define $\mathrm{Irr}_Y^{(s)}(M) := \mathrm{Irr}_Y^{(s)}(\mathcal{M})$, where $\mathcal{M} := \mathcal{D} \otimes_D M$.

Set $\mathrm{d}_s(A, \beta) := \dim H^0(\mathrm{Irr}_Y^{(s)}(M_A(\beta))_p)$ for a generic point $p \in Y = \mathrm{Var}(x_n)$. Applying Théorème 2.3.1 and (2.3.1) in [Laurent and Mebkhout 1999] to this setting yields the equality

$$\mathrm{d}_s(A, \beta) = \mu_{A,0}^{s+\epsilon,\varnothing}(\beta) - \mu_{A,0}^{1+\epsilon,\varnothing}(\beta) + \mu_{A,0}^{1+\epsilon,\{n\}}(\beta) - \mu_{A,0}^{s+\epsilon,\{n\}}(\beta) \tag{7-1}$$

for $\epsilon > 0$ small enough. In particular, if β is not rank-jumping for A, then by Theorem 2.1 and [Fernández-Fernández 2010, Theorem 7.5], $\mathrm{d}_s(A, \beta)$ is equal to

$$\mathrm{d}_s(A) := \mu_A^{s+\epsilon,\varnothing} - \mu_A^{1+\epsilon,\varnothing} + \mu_A^{1+\epsilon,\{n\}} - \mu_A^{s+\epsilon,\{n\}} = \sum_{n \notin \tau \in \Phi_A^{s+\epsilon,d-1} \setminus \Phi_A^{1+\epsilon,d-1}} \mathrm{vol}_{\mathbb{Z}^d}(A_\tau). \tag{7-2}$$

Remark 7.2. Notice that (7-2) also holds for any face $G \preceq A$ in place of A when $a_n \in G$. Moreover, $\mathrm{d}_s(G) = \dim H^0(\mathrm{Irr}_Y^{(s)}(M_G(\beta'))_p$ for a generic point $p \in Y' = \mathrm{Var}(x_n) \subseteq \mathbb{C}^G$ and $\beta' \in \mathbb{C}G$ that is not rank-jumping for G. The genericity condition on p requires that it avoids any other irreducible component of the singular locus of $M_G(\beta')$ (which is independent of β' as a consequence of Theorem 2.2). On the other hand, if $a_n \notin G$, then the coordinates indexed by G of the projective weight vectors $L(s)$ and F are the same. Hence the two induced filtrations over (any cyclic module over) the Weyl algebra in the variables indexed by G are also the same. Thus, $\mu_G^{1+\epsilon,\tau} = \mu_G^{s+\epsilon,\tau}$ for $\tau = \{n\}$ and $\tau = \varnothing$ in this case, so $\mathrm{d}_s(G) = 0$.

Proposition 7.3. *For any $\beta \in \mathbb{C}^d$, there is a lower bound* $\mathrm{d}_s(A, \beta) \geq \mathrm{d}_s(A)$.

Proof. For a $\mathbb{Z}^d$-graded $\mathbb{C}[\partial]$-module N, define $\mathrm{d}_s^{(j)}(N, \beta) := \dim H^0(\mathrm{Irr}_Y^{(s)}(\mathcal{H}_j(N, \beta))_p)$ for a generic point $p \in Y = \mathrm{Var}(x_n)$. Then by the same argument as in (7-1),

$$\mathrm{d}_s^{(j)}(N, \beta) = \mu_{A,j}^{s+\epsilon,\varnothing}(N, \beta) - \mu_{A,j}^{1+\epsilon,\varnothing}(N, \beta) + \mu_{A,j}^{1+\epsilon,\{n\}}(N, \beta) - \mu_{A,j}^{s+\epsilon,\{n\}}(N, \beta) \tag{7-3}$$

for $\epsilon > 0$ small enough. Notice that $\mathrm{d}_s(A, \beta) = \mathrm{d}_s^{(0)}(S_A, \beta)$. By [Schulze and Walther 2008, Corollary 4.13] and (7-3), $\mathrm{d}_s^{(0)}(\tilde{S}_A, \beta) = \mathrm{d}_s(A)$. Moreover, if $j \geq 1$, then $\mathrm{d}_s^{(j)}(\tilde{S}_A, \beta) = 0$ because $\mathcal{H}_j(\tilde{S}_A, \beta) = 0$ by Theorem 1.2.

On the other hand, if N is a toric module with dimension lower than d, it follows that

$$\mathrm{d}_s^{(0)}(N, \beta) \leq \mathrm{d}_s^{(1)}(N, \beta)$$

by the same argument as in the proof of [Schulze and Walther 2008, Lemma 4.29], with the replacement, for each D-module M that appears in that proof, of the role of $\mathrm{CC}^L(M)$ by $\dim H^0(\mathrm{Irr}_Y^{(s)}(M)_p)$ for a generic point $p \in Y = \mathrm{Var}(x_n)$. This is allowable because $H^1(\mathrm{Irr}_Y^{(s)}(M)_p) = 0$ for generic points $p \in Y$ when M is holonomic (see [Mebkhout 1990]). Thus, with the previous ingredients, the proof of [Schulze and Walther 2008, Theorem 4.28] gives the result with $\mathrm{d}_s^{(0)}$ in place of $\mu_{A,0}^{L,\tau}$. □

Corollary 7.4. *For $s > 1$, the dimension $\mathrm{d}_s(A, \beta)$ of the stalk of $\mathrm{Irr}_Y^{(s)}(M_A(\beta))$ at a generic point p of Y can be computed from the combinatorics of $\Phi_A^{s+\epsilon} \setminus \Phi_A^{1+\epsilon}$ for $\epsilon > 0$ small enough and the ranking lattices $\mathbb{E}_G^{\beta}$ at β such that $a_n \in G \preceq A$.*

Proof. It follows from (7-1) and Theorem 4.1 that $\mathrm{d}_s(A, \beta)$ can be computed from the combinatorics of the (A, s')-umbrellas for $s' \in \{1+\epsilon, s+\epsilon\}$ and the ranking lattices $\mathbb{E}^{\beta}$. Thus, by Remark 7.2, it is enough to consider the ranking lattices $\mathbb{E}_G^{\beta}$ at β corresponding to the faces $G \preceq A$ containing a_n. □

We now state further consequences for $\mathrm{d}_s(A, \beta)$.

Corollary 7.5. *If $P^{\beta} = P_G^{\beta}$ for some $G \preceq A$, then*

$$\mathrm{d}_s(A, \beta) = \mathrm{d}_s(A) + |B_G^{\beta}| \cdot (\mathrm{codim}(G) - 1) \cdot \mathrm{d}_s(G).$$

In particular, if $a_n \notin G$ or $\mathrm{codim}(G) = 1$, then $\mathrm{d}_s(A, \beta) = \mathrm{d}_s(A)$.

Proof. It is a direct consequence of (4-3), (4-4), Lemma 4.7, (7-1), (7-2), and Remark 7.2. □

Corollary 7.6. *If $d = 2$, then $\mathrm{d}_s(A, \beta) = \mathrm{d}_s(A)$ for any $\beta \in \mathbb{C}^d$.*

Proof. Since $d = 2$, the matrix A has only two proper faces $G_1, G_2 \preceq A$, which both have codimension 1. Moreover, a_n belongs to at most one of these two facets. Thus, by Corollaries 7.4 and 7.5, it is enough to consider the case when $a_n \in G_1$ and $\max(\mathcal{J}(\beta))$ involves G_1. In this case, $\mathrm{d}_s(A, \beta)$ can be computed as in the simple case, so the formula in Corollary 7.5 can be applied, giving $\mathrm{d}_s(A, \beta) = \mathrm{d}_s(A)$ since $\mathrm{codim}(G_1) = 1$. □

Notice that Corollary 7.6 also follows from [Schulze and Walther 2008, Proposition 4.25] and (7-1).

Corollary 7.7. *If $d = 3$, then $\mathrm{d}_s(A, \beta) > \mathrm{d}_s(A)$ if and only if $\max(\mathcal{J}(\beta))$ involves a face G with $a_n \in G$ and $\dim G = 1$. If this is the case, $\mathrm{d}_s(A, \beta) = \mathrm{d}_s(A) + |B_G^{\beta}| \cdot \mathrm{d}_s(G)$.*

Proof. Again by Corollary 7.4, we only need to consider the ranking lattices $\mathbb{E}_G^{\beta}$ such that $a_n \in G$. Thus, by the reduction given in [Berkesch 2011, Section 5.3], it is enough to prove the result in the following two cases.

The first case is that a_n belongs to a unique face G among those involved in $\max(\mathcal{J}(\beta))$. In this case, the computation follows as in the simple case, and we obtain the same formula as in Corollary 7.5.

In the second case, we may assume that there are exactly two faces G_1 and G_2 involved in $\max(\mathcal{J}(\beta))$ that contain a_n. Since the face $G_1 \cap G_2$ contains a_n and $d = 3$, it follows that G_1 and G_2 are two facets intersecting in a face of codimension 2. In this case, Remark 4.8 shows that $\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau}$ for any filtration L and any $\tau \in \Phi_A^L$, so $\mathrm{d}_s(A, \beta) = \mathrm{d}_s(A)$. □

Lemma 7.8. *Let $s > 1$ be such that the (A, s)-umbrella Φ_A^s has a unique facet τ that is not F-homogeneous, p is a generic point of $Y = \mathrm{Var}(x_n)$, and $\epsilon > 0$ small enough. Then the function*

$$\beta \mapsto d(A, \beta, s) := \dim \mathcal{H}om_{\mathcal{D}}(\mathcal{M}_A(\beta), \mathcal{O}_{\widehat{X|Y}}(s+\epsilon)/\mathcal{O}_{\widehat{X|Y}}(s-\epsilon))_p$$

is upper-semicontinuous.

Proof. Notice first that by the assumption and Remark 7.1, s is a slope of $M_A(\beta)$ along Y and $a_n \in \tau$. Indeed, the assumption implies that $\tau' := \tau \setminus \{n\}$ is the unique facet of $\Phi_A^{s+\epsilon}$ that does not contain a_n and is also not a facet of $\Phi_A^{s-\epsilon}$. On the other hand,

$$d(A, \beta, s) = d_{s+\epsilon}(A, \beta) - d_{s-\epsilon}(A, \beta) = \mu_{A,0}^{s+\epsilon,\varnothing}(\beta) - \mu_{A,0}^{s-\epsilon,\varnothing}(\beta) + \mu_{A,0}^{s-\epsilon,\{n\}}(\beta) - \mu_{A,0}^{s+\epsilon,\{n\}}(\beta).$$

Thus, setting $d(A, s) := \mu_{A,0}^{s+\epsilon,\varnothing} - \mu_{A,0}^{s-\epsilon,\varnothing} + \mu_{A,0}^{s-\epsilon,\{n\}} - \mu_{A,0}^{s+\epsilon,\{n\}}$ yields

$$d(A, s) = \mathrm{vol}_{\mathbb{Z}A}(\Delta_{\tau'}) = \mathrm{rank}(\mathcal{H}_0^{\tau'}(\tilde{S}_A, \beta)),$$

where the first equality follows by the assumption, (2-4), and [Fernández-Fernández 2010, Lemma 7.4]. The second equality follows as in the proof of (6-4), since $A_{\tau'}$ is a rank d submatrix of A. Similarly, for faces G of A such that $a_n \in G$ and $\tau'' := \tau' \cap G$ is a facet of $\Phi_G^{s+\epsilon}$, we also have that $d(G, s) = \mathrm{rank}(\mathcal{H}_0^{\tau''}(\tilde{S}_G, \beta))$. Thus, arguments similar to those in Corollary 6.6 show that $d(A, \beta, s) = \mathrm{rank}(\mathcal{H}_0^{\tau'}(S_A, \beta))$ and that the function $\beta \mapsto d(A, \beta, s) = \mathrm{rank}(\mathcal{H}_0^{\tau'}(S_A, \beta))$ is upper-semicontinuous in β. □

Theorem 7.9. *Assume that for all $s > 1$, a_n/s is in at most one of the hyperplanes off the origin supported in a facet of Δ'* (see Remark 7.1). *Then the function $\beta \mapsto \mathrm{d}_s(A, \beta)$ is upper-semicontinuous for all $s > 1$.*

Proof. Let $1 < s_1 < \cdots < s_r$ be the set of slopes of $M_A(\beta)$ along Y that are lower or equal to s. Then $d_s(A, \beta) = \sum_{j=1}^r d(A, \beta, s_j)$, and the result follows by Lemma 7.8. □

In view of the preceding results we state the following conjecture.

Conjecture 7.10. *The map $\beta \mapsto \mathrm{d}_s(A, \beta)$ is upper-semicontinuous. Moreover, there is an equality*

$$\mathcal{E}_A^n(s) := \{\beta \in \mathbb{C}^d \mid \mathrm{d}_s(A, \beta) > \mathrm{d}_s(A)\} = -\,\mathrm{qdeg}\left(2 \bigoplus_{q=0}^{d-1} \mathrm{Ext}_{\mathbb{C}[\partial]}^{n-q}(S_A^{\{n\}}, \mathbb{C}[\partial])(-\varepsilon_A)\right),$$

where $\varepsilon_A := \sum_{i=1}^n a_i$. In particular, $\mathcal{E}_A^n(s) = \varnothing$ if and only if $S_A^{\{n\}}$ is Cohen–Macaulay.

The values $\mathrm{d}_s(A, \beta)$ and $\mathrm{d}_s(A)$ defined in this section depend on the variety Y along which we are considering the irregularity sheaf of $M_A(\beta)$. Although we assumed $Y = \mathrm{Var}(x_n)$ for simplicity, we can consider any $Y_j := \mathrm{Var}(x_j) \subseteq \mathbb{C}^n$ since reordering the variables is equivalent to reordering the columns of A. Let $\mathrm{d}_s(A, \beta, j)$ and $\mathrm{d}_s(A, j)$ denote the values of $\mathrm{d}_s(A, \beta)$ and $\mathrm{d}_s(A)$ respectively for Y_j in place of Y. In the following example, we compute the difference $\mathrm{d}_s(A, \beta, j) - \mathrm{d}_s(A, j)$ for different j by using Corollary 7.7.

Example 7.11. Let us consider the matrix A in Example 4.10. The hyperplanes contained in the singular locus of $M_A(\beta)$ are exactly Y_j for $j \in \{2, 4, 6, 7\}$ and there is exactly one slope $s_j \geq 1$ of $M_A(\beta)$ along each Y_j. More precisely, by Remark 7.1, $s_2 = \frac{3}{2}$, $s_4 = 3$, $s_6 = 2$, and $s_7 = \frac{7}{6}$. It is clear that $\mathrm{d}_s(A, \beta', j) = 0$ if $1 \leq s < s_j$ for any $\beta' \in \mathbb{C}^d$, so let us assume that $s \geq s_j$ in each case. We have that $\mathrm{d}_s(A, \beta', j) = \mathrm{d}_s(A, j)$ for all β' and $j \in \{2, 7\}$. On the other hand, $\mathrm{d}_s(A, \beta', 4) - \mathrm{d}_s(A, 4)$ is 1 if $\beta' \in \beta + \mathbb{C}G_1$ and 0 otherwise. Finally, $\mathrm{d}_s(A, \beta', 6) - \mathrm{d}_s(A, 6)$ is 1 if $\beta' \in \beta + \mathbb{C}G_2$ and zero otherwise.

One natural problem after the computation of $\mathrm{d}_s(A, \beta) = m$ is to construct an explicit set of Gevrey series $\varphi_1, \ldots, \varphi_m$ along Y at a nonsingular point $p \in Y$ so that their classes in the space $(\mathcal{O}_{\widehat{X|Y}}(s)/\mathcal{O}_{X|Y})_p$ form a basis of $H^0(\mathrm{Irr}_Y^{(s)}(M_A(\beta))_p)$. This was done in [Fernández-Fernández 2010] when β is generic enough. At any parameter β, this problem is much more involved in general. However, it is easy to compute some examples by using a slightly modified version of a method used in [Fernández-Fernández 2013]. In order to do so, recall that the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1 \times n_1}$, $A_2 \in \mathbb{Z}^{d_2 \times n_2}$ is the following $(d_1 + d_2) \times (n_1 + n_2)$ matrix:

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & \mathbf{0}_{d_1 \times n_2} \\ \mathbf{0}_{d_2 \times n_1} & A_2 \end{pmatrix},$$

where $\mathbf{0}_{d \times n}$ denotes the $d \times n$ zero matrix. Let $\beta = (\beta^{(1)}, \beta^{(2)})$ denote a complex vector in $\mathbb{C}^{d_1 + d_2} \cong \mathbb{C}^{d_1} \times \mathbb{C}^{d_2}$. It is easy to show using [Fernández-Fernández 2013, Lemma 2.2] that

$$\mathrm{d}_s(A, \beta, n_1) = \mathrm{d}_s(A_1, \beta^{(1)}, n_1) \cdot \mathrm{rank}(M_{A_2}(\beta^{(2)})).$$

Now, let us take $(A_1, \beta^{(1)})$ such that $M_{A_1}(\beta^{(1)})$ has slopes along $\{x_{n_1} = 0\}$, and let consider the subset of Gevery series $\{g_1, \ldots, g_{r(1)}\} \subseteq \mathcal{O}_{\widehat{X|Y}}(s)$ whose classes form a basis of

$$H^0(\mathrm{Irr}^{(s)}_{\{x_{n_2}=0\}}(M_{A_1}(\beta^{(1)}))_p).$$

Let us take also a pair $(A_2, \beta^{(2)})$ for which a basis $\{f_1, \ldots, f_{r(2)}\}$ of convergent series solutions of $M_{A_2}(\beta^2)$ at a nonsingular point p' is known for a rank-jumping parameter $\beta^{(2)} \in \mathbb{C}^{d_2}$. Then $\{g_i f_j \mid 1 \leq i \leq r(1), 1 \leq j \leq r(2)\}$ is a basis of

$$H^0(\mathrm{Irr}^{(s)}_{\{x_{n_2}=0\}}(M_{A_1 \oplus A_2}(\beta))_{(p,p')}),$$

where $\beta = (\beta^{(1)}, \beta^{(2)})$. Note that

$$\dim H^0(\mathrm{Irr}^{(s)}_{\{x_{n_2}=0\}}(M_{A_1 \oplus A_2}(\beta))_{(p,p')}) = r(1) \cdot r(2) > \mathrm{d}_s(A, n_1) = \mathrm{d}_s(A_1, n_1) \cdot \mu^F_{A_2,0}.$$

In particular, the smallest example of this family is the one obtained by taking $A = A_1 \oplus A_2$ for $A_1 = (1\,2)$ and $A_2 = (\tilde{0}, \tilde{1}, \tilde{3}, \tilde{4})$, where $\tilde{a} = (1, a)^t$ and $\beta = (b, 1, 2)^t$ for any $b \in \mathbb{C} \setminus \mathbb{Z}$. We notice that $M_{A_2}((1, 2)^t)$ was the first example known of an A-hypergeometric system for which the rank is greater than the normalized volume [Sturmfels and Takayama 1998]. Indeed, a basis of $H^0(\mathrm{Irr}^{(s)}_{\{x_{n_2}=0\}}(M_{A_1}(b))_p)$ is $\{\overline{\phi_v}\} \subset (\mathcal{O}_{\widehat{X|Y}}(s)/\mathcal{O}_{X|Y})_p$, where ϕ_v is the Γ-series associated to $v=(b,0)$ (see [Fernández-Fernández 2010]) and $\mathrm{rank}(M_{A_2}(\beta^{(2)})) = \mathrm{vol}_{\mathbb{Z}^2}(A_2)+1=5$ (see [Sturmfels and Takayama 1998], where a basis of solutions is also described). Thus, in this case, $M_A(\beta)$ has the slope $s=2$ along $x_2=0$ and for $s\geq 2$, $\mathrm{d}_s(A,\beta,2)=\mathrm{d}_s(A,2)+1=5$.

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References

[Adolphson 1994] A. Adolphson, "Hypergeometric functions and rings generated by monomials", *Duke Math. J.* **73**:2 (1994), 269–290. MR Zbl

[Berkesch 2011] C. Berkesch, "The rank of a hypergeometric system", *Compos. Math.* **147**:1 (2011), 284–318. MR Zbl

[Berkesch et al. 2018] C. Berkesch, J. Forsgård, and L. F. Matusevich, "On the parametric behavior of A-hypergeometric series", *Trans. Amer. Math. Soc.* **370**:6 (2018), 4089–4109. MR Zbl

[Fernández-Fernández 2010] M.-C. Fernández-Fernández, "Irregular hypergeometric $\mathcal{D}$-modules", *Adv. Math.* **224**:5 (2010), 1735–1764. MR Zbl

[Fernández-Fernández 2013] M.-C. Fernández-Fernández, "Exponential growth of rank jumps for A-hypergeometric systems", *Rev. Mat. Iberoam.* **29**:4 (2013), 1397–1404. MR Zbl

[Gelfand et al. 1987] I. M. Gelfand, M. I. Graev, and A. V. Zelevinsky, "Holonomic systems of equations and series of hypergeometric type", *Dokl. Akad. Nauk SSSR* **295**:1 (1987), 14–19. In Russian; translated in *Soviet Math. Dokl.* **36**:1 (1988), 5–10. MR

[Gelfand et al. 1989] I. M. Gelfand, A. V. Zelevinsky, and M. M. Kapranov, "Hypergeometric functions and toric varieties", *Funkt. Anal. i Prilozhen.* **23**:2 (1989), 12–26. In Russian; translated in *Funct. Anal. Appl.* **23**:2 (1989), 94–106. Correction in *Funkt. Anal. i Prolozhen* **27**:4 (1993), 91; correction translated in *Funct. Anal. Appl.* **27**:4 (1993), 295. MR

[Gelfand et al. 1990] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, "Generalized Euler integrals and A-hypergeometric functions", *Adv. Math.* **84**:2 (1990), 255–271. MR Zbl

[Hochster 1972] M. Hochster, "Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes", *Ann. of Math.* (2) **96** (1972), 318–337. MR Zbl

[Laurent 1987] Y. Laurent, "Polygône de Newton et b-fonctions pour les modules microdifférentiels", *Ann. Sci. École Norm. Sup.* (4) **20**:3 (1987), 391–441. MR Zbl

[Laurent and Mebkhout 1999] Y. Laurent and Z. Mebkhout, "Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module", *Ann. Sci. École Norm. Sup.* (4) **32**:1 (1999), 39–69. MR Zbl

[Matusevich et al. 2005] L. F. Matusevich, E. Miller, and U. Walther, "Homological methods for hypergeometric families", *J. Amer. Math. Soc.* **18**:4 (2005), 919–941. MR Zbl

[Mebkhout 1990] Z. Mebkhout, "Le théorème de positivité de l'irrégularité pour les $\mathcal{D}_X$-modules", pp. 83–132 in *The Grothendieck Festschrift, III*, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Saito et al. 2000] M. Saito, B. Sturmfels, and N. Takayama, *Gröbner deformations of hypergeometric differential equations*, Algor. Comput. Math. **6**, Springer, 2000. MR Zbl

[Schulze and Walther 2008] M. Schulze and U. Walther, "Irregularity of hypergeometric systems via slopes along coordinate subspaces", *Duke Math. J.* **142**:3 (2008), 465–509. MR Zbl

[Schulze and Walther 2009] M. Schulze and U. Walther, "Hypergeometric D-modules and twisted Gauß–Manin systems", *J. Algebra* **322**:9 (2009), 3392–3409. MR Zbl

[Sturmfels and Takayama 1998] B. Sturmfels and N. Takayama, "Gröbner bases and hypergeometric functions", pp. 246–258 in *Gröbner bases and applications* (Linz, Austria, 1998), edited by B. Buchberger and F. Winkler, Lond. Math. Soc. Lecture Note Ser. **251**, Cambridge Univ. Press, 1998. MR Zbl

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Singularity categories of deformations of Kleinian singularities

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Let G be a finite subgroup of $\mathrm{SL}(2, \Bbbk)$ and let $R = \Bbbk[x, y]^G$ be the coordinate ring of the corresponding Kleinian singularity. In 1998, Crawley-Boevey and Holland defined deformations $\mathcal{O}^\lambda$ of R parametrised by weights λ. In this paper, we determine the singularity categories $\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^\lambda)$ of these deformations, and show that they correspond to subgraphs of the Dynkin graph associated to R. This generalises known results on the structure of $\mathcal{D}_{\mathrm{sg}}(R)$. We also provide a generalisation of the intersection theory appearing in the geometric McKay correspondence to a noncommutative setting.

1. Introduction

1A. ***Background.*** Throughout let $\Bbbk$ be an algebraically closed field of characteristic 0. The Kleinian singularities $\Bbbk^2/G$, where G is a finite subgroup of $\mathrm{SL}(2, \Bbbk)$, are ubiquitous in algebraic geometry, representation theory, and singularity theory. In this paper, we shall study the latter of these for a family of (generically noncommutative) algebras.

The notion of the singularity category of a ring R was introduced by Buchweitz [1986] as a particular Verdier quotient of $\mathcal{D}^{\mathrm{b}}(\text{mod-}R)$. More specifically, writing $\mathrm{Perf}(R)$ for the full subcategory of perfect complexes in $\mathcal{D}^{\mathrm{b}}(\text{mod-}R)$, Buchweitz defined the singularity category as the Verdier quotient category

$$\mathcal{D}_{\mathrm{sg}}(R) := \frac{\mathcal{D}^{\mathrm{b}}(\text{mod-}R)}{\mathrm{Perf}(R)}.$$

By construction, this category possesses the structure of a triangulated category. Buchweitz also showed that, when R is Gorenstein, the singularity category is triangle equivalent to $\underline{\mathrm{MCM}}\text{-}R$, the stable category of maximal Cohen–Macaulay R-modules (that this latter category is triangulated also follows from a general result of Happel [1988]). The singularity category of a commutative ring R is also closely related to the category of reduced matrix factorisations of R by [Eisenbud 1980, Corollary 6.3], and under mild hypotheses these categories are in fact equivalent.

From the above definition, it is not difficult to see that $\mathcal{D}_{\mathrm{sg}}(R)$ is trivial precisely when R has finite global dimension. However, in general it is difficult to give an adequate description of the singularity category of an arbitrary Gorenstein ring of infinite global dimension. Recent work includes [Chen 2011; 2018] which describes the singularity category when R has radical square zero or when it is a quadratic

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monomial algebra, and [Kalck 2015] which provides a description when R is a so-called gentle algebra. Moreover, in [Amiot et al. 2015] the authors determine the singularity categories of some commutative Gorenstein isolated singularities.

The standard examples of commutative surface singularities are the Kleinian singularities, which are very well understood. The main aim of this paper is to provide a concrete description of the singularity category of certain noncommutative deformations of the coordinate ring of a Kleinian singularity.

Very little is known about the singularities of noncommutative rings, particularly those which are not finite over their centre. For example, given a singular noncommutative ring S, it is not known whether and under what circumstances one can find commutative rings $R_1, \dots, R_k$ such that we have an equivalence of triangulated categories $\mathcal{D}_{\mathrm{sg}}(S) \simeq \bigoplus_{i=1}^{k} \mathcal{D}_{\mathrm{sg}}(R_i)$. If this is the case, one can think of S as having the same singularities as those of the varieties Spec R_i. Our main result, Theorem 1.1, shows that we have such a decomposition of singularity categories for the algebras of interest in this paper, and can be seen as a first step towards better understanding singularities of noncommutative surfaces.

Crawley-Boevey and Holland [1998] defined a family of algebras $\mathcal{O}^{\lambda}(\tilde{Q})$ depending on the data of an extended Dynkin quiver $\tilde{Q}$ and a so-called weight for $\tilde{Q}$. Write Q for the Dynkin quiver obtained from $\tilde{Q}$ by removing an extending vertex, and write R_Q for the coordinate ring of the corresponding Kleinian singularity. Then the algebras $\mathcal{O}^{\lambda}(\tilde{Q})$ may be thought of as deformations of R_Q in the sense that there exists a filtration $\mathcal{F}$ of $\mathcal{O}^{\lambda}(\tilde{Q})$ satisfying $\mathrm{gr}_{\mathcal{F}}\, \mathcal{O}^{\lambda}(\tilde{Q}) \simeq R_Q$. These deformations are generically noncommutative, a property which depends on the weight λ, and it is easy to determine when this is the case. When $\tilde{Q} = \tilde{\mathbb{A}}_n$, if $\mathcal{O}^{\lambda}(\tilde{Q})$ is noncommutative then it is an example of a generalised Weyl algebra, as studied in [Bavula 1992; Hodges 1993]. If $\mathcal{O}^{\lambda}(\tilde{Q})$ is commutative, a description of its singularity category follows from [Iyama and Wemyss 2014, Theorem 3.2], where quite geometric techniques are employed. Through a completely ring-theoretic approach, we determine $\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^{\lambda}(\tilde{Q}))$ irrespective of whether the deformation is commutative or noncommutative.

Our main result can be stated as follows, where undefined terms will be defined in Section 2.

Theorem 1.1 (Theorem 3.6, Theorem 4.13). *Let $\tilde{Q}$ be an extended Dynkin quiver with vertex set $\{0, 1, \dots, n\}$, where 0 is an extending vertex, and write Q for the full subquiver obtained by deleting vertex 0. Let λ be a weight for $\tilde{Q}$. Then there exists a subset $J = J(\lambda)$ of $\{1, \dots, n\}$ such that, if $Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ is the full subquiver of Q obtained by deleting the vertices in J, so that the $Q^{(i)}$ are connected and therefore necessarily Dynkin, there is a triangle equivalence*

$$\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^{\lambda}(\tilde{Q})) \simeq \bigoplus_{i=1}^{r} \mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}}).$$

One can show that we may restrict our attention to the case where the weight λ is *quasidominant* (see Definition 2.16). For example, when $\Bbbk = \mathbb{C}$, a weight is quasidominant if, for $1 \leqslant i \leqslant n$,

$$\lambda_i \in \{z \in \mathbb{C} \mid \operatorname{Re} z > 0, \text{ or } \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z \geqslant 0\},$$

and where λ_0 can be arbitrary. In this case, the subset J in the above theorem is $J = \{i \in \{1, \dots, n\} \mid \lambda_i = 0\}$.

The result in Theorem 1.1 coincides with the intuition coming from commutative singularity theory which says that deforming a singularity should make it no worse; in our context, deforming a singularity corresponds to making weights at certain vertices of $\tilde{Q}$ nonzero, and the above theorem says that this makes the singularity category simpler, in a precise sense. In the Appendix, we also provide a simple proof which illustrates how the translation functor acts on the triangulated category $\mathcal{D}_{sg}(\mathcal{O}^{\lambda}(\tilde{Q}))$.

Since the first version of this paper appeared online, an alternative proof of Theorem 1.1 has been given in [Kalck and Yang 2018, Theorem 9.4] using relative singularity categories.

Now suppose that the weight $\lambda \in \Bbbk^{n+1}$ is given by $\lambda_0 = 1$ and $\lambda_i = 0$ for $1 \leqslant i \leqslant n$, and in this case write $\lambda = \varepsilon_0$. We then consider $\mathcal{O}^{\lambda}(\tilde{Q})$ to be a noncommutative analogue of R_Q. This viewpoint is partially justified by the following immediate corollary:

Corollary 1.2. *Retain the notation of Theorem 1.1, and suppose that $\lambda = \varepsilon_0$, as above. Then there is a triangle equivalence*

$$\mathcal{D}_{sg}(\mathcal{O}^{\lambda}(\tilde{Q})) \simeq \mathcal{D}_{sg}(R_Q).$$

Another family of results concerning Kleinian singularities is what is often called the geometric McKay correspondence, which concerns the intersection theory of the minimal resolution of Spec R_Q. In the last section, we prove a result which may be seen as a generalisation of this to a noncommutative setting. We give an imprecise statement of this result below, and a more precise statement in Section 5.

Theorem 1.3 (Theorem 5.8). *Let $\tilde{Q}$ be an extended Dynkin quiver with corresponding Dynkin quiver Q and let $\lambda = \varepsilon_0$. Then $\mathcal{O}^{\lambda}(\tilde{Q})$ has a noncommutative resolution, and the intersection theory of the exceptional objects in this resolution is the same as that of the exceptional curves in the minimal resolution of a Kleinian singularity of type corresponding to Q.*

In particular, this result further supports the viewpoint that $\mathcal{O}^{\lambda}(\tilde{Q})$ can be viewed as a noncommutative analogue of R_Q when $\lambda = \varepsilon_0$.

We now take a moment to provide an overview of the proof of Theorem 1.1. For all of our calculations, we work in $\underline{\text{MCM}}$-$\mathcal{O}^{\lambda}(\tilde{Q})$ rather than $\mathcal{D}_{sg}(\mathcal{O}^{\lambda}(\tilde{Q}))$; as mentioned previously, these two categories are triangle equivalent, see Theorem 2.7. The first important observation to make is that we can restrict our attention to weights λ which are *quasidominant*. In Section 3, this restriction allows us to give a concrete description of $\mathcal{D}_{sg}(\mathcal{O}^{\lambda}(\tilde{Q}))$ as a $\Bbbk$-linear category in terms of an auxiliary Krull–Schmidt category. We also find that the isoclasses V_i of indecomposable objects in $\mathcal{D}_{sg}(\mathcal{O}^{\lambda}(\tilde{Q}))$ are indexed by those vertices $i \geqslant 1$ with $\lambda_i = 0$. This auxiliary category allows us to establish the equivalence of Theorem 1.1, but only as a $\Bbbk$-linear equivalence.

In a previous version of this paper, a lengthy case-by-case analysis was then used to show that we had the desired triangle equivalence. A result of Keller which subsequently appeared significantly reduces the amount of work that needs to be done. In fact, it now suffices to show that the direct sum appearing on the right-hand side of the (a priori $\Bbbk$-linear) equivalence of Theorem 1.1 is a decomposition into algebraic triangulated subcategories. This is shown in Section 4.

1B. ***Organisation of the paper.*** This paper is organised as follows. In Section 2, we recall some basic definitions and facts, and introduce the notation used throughout the paper. In Section 3, the singularity categories of $\mathcal{O}^{\lambda}(\tilde{Q})$ are determined as $\Bbbk$-linear categories. We then complete the proof of Theorem 1.1 in Section 4. In Section 5, we provide a noncommutative version of the geometric McKay correspondence, and in the Appendix we detail how the translation functor behaves on objects of $\mathcal{D}_{\text{sg}}(R_Q)$.

2. Preliminaries

We now recall some of the definitions and results that we will make use of throughout this paper. In this section, R will denote an arbitrary ring.

2A. ***Conventions.*** As stated in the introduction, throughout $\Bbbk$ will denote an algebraically closed field of characteristic 0. We write mod-R (respectively, R-mod) for the category of finitely generated right (respectively, left) R-modules; in this paper, we shall use right modules unless otherwise stated. We also write proj-R for the full subcategory of mod-R consisting of finitely generated projective modules. We write $M^* := \operatorname{Hom}_R(M, R)$ for the dual of an R-module M, which is an $(R, \operatorname{End}_R(M))$-bimodule. We write $\operatorname{p.dim} M$ and $\operatorname{i.dim} M$ for the projective and injective dimensions of $M \in$ mod-R, respectively, and $\operatorname{gl.dim} R$ for the global dimension of R.

2B. ***Definitions and basic results.***

Definition 2.1. A *quiver* Q is a directed multigraph, and we write Q_0 for the set of vertices and Q_1 for the set of arrows. We equip Q with head and tail maps $h, t : Q_1 \to Q_0$ which take an arrow to the vertices that are its head and tail respectively. A *nontrivial path* in the quiver is a sequence of arrows $p = \alpha_1\alpha_2 \dots \alpha_\ell$ with $h(\alpha_i) = t(\alpha_{i+1})$ for $1 \leqslant i \leqslant \ell - 1$ (that is, we compose arrows from left to right), and such a path is said to have *length* ℓ. Moreover, for each vertex $i \in Q_0$ there is a *trivial path* e_i of length 0, with head and tail vertex both equal to i.

Definition 2.2. Given a field $\Bbbk$ and a quiver Q, we define the path algebra $\Bbbk Q$ of Q as follows: as a $\Bbbk$-vector space, $\Bbbk Q$ has a basis given by paths in the quiver, and we define multiplication by concatenation of paths:

$$p \cdot q = \begin{cases} pq & \text{if } h(p) = t(q), \\ 0 & \text{otherwise.} \end{cases}$$

If R is a commutative ring, then $\operatorname{Spec} R$ is nonsingular if and only if R has finite global dimension. It is therefore sensible to say that a (possibly noncommutative) ring is *nonsingular* if it has finite global dimension, and *singular* otherwise. Before we are able to define the singularity category of a ring, we must make a few more definitions.

Definition 2.3. Given R-modules M and N, write $\underline{\operatorname{Hom}}_R(M, N) = \operatorname{Hom}_R(M, N)/\sim$, where $f \sim f'$ if and only if $f - f'$ factors through a finitely generated projective module. The *stable module category* of R, denoted $\underline{\text{mod}}$-R, is then the category whose objects are the same as those of mod-R, and for modules

M, N, has morphisms $\underline{\mathrm{Hom}}_R(M, N)$. Given a full subcategory abc-R of mod-R, we write $\underline{\mathrm{abc}}$-R for the full subcategory of $\underline{\mathrm{mod}}$-R whose objects are the same as those of abc-R.

Noting that an element of $\sum_{i=1}^{k} n_i \otimes f_i$ of $N \otimes_R M^*$ gives rise to a homomorphism $M \to N$ via $m \mapsto \sum_{i=1}^{k} n_i f_i(m)$, it is not hard to show that a module homomorphism $f : M \to N$ factors through a projective module if and only if f is the image of some element of $N \otimes_R M^*$. Abusing notation, this allows us to identify $\underline{\mathrm{Hom}}_R(M, N)$ with $\mathrm{Hom}_R(M, N)/(N \otimes_R M^*)$, which will be useful in later calculations. In this paper, we are often in the situation where $R = e\Lambda e$, $M = e_1\Lambda e$, and $N = e_2\Lambda e$, where Λ is some ring and e, e_1, and e_2 are pairwise orthogonal idempotents, and we are able to make identifications

$$M^* \cong e\Lambda e_1, \quad \mathrm{Hom}_R(M, N) \cong e_2\Lambda e_1, \quad \underline{\mathrm{Hom}}_R(M, N) \cong \frac{e_2\Lambda e_1}{e_2\Lambda e\Lambda e_1}.$$

In the stable module category, we have a weaker notion of an isomorphism than in the usual module category. Indeed, [Auslander and Bridger 1969, Proposition 1.44] shows that two R-modules M, N are isomorphic in $\underline{\mathrm{mod}}$-R if and only if there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$ in mod-R.

The *first syzygy* ΩM of $M \in$ mod-R is defined to be the kernel of any surjection $R^n \twoheadrightarrow M$. The observation in the previous paragraph combined with [Rotman 1979, Proposition 8.5] implies that ΩM is uniquely determined in $\underline{\mathrm{mod}}$-R.

Definition 2.4. A ring R is said to be *Gorenstein* if it is noetherian (i.e., left and right noetherian) and both $\mathrm{i.dim}\, R_R$ and $\mathrm{i.dim}\, {}_RR$ are finite. By [Zaks 1969, Lemma A], under these hypotheses the values $\mathrm{i.dim}\, R_R$ and $\mathrm{i.dim}\, {}_RR$ coincide, and we call this common value the (*injective*) *dimension* of R.

Definition 2.5. Suppose that R is Gorenstein. A finitely generated R-module M is said to be *maximal Cohen–Macaulay* (MCM) if it satisfies $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \geqslant 1$. We write MCM-R for the full subcategory of mod-R consisting of maximal Cohen–Macaulay R-modules.

For commutative local rings, the above definition coincides with the usual (commutative) definition of maximal Cohen–Macaulay modules in terms of depth [Buchweitz 1986, Section 4.2]. Maximal Cohen–Macaulay modules have the following elementary properties, proofs of which can be found in [loc. cit.]:

Lemma 2.6. (1) *Any finitely generated projective module is MCM.*

(2) *MCM modules are reflexive.*

(3) *Finite direct sums and direct summands of MCM modules are MCM.*

(4) *An MCM module is either projective or has infinite projective dimension.*

With these definitions in hand, we now recall a theorem which identifies a category that is triangle equivalent to the singularity category in the case of a Gorenstein ring R:

Theorem 2.7 [Buchweitz 1986, Theorem 4.4.1]. *Suppose that R is Gorenstein. Then the full subcategory $\underline{\mathrm{MCM}}$-R of $\underline{\mathrm{mod}}$-R whose objects are* MCM *R-modules is a triangulated category, with translation functor Σ given by $\Sigma M = \Omega^{-1} M$. Moreover, there is a triangle equivalence $\mathcal{D}_{\mathrm{sg}}(R) \simeq \underline{\mathrm{MCM}}$-$R$.*

While the term “singularity category” is more suggestive (which was our main reason for using this terminology in the introduction), since every example that we consider in this paper satisfies the hypotheses of this theorem, we instead focus our attention on determining $\underline{\mathrm{MCM}}\text{-}R$.

Theorem 2.7 is a specific example of a more general result due to Happel, which we now briefly recall. An *exact category* $\mathcal{C}$ is an additive category possessing a class of *conflations* (sometimes called exact sequences) which are triples of objects connected by arrows $X \to Y \to Z$, and which satisfy a number of axioms; see [Chen 2012, Section 2] for more details. An object $P \in \mathcal{C}$ is *projective* if the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ sends conflations to exact sequences, and we say that $\mathcal{C}$ has *enough projectives* if every object $Z \in \mathcal{C}$ fits into a conflation $X \to P \to Z$ with P projective. Dually, one has a notion of an *injective object* and of having *enough injectives*. An exact category $\mathcal{C}$ is said to be *Frobenius* provided that it has enough projectives and enough injectives, and the class of projective objects coincides with the class of injective objects. Given a Frobenius category $\mathcal{C}$, we may form its stable category $\underline{\mathcal{C}}$ in the same way we formed the stable category $\underline{\mathrm{MCM}}\text{-}R$. Then [Happel 1988] shows that this category is triangulated, and if $X \to Y \to Z$ is a conflation in $\mathcal{C}$ then there exists a triangle of the form $X \to Y \to Z \to \Sigma X$ in $\underline{\mathcal{C}}$. If $\mathcal{T}$ is a triangulated category which is triangle equivalent to the stable category of a Frobenius category, then we say that $\mathcal{T}$ is *algebraic*.

If R is a Gorenstein ring, then $\mathrm{MCM}\text{-}R$ is Frobenius and so Happel's result implies that $\underline{\mathrm{MCM}}\text{-}R$ is triangulated; this triangulated structure is precisely the one given in Theorem 2.7. In $\mathrm{MCM}\text{-}R$, every conflation $X \to Y \to Z$ arises from a short exact sequence $0 \to X \to Y \to Z \to 0$ of MCM R-modules.

Finally, we recall two useful results that will be helpful when identifying the maximal Cohen–Macaulay modules of a ring. Given an additive category $\mathcal{C}$ and an object $C \in \mathcal{C}$, we write $\mathrm{add}(C)$ for the full subcategory of $\mathcal{C}$ consisting of direct summands of finite direct sums of C. This is the smallest additive subcategory of $\mathcal{C}$ which contains C and is closed under taking direct summands. The following result is due to Auslander, but we provide a proof.

Proposition 2.8 (Auslander). *Suppose that R is Gorenstein and that $M \in \mathrm{MCM}\text{-}R$ is a generator (for example, this occurs if M has R as a direct summand or if R is simple). If $\mathrm{gl.dim}\,\mathrm{End}_R(M) \leqslant 2$, then* $\mathrm{add}\, M = \mathrm{MCM}\text{-}R$.

Proof. Write $\Lambda = \mathrm{End}_R(M)$. Since mod-$R$ has split idempotents (a fact which holds for any ring R), [Krause 2015, Proposition 2.3] implies that the functor $\mathrm{Hom}_R(M, -) : \text{mod-}R \to \text{mod-}\Lambda$ restricts to an equivalence

$$\mathrm{add}\, M \xrightarrow{\simeq} \text{proj-}\Lambda. \tag{2.9}$$

We also note that since M is a generator, $R^n \in \mathrm{add}\, M$ for any $n \geqslant 1$.

That $\mathrm{add}\, M \subseteq \mathrm{MCM}\text{-}R$ is clear, so suppose that $N \in \mathrm{MCM}\text{-}R$. Since R is noetherian, N^* is finitely presented, so we have an exact sequence of left R-modules of the form

$$R^m \to R^n \to N^* \to 0.$$

Applying $\mathrm{Hom}_R(-, R)$ and noting that N is MCM and therefore reflexive, we obtain an exact sequence

$$0 \to N \to R^n \to R^m.$$

Applying $\mathrm{Hom}_R(M, -)$ then gives an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, R^n) \xrightarrow{\theta} \mathrm{Hom}_R(M, R^m) \to \operatorname{coker} \theta \to 0,$$

where, since M is a generator, $\mathrm{Hom}_R(M, R^n)$ and $\mathrm{Hom}_R(M, R^m)$ are both projective Λ-modules by (2.9). Since $\mathrm{gl.dim}\, \Lambda \leqslant 2$ we have $\mathrm{p.dim} \operatorname{coker} \theta \leqslant 2$, and therefore $\mathrm{Hom}_R(M, N)$ is also a projective Λ-module. By (2.9), it follows that $N \in \operatorname{add} M$. □

Remark 2.10. If R has injective dimension at most 2 then the converse of Proposition 2.8 also holds; see [Simon 2018, Proposition 2.2.11].

When R has injective dimension at most 2, we also have the following:

Lemma 2.11. *Let R be a Gorenstein ring of injective dimension at most 2. Then $M \in \text{mod-}R$ is reflexive if and only if it is maximal Cohen–Macaulay.*

Proof. $(\Leftarrow)$ This is Lemma 2.6 (2), and doesn't require the hypothesis on injective dimension.

$(\Rightarrow)$ Suppose now that M is reflexive. Since R is noetherian, M^* is finitely presented, so we have an exact sequence of the form

$$R^m \to R^n \to M^* \to 0.$$

Applying $\mathrm{Hom}_R(-, R)$ and noting that M is reflexive yields an exact sequence

$$0 \to M \to R^n \xrightarrow{\theta} R^m \to \operatorname{coker} \theta \to 0.$$

But then, by [Rotman 1979, Corollary 6.55], $\mathrm{Ext}^i_R(M, R) \cong \mathrm{Ext}^{i+2}_R(\operatorname{coker} \theta, R) = 0$ for all $i \geqslant 1$, where the last equality follows since $\mathrm{i.dim}\, R \leqslant 2$. That is, M is maximal Cohen–Macaulay. □

2C. *The deformations of Crawley-Boevey and Holland.* In Crawley-Boevey and Holland [1998] introduced the notion of the *deformed preprojective algebra* of a quiver Q, and, if $\tilde{Q}$ is extended Dynkin, a family of $\Bbbk$-algebras $\mathcal{O}^\lambda(\tilde{Q})$ which may be thought of as deformations of the coordinate ring of a Kleinian singularity. We now recall these definitions, noting that our definition of $\mathcal{O}^\lambda(\tilde{Q})$ differs slightly from that of Crawley-Boevey and Holland, but is consistent with their definition by [loc. cit., Theorem 0.1].

Definition 2.12. Let Q be a quiver without loops. The *double* of Q is the quiver $\overline{Q}$ obtained from Q by adding a *reverse arrow* $\bar{\alpha} : j \to i$ for each arrow $\alpha : i \to j$ in Q. We call the arrows in $\overline{Q}$ which are not reverse arrows *ordinary arrows*. Given a *weight* $\lambda \in \Bbbk^{Q_0}$ for Q, the corresponding *deformed preprojective algebra* is the $\Bbbk$-algebra

$$\Pi^\lambda(Q) := \Bbbk\overline{Q}/I$$

where I is the two-sided ideal of $\Bbbk\overline{Q}$ with generators

$$\sum_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} \alpha\bar{\alpha} - \sum_{\substack{\alpha\in Q_1\\ h(\alpha)=i}} \bar{\alpha}\alpha - \lambda_i e_i$$

for each vertex $i \in Q_0$. It is easy to see that we can equivalently define I as being the two-sided ideal with the single generator

$$\sum_{\alpha\in Q_1} (\alpha\bar{\alpha} - \bar{\alpha}\alpha) - \sum_{i\in Q_0} \lambda_i e_i.$$

It is helpful to think of a weight as a label from $\Bbbk$ at each vertex of Q, and we will often refer to λ_i as the weight at vertex i.

Now suppose that $\tilde{Q}$ is extended Dynkin, with vertices and arrows (of its double) labelled as in Figure 1. Throughout this paper, it will be our convention that $\tilde{Q}$ denotes an extended Dynkin quiver, while Q will denote the corresponding Dynkin quiver obtained by removing the extending vertex 0, where the orientation of the arrows comes from Figure 1.

We are now able to define the algebras of interest to us. We write $\mathcal{O}^\lambda(\tilde{Q})$ for the algebra

$$\mathcal{O}^\lambda(\tilde{Q}) := e_0 \Pi^\lambda(\tilde{Q}) e_0.$$

The elements of $\mathcal{O}^\lambda(\tilde{Q})$ may be thought of as linear combinations of (equivalence classes of) paths in the double of $\tilde{Q}$ which start and end at the extending vertex 0.

If $\lambda = \mathbf{0}$, then $\Pi(Q) := \Pi^\lambda(Q)$ is the (undeformed) preprojective algebra of Gelfand and Ponomarev [1979], and in this case we also write $\mathcal{O}(\tilde{Q}) := \mathcal{O}^\lambda(\tilde{Q})$. We will often write Π^λ and $\mathcal{O}^\lambda$ (or Π and $\mathcal{O}$ if $\lambda = \mathbf{0}$) when the corresponding quiver is either unimportant or understood.

For our purposes, it is important to know precisely when the rings $\mathcal{O}^\lambda$ are noncommutative. This depends on the weight λ and also on a vector $\delta \in \mathbb{N}^{\tilde{Q}_0}$, which we now define. Let G be the finite subgroup of $\mathrm{SL}(2, \Bbbk)$ corresponding to $\tilde{Q}$ by the McKay correspondence. Then each vertex of $\tilde{Q}$ corresponds to an irreducible representation W_i of G, and we set $\delta_i := \dim_\Bbbk W_i$. If we number the vertices of $\tilde{Q}$ as in Figure 1, then

$$\tilde{\mathbb{A}}_n: \quad \delta = (\underbrace{1, 1, \dots, 1, 1}_{n+1 \text{ times}})$$

$$\tilde{\mathbb{D}}_n: \quad \delta = (1, 1, \underbrace{2, 2 \dots, 2, 2}_{n-3 \text{ times}}, 1, 1)$$

$$\tilde{\mathbb{E}}_6: \quad \delta = (1, 2, 1, 2, 3, 2, 1)$$

$$\tilde{\mathbb{E}}_7: \quad \delta = (1, 2, 3, 4, 3, 2, 1, 2)$$

$$\tilde{\mathbb{E}}_8: \quad \delta = (1, 2, 3, 4, 5, 6, 4, 2, 3).$$

Theorem 2.13 [Crawley-Boevey and Holland 1998, Theorem 0.1, Theorem 0.4(1)]. *$\mathcal{O}^\lambda$ is commutative if and only if $\lambda \cdot \delta = \sum_{i\in\tilde{Q}_0} \lambda_i \delta_i = 0$. In the case when $\lambda = \mathbf{0}$, $\mathcal{O}$ is isomorphic to the coordinate ring of the Kleinian singularity corresponding to $\tilde{Q}$.*

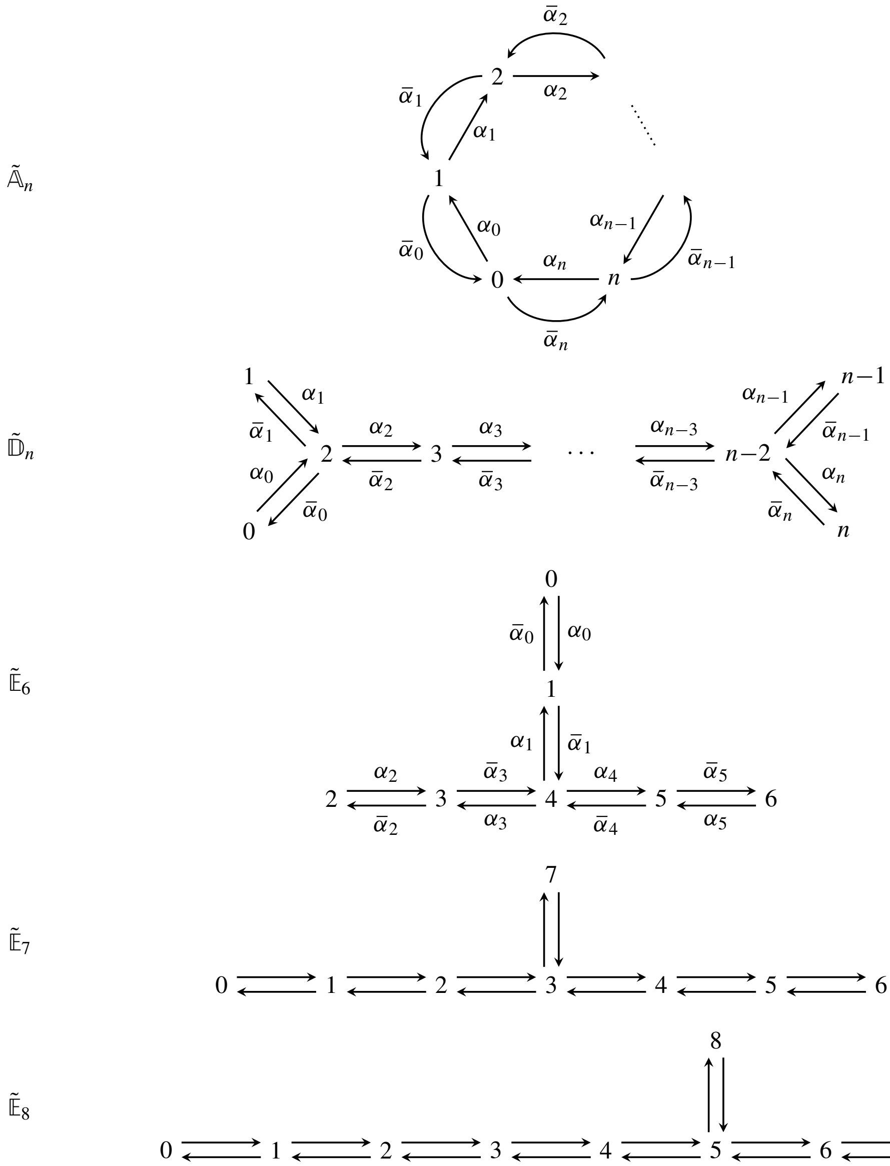

Figure 1. Doubles of extended Dynkin graphs with the labelling of vertices and arrows that will be used throughout this paper. We have labelled the arrows only for those quivers in which we will need to refer to specific paths.

In fact, by [Crawley-Boevey and Holland 1998, Lemma 2.2], one may always assume that $\lambda \cdot \delta$ is either 0 or 1. When λ is a weight with $\lambda \cdot \delta = 0$ (so that $\mathcal{O}^\lambda$ is commutative), if we define $\lambda' = (\lambda_0 + 1, \lambda_1, \dots, \lambda_n)$ then we consider $\mathcal{O}^{\lambda'}$ to be a noncommutative analogue of $\mathcal{O}^\lambda$. It is natural to ask if there is any relationship between the singularity categories of $\mathcal{O}^\lambda$ and $\mathcal{O}^{\lambda'}$. In fact, it follows from Theorem 1.1 that the singularity categories of these $\mathbb{k}$-algebras are triangle equivalent.

When $\tilde{Q} = \tilde{\mathbb{A}}_n$, it can be checked that $\mathcal{O}^\lambda$ has a presentation of the form

$$\frac{\Bbbk\langle x, y, z\rangle}{\left\langle \begin{array}{ll} xz = (z+\lambda\cdot\delta)x, & xy = \prod_{i=0}^{n}\big(z+\sum_{j=1}^{i}\lambda_j\big) \\ yz = (z-\lambda\cdot\delta)y, & yx = \prod_{i=0}^{n}\big(z-\lambda\cdot\delta+\sum_{j=1}^{i}\lambda_j\big) \end{array}\right\rangle}, \tag{2.14}$$

When $\mathcal{O}^\lambda$ is noncommutative, since there is no loss in generality in assuming $\lambda\cdot\delta = 1$, these are precisely the algebras considered by Hodges [1993] and Bavula [1992], where in the latter they were called generalised Weyl algebras.

In [Crawley-Boevey and Holland 1998, Section 7], the authors prove a number of results in the case where the weight λ is *dominant*, a term which we now define. Fix a total ordering $\prec$ on $\Bbbk$ which also satisfies the following:

(1) If $a \prec b$, then $a+c \prec b+c$ for all $c \in \Bbbk$.

(2) On the integers, $\prec$ coincides with the usual order.

(3) For any $a \in \Bbbk$, there exists $m \in \mathbb{Z}$ with $a \prec m$.

For example, when $\Bbbk = \mathbb{C}$ we may define $\prec$ by $z \prec z'$ if and only if $\operatorname{Re} z < \operatorname{Re} z'$, or $\operatorname{Re} z = \operatorname{Re} z'$ and $\operatorname{Im} z < \operatorname{Im} z'$. We then say that a weight $\lambda \in \Bbbk^{Q_0}$ is *dominant* if $\lambda_i \succeq 0$ for all $i \in Q_0$.

When λ is dominant and Q is (extended) Dynkin, and Crawley-Boevey and Holland showed that it is easy to determine certain representation-theoretic properties of $\Pi^\lambda(Q)$. For example, we have the following useful result which we will use frequently in later sections:

Lemma 2.15 [Crawley-Boevey and Holland 1998, Lemma 7.1(1)]. *Suppose that Q is Dynkin, and let λ be a dominant weight for Q. Write Q_λ for the full subquiver supported on those vertices i with $\lambda_i = 0$. Then $\Pi^\lambda(Q) \cong \Pi(Q_\lambda)$. In particular, the projective $\Pi^\lambda(Q)$-modules are the modules $e_i\Pi^\lambda(Q)$, where i is a vertex with $\lambda_i = 0$.*

2D. ***Restriction to quasidominant weights.*** A weaker version of dominance will play an important role in this paper, which we now define.

Definition 2.16. If $\tilde{Q}$ is extended Dynkin, we say that a weight λ is *quasidominant* if $\lambda_i \succeq 0$ for all $i \neq 0$, where $\prec$ is a total ordering on $\Bbbk$ as above.

It turns out that we are able to restrict attention to quasidominant weights for the remainder of this paper. We now state this as an assumption, before explaining why this is the case.

Assumption 2.17. If λ is a weight for an extended Dynkin quiver $\tilde{Q}$, then we always assume that the weight λ is quasidominant unless explicitly stated otherwise.

To explain why this restriction is possible, we first recall a definition. Let Q be a quiver, and let $C = 2I - A$ be the generalised Cartan matrix of Q, where A is the adjacency matrix of the underlying graph of Q. For each loop-free vertex $i \in Q_0$, define the *dual reflection* $r_i : \Bbbk^{Q_0} \to \Bbbk^{Q_0}$ by

$$(r_i\lambda)_j = \lambda_j - C_{ij}\lambda_i.$$

It is easy to see that if $\tilde{Q}$ is extended Dynkin then $\lambda \cdot \delta = (r_i \lambda) \cdot \delta$. We then have the following result, which appears in unpublished work of Boddington and Levy [2007].

Lemma 2.18. *Suppose that λ is a weight for an extended Dynkin quiver $\tilde{Q}$, and let ρ be a sequence of dual reflections at vertices other than the extending vertex* 0. *Then $\mathcal{O}^\lambda \cong \mathcal{O}^{\rho(\lambda)}$.*

This is a strengthening of [Crawley-Boevey and Holland 1998, Lemma 7.9], in which the authors established only a Morita equivalence between these two rings, rather than an isomorphism. Combining Lemma 2.18 with [loc. cit, Lemma 7.8], we have the following result, which justifies the restriction given in Assumption 2.17.

Lemma 2.19. *Suppose that λ is a weight for an extended Dynkin quiver $\tilde{Q}$. Then there exists a quasidominant weight λ' with $\mathcal{O}^\lambda \cong \mathcal{O}^{\lambda'}$.*

We will see later that this assumption allows one to easily read off a number of useful facts about the module category of $\mathcal{O}^\lambda$, and ultimately its singularity category as well. As a first example, if we restrict our attention to quasidominant weights then it is easy to detect whether $\mathcal{O}^\lambda$ is singular.

Lemma 2.20. *If λ is a quasidominant weight for an extended Dynkin quiver $\tilde{Q}$, then $\mathcal{O}^\lambda$ is singular if and only if $\lambda_i = 0$ for some $i \neq 0$.*

Proof. By [Crawley-Boevey and Holland 1998, Theorem 0.4(4)], $\mathcal{O}^\lambda$ is singular if and only if $\lambda \cdot \alpha = 0$ for some *Dynkin root* α. The possible values of these Dynkin roots are not important to us; it suffices to know that they have the form $(0, \alpha') \in \mathbb{Z}^{n+1}$ where, in particular, α' has entirely nonnegative or nonpositive entries, and has at least one nonzero entry. In addition, $\varepsilon_i \in \mathbb{Z}^{n+1}$ for $1 \leqslant i \leqslant n$ is always a Dynkin root, where ε_i is the i-th coordinate vector (here the entries are indexed from 0 to n). Therefore, if $\lambda_i = 0$ for some $i \neq 0$ then $\lambda \cdot \alpha = 0$ for the Dynkin root $\alpha = \varepsilon_i$, while if $\lambda_i \neq 0$ for all $i \neq 0$, then necessarily $\lambda \cdot \alpha \neq 0$ for all Dynkin roots α. The result then follows. □

3. The singularity category of $\mathcal{O}^\lambda(\tilde{Q})$ as a $\Bbbk$-linear category

Our first step in determining $\underline{\mathrm{MCM}}$-$\mathcal{O}^\lambda$ is to determine its structure as an additive category, or indeed as a $\Bbbk$-linear category. We first identify an important module.

Lemma 3.1. *$\Pi^\lambda e_0$ is a finitely generated $\mathcal{O}^\lambda$-module, and it satisfies $\mathrm{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \Pi^\lambda$. Moreover, $\Pi^\lambda e_0$ is maximal Cohen–Macaulay.*

Proof. The first part of the statement follows from [Montgomery and Small 1981, Lemma 1]. To determine the endomorphism ring, first note that, by [Crawley-Boevey and Holland 1998, Lemma 1.4, Corollary 3.5], Π^λ is Morita equivalent to a ring which is a maximal order and hence is itself a maximal order. The claim then follows from the results in [Crawley-Boevey 1999, Section 5.4].

For the final claim, note that $\Pi^\lambda e_0$ is a reflexive $\mathcal{O}^\lambda$-module by [Crawley-Boevey 1999, Section 5.4]. Therefore, since $\Pi^\lambda e_0$ is finitely generated, and since $\mathrm{i.dim}\, \mathcal{O}^\lambda \leqslant 2$ by [Crawley-Boevey and Holland 1998, Theorem 1.6], Lemma 2.11 implies that $\Pi^\lambda e_0$ is maximal Cohen–Macaulay. □

Write $V_i = e_i \Pi^\lambda e_0$; we shall refer to these $\mathcal{O}^\lambda$-modules as *vertex modules*, and they will play an important role in determining $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$. Using Lemma 3.1, we are able to calculate the Hom spaces between the vertex modules.

Corollary 3.2. *We have* $\mathrm{Hom}_{\mathcal{O}^\lambda}(V_i, V_j) = e_j \Pi^\lambda e_i$.

Proof. By Lemma 3.1, $\Pi^\lambda = \mathrm{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \bigoplus_{k,\ell} \mathrm{Hom}_{\mathcal{O}^\lambda}(e_k \Pi^\lambda e_0, e_\ell \Pi^\lambda e_0)$. Multiplying on the left by e_j kills each Hom space with $\ell \neq j$, while multiplying on the right by e_i kills each Hom space with $k \neq i$. It follows that

$$e_j \Pi^\lambda e_i = e_j \Bigg(\bigoplus_{k,\ell} \mathrm{Hom}_{\mathcal{O}^\lambda}(V_k, V_\ell) \Bigg) e_i = e_j \,\mathrm{Hom}_{\mathcal{O}^\lambda}(V_i, V_j) e_i = \mathrm{Hom}_{\mathcal{O}^\lambda}(V_i, V_j),$$

as claimed. □

This allows us to determine the stable endomorphism ring of $\Pi^\lambda e_0$. We fix some notation which will be used throughout the rest of this paper: write Q_λ for the full subquiver of $\tilde{Q}$ with vertex set $I_\lambda := \{i \in \{1, \dots, n\} \mid \lambda_i = 0\}$.

Lemma 3.3. *We have* $\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \Pi(Q_\lambda)$.

Proof. Write $\mu = (\lambda_1, \dots, \lambda_n)$. By Corollary 3.2, we have that

$$(\Pi^\lambda e_0)^* = \bigoplus_i \mathrm{Hom}_{\mathcal{O}^\lambda}(e_i \Pi^\lambda e_0, e_0 \Pi^\lambda e_0) = \bigoplus_i e_0 \Pi^\lambda e_i = e_0 \Pi^\lambda.$$

Then, noting that $\Pi^\lambda e_0 (\Pi^\lambda e_0)^* = \Pi^\lambda e_0 \Pi^\lambda$, we have $\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \Pi^\lambda / (\Pi^\lambda e_0 \Pi^\lambda) \cong \Pi^\mu(Q)$. Since the entries of μ are all $\succeq 0$ by Assumption 2.17 and Q is Dynkin, Lemma 2.15 tells us that $\Pi^\mu(Q)$ is isomorphic to the preprojective algebra supported on the vertices i of Q with $\mu_i = 0$; that is, $\Pi^\mu(Q) \cong \Pi(Q_\lambda)$. Therefore $\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) \cong \Pi(Q_\lambda)$. □

We are also able to determine when a vertex module is projective. It turns out that this is the case precisely when the corresponding vertex is deleted when passing from $\tilde{Q}$ to Q_λ.

Lemma 3.4. *If* $i = 0$ *or* $\lambda_i \neq 0$, *then* V_i *is a projective* $\mathcal{O}^\lambda$*-module.*

Proof. When $i = 0$ this is clear. So suppose that $i \neq 0$ and $\lambda_i \neq 0$. Then, as in the proof of Lemma 3.3, $e_i = 0$ in $\Pi^\lambda / \Pi^\lambda e_0 \Pi^\lambda$ and so $e_i \in \Pi^\lambda e_0 \Pi^\lambda$. But then, using Corollary 3.2, $V_i V_i^* = e_i \Pi^\lambda e_0 \Pi^\lambda e_i \ni e_i^3 = e_i$, where e_i is the identity element of $\mathrm{End}_{\Pi^\lambda}(V_i) = e_i \Pi^\lambda e_i$, and so V_i is projective by the dual basis lemma (see [Lam 1999, (2.9)]). □

It follows that the vertex modules V_i satisfying $\lambda_i \neq 0$ are equal to the zero object in the singularity category, so that $\Pi^\lambda e_0$ and $\bigoplus_{i \in I_\lambda} V_i$ are isomorphic in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$. When working in the stable module category, we will sometimes refer to those vertex modules whose corresponding weight is zero as nonprojective vertex modules.

Proposition 3.5. (1) $\mathrm{MCM}\text{-}\mathcal{O}^\lambda = \mathrm{add}\, \Pi^\lambda e_0$.

(2) $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \mathrm{add}\, \Pi^\lambda e_0 = \mathrm{add}\big(\bigoplus_{i \in I_\lambda} V_i\big)$.

Proof. (1) First note that $\mathcal{O}^\lambda$ is Gorenstein and that, using [Crawley-Boevey and Holland 1998, Theorem 1.5],

$$\mathrm{gl.dim}\,\mathrm{End}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \mathrm{gl.dim}\,\Pi^\lambda \leqslant 2.$$

Since $\Pi^\lambda e_0$ has $\mathcal{O}^\lambda$ as a direct summand, the first claim then follows from Proposition 2.8.

(2) Part (1) immediately implies that $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \mathrm{add}\,\Pi^\lambda e_0 = \mathrm{add}\big(\bigoplus_i V_i\big)$. But projective modules become the zero object when passing to the stable module category, so the result follows by Lemma 3.4. □

We recall that an additive category is said to be *Krull–Schmidt* if every object decomposes into a finite direct sum of objects, each of which has a local endomorphism ring. By [Krause 2015, Theorem 4.2], this decomposition is unique up to reordering.

Theorem 3.6. *The functor* $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -)$ *induces a* $\Bbbk$*-linear equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \mathrm{proj}\text{-}\Pi(Q_\lambda).$$

Proof. By [Krause 2015, Proposition 2.3], the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -) : \underline{\mathrm{mod}}\text{-}\mathcal{O}^\lambda \to \mathrm{mod}\text{-}\,\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0) = \mathrm{mod}\text{-}\Pi(Q_\lambda)$$

induces a fully faithful $\Bbbk$-linear functor $\mathrm{add}\,\Pi^\lambda e_0 \to \mathrm{proj}\text{-}\Pi(Q_\lambda)$, where $\mathrm{add}\,\Pi^\lambda e_0 = \underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ by Proposition 3.5. Since $\Pi(Q_\lambda)$ is finite-dimensional [Białkowski et al. 2007, Proposition 2.1], $\mathrm{mod}\text{-}\Pi(Q_\lambda)$ is Krull–Schmidt and hence so too is $\mathrm{proj}\text{-}\Pi(Q_\lambda)$. Therefore, to establish essential surjectivity of the functor $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, -)$, it suffices to show that we can hit each indecomposable projective $e_i\Pi(Q_\lambda)$, where $i \in I_\lambda$. Indeed, we have

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(\Pi^\lambda e_0, V_i) = \frac{e_i\Pi^\lambda}{e_i\Pi^\lambda e_0\Pi^\lambda} = \frac{e_i\Pi^\lambda}{e_i\Pi^\lambda \cap \Pi^\lambda e_0\Pi^\lambda} = e_i\frac{\Pi^\lambda}{\Pi^\lambda e_0\Pi^\lambda} = e_i\Pi(Q_\lambda),$$

and so the functor is also essentially surjective. We therefore have the claimed equivalence. □

It follows that $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ is nontrivial if and only if $\lambda_i = 0$ for some $i \neq 0$ which, by Lemma 2.20, happens precisely when $\mathcal{O}^\lambda$ is singular; this is consistent with the more general fact that $\mathcal{D}_{\mathrm{sg}}(R)$ is nontrivial if and only if R is singular. Moreover, the vertex modules V_i with $i = 0$, or with $i \neq 0$ and $\lambda_i \neq 0$, are those which are projective and hence vanish in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$. This is reflected by the fact that these are the vertices which are deleted to obtain Q_λ.

As an immediate consequence of (the proof of) Theorem 3.6, we have the following result:

Corollary 3.7. $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ *is a Krull–Schmidt category.*

Remark 3.8. By Proposition 3.5, the objects of $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ are direct summands of finite direct sums of the nonprojective vertex modules. Since these vertex modules are indecomposable and $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ is Krull–Schmidt, in fact every object of $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ is isomorphic to a finite direct sum of vertex modules.

The following two corollaries are then immediate from Theorem 3.6:

Corollary 3.9. *Suppose that $\tilde{Q}$ is an extended Dynkin quiver and $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ is a disjoint union of connected quivers $Q^{(i)}$, which are therefore necessarily Dynkin. Then there is a $\Bbbk$-linear equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^{r} \mathrm{proj}\text{-}\Pi(Q^{(i)}).$$

Corollary 3.10. *Let $\tilde{Q}$ and $\tilde{Q}'$ be extended Dynkin quivers (not necessarily of the same type) and let λ and λ' be quasidominant weights for $\tilde{Q}$ and $\tilde{Q}'$, respectively. If $Q_\lambda \cong Q'_{\lambda'}$ then there is a $\Bbbk$-linear equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q}) \simeq \underline{\mathrm{MCM}}\text{-}\mathcal{O}^{\lambda'}(\tilde{Q}').$$

It is illustrative to apply Theorem 3.6 (and its corollaries) to an example.

Example 3.11. Suppose that $\tilde{Q} = \tilde{\mathbb{A}}_5$, and consider the deformation $\mathcal{O}^\lambda$ where the weight λ is indicated in red on the left-hand quiver in Figure 2.

By Corollary 3.9, there is a $\Bbbk$-linear equivalence $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \mathrm{proj}\text{-}\Pi(\mathbb{A}_3) \oplus \mathrm{proj}\text{-}\Pi(\mathbb{A}_1)$. In more suggestive notation, we can write this equivalence as

$$\mathcal{D}_{\mathrm{sg}}(\mathcal{O}^\lambda) \simeq \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_3}) \oplus \mathcal{D}_{\mathrm{sg}}(R_{\mathbb{A}_1}),$$

and so it is sensible to consider $\mathcal{O}^\lambda$ as having an $\mathbb{A}_3$ singularity and an $\mathbb{A}_1$ singularity.

If we more concretely set $\lambda = (-1, 0, 0, 0, 1, 0)$ (respectively, $\lambda = (0, 0, 0, 0, 1, 0)$) then $\mathcal{O}^\lambda$ is commutative (respectively, noncommutative). In this case we can use (2.14) to write down a presentation for $\mathcal{O}^\lambda$. In particular, we have a $\Bbbk$-linear equivalence

$$\mathcal{D}_{\mathrm{sg}} \frac{\Bbbk[x, y, z]}{\langle xy - z^4(z+1)^2 \rangle} \simeq \mathcal{D}_{\mathrm{sg}} \frac{\Bbbk\langle x, y, z\rangle}{\left\langle \begin{matrix} xz = (z+1)x, & xy = z^4(z+1)^2 \\ yz = (z-1)y, & yx = (z-1)^4 z^2 \end{matrix} \right\rangle}.$$

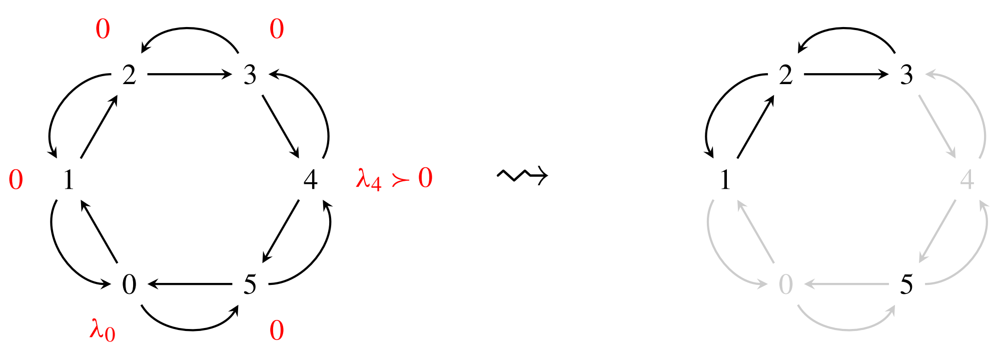

Figure 2. A weight λ for an $\tilde{\mathbb{A}}_5$ quiver, and the corresponding category $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$.

4. The singularity category of $\mathcal{O}^\lambda(\tilde{Q})$ as a triangulated category

We are now able to complete the proof of Theorem 1.1 which gives a complete description of the singularity category of $\mathcal{O}^\lambda(\tilde{Q})$. To do this, we need to show that the induced triangulated structures on the right-hand sides of the $\Bbbk$-linear equivalences

$$\mathcal{D}_{\text{sg}}(\mathcal{O}^\lambda(\tilde{Q})) \simeq \underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})) \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)}), \tag{4.1}$$

$$\bigoplus_{i=1}^{r} \mathcal{D}_{\text{sg}}(R_{Q^{(i)}}) \simeq \bigoplus_{i=1}^{r} \text{proj-}\Pi(Q^{(i)}) \tag{4.2}$$

are the same. We achieve this in by showing that the summands proj-$\Pi(Q^{(i)})$ in (4.1) are in fact triangulated subcategories of $\bigoplus_{i=1}^{r}$ proj-$\Pi(Q^{(i)})$, and that the triangulated structure on each proj-$\Pi(Q^{(i)})$ is essentially unique. To establish the latter of these, we use the following result:

Theorem 4.3 [Keller 2018, Corollary 2]. *Let $\mathcal{T}$ and $\mathcal{T}'$ be Krull–Schmidt $\Bbbk$-linear triangulated categories which are finite, connected, algebraic and standard. If $\mathcal{T}$ and $\mathcal{T}'$ are equivalent as $\Bbbk$-linear categories, then they are in fact equivalent as triangulated categories.*

We note that, if Q is Dynkin, proj-$\Pi(Q)$ (and the $\Bbbk$-linearly equivalent category $\mathcal{D}_{\text{sg}}(R_Q)$) are finite, connected, and standard since they are $\Bbbk$-linearly equivalent to certain orbit categories which are known to have these properties (see [Amiot et al. 2015, Remark 5.9]). Therefore, if we can show that each proj-$\Pi(Q^{(i)})$ is an algebraic triangulated subcategory under the $\Bbbk$-linear equivalence (4.1), then each $\Bbbk$-linear equivalence proj-$\Pi(Q^{(i)}) \simeq \mathcal{D}_{\text{sg}}(R_{Q^{(i)}})$ is in fact a triangle equivalence, which will prove Theorem 1.1 from the introduction.

We must first show that the translation functor Σ induced on the right-hand side of (4.1) *preserves connected components*, in the sense that it restricts to an autoequivalence of each of the subcategories proj-$\Pi(Q^{(i)})$. Writing P_i for the indecomposable projective $\Pi(Q_\lambda)$-module corresponding to vertex i in Q_λ, it follows from the Krull–Schmidt property of proj-$\Pi(Q_\lambda)$ that Σ permutes the P_i. We write σ for the induced permutation of the vertices. This allows us to make the following observation:

Lemma 4.4. *With the above setup, σ is a graph automorphism of Q_λ.*

Proof. First note that the spaces $\text{Hom}_{\Pi(Q_\lambda)}(P_i, P_j) = e_j \Pi(Q_\lambda) e_i$ can be graded by path length, and that vertex i and vertex j are adjacent in Q_λ if and only if there is a degree 1 morphism in $\text{Hom}_{\Pi(Q_\lambda)}(P_i, P_j)$. Applying Σ, this is equivalent to $\text{Hom}_{\Pi(Q_\lambda)}(P_{\sigma(i)}, P_{\sigma(j)})$ containing a degree 1 morphism, which happens if and only if $\sigma(i)$ and $\sigma(i)$ are adjacent in Q_λ. That is, σ is a graph automorphism of Q_λ. □

If the $Q^{(i)}$ are pairwise nonisomorphic, then the fact that the induced translation functor has to be a graph automorphism forces it to preserve connected components, as required. This leaves only the cases where some of the $Q^{(i)}$ are isomorphic, and one might hope to abstractly prove that the translation functor preserves connected components. Unfortunately, the following example shows that one should not expect this to be the case.

Example 4.5. Let T be a Krull–Schmidt $\Bbbk$-linear category with only two indecomposable objects U and V, and suppose these objects satisfy $\mathrm{Hom}_T(U, V) = 0 = \mathrm{Hom}_T(V, U)$ and $\mathrm{End}_T(U) = \Bbbk = \mathrm{End}_T(V)$. For example, this is the case for $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{\mathbb{A}}_3)$ when $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (0, 0, 1, 0)$, since in this case it is $\Bbbk$-linearly equivalent to $\mathrm{proj}\text{-}\Pi(\mathbb{A}_1) \oplus \mathrm{proj}\text{-}\Pi(\mathbb{A}_1)$. This category has two possible triangulated structures: the first has $\Sigma = \mathrm{id}$, and the distinguished triangles are isomorphic to direct sums and rotations of

$$U \xrightarrow{\mathrm{id}} U \to 0 \to U \quad \text{and} \quad V \xrightarrow{\mathrm{id}} V \to 0 \to V,$$

and the second option has $\Sigma U = V$ and $\Sigma V = U$, and the distinguished triangles are isomorphic to direct sums and rotations of

$$U \xrightarrow{\mathrm{id}} U \to 0 \to V.$$

The first example decomposes into a direct sum of two triangulated subcategories, while the second example does not.

While the above example shows that one should not expect to be able to abstractly prove that the translation functor preserves connected components, this is essentially the only counterexample. The following proof is due to Jeremy Rickard, and we thank him for allowing us to reproduce it:

Lemma 4.6. *Suppose that $\mathcal{T}$ is a Krull–Schmidt $\Bbbk$-linear triangulated category with finitely many indecomposables which decomposes as a $\Bbbk$-linear category as*

$$\mathcal{T} = \bigoplus_{i=1}^{n} \mathcal{T}_i.$$

Suppose that the translation functor Σ satisfies $\Sigma\mathcal{T}_i = \mathcal{T}_j$ for some $i \neq j$. Then $\mathcal{T}_i$ and $\mathcal{T}_j$ each have only one isoclass of indecomposable objects.

Proof. Let $\alpha : X \to Y$ be a nonzero morphism between two indecomposable objects of $\mathcal{T}_i$, and complete to a triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X,$$

where $\Sigma X \in \mathcal{T}_j$ by assumption. We claim that every indecomposable summand of Z lies in $\mathcal{T}_j$. To this end, suppose that $Z = Z' \oplus Z''$ where $Z' \in \mathcal{T}_j$ and $Z'' \in \bigoplus_{k \neq j} \mathcal{T}_k$, and write $\gamma = (\gamma', 0)$. The map γ' gives rise to a triangle $Z' \xrightarrow{\gamma'} \Sigma X \to Y' \to \Sigma Z'$ and rotating yields the triangle $X \to \Sigma^{-1} Y' \to Z' \xrightarrow{\gamma'} \Sigma X$. The direct sum of this triangle with the triangle $0 \to Z'' \to Z'' \to 0$ is a triangle isomorphic to the triangle $X \to Y \to Z \to \Sigma X$, and so $Y \cong \Sigma^{-1} Y' \oplus Z''$. By indecomposability of Y, we therefore have $\Sigma^{-1} Y' = 0$ or $Z'' = 0$. If $\Sigma^{-1} Y' = 0$ then $Y \cong Z''$ and $\Sigma X \cong Z'$. Our original triangle becomes

$$X \xrightarrow{\alpha} Z'' \to \Sigma X \oplus Z'' \to \Sigma X$$

which is isomorphic to the direct sum of the triangles $X \to 0 \to \Sigma X \to \Sigma X$ and $0 \to Z'' \to Z'' \to 0$. This means that α is the zero map, contrary to our assumption, and so we must have $Z'' = 0$, establishing the claim. Now, since every indecomposable summand of Z lies in $\mathcal{T}_j$, β is the zero map. Applying $\mathrm{Hom}(Y, -)$, we get an exact sequence

$$\mathrm{Hom}(Y, X) \xrightarrow{\alpha\circ-} \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, Z)$$

where the last term is 0. By exactness, there exists $\alpha' : Y \to X$ with $\alpha\alpha' = \mathrm{id}_Y$. Since $\mathcal{T}$ is Krull–Schmidt the endomorphism ring of X is local, which implies that the idempotent map $\alpha'\alpha$ is a unit and therefore equal to id_X. Therefore $\alpha : X \to Y$ is an isomorphism, and so $\mathcal{T}_i$ (and hence $\mathcal{T}_j$) has only one indecomposable object, up to isomorphism. □

Therefore, to show that the induced translation functor on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$ from the second $\Bbbk$-linear equivalence in (4.1) preserves connected components, we only need to consider the case when there exist $Q^{(i)}$ and $Q^{(j)}$, $i \neq j$, with $Q^{(i)} = \mathbb{A}_1 = Q^{(j)}$. It suffices to show that, for the corresponding objects $V_i, V_j \in \underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$, we have $\Sigma V_i = V_i$ and $\Sigma V_j = V_j$. To this end, we first have the following result:

Proposition 4.7. *Let Q be a non-Dynkin quiver with no oriented cycles, and with vertices labelled $\{0, 1, \dots n\}$. Write $\Pi(Q)$ for the preprojective algebra of Q, and write $V_i = e_i\Pi(Q)e_0$, which is a right $e_0\Pi(Q)e_0$-module. Then, for any $i \neq 0$, there exists a short exact sequence of $e_0\Pi(Q)e_0$-modules*

$$0 \to V_i \to \bigoplus_{j\in\partial i} V_j \to V_i \to 0,$$

where ∂i is the set of vertices adjacent to i in Q.

Proof. By [Brenner et al. 2002, Proposition 4.2], there is an exact sequence of $\Pi(Q)$-modules

$$0 \to e_i\Pi(Q) \to \bigoplus_{j\in\partial i} e_j\Pi(Q) \to e_i\Pi(Q) \to S_i \to 0, \tag{4.8}$$

where S_i is the simple module at vertex i. Noting that $e_0\Pi(Q)$ is a direct summand of $\Pi(Q)$ and hence projective, applying $\mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), -)$ yields an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), e_i\Pi(Q)) \to \bigoplus_{j\in\partial i} \mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), e_j\Pi(Q)) \\ \to \mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), e_i\Pi(Q)) \to \mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), S_i) \to 0. \end{aligned}$$

We also have $\mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), e_k\Pi(Q)) = V_k$ and, since $i \neq 0$, $\mathrm{Hom}_{\Pi(Q)}(e_0\Pi(Q), S_i) = 0$. We therefore have exactness of

$$0 \to V_i \to \bigoplus_{j\in\partial i} V_j \to V_i \to 0,$$

as claimed. □

In particular this result holds for extended Dynkin quivers, where we remark that if we wish to apply it to an $\tilde{\mathbb{A}}_n$ quiver then we must orient the arrows so that there are no oriented cycles; this does not change the isomorphism class of $\Pi(\tilde{\mathbb{A}}_n)$ [Crawley-Boevey and Holland 1998, Lemma 2.2].

Remark 4.9. The above result may or may not fail for Dynkin quivers, depending on how the vertices are labelled. For example, when $Q = \mathbb{A}_3$ where the vertices are labelled as follows,

$$0 \longrightarrow 1 \longrightarrow 2$$

then the complexes of interest to us are

$$0 \to V_1 \to V_0 \oplus V_2 \to V_1 \to 0, \quad \text{and} \quad 0 \to V_2 \to V_1 \to V_2 \to 0.$$

Since $\dim_{\Bbbk} V_0 = 1$, $\dim_{\Bbbk} V_1 = 1$, and $\dim_{\Bbbk} V_2 = 1$, the first of these is exact while the second is not. If instead we label the vertices of Q as follows,

$$1 \longrightarrow 0 \longrightarrow 2$$

then the complexes of interest to us are

$$0 \to V_1 \to V_0 \to V_1 \to 0, \quad \text{and} \quad 0 \to V_2 \to V_0 \to V_2 \to 0,$$

and both of these are exact since $\dim_{\Bbbk} V_0 = 2$, $\dim_{\Bbbk} V_1 = 1$, and $\dim_{\Bbbk} V_2 = 1$.

We now use Proposition 4.7 to show that the induced translation functor on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$ preserves connected components.

Proposition 4.10. *Let $\tilde{Q}$ be an extended Dynkin quiver and λ be a quasidominant weight for $\tilde{Q}$. Write $Q_\lambda = Q^{(1)} \sqcup \cdots \sqcup Q^{(r)}$ as a disjoint union of connected quivers $Q^{(i)}$, which are therefore necessarily Dynkin. Consider the triangulated structure on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$ induced by the $\Bbbk$-linear equivalence*

$$\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$$

of Corollary 3.9, and let Σ be the translation functor. Then each $\text{proj-}\Pi(Q^{(i)})$ is invariant under Σ.

Proof. By Lemma 4.6 and the discussion following it, the only situation in which there exist $\text{proj-}\Pi(Q^{(i)})$ which are not necessarily invariant under Σ is when we have multiple $Q^{(i)}$ equal to $\mathbb{A}_1$. Working in $\underline{\text{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$, this happens if and only if there is some vertex i with $\lambda_i = 0$, and if j is adjacent to i then either $j = 0$ or $\lambda_j \neq 0$; in particular, the modules V_j corresponding to these vertices are projective as $\mathcal{O}^\lambda(\tilde{Q})$-modules by Lemma 3.4. By Proposition 4.7, we have an exact sequence of $e_0\Pi(\tilde{Q})e_0$-modules

$$0 \to V_i \xrightarrow{\phi} \bigoplus_{j \in \partial i} V_j \xrightarrow{\psi} V_i \to 0. \tag{4.11}$$

Now consider (4.11) as a sequence of modules over $\mathcal{O}^\lambda(\tilde{Q})$. It is a complex since the composition $\psi\phi$ is equal to the (undeformed) preprojective relation at vertex i, which is equal to $\lambda_i e_i = 0$. Filtering $\Pi(\tilde{Q})$ and $\mathcal{O}^\lambda(\tilde{Q})$ by path length we obtain a sequence of associated graded modules, which is in fact the exact sequence (4.11). It is standard (see [McConnell and Robson 2001, Proposition 7.6.14]) that this implies that (4.11) is exact as a sequence of modules over $\mathcal{O}^\lambda(\tilde{Q})$. To summarise, we have an exact sequence of $\mathcal{O}^\lambda(\tilde{Q})$-modules

$$0 \to V_i \to \bigoplus_{j \in \partial i} V_j \to V_i \to 0$$

whose middle term is projective. It follows from the definition of the translation functor that $\Sigma V_i = V_i$ in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda(\tilde{Q})$. Thus each $\text{proj-}\Pi(Q^{(i)})$ is invariant under the induced translation functor on $\bigoplus_{i=1}^r \text{proj-}\Pi(Q^{(i)})$. □

We now seek to prove Theorem 1.1. Retaining all of the above notation, for each $1 \leqslant i \leqslant r$, define

$$\mathcal{W}_i := \{V_j \mid j \in Q_0^{(i)}\}, \quad \mathcal{C}_i := \text{add}\Bigg(V_0 \oplus \bigoplus_{j \in Q_0^{(i)}} V_j\Bigg) \quad \text{and} \quad \mathcal{T}_i := \text{add}\Bigg(\bigoplus_{j \in Q_0^{(i)}} V_j\Bigg),$$

where the latter two are viewed as subcategories of $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$ and $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$, respectively. It will also be convenient to write

$$M_i = \bigoplus_{j \in (Q_\lambda)_0 \setminus Q_0^{(i)}} V_j,$$

and to set

$$\mathcal{W}_i^{\mathrm{c}} := \{V_j \mid j \in (Q_\lambda)_0 \setminus Q_0^{(i)}\}, \quad \mathcal{C}_i^{\mathrm{c}} := \text{add}(V_0 \oplus M_i) \quad \text{and} \quad \mathcal{T}_i^{\mathrm{c}} := \text{add}\, M_i.$$

Observe that we can decompose $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$ as

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \bigoplus_{i=1}^r \mathcal{T}_i$$

as $\Bbbk$-linear categories. We wish to show that this is also a decomposition into triangulated subcategories. To do this, we first prove a result which shows that the $\mathcal{C}_i$ are Frobenius subcategories of the Frobenius category $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$. We call a subcategory $\mathcal{B}$ of an exact category $\mathcal{A}$ *extension-closed* if whenever we have a conflation $X \to Y \to Z$ with $X, Z \in \mathcal{B}$ then necessarily $Y \in \mathcal{B}$. Furthermore, an extension-closed subcategory $\mathcal{B}$ is called *admissible* provided that every $B \in \mathcal{B}$ fits into conflations $B \to P \to B'$ and $B'' \to Q \to B$ with $B', B'' \in \mathcal{B}$ and where P, Q are projective in $\mathcal{A}$. We remark that an admissible subcategory of a Frobenius category is itself Frobenius; see [Chen 2012, Section 2].

Lemma 4.12. *For each i, the subcategory $\mathcal{C}_i$ satisfies the following property: if $X \to Y \to Z$ is a conflation in* $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$ *such that two of the three objects are in $\mathcal{C}_i$, then the third object is also in $\mathcal{C}_i$. Consequently, $\mathcal{C}_i$ is a Frobenius subcategory of* $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$.

Proof. We only show that if $X \to Y \to Z$ is a conflation with $X, Y \in \mathcal{C}_i$ then $Z \in \mathcal{C}_i$, with the other cases being similar. So suppose that we have such a conflation. Since $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \text{proj-}\Pi(Q_\lambda)$ and this category is Krull–Schmidt, we have $Z \oplus P \cong U \oplus U' \oplus Q$ in $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$, where $U \in \mathcal{W}_i$, $U' \in \mathcal{W}_i^{\mathrm{c}}$, and P, Q are projective. This conflation gives rise to a triangle $X \to Y \to Z \to \Sigma X$ in $\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda$, and applying the functor $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, -)$ yields an exact sequence

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, Y) \to \underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, \Sigma X).$$

Now $\Sigma X \in \mathcal{C}_i$ by Proposition 4.10 and $Y \in \mathcal{C}_i$ by definition, while $M_i \in \mathcal{C}_i^{\mathrm{c}}$, so both of the flanking terms are 0. This implies that the middle term, which is equal to $\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(M_i, U')$, is also 0. But this means that $U' = 0$, and hence $Z \oplus P \in \mathcal{C}_i$. Since, by definition, $\mathcal{C}_i$ is closed under direct summands, it follows that $Z \in \mathcal{C}_i$ as required.

For the final claim, first notice that the above paragraph tells us that $\mathcal{C}_i$ is extension-closed. Moreover, given an object $C \in \mathcal{C}_i$, since $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$ is Frobenius we can always find conflations $C \to P \to Z$ and $X \to Q \to C$ with $X, Z \in \mathrm{MCM}\text{-}\mathcal{O}^\lambda$ and P, Q projective. Since projective $\mathcal{O}^\lambda$-modules are direct summands of sums of copies of $\mathcal{O}^\lambda$, we have $P, Q \in \mathcal{C}_i$ by definition, and then the previous paragraph tells us that $X, Z \in \mathcal{C}_i$. Therefore $\mathcal{C}_i$ is admissible and hence Frobenius. □

This allows us to prove our main theorem.

Theorem 4.13. *Let $\tilde{Q}$ and Q_λ be as in Corollary 3.9. Then the $\Bbbk$-linear equivalence*

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda \simeq \bigoplus_{i=1}^{r} \mathrm{proj}\text{-}\Pi(Q^{(i)}),$$

of Corollary 3.9 is a triangle equivalence, where the right-hand side is a decomposition into triangulated subcategories satisfying $\mathrm{proj}\text{-}\Pi(Q^{(i)}) \simeq \mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$.

Proof. By Lemma 4.12, we know that $\mathcal{C}_i$ is a Frobenius subcategory of $\mathrm{MCM}\text{-}\mathcal{O}^\lambda$. Using [Arentz-Hansen 2019, Theorem 3.15(2)], it follows that $\mathcal{T}_i$ is equal to the stable category of the Frobenius category $\mathcal{C}_i$ for $1 \leqslant i \leqslant r$, and so the decomposition

$$\underline{\mathrm{MCM}}\text{-}\mathcal{O}^\lambda = \bigoplus_{i=1}^{r} \mathcal{T}_i$$

is in fact a decomposition into triangulated subcategories. If we set $e^{(i)} = \sum_{j \in Q_0^{(i)}} e_j$, [Krause 2015, Proposition 2.3] implies that the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}^\lambda}(e^{(i)}\Pi^\lambda e_0, -) : \underline{\mathrm{mod}}\text{-}\mathcal{O}^\lambda \to \mathrm{mod}\text{-}\,\underline{\mathrm{End}}_{\mathcal{O}^\lambda}(e^{(i)}\Pi^\lambda e_0)$$

restricts to a $\Bbbk$-linear equivalence $\mathcal{T}_i \simeq \mathrm{proj}\text{-}\Pi(Q^{(i)})$. This equivalence also induces an algebraic triangulated structure on $\mathrm{proj}\text{-}\Pi(Q^{(i)})$. Since this category is $\Bbbk$-linearly equivalent to $\mathcal{D}_{\mathrm{sg}}(R_{Q^{(i)}})$, Theorem 4.3 implies that they are triangle equivalent, completing the proof. □

Remark 4.14. In the first version of this paper, [Crawford 2016], a much longer argument was used to establish Theorem 4.13. The original proof made use of the so-called knitting algorithm from [Iyama and Wemyss 2010] to construct short exact sequences of $\mathcal{O}^\lambda$-modules, and explicitly listed many such sequences. The techniques used and the exact sequences given may be of independent interest.

5. A noncommutative geometric McKay correspondence

We now look at a generalisation of the intersection theory of the minimal resolution of a Kleinian singularity Spec R_Q to a noncommutative setting. We begin by recalling the result in the commutative setting, whereby it is often referred to as the geometric McKay correspondence.

Let R_Q be a Kleinian singularity with corresponding extended Dynkin quiver $\tilde{Q}$. Then the affine variety Spec R_Q is an isolated surface singularity which has a unique minimal resolution. It is well-known (see, for example, [Leuschke and Wiegand 2012, Section 6.4]) that the exceptional fibre of this minimal resolution is a union of n irreducible curves γ_i, where n is the number of vertices of Q. Moreover, each γ_i is isomorphic to $\mathbb{P}^1$ and has self-intersection -2, and $\gamma_i \cap \gamma_j$ is either empty or a point. In fact, the dual graph of the exceptional fibre is given by the underlying graph of Q. Let Γ be the $n \times n$ matrix with entries

$$\Gamma_{ij} = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } \gamma_i \text{ and } \gamma_j \text{ intersect}, \\ 0 & \text{otherwise}, \end{cases}$$

so that Γ_{ij} is equal to the intersection multiplicity $\gamma_i \bullet \gamma_j$. With an appropriate labelling of the curves γ_i, we have $\Gamma = -C$ where C is the Cartan matrix corresponding to the Dynkin type of Q; explicitly, $C = 2I - A$, where A is the adjacency matrix of the underlying graph of Q.

Now let $\tilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices, let Q be the quiver obtained by removing the extending vertex, and let $\lambda = \varepsilon_0 = (1, 0, \dots, 0)$; that is, the weight at the extending vertex is 1, and 0 for all of the other vertices. We may then consider $\mathcal{O}^\lambda(\tilde{Q})$ to be a noncommutative analogue of R_Q, the coordinate ring of the corresponding Kleinian singularity; indeed, these rings have equivalent singularity categories by Theorem 1.1. We now seek to generalise the geometric McKay correspondence to $\mathcal{O}^\lambda(\tilde{Q})$ for this particular choice of λ. To do so, we need an appropriate analogue of a resolution of the singular ring $\mathcal{O}^\lambda(\tilde{Q})$; we use the definition given in [Qin et al. 2019], where such a resolution is a noncommutative ring satisfying certain properties which we will recall below. The role of the exceptional curves in the resolution will be played by finite-dimensional simple modules, of which our resolution of $\mathcal{O}^\lambda(\tilde{Q})$ has finitely many, and the intersection multiplicity of any two such modules is provided by [Mori and Smith 2001]. In this section, we will prove the following:

Theorem 5.1 (Theorem 5.14). *Let $\tilde{Q}$ be an extended Dynkin quiver with corresponding Dynkin quiver Q and let $\lambda = \varepsilon_0$. Then $\mathcal{O}^\lambda(\tilde{Q})$ has a noncommutative resolution* (*which in fact can take the form $\mathcal{O}^\mu(\tilde{Q})$ for some weight μ and so is a deformation*), *and the exceptional objects* (*the finite-dimensional simple modules*) *in this resolution may be indexed so that the corresponding intersection matrix is $-C$, where C is the Cartan matrix corresponding to Q.*

5A. ***Noncommutative quasicrepant resolutions.*** We now give a precise definition of the noncommutative resolutions which appear in the above theorem, which is taken from [Qin et al. 2019], and we prove a useful general result. We first recall a definition:

Definition 5.2. Let A be a $\Bbbk$-algebra and $X, Y \in \text{Mod-}A$. We say that X and Y are *n-isomorphic*, written $X \cong_n Y$, if there exists a third module Z and homomorphisms

$$\phi : Z \to X \quad \text{and} \quad \psi : Z \to Y$$

such that the kernels and cokernels of ϕ and ψ each have GK dimension at most n.

We can now give the precise definition of a *noncommutative quasicrepant resolution* from [Qin et al. 2019, Definition 3.15].

Definition 5.3. Let A be a noetherian Auslander–Gorenstein $\Bbbk$-algebra with $\operatorname{GKdim}(A) = d$. Then a *noncommutative quasicrepant resolution* (NQCR) of A is a triple (B, M, N) where B is a noetherian Auslander regular, Cohen–Macaulay $\Bbbk$-algebra with $\operatorname{GKdim}(B) = d$ and where ${}_B M_A$ and ${}_A N_B$ are finitely generated bimodules which are reflexive on both sides and which satisfy

$$M \otimes_A N \cong_{d-2} B \quad \text{and} \quad N \otimes_B M \cong_{d-2} A.$$

NQCRs have the following useful property, which is not proven in [Qin et al. 2019] but provides a useful complement to [loc. cit., Theorem 0.6]. We remark that the following statement remains true if one replaces "noncommutative quasicrepant resolution" by "noncommutative quasiresolution" (see [loc. cit., Definition 3.2]), the latter notion being slightly weaker.

Lemma 5.4. *If B is a noncommutative quasicrepant resolution of A and C is Morita equivalent to B, then C is also a noncommutative quasicrepant resolution of A.*

Proof. Let $d = \operatorname{GKdim} A$. Since B is a NQCR of A, it is Auslander regular and Cohen–Macaulay, has GK dimension d, and there exist bimodules ${}_B M_A$ and ${}_A N_B$ which are finitely generated and reflexive on both sides and which satisfy

$$M \otimes_A N \cong_{d-2} B \quad \text{and} \quad N \otimes_B M \cong_{d-2} A.$$

Moreover, since C is Morita equivalent to B, there exists a progenerator $P \in \text{mod-}B$ with $C \cong \operatorname{End}(P_B)$, and we may view P as a (C, B)-bimodule.

First note that C has GK dimension d by standard Morita theory, and by applying [Yekutieli and Zhang 2002, Propositon 4.3], we deduce that C is also Auslander regular and Cohen–Macaulay. Writing ${}_B Q_C = \operatorname{Hom}({}_C P, C) \cong \operatorname{Hom}(P_B, B)$, define two bimodules ${}_C \tilde{M}_A$ and ${}_A \tilde{N}_C$ as follows:

$$\tilde{M} = P \otimes_B M \quad \text{and} \quad \tilde{N} = N \otimes_B Q.$$

Since we have pairs of mutually inverse equivalences

$$- \otimes_C Q : \text{mod-}B \to \text{mod-}C, \quad - \otimes_B P : \text{mod-}C \to \text{mod-}B,$$
$$P \otimes_B - : B\text{-mod} \to C\text{-mod} \quad Q \otimes_C - : C\text{-mod} \to B\text{-mod}$$

and since M and N are reflexive on both sides, it follows that $\tilde{M}$ and $\tilde{N}$ are reflexive on both sides.

It remains to show that $\tilde{M} \otimes_A \tilde{N} \cong_{d-2} C$ and $\tilde{N} \otimes_C \tilde{M} \cong_{d-2} A$. The latter of these is immediate, since

$$\tilde{N} \otimes_C \tilde{M} = N \otimes_B Q \otimes_C P \otimes_B M \cong N \otimes_B M \cong_{d-2} A,$$

where the first isomorphism follows from [Lam 1999, 18.17 Proposition] and the $(d-2)$-isomorphism follows since (B, M, N) is a NQCR of A. We now wish to show that $\tilde{M} \otimes_A \tilde{N} \cong_{d-2} C$. Since $M \otimes_A N \cong_{d-2} B$, there exists a bimodule ${}_B Z_B$ and morphisms

$$\phi : Z \to M \otimes_A N \quad \text{and} \quad \psi : Z \to B$$

such that the kernels and cokernels of ϕ and ψ have GK dimension at most $d-2$. Define $\tilde{Z} = P \otimes_B Z \otimes_B Q$ and morphisms

$$\begin{aligned} \tilde{\phi} &: \tilde{Z} \to \tilde{M} \otimes_A \tilde{N}, & \tilde{\phi} &= \mathrm{id}_P \otimes \phi \otimes \mathrm{id}_Q, \\ \tilde{\psi} &: \tilde{Z} \to C, & \tilde{\psi}(p \otimes x \otimes q) &= q(p\psi(x)). \end{aligned}$$

We claim that the kernels and cokernels of these maps have GK dimension at most $d-2$, which will complete the proof.

We first consider the kernel and cokernel of $\tilde{\phi}$. Since P and Q are projective (hence flat) on the right and left respectively, we can make an identification $\ker \tilde{\phi} \cong P \otimes_B \ker \phi \otimes_B Q$. Since P and Q are finitely generated modules, [McConnell and Robson 2001, Proposition 8.3.14] implies that $\mathrm{GKdim} \ker \tilde{\phi} \leqslant \mathrm{GKdim} \ker \phi \leqslant d-2$, as required. Flatness of P and Q also allows us to make identifications

$$\mathrm{im}\, \tilde{\phi} \cong P \otimes_B \mathrm{im}\, \phi \otimes_B Q \quad \text{and} \quad \mathrm{coker}\, \tilde{\phi} \cong P \otimes_B \mathrm{coker}\, \phi \otimes_B Q,$$

and so $\mathrm{GKdim}\, \mathrm{coker}\, \tilde{\phi} \leqslant \mathrm{GKdim}\, \mathrm{coker}\, \phi \leqslant d-2$.

We now turn our attention to $\tilde{\psi}$. Observe that we can view $\tilde{\psi}$ as the composition

$$\tilde{\psi} : P \otimes_B Z \otimes_B Q \xrightarrow{\mathrm{id}_P \otimes \psi \otimes \mathrm{id}_Q} P \otimes_B B \otimes_B Q \xrightarrow{m \otimes \mathrm{id}_Q} P \otimes_B Q \xrightarrow{\mu} C$$

where $m : P \otimes_B B \to P$ is the multiplication map and $\mu : P \otimes_B Q \to C$ is the evaluation map. Flatness of Q implies that $m \otimes \mathrm{id}_Q$ is an isomorphism, while μ is an isomorphism because P is a progenerator. Thus $\ker \tilde{\psi} = \ker(\mathrm{id}_P \otimes \psi \otimes \mathrm{id}_Q)$, and again we can identify this with $P \otimes_B \ker \psi \otimes_B Q$, which has GK dimension at most $d-2$ using the same argument as in the previous paragraph. Similarly, we have an identification $\mathrm{im}\, \tilde{\psi} \cong P \otimes_B \mathrm{im}\, \psi \otimes_B Q$, and arguing again as above, we find that $\mathrm{im}\, \tilde{\psi}$ has GK dimension at most $d-2$, completing the proof. □

5B. ***Intersection theory for a family of noncommutative resolutions.*** We return now to the $\Bbbk$-algebra of interest, namely $\mathcal{O}^\lambda$ where $\lambda = \varepsilon_0$. Our first aim is to identify an appropriate NQCR, which we have in fact already done.

Lemma 5.5. *Π^λ is a NQCR of $\mathcal{O}^\lambda$.*

Proof. Since $\mathcal{O}^\lambda$ is Auslander–Gorenstein, Π^λ is Auslander regular and Cohen–Macaulay, and they both have GK dimension 2 [Crawley-Boevey and Holland 1998, Theorem 1.5, Theorem 1.6], it suffices to show that $\Pi^\lambda e_0 \in \text{bimod-}(\Pi^\lambda, \mathcal{O}^\lambda)$ and $e_0\Pi^\lambda \in \text{bimod-}(\mathcal{O}^\lambda, \Pi^\lambda)$ are reflexive on both sides, and that

$$\Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda \cong_0 \Pi^\lambda \quad \text{and} \quad e_0\Pi^\lambda \otimes_{\Pi^\lambda} \Pi^\lambda e_0 \cong_0 \mathcal{O}^\lambda.$$

By Lemma 3.1, $\Pi^\lambda e_0$ is reflexive as a right $\mathcal{O}^\lambda$-module, and since it is a generator for mod-$\mathcal{O}^\lambda$, [Lam 1999, Proposition 18.17] implies it is also reflexive as left Π^λ-module. Similarly, $e_0\Pi^\lambda$ is a reflexive module on both sides.

The 0-isomorphism $e_0\Pi^\lambda \otimes_{\Pi^\lambda} \Pi^\lambda e_0 \cong_0 \mathcal{O}^\lambda$ follows from the fact that these two modules are actually isomorphic. To see that $\Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda \cong_0 \Pi^\lambda$, it suffices to show that the multiplication map

$$m : \Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda \to \Pi^\lambda$$

has finite-dimensional kernel and cokernel. The cokernel of m is $\Pi^\lambda/\Pi^\lambda e_0\Pi^\lambda \cong \Pi(Q_\lambda) = \Pi(Q)$, which is finite-dimensional. To see that $K = \ker m$ is finite-dimensional, factor m as

$$m : \Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda \xrightarrow{\pi} \Pi^\lambda e_0\Pi^\lambda \to \Pi^\lambda$$

where π is surjective and hence $\ker \pi = K$. We then have a short exact sequence

$$0 \to K \to \Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda \xrightarrow{\pi} \Pi^\lambda e_0\Pi^\lambda \to 0.$$

Since $\Pi^\lambda e_0$ is a finitely generated $\mathcal{O}^\lambda$-module, $\Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda$ is a finitely generated Π^λ-module, and so K is also a finitely generated Π^λ-module. Applying $- \otimes_{\Pi^\lambda} \Pi^\lambda e_0$ to the above sequence, which is an exact functor since $\Pi^\lambda e_0$ is a projective left Π^λ-module, we obtain the short exact sequence

$$0 \to K \otimes_{\Pi^\lambda} \Pi^\lambda e_0 \to \Pi^\lambda e_0 \otimes_{\mathcal{O}^\lambda} e_0\Pi^\lambda \otimes_{\Pi^\lambda} \Pi^\lambda e_0 \to \Pi^\lambda e_0\Pi^\lambda \otimes_{\Pi^\lambda} \Pi^\lambda e_0 \to 0.$$

It is easy to see that the above sequence is in fact

$$0 \to Ke_0 \to \Pi^\lambda e_0 \xrightarrow{\sim} \Pi^\lambda e_0 \to 0,$$

and so $Ke_0 = 0$. It follows that K has the structure of a finitely generated right $\Pi^\lambda/\Pi^\lambda e_0\Pi^\lambda$-module, and is therefore finite-dimensional. By definition, we find that Π^λ is a NQCR of $\mathcal{O}^\lambda$. □

We can actually obtain infinitely many noncommutative resolutions of $\mathcal{O}^\lambda$ using the dual reflections r_i of [Crawley-Boevey and Holland 1998], the definition of which was given prior to Lemma 2.18. It is clear that the r_i preserve the $\mathbb{Z}^{n+1}$ lattice inside $\Bbbk^{n+1}$. It was also noted earlier that $\lambda \cdot \delta = r_i\lambda \cdot \delta$ for all $\lambda \in \Bbbk^{\tilde{Q}_0}$ and $i \in \tilde{Q}_0$, so that the r_i preserve the affine hyperplanes $\{\lambda \in \Bbbk^{n+1} \mid \lambda \cdot \delta = c\}$ for each $c \in \Bbbk$; since $\varepsilon_0 \cdot \delta = 1$, we are primarily interested in the case $c = 1$. Crawley-Boevey and Holland [1998, Corollary 5.2] proved the following useful result:

Lemma 5.6. *Let ρ be a composition of dual reflections. Then Π^λ is Morita equivalent to $\Pi^{\rho(\lambda)}$.*

By combining Lemmas 5.4, 5.5 and 5.6, we obtain the following:

Corollary 5.7. *$\Pi^{\rho(\lambda)}$ is a NQCR of $\mathcal{O}^\lambda$ for any composition of dual reflections ρ.*

As stated previously, we need to identify an analogue of the exceptional curves appearing in the minimal resolution of a Kleinian singularity. When $\lambda = \varepsilon_0$, by Lemma 2.15, Π^λ has precisely n isoclasses of finite-dimensional simple modules, and hence by Morita equivalence so does $\Pi^{\rho(\lambda)}$ for any composition of dual reflections ρ. These will play the role of the exceptional objects in our noncommutative resolution.

We also require a notion of intersection multiplicity for the exceptional objects, which is provided by [Mori and Smith 2001]. Given a nonsingular noetherian ring S and $M, N \in \text{mod-}S$ which satisfy $\dim_{\Bbbk} \operatorname{Ext}_S^\ell(M, N) < \infty$ for all $\ell \geqslant 0$, we define the *intersection multiplicity* of M and N to be

$$M \bullet N := \sum_{\ell \geqslant 0} (-1)^{\ell+1} \dim_{\Bbbk} \operatorname{Ext}_S^\ell(M, N)$$

(note that this sum has finitely many terms since S is nonsingular).

We are now in a position to prove a preliminary version of Theorem 5.1:

Theorem 5.8. *Let $\tilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices, and let $\lambda = \varepsilon_0$. Let $\mu = \rho(\lambda)$, where ρ is any composition of dual reflections, so that Π^μ is a NQCR of $\mathcal{O}^\lambda$. Then Π^μ has precisely n finite-dimensional simple modules S_i up to isomorphism, and with a suitable indexing of them, the intersection matrix Γ with entries $\Gamma_{ij} = S_i \bullet S_j$ is $-C$, where C is the Cartan matrix corresponding to Q.*

Proof. The discussion after Corollary 5.7 shows that Π^μ has n finite-dimensional simple modules S_i up to isomorphism, so it remains to prove the result on the intersection multiplicities.

Since Morita equivalence preserves dimensions of Hom and Ext groups, we are able to calculate the intersection numbers of the finite-dimensional Π^μ-modules by performing the calculations over Π^λ instead. Identifying Π^λ-modules with representations of $\bar{\tilde{Q}}$ which satisfy the relations coming from Π^λ, [Crawley-Boevey and Holland 1998, Lemma 7.2(6), Theorem 7.4] tells us that the dimension vector of S_i is $\varepsilon_i \in \mathbb{N}^{n+1}$. It follows that

$$S_i \cong \frac{e_i \Pi^\lambda}{\bigoplus_{\substack{\alpha \in \bar{\tilde{Q}}_1 \\ t(\alpha)=i}} \alpha \Pi^\lambda}. \tag{5.9}$$

Also observe that

$$\operatorname{Hom}_{\Pi^\lambda}(e_i \Pi^\lambda, S_j) = \begin{cases} \Bbbk e_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.10}$$

The proof of [Crawley-Boevey and Holland 1998, Lemma 10.1] shows that, for each $i \neq 0$, there is an exact sequence of Π^λ-modules

$$0 \to e_i \Pi^\lambda \xrightarrow{\phi} \bigoplus_{k \in \partial i} e_k \Pi^\lambda \xrightarrow{\psi} e_i \Pi^\lambda \to S_i \to 0.$$

Since the modules $e_k \Pi^\lambda$ are direct summands of Π^λ and hence projective, this is in fact a projective resolution of S_i. Now let $1 \leqslant j \leqslant n$. Seeking to calculate the extension groups between S_i and S_j, we apply $\operatorname{Hom}_{\Pi^\lambda}(-, S_j)$ to the corresponding deleted resolution to obtain the complex

$$0 \to \operatorname{Hom}_{\Pi^\lambda}(e_i \Pi^\lambda, S_j) \to \bigoplus_{k \in \partial i} \operatorname{Hom}_{\Pi^\lambda}(e_k \Pi^\lambda, S_j) \to \operatorname{Hom}_{\Pi^\lambda}(e_i \Pi^\lambda, S_j) \to 0. \tag{5.11}$$

We now consider three distinct cases when computing the homology of this complex. If $j = i$ then, using (5.10), as a complex of vector spaces (5.11) becomes

$$0 \to \Bbbk \to 0 \to \Bbbk \to 0$$

and so we can immediately read off that

$$\dim_{\Bbbk} \mathrm{Hom}_{\Pi^\lambda}(S_i, S_i) = 1 = \dim_{\Bbbk} \mathrm{Ext}^2_{\Pi^\lambda}(S_i, S_i), \quad \dim_{\Bbbk} \mathrm{Ext}^{\ell}_{\Pi^\lambda}(S_i, S_i) = 0 \quad \text{for } \ell = 1 \text{ or } \ell \geqslant 3,$$

and so $S_i \bullet S_i = -1 + 0 - 1 = -2$. If $j \in \partial i$, then (5.11) becomes

$$0 \to 0 \to \Bbbk \to 0 \to 0$$

and so

$$\dim_{\Bbbk} \mathrm{Ext}^1_{\Pi^\lambda}(S_i, S_i) = 1, \quad \dim_{\Bbbk} \mathrm{Ext}^{\ell}_{\Pi^\lambda}(S_i, S_i) = 0 \text{ for } \ell = 0 \text{ or } \ell \geqslant 2.$$

That is, if i and j are adjacent in $\tilde{Q}$, then

$$S_i \bullet S_j = 0 + 1 + 0 = 1.$$

Finally, if $j \neq i$ and $j \notin \partial i$ then (5.11) becomes

$$0 \to 0 \to 0 \to 0 \to 0$$

and clearly

$$\dim_{\Bbbk} \mathrm{Ext}^{\ell}_{\Pi^\lambda}(S_i, S_i) = 0 \quad \text{for } \ell \geqslant 0,$$

and so $S_i \bullet S_j = 0$ in this case. It follows that the intersection matrix Γ satisfies $\Gamma = -C$. □

The above result should be seen as a noncommutative analogue of the geometric McKay correspondence. However, we can strengthen this result by showing that $\mathcal{O}^\lambda$ possesses a NQCR which is actually a "deformation": that is, a NQCR of the form $\mathcal{O}^\mu$ for some weight μ. Since we are restricting our attention to quasidominant weights, the fact that $\mathcal{O}^\mu$ is nonsingular forces $\mu_i \succ 0$ for all $i \geqslant 1$ (see Lemma 2.20). It is not immediately clear that such a deformation exists; we prove its existence in the next subsection.

5C. *$\mathcal{O}^\lambda$ has a NQCR which is a deformation.* The dual reflections defined earlier also appear in the so-called *numbers game* of [Mozes 1990]. The relationship between this game and our setting is that the moves considered by Mozes can equivalently be described as an application of a dual reflection to a weight λ. This allows us to make use of some of the results from this paper; in particular we are able to prove that, for $\lambda = \varepsilon_0$, NQCRs of $\mathcal{O}^\lambda$ which are also deformations exist.

Lemma 5.12. *Let $\tilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices. Then there exists a sequence of dual reflections ρ such that $\rho(\varepsilon_0)_i > 0$ for all $i \neq 0$; in particular, $\rho(\varepsilon_0)$ is quasidominant.*

Proof. It suffices to show that we can find such a sequence of dual reflections when we work over the field $\mathbb{R}$, since any such sequence will also have the desired effect on ε_0 when we work over our algebraically closed field $\Bbbk$ of characteristic 0. Accordingly, we write $<$ instead of $\prec$ for the total order. Write G for the group generated by the dual reflections, which is simply the Weyl group of type corresponding to Q. Lemma 5.5 of [Mozes 1990], when translated into our notation, says that $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda_i \geqslant 0 \text{ for all } 0 \leqslant i \leqslant n\}$ is a fundamental domain for the action of G on $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda \cdot \delta > 0\}$. Recalling that G preserves the affine hyperplane $V := \{\lambda \in \mathbb{R}^{n+1} \mid \lambda \cdot \delta = 1\}$, it follows that $V = \bigcup_{\rho \in G} \rho U$, where U is the n-simplex $\{\lambda \in \mathbb{R}^{n+1} \mid \lambda_i \geqslant 0 \text{ for all } 0 \leqslant i \leqslant n \text{ and } \lambda \cdot \delta = 1\}$. Let $H = \{\lambda \in V \mid \lambda_i > 0 \text{ for all } i \neq 0\}$, which is a convex subset of V containing open balls of arbitrarily large diameter. Since each ρU has the same finite diameter, there exists some $\rho \in G$ with $\rho U \subseteq H$. In particular, $\rho(\varepsilon_0) \in H$; that is, $\rho(\varepsilon_0)_i > 0$ for all $i \neq 0$. □

Remark 5.13. By playing Mozes' numbers game, one can often determine an explicit sequence of dual reflections ρ satisfying the hypotheses of Lemma 5.12. For example, if $\tilde{Q} = \tilde{\mathbb{A}}_4$, then the numbers game starting with the initial configuration $(-3, 1, 1, 1, 1)$ terminates at ε_0, and so by applying the corresponding dual reflections in reverse we obtain the desired ρ. More generally, [Gashi et al. 2012, Proposition 5.1] tells us that when $\tilde{Q}$ is of type $\tilde{\mathbb{A}}_{2m}$, $\tilde{\mathbb{D}}_{4m}$, $\tilde{\mathbb{D}}_{4m+1}$, $\tilde{\mathbb{E}}_6$ or $\tilde{\mathbb{E}}_8$, where m is a positive integer, then the numbers game starting with the initial configuration $\left(1 - \sum_{i=1}^{n} \delta_i, 1, 1, \ldots, 1\right)$ terminates at ε_0, and so this determines a sequence of dual reflections ρ such that $\rho(\lambda)_i > 0$ for all $i \neq 0$.

We are now in a position to prove Theorem 5.1.

Theorem 5.14. *Let $\tilde{Q}$ be an extended Dynkin quiver with $n+1$ vertices, and let $\lambda = \varepsilon_0$. Then $\mathcal{O}^\lambda$ has a NQCR of the form $\mathcal{O}^\mu$, where $\mathcal{O}^\mu$ has precisely n finite-dimensional simple modules S_i up to isomorphism. With a suitable indexing of the S_i, the intersection matrix Γ with entries $\Gamma_{ij} = S_i \bullet S_j$ is $-C$, where C is the Cartan matrix corresponding to Q.*

Proof. Lemma 5.12 tells us that there exists a sequence of dual reflections ρ such that $\mathcal{O}^\mu$ is nonsingular, where $\mu = \rho(\lambda)$. Since Π^λ is a resolution of $\mathcal{O}^\lambda$ and there are Morita equivalences between Π^λ, Π^μ, and $\mathcal{O}^\mu$ (by [Crawley-Boevey and Holland 1998, Corollary 5.2, Corollary 9.6]), it follows that $\mathcal{O}^\mu$ is a NQCR of $\mathcal{O}^\lambda$. Finally, these Morita equivalences combined with Theorem 5.8 tells us that $\mathcal{O}^\mu$ has precisely n finite-dimensional simple modules S_i up to isomorphism, and since Morita equivalences preserve dimensions of Hom and Ext groups, the claimed intersection multiplicities follow from Theorem 5.8 as well. □

Appendix: Uniqueness of the translation functor on objects of proj-$\Pi(Q)$ when Q is Dynkin

In this appendix, we show that if Q is Dynkin and proj-$\Pi(Q)$ has the structure of a (not necessarily algebraic) triangulated category, then the translation functor Σ is uniquely determined on objects of proj-$\Pi(Q)$. In particular, this tells us how the translation functor acts on objects in Theorem 4.13. Although this is known from that result, we believe that an elementary and relatively short proof of this fact may be of independent interest.

For the remainder of this section, write $P_1, \ldots, P_n$ for the n indecomposable projective right $\Pi(Q)$-modules corresponding to the vertices of Q. Write $W_0, \ldots, W_n$ for the $n+1$ irreducible representations of the finite group G corresponding to Q. Since proj-$\Pi(Q)$ is Krull–Schmidt, it is easy to see that $\Sigma P_i = P_j$ for some j, so write σ for the permutation of the vertices of Q satisfying $\Sigma P_i = P_{\sigma(i)}$. The map $W_i \to W_i^*$

sending a representation to its dual is an involution of $\{W_1, \ldots, W_n\}$ (where we intentionally omit W_0), and we can view this map as an automorphism ν of Q. Throughout this section, all Hom spaces are over $\Pi(Q)$, and we omit this subscript. The aim of this section is to prove the following result, which we achieve by analysing cases.

Theorem A.1. *Consider the category* proj-$\Pi(Q)$ *with some triangulated structure with translation functor* Σ. *Then* $\sigma = \nu$ *as automorphisms of* Q.

Remark A.2. Explicitly, ν is the identity automorphism of Q when Q is $\mathbb{A}_1$, $\mathbb{D}_n$ (n even), $\mathbb{E}_7$, or $\mathbb{E}_8$, and it is the unique graph automorphism of order 2 when Q is $\mathbb{A}_n$ ($n \geqslant 2$), $\mathbb{D}_n$ (n odd), or $\mathbb{E}_6$.

Adapting the proof of Lemma 4.4, σ is necessarily a graph automorphism of Q. Therefore, since the automorphism group of an $\mathbb{A}_1$, $\mathbb{E}_7$, or $\mathbb{E}_8$ graph is trivial, it immediately follows that σ is the identity in these cases. For the remaining cases, we argue that Σ is uniquely determined using Lemma 4.4 and by considering the dimensions of the Hom spaces between the P_i. We record the dimensions of these Hom spaces in the following lemma; the $\mathbb{E}_7$ and $\mathbb{E}_8$ cases are unnecessary and hence omitted, but they can be established in the same way.

Lemma A.3. *Let Q be a Dynkin quiver with n vertices and let $P_1, \ldots, P_n$ be the n indecomposable projective right $\Pi(Q)$-modules corresponding to the vertices of Q. Let $H(Q)$ be the matrix with*

$$H(Q)_{ij} = \dim_{\Bbbk} \operatorname{Hom}(P_j, P_i) = \dim_{\Bbbk} e_i \Pi(Q) e_j.$$

(1) *If $Q = \mathbb{A}_n$ then*

$$H(\mathbb{A}_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 2 & 3 & \cdots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & 3 & 2 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}.$$

(2) *If $Q = \mathbb{D}_n$ then*

$$H(\mathbb{D}_n) = \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(n-2) & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & \lceil \frac{n-1}{2} \rceil & \lfloor \frac{n-1}{2} \rfloor \\ 1 & 2 & 3 & \cdots & n-2 & \lfloor \frac{n-1}{2} \rfloor & \lceil \frac{n-1}{2} \rceil \end{pmatrix}.$$

(3) *If $Q = \mathbb{E}_6$ then*

$$H(\mathbb{E}_6) = \begin{pmatrix} 4 & 2 & 4 & 6 & 4 & 2 \\ 2 & 2 & 3 & 4 & 3 & 2 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 6 & 4 & 8 & 12 & 8 & 4 \\ 4 & 3 & 6 & 8 & 6 & 3 \\ 2 & 2 & 3 & 4 & 3 & 2 \end{pmatrix}.$$

Proof. These can be calculated using [Erdmann and Snashall 1998a, Section 4; 1998b, 3.4; Malkin et al. 2006, Theorem 2.3.b]. □

We now begin our case-by-case argument. In each case, the technique is the same: seeking a contradiction, we show that if σ is a graph automorphism of Q different from the one given in Theorem A.1 then we arrive at a contradiction. We begin with the type $\mathbb{A}$ case.

Proposition A.4. *Let σ be the graph automorphism of $\mathbb{A}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{A}_n)$. *Then σ is the identity when $n = 1$, and it is the unique order 2 graph automorphism when $n \geqslant 2$.*

Proof. We have already established the $n = 1$ case, so suppose $n \geqslant 2$. By Lemma 4.4, σ is either the identity or has order 2 so, seeking a contradiction, suppose that σ is the identity; that is $\Sigma P_i = P_i$ for all i. Consider the nonzero morphism $P_1 \to P_n$ given by left multiplication by $\bar{\alpha}_{n-1}\bar{\alpha}_{n-2}\cdots\bar{\alpha}_1$, which gives rise to a distinguished triangle

$$P_1 \to P_n \to M \to P_1$$

for some $M \in$ proj-$\Pi(\mathbb{A}_n)$. Applying $\operatorname{Hom}(-, P_n)$, this gives rise to an exact sequence

$$\begin{array}{ccccccccc} \operatorname{Hom}(P_n, P_n) & \xrightarrow{\cdot\bar{\alpha}_{n-1}\bar{\alpha}_{n-2}\cdots\bar{\alpha}_1} & \operatorname{Hom}(P_1, P_n) & \xrightarrow{\beta} & \operatorname{Hom}(M, P_n) & \xrightarrow{\gamma} & \operatorname{Hom}(P_n, P_n) & \xrightarrow{\cdot\bar{\alpha}_{n-1}\bar{\alpha}_{n-2}\cdots\bar{\alpha}_1} & \operatorname{Hom}(P_1, P_n) \\ \| & & \| & & & & \| & & \| \\ \Bbbk e_n & & \Bbbk\bar{\alpha}_{n-1}\bar{\alpha}_{n-2}\cdots\bar{\alpha}_1 & & & & \Bbbk e_n & & \Bbbk\bar{\alpha}_{n-1}\bar{\alpha}_{n-2}\cdots\bar{\alpha}_1 \end{array}$$

where we use Lemma A.3 to write down bases for each of the Hom spaces. Now the left-hand map is surjective, so exactness implies that β is the zero map, which forces γ to be injective. Moreover, the right-hand map is injective, so that γ is the zero map. In particular, Lemma A.3 implies that we have $\operatorname{Hom}(M, P_n) = 0$ and so $M = 0$, but this tells us that $P_1 \cong P_n$ which is absurd. Therefore σ must be the unique order 2 graph automorphism of $\mathbb{A}_n$. □

We now turn our attention to the type $\mathbb{E}$ cases.

Proposition A.5. *Let σ be the graph automorphism of $\mathbb{E}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{E}_n)$, *where $n \in \{6, 7, 8\}$. Then σ is the identity when $n \neq 6$, and it is the unique order 2 graph automorphism when $n = 6$.*

Proof. Again, the $\mathbb{E}_7$ and $\mathbb{E}_8$ cases are immediate from Lemma 4.4, so consider $\mathbb{E}_6$. By Lemma 4.4, σ is either the identity or has order 2 so, seeking a contradiction, suppose that σ is the identity. Consider the nonzero morphism $P_2 \to P_6$ given by left multiplication by $\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2$, which gives rise to a distinguished triangle

$$P_2 \to P_6 \to M \to P_2$$

for some $M \in \text{proj-}\Pi(\mathbb{E}_6)$. Applying $\mathrm{Hom}(-, P_6)$, this gives rise to an exact sequence

$$\begin{array}{ccccccccc}
\mathrm{Hom}(P_6, P_6) & \xrightarrow{\cdot\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2} & \mathrm{Hom}(P_2, P_6) & \xrightarrow{\beta} & \mathrm{Hom}(M, P_6) & \xrightarrow{\gamma} & \mathrm{Hom}(P_6, P_6) & \xrightarrow{\cdot\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2} & \mathrm{Hom}(P_2, P_6) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_6 & & \Bbbk\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2 & & & & \Bbbk e_6 & & \Bbbk\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2 \\
\oplus & & \oplus & & & & \oplus & & \oplus \\
\Bbbk\alpha_5\bar{\alpha}_4\alpha_1\bar{\alpha}_1\alpha_4\bar{\alpha}_5 & & \Bbbk\alpha_5\bar{\alpha}_4\alpha_1\bar{\alpha}_1\alpha_4\bar{\alpha}_5\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2 & & & & \Bbbk\alpha_5\bar{\alpha}_4\alpha_1\bar{\alpha}_1\alpha_4\bar{\alpha}_5 & & \Bbbk\alpha_5\bar{\alpha}_4\alpha_1\bar{\alpha}_1\alpha_4\bar{\alpha}_5\alpha_5\bar{\alpha}_4\alpha_3\bar{\alpha}_2
\end{array}$$

where again we use Lemma A.3 to write down bases for each of the Hom spaces. We see that the left-hand map is surjective and so β is the zero map, and exactness implies that γ is injective. Since the right-hand map is injective it follows that γ is the zero map. Therefore by Lemma A.3 $\mathrm{Hom}(M, P_6) = 0$ and so $M = 0$, but this tells us that $P_2 \cong P_6$ which is absurd. Therefore σ must be the unique order 2 graph automorphism of $\mathbb{E}_6$. □

Finally we consider the type $\mathbb{D}$ cases. Since we claim that Σ behaves differently depending on whether n is odd or even, we have to consider these two cases separately; additionally, we consider the $n = 4$ case separately since $\mathrm{Aut}(\mathbb{D}_4) \cong S_3$ instead of it having order 2.

Proposition A.6. *Let σ be the graph automorphism of $\mathbb{D}_4$ induced by the translation functor Σ on* $\text{proj-}\Pi(\mathbb{D}_4)$. *Then σ is the identity.*

Proof. By Lemma 4.4, σ is either the identity, a two-cycle which swaps a pair of vertices $i \neq 2 \neq j$, or it cycles the vertices $1, 3, 4$. We rule out the latter two possibilities.

First suppose that σ is a two-cycle: without loss of generality, $\sigma = (3\,4)$. Consider the nonzero morphism $P_3 \to P_4$ given by left multiplication by $\bar{\alpha}_4\alpha_3$, which gives rise to a distinguished triangle

$$P_3 \to P_4 \to M \to P_4$$

for some $M \in \text{proj-}\Pi(\mathbb{D}_4)$. Applying $\mathrm{Hom}(-, P_3)$, this gives rise to an exact sequence

$$\begin{array}{ccccccccc}
\mathrm{Hom}(P_3, P_3) & \xrightarrow{\cdot\bar{\alpha}_3\alpha_4} & \mathrm{Hom}(P_4, P_3) & \xrightarrow{\beta} & \mathrm{Hom}(M, P_3) & \xrightarrow{\gamma} & \mathrm{Hom}(P_4, P_3) & \xrightarrow{\cdot\bar{\alpha}_4\alpha_3} & \mathrm{Hom}(P_3, P_3) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_3 \oplus \Bbbk\bar{\alpha}_3\alpha_4\bar{\alpha}_4\alpha_3 & & \Bbbk\bar{\alpha}_3\alpha_4 & & & & \Bbbk\bar{\alpha}_3\alpha_4 & & \Bbbk e_3 \oplus \Bbbk\bar{\alpha}_3\alpha_4\bar{\alpha}_4\alpha_3
\end{array}$$

Clearly the left-hand map surjects, so exactness forces β to be the zero map, which in turn implies that γ is injective. The right-hand map is injective, and exactness forces γ to be the zero map. In particular we have $\mathrm{Hom}(M, P_3) = 0$ and so $M = 0$, but this tells us that $P_3 \cong P_4$ which is absurd. Therefore σ is not a two-cycle.

Now suppose that σ is a three-cycle: without loss of generality, $\sigma = (1\,3\,4)$. We now consider the triangle obtained from the morphism $\alpha_1\alpha_3\cdot : P_3 \to P_1$,

$$P_3 \to P_1 \to M \to P_4$$

and seek to obtain contradiction. Applying the functor $\mathrm{Hom}(-, P_3)$, we get exactness of the following sequence:

$$\begin{array}{ccccccccc} \mathrm{Hom}(P_3, P_3) & \xrightarrow{\cdot\bar{\alpha}_3\alpha_4} & \mathrm{Hom}(P_4, P_3) & \xrightarrow{\beta} & \mathrm{Hom}(M, P_3) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1, P_3) & \xrightarrow{\cdot\alpha_1\alpha_3} & \mathrm{Hom}(P_3, P_3) \\ \| & & \| & & & & \| & & \| \\ \Bbbk e_3 \oplus \Bbbk\bar{\alpha}_3\bar{\alpha}_1\alpha_1\alpha_3 & & \Bbbk\bar{\alpha}_3\alpha_4 & & & & \Bbbk\bar{\alpha}_3\bar{\alpha}_1 & & \Bbbk e_3 \oplus \Bbbk\bar{\alpha}_3\bar{\alpha}_1\alpha_1\alpha_3 \end{array}$$

Again the left-hand map is surjective, forcing β to be the zero map and hence γ to be injective. Moreover, the right-hand map is injective, and so γ must be the zero map. In particular we have $\mathrm{Hom}(M, P_3) = 0$ and so $M = 0$, but this tells us that $P_1 \cong P_3$ which is absurd. Therefore σ is not a three-cycle, and hence must be the identity. □

Proposition A.7. *Let $n \geqslant 5$ be odd and let σ be the graph automorphism of $\mathbb{D}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{D}_n)$. *Then σ is the unique graph automorphism of order* 2.

Proof. By Lemma 4.4, σ is either the identity or $(n{-}1\ n)$ so, seeking a contradiction, assume it is the former; that is, $\Sigma P_i = P_i$ for all i. Consider the morphism $P_n \to P_1$ given by left multiplication by $\alpha_1\alpha_2\alpha_3\cdots\alpha_{n-3}\alpha_n$. This gives rise to a distinguished triangle

$$P_n \to P_1 \to M \to P_n$$

for some $M \in$ proj-$\Pi(\mathbb{D}_n)$. Applying $\mathrm{Hom}(-, P_1)$ gives rise to the following exact sequence,

$$\begin{array}{ccccccccc} \mathrm{Hom}(P_1, P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n, P_1) & \xrightarrow{\beta} & \mathrm{Hom}(M, P_1) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1, P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n, P_1) \\ \| & & \| & & & & \| & & \| \\ \Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n & & & & \Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n \end{array}$$

where here p is some path. The left-hand map is surjective, so β is the zero map and therefore γ is injective. The kernel of the right-hand map is one-dimensional, and exactness tells us that γ has rank 1. In particular we have $\dim \mathrm{Hom}(M, P_1) = 1$ and so M is either P_{n-1} or P_n by Lemma A.3. If we instead apply $\mathrm{Hom}(-, P_n)$, the resulting exact sequence is

$$\begin{array}{ccccccccc} \mathrm{Hom}(P_1, P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n, P_n) & \xrightarrow{\beta} & \mathrm{Hom}(M, P_n) & \xrightarrow{\gamma} & \mathrm{Hom}(P_1, P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n} & \mathrm{Hom}(P_n, P_n) \\ \| & & \| & & & & \| & & \| \\ \Bbbk\bar{\alpha}_n\bar{\alpha}_{n-3}\cdots\bar{\alpha}_2\bar{\alpha}_1 & & \Bbbk^{(n-1)/2} & & & & \Bbbk\bar{\alpha}_n\bar{\alpha}_{n-3}\cdots\bar{\alpha}_2\bar{\alpha}_1 & & \Bbbk^{(n-1)/2} \end{array}$$

Since n is odd, the shortest path from vertex n to vertex 1 and back to vertex n is zero in $\Pi(\mathbb{D}_n)$, so the first and the last maps both have rank zero. Therefore β has full rank, forcing the kernel of γ to have dimension $(n-1)/2$. Exactness also forces γ to have rank 1, and therefore $\dim \operatorname{Hom}(M, P_n) = (n-1)/2+1$. Now we have already seen that M is either P_{n-1} or P_n, but $\dim \operatorname{Hom}(P_{n-1}, P_n) = (n-1)/2 = \dim \operatorname{Hom}(P_n, P_n)$, so we have a contradiction. Therefore σ is the unique graph automorphism of order 2. $\square$

Proposition A.8. *Let $n \geqslant 6$ be even and let σ be the graph automorphism of $\mathbb{D}_n$ induced by the translation functor Σ on* proj-$\Pi(\mathbb{D}_n)$. *Then σ is the identity.*

Proof. By Lemma 4.4, σ is either the identity or $(n-1\ n)$ so, seeking a contradiction, assume it is the latter. Consider the morphism $P_n \to P_1$ given by left multiplication by $\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n$, and extend this to a distinguished triangle

$$P_n \to P_1 \to M \to P_{n-1}$$

for some $M \in$ proj-$\Pi(\mathbb{D}_n)$. If we apply $\operatorname{Hom}(-, P_1)$ we get the following exact sequence

$$\begin{array}{ccccccccc}
\operatorname{Hom}(P_1, P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_{n-1}} & \operatorname{Hom}(P_{n-1}, P_1) & \xrightarrow{\beta} & \operatorname{Hom}(M, P_1) & \xrightarrow{\gamma} & \operatorname{Hom}(P_1, P_1) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n} & \operatorname{Hom}(P_n, P_1) \\
\| & & \| & & & & \| & & \| \\
\Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_{n-1} & & & & \Bbbk e_1 \oplus \Bbbk p & & \Bbbk\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n
\end{array}$$

where p is some path. The left-hand map surjects, so $\beta = 0$ and therefore γ injects. The right-hand map has a one-dimensional kernel, so γ has rank 1. It follows that $\dim \operatorname{Hom}(M, P_1) = 1$, which implies that M is either P_{n-1} or P_n. If we instead apply $\operatorname{Hom}(-, P_n)$ we get

$$\begin{array}{ccccccccc}
\operatorname{Hom}(P_1, P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_{n-1}} & \operatorname{Hom}(P_{n-1}, P_n) & \xrightarrow{\theta} & \operatorname{Hom}(M, P_n) & \xrightarrow{\eta} & \operatorname{Hom}(P_1, P_n) & \xrightarrow{\cdot\alpha_1\alpha_2\cdots\alpha_{n-3}\alpha_n} & \operatorname{Hom}(P_n, P_n) \\
\| & & \| & & & & \| & & \| \\
\Bbbk\bar{\alpha}_n\bar{\alpha}_{n-3}\cdots\bar{\alpha}_2\bar{\alpha}_1 & & \Bbbk^{n/2-1} & & & & \Bbbk\bar{\alpha}_n\bar{\alpha}_{n-3}\cdots\bar{\alpha}_2\bar{\alpha}_1 & & \Bbbk^{n/2}
\end{array}$$

Since n is even, the shortest path from vertex n to vertex 1 and then to vertex $n-1$ is zero in $\Pi(\mathbb{D}_n)$, while the shortest path from vertex n to vertex 1 and back to vertex n is nonzero. It follows that the left-hand map is the zero map, while the right-hand map has rank 1. Therefore θ has full rank which implies that the kernel of η has dimension $n/2-1$. Moreover, the right-hand map is injective, so that η has rank 0 and so $\dim \operatorname{Hom}(M, P_n) = n/2-1$. Combining this with our earlier restriction on M, this forces $M = P_{n-1}$, and our distinguished triangle is therefore

$$P_n \to P_1 \to P_{n-1} \to P_{n-1}.$$

Since $\operatorname{Hom}(P_1, P_{n-1})$ is spanned by $\bar{\alpha}_{n-1}\bar{\alpha}_{n-3}\cdots\bar{\alpha}_2\bar{\alpha}_1$ and γ is not the zero map, we can assume that the map $P_1 \to P_{n-1}$ in this triangle is given by left multiplication by (a scalar multiple of) $\bar{\alpha}_{n-1}\bar{\alpha}_{n-3}\cdots\bar{\alpha}_2\bar{\alpha}_1$. Moreover, $\operatorname{Hom}(P_{n-1}, P_{n-1}) = \operatorname{span}\{e_{n-1}, p_2, \dots, p_{n/2}\}$ where the p_i are paths of length $\geqslant 4$. Since θ is not the zero map and the composition $P_1 \to P_{n-1} \to P_{n-1}$ must be zero, the map $P_{n-1} \to P_{n-1}$ in this triangle lies in $\operatorname{span}\{p_2, \dots, p_{n/2}\}$. But then $\theta : \operatorname{Hom}(P_{n-1}, P_n) \to \operatorname{Hom}(P_{n-1}, P_n)$ maps the longest path in $\mathrm{Hom}(P_{n-1}, P_n)$ to zero, contradicting the fact that θ has trivial kernel. It follows that σ is not a two-cycle. □

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References

[Amiot et al. 2015] C. Amiot, O. Iyama, and I. Reiten, "Stable categories of Cohen–Macaulay modules and cluster categories", *Amer. J. Math.* **137**:3 (2015), 813–857. MR Zbl

[Arentz-Hansen 2019] E. Arentz-Hansen, "Classifying subcategories in quotients of exact categories", *Appl. Categ. Structures* **27**:5 (2019), 535–547. MR Zbl

[Auslander and Bridger 1969] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969. MR Zbl

[Bavula 1992] V. V. Bavula, "Generalized Weyl algebras and their representations", *Algebra i Analiz* **4**:1 (1992), 75–97. In Russian; translated in *St. Petersburg Math. J.* **4**:1 (1993), 71–92. MR Zbl

[Białkowski et al. 2007] J. Białkowski, K. Erdmann, and A. Skowroński, "Deformed preprojective algebras of generalized Dynkin type", *Trans. Amer. Math. Soc.* **359**:6 (2007), 2625–2650. MR Zbl

[Boddington and Levy 2007] P. Boddington and P. Levy, "Deformations of Kleinian singularities of type E_6, E_7 and E_8", unpublished paper, 2007.

[Brenner et al. 2002] S. Brenner, M. C. R. Butler, and A. D. King, "Periodic algebras which are almost Koszul", *Algebr. Represent. Theory* **5**:4 (2002), 331–367. MR Zbl

[Buchweitz 1986] R.-O. Buchweitz, *Maximal Cohen–Macaluay modules and Tate-cohomology over Gorenstein rings*, Univ. Hannover, 1986.

[Chen 2011] X.-W. Chen, "The singularity category of an algebra with radical square zero", *Doc. Math.* **16** (2011), 921–936. MR Zbl

[Chen 2012] X.-W. Chen, "Three results on Frobenius categories", *Math. Z.* **270**:1-2 (2012), 43–58. MR Zbl

[Chen 2018] X.-W. Chen, "The singularity category of a quadratic monomial algebra", *Q. J. Math.* **69**:3 (2018), 1015–1033. MR Zbl

[Crawford 2016] S. Crawford, "Singularity categories of deformations of Kleinian singularities", 2016. To appear in *Algebra Number Theory*. arXiv

[Crawley-Boevey 1999] W. Crawley-Boevey, "DMV lectures on representations of quivers, preprojective algebras and deformations of quotient singularities", lecture notes, 1999, Available at https://www.math.uni-bielefeld.de/~wcrawley/dmvlecs.pdf.

[Crawley-Boevey and Holland 1998] W. Crawley-Boevey and M. P. Holland, "Noncommutative deformations of Kleinian singularities", *Duke Math. J.* **92**:3 (1998), 605–635. MR Zbl

[Eisenbud 1980] D. Eisenbud, "Homological algebra on a complete intersection, with an application to group representations", *Trans. Amer. Math. Soc.* **260**:1 (1980), 35–64. MR Zbl

[Erdmann and Snashall 1998a] K. Erdmann and N. Snashall, "On Hochschild cohomology of preprojective algebras, I", *J. Algebra* **205**:2 (1998), 391–412. MR Zbl

[Erdmann and Snashall 1998b] K. Erdmann and N. Snashall, "Preprojective algebras of Dynkin type, periodicity and the second Hochschild cohomology", pp. 183–193 in *Algebras and modules, II* (Geiranger, Norway, 1996), edited by I. Reiten et al., CMS Conf. Proc. **24**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Gashi et al. 2012] Q. R. Gashi, T. Schedler, and D. E. Speyer, "Looping of the numbers game and the alcoved hypercube", *J. Combin. Theory Ser. A* **119**:3 (2012), 713–730. MR Zbl

[Gelfand and Ponomarev 1979] I. M. Gelfand and V. A. Ponomarev, "Model algebras and representations of graphs", *Funkt. Anal. i Prilozhen.* **13**:3 (1979), 1–12. In Russian; translated in *Funct. Anal. Appl.* **13**:3 (1979) 157–166. MR

[Happel 1988] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, Lond. Math. Soc. Lect. Note Ser. **119**, Cambridge Univ. Press, 1988. MR Zbl

[Hodges 1993] T. J. Hodges, "Noncommutative deformations of type-A Kleinian singularities", *J. Algebra* **161**:2 (1993), 271–290. MR Zbl

[Iyama and Wemyss 2010] O. Iyama and M. Wemyss, "The classification of special Cohen–Macaulay modules", *Math. Z.* **265**:1 (2010), 41–83. MR Zbl

[Iyama and Wemyss 2014] O. Iyama and M. Wemyss, "Singular derived categories of $\mathbb{Q}$-factorial terminalizations and maximal modification algebras", *Adv. Math.* **261** (2014), 85–121. MR Zbl

[Kalck 2015] M. Kalck, "Singularity categories of gentle algebras", *Bull. Lond. Math. Soc.* **47**:1 (2015), 65–74. MR Zbl

[Kalck and Yang 2018] M. Kalck and D. Yang, "Relative singularity categories, II: DG models", preprint, 2018. arXiv

[Keller 2018] B. Keller, "A remark on a theorem by Claire Amiot", *C. R. Math. Acad. Sci. Paris* **356**:10 (2018), 984–986. MR Zbl

[Krause 2015] H. Krause, "Krull–Schmidt categories and projective covers", *Expo. Math.* **33**:4 (2015), 535–549. MR Zbl

[Lam 1999] T. Y. Lam, *Lectures on modules and rings*, Grad. Texts Math. **189**, Springer, 1999. MR Zbl

[Leuschke and Wiegand 2012] G. J. Leuschke and R. Wiegand, *Cohen–Macaulay representations*, Math. Surv. Monogr. **181**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Malkin et al. 2006] A. Malkin, V. Ostrik, and M. Vybornov, "Quiver varieties and Lusztig's algebra", *Adv. Math.* **203**:2 (2006), 514–536. MR Zbl

[McConnell and Robson 2001] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, revised ed., Grad. Stud. Math. **30**, Amer. Math. Soc., Providence, RI, 2001. MR Zbl

[Montgomery and Small 1981] S. Montgomery and L. W. Small, "Fixed rings of Noetherian rings", *Bull. Lond. Math. Soc.* **13**:1 (1981), 33–38. MR Zbl

[Mori and Smith 2001] I. Mori and S. P. Smith, "Bézout's theorem for non-commutative projective spaces", *J. Pure Appl. Algebra* **157**:2-3 (2001), 279–299. MR Zbl

[Mozes 1990] S. Mozes, "Reflection processes on graphs and Weyl groups", *J. Combin. Theory Ser. A* **53**:1 (1990), 128–142. MR Zbl

[Qin et al. 2019] X.-S. Qin, Y.-H. Wang, and J. J. Zhang, "Noncommutative quasi-resolutions", *J. Algebra* **536** (2019), 102–148. MR Zbl

[Rotman 1979] J. J. Rotman, *An introduction to homological algebra*, Pure Appl. Math. **85**, Academic, New York, 1979. MR Zbl

[Simon 2018] C. Simon, *Singularities of noncommutative surfaces*, Ph.D. thesis, University of Edinburgh, 2018, Available at http://hdl.handle.net/1842/31543.

[Yekutieli and Zhang 2002] A. Yekutieli and J. J. Zhang, "Dualizing complexes and tilting complexes over simple rings", *J. Algebra* **256**:2 (2002), 556–567. MR Zbl

[Zaks 1969] A. Zaks, "Injective dimension of semi-primary rings", *J. Algebra* **13** (1969), 73–86. MR Zbl

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Iwasawa main conjecture for Rankin–Selberg p-adic L-functions

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In this paper we prove that the p-adic L-function that interpolates the Rankin–Selberg product of a general modular form and a CM form of higher weight divides the characteristic ideal of the corresponding Selmer group. This is one divisibility of the Iwasawa main conjecture for this p-adic L-function. We prove this conjecture using congruences between Klingen–Eisenstein series and cusp forms on the group GU(3, 1), following the strategy of recent work by C. Skinner and E. Urban. The actual argument is, however, more complicated due to the need to work with general Fourier–Jacobi expansions. This theorem is used to deduce a converse of the Gross–Zagier–Kolyvagin theorem and the p-adic part of the precise BSD formula in the rank one case.

1. Introduction

Let p be an odd prime. An important problem in number theory is studying the relations between special values of L-functions and arithmetic objects in p-adic families. The first case was studied by Iwasawa in the 1950's for class groups of number fields, resulting in the asymptotic formula for class numbers in cyclotomic towers of field extensions. Later on, Mazur realized that the idea of Iwasawa's theory can be applied to elliptic curves which provides a powerful way to study their arithmetic (e.g., the BSD conjecture). Such an idea was further generalized to many kinds of Galois representations. Kato [2004] formulated the Iwasawa main conjecture for modular forms on $\mathrm{GL}_2/\mathbb{Q}$ and proved one divisibility by constructing

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Keywords: Iwasawa main conjecture, Rankin–Selberg.

an Euler system. Later on, Skinner and Urban [2014] proved the other side of divisibility in the case when the modular form is ordinary at p, using Eisenstein congruences on the unitary group $\mathrm{U}(2,2)$.

Our paper can be viewed as an extension of Skinner and Urban's approach to the Rankin–Selberg product of a modular form f and an ordinary CM form whose weight is higher than f, using another rank 4 unitary group $\mathrm{U}(3,1)$. Surprisingly, such a result has lots of arithmetic applications which cannot be seen by previous techniques, including the proof by Skinner [2014] of the converse of a theorem of Gross, Zagier and Kolyvagin, and the p-part of the precise BSD formula in the analytic rank one case [Jetchev et al. 2017]. It is also the starting point and a key ingredient of the author's work proving the Iwasawa main conjecture for supersingular elliptic curves [Wan 2014b]. Now we describe the context of our main results.

Let $\mathcal{K}\subset\mathbb{C}$ be an imaginary quadratic field such that p splits in $\mathcal{K}$ as $(p)=v_0\bar{v}_0$. We fix an isomorphism $\iota:\mathbb{C}_p\simeq\mathbb{C}$ and suppose v_0 is determined by ι. There is a unique $\mathbb{Z}_p^2$-extension $\mathcal{K}_\infty/\mathcal{K}$ unramified outside p. Let $\Gamma_{\mathcal{K}}:=\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$. Suppose $\boldsymbol{f}$ is a Hida family of ordinary cuspidal eigenforms new outside p with coefficient ring $\mathbb{I}$, a normal finite extension of the power series ring $\mathbb{Z}_p[\![W]\!]$ of one variable W. Let L be a finite extension of $\mathbb{Q}_p$ with integer ring $\mathcal{O}_L$. Suppose ξ is an L-valued Hecke character of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ whose infinity type is $\left(\frac{\kappa}{2},-\frac{\kappa}{2}\right)$ for some even integer $\kappa\geq 6$ and such that $\mathrm{ord}_{v_0}(\mathrm{cond}(\xi_{v_0}))\leq 1$ and $\mathrm{ord}_{\bar{v}_0}(\mathrm{cond}(\xi_{\bar{v}_0}))\leq 1$. Denote by $\boldsymbol{\xi}$ the $\mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!]$-adic family of Hecke characters containing ξ as some specialization (we make this precise in Section 7B). We write $\hat{\mathcal{O}}_L^{\mathrm{ur}}$ for the completion of the maximal unramified extension of $\mathcal{O}_L$ and $\hat{\mathbb{I}}^{\mathrm{ur}}$ for the normalization of the ring corresponding to an irreducible component of $\mathbb{I}\hat{\otimes}_{\mathcal{O}_L}\hat{\mathcal{O}}_L^{\mathrm{ur}}$.

In Section 2B, we associate with $\boldsymbol{f}$, $\mathcal{K}$ and $\boldsymbol{\xi}$ a dual Selmer group $X_{\boldsymbol{f},\mathcal{K},\boldsymbol{\xi}}$, which is a finite module over the ring $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$. On the analytic side, for a finite set of primes Σ containing all bad primes, we construct using a doubling method the "Σ-primitive" p-adic L-functions $\mathcal{L}^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}}\in\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$, $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}\in\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$, interpolating the algebraic parts of the special L-values $L_{\mathcal{K}}\left(f_\phi,\xi_\phi,\frac{\kappa}{2}\right)$, where f_ϕ and ξ_ϕ are specializations of the families $\boldsymbol{f}$ and $\boldsymbol{\xi}$ (f_ϕ has weight 2 and ξ_ϕ has infinity type $(\kappa/2,-\kappa/2)$). The general case when Σ does not necessarily contain all bad primes, is obtained by putting back the local Euler factors at primes omitted. We let $\mathcal{L}_{f_0,\xi,\mathcal{K}}$ be the specialization of $\mathcal{L}_{\boldsymbol{f},\xi,\mathcal{K}}$ to a single form f_0 of weight 2 and trivial character in the family $\boldsymbol{f}$, which we assume is defined over L. In Section 7E we also recall closely related p-adic L-functions $\mathcal{L}^{\Sigma,\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}$ and $\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}$ constructed by Hida. We also associate with $f=f_0$, $\mathcal{K}$, ξ a dual Selmer group $X_{f,\mathcal{K},\xi}$ over $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. The Iwasawa–Greenberg main conjecture says that the characteristic ideal of $X_{\boldsymbol{f},\mathcal{K},\xi}$ (resp. $X_{f,\mathcal{K},\xi}$) is generated by $\mathcal{L}_{\boldsymbol{f},\xi,\mathcal{K}}$ (resp. $\mathcal{L}_{f,\xi,\mathcal{K}}$).

Let $\bar{\mathbb{Q}}\subset\mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ and let $G_{\mathbb{Q}}=\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the Galois group. Let $G_p\subset G_{\mathbb{Q}}$ be the decomposition group determined by the inclusion $\bar{\mathbb{Q}}\subset\bar{\mathbb{Q}}_p$ coming from ι. We write ϵ for the cyclotomic character and ω for the Techimüller character of $G_{\mathbb{Q}}$. Let g be a cuspidal eigenform on $\mathrm{GL}_2/\mathbb{Q}$ with the associated p-adic Galois representation $\rho_g:G_{\mathbb{Q}}\to\mathrm{GL}_2(\mathcal{O}_L)$. We say g satisfies (**irred**) if

> the residual representation $\bar{\rho}_g$ is absolutely irreducible.

If g is nearly ordinary at p, then $\rho_g|_{G_p}$ is equivalent to an upper triangular representation and we say it satisfies (**dist**) if the characters of $\rho_g|_{G_p}$ on the diagonal are distinct modulo the maximal ideal of $\mathcal{O}_L$.

We will see later (in Section 7E) that if the CM form g_ξ associated to ξ satisfies (**irred**) and (**dist**) then $\mathcal{L}_{\boldsymbol{f},\xi,\mathcal{K}} \in \mathbb{I}[\![\Gamma_\mathcal{K}]\!]$.

In this paper, under certain conditions on $\boldsymbol{f}, \xi, \mathcal{K}$, we prove one inclusion (or divisibility) of the Iwasawa–Greenberg main conjecture for $\mathcal{L}_{\boldsymbol{f},\xi,\mathcal{K}}$. Our first theorem is a three-variable result for Hida families.

Theorem 1.1. *Let $\boldsymbol{f}$ be a Hida family of ordinary eigenforms that are new outside p of square-free tame level N, and suppose $\boldsymbol{f}$ has a weight two specialization f that has trivial nebentypus and is the ordinary stabilization of a new form of level N. Let $\bar{\rho}$ be the $\bmod p$ residual $G_\mathbb{Q}$-representation associated with the Hida family $\boldsymbol{f}$. Let ξ be a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times$ with infinity type $\left(\frac{\kappa}{2}, -\frac{\kappa}{2}\right)$ for some $\kappa \geq 6$. If*:

(a) $p \geq 5$.

(b) $\xi|_{\mathbb{A}_\mathbb{Q}^\times} = \omega \circ \mathrm{Nm}$ *and* $\kappa \equiv 0 \pmod{2(p-1)}$.

(c) $\bar{\rho}_{\boldsymbol{f}}|_{G_\mathcal{K}}$ *is irreducible.*

(d) *There exists $q \mid N$ that does not split in $\mathcal{K}$ and such that $\bar{\rho}_{\boldsymbol{f}}$ is ramified at q.*

(e) *The CM eigenform g_ξ associated to the character ξ satisfies* (**dist**) *and* (**irred**).

(f) *For each nonsplit prime v of $\mathbb{Q}$ we have that*

$$\epsilon\left(\pi_{f,v}, \xi_v, \tfrac{1}{2}\right) = 1.$$

(*As in [Hsieh 2014c] $\epsilon\left(\pi_{f,v}, \xi_v, \frac{1}{2}\right)$ is the local root number for the base change of $\pi_{f,v}$ to $\mathcal{K}_v$ twisted by ξ_v. It differs from the local root number for the Rankin–Selberg product of $\pi_{f,v}$ and $g_{\xi,v}$ by a factor $\chi_{\mathcal{K}/\mathbb{Q},v}(-1)$.*)

(g) *Suppose the conductor of ξ is only divisible by primes split in $\mathcal{K}/\mathbb{Q}$.*

Then $\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}} \in \hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_\mathcal{K}]\!]$ and $(\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}) \supseteq \mathrm{char}_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_\mathcal{K}]\!]}(X_{\boldsymbol{f},\mathcal{K},\xi})$ as ideals of $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_\mathcal{K}]\!]$. Here char *means the characteristic ideal.*

We also have a two variable theorem for a single form.

Theorem 1.2. *Let N, $f = f_0$, κ and ξ be as before. If*

(a) $p \geq 5$;

(b) *the p-adic avatar of $\xi|\cdot|^{\kappa/2}(\omega^{-1} \circ \mathrm{Nm})$ factors through $\Gamma_\mathcal{K}$ and* $\kappa \equiv 0 \pmod{2(p-1)}$;

(c) $\bar{\rho}_f|_{G_\mathcal{K}}$ *is irreducible*;

(d) *there exists $q \parallel N$ that does not split in $\mathcal{K}$.*

Then

$$(\mathcal{L}_{f,\xi,\mathcal{K}}) \supseteq \mathrm{char}_{\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_\mathcal{K}]\!]\otimes_{\mathcal{O}_L} L}(X_{f,\mathcal{K},\xi})$$

is true as fractional ideals of $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_\mathcal{K}]\!] \otimes_{\mathcal{O}_L} L$.

Unlike the previous theorem, in this theorem we allow both global root numbers $+1$ and -1 cases. In particular Theorem 1.2 is not deduced as a consequence of Theorem 1.1. Both theorems are deduced in the proof at the end of this paper. Theorem 1.2 is proved as the specialization of a "weaker version" (since the assumption is weaker than that of Theorem 1.1) of the 3-variable main conjecture, where we inverted all nonzero elements of $\mathbb{I}$. This is where the $\otimes L$ comes in. See the end of the paper for details.

Hida's p-adic L-functions $\mathcal{L}^{\mathrm{Hida}}_{f,\xi,\mathcal{K}}$ are more canonical than the $\mathcal{L}_{f,\xi,\mathcal{K}}$ in that there is a constant in $\overline{\mathbb{Q}}_p^\times$ showing up in our interpolation formula (see Proposition 7.7) that depends on some choices. Under the assumptions of Theorem 1.1 we show that Hida's p-adic L-function is integral: it belongs to $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. Note that in the setting of Theorem 1.2 we do not know if $\mathcal{L}_{f_0,\mathcal{K},\xi}$ is actually in $\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]$.

The assumptions on $\bar{\rho}_f|_{G_{\mathcal{K}}}$ and the local ϵ-factors in Theorem 1.1 are needed to appeal to results of M. Hsieh [2012; 2014c] in proving the nonvanishing modulo p of some special L-values or vanishing of the anticyclotomic μ-invariant. The square-freeness of N is put at the moment for simplicity (mainly to avoid local triple product integrals for supercuspidal representations and we may come back to remove it in the future).

The assumption (g) in Theorem 1.1 is due to lack of reference for [Hida and Tilouine 1993, Conjecture in introduction]. Details are explained in Definition 7.8.

Hypothesis (b) of Theorem 1.2 means that $\mathcal{L}_{f,\xi,\mathcal{K}}$ can be evaluated at the trivial character of $\Gamma_{\mathcal{K}}$, though it is not a point at which it interpolates classical L-values. As a result, Theorem 1.2 has interesting applications for the usual Bloch–Kato Selmer group of f.

The result of this paper is the foundation for several important breakthroughs on arithmetic of elliptic curves and modular forms. Skinner [2014] has recently been able to use Theorem 1.2 to prove a converse of the Gross–Zagier–Kolyvagin theorem: if the Mordell–Weil rank of an elliptic curve over $\mathbb{Q}$ is exactly one and the Shafarevich–Tate group is finite, then its L-function vanishes to exactly order one at the central critical point. The author has been able to prove an anticyclotomic main conjecture of Perrin-Riou when the root number is -1 [Wan 2014a] (by comparing the Selmer group in the theorem with the one studied by Perrin-Riou, using the Poitou–Tate long exact sequence and applying F. Castella's generalization [2019] of a formula of Bertolini–Darmon–Prasanna relating the different p-adic L-functions).

There is also joint work of the author with Skinner and Jetchev that uses Theorem 1.2 to deduce the p-adic part of the precise BSD formula in the rank one case [Jetchev et al. 2017]. We remark that in the above mentioned applications one can not appeal to the main conjecture proved in [Skinner and Urban 2014] since the global sign of the L-functions has to be $+1$ in [loc. cit.].

The methods of this paper can be adapted (with some additional arguments) to the case when f is nonordinary as well. This forms the foundation of the author's recent proof of the Iwasawa main conjecture for supersingular elliptic curves formulated by Kobayashi (see [Wan 2014b]).

Our proofs of Theorems 1.1 and 1.2 use Eisenstein congruences on the unitary group $\mathrm{U}(3, 1)$, which first appeared in [Hsieh 2011]. Recent works with a similar flavor include Skinner and Urban [2014] using the group $\mathrm{U}(2, 2)$, and the work of Hsieh [2014a] for CM characters using the group $\mathrm{U}(2, 1)$. The difference between our results and Skinner and Urban's is that they studied the p-adic L-function of

Rankin–Selberg product of a general modular form and a CM form such that the weight of the CM form is lower, while in our case the weight of the CM form is higher. This is the very reason we work with unitary groups of different signature.

We also mention there are works establishing the other divisibility of the main conjecture using Euler systems [Wan 2014a; Lei et al. 2014] under some more restrictions. Together with Theorems 1.1 and 1.2 these give the full equality of the main conjecture in the case when all hypotheses are satisfied.

For clarity, we briefly discuss our proof of the theorems. The proof follows the main outline of Skinner and Urban's proof in [2014] (which in turn followed the main outline of Wiles' proof of the Iwasawa main conjecture for totally real fields). However, carrying this out requires new arguments. The main steps are: (1) Constructing a p-adic family of Eisenstein series whose constant terms are essentially the p-adic Rankin–Selberg L-function $\mathcal{L}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\boldsymbol{\xi}}$. (2) Proving that the Eisenstein series is coprime to p-adic L-function (that is, modulo any divisor of the p-adic L-function it is still nonzero), which shows that its congruences with cusp forms is "measured" by the p-adic L-function. (3) the Galois argument.

The main differences between our proof and that of Skinner and Urban are in steps (1) and (2). First, we need to work with the unitary group U(3, 1) instead of U(2, 2) which is used in [Skinner and Urban 2014]. The reason is that by our assumption that the CM form has higher weight than f, the L-values interpolated by the p-adic L-function $\mathcal{L}_{\boldsymbol{f},\xi,\mathcal{K}}$ show up in the constant terms of holomorphic Eisenstein series on the group U(3, 1) that are induced from the Klingen parabolic subgroup with Levi $\mathrm{U}(2)\times\mathcal{K}^{\times}$. The cuspidal representation on U(2) is determined by the automorphic representation π_f and a Hecke character of $\mathbb{A}_{\mathcal{K}}^{\times}$ whose restriction to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is the central character of π_f. As a result, the construction of the p-adic families of the Eisenstein series via the pullback formula requires finding the right Siegel section at p (which turns out to be different from the one used in [Skinner and Urban 2014]). To have the right pullback and to make the Fourier–Jacobi coefficient computation not too hard, such choice of section is quite subtle. The idea for our choice is similar to that in [Eischen et al. 2016] and is inspired by the formula for differential operators on p-adic q-expansions on the group GU(3, 3). (The Siegel–Eisenstein series measure used here to construct the p-adic L-function is the special case of [loc. cit.] (see Section 4.3, part II). We also refer to [Eischen 2015, Sections 3 and 4] for a nice exposition and for details about those differential operators. These differential operators are not logically needed for the construction in this paper though.)

Step (2) is the core of the whole argument. In [Skinner and Urban 2014], the Klingen–Eisenstein series on U(2, 2) (which are also special cases of the series constructed in [Wan 2015a]) has a Fourier expansion $E_{\mathrm{Kling}}=\sum_T a_T q^T$, with T running over 2×2 Hermitian matrices. By the pullback formula, we have

$$a_T=\langle \mathrm{FJ}_T\, E_{\mathrm{sieg}}, \varphi_\pi\rangle_{\mathrm{U}(1,1)},$$

where E_{sieg} is a Siegel–Eisenstein series on U(3, 3), $\mathrm{FJ}_T\, E_{\mathrm{sieg}}$ is its T-Fourier–Jacobi coefficient (regarded as a form on U(1, 1)), and φ_π is a form in the U(1, 1) automorphic representation π considered in [Skinner and Urban 2014] (again determined by π_f and a Hecke character of $\mathbb{A}_{\mathcal{K}}^{\times}$). The Siegel–Eisenstein measure is constructed in Proposition 12.3. Computation tells us $\mathrm{FJ}_T\, E_{\mathrm{sieg}}$ is essentially a product of an Eisenstein series and a theta function, and thus this pairing, and hence a_T is essentially a Rankin–Selberg product.

In our case, forms on U(3, 1) only have Fourier–Jacobi expansions (instead of Fourier expansions)

$$F \mapsto \sum_{n\in\mathbb{Q}} a_n(F)q^n := \mathrm{FJ}(F)$$

with $a_n(F) \in H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(n))$, where $\mathcal{Z}^\circ_{[g]}$ is a two-dimensional abelian variety which is the abelian part of the universal semiabelian scheme over a point in the boundary of a toroidal compactification of the Shimura variety for GU(3, 1). The sheaf $\mathcal{L}(n)$ is a line bundle on $\mathcal{Z}^\circ_{[g]}$. We can view each $a_n(F)$ as an automorphic form on the group $\mathrm{U}(2)\cdot N$, where U(2) is the definite unitary group appearing as a factor of the Levi of the Klingen parabolic subgroup of U(3, 1), and N is the unipotent radical of the parabolic subgroup, which is a Heisenberg group. It consists of matrices of the form

$$\begin{pmatrix} 1 & \times & \times & \times \\ & 1 & & \times \\ & & 1 & \times \\ & & & 1 \end{pmatrix}.$$

To study $a_n(F)$ we use a functional $l_{\theta^\star}$ on $H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(n))$. This is just the pairing over N (modulo its center) with an explicit theta function $\theta^\star$ on $\mathrm{U}(2)\cdot N$ (defined in Lemma 6.46). In the following we do the computation at an arithmetic point z.

We divide our argument into five steps:

(1) We first compute the n-th Fourier–Jacobi coefficient of a Siegel–Eisenstein series $E_{\mathrm{sieg,z}}$ (in Section 6J), considered as a form on the Jacobi group $N'\cdot\mathrm{U}(2,2)\subseteq\mathrm{U}(3,3)$ with N' a unipotent subgroup of U(3, 3). It consists of matrices of the form

$$\begin{pmatrix} 1 & \times & \times & \times & \times & \times \\ & 1 & & \times & & \\ & & 1 & \times & & \\ & & & 1 & & \\ & & & \times & 1 & \\ & & & \times & & 1 \end{pmatrix}.$$

This turns out to be a finite sum of products of the form $E_{2,\mathrm{z}}\cdot\Theta_{\mathrm{z}}$ (see Proposition 6.44, with $E_{2,\mathrm{z}}$ a Siegel–Eisenstein series on U(2, 2)) and Θ_{z} a theta function on the Jacobi group (the $E_{\mathrm{sieg},2}$ and $\Theta_{\Phi_{\mathcal{D}}}$ in Proposition 6.44).

(2) Next we restrict this n-th Fourier–Jacobi coefficient to the group

$$(N\cdot\mathrm{U}(2))\times\mathrm{U}(2)\subset\mathrm{U}(3,1)\times\mathrm{U}(2)\cap N'\cdot\mathrm{U}(2,2).$$

Another computation shows that Θ_{z} essentially restricts to a form $\theta_4\times\theta_{2,\mathrm{z}}$ on $(N\cdot\mathrm{U}(2))\times\mathrm{U}(2)$. (The $\theta_{2,\mathrm{z}}$ is varying with the arithmetic point z while the θ_4 is fixed, which justifies dropping the subscript z.) The actual situation is slightly more complicated: it is actually a finite sum of such products. Applying a functional $l_{\theta^\star}$ (which is pairing with a *fixed* theta function $\theta^\star$ to the θ_4-component of each summand above), we show that $\langle\theta_4,\theta^\star\rangle_N$ is a constant function on U(2) by Lemma 4.8 (which we manage to make nonzero), and then end up with a theta function θ_{z} on (the lower) U(2) (a finite linear combination of $\theta_{2,\mathrm{z}}$'s of each summand). See Lemma 6.46 for a precise formula. So using the pullback formula, we get for $E_{\mathrm{Kling},\mathrm{z}}$ the Klingen–Eisenstein series defined in (6-13) (denoted $E_{\mathrm{Kling},\mathcal{D}}$ there),

$$l_{\theta^\star}(a_n(E_{\mathrm{Kling},\mathrm{z}})) = \langle E_{2,\mathrm{z}}|_{\mathrm{U}(2)\times\mathrm{U}(2)},\, f_{\mathrm{z}}\cdot\theta_{\mathrm{z}}\rangle_{1\times\mathrm{U}(2)}$$

regarded as a form on the first U(2), which is the U(2) in the Levi of the Klingen parabolic subgroup. Note that by Lemma 8.23 when z is varying in a p-adic family the $l_{\theta^\star}$ takes values in the Iwasawa algebra (the parameter space).

(3) To study its p-adic properties, we pair it with an auxiliary form h_{z} on U(2) (Section 8B1):

$$\langle\langle E_{2,\mathrm{z}}|_{\mathrm{U}(2)\times\mathrm{U}(2)},\, f_{\mathrm{z}}\cdot\theta_{\mathrm{z}}\rangle_{1\times\mathrm{U}(2)},\, h\rangle_{\mathrm{U}(2)} = (*)\cdot\langle h_{\mathrm{z}},\, f_{\mathrm{z}}\cdot\theta_{\mathrm{z}}\rangle.$$

To obtain this formula, we use the doubling method formula for $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ applied to h_{z}. The $(*)$ is some p-adic L-function factor for h_{z} coming from this (see Proposition 8.25 for details).

(4) We prove that such an expression is interpolated by an element $\boldsymbol{B}_1$ in the Iwasawa algebra (in (8-7)). The pairing on the right-hand side is just a triple product integral $\int_{[\mathrm{U}(2)]} h_{\mathrm{z}}(g)\theta_{\mathrm{z}}(g)f_{\mathrm{z}}(g)\,dg$. The fact that the θ_{z} can be taken to be an eigenform follows from considering the central character (see the proof of Proposition 8.25).

(5) We use Ichino's formula to evaluate

$$\left(\int_{[U(2)]} h_{\mathrm{z}}(g)\theta_{\mathrm{z}}(g)f_{\mathrm{z}}\,dg\right)\left(\int_{[U(2)]}\tilde{h}_{\mathrm{z}}(g)\tilde{\theta}_{3,\mathrm{z}}(g)\tilde{f}_{\mathrm{z}}(g)\,dg\right)$$
$$= \langle h_{\mathrm{z}},\tilde{h}_{\mathrm{z}}\rangle\langle\theta_{\mathrm{z}},\tilde{\theta}_{3,\mathrm{z}}\rangle\langle f_{\mathrm{z}},\tilde{f}_{\mathrm{z}}\rangle\cdot\frac{L^\Sigma\big(\tfrac12,\pi_{f_{\mathrm{z}}}\times\chi_{1,\mathrm{z}}\big)L^\Sigma\big(\tfrac12,\pi_{f_{\mathrm{z}}}\times\chi_{2,\mathrm{z}}\big)}{L^\Sigma(2,\pi_{f_{\mathrm{z}}},\mathrm{ad})L^\Sigma(2,\pi_{\theta_{\mathrm{z}}},\mathrm{ad})L^\Sigma(2,\pi_{h_{\mathrm{z}}},\mathrm{ad})}$$
$$\times\prod_{v\in\Sigma}\frac{I_v(h_{\mathrm{z}}\otimes\theta_{\mathrm{z}}\otimes f_{\mathrm{z}},\tilde{h}_{\mathrm{z}}\otimes\tilde{\theta}_{3,\mathrm{z}}\otimes\tilde{f}_{\mathrm{z}})}{\langle h_{\mathrm{z},v},\tilde{h}_{\mathrm{z},v}\rangle\langle\theta_{\mathrm{z},v},\tilde{\theta}_{3,\mathrm{z},v}\rangle\langle f_{\mathrm{z},v},\tilde{f}_{\mathrm{z},v}\rangle}.$$

Here $\tilde{h}_{\mathrm{z}}$, $\tilde{\theta}_{3,\mathrm{z}}$ and $\tilde{f}_{\mathrm{z}}$ mean some forms or vectors in the contragredient representation of the automorphic representation for h_{z}, θ_{z} and f_{z}. The factor I_v is a local integral defined by Ichino, and $\chi_{1,\mathrm{z}}$ and $\chi_{2,\mathrm{z}}$ are two CM Hecke characters showing up in the computation. We interpolate everything in p-adic families and compare it to the product of several p-adic L-functions of modular forms or Hecke characters (see the proof of Proposition 8.29 for details). Furthermore:

- We can choose h_{z}'s and θ_{z}'s so that these p-adic L-functions are units in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]^\times$ times a fixed number in $\overline{\mathbb{Q}}_p$.
- The ratio of the triple product and the product of these p-adic L-functions is a product of local factors (we show that the triple product is a p-adic analytic function, so the product of these local factors is a p-adic meromorphic function). We make the local choices such that for inert or ramified primes these local factors involve only the Hida-family variable of $\boldsymbol{f}$ (which has nothing to do with $\Gamma_{\mathcal{K}}^+$ or $\Gamma_{\mathcal{K}}^-$). For split primes, we compute these local factors explicitly.

The constructions above finally provide a nonzero element of $\mathbb{I}$, which is sufficient for our use. We are thus able to prove in Proposition 8.29 that height one divisors of $\boldsymbol{B}_1$ are those of $\mathbb{I}$.

After this, we can use the same argument as in [Skinner and Urban 2014] to deduce our main theorem: by a geometric argument we construct a cuspidal family on U(3, 1) congruent modulo the p-adic L-function $\mathcal{L}_{f,\xi,\mathcal{K}}$ to the Eisenstein family constructed as above. Passing to the Galois side, we get a family of Galois representations coming from cuspidal forms that is congruent to the family coming from our Klingen–Eisenstein series, but which is "more irreducible" than the Eisenstein Galois representations. Then an argument (the "lattice construction") of E. Urban gives the required elements in the dual Selmer group.

Remark 1.3. In fact at the point where $\mathcal{L}_{f,\xi,\mathcal{K}}$ takes its central critical value, the Klingen–Eisenstein family does not interpolate a classical Eisenstein series (i.e., not an interpolation point). Therefore even in the case when the global root number for $\mathcal{L}_{f,\xi,\mathcal{K}}$ is -1, so that the constant terms of the Eisenstein family vanish identically along the central critical subfamily, the p-adic Eisenstein series itself can still be nonzero in that subfamily.

Remark 1.4. We emphasize here that $\theta^\star$ is fixed throughout the whole p-adic family (instead of varying). Note also that the space of theta functions with given Archimedean kernel function and level group at finite places is finite-dimensional. The space $H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(n)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is generated by a finite number of such theta functions. We will show in the text that by pairing the $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$-adic Fourier–Jacobi coefficient with one rational theta function (not necessarily p-integral!), we get an element in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We then show that by choosing the datum properly, this element is the product of a unit in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ and a nonzero element in $\overline{\mathbb{Q}}_p$, and proving it is prime to the p-adic L-function we study. Such strategy is notably different from the one adopted in [Hsieh 2014c], where Hsieh argued p-integrally and proved with a stronger result that the Fourier–Jacobi coefficient is already a unit. This is the very reason why we do not need to study the theory of p-integral theta functions.

Remark 1.5. In [Zhang 2013] the special L-value showing up in the Fourier–Jacobi expansion is the near central point of the Rankin–Selberg L-function, while in our case it is the central value of the triple product L-function. Moreover, the Fourier–Jacobi coefficient considered in [loc. cit.] is nonzero only when f is a CM form (see Theorem 4.12 in [loc. cit.]). This is due to the fact that we are pairing the Fourier–Jacobi coefficient with the product of a theta function and an auxiliary form h on U(2), while Zhang paired it with the theta function only (i.e., taking the h in our case to be the constant function). Our strategy has the advantage that these central L-values are accessible to various results of nonvanishing modulo p by Hsieh.

The rest of this paper is organized as follows. In Section 2, we recall some background information and formulate the main conjecture. In Section 3, we discuss automorphic forms and p-adic automorphic forms on various unitary groups. In Section 4 we recall the notion of theta function which plays an important role in studying Fourier–Jacobi expansions as outlined above. In Sections 5 and 6, we make the local and global calculations for Siegel and Klingen–Eisenstein series using the pullback formula of

Shimura. In Section 7, we interpolate our previous calculations p-adically and construct the families. In Section 8, we prove the coprimeness of (the Fourier–Jacobi coefficients of) the Klingen–Eisenstein series and the p-adic L-function. Finally, we deduce the main theorem in Section 9.

2. Background

We first introduce our notation. We will usually take a finite extension $L/\mathbb{Q}_p$ and write $\mathcal{O}_L$ for its integer ring and ϖ_L for a uniformizer. Let $G_{\mathbb{Q}}$ and $G_{\mathcal{K}}$ be the absolute Galois groups of $\mathbb{Q}$ and $\mathcal{K}$. Let $\Gamma_{\mathcal{K}}^{\pm}$ be the subgroups of $\Gamma_{\mathcal{K}}$ such that the complex conjugation c acts by ± 1. We take topological generators $\gamma^{\pm}$ so that $\mathrm{rec}^{-1}(\gamma^{+}) = ((1+p)^{1/2}, (1+p)^{1/2})$ and $\mathrm{rec}^{-1}(\gamma^{-}) = ((1+p)^{1/2}, (1+p)^{-1/2})$ where $\mathrm{rec}: \mathbb{A}_{\mathcal{K}}^{\times} \to G_{\mathcal{K}}^{\mathrm{ab}}$ is the reciprocity map normalized by the geometric Frobenius. Let $\Psi_{\mathcal{K}}$ be the composition

$$G_{\mathcal{K}} \twoheadrightarrow \Gamma_{\mathcal{K}} \hookrightarrow \mathbb{Z}_p[\![\Gamma_{\mathcal{K}}]\!]^{\times}.$$

Define $\Lambda_{\mathcal{K}} := \mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!]$. Recall we defined a branch character ξ in the introduction. We will write σ_{ξ} for the Galois character corresponding to ξ via class field theory. We also let $\mathbb{Q}_{\infty}$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and let $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\mathbb{Q}_{\infty}/\mathbb{Q})$. Define $\Psi_{\mathbb{Q}}$ to be the composition $G_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\mathbb{Q}} \hookrightarrow \mathbb{Z}_p[\![\Gamma_{\mathbb{Q}}]\!]^{\times}$. We also define $\varepsilon_{\mathcal{K}}$ and $\varepsilon_{\mathbb{Q}}$ to be the compositions $\mathcal{K}^{\times}\backslash\mathbb{A}_{\mathcal{K}}^{\times} \xrightarrow{\mathrm{rec}} G_{\mathcal{K}}^{\mathrm{ab}} \to \mathbb{Z}_p[\![\Gamma_{\mathcal{K}}]\!]^{\times}$ and $\mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times} \xrightarrow{\mathrm{rec}} G_{\mathbb{Q}}^{\mathrm{ab}} \to \mathbb{Z}_p[\![\Gamma_{\mathbb{Q}}]\!]^{\times}$ where the second arrows are the $\Psi_{\mathcal{K}}$ and $\Psi_{\mathbb{Q}}$ defined above. Let ω and ϵ be the Techimüller character and the cyclotomic character. We also write $\chi_{\mathcal{K}/\mathbb{Q}}$ to be the quadratic character associated to $\mathcal{K}/\mathbb{Q}$.

Write c for complex conjugation. For a Hecke character χ we write $\chi^c(x) := \chi(c(x))$. For a Galois character χ we define χ^c to be the composition of χ with conjugation by c (regarding c as an element in the Galois group).

2A. *p-adic families for* $\mathrm{GL}_2/\mathbb{Q}$. Let M be a positive integer prime to p and χ a character of $(\mathbb{Z}/pM\mathbb{Z})^{\times}$. Let $\Lambda_{\mathbb{Q}} := \mathbb{Z}_p[\![W]\!]$ (we call $\mathrm{Spec}\,\Lambda_{\mathbb{Q}}(\overline{\mathbb{Q}}_p)$ the weight space). Let $\mathbb{I}$ be a normal domain finite over $\Lambda_{\mathbb{Q}}$. A point $\phi \in \mathrm{Spec}(\mathbb{I})$ is called arithmetic if the image of ϕ in $\mathrm{Spec}\,\Lambda(\overline{\mathbb{Q}}_p)$ is the continuous $\mathbb{Z}_p$-homomorphism sending $(1+W) \mapsto \zeta(1+p)^{k-2}$ for some $k \geq 2$ and ζ a p-power root of unity. We usually write k_{ϕ} for this k, called the weight of ϕ. We also define χ_{ϕ} to be the character of $\mathbb{Z}_p^{\times} \simeq (\mathbb{Z}/p\mathbb{Z})^{\times} \times (1+p\mathbb{Z}_p)$ that is trivial on the first factor and given by $(1+p) \mapsto \zeta$ on the second factor.

Definition 2.1. An $\mathbb{I}$-adic family of modular forms of tame level M and character χ is a formal q-expansion $\boldsymbol{f} = \sum_{n=0}^{\infty} a_n q^n$, $a_n \in \mathbb{I}$, such that for a Zariski dense set of arithmetic points $\phi \in \mathrm{Spec}(\mathbb{I})$ the specialization $f_{\phi} = \sum_{n=0}^{\infty} \phi(a_n) q^n$ of $\boldsymbol{f}$ at ϕ is the q-expansion of a modular form of weight k_{ϕ}, character $\chi\chi_{\phi}\omega^{2-k_{\phi}}$ (where ω is the Techimüller character), and level $Mp^{t_{\phi}}$ for some $t_{\phi} \geq 0$.

The U_p operator is defined in both the spaces of modular forms and families. It is given by

$$U_p\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=0}^{\infty} a_{pn} q^n.$$

Note that $(U_p \cdot \boldsymbol{f})_{\phi} = U_p \cdot \boldsymbol{f}_{\phi}$. Hida's ordinary idempotent e_p is defined by $e_p := \lim_{n\to\infty} U_p^{n!}$. A form f or family $\boldsymbol{f}$ is called ordinary if $e_p f = f$ or $e_p \boldsymbol{f} = \boldsymbol{f}$ (see for instance [Hida 1986, page 550]). A well-known fact is that every ordinary eigenform fits into an ordinary family of eigenforms $\boldsymbol{f}$ ([Hida 1988, Theorem II] for example).

According to the results of Deligne, Langlands, Shimura et al., there is a Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ for f. If the residual representation $\bar{\rho}_f$ is irreducible then one can construct a Galois representation $\rho_{\boldsymbol{f}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I})$ such that it specializes to the Galois representation ρ_{f_ϕ} of $\boldsymbol{f}_\phi$ at each arithmetic specialization $\phi \in \mathrm{Spec}(\mathbb{I})$. We write $T_{\boldsymbol{f}}$ for the representation space of $\rho_{\boldsymbol{f}}$.

2B. ***The main conjecture.*** Before formulating the main conjecture, we first define characteristic ideals and Fitting ideals. We let A be a Noetherian ring. We write $\mathrm{Fitt}_A(X)$ for the Fitting ideal in A of a finitely generated A-module X. This is the ideal generated by the determinant of the $r \times r$ minors of the matrix giving the first arrow in a given presentation of X:

$$A^s \to A^r \to X \to 0.$$

If X is not a torsion A-module, then $\mathrm{Fitt}_A(X) = 0$. This definition does not depend on the choice of the presentation.

Fitting ideals behave well with respect to base change. For $I \subset A$ an ideal

$$\mathrm{Fitt}_{A/I}(X/IX) = \mathrm{Fitt}_A(X) \bmod I.$$

Now suppose A is a Krull domain (a domain which is Noetherian and normal). Then the characteristic ideal is defined by

$$\mathrm{char}_A(X) := \{x \in A : \mathrm{ord}_Q(x) \geq \mathrm{length}_Q(X) \text{ for any height one prime } Q \text{ of } A\}.$$

If X is not a torsion A-module, then we define $\mathrm{char}_A(X) = 0$.

We consider the Galois representation

$$T_{\boldsymbol{f},\mathcal{K},\xi} := T_{\boldsymbol{f}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\mathcal{K}}$$

with the $G_{\mathcal{K}}$ action given by $\rho_{\boldsymbol{f}} \sigma_{\xi^{-c}} \epsilon^{(4-\kappa)/2} \hat{\otimes} \Lambda_{\mathcal{K}}(\Psi_{\mathcal{K}}^{-c})$. (Here c means composing with the complex conjugation in the idele group, and $-$ means taking inverse of the character.) We define the Selmer group (recall κ is assumed to be even)

$$\mathrm{Sel}_{\boldsymbol{f},\mathcal{K},\xi} := \ker\left\{H^1(\mathcal{K}, T_{\boldsymbol{f},\mathcal{K},\xi} \otimes_{\mathbb{I}[[\Gamma_{\mathcal{K}}]]} \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \to H^1(I_{\bar{v}_0}, T_{\boldsymbol{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \times \prod_{v \nmid p} H^1(I_v, T_{\boldsymbol{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*)\right\}$$

where $*$ means the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p)$. We also define the Σ-primitive Selmer groups

$$\mathrm{Sel}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi} := \ker\left\{H^1(\mathcal{K}, T_{\boldsymbol{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \to H^1(I_{\bar{v}_0}, T_{\boldsymbol{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*) \times \prod_{v \notin \Sigma} H^1(I_v, T_{\boldsymbol{f},\mathcal{K},\xi} \otimes \mathbb{I}[[\Gamma_{\mathcal{K}}]]^*)\right\}.$$

We let

$$X_{\boldsymbol{f},\mathcal{K},\xi} := (\mathrm{Sel}_{\boldsymbol{f},\mathcal{K},\xi})^* \quad \text{and} \quad X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi} := (\mathrm{Sel}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi})^*.$$

These are finitely generated $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$-modules (see e.g., [Skinner and Urban 2014, Lemma 3.3]). We take the extension of scalars of them to $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ and still denote them by using the same notations. In Section 7 we construct p-adic L-functions $\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\mathcal{K},\xi}$ and $\mathcal{L}^{\Sigma,\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}$ which are elements in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ or its fraction field. Their interpolation formulas are given in (7-5) (see also Definition 7.8). The three-variable Iwasawa main conjecture is [Greenberg 1994]:

Conjecture 2.2. *$X_{\boldsymbol{f},\mathcal{K},\xi}$ and $X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}$ are torsion $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$-modules and*

$$\mathrm{char}_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]} X_{\boldsymbol{f},\mathcal{K},\xi} = (\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}), \quad \textit{and} \quad \mathrm{char}_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]} X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi} = (\mathcal{L}^{\Sigma,\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}).$$

We can also replace $\boldsymbol{f}$ with a single form f_0 and have the two-variable main conjectures.

Conjecture 2.3. *$X_{f_0,\mathcal{K},\xi}$ and $X^{\Sigma}_{f_0,\mathcal{K},\xi}$ are torsion $\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]$-modules and*

$$\mathrm{char}_{\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]} X_{f_0,\mathcal{K},\xi} = (\mathcal{L}^{\mathrm{Hida}}_{f_0,\mathcal{K},\xi}), \quad \textit{and} \quad \mathrm{char}_{\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]} X^{\Sigma}_{f_0,\mathcal{K},\xi} = (\mathcal{L}^{\Sigma,\mathrm{Hida}}_{f_0,\mathcal{K},\xi}).$$

2C. ***Control of Selmer groups.*** In this subsection we prove a control theorem of Selmer groups which will be used to prove Theorem 1.2. Let $\phi_0 \in \mathrm{Spec}\, \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!](\overline{\mathbb{Q}}_p)$ be the point mapping $\gamma^{\pm}$ to 1 and such that $\phi_0|_{\mathbb{I}}$ corresponds to the form f_0. Let $\wp = \ker \phi_0|_{\mathbb{I}}$ be the point of weight two and trivial character. Then we prove the following proposition.

Proposition 2.4. *Suppose $\bar{\rho}_f|_{G_{\mathcal{K}}}$ is absolutely irreducible. There is an exact sequence of $\mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!]$-modules*

$$M \to X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}/\wp X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi} \to X^{\Sigma}_{f_0,\mathcal{K},\xi} \to 0$$

where $M \otimes_{\mathcal{O}_L} L$ has support of codimension at least 2 *in* $\mathrm{Spec}\, \mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!] \otimes L$.

Proof. We write $\mathbb{I}_{\mathcal{K}}$ for $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ for simplicity. Write $\mathbb{T} = T_{\boldsymbol{f},\mathcal{K},\xi}$ as a $\mathbb{I}_{\mathcal{K}}$-module. Let T be the $\Lambda_{\mathcal{K}}$-module $T_{f_0,\mathcal{K},\xi}$. Recall that $p = v_0\bar{v}_0$. We have an exact sequence

$$0 \to T \otimes_{\Lambda_{\mathcal{K}}} \Lambda^*_{\mathcal{K}} \to \mathbb{T} \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}^*_{\mathcal{K}} \to \mathbb{T} \otimes_{\mathbb{I}_{\mathcal{K}}} (\wp\mathbb{I}_{\mathcal{K}})^* \to 0.$$

Write $G_{\mathcal{K}_\Sigma}$ for the Galois group over $\mathcal{K}$ of the maximal algebraic extension of $\mathcal{K}$ unramified outside Σ. From this we deduce

$$H^1(G_{\mathcal{K}_\Sigma}, T \otimes_{\Lambda} \Lambda^*_{\mathcal{K}}) \xrightarrow{\sim} H^1(G_{\mathcal{K}_\Sigma}, \mathbb{T} \otimes_{\mathbb{I}_{\mathcal{K}}} \mathbb{I}^*_{\mathcal{K}})[\wp]$$

as in [Skinner and Urban 2014, Proposition 3.7]. We also have an exact sequence

$$H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]} (\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]^*)) \xrightarrow{s_1} H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]} (\wp[\![\Gamma_{\mathcal{K}}]\!]^*)) \\ \to H^1(I_{\bar{v}_0}, T \otimes_{\mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!]} (\mathcal{O}_L[\![\Gamma_{\mathcal{K}}]\!]^*)) \to H^1(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]} (\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]^*)).$$

From these we deduce an exact sequence of $\Lambda_{\mathcal{K}}$-modules

$$M := ((\mathrm{coker}\, s_1)^{G_{\bar{v}_0}})^*/\wp\, (\mathrm{coker}\, s_1)^{G_{\bar{v}_0}} \hookrightarrow X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}/\wp X^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi} \to X^{\Sigma}_{f_0,\mathcal{K},\xi} \to 0.$$

Let $\mathcal{K}_{\infty,\bar{v}_0}$ (resp. $\mathcal{K}_{\infty,v_0}$) be the $\mathbb{Z}_p$-extension of $\mathcal{K}$ unramified outside $\bar{v}_0$ (resp. v_0) and let $\Gamma_{\bar{v}_0} = \mathrm{Gal}(\mathcal{K}_{\infty,\bar{v}_0})$ (resp. $\Gamma_{v_0} = \mathrm{Gal}(\mathcal{K}_{v_0}/\mathcal{K})$). Let $\gamma_{\bar{v}_0} \in \Gamma_{\bar{v}_0}$ and $\gamma_{v_0} \in \Gamma_{v_0}$ be topological generators. It is well-known (e.g., see [Skinner and Urban 2014, Section 3.3.5] that we have

$$0 \to T^+ \to T \to T/T^+ \to 0$$

as $G_{\mathbb{Q}_p}$-modules. By the description of the Galois action, there is a $\gamma \in I_{\bar{v}_0}$ such that $\gamma - 1$ acts invertibly on $T^+ \otimes_{\mathbb{I}[\![\Gamma_\mathcal{K}]\!]} (\mathbb{I}[\![\Gamma_\mathcal{K}]\!]^*))$. We take a basis (v_1, v_2) such that v_1 generates T^+ and the action of γ on T is diagonal under this basis. Then it is not hard to see (by looking at the $I_{\bar{v}_0}$-action) that if

$$v \in H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}_\mathcal{K}} \mathbb{I}_\mathcal{K}^*)$$

we have $v \in (\mathbb{I}[\![\Gamma_{v_0}]\!])^* v_2$ and if $v \in H^0(I_{\bar{v}_0}, \mathbb{T} \otimes_{\mathbb{I}_\mathcal{K}} (\wp \mathbb{I}_\mathcal{K})^*)$ then $v \in (\wp \mathbb{I}_\mathcal{K})^* v_2$. From the above discussion, we know that $((\operatorname{coker} s_1)^{G_{\bar{v}_0}})^* / \ker \phi_0' (\operatorname{coker} s_1)^{G_{\bar{v}_0}}$ is supported in

$$\mathrm{Spec}(\mathcal{O}_L[\![\Gamma_v]\!] \otimes L).$$

Moreover by looking at the action of $\mathrm{Frob}_{\bar{v}_0}$ we see it is killed by the function $a_p^{-1} R - 1$ where a_p is the invertible function in $\mathbb{I}$ which gives the U_p-eigenvalue of f and R is the image in Γ_v of $\mathrm{Frob}_{\bar{v}_0}$ under class field theory. But $a_p(\phi_0) \neq 1$ and $R(\phi_0) = 1$ so $a_p^{-1} R - 1$ is nonzero at ϕ_0. So the support of $M \otimes_{\mathcal{O}_L} L$ has support of dimension at most zero and this proves the proposition. □

3. Unitary groups

In this section, we introduce our notation for unitary groups and develop the Hida theory on them. We mainly follow [Hsieh 2014a, Sections 2–4] in our presentation, which in turn, summarizes portions of Shimura's books [1997; 2000]. We define $S_n(R)$ to be the set of $n \times n$ Hermitian matrices with entries in $\mathcal{O}_\mathcal{K} \otimes_\mathbb{Z} R$. We define a map $e_\mathbb{A} = \prod_v e_v : \mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ where for each place v of $\mathbb{Q}$, e_v is the standard additive character as in [Skinner and Urban 2014, 8.1.2] (which again follows Shimura's convention). We refer to [Hsieh 2014a, Section 2.8] for the discussion of the CM period $\Omega_\infty \in \mathbb{C}^\times$ and the p-adic period $\Omega_p \in \hat{\mathbb{Z}}_p^{\mathrm{ur},\times}$.

3A. *Groups.* Let $\delta' \in \mathcal{K}$ be a totally imaginary element such that $-i\delta'$ is positive. Let $d = \mathrm{Nm}(\delta')$ which we assume to be a p-adic unit. Let $\mathrm{U}(2) = \mathrm{U}(2, 0)$ and $\mathrm{GU}(2) = \mathrm{GU}(2, 0)$ be the unitary group and the unitary similitude group, respectively, associated to the skew-Hermitian matrix $\zeta = \begin{pmatrix} \mathfrak{s}\delta' & \\ & \delta' \end{pmatrix}$ for some $\mathfrak{s} \in \mathbb{Z}_+$ prime to p. More precisely $\mathrm{GU}(2)$ is the group scheme over $\mathbb{Z}$ defined by, for any $\mathbb{Z}$ algebra A,

$$\mathrm{GU}(2)(A) = \{g \in \mathrm{GL}_2(A \otimes_\mathbb{Z} \mathcal{O}_\mathcal{K}) \mid {}^t\bar{g}\zeta g = \lambda(g)\zeta,\ \lambda(g) \in A^\times\}.$$

The map $\lambda : \mathrm{GU}(2) \to \mathbb{G}_m$, $g \mapsto \lambda(g)$ is called the similitude character and $\mathrm{U}(2) \subseteq \mathrm{GU}(2)$ is the kernel of λ. Let $G = \mathrm{GU}(3, 1)$ and $\mathrm{U}(3, 1)$ be the similarly defined unitary similitude group and unitary group, respectively, over $\mathbb{Z}$ associated to the skew-Hermitian matrix

$$\begin{pmatrix} & & 1 \\ & \zeta & \\ -1 & & \end{pmatrix}.$$

We denote this Hermitian space as V. Let $P \subseteq G$ be the parabolic subgroup of $\mathrm{GU}(3, 1)$ consisting of those matrices in G of the form

$$\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & \times & \times & \times \\ & & & \times \end{pmatrix}.$$

Let N_P be the unipotent radical of P. Then if $X_{\mathcal{K}}$ is the 1-dimensional space over $\mathcal{K}$,

$$M_P := \mathrm{GL}(X_{\mathcal{K}}) \times \mathrm{GU}(2) \hookrightarrow \mathrm{GU}(V), \quad (a, g_1) \mapsto \mathrm{diag}(a, g_1, \mu(g_1)\bar{a}^{-1})$$

is the Levi subgroup of P. Let $G_P := \mathrm{GU}(2) \subseteq M_P$ be the set of elements $\mathrm{diag}(1, g_1, \mu(g_1))$ as above. Let δ_P be the modulus character for P. We usually use a more convenient character δ such that $\delta^3 = \delta_P$.

Since p splits as $v_0\bar{v}_0$ in $\mathcal{K}$, $\mathrm{GL}_4(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) \times \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{\bar{v}_0}})$. Here

$$\mathrm{U}(3, 1)(\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{GL}_4(\mathcal{O}_{\mathcal{K}_{v_0}}) = \mathrm{GL}_4(\mathbb{Z}_p)$$

is the projection onto the first factor. Let B and N be the upper triangular Borel subgroup of $G(\mathbb{Q}_p)$ and its unipotent radical, respectively. Let

$$K_p = \mathrm{GU}(3, 1)(\mathbb{Z}_p) \simeq \mathrm{GL}_4(\mathbb{Z}_p),$$

and for any $n \geq 1$ let K_0^n be the subgroup of K consisting of matrices upper-triangular modulo p^n. Let $K_1^n \subset K_0^n$ be the subgroup of matrices whose diagonal elements are 1 modulo p^n.

The group $\mathrm{GU}(2)$ is closely related to a division algebra. Put

$$D = \{g \in M_2(\mathcal{K}) \mid g\zeta{}^t\bar{g} = \det(g)\zeta\}.$$

Then D is a definite quaternion algebra over $\mathbb{Q}$ with local invariant $\mathrm{inv}_v(D) = (-\mathfrak{s}, -D_{\mathcal{K}/\mathbb{Q}})_v$ (the Hilbert symbol). The relation between $\mathrm{GU}(2)$ and D is explained by

$$\mathrm{GU}(2) = D^\times \times_{\mathbb{G}_m} \mathrm{Res}_{\mathcal{K}/\mathbb{Q}} \mathbb{G}_m.$$

For each finite place v we write D_v^1 for the set of elements $g_v \in D_v^\times$ such that $|\mathrm{Nm}(g_v)|_v = 1$, where Nm is the reduced norm. In application we will choose the D to be the quaternion algebra ramified exactly at ∞ and the q in the main theorems.

Let Σ be a finite set of primes containing all the primes at which $\mathcal{K}/\mathbb{Q}$ or ξ is ramified, the primes dividing the level of f_0 (as in the introduction), the primes dividing $\mathfrak{s}$, the primes such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact and the prime 2. Let Σ^1 and Σ^2, respectively, be the set of nonsplit primes in Σ such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is noncompact, and compact. We will sometimes write $[D^\times]$ for $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})$.

We similarly write [U(2)], [GU(2, 0)], etcetera. For two automorphic forms f_1, f_2 on U(2) we write $\langle f_1, f_2\rangle = \int_{[\mathrm{U}(2)]} f_1(g) f_2(g)\, dg$. Here the Haar measure is normalized so that at finite places $\mathrm{U}(2)(\mathbb{Z}_\ell)$ has measure 1, and at ∞ the compact set $\mathrm{U}(1)(\mathbb{R})\backslash\mathrm{U}(2)(\mathbb{R})$ has measure 1.

We define $G_n = \mathrm{GU}(n, n)$ for the unitary similitude group for the skew-Hermitian matrix $\left(\begin{smallmatrix} & 1_n \\ -1_n & \end{smallmatrix}\right)$ and $\mathrm{U}(n, n)$ for the corresponding unitary group.

3B. ***Hermitian spaces and automorphic forms.*** Let $(r, s) = (3, 3)$ or $(2, 2)$ or $(3, 1)$ or $(2, 0)$. Then the unbounded Hermitian symmetric domain for $\mathrm{GU}(r, s)$ is

$$X^+ = X_{r,s} = \left\{\tau = \begin{pmatrix} x \\ y \end{pmatrix} \mid x \in M_s(\mathbb{C}),\ y \in M_{(r-s)\times s}(\mathbb{C}),\ i(x^* - x) > i y^* \zeta^{-1} y\right\}.$$

We use x_0 to denote the Hermitian symmetric domain for GU(2), which is just a point. We have the following embedding of Hermitian symmetric domains:

$$\iota : X_{3,1} \times X_{2,0} \hookrightarrow X_{3,3}$$
$$(\tau, x_0) \hookrightarrow Z_\tau,$$

where $Z_\tau = \left(\begin{smallmatrix} x & 0 \\ y & \zeta/2 \end{smallmatrix}\right)$ for $\tau = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

Let $G_{r,s} = \mathrm{GU}(r, s)$ and $H_{r,s} = \mathrm{GL}_r \times \mathrm{GL}_s$. Let $G_{r,s}(\mathbb{R})^+$ be the subgroup of elements of $G_{r,s}(\mathbb{R})$ whose similitude factors are positive. If $s \neq 0$ we define a cocycle

$$J : G_{r,s}(\mathbb{R})^+ \times X^+ \to H_{r,s}(\mathbb{C})$$

by $J(\alpha, \tau) = (\kappa(\alpha, \tau), \mu(\alpha, \tau))$, where for

$$\tau = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} a & b & c \\ g & e & f \\ h & l & d \end{pmatrix}$$

(where α is a block matrix with respect to the partition $(s + (r - s) + s)$),

$$\kappa(\alpha, \tau) = \begin{pmatrix} \bar{h}^t x + \bar{d} & \bar{h}^t y + l\bar{\zeta} \\ -\bar{\zeta}^{-1}(\bar{g}^t x + \bar{f}) & -\bar{\zeta}^{-1}\bar{g}^t y + \bar{\zeta}^{-1}\bar{e}\bar{\zeta} \end{pmatrix}, \quad \mu(\alpha, \tau) = hx + ly + d$$

in the GU(3, 1) case and

$$\kappa(\alpha, \tau) = \bar{h}^t x + \bar{d}, \quad \mu(\alpha, \tau) = hx + d$$

in the GU(3, 3) case. Let $i \in X^+$ be the point $\left(\begin{smallmatrix} i1_s \\ 0 \end{smallmatrix}\right)$. Let K_∞^+ be the compact subgroup of $\mathrm{U}(r, s)(\mathbb{R})$ stabilizing i and let K_∞ be the group generated by K_∞^+ and $\mathrm{diag}(1_{r+s}, -1_s)$. Then

$$K_\infty^+ \to H(\mathbb{C}), \quad k_\infty \mapsto J(k_\infty, i)$$

defines an algebraic representation of K_∞^+. Later on in Section 6E we will also consider a different choice $\boldsymbol{i}$ on the Symmetric domain for $(r, s) = (3, 3)$ or $(2, 2)$.

Definition 3.1. A weight $\underline{k}$ is defined to be an $(r+s)$-tuple

$$\underline{k} = (c_{r+s}, \dots, c_{s+1}; c_1, \dots, c_s) \in \mathbb{Z}^{r+s}$$

with $c_1 \geq \cdots \geq c_{r+s}$, $c_s \geq c_{s+1} + r + s$.

Our convention for identifying a weight with a tuple of integers is different from others in the literature. For example, our c_{s+i} $(1 \leq i \leq r)$ and c_j $(1 \leq j \leq s)$ corresponds to $-a_{r+1-i}$ and $b_{s+1,j}$ in [Hsieh 2014a, Section 3.1].

We refer to [Hsieh 2014a] for the definition of the algebraic representation $L_{\underline{k}}(\mathbb{C})$ of H with the action denoted by $\rho_{\underline{k}}$ (note the different index for weight) and define a model $L^{\underline{k}}(\mathbb{C})$ of the representation $H(\mathbb{C})$ with the highest weight $\underline{k}$ as follows. The underlying space of $L^{\underline{k}}(\mathbb{C})$ is $L_{\underline{k}}(\mathbb{C})$ and the group action is defined by

$$\rho^{\underline{k}}(h) = \rho_{\underline{k}}({}^t h^{-1}), h \in H(\mathbb{C}).$$

We also note that if each $\underline{k} = (0, \dots, 0; \kappa, \dots, \kappa)$ then $L^{\underline{k}}(\mathbb{C})$ is one-dimensional.

For a weight $\underline{k}$, define $\|\underline{k}\|$ by

$$\|\underline{k}\| := -c_{s+1} - \cdots - c_{s+r} + c_1 + \cdots + c_s$$

and $|\underline{k}|$ by

$$|\underline{k}| = (c_1 + \cdots + c_s).\sigma - (c_{s+1} + \cdots + c_{s+r}).\sigma c \in \mathbb{Z}^I.$$

Here I is the set of embeddings $\mathcal{K} \hookrightarrow \mathbb{C}$ and σ is the Archimedean place of $\mathcal{K}$ determined by our fixed embedding $\mathcal{K} \hookrightarrow \mathbb{C}$. Let χ be a Hecke character of $\mathcal{K}$ with infinity type $|\underline{k}|$, i.e., the Archimedean part of χ is given by

$$\chi_\infty(z) = (z^{c_1+\cdots+c_s} \cdot \bar{z}^{-(c_{s+1}+\cdots+c_{s+r})}).$$

Definition 3.2. Let U be an open compact subgroup of $G(\mathbb{A}_f)$. We denote by $M_{\underline{k}}(U, \mathbb{C})$ the space of holomorphic $L^{\underline{k}}(\mathbb{C})$-valued functions f on $X^+ \times G(\mathbb{A}_f)$ such that for $\tau \in X^+$, $\alpha \in G(\mathbb{Q})^+$ and $u \in U$ we have

$$f(\alpha\tau, \alpha g u) = \mu(\alpha)^{-\|\underline{k}\|} \rho^{\underline{k}}(J(\alpha, \tau)) f(\tau, g).$$

Now we consider automorphic forms on unitary groups in the adelic language. The space of automorphic forms of weight $\underline{k}$ and level U with central character χ consists of smooth and slowly increasing functions $F : G(\mathbb{A}) \to L_{\underline{k}}(\mathbb{C})$ such that for every $(\alpha, k_\infty, u, z) \in G(\mathbb{Q}) \times K_\infty^+ \times U \times Z(\mathbb{A})$,

$$F(z\alpha g k_\infty u) = \rho^{\underline{k}}(J(k_\infty, i)^{-1}) F(g) \chi^{-1}(z).$$

We can associate a $L_{\underline{k}}(\mathbb{C})$-valued function on $X^+ \times G(\mathbb{A}_f)/U$ given by

$$f(\tau, g) := \chi_f(\mu(g)) \rho^{\underline{k}}(J(g_\infty, i)) F((f_\infty, g)) \tag{3-1}$$

where $g_\infty \in G(\mathbb{R})$ such that $g_\infty(i) = \tau$. If this function is holomorphic, then we say that the automorphic form F is holomorphic.

3C. *Galois representations associated to cuspidal representations.* In this section, we follow [Skinner and Urban 2014, Theorem 7.1, Lemma 7.2] to discuss the Galois representations associated to cuspidal automorphic representations on $\mathrm{GU}(r,s)(\mathbb{A}_\mathbb{Q})$. Let π be an irreducible automorphic representation of $\mathrm{GU}(r,s)(\mathbb{A}_\mathbb{Q})$ generated by a holomorphic cuspidal eigenform with weight $\underline{k}=(c_{r+s},\dots,c_{s+1};c_1,\dots,c_s)$ and central character χ_π. Let $\Sigma(\pi)$ be a finite set of primes of $\mathbb{Q}$ containing all the primes at which π is unramified and all the primes dividing p. Then for some L finite over $\mathbb{Q}_p$ there is a Galois representation (see [Morel 2010; Shin 2011; Skinner 2012])

$$R_p(\pi):G_\mathcal{K}\to \mathrm{GL}_n(L)$$

($n=r+s$) such that:

(a) $R_p(\pi)^c\simeq R_p(\pi)^\vee\otimes\rho_{p,\chi_\pi^{1+c}}\epsilon^{1-n}$ where $\rho_{p,\chi_\pi^{1+c}}$ denotes the p-adic Galois character associated to χ_π^{1+c} by class field theory and ϵ is the cyclotomic character.

(b) $R_p(\pi)$ is unramified at all finite places not above primes in $\Sigma(\pi)$, and for such a place $w\nmid p$

$$\det(1-R_p(\pi)(\mathrm{Frob}_w)q_w^{-s})=L\left(BC(\pi)_w\otimes\chi_{\pi,w}^c,s+\frac{1-n}{2}\right)^{-1}$$

Here, the Frob_w is the geometric Frobenius and BC means the base change from $\mathrm{U}(r,s)$ to GL_{r+s}. Suppose π_v is nearly ordinary with respect to $\underline{k}$ (see Section 3H) and unramified at all primes v dividing p. Recall $v_0\mid p$ corresponds to $\iota:\mathbb{C}\simeq\mathbb{C}_p$. If we write $\kappa_i=s-i+c_i$ for $1\le i\le s$ and $\kappa_{s+i}=c_{s+i}+s+r-i$ for $s+1\le i\le r+s$, then

$$R_p(\pi)\mid G_{\mathcal{K},v_0}\simeq\begin{pmatrix}\xi_{r+s,v}\epsilon^{-\kappa_{r+s}} & * & * & * \\ & \xi_{r+s-1,v}\epsilon^{\kappa_{r+s-1}} & & * \\ & & \ddots & * \\ & & & \xi_{1,v}\epsilon^{-\kappa_1}\end{pmatrix},$$

an upper-triangular matrix, where $\xi_{i,v}$ are unramified characters. Using fact (a) above, we also know that $R_p(\pi)_{\bar{v}_0}$ is equivalent to an upper triangular representation as well (with the Hodge–Tate weight being $(-(\kappa_1+1-r-s-|\underline{k}|),\dots,-(\kappa_{r+s}+1-r-s-|\underline{k}|))$, in our geometric convention ϵ^{-1} has Hodge–Tate weight one).

3D. *Shimura varieties.* Now we consider the group $\mathrm{GU}(3,1)$. For any open compact subgroup $K=K_pK^p$ of $\mathrm{GU}(3,1)(\mathbb{A}_f)$ whose p-component is $K_p=\mathrm{GU}(3,1)(\mathbb{Z}_p)$, we refer to [Hsieh 2014a, Section 2.1] for the definition and arithmetic model of the associated Shimura variety, which we denote as $S_G(K)_{/\mathcal{O}_{\mathcal{K},(v_0)}}$. The scheme $S_G(K)$ represents the following functor: for any $\mathcal{O}_{\mathcal{K},(v_0)}$-algebra R, $\underline{A}(R)=\{(A,\bar{\lambda},\iota,\bar{\eta}^p)\}$ where A is an abelian scheme over R of relative dimension four with CM by $\mathcal{O}_\mathcal{K}$ given by ι, $\bar{\lambda}$ is an orbit of prime to p polarizations and $\bar{\eta}^p$ is an orbit of prime to p level structures. There is also a theory of compactifications of $S_G(K)$ developed by Lan [2013]. We denote by $\bar{S}_G(K)$ a toroidal compactification and $S_G^*(K)$ the minimal compactification. We refer to [Hsieh 2014a, Section 2.7] for details. The boundary components of $S^*_G(K)$ are in one-to-one correspondence with the set of cusp labels defined below. For $K = K_pK^p$ as above we define the set of cusp labels to be

$$C(K) := (\mathrm{GL}(X_{\mathcal{K}}) \times G_P(\mathbb{A}_f))N_P(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K.$$

This is a finite set. We denote by $[g]$ the class represented by $g \in G(\mathbb{A}_f)$. For each such g whose p-component is 1 we define $K^g_P = G_P(\mathbb{A}_f) \cap gKg^{-1}$ and denote $S_{[g]} := S_{G_P}(K^g_P)$ the corresponding Shimura variety for the group G_P with level group K^g_P. By strong approximation we can choose a set $\underline{C}(K)$ of representatives of $C(K)$ consisting of elements $g = pk^0$ for $p \in P(\mathbb{A}^{(\Sigma)}_f)$ and $k^0 \in K^0$ for K^0 the maximal compact subgroup of $G(\mathbb{A}_f)$ defined in [Hsieh 2014a, Section 1.10].

3E. ***Igusa varieties and p-adic automorphic forms.*** Now we recall briefly the notion of Igusa varieties in [Hsieh 2014a, Section 2.3]. We remark that these materials are special cases in Hida's book [2004, Chapter 8]. Let V be the Hermitian space for the unitary group GU(3, 1) and let M be the standard lattice of V as in [Hsieh 2014a, Section 1.8]. Let $M_p = M \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $\mathrm{Pol}_p = \{N^{-1}, N^0\}$ be a polarization of M_p. Recall that this means that N^{-1} and N^0 are maximal isotropic $\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p$-submodules in M_p such that they are dual to each other with respect to the Hermitian metric on V, and

$$\mathrm{rank}_{\mathbb{Z}_p} N^{-1}_{v_0} = \mathrm{rank}_{\mathbb{Z}_p} N^0_{\bar{v}_0} = 3, \quad \mathrm{rank}_{\mathbb{Z}_p} N^{-1}_{\bar{v}_0} = \mathrm{rank}_{\mathbb{Z}_p} N^0_{v_0} = 1.$$

The Igusa variety $I_G(K^n)$ of level p^n is the scheme over $\mathcal{O}_{\mathcal{K},(v_0)}$ representing the quintuple $\underline{A}(R) = \{(A, \bar{\lambda}, \iota, \bar{\eta}^p, j)\}$ where the $A, \bar{\lambda}, \iota, \bar{\eta}^p$ are as in the definition for Shimura varieties of GU(3, 1) above, and an injection of group schemes

$$j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$$

over R which is compatible with the $\mathcal{O}_{\mathcal{K}}$-action on both sides. Note that the existence of j implies that A must be ordinary along the special fiber. There is also a theory of Igusa varieties over $\bar{S}_G(K)$. Let $\underline{\omega}$ be the automorphic vector bundle on $S_G(K)$ as defined in [Hsieh 2014a, Subsection 2.7.3]. As in [loc. cit.] let $\overline{H}_{p-1} \in H^0(S_G(K)_{/\bar{\mathbb{F}}}, \det(\underline{\omega})^{p-1})$ be the Hasse invariant. Over the minimal compactification, some power (say the t-th) of the Hasse invariant can be lifted to $\mathcal{O}_{v_0}$, by the ampleness of $\det \underline{\omega}$. We denote such a lift by E. By the Koecher principle we can regard E as in $H^0(\bar{S}_G(K), \det(\underline{\omega}^{t(p-1)}))$. Let $\mathcal{O}_m := \mathcal{O}_{\mathcal{K},v_0}/p^m\mathcal{O}_{\mathcal{K},v_0}$. Set $T_{0,m} := \bar{S}_G(K)[1/E]_{/\mathcal{O}_m}$. For any positive integer n define $T_{n,m} := I_G(K^n)_{/\mathcal{O}_m}$ and $T_{\infty,m} = \varprojlim_n T_{n,m}$. Then $T_{\infty,m}$ is a Galois cover over $T_{0,m}$ with Galois group $\boldsymbol{H} \simeq \mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$. Let $\boldsymbol{N} \subset \boldsymbol{H}$ be the upper triangular unipotent radical. Define

$$V_{n,m} = H^0(T_{n,m}, \mathcal{O}_{T_{n,m}}).$$

Let $V_{\infty,m} = \varinjlim_n V_{n,m}$ and $V_{\infty,\infty} = \varprojlim_m V_{\infty,m}$ be the space of p-adic automorphic forms on GU(3, 1) with tame level K. We also define $W_{n,m} = V^{\boldsymbol{N}}_{n,m}$, $W_{\infty,m} = V^{\boldsymbol{N}}_{\infty,m}$ and $\mathcal{W} = \varinjlim_n \varinjlim_m W_{n,m}$. We define $V^0_{n,m}$, etcetera, to be the cuspidal part of the corresponding spaces.

We can make similar definitions for the definite unitary similitude groups G_P as well and define $V_{n,m,P}$, $V_{\infty,m,P}$, $V_{\infty,\infty,P}$, $V^{\boldsymbol{N}}_{n,m,P}$, $\mathcal{W}_P$, and so forth.

Let K_0^n and K_1^n be the subgroup of $\boldsymbol{H}$ consisting of matrices which are in $B_3 \times \mathrm{GL}_1$ or $N_3 \times \mathrm{GL}_1$ modulo p^n, for B_3 and N_3 being the upper triangular Borel subgroup of GL_3 and its unipotent radical, respectively. (These notations are already used for level groups of automorphic forms. The reason for using the same notation here is that automorphic forms with level group $K_\bullet^n$ are p-adic automorphic forms of level group $K_\bullet^n$.) We sometimes denote $I_G(K_1^n) = I_G(K^n)/K_1^n$ and $I_G(K_0^n) = I_G(K^n)/K_0^n$.

We can define the Igusa varieties for G_P as well. For $\bullet = 0, 1$ we let $K_{P,\bullet}^{g,n} := gK_\bullet^n g^{-1} \cap G_P(\mathbb{A}_f)$ and let $I_{[g]}(K_\bullet^n) := I_{G_P}(K_{P,\bullet}^{g,n})$ be the corresponding Igusa variety over $S_{[g]}$. We denote $A_{[g]}^n$ the coordinate ring of $I_{[g]}(K_1^n)$. Let $A_{[g]}^\infty = \varinjlim_n A_{[g]}^n$ and let $\hat{A}_{[g]}^\infty$ be the p-adic completion of $A_{[g]}^\infty$. This is the space of p-adic automorphic forms for the group $\mathrm{GU}(2, 0)$ of level group $gKg^{-1} \cap G_P(\mathbb{A}_f)$.

For unitary groups. Assume the tame level group K is neat. Let c be an element in $\mathbb{Q}_+ \backslash \mathbb{A}_{\mathbb{Q},f}^\times / \mu(K)$. We refer to [Hsieh 2014a, Section 2.5] for the notion of c-Igusa schemes $I_{\mathrm{U}(2)}^0(K, c)$ for the unitary groups $\mathrm{U}(2, 0)$ (not the similitude group). It parametrizes quintuples $(A, \lambda, \iota, \bar{\eta}^{(p)}, j)_{/S}$ similar to the Igusa schemes for unitary similitude groups but requires λ to be a prime to p c-polarization of A such that $(A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}, j)$ is a quintuple as in the definition of Shimura varieties for $\mathrm{GU}(2)$. For g_c with $\mu(g) \in \mathbb{A}_{\mathbb{Q}}^\times$ in the class of c. Let ${}^cK = g_c K g_c^{-1} \cap U(2)(\mathbb{A}_{\mathbb{Q},f})$. Then the space $I_{\mathrm{U}(2)}^0(K, c)$ is isomorphic to the space of forms on $I_{\mathrm{U}(2)}^0({}^cK, 1)$ (see [loc. cit.]).

Embedding of Igusa schemes. In order to use the pullback formula geometrically we need a map from the Igusa scheme of $\mathrm{U}(3, 1) \times \mathrm{U}(0, 2)$ to that of $\mathrm{U}(3, 3)$ (or from the Igusa scheme of $\mathrm{U}(2, 0) \times \mathrm{U}(0, 2)$ to that of $\mathrm{U}(2, 2)$) given by

$$i([(A_1, \lambda_1, \iota_1, \eta_1^p K_1, j_1)], [(A_2, \lambda_2, \iota_2, \eta_2^p K_2, j_2)]) = [(A_1 \times A_2, \lambda_1 \times \lambda_2, \iota_1 \times \iota_2, (\eta_1^p \times \eta_2^p) K_3, j_1 \times j_2)]. \tag{3-2}$$

3F. *Fourier–Jacobi expansions.*

Analytic Fourier–Jacobi coefficients. Let $\beta \in \mathbb{Q}_+$. Over $\mathbb{C}$ we have the β-analytic Fourier–Jacobi coefficient for a holomorphic automorphic form f on $G = \mathrm{GU}(3, 1)$ given by

$$\mathrm{FJ}_\beta(f, g) = \int_{\mathbb{Q}\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & & n \\ & 1_2 & \\ & & 1 \end{pmatrix} g\right) e_{\mathbb{A}}(-\beta n)\, dn.$$

The Haar measure is normalized so that the set $(\mathbb{Q}\backslash\mathbb{A})$ has measure 1.

p-adic cusps. As in [Hsieh 2014a] each pair $(g, j) \in C(K) \times \boldsymbol{H}$ can be regarded as a p-adic cusp, i.e., cusps of the Igusa tower. In the following we give the algebraic Fourier–Jacobi expansion at p-adic cusps.

Algebraic theory for Fourier–Jacobi expansions. We follow [Hsieh 2014a, pages 16–17] to give some background about the algebraic theory for Fourier–Jacobi expansion on the group $G = \mathrm{GU}(r, 1)$. These are special cases developed by Lan [2012; 2013]. Recall $[g]$ is a cusp label corresponding to class $g \in G(\mathbb{A}_f)$. One defines $\mathcal{Z}_{[g]}$ a group scheme over $S_{[g]}$ using the universal abelian variety as in [loc. cit.]

and denote $\mathcal{Z}^\circ_{[g]}$ the connected component over $S_{[g]}$. There is a line bundle $\mathcal{L}(\beta)$ on $\mathcal{Z}_{[g]}$ determined by β [Hsieh 2014a, Subsection 2.7.4].

Now let $f \in H^0(I_G(K_1^n)_{/R}, \omega_\kappa)$ be a scalar weight $\kappa \geq 6$ (i.e., of weight $(0, 0, 0; \kappa)$) modular form over an $\mathcal{O}$ algebra R, then by [Hsieh 2014a, Subsection 3.6.2] there is a Fourier–Jacobi expansion of f at the p-adic cusp (g, h) for $h \in \boldsymbol{H}$

$$\mathrm{FJ}^h_{[g]}(f) = \sum_{\beta \in \mathscr{S}_{[g]}} a^h_{[g]}(\beta, f) q^\beta$$

where

$$a^h_{[g]}(\beta, f) \in (\hat{A}^\infty_{[g]} \otimes_{\mathcal{O}} R) \otimes_{A_{[g]}} H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$$

and $\mathscr{S}_{[g]}$ is a sublattice of $\mathbb{Q}$ determined by the level subgroup. This is given by evaluating f at the Mumford family $(\mathfrak{M}, h^{-1} j_{\mathfrak{M}}, \omega_{\mathfrak{M}})$ where $j_{\mathfrak{M}}$ is a fixed level structure (see [Hsieh 2014a, Subsection 2.7.4]). (Note that we do not have the subscript N^1_H there since it is a scalar weight κ.)

Siegel operators. We have a Siegel operator Φ at the p-adic cusp (g, h) defined by

$$\begin{aligned} \Phi^h_{[g]} : H^0(I_G(K_1^n)_{/R}, \omega_\kappa) &\to A^n_{[g]} \otimes_{\mathcal{O}} R \\ f &\mapsto \Phi^h_{[g]}(f) := a^h_{[g]}(0, f). \end{aligned}$$

The Siegel operator at $[g]$ can be defined analytically as follows: for any $g \in G(\mathbb{A}_f)$ we define

$$\Phi_{P,g}(f) = \int_{N_P(\mathbb{Q}) \backslash N_P(\mathbb{A}_\mathbb{Q})} f(ng)\, dn. \tag{3-3}$$

We fix the Haar measure on $N_P(\mathbb{Q}) \backslash N_P(\mathbb{A}_\mathbb{Q})$ as in [Skinner and Urban 2014, Section 8.2]. The relation between the algebraic and analytic Siegel operator is given in [Hsieh 2014a, (3.12)].

3G. ***Weight space for* GU(3, 1).** Let $H = \mathrm{GL}_3 \times \mathrm{GL}_1$ and $T \subseteq H$ be the diagonal torus. Then $\boldsymbol{H} \simeq H(\mathbb{Z}_p)$. We let $\Lambda_2 = \Lambda$ be the completed group algebra $\mathbb{Z}_p[\![T(1 + p\mathbb{Z}_p)]\!]$. This is (noncanonically) isomorphic to a formal power series ring with four variables. There is an action of $T(\mathbb{Z}_p)$ on the Igusa scheme given by its action on the embedding $j : \mu_{p^n} \otimes_{\mathbb{Z}} N^0 \hookrightarrow A[p^n]$. (See [Hsieh 2014a, Definition 3.4], which in turn, follows Hida's convention [2004, Section 8.2].) This gives the spaces of p-adic modular forms for GU(3, 1) a structure of Λ-algebra. A $\overline{\mathbb{Q}}_p$-point ϕ of Spec Λ is call arithmetic if it is determined by a character $[\underline{k}] \cdot [\zeta]$ of $T(1 + p\mathbb{Z}_p) \simeq (1 + p\mathbb{Z}_p)^4$ where $\underline{k}$ is a weight and $\zeta = (\zeta_1, \zeta_2, \zeta_3; \zeta_4)$ for $\zeta_i \in \mu_{p^\infty}$. Here $[\underline{k}]$ is the character by regarding $\underline{k}$ as a character of $T(1+\mathbb{Z}_p)$ by $[\underline{k}](t_1, t_2, t_3, t_4) = (t_1^{-c_4} t_2^{-c_3} t_3^{-c_2} t_4^{-c_1})$ and $[\zeta]$ is the finite order character given by mapping $(1 + p)$ to ζ_i at the corresponding entry of $T(\mathbb{Z}_p)$. We often write this point $\underline{k}_\zeta$. We also define $\omega^{[\underline{k}]}$ as a character of the torsion part of $T(\mathbb{Z}_p)$ (canonically isomorphic to $(\mathbb{F}_p^\times)^4$) given by $\omega^{[\underline{k}]}(t_1, t_2, t_3, t_4) = \omega(t_1^{-c_4} t_2^{-c_3} t_3^{-c_2} t_4^{-c_1})$.

We can define the weight ring Λ_P for the definite unitary group G_P as well.

3H. ***Nearly ordinary forms.*** Here for convenience we again follow Hsieh's treatment of Hida theory, but point out that the results are actually due to Hida [2002]. We refer to [Hsieh 2014a, 3.8.3 and 4.3] for the definition of the U_p operator and Hida's idempotent e acting on the space $V^N_{\infty,\infty}$ of p-adic automorphic forms on GU(3, 1) and the nearly ordinary subspace of the space of p-adic modular forms. The space of nearly ordinary automorphic forms (cusp forms) is denoted as $\mathcal{W}_{\mathrm{ord}}$ ($\mathcal{W}^0_{\mathrm{ord}}$). For $q = 0$ or $\varnothing$ we let $\boldsymbol{V}^q_{\mathrm{ord}}$ be the Pontryagin dual of $\mathcal{W}^q_{\mathrm{ord}}$. Then we have the following theorem [Hsieh 2014a, Theorem 4.21]:

Theorem 3.3. *Let $q = 0$ or $\varnothing$. Then*:

(1) $\boldsymbol{V}^q_{\mathrm{ord}}$ *is a free Λ module of finite rank.*

(2) *For any $\underline{k}$ very regular we have natural isomorphisms*

$$\mathcal{M}^q_{\mathrm{ord}}(K, \Lambda) \otimes \Lambda/P_{\underline{k}} \xrightarrow{\sim} eM^q_{\underline{k}}(K, \mathcal{O}_p)$$

where $\mathcal{M}^q_{\mathrm{ord}}(K, \Lambda)$ is defined in Definition 3.5. Here we identify $eM^q_{\underline{k}}(K, \mathcal{O}_{v_0})$ with its image in the space of p-adic automorphic forms of weight $\underline{k}$ under $\beta_{\underline{k}}$ for the map $\beta_{\underline{k}}$ defined in [Hsieh 2014a, (3.3)].

Remark 3.4. If $\mathcal{K}$ is a general CM field, then the statement of the corresponding result is more complicated; see [Hsieh 2014a, Section 4.5].

3I. Λ-*adic forms.*

Definition 3.5. For any finite Λ algebra R, and $q = 0$ or $\varnothing$ we define the space of R-adic ordinary forms to be

$$\mathcal{M}^q_{\mathrm{ord}}(K, R) := \mathrm{Hom}_\Lambda(\boldsymbol{V}^q_{\mathrm{ord}}, R).$$

Similarly, if R is a Λ_P-algebra, then we define

$$\mathcal{M}_{\mathrm{ord},[g],P}(K_{P,[g]}, R) := \mathrm{Hom}_{\Lambda_P}(\boldsymbol{V}_{\mathrm{ord},P,[g]}, R).$$

Here the subscript $[g]$ means that the prime to p level group is K^g_P as defined previously.

For any $f \in \mathcal{M}_{\mathrm{ord}}(K, R)$ we have an R-adic Fourier–Jacobi expansion

$$\mathrm{FJ}^h_{[g]}(f) = \sum_{\beta \in \mathcal{S}_{[g]}} a^h_{[g]}(\beta, f) q^\beta \tag{3-4}$$

obtained from the Fourier–Jacobi expansion on $\mathcal{W}^q_{\mathrm{ord}}$, where $a^h_{[g]}(\beta, f) \in R \hat{\otimes} \hat{A}^\infty_{[g]} \otimes_{A_{[g]}} H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$ (see [Hsieh 2014a, Subsection 4.6.1]). We also have an R-adic Siegel operator which we denote as $\hat{\Phi}^h_{[g]}$. Let

$$w_3' = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \in \mathrm{GL}_4(\mathbb{Z}_p) \simeq \mathrm{U}(3,1)(\mathbb{Z}_p).$$

(Notice that we used the place v_0 to identify $\mathrm{GL}_4(\mathbb{Z}_p)$ with $\mathrm{U}(3,1)(\mathbb{Z}_p)$ here. We use w_3' instead of w_3 as in [Hsieh 2014a, page 35] to distinguish it from $w_3 \in \mathrm{U}(3,3)$.) Now we have the following important theorem:

Theorem 3.6 [Hsieh 2014a, Theorem 4.26]. *Let R be as before. We have the following short exact sequence*

$$0 \to \mathcal{M}^0_{\mathrm{ord}}(K, R) \to \mathcal{M}_{\mathrm{ord}}(K, R) \xrightarrow{\hat{\Phi}^{w_3'} = \oplus \hat{\Phi}^{w_3'}_{[g]}} \oplus_{g \in C(K)} \mathcal{M}_{\mathrm{ord}}(K_P^g, R) \to 0.$$

We need one more theorem which gives another definition of nearly ordinary p-adic modular forms using Fourier–Jacobi expansions.

Definition 3.7. Let R be a finite torsion free Λ-algebra. Let $X(K)$ be the set $\{(g, h)\}$ where g runs over a set of representatives of cusp labels $C(K)$ and h runs over $\boldsymbol{T}$ which is the diagonal torus of $\boldsymbol{H}$. Let $\mathcal{N}_{\mathrm{ord}}(K, R)$ be the set of formal Fourier–Jacobi expansions

$$F = \Big\{ \sum_{\beta \in \mathcal{S}_{[g]}} a(\beta, F) q^\beta, a(\beta, F) \in (R \hat{\otimes} \hat{A}^\infty_{[g]})^\Lambda \otimes H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta)) \Big\}_{g \in X(K)}$$

(here $\hat{\otimes}$ means completed tensor product, and the superscript Λ in $(R \hat{\otimes} \hat{A}^\infty_{[g]})^\Lambda$ means that the Λ-action as a nebentypus character is compatible with the Λ-algebra structure of R), such that for a Zariski dense set $\mathcal{X}_F$ of points $\phi \in \mathrm{Spec}(R)$ such that the induced point in $\mathrm{Spec}(\Lambda)$ is some arithmetic weight $\underline{k}_\zeta$, the specialization F_ϕ of F is the Fourier–Jacobi expansion of a nearly ordinary modular form with prime to p level group K, weight $\underline{k}$ and nebentype at p given by $[\underline{k}][\underline{\zeta}]\omega^{-[\underline{k}]}$.

Then we have the following theorem [Hsieh 2014a, Theorem 4.25]:

Theorem 3.8. $$\mathcal{M}_{\mathrm{ord}}(K, R) = \mathcal{N}_{\mathrm{ord}}(K, R).$$

Remark 3.9. The proof uses the p-adic q-expansion principle for $\mathrm{GU}(r, 1)$, which is proved by Hida [2009, Theorem 0.1] (also recalled in [Hsieh 2014a, Subsection 3.6.4]). The q-expansion principle follows from the irreducibility of the Igusa scheme. As mentioned by Hida [2014a] the Igusa scheme is not quite irreducible; in fact the component group is isomorphic to the quotient of $\mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$ over his M_1, which is the subgroup consisting of matrices (g_1, g_2) with $\det g_1 = \det g_2$. (Hida proved the monodromy group, i.e., the image of $\pi_1(\bar{S}_K(G)_{/\mathbb{F}_p}, s)$ in $\mathrm{GL}_3(\mathbb{Z}_p) \times \mathrm{GL}_1(\mathbb{Z}_p)$ is exactly this M_1.) By our definition of $X(K)$ above, it clearly contains a representative of this quotient. So we still have the q-expansion principle.

4. Background for theta functions

Now we recall briefly the basic notions of theta functions and theta liftings, following closely to [Zhang 2013] with some modifications.

4A. ***Heisenberg group.*** Let W be a finite-dimensional vector space over $\mathbb{Q}_v$ with a nondegenerate alternating form $\langle \cdot, \cdot \rangle$. We define

$$H(W) := \{(w, t) \mid w \in W, t \in \mathbb{Q}_v\}$$

with multiplication law: $(w_1, t_1)(w_2, t_2) = \big(w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle\big)$.

4B. ***Schrödinger representation.*** Fix an additive character ψ of $\mathbb{Q}_v$ and a complete polarization as $W = X \oplus Y$ of W where X and Y are maximal totally isotropic subspaces of W. Let $S(X)$ be a space of Bruhat–Schwartz functions on X, and define a representation ρ_ψ of $H(W)$ on $S(X)$ by

$$\rho_\psi(x) f(z) = f(x+z), x \in X$$
$$\rho_\psi(y) f(z) = \psi(\langle z, y\rangle) f(z), y \in Y$$
$$\rho_\psi(t) f(z) = \psi(t) f(z), t \in \mathbb{Q}_v$$

This is called the Schrödinger representation. By the theorem of Stone and von Neumann, ρ_ψ is the unique irreducible smooth representation on which $\mathbb{Q}_v = \{(0, t) \mid t \in \mathbb{Q}_v\}$ acts via the character ψ.

4C. ***Metaplectic groups and Weil representations.*** Let $\mathrm{Sp}(W)$ be the symplectic group preserving the alternating form $\langle \cdot , \cdot \rangle$ on W. Then $\mathrm{Sp}(W)$ acts on $H(W)$ by $(w, t)g = (gw, t)$ (we use column vectors for $w \in W$ and the left action of $\mathrm{Sp}(W)$ as [Zhang 2013]). By the uniqueness of ρ_ψ, there is an operator $\omega_\psi(g)$ on $S(X)$, determined up to scalar, such that

$$\rho_\psi(gw, t)\omega_\psi(g) = \omega_\psi(g)\rho_\psi(w, t)$$

for any $(w, t) \in H(W)$. Define the metaplectic group $\widetilde{\mathrm{Sp}}_\psi(W) = \{(g, \omega_\psi(g)) \text{ as above }\}$ which we often abbreviate as $\widetilde{\mathrm{Sp}}$ for short. Thus $\widetilde{\mathrm{Sp}}(W)$ has an action ω_ψ on $S(X)$ called the Weil representation.

Now suppose $\psi = \prod_v \psi_v$ is a global additive character of $\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}$ and W is a finite-dimensional vector space over $\mathbb{Q}$ equipped with an alternating pairing $\langle \cdot , \cdot \rangle$. We can put the above construction together for all v's to get a representation of $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ on $S(X(\mathbb{A}))$. This can be viewed as a projective representation of $\mathrm{Sp}(W)$ (a representation with image in the infinite dimensional projective linear group). We now give formulas for this representation. Let $\{e_1, \dots, e_n; f_1, \dots, f_n\}$ be a basis of $W = X \oplus Y$ such that $\langle e_i, f_j\rangle = \delta_{ij}$. With respect to this basis, we write elements in X in row vectors, and the projective representation of $\widetilde{\mathrm{Sp}}(W)(\mathbb{A}_\mathbb{Q})$ on $\mathrm{Proj}\, S(X(\mathbb{A}))$ is given by the formulas

- $\omega_\psi\big(\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix}\big)\phi(x) = |\det A|^{1/2}\phi(xA)$;
- $\omega_\psi\big(\begin{pmatrix} 1 & B \\ & 1\end{pmatrix}\big)\phi(x) = \psi(xB{}^tx/2)\phi(x)$;
- $\omega_\psi\big(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\big)\phi(x) = \gamma\hat{\phi}(x)$ where $\hat{\phi}$ means the Fourier transform of ϕ with respect to the additive character ψ. The γ is an 8-th root of unity which is called the Weil constant.

4D. ***Dual reductive pairs.*** A dual reductive pair is a pair of subgroups (G, G') in the symplectic group $\mathrm{Sp}(W)$ satisfying:

(1) G is the centralizer of G' in $\mathrm{Sp}(W)$ and vice versa.

(2) The action of G and G' are completely reducible on W.

We are mainly interested in the following dual reductive pairs of unitary groups. Let $\mathcal{K}$ be a quadratic imaginary extension of $\mathbb{Q}$, $(V_1, (\cdot , \cdot)_1)$ be a skew Hermitian space over $\mathcal{K}$ and $(V_2, (\cdot , \cdot)_2)$ a Hermitian space over $\mathcal{K}$. Then the unitary groups $\mathrm{U}(V_1)$ and $\mathrm{U}(V_2)$ form a dual reductive pair in $\mathrm{Sp}(W)$, where

$W = V_1 \otimes V_2$ is given the alternating form $\frac{1}{2}\operatorname{tr}_{\mathcal{K}/\mathbb{Q}}((\cdot,\cdot)_1 \otimes \overline{(\cdot,\cdot)_2})$ over $\mathbb{Q}$. The embedding of the dual reductive pair $(\mathrm{U}(V_1), \mathrm{U}(V_2))$ into $\mathrm{Sp}(W)$ is

$$e : \mathrm{U}(V_1) \times \mathrm{U}(V_2) \to \mathrm{Sp}(W), \quad e(g_1, g_2)\cdot(v_1 \otimes v_2) = v_1 g_1 \otimes g_2^{-1} v_2.$$

4E. *Splittings.* Suppose $\dim_{\mathcal{K}} V_1 = n$ and $\dim_{\mathcal{K}} V_2 = m$. If χ_1 and χ_2 are Hecke characters of $\mathcal{K}^\times$ such that $\chi_1|_{\mathbb{A}_{\mathbb{Q}}^\times} = \chi_{\mathcal{K}/\mathbb{Q}}^n$ and $\chi_2|_{\mathbb{A}_{\mathbb{Q}}^\times} = \chi_{\mathcal{K}/\mathbb{Q}}^m$, then there is a splitting (see [Harris et al. 1996, Section 1])

$$s : \mathrm{U}(V_1) \times \mathrm{U}(V_2) \to \widetilde{\mathrm{Sp}}(W)$$

determined by χ_2 and χ_1. This enables us to define the Weil representations of $\mathrm{U}(V_1) \times \mathrm{U}(V_2)$ on $S(X(\mathbb{A}))$ which we denote as $\omega_{\chi_1,\chi_2} = \omega_{\chi_1} \otimes \omega_{\chi_2}$.

4F. *Theta functions.* Now let us define theta functions.

Definition 4.1. Let $\phi \in S(X(\mathbb{A}_{\mathbb{Q}}))$. Define the theta kernel function

$$\theta(\phi) = \sum_{l \in X(\mathbb{Q})} \phi(l).$$

Let $J = H(W) \ltimes \mathrm{Sp}(W)$ ($\tilde{J} = H(W) \ltimes \widetilde{\mathrm{Sp}}(W)$) be the Jacobi group with $\mathrm{Sp}(W)$ acting on $H(W)$ by $(w,t)\cdot g = (wg, t)$ ($\widetilde{\mathrm{Sp}}(W)$ acts on $H(W)$ by $(w,t)\cdot \tilde{g} = (wg, t)$, where g is the image of $\tilde{g}$ in $\mathrm{Sp}(W)$). We define a theta kernel on $\tilde{J}(\mathbb{A}_{\mathbb{Q}})$ as below.

Definition 4.2. Let $\tilde{g} \in \widetilde{\mathrm{Sp}}(W)$ and $(w,t) \in H(W)$, define

$$\theta_\phi((w,t)\tilde{g}) = \sum_{l \in X(\mathbb{Q})} \rho_\psi(w,t)\tilde{g}.\phi(l).$$

Using the Weil representation of the dual reductive pair above (with the choices of the splitting characters) we define the theta kernel for the theta correspondence as follows:

Definition 4.3.
$$\theta_\phi(g_1, g_2) = \theta(\omega(g_1, g_2)\phi).$$

4G. *Intertwining maps.* Here, we study the intertwining maps between theta series corresponding to different polarizations (X, Y) of W. Suppose $r \in \mathrm{Sp}(W)$, then (rX, rY) gives another polarization of W, and all polarizations are obtained this way. If $\phi \in S(X)$ then we define an intertwining map (local or global) $\delta_\psi : S(X) \to S(rX)$ by

$$\delta_\psi \phi(xr) = \omega_\psi(r^{-1})\phi(x) \tag{4-1}$$

for $x \in X$. We can see that δ_ψ is an isometry intertwining the actions of $\tilde{J}$.

Let W^- be the skew Hermitian space which is isomorphic to W as $\mathbb{Q}_v$-vector spaces but equipped with the alternating pairing $-\langle \cdot, \cdot \rangle$. For a polarization (X, Y) of W we present the intertwining formula for the two polarizations $(X \oplus X^-) \oplus (Y \oplus Y^-)$ and $\{w \oplus w, w \in W\} \oplus \{w \oplus -w, w \in W\}$ of $W \oplus W^-$.

We write the formula for the map $\delta_\psi : S(X(\mathbb{Q}_v) \oplus X^-(\mathbb{Q}_v)) \to S(W(\mathbb{Q}_v))$ and its inverse

$$\begin{aligned}\delta_\psi(\phi)(x_1, y) &= \int \psi(2\langle x_2, y\rangle)\phi(x_1 + x_2, x_2 - x_1)\, dx_2 \\ \delta_\psi^{-1}(\phi)(x_1, x_2) &= \int \psi(\langle -x_1 - x_2, y\rangle)\phi\big(x_1 - x_2, \tfrac{1}{2}y\big)\, dy.\end{aligned} \tag{4-2}$$

Another easy property is that if the two polarizations (X, Y) and (rX, rY) are globally defined, then the theta kernels Θ_ϕ and $\Theta_{\delta_\psi(\phi)}$ are defined and

$$\Theta_\phi(u, ng) = \Theta_{\delta_\psi(\phi)}(u, (rn)g).$$

4H. ***Special cases.*** Here we give two special cases which are used later. Case One is used in Section 8C to construct families of theta functions on U(2). Case Two is used in the computation of Fourier–Jacobi coefficients for the Siegel–Eisenstein series on U(3, 3) as a finite sum of products of Siegel–Eisenstein series and theta functions.

4H1. *Case one.* We write V for the two-dimensional Hermitian space over $\mathcal{K}$ for ζ/δ with respect to the basis (v_1, v_2), V^- for the Hermitian space for $-\zeta/\delta$ with respect to the basis (v_1^-, v_2^-), and V_1 for the one-dimensional skew Hermitian space with the metric δ with respect to the basis v. Let $W = V \otimes V_1$ and $W^- = V^- \otimes V_1$. We define several polarizations for the Hermitian space $\mathbb{W} := W \oplus W^-$ (the alternating pairing being the direct sum of those for W and W^-).

Definition 4.4.

$$\begin{aligned}X &:= \mathbb{Q}v_1 \otimes v \oplus \mathbb{Q}v_2 \otimes v \\ X^- &:= \mathbb{Q}v_1^- \otimes v \oplus \mathbb{Q}v_2^- \otimes v \\ Y &:= \mathbb{Q}\delta v_1 \otimes v \oplus \mathbb{Q}\delta v_2 \otimes v \\ Y^- &:= \mathbb{Q}\delta v_1^- \otimes v \oplus \mathbb{Q}\delta v_2^- \otimes v.\end{aligned}$$

Fix the additive character $\psi = \prod \psi_v$. Thus $W = X \oplus Y$ and $W^- = X^- \oplus Y^-$ are globally defined polarizations. For a split prime v we write $v = w\overline{w}$ for its decomposition in $\mathcal{K}$. We will often use an auxiliary polarization $W_v = X_v' \oplus Y_v'$ of $W_v = W \otimes_{\mathcal{K}} \mathcal{K}_v$ with respect to $\mathcal{K}_v \simeq \mathcal{K}_w \times \mathcal{K}_{\overline{w}} \simeq \mathbb{Q}_v^2$ and $W_v = X_v^{'-} \oplus Y_v^{'-}$ that is defined by $X_v' = \mathcal{K}_w v_1 \otimes v \oplus \mathcal{K}_w v_2 \otimes v$, $Y_v' = \mathcal{K}_{\overline{w}} v_1 \otimes v \oplus \mathcal{K}_{\overline{w}} v_2 \otimes v$ and similar for $X_v^{'-}$, $Y_v^{'-}$. This polarization is better suited for computing the Weil representation. For split primes v let $\delta_\psi'' : S(X_v') \to S(X_v)$ and $\delta_\psi^{-,''} : S(X_v^{-,'}) \to S(X_v^-)$ be the intertwining operators between the Schwartz functions defined above.

Let $\mathbb{W}^d = \{w \oplus w,\ w \in W\}$ (d stands for diagonal). We denote the intertwining maps

$$\delta_\psi : S(X_v \oplus X_v^-) \to S(\mathbb{W}_v^d)$$

and if v splits,

$$\delta_\psi' : S(X_v' \oplus X_v^{'-}) \to S(\mathbb{W}_v^d).$$

Recall the formulas given in (4-2).

Remark 4.5. In application in Section 6 we compute the intertwining operator

$$\delta_\psi : S(X_v \oplus X_v^-) \to S(\mathbb{W}_v^d)$$

(for $\mathbb{W} = (V \oplus V^-) \otimes V_1$) in this special case and the Weil representations restricting to semidirect products $H(\mathbb{W}) \ltimes (\mathrm{U}(V \oplus V^-) \times \mathrm{U}(V_1))$ (recall $\mathrm{U}(V \oplus V^-) \times \mathrm{U}(V_1) \hookrightarrow \mathrm{Sp}(\mathbb{W})$). We provide the matrix forms of these semidirect products that will be used in Section 6. Let U_1 and U_2 be unitary groups associated to the matrices

$$\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & 1_3 \\ -1_3 & \end{pmatrix}$$

respectively, and let U_1' and U_2' be the unitary groups associated to

$$\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix},$$

considered as subgroups of U_1 and U_2 respectively in the obvious way. Let N_1 be the subgroup of U_1 consisting of matrices of the form

$$\begin{pmatrix} 1 & x_1 & * & x_2 \\ & 1_2 & \zeta x_1^* & \\ & & 1 & \\ & & -\zeta x_2^* & 1_2 \end{pmatrix},$$

and $N_2 \subset \mathrm{U}_2$ the subgroup consisting of matrices of the form

$$\begin{pmatrix} 1 & x & t + \frac{1}{2}(xy^* - yx^*) & y \\ & 1_2 & y^* & 0 \\ & & 1 & \\ & & -x^* & I_2 \end{pmatrix}.$$

The corresponding semidirect products mentioned above are $J_1 = N_1 \mathrm{U}_1'$ and $J_2 = N_2 \mathrm{U}_2'$. These are used later in computing Fourier–Jacobi coefficients (see Definition 6.5 for J_2).

4H2. *Case two.* Now we discuss another special situation which will be used in the Fourier–Jacobi coefficient computations for the Siegel–Eisenstein series on $\mathrm{GU}(3, 3)$.

The local set-up. Let v be a place of $\mathbb{Q}$. Let $h \in S_1(\mathbb{Q}_v)$, $h \neq 0$. Let U_h be the unitary group of this matrix and let V_v be the corresponding one-dimensional Hermitian space. Let

$$V_{2,v} = \mathcal{K}_v^2 \oplus \mathcal{K}_v^2 = \boldsymbol{X}_v \oplus \boldsymbol{Y}_v$$

be the Hermitian space associated to $\mathrm{U}_2 = \mathrm{U}(2, 2)$ with the alternating pairing denoted as $\langle \cdot, \cdot \rangle_2$. Let $\boldsymbol{W} = V_v \otimes_{\mathcal{K}_v} V_{2,v}$. Then

$$(-, -) := \mathrm{Tr}_{\mathcal{K}_v/\mathbb{Q}_v}(\langle -, - \rangle_h \otimes_{\mathcal{K}_v} \langle -, - \rangle_2)$$

is a $\mathbb{Q}_v$ linear pairing on $\boldsymbol{W}$ that makes $\boldsymbol{W}$ into an eight-dimensional symplectic space over $\mathbb{Q}_v$. The canonical embedding of $\mathrm{U}_h \times \mathrm{U}_2$ into $\mathrm{Sp}(\boldsymbol{W})$ realizes the pair $(\mathrm{U}_h, \mathrm{U}_2)$ as a dual pair in $\mathrm{Sp}(\boldsymbol{W})$. Let λ_v be a character of $\mathcal{K}_v^\times$ such that $\lambda_v|_{\mathbb{Q}_v^\times} = \chi_{\mathcal{K}/\mathbb{Q},v}$. As noted earlier, there is a splitting $\mathrm{U}_h(\mathbb{Q}_v) \times \mathrm{U}_2(\mathbb{Q}_v) \hookrightarrow \widetilde{\mathrm{Sp}}(\boldsymbol{W}, \mathbb{Q}_v)$ of the metaplectic cover $\widetilde{\mathrm{Sp}}(\boldsymbol{W}, \mathbb{Q}_v) \to \mathrm{Sp}(\boldsymbol{W}, \mathbb{Q}_v)$ determined by the character λ_v. This gives the Weil representation $\omega_{\lambda_v,1}$, which we denote here as $\omega_{h,\lambda_v}(u, g)$ of $\mathrm{U}_h(\mathbb{Q}_v) \times \mathrm{U}_2(\mathbb{Q}_v)$ where $u \in \mathrm{U}_h(\mathbb{Q}_v)$ and $g \in \mathrm{U}_2(\mathbb{Q}_v)$, via the Weil representation of $\widetilde{\mathrm{Sp}}(\boldsymbol{W}, \mathbb{Q}_v)$ on the space of Schwartz functions $\mathcal{S}(V_v \otimes_{\mathcal{K}_v} \boldsymbol{X}_v)$ (we use the polarization $\boldsymbol{W} = V_v \otimes_{\mathcal{K}_v} \boldsymbol{X}_v \oplus V_v \otimes_{\mathcal{K}_v} \boldsymbol{Y}_v$). Moreover, we write $\omega_{h,\lambda_v}(g)$ to mean $\omega_{h,\lambda_v}(1, g)$. For $X \in M_{1\times 2}(\mathcal{K}_v)$, we define $\langle X, X\rangle_h := {}^t\overline{X}hX$ (note that this is a 2×2 matrix). We record here some useful formulas for ω_{h,λ_v} which are generalizations of the formulas in [Skinner and Urban 2014, Section 10.1].

$$\begin{aligned}
\omega_{h,\lambda_v}(u, g)\Phi(X) &= \omega_{h,v,\lambda_v}(1, g)\Phi(u^{-1}X),\\
\omega_{h,\lambda_v}(\mathrm{diag}(A, {}^t\bar{A}^{-1}))\Phi(X) &= \lambda(\det A)|\det A|_{\mathcal{K}}^{1/2}\Phi(XA),\\
\omega_{h,\lambda_v}(r(S))\Phi(X) &= \Phi(X)e_v(\mathrm{tr}\langle X, X\rangle_h S),\\
\omega_{h,\lambda_v}(\eta)\Phi(X) &= |\det h|_v \int \Phi(Y)e_v(\mathrm{Tr}_{\mathcal{K}_v/F_v}(\mathrm{tr}\langle Y, X\rangle_h))\, dY.
\end{aligned}$$

The global set-up. Let $h \in S_1(\mathbb{Q})$, $h > 0$. We can define global versions of U_h, GU_h, $\boldsymbol{W}$, and $(-,-)$, similar as the local case above. Fixing an idele class character $\lambda = \otimes\lambda_v$ of $\mathbb{A}_{\mathcal{K}}^\times/\mathcal{K}^\times$ such that $\lambda|_{\mathbb{Q}^\times} = \chi^1_{\mathcal{K}/\mathbb{Q}}$, the associated local splitting described above then determines a global splitting $\mathrm{U}_h(\mathbb{A}_\mathbb{Q}) \times \mathrm{U}_1(\mathbb{A}_\mathbb{Q}) \hookrightarrow \widetilde{\mathrm{Sp}}(\boldsymbol{W}, \mathbb{A}_\mathbb{Q})$ and hence an action $\omega_h := \otimes\omega_{h,\lambda_v}$ of $\mathrm{U}_h(\mathbb{A}_\mathbb{Q}) \times \mathrm{U}_1(\mathbb{A}_\mathbb{Q})$ on the Schwartz space $S(V_{\mathbb{A}_\mathcal{K}} \otimes \boldsymbol{X})$. In application, we require the infinity type of λ to be $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

For any Schwartz function $\Phi \in S(V_{\mathbb{A}_\mathcal{K}} \otimes \boldsymbol{X})$ we define the theta function

$$\Theta_h(\Phi, -) : J(\mathbb{A}) \to \mathbb{C}.$$

associated to it by

$$\Theta_h(\Phi, u, g) = \sum_{x \in V \otimes X} \omega_{h,\lambda}(g)\Phi(x),\ g \in J(\mathbb{A}).$$

4I. ***Theta functions with complex multiplication.*** We consider the situation of theta correspondences for $\mathrm{U}(\zeta) = \mathrm{U}(V)$ and $\mathrm{U}(V_1)$. Let V be a two-dimensional Hermitian vector space over $\mathcal{K}$. Let L be an $\mathcal{O}_\mathcal{K}$ lattice such that it gives an abelian variety $\mathcal{A}_L = \mathbb{C}^2/L$.

Let H be a Riemann form on V and $\epsilon : L \to U$ be a map where U is the unit circle of $\mathbb{C}$ (in application the ϵ is given by the formula after [Zhang 2013, (38)], there is a line bundle $\mathfrak{L}_{H,\epsilon}$ on $\mathcal{A}_\mathcal{L}$ associated to H and ϵ as follows: define an analytic line bundle $\mathfrak{L}_{H,\epsilon} \simeq \mathbb{C} \times \mathbb{C}^2/L$ with the action of L given by

$$l \cdot (w, x) = \left(w + l, \epsilon(l)e\left(\tfrac{1}{2i}H\left(l, w + \tfrac{l}{2}\right)\right)x\right), \quad l \in L, (w, x) \in \mathbb{C}^2 \times \mathbb{C}$$

where $e(x) = e^{2\pi i x}$. The space of global sections of this line bundle is canonically identified with the space $T(H, \epsilon, L)$ of theta functions consisting of holomorphic functions f on V such that

$$f(w + l) = f(w)\epsilon(l)e\left(\tfrac{1}{2i}H\left(l + w + \tfrac{l}{2}\right)\right), \quad w \in V, l \in L.$$

There are arithmetic models for the above abelian variety and line bundle. Shimura defined subspaces $T^{\mathrm{ar}}(H, \epsilon, L) \subset T(H, \epsilon, L)$ of arithmetic theta functions by requiring that the values at all CM points are in $\bar{\mathbb{Q}}$ which under the canonical identification, are identified with rational sections of the line bundle (see [Lan 2012, Section 2.6], and also [Zhang 2013, Appendix B] for a discussion in terms of theta kernel functions. Note that we only need to consider arithmetic sections, but not integral sections.)

Adelic theta functions. Now we consider theta functions for U(3, 1). Let the Hermitian form on V be defined by

$$\langle v_1, v_2\rangle = v_1\zeta v_2^* - v_2\zeta v_1^*.$$

Let U_f be some compact open subgroup of $U(\zeta)(\mathbb{A}_f)$.

Definition 4.6. We define the space $T_{\mathbb{A}}(m, L, U_f)$ of adelic theta functions as the space of functions

$$\Theta : N(\mathbb{Q})\mathrm{U}(\zeta)(\mathbb{Q})\backslash N(\mathbb{A})\mathrm{U}(\zeta)(\mathbb{A})/\mathrm{U}(\zeta)_\infty \mathrm{U}_f N(L)_f \to \mathbb{C},$$

where $N = N_P \subset U(3, 1)$ and $U(\zeta) \hookrightarrow U(3, 1)$ as before;

$$N(L)_f = \big\{(w, t) \mid x \in \hat{L}, t + \tfrac{1}{2}w\zeta w^* \in \mu(L)\hat{\mathcal{O}}_{\mathcal{K}}\big\},$$

where $\mu(L)$ is the ideal generated by $w\zeta w^*$ for $w \in L$ and Θ satisfies

$$\Theta((0, t)r) = e(mt)\Theta(r), r \in N(\mathbb{A})U(\zeta)(\mathbb{A}).$$

Since $\mathrm{U}(\zeta)$ is anisotropic, the set $\mathrm{U}(\zeta)\backslash U(\zeta)(\mathbb{A})/\mathrm{U}(\zeta)_\infty\mathrm{U}_f$ consists of finitely many points $\{x_1, \dots, x_s\} \subset \mathrm{U}(\zeta)(\mathbb{A}_f)$. We assume that for each x_i the p-component is within $\mathrm{GL}_2(\mathbb{Z}_p)$ under the first projection $\mathrm{U}(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$. In this paper, we consider the case satisfying the following:

Assumpution. The lattice L is such that $m(\delta_{\mathcal{K}/\mathbb{Q}}^{-1})l\zeta{}^t\bar{l}$ is always integral for any i and $l \in x_iL$.

Then for any

$$\Theta \in T_{\mathbb{A}}(m, L, U_f),$$

write $\Theta_i(n) = \Theta(nx_i)$ for $n \in N(\mathbb{A})$ as functions on $N(\mathbb{A})$. Then for each i we define the function

$$\theta_i(w_\infty) = e\big(-m\tfrac{1}{2}w\zeta w^*\big)\Theta_i((w_\infty, 0)).$$

If this function is holomorphic then it is a classical theta function in $T(H, \epsilon, x_iL)$ where H and ϵ are defined as follows. The

$$H(v, v) := -2miv\zeta{}^t\bar{v}.$$

As in [Hsieh 2014a, Subsection 7.2.3], we choose a finite idele $u = (u_v) \in \mathbb{A}_{\mathcal{K},f})$ such that $\mathcal{O}_{\mathcal{K}_v} = \mathbb{Z}_v \oplus \mathbb{Z}_v u_v$ for each finite place v. For each $l \in x_iL$ let x_l and y_l be the unique elements in $\mathbb{A}_{\mathbb{Q},f}$ such that

$$\tfrac{1}{2}l\zeta{}^t\bar{l} = x_l + y_l u.$$

Let ψ be the standard additive character $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ such that at the Archimedean place it is given by $\psi(x_\infty) = e^{2\pi i x_\infty}$. Let $\psi_m(x) = \psi(m\,\mathrm{tr}_{\mathcal{K}/\mathbb{Q}}(x)$ for $x \in \mathbb{A}_{\mathcal{K}}$. Define $\epsilon(l) = \psi_m(-x_l)$. Then under our assumptions the ϵ is well defined and takes a value of ± 1.

If it is the case that for all i the θ_i is holomorphic, then we say it is a holomorphic adelic theta function and write the corresponding space as $T^{\mathrm{Hol}}_{\mathbb{A}}(m, L, U_f)$.

We make the following identification (recall we defined $\mathcal{Z}^\circ_{[g]}$ in Section 3F)

$$H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta)) \otimes \mathbb{C} \simeq T^{\mathrm{Hol}}_{\mathbb{A}}(\beta, L, U_f)$$

for certain level group $U_f = U_p \times \prod_{v\nmid p} U_v$.

A functional. Recall that we constructed a theta function θ_ϕ on the Jacobi group $J(V)$ from the Schwartz function ϕ. As mentioned in the introduction, we only need to develop a rational theory on theta functions instead of p-integral theory. Upon choosing $v_1 \in V_1$ such that $\langle v_1, v_1\rangle = 1$ we have an isomorphism $V \simeq W = V \otimes V_1$. We also consider $W^- = V^- \otimes V_1$. It is the space W but with the metric being the negative of W. Let $H^- = H(W^-)$ be the corresponding Heisenberg group. We have an isomorphism of H and H^- (as Heisenberg groups) given by

$$(w, t) \to (w, -t).$$

We construct a theta function $\theta^\star = \theta_{\phi_1}$ on $H^- \rtimes \mathrm{U}(V^-)$ for the Schwartz function ϕ_1. We have chosen a set $\{x_1, \dots, x_s\}$ above. We write

$$\langle \theta_{\phi_1}, \theta_\phi\rangle(x_i) := \int_{[N]} \theta_{\phi_1}(nx_i)\theta_\phi(nx_i)\,dn.$$

and

$$(\phi_1, \phi)_{x_i} = \int_{X(\mathbb{A}_{\mathbb{Q}})} \omega_{\lambda^{-1}}(x_i)(\phi_1)(x)\omega_\lambda(x_i)(\phi)(x)\,dx.$$

Then it is easy to check for each x_i,

$$\langle \theta_{\phi_1}, \theta_\phi\rangle(x_i) = (\phi_1, \phi)_{x_i}. \tag{4-3}$$

We first construct a functional $l'_{\theta^\star}$ on the space $H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$, which we identify with some $T(\beta, \epsilon, L)$ for appropriate ϵ and L as before (we save the notation $l_{\theta^\star}$ for later use) with values in $A_{[g]}$.

Lemma 4.7. *The general elements in $H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$ is a linear combination of $\frac{\theta_\phi}{\Omega_\infty}$'s with coefficients in $A_{[g]}$, where $\phi = \prod_v \phi_v$ are kernel functions such that ϕ_∞ is defined as in Definition 6.12, and ϕ_v takes $\bar{\mathbb{Q}}$-values for $v < \infty$.*

Proof. This can be seen by interpreting the theta functions defined before [Zhang 2013, Theorem B.2] in terms of Weil representations presented here. Note that the CM period Ω_∞ is missing in [loc. cit., Theorem B.2]. The algebraicity follows from [Shimura 1976, Theorem 2.5]. In fact in [loc. cit.] the period is $h(z_0)$ for some weight $\frac{1}{2}$ form h on Sp_4, as our Hermitian space is two-dimensional, and z_0 is a CM point with $h(z_0) \neq 0$. This $h(z_0)$ is just Ω_∞ up to multiplying by a nonzero algebraic number. □

We define

$$l'_{\theta^\star}\left(\frac{\theta_\phi}{\Omega_\infty}\right)(x_i) := \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \theta_{\omega_{\lambda^{-1}}(x_i)(\phi_1)}(n)\theta_{\omega_\lambda(x_i)(\phi)}(n)\,dn$$
$$= \int_{X(\mathbb{A}_\mathbb{Q})} \omega_{\lambda^{-1}}(x_i)(\phi_1)(x)\omega_\lambda(x_i)(\phi)(x)\,dx. \tag{4-4}$$

The last equality is easily seen, and we denote the last term as $(\omega_\lambda^{-1}(x_i)(\phi_1), \omega_\lambda(x_i)(\phi))$. Note that in $N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})$ we identified H with H^- using the above isomorphism. (In the literature, people usually consider $\int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \theta_\phi(n)\bar{\theta}_{\phi_1}(n)\,dn$ for θ_ϕ and θ_{ϕ_1} on the same space of theta functions. We use a different convention for the sake of simplicity.) So by taking appropriate ϕ_1 the $l'_{\theta^\star}$ is a rational functional. We extend the definition of $l'_{\theta^\star}$ to whole $H^0(\mathcal{Z}[g]^\circ, \mathcal{L}(\beta))$ linearly. Thus, it is well defined.

Lemma 4.8. *The $l'_{\theta^\star}$ takes values in the space of constant functions on any theta function θ_ϕ as above.*

Proof. We note that for any ϕ,

$$(\omega_{\lambda^{-1}}(x_i)(\phi_1), \omega_\lambda(x_i)(\phi)) = (\phi_1, \phi). \tag{4-5}$$

This is a standard fact and can be seen by simply unfolding the definition and integration. The lemma follows from the above equation. □

Remark 4.9. Later we will use this functional on Fourier–Jacobi coefficients for U(3, 1). We can view it as a function on $G_P N_P(\mathbb{A})$ by $\mathrm{FJ}_{[g]}(p, f) = \mathrm{FJ}_P(pg, f)$ for $p \in G_P N_P(\mathbb{A})$ and thus an adelic theta function. Lan [2012] has proved the following compatibility of the analytically and algebraically defined Fourier–Jacobi expansions using the usual identification of the global sections of $\mathcal{L}(\beta)$ with (classical or adelic) theta functions, keeping the rational structures

$$\mathrm{FJ}_{[hg]}(-, f) = \mathrm{FJ}^h_{[g]}(f)(-).$$

Note that the period factor appearing in [Hsieh 2014a, Section 3.6.5] is 1 since we are in the scalar weight κ.

5. Klingen–Eisenstein series

From now on throughout this paper we define $z_\kappa = (\kappa - 3)/2$ and $z'_\kappa = (\kappa - 2)/2$.

5A. ***Archimedean picture.*** Let (π_∞, V_∞) be a finite-dimensional representation of $D_\infty^\times$. Let ψ_∞ and τ_∞ be characters of $\mathbb{C}^\times$ such that $\psi_\infty|_{\mathbb{R}^\times}$ is the central character of π_∞. We assume here that $\tau_\infty(z) = z^{-\kappa/2}\bar{z}^{\kappa/2}$ and ψ_∞ is trivial. Then there is a unique representation π_ψ of GU(2)($\mathbb{R}$) determined by π_∞ and ψ_∞ such that the central character is ψ_∞. These determine a representation $\pi_\psi \times \tau_\infty$ of $M_P(\mathbb{R}) \simeq \mathrm{GU}(2)(\mathbb{R}) \times \mathbb{C}^\times$. Here for $g \in \mathrm{GU}(2)(\mathbb{R})$ and $x \in \mathbb{C}^\times$, we identify with it an element

$$m(g, x) = \begin{pmatrix} \mu(g)x^{-1} & & \\ & g & \\ & & x \end{pmatrix} \in M_P(\mathbb{R}).$$

We extend this to a representation ρ_∞ of $P(\mathbb{R})$ by requiring that $N_P(\mathbb{R})$ acts trivially. Let $I(V_\infty) = \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\rho_\infty$ (smooth induction) and $I(\rho_\infty) \subset I(V_\infty)$ be the subspace of K_∞ -finite vectors. (Elements of $I(V_\infty)$ can be realized as functions on K_∞.) For any $F \in I(V)$ and $z \in \mathbb{C}^\times$ we define a function F_z on $G(\mathbb{R})$ by

$$F_z(g) := \delta(m)^{3/2+z}\rho(m)f(k), g = mnk \in P(\mathbb{R})K_\infty.$$

There is an action $\sigma(\rho, z)$ on $I(V_\infty)$ by

$$(\sigma(\rho, z)(g))(k) = F_z(kg).$$

We let $\rho_\infty^\vee$ and $I(\rho_\infty^\vee)$ be the corresponding objects by replacing π_∞, ψ_∞, τ_∞ with $\pi_\infty \otimes (\tau_\infty \circ \mathrm{Nm})$, $\psi_\infty\tau_\infty\tau_\infty^c$, $\bar{\tau}_\infty^c$. Let

$$w = \begin{pmatrix} &&&1\\ &1&&\\ &&1&\\ -1&&& \end{pmatrix}.$$

Then there is an intertwining operator $A(\rho_\infty, z, -) : I(\rho_\infty) \to I(\rho_\infty^\vee)$ by

$$A(\rho_\infty, z, F)(k) := \int_{N_P(\mathbb{R})} F_z(wnk)\, dn.$$

In this paper, we use the case when π_∞ is the trivial representation. By the Frobenius reciprocity law there is a unique (up to scalar) vector $\tilde{v} \in I(\rho)$ such that $k.\tilde{v} = \det\mu(k, i)^{-\kappa}\tilde{v}$ for any $k \in K_\infty^+$. We fix v and scale $\tilde{v}$ such that $\tilde{v}(1) = v$. In $\pi^\vee$ it has the action of K_∞^+ given by multiplying by $\det\mu(k, i)^{-\kappa}$. There is a unique vector $\tilde{v}^\vee \in I(\rho^\vee)$ such that the action of K_∞^+ is given by $\det\mu(k, i)^{-\kappa}$ and $\tilde{v}^\vee(w) = v$. Then by uniqueness there is a constant $c(\rho, z)$ such that $A(\rho, z, \tilde{v}) = c(\rho, z)\tilde{v}^\vee$.

Definition 5.1. We define $F_\kappa \in I(\rho)$ to be the $\tilde{v}$ as above.

We record the following lemma proved in [Wan 2015a, Section 5.4.2]:

Lemma 5.2. *Let $\kappa \geq 6$ and $z_\kappa = (\kappa - 3)/2$. Then $c(\rho, z_\kappa) = 0$.*

5B. *ℓ-adic picture.* Let (π_ℓ, V_ℓ) be an irreducible admissible representation of $D^\times(\mathbb{Q}_\ell)$ and π_ℓ is unitary and tempered if D is split at ℓ. Let ψ and τ be characters of $\mathcal{K}_\ell^\times$ such that $\psi|_{\mathbb{Q}_\ell^\times}$ is the central character of π_ℓ. Then similar to the Archimedean case, there is a unique irreducible admissible representation π_ψ of $\mathrm{GU}(2)(\mathbb{Q}_\ell)$ determined by π_ℓ and ψ_ℓ. As before we have a representation $\pi_\psi \times \tau$ of $M_P(\mathbb{Q}_\ell)$ and extend it to a representation ρ_ℓ of $P(\mathbb{Q}_\ell)$ by requiring that $N_P(\mathbb{Q}_\ell)$ acts trivially. Let $I(\rho_\ell) = \mathrm{Ind}_{P(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}\rho_\ell$ be the admissible induction.

Define F_z for $F \in I(\rho_\ell)$ and $\rho_\ell^\vee$, $I(\rho_\ell^\vee)$, $A(\rho_\ell, z, F)$ etcetera as before. For $v \notin \Sigma$ we have $D^\times(\mathbb{Q}_\ell) \simeq \mathrm{GL}_2(\mathbb{Q}_\ell)$. Moreover, we can choose isomorphism as a conjugation by elements in $\mathrm{GL}_2(\mathcal{O}_{\mathcal{K},\ell})$ (note that both groups are subgroups of $\mathrm{GL}_2(\mathcal{K}_\ell)$).

Definition 5.3. When π_ℓ, ψ_ℓ, τ_ℓ are unramified and $\varphi_\ell \in V_\ell$ are spherical vectors, then there is a unique vector $F^0_{\varphi_\ell} \in I(\rho_\ell)$ which is invariant under $G(\mathbb{Z}_\ell)$ and $F^0_{\varphi_\ell}(1) = \varphi_\ell$.

5C. ***Global picture.***

Definition 5.4. We define an Eisenstein data $\mathcal{D} = (\Sigma, \pi, \psi, \tau, \varphi)$ to consist of the following:

- A finite set of primes Σ containing all bad primes.
- An irreducible unitary cuspidal automorphic representation $(\pi = \otimes_v \pi_v, V)$ of $D^\times(\mathbb{A}_\mathbb{Q})$ and a vector $\varphi = \prod_v \varphi_v \in \pi$, which is ordinary at p.
- The $\psi = \prod \psi_v$ and $\tau = \prod \tau_v$, CM characters of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ of infinite types $(0, 0)$ and $\left(-\frac{\kappa}{2}, \frac{\kappa}{2}\right)$, respectively, such that $\psi|_{\mathbb{A}_\mathbb{Q}^\times}$ is the central character of π. We define $\xi := \psi/\tau$.

Remark 5.5. In application the φ we use later on is not of the form $\prod_v \varphi_v$, but is a finite sum of such functions. However all theory extends to this situation easily by linear combination.

We define $I(\rho)$ to be the restricted tensor product of $\otimes_v I(\rho_v)$ with respect to the unramified vectors $F^0_{\varphi_\ell}$ for some $\varphi = \otimes_v \varphi_v \in \pi$. We can define F_z, $I(\rho^\vee)$ and $A(\rho, z, F)$ similar to the local case. The F_z takes values in V which can be realized as automorphic forms on $D^\times(\mathbb{A}_\mathbb{Q})$. We also write F_z for the scalar-valued functions $F_z(g) := F_z(g)(1)$ and define the Klingen–Eisenstein series

$$E(F, z, g) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} F_z(\gamma g).$$

This is absolutely convergent if $\operatorname{Re} z \gg 0$ and has meromorphic continuation to all $z \in \mathbb{C}$.

5D. ***Good Klingen–Eisenstein sections.*** We specify good choices for Klingen–Eisenstein sections at local places. We write

$$w = \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{pmatrix}.$$

- For the Archimedean place we define $F_{\mathcal{D},\infty} := F_\kappa$.
- For finite places v outside Σ, we define $F_{\mathcal{D},v} := F^0_{\varphi_v}$.
- For finite places v inside Σ, let y be an element in $\mathcal{O}_v$ divisible by some high power of the uniformizer ϖ_v at v (to be made precise in the next chapter). Let $\mathfrak{Y}$ be the set of matrices $A \in \mathrm{U}(2)(\mathbb{Q}_v)$ such that $M = A - 1$ satisfies

$$M(1 + y\bar{y}N) = \zeta y\bar{y}$$

for some $N \in M_2(\mathcal{O}_v)$. Let φ be some vector invariant under the action of $\mathfrak{Y}$. Let $K_v^{(2)}$ be the subgroup of $G(\mathbb{Q}_v)$ of the form

$$\begin{pmatrix} 1 & f & c \\ & 1_2 & g \\ & & 1 \end{pmatrix}$$

where

$$g = -\zeta^t \bar{f}, \quad c - \tfrac{1}{2} f \zeta^t \bar{f} \in \mathbb{Z}_\ell, \quad f \in (y\bar{y}), \quad g \in (\zeta y\bar{y}), \quad c \in \mathcal{O}_v.$$

We define $F_{y,v}$ to be supported in $PwK_v^{(2)}$ and is invariant under the right action of $K_v^{(2)}$, and such that

$$F_{y,v}(w)=\tau(y\bar{y})|(y\bar{y})^2|_v^{-z-3/2}\,\mathrm{Vol}(\mathfrak{Y})\cdot\varphi.$$

This $F_{y,v}$ is the Klingen–Eisenstein section $F_{\mathcal{D},v}$ that we choose.

- For the p-adic places we use the following:

Definition 5.6. Define $F^{0,\bullet}_{\mathcal{D},p}\in I_p(\rho)$ to be the Klingen section described as follows It is supported in $P(\mathbb{Q}_p)w_3'B_t(\mathbb{Z}_p)$ where $B_t(\mathbb{Z}_p)$ consists of matrices in $\mathrm{GL}_4(\mathbb{Z}_p)$ which are upper triangular modulo p^t,

$$w_3'=\begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1\end{pmatrix}\in\mathrm{GL}_4(\mathbb{Z}_p),$$

and is right invariant under $N_t(\mathbb{Z}_p)$ for $N_t(\mathbb{Z}_p)\subseteq B_t(\mathbb{Z}_p)$ consisting of matrices which are 1 along the diagonal modulo p^t; see [Wan 2015a, Section 4], note the differences in the indices discussed there (Subsections 4.D.2 and 4.D.4 in [loc. cit.]). The w_3' here is the w_{Borel} there.). Moreover we require that the value of $F^{0,\bullet}_{\mathcal{D},p}$ on w_3' is given by φ. We define the p-part level group $K_t\subset\mathrm{GL}_4(\mathbb{Z}_p)$ to consist of matrices congruent to some elements in $B(\mathbb{Z}_p)$ modulo p^t.

Now we briefly recall Hida's U_p operator for U(3, 1). We refer to [Hsieh 2014a, Section 3.8] for geometric backgrounds for U_p operators, and that it is compatible with our representation theoretic description below. Let $\lambda_1,\dots,\lambda_4$ be 4 characters of $\mathbb{Q}_p^\times$, $\pi=\mathrm{Ind}_B^{\mathrm{GL}_n}(\lambda_1,\dots,\lambda_4)$.

Definition 5.7. Let $\underline{k}=(c_4;c_1,\dots,c_3)$ be a weight. We say $(\lambda_1,\dots,\lambda_n)$ is nearly ordinary with respect to $\underline{k}$ if the set

$$\{\mathrm{val}_p\,\lambda_1(p),\dots,\mathrm{val}_p\,\lambda_4(p)\}=\left\{c_1-\tfrac{3}{2},c_2+\tfrac{3}{2},c_3+\tfrac{1}{2},c_4-\tfrac{1}{2}\right\}$$

We denote the set as $\{\kappa_1,\dots,\kappa_{r+s}\}$ so that $\kappa_1>\dots>\kappa_{r+s}$.

We define the p-component of the Eisenstein data is *generic* if it satisfies Definition 6.30. Let $\mathcal{A}_p:=\mathbb{Z}_p[t_1,t_2,\dots,t_n,t_n^{-1}]$ be the Atkin–Lehner ring of $G(\mathbb{Q}_p)$, where t_i is defined by

$$t_i=N(\mathbb{Z}_p)\alpha_iN(\mathbb{Z}_p),\quad \alpha_i=\begin{pmatrix}1_{n-i} & \\ & p1_i\end{pmatrix}.$$

Then t_i acts on $\pi^{N(\mathbb{Z}_p)}$ by

$$v\mid t_i=\sum_{x\in N|\alpha_i^{-1}N\alpha_i}x_i\alpha_i^{-1}v.$$

We also define a normalized action with respect to the weight $\underline{k}$ following [Hida 2004]

$$v\parallel t_i:=\delta(\alpha_i)^{-1/2}p^{\kappa_1+\dots+\kappa_i}v\mid t_i.$$

(The δ is the modulus character).

Definition 5.8. A vector $v \in \pi$ is called nearly ordinary if it is an eigenvector for all $\|t_i$'s with eigenvalues that are p-adic units.

Now we define

$$E_{\mathrm{Kling},\mathcal{D}}(g) := E\Big(\prod_{v \nmid p} F_{\mathcal{D},v} \times F^{0,\bullet}_{\mathcal{D},p}, z_\kappa, g\Big). \tag{5-1}$$

The following proposition is easily proved as in [Skinner and Urban 2014, Proposition 9.8].

Proposition 5.9. *The classical automorphic form corresponding to $E_{\mathrm{Kling},\mathcal{D}}$ defined by (3-1) is a holomorphic automorphic form on* $\mathrm{U}(3, 1)$ *with weight* $\underline{k} = (\kappa; 0, 0, 0)$.

Suppose π_p is nearly ordinary, in the sense that it is of the form $\pi(\chi_{1,p}, \chi_{2,p})$ with $\mathrm{val}_p(\chi_{1,p}(p)) = -\frac{1}{2}$, $\mathrm{val}_p(\chi_{2,p}(0)) = \frac{1}{2}$. Then it is easy to check that the representation $I(\rho_p)$ is nearly ordinary with respect to the weight $\underline{k}$. Suppose moreover that the p-component of the Eisenstein datum is generic. Then we have

Proposition 5.10. *The $F^{0,\bullet}_p$ is an eigenvector for all the actions $\|t_i$ with eigenvalues*

$$\lambda_1 \cdots \lambda_i(p^{-1}) p^{\kappa_1 + \cdots + \kappa_i},$$

which are clearly p-adic units.

Proof. The proof is a little convoluted and given in [Wan 2015a, Subsection 4.4.1]. It uses the intertwining operator which maps $F^{0,\bullet}_p$ to the section supported on the big cell (denoted f^ℓ there; see Lemma 4.16 of [loc. cit.]), whose eigenvalues are easy to compute. It is proved in [loc. cit., Lemma 4.17] that the f^ℓ is indeed an ordinary operator. In the generic case, the intertwining operator gives an isomorphism between the corresponding principal series representations of $\mathrm{GL}_4(\mathbb{Q}_p)$. Although our definition of being "generic" is different from [loc. cit.], however the argument of [loc. cit., Lemma 4.17] still works. Then as in the proof of [loc. cit., Lemma 4.19], one checks the $F^{0,\bullet}_p$ and the f^ℓ have the same action by the level group K^t, and that such vector is unique up to scalar in the corresponding principal series representation, identifying the $F^{0,\bullet}_p$ and the f^ℓ under the intertwining operator. These altogether implies that $F^{0,\bullet}_p$ is indeed a nearly ordinary vector.

In our $\mathrm{U}(3, 1)$ situation the argument is also given by Hsieh [2011, Section 6.2]. □

Remark 5.11. It is worth pointing out that the definition is quite different from the $\mathrm{U}(2, 2)$ case in [Skinner and Urban 2014]. The nearly ordinary section is supported on the set containing the Weyl element w_3' instead of the identity. This description also coincides with property that in [Hsieh 2011, Lemma 6.6] that the only Weyl element in which the ordinary section is nonzero is w_3'.

Definition 5.12. Throughout this paper, we fix the tame level subgroup $K^{(3,1)}_{\mathcal{D}}$ of $\mathrm{U}(3, 1)(\mathbb{A}^{p\infty})$, under which our $E_{\mathrm{Kling},\mathcal{D}}$ is invariant. We can do so by simply taking it to be the set of matrices congruent to 1 modulo the $(y\bar{y})^2$ at each finite place not dividing p for the y above. We also define the p-component of the level group as the $N_t(\mathbb{Z}_p)$ above, where t is the one in the definition for "generic".

5E. *Constant terms.*

Definition 5.13. For any parabolic subgroup R of $\mathrm{GU}(3,1)$ and an automorphic form φ we define φ_R to be the constant term of φ along R given by the following

$$\varphi_R(g)=\int_{N_R(\mathbb{Q})\backslash N_R(\mathbb{A}_{\mathbb{Q}})}\varphi(ng)\,dn$$

where N_R is the unipotent radical of R.

The following lemma is well-known (see [Mœglin and Waldspurger 1995, Section II.1.7]).

Lemma 5.14. *Let R be a standard $\mathbb{Q}$-parabolic subgroup of* $\mathrm{GU}(3,1)$. *Suppose* $\mathrm{Re}(z)>\frac{3}{2}$.

(i) *If $R\neq P$ then $E(f,z,g)_R=0$.*

(ii) $E(f,z,-)_P=f_z+A(\rho,f,z)_{-z}$.

5F. *Hecke operators.* Let $K'=K'_{\Sigma\backslash\{p\}}K^{\Sigma}\subset G(\mathbb{A}_f^p)$ be an open compact subgroup with $K^{\Sigma}=G(\hat{\mathbb{Z}}^{\Sigma})$ and such that $K:=K'K_p^0$ is neat. For each v outside Σ we have $\mathrm{GU}(3,1)(\mathbb{Q}_v)\simeq\mathrm{GU}(2,2)(\mathbb{Q}_v)$ with the isomorphism given by conjugation by some elements in $\mathrm{GL}_4(\mathcal{O}_{\mathcal{K},v})$. So we only need to study the unramified Hecke operators for $\mathrm{GU}(2,2)$ with respect to $\mathrm{GU}(2,2)(\mathbb{Z}_v)$. We follow closely to [Skinner and Urban 2014, Sections 9.5 and 9.6].

Unramified inert case. Let v be a prime of $\mathbb{Q}$ inert in $\mathcal{K}$. Recall that as in [Skinner and Urban 2014, Section 9.5.2] that $Z_{v,0}$ is the Hecke operator associated to the matrix $z_0:=\mathrm{diag}(\varpi_v,\varpi_v,\varpi_v,\varpi_v)$ by the double coset Kz_0K where K is the maximal compact subgroup of $G(\mathbb{Z}_v)$. Let $t_0:=\mathrm{diag}(\varpi_v,\varpi_v,1,1)$, $t_1:=\mathrm{diag}(1,\varpi_v,1,\varpi_v^{-1})$ and $t_2:=\mathrm{diag}(\varpi_v,1,\varpi_v^{-1},1)$. As in [loc. cit., 9.5.2] we define

$$\mathcal{R}_v:=\mathbb{Z}[X_v,q^{1/2},q^{-1/2}]$$

where X_v is $T(\mathbb{Q}_v)/T(\mathbb{Z}_v)$ and write $[t]$ for the image of t in X_v. Let $\mathcal{H}_K$ be the abstract Hecke ring with respect to the level group K. There is a Satake map: $\mathcal{S}_K:\mathcal{H}_K\to\mathcal{R}_v$ given by

$$\mathcal{S}_K(KgK)=\sum\delta_B^{1/2}(t_i)[t_i]$$

if $KgK=\sqcup t_in_iK$ for $t_i\in T(\mathbb{Q}_v)$, $n_i\in N_B(\mathbb{Q}_v)$ and extend linearly. We define the Hecke operators T_i for $i=1,2,3,4$ by requiring that

$$1+\sum_{i=1}^{4}\mathcal{S}(T_i)X^i=\prod_{i=1}^{2}(1-q_v^{3/2}[t_i]X)(1-q_v^{3/2}[t_i]^{-1}X)$$

be an equality of polynomials of the variable X. We also define

$$Q_v(X):=1+\sum_{i=1}^{4}T_i(Z_0X)^i.$$

Unramified split case. Suppose v is a prime of $\mathbb{Q}$ split in $\mathcal{K}$. In this case we define $z_0^{(1)}$ and $z_0^{(2)}$ to be $(\mathrm{diag}(\varpi_v,\varpi_v,\varpi_v,\varpi_v),1)$ and $(1,\mathrm{diag}(\varpi_v,\varpi_v,\varpi_v,\varpi_v))$ and define the Hecke operators $Z_0^{(1)}$

and $Z_0^{(2)}$ as above but replacing z_0 by $z_0^{(1)}$ and $z_0^{(2)}$. Let $t_1^{(1)} := \mathrm{diag}(1, (\varpi_v, 1), 1, (1, \varpi_v^{-1}))$ and $t_2^{(1)} := \mathrm{diag}((\varpi_v, 1), 1, (1, \varpi_v^{-1}), 1)$. Define $t_i^{(2)} := \bar{t}_i^{(1)}$ and $t_i = t_i^{(1)} t_i^{(2)}$ for $i = 1, 2$. We define R_v and $\mathcal{S}_K$ in the same way as the inert case. Then we define Hecke operators $T_i^{(j)}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ by requiring that the following

$$1 + \sum_{i=1}^{4} \mathcal{S}_K(T_i^{(j)}) X^i = \prod_{i=1}^{2} (1 - q_v^{3/2} [t_i^{(j)}] X)(1 - q_v^{3/2} [t_i^{(j')}]^{-1} X)$$

be equalities of polynomials of the variable X. Here $j' = 3 - j$ and $[t_i^{(j)}]$'s are defined similarly to the inert case. Now suppose $v = w\bar{w}$ for a place w of $\mathcal{K}$ and a place v of $\mathbb{Q}$. Define $i_w = 1$ and $i_{\bar{w}} = 2$. Then we define

$$Q_w(X) := 1 + \sum_{i=1}^{4} T_i^{(i_w)} (Z_0^{(3-i_w)} X)^i \quad \text{and} \quad Q_{\bar{w}}(X) := 1 + \sum_{i=1}^{4} T_i^{(i_{\bar{w}})} (Z_0^{(3-i_{\bar{w}})} X)^i.$$

5G. *Galois representations.* For the holomorphic Klingen–Eisenstein series, we can also associate a reducible Galois representation with the same recipe as in Section 3C. Write τ' for the restriction of τ to $\mathbb{A}_{\mathbb{Q}}$ and let $\sigma_{\tau'}$ be the corresponding Galois character of $G_{\mathbb{Q}}$ via class field theory. The resulting Galois representation associated to the Klingen–Eisenstein series we defined above can be seen as follows:

$$\sigma_{\tau'} \sigma_{\psi^c} \epsilon^{-\kappa} \oplus \sigma_{\psi^c} \epsilon^{-3} \oplus \rho_{\pi_f} . \sigma_{\tau^c} \epsilon^{-(\kappa+3)/2}.$$

Note that $\kappa + 3$ is an odd number; however, π_f is a unitary representation whose L-function is the usual L-function for f shifted by $\frac{1}{2}$. So it makes sense to write in the above way. This can be obtained in the same manner as [Skinner and Urban 2014, Sections 9.5 and 9.6], by studying the Hecke operators defined above. Indeed the Galois representation is determined by its local Euler factors at unramified places, which has been worked out in [loc. cit., in particular Proposition 9.14].

6. Siegel–Eisenstein series and pullback

6A. *Generalities.*

Local picture. Our discussion in this section follows [Skinner and Urban 2014, Sections 11.1–11.3] closely. Let $Q = Q_n$ be the Siegel parabolic subgroup of G_n consisting of matrices $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$. It consists of matrices whose lower-left $n \times n$ block is zero. For a place v of $\mathbb{Q}$ and a character χ of $\mathcal{K}_v^\times$ we let $I_n(\chi_v)$ be the space of smooth $K_{n,v}$-finite functions (here $K_{n,v}$ means the maximal compact subgroup $G_n(\mathbb{Z}_v)$) $f : K_{n,v} \to \mathbb{C}$ such that $f(qk) = \chi_v(\det D_q) f(k)$ for all $q \in Q_n(\mathbb{Q}_v) \cap K_{n,v}$ (we write q as block matrix $\begin{pmatrix} A_q & B_q \\ 0 & D_q \end{pmatrix}$). For $z \in \mathbb{C}$ and $f \in I(\chi)$ we also define a function $f(z, -) : G_n(\mathbb{Q}_v) \to \mathbb{C}$ by $f(z, qk) := \chi(\det D_q)) |\det A_q D_q^{-1}|_v^{z+n/2} f(k)$, $q \in Q_n(\mathbb{Q}_v)$ and $k \in K_{n,v}$.

For $f \in I_n(\chi_v)$, $z \in \mathbb{C}$, and $k \in K_{n,v}$, the intertwining integral is defined by

$$M(z, f)(k) := \bar{\chi}_v^n(\mu_n(k)) \int_{N_{Q_n}(F_v)} f(z, w_n r k)\, dr$$

where N_{Q_n} is the unipotent radical of Q_n, and $w_n := \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$. For z in compact subsets of $\{\mathrm{Re}(z) > n/2\}$ this integral converges absolutely and uniformly, with the convergence being uniform in k. In this case it is easy to see that $M(z, f) \in I_n(\bar{\chi}_v^c)$. Let $\mathcal{U} \subseteq \mathbb{C}$ be an open set. By a meromorphic section of $I_n(\chi_v)$ on $\mathcal{U}$ we mean a function $\varphi : \mathcal{U} \mapsto I_n(\chi_v)$ taking values in a finite-dimensional subspace $V \subset I_n(\chi_v)$ and such that $\varphi : \mathcal{U} \to V$ is meromorphic. A standard fact from the theory of Eisenstein series says that this has a continuation to a meromorphic section on all of $\mathbb{C}$.

Global picture. For a Hecke character $\chi = \otimes\chi_v$ of $\mathbb{A}_{\mathcal{K}}^\times$ we define a space $I_n(\chi)$ to be the restricted tensor product defined using the spherical vectors $f_v^{\mathrm{sph}} \in I_n(\chi_v)$ (invariant under $K_{n,v}$) such that $f_v^{\mathrm{sph}}(K_{n,v}) = 1$, at the finite places v where χ_v is unramified.

For $f \in I_n(\chi)$ we consider the Eisenstein series

$$E(f; z, g) := \sum_{\gamma \in Q_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} f(z, \gamma g).$$

This series converges absolutely and uniformly for (z, g) in compact subsets of

$$\left\{\mathrm{Re}(z) > \tfrac{n}{2}\right\} \times G_n(\mathbb{A}_{\mathbb{Q}}).$$

The defined automorphic form is called Siegel–Eisenstein series.

The Eisenstein series $E(f; z, g)$ has a meromorphic continuation in z to all of $\mathbb{C}$ in the following sense. If $\varphi : \mathcal{U} \to I_n(\chi)$ is a meromorphic section, then we put $E(\varphi; z, g) = E(\varphi(z); z, g)$. This is defined at least on the region of absolute convergence and it is well-known that it can be meromorphically continued to all $z \in \mathbb{C}$.

Now for $f \in I_n(\chi)$, $z \in \mathbb{C}$, and $k \in \prod_{v\nmid\infty} K_{n,v} \prod_{v\mid\infty} K_\infty$, there is a similar intertwining integral $M(z, f)(k)$ as above but with the integral being over $N_{Q_n}(\mathbb{A}_{\mathbb{Q}})$. This again converges absolutely and uniformly for z in compact subsets of $\{\mathrm{Re}(z) > n/2\} \times K_n$. Thus $z \mapsto M(z, f)$ defines a holomorphic section $\{\mathrm{Re}(z) > n/2\} \to I_n(\bar{\chi}^c)$. This intertwining operator has a continuation to a meromorphic section on $\mathbb{C}$. For $\mathrm{Re}(z) > n/2$, we have

$$M(z, f) = \otimes_v M(z, f_v),\ f = \otimes f_v.$$

The functional equation for Siegel–Eisenstein series is

$$E(f, z, g) = \chi^n(\mu(g)) E(M(z, f); -z, g)$$

in the sense that both sides can be meromorphically continued to all $z \in \mathbb{C}$ and the equality is understood as an equality of meromorphic functions of $z \in \mathbb{C}$.

6B. ***Embeddings.*** We define some embeddings of a subgroup of $\mathrm{GU}(3, 1) \times \mathrm{GU}(0, 2)$ into some larger groups. This is used in the doubling method. First we define $\mathrm{GU}(3, 3)'$ to be the unitary similitude group associated to

$$\begin{pmatrix} & & 1 & \\ & \zeta & & \\ -1 & & & \\ & & & -\zeta \end{pmatrix}$$

and $\mathrm{GU}(2,2)'$ to be the unitary group associated to

$$\begin{pmatrix} \zeta & \\ & -\zeta \end{pmatrix}.$$

We define embeddings

$$\alpha : \{g_1 \times g_2 \in \mathrm{GU}(3,1) \times \mathrm{GU}(0,2),\, \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(3,3)'$$

and

$$\alpha' : \{g_1 \times g_2 \in \mathrm{GU}(2,0) \times \mathrm{GU}(0,2),\, \mu(g_1) = \mu(g_2)\} \to \mathrm{GU}(2,2)'$$

by $\alpha(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$ and $\alpha'(g_1, g_2) = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}$. We also define isomorphisms

$$\beta : \mathrm{GU}(3,3)' \xrightarrow{\sim} \mathrm{GU}(3,3) \quad \text{and} \quad \beta' : \mathrm{GU}(2,2)' \xrightarrow{\sim} \mathrm{GU}(2,2)$$

by

$$g \mapsto S^{-1} g S \quad \text{or} \quad g \mapsto S'^{-1} g S'$$

where

$$S = \begin{pmatrix} 1 & & & \\ & 1 & & -\zeta/2 \\ & & 1 & \\ & -1 & & -\zeta/2 \end{pmatrix} \quad \text{and} \quad S' = \begin{pmatrix} 1 & -\zeta/2 \\ -1 & -\zeta/2 \end{pmatrix}$$

We write γ and γ' for the embeddings $\beta \circ \alpha$ and $\beta' \circ \alpha'$, respectively.

We define an element $\Upsilon \in \mathrm{U}(3,3)(\mathbb{Q}_p)$ such that $\Upsilon_{v_0} = S_{v_0}^{-1}$ and $\Upsilon'_{v_0} = S_{v_0}^{-1,'}$, where S is defined as above. We know that under the complex uniformization, taking the change of polarization into consideration the map (3-2) is given by

$$i([\tau, g], [x_0, h]) = [Z_\tau, (g, h)\Upsilon] \tag{6-1}$$

(see [Hsieh 2014a, Section 2.6].)

6C. *Pullback formula.* We recall the pullback formula of Shimura. Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$. Given a cusp form φ on $\mathrm{GU}(2)$ we consider

$$F_\varphi(f; z, g) := \int_{\mathrm{U}(2)(\mathbb{A}_\mathbb{Q})} f(z, S^{-1}\alpha(g, g_1 h) S)\bar{\chi}(\det g_1 g)\varphi(g_1 h)\, dg_1,$$

$$f \in I_3(\chi), \quad g \in \mathrm{GU}(3,1)(\mathbb{A}_\mathbb{Q}), \quad h \in \mathrm{GU}(2)(\mathbb{A}_\mathbb{Q}), \quad \mu(g) = \mu(h)$$

or

$$F'_{\varphi}(f';z,g)=\int_{\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} f'(z,S'^{-1}\alpha(g,g_1h)S')\bar{\chi}(\det g_1g)\varphi(g_1h)\,dg_1$$
$$f'\in I_2(\chi),\quad g\in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}),\quad h\in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}),\quad \mu(g)=\mu(h)$$

This is independent of h. We see that the above integrals can be factorized as local integrals, which we denote as $F_{\varphi_v}(f_v;z,g_v)$ and $F'_{\varphi_v}(f'_v;z,g_v)$, respectively. The pullback formulas are the identities in the following proposition.

Proposition 6.1. *Let χ be a unitary idele class character of $\mathbb{A}_{\mathcal{K}}^\times$.*

(i) *If $f'\in I_2(\chi)$, then $F'_{\varphi}(f';z,g)$ converges absolutely and uniformly for (z,g) in compact subsets of $\{\mathrm{Re}(z)>1\}\times \mathrm{GU}(2,0)(\mathbb{A}_{\mathbb{Q}})$, and for any $h\in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(h)=\mu(g)$*

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} E(f';z,S'^{-1}\alpha(g,g_1h)S')\bar{\chi}(\det g_1h)\varphi(g_1h)\,dg_1=F'_{\varphi}(f';z,g).$$

(ii) *If $f\in I_3(\chi)$, then $F_{\varphi}(f;z,g)$ converges absolutely and uniformly for (z,g) in compact subsets of $\left\{\mathrm{Re}(z)>\frac{3}{2}\right\}\times \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(h)=\mu(g)$*

$$\int_{\mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})} E(f;z,S^{-1}\alpha(g,g_1h)S)\bar{\chi}(\det g_1h)\varphi(g_1h)\,dg_1=\sum_{\gamma\in P(\mathbb{Q})\backslash \mathrm{GU}(3,1)(\mathbb{Q})} F_{\varphi}(f;z,\gamma g),$$

with the series converging absolutely and uniformly for (z,g) in compact subsets of

$$\left\{\mathrm{Re}(z)>\tfrac{3}{2}\right\}\times \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}}).$$

This is a special case of [Wan 2015a, Proposition 3.5], which summarizes results proved in [Shimura 1997].

6D. *Fourier–Jacobi expansion.* From now on we fix a splitting character λ of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$ of infinity type $\left(-\frac{1}{2},\frac{1}{2}\right)$ which is unramified at p and unramified outside Σ and such that $\lambda|_{\mathbb{A}_{\mathbb{Q}}^\times}=\chi_{\mathcal{K}/\mathbb{Q}}$. Let τ be a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$ of infinity type $\left(-\frac{\kappa}{2},\frac{\kappa}{2}\right)$.

Definition 6.2. For $\beta\in S_n(\mathbb{Q})$ and φ a holomorphic automorphic form on $\mathrm{GU}(n,n)$ we define the β-th Fourier coefficient

$$\varphi_\beta(g)=\int_{S_n(\mathbb{Q})\backslash S_n(\mathbb{A})}\varphi\left(\begin{pmatrix}1_n & S\\ & 1_n\end{pmatrix}g\right)e_{\mathbb{A}}(-\mathrm{Tr}\,\beta S)\,dS.$$

For a prime v and $f_v\in I_n(\tau)$ we also define the local Fourier coefficient at $g_v\in \mathrm{GU}(n,n)(\mathbb{Q}_v)$ as

$$f_{v,\beta}(z,g_v)=\int_{S_n(\mathbb{Q}_v)} f_v\left(z,\omega_n\left(\begin{pmatrix}1_n & S_v\\ & 1_n\end{pmatrix}g_v\right)\right)e_v(-\mathrm{Tr}\,\beta S_v)\,dS_v.$$

For φ a holomorphic automorphic form on $\mathrm{GU}(3,3)$ and $\beta\in\mathbb{Q}^+$ we define

$$\mathrm{FJ}_\beta(\varphi)(g)=\int_{\mathbb{Q}\backslash\mathbb{A}}\varphi\left(\begin{pmatrix}1_3 & \begin{matrix}S & 0\\ 0 & 0\end{matrix}\\ & 1_3\end{pmatrix}g\right)e_{\mathbb{A}}(-\mathrm{Tr}\,\beta S)\,dS.$$

For $E(f;z,g)$ with $f\in I_3(\tau)$ we define

$$\mathrm{FJ}_\beta(f;z,g)=\mathrm{FJ}_\beta(E(f;z,-))(g).$$

The following formula is proved in [Wan 2015a, Subsection 3.3.1].

Proposition 6.3. *Suppose $f\in I_3(\tau)$ and $\beta\in\mathbb{Q}_+$. If $E(f;z,g)$ is the Siegel–Eisenstein series on $\mathrm{GU}(3,3)$ defined by f for some $\mathrm{Re}(z)$ sufficiently large, then the β-th Fourier–Jacobi coefficient $E_\beta(f;z,g)$ satisfies*

$$E_\beta(f;z,g)=\sum_{\gamma\in Q_2(\mathbb{Q})\backslash \mathrm{GU}_2(\mathbb{Q})}\sum_{x\in\mathcal{K}^2}\int_{S_1(\mathbb{A})} f\left(w_3\begin{pmatrix}1_3 & \begin{matrix} S & x\\ {}^t\bar{x} & 0\end{matrix}\\ & 1_3\end{pmatrix}\boldsymbol{j}(1,\gamma)g\right)e_{\mathbb{A}}(-\beta S)\,dS \tag{6-2}$$

where

$$\boldsymbol{j}:\mathrm{U}(1,1)\times\mathrm{U}(2,2)\hookrightarrow\mathrm{U}(3,3)$$

is given by

$$\boldsymbol{j}(g_1,g_2)=\begin{pmatrix}A & & B & \\ & D' & & C'\\ C & & D & \\ & B' & & A'\end{pmatrix}$$

if $g_1=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)\in\mathrm{U}(1,1)$, $g_2=\left(\begin{smallmatrix}A'&B'\\C'&D'\end{smallmatrix}\right)\in\mathrm{U}(2,2)$.

Definition 6.4. If $g_v\in\mathrm{U}(2,2)(\mathbb{Q}_v)$, $x\in\mathcal{K}_v^2$ and $a\in\mathcal{K}_v^\times$, we define

$$\mathrm{FJ}_\beta(f_v;z,x,g_v,a)=\int_{S_1(\mathbb{Q}_v)} f_{v,z}\left(w_3\begin{pmatrix}1_3 & \begin{matrix} S & x\\ {}^t\bar{x} & 0\end{matrix}\\ & 1_3\end{pmatrix}\boldsymbol{j}(\mathrm{diag}(a,{}^t\bar{a}^{-1}),g_v)\right)e_{\mathbb{Q}_v}(-\mathrm{Tr}\,\beta S)\,dS.$$

We have

$$\mathrm{FJ}_\beta(f_v;z,x,g,a)=\tau_v(\det a)|\det a\bar{a}|_{\mathbb{A}}^{-(z+1/2)}\,\mathrm{FJ}_{{}^t\bar{a}\beta x}(f_v;z,a^{-1}x,g,1).$$

Definition 6.5. For $x,y\in\mathcal{K}_v^2$ and $t\in\mathbb{Q}_v$, we write $n(x,y,t)$ for

$$\begin{pmatrix}1 & y & t+\frac{1}{2}(yx^*-xy*) & x\\ & 1_2 & x^* & 0_2\\ & & 1 & \\ & & -y^* & 1_2\end{pmatrix}.$$

So that it becomes a Heisenberg group if we give the pairing $\langle(x_1,y_1),(x_2,y_2)\rangle=y_1x_2^*+x_2y_1^*-y_2x_1^*-x_1y_2^*$. (see [Zhang 2013, Section 4]).

Lemma 6.6. *We write*

$$\mathrm{FJ}_\beta(f_v;z,n(x,y,t)\alpha(1,u))=\int_{S_1(\mathbb{Q}_v)} f\left(w_3\begin{pmatrix}1_3 & \begin{matrix} S & 0\\ 0 & 0\end{matrix}\\ & 1_3\end{pmatrix}(x,y,t)\boldsymbol{j}(1,u)\right)e_{\mathbb{A}}(-\mathrm{Tr}\,\beta S)\,dS$$

for $u\in U(2,2)(\mathbb{Q}_v)$ and $n(x,y,t)$ as above. Suppose for some place v we have

$$\mathrm{FJ}_\beta(f_v; z, n(x,0,0)\alpha(1,u)) = f(u,z)\omega_{\beta,\lambda}(u)\phi(x)$$

for any u, $n(x,0,0)$ as above, and some Schwartz function $\phi\in S(W^d)$ and some $f\in I((\tau/\lambda)_v, z)$. Then we have

$$\mathrm{FJ}_\beta(f_v; z, (x,y,t)\alpha(1,u)) = f(u,z)\omega_{\beta,\lambda}(n(x,y,t)\cdot u)\phi(0)$$

for any $n(x,y,t)$.

Note that comparing with [Ikeda 1994] we have switched the roles played by x and y.

Proof. Since

$$\begin{pmatrix}1_3 & \begin{smallmatrix} S & x\\ {}^t\bar{x} & \end{smallmatrix}\\ & 1_3\end{pmatrix}\begin{pmatrix}1_1 & & & \\ & \bar{A}^{-1} & & \\ & & 1 & \\ & B\bar{A}^{-1} & & A\end{pmatrix} = \begin{pmatrix}1 & xB\bar{A}^{-1} & & \\ & \bar{A}^{-1} & & \\ & & 1 & \\ & B\bar{A}^{-1} & -B\bar{x} & A\end{pmatrix}\begin{pmatrix}1_3 & \begin{smallmatrix} S-xB{}^t\bar{x} & xA\\ \bar{A}{}^t\bar{x} & \end{smallmatrix}\\ & 1_3\end{pmatrix},$$

it follows that

$$\mathrm{FJ}_\beta\left(f_v; z, x, \begin{pmatrix}A & B\bar{A}^{-1}\\ & \bar{A}^{-1}\end{pmatrix}g, a\right) = \tau_v^c(\det A)^{-1}|\det A\bar{A}|_v^{z+3/2}e_v(-\mathrm{Tr}({}^t\bar{a}\beta aB))\,\mathrm{FJ}_\beta(f; z, xA, g, a).$$

Now the lemma is a consequence of

$$\begin{pmatrix}1 & -y & & \\ & 1_2 & & \\ & & 1 & \\ & & y^* & 1_2\end{pmatrix} = \begin{pmatrix}1 & y & t+\frac{1}{2}(yx^*-xy^*) & x\\ & 1_2 & x^* & 0_2\\ & & 1 & \\ & & -y^* & 1_2\end{pmatrix}\begin{pmatrix}1 & & t-\frac{1}{2}(yx^*+xy^*) & x\\ & 1_2 & x^* & 0_2\\ & & 1 & \\ & & & 1_2\end{pmatrix}$$

where we write y^* for ${}^t\bar{y}$. □

6E. *Archimedean cases.* We let $\boldsymbol{i} := \left(\begin{smallmatrix} i & \\ & \zeta/2\end{smallmatrix}\right)$ or $\frac{\zeta}{2}$ depending on the size (3×3) or (2×2). Let $J_n(g,Z) := \det(C_g i + D_g)$ for $g = \left(\begin{smallmatrix}A_g & B_g\\ C_g & D_g\end{smallmatrix}\right)$ be the automorphic factor for $\mathrm{U}(n,n)$. The Siegel section we choose is $f_{\mathrm{sieg},\mathcal{D},\infty} = f_{\mathrm{sieg},\infty} := f_\kappa(g,z) := J_3(g,\boldsymbol{i})^{-\kappa}|J_3(g,\boldsymbol{i})|^{\kappa-2z-3}$ and $f'_{\mathrm{sieg},\mathcal{D},\infty} = f'_{\mathrm{sieg},\infty} := f'_\kappa(g,z) = J_2(g,\boldsymbol{i})^{-\kappa}|J_2(g,\boldsymbol{i})|^{\kappa-2z-2}$. Recall for $\varphi\in\pi_\infty$ we define the pullback sections

$$F_\kappa(z,g) := \int_{\mathrm{U}(2)(\mathbb{R})} f_\kappa(z, S^{-1}\alpha(g,g_1)S)\bar{\tau}(\det g_1)\pi(g_1)\varphi\, dg_1$$

and

$$F'_\kappa(z,g) := \int_{\mathrm{U}(2)(\mathbb{R})} f'_\kappa(z, S'^{-1}\alpha(g,g_1)S')\bar{\tau}(\det g_1)\pi(g_1)\varphi\, dg_1$$

If we define an auxiliary $f^\circ_{\kappa,n}(z,g) = J_n(g, i1_n)^{-\kappa}|J_n(g,i1_n)|^{\kappa-2z-n}$ for $n=2,3$, then $f_\kappa(g,z) = f^\circ_{\kappa,3}(gg_0)$ and $f'_\kappa(g,z) = f^\circ_{\kappa,2}(gg_0)$ for

$$g_0 = \mathrm{diag}\left(1, \frac{\mathfrak{s}^{1/2}d^{1/4}}{\sqrt{2}}, \frac{d^{1/4}}{\sqrt{2}}, 1, \left(\frac{\mathfrak{s}^{1/2}d^{1/4}}{\sqrt{2}}\right)^{-1}, \left(\frac{d^{1/4}}{\sqrt{2}}\right)^{-1}\right)$$

or

$$g_0 = \mathrm{diag}\left(\frac{\mathfrak{s}^{1/2}d^{1/4}}{\sqrt{2}}, \frac{d^{1/4}}{\sqrt{2}}, \left(\frac{\mathfrak{s}^{1/2}d^{1/4}}{\sqrt{2}}\right)^{-1}, \left(\frac{d^{1/4}}{\sqrt{2}}\right)^{-1}\right)$$

depending on the sizes.

Lemma 6.7. *The integrals are absolutely convergent for* $\mathrm{Re}(z)$ *sufficiently large and for such* z, *we have*:

(i) $F_{\mathcal{D},\mathrm{Kling},\infty}(z, g) := F_\kappa(z, g) = F_{\kappa,z}(g)$.

(ii) $F'_{\mathcal{D},\infty} := F'_\kappa(z, g) = \pi(g)\varphi$, *where* $F_{\kappa,z}$ *is defined in Definition 5.1 using* φ *as the* v *there.*

Fourier coefficients. The following lemma is [Skinner and Urban 2014, Lemma 11.4].

Lemma 6.8. *Suppose* $\beta \in S_n(\mathbb{R})$. *Then the function* $z \to f_{\kappa,\beta}(z, g)$ *has a meromorphic continuation to all of* $\mathbb{C}$. *Furthermore, if* $\kappa \geq n$ *then* $f_{\kappa,n,\beta}(z, g)$ *is holomorphic at* $z_\kappa := (\kappa - n)/2$. *For* $y \in \mathrm{GL}_n(\mathbb{C})$, $f^\circ_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = 0$ *if* $\det\beta \leq 0$ *and if* $\det\beta > 0$ *then*

$$f^\circ_{\kappa,n,\beta}(z_\kappa, \mathrm{diag}(y, {}^t\bar{y}^{-1})) = \frac{(-2)^{-n}(2\pi i)^{n\kappa}(2/\pi)^{n(n-1)/2}}{\prod_{j=0}^{n-1}(\kappa - j - 1)!} e(i\,\mathrm{Tr}(\beta y {}^t\bar{y}))\det(\beta)^{\kappa-n}\det\bar{y}^\kappa.$$

The local Fourier coefficient for f_κ can be easily deduced from that for f°_κ.

Fourier–Jacobi coefficients. The following lemma can be found in [Wan 2015a, Lemma 4.4].

Lemma 6.9. *Let* $z_\kappa = (\kappa - 3)/2$, $\beta \in \mathbb{R}_+$. *Then*:

(i) $\mathrm{FJ}_\beta(z_\kappa, f^\circ_{\kappa,3}, x, \eta, 1) = f^\circ_{\kappa,1,\beta}(z_\kappa + 1, 1)e(i\,\mathrm{Tr}({}^t\bar{x}\beta x))$. *Recall that* $\eta = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix}$.

(ii) *If* $g \in \mathrm{U}(2, 2)(\mathbb{R})$, *then*

$$\mathrm{FJ}_{\beta,\kappa}(z_\kappa, f^\circ_{\kappa,3}, x, g, 1) = e(i\,\mathrm{Tr}\,\beta)c_1(\beta, \kappa) f^\circ_{\kappa-1,2}(z_\kappa, g')\omega_{\beta,\lambda_\infty}(g')\Phi_{\beta,\infty}(x).$$

where $g' = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} g \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}$, $c_1(\beta, \kappa) = ((-2)^{-1}(2\pi i)^\kappa/(\kappa-1)!)\det\beta^{\kappa-1}$ *and* $\Phi_{\beta,\infty} = e^{-2\pi\,\mathrm{Tr}(\langle x,x\rangle_\beta)}$. *Recall that the* $\langle x, x\rangle_\beta$ *is a* 2 *by* 2 *matrix.*

Lemma 6.10. *We have*

$$\mathrm{FJ}_\beta(f_\kappa, x, g, 1) = e(i\,\mathrm{Tr}\,\beta)c_1(\beta, \kappa)J(g, \boldsymbol{i})^{-\kappa}\omega_{\beta,\lambda_\infty}(g'g_0)\Phi_{\beta,\infty}(x)$$

for all $g \in \mathrm{U}(2, 2)(\mathbb{R})$, $x \in \mathbb{C}^2$.

Proof. Note that

$$f_\kappa(g, z_\kappa) = J(g, \boldsymbol{i})^{-\kappa} = J(gg_0, i)^{-\kappa}J(g_0, i)^\kappa = \left(\frac{\sqrt{d}}{2}\right)^{-\kappa} J(gg_0, i)^{-\kappa}.$$ □

Lemma 6.11. *Let* $x_1 = (x_{11}, x_{12})$, $x_2 = (x_{21}, x_{22})$ *where the* $x_{ij} \in \mathbb{R}$. *Then*

$$\delta_\psi^{-1}(\omega_{1,\lambda}(\eta g_0)\Phi_{1,\infty})(x_1, x_2) = \tfrac{1}{4}(\mathfrak{s}^{1/2}d^{1/2})e^{-2\pi\sqrt{d}(\mathfrak{s}x_{11}^2+x_{12}^2)}e^{-2\pi\sqrt{d}(\mathfrak{s}x_{21}^2+x_{22}^2)}$$

Proof. Straightforward from the expression for $\Phi_{1,\infty}$ and δ_ψ. □

We summarize the key definitions associated to the Archimedean datum below.

Definition 6.12. Recall we defined $f_{\mathrm{sieg},\mathcal{D},\infty} = f_{\mathrm{sieg},\infty} := f_\kappa(g,z) := J_3(g,\boldsymbol{i})^{-\kappa}|J_3(g,\boldsymbol{i})|^{\kappa-2z-3}$ and $f'_{\mathrm{sieg},\mathcal{D},\infty} = f'_{\mathrm{sieg},\infty} := f'_\kappa(g,z) = J_2(g,\boldsymbol{i})^{-\kappa}|J_2(g,\boldsymbol{i})|^{\kappa-2z-2}$. Now set

$$\Phi_{\mathcal{D},\infty} = \omega_1(g_0)\Phi_{1,\infty}, \qquad f_{2,\mathcal{D},\infty}(g) = f'_{\kappa-1}(gg_0),$$
$$\Phi''_{\mathcal{D},\infty} = \omega_1(\eta g_0)\Phi_{\mathcal{D},\infty}, \quad f''_{2,\mathcal{D},\infty}(g) = f'_{\kappa-1}(g\eta g_0),$$
$$\phi_{1,\infty}(x_1,x_2) = \phi_{2,\infty}(x_1,x_2) = \tfrac{1}{2}(\mathfrak{s}^{1/4}d^{1/4})e^{-2\pi\sqrt{d}(\mathfrak{s}x_1^2+x_2^2)}, \quad x_1,x_2\in\mathbb{R}.$$

Finally we record a lemma.

Lemma 6.13. *Let $Z \in X_{2,2}$ and $\Phi_{\beta,Z}(x) = e(\mathrm{tr}(\langle x,x\rangle_\beta Z))$. For any $g \in \mathrm{U}(2,2)(\mathbb{R})$,*

$$\omega_{\beta,\lambda_\infty}(g)\Phi_{\beta,Z} = \det J(g,Z)^{-1}\Phi_{\beta,g(Z)}.$$

Proof. Similar to [Skinner and Urban 2014, Lemma 10.1]. □

Example 6.14. We work out an example for the theta function constructed via the Weil representation whose Archimedean Schwartz function is given by Definition 6.12. We check that it is nothing but the adelic theta function defined before. We take the w_2 in [Zhang 2013, Appendix B] to be the identity. The z there is thus equal to the w_1 there. We first note that

$$N \ni \begin{pmatrix} 1 & z & t+z\zeta z^*/2 \\ & 1_2 & \zeta z^* \\ & & 1 \end{pmatrix}$$

acting on i gives $\binom{i+t+z\zeta z^*/2}{\zeta z^*}$. Thus the complex structure on $N/Z(N)_\infty$ is given by the complex conjugation of z. (Note that the z is not the z in [loc. cit.] — instead it plays the role of $\bar{u}$ there.)

Now write $z = (z_1, z_2)$, and write $x = (x_1, x_2) \in \mathbb{Q}^2$. A straightforward computation using the formulas for Weil representation of $H(W)$ implies that the *classical* theta function is a sum

$$\sum_x \prod_{v<\infty} \phi_v(x) e^{-(x_1^2+2x_1\bar{z}_1+\bar{z}_1^2/2)2\pi\sqrt{d}} \cdot e^{-(x_2+2x_2\bar{z}_2+\bar{z}_2^2/2)2\pi\mathfrak{s}\sqrt{d}}$$

for x running over some lattice of $\mathbb{Q}^2$. This is clearly holomorphic with respect to the complex structure. In fact comparing with the notations in [loc. cit.], taking the u there to be $2(\bar{z}_1,\bar{z}_2)\zeta$ and z there to be 2ζ, this theta function is nothing but the one considered in [loc. cit.].

In applications later on, we also take Schwartz functions in $\phi_f \in \mathcal{S}(\mathbb{A}_f)$ and consider the associated theta function Θ_ϕ for $\phi = \phi_{1,\infty} \times \phi_f$. Suppose the Θ_ϕ is right invariant under the open compact level group $K \subseteq NU(\zeta)(\mathbb{A}_f)$. Then we define L to be a certain lattice contained in

$$(N(\mathbb{Q}) \cap K)/(Z(N)(\mathbb{Q}) \cap K)$$

satisfying the assumption after Definition 4.6, and $U_f = K \cap \mathrm{U}(\zeta)(\mathbb{A}^\infty)$. The associated classical theta functions θ_ϕ is indeed in $T^{\mathrm{Hol}}_{\mathbb{A}}(1, L, U_f)$.

6F. *Unramified cases.* Let v be a prime outside Σ (in particular $v\nmid p$). Then the Siegel sections $f_{\mathrm{sieg},\mathcal{D},v} = f_{v,\mathrm{sieg}} = f_v^{\mathrm{sph}}$ and $f'_{\mathrm{sieg},\mathcal{D},v} = f'_{v,\mathrm{sieg}} = f_v^{\mathrm{sph},'}$ is defined to be the unique section that is invariant under $\mathrm{GU}(n,n)(\mathbb{Z}_v)$ $(n=3,2)$ and is 1 at identity.

Lemma 6.15. *Suppose π, ψ and τ are unramified and $\varphi \in \pi$ is a new vector. If* $\mathrm{Re}(z) > \frac{3}{2}$ *then the pullback integral converges and*

$$F_{\mathrm{Kling},\mathcal{D},v} := F_\varphi(f_v^{\mathrm{sph}}; z, g) = \frac{L(\tilde{\pi}, \xi, z+1)}{\prod_{i=0}^{1} L(2z+3-i, \bar{\tau}'\chi_{\mathcal{K}}^i)} F_{\rho,z}(g)$$

where F_ρ is the spherical section defined using $\varphi \in \pi$. Also

$$F'_{\mathcal{D},v} := F'_\varphi(f_v^{\mathrm{sph},'}, z, g) = \frac{L\big(\tilde{\pi}, \xi, z+\frac{1}{2}\big)}{\prod_{i=0}^{1} L(2z+2-i, \bar{\tau}'\chi_{\mathcal{K}}^i)} \pi(g)\varphi.$$

This is a special case of [Lapid and Rallis 2005, Proposition 3.3].

Fourier coefficients.

Definition 6.16. Let Φ_0 be the characteristic function of $\mathcal{O}_{\mathcal{K}}^2$.

Lemma 6.17. *Let $\beta \in S_n(\mathbb{Q}_v)$ and let $r :=$* $\mathrm{rank}(\beta)$. *Then for $y \in \mathrm{GL}_n(\mathcal{K}_v)$,*

$$\begin{aligned} &f_{v,\beta}^{\mathrm{sph}}(z, \mathrm{diag}(y, {}^t\bar{y}^{-1})) \\ &\quad = \tau(\det y)|\det y\bar{y}|_v^{-z+n/2} D_v^{-n(n-1)/4} \times \frac{\prod_{i=r}^{n-1} L(2z+i-n+1, \bar{\tau}'\chi_{\mathcal{K}}^i)}{\prod_{i=0}^{n-1} L(2z+n-i, \bar{\tau}'\chi_{\mathcal{K}}^i)} h_{v,{}^t\bar{y}\beta y}(\bar{\tau}'(q_v)q_v^{-2z-n}). \end{aligned}$$

where $h_{v,{}^t\bar{y}\beta y} \in \mathbb{Z}[X]$ is a monic polynomial depending on v and ${}^t\bar{y}\beta y$ but not on τ. If $\beta \in S_n(\mathbb{Z}_v)$ and $\det\beta \in \mathbb{Z}_v^\times$, *then we say that β is v-primitive and in this case $h_{v,\beta} = 1$.*

Proof. This is a computation of Shimura [1997, Propositions 18.14 and 19.2]. See also [Skinner and Urban 2014, Lemma 11.7]. ☐

Fourier–Jacobi coefficients.

Lemma 6.18. *Suppose $v \notin \Sigma$ and not dividing p. Let $\beta \in S_1(\mathbb{Q}_v)$ such that $\beta \neq 0$. Let $y \in \mathrm{GL}_2(\mathcal{K}_v)$ be such that ${}^t\bar{y}\beta y \in S_1(\mathbb{Z}_v)$, let λ be an unramified character of $\mathcal{K}_v^\times$ such that $\lambda|_{\mathbb{Q}_v^\times} = 1$. If $\beta \in \mathrm{GL}_1(\mathcal{O}_{\mathcal{K},v})$, then for $u \in \mathrm{U}_\beta(\mathbb{Q}_v)$, we have*

$$\mathrm{FJ}_\beta(f_3^{\mathrm{sph}}; z, x, g, u) = \tau(\det u)|\det u\bar{u}|_v^{-z+1/2} \frac{f_2^{\mathrm{sph}}(z, g')(\omega_{\beta,\lambda_v}(u, g')\Phi_0)(x)}{L(2z+3, \bar{\tau}')}.$$

Here $g' = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} g \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}$, and the f_2^{sph} on the right is in $I(\tau/\lambda)$.

This is a formal generalization of [Skinner and Urban 2014, Lemma 11.8].

Definition 6.19. Recall we have defined $f_{\mathrm{sieg},\mathcal{D},v} = f_{v,\mathrm{sieg}} = f_v^{\mathrm{sph}}$ and $f'_{\mathrm{sieg},\mathcal{D},v} = f'_{\mathrm{sieg},v} = f_v^{\mathrm{sph},'}$. We also define $\phi_{1,v}$ and $\phi_{2,v}$ to be the Schwartz function on X_v which is the characteristic function of $\mathbb{Z}_v^2$. We define $f_{2,\mathcal{D},v} = f_v^{\mathrm{sph},'}$, $\Phi_{\mathcal{D},v} = \Phi_0$ and $\Phi''_{\mathcal{D},v} = \Phi_0$.

6G. ***Ramified cases.*** Let $f^\dagger \in I_n(\tau)$ ($n = 2$ or 3) be the Siegel section supported on $Q(\mathbb{Q}_v)w_nQ(\mathbb{Z}_v)$, which takes value 1 on $w_nN_Q(\mathbb{Z}_v)$. The Siegel section we choose is $I_3(\tau) \ni f_{\mathrm{sieg},v} = f_{\mathrm{sieg},\mathcal{D},v} = f^\dagger(g\tilde{\gamma}_v)$ where $\tilde{\gamma}_v$ is

$$\begin{pmatrix} 1 & & & \\ & 1_2 & & 1_2/(y\bar{y}) \\ & & 1 & \\ & & & 1_2 \end{pmatrix}$$

where $y \in \mathcal{O}_v$ is some fixed element such that the valuation is sufficiently large (can be made precise in the text). We also define

$$I_2(\tau) \ni f_{\mathrm{sieg},\mathcal{D},v} = f'_{\mathrm{sieg},v} = f^\dagger(g\tilde{\gamma}'_v)$$

where $\tilde{\gamma}'_v = \left(\begin{smallmatrix} 1_2 & 1_2/y\bar{y} \\ & 1_2 \end{smallmatrix}\right)$.

Pullback formulas. Recall the notations in Section 5D.

Lemma 6.20. *Let φ be some vector invariant under the action of $\mathfrak{Y}$ defined before, then*

$$F_{\mathrm{Kling},\mathcal{D},v} := F_\varphi(z, w) = \tau(y\bar{y})|(y\bar{y})^2|_v^{-z-3/2}\,\mathrm{Vol}(\mathfrak{Y})\cdot\varphi.$$

Also

$$F'_{\mathcal{D},v} := F'_\varphi(f'_{v,\mathrm{sieg}}; z, g) = \tau(y\bar{y})|(y\bar{y})^2|_v^{-z-1}\,\mathrm{Vol}(\mathfrak{Y})\cdot\pi(g)\varphi.$$

The proof is a special case of [Wan 2015a, Lemma 4.9 and 4.10].

Fourier coefficients.

Lemma 6.21. (i) *Let $\beta \in S_3(\mathbb{Q}_\ell)$. Then $f_{v,\beta}(z, 1) = 0$ if $\beta \notin S_3(\mathbb{Z}_\ell)^*$. If $\beta \in S_3(\mathbb{Z}_\ell)^*$ then*

$$f_{v,\beta}(z, \mathrm{diag}(A, {}^t\bar{A}^{-1})) = D_\ell^{-3/2}\tau(\det A)|\det A\bar{A}|_\ell^{-z+3/2}e_\ell\left(\frac{\beta_{22}+\beta_{33}}{y\bar{y}}\right)$$

where D_ℓ is the discriminant of $\mathcal{K}_\ell$.

(ii) *If $\beta \in S_2(\mathbb{Q}_\ell)$, then $f_{v,\beta}(z, 1) = 0$ if $\beta \in S_2(\mathbb{Z}_\ell)^*$. If $\beta \in S_2(\mathbb{Z}_\ell)^*$ then*

$$f'_{v,\beta}(z, \mathrm{diag}(A, {}^t\bar{A}^{-1})) = D_\ell^{-1/2}\tau(\det A)|\det A\bar{A}|_\ell^{-z+r/2}e_\ell\left(\frac{\beta_{11}+\beta_{22}}{y\bar{y}}\right).$$

The proof is a special case of [Wan 2015a, Lemma 4.12].

Fourier–Jacobi coefficients.

Lemma 6.22. *If $\beta \notin S_1(\mathbb{Z}_v)^*$ then $\mathrm{FJ}_\beta(f^\dagger; z, x, g, 1) = 0$. If $\beta \in S_1(\mathbb{Z}_v)^*$ then*

$$\mathrm{FJ}_\beta(f^\dagger; z, x, g, 1) = f^\dagger(z, g'\eta)\omega_{\beta,\lambda_v}(h, g'\eta^{-1})\Phi_0(x).\,\mathrm{Vol}(S_1(\mathbb{Z}_v)),$$

where $g' = \left(\begin{smallmatrix} 1_2 & \\ & -1_2 \end{smallmatrix}\right)g\left(\begin{smallmatrix} 1_2 & \\ & -1_2 \end{smallmatrix}\right)$.

The proof is a special case of [Wan 2015a, Lemma 4.13].

Definition 6.23. Let $A = 1_2/(y\bar{y})$. Thus

$$\mathrm{FJ}_\beta(f_{\mathrm{sieg},v}; z, x, g, h) = f^\dagger\left(z, g'\begin{pmatrix}1 & \\ -A & 1\end{pmatrix}\eta\right)\left(\omega_{\beta,\lambda_v}\left(h, g'\begin{pmatrix}1 & \\ -A & 1\end{pmatrix}\eta\right)\right)\Phi_0(x)$$

for $h \in \mathrm{U}_\beta(\mathbb{Q}_\ell)$. We define $\Phi''_{\mathcal{D},v} := \omega_\beta\big(\big(\begin{smallmatrix}1 & A\\ & 1\end{smallmatrix}\big)\big)\Phi_0$ and $\Phi_{\mathcal{D},v} = \omega_{\beta,\lambda_v}\big(\big(\begin{smallmatrix}1 & \\ -A & 1\end{smallmatrix}\big)\eta\big)\Phi_0$. We also define $f_{2,\mathcal{D},v} = \rho\big(\big(\begin{smallmatrix}1 & \\ -A & 1\end{smallmatrix}\big)\eta\big)f^\dagger \in I_2(\tau/\lambda)$.

Split case. Suppose $v = w\bar{w}$ is a split prime. Recall we have the local polarization $X'_v \oplus Y'_v$. Now we write $x'_1 = (x'_{11}, x'_{12})$ and $x'_2 = (x'_{21}, x'_{22})$ with respect to $\mathcal{K}_v \simeq \mathcal{K}_w \times \mathcal{K}_{\bar{w}}$. The following lemma follows from a straightforward computation and will be used later.

Lemma 6.24. *Let $\chi_{\theta,v}$ be a character of $\mathbb{Z}_v^\times$ such that*

$$\mathrm{cond}(\lambda_v) < \mathrm{cond}(\chi_{\theta,v}) < \mathrm{ord}_v(y\bar{y}).$$

Then it is possible to choose a Schwartz function ϕ'_1 such that the function

$$\phi'_2(x'_2) := \int_{X'_1} \delta'^{-1}_\psi(\Phi''_v)(x'_1, x'_2)\phi'_1(x'_1)\,dx'_1$$

is given by

$$\phi'_2(x'_2) = \begin{cases}\lambda_v\chi_{\theta,v}(x'_{22}) & x'_{21} \in \mathbb{Z}_v,\ x'_{22} \in \mathbb{Z}_v^\times,\\ 0 & \textit{otherwise}.\end{cases}$$

Moreover we can ensure that when we are moving our datum in p-adic families, this ϕ'_1 is not going to change.

Remark 6.25. Note here we have flexibility on choosing the $\chi_{\theta,v}$ for split v. It is an analogue of the fact that in the doubling method, if we choose the Siegel section as in the beginning of this subsection, then the local pullback integral is nonzero if the test vector has appropriate conductor. This flexibility is important for our argument in Section 8C.

We define

$$\phi_{1,v} = \delta''_\psi(\phi'_1),\ \phi_{2,v} = \delta^{-,''}_\psi(\phi'_2). \tag{6-3}$$

Nonsplit case.

Lemma 6.26. *We consider the action of the compact abelian group* $\mathrm{U}(1)(\mathbb{Q}_v)$ *on* $\delta_\psi^{-1}(\Phi''_v)$ *by the Weil representation (using the splitting character λ_v) of*

$$1 \times \mathrm{U}(1)(\mathbb{Q}_v) \hookrightarrow 1 \times \mathrm{U}(2)(\mathbb{Q}_v) \hookrightarrow \mathrm{U}(2)(\mathbb{Q}_v) \times \mathrm{U}(2)(\mathbb{Q}_v) \hookrightarrow \mathrm{U}(2,2)(\mathbb{Q}_v).$$

We can write $\delta_\psi^{-1}(\Phi''_v)$ as a sum of eigenfunctions of this action. Let $m = \max\{\mathrm{ord}_v(\mathrm{cond}\,\lambda_v), 3) + 1$. If $\mathrm{ord}_v(y\bar{y}) > m$, then there is such a nonzero eigenfunction $\phi_{2,v}$ whose eigenvalue is a character $\lambda_v^2\chi_{\theta,v}$

for $\chi_{\theta,v}$ of conductor at least ϖ_v^m, such that there is a $\overline{\mathbb{Q}}_p$-valued Schwartz function $\phi_{1,v}$ and a set of constants $C_{v,i} \in \overline{\mathbb{Q}}_p$ and $u_{v,i} \in \mathrm{U}(1)(\mathbb{Q}_v)$'s such that the function

$$\phi_{2,v}(x_2) = \int_{X_1(\mathbb{Q}_v)} \sum_i \delta_\psi^{-1}(C_{v,i}\omega_{\beta,\lambda_v}(u_{v,i}, 1)\Phi''_v)(x_1, x_2)\phi_{1,v}(x_1)\, dx_1$$

(*here* $1 \in \mathrm{U}(2,2)(\mathbb{Q}_v)$).

Proof. Consider the embedding $\mathrm{U}(1,1) \hookrightarrow \mathrm{U}(2,2)$ by

$$j : g \mapsto \begin{pmatrix} \mathfrak{s}^{-1} & & & \\ & 1 & & \\ & & \mathfrak{s}^{-1} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} a_g & & b_g & \\ & a_g & & b_g \\ d_g & & d_g & \\ & c_g & & d_g \end{pmatrix} \begin{pmatrix} \mathfrak{s} & & & \\ & 1 & & \\ & & \mathfrak{s} & \\ & & & 1 \end{pmatrix}$$

for $g = \left(\begin{smallmatrix} a_g & b_g \\ c_g & d_g \end{smallmatrix}\right)$. Define $f_{\Phi''_v}\left(g, \frac{1}{2}\right) = (\omega_{\beta,\lambda_v}(j(g))\Phi''_v)(0) \in I_2(\tau/\lambda_v)$. We define $i : \mathrm{U}(1)\times\mathrm{U}(1) \hookrightarrow \mathrm{U}(1,1)$ by

$$i(g_1, g_2) = \begin{pmatrix} 1/2 & -1/2 \\ -\delta^{-1} & -\delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ & g_1 \end{pmatrix} \begin{pmatrix} 1 & -\delta/2 \\ -1 & -\delta/2 \end{pmatrix}. \tag{6-4}$$

For $g_1 \in \mathrm{U}(1)(\mathbb{Q}_v)$,

$$f_{\Phi''_v}\left(i(1, g_1), \tfrac{1}{2}\right) = (\omega_{\beta,\lambda_v}(j \circ i(1, g_1))\Phi''_v)(0) = (\delta_\psi \omega_{\lambda_v}(1, g_1)\delta_\psi^{-1}\Phi''_v)(0).$$

Here in the first and last expression $1 \in \mathrm{U}(2)(\mathbb{Q}_v)$ and g_1 is viewed as the element in the center of $\mathrm{U}(2)(\mathbb{Q}_v)$. Thus we are reduced to proving the following lemma. □

Lemma 6.27. *Let $g_1 = 1 + \varpi_v^m.a \in \mathrm{U}(1)(\mathbb{Q}_v)$ for m as in the above lemma and $a \in \mathcal{O}_{\mathcal{K},v}$, if $n = y\bar{y}$ is such that* $\mathrm{ord}_v\, n > m$ *then* $f_{\Phi''_v}\left(i(1, g_1); \frac{1}{2}\right) \neq f_{\Phi''_v}\left(1; \frac{1}{2}\right)$.

Proof. We have the following:

$$\begin{aligned} \begin{pmatrix} 1/2 & -1/2 \\ -\delta^{-1} & -\delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ & g_1 \end{pmatrix} \begin{pmatrix} 1 & -\delta/2 \\ -1 & -\delta/2 \end{pmatrix} &= \begin{pmatrix} 1/2 & -g_1/2 \\ -\delta^{-1} & -\delta^{-1}g_1 \end{pmatrix} \begin{pmatrix} 1 & -\delta/2 \\ -1 & -\delta/2 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 + g_1/2 & -\delta/4 + \delta g_1/4 \\ -\delta^{-1} + \delta^{-1}g_1 & 1/2 + g_1/2 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1/n \\ & 1 \end{pmatrix} = \begin{pmatrix} a & a/n + b \\ c & c/n + d \end{pmatrix} = \begin{pmatrix} n(ad - bc)/(c + nd) & a/n + b \\ 0 & c/n + d \end{pmatrix} \begin{pmatrix} 1 & \\ n/(1 + nd/c) & 1 \end{pmatrix}.$$

Now the lemma follows readily. □

Definition 6.28. From now on we fix the choices and define the local characters $\chi_{\theta,v}$ as in Lemma 6.24 and Lemma 6.26.

As before we summarize the definitions associated to our Eisenstein datum at v.

Definition 6.29. Recall we defined $f_{\mathrm{sieg},\mathcal{D},v}=f_{\mathrm{sieg},v}=f^{\dagger}(g\tilde{\gamma}_v)$ where

$$\tilde{\gamma}_v=\begin{pmatrix}1&&&\\&1_2&&\frac{1}{y\bar{y}}1_2\\&&1&\\&&&1_2\end{pmatrix},$$

and defined

$$f'_{\mathrm{sieg},\mathcal{D},v}=f'_{\mathrm{sieg},v}=f^{\dagger}(g\tilde{\gamma}'_v)$$

where $\tilde{\gamma}'_v=\left(\begin{smallmatrix}1_2&1_2/(y\bar{y})\\&1_2\end{smallmatrix}\right)$. Let

$$f_{2,\mathcal{D},v}=\rho\left(\begin{pmatrix}1&\\-A&1\end{pmatrix}\eta\right)f^{\dagger}\in I_2\left(\frac{\tau}{\lambda}\right).$$

We define $\Phi_{\mathcal{D},v}=\omega_{\beta,\lambda_v}\left(\left(\begin{smallmatrix}1&\\-A&1\end{smallmatrix}\right)\eta\right)\Phi_0$. We also define the $\phi_{1,v}$ and $\phi_{2,v}$ as in (6-3) or as in Lemma 6.26, depending on whether v is split or not.

6H. *p-adic cases.* We recall some results in [Wan 2015a, Section 4.D] with some modifications. Recall that we have the triple (π_p,ψ_p,τ_p) and $\xi_p:=\psi_p/\tau_p$ in the p-component of the Klingen–Eisenstein datum $\mathcal{D}$, where χ_p is the central character of π_p and $\psi_p|_{\mathbb{Q}_p^\times}=\chi_p$. Suppose π_p is nearly ordinary in the sense that $\pi_p=\pi(\chi_{1,p},\chi_{2,p})$ such that $\mathrm{ord}_p(\chi_{1,p}(p))=-\frac{1}{2}$ and $\mathrm{ord}_p(\chi_{2,p}(p))=\frac{1}{2}$. We write $\tau_p=(\tau_1,\tau_2)$ and $\xi_p=(\xi_1,\xi_2)$.

Definition 6.30. The triple (π_p,ψ_p,τ_p) is generic if there is a $t\geq 2$ such that $\xi_1,\xi_2,\chi_p,\chi_p^{-1}\xi_1,\chi_p^{-1}\xi_2$ all have conductor p^t.

Although the definition for generic points is different from [Wan 2015a, Definition 4.21], the argument there goes through since the only place using this definition is Lemma 4.19 there, which can be proved completely in the same way under our definition for generic. We define: $\xi_1^{\dagger}=\chi_{1,p}\bar{\xi}_2$, $\xi_2^{\dagger}=\chi_{2,p}\bar{\xi}_2$. Let $K^{(3,3)}\subseteq\mathrm{GL}_6(\mathbb{Z}_p)$ be the subgroup consisting of matrices congruent to upper triangular matrices modulo p^t. We define f_t to be the Siegel section supported in $Q(\mathbb{Q}_v)K_t$, invariant under $K_t^{(3,3)}$ and takes value 1 on the identity. We define

$$f_{\mathrm{sieg},\mathcal{D},p}=f_{\mathrm{sieg},p}(g):=\mathfrak{g}(\tau'_p)^{-3}c_3^{-1}(\bar{\tau}'_p,-z)p^{-3t}\mathfrak{g}(\xi_1^{\dagger})\xi_1^{\dagger}(-1)\mathfrak{g}(\xi_2^{\dagger})\xi_2^{\dagger}(-1)$$
$$\times\sum_{\substack{a,b\in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p\\ m,n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}}\bar{\xi}_1^{\dagger}(p^ta)\bar{\xi}_2^{\dagger}(p^tb)f_t\left(g\Upsilon\begin{pmatrix}1&&&&a+bmn&bm\\&1&&&bn&b\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right).\tag{6-5}$$

and

$$f^{\square}_{\mathrm{sieg},\mathcal{D},p}=f^{\square}_{\mathrm{sieg},p}(g):=\mathfrak{g}(\tau'_p)^{-3}c_3^{-1}(\bar{\tau}'_p,-z)p^{-3t}\mathfrak{g}(\xi_1^{\dagger})\xi_1^{\dagger}(-1)\mathfrak{g}(\xi_2^{\dagger})\xi_2^{\dagger}(-1)$$
$$\times\sum_{\substack{a,b\in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p,\\ m\in\mathbb{Z}_p/p^t\mathbb{Z}_p}}\bar{\xi}_1^{\dagger}(p^ta)\bar{\xi}_2^{\dagger}(p^tb)f_t\left(g\Upsilon\begin{pmatrix}1&&&&a&bm\\&1&&&&b\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}\right).\tag{6-6}$$

We also define

$$f'_{\mathrm{sieg},\mathcal{D},p}=f'_{\mathrm{sieg},p}(g):=\mathfrak{g}(\tau'_p)^{-3}c_2^{-1}(\bar{\tau}'_p,-z)p^{-3t}\mathfrak{g}(\xi_1^\dagger)\xi_1^\dagger(-1)\mathfrak{g}(\xi_2^\dagger)\xi_2^\dagger(-1)$$
$$\times\sum_{a,b\in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p,m,n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}\bar{\xi}_1^\dagger(p^ta)\bar{\xi}_2^\dagger(p^tb)f_t\left(g\Upsilon\begin{pmatrix}1&&a+bmn&bm\\&1&bn&b\\&&1&\\&&&1\end{pmatrix}\right).\tag{6-7}$$

Here $\Upsilon\in\mathrm{U}(3,3)(\mathbb{Q}_p)$ is such that

$$\Upsilon=\begin{pmatrix}1&&&\\&\frac{1}{2}.1_2&&-\frac{1}{2}1_2\\&&1&\\&-\zeta^{-1}&&-\zeta^{-1}\end{pmatrix}$$

via the first projection $\mathrm{U}(3,3)(\mathbb{Q}_p)\simeq\mathrm{GL}_6(\mathbb{Q}_p)$ and $c_n(\tau',z)=\tau'(p^{nt})p^{2ntz-tn(n+1)/2}$.

Among these sections the $f_{\mathrm{sieg},\mathcal{D},p}$ and $f'_{\mathrm{sieg},\mathcal{D},p}$ correspond to Siegel–Eisenstein series that we interpolate. The relations between these sections are

$$f_{\mathrm{sieg},\mathcal{D},p}(g)=\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}f^{\square}_{\mathrm{sieg},\mathcal{D},p}\left(g\gamma\left(1,\begin{pmatrix}1&\\n&1\end{pmatrix}_p\right)\right).\tag{6-8}$$

The reason for introducing the $\square$ sections is to help computing the Fourier–Jacobi expansion.

Pullback formulas. We refer to [Wan 2015a, Subsection 4.D.1] for the discussion of nearly ordinary vectors, which means the vector whose U_p-eigenvalues are p-adic units. Let $\varphi=\varphi^{\mathrm{ord}}\in\pi_p$ be a nearly ordinary vector.

Definition 6.31. We define the p-adic Klingen section $F^0(g)$ to be the $F_p^{0,\bullet}$ defined in Definition 5.6, multiplied by

$$\mathfrak{g}(\tau'_p)^{-1}\tau'_p(p^{-t})p^{\kappa-2}p^{(\kappa-3)t}\xi_{1,p}^2\chi_{1,p}^{-1}\chi_{2,p}^{-1}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi_{1,p}^{-1})\mathfrak{g}(\xi_{1,p}\chi_{2,p}^{-1}).$$

Then by the computations in [Wan 2015a] we have the following (see the end of [loc. cit., Section 4]):

Lemma 6.32. (1) $F_\varphi(f_{\mathrm{sieg},p},z_\kappa,g)=F^0(g):=F_{\mathrm{Kling},p}(g)$.

(2) $F'_\varphi(f'_{\mathrm{sieg},p},z_\kappa,g)=p^{(\kappa-3)t}\xi_{1,p}^2\chi_{1,p}^{-1}\chi_{2,p}^{-1}(p^{-t})\mathfrak{g}(\xi_{1,p}\chi_{1,p}^{-1})\mathfrak{g}(\xi_{1,p}\chi_{2,p}^{-1}).\pi(g)\varphi$.

Proof. For reader's convenience we briefly recall the proof of [Wan 2015a] in case (1) of this lemma. In our $\mathrm{U}(3,1)$ case the proof is actually much simpler than the general case considered in [loc. cit.]. The proof uses the trick of [Skinner and Urban 2014, Proposition 11.13] of using the local and global functional equations to reduce the local pullback integral for $f_{\mathrm{sieg},p}$ to that of another Siegel section $f^\dagger_{\mathrm{sieg},p}$ defined below. The computation of the pullback section of $f^\dagger_{\mathrm{sieg},p}$ is easier than that of $f_{\mathrm{sieg},p}$. We define an auxiliary Siegel section $\tilde{f}^\dagger_{\mathrm{sieg},p}$ to be supported in $Q_3(\mathbb{Q}_p)wQ_3(\mathbb{Z}_p)$, and such that it takes value 1 on

$wN_{Q_3}(\mathbb{Z}_p)$. We define

$$f^{\dagger}_{\mathrm{sieg},p}(g,z) := \mathfrak{g}(\tau'_p)^{-3} c_3^{-1}(\bar{\tau}'_p, -z) p^{-3t} \mathfrak{g}(\xi_1^{\dagger})\xi_1^{\dagger}(-1)\mathfrak{g}(\xi_2^{\dagger})\xi_2^{\dagger}(-1)$$
$$\times \sum_{\substack{a,b\in p^{-t}\mathbb{Z}_p^{\times}/\mathbb{Z}_p,\\ m,n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}} \bar{\xi}_1^{\dagger}(p^t a)\bar{\xi}_2^{\dagger}(p^t b)\tilde{f}^{\dagger}_{\mathrm{sieg},p}\left(g\Upsilon\left(\begin{pmatrix} 1 & & & & a+bmn & bm \\ & 1 & & & bn & b \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, z\right)\right).$$

We can show by direct checking that the $F_{\varphi}(f^{\dagger}_{\mathrm{sieg},p}, z, g)$ is invariant under $N_t(\mathbb{Z}_p)$ defined above (as a function of g). This can be seen by checking that the pullback of the Siegel section $f^{\dagger}_{\mathrm{sieg},p}$ is already invariant under $N_t(\mathbb{Z}_p)$. Moreover the value of $F_{\varphi}(f^{\dagger}_{\mathrm{sieg},p}, z, g)$ at ww'_3 can also be computed directly [Wan 2015a, Lemma 4.38]. Returning to the pullback section of $f_{\mathrm{sieg},p}$, we note that it is the image of $f^{\dagger}_{\mathrm{sieg},p}$ under the intertwining operator $M(z,-)$. We apply [Wan 2015a, Proposition 4.40] (which is just a variant of [Skinner and Urban 2014, Proposition 11.13]). By the uniqueness up to scalar of the vector with the same action of the level group $B_t(\mathbb{Z}_p)$ (see [Wan 2015a, Lemma 4.19], which is just a variant of [Skinner and Urban 2014, Proposition 9.5]. Note that the result in [Wan 2015a] also works under our assumption of genericity), we know that $F_{\varphi}(f_{\mathrm{sieg},p}, z_{\kappa}, g) = F^0(g)$ up to scalar. Then applying [loc. cit., Proposition 4.40] again we can evaluate the $F_{\varphi}(f_{\mathrm{sieg},p}, z_{\kappa}, g)$ at w'_3 and get the lemma. Note that this simplified proof does not apply to the general case of [loc. cit.] including the $\mathrm{U}(2,2)$ case of [Skinner and Urban 2014], since there by looking at the Siegel–Eisenstein section we can only determine the action of a level group which is smaller than the level group corresponding to the ordinary vector. □

Fourier coefficients.

Definition 6.33. We define the function $\Phi_{\xi^{\dagger}}$ as the function on the set of (2×2) $\mathbb{Q}_p$-matrices as follows. If $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is such that both its determinant and a are in $\mathbb{Z}_p^{\times}$ then $\Phi_{\xi^{\dagger}}(x) = \xi_1^{\dagger}(a)\xi_2^{\dagger}(\det x/a)$. Otherwise $\Phi_{\xi^{\dagger}}(x) = 0$. The following lemma is proved in [Wan 2015a, Lemma 4.46].

Lemma 6.34. *We have*

$$f_{\mathrm{sieg},p,\beta}(1) = \bar{\tau}'_p(\det\beta)|\det\beta|_p^{\kappa-3}\Phi_{\xi^{\dagger}}\left(\begin{pmatrix}\beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32}\end{pmatrix}\right)$$

for

$$\beta = \begin{pmatrix}\beta_{11} & \beta_{12} & \beta_{13}\\ \beta_{21} & \beta_{22} & \beta_{23}\\ \beta_{31} & \beta_{32} & \beta_{33}\end{pmatrix}$$

with $\beta_{11}, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{33} \in \mathbb{Z}_p$, *and is* 0 *otherwise.*

Proof. This follows from straightforward computation using:

- The Fourier coefficients of the section f_t as computed in [Skinner and Urban 2014, Lemma 11.12].
- The computation of the Fourier transform of the function $\Phi_{\xi^{\dagger}}$ defined above as detailed in [Wan 2015a, Lemma 4.28] (note that the proof also works under our assumption of genericity). We compare it with the definition of $f_{\mathrm{sieg},p}$ using f_t. □

Fourier–Jacobi coefficients. For $\beta \in S_1(\mathbb{Q}_v) \cap \mathrm{GL}_1(\mathbb{Z}_v)$ we compute the Fourier–Jacobi coefficient for f_t at β. We have the following [Wan 2015a, Lemma 4.54]

Lemma 6.35. *Let* $x := \begin{pmatrix} 1 & \\ D & 1 \end{pmatrix}$ *(this is a block matrix with respect to* $(2+2)$*).*

(a) $\mathrm{FJ}_\beta(f_t; -z, v, x\eta^{-1}, 1) = 0$ *if* $D \notin p^t M_2(\mathbb{Z}_p)$*.*

(b) *If* $D \in p^t M_2(\mathbb{Z}_p)$ *then* $\mathrm{FJ}_\beta(f_t; -z, v, x\eta^{-1}, 1) = c(\beta, \tau, z)\Phi_0(v)$*, where*

$$c(\beta, \tau, z) := \bar{\tau}(-\det \beta)|\det \beta|_v^{2z+2} \mathfrak{g}(\tau')\mathfrak{g}(\tau'_p)\tau'_p(p^t)p^{-2tz-3t}.$$

Note the formula

$$\mathfrak{g}(\tau'_p)^3\tau'_p(p^{3t})p^{-6tz-6t} = \mathfrak{g}(\tau'_p)\tau'_p(p^t)p^{-2tz-3t}\mathfrak{g}(\tau'_p)^2\tau'_p(p^{2t})p^{-4tz-3t}.$$

Definition 6.36. Let $K'_t \subseteq \mathrm{GL}_4(\mathbb{Z}_p)$ be the subgroup consisting of elements congruent to upper triangular matrices modulo p^t. We define the Siegel section f'_t on $I_2(\frac{\tau}{\lambda})$ to be the section supported on $Q_2(\mathbb{Q}_p)K'_t$, invariant under the right action of $N_B(\mathbb{Z}_p)$ and takes value 1 on the identity element. Let A_x be the matrix $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$. We define $f_{2,p} = \sum_b \mathfrak{g}(\xi_2^\dagger)p^{-t}\bar{\xi}_2^\dagger(-p^t b)\mathfrak{g}(\tau'_p)^{-2}c_2^{-1}(\bar{\tau}'_p, -z)\rho\left(\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}\eta\right)f'_t$. Let $\phi_{\xi_1^\dagger}(x) := \xi_1^\dagger(x^{-1})\mathbb{1}_{\mathbb{Z}_p^\times}(x)$ where $\mathbb{1}$ denotes the characteristic function. Define

$$\Phi_p(v_1, v_2, v_3, v_4) = \mathbb{1}_{\mathbb{Z}_p \oplus \mathbb{Z}_p}(v_1, v_2)\hat{\phi}_{\xi_1^\dagger}(v_3)\hat{\mathbb{1}}_{p^t\mathbb{Z}_p}(v_4)$$

where the $\hat{\phi}$ means taking Fourier transform of ϕ. The precise formula is given by

$$\Phi_{\mathcal{D},p}(v_1, v_2, v_3, v_4) = \begin{cases} \mathfrak{g}(\xi_1^\dagger)p^{-2t}\bar{\xi}_1^\dagger(a), & v_1, v_2 \in \mathbb{Z}_p,\ v_4 \in p^{-t}\mathbb{Z}_p,\ v_3 \in \frac{a}{p^t} + \mathbb{Z}_p \text{ for some } a \in \mathbb{Z}_p^\times, \\ 0, & \text{otherwise.} \end{cases} \tag{6-9}$$

These are defined just for computing Fourier–Jacobi coefficients and not used for the pullback formula.

We prove the following lemma:

Lemma 6.37. *We have*

$$\mathfrak{g}(\tau'_p)^{-3}c_3(\bar{\tau}'_p, -z_\kappa)p^{-3t}\sum_{a,b,m}\mathrm{FJ}_1\left(\rho\begin{pmatrix} 1_3 & \begin{smallmatrix} a & bm \\ & b \end{smallmatrix} \\ & 1_3 \end{pmatrix}f_{t,p}; z_\kappa, v, g\right)\bar{\xi}_1^\dagger(-p^t a)\bar{\xi}_2^\dagger(-p^t b)\mathfrak{g}(\xi_1^\dagger)\mathfrak{g}(\xi_2^\dagger)$$
$$= \left(\mathfrak{g}(\tau'_p)^{-2}c_2^{-1}(\bar{\tau}'_p, -z)p^{-t}\mathfrak{g}(\xi_2^\dagger)\sum_b \bar{\xi}_2^\dagger(p^t b)f'_t\left(g\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}\eta\right)\right)(\omega_{\beta,\lambda_p}(g)\Phi_{\mathcal{D},p}(v_1, v_2, v_3, v_4)).$$

Recall $A_x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$*. Also under the projection* $\mathrm{U}(3,3)(\mathbb{Q}_p) \simeq \mathrm{GL}_6(\mathbb{Q}_p)$*, the* v_1, v_2, v_3, v_4 *appear as*

$$\begin{pmatrix} 1 & & & & v_3 & v_4 \\ & 1 & & v_1 & & \\ & & 1 & v_2 & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

The Weil representation is the one in Section 4H case two. We use ρ *to denote the right action of* $\mathrm{GU}(3,3)(\mathbb{Q}_p)$ *on the Siegel sections. Note that for the Schwartz function* $\Phi_{\mathcal{D},p}$*, we used the identification* $\mathcal{K}_p^2 \simeq \mathbb{Q}_p^4$ *given by* $(x_1, x_2) \to ((v_3, v_1), (v_4, v_2))$*.*

Proof. First we fix b and consider the Fourier–Jacobi coefficient of

$$\sum_{a,m} \rho\left(\begin{pmatrix} 1 & & & & a & bm \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right) f_t \tag{6-10}$$

note that

$$w_3 \begin{pmatrix} & & t & v_3 & v_4 \\ 1_3 & v_1 & D_1 & D_2 \\ & v_2 & D_3 & D_4 \\ & & 1_3 & \end{pmatrix} \alpha(1,\eta^{-1}) \begin{pmatrix} 1 & & & & a & bm \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$= w_3 \begin{pmatrix} 1 & -a & -bm & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} & t' & v_3' & v_4' \\ 1_3 & v_1 & D_1 & D_2 \\ & v_2 & D_3 & D_4 \\ 1_3 & & & \end{pmatrix} \alpha(1,\eta^{-1})$$

where $v_3' = v_3 + D_1.a + D_3.bm$, $v_4' = v_4 + D_2.a + D_4.bm$ and $t' = t + v_1.a + v_2.bm$. From this, a calculation using the above lemma (similar to [Skinner and Urban 2014, page 203]) shows that the Fourier–Jacobi expansion of (6-10) at g is

$$(\mathfrak{g}(\tau_p'))^{-2} c_2^{-1}(\bar{\tau}'_p, -z) p^{-t} \mathfrak{g}(\xi_2^\dagger) \rho(\eta) f_t'(g')(\omega_{\beta,\lambda_p}(g')\Phi_{\mathcal{D},p})(v_1, v_2, v_3, v_4).$$

So the Fourier–Jacobi expansion of

$$\sum_{a,b,m} \rho\left(\begin{pmatrix} 1 & & & & a & bm \\ & 1 & & & & b \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}\right) f_t$$

is

$$\sum_b (\mathfrak{g}(\tau'_p)^{-2} c_2^{-1}(\bar{\tau}'_p, -z) p^{-t} \mathfrak{g}(\xi_2) \rho\left(\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}\right) \\ \times \rho(\eta) f'_t(g')(\omega(g')\omega_{\beta,\lambda_p}\left(g'\begin{pmatrix} 1 & \\ A_{-b} & 1 \end{pmatrix}\right)\Phi_{\mathcal{D},p})(v_1, v_2, v_3, v_4).$$

Note that $\omega_{\beta,\lambda_p}(\left(\begin{smallmatrix} 1 & \\ A_{-b} & 1 \end{smallmatrix}\right))\Phi_{\mathcal{D},p} = \Phi_{\mathcal{D},p}$. We get the required Fourier–Jacobi coefficient. □

We record some formulas

$$\rho(\eta) f_{2,p} = f''_{2,p} = \frac{\tau_p}{\lambda_p}(-1)\mathfrak{g}(\xi_2^\dagger) \sum_b \bar{\xi}_2^\dagger(b)\mathfrak{g}(\tau'_p)^{-2} c_2^{-1}(\bar{\tau}'_p, -z)\rho\left(\begin{pmatrix} 1 & A_b \\ & 1 \end{pmatrix}\right) f'_t,$$

$$\Phi''_{\mathcal{D},p} := (\omega_{\beta,\lambda_p}(\eta)\Phi_{\mathcal{D},p})(v_1, v_2, v_3, v_4) = \begin{cases} \xi_1^\dagger(-v_1) & v_3, v_4 \in \mathbb{Z}_p, v_1 \in \mathbb{Z}_p^\times, v_2 \in p^t\mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases} \tag{6-11}$$

We define two Schwartz functions on $X_p^{-,'}$. Let $\phi'_{1,p}$ be the characteristic function of $\mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$. Let $\phi'_{2,p}(x) = p^{-2t}\mathfrak{g}(\xi_1^\dagger)\bar{\xi}_1^\dagger(-p^t x_1)$ if $x = (x_1, x_2)$ for $x_1 \in p^{-t}\mathbb{Z}_p^\times$, $x_2 \in p^{-t}\mathbb{Z}_p$, and is zero otherwise. xmesn

Definition 6.38. We summarize our definitions at the p-adic place. Recall we defined $f_{\mathrm{sieg},\mathcal{D},p}$ and $f'_{\mathrm{sieg},\mathcal{D},p}$ as in (6-5) and (6-7). The definition for $f_{2,\mathcal{D},p}$ is given in (6-12) below.

We defined $\Phi_{\mathcal{D},p}$ as in (6-9) and defined $\Phi''_{\mathcal{D},p}$ as in (6-11). We also define $\phi_{1,p} = \delta''_\psi(\phi'_{1,p})$ and $\phi_{2,p} = \delta_\psi^{-,''}(\phi'_{2,p})$. It is easily checked that

$$\delta_{\psi,p}^{',-1}(\omega_{\beta,\lambda_p}(\Upsilon)(\omega_{\beta,\lambda_p}(\eta)\Phi_{\mathcal{D},p})) = \phi'_{1,p} \boxtimes \phi'_{2,p}.$$

(One compares these with later computations after Definition 8.4.)

For convenience of the reader we explain how this computation is done. We first take standard complex basis ${}^t(e_1, e_2; e_1^-, e_2^-)$ for the Hermitian space corresponding to $\mathrm{U}(\zeta) \times \mathrm{U}(-\zeta)$. To save space we write e for ${}^t(e_1, e_2)$ and similarly for e^-. Then the embedding

$$\mathrm{U}(\zeta) \times \mathrm{U}(-\zeta) \hookrightarrow \mathrm{U}(2, 2)$$

is reflected by the change of basis

$$\begin{pmatrix} e \\ e^- \end{pmatrix} \mapsto {}^t S \begin{pmatrix} e \\ e^- \end{pmatrix} = \begin{pmatrix} 1_2 & -1_2 \\ -\zeta/2 & -\zeta/2 \end{pmatrix}\begin{pmatrix} e \\ e^- \end{pmatrix}.$$

We further take symplectic basis according to the polarization via $\mathcal{O}_p \simeq \mathbb{Z}_p \times \mathbb{Z}_p$. Then under this basis the Υ (as homomorphism of 8-dimensional symplectic spaces) is given by the matrix

$$\begin{pmatrix} 1_2/2 & & & -1_2/2 \\ & -1_2 & 1_2 & \\ & 1_2 & 1_2 & \\ 1_2/2 & & & 1_2/2 \end{pmatrix}.$$

Using (4-1), the matrix for the intertwining operator $\delta'^{,-1}_{\psi,p}$ is

$$\begin{pmatrix} 1_2 & & & 1_2 \\ -1_2 & & & 1_2 \\ & 1_2/2 & 1_2/2 & \\ & 1_2/2 & -1_2/2 & \end{pmatrix} = \begin{pmatrix} 1_2 & & & \\ & & & 1_2 \\ & & 1_2 & \\ & -1_2 & & \end{pmatrix} \begin{pmatrix} 1_2 & & & 1_2 \\ & -1/2 & 1/2 & \\ & 1/2 & 1/2 & \\ -1_2 & & & 1_2 \end{pmatrix}.$$

We get the composed map is given by the matrix

$$\begin{pmatrix} 1_2 & & & \\ & & & 1_2 \\ & & 1_2 & \\ & -1_2 & & \end{pmatrix}$$

and thus the formula.

6I. ***Pullback formulas again.*** In this section, we prove the local pullback formulas for $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ which will be used to decompose the restriction to $\mathrm{U}(2)\times\mathrm{U}(2)$ of the Siegel–Eisenstein series associated to the character τ/λ on $\mathrm{U}(2,2)$ showing up in the Fourier–Jacobi expansion of E_{sieg} on $\mathrm{U}(3,3)$. Fortunately, the local calculations are the same as in the previous sections for f'_{sieg} and F'_φ's except for the case $v=p$.

<u>p-adic case</u>. We temporarily denote the p-component of an automorphic representation π_h of some weight two cuspidal ordinary eigenform h on $\mathrm{U}(2)(\mathbb{Q}_p)$ as $\pi(\chi_1,\chi_2)$ with $v_p(\chi_1(p))=-\frac{1}{2}$ and $v_p(\chi_2(p))=\frac{1}{2}$. We also temporarily write τ for τ/λ in this subsection. We let $\tau_p=(\tau_1,\tau_2)$ and require $\chi_1\tau_2^{-1}$ and $\chi_2\tau_2^{-1}\xi_2^\dagger$ are unramified. We let $\varphi=\varphi^{ss}\in\pi_{h,p}$ for $\varphi^{ss}=\pi_p\big(\big(\begin{smallmatrix} & 1\\ p^t & \end{smallmatrix}\big)\big)\varphi^{\mathrm{ord}}$ for some nearly ordinary vector φ^{ord}. Define

$$f_{2,\mathcal{D},p}(g):=\mathfrak{g}(\tau'_p)^{-2}c_2(\bar\tau'_p,-z_\kappa/2)^{-1}p^{-t}\mathfrak{g}(\xi_2^\dagger)\sum_{b\in p^{-t}\mathbb{Z}_p^\times/\mathbb{Z}_p}\bar\xi_2^\dagger(bp^t)\rho\left(\begin{pmatrix}1 & A_b\\ & 1\end{pmatrix}\right)f'_t(g\Upsilon'). \tag{6-12}$$

It is hard to evaluate the integral directly. So we use the trick of using the functional equation as in [Skinner and Urban 2014, Proposition 11.13] (which is also used in [Wan 2015a]). We first evaluate the integral for the auxiliary $f^\dagger_{2,p}$

$$F_\varphi(f^\dagger_{2,p};z,g):=\int_{\mathrm{U}(2)(\mathbb{Q}_p)}f^\dagger_{2,p}(z,S^{-1}\alpha(g_1,g)S)\bar\tau(\det g)\pi_h(g_1)\varphi\,dg_1$$

at

$$g=w=\begin{pmatrix} & & & 1\\ & 1 & & \\ & & 1 & \\ -1 & & & \end{pmatrix}$$

where $\varphi \in \pi_{h,p}$ and

$$f^{\dagger}_{2,p}(g) := p^{-t}\mathfrak{g}(\xi^{\dagger}_2)\sum_b \bar{\xi}^{\dagger}_2(bp^t)\rho\left(\begin{pmatrix}1 & A_b\\ & 1\end{pmatrix}\right)f^{\dagger}(g\Upsilon), \quad \tilde{f}^{\dagger}_{\mathrm{sieg},p} \in I_2(\bar{\tau}^c).$$

The $f_{2,p}$ and $f^{\dagger}_{2,p}$ are related by the intertwining operator, as used in the proof of Lemma 6.32. For $A \in \mathrm{U}(2)(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$ note that

$$S^{-1}\operatorname{diag}(A,1)\begin{pmatrix}1 & A_b\\ & 1\end{pmatrix} = \begin{pmatrix}-1 & -A\\ & -A\end{pmatrix}\begin{pmatrix}1 & \\ -A_b - A^{-1} & 1\end{pmatrix}w.$$

So in order for this to be in supp $f^{\dagger}$ we must have $A^{-1} + A_b \in M_2(\mathbb{Z}_p)$. So A^{-1} can be written as $\left(\begin{smallmatrix}1 & \\ n & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} & u\\ v & \end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \\ m & 1\end{smallmatrix}\right)$ for $v \in \mathbb{Z}_p\backslash\{0\}$, $u \in -b + \mathbb{Z}_p$ and $m, n \in p^t\mathbb{Z}_p$. Thus

$$A = \begin{pmatrix}1 & m\\ & 1\end{pmatrix}\begin{pmatrix}v^{-1} & \\ & u^{-1}\end{pmatrix}\begin{pmatrix} & 1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & \\ n & 1\end{pmatrix}.$$

A direct computation gives the integral equals

$$\chi_2(-1)\chi_1\chi_2(p^t)p^{t(z+1)}\bar{\tau}^c((-p^t,-1))\sum_{i=1}^{\infty}(\chi_1(p^{-1})\bar{\tau}^c((p^{-1},1))p^{-z-1/2})^i\phi^{\mathrm{ord}}$$
$$= p^{t(z+1)}\bar{\tau}^c((-p^t,-1))L_p\big(\pi,\tau,z+\tfrac{1}{2}\big)\chi_1\chi_2(p^t)\chi_2(-1).$$

The π in the L-factor means the base change of π from U(2) to GL_2. Note that it is not convergent at $z = -z_\kappa$ and is defined by analytic continuation at that point.

Now we apply the functional equation trick to evaluate the pullback integral for $f_{2,\mathcal{D},p}$, similar to the proof of Lemma 6.32. As in [Skinner and Urban 2014, Proposition 11.28] the local constant showing up when applying the intertwining operator at $z = -z_\kappa$ is

$$\epsilon\big(\pi,\tau,-z_\kappa+\tfrac{1}{2}\big) = \mathfrak{g}(\bar{\tau}_1\bar{\chi}_1)\tau_1\chi_1(p^t)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_2)\tau_1\chi_2(p^t)\mathfrak{g}(\bar{\tau}_2\chi_2)\tau_2\chi_2^{-1}(p^t)p^{3\kappa/2-6}.$$

To sum up our original local integral for $f_{2,p}$ equals

$$L_p\big(\pi_h,\bar{\tau}^c,z_\kappa+\tfrac{1}{2}\big)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_1)\tau_1\chi_1(p^t)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_2)\tau_1\chi_2(p^t)p^{(2\kappa-5)t}\chi_1(p^t)p^{t/2}\varphi^{\mathrm{ord}}.$$

Note that $\langle\tilde{\varphi}^{\mathrm{ord}},\varphi^{ss}\rangle = \langle\tilde{\phi}^{\mathrm{ord}},\varphi_{\mathrm{low}}\rangle\cdot\chi_1(p^t)p^{t/2}$ where we define $\varphi_{\mathrm{low}} = \pi_p\big(\left(\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right)\big)\varphi^{\mathrm{ord}}$. Thus if we replace φ^{ss} by φ_{low} in the definition for pullback integral then it equals

$$L_p\big(\pi_h,\bar{\tau}^c,z_\kappa+\tfrac{1}{2}\big)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_1)\tau_1\chi_1(p^t)\mathfrak{g}(\bar{\tau}_1\bar{\chi}_2)\tau_1\chi_2(p^t)p^{(2\kappa-5)t}\varphi^{\mathrm{ord}}.$$

Noting again that later when we are defining $E_{\mathrm{sieg},2}$ the τ here should be τ/λ.

6J. *Global Computations.*

6J1. *Good Siegel–Eisenstein series.*

Definition 6.39. As for Klingen–Eisenstein series case, throughout this paper, we fix the tame level subgroup $K^{(3,3)}$ of $\mathrm{U}(3,3)(\mathbb{A}^{p\infty})$, under which our $E_{\mathrm{sieg},\mathcal{D}}$ is invariant. We can do so by simply taking it to be the set of matrices congruent to 1 modulo the $(y\bar{y})^2$ at each finite place not dividing p for the y defined above. We define the p-component of the level group for $E_{\mathrm{sieg},\mathcal{D}}$ at p to consist of elements congruent to 1 modulo p^{2t}.

We also fix a tame level subgroup $K^{(2,0)}$ of $\mathrm{U}(2,0)(\mathbb{A}^{p\infty})$ consisting of matrices congruent to 1 modulo the $(y\bar{y})^2$ above at each finite place away from p.

We first define two normalization factors as in [Wan 2015a, Subsection 5.3.1]

$$B_{\mathcal{D}} := \left(\frac{(-2)^{-3}(2\pi i)^{3\kappa}(2/\pi)^3}{\prod_{j=0}^{2}(\kappa-j-1)!}\right)^{-1}\prod_{i=0}^{2} L^{\Sigma}(2z_\kappa+3-i,\bar{\tau}'\chi_{\mathcal{K}}^{i}),$$

$$B'_{\mathcal{D}} := \left(\frac{(-2)^{-2}(2\pi i)^{2\kappa}(2/\pi)}{\prod_{j=0}^{1}(\kappa-j-1)!}\right)^{-1}\prod_{i=0}^{1} L^{\Sigma}(2z_\kappa+2-i,\bar{\tau}'\chi_{\mathcal{K}}^{i}).$$

Here recall $z_\kappa=(\kappa-3)/2$ and $z'_\kappa=(\kappa-2)/2$.

Definition 6.40. We define

$$E_{\mathrm{sieg},\mathcal{D}}(z,g)=E_{\mathrm{sieg}}(z,f_{\mathrm{sieg}},g)$$

on $\mathrm{GU}(3,3)$ for $f_{\mathrm{sieg},p}=f_{\mathrm{sieg},\mathcal{D}}=B_{\mathcal{D}}\prod_v f_{\mathrm{sieg},\mathcal{D},v}$ and

$$E'_{\mathrm{sieg},\mathcal{D}}(z,g)=E'_{\mathrm{sieg}}(z,f'_{\mathrm{sieg},\mathcal{D}},g)$$

on $\mathrm{GU}(2,2)$ for $f'_{\mathrm{sieg}}=B'_{\mathcal{D}}\prod_v f'_{\mathrm{sieg},v}$. (Note that compared to [Wan 2015a], the normalization factors at p here are already included in our definitions of p-adic Siegel sections.) We also define

$$E^{\square}_{\mathrm{sieg},\mathcal{D}}(g)=E(z,f^{\square}_{\mathrm{sieg},\mathcal{D}},g)$$

where $f^{\square}_{\mathrm{sieg},\mathcal{D}}$ is the same as $f_{\mathrm{sieg},\mathcal{D}}$ at all primes not dividing p and is $f^{\square}_{\mathrm{sieg},\mathcal{D},p}$ at p.

6J2. *Pullbacks.* For some $g_1\in\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ (which we specify in Definition 8.20) we define $E_{\mathrm{Kling},\mathcal{D}}$ by

$$E_{\mathrm{Kling},\mathcal{D}}(z,g)=\frac{1}{\Omega_\infty^{2\kappa}}\int_{[\mathrm{U}(2)]}E_{\mathrm{sieg},\mathcal{D}}(z,\gamma(g,hg')\Upsilon)\bar{\tau}(\det g')\pi(g_1)\varphi_{\mathcal{D}}(g')\,dg'. \tag{6-13}$$

Define

$$\varphi'_{\mathcal{D}}(z,g)=\frac{1}{\Omega_\infty^{2\kappa}}\int_{[\mathrm{U}(2)]}E'_{\mathrm{sieg},\mathcal{D}}(z,\gamma(g,hg')\Upsilon')\bar{\tau}(\det g')\pi(g_1)\varphi_{\mathcal{D}}(g')\,dg'. \tag{6-14}$$

Here we use the local components of a Klingen–Eisenstein data $\mathcal{D}$ in the construction. The period factor showing up comes from the geometric pullback map (see [Hsieh 2014a, Section 2.8, Subsection 5.6.5]). In fact comparing the definition for the geometric pullback map and the pullback formula for automorphic forms, in order to get rational automorphic forms on $U(2) \times U(2)$ or $U(3,1) \times U(2)$ via pullbacks, such CM periods have to be divided out. The $\varphi_{\mathcal{D}}$ is defined as follows. First, recall that given a CM character ψ and a form on $D^\times$ whose central character is $\psi|_{\mathbb{A}_{\mathbb{Q}}^\times}$ we can produce a form on $U(2)$ whose central character is the restriction of ψ. So we often construct forms on $D^\times$ and get forms on $U(2)$ this way. In Section 8C we construct a Dirichlet character ϑ.

Definition 6.41. We define f_ϑ as in the end of Section 8D, and an element g_1 in Definition 8.20. Our φ is defined as $\pi(g_1) f_\vartheta$.

Proposition 6.42. *The $E_{\mathrm{Kling},\mathcal{D}}(z_\kappa, -)$ defined above is the Klingen–Eisenstein series constructed using the Klingen section $F_{\mathrm{Kling},\mathcal{D}} = \prod_v F_{\mathrm{Kling},\mathcal{D},v}$ for $F_{\mathrm{Kling},\mathcal{D},v}$ the Klingen–Eisenstein sections defined in previous subsections. We also have that*

$$\varphi'_{\mathcal{D}}(z'_\kappa, -) = \prod_v F'_{\varphi_v}(f'_{\mathrm{sieg},v}, z'_\kappa, 1).$$

Note that we have used $\pi(g_1)\varphi_{\mathcal{D}}$ in the place of $\varphi_{\mathcal{D}}$ in (5-1), and there is a normalization factor appearing in the p-adic Klingen section here.

Proof. It is a straightforward consequence of Shimura's pullback formula, and our local pullback computations in previous sections. $\square$

We record the following easy lemma, which explains the motivation for the definition of f_Σ: to pick up a certain Iwahori-invariant vector from the unramified representation π_v for $v \in \Sigma \backslash \{v, v \mid N\}$.

Lemma 6.43. *Consider the model for the unramified principal series representation*

$$\pi(\chi_{1,v}, \chi_{2,v}) = \left\{ f : K_v \to \mathbb{C},\ f(qk) = \chi_{1,v}(a)\chi_{2,v}(d)\delta_B(q) f(k),\ q = \begin{pmatrix} a & b \\ & d \end{pmatrix} \in B(\mathbb{Z}_v) \right\}.$$

Let f_{ur} be the constant function 1 on K_v, f_0 be the function supported and takes value 1 on K_1 for $K_1 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \varpi_v \mid c \right\}$. Then

$$(\chi_{2,v}(\varpi_v) q_v^{-1/2} - \chi_1(\varpi_v) q_v^{1/2}) f_0 = \pi\left(\begin{pmatrix} & 1 \\ \varpi_v & \end{pmatrix} \right) f_{\mathrm{ur}} - \chi_{1,v}(\varpi_v) q_v^{1/2} f_{\mathrm{ur}}.$$

<u>Fourier–Jacobi coefficients</u>.

Proposition 6.44. *The Fourier–Jacobi coefficient for $\beta = 1$ is given by*

$$\mathrm{FJ}_1(E_{\mathrm{sieg},\mathcal{D}})(z_\kappa, \mathrm{diag}(u, 1_2, u, 1_2) n' \boldsymbol{j}(1, g))$$
$$= \sum_{n \in \mathbb{Z}_p/p^t\mathbb{Z}_p} E_{\mathrm{sieg},2}\left(z_\kappa, f_{2,\mathcal{D}}, g'\gamma\left(1, \begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}_p\right)\right) \times \Theta_{\Phi_{\mathcal{D}}}\left(u, n'g'\gamma\left(1, \begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}_p\right)\right)$$

for $n' \in N_2$ (N_2 defined in Section 4G), $g \in U(2,2)(\mathbb{A}_{\mathbb{Q}})$, $g' = \left(\begin{smallmatrix} 1_2 & \\ & -1_2 \end{smallmatrix}\right) g \left(\begin{smallmatrix} 1_2 & \\ & -1_2 \end{smallmatrix}\right)$, and $u \in U(1)(\mathbb{A}_{\mathbb{Q}})$. Here $f_{2,\mathcal{D}} = \prod_v f_{2,\mathcal{D},v}$ and $\Phi_{\mathcal{D}} = \prod_v \Phi_{\mathcal{D},v}$ are given in Definitions 6.12, 6.19, 6.23 and 6.36. The $E_{\mathrm{sieg},2}$ is considered as a function on $U(2,2)(\mathbb{A}_{\mathbb{Q}})$.

Proof. This follows from our computations for local Fourier–Jacobi coefficients and Lemma 6.6. □

So far we have used the embedding

$$\boldsymbol{j}(1,g)=\begin{pmatrix}1&&&\\&D_g&&C_g\\&&1&\\&B_g&&A_g\end{pmatrix}$$

for $g=\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$, to keep accordance with the convention of [Skinner and Urban 2014] and [Wan 2015a]. In our actual applications later on we will use another embedding α'' as below. Now that we have

$$\mathrm{FJ}_1(E^{\square}_{\mathrm{sieg},\mathcal{D}},z_\kappa,g,x,u)=E_{\mathrm{sieg},2}(z_\kappa,f_{2,\mathcal{D}},g')\Theta_{\Phi_{\mathcal{D}}}(u,g'),$$

we consider another embedding

$$\boldsymbol{j}''(1,g)=\begin{pmatrix}1&&&\\&A_g&&B_g\\&&1&\\&C_g&&D_g\end{pmatrix}.$$

(The g' is as defined in Proposition 6.44). Let $\Phi''_{\mathcal{D}}=\omega_\beta(\eta)\Phi_{\mathcal{D}}$ and $f''_{2,\mathcal{D}}=\rho(\eta)f_{2,\mathcal{D}}$ and we let FJ''_β be defined as FJ_β but replacing $\boldsymbol{j}$ by $\boldsymbol{j}''$. By observing that $E_{\mathrm{sieg},\mathcal{D},2}$ and Θ are automorphic forms and thus invariant under left multiplication by η^{-1}, we get

$$\mathrm{FJ}''_\beta(E^{\square}_{\mathrm{sieg},\mathcal{D}},z_\kappa,\mathrm{diag}(u,1_2,u,1_2)n'\boldsymbol{j}''(1,g))=E_{\mathrm{sieg},\mathcal{D},2}(z_\kappa,f''_{2,\mathcal{D}},g)\Theta_{\Phi''_{\mathcal{D}}}(u,n'g). \tag{6-15}$$

Definition 6.45. Let $\prod_{v\nmid p}U_v$ be the intersection of $g^{-1}K^{(3,1)}g$ (the $K^{(3,1)}$ is defined in Definition 6.39) with $\mathrm{U}(2,0)(\mathbb{A}^{p\infty})$, and U_p consists of matrices in $\mathrm{U}(2,0)(\mathbb{Z}_p)$ which are upper triangular modulo the p^t in Section 6H. The L is defined to be the intersection of $\mathcal{K}^2$ (the quotient of $N_P(\mathbb{Q})$ over the center of it) with the image of $(K^{(3,1)}\times\mathrm{U}(3,1)(\mathbb{Z}_p))\cap N_P(\mathbb{A}_f)$. Define $\phi_1=\prod_v\phi_{1,v}$ and $\theta^\star_{\mathcal{D}}=\theta_{\phi_1}$ as an element in the $\mathbb{C}$-dual space of

$$H^0(\mathcal{Z}^\circ_{[g]},\mathcal{L}(\beta))\otimes\mathbb{C}\simeq T^{\mathrm{Hol}}_{\mathbb{A}}(\beta,L,U_f)$$

as in the end of Section 4I for $m=1$, $U_f\subseteq\mathrm{U}(2)(\mathbb{A}_f)$ an open compact subgroup defined as $K^{(2,0)}\times\mathrm{GL}_2(\mathbb{Z}_p)$. The $L\subset\mathcal{K}^{\oplus 2}$ being the ideal generated by $(y\bar{y})^2$. (These level groups are fixed throughout the family).

We will usually write $E_{\mathrm{sieg},\mathcal{D},2}$ for $E_{\mathrm{sieg},\mathcal{D},2}(z_\kappa,f''_{2,\mathcal{D}},g)$ for short.

Lemma 6.46. *Suppose $\delta_\psi^{-1}(\Phi''_\infty)=\phi_{1,\infty}\boxtimes\phi_{2,\infty}$, and for each $v<\infty$*

$$\phi_{2,v}(x)=\int_{X_v}\delta_\psi^{-1}(\Phi''_v)(x',x)\phi_{1,v}(x')\,dx'. \tag{6-16}$$

Then

$$l'_{\theta^\star_{\mathcal{D}}}(\gamma^{-1}(\Theta_{\Phi''_{\mathcal{D}}}))(h)=\theta_{\phi_2}(h). \tag{6-17}$$

Here we consider $\gamma^{-1}(\Theta_{\Phi''_{\mathcal{D}}})$ *as a function on* $(N\mathrm{U}(2))\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(3,1)\times\mathrm{U}(2)$ *and apply* $l'_{\theta^\star_{\mathcal{D}}}$ *to it on the* $N\mathrm{U}(2)$ *part.*

Proof. We write $\delta_\psi^{-1}\Phi''$ for each finite place v as a finite sum of expressions of the form $\phi'_{1,i,v}\boxtimes\phi_{2,i,v}$ and let $\phi'_{1,i}=\prod_{v\nmid\infty}\phi'_{1,v}\times\phi_{1,\infty}$ and $\phi_{2,i}=\prod_{v\nmid\infty}\phi_{2,i,v}\times\phi_{2,\infty}$. We have a finite sum

$$\Theta_{\Phi''_{\mathcal{D}}}(vh_1,h_2)=\sum_{i=1}^{r}\theta_{\phi'_{1,i}}(vh_1)\theta_{\phi_{2,i}}(h_2)$$

and that the function

$$T_i(h):=\int_{[V]}\theta_{\phi'_{1,i}}(vh)\theta^\star_{\mathcal{D}}(vh)\,dv$$

is a constant function by Lemma 4.8. Then the expression $\sum_{i=1}^r T_i\cdot\theta_{\phi_{2,i}}$ is clearly equal to θ_{ϕ_2} by (4-3) and (4-5). □

Corollary 6.47. *We have*

$$l'_{\theta^\star_{\mathcal{D}}}\biggl(\mathrm{FJ}_1\biggl(\prod_v\sum_i\rho(\mathrm{diag}(u_{v,i},1_2,u_{v,i}))\biggr)E_{\mathrm{Kling},\mathcal{D}}\biggr)(g)$$
$$=\biggl\langle\frac{1}{\Omega_\infty^{2\kappa-2}}\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}E_{\mathrm{sieg},\mathcal{D},2}\biggl(\alpha\biggl(g,-\begin{pmatrix}1&\\n&1\end{pmatrix}_p\biggr)\biggr)\cdot\frac{1}{\Omega_\infty}\theta_{2,\mathcal{D}}\biggl(-\begin{pmatrix}1&\\n&1\end{pmatrix}_p\biggr),\varphi_{\mathcal{D}}(-)\biggr\rangle$$

where $\theta^\star_{\mathcal{D}}$ *and* $\theta_{2,\mathcal{D}}$ *are the theta functions on* $U(-\zeta)(\mathbb{A}_{\mathbb{Q}})$ *defined using the Schwartz functions* $\phi_1=\prod_v\phi_{1,v}$ *and* $\phi_2=\prod_v\phi_{2,v}$ *for* $\phi_{1,v}$ *and* $\phi_{2,v}$*'s defined as before (Definitions 6.12, 6.38, 4.1 and the corresponding definition for ramified places). Note that the* ϕ_2 *is defined using* ϕ_1*, which explains the dependence of the right hand side on* ϕ_1*. The inner product is over the group* $1\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(3,3)$*. Moreover, suppose* $\varphi_p\in\pi_p$ *is chosen such that* φ_p *is the ordinary vector, then the above expression is*

$$\frac{1}{\Omega_\infty^{2\kappa-1}}\langle E_{\mathrm{sieg},\mathcal{D},2}(\alpha(g,-))\cdot\theta_{2,\mathcal{D}}(-),\varphi_{\mathcal{D}}(-)\rangle.$$

Proof. It follows from the Proposition 6.44, Lemma 6.46, noting that the pairing $l'_{\theta^\star_{\mathcal{D}}}$ is essentially applied to the Θ_Φ factor on the right hand side of Proposition 6.44 (recall also the meaning for intertwining maps defined in Section 4G). The last sentence follows from describing the pairing between π_p and $\pi_p^\vee$. □

7. *p*-adic interpolation

7A. ***Congruence module and the canonical period.*** We now discuss the theory of congruences of modular forms on $\mathrm{GL}_2/\mathbb{Q}$, following [Skinner and Urban 2014, Section 12.2]. Let R be a finite discrete valuation ring extension of $\mathbb{Z}_p$ and ε a finite order character of $\hat{\mathbb{Z}}^\times$ whose p-component has conductor dividing p. Let $M_\kappa^{\mathrm{ord}}(Mp^r,\varepsilon;R)$ be the space of ordinary modular forms on $\mathrm{GL}_2/\mathbb{Q}$ with level $N=Mp^r$, character ε and coefficient R. Let $S_\kappa^{\mathrm{ord}}(Mp^r,\varepsilon;R)$ be the subspace of cusp forms. Let $\mathbb{T}_\kappa^{\mathrm{ord}}(N,\varepsilon;R)$ ($\mathbb{T}_\kappa^{\mathrm{ord},0}(Mp^r,\varepsilon;R)$) be the R-subalgebra of $\mathrm{End}_R(M_\kappa^{\mathrm{ord}}(Mp^r,\varepsilon;R))$ (respectively,

$\mathrm{End}_R(S_\kappa^{\mathrm{ord}}(Mp^r, \varepsilon; R)))$ generated by the Hecke operators T_v (these are Hecke operators defined using the double cosets $\Gamma_1(N)_v\begin{pmatrix}\varpi_v & \\ & 1\end{pmatrix}\Gamma_1(N)_v$ for the v's) for all v. Let any $f \in S_\kappa^{\mathrm{ord}}(N, \varepsilon; R)$ be an ordinary eigenform. Then we have an $1_f \in \mathbb{T}_\kappa^{\mathrm{ord},0}(N, \varepsilon; R) \otimes_R F_R = \mathbb{T}'_\kappa \times F_R$ the projection onto the second factor.

Let $\mathfrak{m}_f$ be the maximal ideal of the Hecke algebra corresponding to f. The $\mathbb{T}^{\mathrm{ord},0}(M, \varepsilon; R) \cap (0 \otimes F_R)$ is free of rank one over R. We let ℓ_f be a generator and so $\ell_f = \eta_f 1_f$ for some $\eta_f \in R$. This η_f is called the congruence number of f.

Now let $\mathbb{I}$ be as in the introduction. Suppose $\boldsymbol{f} \in M^{\mathrm{ord}}(M, \varepsilon; \mathbb{I})$ is an ordinary $\mathbb{I}$-adic cuspidal eigenform. Then as above $\mathbb{T}^{\mathrm{ord},0}(M, \varepsilon; \mathbb{I}) \otimes F_{\mathbb{I}} \simeq \mathbb{T}' \times F_{\mathbb{I}}$, $F_{\mathbb{I}}$ being the fraction field of $\mathbb{I}$ where projection onto the second factor gives the eigenvalues for the actions on $\boldsymbol{f}$. Again let $1_{\boldsymbol{f}}$ be the idempotent corresponding to projection onto the second factor. Then for a $\boldsymbol{g} \in S^{\mathrm{ord}}(M, \varepsilon; \mathbb{I}) \otimes_{\mathbb{I}} F_{\mathbb{I}}$, $1_{\boldsymbol{f}}\boldsymbol{g} = c\boldsymbol{f}$ for some $c \in F_{\mathbb{I}}$. In the case when the localized Hecke algebra $\mathbb{T}^{\mathrm{ord},0}(M, \varepsilon; R)_{\mathfrak{m}_f}$ is a Gorenstein $\mathbb{I}$-algebra (which is indeed the case under assumptions (**dist**) and (**irred**)), $\mathbb{T}^{\mathrm{ord},0}(M, \varepsilon; \mathbb{I}) \cap (0 \otimes F_{\mathbb{I}})$ is a rank one $\mathbb{I}$-module. Under the Gorenstein property for $\mathbb{T}_{\boldsymbol{f}}$, we can define $\ell_{\boldsymbol{f}}$ and $\eta_{\boldsymbol{f}}$.

Definition 7.1. From now on, we will define D to be the unique quaternion algebra ramified exactly at ∞ and the q in our main theorems in the introduction. We choose the group $\mathrm{U}(2)$ with $D^\times$ being its associated quaternion algebra. It is clear that this is possible.

We also make the following definition for p-adic families of forms on $D^\times$.

Definition 7.2. For any complete local $\mathcal{O}_L[\![W]\!]$-algebra R (need not be finite over $\mathcal{O}_L[\![W]\!]$) we define the space of R-adic families on $G = D^\times$ with level group $K_D \subset D^\times(\mathbb{A}_f)$ which is $\mathrm{GL}_2(\mathbb{Z}_p)$ at p to be the space of continuous functions

$$f : D^\times(\mathbb{Q})\backslash D_f^\times(\mathbb{A})/K_D^{(p)} \to R$$

such that

$$f\left(x\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\right) = f(x)\langle d\rangle_W \chi(d)$$

for $a, d \in \mathbb{Z}_p^\times$, $b \in \mathbb{Z}_p$, where χ is a fixed character of $\mathbb{Z}_p^\times$ trivial on $1 + p\mathbb{Z}_p$, $\langle d\rangle_W \in \mathcal{O}_L[\![W]\!] \simeq \mathcal{O}_L[\![\Gamma_{\mathbb{Q}}]\!]$ is the image of d under the local reciprocity law. Here we make an identification of W with $\mathrm{rec}_p(1 + p)$ which is a topological generator of $\Gamma_{\mathbb{Q}}$. We also equip $\mathrm{GL}_2(\mathbb{Z}_p) \subset D^\times(\mathbb{A}_f)$ with the topology as a p-adic Lie group. We write

$$M(S_G(K_D^{(p)}), R)$$

for the space of all such forms.

One can define Hecke operators T_ℓ at primes ℓ where K is $\mathrm{GL}_2(\mathbb{Z}_\ell)$, and the U_p operator defined by

$$U_p f(g) = \sum_{n \in \mathbb{Z}_p/p\mathbb{Z}_p} f\left(g\begin{pmatrix} 1 & n \\ & 1\end{pmatrix}_p \begin{pmatrix} p & \\ & 1\end{pmatrix}_p\right).$$

We make similar definitions for R-adic families on the definite unitary group $G = \mathrm{U}(2)$ of some prime to p level group $K^{(p)}$, which we denote as $M(S_G(K^{(p)}), R)$. It is also possible to add one more variable allowing twisting by characters (so that the nebentypus will be a character of $\mathrm{diag}(a, d)$ instead of just d) as follows. Let $\Lambda_{\mathrm{U}(2)} = \Lambda_{2,0} = \mathbb{Z}_p[\![T_1, T_2]\!]$. Let $\mathcal{M}_{\mathrm{ord}}(K^{(2,0)}, \Lambda_{2,0})$ be the space of $\Lambda_{2,0}$-adic ordinary modular forms on $\mathrm{U}(2,0)$, consisting of functions

$$\boldsymbol{f} : \mathrm{U}(2,0)\backslash \mathrm{U}(2,0)(\mathbb{A}_{\mathbb{Q}})/K^{(2,0)}\mathrm{U}(2,0)_\infty \to \Lambda_{2,0}$$

such that

$$\boldsymbol{f}\left(g\begin{pmatrix} a & \\ & d\end{pmatrix}\right) = \boldsymbol{f}(g)\cdot\langle a\rangle_{T_1}\langle d\rangle_{T_2},\ a, d \in \mathbb{Z}_p^\times.$$

For later use in Section 7D, we will also define a space $\check{\mathcal{M}}_{\mathrm{ord}}(K^{(2,0)}, \Lambda_{2,0})$ be the space of Λ-adic ordinary modular forms consisting of functions $\boldsymbol{f} : \mathrm{U}(2,0)\backslash \mathrm{U}(2,0)(\mathbb{A}_{\mathbb{Q}})/K^{(2,0)}\mathrm{U}(2,0)_\infty \to \Lambda_{2,0}$ such that

$$\boldsymbol{f}\left(g\begin{pmatrix} a & \\ & d\end{pmatrix}\right) = \boldsymbol{f}(g)\cdot\langle d\rangle_{T_1}^{-1}\langle a\rangle_{T_2}^{-1}.$$

The ordinary family $\boldsymbol{f}$ on $D^\times$. Let $\boldsymbol{f}$ be a Hida family of ordinary cuspidal eigenforms new outside p as in Theorem 1.1. Suppose $\mathbb{T}_{\mathfrak{m}_f}$ is Gorenstein. Thus we have the integral projector $\ell_{\boldsymbol{f}}$. We construct from it a Hida family of ordinary forms on $D^\times(\mathbb{A}_{\mathbb{Q}})$, also denoted as $\boldsymbol{f}$. We refer to [Hida 1988, Sections 2 and 3] for the definition and theory of ordinary forms on quaternion algebras. By our assumption in Theorem 1.1 and our choice for D, we may choose f_0 a form on $D^\times(\mathbb{A}_{\mathbb{Q}})$ which is in the Jacquet–Langlands correspondence of f in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ with values in $\mathcal{O}_L$ and f_0 is not divisible by π_L (the uniformizer of the maximal ideal of $\mathcal{O}_L$). Under the assumption of Theorem 1.1, we do the following we first take an ordinary Hida family $\boldsymbol{g}$ such that $\ell_{\boldsymbol{f}}\boldsymbol{g}$ is nonzero. Note that $\mathbb{I}$ is a Dedekind domain. So we can divide $\ell_{\boldsymbol{f}}\boldsymbol{g}$ by an element in $\mathbb{I}$ such that the quotient (which we still denote as $\boldsymbol{f}$ to save notations) is integral (i.e., $\mathbb{I}$-valued) and there is no nontrivial common divisor for the values of $\boldsymbol{f}$.

7B. *Eisenstein datum.*

Definition 7.3. An Eisenstein datum $\boldsymbol{D} = (\Sigma, L, \mathbb{I}, \boldsymbol{f}, \psi, \tau)$ consists of:

- A finite set of primes Σ containing all bad primes.
- A finite extension $L/\mathbb{Q}_p$.
- A finite normal $\mathcal{O}_L[\![W]\!]$-algebra $\mathbb{I}$.
- An $\mathbb{I}$-adic Hida family $\boldsymbol{f}$ of cuspidal ordinary eigenforms on $D^\times$ new outside p, of square-free tame conductor N such that some weight 2 specialization f_0 has trivial character.
- Two L-valued Hecke characters ψ and τ of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$ whose p-part conductors divide p, and whose infinity types are $(0,0)$ and $\left(-\frac{\kappa}{2}, \frac{\kappa}{2}\right)$, respectively, such that $\psi|_{\mathbb{A}_{\mathbb{Q}}^\times}$ is equal to the central character of π_{f_0}. Let $\xi = \psi/\tau$. We define p-adic deformations $\boldsymbol{\psi}, \boldsymbol{\tau}$ of them in (7-1).

Note that for any arithmetic point, the specialization $\mathcal{D}_\phi$ of $\boldsymbol{D}$ gives an Eisenstein datum in the sense of Definition 5.4. Now we need to modify our $\mathbb{I}$. By taking an irreducible component of the normalization of a series of quadratic extensions of $\mathbb{I}$ we may assume that for each $v \in \Sigma$ not dividing N, we can find two functions $\alpha_v, \beta_v \in \mathbb{I}$ interpolating the Satake parameters of π_v. This enables us to do the constructions in the global computations in Lemma 6.43 in families. We still denote $\mathbb{I}$ for the new ring for simplicity. At the end of this paper, we will see how to deduce the main conjecture for the original $\mathbb{I}$ from that for the new $\mathbb{I}$.

Let $\hat{\mathbb{I}}^{\mathrm{ur}}_{\mathcal{K}} := \hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$ (see Introduction for the notion of $\hat{\mathbb{I}}^{\mathrm{ur}}$). We define $\alpha : \mathcal{O}_L[[\Gamma_{\mathcal{K}}]] \to \hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma^-_{\mathcal{K}}]]$ and $\beta : \mathcal{O}_L[[\Gamma_{\mathcal{K}}]] \to \hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$ by

$$\alpha(\gamma_+) = (1+W)^{1/2}, \quad \alpha(\gamma_-) = \gamma_-, \quad \beta(\gamma_+) = \gamma_+, \quad \beta(\gamma_-) = \gamma_-.$$

We let

$$\boldsymbol{\psi} = \psi \cdot \alpha \circ \Psi_{\mathcal{K}}, \quad \boldsymbol{\xi} = (\beta \circ \Psi_{\mathcal{K}}) \cdot \xi, \quad \boldsymbol{\tau} = \boldsymbol{\psi}/\boldsymbol{\xi}. \tag{7-1}$$

Define $\psi_\phi := \phi \circ \boldsymbol{\psi}$ and $\xi_\phi = \phi \circ \boldsymbol{\xi}$. Let $\Lambda_{\boldsymbol{D}} = \hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]][[\Gamma^-_{\mathcal{K}}]]$. We give $\Lambda_{\boldsymbol{D}}$ a Λ_2-algebra structure by first defining a homomorphism $\Gamma_2 = (1+p\mathbb{Z}_p)^4 \to \Gamma_{\mathcal{K}} \times \Gamma_{\mathcal{K}}$ given by

$$(a, b, c, d) \to \mathrm{rec}_{\mathcal{K}}(db, a^{-1}c^{-1}) \times \mathrm{rec}_{\mathcal{K}}(d^{-1}, c)$$

and then compose with $\alpha \otimes \beta$.

We remark here that only the subring $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$ of $\Lambda_{\boldsymbol{D}}$ really matters; the $\Gamma^-_{\mathcal{K}}$ variable corresponds to twisting everything by the same character and does not affect the p-adic L-functions and the Selmer groups.

Definition 7.4. A point $\phi \in \operatorname{Spec} \Lambda_{\boldsymbol{D}}$ is called arithmetic if $\phi(1+W), \phi(\gamma), \phi(\gamma^-)$ for $\gamma \in \Gamma_{\mathcal{K}}$ and $\gamma^- \in \Gamma^-_{\mathcal{K}}$ are all p-power roots of unity. We call it generic if the p-part of $(f_\phi, \psi_\phi, \tau_\phi)$ is generic in the sense defined in Definition 6.30. Note that whether ϕ is generic only depends on its image in $\operatorname{Spec} \Lambda_2$. It also only depends on the subring $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$. So it makes sense to talk about generic points on these weight spaces as well. We let $\mathcal{X}_{\boldsymbol{D}}$ be the set of arithmetic points and $\mathcal{X}^{\mathrm{gen}}_{\boldsymbol{D}}$ be the set of generic arithmetic points. Later when we are working with families, it is easily seen that these points are Zariski dense due to the fact that $p \geq 5$ (in fact this is the only place where we used the fact $p \geq 5$).

Let us write down the weight map $j_0 : \Lambda_{2,0} \to \Lambda_{\boldsymbol{D}}$ for the family (which we still denote as $\boldsymbol{f} \in \mathcal{M}_{\mathrm{ord}}(K^{(2,0)}, \Lambda_{2,0}) \otimes_{j_0} \Lambda_{\boldsymbol{D}}$) on $\mathrm{U}(2,0)$ constructed from the Hida family $\boldsymbol{f}$ and the character $\boldsymbol{\psi}$. In fact

$$j_0(1+T_1) = \boldsymbol{\psi}_2^{-1}(1+p), \quad j_0(1+T_2) = \boldsymbol{\chi}_{\boldsymbol{f}} \boldsymbol{\psi}_2|_{\mathbb{Z}_p^\times}(1+p).$$

Here $\boldsymbol{\chi}_{\boldsymbol{f}}$ is the central character of $\boldsymbol{f}$, and we write $\boldsymbol{\psi}_2$ for the restriction of $\boldsymbol{\psi}$ to $\mathcal{K}^\times_{\bar{v}_0} \simeq \mathbb{Q}_p^\times$.

7C. *Siegel–Eisenstein measure.*

Proposition 7.5. *There are Λ_2-adic formal Fourier expansions $\boldsymbol{E}_{\boldsymbol{D},\mathrm{sieg}}$ and $\boldsymbol{E}'_{\boldsymbol{D},\mathrm{sieg}}$ such that*

$$\boldsymbol{E}_{\boldsymbol{D},\mathrm{sieg},\phi} = E_{\mathrm{sieg},\mathcal{D}_\phi}\Big(\prod_v f_{\mathrm{sieg},v}, z_\kappa, -\Big), \quad \boldsymbol{E}'_{\boldsymbol{D},\mathrm{sieg},\phi} = E'_{\mathrm{sieg},\mathcal{D}_\phi}\Big(\prod_v f'_{\mathrm{sieg},v}, z'_\kappa, -\Big)$$

in terms of formal Fourier expansions. Here the datum $\mathcal{D}_\phi = (f_\phi, \xi_\phi, \psi_\phi)$ is the specialization of $\boldsymbol{D}$ at ϕ.

Proof. It is a special case of [Wan 2015a, Lemma 5.7] and follows from our computations of the local Fourier coefficients for the Siegel–Eisenstein series. Recall $\boldsymbol{\tau} = \boldsymbol{\psi}/\boldsymbol{\xi}$ and write τ_ϕ for the localization of $\boldsymbol{\tau}$ at an arithmetic point ϕ. On the other hand, we have families of characters $\boldsymbol{\xi}_1^\dagger$ and $\boldsymbol{\xi}_2^\dagger$ of $\mathbb{Z}_p^\times$ interpolating the restriction of characters $\xi_{\phi,1}^\dagger$ and $\xi_{\phi,2}^\dagger$ to $\mathbb{Z}_p^\times$. It follows from our computations in Section 6 for local Fourier coefficients and [Skinner and Urban 2014, Lemma 11.2], that we can form the β-th Fourier coefficient of

$$E_{\boldsymbol{D},\mathrm{sieg}}(f_{\mathrm{sieg}}; z_\kappa, g)$$

at $\mathrm{diag}(y, {}^t\bar{y})$ for $y \in \mathrm{GL}_3(\mathbb{A}_{\mathcal{K},f})$ is given by

$$\prod_{\ell\in\Sigma,\ell\nmid p} D_\ell^{-3/2}\boldsymbol{\tau}(\det y_\ell)|\det y_\ell\bar{y}_\ell|_\ell^{-\kappa/2} e_\ell\left(\frac{\beta_{22}+\beta_{33}}{y_\ell\bar{y}_\ell}\right)$$
$$\times\prod_{\ell\nmid p}\boldsymbol{\tau}'_\ell(\det\beta)\times\prod_{\ell\notin\Sigma} h_{\ell,{}^t\bar{y}_\ell\beta y_\ell}(\bar{\boldsymbol{\tau}}'(\ell)\ell^{-\kappa})\times(\det\beta|\det\beta|_p)^{\kappa-3}\times\boldsymbol{\xi}_1^\dagger(\beta_{21})\boldsymbol{\xi}_2^\dagger$$
$$\times\left(\frac{\beta_{21}\beta_{32}-\beta_{22}\beta_{31}}{\beta_{21}}\right)\cdot\mathrm{char}(\mathbb{Z}_p,\beta_{11})\,\mathrm{char}(\mathbb{Z}_p,\beta_{12})\,\mathrm{char}(\mathbb{Z}_p,\beta_{13})\,\mathrm{char}(\mathbb{Z}_p,\beta_{23})\,\mathrm{char}(\mathbb{Z}_p,\beta_{33})$$

if $\det\beta > 0$ and β_{21} and $\det\left(\begin{smallmatrix}\beta_{21} & \beta_{22}\\ \beta_{31} & \beta_{32}\end{smallmatrix}\right)$ are both in $\mathbb{Z}_p^\times$, and is equal to 0 otherwise. Here, $\mathrm{char}(\mathbb{Z}_p, x)$ is the characteristic function for $\mathbb{Z}_p$ for the variable x. That the Fourier coefficient is zero if $\det\beta \le 0$ follows from our computations in Lemma 6.8. Note the definition of $\Phi_{\xi_\phi^\dagger}$ in Definition 6.33 for the part in the third row involving the p-adic place. This whole expression is clearly interpolated by an element in the Iwasawa algebra. The case for

$$E'_{\boldsymbol{D},\mathrm{sieg}}(f'_{\mathrm{sieg}}; z, g)$$

is similar.

Now we can obtain the Siegel–Eisenstein measure from the abstract Kummer congruence as detailed in [Hsieh 2011, Lemma 3.15]. From the $\bmod p$ q-expansion principle and that all Fourier coefficients above are interpolated by elements in the Iwasawa algebra, we see that $\boldsymbol{E}_{\boldsymbol{D},\mathrm{sieg}}$ and $\boldsymbol{E}'_{\boldsymbol{D},\mathrm{sieg}}$ indeed give a measure with values in the space of p-adic automorphic forms on $\mathrm{GU}(3,3)$ (see also [Hsieh 2011, Theorem 3.16]). □

This formal Fourier expansion gives a measure on $\Gamma_{\mathcal{K}} \times \mathbb{Z}_p$ with values in the space of p-adic automorphic forms on $\mathrm{GU}(3,3)$, which we denote as $\mathcal{E}_{\mathcal{D},\mathrm{sieg}}$ and $\mathcal{E}'_{\mathcal{D},\mathrm{sieg}}$, respectively.

7D. *Interpolating Petersson inner products.* In this subsection we make an additional construction of pairing Hida families on definite unitary groups following [Hsieh 2017], which is crucial for our later construction.

Definition 7.6. For a *neat* tame level group $K \subset \mathrm{U}(2)(\mathbb{A}^{p\infty})$ we use the notation $\boldsymbol{B}_K\langle -,-\rangle$ to denote the $\Lambda_{\mathrm{U}(2)}$-pairing

$$\boldsymbol{B}_K : \mathcal{M}_{\mathrm{ord}}(K,\Lambda_{\mathrm{U}(2)}) \times \check{\mathcal{M}}_{\mathrm{ord}}(K,\Lambda_{\mathrm{U}(2)}) \to \Lambda_{\mathrm{U}(2)}$$

such that for any $\boldsymbol{f} \in \mathcal{M}_{\mathrm{ord}}(K, \Lambda_{\mathrm{U}(2)})$, $\boldsymbol{g} \in \breve{\mathcal{M}}_{\mathrm{ord}}(K, \Lambda_{\mathrm{U}(2)})$ and $\phi \in \operatorname{Spec} \Lambda_{\mathrm{U}(2)}(\mathbb{C}_p)$ a weight two point, for any n we define

$$\boldsymbol{B}_{K,n}\langle \boldsymbol{g}, \boldsymbol{f}\rangle := \sum_{[x_i]\in \mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)/KU_0(p^n)} U_p^{-n} \boldsymbol{f}(x_i)\boldsymbol{g}\left(x_i \begin{pmatrix} & 1 \\ p^n & \end{pmatrix}\right) \pmod{(1+T_1)^{p^n}-1, (1+T_2)^{p^n}-1}.$$

Then one checks

$$\boldsymbol{B}_{K,n+1} \equiv \boldsymbol{B}_{K,n} \pmod{(1+T_1)^{p^n}-1, (1+T_2)^{p^n}-1}.$$

We define

$$\boldsymbol{B}_K\langle \boldsymbol{g}, \boldsymbol{f}\rangle = \lim_n \boldsymbol{B}_{K,n}\langle \boldsymbol{g}, \boldsymbol{f}\rangle.$$

By definition we have

$$\phi(\boldsymbol{B}_K\langle \boldsymbol{g}, \boldsymbol{f}\rangle) = \sum_{[x_i]\in \mathrm{U}(2)(\mathbb{Q})\backslash \mathrm{U}(2)/KU_0(p^n)} U_p^{-n} \boldsymbol{f}_\phi(x_i)\boldsymbol{g}_\phi\left(x_i \begin{pmatrix} & 1 \\ p^n & \end{pmatrix}\right)$$

and hence

$$\begin{aligned}\phi(\boldsymbol{B}_K\langle \boldsymbol{g}, \boldsymbol{f}\rangle) &= \operatorname{vol}(KU_0(p^n))^{-1} \int_{[\mathrm{U}(2)]} U_p^{-n} \boldsymbol{f}_\phi(h)\boldsymbol{g}_\phi\left(h \begin{pmatrix} & 1 \\ p^n & \end{pmatrix}\right) dh \\ &= \operatorname{vol}(KU_0(p^n))^{-1} \int_{[\mathrm{U}(2)]} \boldsymbol{f}_\phi\left(h \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}_p\right) \boldsymbol{g}_\phi(h)\, dh\end{aligned}$$

if ϕ corresponds to an ordinary form whose p-part conductor is p^n. In the following we will fix the tame level group $K^{(2,0)}$ as defined before, and will sometimes suppress the subscript K in B_K.

7E. *p-adic L-functions.* We have the following proposition for p-adic L-functions. Recall Σ is a finite set of primes containing all bad primes.

Proposition 7.7. *Notations are as before. There is an element $\mathcal{L}^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}}$ in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$, and a p-integral element $C^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}} \in \overline{\mathbb{Q}}_p^\times$ such that for any generic arithmetic point ϕ of conductor p^t, we have*

$$\phi\left(\frac{\mathcal{L}^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}}}{\Omega_p^{2\kappa}}\right) = C^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}} \frac{2\pi i p^{(\kappa-3)t}\xi_{1,p}^2\chi_{1,p}^{-1}\chi_{2,p}^{-1}(p^{-t})\mathfrak{g}(\xi_{v_0}\chi_{1,p}^{-1})\mathfrak{g}(\xi_{v_0}\chi_{2,p}^{-1})L^{\Sigma}\left(\mathcal{K}, \pi_{f_\phi}, \bar{\chi}_\phi\xi_\phi, \frac{\kappa}{2}-\frac{1}{2}\right)(\kappa-1)!(\kappa-2)!}{\Omega_\infty^{2\kappa}}. \tag{7-2}$$

Here $\chi_{1,p}, \chi_{2,p}$ is such that the unitary representation $\pi_{f_\phi} \simeq \pi(\chi_{1,p}, \chi_{2,p})$ with $\mathrm{val}_p(\chi_{1,p}(p)) = -\frac{1}{2}$, $\mathrm{val}(\chi_{2,p}(p)) = \frac{1}{2}$. We remark that the fraction on the right hand side is an algebraic number, by the definition of the periods. Moreover, by making different choices for the Néron differential of the CM elliptic curve, the Ω_∞ and Ω_p are changed by multiplying by the same nonzero algebraic number. Let the local ϵ-factor at p for a ramified character λ of $\mathbb{Q}_p^\times$ with conductor c be defined by

$$\epsilon_p(\lambda, s) = \int_{c^{-1}\mathbb{Z}_p^\times} \lambda^{-1}(a)|a|_p^{-s} e_p(\operatorname{Tr} a)\, da. \tag{7-3}$$

Note $1-(\kappa-1)/2=(3-\kappa)/2$. *The above interpolation formula can be written in terms of local* ϵ*-factors by*

$$\phi\left(\frac{\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}}{\Omega_p^{2\kappa}}\right) = C^{\Sigma}_{f,\xi,\mathcal{K}} \frac{2\pi i \epsilon_p\big(\frac{3}{2}-\frac{\kappa}{2},\xi_{v_0}^{-1}\chi_{1,p}\big)\epsilon_p\big(\frac{3}{2}-\frac{\kappa}{2},\xi_{v_0}^{-1}\chi_{2,p}\big)L^{\Sigma}\big(\mathcal{K},\pi_{f_\phi},\bar{\chi}_\phi\xi_\phi,\frac{\kappa}{2}-\frac{1}{2}\big)(\kappa-1)!(\kappa-2)!}{\Omega_\infty^{2\kappa}}. \tag{7-4}$$

Proof. Suppose we are under the assumption of Theorem 1.1. Take g_0 to be a point on the Igusa scheme for GU(2) defined over $\hat{\mathcal{O}}_L^{\mathrm{ur}}$ such that $\boldsymbol{f}(g_0)$ is nonzero in $\hat{\mathbb{I}}^{\mathrm{ur}}$ and take a $h_0\in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(g_0)=\mu(h_0)$. It is noted in [Hsieh 2014a, Section 2.8] that

$$I_{\mathrm{GU}(2)}(K_1^n)(\hat{\mathcal{O}}_L^{\mathrm{ur}})=\mathrm{GU}(2)(\mathbb{Q})^+\backslash \mathrm{GU}(2)(\mathbb{A}_f)/K_1^n.$$

In the following we write $\hat{F}$ for the p-adic modular form associated to a form F. We define $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}$ such that

$$\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}=\boldsymbol{B}\langle e_{\mathrm{ord}}^{\mathrm{U}(2)}\hat{\boldsymbol{E}}'_{\boldsymbol{D},\mathrm{sieg}}(\underline{A}_{g_0},-)\bar{\boldsymbol{\tau}}\circ\det(-),\pi(h_0)\boldsymbol{f}\rangle/\boldsymbol{f}(g_0)$$

where $\underline{A}$ is the quintuple associated to g_0, and we regard $\hat{\boldsymbol{E}}'_{\boldsymbol{D},\mathrm{sieg}}(-,-)$ as a measure of forms on $I_{\mathrm{U}(2,0)}(K^{(2,0)})\times I_{\mathrm{U}(2)}(K^{(2,0)})$ under the embedding i as in (3-2). The $e_{\mathrm{ord}}^{\mathrm{U}(2)}$ means applying the ordinary projector to the U(2)-factor. Therefore for any generic arithmetic point ϕ of conductor p^t,

$$\mathrm{Vol}(K^\infty)^{-1}f_\phi(g_0)^{-1}\sum_{\underline{B}\in I_{\mathrm{U}(2)}(K^{(2,0)}K_0(p^t))}\hat{E}'_{\mathcal{D}_\phi,\mathrm{sieg}}(\underline{A}_{g_0},\underline{B})\bar{\tau}_\phi\circ\det(\underline{B})\times\pi(h_0)\hat{f}_\phi(\underline{B}).$$

The $\boldsymbol{B}\langle\cdot,\cdot\rangle$ is in terms of Definition 7.6. Let the character $\boldsymbol{\tau}=\boldsymbol{\psi}/\boldsymbol{\xi}$. The function $\boldsymbol{\tau}(\det g_2)$ means the function taking value $\boldsymbol{\tau}(\det g_2)$ at the point (g_1,g_2) in the above set. The integration is in the sense of Section 7C with respect to the level group $h_0^{-1}(K_{\mathcal{D}}\cap(1\times\mathrm{GU}(2)(\mathbb{A}_f))h_0$ (in fact by pullback we get a measure of forms on the h_0-Igusa schemes, see the last part of Section 3E). This $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}$ satisfies the proposition (by Lemmas 6.7, 6.15, 6.20, 6.32).

This construction only implies the p-adic L-function is in $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]\otimes\mathrm{Frac}(\mathbb{I})$. To see the integrality, we take different choices for g_0 and note that by our choices for $\boldsymbol{f}$, its values have no nontrivial common divisors in $\mathbb{I}$.

If we are under assumption of Theorem 1.2 then we just pick up a g_0 such that $\boldsymbol{f}$ has nonzero specialization at g_0. Note that the period factors $\Omega_\infty^{2\kappa}$ and $\Omega_p^{2\kappa}$ come from the pullback as discussed in [Hsieh 2014a, Section 2.8, Subsection 5.6.5]. □

Definition 7.8. Now we define Hida's p-adic L-function $\mathcal{L}^{\mathrm{Hida}}_{f,\xi,\mathcal{K}}\in\Lambda_{\boldsymbol{D}}$. As in the main theorems we assume the ξ has split conductor.

We consider the Hida family of ordinary CM eigenforms $g_{\boldsymbol{\xi}}$ associated to ξ (for simplicity we consider here ordinary forms by twisting the adelic nearly ordinary form corresponding to ξ_ϕ's by the unique Dirichlet character of p-power conductor, which makes it an ordinary form, i.e., being invariant under the right action of $\left(\begin{smallmatrix}\mathbb{Z}_p^\times & \\ & 1\end{smallmatrix}\right)$). For our purpose we further twist the automorphic representation of g_ξ by a choice of a fixed finite order Dirichlet character $\chi_{g_\xi}^{\mathrm{tw}}$, unramified outside the prime to p places where ξ is ramified, such that at each bad prime $\ell \neq p$, the resulting automorphic representation at ℓ is minimal in the sense of [Hida and Tilouine 1993, Section 7] (namely it is principal series induced from two characters in which at least one is unramified). The reason is to ensure that we can compare the Petersson inner product of g_ϕ's with certain Katz p-adic L-functions without different Euler factors. (Alternatively we can also multiply the ξ by $\chi_{g_\xi}^{\mathrm{tw}} \circ \mathrm{Nm}$ and construct the primitive ordinary Hida family associated to this product character $\xi' = \xi \cdot \chi_{g_\xi}^{\mathrm{tw}} \circ \mathrm{Nm}$.) We denote the resulting primitive Hida family as $g_{\xi'}$. Note that

$$\xi\xi^{-c} = \xi'\xi'^{-c}.$$

We consider the p-adic L-function $\mathfrak{D}$ constructed in [Hida 1991, Theorem I] choosing $\boldsymbol{f}$ there to be the $g_{\xi'}$ and $\boldsymbol{g}^\rho$ there to be the ordinary eigenforms of our $\boldsymbol{f}$, twisted by the Dirichlet character $\chi_{g_\xi}^{\mathrm{tw}}$ above.

Instead of the CM period Ω_∞, the period factor in Hida's construction is the Petersson period (see [loc. cit., Theorem I, Lemma 5.3(vi)])

$$W'(g_\phi)^{-1}\left\langle g_\phi, g_\phi|_{\kappa+1}\begin{pmatrix} & -1\\ N_{g_\phi} & \end{pmatrix}\right\rangle \eta_{g_\phi}(p^{-t_\phi}),$$

where g_ϕ is a specialization of g_ξ of weight $\kappa+1$, N_{g_ϕ} is the conductor of g_ϕ, the p^{t_ϕ} is the p-part of its conductor, the p-component of the automorphic representation associated to the unitarization of g_ϕ (in the sense of [loc. cit., Introduction]) is $\pi(\eta_{g_\phi}, \eta'_{g_\phi})$ with $\mathrm{ord}_p\, \eta_{g_\phi}(p) < \mathrm{ord}_p\, \eta'_{g_\phi}(p)$. Note that the $S(P)^{-1}$ and the Gauss sum in the denominator of the second row of the definition of $W(P, Q)$ in [loc. cit., Theorem I] are included as part of the period (see Lemma 5.3(vi) of [loc. cit.]). The above expression equals (by [Hida and Tilouine 1993, Theorem 7.1, (8.8b)] and [Hida 1991, Lemma 5.3(vi)])

$$\langle g_\phi, g_\phi\rangle \cdot \eta'_{g_\phi}\eta_{g_\phi}^{-1}(p^{t_\phi})\mathfrak{g}(\eta'^{-1}_{g_\phi}\eta_{g_\phi}) \cdot p^{-t_\phi} = \mathfrak{L}(\mathrm{ad}, g_\phi, 1)\eta'_{g_\phi}\eta_{g_\phi}^{-1}(p^{t_\phi})\mathfrak{g}(\eta'^{-1}_{g_\phi}\eta_{g_\phi})\kappa! 2^{-2\kappa-1}\pi^{-\kappa-2}$$

where the $\mathfrak{L}(\mathrm{ad}, -)$ is in the sense of [Hida and Tilouine 1993, Section 7]. By our assumption on that ξ has split conductor, we see that the $\Delta(1)$ in [loc. cit., (0.7a)] equals 1. So

$$\mathfrak{L}(\mathrm{ad}, g_\phi, 1) = L(1, \chi_{\mathcal{K}})L(\xi_\phi\xi_\phi^{-c}, 1).$$

On the other hand, there is a Katz p-adic L-function (see [loc. cit., (8.2)] for this Katz measure, we remove the factor $\mathrm{Im}(\delta)^{\kappa-1}$ there since it is a unit Iwasawa element and has no effect to us.) $\mathcal{L}_{\mathcal{K},\xi}^{\mathrm{Katz}} \in \hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ interpolating the values

$$\eta'_{g_\phi}\eta_{g_\phi}^{-1}(p^{t_\phi})\frac{\pi^{\kappa-1}\kappa! L(\xi_\phi\xi_\phi^{-c}, 1) \cdot \mathfrak{g}(\eta'^{-1}_{g_\phi}\eta_{g_\phi})}{(2i)^{2\kappa}\Omega_\infty^{2\kappa}}\Omega_p^{2\kappa}$$

Let $\mathrm{Cl}_{\mathcal{K}}$ be the class number of $\mathcal{K}$. By class number formula

$$\frac{L(1, \chi_{\mathcal{K}})}{\pi} = \mathrm{Cl}_{\mathcal{K}} \cdot D_{\mathcal{K}}^{-1/2}.$$

We multiply Hida's p-adic L-function $\mathcal{D}$ by $2i^{2\kappa}\,\mathrm{Cl}_{\mathcal{K}}\,D_{\mathcal{K}}^{-1/2}\cdot\mathcal{L}^{\mathrm{Katz}}_{\mathcal{K},\xi}$, and further divide it by the first row in the definition of $W(P,Q)$ of [Hida 1991, Theorem I]. Note that this last factor is a unit Iwasawa element. We denote this result as $\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}$. One checks readily the interpolation formula for it is

$$\phi\left(\frac{\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}}{\Omega_p^{2\kappa}}\right)=\frac{2\pi i\epsilon_p\big(\frac{3}{2}-\frac{\kappa}{2},\xi_{v_0}^{-1}\chi_{1,p}\big)\epsilon_p\big(\frac{3}{2}-\frac{\kappa}{2},\xi_{v_0}^{-1}\chi_{2,p}\big)L\big(\mathcal{K},\pi_{f_\phi},\bar{\chi}_\phi\xi_\phi,\frac{\kappa}{2}-\frac{1}{2}\big)(\kappa-1)!(\kappa-2)!}{\Omega_\infty^{2\kappa}}. \tag{7-5}$$

(Recall we have divided out the product of prime to p root numbers in *loc.cit.* which are p-units and moves p-adic analytically.) If ξ is such that g_ξ satisfies the (dist) and (irred) in the introduction, then the local Hecke algebra for $g_{\boldsymbol{\xi}}$ is Gorenstein. By the main conjecture proved in [Hida and Tilouine 1993; 1994; Hida 2006] (see [Hida 2006, Theorem and page 468, (F)], the $\mathrm{Cl}_{\mathcal{K}}\cdot\mathcal{L}^{\mathrm{Katz}}_{\mathcal{K},\xi}$ generates the congruence module for $\boldsymbol{g}$ and our $\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\xi,\mathcal{K}}$ is integral (i.e., in $\Lambda_{\boldsymbol{D}}$)). We explain here the weight map parametrizing the Hida family $\boldsymbol{g}$ with coefficient ring $\mathcal{O}_L[\![\Gamma^-_{\mathcal{K}}]\!]$. Let $\mathcal{K}^-$ be the anticyclotomic $\mathbb{Z}_p$ extension of $\mathcal{K}$ and $\mathcal{K}^{\mathrm{ur}}$ be the maximal subextension of $\mathcal{K}^-/\mathcal{K}$ unramified everywhere. Let p^a be the index of $\mathcal{K}^{\mathrm{ur}}/\mathcal{K}$ and $\Gamma^{-,\prime}_{\mathcal{K}}$ be the corresponding subgroup of $\Gamma^-_{\mathcal{K}}$. Then local class field theory gives an isomorphism

$$\sigma_{v_0}:(1+p\mathbb{Z}_p)^\times\simeq\Gamma^{-,\prime}_{\mathcal{K}}.$$

Write $\mathcal{O}_L[\![W]\!]$ for the weight algebra of the Hida family $\boldsymbol{g}$, then the weight map is

$$\mathcal{O}_L[\![W]\!]\simeq\mathcal{O}_L[\![\Gamma^{-,\prime}_{\mathcal{K}}]\!]\hookrightarrow\mathcal{O}_L[\![\Gamma^-_{\mathcal{K}}]\!],$$

where the first map is determined by

$$(1+W)\mapsto(1+p)^{-1}\sigma_{\bar{v}_0}(1+p),$$

and the last term is a finite free module over the second term.

Given a finite set of primes Σ we can define the Σ-primitive Hida p-adic L-function $\mathcal{L}^{\Sigma,\mathrm{Hida}}_{\boldsymbol{f},\mathcal{K},\xi}$ by removing local Euler factors at Σ. Obviously, it is just multiplying $\mathcal{L}^{\mathrm{Hida}}_{\boldsymbol{f},\mathcal{K},\xi}$ by a finite number of nonzero elements in $\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$. Note that Hida proved the interpolation formula for general arithmetic points. We may compare (7-2) and (7-5). If we write $\mathcal{L}^\Sigma_{f_0,\xi,\mathcal{K}}$ for the specialization of $\mathcal{L}^\Sigma_{\boldsymbol{f},\xi,\mathcal{K}}$ to $\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]$ at f_0, then we get the interpolation formula

$$\phi\left(\frac{\mathcal{L}^\Sigma_{f_0,\xi,\mathcal{K}}}{\Omega_p^{2\kappa}}\right)=C^\Sigma_{\boldsymbol{f},\xi,\mathcal{K}}\frac{2\pi i\epsilon_p\big(\frac{3}{2}-\frac{\kappa}{2},\xi_{v_0}^{-1}\chi_{1,p}\big)\epsilon_p\big(\frac{3}{2}-\frac{\kappa}{2},\xi_{v_0}^{-1}\chi_{2,p}\big)L^\Sigma\big(\mathcal{K},\pi_{f_0},\xi_\phi,\frac{\kappa}{2}-\frac{1}{2}\big)(\kappa-1)!(\kappa-2)!}{\Omega_\infty^{2\kappa}}. \tag{7-6}$$

for ξ_ϕ's of conductor (p^t,p^t) at p.

Anticyclotomic μ-invariants. Now assume we are under assumption of Theorem 1.1 in the introduction. We define ϕ_0 to be the $\bar{\mathbb{Q}}_p$-point in $\mathrm{Spec}\,\mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ sending $\gamma^\pm$ to 1 and such that $\phi_0|_{\hat{\mathbb{I}}^{\mathrm{ur}}}$ correspond to f_0. Our assumptions on ξ and κ ensure that our p-adic families pass through this point. (This is not an arithmetic point in Definition 7.3, however it still interpolates the algebraic part of the special L-value by [Hida 1991].) Consider the one-dimensional subspace of Spec $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$ of anticyclotomic twists by characters of order and conductor powers of p that passes through ϕ_0. We look at the ratio between the specialization of Hida's p-adic L-function $\mathcal{L}^{\mathrm{Hida}}_{f,\xi,\mathcal{K}}$ to this subspace and the anticyclotomic p-adic L-function considered by [Hsieh 2014c] (note that the local sign assumptions there are satisfied). We explain the fudge factors: recall the $S(P)^{-1}$ and the Gauss sum in the denominator of the definition of $W(P,Q)$ of [Hida 1991, Theorem I] are already included in the period studied above. Also the first row of the definition of $W(P,Q)$ are also divided out in our definition for $\mathcal{L}^{\mathrm{Hida}}_{f,\xi,\mathcal{K}}$. The remaining factors are: the $C(\pi,\lambda)$ and $\mathfrak{F}$ in [Hsieh 2014c] which is a *fixed* p-adic unit, and the powers of $\mathrm{Im}(\delta)$ and 2 which are unit Iwasawa elements. So by result proved in [loc. cit.] the anticyclotomic p-adic L-function has μ-invariant 0. Thus it is easy to see that any height one prime P of $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$ containing $\mathcal{L}^{\mathrm{Hida}}_{f,\mathcal{K},\xi}$ can not be the pullback of a height 1 prime of $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma^+_{\mathcal{K}}]]$. Therefore for any height 1 prime containing $\mathcal{L}^{\mathrm{Hida}}_{f,\mathcal{K},\xi}$,

$$\mathrm{ord}_P(\mathcal{L}^{\mathrm{Hida}}_{f,\mathcal{K},\xi}) = \mathrm{ord}_P(\mathcal{L}_{f,\xi,\mathcal{K}})$$

and $\mathrm{ord}_P(\mathcal{L}_{\bar{\chi}_f\xi}) = 0$. The $\mathcal{L}_{f,\xi,\mathcal{K}}$ is obtained by putting back the Euler factors at primes in Σ on $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}$. (There might be factors coming from Euler factors at nonsplit primes contributing to the anticyclotomic μ-invariant of the Σ-primitive p-adic L-functions, however. We will explain how to treat those factors when proving our main theorem.)

7F. *p-adic Eisenstein series.*

Proposition 7.9. *There is a $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$-adic formal Fourier–Jacobi expansion*

$$\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}} \in \mathcal{M}_{\mathrm{ord}}(K, \Lambda_{\boldsymbol{D}})$$

such that for each generic arithmetic point $\phi \in$ Spec $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$, the specialization $\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling},\phi}$ is the Fourier–Jacobi expansion of the nearly ordinary Klingen–Eisenstein series $E_{\mathrm{Kling},\boldsymbol{D}_\phi}$ we constructed in (6-13) using the Eisenstein datum at ϕ. Moreover, recall the fundamental exact sequence in Theorem 3.6 and the Siegel operator $\hat{\Phi}^h_{[g]}$ there, then the constant term $\hat{\Phi}^{w_3'}_{[g]}(\boldsymbol{E}_{\mathrm{Kling},\boldsymbol{D}})$'s are divisible by $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}.\mathcal{L}^{\Sigma}_{\bar{\chi}\xi'}$, where $\mathcal{L}^{\Sigma}_{\bar{\chi}\xi'}$ is the element in $\mathbb{I}[[\Gamma^+_{\mathcal{K}}]]$ which is the Dirichlet p-adic L-function interpolating the algebraic part of the special values $L^{\Sigma}(\bar{\chi}_\phi\xi'_\phi, \kappa_\phi - 2)$.

Proof. Our construction is more similar to [Hsieh 2011, Theorem 4.4] than to [Skinner and Urban 2014, Theorem 12.11]. Recall Definition 3.7 the notion of $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}]]$-adic Fourier–Jacobi expansion. It is a special case of [Wan 2015a, Theorem 1.1(3)]. In our cases, the local choices are slightly different but the arguments are the same, which we give below. For a $[g]$ we take a basis $(\theta'_{1,\beta},\dots,\theta'_{m_\beta,\beta})$ of the $\mathcal{O}_L$-dual space of $H^0(\mathcal{Z}^\circ_{[g]},\mathcal{L}(\beta))$ consisting of theta functions. Write $(\theta''_{1,\beta},\dots,\theta''_{m_\beta,\beta})$ for the dual basis. Suppose θ is one of the θ'_is. For any $g \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}}) \subset \mathrm{GU}(3,1)(\mathbb{A}_{\mathbb{Q}})$ we take $h \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q}})$ such that $\mu(g) = \mu(h)$. Recall we have denoted the β-Fourier–Jacobi coefficient $a^h_{[g]}(\beta, F)$ for forms on $\mathrm{GU}(3,1)$.

We write $a^h_{[g],\theta}(\beta, F)$ for the pairing of it with θ. We define

$$a^1_{[g],\theta}(\beta, \boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}) = \boldsymbol{B}_{K^{(2,0)}}\langle e^{\mathrm{U}(2)}_{\mathrm{ord}} a^1_{[g],\theta}(\beta, \hat{\boldsymbol{E}}_{\boldsymbol{D},\mathrm{sieg}}(-,-))\bar{\boldsymbol{\tau}} \circ \det(-), \pi(h)\boldsymbol{f}\rangle \tag{7-7}$$

As before the $e^{\mathrm{U}(2)}_{\mathrm{ord}}$ means applying the ordinary projector for the U(2) factor. We regard $\hat{\boldsymbol{E}}_{\boldsymbol{D},\mathrm{sieg}}(-,-)$ as evaluated on $(\underline{A}, \underline{B}) \in I_{\mathrm{U}(3,1)}(K^{(3,1)}) \times I_{\mathrm{U}(2)}(K^{(2,0)})$ under the embedding (3-2). The $\bar{\boldsymbol{\tau}}(\det -)$ is regarded as a function for the U(2) factor. In view of the algebraic embedding of Igusa schemes, the pullback of the Siegel–Eisenstein measure gives a measure with values in the space of p-adic automorphic forms on the group $\{g, h \in \mathrm{GU}(3,1) \otimes \mathrm{GU}(2), \det g = \det h\}$. Fix the g, applying the β-th Fourier–Jacobi coefficient operator to the GU(3, 1)-factor and take the θ-component we get a $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$-valued family of forms on (the lower) GU(2) in the sense of Definition 7.2, which is the integrand of (7-7). Then we form the pairing $\langle\cdot,\cdot\rangle$ of (7-7) in the sense of Section 7D with respect to the level group $h^{-1}(K_{\mathcal{D}} \cap (1 \times \mathrm{GU}(2)(\mathbb{A}_f))h$ (In fact, by pullback, we get a measure of forms on the h-Igusa schemes.) We obtain the family of Fourier–Jacobi expansions. It is clear from the construction that this interpolates the Fourier–Jacobi expansions of the ordinary Klingen–Eisenstein series we constructed at arithmetic points (see (6-13)). We get the Fourier–Jacobi expansion at g as in Definition 3.7

$$\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}(g) = \sum_\beta a^1_{[g]}(\beta, E_{\boldsymbol{D},\mathrm{Kling}})q^\beta$$

where

$$a^1_{[g]}(\beta, \boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}) = \sum_i a^1_{[g],\theta'_{i,\beta}}(\beta, \boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}) \otimes \theta''_{i,\beta}$$

with $\theta''_{i,\beta} \in H^0(\mathcal{Z}^\circ_{[g]}, \mathcal{L}(\beta))$. At a generic arithmetic point ϕ of conductor p^t we have

$$a^1_{[g],\theta}(\beta, E_{\mathcal{D}_\phi,\mathrm{Kling}}) = \sum_{\underline{B}\in I_{\mathrm{U}(2)}(K^{(2,0)}K_0(p^t))} a^1_{[g],\theta}(\beta, \hat{E}_{\mathcal{D}_\phi,\mathrm{sieg}}(-,\underline{B}))\cdot\bar{\tau}_\phi\circ\det(\underline{B}) \times \pi(h)\varphi_\phi(\underline{B})\rangle.$$

Next we explain the assertion on constant terms. The constant terms is simply interpolating the β-th Fourier–Jacobi coefficient for $\beta = 0$ (i.e., the Siegel operator $\Phi_{P,g}(\boldsymbol{E}_{\mathrm{Kling},\boldsymbol{D},\phi})$). Let's consider the case when g_v's are in the support of $F_{\mathrm{Kling},v}$ for $v \nmid p$, and $g_p = \omega'_3$. We claim that

$$\Phi^{w'_3}_{[g]}(\boldsymbol{E}_{\mathrm{Kling},\boldsymbol{D}}) = C_{\boldsymbol{D}}\mathcal{L}^\Sigma_{\boldsymbol{f},\xi,\mathcal{K}}\cdot\mathcal{L}^\Sigma_{\xi\xi'}\boldsymbol{f}$$

for $C_{\boldsymbol{D}}$ being the product of the constants in the local pullback sections at primes outside p. It is a fixed nonzero number throughout the family.

To see the claim, specializing to an arithmetic point ϕ, this is simply the constant term computation in Section 5E for R being the Klingen parabolic subgroup P. This constant term is given by Lemma 5.14. On the other hand, from the Archimedean computation in [Wan 2015a, Corollary 5.11] we see the contribution $A(\rho, f, z)_{-z}$ is actually 0 in our case. So we only need to work out the pullback Klingen–Eisenstein section. Now it is an easy consequence of our computations in Section 6 of local pullback integrals (Lemmas 6.7, 6.15, 6.20, 6.32), together with the normalization factors in Section 6J. □

It follows that the formal Fourier–Jacobi expansion $\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}$ comes from a family in $\mathcal{M}^{\mathrm{ord}}(K^p, \Lambda_{\boldsymbol{D}})$, which we still denote as $\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}$. (In fact, Theorem 3.8 is still true after replacing $A = \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!]$ by $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$.)

8. p-adic properties of Fourier–Jacobi coefficients

Notation. To avoid confusion in this section, we use $\mathrm{z} \in \operatorname{Spec} \mathbb{I}[\![\Gamma_{\mathcal{K}}]\!](\overline{\mathbb{Q}}_p)$ instead of ϕ to denote arithmetic points on the weight space. The ϕ will usually denote Schwartz functions in theta correspondences. Such convention is only valid in this section.

The purpose of this section is to prove Proposition 8.29.

8A. ***Preliminaries.***

Some local representation theories. (Noncompact case) Let v be a nonsplit prime where $\mathrm{U}(2)(\mathbb{Q}_v) \simeq \mathrm{U}(1,1)(\mathbb{Q}_v)$. Then $D_v^\times \simeq \mathrm{GL}_2(\mathbb{Q}_v)$. For some irreducible admissible representation $\pi^{\mathrm{U}(2)_v}$ of $\mathrm{U}(2)_v$ we can find an irreducible representation $\pi^{\mathrm{GU}(2)_v}$ of $\mathrm{GU}(2)_v$ such that $\pi^{\mathrm{U}(2)_v}$ is a summand of $\pi^{\mathrm{GU}(2)_v}$ restricting to $\mathrm{U}(2)_v$ (note here the superscripts do not mean invariant subspaces). Thus we have

$$\pi^{\mathrm{GU}(2)_v}|_{\mathrm{U}(2)_v} = \pi^{\mathrm{U}(2)_v} \oplus {}^{\alpha}\pi^{\mathrm{U}(2)_v} \text{ or } \pi^{\mathrm{U}(2)_v}$$

for irreducible representations $\pi^{\mathrm{U}(2)(\mathbb{Q}_v)}$ of $\mathrm{U}(2)(\mathbb{Q}_v)$. Here α is some element such that $\mathrm{Nm}(\alpha) \notin \mathrm{Nm}(\mathcal{K}_v/\mathbb{Q}_v)$. The ${}^{\alpha}$ means the representation composed with the automorphism given by conjugation by α. Also the restriction of $\pi^{\mathrm{GU}(2)_v}$ to $D_v^\times$ is clearly irreducible.

(Compact Case) If $D_v^\times$ modulo center is compact, then we let α be some element such that $\mathrm{Nm}(\alpha) \notin \mathrm{Nm}(\mathcal{K}_v/\mathbb{Q}_v)$. For $\pi^{\mathrm{U}(2)_v}$ we similarly have $\pi^{\mathrm{GU}(2)_v}, \pi^{D_v^\times}$. These can all be considered as finite-dimensional representations of finite groups. The $\pi^{\mathrm{GU}(2)_v} = \pi^{D_v^\times}$ as vector spaces and $\pi^{\mathrm{GU}(2)_v}|_{\mathrm{U}(2)_v} = \pi^{\mathrm{U}(2)_v}$ or $\pi^{\mathrm{U}(2)_v} \oplus {}^{\alpha}\pi^{\mathrm{U}(2)_v}$.

In both cases, we write ι_α for the isomorphism between $\pi^{\mathrm{U}(2)_v}$ and ${}^{\alpha}\pi^{\mathrm{U}(2)_v}$ given by right action by α (as vector spaces, the group actions may differ by a conjugation however.)

Forms on $D^\times$ and $\mathrm{U}(2)$. We first define $\breve{D}^\times \subset D^\times(\mathbb{A}_{\mathbb{Q}})$ as the index 2 subgroup consisting of elements whose reduced norms are in $\mathbb{Q}^\times \mathrm{Nm}(\mathbb{A}_{\mathcal{K}}^\times)$ and let $\breve{D}^\times(\mathbb{Q}_v)$ be the set of elements whose determinants are in $\mathrm{Nm}(\mathcal{K}_v^\times)$. Suppose φ is a form on $\mathrm{U}(2)(\mathbb{Q})\backslash\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$, χ is a Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$. Suppose the central action of $\mathrm{U}(1)(\mathbb{Z}_p)$ on φ is given by $\chi|_{\mathrm{U}(1)(\mathbb{Z}_p)}$, we can define a form $\varphi^D \boxtimes \chi$ on $D^\times(\mathbb{A}_{\mathbb{Q}})$ as follows. We first define φ'_χ on $\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ as

$$\varphi'_\chi(g) := \int_{[U(1)]} \varphi(gt)\chi^{-1}(t)\,dt.$$

We now define a form on $D^\times(\mathbb{A}_{\mathbb{Q}})$. Recall that the image of reduced norm map from $D^\times(\mathbb{Q})$ consists of all positive elements $\mathbb{Q}^+$ in $\mathbb{Q}^\times$ [Weil 1974, page 206]. Note that

$$\mathbb{A}_{\mathbb{Q}}^{\times,+}/\mathbb{Q}^+ \mathrm{Nm}(\mathbb{A}_{\mathcal{K}}^\times) \to \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \mathrm{Nm}(\mathbb{A}_{\mathcal{K}}^\times)$$

is an isomorphism, where $\mathbb{A}_{\mathbb{Q}}^{+}$ is the set of ideles with positive Archimedean component, which is also the image of the reduced norm map from $D^\times(\mathbb{A}_{\mathbb{Q}})$. Thus for $g \in \breve{D}^\times$, write $g = bag'$, $b \in D^\times(\mathbb{Q})$, $a \in \mathbb{A}_{\mathcal{K}}^\times$, $g' \in \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$, define

$$\varphi^D \boxtimes \chi(g) = \varphi_\chi(g)\chi(a).$$

Note that this is well defined since $\mathbb{Q}^\times \cap \mathrm{Nm}(\mathbb{A}_{\mathcal{K}}^\times/\mathbb{A}_{\mathbb{Q}}^\times) = \mathrm{Nm}(\mathcal{K}^\times/\mathbb{Q}^\times)$. For g outside $\breve{D}^\times$ we define $\varphi^D \boxtimes \chi(g) = 0$. When χ is clear from the context we simply drop the subscript χ.

Lemma 8.1. *Let π_ξ be the irreducible automorphic representation of* $\mathrm{GL}_2/\mathbb{Q}$ *associated to a CM character ξ of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$. If φ_ξ is in an irreducible automorphic representation of* $\mathrm{U}(2)$ *whose restriction to* $\mathrm{SU}(2)$ *is in the restriction of the automorphic representation of $D^\times(\mathbb{A}_{\mathbb{Q}})$ corresponding to π_ξ under the Jacquet–Langlands correspondence, then $\varphi_\xi^D \boxtimes \xi$ itself is in π_ξ.*

Proof. Clearly the φ_ξ^D and π_ξ have the same Hecke eigenvalues at split v's. Note that the set of primes of $\mathcal{K}$ sitting over split primes of $\mathcal{K}/\mathbb{Q}$ has Dirichlet density one. Write φ_ξ^D as a sum of forms in irreducible automorphic representations. Then for any such automorphic representation π_i, the corresponding Galois representation ρ_{π_i} satisfies

$$\rho_{\pi_i}|_{G_{\mathcal{K}}} \simeq \xi \oplus \xi^c.$$

This implies each π_i is isomorphic to π_ξ. □

We relate the integrals over $[\mathrm{U}(2)]$ to that over a subset of $[\mathbb{Q}^\times\backslash D^\times](\mathbb{A}_{\mathbb{Q}})$. It is elementary to check that there is a constant $C_{\mathrm{U}(2)}^D$ depending only on the groups $D^\times$ and $\mathrm{U}(2)$ such that if $\chi = 1$ then

$$\int_{[\mathrm{U}(2)]} \varphi_{\mathrm{U}(2)}(g)\,dg = C_{\mathrm{U}(2)}^D \int_{D^\times(\mathbb{Q})\mathbb{A}_{\mathbb{Q}}^\times\backslash \breve{D}^\times(\mathbb{A}_{\mathbb{Q}})} \varphi^D \boxtimes \chi(g)\,dg. \tag{8-1}$$

Here, we normalize the Haar measure so that the measure $\mathrm{U}(1)\backslash[\mathrm{U}(2)] = 1$ and the measure of $[D^\times]$ modulo center is also 1.

8B. *Constructing auxiliary families of theta functions.*

Convention. From now on, we usually do the computations at a generic arithmetic point $\mathrm{z} \in \mathrm{Spec}\,\Lambda_{\boldsymbol{D}}(\overline{\mathbb{Q}}_p)$. We usually write bold symbols for p-adic families constructed (e.g., $\boldsymbol{h}$), and write their specializations using nonbold symbols (e.g., h_{z} for specializations of $\boldsymbol{h}$).

In this section we fix finite order CM characters η and η' of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$. This notation is only used in this subsection so as not to confuse with our use of η in previous sections.

Before continuing, we need to introduce some more families of characters. Let $\Gamma_{v_0} \simeq \mathbb{Z}_p$ be the quotient of $\Gamma_{\mathcal{K}}$ corresponding to the maximal subextension of $\mathcal{K}_\infty$ unramified outside v_0. Define $\Gamma_{\bar{v}_0}$ similarly. We need to further enlarge our parameter space. We write u_{v_0} and $u_{\bar{v}_0}$ for topological generators of $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}_{v_0})$ and $\mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}_{\bar{v}_0})$ respectively. Let $\mathcal{K}^{\mathrm{ur}}$ be the maximal (finite) everywhere unramified subextension of $\mathcal{K}_\infty/\mathcal{K}$. Then u_{v_0} and $u_{\bar{v}_0}$ generates $\Gamma'_{\mathcal{K}} := \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}^{\mathrm{ur}})$. We define an abstract group $\Gamma''_{\mathcal{K}} := (\mathbb{Z}_p/p^a)u_{v_0} \oplus (\mathbb{Z}_p/p^a)u_{\bar{v}_0}$ with $p^a = \sharp\,\mathrm{Gal}(\mathcal{K}^{\mathrm{ur}}/\mathcal{K})$. Then we have

$$\Gamma'_{\mathcal{K}} \subseteq \Gamma_{\mathcal{K}} \subseteq \Gamma''_{\mathcal{K}}$$

with each containment of index p^a. We consider the natural projection $\Gamma'_{\mathcal{K}} \to \mathbb{Z}_p u_{v_0}$, which extends canonically to a surjection $\Gamma_{\mathcal{K}} \to (\mathbb{Z}_p/p^a)u_{v_0} \subseteq \Gamma''_{\mathcal{K}}$. Recall we have defined a $\Lambda_{\boldsymbol{D}}$-adic character $\boldsymbol{\xi}$. Let $\Lambda''_{\boldsymbol{D}} := \Lambda_{\boldsymbol{D}} \otimes_{\mathbb{Z}_p[\![\Gamma_{\mathcal{K}}]\!]} \mathbb{Z}_p[\![\Gamma''_{\mathcal{K}}]\!]$. This is an enlarged parameter space. Then we can define a $\Lambda''_{\boldsymbol{D}}$-valued character $\boldsymbol{\xi}_2$ as the composition of $\boldsymbol{\xi}$ with the surjection $\Gamma_{\mathcal{K}} \to (\mathbb{Z}_p/p^a)u_{v_0} \subseteq \Gamma''_{\mathcal{K}}$ above. It is easy to see that at any arithmetic point z the specialization of $\boldsymbol{\xi}_2$ is unramified at v_0, and its restriction at $\mathcal{O}^\times_{\bar{v}_0}$ is the same as that for $\boldsymbol{\xi}_{\mathrm{z}}$. Using the same construction, we can define a $\Lambda''_{\boldsymbol{D}}$-valued character $\boldsymbol{\eta}$ be such that the specialization to ϕ_0 (ϕ_0 is defined in Section 7E) is η defined before and the specialization to each arithmetic point z satisfies $(\boldsymbol{\eta})_{\mathrm{z},p}|_{(\mathbb{Z}_p^\times,\mathbb{Z}_p^\times)} = (1, \bar{\xi}^\dagger_{\mathrm{z},v_0}|_{\mathbb{Z}_p^\times})$ (recall the definition for $\xi^\dagger_1$ in Section 6H, the triple there is the p-component of the specialization $(f_{\mathrm{z}}, \psi_{\mathrm{z}}, \tau_{\mathrm{z}})$ here).

Similarly starting with the character η' before we can define another family of characters of $\mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K}}$ with values in the enlarged $\mathbb{I}$ (taking tensor product of the original $\mathbb{I}$ with some degree p^a extension of $\mathcal{O}_L[\![W]\!]$ and take a reduced irreducible component and normalization) upon appropriate choice of identification of $\mathbb{Z}_p[\![W]\!]$ with $\mathbb{Z}_p[\![u_{v_0}]\!]$, such that at any arithmetic point z the specialization is unramified at v_0 and is equal to $\chi_{f,\mathrm{z}}\tau_{\mathrm{z},\bar{v}_0}\psi^{-1}_{\mathrm{z},\bar{v}_0}$ when restricting to $\mathcal{O}^\times_{v_0} \simeq \mathbb{Z}_p^\times$. Then we can define a $\Lambda''_{\boldsymbol{D}}$-adic character $\boldsymbol{\eta}''$ such that its specialization to ϕ_0 is η', and that $(\boldsymbol{\eta}'')_{\mathrm{z},\bar{v}_0}|_{\mathbb{Z}_p^\times} = \chi_{f,\mathrm{z}}\tau_{\mathrm{z},\bar{v}_0}\psi^{-1}_{\mathrm{z},\bar{v}_0}|_{\mathbb{Z}_p^\times}$, $(\boldsymbol{\eta}'')_{\mathrm{z},v_0}|_{\mathbb{Z}_p^\times} = 1$. Moreover there is a character $\boldsymbol{\chi}$ of $\mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K}}$ which factors through $\Gamma_{\mathcal{K}}$ (again we use class field theory), such that $\boldsymbol{\chi}_{\mathrm{z},\bar{v}_0}|_{\mathbb{Z}_p^\times} = \chi^{-1}_{f,\mathrm{z}}\psi_{\mathrm{z},\bar{v}_0}|_{\mathbb{Z}_p^\times}$ and $\boldsymbol{\chi}_{\mathrm{z},v_0}|_{\mathbb{Z}_p^\times} = \chi^{-1}_{f,\mathrm{z}}\psi_{\mathrm{z},\bar{v}_0}|_{\mathbb{Z}_p^\times}$. Define $\boldsymbol{\eta}' := \boldsymbol{\eta}'' \cdot \boldsymbol{\chi}$.

Note that we have enlarged our parameter space. At the end of Section 9 we will first prove the main theorems for this enlarged Iwasawa algebra and then go back to prove it for the original one.

Rallis inner product formula.

$$\begin{array}{ccc} \mathrm{U}(1,1)(\omega_{\lambda^2}) & & \mathrm{U}(2)(\omega_\lambda)\times\mathrm{U}(2)(\omega_\lambda) \\ \uparrow & & \uparrow \\ \mathrm{U}(1)(\omega_{\lambda^2})\times\mathrm{U}(1)(\omega_{\lambda^2}) & & \mathrm{U}(2)(\omega_{\lambda^2}) \end{array}$$

Recall we fixed a splitting character λ of $\mathcal{K}^\times\backslash\mathbb{A}^\times_{\mathcal{K}}$ of infinity type $\left(-\frac{1}{2}, \frac{1}{2}\right)$ such that its restriction to $\mathbb{A}^\times_{\mathbb{Q}}$ is $\chi_{\mathcal{K}/\mathbb{Q}}$. We use the background for dual reductive pair, splitting characters and theta correspondences in [Harris et al. 1996, Sections 1–3] freely. We consider the seesaw pair above. The U(2) above is for the Hermitian matrix $\left(\begin{smallmatrix}\mathfrak{s} & \\ & 1\end{smallmatrix}\right)$ and the U(1)'s are for the skew-Hermitian matrices δ and $-\delta$. The embedding $\mathrm{U}(1)\times\mathrm{U}(1) \hookrightarrow \mathrm{U}(1,1)$ is given by the i defined in the proof of Lemma 6.26. The splitting characters used are indicated in the brackets beside the groups. We want to consider the component of theta correspondence such that the first U(1) on the lower left corner acts by $\lambda^2\eta_{\mathrm{z}}$ and the second U(1) acts by η_{z}^{-1}. We consider a theta function on U(2, 2) by some Schwartz function ϕ such that $\phi = \delta_\psi(\phi_3 \boxtimes \phi_2)$ for some ϕ_3 and ϕ_2 (recall the notion of intertwining operators in Section 4G. We consider

$$\int_{[\mathrm{U}(2)]}\int_{[\mathrm{U}(1)]\times[\mathrm{U}(1)]} \theta_\phi(u_1, u_2, g)\lambda^{-2}\eta_{\mathrm{z}}^{-1}(u_1)\eta_{\mathrm{z}}(u_2)\bar{\lambda}(\det g)\, du_1du_2dg. \tag{8-2}$$

The splitting characters are consequences of [Harris et al. 1996, Lemma A.7] and the discussion on "reflection principle" right after it. Here the $\bar{\lambda}(\det g)$ shows up due to the splitting ω_{λ^2} on U(2) since in the Siegel–Weil formula [Ichino 2007], the splitting character on $\mathrm{U}(2)$ is the trivial character. Thus the restriction of the theta kernel to $\mathrm{U}(1,1)\times\mathrm{U}(2)$ is $\lambda(\det g)$ times the restriction to it of the theta kernel appearing in the Siegel–Weil formula of [Ichino 2007]. So this is exactly the same formula considered in [Harris et al. 2005, Section 6].

On one hand, one can check that this is nothing but the inner product of the theta liftings $\theta_{\phi_3,\lambda}(\lambda\eta_{\mathrm{z}})$ and $\theta_{\phi_2,\lambda}(\lambda^{-1}\eta_{\mathrm{z}}^{-1})\cdot(\bar{\lambda}\circ\det)$ (by writing $\theta_{\phi,\lambda}$ we take the splitting character for $\mathrm{U}(1)$ to be trivial and for $\mathrm{U}(2)$ to be λ. We need to notice the different choices of splitting characters). On the other hand, if we change the order of integration using the Siegel–Weil formula for $\mathrm{U}(1,1)\times\mathrm{U}(2)$ as proved by Ichino [2007], this equals

$$\int_{[\mathrm{U}(1)]\times[\mathrm{U}(1)]} E\big(f_{\delta_\psi(\phi)},\tfrac{1}{2},i(u_1,u_2)\big)\lambda^{-2}\eta_{\mathrm{z}}^{-1}(u_1)\eta_{\mathrm{z}}(u_2)\,du_1du_2 \tag{8-3}$$

Here i is defined right before Lemma 6.27 and $f_{\delta_\psi(\phi)}$ is the Siegel section defined by

$$f_{\delta_\psi(\phi)}(g):=\omega_{\lambda^2}(j(g))\delta_\psi(\phi)(0),\quad g\in\mathrm{U}(1,1)$$

where j is defined in the proof of Lemma 6.26. Thus we reduced the Petersson inner product of theta liftings to the pullback formula of the Siegel–Eisenstein series on $\mathrm{U}(1,1)$.

Functorial properties of theta liftings. For any Hecke character χ of $\mathrm{U}(1)$ (in application $\chi(z_\infty)=z_\infty^{\pm1}$ for $z_\infty\in\mathrm{U}(\mathbb{R})$), we describe the L-packet of theta correspondence $\theta_\lambda(\chi)$ (possibly zero) of χ to $\mathrm{U}(2)$ where λ is a Hecke character of $\mathbb{A}_{\mathcal{K}}^\times$ such that $\lambda|_{\mathbb{A}_{\mathbb{Q}}^\times}=\omega_{\mathcal{K}/\mathbb{Q}}$. We pick a Hecke character $\breve{\chi}$ such that $\breve{\chi}|_{\mathrm{U}(1)(\mathbb{A}_{\mathbb{Q}})}=\chi^{-1}$. In our application the automorphic representation of θ_χ is supercuspidal at all primes where D is compact modulo center. So we have the Jacquet–Langlands correspondence $\pi_{\breve{\chi}}$ on $D^\times$ of the automorphic representation of $\mathrm{GL}_2/\mathbb{Q}$ generated by the CM form $\theta_{\breve{\chi}}$ corresponding to $\breve{\chi}$. We form an automorphic representation in the way we introduced before: $\pi_{\breve{\chi}}\boxtimes\breve{\chi}\lambda$ of $\mathrm{GU}(2)$. Then by looking at the local L-packets (see [Harris et al. 1996, Section 7]) $\theta_\lambda(\chi)$ is a subspace of the restriction of this representation to $\mathrm{U}(2)$. (This restriction is not necessarily irreducible.) The representations at split primes are irreducible. Therefore, we still have not specified the automorphic representation on $\mathrm{U}(2)$.)

Constructing families of theta liftings. Let v be a prime inert or ramified in $\mathcal{K}$. Thanks to the recent work [Gan and Takeda 2016], we know that the Howe duality conjecture is true for any characteristic. Recall as in Definition 4.4, consider the theta lifting from $\mathrm{U}(1)$ to $\mathrm{U}(2)$ at v (the $\mathrm{U}(2)(\mathbb{Q}_v)$ might be $\mathrm{U}(1,1)(\mathbb{Q}_v)$ or compact). Write $S(X_v,\eta_{\mathrm{z},v}^{-1})$ for the summand of $S(X_v)$ such that $\mathrm{U}(1)$ acts by $\eta_{\mathrm{z},v}^{-1}$. Given a Schwartz function ϕ^v on $\otimes_{w\neq v}S(X_w)$ we consider the map $S(X_v,\eta_{\mathrm{z},v}^{-1})\to\pi_{\theta_{\eta_{\mathrm{z}}}}$ (the $\pi_{\theta_{\eta_{\mathrm{z}}}}$ is the automorphic representation of $\mathrm{U}(2)$ by Howe duality corresponding to η_{z}^{-1}) by $\iota_v:\phi_v\to\Theta_{\phi^v\otimes\phi_v}(\eta_{\mathrm{z}}^{-1})$. By the Howe duality conjecture, we know that there is a maximal proper subrepresentation V_v of $S(X_v,\eta_{\mathrm{z},v}^{-1})$ such that $S(X_v,\eta_{\mathrm{z},v}^{-1})/V_v$ is irreducible and isomorphic to the local theta correspondence $\pi_{\theta,\mathrm{z},v}$ of $\eta_{\mathrm{z},v}^{-1}$ by the local and global compatibility of theta lifting (see [Prasad 1993, Theorem 8.5]). Suppose there is some ϕ_v so that $\iota_v(\phi_v)\neq0$ (a finite sum of pure tensors in $\pi_{\theta_{\eta_{\mathrm{z}}}}$). We consider the representation of $\mathrm{U}(2)(\mathbb{Q}_v)$ on $\pi_{\theta_{\eta_{\mathrm{z}}}}(\mathrm{U}(2)(\mathbb{Q}_v))(\iota_v(\phi_v))$. This is a subrepresentation of a direct sum of finite number of $\pi_{\theta_{\eta_{\mathrm{z}}},v}$'s. The ι_v

gives a homomorphism of representations of $\mathrm{U}(2)(\mathbb{Q}_v)$ from $S(X_v)$ to $\pi(\mathrm{U}(2)(\mathbb{Q}_v))(\iota(\phi_v)) \hookrightarrow \oplus\pi_{\theta_{\eta_z},v}$. Note that the automorphism group of the representation $\pi_{\theta_{\eta_z},v}$ consists of scalar multiplications. Thus it is easy to see that the kernel of the above embedding is exactly V_v and we have the following lemma:

Lemma 8.2. *Fix the ϕ^v as above. Let v_{ϕ_v} be the image of ϕ_v in $\pi_{\theta_{\eta_z},v}$ under the Howe duality isomorphism to $S(X_v, \eta_{z,v}^{-1})/V_v$. Then $\iota_v(\phi_v) \in \pi_{\theta_{\eta_z}}$ can be written as a finite sum of pure tensors of the form*

$$v_{\phi_v} \otimes \Big(\sum_i \prod_{w\neq v} \phi_{w,i}\Big)$$

for $\phi_{w,i} \in \pi_{\theta_{\eta_z},w}$.

We define the weight map $j_1 : \Lambda_{2,0} \to \Lambda''_{\boldsymbol{D}}$ is given by

$$(1+T_1) \mapsto 1, \quad (1+T_2) \mapsto \boldsymbol{\tau}_{\bar{v}_0}^{-1}\boldsymbol{\psi}_{\bar{v}_0}|_{\mathbb{Z}_p^\times}(1+p)$$

where we write $\boldsymbol{\tau}_2$ for the restriction of $\boldsymbol{\tau}$ to $\mathcal{K}_{\bar{v}_0}^\times \simeq \mathbb{Q}_p^\times$, and similarly to $\boldsymbol{\psi}_2$.

Proposition 8.3. *Suppose η is such that for each nonsplit prime $v \in \Sigma$, $\eta|_{\mathrm{U}(1)(\mathbb{Q}_v)}$ equals the $\chi_{\theta,v}$ defined in Section 6G; for each split prime $v\nmid p$ in Σ with $v = w\bar{w}$, we have η_w is unramified and $\eta_{\bar{w}}$ is the $\chi_{\theta,v}$ defined in Section 6G. We can construct a family $\boldsymbol{\theta} = \boldsymbol{\theta}^-_{\boldsymbol{\eta},\phi_2} \in \mathcal{M}_{\mathrm{ord}}(K^{(2,0)}, \Lambda_{2,0}) \otimes_{j_1} \Lambda''_{\boldsymbol{D}}$ whose specialization to $\mathrm{z} \in \mathrm{Spec}\,\Lambda''_{\boldsymbol{D}}$ of conductor p^t equals*

$$\theta_{\mathrm{z}}(g) := \sum_{i=1}^{h_{\mathcal{K}}} \sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p} \eta_{\mathrm{z}}(\breve{u}_i)^{-1} \frac{\Omega_p\theta_{2,\mathcal{D}_{\mathrm{z}}}\Big(g\begin{pmatrix}1&0\\n&1\end{pmatrix}_p\breve{u}_i\Big)}{\Omega_\infty},$$

where the subscript ϕ_2 stands for the Schwartz functions $\phi_{2,\mathrm{z}}$ we define in the proof below, the $\theta_{2,\boldsymbol{D}_{\mathrm{z}}}$ is the one appearing in Corollary 6.47 with the $\mathcal{D}$ being the specialization $\boldsymbol{D}_{\mathrm{z}}$ of $\boldsymbol{D}$. The superscript $-$ is to indicate that the theta function is constructed through pullback under the map $1\times\mathrm{U}(2) \hookrightarrow \mathrm{U}(2,2)$. (Later on we will constructed theta functions via pullback under the map $\mathrm{U}(2)\times 1 \hookrightarrow \mathrm{U}(2,2)$, which we use a superscript $+$ to indicate).

Proof. For an eigenform θ such constructed we sometimes write π_θ for the automorphic representation of $\mathrm{U}(2)$ of θ. First, we give the choices for the Schwartz function ϕ for the construction using the embedding $1\times\mathrm{U}(2) \hookrightarrow \mathrm{U}(2,2)$.

Local computations. In the following we define some Schwartz function. The $\phi_{2,\mathrm{z},v}$'s depend on the arithmetic point z (in fact only varying at the p-adic place), while the $\phi_{3,v}$ are fixed throughout the family and we thus suppress the subscript z.

Case 0. At finite places outside Σ we choose the obvious spherical kernel functions.

Case 1. If $v=\infty$, we let $\phi_{v,\mathrm{z}} = \omega_\lambda\big(\big(\begin{smallmatrix}&1_2\\-1_2&\end{smallmatrix}\big)g_0\big)\Phi_\infty$. Recall that $\Phi_\infty = e^{-2\pi\,\mathrm{Tr}(\langle x,x\rangle_1)}$ and $g_0 = \mathrm{diag}(\mathfrak{s}^{1/2}d^{1/4}/\sqrt{2}, d^{1/4}/\sqrt{2}, (\mathfrak{s}^{1/2}d^{1/4}/\sqrt{2})^{-1}, (d^{1/4}/\sqrt{2})^{-1})$. Let $\phi_{3,v} = \frac{1}{2}(\mathfrak{s}d)^{1/4}e^{-2\pi\sqrt{d}(\mathfrak{s}x_{11}^2+x_{12}^2)}$ and $\phi_{2,v,\mathrm{z}} = \frac{1}{2}(\mathfrak{s}d)^{1/4}e^{-2\pi\sqrt{d}(\mathfrak{s}x_{21}^2+x_{22}^2)}$. By our computation in Section 6.5 we have $\delta_\psi(\phi_{3,v}\boxtimes\phi_{2,v,\mathrm{z}}) = \phi_{v,\mathrm{z}}$.

Case 2. If $v \in S$ is split and $v \nmid p$ we recall that we have two different polarizations $W = X_v \oplus Y_v = X'_v \oplus Y'_v$ where the first one is globally defined which we use to define theta function and the second is defined using $\mathcal{K}_v \equiv \mathbb{Q}_v \times \mathbb{Q}_v$ which is more convenient for computing the actions of level groups. We have defined intertwining operators δ''_ψ between $S(X_v)$ and $S(X'_v)$ intertwining the corresponding Weil representations. Consider the theta correspondence of U(1) to U(2) on $S(X_v)$ and $S(X_v^-)$. We write $X_v \ni x'_{3,v} = (x''_{3,v}, x'''_{3,v})$ and $X_v^{'-} \ni x'_{2,v} = (x''_{2,v}, x'''_{2,v})$. We define

$$\phi'_{3,v}(x''_{3,v}, x'''_{3,v}) = \begin{cases} (\lambda\eta_{\mathrm{z}})_v^{-1}(x''_{3,v}) & x''_{3,v} \in \mathbb{Z}_v^\times, x'''_{3,v} \in \mathbb{Z}_v, \\ 0 & \text{otherwise.} \end{cases}$$

where $\varpi_v^{t_v}$ is the conductor of $\eta_{\mathrm{z},v}$ and

$$\phi'_{2,\mathrm{z},v}(x''_{2,v}, x'''_{2,v}) = \begin{cases} (\lambda\eta_{\mathrm{z}})_v(x''_{2,v}) & x''_{2,v} \in \mathbb{Z}_v^\times, x'''_{2,v} \in \mathbb{Z}_v, \\ 0 & \text{otherwise.} \end{cases}$$

We define $\phi_{v,\mathrm{z}} \in S(\mathbb{W}^d)$ by $\phi_{v,\mathrm{z}} = \delta'_\psi(\phi'_{3,\mathrm{z},v} \boxtimes \phi'_{2,\mathrm{z},v})$ and define $\phi_{2,\mathrm{z},v} = \delta_\psi^{-,''}(\phi'_{2,\mathrm{z},v})$, $\phi_{3,v} = \delta''_\psi(\phi'_{3,v})$. Then if $f_v \in I_v(\lambda^2)$ is the Siegel section corresponding to $\phi_{v,\mathrm{z}}$ in the Rallis inner product formula, we have

$$f(i(1, u_2)) = \langle \phi_{3,v}, \omega(u_2, 1)\phi_{2,\mathrm{z},v} \rangle$$

by the formula for the intertwining operator. This is zero unless $u_2 \in \mathbb{Z}_v^\times$ ($\mathrm{U}(1)(\mathbb{Q}_v) \simeq \mathbb{Q}_v^\times$) and equals $\lambda^2\eta_{\mathrm{z}}(u_2)$ for those u_2's. To sum up, for such v the local integral in the Rallis inner product formula is a nonzero constant $(q_v - 1/q_v)^2$. We modify the definitions for $\phi_{2,\mathrm{z},v}$ and $\phi_{3,v}$ by multiplying them by an element c'_v in $\mathbb{Z}_p^\times$ to make the resulting $\phi_{v,\mathrm{z}}$ integral. Note that our definition for the Schwartz functions $\phi_{2,\mathrm{z},v}$, $\phi_{3,v}$ and thus $\phi_{v,\mathrm{z}}$ are independent of the choice of z. Let $c_v = (c'_v)^2(q_v - 1/q_v)^2$. Later when we are moving things p-adically, this constant is not going to change.

Case 3. For $v \in S$ ramified or inert such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is not compact. In this case $\mathrm{U}(1)(\mathbb{Q}_v)$ is a compact abelian group. We let $\phi_{2,\mathrm{z},v}$ be the Schwartz function $\phi_{2,v}$ on $S(X_v^-)$ constructed in Section 6G, with the Eisenstein datum $\mathcal{D} = \boldsymbol{D}_{\mathrm{z}}$. Let $\phi_{3,v}$ be a p-integral valued Schwartz function on $S(X_v)$ such that $(\phi_{3,v}, \phi_{2,\mathrm{z},v}) = \int_{X_v \simeq X_v^-} \phi_{3,v}(x)\phi_{2,\mathrm{z},v}(x)\,dx \neq 0$ and that the action of $\mathrm{U}(1)(\mathbb{Q}_v)$ via the Weil representation is given by a certain character. (It is easy to see that this character is $\lambda_v^2\eta_{\mathrm{z},v}$, thus the action of the center of $\mathrm{U}(2)(\mathbb{Q}_v)$ via $\mathrm{U}(2) \times 1$ is given by $\eta_{\mathrm{z},v}^{-1}$.) We define $\phi_{v,\mathrm{z}} = \delta_\psi(\phi_{3,v} \boxtimes \phi_{2,\mathrm{z},v}) \in S(\mathbb{W}_v^d)$. We note that at all arithmetic points z, the character $\eta_{\mathrm{z},v}$ as a representation of $\mathrm{U}(1)(\mathbb{Q}_v)$ since the character is changed by an unramified character and the group $\mathrm{U}(1)(\mathbb{Q}_v)$ is compact. We further multiply the $\phi_{3,v}$ and $\phi_{2,\mathrm{z},v}$ by an element c'_v of $\mathbb{Z}_p^\times$ to make them and the resulting $\phi_{\mathrm{z},v}$ integral. Let

$$c_v = (c'_v)^2\,\mathrm{Vol}(\mathrm{U}(1)(\mathbb{Q}_v))(\phi_{3,v}, \phi_{2,\mathrm{z},v}) = \int_{X_v \simeq X_v^-} \phi_{3,v}(x)\phi_{2,\mathrm{z},v}(x)\,dx$$

The $\phi_{3,v}$ and $\phi_{2,\mathrm{z},v}$, and therefore the c_v and $\phi_{v,\mathrm{z}}$ are fixed throughout the family.

Case 4. For v such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact. Note that the local representation $\pi_{\theta_{\eta_{\mathrm{z}}},v}$ is finite-dimensional with some level group K_v. Again let $\phi_{w,\mathrm{z},v}$ be the Schwartz function $\phi_{2,v}$ defined in Section 6G with the

Eisenstein datum being $\mathcal{D} = \boldsymbol{D}_{\mathrm{z}}$. We write v_1 for the image of $\phi_{2,\mathrm{z},v}$ in $\pi_{\theta_{\mathrm{z}},v}$ under the Howe duality. We fix an $\mathrm{U}(2)(\mathbb{Q}_v)/K_v$-invariant measure of $\pi_{\theta_{\mathrm{z}},v}$ and extend v_1 to $\{v_1, \ldots, v_{d_v}\}$ an orthonormal basis of $\pi_{\theta_{\mathrm{z}},v}$. Let $(\tilde{v}_1, \ldots)$ be the dual basis. Let $\phi_{3,v}$ be a p-integral valued Schwartz function on $S(X_v)$ pairing nontrivially with $\phi_{2,\mathrm{z},v}$, such that the action of $\mathrm{U}(1)(\mathbb{Q}_v)$ via the Weil representation is given by a certain character. (As before this character is $\lambda_v^2\eta_{\mathrm{z},v}$, thus the action of the center of $\mathrm{U}(2)(\mathbb{Q}_v)$ via $\mathrm{U}(2)\times 1$ is given by $\eta_{\mathrm{z},v}^{-1}$.) We require also that the image of ϕ_3 in the representation of $\mathrm{U}(2)(\mathbb{Q}_v)$ (which is the dual of $\pi_{\theta_{\mathrm{z}},v}$) is $\tilde{v}_1$. We define $\phi_{v,\mathrm{z}} = \delta_\psi(\phi_{3,v}\boxtimes\phi_{2,\mathrm{z},v}) \in S(\mathbb{W}_v^d)$. We further multiply the $\phi_{3,v}$ and $\phi_{2,\mathrm{z},v}$ by an element c_v' of $\mathbb{Z}_p^\times$ to make them and the resulting $\phi_{v,\mathrm{z}}$ integral. Define $c_v = (c_v')^2$.

Case 5. We write $\eta_{\mathrm{z},p}$ for $\eta_{\mathrm{z}}|_{\mathbb{Q}_p^\times}$. For $v = p$, $W_p = X_p' \oplus Y_p'$, we write elements

$$x_p' = (x_{p,1}', x_{p,2}') \in X_p',\ y_p' = (y_{p,1}', y_{p,2}') \in Y_p'.$$

We define $\phi_{p,\mathrm{z}}(x_p', y_p') = \eta_{\mathrm{z},p}(y_{p,1}')$ if $y_{p,1}' \in \mathbb{Z}_p^\times$ and $x_{p,1}', x_{p,2}', y_{p,2}' \in \mathbb{Z}_p$ and $\phi_{p,\mathrm{z}}(x_p', y_p') = 0$ otherwise.

Definition 8.4. For later interpolation we define a $\Lambda_{\boldsymbol{D}}''$ valued function $\boldsymbol{\phi}_p(x_p', y_p') = \boldsymbol{\eta}_p(y_{p,1}')$ if $y_{p,1}' \in \mathbb{Z}_p^\times$ and $x_{p,1}', x_{p,2}', y_{p,2}' \in \mathbb{Z}_p$ and $\boldsymbol{\phi}_p(x_p', y_p') = 0$ otherwise.

We also write $x_{3,p}' = (x_{3,p}'', x_{3,p}''') \in X_p'$, $x_{2,p}' = (x_{2,p}'', x_{2,p}''') \in X_p'^{-}$ (note that we use $x_{2,p}'$ to distinguish from $x_{p,2}'$ above). A straightforward computation gives

$$\delta_{\psi,p}^{\prime,-1}(\omega_\lambda(\Upsilon)(\phi_{p,\mathrm{z}}))(x_{3,p}', x_{2,p}') = \frac{\mathfrak{g}(\eta_{\mathrm{z},p})}{p^t}\eta_{\mathrm{z},p}^{-1}(-x_{2,p}''p^t)$$

if $x_{2,p}'' \in p^{-t}\mathbb{Z}_p^\times$ and $x_{3,p}'', x_{3,p}''', x_{2,p}''' \in \mathbb{Z}_p$, and equals 0 otherwise. We write $\phi_{3,p}'$ to be the characteristic function of $\mathbb{Z}_p^2$ on X_p' and define $\phi_{2,\mathrm{z},p}'(x_{2,p}') = \frac{\mathfrak{g}(\eta_{\mathrm{z},p})}{p^t}\eta_{\mathrm{z},p}^{-1}(-x_{2,p}''p^t)$ if $x_{2,p}'' \in p^{-t}\mathbb{Z}_p^\times$ and $x_{2,p}''' \in \mathbb{Z}_p$, and is 0 otherwise. We define $\phi_{3,p}$ and $\phi_{2,\mathrm{z},p}$ as the images of $\phi_{3,p}'$ and $\phi_{2,\mathrm{z},p}'$ under $\delta_{\psi,p}^{-1}\circ\delta_{\psi,p}'$. We note that the $\phi_{2,\mathrm{z},p}$ here is not the $\phi_{2,p}$ constructed in Section 6. In fact

$$\phi_{2,\mathrm{z},p} = \sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p} \omega_\lambda\left(\begin{pmatrix} 1 & 0 \\ n & 1\end{pmatrix}_p\right)\phi_{2,p}.$$

This can be seen by comparing the Φ_p'' in Section 6 with the $\phi_{p,\mathrm{z}}$ here. This is exactly where the $\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}\omega_\lambda\left(\left(\begin{smallmatrix} 1 & 0 \\ n & 1\end{smallmatrix}\right)_p\right)$ appears in the statement of the proposition.

Global case.

Definition 8.5. Let $\phi_{\mathrm{z}} = \prod_v \phi_{v,\mathrm{z}}$ with ϕ_v defined in Case 0 through Case 5 before. We define a $\Lambda_{\boldsymbol{D}}''$-adic formal q-expansion $\boldsymbol{\Theta}$ on $\mathrm{U}(2,2)$ interpolating our theta kernel functions on $\mathrm{U}(2,2)$

$$\sum_{x\in\mathcal{K}^2}\prod_{v\nmid p\infty}\phi_v(x)\times\boldsymbol{\phi}_p(x)e^{2\pi i\,\mathrm{tr}({}^t\bar{x}Zx)}.$$

(The $\boldsymbol{\phi}_p$ is defined in Definition 8.4 and note the ϕ_v's are the $\phi_{v,\mathrm{z}}$'s, which are fixed throughout the family for $v\nmid p\infty$, justifying suppressing the subscript z.)

This is easily seen using Lemma 6.13 and the computations in [Skinner and Urban 2014, Section 10.3]. (We note that the $\phi_{\infty,\mathrm{z}}$ here is $\Phi_{1,\infty}$ right translated by g_0. On the other hand, our distinguished point $\boldsymbol{i}$ is chosen as $\zeta \in X_{2,2}$. Note also that the $\phi_{2,\mathrm{z},v}$ and $\phi_{3,v}$ are independent of z for $v \nmid p$.) As in the pullback formula for Siegel–Eisenstein family under $\mathrm{U}(2)\times\mathrm{U}(2) \hookrightarrow \mathrm{U}(2,2)$, the $\boldsymbol{\Theta}(-\Upsilon)$ also pulls back to p-adic analytic family of forms on $\mathrm{U}(2)\times\mathrm{U}(2)$. The following lemma is immediate from our computations from Case 0 to Case 5.

Lemma 8.6. *The pullback of the specialization $\boldsymbol{\Theta}_{\mathrm{z}}$ to $\mathrm{U}(2)\times\mathrm{U}(2)$ is in fact $\theta_{\phi_3}\boxtimes\theta_{\phi_{2,\mathrm{z}}}$ for $\phi_{2,\mathrm{z}}=\prod_v \phi_{2,\mathrm{z},v}$ and $\phi_3=\prod_v \phi_{3,v}$. (We omit the subscript z for ϕ_3 since it is fixed along the family by definition.)*

It follows easily using the Rallis inner product formula (see Proposition 8.9 and its proof, note the arithmetic specialization of the Katz p-adic L-function there is nonzero) and our choices for the Schwartz functions that for some $u_{\mathrm{aux}} \in \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$, $\theta_{\phi_3}(u_{\mathrm{aux}}) \neq 0$. Thus $(\theta_{\phi_3}(u_{\mathrm{aux}})\Omega_p/\Omega_\infty)\cdot(\theta_{\phi_{2,\mathrm{z}}}\Omega_p/\Omega_\infty)$ is a $\Lambda''_{\boldsymbol{D}}$-adic family of forms on $1\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$. Now we take a representative $(\breve{u}_1,\dots\breve{u}_{h_{\mathcal{K}}})$ of $\mathrm{U}(1)(\mathbb{Q})\mathrm{U}_1(\mathbb{R})\backslash\mathrm{U}(1)(\mathbb{A}_{\mathbb{Q}})/\mathrm{U}(1)(\hat{\mathbb{Z}})$ considered as elements of the center of $\mathrm{U}(2)$.

Definition 8.7. Write $c_\theta := (\theta_{\phi_3}(u_{\mathrm{aux}})\Omega_p/\Omega_\infty)$. We denote $\boldsymbol{\theta}=\boldsymbol{\theta}^-_{\boldsymbol{\eta},\phi_2}\in\mathcal{M}_{\mathrm{ord}}(K^{(2,0)},\Lambda_{2,0})\otimes_{j_1}\Lambda''_{\boldsymbol{D}}$ for the $\Lambda''_{\boldsymbol{D}}$-adic family constructed by

$$\boldsymbol{\theta}(g)=\sum_{i=1}^{h_{\mathcal{K}}}\boldsymbol{\eta}(\breve{u}_j)^{-1}\omega_{\lambda^{-1}}(\breve{u}_j)\frac{\Omega_p^2\boldsymbol{\Theta}(u_{\mathrm{aux}},g)}{\Omega_\infty^2}\bar{\lambda}(\det g).$$

Its specialization to z is

$$\theta_{\mathrm{z}}(g):=c_\theta\cdot\frac{\Omega_p}{\Omega_\infty}\sum_{i=1}^{h_{\mathcal{K}}}\eta_{\mathrm{z}}(\breve{u}_j)^{-1}\omega_{\lambda^{-1}}(\breve{u}_j)(\theta_{\phi_{2,\mathrm{z}}}\cdot\bar{\lambda})(g). \tag{8-4}$$

The property required by the proposition follows from comparing our choices of theta kernel function with the (local and global) computations for θ_2 in Section 6. □

We can do the same thing to construct a $\Lambda''_{\boldsymbol{D}}$-adic family of forms on $\mathrm{U}(2)\times 1\hookrightarrow\mathrm{U}(2,2)$. This time we define ϕ such that for $v\neq p$ the local components are as before. If $v=p$ recall that $W_p=X'_p\oplus Y'_p$. If $x'_p=(x'_{p,1},x'_{p,2})$ and $y'_p=(y'_{p,1},y'_{p,2})$ we define $\phi_{\mathrm{z},p}(x'_p,y'_p)=\chi_{\theta,\mathrm{z},p}(x'_{1,p})$ for $x'_{1,p}\in\mathbb{Z}_p^\times$ and $x'_{2,p},y'_{1,p},y'_{2,p}\in\mathbb{Z}_p$ and $\phi_{\mathrm{z},p}(x'_p,y'_p)=0$ otherwise. Direct computation by plugging in the intertwining operator gives, if we write $x'_{1,p}=(x''_{1,p},x'''_{1,p})$ and $x'_{2,p}=(x''_{2,p},x'''_{2,p})$, then $\delta'^{,-1}_{\psi,p}(\omega_\lambda(\Upsilon)\phi_{\mathrm{z},p})(x'_{1,p},x'_{2,p})=\chi_{\theta,\mathrm{z},p}(x''_{1,p})$ if $x''_{1,p}\in\mathbb{Z}_p^\times$ and $x''_{1,p},x''_{2,p},x'''_{2,p}\in\mathbb{Z}_p$, and equals 0 otherwise. We also get new $\phi_{2,\mathrm{z}}$ and $\phi_{3,\mathrm{z}}$ in this case from the ϕ_v's defined here.

As before we move $\chi_{\theta,\mathrm{z}}$ p-adically. This time our θ_{ϕ_2} is fixed and nonzero at some point $u'_{\mathrm{aux}}\in\mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ and $\theta_{\phi_{3,\mathrm{z}}}$ is moving p-adic analytically. We write $c_{\tilde{\theta}_3}=\theta_{\phi_2}(u'_{\mathrm{aux}})\cdot\Omega_p/\Omega_\infty$. Thus $(\theta_{\phi_2}(u'_{\mathrm{aux}})\Omega_p^2/\Omega_\infty^2)\theta_{\phi_{3,\mathrm{z}}}$ is a $\Lambda_{\mathcal{D}}$-adic form on $\mathrm{U}(2)\times 1\subset\mathrm{U}(2,2)$.

As before we define $\boldsymbol{\Theta}$ as in Definition 8.5, and write

$$\tilde{\boldsymbol{\theta}}_3(g)=\boldsymbol{\theta}^{+}_{\boldsymbol{\eta},\phi_3}(g)=\frac{\Omega_p}{\Omega_\infty}\sum_{i=1}^{h_{\mathcal{K}}}\boldsymbol{\eta}_\theta(\breve{u}_j)\omega_{\lambda^{-1}}(\breve{u}_j)\frac{\boldsymbol{\Theta}(u'_{\mathrm{aux}},g)\Omega_p^2}{\Omega_\infty^2}\bar{\lambda}(\det g).$$

The superscript $+$ stands for the fact that the theta function is constructed via the pullback $\mathrm{U}(2)\times 1\hookrightarrow \mathrm{U}(2,2)$. Its specialization $\tilde{\theta}_{3,\mathrm{z}}$ to z satisfies

$$\tilde{\theta}_{3,\mathrm{z}}(g)=c_{\tilde{\theta}_3}\frac{\Omega_p}{\Omega_\infty}\sum_{i=1}^{h_{\mathcal{K}}}\theta_{\phi_{3,\mathrm{z}}}(g\breve{u}_i)\eta_{\mathrm{z}}(\breve{u}_i).$$

Definition 8.8. We define forms θ^D_{z} and $\tilde{\theta}^D_{3,\mathrm{z}}$ on $D^\times(\mathbb{A}_{\mathbb{Q}})$ as $\theta^D_{\mathrm{z}}\boxtimes\eta_{\mathrm{z}}$ and $\tilde{\theta}^D_{3,\mathrm{z}}\boxtimes\bar{\eta}_3$ respectively and characters η_{z} and $\bar{\eta}_{\mathrm{z}}$, as at the beginning of Section 8A. Sometimes we drop the superscript D when it is clear from the context. The key functorial property of it (and some other automorphic forms constructed) is summarized in Section 8B1.

As before we let $\mathcal{L}^{\mathrm{Katz}}_{\lambda\eta}$ be the Katz p-adic L-function interpolating the values

$$\Omega_p^2\frac{L(\lambda^2\chi_{\theta,\mathrm{z}}\chi_{\theta,\mathrm{z}}^{-c},1)}{(\lambda^2\eta_{\mathrm{z}}\eta_{\mathrm{z}}^{-c})_{2,p}G(\lambda^2\eta_{\mathrm{z}}\eta_{\mathrm{z}}^{-c})\Omega_\infty^2}.$$

We now compute the Petersson inner product of θ_{z} and $\tilde{\theta}_{3,\mathrm{z}}$ at a generic arithmetic point z.

Proposition 8.9. *We have*

$$\boldsymbol{B}_{K^{(0,2)}}(\boldsymbol{\theta}^{-}_{\boldsymbol{\eta},\phi_2},\boldsymbol{\theta}^{+}_{\boldsymbol{\eta},\phi_3})=c_\theta\cdot c_{\tilde{\theta}_3}\prod_v c_v\cdot\mathcal{L}^{\mathrm{Katz}}_{\lambda\eta}$$

for the c_v in Case 2, Case 3 and Case 4 as before. Note that these are constants fixed throughout the family.

Proof. We do the computation at a generic arithmetic point z, and apply (8-2) and ((8-2)2). By the doubling method for $\mathrm{U}(1)\times\mathrm{U}(1)\hookrightarrow\mathrm{U}(1,1)$ we are reduced to local pullback formulas.

We first construct the local theta kernels $\phi_{v,\mathrm{z}}$'s. At places outside p, the choices are the same as Case 0 through Case 4 above. At the p-adic place, we construct the Schwartz function in $S(W_p)=S(X'_p\oplus Y'_p)$ first and apply the intertwining operators δ_ψ^{-1}. We write

$$x'_p=(x'_{p,1},x'_{p,2})\in X'_p\quad\text{and}\quad y'_p=(y'_{p,1},y'_{p,2})\in Y'_p.$$

We define $\phi_{\mathrm{z},p}(x_p,y_p)=\eta_{\mathrm{z},p}(x'_{p,1},y'_{p,1})$ if $x'_{p,1},y'_{p,1}\in\mathbb{Z}_p^\times$ and $x'_{p,2},y'_{p,2}\in\mathbb{Z}_p$, and equals 0 otherwise. Now we compute the local pullback integrals.

- For the Archimedean place, this pullback integral is 1 as in [Wan 2015a, Section 4.1] ($n=1$, and $\kappa=2$ there).

• For $v \nmid p$, by our choices the corresponding local integrals are nonzero constants which are fixed along the p-adic families (see the computations in Case 0 through Case 4 above). The product of such local integrals is $\prod_v c_v$.

• For $v = p$, as before we have $\phi_{3,z,v}$ and $\phi_{2,z,v}$'s (this time both Schwartz functions depend on the arithmetic point z!), and can compute that $\delta_\psi^{',-1}(\phi_{z,p}) = \phi_{3,z,p} \times \phi_{2,z,p} \in S(X'_{1,p}) \times S(X_{2,p'})$ where $\phi_{3,z,p}(x''_{1,p}, x'''_{1,p}) = \eta_{z,p}(x''_{1,p})$ if $x''_{1,p} \in \mathbb{Z}_p$, $x'''_{1,p} \in \mathbb{Z}_p$ and equals 0 otherwise,

$$\phi_{2,z,p}(x''_{2,p}, x'''_{2,p}) = \frac{\mathfrak{g}(\eta_{z,p})}{p^t}\eta_{z,p}^{-1}(x''_{2,p})$$

if $x''_{2,p} \in p^{-t}\mathbb{Z}_p^\times$ and $x'''_{2,p} \in \mathbb{Z}_p$, and equals 0 otherwise. So these are exactly the theta functions whose inner product we want to compute. Now we compute the Siegel section $f \in I_1(\lambda^2)$ on $\mathrm{U}(1,1)(\mathbb{Q}_p)$. From the form of the Schwartz functions, it is easy to see that the Siegel–Weil section $f_{\phi_{z,p}} = f_{1,p} f_{2,p}$ for $f_{2,p} \in I_1(\lambda)$ to be the spherical function which takes value 1 on the identity and $f_{1,p} \in I_1(\lambda)$ is the Siegel–Weil section on $\mathrm{U}(1,1)$ of $\mathrm{U}(1,1) \times \mathrm{U}(1)$ for the Schwartz function $\phi_p \in S(\mathcal{K}_p)$ which, with respect to $\mathcal{K}_v \equiv \mathbb{Q}_v \times \mathbb{Q}_v$, is $\phi_p(x_1, x_2) = \eta_{z,p}(x_1 \cdot x_2)$ for $x_1, x_2 \in \mathbb{Z}_p^\times$, and $\phi_p(x_1, x_2) = 0$ otherwise. However, this section is nothing but the Siegel section $f^\dagger$ we constructed in [Wan 2015a, Section 4] for the one-dimensional unitary group case. Thus, the local integral is easily computed to be

$$\frac{\mathfrak{g}(\eta_{z,p})}{p^t}.\lambda_p^{-2}((p^{-t}, p^t))p^t\lambda_p^2(p^{-t}, 1) = \lambda_p^2((1, p^t))\mathfrak{g}(\eta_{z,p}).$$

Recall the c_θ and $c_{\tilde{\theta}_3}$ we used in the construction of $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}_3$. To sum up, we obtain the proposition from above computations. □

8B1. *Constructing* $\boldsymbol{h}$. We repeat the above process to construct another family $\boldsymbol{h}$ from the family $\boldsymbol{\eta}'$, which will be used in computing the Fourier–Jacobi expansion as well. We put an assumption:

suppose for each nonsplit bad prime v such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact, $\eta|_{\mathrm{U}(1)(\mathbb{Q}_v)} = \eta'|_{\mathrm{U}(1)(\mathbb{Q}_v)}$.

Again we compute at a generic arithmetic point $z \in \mathrm{Spec}\,\Lambda''_{\boldsymbol{D}}(\bar{\mathbb{Q}}_p)$. First we construct forms h'_z and $\tilde{h}'_{3,z}$ using theta lifting in the same way as we constructed θ_z and $\tilde{\theta}_{z,3}$, except with η''_z in place of η_z and slightly different theta kernels described as follows. (Therefore, in our application, the forms h'_z's are still CM forms.) More precisely, we make the Schwartz functions as follows:

- In case 0 and case 1 (unramified and Archimedean cases) our Schwartz functions $\phi_{v,z}$, $\phi_{2,z,v}$, $\phi_{3,v}$ are chosen by the same formula.
- In case 2 (split bad primes) the Schwartz functions $\phi'_{2,z,v}$ and $\phi'_{3,v}$ are chosen as before except replacing $\eta_{z,v}$ by $\eta'_{z,v}$. We define $\phi_{v,z} = \delta'_\psi(\phi'_{3,v} \boxtimes \phi'_{2,z,v})$ and define $\phi_{2,z,v} = \delta_\psi^{-,''}(\phi'_{2,z,v})$, $\phi_{3,v} = \delta''_\psi(\phi'_{3,v})$. We further multiply the Schwartz functions by a (fixed) nonzero element in $\mathbb{Z}_p$ to make the $\phi_{3,v}$, $\phi_{2,z,v}$, and $\phi_{v,z}$ are integral.
- In case 3 (nonsplit bad primes with $\mathrm{U}(2)(\mathbb{Q}_v)$ not compact). In this case we recall the local theta correspondence from $\mathrm{U}(1)$ to $\mathrm{U}(1,1)$ is always nonvanishing from any character of $\mathrm{U}(1)(\mathbb{Q}_v)$ since it

is in the "stable range" (see [Kudla 1996, Propositions 4.3 and 4.5]). We take some Schwartz function $\phi_{2,z,v}$ and $\phi_{3,v}$ with nonzero pairing so that they are eigenvectors under the action of $\mathrm{U}(1)(\mathbb{Q}_v)$, and the eigenvalue for $\phi_{3,v}$ is $\lambda_v^2\eta'_{z,v}$. (The existence is guaranteed by the previous "stable range" result.) These Schwartz functions are fixed throughout the family. Define $\phi_{v,z}=\delta_\psi(\phi_{3,v}\boxtimes\phi_{2,z,v})\in S(\mathbb{W}_v^d)$ as before. We further multiply the Schwartz functions by a (fixed) nonzero element in $\mathbb{Z}_p$ to make the $\phi_{3,v}$, $\phi_{2,z,v}$, and $\phi_{v,z}$ are integral.

- In case 4 (nonsplit bad primes with $\mathrm{U}(2)(\mathbb{Q}_v)$ being compact) Recall we assumed $\eta|_{\mathrm{U}(1)(\mathbb{Q}_v)}=\eta'|_{\mathrm{U}(1)(\mathbb{Q}_v)}$. As discussed at the beginning of Section 8A, it is easy to see that the $\pi_{h'_z,v}$ is either $\pi^\vee_{\theta_{\eta_z},v}\otimes\eta_z$ or ${}^\alpha\pi^\vee_{\theta_{\eta_z},v}\otimes\eta_z$. Under this identification we choose the local Schwartz function $\phi_{2,z,v}$ at v so that the image in $\pi_{h'_z,v}$ is $\tilde v_1$ or $\iota_\alpha(\tilde v_1)$ depending on whether $\pi^\vee_{\theta_{\eta_z},v}\otimes\eta_z$ or ${}^\alpha\pi^\vee_{\theta_{\eta_z},v}\otimes\eta_z$ (notations as in Section 8A), and the Schwartz function $\phi_{3,v}$ whose image is v_1 or $\iota_\alpha(v_1)$. Define $\phi_{v,z}=\delta_\psi(\phi_{3,v}\boxtimes\phi_{2,z,v})\in S(\mathbb{W}_v^d)$ as before. We further multiply the Schwartz functions by a (fixed) nonzero element in $\mathbb{Z}_p$ to make the $\phi_{3,v}$, $\phi_{2,z,v}$, and $\phi_{v,z}$ are integral.
- In case 5 (p-adic places) we choose $\phi_{p,z}$, $\phi_{2,z,v}$ and $\phi_{3,v}$ as in case 5 before Definition 8.4 except replacing $\eta_{z,p}$ by $\eta''_{z,p}$.

Let $\boldsymbol{H}'$ be the family of theta functions on $\mathrm{U}(2,2)$ defined as the $\boldsymbol{\Theta}$ in Definition 8.5 as

$$\sum_{x\in\mathcal{K}^2}\prod_{v\nmid p\infty}\phi_v(x)\times\boldsymbol{\phi}_p(x)e^{2\pi i\,\mathrm{tr}({}^t\bar{x}Zx)}$$

but replacing the local Schwartz functions there by the ones defined here. As in Definition 8.7, we denote $\boldsymbol{h}'$ for the $\Lambda''_{\boldsymbol{D}}$-adic family constructed by

$$\boldsymbol{h}'(g)=\boldsymbol{\theta}^-_{\boldsymbol{\eta}'',\phi_2}=\sum_{i=1}^{h_{\mathcal{K}}}\boldsymbol{\eta}''(\breve{u}_j)^{-1}\omega_{\lambda^{-1}}(\breve{u}_j)\frac{\boldsymbol{H}'(u_{\mathrm{aux}},g)\Omega_p^2}{\Omega_\infty^2}\bar{\lambda}(\det g).$$

Note that its central character is $\boldsymbol{\eta}''$. Write $c_{h'}=\theta_{\phi_3}(u'_{\mathrm{aux}})\cdot\Omega_p/\Omega_\infty$. Clearly it is interpolating theta functions

$$h'_z:=c_{h'}\sum_{i=1}^{h_{\mathcal{K}}}\eta''_z(\breve{u}_j)^{-1}\omega_{\lambda^{-1}}(\breve{u}_j)(\theta_{\prod_v\phi_{2,z,v}}\cdot\bar{\lambda}). \tag{8-5}$$

Again we define h_z^D on $D^\times(\mathbb{A}_\mathbb{Q})$ as $h'^D_z\boxtimes\eta'_z$ (note the character is not η''_z) using the procedure at the beginning of Section 8A. Clearly we can also form the corresponding family which we denote as $\boldsymbol{h}^D=\boldsymbol{h}'^D\boxtimes\boldsymbol{\eta}'$.

The automorphic representation for h_z^D (θ_z^D) is the Jacquet–Langlands correspondence of the CM form associated to $\lambda\eta'_z$ ($\lambda\eta_z$ respectively). We define h_z (or $\boldsymbol{h}$) on $\mathrm{GU}(2)(\mathbb{A}_\mathbb{Q})$ using h_z^D (or $\boldsymbol{h}^D$) and the character $\eta_z^{-1}\psi_z$ (not η'_z! This is crucial for our p-adic analysis of Fourier–Jacobi coefficients) (or $\boldsymbol{\eta}^{-1}\boldsymbol{\psi}$) as

$$h_z(ag)=\eta_z^{-1}\psi_z(a)h_z^D(g),\quad \boldsymbol{h}(ag)=\boldsymbol{\eta}^{-1}\boldsymbol{\psi}(a)\boldsymbol{h}^D(g)$$

for $a \in \mathbb{A}_{\mathcal{K}}^\times$ and $g \in D^\times(\mathbb{A}_\mathbb{Q})$. The h_z's are certainly interpolated by the family $\boldsymbol{h}$. We write $\pi_{h,z}$ for the corresponding automorphic representation. Similarly let $\tilde{h}^D_{3,z}$ be the form $\tilde{h}'^D_{3,z} \boxtimes \eta'^{-1}_{h,z}$, and let $\tilde{h}_{3,z}$ be the form on GU(2) constructed from $\tilde{h}^D_{3,z}$ using the character $\eta_z\psi_z^{-1}$. Let $\tilde{\boldsymbol{h}}_3$ be the corresponding family.

Now we give the map of the weight spaces for $\boldsymbol{h}$ and the Fourier–Jacobi coefficients of the $\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}$. In fact the nebentypus of the $\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling},z}$ is given by $\mathrm{diag}(\psi_{2,z}^{-1}, \chi_{f_z}\psi_{2,z}^{-1}, \tau_{2,z}^{-1}, \tau_{1,z})$. Therefore the nebentypus of $el_{\theta^\star}\,\mathrm{FJ}_\beta(\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling},z})$ as a form on U(2) is given by $\chi_{f_z}\psi_{2,z}^{-1}, \tau_{2,z}^{-1}$. So it is interpolated by a family in $\mathcal{M}_{\mathrm{ord}}(K^{(2,0)}, \Lambda_{2,0}) \otimes_{j_2} \Lambda''_{\boldsymbol{D}}$ with the weight map $j_2 : \Lambda_{2,0} \to \Lambda''_{\boldsymbol{D}}$ given by

$$(1+T_1) \mapsto \boldsymbol{\chi}_{f,p}\boldsymbol{\psi}_2^{-1}|_{\mathbb{Z}_p^\times}(1+p),\ (1+T_2) \mapsto \boldsymbol{\tau}_2^{-1}(1+p). \tag{8-6}$$

Here $\boldsymbol{\chi}_{f,p}$ is the p-part of the central character of $\boldsymbol{f}$, and we write $\boldsymbol{\psi}_2$ for the restriction of $\boldsymbol{\psi}$ to $\mathcal{K}_{\bar{v}_0}^\times \simeq \mathbb{Q}_p^\times$, and similarly for $\boldsymbol{\tau}_2$. It is also straightforward to check that $\boldsymbol{h} \in \check{\mathcal{M}}_{\mathrm{ord}}(K^{(2,0)}, \Lambda_{2,0}) \otimes_{j_2} \Lambda''_{\boldsymbol{D}}$ for the same j_2.

Convention. Oftentimes, when we constructed $\theta_z \in \pi_{\theta_z}$ with central character $\chi_{\theta,z}$ then by $\tilde{\theta}_z \in \pi_{\tilde{\theta}_z}$ we mean $\theta_z \cdot (\chi_{\theta,z}^{-1} \circ \det)$. If we have constructed $\tilde{\theta}_{3,z}$ with central character $\chi_{\theta,z}^{-1}$, then by $\theta_{3,z}$ we mean $\tilde{\theta}_{3,z} \cdot (\chi_{\theta,z} \circ \det)$. We use the same conventions for h_z's as well.

We have the following immediate corollary:

Corollary 8.10. *The θ_z, $\tilde{\theta}_{3,z}$, θ_z^D, $\tilde{\theta}^D_{3,z}$, h_z, $\tilde{h}_{3,z}$, h_z^D, $\tilde{h}^D_{3,z}$ constructed before are pure tensors in the corresponding automorphic representations.*

Proof. This follows immediately from Lemma 8.2 and the constructions above. □

8C. *Choosing some characters.* In this section we make choices for some Hecke characters for the η and η' in the previous section. These are important in our study for Fourier–Jacobi coefficients for Klingen–Eisenstein series later on.

We first give a result of Pin-Chi Hung [2017, Theorem C]. Let χ be a finite order Hecke character of $\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times$ of conductor $M\mathcal{O}_{\mathcal{K}}$ for some $M > 0$. Let $f \in S_k(\Gamma_0(N))$ be an elliptic cusp form of even weight k, level $\Gamma_0(N)$ with q-expansion

$$f(q) = \sum_{n\ge 0} a_n(f)q^n.$$

We decompose $N = N^+N^-$, where N^+ is a product of primes split in $\mathcal{K}$ and N^- is a product of primes ramified or inert in $\mathcal{K}$. Suppose N^- is square-free and $N^- = N_f^- N_\chi^-$ where N_f^- is a product of an odd number of primes coprime to M and N_χ^- is a divisor of M. Let ℓ be a rational prime split in $\mathcal{K}$. Let $\mathcal{K}_\ell^-$ be the unique abelian anticyclotomic $\mathbb{Z}_\ell$-extension of $\mathcal{K}$ and Γ^- be the Galois group $\mathrm{Gal}(\mathcal{K}_\ell^-/\mathcal{K})$.

Theorem 8.11. *Suppose $\ell^2 \nmid N$. Let p be a rational prime such that*

- *$p \nmid \ell N D_{\mathcal{K}}$ and $p \ge k-2$,*
- *for every nonsplit $q \mid M$, $q+1$ is not divisible by p,*
- *for every $q \mid N_f^-$ ramified in $\mathcal{K}$, $a_q(f) = \chi(\mathfrak{q})(=\pm 1)$, where $q = \mathfrak{q}^2$,*
- *the residual Galois representation $\bar{\rho}_{f,\lambda}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K})}$ is absolutely irreducible.*

Then there is a finite extension $L/\mathbb{Q}_p$ with integer ring $\mathcal{O}_L$ and uniformizer λ. We have for all but finitely many characters $\nu : \Gamma^- \to \mu_{\ell^\infty}$, we have

$$\frac{L(f/\mathcal{K}, \chi\nu, k/2)}{\Omega_{f,N^-}} \not\equiv 0 \pmod{\lambda}.$$

Here the Ω_{f,N^-} is a period factor defined in [loc. cit.].

Now we choose the characters needed. From now on we fix once for all a split prime ℓ outside Σ and write a new "Σ" for $\Sigma \cup \{\ell\}$. We choose χ_θ a Hecke character of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ as follows: $\chi_{\theta,\infty}$ is trivial. At p we require that χ be unramified. For $v \in \Sigma$ nonsplit in $\mathcal{K}/\mathbb{Q}$, then we let $\chi_{\theta,v}|_{\mathrm{U}(1)}$ to be the character chosen in Section 6. For split $v \in \Sigma$, $v \nmid p, \ell$, $v = w\bar{w}$, suppose $\mathrm{cond}(\pi_v) = (\varpi_v^{t_{1,v}})$, we require that $\chi_{\theta,w}$ be unramified and $\mathrm{cond}(\chi_{\theta,\bar{w}}) = (\varpi_v^{t_{2,v}})$ for $t_{2,v} \geq 2t_{1,v} + 2$. We choose a character χ_{aux} of $\mathcal{K}^\times \backslash \mathbb{A}_\mathcal{K}^\times$ as follows: $\chi_{\mathrm{aux}}|_{\mathbb{A}_\mathbb{Q}^\times} = 1$, it is trivial at ∞, and is only ramified at primes in Σ not dividing p such that $\mathrm{U}(2)(\mathbb{Q}_v)$ is not compact. For split such v we require that

$$\mathrm{cond}(\chi_{\mathrm{aux}}) = \begin{cases} (\varpi_v^{t_{2,v}-t_{1,v}}) & \text{if } t_{1,v} \neq 0, \\ (\varpi_v^{t_{2,v}-1}) & \text{if } t_{1,v} = 0. \end{cases}$$

At nonsplit such primes we require that

$$\mathrm{cond}(\chi_{\mathrm{aux},v}) > \mathrm{cond}(\pi_v), \mathrm{cond}_v(\lambda^2 \chi_\theta^{-c} \chi_\theta \chi_{\mathrm{aux}}) > \mathrm{cond}(\pi_v)$$

and $\chi_{\mathrm{aux},v}|_{\mathbb{Q}_v^\times}$ has a smaller conductor than $\chi_{\mathrm{aux},v}$ (here $>$ means the conductor of the former is of higher power of the uniformizer than the latter). Also for each prime q such that $\mathrm{U}(2)(\mathbb{Q}_q)$ is compact and q is ramified as w^2 in $\mathcal{K}$, suppose $\pi_q \simeq \mathrm{Steinberg} \otimes \chi_{q,1}$ for some unramified quadratic character $\chi_{q,1}$ we require that $\chi_{q,1}(q) = \chi_{\mathrm{aux}}(\varpi_w)$ (these are used in the next paragraph to make sure that the special L-values are of the correct local signs when applying Theorem 8.11). To ensure the existence, we may need to enlarge the Σ by including one prime ℓ' which is prime to N and inert in $\mathcal{K}$. Let $\chi_h = \chi_\theta^{-c} \chi_{\mathrm{aux}}$.

We further require that

$$\frac{L\big(\pi_f, \lambda^2 \chi_\theta \chi_h, \tfrac{1}{2}\big)}{\pi^3 \Omega_\infty^4} \mathrm{Eul}_p\big(\pi_f, \lambda^2 \chi_\theta \chi_h, \tfrac{1}{2}\big), \quad \frac{\Gamma(\kappa-1) L\big(\chi_{\mathrm{aux}} \tau^{-c}, \tfrac{\kappa-2}{2}\big)}{\pi^{\kappa-1} \Omega_\infty^\kappa} \mathrm{Eul}_p\left(\chi_{\mathrm{aux}} \tau^{-c}, \frac{\kappa-2}{2}\right),$$

and

$$\frac{\Gamma(\kappa-2) L\big(\lambda^2 \chi_\theta^{-c} \chi_\theta \chi_{\mathrm{aux}} \tau^{-1}, \tfrac{\kappa-2}{2}\big)}{\pi^{\kappa-2} \Omega_\infty^{\kappa-2}} \mathrm{Eul}_p\left(\lambda^2 \chi_\theta^{-c} \chi_\theta \chi_{\mathrm{aux}} \tau^{-1}, \frac{\kappa-2}{2}\right)$$

are p-adic units where the Eul_p are the local Euler factors for the corresponding p-adic L-functions at p when everything is unramified at p (we refer to [Hsieh 2014c, (0.2); 2014b, (4.16)] for their precise definitions). The first uses [Hsieh 2014c]. Our assumptions above on conductors imply that at all nonsplit primes the local root numbers in Theorem A of [Hsieh 2014c] are all $+1$ (this uses [Jacquet and Langlands 1970, Proposition 3.8], as also mentioned in [Brakočević 2010, Introduction]). Then we take a split prime $\ell \nmid Np$ and apply that theorem to see that there exists a twist by anticyclotomic character of ℓ-power conductor which satisfies the requirements. The second and third uses [Hsieh

2012] (we are in the residually non-self-dual case there) and again we can achieve the requirements by twisting by an appropriate anticyclotomic character of conductor powers of ℓ. Further, we assume that $1-a_p(f)^{-1}\chi_{\theta,p,2}\chi_{h,p,1}(p)$, $1-a_p(f)\chi_{\theta,p,1}\chi_{h,p,2}(p)^{-1}$ and $1-\lambda_{p,2}^2\chi_{h,p,2}\chi_{\theta,p,2}\tau_{p,2}^{-1}(p)p^{-(\kappa-2)/2}$ are p-adic units. At each prime v of $\mathcal{K}$ above a prime where $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact, we require that $1-\chi_{\mathrm{aux}}\tau^{-c}(q_v)q_v^{-(\kappa-2)/2}$ be a p-adic unit. We also require that $L\big(\pi_f,\chi_\theta^c\chi_h,\frac{1}{2}\big)/(\pi^2\Omega_f^+\Omega_f^-)$ is nonzero (do not need to be nonzero modulo p!) using the Theorem 8.11 recalled above (by choosing a different "p'" and prove nonvanishing the new p'. Note that the mod p' residual representation $\bar{\rho}_f|_{G_{\mathcal{K}}}$ is absolutely irreducible for all but finitely many primes p').

Definition 8.12. Now we take the η and η' in the previous section to be the χ_θ and χ_h above (indicating they are used for forms θ_z and h_z's. Clearly they satisfy all the requirements there.).

Now we define a character ϑ of $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times$ (the reason for doing so is just a cheap way to use the new form theory at split primes to pick different vectors inside an automorphic representation of U(2)). We require that these be ramified only at split primes in $\Sigma\backslash\{p\}$. At such $v=w\bar{w}$ and require that $\vartheta|_{\mathbb{Z}_v^\times}=\chi_h|_{\mathcal{O}_{\mathcal{K},w}^\times}$. These uniquely determine the character ϑ.

8D. *Triple product formula.*

Background for Ichino's formula. Let π_1,π_2,π_3 be three irreducible cuspidal automorphic representations for $\mathrm{GL}_2/\mathbb{Q}$ such that the product of their central characters is trivial and the archimedean components are holomorphic discrete series of weight two. Let π_i^D be the Jacquet–Langlands correspondence of them to $D^\times$ (assume they do exist). Let $\phi_i\in\pi_i^D$ and $\tilde{\phi}\in\tilde{\pi}_i^D$. Write $\Pi=\prod_{i=1}^3\pi_i$, $\phi=\prod_i\phi_i\in\Pi$, $\tilde{\phi}=\prod_i\tilde{\phi}_i\in\tilde{\Pi}$, and r the natural eight-dimensional representation of $\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2$. We write

$$I(\phi\otimes\tilde{\phi})=\bigg(\int_{[D]}\phi_1(g)\phi_2(g)\phi_3(g)\,dg\bigg)\bigg(\int_{[D]}\tilde{\phi}_1(g)\tilde{\phi}_2(g)\tilde{\phi}_3(g)\,dg\bigg).$$

Now look at the local picture. Suppose $\phi_i=\otimes_v\phi_{i,v}$ and $\tilde{\phi}_i=\otimes_v\tilde{\phi}_{i,v}$. We fix $\langle\cdot,\cdot\rangle$ a $D_v^\times$ invariant pairing between π_i^D and $\tilde{\pi}_i^D$. Let ζ be the Riemann zeta-function. Define

$$I_v(\phi_v\otimes\tilde{\phi}_v)=\zeta_v(2)^{-2}\frac{L_v(1,\Pi_v,Ad)}{L_v(1/2,\Pi_v,r)}\cdot\int_{\mathbb{Q}_v^\times\backslash D^\times(\mathbb{Q}_v)}\prod_i\langle\pi_v^D\phi_v(x_v),\tilde{\phi}_v\rangle\,d^\times x_v.$$

Note that this depends on the choice of the pairing.

Let Σ be a finite set of primes including all bad primes, then we have the following formula of Ichino [2008]:

$$\frac{I(\phi\otimes\tilde{\phi})}{\prod_i\langle\phi_i,\tilde{\phi}_i\rangle}=\frac{C}{8}\zeta^2(2)\frac{L^\Sigma(\frac{1}{2},\Pi,r)}{L^\Sigma(1,\Pi,Ad)}\prod_{v\in\Sigma}\frac{I_v(\phi_v\otimes\tilde{\phi}_v)}{\langle\phi_v,\tilde{\phi}_v\rangle}$$

where C is the Tamagawa number for $D^\times$. This does not depend on the choice of the pairing.

In application, our $\langle\phi,\tilde{\phi}\rangle$ is usually 0, thus we need a slight variant of the above formula. Suppose we have elements $g_i'=\prod_v g_{i,v}'$ such that $\langle\phi_i,\pi(g_i')\tilde{\phi}_i\rangle\neq 0$ for $i=1,2,3$, where $g_{i,v}'$ are elements in the group algebra $\overline{\mathbb{Q}}_p[D^\times(\mathbb{Q}_v)]$. Then

$$\frac{I(\phi\otimes\tilde{\phi})}{\prod_i\langle\phi_i,\pi(g_i')\tilde{\phi}_i\rangle}=\frac{C}{8}\zeta^2(2)\frac{L^\Sigma(\frac{1}{2},\Pi,r)}{L^\Sigma(1,\Pi,Ad)}\prod_{v\in\Sigma}\frac{I_v(\phi_v\otimes\tilde{\phi}_v)}{\langle\phi_v,\pi(g_v')\tilde{\phi}_v\rangle}$$

with $g_v=\prod_v g_{i,v}$.

Local triple product computations. We remark that in [Hsieh 2017, Sections 5 and 6] the local test vectors and triple product integrals are worked out in full generality. However here for the special cases needed in this paper we include the computations for convenience of the reader.

Split case principal series. Suppose v is a split prime of $\mathcal{K}/\mathbb{Q}$ with q_v being the cardinality of its residue field. We assume $\pi_{1,v}$ and $\pi_{2,v}$ are principal series representation and $\pi_{3,v}$ is either principal series representation or special representation with square-free conductor. For $K=\mathrm{GL}_2(\mathbb{Z}_v)$ the maximal compact subgroup of $\mathrm{GL}_2(\mathbb{Q}_v)$, we use the realizations of induced representations as functions on K:

$$\mathrm{Ind}_{B(\mathbb{Q}_v)}^{\mathrm{GL}_2(\mathbb{Q}_v)}(\chi_{1,v},\chi_{2,v})=\{v:K\to\mathbb{C},\ v(qk)=\chi(q)v(K),\ q\in B(\mathbb{Q}_v)\cap K\}$$

where $\chi(q)=\chi_{1,v}(a)\chi_{2,v}(d)\delta_B(q)$ for $q=\begin{pmatrix}a&b\\&d\end{pmatrix}$. We realize the inner products as

$$\langle v_1,v_2\rangle=\int_K v_1(k)v_2(k)\,dk$$

for $v_1\in\mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_{1,v},\chi_{2,v})$, $v_2\in\mathrm{Ind}_B^{\mathrm{GL}_2}(\chi_{1,v}^{-1},\chi_{2,v}^{-1})$. For a positive integer t, let $K_t\subset K$ consist of matrices in $B(\mathbb{Z}_v)$ modulo ϖ_v^t. For $f\in\pi(\chi_1,\chi_2)$, $\tilde{f}\in\pi(\chi_1^{-1},\chi_2^{-1})$, we define the matrix coefficient $\Phi_{f,\tilde{f}}(g)=\langle\pi(g)f,\tilde{f}\rangle$. Let $\sigma_n=\begin{pmatrix}\varpi_v^n&\\&1\end{pmatrix}$.

Lemma 8.13. *Suppose $t\geq 1$, $\mathrm{cond}(\chi_1\chi_2^{-1})=(\varpi_v^t)$. Let $w=\begin{pmatrix}&1\\1&\end{pmatrix}$ in this lemma. If*

$$f_\chi(k_v)=\begin{cases}\chi_1(a)\chi_2(d) & k_v\in K_t,\\ 0, & \textit{otherwise},\end{cases}$$

and $\tilde{f}_{\chi^{-1}}$ is defined similar to f_χ but with χ replaced by χ^{-1}. Then $\Phi_{f_\chi,\tilde{f}_{\chi^{-1}}}(g)=0$ on

$$\cup_n K_1w\sigma_nK_1\cup_n K_1\sigma_nwK_1.$$

On $\cup_nK_1\sigma_nK_1\cup_n K_1w\sigma_n\omega K_1$, it is supported in

$$\cup_n\begin{pmatrix}1&\\&\varpi_v^n\end{pmatrix}\begin{pmatrix}1&\\ \varpi_v^{t-n}\mathbb{Z}_v&1\end{pmatrix}K_t\cup_n\begin{pmatrix}\varpi_v^n&\\&1\end{pmatrix}\begin{pmatrix}1&\varpi_v^{-n}\mathbb{Z}_v\\&1\end{pmatrix}K_t.$$

The corresponding values at $\begin{pmatrix}1&\\&\varpi_v^n\end{pmatrix}$ and $\begin{pmatrix}\varpi_v^n&\\&1\end{pmatrix}$ are $\mathrm{Vol}(K_t)\alpha_2^nq_v^{-n/2}$ and $\mathrm{Vol}(K_t)\alpha_1^nq_v^{-n/2}$ respectively, where $\alpha_i=\chi_i(\varpi_v)$ for $i=1,2$.

Proof. It is easy to check by considering the supports of f_χ and $\tilde{f}_\chi$ that $\Phi_{f_\chi,\tilde{f}_\chi}(g)=0$ on $\coprod_{n\geq 0}K_1\sigma_nwK_1$. ($K_1\sigma_nwK_1$ does not intersect $\mathrm{supp}\,f_\chi$.)

Now suppose $g \in K_1 w\sigma_n K_1$ for $n \geq 1$, without loss of generality we assume

$$g = \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} w\sigma_n \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} = g = \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{c}{\varpi_v} & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} w$$

for $\varpi_v \mid b$, $\varpi_v \mid c$. Plugging in the formula for matrix coefficients, write $K_t \ni g' = \left(\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c' & 1 \end{smallmatrix}\right)$ for $a', d' \in \mathbb{Z}_v^\times$, $b' \in \mathbb{Z}_v$, $c' \in \varpi_v^t \mathbb{Z}_v$.

$$\begin{aligned} g'g &= \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ c + c' & 1 \end{pmatrix} w\sigma_n \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \\ &= \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix} \begin{pmatrix} 1 & \\ (c'+c)/\varpi_v^n & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} w \\ &= \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ & \varpi_v^n \end{pmatrix} \begin{pmatrix} 1 & 1/c'' \\ & 1 \end{pmatrix} \begin{pmatrix} -1/c'' & \\ & c'' \end{pmatrix} \begin{pmatrix} 1 & \\ (bc''+1)/c'' & 1 \end{pmatrix} \end{aligned}$$

Here we write $c'' = (c'+c)/\varpi_v^n$. We need to fix g and do the integration for g'. The first observation is that we only need to consider integration with respect to c'. Next we can integrate for those c' such that $p^t \mid b + \frac{1}{c''}$. We divide the problem into four cases according to whether $\varpi_v^t \mid c$ and whether $\varpi_v^t \mid b$. In any case, it is not difficult to check that the integration is 0 since $\mathrm{cond}(\chi_1\chi_2^{-1}) = (\varpi_v^t)$. We leave the verification for $g \in K_1\sigma_n K_1$ and $K_1 w\sigma w K_1$ to the reader. $\square$

Lemma 8.14. *Suppose* $\mathrm{cond}(\chi_1\chi_2^{-1}) = (\varpi_v^t)$, $t > 0$ *and* ϑ *is a character with conductor* (ϖ_v^s), $s > t$. *Define*

$$f = f_{\chi,\vartheta}(k_v) = \begin{cases} \chi_1(a)\chi_2(d)\vartheta(c/\varpi_v^t) & a, d \in \mathbb{Z}_v^\times, b \in \mathbb{Z}_v, c \in \varpi_v^t \mathbb{Z}_v^\times, \\ 0 & \textit{otherwise}, \end{cases}$$

for $k_v = \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right)$. *We similarly define*

$$\tilde{f} = \tilde{f}_{\chi^{-1},\vartheta^{-1}}.$$

Then on $\cup_n K_1\sigma_n K_1 \cup_n K_1 w\sigma_n w K_1$, $\Phi_{f,\tilde{f}}(g)$ *is supported in* K_1. *Moreover if* $g \in K_{t+s}$ *with* $\Phi_{f,\tilde{f}}(g) \neq 0$, *then its upper right entry is divisible by* ϖ_v^{s-t}, *and* $\Phi_{f,\tilde{f}}(g) = (q_v - 1)q_v^{-t}\,\mathrm{Vol}(K_1)$.

Proof. Suppose $\Phi_{f_{\chi,\vartheta},\tilde{f}_{\chi^{-1},\vartheta^{-1}}}(g) \neq 0$. In the following proof we write Φ for $\Phi_{f_{\chi,\vartheta},\tilde{f}_{\chi^{-1},\vartheta^{-1}}}$ for short. If $g \in K_1\left(\begin{smallmatrix} \varpi_v^n & \\ & 1 \end{smallmatrix}\right)K_1$ for some $n \geq 0$, then as before we have $g \in \left(\begin{smallmatrix} \varpi_v^n & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \varpi_v^n \mathbb{Z}_v \\ & 1 \end{smallmatrix}\right)K_t$. For $g' \in K_t$, write $g' = \left(\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c' & 1 \end{smallmatrix}\right)$. Without loss of generality assume

$$g = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$$

for $b \in \mathbb{Z}_v$, $\varpi_v^t \mid c$. Then

$$g'g = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix} \begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}.$$

Plugging in the formula for matrix coefficients we need to integrate for a', b', c', d'. Again we only need to consider integral with respect to $c' \in \varpi_v^t \mathbb{Z}_v$. Thus,

$$\begin{pmatrix} 1 & \\ c' & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & b/(c'b+1) \\ & 1 \end{pmatrix}\begin{pmatrix} 1/(bc'+1) & \\ & bc'+1 \end{pmatrix}\begin{pmatrix} \varpi_v^n & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ c'\varpi_v^n/(c'b+1) & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}.$$

If $n \geq 1$, then the integral is 0. If $n = 0$ and $\Phi(g) \neq 0$, then $g \in K_t$. Suppose $g = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right)$ with c divisible by ϖ_v^{s+t} then we easily see that b is divisible by ϖ_v^{s-t} and $\Phi(g)$ is given as in the lemma. For $g \in K_1 w \left(\begin{smallmatrix} \varpi_v^n & \\ & 1 \end{smallmatrix}\right) w K_1$ again if $\Phi(g) \neq 0$ then $g \in \left(\begin{smallmatrix} 1 & \\ & \varpi_v^n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ \varpi_v^n \mathbb{Z}_v & 1 \end{smallmatrix}\right) K_t$. Without loss of generality write $g \in \left(\begin{smallmatrix} 1 & \\ & \varpi_v^n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right)$ for $c \in \varpi_v^{t-n}\mathbb{Z}_v$, $K_t \ni g' = \left(\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c' & 1 \end{smallmatrix}\right)$, $g'g = \left(\begin{smallmatrix} a' & b' \\ & d' \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ & \varpi_v^n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c'/(\varpi_v^n)+c & 1 \end{smallmatrix}\right)$. If $n \geq 1$, then one can check that the integration is again 0. □

Suppose $\chi_1^{-1}\chi_2$ is unramified and ϑ has conductor (ϖ_v^s), $s > 0$, then we define $f_{\chi,\vartheta} \in \pi$ by

$$f_{\chi,\vartheta}(k_v) = \begin{cases} \chi_1(a)\chi_2(d)\vartheta(c/\varpi_v) & g = \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) \in B(\mathbb{Z}_v),\ c \in \varpi_v \mathbb{Z}_v^\times, \\ 0 & \text{otherwise.} \end{cases}$$

We define similarly $\tilde{f}_{\tilde{\chi},\tilde{\vartheta}} \in \tilde{\pi}$ by replacing χ, ϑ by $\chi^{-1}, \vartheta^{-1}$.

Lemma 8.15. *Write $f = f_{\chi,\vartheta}$, $\tilde{f} = \tilde{f}_{\tilde{\chi},\tilde{\vartheta}} \in \tilde{\pi}$ then on $\sqcup_n K_0\sigma_n K_0 \sqcup_n K_0 w \sigma_n w K_0$ it is supported in K_1. Moreover if $g \in K_{s+1}$ and $\Phi_{f,\tilde{f}}(g) \neq 0$, then the upper right entry of it is divisible by ϖ_v^{s+1}, and*

$$\Phi_{f,\tilde{f}}(g) = \frac{q_v - 1}{q_v}\,\mathrm{Vol}(K_1)$$

Proof. Similar to the above lemma. □

Lemma 8.16. *Suppose χ_1 and χ_2 are both unramified. Let f^{sph} be the spherical vector which takes value 1 at identity in the model above. Then*

$$\sum_{a \in \varpi_v\mathbb{Z}_v/(\varpi_v^{1+s}\mathbb{Z}_v)} \vartheta_v\Big(-\frac{a}{\varpi_v}\Big)\pi\left(\begin{pmatrix} 1 & \\ a & 1 \end{pmatrix}\begin{pmatrix} \varpi_v^{-s} & \\ & 1 \end{pmatrix}\right)$$
$$\times (1 - q_v\chi_1/\chi_2(\varpi_v))^{-1}(1 - q_v^{-1/2}\chi_2^{-1}(\varpi_v)\pi_v\left(\begin{pmatrix} \varpi_v & \\ & 1 \end{pmatrix}\right) f^{\mathrm{sph}}$$

equals $\chi_1(\varpi_v^{-s})q_v^{-s/2} f_{\chi,\vartheta}$.

Proof. Straightforward computations. □

Next we evaluate the local triple product integral for certain sections. The following lemma follows from the lemmas above.

Lemma 8.17. *Let $\chi_{f,1}, \chi_{f,2}, \chi_{\theta,1}, \chi_{\theta,2}, \chi_{h,1}, \chi_{\theta,2}, \vartheta$ be characters of $\mathbb{Q}_v^\times$ and $t_1 < s < t_2$ be nonnegative integers such that if $t_1 \neq 0$ then $t_1 + s = t_2$, and if $t_1 = 0$ then $s + 1 = t_2$. Suppose* $\mathrm{cond}(\chi_{f,1}\chi_{f,2}^{-1}) = (\varpi_v^{t_1})$ *and* $\mathrm{cond}(\vartheta) = (\varpi_v)^s$ *and* $\mathrm{cond}(\chi_{\theta,1}\chi_{\theta,2}^{-1}) = \mathrm{cond}(\chi_{h,1}\chi_{h,2}^{-1}) = (\varpi_v^{t_2})$. *Assume*: $\chi_{f,1}\cdot\chi_{\theta,1}\cdot\chi_{h,1}\cdot\vartheta = 1$ *and*

$\chi_{f,2}\cdot\chi_{\theta,2}\cdot\chi_{h,2}\cdot\vartheta^{-1}=1$. *We also define* $f_{\chi_f,\vartheta}\in\pi(\chi_{f,1},\chi_{f,2})$, $f_{\chi_\theta}\in\pi(\chi_{\theta,1},\chi_{\theta,2})$, $f_{\chi_h}\in\pi(\chi_{h,1},\chi_{h,2})$ *as above. Similarly for* $\tilde{f}_{\tilde{\chi}_f,\tilde{\vartheta}}$, $\tilde{f}_{\tilde{\chi}_\theta}$, $\tilde{f}_{\tilde{\chi}_h}$. *Then Ichino's local triple product is*

$$I_v(f_{\chi_f,\vartheta}\otimes f_{\chi_\theta}\otimes f_{\chi_h},\tilde{f}_{\tilde{\chi}_f,\tilde{\vartheta}}\otimes\tilde{f}_{\tilde{\chi}_\theta}\otimes\tilde{f}_{\tilde{\chi}_h})=\frac{(q_v-1)^2}{q_v^{2s+1}}\operatorname{Vol}(K_1)^2\operatorname{Vol}(K_{t_2})^2.$$

(*In this lemma, the* χ_h, χ_θ *are defined using* $\chi_{h,1}$, $\chi_{h,2}$, $\chi_{\theta,1}$, $\chi_{\theta,2}$ *similarly as in Lemma 8.13.*)

Special representations. We consider the induced representation $\pi(\chi_1,\chi_2)=\{f:K_v\to K,\ f(qk)=\chi_1(a)\chi_2(d)\delta_B(q),\ q=\left(\begin{smallmatrix}a&b\\&d\end{smallmatrix}\right)\in B(\mathbb{Z}_v)\}$ where $\chi_1=\chi_2|\cdot|$. The special representation $\sigma(\chi_1,\chi_2)\subset\pi(\chi_1,\chi_2)$ consists of functions f such that $\int_K f(k)\,dk=0$. We consider the case when π_3 is the special representation $\sigma(\chi_{v,1},\chi_{v,2})\subset\operatorname{Ind}_{B(\mathbb{Q}_v)}^{\mathrm{GL}_2(\mathbb{Q}_v)}(\chi_{v,1},\chi_{v,2})$ at v with a square-free conductor. Here $\chi_{v,i}$ are unramified characters. Similar to the unramified principal series case, we use the model of induced representations. It is easy to see that the $f_{\chi,\vartheta}$ defined above is inside $\sigma(\chi_{1,v},\chi_{2,v})$. Note that in the model for $\pi(\chi_1^{-1},\chi_2^{-1})$, there is a one-dimensional subrepresentation and the quotient is $\sigma(\chi_{1,v}^{-1}),\chi_{2,v}^{-1})$. The inner product of $\sigma(\chi_{1,v},\chi_{2,v})$ and $\sigma(\chi_{1,v}^{-1},\chi_{2,v}^{-1})$ is still given by $\langle v_1,v_2\rangle=\int_K v_1(k)v_2(k)\,dk$. The formula for the triple product integral is the same as the one in the case of principal series representations. Let $f_{\mathrm{new},v}\in\sigma(\chi_1,\chi_2)$ be the f such that $f(k)=q_v$ for $k\in K_1$ and $f(k)=-1$ otherwise. Clearly it is the new vector in the special representation. Then we have the following lemma:

Lemma 8.18. *We have*

$$\sum_{a\in\varpi_v\mathbb{Z}_v/(\varpi_v^{1+s}\mathbb{Z}_v)}\vartheta_v\Big(-\frac{a}{\varpi_v}\Big)\pi\left(\begin{pmatrix}1&\\a&1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s}&\\&1\end{pmatrix}\right)f_{\mathrm{new},v}$$

is $\chi_1(\varpi_v^{-s})q_v^{1-s/2}\cdot f_{\chi,\vartheta}$ *where* $f_{\chi,\vartheta}$ *is defined above.*

Proof. Let f_0 and f_1 be the characteristic functions on K_1 and K_1wK_1. Then $f_{\mathrm{new},v}=q_vf_0-f_1$. A computation shows that

$$\sum_{a\in\varpi_v\mathbb{Z}_v/(\varpi_v^{1+s}\mathbb{Z}_v)}\vartheta_v\Big(-\frac{a}{\varpi_v}\Big)\pi\left(\begin{pmatrix}1&\\a&1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s}&\\&1\end{pmatrix}\right)f_0=\chi_1(\varpi_v^{-s})q_v^{-s/2}f_\vartheta,$$

$$\sum_{a\in\varpi_v\mathbb{Z}_v/(\varpi_v^{1+s}\mathbb{Z}_v)}\vartheta_v\Big(-\frac{a}{\varpi_v}\Big)\pi\left(\begin{pmatrix}1&\\a&1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s}&\\&1\end{pmatrix}\right)f_1=0.$$

The lemma follows. □

Remark 8.19. The reason why the local integrals at split primes showing up in the triple product formula later on are the ones considered in this subsection is a consequence of the computations in Section 6G.

Now we consider nonsplit primes.

Nonsplit case 1. The $\mathrm{U}(2)(\mathbb{Q}_v)$ is compact. This case is easier since we are in the representation theory for finite groups. By construction the representation $\pi_{f,v}^D$ is one dimensional. Recall our construction for $\boldsymbol{h}$ in Section 8B. For such v, we let $g_{2,v}$ and $g_{4,v}$ be either the identity or α there depending on whether $\pi'_{h_z,v}$

is $\pi^\vee_{\theta_{\eta_z},v}\otimes\eta_z$ or ${}^\alpha\pi^\vee_{\theta_{\eta_z},v}\otimes\eta_z$. Let $g_{1,v}$ and $g_{3,v}$ are the identify elements. By our assumptions on $\chi_{\theta,v}$, $\chi_{h,v}$ and $\pi_{f,v}$ and our chosen vectors $h,\theta,\tilde{\theta}_3,\tilde{h}_3$ and that $\pi^D_{f,v}$ is one-dimensional, it is easily checked from the construction (note that by the choices in Section 8C, the local triple product root number at v is -1)

$$\pi_{h^D,v}\simeq\pi^\vee_{\theta^D,v}\otimes\pi^{D,\vee}_{f,v}.$$

We conclude that by the inner product formula of matrix coefficients of representations of finite groups

$$\frac{I_v(\pi_{h,v}(g_{2,v})h^D_v\otimes v^D_{\phi_v}\otimes\pi_{f,v}(g_{1,v})f^D_{\vartheta,v},\tilde{\pi}_{h,v}(g_{4,v})\tilde{h}^D_v\otimes\tilde{v}^D_{\phi_v}\otimes\tilde{\pi}_{f,v}(g_{3,v})\tilde{f}^D_{\tilde{\vartheta},v})}{\langle\pi_{h,v}(g_{2,v})h^D_v,\tilde{\pi}_{h,v}(g_{4,v})\tilde{h}^D_v\rangle\langle v^D_{\phi_v},\tilde{v}^D_{\phi_v}\rangle\langle\pi_{f,v}(g_{1,v})f^D_{\vartheta,v},\tilde{\pi}_{f,v}(g_{3,v})\tilde{f}^D_{\tilde{\vartheta},v}\rangle}=\frac{1}{d_{\pi_{h,v}}}.$$

where $d_{\pi_{h,v}}$ is the dimension of the representation $\pi_{h,v}$. When we are moving our datum p-adic analytically, this integral is not going to change.

Nonsplit case 2. The $\mathrm{U}(2)(\mathbb{Q}_v)$ is not compact. Recall that we fix a generic arithmetic point. By [Prasad 1990, Theorem 1.4] we know that there are

$$g_{1,v},g_{2,v},g_{3,v},g_{4,v}\in D^\times(\mathbb{Q}_v)\subset\mathrm{GU}(2)(\mathbb{Q}_v)$$

such that

$$I_v(\pi_{h,v}(g_{2,v})h^D_v\otimes v^D_{\phi_v}\otimes\pi_{f,v}(g_{1,v})f^D_{\vartheta,v},\tilde{\pi}_{h,v}(g_{4,v})\tilde{h}^D_v\otimes\tilde{v}^D_{\phi_v}\otimes\tilde{\pi}_{f,v}(g_{3,v})\tilde{f}^D_{\tilde{\vartheta},v})\neq 0.$$

(We write $v^D_{\phi_v}$ for the image of v_{ϕ_v} in the corresponding $D^\times(\mathbb{Q}_v)$ representation and similarly for $\tilde{v}^D_{\phi_v}$ (notations as in Lemma 8.2. To apply Prasad's result, note that the local sign for this triple product is $+1$ by our choices of high conductors.)

Definition 8.20. We define

$$g_i=\prod_v g_{i,v}$$

for $i=1,2,3,4$. (We take $g_{i,v}=1$ if v is split in $\mathcal{K}/\mathbb{Q}$.)

The local triple product integrals are nonzero by our computations. By Ichino's formula and our requirement on special L-values (note that the product of the central characters for f_z,θ_z,h_z is trivial by construction) the global trilinear form is also nonzero. So by our definitions for h^D_z, θ^D_z, etcetera in Section 8A, we know that $\prod_v g_{2,v}$ has to be in $\breve{D}^\times$. Thus up to a nonzero constant fixed throughout the family, we have

$$\int_{[D^\times]}(\pi(g_2)h^D_z)(g)\theta^D_z(g)(\pi(g_1)f_{z,\vartheta})(g)\,dg=\int_{[\mathrm{U}(2)]}(\pi(g_2)h_z)(g)\theta_z(g)(\pi(g_1)f_{z,\vartheta})(g)\,dg.$$

We have similarly $\prod_v g_{4,v}\in\breve{D}^\times$ and up to a nonzero constant fixed throughout the family,

$$\int_{[D^\times]}(\pi(g_4)\tilde{h}^D_{3,z})(g)(\pi_{f_z}(g_3)\tilde{f}_{z,\tilde{\vartheta}})(g)\tilde{\theta}^D_{3,z,\mathrm{low}}(g)\,dg=\int_{[\mathrm{U}(2)]}(\pi(g_4)\tilde{h}_{3,z})(g)(\pi_{f_z}(g_3)\tilde{f}_{z,\tilde{\vartheta}})(g)\tilde{\theta}_{3,z,\mathrm{low}}(g)\,dg.$$

These $g_{i,v}$ are chosen only at the arithmetic point z. We fix them when moving the Eisenstein datum in p-adic families.

Lemma 8.21. *Let v be a nonsplit prime. Then for different arithmetic points* z *with fixed* $\mathrm{z}|_{\mathbb{I}}$, *the* $\pi^D_{h,\mathrm{z},v}$ *and* $\pi^D_{\theta,\mathrm{z},v}$ *only differ by twisting by unramified characters which are inverse to each other.*

Proof. In fact, at nonsplit primes, the local Weil representations on unitary groups are unchanged throughout the family since the characters $\chi_{\theta,\mathrm{z},v}|_{\mathrm{U}(1)}$ are unchanged. (They differ by multiplying by unramified characters and $\mathrm{U}(1)(\mathbb{Q}_v)$ is compact.) So for different z and z′, the difference between $\pi^D_{h,\mathrm{z},v}$ and $\pi^D_{h,\mathrm{z}',v}$ only comes from the characters extending the form on U(2) to GU(2), and similarly for $\pi^D_{\theta,\mathrm{z},v}$. Note moreover that from construction the product of the central characters of $\pi^D_{h,\mathrm{z},v}$, $\pi^D_{\theta,\mathrm{z},v}$ and π_{f_z} is trivial. The lemma thus follows. □

Thus if the theta kernel we used to define h_z and θ_z are fixed, then the local triple product integrals

$$\frac{I_v\big(\pi_{h,\mathrm{z},v}(g_{2,v})h^D_{\mathrm{z},v}\otimes v^D_{\phi_v}\otimes\pi_{f_\mathrm{z},v}(g_{1,v})f^D_{\mathrm{z},\vartheta,v},\ \tilde{\pi}_{h,\mathrm{z},v}(g_{4,v})\tilde{h}^D_{\mathrm{z},v}\otimes\tilde{v}^D_{\phi_v}\otimes\tilde{\pi}_{f_\mathrm{z},v}(g_{3,v})\tilde{f}^D_{\mathrm{z},\tilde{\vartheta},v}\big)}{\langle h^D_{\mathrm{z},v},\tilde{h}^D_{3,\mathrm{z},v}\rangle\langle v^D_1,\tilde{v}^D_1\rangle\langle f^D_{\mathrm{z},\vartheta,v},\tilde{f}^D_{\mathrm{z},\tilde{\vartheta},v}\rangle}$$

does not change (note that the $\langle f^D_{\mathrm{z},\vartheta,v},\tilde{f}^D_{\mathrm{z},\tilde{\vartheta},v}\rangle$ has a nonzero inner product). This observation is crucial in proving Proposition 8.29 later on.

We now define $\mathbb{I}$-adic families from the family $\boldsymbol{f}$ defined in Section 7A using Lemma 8.16 and 8.18. However, in order to do so we may have to replace $\mathbb{I}$ by a larger normal domain finite over $\mathbb{I}$, so that the $\chi_{1,v}(\varpi_v)$ and $\chi_{2,v}(\varpi_v)$'s at primes in Σ where π_f is unramified will be elements of this newly defined $\mathbb{I}$.

Definition 8.22. We define $\boldsymbol{f}_\vartheta$ as

$$\prod_v\sum_{a\in\frac{\varpi_v\mathbb{Z}_v}{\varpi_v^{1+s}\mathbb{Z}_v}}\vartheta_v\Big(-\frac{a}{\varpi_v}\Big)\pi\left(\begin{pmatrix}1&\\a&1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s}&\\&1\end{pmatrix}\right)(1-q_v\chi_1/\chi_2(\varpi_v))^{-1}(1-q_v^{-1/2}\chi_2^{-1}(\varpi_v)\pi_v\left(\begin{pmatrix}\varpi_v&\\&1\end{pmatrix}\right)$$
$$\times\prod_v\sum_{a\in\frac{\varpi_v\mathbb{Z}_v}{\varpi_v^{1+s}\mathbb{Z}_v}}\vartheta_v\Big(-\frac{a}{\varpi_v}\Big)\pi\left(\begin{pmatrix}1&\\a&1\end{pmatrix}\begin{pmatrix}\varpi_v^{-s}&\\&1\end{pmatrix}\right)\boldsymbol{f}.$$

We define $\tilde{\boldsymbol{f}}=\boldsymbol{f}\otimes\boldsymbol{\chi}_{\boldsymbol{f}}^{-1}$ and similarly define $\tilde{\boldsymbol{f}}_{\tilde{\vartheta}}$.

We also denote its specialization at an arithmetic point $\mathrm{z}\in\operatorname{Spec}\Lambda''_{\boldsymbol{D}}(\bar{\mathbb{Q}}_p)$ by $f_{\mathrm{z},\vartheta}$.

8E. *Evaluating the integral.* Recall that we have defined a theta function $\theta^\star$ in Corollary 6.47 (we suppress the subscript $\mathcal{D}$ as it is fixed throughout the family). Now we construct: for any $F\in\mathcal{M}_{\mathrm{ord}}(K^{(3,1)},\Lambda''_{\boldsymbol{D}})$ (weight map as in Section 7B),

$$l_{\theta^\star}(F):=l'_{\theta^\star}a^1_{[1]}\left(1,\prod_{v\in\Sigma^1\cup\Sigma^2}\Big(\sum_i C_{v,i}\rho\left(\begin{pmatrix}&&u_{v,i}\\&I_2&\\u_{v,i}&&\end{pmatrix}\right)\Big)(F)\right)$$

whose value is in $\mathcal{M}(K^{(2,0)},\Lambda_{2,0})\otimes_{j_2}\Lambda''_{\boldsymbol{D}}$. Here $C_{v,i}$ are defined in Lemma 6.26. The $a^1_{[1]}$ is the family version of the Fourier–Jacobi expansion in the sense of Definition 3.7, and $l'_{\theta^\star}$ is applied to the theta function part of the Fourier–Jacobi expansion as in (4-4). See the proof of Proposition 7.9. Note that we can easily make sure that $\theta^\star$ is a member in the basis θ_i''s there. Recall that for the specialization F_z of F,

$$l'_{\theta^\star}a^1_{[1]}(1, F_z)(h)) := \int_{[W]} a^1_{[1]}(1, F_z(wh))\theta^\star(wh)\, dw$$

with the Heisenberg group $W \hookrightarrow \mathrm{U}(3, 1)$

$$w \mapsto \begin{pmatrix} 1 & w & \frac{1}{2}\langle w, w\rangle \\ & 1 & \zeta w^* \\ & & 1 \end{pmatrix}.$$

It is clear that $l'_{\theta^\star}\,\mathrm{FJ}_\beta(F)(h)$ is an automorphic form on $\mathrm{U}(2)$. We can also define $l_{\theta^\star}$ on a single form on $\mathrm{U}(3, 1)$ instead of on families, using the same formula. It is clear that

$$z(l_{\theta^\star}(F)) = l_{\theta^\star}(F_z).$$

Lemma 8.23. *The $l_{\theta^\star}(F) \in \mathcal{M}(K^{(2,0)}, \Lambda_{2,0}) \otimes_{j_2} (\Lambda''_{\boldsymbol{D}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$.*

Proof. Recall that in Definition 3.7 there is a $\Lambda_{\mathcal{D}}$-adic Fourier–Jacobi expansion for families on $\mathrm{U}(3, 1)$. As before we just take a basis of $\mathcal{O}_L$-dual space $(\theta_1', \dots, \theta_m')$ of the finite-dimensional space $H^0(\mathcal{B}, \mathcal{L}(\beta))$. Pairing the $\Lambda''_{\boldsymbol{D}}$-adic Fourier–Jacobi coefficient of F with these θ_i' we get a $\Lambda''_{\boldsymbol{D}}$-adic family. But our $l'_{\theta^\star}$ is in the L-linear combination of the θ_i''s. Thus we get a $\Lambda''_{\boldsymbol{D}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$-adic family on $\mathrm{U}(2)$. □

We also define

$$\boldsymbol{B}_1 := \boldsymbol{B}_{K^{(2,0)}}\langle e_{\mathrm{ord}} l_{\theta^\star}(\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}), \pi(g_2)\boldsymbol{h}\rangle. \tag{8-7}$$

Definition 8.24. For any nearly ordinary form f or family $\boldsymbol{f}$ (we use the same notations for $\boldsymbol{h}, \boldsymbol{\theta}$, etcetera) we define $f_{\mathrm{low}}(g) := f\big(g\big(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\big)_p\big)$(under the identification $D_p^\times \simeq \mathrm{GL}_2(\mathbb{Q}_p)$ given by the v_0 projection). Also, if $\boldsymbol{\chi}$ is the (family of) central characters of $\boldsymbol{f}$ we define $\tilde{\boldsymbol{f}} = \boldsymbol{f}.(\boldsymbol{\chi}^{-1} \circ \mathrm{Nm})$. We also define $f^{ss}(g) = f(g\big(\begin{smallmatrix} & 1 \\ p^t & \end{smallmatrix}\big)_p)$.

We will compute the specializations

$$z(\boldsymbol{B}_1) = \langle l_{\theta^\star}(\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling},z}), \pi(g_2)h_{z,\mathrm{low}}\rangle\,\mathrm{Vol}(K_z)^{-1}.$$

(The K_z is the level group at the arithmetic point z. Note that the tame level group is fixed at the end of Section 6 throughout).

We can evaluate this expression by Ichino's formula for triple products.

We have the following

Proposition 8.25. *We use the notations of Section 8B and Definition 8.22. Then there is a $\mathcal{C}$, the product of a constant in $\overline{\mathbb{Q}}_p^\times$ which is fixed along the family and a unit in $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma''_{\mathcal{K}}]]$ (precise definition is given in the* *following proof), such that for any generic arithmetic point* z *of conductor* p^t,

$$\mathrm{z}(\boldsymbol{B}_1)=\mathcal{C}_{\mathrm{z}}p^tC^D_{\mathrm{U}(2)}\mathrm{z}(\mathcal{L}_5)\mathrm{z}(\mathcal{L}_6)\times\left(\int_{\mathbb{A}^\times_{\mathbb{Q}}D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})}\pi(g_2)h^D_{\mathrm{z}}(g)\pi(g_1)f_{\mathrm{z},\vartheta}(g)\theta^D_{\mathrm{z,low}}(g)\,dg\right)$$

where $\mathcal{L}_5$ *and* $\mathcal{L}_6$ *are Katz p-adic L-functions in* $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma''_{\mathcal{K}}]]$ *interpolating the L-values*

$$\Omega_p^\kappa\frac{2\pi i\Gamma(\kappa-1)}{\Omega^\kappa_\infty\mathfrak{g}(\tau_{\mathrm{z},1,p})}L\left(\chi_{\mathrm{aux}}\tau_{\mathrm{z}}^{-c},\frac{\kappa-2}{2}\right)\cdot p^{t(\kappa-1)}\cdot(\chi_{\mathrm{aux},1,p}\tau_{\mathrm{z},1,p}(p)p^{-\kappa/2})^t$$

and

$$\Omega_p^{\kappa-2}\frac{\Gamma(\kappa-2)}{\Omega^{\kappa-2}_\infty\mathfrak{g}(\tau_{\mathrm{z},1,p}\chi_{\theta,\mathrm{z},2}\chi^{-1}_{\theta,\mathrm{z},1})}\\ \times L\left(\lambda^2\chi_{\theta,\mathrm{z}}^{-c}\chi_{\theta,\mathrm{z}}\chi_{\mathrm{aux}}\tau_{\mathrm{z}}^{-1},\frac{\kappa-2}{2}\right)p^{t(\kappa-2)}(\tau_{\mathrm{z},1,p}\chi^{-1}_{\theta,\mathrm{z},1,p}\chi_{\theta,\mathrm{z},2,p}\lambda^2_{1,p}(p)p^{-\kappa-2/2})^t$$

respectively.

Remark 8.26. Although our coefficient ring is $\Lambda''_{\boldsymbol{D}}$, however in fact the functional does take values in the subring $\mathbb{I}[[\Gamma''_{\mathcal{K}}]]$. Indeed the additional variable $\Gamma^-_{\mathcal{K}}$ of $\Lambda''_{\boldsymbol{D}}$ corresponds to twisting the Klingen–Eisenstein family by p-adic anticyclotomic characters.

Proof. We first remark that it indeed makes sense to talk about the Katz p-adic L-functions $\mathcal{L}_5$ and $\mathcal{L}_6$ interpolating those values, since the characters τ_{z} and $\chi_{\theta,\mathrm{z}}$ are indeed interpolated as characters with values in $\mathbb{I}[[\Gamma''_{\mathcal{K}}]]$.

By Corollary 6.47, we have $\mathrm{z}(B_1)$ equals

$$\frac{p^{2t}}{\Omega^{2\kappa-1}_\infty}\times\int_{[\mathrm{U}(2)]\times[\mathrm{U}(2)]}\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}E_{\mathrm{sieg},\boldsymbol{D}_{\mathrm{z}},2}\left(u_1,u_2\begin{pmatrix}1&\\n&1\end{pmatrix}_p\right)(\pi(g_2)h_{\mathrm{z,low}})(u_1)\theta_{2,\boldsymbol{D}_{\mathrm{z}}}\left(u_2\begin{pmatrix}1&\\n&1\end{pmatrix}_p\right)\\ \times(\pi(g_1)f_{\mathrm{z},\vartheta})(u_2)\,du_1du_2.$$

We integrate over u_1 first and factorize the $E_{\mathrm{sieg},\mathcal{D}_{\mathrm{z}},2}$ via the embedding $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ of $E_{\mathrm{sieg},\mathcal{D}_{\mathrm{z}},2}$. By the local pullback formulas we computed in Section 6I, we get

$$\mathcal{C}_{\mathrm{z}}p^tC^D_{\mathrm{U}(2)}\mathrm{z}(\mathcal{L}_5)\mathrm{z}(\mathcal{L}_6)\times\left(\int_{\mathbb{A}^\times_{\mathbb{Q}}D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})}\pi(g_2)h^D_{\mathrm{z}}(g)\pi(g_1)f_{\mathrm{z},\vartheta}(g)\theta^D_{\phi_2}(g)\bar{\lambda}(\det g)\,dg\right)$$

where $\mathcal{C}_{\mathrm{z}}$ is the product of the local pullback integrals in Section 6I (and also the local pullback computations in Sections 6E through 6G) at primes outside p, local Euler factors at p for $\mathcal{L}_5$, $\mathcal{L}_6$ (see [Hsieh 2014b, (4.16)] for the Euler factor at p), Euler factor for $\mathcal{L}_5$ at primes in Σ^2 and Euler factors for $\mathcal{L}_6$ at p. The first is a fixed constant which does not move with z, the rest are interpolated by units in the Iwasawa algebra by our choice of characters in Section 8C. Thus the $\mathcal{C}_{\mathrm{z}}$ are clearly interpolated by an element $\mathcal{C}$ mentioned in the proposition. Note also that the Euler factors at other places are trivial.

We are now ready to deduce the expression in the proposition. Although the θ_{z} part appearing above is $\theta_{\phi_2,\mathrm{z}}\otimes\bar{\lambda}$ constructed in Section 6 (not an eigenform), however, in view of the central character of h_{z}

and f_{z}, only the eigencomponent θ_{z} of $\theta_{\phi_2,\mathrm{z}}\otimes\bar{\lambda}$ with the correct central character matters. Also by the construction this θ_{z} part is a multiple of θ_{z}^{ss} which, after applying the operator $\sum_{n\in\mathbb{Z}_p/p^t\mathbb{Z}_p}\pi_{\theta,\mathrm{z}}\left(\left(\begin{smallmatrix}1&\\n&1\end{smallmatrix}\right)_p\right)$ is $\theta_{\mathrm{z},\mathrm{low}}$ by construction (see Proposition 8.3). So by considering the local pairing between $\pi_{\theta_{\mathrm{z}},p}$ and $\pi^\vee_{\theta_{\mathrm{z}},p}$, it is easy to see that if we replace this multiple of θ_{z}^{ss} (see Definition 8.24) by $\theta_{\mathrm{z},\mathrm{low}}$ we do not change the whole integral. Thus we finally arrived at

$$\mathcal{C}_{\mathrm{z}}p^tC^D_{\mathrm{U}(2)}\mathrm{z}(\mathcal{L}_5)\mathrm{z}(\mathcal{L}_6)\times\left(\int_{\mathbb{A}_{\mathbb{Q}}^\times D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_{\mathbb{Q}})}\pi(g_2)h^D_{\mathrm{z}}(g)\pi(g_1)f_{\mathrm{z},\vartheta}(g)\theta^D_{\mathrm{z},\mathrm{low}}(g)\,dg\right).\qquad\square$$

By our choices for characters, the corresponding non-Σ-primitive p-adic L-functions $\mathcal{L}_5$ and $\mathcal{L}_6$ are units in $\Lambda''_{\boldsymbol{D}}$.

In the following, we often omit the superscript D for simplicity. Up to a constant in $\overline{\mathbb{Q}}_p^\times$ (which does not change along the family) we have

$$\begin{aligned}\mathrm{z}(\boldsymbol{B}_1)\cdot p^t&\left(\int(\pi(g_4)\tilde{h}_{3,\mathrm{z}})(g)(\pi_{f_{\mathrm{z}}}(g_3)\tilde{f}_{\mathrm{z},\tilde{\vartheta}})(g)\tilde{\theta}_{3,\mathrm{z},\mathrm{low}}(g)\,dg\right)\\&=(\lambda_{p,2}\chi_{\theta,\mathrm{z},2})^{-t}(p)(\chi_{\theta,\mathrm{z}1}\lambda_{p,1})^t(p)p^{3t}\left(\int(\pi(g_2)h_{\mathrm{z}})(g)(\pi_{f_{\mathrm{z}}}(g_1)f_{\mathrm{z},\vartheta})(g)\theta^{ss}_{\mathrm{z}}(g)\,dg\right)\\&\qquad\times\left(\int(\pi_{\tilde{h}_{\mathrm{z}}}(g_4)\tilde{h}_{3,\mathrm{z}})(g)(\pi_{\tilde{f}_{\mathrm{z}}}(g_3)\tilde{f}_{\mathrm{z},\tilde{\vartheta}})(g)\tilde{\theta}^{ss}_{3,\mathrm{z}}(g)\,dg\right)\times\mathrm{z}(\mathcal{L}_5^\Sigma\mathcal{L}_6^\Sigma)\\&=\lambda_{p,2}^{-2t}(p)\chi_{\theta,\mathrm{z},2}^{-2t}(p)p^{3t}\left(\int(\pi(g_2)h_{\mathrm{z}})(g)(\pi_{f_{\mathrm{z}}}(g_1)f_{\mathrm{z},\vartheta})(g)\theta^{ss}_{\mathrm{z}}(g)\,dg\right)\\&\qquad\times\left(\int(\pi(g_4)\tilde{h}_{3,\mathrm{z}})^{ss}(g)(\pi_{\tilde{f}_{\mathrm{z}}}(g_3)\tilde{f}_{\mathrm{z},\tilde{\vartheta}})^{ss}(g)\tilde{\theta}_{3,\mathrm{z}}(g)\,dg\right)\times\mathrm{z}(\mathcal{L}_5^\Sigma\mathcal{L}_6^\Sigma).\end{aligned}\tag{8-8}$$

(Note that by our discussion in Section 7D, we know up to multiplying by an element in $\overline{\mathbb{Q}}_p^\times$, the

$$\left(p^t\int(\pi(g_4)\tilde{h}_{3,\mathrm{z}})(g)(\pi_{f_{\mathrm{z}}}(g_3)\tilde{f}_{\mathrm{z},\tilde{\vartheta}})(g)\tilde{\theta}_{3,\mathrm{z},\mathrm{low}}(g)\,dg\right)\tag{8-9}$$

is interpolated by an element

$$\boldsymbol{B}_{K^{(2,0)}}\langle\pi(\tilde{g_4})\boldsymbol{h}_3\cdot\pi_{f_{\mathrm{z}}}(g_3)\tilde{\boldsymbol{f}}_\vartheta,\tilde{\boldsymbol{\theta}}_3\rangle$$

in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!]$.)

Definition 8.27. We define $\chi_{f_{\mathrm{z}},s}$ and $\chi_{f_{\mathrm{z}},o}$ so that $\pi_{f_{\mathrm{z}},p}$ (the p-component of the automorphic representation π_f associated to f) is the principal series representation $\pi(\chi_{f_{\mathrm{z}},s},\chi_{f_{\mathrm{z}},o})$ where $\mathrm{val}_p(\chi_{f_{\mathrm{z}},s})=\frac{1}{2}$ and $\mathrm{val}_p(\chi_{f_{\mathrm{z}},o})=-\frac{1}{2}$, respectively.

We write the p-component of $\lambda^2\chi_{\theta,\mathrm{z}}\chi_{h,\mathrm{z}}$ as $((\lambda^2\chi_{\theta,\mathrm{z}}\chi_{h,\mathrm{z}})_1,(\lambda^2\chi_{\theta,\mathrm{z}}\chi_{h,\mathrm{z}})_2)$ and f_{z} the normalized GL_2 ordinary form new outside p. We also define the following: the p-adic L-function $\mathcal{L}_1$ such that for any generic z

$$\mathcal{L}_1(\mathrm{z})=\frac{2\pi i\mathfrak{g}(\chi_{\theta,\mathrm{z},2})\mathfrak{g}(\chi^{-1}_{h,\mathrm{z},1})L\big(f_{\mathrm{z}},\lambda^2\chi_{\theta,\mathrm{z}}\chi_{h,\mathrm{z}},\frac{1}{2}\big)(\chi_{f_{\mathrm{z}},s}(p)\chi_{f_{\mathrm{z}},o}(p))^{-t}((\lambda^2\chi_{\theta,\mathrm{z}}\chi_{h,\mathrm{z}})_2(p).p)^{-2t}p^t}{\Omega_\infty^4}\Omega_p^4;$$

the p-adic L-function $\mathcal{L}_2$ such that for any generic z

$$\mathcal{L}_2(\mathrm{z})=\frac{(1-a_p(f_{\mathrm{z}})^{-1}\chi_{\theta,\mathrm{z},p,1}\chi_{h,\mathrm{z},p,2}(p^{-1}))}{(1-a_p(f_{\mathrm{z}})\chi_{\theta,\mathrm{z},p,1}\chi_{h,\mathrm{z},p,2}(p)p^{-1})}\times\frac{\mathfrak{g}(\chi_{f_{\mathrm{z}}}^{-1})L^{(p)}\big(f_{\mathrm{z}},\chi_{\theta,\mathrm{z}}^{c}\chi_{h,\mathrm{z}},\tfrac{1}{2}\big)(\chi_{f_{\mathrm{z}},o}(p)p^{1/2}(\chi_{h_{\mathrm{z}}}\chi_{\theta,\mathrm{z}}^{c})_1(p))^{-t}}{(2\pi i)^2\big\langle f_{\mathrm{z}},f_{\mathrm{z}}^{c}\mid\big(\begin{smallmatrix}&-1\\N&\end{smallmatrix}\big)\big\rangle_{\Gamma_0(N)}};$$

the p-adic L-function $\mathcal{L}_3$ such that for any generic z,

$$\mathcal{L}_3(\mathrm{z})=\frac{\zeta_{\chi_{\mathcal{K}}}(1)}{\pi}\frac{\mathfrak{g}(\chi_{\theta,\mathrm{z}})L(\lambda^2\chi_{\theta,\mathrm{z}}\chi_{\theta,\mathrm{z}}^{-c},1)((\lambda^2\chi_{\theta,\mathrm{z}}\chi_{\theta,\mathrm{z}}^{-c})_2(p)p)^{-t}p^t}{\Omega_\infty^2}\Omega_p^2;$$

the p-adic L-function $\mathcal{L}_4$ such that for any generic z,

$$\mathcal{L}_4(\mathrm{z})=\frac{\zeta_{\chi_{\mathcal{K}}}(1)}{\pi}.\frac{\mathfrak{g}(\chi_{h,\mathrm{z},1}^{-1})L(\lambda^2\chi_{h,\mathrm{z}}\chi_{h,\mathrm{z}}^{-c},1)((\lambda^2\chi_{h,\mathrm{z}}\chi_{h,\mathrm{z}}^{-c})_2(p).p)^{-t}.p^t}{\Omega_\infty^2}\Omega_p^2.$$

We refer to [Hsieh 2014b; 2014c] for the justification of their interpolation formulas. These values are interpolated by some p-adic L-functions in $\hat{\mathbb{I}}_{\mathcal{K}}^{\mathrm{ur}}$. Note that by [Hida 1991, Lemma 5.3(vi)] and [Hida and Tilouine 1993, Theorem 7.1],

$$\frac{\mathfrak{g}(\chi_f^{-1})L(\mathrm{ad},f_{\mathrm{z}},1)M\varphi(M)(p-1)(\chi_{f_{\mathrm{z}},s}(p))^t}{2^3\pi^3W'(f_{\mathrm{z}})^{-1}\big\langle f_{\mathrm{z}},f_{\mathrm{z}}\mid\big(\begin{smallmatrix}&-1\\N&\end{smallmatrix}\big)\big\rangle_{\Gamma_0(N)}}=1$$

where $W'(f_{\mathrm{z}})$ is the prime to p part of the root number of f_{z}, which is an element in $\mathbb{I}^\times$ (i.e., a unit).

Note also that $\mathcal{L}_2$ is in fact a nonzero element in $\mathrm{Frac}(\mathbb{I})$ (i.e., does not depend on the variable $\Gamma_{\mathcal{K}}$) by checking the p-components of $\chi_{\theta,\mathrm{z}}^{c}\chi_{h,\mathrm{z}}$ and nonzero by our choice of characters in Section 8C. It can actually be written as the ratio of two elements whose specializations to all but finitely many generic arithmetic points are nonzero. By our choices for χ_θ and χ_h we know that $\mathcal{L}_1$ is in $\hat{\mathbb{I}}^{\mathrm{ur}}[[\Gamma_{\mathcal{K}}'']]^\times$. We consider the expression

$$C_{\mathrm{U}(2)}^{D}\frac{\langle f_{\mathrm{z}},\tilde{f}_{\mathrm{z}}^{ss}\rangle\langle h_{\mathrm{z}},\tilde{h}_{\mathrm{z}}^{ss}\rangle\langle\tilde{\theta}_{\mathrm{z}},\theta_{\mathrm{z}}^{ss}\rangle\chi_{\theta,\mathrm{z},1}^{-1}(p^t)\mathrm{z}(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_5\mathcal{L}_6)}{\mathrm{z}(\mathcal{L}_3\mathcal{L}_4)}(\chi_{f,\mathrm{z},s}(p)\lambda_{p,1}^2(p)\chi_{\theta,\mathrm{z},1}(p)\chi_{h,\mathrm{z},1}(p)p^{3/2})^tp^{3t}$$
$$=C_{\mathrm{U}(2)}^{D}\frac{\langle f_{\mathrm{z}},\tilde{f}_{\mathrm{z,low}}\rangle\langle h_{\mathrm{z}},\tilde{h}_{\mathrm{z,low}}\rangle\langle\theta_{\mathrm{z}},\tilde{\theta}_{\mathrm{z,low}}\rangle\mathrm{z}(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_5\mathcal{L}_6)}{\mathrm{z}(\mathcal{L}_3\mathcal{L}_4)}p^{3t}.$$

The above element is clearly interpolated by an element

$$\mathcal{G}\in\Lambda_{\boldsymbol{D}}'' \tag{8-10}$$

We first give the following lemma for the local triple product integral at p.

Lemma 8.28. *At a generic point* z, *the local triple product integral for the expression in* (8-8) *at* p *is given by*

$$\frac{I_p(\phi_p\otimes\tilde{\phi}_p)}{\langle\phi_p,\tilde{\phi}_p\rangle}=\frac{p^{-t}(1-p)}{1+p}\frac{1}{1-a_p(f_{\mathrm{z}})\chi_{\theta,\mathrm{z},p,1}\chi_{h,\mathrm{z},p,2}(p)p^{-1}}\cdot\frac{1}{1-a_p(f_{\mathrm{z}})^{-1}\chi_{\theta,\mathrm{z},p,2}\chi_{h,\mathrm{z},p,1}(p)}.$$

Proof. This follows from Lemma 8.13. □

We observe that by definition this expression is a nonzero element in $\mathrm{Frac}(\mathbb{I})$.

Thus we have the following proposition:

Proposition 8.29. *Any height* 1 *prime of* $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ *containing* $\boldsymbol{B}_{K^{(2,0)}}\langle l_{\theta^\star}(\boldsymbol{E}_{\mathrm{Kling},\boldsymbol{D}}), \pi(g_2')\boldsymbol{h}\rangle$ *must be the pullback of a height* 1 *prime of* $\hat{\mathbb{I}}^{\mathrm{ur}}$.

Proof. Let us recall what we have achieved so far. We have computed the Fourier–Jacobi coefficients of the Siegel–Eisenstein series in Proposition 6.44. Using pullback formulas and computations on theta functions, we further computed the $\theta^\star$-part of the β-th Fourier–Jacobi coefficient of our ordinary Klingen–Eisenstein series in Corollary 6.47. This is a function on the definite unitary group $\mathrm{U}(2)$. Moreover, we constructed an ordinary family $\boldsymbol{h}$ of CM forms on $\mathrm{U}(2)$ in Section 8B1. Then we form the pairing of this $\theta^\star$ Fourier–Jacobi coefficient with $\boldsymbol{h}$ (see the beginning of this subsection), and resulted an element $\boldsymbol{B}_1$ which is an element in $\Lambda''_{\boldsymbol{D}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ by its construction and Lemma 8.23. In Proposition 8.25 we used the doubling method for $\mathrm{U}(2)\times\mathrm{U}(2)\hookrightarrow\mathrm{U}(2,2)$ to obtain an expression for the specializations of $\boldsymbol{B}_1$ at arithmetic points z.

To prove the proposition, it is enough to show that the product of $\boldsymbol{B}_1$ and the element (8-9) in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!]$ satisfies the property stated in the proposition. We examine the ratio between (8-10) and the expression for the element interpolating

$$\frac{1}{C}p^t\mathrm{z}(\boldsymbol{B}_1)\biggl(\int(\pi(g_4)\tilde{h}_{3,\mathrm{z}})(g)(\pi_{f_{\mathrm{z}}}(g_3)\tilde{f}_{\mathrm{z}})(g)\tilde{\theta}_{3,\mathrm{z},\mathrm{low}}(g)\,dg\biggr) \tag{8-11}$$

using Ichino's formula. (Recall C here is the Tamagawa number for $D^\times$ appearing in Ichino's formula.) By our calculations for the local triple product integrals, and the Petersson inner product of $\langle\theta,\tilde{\theta}_3\rangle$ and $\langle h,\tilde{h}_3\rangle$, the ratio is a product of

- Euler factors for $\zeta_{\chi_{\mathcal{K}}}(1)$ at Σ;
- the local Euler factors for $L(ad, f_{\mathrm{z}}, 1)$ at $\Sigma\backslash\{p\}$;
- p^t times the local triple product integral for $v=p$, which are nonzero elements in $\mathrm{Frac}(\mathbb{I})$;
- $\langle f^D_{\mathrm{z},\vartheta}, \tilde{f}^D_{\vartheta,\mathrm{z},\mathrm{low}}\rangle/\langle f^D_{\mathrm{z}}, \tilde{f}^D_{\mathrm{z},\mathrm{low}}\rangle$ which is interpolated by a nonzero element in $\mathbb{I}$;
- the local Euler factors of $\mathrm{z}(\mathcal{L}_5)$ and $\mathrm{z}(\mathcal{L}_6)$ at Σ^2 and p which are units by our choices;
- the local Euler factors at Σ^2 of $L\bigl(f_{\mathrm{z}}, \chi^c_{\theta,\mathrm{z}}\chi_{h,\mathrm{z}}, \tfrac{1}{2}\bigr)$ which are nonzero elements in $\mathbb{I}$;
- the local triple product integrals for $v\nmid p$.

The first two items are clearly interpolated by nonzero elements in $\mathrm{Frac}(\hat{\mathbb{I}})$. The last item we listed above has two parts: at split primes and nonsplit primes. The integrals at split primes are nonzero numbers in $\bar{\mathbb{Q}}_p^\times$ which are fixed throughout the family. At nonsplit primes, we do not know much about it. We only know that at a generic arithmetic point z, this integral is not zero, and it only depends on $\mathrm{z}|_{\mathbb{I}}$ at generic points, as observed in Lemma 8.21. We may assume that at this z the expression (8-10) is nonzero and not a pole (in fact just need $\mathcal{L}_2$ to be a nonzero finite number here). Thus the expression (8-11) is not identically zero. So the ratio of (8-11) over (8-10) is a nonzero element of $\mathrm{Frac}(\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!])$. If we evaluate this ratio at the generic arithmetic points, it depends only on $\mathrm{z}|_{\mathbb{I}}$. And also it is nonzero somewhere. From this, it is not difficult to prove that (say using the following lemma) the ratio is a nonzero element of $\mathrm{Frac}(\hat{\mathbb{I}}^{\mathrm{ur}})$. (In fact we apply the following lemma to (8-11)/(8-10). Recall that by our choices for characters, the $\mathcal{L}_1$, $\mathcal{L}_5$, $\mathcal{L}_6$ are units in $\Lambda_{\mathcal{D}}$, and $\mathcal{L}_2$ is a nonzero element in $\mathrm{Frac}(\hat{\mathbb{I}}^{\mathrm{ur}})$. Moreover, the local integrals showing up in the Rallis inner product formula for θ_{z} and h_{z} at Σ, which are nonzero and fixed along the family. This gives the ratio between $\langle\theta_{\mathrm{z}}, \tilde{\theta}_{\mathrm{z,low}}\rangle\cdot\langle h_{\mathrm{z}}, \tilde{h}_{\mathrm{z,low}}\rangle$ and $\mathrm{z}(\mathcal{L}_3\cdot\mathcal{L}_4)$. So the proposition follows clearly. □

Lemma 8.30. *Suppose A is an element in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!]\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. If for any generic arithmetic points $\mathrm{z}, \mathrm{z}' \in \hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ such that $\mathrm{z}|_{\hat{\mathbb{I}}^{\mathrm{ur}}} = \mathrm{z}'|_{\hat{\mathbb{I}}^{\mathrm{ur}}}$, we have $\mathrm{z}(A) = \mathrm{z}'(A)$. Then $A \in \hat{\mathbb{I}}^{\mathrm{ur}}$.*

Proof. This lemma is easily proved by observing that if ζ_1, ζ_2 are p^t-roots of unity and ϕ is a generic arithmetic point with conductors being $p^{t'}$ such that $t' > t$, then the composition ϕ' of ϕ with the ring automorphism $\iota_{\zeta_1,\zeta_2} : \hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!] \to \hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!]$ given by identity on $\hat{\mathbb{I}}^{\mathrm{ur}}$ and $\gamma^+ \mapsto \gamma^+\zeta_1$, $\gamma^- \mapsto \gamma^-\zeta_2$ is still a generic arithmetic point. Let A be the element considered in the lemma. Then $A - A\circ\iota_{\zeta_1,\zeta_2}$ is 0 at a Zariski dense set of points, and is thus identically zero. The arbitrariness of ζ_1, ζ_2 implies the lemma. □

9. Proof of the theorems

9A. *Eisenstein ideals.* Let $K_{\boldsymbol{D}}$ be an open compact subgroup of $\mathrm{GU}(3, 1)(\mathbb{A}_{\mathbb{Q}})$ maximal at p and all primes outside Σ such that the Klingen–Eisenstein series we construct is invariant under $K_{\boldsymbol{D}}^{(p)}$. We consider the ring $\mathbb{T}_{\boldsymbol{D}}$ of reduced Hecke algebras acting on the space of $\Lambda''_{\boldsymbol{D}}$-adic nearly ordinary cuspidal forms with level group $K_{\boldsymbol{D}}$. It is generated by the Hecke operators $Z_{v,0}$, $Z_{v,0}^{(i)}$, $T_{i,v}$, $T_{i,v}^{(j)}$ defined before, together with the U_p-operator and then taking the maximal reduced quotient. It is well-known that one can interpolate the pseudo Galois characters attached to nearly ordinary cusp forms to get a pseudocharacter $R_{\boldsymbol{D}}$ of $G_{\mathcal{K}}$ with values in $\mathbb{T}_{\boldsymbol{D}}$ (see [Skinner and Urban 2014, Section 7.2] for details). We define the ideal $I_{\boldsymbol{D}}$ of $\mathbb{T}_{\boldsymbol{D}}$ to be generated by $\{t - \lambda(t)\}_t$ for t's in the abstract Hecke algebra and $\lambda(t)$ is the Hecke eigenvalue of t on $\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}$. Then it is easy to see that the structure map $\Lambda''_{\boldsymbol{D}} \to \mathbb{T}_{\boldsymbol{D}}/I_{\boldsymbol{D}}$ is surjective. Suppose the inverse image of $I_{\boldsymbol{D}}$ in $\Lambda''_{\boldsymbol{D}}$ is $\mathcal{E}_{\boldsymbol{D}}$. We call it the Eisenstein ideal. It measures the congruences between the Hecke eigenvalues of cusp forms and Klingen–Eisenstein series. We have

$$R_{\boldsymbol{D}} \pmod{I_{\boldsymbol{D}}} \equiv \mathrm{tr}\,\rho_{\boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}}} \pmod{\mathcal{E}_{\boldsymbol{D}}}.$$

Now we prove the following lemma:

Lemma 9.1. *Let P be a height 1 prime of $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma''_{\mathcal{K}}]\!]$ which is not the pullback of a height 1 prime of $\hat{\mathbb{I}}^{\mathrm{ur}}$. Then*

$$\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}}) \le \mathrm{ord}_P(\mathcal{E}_{\boldsymbol{D}}).$$

Proof. Suppose $t := \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}) > 0$. By the fundamental exact sequence Theorem 3.6 there is an $\boldsymbol{H} = \boldsymbol{E}_{\boldsymbol{D},\mathrm{Kling}} - \mathcal{L}^{\Sigma}_{\boldsymbol{f},\xi,\mathcal{K}}F$ for some $\Lambda''_{\boldsymbol{D}}$-adic form F such that $\boldsymbol{H}$ is a cuspidal family. We write ℓ for the $\Lambda''_{\boldsymbol{D}}$-adic functional $\ell(G) = \langle l_{\theta^\star}(G), \pi(g_2')\boldsymbol{h}\rangle$ constructed in Section 8E on the space of $\Lambda''_{\boldsymbol{D}}$-adic forms. By our assumption on P we have proved that $\ell(\boldsymbol{H}) \not\equiv 0 \pmod{P}$. Consider the $\Lambda''_{\boldsymbol{D}}$-linear map

$$\mu : \mathbb{T}_{\boldsymbol{D}} \to \Lambda''_{\boldsymbol{D},P}/P^t\Lambda''_{\boldsymbol{D},P}$$

given by: $\mu(t) = \ell(t.\boldsymbol{H})/\ell(\boldsymbol{H})$ for t in the Hecke algebra. Then

$$\ell(t.\boldsymbol{H}) \equiv \ell(t\boldsymbol{E_D}) \equiv \lambda(t)\ell(\boldsymbol{E_D}) \equiv \lambda(t)\ell(\boldsymbol{H}) \pmod{P^t}$$

so $I_{\boldsymbol{D}}$ is contained in the kernel of μ. Thus it induces: $\Lambda''_{\boldsymbol{D},P}/\mathcal{E}_{\boldsymbol{D}}\Lambda''_{\boldsymbol{D},P} \twoheadrightarrow \Lambda''_{\boldsymbol{D},P}/P^t\Lambda''_{\boldsymbol{D},P}$ which proves the lemma. □

9B. *Galois theoretic argument.* In this section, for ease of reference, we repeat the set-up and certain results from [Skinner and Urban 2014, Chapter 4] with some modifications, which are used to construct elements in the Selmer group.

Let G be a group and C a ring. Let $r : G \to \mathrm{Aut}_C(V)$ be a representation of G with $V \simeq C^n$. This can be extended to $r : C[G] \to \mathrm{End}_C(V)$. For any $x \in C[G]$, define: $\mathrm{Ch}(r, x, T) := \det(\mathrm{id} - r(x)T) \in C[T]$.

Let (V_1, σ_1) and (V_2, σ_2) be two C representations of G. Assume both are defined over a local henselian subring $B \subseteq C$, we say σ_1 and σ_2 are residually disjoint modulo the maximal ideal $\mathfrak{m}_B$ if there exists $x \in B[G]$ such that $\mathrm{Ch}(\sigma_1, x, T) \bmod \mathfrak{m}_B$ and $\mathrm{Ch}(\sigma_2, x, , T) \bmod \mathfrak{m}_B$ are relatively prime in $\kappa_B[T]$, where $\kappa_B := B/\mathfrak{m}_B$.

Let H be a group with a decomposition $H = G \rtimes \{1, c\}$ with $c \in H$ an element of order two normalizing G. For any C representations (V, r) of G we write r^c for the representation defined by $r^c(g) = r(cgc)$ for all $g \in G$.

Polarizations. Let $\theta : G \to \mathrm{GL}_L(V)$ be a representation of G on a vector space V over field L and let $\psi : H \to L^\times$ be a character. We assume that θ satisfies the ψ-polarization condition

$$\theta^c \simeq \psi \otimes \theta^\vee.$$

By a ψ-polarization of θ we mean an L-bilinear pairing $\Phi_\theta : V \times V \to L$ such that

$$\Phi_\theta(\theta(g)v, v') = \psi(g)\Phi_\theta(v, \theta^c(g)^{-1}v').$$

Let $\Phi^t_\theta(v, v') := \Phi_\theta(v', v)$, which is another ψ-polarization. We say that ψ is compatible with the polarization Φ_θ if

$$\Phi^t_\theta = -\psi(c)\Phi_\theta.$$

Suppose that:

(1) A_0 is a profinite $\mathbb{Z}_p$ algebra and a Krull domain.

(2) $P \subset A_0$ is a height one prime and $A = \hat{A}_{0,P}$ is the completion of the localization of A_0 at P. This is a discrete valuation ring.

(3) R_0 is local reduced finite A_0-algebra.

(4) $Q \subset R_0$ is prime such that $Q \cap A_0 = P$ and $R = \hat{R}_{0,Q}$.

(5) There exist ideals $J_0 \subset A_0$ and $I_0 \subset R_0$ such that $I_0 \cap A_0 = J_0$, $A_0/J_0 = R_0/I_0$, $J = J_0A$, $I = I_0R$, $J_0 = J \cap A_0$ and $I_0 = I \cap R_0$.

(6) G and H are profinite groups; we have a subgroup $G_{\bar{v}_0} \subset G$.

Set up. Suppose we have the following data:

(1) A continuous character $\nu : H \to A_0^\times$.

(2) A continuous character $\xi : G \to A_0^\times$ such that $\bar{\chi} \neq \bar{\nu}\bar{\chi}^{-c}$; Let $\chi' := \nu\chi^{-c}$.

(3) A representation $\rho : G \to \mathrm{Aut}_A(V)$, $V \simeq A^n$, which is a base change from a representation over A_0, such that:
   (a) $\rho^c \simeq \rho^\vee \otimes \nu$.
   (b) $\bar{\rho}$ is absolutely irreducible.
   (c) ρ is residually disjoint from χ and χ'.

(4) A representation $\sigma : G \to \mathrm{Aut}_{R\otimes_A F}(M)$, $M \simeq (R \otimes_A F)^m$ with $m = n+2$, which is defined over the image of R_0 in R, such that:
   (a) $\sigma^c \simeq \sigma^\vee \otimes \nu$.
   (b) $\mathrm{tr}\,\sigma(g) \in R$ for all $g \in G$.
   (c) For any $v \in M$, $\sigma(R[G])v$ is a finitely generated R-module.

(5) A proper ideal $I \subset R$ such that $J := A \cap I \neq 0$, the natural map $A/J \to R/I$ is an isomorphism, and
$$\mathrm{tr}\,\sigma(g) \equiv \chi'(g) + \mathrm{tr}\,\rho(g) + \chi(g) \mod I$$
for all $g \in G$.

(6) ρ is irreducible and ν is compatible with ρ.

(7) (local conditions for σ) There is a $G_{\bar{v}_0}$-stable sub-$R \otimes_A F$-module $M^+_{\bar{v}_0} \subseteq M$ such that $M^+_{\bar{v}_0}$ and $M^-_{\bar{v}_0} := M/M^+_{\bar{v}_0}$ are free $R \otimes_A F$ modules. We also require that $M^+_{\bar{v}_0}$ and $M^-_{\bar{v}_0}$ are disjoint modulo I. (In our applications this is always satisfied although the $\bar{\mathbb{F}}_p$-representations of $\bar{\rho}_f$, $\bar{\chi}'$, $\bar{\chi}$ are not necessarily mutually disjoint when restricting to $G_{\bar{v}_0}$.)

(8) (compatibility with the congruence condition) Assume that for all $x \in R[G_{\bar{v}}]$, we have the congruence relation
$$\mathrm{Ch}(M^+_{\bar{v}_0}, x, T) \equiv (1 - T\chi(x)) \mod I,$$
then we automatically have
$$\mathrm{Ch}(M^-_{\bar{v}_0}, x, T) \equiv \mathrm{Ch}(V_{\bar{v}_0}, x, T)(1 - T\chi'(x)) \mod I.$$

(9) For each F-algebra homomorphism $\lambda : R \otimes_A F \to K$, K a finite field extension of F, the representation $\sigma_\lambda : G \to \mathrm{GL}_m(M \otimes_{R\otimes F} K)$ obtained from σ via λ is either absolutely irreducible or contains an absolutely irreducible two-dimensional sub-K-representation σ'_λ such that $\mathrm{tr}\,\sigma'_\lambda(g) \equiv \chi(g) + \chi'(g) \mod I$.

One defines the Selmer groups $X_H(\chi'/\chi) := \ker\{H^1(H, A_0^*(\chi'/\chi)) \to H^1(G_{\bar{v}_0}, A_0^*(\chi'\chi))\}^*$ and $X_G(\rho_0 \otimes \chi^{-1}) := \ker\{H^1(G, V_0 \otimes_{A_0} A_0^*(\chi^{-1})) \to H^1(G_{\bar{v}_0}, V_0^- \otimes_{A_0} A_0^*(\chi^{-1}))\}^*$. Let $\mathrm{Ch}_H(\chi'/\chi)$ and $\mathrm{Ch}_G(\rho_0 \otimes \chi^{-1})$ be their characteristic ideals as A_0-modules.

Proposition 9.2. *Under the above assumptions, if* $\mathrm{ord}_P(\mathrm{Ch}_H(\chi'/\chi)) = 0$ *then*
$$\mathrm{ord}_P(\mathrm{Ch}_G(\rho_0 \otimes \chi^{-1})) \geq \mathrm{ord}_P(J).$$

Proof. This can be proved in the same way as [Skinner and Urban 2014, Corollary 4.16]. The only difference is the Selmer condition at p, which we use the description of Section 3C to guarantee. Note that the part corresponding to ρ_0 corresponds to the upper-left two by two block here while in [loc. cit.] the ρ_f contains the highest and the lowest Hodge–Tate weights. $\square$

Before proving the main theorem we first prove a useful lemma, which appears in an earlier version of [loc. cit.].

Lemma 9.3. *Let $Q \subset \mathbb{I}[[\Gamma''_{\mathcal{K}}]]$ be a height one prime such that* $\mathrm{ord}_Q(\mathcal{L}^{\Sigma}_{\boldsymbol{f},\mathcal{K},\xi}) \geq 1$ *and* $\mathrm{ord}_Q(\mathcal{L}_{\chi_f\bar{\xi}'}) = 0$, *then* $\mathrm{ord}_Q(\mathcal{L}^{\Sigma}_{\chi_f\bar{\xi}'}) = 0$.

Proof. Let $\theta = \chi_{f_0}\bar{\xi}'$. If $\mathrm{ord}_Q(\mathcal{L}^{\Sigma}_{\chi_f\bar{\xi}'}) \geq 1$, then for some $\ell \in \Sigma\backslash\{p\}$,

$$\prod_{\ell\in\Sigma\backslash\{p\}} (1-\theta^{-1}(\gamma_+^{-1}(1+W))^e\ell^{2-\kappa}) \in Q,$$

where $e \in \mathbb{Z}_p$ be such that $\ell = \omega^{-1}(\ell)(1+p)^e$. Thus

$$\theta(\ell) \equiv \gamma_+^{-e}\omega(\ell)^{\kappa-2}(1+p)^{e(2-\kappa)} \pmod{Q}.$$

Thus there is some integer f such that

$$1 \equiv (\gamma_+(1+W)^{-1}(1+p)^{\kappa-2})^{-fe} \pmod{Q}$$

which implies that for some p-power root of unity ζ_+, Q is contained in the kernel of any ϕ' such that $\phi'(\gamma_+(1+W)^{-1}) = \zeta_+(1+p)^{2-\kappa}$. This implies, by [Hida 1991, Theorem I] for the interpolation formula, that at the central critical point $L_{\mathcal{K}}(f_\phi, \theta_1, 1) = 0$ where θ_1 is some fixed CM character of infinity type $\left(\frac{\kappa}{2}, -\frac{\kappa}{2}\right)$ and ϕ any arithmetic point. But then we can specialize $\boldsymbol{f}$ to some point ϕ'' of weight 4 (this is not an arithmetic point in our definition, but is an interpolation point, by [loc. cit.]). By temperedness for $f_{\phi''}$, the specialization is not 0. This is a contradiction. $\square$

Now we apply the above result to prove the theorem.

- $H := G_{\mathbb{Q},\Sigma}$, $G = G_{\mathcal{K},\Sigma}$, c is the complex conjugation.
- $A_0 = \Lambda''_{\boldsymbol{D}}$, $A := \Lambda''_{\boldsymbol{D},P}$.
- $J_0 := \mathcal{E}_{\boldsymbol{D}}$, $J := \mathcal{E}_{\boldsymbol{D}}A$.
- $R_0 := \mathbb{T}_{\boldsymbol{D}}$, $I_0 := I_{\boldsymbol{D}}$.
- $Q \subset R_0$ is the inverse image of P modulo $\mathcal{E}_{\boldsymbol{D}}$ under $\mathbb{T}_{\boldsymbol{D}} \to \mathbb{T}_{\boldsymbol{D}}/I_{\boldsymbol{D}} = \Lambda_{\boldsymbol{D}}/\mathcal{E}_{\boldsymbol{D}}$.
- $R := T_{\boldsymbol{f}} \otimes_{\mathbb{I}} \Lambda''_{\boldsymbol{D}}$, $\rho_0 := \rho_{\pi_f}\sigma_{(\boldsymbol{\psi}/\boldsymbol{\xi})^c}\epsilon^{-(\kappa+3)/2}$.
- $V = V_0 \otimes_{A_0} A$, $\rho = \rho_0 \otimes_{A_0} A$.
- $\chi = \sigma_{\boldsymbol{\psi}^c}$, $\chi' = \sigma_{\boldsymbol{\psi}^c}\sigma_{(\boldsymbol{\psi}/\boldsymbol{\xi})'}\epsilon^{-\kappa}$. $\nu = \chi^c\chi'$.
- $M := (R \otimes_A F_A)^4$, F_A is the fraction field of A.
- σ is the representation on M obtained as the pseudorepresentation associated to $\mathbb{T}_{\boldsymbol{D}}$, as in [Skinner and Urban 2014, Proposition 7.2.1].

Proof of Theorem 1.2. We first note that we need only to prove the corresponding inclusion for the Σ-primitive Selmer groups and L-functions since locally the sizes of the unramified extensions at primes outside p are controlled by the local Euler factors of the p-adic L-functions since $\mathbb{Q}_\infty \subseteq \mathcal{K}_\infty$. (See [Greenberg and Vatsal 2000, Proposition 2.4].)

Recall that we have enlarged our $\mathbb{I}$ at the beginning of Section 8B and the end of Section 8D which we denote as $\mathbb{J}$ in this proof. We first prove the main theorem with $\hat{\mathbb{I}}^{\mathrm{ur}}$ replaced by $\hat{\mathbb{J}}^{\mathrm{ur}}$. Under the assumption of Theorem 1.1, as in [Skinner and Urban 2014, Proposition 12.9] we know that by the discussion for the anticyclotomic μ-invariant at the end of Section 7E, $\mathcal{L}_{f,\xi,\mathcal{K}}$ is not contained in any height one prime which is the pullback of a prime in $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^+]\!]$. Note that there might be height one primes dividing the Euler factors at nonsplit primes which are pullbacks of height one primes of $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}^+]\!]$. Such issue has been overlooked in [Skinner and Urban 2014]. These primes are treated in [Wan 2015b, Lemma 87 and Theorem 101] and can be treated in the same way here. In the following, we treat the height one primes which are not such pullbacks. By Lemma 9.1 for any such height one prime P of $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!]$,

$$\mathrm{ord}_P(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}) = \mathrm{ord}_P(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}) \le \mathrm{ord}_P(\mathcal{E}_{\boldsymbol{D}}).$$

Applying Proposition 9.2, we prove the first part of the theorem for $\hat{\mathbb{J}}^{\mathrm{ur}}$ in place of $\mathbb{I}$.

We replace $\hat{\mathbb{J}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!]$ by $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. We write $\mathcal{L}$ for $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}$. Recall Fitting ideal respects base change. We claim that for any $x \in \mathrm{Fitt}(X)$, $x\mathcal{L}^{-1} \in \hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. In fact, from what we proved for $\mathrm{Fitt}(V \otimes_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]} \hat{\mathbb{J}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!])$ as ideals of $\hat{\mathbb{J}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!]$, we have $x\mathcal{L}^{-1} \in \hat{\mathbb{J}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!] \cap F_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!]}$ where $F_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]}$ is the fraction field of $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. Since $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$ is normal and $\hat{\mathbb{J}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}'']\!]$ is finite over $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$, we have $x\mathcal{L}^{-1} \in \hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. Thus $\mathrm{Fitt}(X) \subseteq (\mathcal{L})$, which in turn implies that $\mathrm{char}(X) \subseteq (\mathcal{L})$. This proves Theorem 1.1.

Now assume we are under the assumptions of Theorem 1.2. Note that in this case $\mathcal{L}_{\chi\bar{\xi}'} = 1$. Thus by the Lemma 9.3 $\mathcal{L}^{\Sigma}_{f,\xi,\mathcal{K}}$ is coprime to $\mathcal{L}^{\Sigma}_{\chi\bar{\xi}'}$. Suppose $P_1, \ldots, P_t$ are the height one primes of $\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}$ that are pullbacks of height one primes in $\hat{\mathbb{I}}^{\mathrm{ur}}$. Note that none of the primes passes through ϕ_0 since the two-variable p-adic L-function for f_0 is not identically 0. We consider the ring $\hat{\mathbb{I}}^{\mathrm{ur}}_{p,P_1,\ldots,P_t}[\![\Gamma_{\mathcal{K}}]\!]$ where the subscripts denote localizations. Then the argument as in the proof of Theorem 1.1 proves that there is a number a such that $(\mathcal{L}^{\Sigma}_{f,\mathcal{K},\xi}) \supseteq (P_1 \cdots P_t)^a \,\mathrm{Fitt}_{\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]}(X^{\Sigma})$ as ideals of $\hat{\mathbb{I}}^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!]$. Specialize to ϕ_0', using Proposition 2.4, we find

$$(\mathcal{L}^{\Sigma}_{f_0,\mathcal{K},\xi}) \supseteq \mathrm{Fitt}_{\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]\otimes L}(X_{f_0}). \qquad \square$$

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References

[Brakočević 2010] M. Brakočević, "Non-vanishing modulo p of central critical Rankin–Selberg L-values with anticyclotomic twists", preprint, 2010. arXiv

[Castella 2019] F. Castella, "On the p-adic variation of Heegner points", *Journal of the Institute of Mathematics of Jussieu* (online publication February 2019), 1–38.

[Eischen 2015] E. E. Eischen, "A p-adic Eisenstein measure for unitary groups", *J. Reine Angew. Math.* **699** (2015), 111–142. MR Zbl

[Eischen et al. 2016] E. Eischen, M. Harris, J. Li, and C. Skinner, "p-adic L-functions for unitary groups, Part II: The zeta-integral calculations", preprint, 2016. arXiv

[Gan and Takeda 2016] W. T. Gan and S. Takeda, "A proof of the Howe duality conjecture", *J. Amer. Math. Soc.* **29**:2 (2016), 473–493. MR Zbl

[Greenberg 1994] R. Greenberg, "Iwasawa theory and p-adic deformations of motives", pp. 193–223 in *Motives* (Seattle, WA, 1991), edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Greenberg and Vatsal 2000] R. Greenberg and V. Vatsal, "On the Iwasawa invariants of elliptic curves", *Invent. Math.* **142**:1 (2000), 17–63. MR Zbl

[Harris et al. 1996] M. Harris, S. S. Kudla, and W. J. Sweet, "Theta dichotomy for unitary groups", *J. Amer. Math. Soc.* **9**:4 (1996), 941–1004. MR Zbl

[Harris et al. 2005] M. Harris, J.-S. Li, and C. M. Skinner, "The Rallis inner product formula and p-adic L-functions", pp. 225–255 in *Automorphic representations, L-functions and applications: progress and prospects*, edited by J. W. Cogdell et al., Ohio State Univ. Math. Res. Inst. Publ. **11**, de Gruyter, Berlin, 2005. MR Zbl

[Hida 1986] H. Hida, "Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms", *Invent. Math.* **85**:3 (1986), 545–613. MR Zbl

[Hida 1988] H. Hida, "On p-adic Hecke algebras for GL_2 over totally real fields", *Ann. of Math.* (2) **128**:2 (1988), 295–384. MR Zbl

[Hida 1991] H. Hida, "On p-adic L-functions of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ over totally real fields", *Ann. Inst. Fourier* (*Grenoble*) **41**:2 (1991), 311–391. MR Zbl

[Hida 2002] H. Hida, "Control theorems of coherent sheaves on Shimura varieties of PEL type", *J. Inst. Math. Jussieu* **1**:1 (2002), 1–76. MR Zbl

[Hida 2004] H. Hida, *p-adic automorphic forms on Shimura varieties*, Springer, 2004. MR Zbl

[Hida 2006] H. Hida, "Anticyclotomic main conjectures", *Doc. Math.* Extra Vol. (2006), 465–532. MR Zbl

[Hida 2009] H. Hida, "Irreducibility of the Igusa tower", *Acta Math. Sin.* (*Engl. Ser.*) **25**:1 (2009), 1–20. MR Zbl

[Hida and Tilouine 1993] H. Hida and J. Tilouine, "Anti-cyclotomic Katz p-adic L-functions and congruence modules", *Ann. Sci. École Norm. Sup.* (4) **26**:2 (1993), 189–259. MR Zbl

[Hida and Tilouine 1994] H. Hida and J. Tilouine, "On the anticyclotomic main conjecture for CM fields", *Invent. Math.* **117**:1 (1994), 89–147. MR Zbl

[Hsieh 2011] M.-L. Hsieh, "Ordinary p-adic Eisenstein series and p-adic L-functions for unitary groups", *Ann. Inst. Fourier* (*Grenoble*) **61**:3 (2011), 987–1059. MR Zbl

[Hsieh 2012] M.-L. Hsieh, "On the non-vanishing of Hecke L-values modulo p", *Amer. J. Math.* **134**:6 (2012), 1503–1539. MR Zbl

[Hsieh 2014a] M.-L. Hsieh, "Eisenstein congruence on unitary groups and Iwasawa main conjectures for CM fields", *J. Amer. Math. Soc.* **27**:3 (2014), 753–862. MR Zbl

[Hsieh 2014b] M.-L. Hsieh, "On the μ-invariant of anticyclotomic p-adic L-functions for CM fields", *J. Reine Angew. Math.* **688** (2014), 67–100. MR Zbl

[Hsieh 2014c] M.-L. Hsieh, "Special values of anticyclotomic Rankin–Selberg L-functions", *Doc. Math.* **19** (2014), 709–767. MR Zbl

[Hsieh 2017] M.-L. Hsieh, "Hida families and p-adic triple product L-functions", preprint, 2017. arXiv

[Hung 2017] P.-C. Hung, "On the non-vanishing mod ℓ of central L-values with anticyclotomic twists for Hilbert modular forms", *J. Number Theory* **173** (2017), 170–209. MR Zbl

[Ichino 2007] A. Ichino, "On the Siegel–Weil formula for unitary groups", *Math. Z.* **255**:4 (2007), 721–729. MR Zbl

[Ichino 2008] A. Ichino, "Trilinear forms and the central values of triple product L-functions", *Duke Math. J.* **145**:2 (2008), 281–307. MR Zbl

[Ikeda 1994] T. Ikeda, "On the theory of Jacobi forms and Fourier–Jacobi coefficients of Eisenstein series", *J. Math. Kyoto Univ.* **34**:3 (1994), 615–636. MR Zbl

[Jacquet and Langlands 1970] H. Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Mathematics **114**, Springer, 1970. MR Zbl

[Jetchev et al. 2017] D. Jetchev, C. Skinner, and X. Wan, "The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one", *Camb. J. Math.* **5**:3 (2017), 369–434. MR Zbl

[Kato 2004] K. Kato, "p-adic Hodge theory and values of zeta functions of modular forms", pp. 117–290 in *Cohomologies p-adiques et applications arithmétiques*, vol. III, edited by P. Berthelot et al., Astérisque **295**, Société Mathématique de France, Paris, 2004. MR Zbl

[Kudla 1996] S. S. Kudla, "Notes on the local theta correspondence", lecture notes, 1996, http://www.math.toronto.edu/skudla/castle.pdf.

[Lan 2012] K.-W. Lan, "Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties", *J. Reine Angew. Math.* **664** (2012), 163–228. MR Zbl

[Lan 2013] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series **36**, Princeton University Press, 2013. MR Zbl

[Lapid and Rallis 2005] E. M. Lapid and S. Rallis, "On the local factors of representations of classical groups", pp. 309–359 in *Automorphic representations, L-functions and applications: progress and prospects*, edited by J. W. Cogdell et al., Ohio State Univ. Math. Res. Inst. Publ. **11**, de Gruyter, Berlin, 2005. MR Zbl

[Lei et al. 2014] A. Lei, D. Loeffler, and S. L. Zerbes, "Euler systems for Rankin–Selberg convolutions of modular forms", *Ann. of Math.* (2) **180**:2 (2014), 653–771. MR Zbl

[Mœglin and Waldspurger 1995] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathematics **113**, Cambridge University Press, 1995. MR Zbl

[Morel 2010] S. Morel, *On the cohomology of certain noncompact Shimura varieties*, Annals of Mathematics Studies **173**, Princeton University Press, 2010. MR Zbl

[Prasad 1990] D. Prasad, "Trilinear forms for representations of GL(2) and local ϵ-factors", *Compositio Math.* **75**:1 (1990), 1–46. MR Zbl

[Prasad 1993] D. Prasad, "Weil representation, Howe duality, and the theta correspondence", pp. 105–127 in *Theta functions: from the classical to the modern*, edited by M. R. Murty, CRM Proc. Lecture Notes **1**, Amer. Math. Soc., Providence, RI, 1993. MR Zbl

[Shimura 1976] G. Shimura, "Theta functions with complex multiplication", *Duke Math. J.* **43**:4 (1976), 673–696. MR Zbl

[Shimura 1997] G. Shimura, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics **93**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[Shimura 2000] G. Shimura, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs **82**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Shin 2011] S. W. Shin, "Galois representations arising from some compact Shimura varieties", *Ann. of Math.* (2) **173**:3 (2011), 1645–1741. MR Zbl

[Skinner 2012] C. Skinner, "Galois representations associated with unitary groups over $\mathbb{Q}$", *Algebra Number Theory* **6**:8 (2012), 1697–1717. MR Zbl

[Skinner 2014] C. Skinner, "A converse to a theorem of Gross, Zagier, and Kolyvagin", preprint, 2014. arXiv

[Skinner and Urban 2014] C. Skinner and E. Urban, "The Iwasawa main conjectures for GL_2", *Invent. Math.* **195**:1 (2014), 1–277. MR Zbl

[Wan 2014a] X. Wan, "Heegner point Kolyvagin system and Iwasawa main conjecture", preprint, 2014. arXiv

[Wan 2014b] X. Wan, "Iwasawa main conjecture for supersingular elliptic curves and BSD conjecture", preprint, 2014. arXiv

[Wan 2015a] X. Wan, "Families of nearly ordinary Eisenstein series on unitary groups", *Algebra Number Theory* **9**:9 (2015), 1955–2054. MR Zbl

[Wan 2015b] X. Wan, "The Iwasawa main conjecture for Hilbert modular forms", *Forum Math. Sigma* **3** (2015), art. id. e18, 95 pp. MR Zbl

[Weil 1974] A. Weil, *Basic number theory*, 3rd ed., Die Grundlehren der Mathematischen Wissenschaften **144**, Springer, 1974. MR Zbl

[Zhang 2013] B. Zhang, "Fourier–Jacobi coefficients of Eisenstein series on unitary groups", *Algebra Number Theory* **7**:2 (2013), 283–337. MR Zbl

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Positivity results for spaces of rational curves

Roya Beheshti and Eric Riedl

Let X be a very general hypersurface of degree d in $\boldsymbol{P}^n$. We investigate positivity properties of the spaces $R_e(X)$ of degree e rational curves in X. We show that for small e, $R_e(X)$ has no rational curves meeting the locus of smooth embedded curves. We show that for $n \le d$, there are no rational curves other than lines in the locus $Y \subset X$ swept out by lines. We exhibit differential forms on a smooth compactification of $R_e(X)$ for every e and $n-2 \ge d \ge \frac{1}{2}(n+1)$.

1. Introduction

We work over $\boldsymbol{C}$, the field of complex numbers. Let X be a smooth hypersurface of degree d in $\boldsymbol{P}^n$, and for $e \ge 1$, denote by $R_e(X)$ the space of smooth rational curves of degree e on X. In this paper, we study some geometric properties of $R_e(X)$.

Question 1.1. *For which d, n and e with $n \ge d$ does the very general hypersurface $X \subset \boldsymbol{P}^n$ of degree d have a rational curve in $R_e(X)$?*

One major motivation for considering Question 1.1 is to study rational surfaces in Fano hypersurfaces. A rational curve in $R_e(X)$ gives a rational surface in X, and conversely a rational surface in X gives a nonconstant map from $\boldsymbol{P}^1$ to a compactification of $R_e(X)$ for some $e \ge 1$. It is known that if $d \ll \sqrt{n}$ every smooth hypersurface of degree d contains rational surfaces, but it is not known if the same holds for higher degree Fano hypersurfaces. It is conjectured that when $d = n \ge 5$, X is not covered by rational surfaces:

Conjecture 1.2. *A very general hypersurface of degree n in $\boldsymbol{P}^n$ is not covered by rational surfaces if $n \ge 5$.*

Conjecture 1.2 has important implications in understanding birational properties of varieties. Recall that a variety is unirational if it is rationally dominated by projective space, and it is rationally connected if there is a rational curve through two general points. It is uniruled if there is a rational curve through a general point. Every unirational variety is rationally connected, and indeed, is swept out by rational surfaces. It is expected that there exist rationally connected varieties which are not unirational, but to date, no examples of this have been proven. Proving Conjecture 1.2 would prove that a very general

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hypersurface of degree n in $\boldsymbol{P}^n$ is not unirational. Since every hypersurface of degree $d \leq n$ is rationally connected, this would give an example of a variety which is rationally connected but not unirational.

Riedl and Yang [2016] answered Question 1.1 for $e = 1$ by showing that if $n \leq \frac{1}{6}(d^2 + 3d + 6)$ there is no rational curve in $R_1(X)$, the space of lines on X. Beheshti [2012] proved that $R_e(X)$ is not uniruled for $\frac{1}{2}(n+1) \leq d \leq n-3$. Beheshti and Starr [2008] considered the special case $d = n$ and proved that X is not swept out by del Pezzo surfaces or rational surfaces ruled by curves of degree up to n. However, Question 1.1 remains open and is presumably quite difficult in general.

In this paper, we prove a few results about the positivity of $R_e(X)$. In Section 2, we consider the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_e(X)$, a compactification of $R_e(X)$, and show the following:

Theorem 1.3. *Fix an integer $n \geq 20$ and e, and suppose $d \leq n$ satisfies*

$$d^2 + (2e-1)d \geq e(e+1)n - 3e(e-1) + 2 \qquad \text{if } e \geq 3,$$

$$d^2 + (2e+1)d \geq (e+1)(e+2)n - 3e(e+1) + 2 \qquad \text{if } e = 1, 2.$$

Then for a very general hypersurface X of degree d in $\boldsymbol{P}^n$, there is no nonconstant morphism $\boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(X)$ whose image intersects the locus of embedded rational curves in X.

This is a substantial generalization of results in [Riedl and Yang 2016] since it applies to curves with $e > 1$. Unlike results in [Beheshti and Starr 2008], it applies to hypersurfaces with $n > d$, and proves that there are no rational curves meeting the locus of smooth curves in $R_e(X)$, instead of merely proving that the locus of $R_e(X)$ covered by rational curves does not sweep out X. It also generalizes results of [Beheshti 2012].

Starr [2003, Corollary 10.10] computed the $\boldsymbol{Q}$-divisor class of the first chern class of the dualizing sheaf of $\overline{\mathcal{M}}_e(X)$ for a general hypersurface X, and showed that the canonical divisor of $\overline{\mathcal{M}}_e(X)$ is big for n less than about d^2. If $\overline{\mathcal{M}}_e(X)$ were irreducible, of the expected dimension, and had canonical singularities, then $\overline{\mathcal{M}}_e(X)$ would be of general type for d large, $n \geq d+6$, and $2n+2 \leq d^2+d$. Results of Harris, Roth, and Starr [Harris et al. 2004], Beheshti and Kumar [2013], and Riedl and Yang [2019] prove that $\overline{\mathcal{M}}_e(X)$ is irreducible and of the expected dimension for $d \leq n-2$. However, $\overline{\mathcal{M}}_e(X)$ is not always known to have canonical singularities. The best current results are for $e + d \leq n$, due to [Starr 2003].

Theorem 1.3, along with results of [Starr 2003; Beheshti 2012], suggests that there should be nonzero pluri-canonical forms on $\overline{\mathcal{M}}_e(X)$ if $d \geq \frac{1}{2}(n+1)$ or if $d < \frac{1}{2}(n+1)$ and the inequality of Theorem 1.3 is satisfied. De Jong and Starr [2004] gave a general construction of differential forms on any desingularization of $\overline{M}_e(X)$, the coarse moduli scheme of $\overline{\mathcal{M}}_e(X)$, and used this to construct nonzero pluri-canonical forms on these schemes when X is a general cubic fourfold and $e \geq 5$ is an odd number. In Section 3, we use their construction and show:

Proposition 1.4. *If $\frac{1}{2}(n+1) \leq d \leq n-2$ and X is any smooth hypersurface of degree d in $\boldsymbol{P}^n$, then there are nonzero differential forms on any desingularization of $\overline{M}_e(X)$ for every $e \geq 1$.*

It is interesting to investigate whether these forms can be used to construct nonzero pluri-canonical forms when $d \geq \frac{1}{2}(n+1)$.

Finally, in Section 4 we generalize results of [Riedl and Yang 2016] in a different direction, and show:

Theorem 1.5. *If $n \le \frac{1}{6}(d(d+3)) + 1 - \frac{1}{3}k$, then a very general degree d hypersurface X in $\boldsymbol{P}^n$ contains no k-gonal curves in $F_1(X)$.*

If $n \le d$, the lines on X sweep out a proper subvariety $Y \subset X$. Results of Clemens and Ran [2004] seem to suggest that Y is the "most negative" subvariety of X. They prove that for $d \ge \frac{1}{2}(3n+1)$, any subvarieties of X without effective canonical bundle (such as rational curves) must lie in Y. The following corollary of Theorem 1.5 proves that in contrast to this, Y never contains rational curves for $n \le d$.

Corollary 1.6. *Let $X \subset \boldsymbol{P}^n$ be a very general degree d hypersurface with $d \ge n$. Let $Y \subset X$ be the locus swept out by lines. Then Y contains no k-gonal curves other than lines if $k \le (d^2+3d+6-6n)/(2(n-1))$. In particular, Y contains no rational curves other than lines.*

2. Rational curves in $\boldsymbol{R_e}(X)$

For a hypersurface $X \subset \boldsymbol{P}^n$, we denote by $\overline{\mathcal{M}}_e(X)$ the Kontsevich moduli space of stable maps of degree e from curves of genus 0 to X. The goal of this section is to prove the following.

Theorem 2.1. *Fix an integer e and n $(n \ge 20)$, and suppose $d \le n$ satisfies*

$$d^2 + (2e-1)d \ge e(e+1)n - 3e(e-1) + 2 \qquad \text{if } e \ge 3,$$
$$d^2 + (2e+1)d \ge (e+1)(e+2)n - 3e(e+1) + 2 \qquad \text{if } e = 1, 2.$$

Then for a very general hypersurface X of degree d in $\boldsymbol{P}^n$, there is no nonconstant morphism $\boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(X)$ whose image intersects the locus of embedded smooth rational curves in X.

Before proving Theorem 2.1, we set some notation. Let S be a smooth rational surface admitting a map to $\boldsymbol{P}^1$ with general fiber C isomorphic to $\boldsymbol{P}^1$. Let $f : S \to \boldsymbol{P}^n$ be a generically finite morphism whose image is contained in a smooth hypersurface X of degree d in $\boldsymbol{P}^n$. We denote by $N_{f,\boldsymbol{P}^n}$ the normal sheaf of f, i.e., the cokernel of the map $T_S \to f^*T_{\boldsymbol{P}^n}$. Similarly, we denote by $N_{f,X}$ the normal sheaf of f considered as a morphism from S to X. There is a short exact sequence

$$0 \to N_{f,X} \to N_{f,\boldsymbol{P}^n} \to f^*\mathcal{O}_{\boldsymbol{P}^n}(d) \to 0 \tag{1}$$

The strategy of the proof will be to compute, for various values of t, the Euler characteristic of $N_{f,X}(t) \otimes I_{C/S}$. The main technique will be to argue that for X a general hypersurface of the appropriate degree and for t carefully chosen, the Euler characteristic is positive. We contrast this with the following direct computation of the Euler characteristic.

Proposition 2.2. *Assume B is a smooth projective curve and S a smooth surface admitting a fibration $\pi : S \to B$ with rational fibers. Let $f : S \to X$ be a map from S to a smooth degree d hypersurface $X \subset \boldsymbol{P}^n$, and let C be a general fiber of π. Suppose $f|_C$ is an embedding. Let $H = f^*\mathcal{O}(1)$ and K be the canonical class of S. Then we have*

$$\chi(N_{f,X}(t) \otimes I_{C/S}) = \tfrac{1}{2}b_t H^2 + \tfrac{1}{2}c_t H \cdot K - 2K^2 + d_t,$$

where

$$b_t = (n-3)t^2 + 2(n+1-d)t + n + 1 - d^2,$$
$$c_t = -(n-5)t + d - n - 1,$$
$$d_t = (-(n-3)t - n - 1 + d)\, H \cdot C + (n+9)\,\chi(\mathcal{O}_S) - n + 5.$$

Proof. This is a calculation involving additivity of the Euler characteristic. Using the exact sequence (1), we see

$$\chi(N_{f,X} \otimes I_{C/S}(t)) = \chi(N_{f,\boldsymbol{P}^n} \otimes I_{C/S}(t)) - \chi(f^*\mathcal{O}_{\boldsymbol{P}^n}(d+t) \otimes I_{C/S}).$$

Using the sequence defining $N_{f,\boldsymbol{P}^n}$ and the Euler sequence for $\boldsymbol{P}^n$, we see that

$$\begin{aligned}\chi(N_{f,X}\otimes I_{C/S}(t)) &= \chi(T_{\boldsymbol{P}^n}\otimes I_{C/S}(t)) - \chi(T_S \otimes I_{C/S}(t)) - \chi(f^*\mathcal{O}_{\boldsymbol{P}^n}(d+t)\otimes I_{C/S}) \\ &= (n+1)\chi(f^* I_{C/S}(t+1)) - \chi(I_{C/S}(t)) - \chi(f^*\mathcal{O}_{\boldsymbol{P}^n}(d+t)\otimes I_{C/S}) - \chi(T_S \otimes I_{C/S}(t)).\end{aligned}$$

Using Hirzebruch–Riemann–Roch, this becomes

$$\begin{aligned}&(n+1)\frac{((t+1)H-C)\cdot((t+1)H-C-K)}{2} - \frac{(tH-C)\cdot(tH-C-K)}{2} \\ &\quad - \frac{((t+d)H-C)\cdot((t+d)H-C-K)}{2} + (n-1)\chi(\mathcal{O}_S) - \chi(T_S \otimes I_{C/S}(t)). \qquad (2)\end{aligned}$$

We have

$$c_1(T_S \otimes I_{C/S}(t)) = c_1(T_S) + 2(-C+tH) = -K + 2(-C+tH),$$
$$c_2(T_S \otimes I_{C/S}(t)) = c_2(T_S) - K \cdot (-C+tH) + (-C+tH)^2.$$

Via the splitting principle, we can introduce variables α and β with $\alpha + \beta = c_1(T_S \otimes I_{C/S}(t))$ and $\alpha\beta = c_2(T_S \otimes I_{C/S}(t))$. By Hirzebruch–Riemann–Roch, we get

$$\begin{aligned}\chi(T_S \otimes I_{C/S}(t)) &= 2\chi(\mathcal{O}_S) + \tfrac{1}{2}\alpha(\alpha - K) + \tfrac{1}{2}\beta(\beta - K) \\ &= 2\chi(\mathcal{O}_S) - \alpha\beta + \tfrac{1}{2}((\alpha+\beta)^2 - (\alpha+\beta)\cdot K).\end{aligned}$$

Plugging in the chern classes of $T_S \otimes I_{C/S}(t)$ for $\alpha+\beta$ and $\alpha\beta$, we obtain

$$\chi(T_S \otimes I_{C/S}(t)) = K^2 - c_2(T_S) - 2tH\cdot K + 2K\cdot C + t^2H^2 - 2tH\cdot C + 2\chi(\mathcal{O}_S).$$

We have $K \cdot C = -2$. So putting the above equation in (2), collecting like terms, and using Noether's formula $c_2(T_S) = 12\chi(\mathcal{O}_S) - K^2$, we obtain the result. □

Corollary 2.3. *Assume $t \geq 1$, $n \geq 20$ and $d \leq n$. If S contains no (-1)-curves contracted by both f and π, and*

$$d^2 + d(2t+1) \geq n(t+2)(t+1) - 3t^2 - 3t + 2,$$

then $\chi(N_{f,X} \otimes I_{C/S}(t)) < 0$.

Proof. This is an intersection theory calculation on S. First observe that $2H+2C+K$ is basepoint free, and hence, nef. If it were not, then by Reider's theorem [1988], there would be an effective divisor E on S with either $E\cdot(2H+2C)=1$ and $E^2=0$ or $E\cdot(2H+2C)=0$ and $E^2=-1$. The first case is impossible since $E\cdot(2H+2C)$ must be even. In the second case, we would have $H\cdot E=0=C\cdot E$, which implies that E is a (-1)-curve contracted by both π and f, and contradicts our assumption. It follows that $2H+2C+K$ is nef. So

$$0\le(2H+2C+K)^2=4H^2+K^2+4H\cdot K+8H\cdot C-8 \tag{3}$$

since $K\cdot C=-2$ and $C^2=0$. Note that $H^1(f^*\mathcal{O}(-1))=0$ by Kodaira vanishing, so by Hirzebruch–Riemann–Roch and the fact that $\chi(\mathcal{O}_S)\le 1$ we see that

$$H\cdot(H+K)=2\chi(f^*\mathcal{O}_X(-1))-2\chi(\mathcal{O}_S)\ge-2\chi(\mathcal{O}_S)\ge-2. \tag{4}$$

By Proposition 2.2 and the relation (3) we see that

$$\begin{aligned}\chi(N_{f,X}(t)\otimes I_{C/S})&=\tfrac12 b_tH^2+\tfrac12 c_tH\cdot K-2K^2+d_t\\&\le\tfrac12 b_tH^2+\tfrac12 c_tH\cdot K+8H\cdot(H+K)+16H\cdot C-16+d_t\\&=\tfrac12(b_t-c_t)H^2+\tfrac12(c_t+16)(H\cdot(H+K)+2)+d_t-c_t+16H\cdot C-32.\end{aligned}$$

Note that since by our assumption $d\le n$ and $n\ge 20$, we have $c_t+16\le 0$, so the above inequality and (4) give

$$\chi(N_{f,X}(t)\otimes I_{C/S})\le\tfrac12(b_t-c_t)H^2+d_t-c_t+16H\cdot C-32.$$

We see that $b_t-c_t\le 0$ precisely when $(n-3)t^2+(3n-3-2d)t+2n+2-d^2-d\le 0$, or

$$d^2+d(2t+1)\ge n(t+2)(t+1)-3t^2-3t+2.$$

It remains to show that $d_t-c_t+16H\cdot C-32<0$. By Proposition 2.2 we have

$$\begin{aligned}&d_t-c_t+16H\cdot C-32\\&\qquad=-(n-d+t(n-3)-15)(H\cdot C-1)-2t+(n+9)\chi(\mathcal{O}_S)-(n-5)-16\\&\qquad\le-(n-d+t(n-3)-15)(H\cdot C-1)-2t-2.\end{aligned}$$

We see that this is negative if $n-d+t(n-3)-15\ge 0$, which happens for $n\ge 20$ since $t\ge 1$ and $d\le n$. □

Proposition 2.4. *Fix $e\ge 1$. Let $m=e-1$ if $e\ge 3$ and $m=e$ if $e=1,2$. Assume that for a very general hypersurface X of degree d, there is a map $\phi:\boldsymbol{P}^1\to\overline{\mathcal{M}}_e(X)$ whose image intersects the locus of embedded smooth rational curves. Then there is a smooth surface S with two morphisms $f:S\to X$ and $\pi:S\to\boldsymbol{P}^1$ such that f maps a general fiber C of π isomorphically to a smooth rational curve of degree e in X and*

(a) *the image of the pull-back map $H^0(X,\mathcal{O}_X(d))\to H^0(S,f^*\mathcal{O}_{\boldsymbol{P}^n}(d))$ is contained in the image of the map*

$$H^0(S,N_{f,\boldsymbol{P}^n})\to H^0(S,f^*\mathcal{O}_{\boldsymbol{P}^n}(d)),$$

(b) *the restriction map* $H^0(S, N_{f,X}(m)) \to H^0(C, N_{f,X}(m)|_C)$ *is surjective, and*

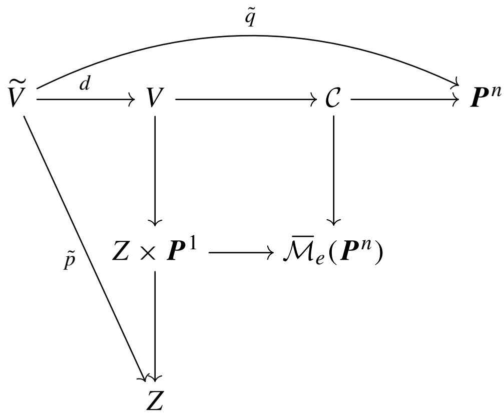
(c) *the Euler characteristic* $\chi(N_{f,X}(m) \otimes I_{C/S}) \geq 0$.

Proof. Our assumptions imply that there is an irreducible quasiprojective variety Z and a morphism $\phi : Z \times \boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(\boldsymbol{P}^n)$ such that the following hold:

- For each $z_0 \in Z$, the morphism
$$\phi_{z_0} := \phi(z_0, b) : \boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(\boldsymbol{P}^n)$$
is a nonconstant morphism whose image intersects the locus of embedded smooth rational curves.
- For a very general X, there is $z \in Z$ such that the image of ϕ_z is contained in $\overline{\mathcal{M}}_e(X)$.

Replacing Z with an open subset we may assume Z is nonsingular.

Let $\boldsymbol{P}^N$ be the projective space parametrizing hypersurfaces of degree d in $\boldsymbol{P}^n$ and $U \subset \boldsymbol{P}^N$ the open subset parametrizing smooth hypersurfaces. Denote by $I \subset Z \times U$ the incidence correspondence parametrizing pairs $(z, [X])$ such that the image of $\phi_z : \boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(\boldsymbol{P}^n)$ is contained in $\overline{\mathcal{M}}_e(X)$. Denote by π_1 and π_2 the projection maps from I to Z and U respectively. By our assumption π_2 is dominant. Replacing I with an irreducible component which maps dominantly to U under π_2, we may assume I is irreducible. Let V be the pullback of the universal curve $\mathcal{C} \to \overline{\mathcal{M}}_e(\boldsymbol{P}^n)$ to $Z \times \boldsymbol{P}^1$. Denote by $q : V \to \boldsymbol{P}^n$ the pullback of the universal map $\mathcal{C} \to \boldsymbol{P}^n$ to V and by $p : V \to Z$ the projection to the first factor. Let $d : \widetilde{V} \to V$ be a desingularization, and set $\tilde{p} = p \circ d$ and $\tilde{q} = q \circ d$.



Let $(z, [X]) \in I$ be a general point. Denote the fiber of $\tilde{p}$ over z by S. Since z is general in Z, by generic smoothness, S is a smooth surface and if f is the restriction of $\tilde{q}$ to S, then $f : S \to X$ maps a general fiber of $S \to \boldsymbol{P}^1$ isomorphically onto a curve in X.

Denote by $N_{(\tilde{p},\tilde{q})}$ the normal sheaf of the map $(\tilde{p}, \tilde{q}) : \widetilde{V} \to Z \times \boldsymbol{P}^n$. We get a sequence of maps

$$\rho : \ T_{Z,z} \to H^0(S, \tilde{p}^* T_Z|_S) \to H^0(S, (\tilde{p}, \tilde{q})^* T_{Z \times \boldsymbol{P}^n}|_S) \to H^0(S, N_{(\tilde{p},\tilde{q})}|_S).$$

Note $(\tilde{p}, \tilde{q})$ is generically finite and z is general, therefore $T_{\widetilde{V}}|_S \to (\tilde{p}, \tilde{q})^* T_{Z \times \boldsymbol{P}^n}|_S$ is injective. So $N_{(\tilde{p},\tilde{q})}|_S$ is isomorphic to $N_{f,\boldsymbol{P}^n}$:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & T_S & \longrightarrow & f^*T_{\boldsymbol{P}^n} & \longrightarrow & N_{f,\boldsymbol{P}^n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{\widetilde{V}}|_S & \longrightarrow & (\tilde{p}, \tilde{q})^* T_{Z \times \boldsymbol{P}^n}|_S & \longrightarrow & N_{(\tilde{p},\tilde{q})}|_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N_{S/\widetilde{V}} & \xrightarrow{=} & \tilde{p}^* T_Z|_S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

There is a commutative diagram

$$
\begin{array}{ccc}
 & T_{I,(z,[X])} & \\
 \swarrow{\scriptstyle d\pi_1} & & \searrow{\scriptstyle d\pi_2} \\
T_{Z,z} & & T_{\boldsymbol{P}^N,[X]} = H^0(X, \mathcal{O}_X(d)) \\
\downarrow{\scriptstyle \rho} & & \downarrow \\
H^0(S, N_{f,\boldsymbol{P}^n}) & \longrightarrow & H^0(S, f^*\mathcal{O}_X(d))
\end{array}
$$

Since π_2 is dominant, $d\pi_2$ is surjective, and part (a) follows.

To prove part (b), we tensor sequence (1) with $\mathcal{O}_X(m)$ to get the short exact sequence

$$0 \to N_{f,X}(m) \to N_{f,\boldsymbol{P}^n}(m) \to f^*\mathcal{O}_X(d+m) \to 0.$$

Let C be a general fiber of the map $\pi : S \to \boldsymbol{P}^1$. Restricting the above short exact sequence to C, we get a short exact sequence of $\mathcal{O}_C$-modules

$$0 \to N_{f,X}(m)|_C \to N_{f,\boldsymbol{P}^n}(m)|_C \to f^*\mathcal{O}_X(d+m)|_C \to 0.$$

We have the following commutative diagram:

$$\begin{array}{ccccc} H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(m+1)^{n+1}) & \longrightarrow & H^0(\boldsymbol{P}^n, T_{\boldsymbol{P}^n}(m)) & \longrightarrow & H^0(S, N_{f,\boldsymbol{P}^n}(m)) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C, \mathcal{O}_{\boldsymbol{P}^n}(m+1)^{n+1}|_C) & \longrightarrow & H^0(C, T_{\boldsymbol{P}^n}(m)|_C) & \longrightarrow & H^0(C, N_{f,\boldsymbol{P}^n}(m)|_C) \end{array}$$

Since $f(C)$ is $(m+1)$-normal, the left vertical map is surjective. Using the Euler sequence and the fact that $H^1(C, \mathcal{O}_C(m)) = 0$ and $H^1(C, T_S(m)|_C) = 0$, we see that the two lower horizontal maps are also surjective. So we have a surjective map

$$H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(m+1)^{n+1}) \to H^0(C, N_{f,\boldsymbol{P}^n}(m)|_C).$$

Pick $\alpha \in H^0(C, N_{f,X}(m)|_C)$, and denote the image of α in $H^0(C, N_{f,\boldsymbol{P}^n}(m)|_C)$ by β. Let $\tilde{\beta}$ be a lift of β to $H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(m+1)^{n+1})$ under the above map and $\bar{\beta}$ the image of $\tilde{\beta}$ in $H^0(S, N_{f,\boldsymbol{P}^n}(m))$. Let $\gamma \in H^0(X, \mathcal{O}_X(m+d))$ be the image of $\tilde{\beta}$ under the composition

$$\begin{aligned} H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(m+1)^{n+1}) &\to H^0(\boldsymbol{P}^n, T_{\boldsymbol{P}^n}(m)) \\ &\to H^0(X, T_{\boldsymbol{P}^n}(m)|_X) \to H^0(X, \mathcal{O}_X(m+d)), \end{aligned}$$

where the last map is induced by the sequence

$$0 \to T_X \to T_{\boldsymbol{P}^n}|_X \to \mathcal{O}_X(d) \to 0.$$

Let $\tilde{\gamma}$ be a preimage of γ in $H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(d+m))$. Then $\bar{\beta}|_C = \beta$ and $f^*\tilde{\gamma}|_C = 0$. Since $\tilde{\gamma}|_{f(C)} = 0$, we can view $\tilde{\gamma}$ as an element of $H^0(\boldsymbol{P}^n, I_{f(C)/\boldsymbol{P}^n}(d+m))$. Since $I_{f(C)/\boldsymbol{P}^n}$ is an m-regular sheaf, the multiplication map

$$H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(d)) \otimes H^0(\boldsymbol{P}^n, I_{f(C)/\boldsymbol{P}^n}(m)) \to H^0(\boldsymbol{P}^n, I_{f(C)/\boldsymbol{P}^n}(d+m))$$

is surjective, so $\tilde{\gamma}$ can be written as

$$\tilde{\gamma} = \tilde{\gamma}_1\tilde{\eta}_1 + \cdots + \tilde{\gamma}_k\tilde{\eta}_k,$$

where $\tilde{\gamma}_i \in H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(d))$ and $\tilde{\eta}_i \in H^0(\boldsymbol{P}^n, I_{f(C)/\boldsymbol{P}^n}(m))$ for each i. So if we view $\tilde{\eta}_i$ as an element of $H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(m))$ we see that $f^*\tilde{\eta}_i|_C = 0$ for each i. By part (a) the image of the map

$$f^* : H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(d)) \to H^0(S, f^*\mathcal{O}_{\boldsymbol{P}^n}(d))$$

is contained in the image of the map

$$h : H^0(S, N_{f,\boldsymbol{P}^n}) \to H^0(S, f^*\mathcal{O}_{\boldsymbol{P}^n}(d)).$$

So $f^*\tilde{\gamma}_i = h(\bar{\mu}_i)$ for some $\bar{\mu}_i \in H^0(S, N_{f,\boldsymbol{P}^n})$. Let

$$\bar{\mu} = \bar{\mu}_1 f^*\tilde{\eta}_1 + \cdots + \bar{\mu}_k f^*\tilde{\eta}_k \in H^0(S, N_{f,\boldsymbol{P}^n}(m)).$$

Then $\bar{\mu}|_C = 0$ and since $h(\bar{\mu} - \bar{\beta}) = 0$, we have that $\bar{\mu} - \bar{\beta}$ is the image of a section of $N_{f,X}(m)$ whose restriction to C is α.

To prove (c), note that by the Leray spectral sequence, to show the Euler characteristic is nonnegative, it is enough to show that for a general fiber C of π, (1) $H^1(\boldsymbol{P}^1, \pi_*(N_{f,X}(m) \otimes I_{C/S})) = 0$ and (2) $R^1\pi_*(N_{f,X}(m) \otimes I_{C/S}) = 0$.

By part (b), the map $H^0(S, N_{f,X}(m)) \to H^0(C, N_{f,X}(m)|_C)$ is surjective, and $H^1(C, N_{f,X}(m)|_C) = 0$, so

$$H^1(S, N_{f,X}(m) \otimes I_{C/S}) = H^1(S, N_{f,X}(m)).$$

Thus

$$H^1(\boldsymbol{P}^1, \pi_* N_{f,X}(m) \otimes \mathcal{O}_{\boldsymbol{P}^1}(-1)) = H^1(\boldsymbol{P}^1, \pi_* N_{f,X}(m)),$$

so $H^1(\boldsymbol{P}^1, \pi_* N_{f,X}(m) \otimes \mathcal{O}_{\boldsymbol{P}^1}(-1)) = 0$. This shows (1).

To show (2), we note that since X is very general, for any morphism $g : \boldsymbol{P}^1 \to X$, $g^* T_X(1)$ is globally generated by the main result of [Clemens 1986]; see also [Ein 1988; Voisin 1996]. So the restriction of $f^* T_X(m)$ to every irreducible component of every fiber of π is globally generated. So $R^1\pi_* T_X(m) = 0$ and $R^1\pi_* N_{f,X}(m) = 0$. □

Proof of Theorem 2.1. Assume to the contrary that for a very general X, there is a nonconstant map $\boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(X)$ whose image intersects the locus of embedded smooth rational curves in X. Let $m = e - 1$ if $e \geq 3$ and $m = e$ if $e = 1$ or $e = 2$. By Proposition 2.4, there is a surface S and map $f : S \to X$ and $\pi : S \to \boldsymbol{P}^1$ such that $\chi(N_{f,X}(m) \otimes I_{C/S}) \geq 0$. Blowing down, we may assume that there is no (-1)-curve in any fiber of π which is contracted by f. This contradicts Corollary 2.3. □

A modification of the proof of Proposition 2.4 shows that the statement of Theorem 2.1 remains true for morphisms $\boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(X)$ whose image intersects the locus of embedded reducible nodal rational curves in X. We sketch the proof here. Fix $e \geq 1$, let $m = e - 1$ if $e \geq 3$ and $m = e$ if $e = 1$ or 2, and assume that for a very general hypersurface X of degree d, there is a nonconstant map $\phi : \boldsymbol{P}^1 \to \overline{\mathcal{M}}_e(X)$ whose image intersects the locus of embedded nodal rational curves in X. Then there are positive integers r and s such that $rs \leq e$ and for a very general hypersurface X of degree d, there is a smooth curve B, a degree r morphism $B \to \boldsymbol{P}^1$, and a morphism $B \to \overline{\mathcal{M}}_s(X)$ such that the image intersects the locus of embedded smooth rational curves of degree s in X.

This implies that there are smooth, irreducible, and quasiprojective varieties P and Z (P is just a point in the case of Proposition 2.4), a proper morphism $p_1 : W \to P$ whose fibers are smooth projective curves which are degree r covers of $\boldsymbol{P}^1$ ($r = 1$ and W is the projective line in the case of Proposition 2.4), and morphisms $p_2 : Z \to P$ and $\phi : Z \times_P W \to \overline{\mathcal{M}}_s(\boldsymbol{P}^n)$ with the following property: for every $z \in Z$,

$$\phi_z : p_1^{-1}(p_2(z)) \to \overline{\mathcal{M}}_s(\boldsymbol{P}^n)$$

is a morphism which intersects the locus of embedded smooth rational curves, and for a very general X, there is z such that ϕ_z parametrizes stable maps which are mapped to X. We proceed now as in the proof of

Proposition 2.4 and let $I \subset Z \times U$ be a dominating irreducible component of the incidence correspondence where U is the locus of smooth hypersurfaces of degree d in $\boldsymbol{P}^n$. We also let V, S, and z be as before.

We conclude that if $B = p_1^{-1}(p_2(z))$, then there is a morphism of degree r $g : B \to \boldsymbol{P}^1$, and there are morphisms $f : S \to X$ and $\pi : S \to B$ such that f maps a general fiber C of π isomorphically onto a smooth rational curve of degree s on X, $rs \le e$, and

(a) the image of the map $H^0(X, \mathcal{O}_{\boldsymbol{P}^n}(d)) \to H^0(S, f^*\mathcal{O}_{\boldsymbol{P}^n}(d))$ is contained in the image of the map

$$H^0(S, N_{f,\boldsymbol{P}^n}(d)) \to H^0(S, f^*\mathcal{O}_{\boldsymbol{P}^n}(d)),$$

(b) the map $H^0(S, N_{f,X}(m)) \to H^0(D, N_{f,X}(m)|_D)$ is surjective where D is a general fiber of $g \circ \pi : S \to \boldsymbol{P}^1$, and

(c) the Euler characteristic $\chi(N_{f,X}(m) \otimes I_{D/S}) \ge 0$.

Part (b) follows from the proof of Proposition 2.4 since the image of D is e-regular in X. Part (c) follows from a similar argument as in Proposition 2.4 and the fact that if F is a sheaf on B with $H^1(B, F \otimes I_{g^{-1}(p)}) = H^1(B, F)$ for a general $p \in \boldsymbol{P}^1$, then $H^1(B, F) = 0$. Applying Corollary 2.3 to S gives the desired result.

3. Differential forms on Kontsevich moduli space

Let X be a smooth hypersurface of degree d in $\boldsymbol{P}^n_{\boldsymbol{C}}$. Let $\overline{\mathcal{M}}_e(X)$ be the Kontsevich moduli space of stable maps of degree e from curves of genus zero to X, and let $\overline{M}_e(X)$ be the corresponding coarse moduli scheme. There is a universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}_e(X)$ and an evaluation map $ev : \mathcal{C} \to X$.

In this section, we use the construction of de Jong and Starr [2004] to show that there are nonzero differential forms on any desingularization of $\overline{M}_e(X)$ when $\frac{1}{2}(n+1) \le d \le n-3$. By [de Jong and Starr 2004, Corollary 4.3], for every $i, j \ge 1$, there is a $\boldsymbol{C}$-linear map

$$\alpha_{i,j} : H^i(X, \Omega_X^j) \to H^{i-1}(\overline{\mathcal{M}}_e(X), \Omega^{j-1}_{\overline{\mathcal{M}}_e(X)}).$$

We consider the map $\alpha_{1,n-2}$, so the above map gives $(n-3)$-forms on the Kontsevich moduli stack. Let $\overline{N}_e(X)$ be a desingularization of $\overline{M}_e(X)$. By [de Jong and Starr 2004, Proposition 3.6], for every $j \ge 0$, there is a linear map

$$H^0(\overline{\mathcal{M}}_e(X), \Omega^j_{\overline{\mathcal{M}}_e(X)}) \to H^0(\overline{N}_e(X), \Omega^j_{\overline{N}_e(X)}).$$

Composing this with $\alpha_{1,n-2}$, we get a map from $H^1(X, \Omega_X^{n-2})$ to the space of $(n-3)$-forms on $\overline{N}_e(X)$.

Proposition 3.1. *Assume $\frac{1}{2}(n+1) \le d \le n-3$ and X is a smooth hypersurface of degree d in $\boldsymbol{P}^n$. If $\overline{N}_e(X)$ is a desingularization of $\overline{M}_e(X)$, then the map $H^1\big(X, \bigwedge^{n-2}\Omega_X\big) \to H^0\big(\overline{N}_e(X), \bigwedge^{n-3}\Omega_{\overline{N}_e(X)}\big)$ is nonzero for every e.*

Proof. Fix e and X, and set $\mathcal{M} = \overline{\mathcal{M}}_e(X)$ and $N = \overline{N}_e(X)$ for simplicity. By [de Jong and Starr 2004, Corollaries 4.2 and 4.3] the map $\alpha_{1,n-2}$ factors through the maps

$$H^1(X, \Omega_X^{n-2}) \to H^1(\mathcal{C}, \Omega_{\mathcal{C}}^{n-2}) \to H^1(\mathcal{C}, \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi) \to H^0(\mathcal{M}, \Omega_{\mathcal{M}}^{n-3}), \tag{5}$$

where

- the first map comes from the map $ev^*\Omega_X \to \Omega_{\mathcal{C}}$,
- the second map comes from a map of $\mathcal{O}_{\mathcal{C}}$-modules

$$\Omega_{\mathcal{C}}^{n-2} \to \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi$$

which fits into the following short exact sequence over the locus U of embedded smooth curves

$$\pi^*\Omega_{\mathcal{M}}^{n-2}|_U \to \Omega_{\mathcal{C}}^{n-2}|_U \to \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_U \to 0,$$

- and the last map comes from the Leray spectral sequence and the fact that $R^1\pi_*\omega_\pi = \mathcal{O}_{\mathcal{M}}$.

Since $d < n$, there is an irreducible component of $\mathcal{M}$ whose general point parametrizes an embedded smooth free rational curve of degree e on X. Let C be a such a curve. We denote the stable map corresponding to the isomorphism from $\boldsymbol{P}^1$ onto C by $[C] \in \mathcal{M}$, and identify the fiber of $\pi : \mathcal{C} \to \mathcal{M}$ over $[C]$ with C. Since C is free, $\mathcal{M}$ is smooth at $[C]$ and $T_{\mathcal{M}}|_{[C]} = H^0(C, N_{C/X})$. Restricting sequence (5) to C, we get the diagram

$$\begin{array}{ccc} H^1(X, \Omega_X^{n-2}) & \xrightarrow{\alpha_{1,n-2}} & H^0(\mathcal{M}, \Omega_{\mathcal{M}}^{n-3}) \\ \downarrow & & \downarrow \\ H^1(C, \Omega_X^{n-2}|_C) \to H^1(C, \Omega_{\mathcal{C}}^{n-2}|_C) \to H^1(C, \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_C) & \xrightarrow{\simeq} & \Omega_{\mathcal{M}}^{n-3}|_{[C]} \end{array}$$

and we have $\pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_C = \bigwedge^{n-3} I_{C/\mathcal{C}} \otimes \Omega_C$. In order to show the statement, it suffices to prove that in the above diagram the composition of the maps

$$\begin{aligned} H^1(X, \Omega_X^{n-2}) \to H^1(C, \Omega_X^{n-2}|_C) \to H^1(C, \Omega_{\mathcal{C}}^{n-2}|_C) \\ \to H^1(C, \pi^*\Omega_{\mathcal{M}}^{n-3} \otimes \omega_\pi|_C) = H^1(C, \textstyle\bigwedge^{n-3} I_{C/\mathcal{C}} \otimes \Omega_C) \end{aligned}$$

is nonzero. Since $\Omega_{\mathcal{M}}^{n-3}|_{[C]} = \Omega_N^{n-3}|_{[C]}$, this would show the assertion of the theorem. From the short exact sequence

$$0 \to I_{C/X} \otimes \mathcal{O}_C \to \Omega_X|_C \to \Omega_C \to 0,$$

we get the short exact sequence

$$0 \to \textstyle\bigwedge^{n-2} I_{C/X} \otimes \mathcal{O}_C \to \Omega_X^{n-2}|_C \to \bigwedge^{n-3} I_{C/X} \otimes \Omega_C \to 0.$$

Similarly, there is an exact sequence

$$0 \to \textstyle\bigwedge^{n-2} I_{C/\mathcal{C}} \otimes \mathcal{O}_C \to \Omega_{\mathcal{C}}^{n-2}|_C \to \bigwedge^{n-3} I_{C/\mathcal{C}} \otimes \Omega_C \to 0,$$

and a commutative diagram

$$\begin{array}{ccc} H^1(C, \Omega_X^{n-2}|_C) & \longrightarrow & H^1(C, \bigwedge^{n-3} I_{C/X} \otimes \Omega_C) \\ \downarrow & & \downarrow \\ H^1(C, \Omega_{\mathcal{C}}^{n-2}|_C) & \longrightarrow & H^1(C, \bigwedge^{n-3} I_{C/\mathcal{C}} \otimes \Omega_C) \end{array}$$

So to show the assertion, we show that the composition of the maps

$$\begin{aligned} H^1(X, \Omega_X^{n-2}) &\to H^1(C, \Omega_X^{n-2}|_C) \\ &\to H^1(C, \textstyle\bigwedge^{n-3} I_{C/X} \otimes \Omega_C) \to H^1(C, \bigwedge^{n-3} I_{C/\mathcal{C}} \otimes \Omega_C) \end{aligned} \tag{6}$$

is nonzero. Note that

$$\textstyle\bigwedge^{n-3} I_{C/\mathcal{C}} \otimes \mathcal{O}_C = \Omega_{\mathcal{M}}^{n-3}|_{[C]} \otimes \mathcal{O}_C = \bigwedge^{n-3} T_{\mathcal{M}}^{\vee}|_{[C]} \otimes \mathcal{O}_C = \bigwedge^{n-3} H^0(C, N_{C/X})^{\vee} \otimes \mathcal{O}_C.$$

So by Serre duality, the last map in sequence (6) is the dual of the map

$$\textstyle\bigwedge^{n-3} H^0(C, N_{C/X}) \to H^0(C, \bigwedge^{n-3} N_{C/X}).$$

Since C is free, the above map is surjective, so the last map in sequence (6) is injective. Hence it is enough to show that under our assumptions, the composition of the maps

$$H^1(X, \Omega_X^{n-2}) \to H^1(C, \Omega_X^{n-2}|_C) \to H^1(C, \textstyle\bigwedge^{n-3} I_{C/X} \otimes \Omega_C)$$

is nonzero.

To prove this we consider the short exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\boldsymbol{P}^n}|_X \to \Omega_X \to 0$$

which gives the following short exact sequence

$$0 \to \Omega_X^{n-2} \to \Omega_{\boldsymbol{P}^n}^{n-1}|_X \otimes \mathcal{O}_X(d) \to \mathcal{O}_X(2d-n-1) \to 0. \tag{7}$$

There is also a short exact sequence on C,

$$0 \to \textstyle\bigwedge^{n-3} I_{C/X} \otimes \Omega_C \to \bigwedge^{n-2} I_{C/\boldsymbol{P}^n} \otimes \Omega_C \otimes \mathcal{O}_C(d) \to \mathcal{O}_C(2d-n-1) \to 0, \tag{8}$$

and sequence (7) maps to sequence (8). Taking the long exact sequence of cohomology we get a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(2d-n-1)) & \longrightarrow & H^1(X, \Omega_X^{n-2}) \\ \downarrow & & \downarrow \\ H^0(C, \mathcal{O}_C(2d-n-1)) & \longrightarrow & H^1(C, \bigwedge^{n-3} I_{C/X} \otimes \Omega_C) \end{array}$$

and since $2d-n-1\geq 0$, the left vertical map is nonzero. To show the desired result, we show that the bottom map is injective. This follows if we show $H^0(C, \bigwedge^{n-2} I_{C/\boldsymbol{P}^n}\otimes\Omega_C\otimes\mathcal{O}_C(d))=0$. Let

$$N_{C/\boldsymbol{P}^n}=\mathcal{O}_{\boldsymbol{P}^1}(a_1)\oplus\cdots\oplus\mathcal{O}_{\boldsymbol{P}^1}(a_{n-1}).$$

We have $\bigwedge^{n-2} I_{C/\boldsymbol{P}^n}\otimes\Omega_C\otimes\mathcal{O}_C(d)=N_{C/\boldsymbol{P}^n}\otimes\mathcal{O}_{\boldsymbol{P}^1}(e(d-n-1))$, so we need to show $a_i<e(n+1-d)$ for each i. Since $\sum_i a_i=e(n+1)-2$ and each a_i is at least e, we see that $a_i\leq 3e-2$ for all i. Thus, $a_i<e(n+1-d)$ provided $d\leq n-2$. This proves the result. □

4. Gonality and the space swept out by lines

Our goal is to prove the following:

Theorem 4.1. *Let $X\subset\boldsymbol{P}^n$ be a very general degree $d\geq n$ hypersurface. Let $Y\subset X$ be the locus swept out by lines. Then Y contains no k-gonal curves other than lines if $k\leq(d^2+3d+6-6n)/(2(n-1))$. In particular, Y contains no rational curves other than lines.*

We begin with a description of the Fano scheme $F_1(X)$.

Proposition 4.2. *Let $X\subset\boldsymbol{P}^n$ be a general hypersurface. If $2n-d-3\geq 0$, then the Fano scheme $F_1(X)$ of lines on X is smooth of dimension $2n-d-3$. If $2n-d-3\geq 0$, then the canonical bundle of $F_1(X)$ is $\left(\frac{1}{2}d(d+1)-n-1\right)\sigma_1$, where σ_1 is the restriction of the divisor on $\mathbb{G}(1,n)$ of lines meeting a fixed codimension 2 space.*

Now we recall the following definitions and results of Bastianelli, De Poi, Ein, Lazarsfeld and Ullery [Bastianelli et al. 2017]. A divisor D is *birationally very ample* to order k (BVA_k) if $D=E+kA$, where E is effective and A is very ample.

Theorem 4.3 [Bastianelli et al. 2017, Theorem 1.10]. *If a smooth variety Z which satisfies K_Z is BVA_k, then Z is not swept out by $(k+1)$-gonal curves.*

Corollary 4.4. *If X in $\boldsymbol{P}^n$ is a general degree d hypersurface with $n\leq\frac{1}{2}d(d+1)-k$, then $F_1(X)$ is not swept out by k-gonal curves.*

We need a few basic results about the space of degree d hypersurfaces containing a fixed variety.

Lemma 4.5. *Any set of k distinct points in $\boldsymbol{P}^n$ with $k\leq d+1$ imposes k conditions on the space of hypersurfaces of degree d.*

Proof. By a degeneration argument, it suffices to prove the result for k points that lie along a line ℓ. Then, we can consider the map $\alpha: H^0(\boldsymbol{P}^n,\mathcal{O}_{\boldsymbol{P}^n}(d))\to H^0(\mathcal{O}_{\boldsymbol{P}^1}(d))$. Since ℓ is normal, we see that α is a surjective linear map. Hence, if we fix k points on ℓ with $k\leq d+1$, then the space of degree d hypersurfaces in $\boldsymbol{P}^n$ containing those k points is the preimage under α of all sections of $\mathcal{O}_\ell(d)$ that vanish on those k points. The result follows. □

We also need the following standard lemma, described in [Riedl and Yang 2019].

Lemma 4.6. *Let Z be a variety of dimension k. The space of hypersurfaces containing Z is codimension at least $\binom{n+d}{d}$ in the space of all hypersurfaces.*

We need the following proposition from [Riedl and Yang 2016].

Proposition 4.7. *Let $C \subsetneq \mathbb{G}(k-1, n)$ be a nonempty variety of $(k-1)$-planes and let $B \subset \mathbb{G}(k, n)$ be the set of k-planes containing the planes in C. Then if the codimension of C in $\mathbb{G}(k-1, n)$ is ϵ, the codimension of B in $\mathbb{G}(k, n)$ is at most $\epsilon - 1$.*

We use the following corollary which follows immediately from Proposition 4.7.

Corollary 4.8. *Let $\ell \subset \boldsymbol{P}^n$ be a line and let S_k be the variety of k-planes containing ℓ. If $C \subset S_{k-1}$ is a nonempty variety of $(k-1)$-planes of codimension $\epsilon > 0$, and $B \subset S_k$ is the set of k-planes that contain a plane of C, then the codimension of B in S_k is at most $\epsilon - 1$.*

We also discuss the notion of a *parametrized k-plane*. A parametrized k-plane is a map $\Lambda : \boldsymbol{P}^k \to \boldsymbol{P}^n$ defined by linear equations. The set of all parametrized k-planes is naturally a PGL_k bundle over the Grassmannian of k-planes in $\boldsymbol{P}^n$. Given a parametrized k-plane Λ, we have a natural map $H^0(\boldsymbol{P}^n, \mathcal{O}_{\boldsymbol{P}^n}(d)) \to H^0(\boldsymbol{P}^k, \mathcal{O}_{\boldsymbol{P}^k}(d))$ given by pulling polynomials back to $\boldsymbol{P}^k$ along Λ. We say that $X' \subset \boldsymbol{P}^k$ is a parametrized k-plane section of $X \subset \boldsymbol{P}^n$ if X' is cut out by the restriction of f to $\boldsymbol{P}^k$, where $X = V(f)$.

We now show that $F_1(X)$ contains no k-gonal curves for certain ranges of n, d and k.

Theorem 4.9. *If $n \le \frac{1}{6}d(d+3) + 1 - \frac{1}{3}k$, then a very general degree d hypersurface X in $\boldsymbol{P}^n$ contains no k-gonal curves in $F_1(X)$.*

Proof. Let $\mathcal{UL}_{n,d}$ be the set of pairs (ℓ, X) of lines ℓ lying in a hypersurface X of degree d in $\boldsymbol{P}^n$. Let $R_{n,d,k}$ be the space of (ℓ, X) such that $F_1(X)$ has a k-gonal curve passing through $[\ell]$. We see that $R_{n,d,k}$ will be a countable union of varieties. If for some tuple (n, d, k), the codimension of (each component of) $R_{n,d,k}$ in $\mathcal{UL}_{n,d}$ is at least $2n - d - 3$, then a general hypersurface X of degree d in $\boldsymbol{P}^n$ contains no k-gonal curves in $F_1(X)$.

Let $d \ge 3$ be an integer, let $m = \frac{1}{2}d(d+1) - k$, and let $X \subset \boldsymbol{P}^m$ be a general hypersurface of degree d in $\boldsymbol{P}^m$. By Corollary 4.4, we see that $R_{m,d,k} \subset \mathcal{UL}_{m,d}$ has codimension at least 1, so we can find some pair (ℓ_0, X_0) where $F_1(X_0)$ has no k-gonal curves through $[\ell_0]$.

Let (ℓ_1, X_1) be a general point of a component of $R_{n,d,k}$. We find a subvariety $S \subset \mathcal{UL}_{n,d}$ containing (ℓ_1, X_1) such that $S \cap R_{n,d,k}$ is of codimension at least $2n-d-3$ in S. Since (ℓ_1, X_1) could have been on any component, it follows that $R_{n,d,k}$ has codimension at least $2n - d - 3$ in $\mathcal{UL}_{n,d}$. We now construct S.

We claim that we can find a hypersurface $Y \subset \boldsymbol{P}^M$ containing a line ℓ_2 such that (ℓ_0, X_0) and (ℓ_1, X_1) are both parametrized linear sections of (Y, ℓ_2). To see this, choose coordinates $x_0, x_1, \ldots, x_n$ on $\boldsymbol{P}^n$ so that ℓ_1 is given by the vanishing of $x_2, \ldots, x_n$, and choose coordinates $x_0, x_1, y_2, \ldots, y_m$ on $\boldsymbol{P}^m$ so that ℓ_0 is given by the vanishing of $y_2, \ldots, y_m$. Write $X_0 = V(f_0)$ and $X_1 = V(f_1)$. Then we may take $Y = V(f_0 + f_1)$ in $\boldsymbol{P}^{m+n-1}$, which has the desired properties.

Let S_n be the closure of the set of parametrized n-plane sections of (Y, ℓ_2). We see by the fact that (ℓ_0, X_0) has no k-gonal curves in $F_1(X_0)$ passing through $[\ell_0]$ that $S_m \cap R_{m,d,k}$ is codimension at least one in S_m. By Corollary 4.8, we see that $S_{m-1} \cap R_{m-1,d,k}$ has codimension at least 2 in S_{m-1}. Thus, $S_n \cap R_{n,d,k}$ has codimension at least $m - n + 1$ in S_n. Therefore, if $m - n + 1 \geq 2n - d - 3$, we see that there is no k-gonal curves in $F_1(X)$ for X a general hypersurface of degree d in $\boldsymbol{P}^n$. This holds if

$$n \leq \tfrac{1}{3}(m + d + 4) = \tfrac{1}{6}d(d + 3) + \tfrac{4}{3} - \tfrac{1}{3}k. \qquad \square$$

Proposition 4.10. *If $X \subset \boldsymbol{P}^n$ is a general degree $d \geq n$ hypersurface, then any $p \in X$ has at most $n - 1$ lines in X passing through p.*

Proof. It suffices to prove the result for $d = n$. Let $\mathcal{U}$ be the incidence correspondence of pairs (p, X) where $X \subset \boldsymbol{P}^n$ is a degree n hypersurface and $p \in X$ is a point. Let $\mathcal{L}_m \subset \mathcal{U}$ be the set of pairs (p, X) such that X contains m lines passing through p. We wish to show that $\mathcal{L}_m$ has codimension at least n in $\mathcal{U}$ for $m \geq n$, from which the result will follow. Consider a point $p \in \boldsymbol{P}^n$. Choose coordinates on $\boldsymbol{P}^n$ so that p is the point $[1, 0, \ldots, 0]$. Then given an $X = V(f)$ containing p, we can expand the equation of f around p, writing $f = f_1 x_0^{d-1} + f_2 x_0^{d-2} + \cdots + f_d$. Then the space of lines in X passing through p is $\Sigma_p = V(f_1, \ldots, f_d) \subset \boldsymbol{P}^{n-1}$, where Σ_p is naturally contained in the $\boldsymbol{P}^{n-1}$ of lines in $\boldsymbol{P}^n$ passing through p. We consider the codimension of the locus of f for which $V(f_1, \ldots, f_i)$ has larger than expected dimension. Many of the ideas here are adapted from [Harris et al. 2004; Riedl and Yang 2019].

If $i \leq n - 1$, the locus of f where $V(f_i)$ contains a component of $V(f_1, \ldots, f_{i-1})$ has codimension at least

$$\binom{n-i+i}{i} = \binom{n}{i} \geq n,$$

so the locus $S \subset \mathcal{U}$ of the set of pairs (p, X) where there is a positive dimensional family of lines through p in X has codimension at least n. Thus, we may assume $V(f_1, \ldots, f_{n-1})$ is a finite set of points. For f_n to contain k of those points is k conditions by Lemma 4.5, so the locus $\mathcal{L}_m \subset \mathcal{U}$ is codimension at least m. Thus, a general hypersurface X of degree d in $\boldsymbol{P}^n$ cannot have n lines passing through a single point of X. The result follows. $\square$

Proof of Corollary 1.6. We simply put together the pieces. Let $\mathcal{U} \to F_1(X)$ be the universal line on X, mapping to $F_1(X)$ via a map π_1. Then $\mathcal{U}$ maps surjectively to Y via a map π_2. Note that π_2 is finite by Proposition 4.10. Suppose $C \subset Y$ is a k-gonal curve, and let D be an irreducible component of $\pi_2^{-1}(C)$. Then by Proposition 4.10, the degree of $\pi_2|_D$ is at most $n - 1$, and so D has gonality at most $k(n - 1)$. If D is contracted by π_1, then C must have been a line in X. If D is not contracted by π_1, then its image is a curve in $F_1(X)$ of gonality at most $k(n - 1)$. By Theorem 4.9, this means $k(n-1) > \frac{1}{2}(d^2 + 3d + 6) - 3n = \frac{1}{2}(d^2 + 3d + 6 - 6n)$. Thus, $k > (d^2 + 3d + 6 - 6n)/(2(n - 1))$, so Y contains no k-gonal curves other than lines for $k \leq (d^2 + 3d + 6 - 6n)/(2(n - 1))$. In particular, this bound for k is at least one for every $d \geq n \geq 3$. $\square$

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References

[Bastianelli et al. 2017] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, and B. Ullery, "Measures of irrationality for hypersurfaces of large degree", *Compos. Math.* **153**:11 (2017), 2368–2393. MR Zbl

[Beheshti 2012] R. Beheshti, "Nonuniruledness results for spaces of rational curves in hypersurfaces", *Algebra Number Theory* **6**:4 (2012), 669–687. MR Zbl

[Beheshti and Kumar 2013] R. Beheshti and N. M. Kumar, "Spaces of rational curves on complete intersections", *Compos. Math.* **149**:6 (2013), 1041–1060. MR Zbl

[Beheshti and Starr 2008] R. Beheshti and J. M. Starr, "Rational surfaces in index-one Fano hypersurfaces", *J. Algebraic Geom.* **17**:2 (2008), 255–274. MR Zbl

[Clemens 1986] H. Clemens, "Curves on generic hypersurfaces", *Ann. Sci. École Norm. Sup.* (4) **19**:4 (1986), 629–636. MR Zbl

[Clemens and Ran 2004] H. Clemens and Z. Ran, "Twisted genus bounds for subvarieties of generic hypersurfaces", *Amer. J. Math.* **126**:1 (2004), 89–120. MR Zbl

[Ein 1988] L. Ein, "Subvarieties of generic complete intersections", *Invent. Math.* **94**:1 (1988), 163–169. MR Zbl

[Harris et al. 2004] J. Harris, M. Roth, and J. Starr, "Rational curves on hypersurfaces of low degree", *J. Reine Angew. Math.* **571** (2004), 73–106. MR Zbl

[de Jong and Starr 2004] A. J. de Jong and J. Starr, "Cubic fourfolds and spaces of rational curves", *Illinois J. Math.* **48**:2 (2004), 415–450. MR Zbl

[Reider 1988] I. Reider, "Vector bundles of rank 2 and linear systems on algebraic surfaces", *Ann. of Math.* (2) **127**:2 (1988), 309–316. MR Zbl

[Riedl and Yang 2016] E. Riedl and D. Yang, "Rational curves on general type hypersurfaces", 2016. To appear in *J. Differential Geom.* arXiv

[Riedl and Yang 2019] E. Riedl and D. Yang, "Kontsevich spaces of rational curves on Fano hypersurfaces", *J. Reine Angew. Math.* **748** (2019), 207–225. MR Zbl

[Starr 2003] J. Starr, "The Kodaira dimension of spaces of rational curves on low degree hypersurfaces", preprint, 2003. arXiv

[Voisin 1996] C. Voisin, "On a conjecture of Clemens on rational curves on hypersurfaces", *J. Differential Geom.* **44**:1 (1996), 200–213. MR Zbl

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Generalized Schur algebras

Alexander Kleshchev and Robert Muth

We define and study a new class of bialgebras, which generalize certain Turner double algebras related to generic blocks of symmetric groups. Bases and generators of these algebras are given. We investigate when the algebras are symmetric which is relevant to block theory of finite groups. We then establish a double centralizer property related to blocks of Schur algebras.

1. Introduction

Let $\Bbbk$ be a commutative domain of characteristic 0 and A be a unital $\Bbbk$-superalgebra, which is free as a $\Bbbk$-supermodule. Let $\mathfrak{a}$ be a unital subalgebra of the even part $A_{\bar{0}}$, which is a direct summand of $A_{\bar{0}}$ as a $\Bbbk$-module. Some of the unitality conditions will be relaxed in the main body of the paper but in this introduction we will consider a special case.

We define and study *generalized Schur (super)algebras*

$$T^A_{\mathfrak{a}}(n,d) \subseteq S^A(n,d).$$

The algebra $S^A(n,d)$ is defined as the algebra of invariants $(M_n(A)^{\otimes d})^{\mathfrak{S}_d}$, and so in the case $A = \Bbbk$ we get that $S^{\Bbbk}(n,d)$ is the classical Schur algebra. If $\mathfrak{a} = A_{\bar{0}}$, then $T^A_{\mathfrak{a}}(n,d) = S^A(n,d)$, but in general the subalgebra $T^A_{\mathfrak{a}}(n,d) \subseteq S^A(n,d)$ is proper, although it is always a full sublattice in $S^A(n,d)$. Thus extending scalars to a field $\mathbb{K}$ of characteristic 0 produces the same algebras: $T^A_{\mathfrak{a}}(n,d)_{\mathbb{K}} = S^A(n,d)_{\mathbb{K}}$. However, importantly, extending scalars to a field $\mathbb{F}$ of positive characteristic will in general yield *nonisomorphic* algebras $T^A_{\mathfrak{a}}(n,d)_{\mathbb{F}}$ and $S^A(n,d)_{\mathbb{F}}$ of the same dimension. It turns out that in many situations it is the more subtly defined algebra $T^A_{\mathfrak{a}}(n,d)_{\mathbb{F}}$ that plays an important role.

As a special case of our construction, we recover the *Turner double algebras* $D^A(n,d)$ studied in [Turner 2008a; 2008b; 2009; Evseev and Kleshchev 2017]. In fact, we show in Section 5D2 that

$$D^A(n,d) \cong T^{\mathrm{E}(A)}_{A_{\bar{0}}}(n,d),$$

where $\mathrm{E}(A)$ is the trivial extension algebra of A. Turner double algebras are important because of their connection to generic blocks of symmetric groups via Turner's conjecture, recently proved in [Evseev and Kleshchev 2018]. Importantly, even when one is interested in "purely even" objects such as blocks of symmetric groups, the superstructure of A plays a crucial role in defining and studying Turner doubles and the generalized Schur algebra $T^A_{\mathfrak{a}}(n,d)$; see Remark 3.1.

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To be more precise, for an appropriate zigzag algebra $\bar{Z}$ and a subalgebra $\bar{\mathfrak{z}} \subseteq \bar{Z}$, the generalized Schur algebra $T^{\bar{Z}}_{\bar{\mathfrak{z}}}(n, d)$ is Morita equivalent to weight d RoCK blocks of symmetric groups. In this way, $T^{\bar{Z}}_{\bar{\mathfrak{z}}}(n, d)$ can be considered as a "local" object replacing wreath products of Brauer tree algebras in the context of the Broué abelian defect group conjecture for blocks of symmetric groups with *nonabelian* defect groups.

However, it is known that Turner doubles cannot provide a similar "local" description for blocks of classical Schur algebras because the former are always symmetric algebras while the latter in general are not. We believe that our more general construction of $T^A_{\mathfrak{a}}(n, d)$ fixes the problem. In [Kleshchev and Muth 2018], we formulate an explicit conjecture for RoCK blocks of classical Schur algebras in terms of the generalized Schur algebras $T^Z_{\mathfrak{z}}(n, d)$, where Z is an *extended* zigzag algebra.

Furthermore, we prove in [Kleshchev and Muth 2018] that, under reasonable additional assumptions on $\mathfrak{a}$, the algebra $T^A_{\mathfrak{a}}(n, d)$ is quasihereditary if A is quasihereditary. This provides us with a method to produce new interesting quasihereditary algebras from old. In particular, the algebra $T^Z_{\mathfrak{z}}(n, d)$ from the previous paragraph is quasihereditary, as should be expected if it is to be Morita equivalent to a block of the Schur algebra.

We now describe the contents of the paper in more detail. Section 2 is preliminary. In Section 3, given a basis B for A which extends a basis for $\mathfrak{a}$, we describe a natural basis for $S^A(n, d)$ in terms of certain elements $\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$, where $\boldsymbol{b} \in B^d$, $\boldsymbol{r}, \boldsymbol{s} \in [1, n]^d$. This is an analogue of Schur's basis of the classical Schur algebra. By rescaling this natural basis using certain products of factorials defined in Section 2, we define the full sublattice $T^A_{\mathfrak{a}}(n, d) \subseteq S^A(n, d)$. Our first main result is:

Theorem 1. *We have that $T^A_{\mathfrak{a}}(n, d) \subseteq S^A(n, d)$ is a unital subalgebra.*

There is another description of $T^A_{\mathfrak{a}}(n, d)$ as a subalgebra of $S^A(n, d)$, which shows in particular that $T^A_{\mathfrak{a}}(n, d)$ is independent of the choice of basis B above:

Theorem 2. *We have that $T^A_{\mathfrak{a}}(n, d)$ is the subalgebra of $S^A(n, d)$ generated by $S^{\mathfrak{a}}(n, d)$ and the elements of the form*

$$\sum_{e=0}^{d-1} 1^{\otimes d-1-e} \otimes \xi \otimes 1^{\otimes e},$$

where $\xi \in M_n(A)$ and $1 := 1_{M_n(A)}$.

A slightly stronger result appears as Theorem 4.13. In order to prove this result, we first investigate some coproducts and $*$-products. Recall that $\bigoplus_{d\geq 0} M_n(A)^{\otimes d}$ has a natural coproduct ∇; see Section 3C. We then prove:

Theorem 3. *The coproduct ∇ restricts to coproducts on*

$$S^A(n) := \bigoplus_{d\geq 0} S^A(n, d) \quad \textit{and} \quad T^A_{\mathfrak{a}}(n) := \bigoplus_{d\geq 0} T^A_{\mathfrak{a}}(n, d).$$

In Section 4, we show that the $*$-product (or shuffle product) on

$$\bigoplus_{d\geq 0} M_n(A)^{\otimes d}$$

restricts to a product on $S^A(n)$ and $T^A_{\mathfrak{a}}(n)$, which, together with ∇, gives these objects a superbialgebra structure. We then prove that $T^A_{\mathfrak{a}}(n)$ is generated under the $*$-product by $S^{\mathfrak{a}}(n)$ and $M_n(A)$. This allows us to prove Theorem 2.

In Section 5 we first discuss some properties of idempotents and idempotent truncations in $T^A_{\mathfrak{a}}(n,d)$. Given an idempotent $e\in\mathfrak{a}$, we define an idempotent $\xi^e\in T^A_{\mathfrak{a}}(n,d)$ and prove in Lemma 5.12 that

$$\xi^e T^A_{\mathfrak{a}}(n,d)\xi^e \cong T^{eAe}_{e\mathfrak{a}e}(n,d).$$

Section 5 is completed with some important examples of generalized Schur algebras. We discuss how $T^A_{\mathfrak{a}}(n,d)$ generalizes the Turner double construction and look at the case where A is the extended zigzag algebra.

In Section 6 we study the symmetricity of $T^A_{\mathfrak{a}}(n,d)$. This is important since blocks of finite groups are symmetric algebras and, inspired by [Evseev and Kleshchev 2018], we hope that in some situations $T^A_{\mathfrak{a}}(n,d)$ could provide a local description of some interesting blocks. As the example $A=\Bbbk$ shows, it is certainly not enough to assume that A is symmetric to guarantee that so is $T^A_{\mathfrak{a}}(n,d)$. A natural assumption we have to make is that the symmetrizing form $\mathtt{t}$ is $(A,\mathfrak{a})$-*symmetrizing*, i.e., $(\mathfrak{a},\mathfrak{a})_{\mathtt{t}}=0$ and the $\Bbbk$-complement $\mathfrak{c}$ of $\mathfrak{a}$ in $A_{\bar{0}}$ can be chosen so that the restriction of $(\,\cdot\,,\,\cdot\,)_{\mathtt{t}}$ to $\mathfrak{a}\times\mathfrak{c}$ is a perfect pairing. Then we construct an explicit symmetrizing form $\mathtt{t}^T$ on $T^A_{\mathfrak{a}}(n,d)$ and prove in Corollary 6.7:

Theorem 4. *If $\mathtt{t}$ is an $(A,\mathfrak{a})$-symmetrizing form on A, then the algebra $T^A_{\mathfrak{a}}(n,d)$ is symmetric, with symmetrizing form $\mathtt{t}^T$.*

In Section 7 we investigate double centralizer properties. Let S be a $\Bbbk$-algebra and $e\in S$ be an idempotent. We say that e is a *double centralizer idempotent for S* if the natural map $S\to \mathrm{End}_{eSe}(Se)$ is an isomorphism. Given $e\in\mathfrak{a}$, which is a double centralizer idempotent for A, it is not in general true that ξ^e is a double centralizer idempotent for $T^A_{\mathfrak{a}}(n,d)$; see Remark 7.19. However, in Theorem 7.2, we prove the following positive result:

Theorem 5. *Let $e\in A$ be a double centralizer idempotent for A and $d\leq n$. Then ξ^e is a double centralizer idempotent for $S^A(n,d)$. In particular, if $\mathbb{K}$ is the quotient field of $\Bbbk$, then ξ^e is a double centralizer idempotent for $S^A(n,d)_{\mathbb{K}}=T^A_{\mathfrak{a}}(n,d)_{\mathbb{K}}$.*

Finally, in Theorem 7.17, we deal with the all-important zigzag case over the arbitrary $\Bbbk$:

Theorem 6. *Let Z be the extended zigzag algebra with the standard idempotents $e_0,e_1,\dots,e_\ell$. We set $e:=e_0+\dots+e_{\ell-1}$, so that eZe is the zigzag algebra. Then e is a double centralizer idempotent for Z, and ξ^e is a double centralizer idempotent for $T^Z_{\mathfrak{z}}(n,d)$ provided $d\leq n$.*

2. Preliminaries

Throughout the paper $\Bbbk$ is always a commutative domain of characteristic 0.

2A. ***Superalgebras and supermodules.*** Let V be a $\Bbbk$-*supermodule*, i.e., V is endowed with a $\Bbbk$-module decomposition $V = V_{\bar 0} \oplus V_{\bar 1}$ (the superstructure could be trivial, i.e., we could have $V = V_{\bar 0}$). If $\varepsilon \in \mathbb{Z}/2$ and $v \in V_\varepsilon$, we call v *homogeneous* and write $\bar v := \varepsilon$. For a set S of homogeneous elements of V and $\varepsilon \in \mathbb{Z}/2$ we denote

$$S_\varepsilon := S \cap V_\varepsilon. \tag{2.1}$$

A map $f : V \to W$ of $\Bbbk$-supermodules is called *homogeneous* if $f(V_\varepsilon) \subseteq W_\varepsilon$ for all ε. A $\Bbbk$-supermodule V is *free* if so is each V_ε. Let V be a free $\Bbbk$-supermodule. A *homogeneous basis* of V is a $\Bbbk$-basis all of whose elements are homogeneous. A (not necessarily unital) $\Bbbk$-algebra A is called a $\Bbbk$-*superalgebra*, if A is a $\Bbbk$-supermodule and $A_\varepsilon A_\delta \subseteq A_{\varepsilon+\delta}$ for all ε, δ.

Throughout we work with a fixed superalgebra A which is free as a $\Bbbk$-supermodule (not necessarily of finite rank). Moreover, we fix a $\Bbbk$-subalgebra $\mathfrak{a} \subseteq A_{\bar 0}$ such that $\mathfrak{a}$ and $A/\mathfrak{a}$ are both free as $\Bbbk$-modules. Such a pair $(A, \mathfrak{a})$ will be called a *good pair*. It is called a *unital good pair* if both A and $\mathfrak{a}$ are unital and $1_{\mathfrak{a}} = 1_A$.

For our fixed good pair $(A, \mathfrak{a})$, we pick a $\Bbbk$-module complement $\mathfrak{c}$ for $\mathfrak{a}$ in $A_{\bar 0}$ and $\Bbbk$-bases $B_{\mathfrak{a}}$, $B_{\mathfrak{c}}$, $B_{\bar 1}$ for $\mathfrak{a}$, $\mathfrak{c}$, $A_{\bar 1}$, respectively, so that

$$B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar 1} \tag{2.2}$$

is a homogeneous basis for A. We call such a basis an $(A, \mathfrak{a})$-*basis*.

Remark 2.3. To make things somewhat more concrete throughout this section, one may keep in mind the *extended zigzag algebra*, which is a motivating example for the construction in this paper. See Section 5D3 for an explicit description of this algebra and its $(A, \mathfrak{a})$-basis.

Define the structure constants $\kappa^b_{a,c}$ of A from

$$ac = \sum_{b \in B} \kappa^b_{a,c} b \qquad (a, c \in A). \tag{2.4}$$

More generally, for

$$\boldsymbol{b} = (b_1, \dots, b_d) \in B^d \quad \text{and} \quad \boldsymbol{a} = (a_1, \dots, a_d),\ \boldsymbol{c} = (c_1, \dots, c_d) \in A^d,$$

we define

$$\kappa^{\boldsymbol{b}}_{\boldsymbol{a},\boldsymbol{c}} := \kappa^{b_1}_{a_1,c_1} \cdots \kappa^{b_d}_{a_d,c_d}. \tag{2.5}$$

Finally, we denote by H the set of all nonzero homogeneous elements of A.

The matrix algebra $M_n(A)$ is naturally a superalgebra. For $1 \le r, s \le n$ and $a \in A$, we denote

$$\xi^a_{r,s} := aE_{r,s} \in M_n(A). \tag{2.6}$$

Then

$$\{\xi^b_{r,s} \mid 1 \le r, s \le n,\ b \in B\} \tag{2.7}$$

is a homogeneous basis of $M_n(A)$, and by (2.4) we have

$$\xi^a_{r,s}\xi^c_{t,u} = \delta_{s,t}\sum_{b\in B}\kappa^b_{a,c}\xi^b_{r,u} \qquad (a,c\in A,\ 1\le r,s,t,u\le n). \tag{2.8}$$

2B. ***Combinatorics.*** For $r, s \in \mathbb{Z}$ we denote $[r,s] := \{t \in \mathbb{Z} \mid r \le t \le s\}$. We fix $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\ge 0}$. For a set X, the elements of X^d are referred to as *words* (of length d) with letters in the alphabet X. The words are usually written as $x_1x_2\cdots x_d \in X^d$. For $\boldsymbol{x} \in X^d$ and $\boldsymbol{x}' \in X^{d'}$ we denote by $\boldsymbol{x}\boldsymbol{x}' \in X^{d+d'}$ the concatenation of $\boldsymbol{x}$ and $\boldsymbol{x}'$. For $x \in X$, we denote $x^d := x\cdots x \in X^d$.

The symmetric group $\mathfrak{S}_d$ acts on the right on X^d by place permutations:

$$(x_1\cdots x_d)\sigma = x_{\sigma 1}\cdots x_{\sigma d}.$$

For $\boldsymbol{x}, \boldsymbol{x}' \in X^d$, we write $\boldsymbol{x} \sim \boldsymbol{x}'$ if $\boldsymbol{x}\sigma = \boldsymbol{x}'$ for some $\sigma \in \mathfrak{S}_d$. If $X_1,\dots,X_N$ are sets, then $\mathfrak{S}_d$ acts on $X_1^d \times \cdots \times X_N^d$ diagonally:

$$(\boldsymbol{x}^1,\dots,\boldsymbol{x}^N)\sigma = (\boldsymbol{x}^1\sigma,\dots,\boldsymbol{x}^N\sigma).$$

The set of the corresponding orbits is denoted $(X_1^d \times \cdots \times X_N^d)/\mathfrak{S}_d$, and the orbit of $(\boldsymbol{x}^1,\dots,\boldsymbol{x}^N)$ is denoted $[\boldsymbol{x}^1,\dots,\boldsymbol{x}^N]$. We write

$$(\boldsymbol{x}^1,\dots,\boldsymbol{x}^N) \sim (\boldsymbol{y}^1,\dots,\boldsymbol{y}^N)$$

if $[\boldsymbol{x}^1,\dots,\boldsymbol{x}^N] = [\boldsymbol{y}^1,\dots,\boldsymbol{y}^N]$.

Let P be a set of homogeneous elements of A. Our main examples will be $P = B$ and $P = H$ (the set of all nonzero homogeneous elements of A). We have $P = P_{\bar 0} \sqcup P_{\bar 1}$. Define $\mathrm{Tri}^P(n,d)$ to be the set of all triples

$$(\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}) = (p_1\cdots p_d,\ r_1\cdots r_d,\ s_1\cdots s_d) \in P^d \times [1,n]^d \times [1,n]^d$$

such that for any $1 \le k \ne l \le d$ we have $(p_k,r_k,s_k) = (p_l,r_l,s_l)$ only if $p_k \in P_{\bar 0}$. Then $\mathrm{Tri}^P(n,d) \subseteq P^d \times [1,n]^d \times [1,n]^d$ is $\mathfrak{S}_d$-invariant and so we have the orbit set $\mathrm{Tri}^P(n,d)/\mathfrak{S}_d$.

For $(\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^P(n,d)$, we consider the stabilizer

$$\mathfrak{S}_{\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}} := \{\sigma \in \mathfrak{S}_d \mid (\boldsymbol{p},\boldsymbol{r},\boldsymbol{s})\sigma = (\boldsymbol{p},\boldsymbol{r},\boldsymbol{s})\},$$

and denote by ${}^{\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}$ a set of the shortest coset representatives for $\mathfrak{S}_{\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}}\backslash\mathfrak{S}_d$. Then $\{(\boldsymbol{p},\boldsymbol{r},\boldsymbol{s})\sigma \mid \sigma \in {}^{\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}\}$ is the set of distinct elements in the orbit $[\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}]$.

We fix a total order "$<$" on $P \times [1,n] \times [1,n]$. Then we also have a total order on $\mathrm{Tri}^P(n,d)$ defined as follows: $(\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}) < (\boldsymbol{p}',\boldsymbol{r}',\boldsymbol{s}')$ if and only if there exists $l \in [1,d]$ such that $(p_k,r_k,s_k) = (p'_k,r'_k,s'_k)$ for all $k < l$ and $(p_l,r_l,s_l) < (p'_l,r'_l,s'_l)$. Denote

$$\mathrm{Tri}^P_0(n,d) = \{(\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^P(n,d) \mid (\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}) \le (\boldsymbol{p},\boldsymbol{r},\boldsymbol{s})\sigma \text{ for all } \sigma \in \mathfrak{S}_d\}. \tag{2.9}$$

We have a bijection

$$\mathrm{Tri}^P_0(n,d) \xrightarrow{\sim} \mathrm{Tri}^P(n,d)/\mathfrak{S}_d, \quad (\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}) \mapsto [\boldsymbol{p},\boldsymbol{r},\boldsymbol{s}].$$

For $(\boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^P(n, d)$, $\boldsymbol{p}' \in P^d$ and $\sigma \in \mathfrak{S}_d$, we define

$$\begin{aligned}\langle \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}\rangle &:= \sharp\{(k,l) \in [1,d]^2 \mid k < l,\ p_k, p_l \in P_{\bar{1}},\ (p_k, r_k, s_k) > (p_l, r_l, s_l)\},\\ \langle \boldsymbol{p}, \boldsymbol{p}'\rangle &:= \sharp\{(k,l) \in [1,d]^2 \mid k > l,\ p_k, p_l' \in P_{\bar{1}}\}.\\ \langle \sigma; \boldsymbol{p}\rangle &:= \sharp\{(k,l) \in [1,d]^2 \mid k < l,\ \sigma^{-1}k > \sigma^{-1}l,\ p_k, p_l \in P_{\bar{1}}\}.\end{aligned}$$

Note that

$$(-1)^{\langle \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}\rangle + \langle \boldsymbol{p}\sigma, \boldsymbol{r}\sigma, \boldsymbol{s}\sigma\rangle} = (-1)^{\langle \sigma; \boldsymbol{p}\rangle}. \tag{2.10}$$

Let us now specialize to the case $P = B$.

Lemma 2.11. *Let $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{t}), (\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}) \in \mathrm{Tri}^B(n, d)$. Assume that, for some $1 \le k < d$, either $\bar{a}_k = \bar{c}_k$ or $\bar{a}_{k+1} = \bar{c}_{k+1}$. Then*

$$(-1)^{\langle \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{t}\rangle + \langle \boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}\rangle + \langle \boldsymbol{a}, \boldsymbol{c}\rangle} = (-1)^{\langle \boldsymbol{a}s_k, \boldsymbol{r}s_k, \boldsymbol{t}s_k\rangle + \langle \boldsymbol{c}s_k, \boldsymbol{t}s_k, \boldsymbol{u}s_k\rangle + \langle \boldsymbol{a}s_k, \boldsymbol{c}s_k\rangle}.$$

Proof. We consider three cases:

Case 1: At least two of $a_k, c_k, a_{k+1}, c_{k+1}$ are even. In this case s_k does not exchange the positions of two odd elements in $\boldsymbol{a}$ or $\boldsymbol{c}$, so $\langle \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{t}\rangle = \langle \boldsymbol{a}s_k, \boldsymbol{r}s_k, \boldsymbol{t}s_k\rangle$ and $\langle \boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}\rangle = \langle \boldsymbol{c}s_k, \boldsymbol{t}s_k, \boldsymbol{u}s_k\rangle$. We also note that a_{k+1} and c_k cannot both be odd, and a_k and c_{k+1} cannot both be odd, so $\langle \boldsymbol{a}, \boldsymbol{c}\rangle = \langle \boldsymbol{a}s_k, \boldsymbol{c}s_k\rangle$.

Case 2: Exactly one of $a_k, c_k, a_{k+1}, c_{k+1}$ is even. By symmetry we may assume that a_k, a_{k+1} are odd and one of c_k, c_{k+1} is even. Then we have

$$\begin{aligned}(-1)^{\langle \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{t}\rangle} &= -(-1)^{\langle \boldsymbol{a}s_k, \boldsymbol{r}s_k, \boldsymbol{t}s_k\rangle},\\ (-1)^{\langle \boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}\rangle} &= (-1)^{\langle \boldsymbol{c}s_k, \boldsymbol{t}s_k, \boldsymbol{u}s_k\rangle},\\ (-1)^{\langle \boldsymbol{a}, \boldsymbol{c}\rangle} &= -(-1)^{\langle \boldsymbol{a}s_k, \boldsymbol{c}s_k\rangle}.\end{aligned}$$

Case 3: All of $a_k, c_k, a_{k+1}, c_{k+1}$ are odd. Then we have

$$\begin{aligned}(-1)^{\langle \boldsymbol{a}, \boldsymbol{r}, \boldsymbol{t}\rangle} &= -(-1)^{\langle \boldsymbol{a}s_k, \boldsymbol{r}s_k, \boldsymbol{t}s_k\rangle},\\ (-1)^{\langle \boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}\rangle} &= -(-1)^{\langle \boldsymbol{c}s_k, \boldsymbol{t}s_k, \boldsymbol{u}s_k\rangle},\\ (-1)^{\langle \boldsymbol{a}, \boldsymbol{c}\rangle} &= (-1)^{\langle \boldsymbol{a}s_k, \boldsymbol{c}s_k\rangle}.\end{aligned}$$

□

Let $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)$. For $b \in B$ and $r, s \in [1, n]$, we denote

$$[\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s} := \sharp\{k \in [1,d] \mid (b_k, r_k, s_k) = (b, r, s)\}, \tag{2.12}$$

and define

$$[\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^! := \prod_{b \in B,\, r,s \in [1,n]} [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s}! = \prod_{b \in B_{\bar{0}},\, r,s \in [1,n]} [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s}! \tag{2.13}$$

(if B is infinite, these are infinite products and all but finitely many factors are 1). Note that

$$|\mathfrak{S}_{\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}}| = [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^!. \tag{2.14}$$

Moreover, we define

$$[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{a}} := \prod_{b\in B_{\mathfrak{a}},\ r,s\in[1,n]} [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^b_{r,s}!\,, \tag{2.15}$$

$$[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}} := \prod_{b\in B_{\mathfrak{c}},\ r,s\in[1,n]} [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^b_{r,s}!\,. \tag{2.16}$$

3. Generalized Schur algebras

Throughout the section, $(A,\mathfrak{a})$ is a fixed good pair with an $(A,\mathfrak{a})$-basis $B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar{1}}$ as in (2.2). Recall that H denotes the set of all nonzero homogeneous elements of A. We also fix $n \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$.

In this section, we construct generalized Schur algebras $T^A_{\mathfrak{a}}(n,d) \subseteq S^A(n,d)$. The definition of the algebra $S^A(n,d)$ is straightforward, while $T^A_{\mathfrak{a}}(n,d)$ is obtained by making a subtle choice of a full-rank sublattice in $S^A(n,d)$ which depends on $\mathfrak{a}$. If $\mathfrak{a} = A_{\bar{0}}$, then $T^A_{\mathfrak{a}}(n,d) = S^A(n,d)$, but in general the algebras are different. In Section 3C, we investigate a natural coproduct on $S^A(n) := \bigoplus_{d\in\mathbb{Z}_{\geq 0}} S^A(n,d)$ and show that

$$T^A_{\mathfrak{a}}(n) := \bigoplus_{d\in\mathbb{Z}_{\geq 0}} T^A_{\mathfrak{a}}(n,d) \subseteq S^A(n)$$

is a subcoalgebra.

3A. ***The algebra $S^A(n,d)$.*** Let $M_n(A)$ be the $\Bbbk$-(super)algebra of $n\times n$ matrices with entries in A and recall the notation (2.6). There is a right action of $\mathfrak{S}_d$ on $M_n(A)^{\otimes d}$ with (super)algebra automorphisms, such that for all $a_1,\dots,a_d \in H$, $r_1,s_1,\dots,r_d,s_d \in [1,n]$ and $\sigma \in \mathfrak{S}_d$, we have

$$(\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d})^\sigma = (-1)^{\langle \sigma;\boldsymbol{a}\rangle} \xi^{a_{\sigma 1}}_{r_{\sigma 1},s_{\sigma 1}} \otimes \cdots \otimes \xi^{a_{\sigma d}}_{r_{\sigma d},s_{\sigma d}}.$$

The algebra $S^A(n,d)$ is defined as the corresponding algebra of invariants

$$S^A(n,d) := (M_n(A)^{\otimes d})^{\mathfrak{S}_d}.$$

Note that $S^A(n,d)$ is unital if and only if so is A.

Remark 3.1. It is important to note that the superstructure plays a crucial role here, even if one chooses to ultimately ignore the superstructure on $S^A(n,d)$. Namely, choosing a different superstructure on the same algebra A will lead to different (even different dimensional) algebras of invariants $S^A(n,d)$; see for instance Example 5.28.

For $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^H(n,d)$, we define elements

$$\begin{aligned}\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} &:= \sum_{\sigma\in {}^{\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}} (\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d})^\sigma \\ &= \sum_{(\boldsymbol{a}',\boldsymbol{r}',\boldsymbol{s}')\sim(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s})} (-1)^{\langle \boldsymbol{a},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{r}',\boldsymbol{s}'\rangle}\, \xi^{a'_1}_{r'_1,s'_1} \otimes \cdots \otimes \xi^{a'_d}_{r'_d,s'_d}.\end{aligned} \tag{3.2}$$

in $S^A(n,d)$, where we have used (2.10) to obtain the last equality. The following is clear (as noted in [Evseev and Kleshchev 2017, Lemma 6.10]):

Lemma 3.3. *We have that* $\{\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}] \in \mathrm{Tri}^B(n,d)/\mathfrak{S}_d\}$ *is a basis of* $S^A(n,d)$.

Lemma 3.4. *If* $(\boldsymbol{a}',\boldsymbol{r}',\boldsymbol{s}') \sim (\boldsymbol{a},\boldsymbol{r},\boldsymbol{s})$ *are elements of* $\mathrm{Tri}^H(n,d)$, *then*

$$\xi^{\boldsymbol{a}'}_{\boldsymbol{r}',\boldsymbol{s}'} = (-1)^{\langle \boldsymbol{a},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{r}',\boldsymbol{s}'\rangle} \xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}.$$

Proof. This follows from (3.2). □

For $(\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}),\ (\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}) \in \mathrm{Tri}^H(n,d)$ and $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$, define the structure constants $f^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}}$ from

$$\xi^{\boldsymbol{a}}_{\boldsymbol{p},\boldsymbol{q}}\, \xi^{\boldsymbol{c}}_{\boldsymbol{u},\boldsymbol{v}} = \sum_{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}] \in \mathrm{Tri}^B(n,d)/\mathfrak{S}_d} f^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}}\, \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}. \tag{3.5}$$

Note by Lemma 3.4 that if $(\boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}') \sim (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})$ then

$$f^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}} = (-1)^{\langle \boldsymbol{b},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}'\rangle} f^{\boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}'}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}}.$$

Recalling the notation (2.5), the following generalization of Green's product rule [Green 2007, (2.3b)] follows from [Evseev and Kleshchev 2017, (6.14)].

Proposition 3.6. *Let* $(\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}),\ (\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}) \in \mathrm{Tri}^H(n,d)$ *and* $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$. *Then*

$$f^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}} = \sum_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}} (-1)^{\langle \boldsymbol{a},\boldsymbol{p},\boldsymbol{q}\rangle + \langle \boldsymbol{c},\boldsymbol{u},\boldsymbol{v}\rangle + \langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle}\, \kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'},$$

where the sum is over all $\boldsymbol{a}',\boldsymbol{c}' \in H^d$ *and* $\boldsymbol{t} \in [1,n]$ *such that* $(\boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}) \sim (\boldsymbol{a},\boldsymbol{p},\boldsymbol{q})$ *and* $(\boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}) \sim (\boldsymbol{c},\boldsymbol{u},\boldsymbol{v})$.

We can collect some of the equal terms in the formula above to rewrite it in the following form:

Corollary 3.7. *Let* $(\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}),\ (\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}) \in \mathrm{Tri}^H(n,d)$, $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$, *and let* X *be the set of all* $(\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}) \in H^d \times H^d \times [1,n]$ *such that* $(\boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}) \sim (\boldsymbol{a},\boldsymbol{p},\boldsymbol{q})$, $(\boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}) \sim (\boldsymbol{c},\boldsymbol{u},\boldsymbol{v})$, *and* $\bar{a}'_k + \bar{c}'_k = \bar{b}_k$ *for all* $k \in [1,d]$.

(i) *If* $(\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}) \in X$ *then* $(\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t})\sigma \in X$ *for any* $\sigma \in \mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}$. *Let* $[\![\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}]\!] := \{(\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t})\sigma \mid \sigma \in \mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}\} \subseteq X$ *denote the corresponding* $\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}$-*orbit.*

(ii) $\kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'}$ *and the parity of* $\langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle$ *depend only on the orbit* $[\![\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}]\!]$.

(iii) *The structure constant* $f^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}}$ *equals*

$$\sum_{[\![\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}]\!] \in X/\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}} (-1)^{\langle \boldsymbol{a},\boldsymbol{p},\boldsymbol{q}\rangle + \langle \boldsymbol{c},\boldsymbol{u},\boldsymbol{v}\rangle + \langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle} [\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}} : \mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}} \cap \mathfrak{S}_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}}]\, \kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'}.$$

Proof. Let $\sigma \in \mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}$ and $(\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}) \in X$.

(i) To show that $(\boldsymbol{a}'\sigma, \boldsymbol{c}'\sigma, \boldsymbol{t}\sigma) \in X$, note that

$$(\boldsymbol{a}'\sigma, \boldsymbol{r}, \boldsymbol{t}\sigma) = (\boldsymbol{a}'\sigma, \boldsymbol{r}\sigma, \boldsymbol{t}\sigma) \sim (\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q})$$

and similarly $(\boldsymbol{c}'\sigma, \boldsymbol{t}\sigma, \boldsymbol{s}) \sim (\boldsymbol{c}, \boldsymbol{u}, \boldsymbol{v})$. Finally, we have $\bar{a}'_{\sigma k} + \bar{c}'_{\sigma k} = \bar{b}_{\sigma k} = \bar{b}_k$ for all k.

(ii) We have $\kappa^{\boldsymbol{b}}_{\boldsymbol{a}'\sigma,\boldsymbol{c}'\sigma} = \kappa^{\boldsymbol{b}\sigma}_{\boldsymbol{a}'\sigma,\boldsymbol{c}'\sigma} = \kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'}$, giving the first statement of (ii). To complete the proof of (ii), we now show that

$$(-1)^{\langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle} = (-1)^{\langle \boldsymbol{a}'\sigma,\boldsymbol{r},\boldsymbol{t}\sigma\rangle + \langle \boldsymbol{c}'\sigma,\boldsymbol{t}\sigma,\boldsymbol{s}\rangle + \langle \boldsymbol{a}'\sigma,\boldsymbol{c}'\sigma\rangle}.$$

Write σ as a reduced product of simple transpositions $\sigma = s_{l_1} \cdots s_{l_m}$ (it is not in general true that $s_{l_1}, \dots, s_{l_m} \in \mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}$). Since $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}(n, d)$, we have $\sigma k = k$ for all k such that b_k is odd. Therefore for all $1 \le j \le m$, at least one of $(\boldsymbol{b}s_{l_1} \cdots s_{l_{j-1}})_{l_j}$, $(\boldsymbol{b}s_{l_1} \cdots s_{l_{j-1}})_{l_j+1}$ is even — i.e., no two odd elements are ever exchanged by the simple transpositions that comprise σ.

For $1 \le j \le m$, either $(\boldsymbol{a}'s_{l_1} \cdots s_{l_{j-1}})_{l_j}$ and $(\boldsymbol{c}'s_{l_1} \cdots s_{l_{j-1}})_{l_j}$ are of the same parity, or $(\boldsymbol{a}'s_{l_1} \cdots s_{l_{j-1}})_{l_j+1}$ and $(\boldsymbol{c}'s_{l_1} \cdots s_{l_{j-1}})_{l_j+1}$ are of the same parity, by the above paragraph and the fact that $\bar{a}'_k + \bar{c}'_k = \bar{b}_k$ for all k. Therefore we may repeatedly apply Lemma 2.11 to get

$$\begin{aligned}
(-1)^{\langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle} &= (-1)^{\langle \boldsymbol{a}'s_{l_1},\boldsymbol{r}s_{l_1},\boldsymbol{t}s_{l_1}\rangle + \langle \boldsymbol{c}'s_{l_1},\boldsymbol{t}s_{l_1},\boldsymbol{s}s_{l_1}\rangle + \langle \boldsymbol{a}'s_{l_1},\boldsymbol{c}'s_{l_1}\rangle} \\
&= (-1)^{\langle \boldsymbol{a}'s_{l_1}s_{l_2},\boldsymbol{r}s_{l_1}s_{l_2},\boldsymbol{t}s_{l_1}s_{l_2}\rangle + \langle \boldsymbol{c}'s_{l_1}s_{l_2},\boldsymbol{t}s_{l_1}s_{l_2},\boldsymbol{s}s_{l_1}s_{l_2}\rangle + \langle \boldsymbol{a}'s_{l_1}s_{l_2},\boldsymbol{c}'s_{l_1}s_{l_2}\rangle} \\
&= \cdots \\
&= (-1)^{\langle \boldsymbol{a}'\sigma,\boldsymbol{r}\sigma,\boldsymbol{t}\sigma\rangle + \langle \boldsymbol{c}'\sigma,\boldsymbol{t}\sigma,\boldsymbol{s}\sigma\rangle + \langle \boldsymbol{a}'\sigma,\boldsymbol{c}'\sigma\rangle} \\
&= (-1)^{\langle \boldsymbol{a}'\sigma,\boldsymbol{r},\boldsymbol{t}\sigma\rangle + \langle \boldsymbol{c}'\sigma,\boldsymbol{t}\sigma,\boldsymbol{s}\rangle + \langle \boldsymbol{a}'\sigma,\boldsymbol{c}'\sigma\rangle},
\end{aligned}$$

completing the proof of (ii).

(iii) As $\kappa^{b_k}_{a'_k,c'_k}$ equals 0 unless $\bar{a}'_k + \bar{c}'_k = \bar{b}_k$, we may assume that the summation in Proposition 3.6 is over all $(\boldsymbol{a}', \boldsymbol{c}', \boldsymbol{t}) \in X$. By (i), (ii) and Proposition 3.6, we have

$$f^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{a},\boldsymbol{p},\boldsymbol{q};\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}} = \sum_{[\![\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}]\!] \in X/\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}} \#[\![\boldsymbol{a}', \boldsymbol{c}', \boldsymbol{t}]\!]\, (-1)^{\langle \boldsymbol{a},\boldsymbol{p},\boldsymbol{q}\rangle + \langle \boldsymbol{c},\boldsymbol{u},\boldsymbol{v}\rangle + \langle \boldsymbol{a}',\boldsymbol{r},\boldsymbol{t}\rangle + \langle \boldsymbol{c}',\boldsymbol{t},\boldsymbol{s}\rangle + \langle \boldsymbol{a}',\boldsymbol{c}'\rangle}\, \kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'}.$$

It remains to note that $\#[\![\boldsymbol{a}', \boldsymbol{c}', \boldsymbol{t}]\!] = |\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}/\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}} \cap \mathfrak{S}_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}}|$. □

Let τ be a homogeneous anti-involution on A. Then τ induces a homogeneous anti-involution

$$\tau_n : M_n(A) \to M_n(A), \quad \xi^a_{r,s} \mapsto \xi^{\tau(a)}_{s,r},$$

which in turn induces an anti-involution

$$\tau_{n,d} : S^A(n, d) \to S^A(n, d), \quad \xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} \mapsto \xi^{\boldsymbol{a}^\tau}_{\boldsymbol{s},\boldsymbol{r}}, \tag{3.8}$$

where for $\boldsymbol{a} = a_1 \cdots a_d \in H^d$, we have denoted $\boldsymbol{a}^\tau := \tau(a_1) \cdots \tau(a_d) \in H^d$.

3B. ***The algebra $T^A_{\mathfrak{a}}(n,d)$.*** Recalling the notation (2.16), for $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in \mathrm{Tri}^B(n,d)$, we set

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} := [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}}\,\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}. \tag{3.9}$$

Define the $\Bbbk$-submodule $T^A_{\mathfrak{a}}(n,d)\subseteq S^A(n,d)$ to be

$$T^A_{\mathfrak{a}}(n,d) := \mathrm{span}\big(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in \mathrm{Tri}^B(n,d)\big).$$

It will turn out that $T^A_{\mathfrak{a}}(n,d)$ depends only on $\mathfrak{a}$, not on $\mathfrak{c}$ or B; see Proposition 4.11.

Lemma 3.10. *We have that* $\{\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in \mathrm{Tri}^B(n,d)/\mathfrak{S}_d\}$ *is a basis of* $T^A_{\mathfrak{a}}(n,d)$.

Proof. Follows from the definition and Lemma 3.3. □

Lemma 3.11. *Let* $a_1,\dots,a_d\in\mathfrak{a}\cup A_{\bar 1}$ *and* $\boldsymbol{r},\boldsymbol{s}\in[1,n]^d$. *Then* $\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}\in T^A_{\mathfrak{a}}(n,d)$.

Proof. By assumption, for $1\le l\le d$, either $a_l=\sum_{b\in\mathfrak{a}} c_{l,b}b$ or $a_l=\sum_{b\in B_{\bar 1}} c_{l,b}b$, with $c_{l,b}\in\Bbbk$. It follows that $\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}$ is a linear combination of the elements $\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$ such that $\boldsymbol{b}$ is of the form $b_1\cdots b_d$ with $b_l\in B_{\mathfrak{a}}\cup B_{\bar 1}$ for all $l=1,\dots,d$. But for such $\boldsymbol{b}$, we have $\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}=\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\in T^A_{\mathfrak{a}}(n,d)$. □

Proposition 3.12. *We have that* $T^A_{\mathfrak{a}}(n,d)\subseteq S^A(n,d)$ *is a* $\Bbbk$*-subalgebra. It is a unital subalgebra if* $(A,\mathfrak{a})$ *is a unital good pair.*

Proof. By Lemma 3.11, if $1_A\in\mathfrak{a}$, then the identity $1_A\otimes\cdots\otimes 1_A\in S^A(n,d)$ belongs to $T^A_{\mathfrak{a}}(n,d)$, so we only have to prove the first statement of the lemma.

We now fix $(\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}),(\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}),(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B(n,d)$ and apply Corollary 3.7. Using the notation as in the corollary, assume that $(\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t})\in X$ is such that $\kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'}\neq 0$. In view of Corollary 3.7(iii), it suffices to prove that the integer

$$M := [\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}]^!_{\mathfrak{c}}\cdot[\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}]^!_{\mathfrak{c}}\cdot|\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}/\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}\cap\mathfrak{S}_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}}|$$

is divisible by $[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}}$. For $b,a',c'\in B$ and $r,s,t\in[1,n]$, define

$$m^{a',b,c'}_{r,s,t} := \#\{k\in[1,d]\mid a'_k=a',\ b_k=b,\ c'_k=c',\ r_k=r,\ s_k=s,\ t_k=t\}.$$

Then, using that $(\boldsymbol{a}',\boldsymbol{r},\boldsymbol{t})\sim(\boldsymbol{a},\boldsymbol{p},\boldsymbol{q})$, $(\boldsymbol{c}',\boldsymbol{t},\boldsymbol{s})\sim(\boldsymbol{c},\boldsymbol{u},\boldsymbol{v})$, we obtain

$$|\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}\cap\mathfrak{S}_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}}| = \prod_{a',b,c'\in B,\ r,s,t\in[1,n]} m^{a',b,c'}_{r,s,t}! \tag{3.13}$$

$$[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^b_{r,s} = \sum_{a',c'\in B,\ t\in[1,n]} m^{a',b,c'}_{r,s,t} \qquad (b\in B,\ r,s\in[1,n]), \tag{3.14}$$

$$[\boldsymbol{a},\boldsymbol{p},\boldsymbol{q}]^a_{p,q} = \sum_{b,c'\in B,\ t\in[1,n]} m^{a,b,c'}_{p,t,q} \qquad (a\in B,\ p,q\in[1,n]), \tag{3.15}$$

$$[\boldsymbol{c},\boldsymbol{u},\boldsymbol{v}]^c_{u,v} = \sum_{a',b\in B,\ t\in[1,n]} m^{a',b,c}_{t,v,u} \qquad (c\in B,\ u,v\in[1,n]). \tag{3.16}$$

By (3.14), for every $b \in B$ and $r, s \in [1, n]$, we have that

$$y^b_{r,s} := \frac{[\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s}!}{\prod_{a',c' \in B,\ t \in [1,n]} m^{a',b,c'}_{r,s,t}!} \in \mathbb{Z}.$$

So

$$C := \prod_{b \in B_{\mathfrak{a}} \cup B_{\bar{1}},\ r,s \in [1,n]} y^b_{r,s} \in \mathbb{Z},$$

and by (3.13), we have

$$|\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}} / \mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}} \cap \mathfrak{S}_{\boldsymbol{a}',\boldsymbol{c}',\boldsymbol{t}}| = \prod_{b \in B,\ r,s \in [1,n]} y^b_{r,s} = C \cdot \prod_{b \in B_{\mathfrak{c}},\ r,s \in [1,n]} y^b_{r,s}.$$

Now we claim that $b \in B_{\mathfrak{c}}$ and $m^{a',b,c'}_{r,s,t} > 1$ imply $a' \in B_{\mathfrak{c}}$ or $c' \in B_{\mathfrak{c}}$. Indeed, if $a' \in B_{\bar{1}}$, then using (3.15), we get $m^{a',b,c'}_{r,s,t} \leq [\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}]^{a'}_{r,t} \leq 1$ as $(\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}) \in \mathrm{Tri}^B(n, d)$, which is a contradiction. Thus $a' \in B_{\bar{0}}$. Similarly, $c' \in B_{\bar{0}}$. If $a', c' \in B_{\mathfrak{a}}$, then $\kappa^b_{a',c'} = 0$ since $\mathfrak{a}$ is closed under multiplication. Since $m^{a',b,c'}_{r,s,t} > 0$, this implies $\kappa^{\boldsymbol{b}}_{\boldsymbol{a}',\boldsymbol{c}'} = 0$, which contradicts our choice of $(\boldsymbol{a}', \boldsymbol{c}', \boldsymbol{t})$, proving the claim.

By the claim, for $b \in B_{\mathfrak{c}}$ and $r, s \in [1, n]$, we may write

$$y^b_{r,s} = \frac{[\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s}!}{\left(\prod_{a' \in B_{\mathfrak{c}},\, c' \in B,\, t \in [1,n]} m^{a',b,c'}_{r,s,t}!\right)\left(\prod_{a' \in B_{\mathfrak{a}} \cup B_{\bar{1}},\, c' \in B_{\mathfrak{c}},\, t \in [1,n]} m^{a',b,c'}_{r,s,t}!\right)}.$$

So M equals

$$\left(\prod_{\substack{a' \in B_{\mathfrak{c}} \\ r,t \in [1,n]}} [\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}]^{a'}_{r,t}!\right) \cdot \left(\prod_{\substack{c' \in B_{\mathfrak{c}} \\ t,s \in [1,n]}} [\boldsymbol{c}, \boldsymbol{u}, \boldsymbol{v}]^{c'}_{t,s}!\right) \cdot C \cdot \prod_{b \in B_{\mathfrak{c}},\, r,s \in [1,n]} y^b_{r,s}$$

$$= \left(\prod_{\substack{a' \in B_{\mathfrak{c}} \\ r,t \in [1,n]}} \frac{[\boldsymbol{a}, \boldsymbol{p}, \boldsymbol{q}]^{a'}_{r,t}!}{\prod_{c' \in B,\, b \in B_{\mathfrak{c}},\, s \in [1,n]} m^{a',b,c'}_{r,s,t}!}\right)\left(\prod_{\substack{c' \in B_{\mathfrak{c}} \\ t,s \in [1,n]}} \frac{[\boldsymbol{c}, \boldsymbol{u}, \boldsymbol{v}]^{c'}_{t,s}!}{\prod_{a' \in B_{\mathfrak{a}} \cup B_{\bar{1}},\, b \in B_{\mathfrak{c}},\, r \in [1,n]} m^{a',b,c'}_{r,s,t}!}\right)$$

$$\times C \cdot \prod_{b \in B_{\mathfrak{c}},\, r,s \in [1,n]} [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s}!.$$

Note that the first factor is an integer by (3.15), and the second factor is an integer by (3.16). We have thus proved that M is divisible by

$$\prod_{b \in B_{\mathfrak{c}},\, r,s \in [1,n]} [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^b_{r,s}! = [\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}]^!_{\mathfrak{c}},$$

completing the proof. □

Remark 3.17. When $\mathfrak{a} = A_{\bar{0}}$, we have $T^A_{\mathfrak{a}}(n, d) = S^A(n, d)$. In the general situation however, when $\mathfrak{a} \subsetneq A_{\bar{0}}$ and $d \geq 2$, the subalgebra $T^A_{\mathfrak{a}}(n, d) \subseteq S^A(n, d)$ is a proper sublattice in $S^A(n, d)$. An explicit example of this distinction, when A is the *extended zigzag algebra*, is provided in Example 5.28.

Remark 3.18. We sometimes refer to the algebras $S^A(n,d)$ and $T^A_{\mathfrak{a}}(n,d)$ as generalized Schur algebras since for the case $A=\Bbbk$ they both return the classical Schur algebra $S^{\Bbbk}(n,d)$. In [Kleshchev and Muth 2018], we sometimes also refer to $T^A_{\mathfrak{a}}(n,d)$ informally as the "schurification" of the pair $(A,\mathfrak{a})$. We point out that various other generalizations of the classical Schur algebras were studied in the literature; see especially [Donkin 1986; 1987; 1998; Doty 1998].

Remark 3.19. Let us remark on our reasoning for studying the more subtly defined construction $T^A_{\mathfrak{a}}(n,d)$, as opposed to just the invariant algebra $S^A(n,d)$.

First, the elements (3.9) arising in the definition of $T^A_{\mathfrak{a}}(n,d)$ are motivated in part by analogous elements in natural examples, such as Turner's double algebra. Indeed, a specific choice of good pair $(A,\mathfrak{a})$ allows $T^A_{\mathfrak{a}}(n,d)$ to recover Turner's double D, whereas $S^A(n,d)$ instead recovers Turner's divided power algebra $'D$, as defined in [Evseev and Kleshchev 2017]. Moreover, it is certain $T^A_{\mathfrak{a}}(n,d)$'s — and not $S^A(n,d)$'s — that appear as "local descriptions" of blocks of symmetric groups (see [Evseev and Kleshchev 2018]), classical Schur algebras (conjecturally; see [Kleshchev and Muth 2018]), and (potentially) other interesting classical objects. We refer the reader to Section 5D2 for more details on the relationship between generalized Schur algebras and Turner's double algebras.

Additionally, the algebra $T^A_{\mathfrak{a}}(n,d)$ tends to have more natural properties than $S^A(n,d)$. For example, the following properties hold for $T^A_{\mathfrak{a}}(n,d)$, but do not generally hold for $S^A(n,d)$:

(i) If A is symmetric (and $(A,\mathfrak{a})$ satisfies certain conditions), then $T^A_{\mathfrak{a}}(n,d)$ is symmetric as well; see Corollary 6.7.

(ii) If A is cellular/quasihereditary (and $(A,\mathfrak{a})$ satisfies certain conditions), then $T^A_{\mathfrak{a}}(n,d)$ is cellular/quasihereditary when $n\geq d$; see [Kleshchev and Muth 2018].

(iii) The algebra $T^A_{\mathfrak{a}}(n,d)$ is generated by very special elements (see Theorem 4.13), which is crucial in many situations; see, e.g., the proof of Corollary 8.24 in [Evseev and Kleshchev 2018].

3C. *Coproduct on generalized Schur algebras.* In this subsection, it will be convenient to use the following notation. Let $\mathcal{T}=(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in \mathrm{Tri}^B(n,d)$. We write

$$\xi_{\mathcal{T}}:=\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}},\quad \eta_{\mathcal{T}}:=\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}},\quad {}^{\mathcal{T}}\mathscr{D}:={}^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}\mathscr{D},\quad [\mathcal{T}]^!_{\mathfrak{c}}:=[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}},\quad \mathcal{T}\sigma:=(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\sigma,\quad \text{etc.}$$

If $d=d_1+d_2$, $\mathcal{T}^1=(\boldsymbol{b}^1,\boldsymbol{r}^1,\boldsymbol{s}^1)\in\mathrm{Tri}^B(n,d_1)$ and $\mathcal{T}^2=(\boldsymbol{b}^2,\boldsymbol{r}^2,\boldsymbol{s}^2)\in\mathrm{Tri}^B(n,d_2)$, we denote

$$\mathcal{T}^1\mathcal{T}^2:=(\boldsymbol{b}^1\boldsymbol{b}^2,\boldsymbol{r}^1\boldsymbol{r}^2,\boldsymbol{s}^1\boldsymbol{s}^2)\in B^d\times[1,n]^d\times[1,n]^d.$$

In general $\mathcal{T}^1\mathcal{T}^2$ does not need to be an element of $\mathrm{Tri}^B(n,d)$.

Recall the notation (2.9). For $\mathcal{T}\in\mathrm{Tri}^B_0(n,d)$ and $0\leq l\leq d$, define the sets of *l-splits of $\mathcal{T}$* and *splits of $\mathcal{T}$* as

$$\mathrm{Spl}_l(\mathcal{T}):=\big\{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Tri}^B_0(n,l)\times\mathrm{Tri}^B_0(n,d-l)\mid \mathcal{T}^1\mathcal{T}^2\sim\mathcal{T}\big\},$$
$$\mathrm{Spl}(\mathcal{T}):=\bigsqcup_{0\leq l\leq d}\mathrm{Spl}_l(\mathcal{T}).$$

For $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}_l(\mathcal{T})$, let $\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}$ be the unique element of ${}^{\mathcal{T}}\mathscr{D}$ such that

$$\mathcal{T}\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2} = \mathcal{T}^1\mathcal{T}^2.$$

Let $\iota_l : \mathfrak{S}_l \times \mathfrak{S}_{d-l} \to \mathfrak{S}_d$ be the standard inclusion. Let $\mathcal{T} \in \mathrm{Tri}^B_0(n, d)$. Note that for every $\sigma \in {}^{\mathcal{T}}\mathscr{D}$ there exist unique $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}_l(\mathcal{T})$, $\sigma_1 \in {}^{\mathcal{T}^1}\mathscr{D}$ and $\sigma_2 \in {}^{\mathcal{T}^2}\mathscr{D}$ such that $\sigma = \sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}\iota_l(\sigma_1, \sigma_2)$. In other words, the map

$$\bigsqcup_{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}_l(\mathcal{T})} {}^{\mathcal{T}^1}\mathscr{D} \times {}^{\mathcal{T}^2}\mathscr{D} \to {}^{\mathcal{T}}\mathscr{D} \tag{3.20}$$

sending $(\sigma_1, \sigma_2) \in {}^{\mathcal{T}^1}\mathscr{D} \times {}^{\mathcal{T}^2}\mathscr{D}$ to $\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}\iota_l(\sigma_1, \sigma_2)$, is a bijection.

Note that for $(\sigma_1, \sigma_2) \in {}^{\mathcal{T}^1}\mathscr{D} \times {}^{\mathcal{T}^2}\mathscr{D}$ we have

$$\langle \sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}\iota_l(\sigma_1, \sigma_2); \boldsymbol{b}\rangle = \langle \sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}; \boldsymbol{b}\rangle + \langle \sigma_1; \boldsymbol{b}^1\rangle + \langle \sigma_2; \boldsymbol{b}^2\rangle. \tag{3.21}$$

Recall from [Evseev and Kleshchev 2017, §3.3] that

$$\bigoplus_{d\geq 0} M_n(A)^{\otimes d}$$

is a supercoalgebra with the coproduct ∇ defined by

$$\begin{aligned} \nabla : M_n(A)^{\otimes d} &\to \bigoplus_{l=0}^{d} M_n(A)^{\otimes l} \otimes M_n(A)^{\otimes (d-l)} \\ \xi_1 \otimes \cdots \otimes \xi_d &\mapsto \sum_{l=0}^{d} (\xi_1 \otimes \cdots \otimes \xi_l) \otimes (\xi_{l+1} \otimes \cdots \otimes \xi_d). \end{aligned}$$

Let

$$S^A(n) := \bigoplus_{d\geq 0} S^A(n, d) \quad \text{and} \quad T^A_{\mathfrak{a}}(n) := \bigoplus_{d\geq 0} T^A_{\mathfrak{a}}(n, d). \tag{3.22}$$

We next prove that these are subsupercoalgebras of $\bigoplus_{d\geq 0} M_n(A)^{\otimes d}$. The following result is actually contained in [Evseev and Kleshchev 2017], but we give a proof using our current notation for reader's convenience.

Lemma 3.23 [Evseev and Kleshchev 2017, (6.12)]. *If $\mathcal{T} = (\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B_0(n, d)$, then*

$$\nabla(\xi_{\mathcal{T}}) = \sum_{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}(\mathcal{T})} (-1)^{\langle \sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2}; \boldsymbol{b}\rangle} \xi_{\mathcal{T}^1} \otimes \xi_{\mathcal{T}^2}.$$

In particular, $S^A(n)$ is a subsupercoalgebra of $\bigoplus_{d\geq 0} M_n(A)^{\otimes d}$.

Proof. Writing $\sum_{\mathrm{Spl}_l(\mathcal{T})}$ for the sum over all $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}_l(\mathcal{T})$ with

$$\mathcal{T}^1 = (\boldsymbol{b}^1, \boldsymbol{r}^1, \boldsymbol{s}^1) \quad \text{and} \quad \mathcal{T}^2 = (\boldsymbol{b}^2, \boldsymbol{r}^2, \boldsymbol{s}^2),$$

we have that $\nabla(\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})$ equals

$$\sum_{\sigma\in{}^{\mathcal{T}}\mathscr{D}}(-1)^{\langle\sigma;\boldsymbol{b}\rangle}\nabla(\xi^{b_{\sigma 1}}_{r_{\sigma 1},s_{\sigma 1}}\otimes\cdots\otimes\xi^{b_{\sigma d}}_{r_{\sigma d},s_{\sigma d}})$$

$$=\sum_{\sigma\in{}^{\mathcal{T}}\mathscr{D}}(-1)^{\langle\sigma;\boldsymbol{b}\rangle}\sum_{l=0}^{d}(\xi^{b_{\sigma 1}}_{r_{\sigma 1},s_{\sigma 1}}\otimes\cdots\otimes\xi^{b_{\sigma l}}_{r_{\sigma l},s_{\sigma l}})\otimes(\xi^{b_{\sigma(l+1)}}_{r_{\sigma(l+1)},s_{\sigma(l+1)}}\otimes\cdots\otimes\xi^{b_{\sigma d}}_{r_{\sigma d},s_{\sigma d}})$$

$$=\sum_{l=0}^{d}\sum_{\mathrm{Spl}_l(\mathcal{T})}\sum_{\substack{\sigma_1\in{}^{\mathcal{T}^1}\mathscr{D}\\ \sigma_2\in{}^{\mathcal{T}^2}\mathscr{D}}}(-1)^{\langle\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2};\boldsymbol{b}\rangle}(\xi^{b^1_1}_{r^1_1,s^1_1}\otimes\cdots\otimes\xi^{b^1_l}_{r^1_l,s^1_l})^{\sigma_1}\otimes(\xi^{b^2_1}_{r^2_1,s^2_1}\otimes\cdots\otimes\xi^{b^2_{d-l}}_{r^2_{d-l},s^2_{d-l}})^{\sigma_2}$$

$$=\sum_{l=0}^{d}\sum_{\mathrm{Spl}_l(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})}(-1)^{\langle\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2};\boldsymbol{b}\rangle}\xi_{\mathcal{T}^1}\otimes\xi_{\mathcal{T}^2},$$

where we have used the bijection (3.20) and the sign identity (3.21) for the second equality above. □

Corollary 3.24. *If $\mathcal{T}=(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B_0(n,d)$, then*

$$\nabla(\eta_{\mathcal{T}})=\sum_{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}(\mathcal{T})}(-1)^{\langle\sigma^{\mathcal{T}}_{\mathcal{T}^1,\mathcal{T}^2};\boldsymbol{b}\rangle}\frac{[\mathcal{T}]^!_{\mathfrak{c}}}{[\mathcal{T}^1]^!_{\mathfrak{c}}[\mathcal{T}^2]^!_{\mathfrak{c}}}\eta_{\mathcal{T}^1}\otimes\eta_{\mathcal{T}^2},$$

with $[\mathcal{T}]^!_{\mathfrak{c}}/([\mathcal{T}^1]^!_{\mathfrak{c}}[\mathcal{T}^2]^!_{\mathfrak{c}})\in\mathbb{Z}$. In particular, $T^A_{\mathfrak{a}}(n)$ is a subsupercoalgebra of $\bigoplus_{d\geq 0}M_n(A)^{\otimes d}$.

Proof. By Lemma 3.23, we just have to check that $[\mathcal{T}^1]^!_{\mathfrak{c}}[\mathcal{T}^2]^!_{\mathfrak{c}}$ divides $[\mathcal{T}]^!_{\mathfrak{c}}$ whenever $(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}_l(\mathcal{T})$. But in this situation we have that $[\mathcal{T}]^b_{r,s}=[\mathcal{T}^1]^b_{r,s}+[\mathcal{T}^2]^b_{r,s}$ for all $b\in B$ and $1\leq r,s\leq n$, which implies the required divisibility. □

4. Superbialgebra structure

Recall the definition of $S^A(n)$ and $T^A_{\mathfrak{a}}(n)$ from (3.22). In this section we study the star-product on $S^A(n)$ and $T^A_{\mathfrak{a}}(n)$ which together with the coproduct ∇ from Section 3C makes them into superbialgebras. For $S^A(n)$ this is well-known; see, for example, [Evseev and Kleshchev 2017, Lemma 3.12].

4A. *Star-product.* For $d,e\in Z_{\geq 0}$, let ${}^{(d,e)}\mathscr{D}$ be the set of the shortest coset representatives for $(\mathfrak{S}_d\times\mathfrak{S}_e)\backslash\mathfrak{S}_{d+e}$. Given $\xi_1\in M_n(A)^{\otimes d}$ and $\xi_2\in M_n(A)^{\otimes e}$, we define

$$\xi_1*\xi_2:=\sum_{\sigma\in{}^{(d,e)}\mathscr{D}}(\xi_1\otimes\xi_2)^{\sigma}. \tag{4.1}$$

It is well-known that this $*$-product makes $\bigoplus_{d\geq 0}M_n(A)^{\otimes d}$ into an associative supercommutative superalgebra.

Lemma 4.2. *For $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B(n,d)$ and $(\boldsymbol{c},\boldsymbol{t},\boldsymbol{u})\in\mathrm{Tri}^B(n,e)$, we have*

(i) $\xi^{b_1}_{r_1,s_1}*\cdots*\xi^{b_d}_{r_d,s_d}=[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!\,\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$,

(ii) $\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}} = \dfrac{[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!}{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^![\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!}\xi^{\boldsymbol{bc}}_{\boldsymbol{rt},\boldsymbol{su}}$,

(iii) $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}} = \dfrac{[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!_{\mathfrak{a}}}{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{a}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{a}}}\eta^{\boldsymbol{bc}}_{\boldsymbol{rt},\boldsymbol{su}}$,

where $[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!/([\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^![\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!)$ *and* $[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!_{\mathfrak{a}}/([\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{a}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{a}})$ *are integers, and the right-hand sides of* (ii) *and* (iii) *are taken to be zero when* $(\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}) \notin \mathrm{Tri}^B(n, d+e)$.

Proof. We have that $\xi^{b_1}_{r_1,s_1} * \cdots * \xi^{b_d}_{r_d,s_d}$ is equal to

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_d}(\xi^{b_1}_{r_1,s_1}\otimes\cdots\otimes\xi^{b_d}_{r_d,s_d})^{\sigma} &= \sum_{\sigma\in{}^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}}\ \sum_{\sigma'\in\mathfrak{S}_{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}}(\xi^{b_1}_{r_1,s_1}\otimes\cdots\otimes\xi^{b_d}_{r_d,s_d})^{\sigma'\sigma}\\ &= [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!\sum_{\sigma\in{}^{\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}}(\xi^{b_1}_{r_1,s_1}\otimes\cdots\otimes\xi^{b_d}_{r_d,s_d})^{\sigma}\\ &= [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!\,\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}},\end{aligned}$$

proving (i). Thus

$$\begin{aligned}[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^![\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!\,\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}} &= (\xi^{b_1}_{r_1,s_1} * \cdots * \xi^{b_d}_{r_d,s_d}) * (\xi^{c_1}_{t_1,u_1} * \cdots * \xi^{c_e}_{t_e,u_e})\\ &= \xi^{b_1}_{r_1,s_1} * \cdots * \xi^{b_d}_{r_d,s_d} * \xi^{c_1}_{t_1,u_1} * \cdots * \xi^{c_e}_{t_e,u_e}\\ &= [\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!\,\xi^{\boldsymbol{bc}}_{\boldsymbol{rt},\boldsymbol{su}},\end{aligned}$$

where the last line is interpreted as 0 if $(\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}) \notin \mathrm{Tri}^B(n, d+e)$. Therefore

$$\begin{aligned}[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{a}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{a}}\,\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}} &= [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^![\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!\,\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}\\ &= [\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!\,\xi^{\boldsymbol{bc}}_{\boldsymbol{rt},\boldsymbol{su}}\\ &= [\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!_{\mathfrak{a}}\,\eta^{\boldsymbol{bc}}_{\boldsymbol{rt},\boldsymbol{su}}.\end{aligned}$$

Now (ii) and (iii) follow by noting that

$$[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^b_{r,s} = [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^b_{r,s} + [\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^b_{r,s}$$

for all b, r, s. □

Corollary 4.3. *$S^A(n)$ and $T^A_{\mathfrak{a}}(n)$ are subsuperalgebras of $\bigoplus_{d\ge 0} M_n(A)^{\otimes d}$ with respect to the $*$-product.*

Corollary 4.3 and [Evseev and Kleshchev 2017, Lemma 3.12] now imply:

Corollary 4.4. *With respect to the coproduct ∇ and the product $*$, $S^A(n)$ and $T^A_{\mathfrak{a}}(n)$ are superbialgebras.*

We will also need the following result, where the Sweedler notation $\nabla(x) = \sum x_{(1)} \otimes x_{(2)}$ is used:

Lemma 4.5 [Evseev and Kleshchev 2017, Lemma 4.2]. *Let $x, y, z, u \in S^A(n, d)$. Then*

$$(x * y)(z * u) = \sum(-1)^s (x_{(1)}z_{(1)}) * (y_{(1)}z_{(2)}) * (x_{(2)}u_{(1)}) * (y_{(2)}u_{(2)}),$$

where $s = (\bar{x}_{(2)} + \bar{y}_{(2)})\bar{z} + \bar{y}_{(1)}(\bar{x}_{(2)} + \bar{z}_{(1)}) + \bar{y}_{(2)}\bar{u}_{(1)}$.

4B. *Separation.* Let $q \in \mathbb{Z}_{>0}$ and $\delta = (d_1, \dots, d_q) \in \mathbb{Z}^q_{\geq 0}$ with $d_1 + \dots + d_q = d$. Then $\mathfrak{S}_\delta := \mathfrak{S}_{d_1} \times \dots \times \mathfrak{S}_{d_q} \leq \mathfrak{S}_d$. Suppose that for each $m = 1, \dots, q$, we are given

$$(\boldsymbol{a}^{(m)}, \boldsymbol{r}^{(m)}, \boldsymbol{s}^{(m)}),\ (\boldsymbol{c}^{(m)}, \boldsymbol{t}^{(m)}, \boldsymbol{u}^{(m)}) \in \mathrm{Tri}^H(n, d_m).$$

We write

$$\boldsymbol{a}^{(m)} = a_1^{(m)} \cdots a_{d_m}^{(m)}, \quad \boldsymbol{r}^{(m)} = r_1^{(m)} \cdots r_{d_m}^{(m)}, \quad \text{etc.}$$

Let

$$\boldsymbol{a} = \boldsymbol{a}^{(1)} \dots \boldsymbol{a}^{(q)}, \quad \boldsymbol{r} = \boldsymbol{r}^{(1)} \dots \boldsymbol{r}^{(q)}, \quad \text{etc.}$$

We also write

$$\boldsymbol{a} = a_1 \cdots a_d, \quad \boldsymbol{r} = r_1 \cdots r_d, \quad \text{etc.}$$

The triple $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s})$ is called δ-*separated* if $1 \leq m \neq l \leq q$ implies

$$(a_t^{(m)}, r_t^{(m)}, s_t^{(m)}) \neq (a_u^{(l)}, r_u^{(l)}, s_u^{(l)})$$

for all $1 \leq t \leq d_m$ and $1 \leq u \leq d_l$. We then automatically have $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^H(n, d)$.

Lemma 4.6. *If $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s})$ is δ-separated, then*

$$\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} = \xi^{\boldsymbol{a}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} * \dots * \xi^{\boldsymbol{a}^{(q)}}_{\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}} \quad \textit{and} \quad \eta^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} = \eta^{\boldsymbol{a}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} * \dots * \eta^{\boldsymbol{a}^{(q)}}_{\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}}.$$

Proof. Recalling the notation (2.9), for each $1 \leq t \leq q$, there exists $(\hat{\boldsymbol{a}}^{(t)}, \hat{\boldsymbol{r}}^{(t)}, \hat{\boldsymbol{s}}^{(t)}) \in \mathrm{Tri}^H_0(n, d_t)$ such that $(\hat{\boldsymbol{a}}^{(t)}, \hat{\boldsymbol{r}}^{(t)}, \hat{\boldsymbol{s}}^{(t)}) \sim (\boldsymbol{a}^{(t)}, \boldsymbol{r}^{(t)}, \boldsymbol{s}^{(t)})$. Write $\hat{\boldsymbol{a}} := \hat{\boldsymbol{a}}^{(1)} \cdots \hat{\boldsymbol{a}}^{(q)}$, $\hat{\boldsymbol{r}} := \hat{\boldsymbol{r}}^{(1)} \cdots \hat{\boldsymbol{r}}^{(q)}$, $\hat{\boldsymbol{s}} := \hat{\boldsymbol{s}}^{(1)} \cdots \hat{\boldsymbol{s}}^{(q)}$. Then $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{r}}, \hat{\boldsymbol{s}}) \sim (\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s})$, and $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{r}}, \hat{\boldsymbol{s}})$ is δ-separated. Moreover we have that $\mathfrak{S}_{\hat{\boldsymbol{a}},\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}} \leq \mathfrak{S}_\delta$, and both groups are standard parabolic subgroups of $\mathfrak{S}_d$. Using (3.2), we get

$$\begin{aligned}\xi^{\hat{\boldsymbol{a}}}_{\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}} &= \sum_{\sigma} (\xi^{\hat{a}_1}_{\hat{r}_1,\hat{s}_1} \otimes \dots \otimes \xi^{\hat{a}_d}_{\hat{r}_d,\hat{s}_d})^{\sigma} = \sum_{\sigma',\sigma''} (\xi^{\hat{a}_1}_{\hat{r}_1,\hat{s}_1} \otimes \dots \otimes \xi^{\hat{a}_d}_{\hat{r}_d,\hat{s}_d})^{\sigma'\sigma''} \\ &= \sum_{\sigma''} (\xi^{\hat{\boldsymbol{a}}^{(1)}}_{\hat{\boldsymbol{r}}^{(1)},\hat{\boldsymbol{s}}^{(1)}} \otimes \dots \otimes \xi^{\hat{\boldsymbol{a}}^{(q)}}_{\hat{\boldsymbol{r}}^{(q)},\hat{\boldsymbol{s}}^{(q)}})^{\sigma''} = \xi^{\hat{\boldsymbol{a}}^{(1)}}_{\hat{\boldsymbol{r}}^{(1)},\hat{\boldsymbol{s}}^{(1)}} * \dots * \xi^{\hat{\boldsymbol{a}}^{(q)}}_{\hat{\boldsymbol{r}}^{(q)},\hat{\boldsymbol{s}}^{(q)}},\end{aligned}$$

where σ runs over ${}^{\hat{\boldsymbol{a}},\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}}\mathscr{D}$, σ' runs over all shortest coset representatives for $\mathfrak{S}_{\hat{\boldsymbol{a}},\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}} \backslash \mathfrak{S}_\delta$ and σ'' runs over all shortest coset representatives for $\mathfrak{S}_\delta \backslash \mathfrak{S}_d$.

Since $(\hat{\boldsymbol{a}}^{(t)}, \hat{\boldsymbol{r}}^{(t)}, \hat{\boldsymbol{s}}^{(t)}) \sim (\boldsymbol{a}^{(t)}, \boldsymbol{r}^{(t)}, \boldsymbol{s}^{(t)})$ for all $1 \leq t \leq q$, we have

$$(-1)^{\langle \boldsymbol{a},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \hat{\boldsymbol{a}},\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}\rangle} = (-1)^{\langle \boldsymbol{a}^{(1)},\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}\rangle + \dots + \langle \boldsymbol{a}^{(q)},\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}\rangle + \langle \hat{\boldsymbol{a}}^{(1)},\hat{\boldsymbol{r}}^{(1)},\hat{\boldsymbol{s}}^{(1)}\rangle + \dots + \langle \hat{\boldsymbol{a}}^{(q)},\hat{\boldsymbol{r}}^{(q)},\hat{\boldsymbol{s}}^{(q)}\rangle}.$$

Then, using Lemma 3.4, we have

$$\begin{aligned}\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} &= (-1)^{\langle \boldsymbol{a},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \hat{\boldsymbol{a}},\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}\rangle} \xi^{\hat{\boldsymbol{a}}}_{\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}} = (-1)^{\langle \boldsymbol{a},\boldsymbol{r},\boldsymbol{s}\rangle + \langle \hat{\boldsymbol{a}},\hat{\boldsymbol{r}},\hat{\boldsymbol{s}}\rangle} \xi^{\hat{\boldsymbol{a}}^{(1)}}_{\hat{\boldsymbol{r}}^{(1)},\hat{\boldsymbol{s}}^{(1)}} * \dots * \xi^{\hat{\boldsymbol{a}}^{(q)}}_{\hat{\boldsymbol{r}}^{(q)},\hat{\boldsymbol{s}}^{(q)}} \\ &= (-1)^{\langle \boldsymbol{a}^{(1)},\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}\rangle + \langle \hat{\boldsymbol{a}}^{(1)},\hat{\boldsymbol{r}}^{(1)},\hat{\boldsymbol{s}}^{(1)}\rangle} \xi^{\hat{\boldsymbol{a}}^{(1)}}_{\hat{\boldsymbol{r}}^{(1)},\hat{\boldsymbol{s}}^{(1)}} \\ &\qquad\qquad * \dots * (-1)^{\langle \boldsymbol{a}^{(q)},\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}\rangle + \langle \hat{\boldsymbol{a}}^{(q)},\hat{\boldsymbol{r}}^{(q)},\hat{\boldsymbol{s}}^{(q)}\rangle} \xi^{\hat{\boldsymbol{a}}^{(q)}}_{\hat{\boldsymbol{r}}^{(q)},\hat{\boldsymbol{s}}^{(q)}} \\ &= \xi^{\boldsymbol{a}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} * \dots * \xi^{\boldsymbol{a}^{(q)}}_{\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}},\end{aligned}$$

as desired. The result for η's follows from the result on ξ's. □

Lemma 4.7. *Let* $(\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{s})$ *and* $(\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u})$ *be* δ*-separated and suppose that*

$$(\xi^{\boldsymbol{a}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} \otimes \cdots \otimes \xi^{\boldsymbol{a}^{(q)}}_{\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}})^{\sigma} (\xi^{\boldsymbol{c}^{(1)}}_{\boldsymbol{t}^{(1)},\boldsymbol{u}^{(1)}} \otimes \cdots \otimes \xi^{\boldsymbol{c}^{(q)}}_{\boldsymbol{t}^{(q)},\boldsymbol{u}^{(q)}})^{\sigma'} = 0$$

whenever σ *and* σ' *are distinct elements of* ${}^{\delta}\mathscr{D}$*. Then*

$$\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} \xi^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}} = \pm(\xi^{\boldsymbol{a}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} \xi^{\boldsymbol{c}^{(1)}}_{\boldsymbol{t}^{(1)},\boldsymbol{u}^{(1)}}) * \cdots * (\xi^{\boldsymbol{a}^{(q)}}_{\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}} \xi^{\boldsymbol{c}^{(q)}}_{\boldsymbol{t}^{(q)},\boldsymbol{u}^{(q)}}).$$

Moreover, if $a_1, \dots, a_d$ *or* $c_1, \dots, c_d$ *are all even, then the sign of the right-hand side is* $+$.

Proof. By Lemma 4.6, $\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} \xi^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}$ equals

$$\Big(\sum_{\sigma \in {}^{\delta}\mathscr{D}} (\xi^{\boldsymbol{a}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} \otimes \cdots \otimes \xi^{\boldsymbol{a}^{(q)}}_{\boldsymbol{r}^{(q)},\boldsymbol{s}^{(q)}})^{\sigma} \Big) \Big(\sum_{\sigma' \in {}^{\delta}\mathscr{D}} (\xi^{\boldsymbol{c}^{(1)}}_{\boldsymbol{t}^{(1)},\boldsymbol{u}^{(1)}} \otimes \cdots \otimes \xi^{\boldsymbol{c}^{(q)}}_{\boldsymbol{t}^{(q)},\boldsymbol{u}^{(q)}})^{\sigma'} \Big),$$

and the result follows. □

The following result allows one to reduce the study of $S^A(n, d)$ to the blocks of A, and similarly for $T^A_{\mathfrak{a}}(n, d)$.

Lemma 4.8. *Let* $m \in \mathbb{Z}_{>0}$*. For* $t \in [1, m]$ *assume that* $(A_t, \mathfrak{a}_t)$ *is a good pair. Write* $A := \bigoplus_{t=1}^m A_t$ *and* $\mathfrak{a} := \bigoplus_{t=1}^m \mathfrak{a}_t$*. Then we have*

$$S^A(n, d) \cong \bigoplus_{\nu \in \Lambda(m,d)} \bigotimes_{t=1}^{m} S^{A_t}(n, \nu_t) \quad \text{and} \quad T^A_{\mathfrak{a}}(n, d) \cong \bigoplus_{\nu \in \Lambda(m,d)} \bigotimes_{t=1}^{m} T^{A_t}_{\mathfrak{a}_t}(n, \nu_t)$$

as $\Bbbk$*-superalgebras.*

Proof. For $t \in [1, m]$, let B_t be the designated $(A_t, \mathfrak{a}_t)$-basis, and set $B = \bigsqcup_{t=1}^m B_t$ as the designated $(A, \mathfrak{a})$-basis. It follows from Lemma 4.6 that, for any $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)$, we have

$$\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \pm \xi^{\boldsymbol{b}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} * \cdots * \xi^{\boldsymbol{b}^{(m)}}_{\boldsymbol{r}^{(m)},\boldsymbol{s}^{(m)}}$$

for some $\nu \in \Lambda(m, d)$ and $(\boldsymbol{b}^{(t)}, \boldsymbol{r}^{(t)}, \boldsymbol{s}^{(t)}) \in \mathrm{Tri}^{B_t}(n, \nu_t)$ for $t \in [1, m]$. So we may write

$$S^A(n, d) = \bigoplus_{\nu \in \Lambda(m,d)} S^{A_1}(n, \nu_1) * \cdots * S^{A_m}(n, \nu_m)$$

Inductive application of Lemma 4.5 shows that this is a decomposition of $S^A(n, d)$ into subalgebras. Moreover, it follows as well from Lemma 4.5 that for all $\nu \in \Lambda(m, d)$ we have

$$S^{A_1}(n, \nu_1) * \cdots * S^{A_m}(n, \nu_m) \cong S^{A_1}(n, \nu_1) \otimes \cdots \otimes S^{A_m}(n, \nu_m)$$

as $\Bbbk$-superalgebras, proving the claim for $S^A(n, d)$. The proof of the claim for $T^A_{\mathfrak{a}}(n, d)$ proceeds exactly as above, since Lemma 4.6 provides an analogous result for η's. □

4C. *Generation.* We define

$$Y := \mathrm{span}(\xi^b_{r,s} \mid r, s \in [1,n],\ b \in B_{\mathfrak{c}} \sqcup B_{\bar{1}}) \subseteq M_n(A),$$
$$\mathtt{Star}^d\, Y := \underbrace{Y * \cdots * Y}_{d \text{ times}} \subseteq T^A_{\mathfrak{a}}(n,d),$$

where the second inclusion comes from Corollary 4.3. Note also that $S^{\mathfrak{a}}(n,d) \subseteq T^A_{\mathfrak{a}}(n,d)$ since by definition, for $\boldsymbol{b} \in B^d_{\mathfrak{a}}$, we have $\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$. The following is a generalization of [Evseev and Kleshchev 2017, Lemma 4.30].

Lemma 4.9. *We have*

$$T^A_{\mathfrak{a}}(n,d) = \bigoplus_{e=0}^{d} S^{\mathfrak{a}}(n,d-e) * \mathtt{Star}^e\, Y.$$

Proof. As $S^{\mathfrak{a}}(n,d-e) \subseteq T^A_{\mathfrak{a}}(n,d-e)$ and $Y \subseteq T^A_{\mathfrak{a}}(n,1)$, the right-hand side is contained in the left-hand side thanks to Corollary 4.3. For the converse containment, we only need to prove that every $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$ with $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$ is contained in the right-hand side. For any $b \in B$ and $r, s \in [1,n]$, denote $m^b_{r,s} := [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^b_{r,s}$ and set

$$e := \sum_{\mathrm{b} \in B_{\mathfrak{c}} \sqcup B_{\bar{1}},\, r,s \in [1,n]} m^{\mathrm{b}}_{r,s}.$$

Using the fact that $m^b_{r,s} \in \{0,1\}$ for all $b \in B_{\bar{1}}$, Lemma 4.6 and the definition of $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$, we see that

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \pm\Big(\mathop{\ast}_{\mathrm{b} \in B_{\mathfrak{a}},\, r,s \in [1,n]} \big((\xi^{\mathrm{b}}_{r,s})^{\otimes m^{\mathrm{b}}_{r,s}}\big)\Big) * \Big(\mathop{\ast}_{\mathrm{b} \in B_{\mathfrak{c}} \sqcup B_{\bar{1}},\, r,s \in [1,n]} \big((\xi^{\mathrm{b}}_{r,s})^{* m^{\mathrm{b}}_{r,s}}\big)\Big), \tag{4.10}$$

with the first term in $S^{\mathfrak{a}}(n,d-e)$ and the second term in $\mathtt{Star}^e\, Y$. □

Proposition 4.11. *The algebra $T^A_{\mathfrak{a}}(n,d)$ depends only on the subalgebra $\mathfrak{a}$, and not on the choice of the $(A,\mathfrak{a})$-basis B.*

Proof. Let $B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar{1}}$ and $B' = B'_{\mathfrak{a}} \sqcup B'_{\mathfrak{c}} \sqcup B'_{\bar{1}}$ be distinct choices of $(A,\mathfrak{a})$-bases, $Y = \mathrm{span}(\xi^b_{r,s} \mid r, s \in [1,n],\ b \in B_{\mathfrak{c}} \sqcup B_{\bar{1}})$, and $Y' = \mathrm{span}(\xi^b_{r,s} \mid r,s \in [1,n],\ b \in B'_{\mathfrak{c}} \sqcup B'_{\bar{1}})$. As $B'_{\mathfrak{c}} \subseteq \mathrm{span}(B_{\mathfrak{c}} \sqcup B_{\mathfrak{a}})$, we deduce that

$$\mathtt{Star}^e\, Y' \subseteq \bigoplus_{f=0}^{e} \mathtt{Star}^{e-f}\, Y * \mathtt{Star}^f\, \mathfrak{a}.$$

Therefore by Lemma 4.9, the algebra ${}'T^A_{\mathfrak{a}}(n,d)$ defined using the basis B' is contained in the algebra $T^A_{\mathfrak{a}}(n,d)$ defined using the basis B. Similarly, $T^A_{\mathfrak{a}}(n,d) \subseteq {}'T^A_{\mathfrak{a}}(n,d)$. □

Let τ be an anti-involution on A, such that $\tau(\mathfrak{a}) = \mathfrak{a}$. Then it is easy to see, using Proposition 4.11, that the involution $\tau_{n,d}$ on $S^A(n,d)$ defined in (3.8) restricts to the involution of $T^A_{\mathfrak{a}}(n,d)$. Moreover, if $\tau(B_{\mathfrak{a}}) = B_{\mathfrak{a}}$, $\tau(B_{\mathfrak{c}}) = B_{\mathfrak{c}}$ and $\tau(B_{\bar{1}}) = B_{\bar{1}}$, then we have

$$\tau_{n,d} : T^A_{\mathfrak{a}}(n,d) \to T^A_{\mathfrak{a}}(n,d),\ \ \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mapsto \eta^{\boldsymbol{b}^\tau}_{\boldsymbol{s},\boldsymbol{r}}. \tag{4.12}$$

The following theorem generalizes [Evseev and Kleshchev 2017, Theorem 4.31].

Theorem 4.13. *Suppose that $(A, \mathfrak{a})$ is a unital good pair and let $1 := 1_{M_n(A)}$. Then $T^A_{\mathfrak{a}}(n, d)$ is the subalgebra of $S^A(n, d)$ generated by $S^{\mathfrak{a}}(n, d)$ and $1^{\otimes d-1} * Y := \{1^{\otimes d-1} * y \mid y \in Y\}$.*

Proof. Let U be the subalgebra of $T^A_{\mathfrak{a}}(n, d)$ generated by $S^{\mathfrak{a}}(n, d)$ and $1^{\otimes d-1} * Y$. We show by induction on $e = 0, \dots, d$ that U contains every element of the form $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * 1^{\otimes(d-e)}$, where $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, e)$. This proves the theorem in the case $d = e$.

The base case $e = 0$ is clear. Let $0 < e \le d$. Let $(\boldsymbol{b}', \boldsymbol{r}', \boldsymbol{s}') \in \mathrm{Tri}^B(n, e)$. We will show, using the inductive assumption, that $\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'} * 1^{\otimes(d-e)} \in U$. If $\boldsymbol{b}' \in B^e_{\mathfrak{a}}$, then $\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'} * 1^{\otimes(d-e)} \in S^{\mathfrak{a}}(n, d) \subseteq U$, and we are done. So we may assume that $(\boldsymbol{b}', \boldsymbol{r}', \boldsymbol{s}') = (\boldsymbol{b}b, \boldsymbol{r}r, \boldsymbol{s}s)$, for some $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, e-1)$, $b \in B_{\mathfrak{c}} \cup B_{\bar{1}}$ and $r, s \in [1, n]$.

By the induction assumption, $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * 1^{\otimes(d-e+1)} \in U$. We also have $1^{\otimes(d-1)} * \xi^b_{r,s} \in 1^{\otimes(d-1)} * Y \subseteq U$. Thus the following product is contained in U:

$$
\begin{aligned}
(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * 1^{\otimes(d-e+1)})(1^{\otimes(d-1)} * \xi^b_{r,s}) &= \pm\, (\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(1)} * 1^{\otimes(d-e+1)} * (\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(2)}\xi^b_{r,s} \pm \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * 1^{\otimes(d-e)} * \xi^b_{r,s} \\
&= \pm\, (\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(1)} * (\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(2)}\xi^b_{r,s} * 1^{\otimes(d-e+1)} \pm \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^b_{r,s} * 1^{\otimes(d-e)},
\end{aligned}
$$

where the equalities come from Lemma 4.5 and the supercommutativity of $*$. Note that by Corollary 4.4, we have that $(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(1)} * (\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(2)}\xi^b_{r,s}$ belongs to $T^A_{\mathfrak{a}}(n, e-1)$, and thus may be written as a linear combination of elements of the form $\eta^{\boldsymbol{b}''}_{\boldsymbol{r}'',\boldsymbol{s}''}$, where $(\boldsymbol{b}'', \boldsymbol{r}'', \boldsymbol{s}'') \in \mathrm{Tri}^B(n, e-1)$. Therefore the induction assumption implies that the term $(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(1)} * (\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})_{(2)}\xi^b_{r,s} * 1^{\otimes(d-e+1)}$ belongs to U, which in turn implies that $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^b_{r,s} * 1^{\otimes(d-e)} \in U$. But since $b \in B_{\mathfrak{c}} \cup B_{\bar{1}}$, we have as in (4.10) that $\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'} = \pm\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^b_{r,s}$, so

$$
\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'} * 1^{\otimes(d-e)} = \pm\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \xi^b_{r,s} * 1^{\otimes(d-e)} \in U,
$$

completing the induction step, and the proof. □

5. Miscellaneous properties and examples

Throughout the section, $(A, \mathfrak{a})$ is a fixed good pair with an $(A, \mathfrak{a})$-basis $B = B_{\mathfrak{a}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar{1}}$ as in (2.2). Much of this section deals with various idempotent truncations. If $e \in A$ is an idempotent, we say that B is *e-admissible* if $ebe = b$ or $ebe = 0$ for all $b \in B$. We say that we say that B is *right e-admissible* if $be = b$ or $be = 0$ for all $b \in B$.

5A. *Idempotents and characters.* Throughout the section, let $e_0, \dots, e_\ell \in \mathfrak{a}$ be a set of orthogonal idempotents. We do not assume that $\sum_{i=0}^{\ell} e_i = 1$, and usually we do not make any admissibility assumptions on B. Set $I = [0, \ell]$.

Let $\Lambda(n) := \mathbb{Z}^n_{\ge 0}$ and $\Lambda^I(n) := \Lambda(n)^I$. We think of the elements of $\Lambda(n)$ as compositions $\lambda = (\lambda_1, \dots, \lambda_n)$ and the elements of $\Lambda^I(n)$ as tuples $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)})$ of compositions. For such $\lambda \in \Lambda(n)$ and $\boldsymbol{\lambda} \in \Lambda^I(n)$, we set $|\lambda| := \sum_{r=1}^{n} \lambda_r$, $|\boldsymbol{\lambda}| := \sum_{i \in I} |\lambda^{(i)}|$, and, for any $d \in \mathbb{Z}_{\ge 0}$, we define

$$
\Lambda(n, d) := \{\lambda \in \Lambda(n) \mid |\lambda| = d\}, \quad \Lambda^I(n, d) := \{\boldsymbol{\lambda} \in \Lambda(n) \mid |\boldsymbol{\lambda}| = d\}.
$$

The group $\mathfrak{S}_n$ acts on $\Lambda(n)$ via

$$\sigma\lambda := (\lambda_{\sigma^{-1}1}, \dots, \lambda_{\sigma^{-1}n}).$$

The group $\mathfrak{S}_n^I := \prod_{i\in I} \mathfrak{S}_n$ acts on $\Lambda^I(n)$ via

$$\boldsymbol{\sigma\lambda} := (\sigma^{(0)}\lambda^{(0)}, \dots, \sigma^{(\ell)}\lambda^{(\ell)}),$$

for $\boldsymbol{\sigma} = (\sigma^{(0)}, \dots, \sigma^{(\ell)}) \in \mathfrak{S}_n^I$ and $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(\ell)}) \in \Lambda^I(n)$.

To $\lambda \in \Lambda(n, d)$ we associate the word $\boldsymbol{l}^\lambda = 1^{\lambda_1} \cdots n^{\lambda_n} \in [1, n]^d$. For any idempotent $f \in A$ we have an idempotent

$$\xi_\lambda^f := \xi_{\boldsymbol{l}^\lambda,\boldsymbol{l}^\lambda}^{f^d} \in S^A(n, d).$$

Note using Lemma 3.11 that $\xi_\lambda^f \in T_{\mathfrak{a}}^A(n, d)$ if $f \in \mathfrak{a}$. Define

$$\xi^f := \sum_{\lambda\in\Lambda(n,d)} \xi_\lambda^f. \tag{5.1}$$

If, for any $a \in A$ we define

$$E^a := \sum_{r=1}^{n} \xi_{r,r}^a \in M_n(A), \tag{5.2}$$

then

$$\xi^f = E^f \otimes \cdots \otimes E^f. \tag{5.3}$$

If A is unital, we denote

$$\xi_\lambda := \xi_\lambda^{1_A}. \tag{5.4}$$

Then $1_{S^A(n,d)} = \sum_{\lambda\in\Lambda(n,d)} \xi_\lambda$ is an orthogonal idempotent decomposition. If the pair $(A, \mathfrak{a})$ is unital, then $\xi_\lambda \in T_{\mathfrak{a}}^A(n, d)$ for all $\lambda \in \Lambda(n, d)$. For $\boldsymbol{\mu} \in \Lambda^I(n, d)$, define

$$e_{\boldsymbol{\lambda}} := \xi_{\lambda^{(0)}}^{e_0} * \cdots * \xi_{\lambda^{(\ell)}}^{e_\ell} \in T_{\mathfrak{a}}^A(n, d). \tag{5.5}$$

For $a \in A$ and $\sigma \in \mathfrak{S}_n$, let $\xi_\sigma^a := \sum_{r=1}^n \xi_{\sigma(r),r}^a \in M_n(A)$ be the permutation matrix corresponding to σ multiplied by a. For $\boldsymbol{\sigma} = (\sigma^{(0)}, \dots, \sigma^{(\ell)}) \in \mathfrak{S}_n^I$, we set

$$\xi_{\boldsymbol{\sigma}} := \sum_{(\delta_0,\dots,\delta_\ell)\in\Lambda^I(d)} (\xi_{\sigma^{(0)}}^{e_0})^{\otimes\delta_0} * \cdots * (\xi_{\sigma^{(\ell)}}^{e_\ell})^{\otimes\delta_\ell} \in T_{\mathfrak{a}}^A(n, d).$$

Lemma 5.6. *For all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathfrak{S}_n^I$, we have $\xi_{\boldsymbol{\sigma}}\xi_{\boldsymbol{\tau}} = \xi_{\boldsymbol{\sigma\tau}}$.*

Proof. This follows easily from Lemma 4.7. □

Lemma 5.7. *If $\boldsymbol{\lambda} \in \Lambda^I(n, d)$ and $\boldsymbol{\sigma} \in \mathfrak{S}_n^I$, then we have $\xi_{\boldsymbol{\sigma}} e_{\boldsymbol{\lambda}} \xi_{\boldsymbol{\sigma}^{-1}} = e_{\boldsymbol{\sigma\lambda}}$.*

Proof. Let $d_i = |\lambda^{(i)}|$ for all $i \in I$. Using Lemma 4.7, we get

$$\xi_{\boldsymbol{\sigma}} e_{\boldsymbol{\lambda}} \xi_{\boldsymbol{\sigma}^{-1}} = \big((\xi_{\sigma^{(0)}}^{e_0})^{\otimes d_0} \xi_{\lambda^{(0)}}^{e_0} (\xi_{(\sigma^{(0)})^{-1}}^{e_0})^{\otimes d_0}\big) * \cdots * \big((\xi_{\sigma^{(\ell)}}^{e_\ell})^{\otimes d_\ell} \xi_{\lambda^{(\ell)}}^{e_\ell} (\xi_{(\sigma^{(\ell)})^{-1}}^{e_\ell})^{\otimes d_\ell}\big).$$

For all $i \in I$, we have

$$\begin{aligned}(\xi^{e_i}_{\sigma^{(i)}})^{\otimes d_i}\xi^{e_i}_{\lambda^{(i)}}(\xi^{e_i}_{(\sigma^{(i)})^{-1}})^{\otimes d_i} &= (\xi^{e_i}_{\sigma^{(i)}})^{\otimes d_i}\big((\xi^{e_i}_{1,1})^{\otimes\lambda^{(i)}_1} * \cdots * (\xi^{e_i}_{n,n})^{\otimes\lambda^{(i)}_n}\big)(\xi^{e_i}_{(\sigma^{(i)})^{-1}})^{\otimes d_i}\\ &= (\xi^{e_i}_{\sigma^{(i)}}\xi^{e_i}_{1,1}\xi^{e_i}_{(\sigma^{(i)})^{-1}})^{\otimes\lambda^{(i)}_1} * \cdots * (\xi^{e_i}_{\sigma^{(i)}}\xi^{e_i}_{n,n}\xi^{e_i}_{(\sigma^{(i)})^{-1}})^{\otimes\lambda^{(i)}_n}\\ &= (\xi^{e_i}_{\sigma^{(i)}1,\sigma^{(i)}1})^{\otimes\lambda^{(i)}_1} * \cdots * (\xi^{e_i}_{\sigma^{(i)}n,\sigma^{(i)}n})^{\otimes\lambda^{(i)}_n}\\ &= (\xi^{e_i}_{1,1})^{\otimes\lambda^{(i)}_{(\sigma^{(i)})^{-1}1}} * \cdots * (\xi^{e_i}_{n,n})^{\otimes\lambda^{(i)}_{(\sigma^{(i)})^{-1}n}}\\ &= \xi^{e_i}_{\sigma^{(i)}\lambda^{(i)}},\end{aligned}$$

where we have used the commutativity of $*$-product on even elements for the penultimate equality. So the result follows. □

We consider $\Lambda^I(n)$ as an abelian monoid, where $\boldsymbol{\lambda} = \boldsymbol{\mu} + \boldsymbol{\nu}$ when $\lambda^{(i)}_r = \mu^{(i)}_r + \nu^{(i)}_r$ for all $i \in I$ and $r \in [1, n]$.

Lemma 5.8. *For $\boldsymbol{\lambda} \in \Lambda^I(n, d)$, we have*

$$\nabla(e_{\boldsymbol{\lambda}}) = \sum_{\substack{\boldsymbol{\mu},\,\boldsymbol{\nu}\in\Lambda^I(n)\\ \boldsymbol{\mu}+\boldsymbol{\nu}=\boldsymbol{\lambda}}} e_{\boldsymbol{\mu}} \otimes e_{\boldsymbol{\nu}}.$$

Proof. The result follows by Corollary 3.24. Indeed, recalling the notation of Section 3C, note that $e_{\boldsymbol{\lambda}} = \xi_{\mathcal{T}}$, where $\mathcal{T} = (\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s})$, with $\boldsymbol{b} = e_0^{|\lambda^{(0)}|}\cdots e_\ell^{|\lambda^{(\ell)}|}$, $\boldsymbol{r} = \boldsymbol{s} = \boldsymbol{l}^{\lambda^{(0)}}\cdots\boldsymbol{l}^{\lambda^{(\ell)}}$. Then $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}(\mathcal{T})$ if and only if $\mathcal{T}^1 = (\boldsymbol{b}^1, \boldsymbol{r}^1, \boldsymbol{s}^1)$ with $\boldsymbol{b}^1 = e_0^{|\mu^{(0)}|}\cdots e_\ell^{|\mu^{(\ell)}|}$, $\boldsymbol{r}^1 = \boldsymbol{s}^1 = \boldsymbol{l}^{\mu^{(0)}}\cdots\boldsymbol{l}^{\mu^{(\ell)}}$ and $\mathcal{T}^2 = (\boldsymbol{b}^2, \boldsymbol{r}^2, \boldsymbol{s}^2)$, with $\boldsymbol{b}^2 = e_0^{|\nu^{(0)}|}\cdots e_\ell^{|\nu^{(\ell)}|}$, $\boldsymbol{r}^2 = \boldsymbol{s}^2 = \boldsymbol{l}^{\nu^{(0)}}\cdots\boldsymbol{l}^{\nu^{(\ell)}}$ such that $\boldsymbol{\mu} + \boldsymbol{\nu} = \boldsymbol{\lambda}$. □

Define

$$R := \mathbb{Z}[t]/(t^2 - 1),$$

and denote the image of t in the quotient ring by π, so that π^ε makes sense for $\varepsilon \in \mathbb{Z}/2$. Writing the operation in the monoid $\Lambda^I(n, d)$ multiplicatively, denote by $R\Lambda^I(n, d)$ the corresponding R-monoid algebra. This algebra inherits the $\mathfrak{S}^I_n$-action from that on $\Lambda^I(n, d)$. Since this action is by algebra automorphisms, we have the invariant algebra $(R\Lambda^I(n, d))^{\mathfrak{S}^I_n}$.

If V is a free $\Bbbk$-module of finite rank, we denote its rank by $\dim V$. If V be a free $\Bbbk$-supermodule of finite rank, its super-rank is defined to be $\dim_\pi V := \dim V_{\bar 0} + (\dim V_{\bar 1})\pi \in R$. Let W be a $T^A_{\mathfrak{a}}(n, d)$-supermodule. If $e_{\boldsymbol{\lambda}} W$ is free of finite rank as a $\Bbbk$-supermodule for all $\boldsymbol{\lambda} \in \Lambda^I(n, d)$, we say that W is a *supermodule with free weight spaces*. In this case, the (formal) *character* of W is defined to be

$$\mathrm{ch}_\pi W := \sum_{\boldsymbol{\lambda}\in\Lambda^I(n,d)} (\dim_\pi e_{\boldsymbol{\lambda}} W)\,\boldsymbol{\lambda} \in R\Lambda^I(n, d).$$

Lemma 5.9. *If W is a $T^A_{\mathfrak{a}}(n, d)$-supermodule with free weight spaces then $\mathrm{ch}_\pi W \in (R\Lambda^I(n, d))^{\mathfrak{S}^I_n}$.*

Proof. By Lemma 5.7, we have that $e_{\boldsymbol{\mu}} W \cong e_{\boldsymbol{\lambda}} W$ as $\Bbbk$-supermodules whenever $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are in the same $\mathfrak{S}^I_n$-orbit. □

Finally, Lemma 5.8 gives us:

Lemma 5.10. *Let W_1 be a $T^A_{\mathfrak{a}}(n,d_1)$-supermodule with free weight spaces and W_2 be a $T^A_{\mathfrak{a}}(n,d_2)$-supermodule with free weight spaces. We consider $W_1 \otimes W_2$ as a $T^A_{\mathfrak{a}}(n,d_1+d_2)$-supermodule via the coproduct ∇. Then $W_1 \otimes W_2$ is a supermodule with free weight spaces, and*

$$\mathrm{ch}_\pi(W_1 \otimes W_2) = \mathrm{ch}_\pi(W_1)\ \mathrm{ch}_\pi(W_2).$$

Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^I(n)$. We call $\boldsymbol{\lambda}, \boldsymbol{\mu}$ *nonoverlapping* if for every $i \in I$ and $r \in [1,n]$ we have that $\lambda_r^{(i)} \neq 0$ implies $\mu_r^{(i)} = 0$.

Proposition 5.11. *Suppose that B is right e_i-admissible for all $i \in I$. Let $\boldsymbol{\lambda} \in \Lambda^I(n,c)$, $\boldsymbol{\mu} \in \Lambda^I(n,d)$ and suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}\boldsymbol{\mu}$ are nonoverlapping for some $\boldsymbol{\sigma} \in \mathfrak{S}_n^I$. Then we have isomorphisms*

$$S^A(n,c)e_{\boldsymbol{\lambda}} \otimes S^A(n,d)e_{\boldsymbol{\mu}} \cong S^A(n,c+d)e_{\boldsymbol{\lambda}+\boldsymbol{\sigma}\boldsymbol{\mu}},$$
$$T^A_{\mathfrak{a}}(n,c)e_{\boldsymbol{\lambda}} \otimes T^A_{\mathfrak{a}}(n,d)e_{\boldsymbol{\mu}} \cong T^A_{\mathfrak{a}}(n,c+d)e_{\boldsymbol{\lambda}+\boldsymbol{\sigma}\boldsymbol{\mu}},$$

of $S^A(n,c+d)$- and $T^A_{\mathfrak{a}}(n,c+d)$-modules, respectively.

Proof. We prove the result for $T^A_{\mathfrak{a}}$; the proof for S^A is similar. Since $T^A_{\mathfrak{a}}(n,d)e_{\boldsymbol{\mu}} \cong T^A_{\mathfrak{a}}(n,d)e_{\boldsymbol{\sigma}\boldsymbol{\mu}}$ by Lemma 5.7, we may assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are nonoverlapping and prove that

$$T^A_{\mathfrak{a}}(n,c)e_{\boldsymbol{\lambda}} \otimes T^A_{\mathfrak{a}}(n,d)e_{\boldsymbol{\mu}} \cong T^A_{\mathfrak{a}}(n,c+d)e_{\boldsymbol{\lambda}+\boldsymbol{\mu}}.$$

Set $B(i) := \{b \in B \mid be_i = b\}$ for all $i \in I$. For $\boldsymbol{\nu} \in \Lambda^I(n,f)$, let $\mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)$ be the set of all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,f)$ such that

$$\#\{k \mid b_k \in B(i), s_k = t\} = \nu_t^{(i)} \quad \text{for all } i \in I, t \in [1,n].$$

Then for all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,f)$ we have

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} e_{\boldsymbol{\nu}} \neq 0 \iff \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} e_{\boldsymbol{\nu}} = \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \iff (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f),$$

so

$$\{\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}] \in \mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)/\mathfrak{S}_f\}$$

is a basis for $T^A_{\mathfrak{a}}(n,f)e_{\boldsymbol{\nu}}$. By the nonoverlapping condition, we may choose a total order on $B \times [1,n] \times [1,n]$ such that $(b,r,s) > (b',r',s')$ whenever $b \in B(i)$ and $b' \in B(j)$ for some $i,j \in I$ with $\lambda_s^{(i)} > 0$ and $\mu_{s'}^{(j)} > 0$. Let $\mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)_0 \subseteq \mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)$ be the subset of triples which are lexicographically maximal under this total order. The set $\mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)/\mathfrak{S}_f$ is in bijection with $\mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)_0$, so

$$\{\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B_{\boldsymbol{\nu}}(n,f)_0\}$$

is a basis for $T^A_{\mathfrak{a}}(n,f)e_{\boldsymbol{\nu}}$.

We have a one-to-one correspondence

$$\mathrm{Tri}^B_{\boldsymbol{\lambda}}(n,c)_0 \times \mathrm{Tri}^B_{\boldsymbol{\mu}}(n,d)_0 \leftrightarrow \mathrm{Tri}^B_{\boldsymbol{\lambda}+\boldsymbol{\mu}}(n,c+d)_0,$$

given by

$$((\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}),(\boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}'))\mapsto(\boldsymbol{b}\boldsymbol{b}',\boldsymbol{r}\boldsymbol{r}',\boldsymbol{s}\boldsymbol{s}').$$

Thus we have a $\Bbbk$-linear isomorphism

$$\varphi: T^A_{\mathfrak{a}}(n,c)e_{\boldsymbol{\lambda}}\otimes T^A_{\mathfrak{a}}(n,d)e_{\boldsymbol{\mu}}\xrightarrow{\sim} T^A(n,c+d)e_{\boldsymbol{\lambda}+\boldsymbol{\mu}}$$

defined via

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\otimes\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'}\mapsto\eta^{\boldsymbol{b}\boldsymbol{b}'}_{\boldsymbol{r}\boldsymbol{r}',\boldsymbol{s}\boldsymbol{s}'}$$

for all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B_{\boldsymbol{\lambda}}(n,c)_0$ and $(\boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}')\in\mathrm{Tri}^B_{\boldsymbol{\mu}}(n,d)_0$. Moreover, in this situation $(\boldsymbol{b}\boldsymbol{b}',\boldsymbol{r}\boldsymbol{r}',\boldsymbol{s}\boldsymbol{s}')$ is (c,d)-separated by the nonoverlapping condition, so $\eta^{\boldsymbol{b}\boldsymbol{b}'}_{\boldsymbol{r}\boldsymbol{r}',\boldsymbol{s}\boldsymbol{s}'}=\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}*\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'}$ by Lemma 4.6. Thus we may describe the isomorphism more generally via the star map:

$$\varphi: T^A_{\mathfrak{a}}(n,c)e_{\boldsymbol{\lambda}}\otimes T^A_{\mathfrak{a}}(n,d)e_{\boldsymbol{\mu}}\xrightarrow{\sim} T^A_{\mathfrak{a}}(n,c+d)e_{\boldsymbol{\lambda}+\boldsymbol{\mu}},\qquad x\otimes y\mapsto x*y.$$

Finally, φ is an isomorphism of $T^A_{\mathfrak{a}}(n,c+d)$-modules thanks to Lemma 4.5. □

5B. ***Idempotent truncation.*** Let $e\in\mathfrak{a}$ be an idempotent and $\xi^e\in T^A_{\mathfrak{a}}(n,d)$ be the idempotent of (5.1). Set

$$\bar{A}:=eAe\quad\text{and}\quad\bar{\mathfrak{a}}:=e\mathfrak{a}e.$$

By definition, $\bar{A}$ is a subalgebra of A and $\bar{\mathfrak{a}}$ is a subalgebra of $\mathfrak{a}$. So we can consider $S^{\bar{A}}(n,d)$ and hence $T^{\bar{A}}_{\bar{\mathfrak{a}}}(n,d)$ as subalgebras of $S^A(n,d)$.

Lemma 5.12. *Let $e\in\mathfrak{a}$ be an idempotent. Suppose that B is e-admissible. Then*:

(i) $S^{\bar{A}}(n,d)=\xi^e S^A(n,d)\xi^e$.

(ii) $T^{\bar{A}}_{\bar{\mathfrak{a}}}(n,d)=\xi^e T^A_{\mathfrak{a}}(n,d)\xi^e$.

Proof. By assumption, we have an $(A,\mathfrak{a})$-basis $B=B_{\mathfrak{a}}\sqcup B_{\mathfrak{c}}\sqcup B_{\bar{1}}$ such that $ebe=b$ or $ebe=0$ for all $b\in B$. Defining

$$\bar{B}_{\mathfrak{a}}:=\{b\in B_{\mathfrak{a}}\mid ebe=b\},$$
$$\bar{B}_{\mathfrak{c}}:=\{b\in B_{\mathfrak{c}}\mid ebe=b\},$$
$$\bar{B}_{\bar{1}}:=\{b\in B_{\bar{1}}\mid ebe=b\},$$

we have that $\bar{B}:=\bar{B}_{\mathfrak{a}}\sqcup\bar{B}_{\mathfrak{c}}\sqcup\bar{B}_{\bar{1}}$ is an $(\bar{A},\bar{\mathfrak{a}})$-basis for $\bar{A}$. Then, for all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B(n,d)$, we have

$$\xi^e\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\xi^e=\xi^{eb_1e,\dots,eb_de}_{\boldsymbol{r},\boldsymbol{s}}=\begin{cases}\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} & \text{if } \boldsymbol{b}\in\bar{B}^d,\\ 0 & \text{otherwise,}\end{cases}$$

which implies the result. □

For $\boldsymbol{r}\in[1,n]^d$ we define

$$\omega^{\boldsymbol{r}}=(\omega_1,\dots,\omega_n)\in\Lambda(n,d)$$

via $\omega_r:=\{k\in[1,d]\mid r_k=r\}$ for all $r\in[1,n]$. Recall the idempotent ξ_λ from (5.4).

Lemma 5.13. *Let A be unital. If $\lambda \in \Lambda(n,d)$ and $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^H(n,d)$ then*

$$\xi_\lambda \xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} = \delta_{\lambda,\omega^{\boldsymbol{r}}} \xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} \quad \text{and} \quad \xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}} \xi_\lambda = \delta_{\lambda,\omega^{\boldsymbol{s}}} \xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}.$$

Proof. Immediate from Proposition 3.6. □

Let $N \geq n$. Set

$$\Lambda^N_n(d) := \{\lambda \in \Lambda(N,d) \mid \lambda_{n+1} = \cdots = \lambda_N = 0\} \subseteq \Lambda(N,d),$$

and define the idempotent

$$\xi^N_n(d) := \sum_{\lambda \in \Lambda^N_n(d)} \xi_\lambda \in S^A(N,d). \tag{5.14}$$

If $(A,\mathfrak{a})$ is unital, then $\xi^N_n(d) \in T^A_{\mathfrak{a}}(N,d)$.

Lemma 5.15. *Let A be unital, $N \geq n$ and $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(N,d)$.*

(i) *We have*

$$\xi^N_n(d)\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \delta_{\omega^{\boldsymbol{r}} \in \Lambda^N_n(d)} \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \quad \text{and} \quad \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \xi^N_n(d) = \delta_{\omega^{\boldsymbol{s}} \in \Lambda^N_n(d)} \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}.$$

In particular, the map

$$S^A(n,d) \to S^A(N,d), \quad \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mapsto \xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \qquad \big((\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)\big)$$

is a (unital) algebra isomorphism

$$S^A(n,d) \xrightarrow{\sim} \xi^N_n(d) S^A(N,d) \xi^N_n(d).$$

(ii) *If $(A,\mathfrak{a})$ is a unital good pair then $\xi^N_n(d) \in T^A_{\mathfrak{a}}(N,d)$,*

$$\xi^N_n(d)\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \delta_{\omega^{\boldsymbol{r}} \in \Lambda^N_n(d)} \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \quad \text{and} \quad \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \xi^N_n(d) = \delta_{\omega^{\boldsymbol{s}} \in \Lambda^N_n(d)} \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}.$$

In particular, the map

$$T^A_{\mathfrak{a}}(n,d) \to T^A_{\mathfrak{a}}(N,d), \quad \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mapsto \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \qquad \big((\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)\big)$$

is a (unital) algebra isomorphism

$$T^A_{\mathfrak{a}}(n,d) \xrightarrow{\sim} \xi^N_n(d) T^A_{\mathfrak{a}}(N,d) \xi^N_n(d).$$

Proof. Follows from Lemma 5.13. □

Corollary 5.16. *If $d \leq n \leq N$, then $V \mapsto \xi^N_n(d)V$ defines equivalences of categories*

$$S^A(N,d)\text{-mod} \xrightarrow{\sim} S^A(n,d)\text{-mod} \quad \text{and} \quad T^A_{\mathfrak{a}}(N,d)\text{-mod} \xrightarrow{\sim} T^A_{\mathfrak{a}}(n,d)\text{-mod}.$$

Proof. To prove the result for S^A, in view of Lemma 5.15, we just have to prove that

$$S^A(N,d)\xi^N_n(d)S^A(N,d) = S^A(N,d).$$

The last equality will follow if we can show that each ξ_λ with $\lambda \in \Lambda(N, d)$ is in the left-hand side. By the assumption that $d \le n$, there is $\sigma \in \mathfrak{S}_n$ such that all nonzero parts of $\sigma\lambda$ are among its first n parts, and so

$$\xi_{\sigma\lambda} = \xi_{\sigma\lambda}\xi_n^N(d) \in S^A(N, d)\xi_n^N(d)S^A(N, d).$$

By Lemma 5.7, we have that $\xi_\sigma\xi_\lambda\xi_\sigma^{-1} = \xi_{\sigma\lambda}$, or

$$\xi_\lambda = \xi_\sigma^{-1}\xi_{\sigma\lambda}\xi_\sigma \in S^A(N, d)\xi_n^N(d)S^A(N, d),$$

and we are done. The proof for $T_\mathfrak{a}^A$ is the same, using the fact that $\xi_\sigma \in T_\mathfrak{a}^A(n, d)$. □

Remark 5.17. Let $d \le n$ and $\omega := (1, \dots, 1, 0, \dots, 0) \in \Lambda(n, d)$. Then we have

$$\xi_\omega = \xi_{1,1}^1 * \cdots * \xi_{d,d}^1 \in S^A(n, d).$$

It is proved in [Evseev and Kleshchev 2017, Lemma 5.15] that the idempotent truncation $\xi_\omega S^A(n, d)\xi_\omega$ is naturally isomorphic to the wreath product superalgebra $A \wr \mathfrak{S}_d$. This is the analogue of the classical result on Green idempotent truncation [Green 2007, (6.1d)]. If the pair $(A, \mathfrak{a})$ is unital, we have $\xi_\omega \in T_\mathfrak{a}^A(n, d)$ and it is easy to see that $\xi_\omega T_\mathfrak{a}^A(n, d)\xi_\omega = \xi_\omega S^A(n, d)\xi_\omega$.

5C. ***Tensor product, truncation and induction.*** In this subsection, we drop indices and write $T(n, d)$ for $T_\mathfrak{a}^A(n, d)$ and $S(n, d)$ for $S^A(n, d)$. Throughout the subsection, we fix $a \in \mathbb{Z}_{\ge 1}$, a composition $\delta = (d_1, \dots, d_a) \in \Lambda(a, d)$, and a composition $\nu = (n_1, \dots, n_a) \in \Lambda(a, n)$ with $n_1, \dots, n_a > 0$. We denote

$$T(n, \delta) := T(n, d_1) \otimes \cdots \otimes T(n, d_a) \quad \text{and} \quad T(\nu, \delta) := T(n_1, d_1) \otimes \cdots \otimes T(n_a, d_a).$$

Let

$$\nabla^{(a)} := (\mathrm{id}^{\otimes a-2} \otimes \nabla) \circ \cdots \circ (\mathrm{id} \otimes \nabla) \circ \nabla \; : \; T(n, d) \to \bigoplus_{\gamma \in \Lambda(a,d)} T(n, \gamma)$$

be the iterated coproduct. Projecting onto the summand $T(n, \delta)$ yields the algebra homomorphism

$$\nabla_\delta : T \to T(n, \delta).$$

Using ∇_δ, we can consider $T(n, \delta)$ as a $(T(n, d), T(n, \delta))$-bimodule, so that

$$U_1 \otimes \cdots \otimes U_a \cong T(n, \delta) \otimes_{T(n,\delta)} (U_1 \boxtimes \cdots \boxtimes U_a) \tag{5.18}$$

for $U_1 \in T(n, d_1)$-mod, ..., $U_a \in T(n, d_a)$-mod.

Recall the idempotent $\xi_n^N(d) \in T(N, d)$ from (5.14). The first result relates tensor product and truncation.

Proposition 5.19. *Let $n \le N$ and $V_k \in T(N, d_k)$-mod for $k = 1, \dots, a$. Then there is a functorial isomorphism of $T(n, d)$-modules*

$$\xi_n^N(d)(V_1 \otimes \cdots \otimes V_a) \cong (\xi_n^N(d_1)V_1) \otimes \cdots \otimes (\xi_n^N(d_a)V_a).$$

A similar statement holds for S in place of T.

Proof. Note using Lemma 5.8 that

$$\nabla_\delta(\xi_n^N(d)) = \sum_{\lambda\in\Lambda_n^N(d)} \nabla_\delta(\xi_\lambda) = \sum_{\mu_1\in\Lambda_n^N(d_1),\dots,\mu_a\in\Lambda_n^N(d_a)} \xi_{\mu_1}\otimes\cdots\otimes\xi_{\mu_a}$$
$$= \xi_n^N(d_1)\otimes\cdots\otimes\xi_n^N(d_a).$$

Therefore

$$\xi_n^N(d)(V_1\otimes\cdots\otimes V_a) = (\xi_n^N(d_1)V_1)\otimes\cdots\otimes(\xi_n^N(d_a)V_a),$$

and the result follows. □

In the rest of this subsection, we concentrate on $T(n,d)$, although similar results hold for $S(n,d)$. We now define certain induction operation and relate it to tensor product. Set

$$m_k := \sum_{r=1}^{k-1} n_r \qquad (k = 1,\dots,a+1).$$

Denote

$$\Lambda(\nu;\delta) = \left\{\lambda\in\Lambda(n,d)\;\middle|\;\sum_{r=m_k+1}^{m_{k+1}}\lambda_r = d_k \text{ for all } k=1,\dots,a\right\},$$

and define the idempotent

$$\xi(\nu;\delta) := \sum_{\lambda\in\Lambda(\nu;\delta)}\xi_\lambda \in T(n,d).$$

For $\boldsymbol{r} = r_1\cdots r_t\in\mathbb{Z}^t$ and $m\in\mathbb{Z}_{\geq 0}$, we define

$$\boldsymbol{r}(+m) := (r_1+m)\cdots(r_t+m)\in\mathbb{Z}^t.$$

Now let $\boldsymbol{r}^k\in[1,n_k]^{d_k}$ for $k=1,\dots,a$, and $\boldsymbol{r} := \boldsymbol{r}^1\cdots\boldsymbol{r}^a\in[1,n]^d$. We define

$$\boldsymbol{r}(+\nu) := \boldsymbol{r}^1(+m_1)\boldsymbol{r}^2(+m_2)\cdots\boldsymbol{r}^a(+m_a)\in[1,n]^d.$$

If $(\boldsymbol{b}^k,\boldsymbol{r}^k,\boldsymbol{s}^k)\in\mathrm{Tri}^B(n_k,d_k)$ for $k=1,\dots,a$, and $\boldsymbol{b} := \boldsymbol{b}^1\cdots\boldsymbol{b}^a$, $\boldsymbol{r} := \boldsymbol{r}^1\cdots\boldsymbol{r}^a$, $\boldsymbol{s} := \boldsymbol{s}^1\cdots\boldsymbol{s}^a$, then $(\boldsymbol{b},\boldsymbol{r}(+\nu),\boldsymbol{s}(+\nu))\in\mathrm{Tri}^B(n,d)$ is δ-separated. So by Lemma 4.6,

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r}(+\nu),\boldsymbol{s}(+\nu)} = \eta^{\boldsymbol{b}^1}_{\boldsymbol{r}(+m_1),\boldsymbol{s}(+m_1)} * \eta^{\boldsymbol{b}^2}_{\boldsymbol{r}(+m_2),\boldsymbol{s}(+m_2)} * \cdots * \eta^{\boldsymbol{b}^a}_{\boldsymbol{r}(+m_a),\boldsymbol{s}(+m_a)}. \tag{5.20}$$

Similarly, $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}(+\nu))\in\mathrm{Tri}^B(n,d)$ is δ-separated, and

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}(+\nu)} = \eta^{\boldsymbol{b}^1}_{\boldsymbol{r},\boldsymbol{s}(+m_1)} * \eta^{\boldsymbol{b}^2}_{\boldsymbol{r},\boldsymbol{s}(+m_2)} * \cdots * \eta^{\boldsymbol{b}^a}_{\boldsymbol{r},\boldsymbol{s}(+m_a)}. \tag{5.21}$$

Lemma 5.22. *The map*

$$T(\nu,\delta)\to T(n,d),\qquad \eta^{\boldsymbol{b}^1}_{\boldsymbol{r}^1,\boldsymbol{s}^1}\otimes\cdots\otimes\eta^{\boldsymbol{b}^a}_{\boldsymbol{r}^a,\boldsymbol{s}^a}\mapsto\eta^{\boldsymbol{b}}_{\boldsymbol{r}(+\nu),\boldsymbol{s}(+\nu)}$$

is an algebra homomorphism, mapping the identity element of $T(\nu,\delta)$ onto $\xi(\nu;\delta)$.

Proof. This follows easily from (5.20) and Lemma 4.5. □

In view of the lemma, we consider

$$T(n,d)\xi(\nu,\delta)$$

as a $(T(n,d),T(\nu,\delta))$-bimodule. Given a $T(\nu,\delta)$-module V, we now define

$$I^{n,d}_{\nu,\delta}V := T(n,d)\xi(\nu;\delta)\otimes_{T(\nu,\delta)} V.$$

This yields the functor

$$I^{n,d}_{\nu,\delta} : T(\nu,\delta)\text{-mod} \to T(n,d)\text{-mod}\,.$$

The following proposition generalizes [Brundan and Kleshchev 1999, 2.7].

Proposition 5.23. *Suppose that for all $k=1,\dots,a$ we have $d_k\le n_k$ and let $V_k\in T(n,d_k)$-mod. Then we have a functorial isomorphism*

$$V_1\otimes\cdots\otimes V_a \cong I^{n,d}_{\nu;\delta}\big((\xi^n_{n_1}V_1)\boxtimes\cdots\boxtimes(\xi^n_{n_a}V_a)\big).$$

Proof. In this proof k always runs through $\{1,\dots,a\}$. Denote $T:=T(n,d)$, $T_k:=T(n,d_k)$, $T'_k:=T(n_k,d_k)$, so that $T(n,\delta)=T_1\otimes\cdots\otimes T_a$ and $T(\nu,\delta)=T'_1\otimes\cdots\otimes T'_a$.

Since $V_k\mapsto \xi^n_{n_k}V_k$ is an equivalence by Corollary 5.16, denoting $W_k:=\xi^n_{n_k}V_k$, we have $V_k\cong T_k\xi^N_{n_k}\otimes_{T'_k} W_k$, and it suffices to prove

$$(T_1\xi^n_{n_1}\otimes_{T'_1}W_1)\otimes\cdots\otimes(T_a\xi^n_{n_a}\otimes_{T'_a}W_a)\cong I^{n,d}_{\nu;\delta}(W_1\boxtimes\cdots\boxtimes W_a). \tag{5.24}$$

We now apply (5.18) with $U_k:=T_k\xi^n_{n_k}\otimes_{T'_k}W_k$ to see that the left-hand side of (5.24) is obtained from $W_1\boxtimes\cdots\boxtimes W_a$ by tensoring with the $(T,T(\nu,\delta))$-bimodule

$$M':=M\otimes_{T(n,\delta)}(T_1\xi^n_{n_1}\otimes\cdots\otimes T_a\xi^n_{n_a})\cong T_1\xi^n_{n_1}\otimes\cdots\otimes T_a\xi^n_{n_a}.$$

On the other hand, the right-hand side of (5.24) is obtained from $W_1\boxtimes\cdots\boxtimes W_a$ by tensoring with the $(T,T(\nu,\delta))$-bimodule $T\xi(\nu;\delta)$. So we just need to prove that the $(T,T(\nu,\delta))$-bimodules M' and $T\xi(\nu;\delta)$ are isomorphic.

Define

$$\mathrm{Tri}^B((n,n_k),d_k):=\{(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B(n,d_k)\mid \boldsymbol{s}\in[1,n_k]^{d_k}\}.$$

Then, for all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B(n,d_k)$, we have that

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\xi^n_{n_k}=\begin{cases}\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} & \text{if } (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s})\in\mathrm{Tri}^B((n,n_k),d_k),\\ 0 & \text{otherwise.}\end{cases}$$

Therefore $\{\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\mid[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]\in\mathrm{Tri}^B((n,n_k),d_k)/\mathfrak{S}_{d_k}\}$ is a basis for $T_k\xi^n_{n_k}$, and we may define a $\Bbbk$-linear map

$$\varphi: M'\to T\xi(\nu;\delta),\qquad \eta^{\boldsymbol{b}^1}_{\boldsymbol{r}^1,\boldsymbol{s}^1}\otimes\cdots\otimes\eta^{\boldsymbol{b}^a}_{\boldsymbol{r}^a,\boldsymbol{s}^a}\mapsto\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}(+\nu)}\xi(\nu;\delta)=\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}(+\nu)},$$

where $[\boldsymbol{b}^k,\boldsymbol{r}^k,\boldsymbol{s}^k]\in\mathrm{Tri}^B((n,n_k),d_k)/\mathfrak{S}_{d_k}$ for all k, $\boldsymbol{b}=\boldsymbol{b}^1\cdots\boldsymbol{b}^a$, $\boldsymbol{r}=\boldsymbol{r}^1\cdots\boldsymbol{r}^a$, and $\boldsymbol{s}=\boldsymbol{s}^1\cdots\boldsymbol{s}^a$. It follows from (5.21) and Lemmas 4.5 and 5.22 that φ is a map of $(T,T(\nu,\delta))$-bimodules, and it remains to prove that φ is an isomorphism.

For $\boldsymbol{s} \in [1,n]^d$, we define $\beta(\boldsymbol{s}) \in \Lambda(n,d)$ via $\beta(\boldsymbol{s})_t := \#\{u \in [1,d] \mid s_u = t\}$, for all $t \in [1,n]$. Then for $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$ we have

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\xi(\nu;\delta) = \begin{cases} \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} & \text{if } \beta(\boldsymbol{s}) \in \Lambda(\nu;\delta), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, setting

$$\mathrm{Tri}^B(\nu;\delta) := \{(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d) \mid \beta(\boldsymbol{s}) \in \Lambda(\nu;\delta)\},$$

we have that $\{\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}] \in \mathrm{Tri}^B(\nu;\delta)/\mathfrak{S}_d\}$ is a basis for $T\xi(\nu;\delta)$. It is straightforward to check that the map

$$\prod_{k=1}^{a}(\mathrm{Tri}^B((n,n_k),d_k)/\mathfrak{S}_{d_k}) \to \mathrm{Tri}^B(\nu;\delta)/\mathfrak{S}_d,$$
$$([\boldsymbol{b}^1,\boldsymbol{r}^1,\boldsymbol{s}^1],\dots,[\boldsymbol{b}^a,\boldsymbol{r}^a,\boldsymbol{s}^a]) \mapsto [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}(+\nu)],$$

where $\boldsymbol{b} = \boldsymbol{b}^1\cdots\boldsymbol{b}^a$, $\boldsymbol{r} = \boldsymbol{r}^1\cdots\boldsymbol{r}^a$, $\boldsymbol{s} = \boldsymbol{s}^1\cdots\boldsymbol{s}^a$, is a well-defined bijection. Hence φ restricts to a bijection (up to signs) of bases, and so φ is an isomorphism. □

5D. *Examples.* We finish this section with some examples.

5D1. *Schur superalgebras.* Let $A = M_{p|q}(\Bbbk)$. For $r,s \in [1,p+q]$, let $E_{r,s}$ be the matrix with 1 in the (r,s)-th component, and zeros elsewhere. We have

$$\bar{E}_{r,s} := \begin{cases} \bar{0} & \text{if } r,s \le p \text{ or } r,s > p, \\ \bar{1} & \text{otherwise.} \end{cases}$$

It follows from [Marko and Zubkov 2006, Theorem 2] that the Schur superalgebra $S(p|q,d)$ is not quasihereditary (over $\Bbbk$) unless $q = 0$ or $p = q = d = 1$. Choosing $\mathfrak{a} := \mathrm{span}(E_{r,s} \mid r,s \le p)$ we get a nonunital good pair $(A,\mathfrak{a})$ and the corresponding nonunital generalized Schur superalgebra $T^A_{\mathfrak{a}}(n,d)$. We prove in [Kleshchev and Muth 2018] that $T^A_{\mathfrak{a}}(n,d)$ is quasihereditary if $d \le n$.

5D2. *Trivial extension algebras.* Let C be a unital superalgebra which is free of finite rank as a $\Bbbk$-supermodule. The dual $C^* := \mathrm{Hom}_{\Bbbk}(V,\Bbbk)$ is a $\Bbbk$-supermodule in a natural way. We have the pairing $\langle\cdot,\cdot\rangle$ between C and C^* with $\langle a,\alpha\rangle = \langle\alpha,a\rangle := \alpha(a)$ for $a \in C$ and $\alpha \in C^*$. We consider C^* as a C-bimodule with respect to the dual regular actions given by

$$\langle\alpha\cdot a,b\rangle = \langle\alpha,ab\rangle, \quad \langle b,a\cdot\alpha\rangle = \langle ba,\alpha\rangle \qquad (a,b \in C,\ \alpha \in C^*).$$

The *trivial extension superalgebra* $\mathrm{E}(C)$ of C is $\mathrm{E}(C) = C \oplus C^*$ as a $\Bbbk$-supermodule, with multiplication

$$(a,\alpha)(b,\beta) = (ab, a\cdot\beta + \alpha\cdot b) \qquad (a,b \in C,\ \alpha,\beta \in C^*).$$

Note that $C_{\bar{0}}$ is a unital subalgebra of $\mathrm{E}(C)_{\bar{0}}$. The pair $(\mathrm{E}(C),C_{\bar{0}})$ is an example of a unital good pair $(A,\mathfrak{a})$. In this case it is natural to take $\mathfrak{c} = C^*_{\bar{0}}$. The basis $B_{\mathfrak{a}}$ is a basis of $C_{\bar{0}}$, the basis $B_{\bar{1}}$ is a basis of $\mathrm{E}(C)_{\bar{1}} = C_{\bar{1}} \oplus C^*_{\bar{1}}$, and the basis $B_{\mathfrak{c}}$ is a basis of $C^*_{\bar{0}}$.

Let $n \in \mathbb{Z}_{>0}$. For $\alpha \in C^*$ and $1 \le r, s \le n$, we have the element $x^{\alpha}_{r,s} \in M_n(C)^*$ defined from

$$\langle x^{\alpha}_{r,s}, \xi^{a}_{t,u} \rangle = \delta_{r,t}\delta_{s,u}\langle \alpha, a\rangle \qquad (1 \le t, u \le n,\ a \in C).$$

It is pointed out in [Evseev and Kleshchev 2017, Lemma 3.21] that there is an isomorphism of superalgebras

$$M_n(\mathrm{E}(C)) \xrightarrow{\sim} \mathrm{E}(M_n(C)), \quad \xi^{(a,\alpha)}_{r,s} \mapsto (\xi^{a}_{r,s}, x^{\alpha}_{s,r}). \tag{5.25}$$

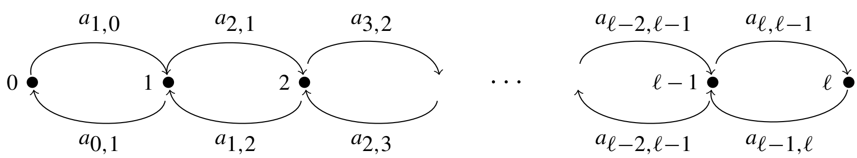
By [loc. cit., Theorem 4.27], we have an explicit isomorphism

$$S^{\mathrm{E}(C)}(n,d) \cong {}'D^{C}(n,d), \tag{5.26}$$

where ${}'D^C(n,d)$ is the *divided power Turner's double algebra* defined in [loc. cit., §4.6]. It now follows from Theorem 4.13 and [loc. cit., Theorem 4.31] that under the isomorphism (5.26) the subalgebra $T^{\mathrm{E}(C)}_{C_{\bar{0}}}(n,d) \subseteq S^{\mathrm{E}(C)}(n,d)$ gets identified with the *Turner double* subalgebra $D^C(n,d) \subseteq {}'D^C(n,d)$ of [loc. cit., §4.6]:

$$T^{\mathrm{E}(C)}_{C_{\bar{0}}}(n,d) \cong D^{C}(n,d). \tag{5.27}$$

5D3. *Zigzag algebras.* Fix $\ell \ge 1$ and let Γ be the quiver with vertex set $I := \{0, 1, \dots, \ell\}$ and arrows $\{a_{j,j-1}, a_{j-1,j} \mid j = 1, \dots, \ell\}$ as in the picture:



The *extended zigzag algebra* Z is the path algebra $\Bbbk\Gamma$ modulo the following relations:

(i) All paths of length three or greater are zero.

(ii) All paths of length two that are not cycles are zero.

(iii) All length-two cycles based at the same vertex are equivalent.

(iv) $a_{\ell,\ell-1}a_{\ell-1,\ell} = 0$.

Length zero paths yield the idempotents $\{e_0, \dots, e_\ell\}$ with $e_i a_{i,j} e_j = a_{i,j}$ for all admissible i, j. The algebra Z is graded by the path length: $Z = Z^0 \oplus Z^1 \oplus Z^2$. We consider Z as a superalgebra with $Z_{\bar{0}} = Z^0 \oplus Z^2$ and $Z_{\bar{1}} = Z^1$. Let $\mathfrak{z} := \mathrm{span}(e_0, \dots, e_\ell)$. Define also $J := \{0, 1, \dots, \ell-1\}$ and for all $j \in J$, set $c_j := a_{j,j+1}a_{j+1,j}$. We have a basis $B = B_{\mathfrak{z}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar{1}}$ of Z as in (2.2), with

$$B_{\bar{1}} = \{a_{j,j+1}, a_{j+1,j} \mid j \in J\}, \quad B_{\mathfrak{z}} = \{e_i \mid i \in I\}, \quad B_{\mathfrak{c}} := \{c_j \mid j \in J\}.$$

Let $e := e_0 + \cdots + e_{\ell-1} \in Z$. The *zigzag algebra* is $\bar{Z} := eZe \subset Z$. We also have $\bar{\mathfrak{z}} := e\mathfrak{z}e = \mathrm{span}(e_0, \dots, e_{\ell-1})$. For $n \ge d$, the generalized Schur algebra $T^{\bar{Z}}_{\bar{\mathfrak{z}}}(n,d)$ in this case is Morita equivalent to weight d RoCK blocks of symmetric groups (and the corresponding Hecke algebras), as conjectured by Turner [2009] and proved in [Evseev and Kleshchev 2018]. This was our motivating example.

In [Kleshchev and Muth 2018] we construct an explicit cellular basis of $T^{\bar{Z}}_{\bar{\mathfrak{z}}}(n,d)$, while no such basis is known for $S^{\bar{Z}}(n,d)$ (and it probably does not exist in general). We also prove that $T^{Z}_{\mathfrak{z}}(n,d)$ is quasihereditary, while $S^{Z}(n,d)$ in general is not.

Example 5.28. To make things a bit more concrete and clarify the distinction between $T^{Z}_{\mathfrak{z}}(n,d)$ and $S^{Z}(n,d)$, let us describe the bases for each in the case $d=2$. By Lemma 3.3, $S^{Z}(n,2)$ has basis

$$\{\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^{B}(n,2)/\mathfrak{S}_2\}.$$

Note that the superstructure of Z plays a crucial role in defining $S^{Z}(n,d)$ here; for instance, if we took Z to be purely even, we would have $(a_{01}a_{01}, 11, 11) \in \mathrm{Tri}^{B}(n,2)$, whereas $(a_{01}a_{01}, 11, 11) \notin \mathrm{Tri}^{B}(n,2)$ with the given superstructure. Thus, the differing superstructure on the same underlying algebra would yield distinct associated generalized Schur algebras.

Let $M \subset \mathrm{Tri}^{B}(n,2)$ be the set of triples of the form (c_jc_j, rr, ss), where $j \in J$, $r,s \in [1,n]$. Then for $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^{B}(n,2)$, we have

$$[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^{!}_{\mathfrak{c}} = \begin{cases} 2 & \text{if } (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in M, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, by Lemma 3.10, $T^{Z}_{\mathfrak{z}}(n,2)$ has basis

$$\big\{2\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in M\big\} \cup \big\{\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} \mid (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^{B}(n,2)/\mathfrak{S}_2,\ (\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \notin M\big\},$$

so $T^{Z}_{\mathfrak{z}}(n,2)$ is a proper sublattice in $S^{Z}(n,2)$.

6. Symmetricity

6A. ***Central and symmetrizing forms.*** Assume in this section that A has finite rank as a $\Bbbk$-supermodule. We say an even $\Bbbk$-linear map $\mathtt{t}: A \to \Bbbk$ is a *central form* for A provided $\mathtt{t}(ab) = \mathtt{t}(ba)$ for all $a,b \in A$. In this case we have an associative symmetric bilinear form $(\,\cdot\,,\,\cdot\,)_{\mathtt{t}}$ with $(a,b)_{\mathtt{t}} = \mathtt{t}(ab)$ and a homomorphism

$$A \to A^* := \mathrm{Hom}_{\Bbbk}(A,\Bbbk), \quad a \mapsto (a,\,\cdot\,)_{\mathtt{t}}$$

of (A,A)-superbimodules. We say that the central form $\mathtt{t}$ is a *symmetrizing form* if this homomorphism is an isomorphism, i.e., if $(\,\cdot\,,\,\cdot\,)_{\mathtt{t}}$ is a perfect pairing.

If A is equipped with a symmetrizing form, we say A is *symmetric*. We want to show that if A is symmetric then so is $T^{A}_{\mathfrak{a}}(n,d)$. We will do this under some natural assumptions. The following lemma is easily checked.

Lemma 6.1. *Assume that* $\mathtt{t}$ *is a central form on A. Then the algebra $M_n(A)^{\otimes d}$ has a central form* $\mathtt{t}^{M}: M_n(A)^{\otimes d} \to \Bbbk$ *given by*

$$\mathtt{t}^{M}(\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d}) = \delta_{\boldsymbol{r},\boldsymbol{s}}\mathtt{t}(x_1)\cdots\mathtt{t}(a_d),$$

for all $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}) \in A^d \times [1,n]^d \times [1,n]^d$.

Lemma 6.2. *Assume that* $\mathtt{t}$ *is a central form on* A. *Then the algebra* $S^A(n,d)$ *has a central form* $\mathtt{t}^S : S^A(n,d) \to \Bbbk$ *given by*

$$\mathtt{t}^S(\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}) := \frac{d!}{[\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}]^!}\delta_{\boldsymbol{r},\boldsymbol{s}}\mathtt{t}(a_1)\cdots\mathtt{t}(a_d).$$

for all $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^H(n,d)$. *Moreover,* $\mathtt{t}^M|_{S^A(n,d)} = \mathtt{t}^S$.

Proof. Let $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^H(n,d)$. Since $\mathtt{t}(x) = 0$ whenever $x \in A_{\bar{1}}$, we have

$$\mathtt{t}^M((\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d})^\sigma) = \mathtt{t}^M(\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d}),$$

for all $\sigma \in \mathfrak{S}_d$.

Then for all $(\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^H(n,d)$, we have

$$\begin{aligned}
\mathtt{t}^M(\xi^{\boldsymbol{a}}_{\boldsymbol{r},\boldsymbol{s}}) &= \mathtt{t}^M\Big(\sum_{\sigma\in {}^{\boldsymbol{x},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}} (\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d})^\sigma\Big) \\
&= \sum_{\sigma\in {}^{\boldsymbol{x},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}} \mathtt{t}^M((\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d})^\sigma) \\
&= \sum_{\sigma\in {}^{\boldsymbol{x},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}} \mathtt{t}^M(\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d}) \\
&= |{}^{\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}}\mathscr{D}|\mathtt{t}^M(\xi^{a_1}_{r_1,s_1} \otimes \cdots \otimes \xi^{a_d}_{r_d,s_d}) \\
&= \frac{d!}{[\boldsymbol{a},\boldsymbol{r},\boldsymbol{s}]^!}\delta_{\boldsymbol{r},\boldsymbol{s}}\mathtt{t}(x_1)\cdots\mathtt{t}(x_d),
\end{aligned}$$

giving the result. □

If $\Bbbk$ is a field of characteristic 0 or greater than d, one can check that the form $\mathtt{t}^S$ is symmetrizing. But this is certainly false over fields of positive characteristics less than d. In fact, for such fields even the classical Schur algebra $S^{\Bbbk}(n,d)$ is not symmetric.

6B. *Symmetricity of $T^A_{\mathfrak{a}}(n,d)$.* Throughout the subsection we assume that $(A,\mathfrak{a})$ is a unital good pair, and that A is symmetric, with symmetrizing form $\mathtt{t}$. We say that $\mathtt{t}$ is $(A,\mathfrak{a})$-*symmetrizing* if $(\mathfrak{a},\mathfrak{a})_{\mathtt{t}} = 0$ and the $\Bbbk$-complement $\mathfrak{c}$ of $\mathfrak{a}$ in $A_{\bar{0}}$ can be chosen so that the restriction of $(\cdot\,,\,\cdot)_{\mathtt{t}}$ to $\mathfrak{a}\times\mathfrak{c}$ is a perfect pairing. If $\mathtt{t}$ is an $(A,\mathfrak{a})$-*symmetrizing* form, we always assume that the complement $\mathfrak{c}$ has this property.

Let $\mathtt{t}$ be an $(A,\mathfrak{a})$-*symmetrizing* form. For $a \in \mathfrak{a}$, we have $\mathtt{t}(a) = (a,1)_{\mathtt{t}} = 0$ so $\mathtt{t}(\mathfrak{a}) = 0$. There exists an $(A,\mathfrak{a})$-basis $B = B_{\mathfrak{a}} \cup B_{\mathfrak{c}} \cup B_{\bar{1}}$ of A such that the dual basis $B^* = \{b^* \mid b \in B\}$ of A with respect to $(\cdot\,,\,\cdot)_{\mathtt{t}}$ satisfies the following property: setting

$$B^*_{\mathfrak{a}} := \{b^* \mid b \in B_{\mathfrak{a}}\}, \quad B^*_{\mathfrak{c}} := \{b^* \mid b \in B_{\mathfrak{c}}\}, \quad B^*_{\bar{1}} := \{b^* \mid b \in B_{\bar{1}}\}.$$

we have that $B^*_{\mathfrak{a}} = B_{\mathfrak{c}}$, $B^*_{\mathfrak{c}} = B_{\mathfrak{a}}$, and $B^*_{\bar{1}}$ is a basis for $A_{\bar{1}}$. Then $B^* := B^*_{\mathfrak{c}} \cup B^*_{\mathfrak{a}} \cup B^*_{\bar{1}}$ is an $(A,\mathfrak{a})$-basis as well. If $\mathtt{t}$ is an $(A,\mathfrak{a})$-*symmetrizing* form, we always assume that B has been chosen to satisfy these properties.

Lemma 6.3. *Let* t *be an* $(A,\mathfrak{a})$-***symmetrizing*** *form. Then the algebra* $T^A_{\mathfrak{a}}(n,d)$ *has a central form* $\mathsf{t}^T : T^A_{\mathfrak{a}}(n,d) \to \Bbbk$ *given by*

$$\mathsf{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}) := \delta_{\boldsymbol{r},\boldsymbol{s}}\mathsf{t}(b_1)\cdots\mathsf{t}(b_d).$$

for all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$. *Moreover,* $\mathsf{t}^S|_{T^A_{\mathfrak{a}}(n,d)} = d!\,\mathsf{t}^T$.

Proof. Recall t^S from Lemma 6.2. For $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$ we have

$$\mathsf{t}^S(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}) = \mathsf{t}^S\big([\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}}\xi^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\big) = [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}}\cdot\frac{d!}{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!}\delta_{\boldsymbol{r},\boldsymbol{s}}\mathsf{t}(b_1)\cdots\mathsf{t}(b_d).$$

But, since $\mathsf{t}(b)=0$ whenever $b \in B_{\mathfrak{a}} \cup B_{\bar{1}}$, we have $[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{c}} = [\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!$ whenever $\mathsf{t}^S(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}) \neq 0$. So for all $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d)$ we have

$$\mathsf{t}^S(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}) = d!\,\delta_{\boldsymbol{r},\boldsymbol{s}}\mathsf{t}(b_1)\cdots\mathsf{t}(b_d) = d!\,\mathsf{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}).$$

As $\Bbbk$ is a characteristic zero domain and t^S is central by Lemma 6.2, t^T is also central. □

Lemma 6.4. *For all* $x \in T^A_{\mathfrak{a}}(n,d_1)$, $y \in T^A_{\mathfrak{a}}(n,d_2)$, *we have* $\mathsf{t}^T(x*y) = \mathsf{t}^T(x)\mathsf{t}^T(y)$.

Proof. Take $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \in \mathrm{Tri}^B(n,d_1)$ and $(\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}) \in \mathrm{Tri}^B(n,d_2)$. We have

$$\mathsf{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}})\mathsf{t}^T(\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = \delta_{\boldsymbol{r},\boldsymbol{s}}\delta_{\boldsymbol{t},\boldsymbol{u}}\mathsf{t}(b_1)\cdots\mathsf{t}(b_{d_1})\mathsf{t}(c_1)\cdots\mathsf{t}(c_{d_2}).$$

On the other hand, by Lemma 4.2(iii), we have

$$\begin{aligned}\mathsf{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) &= \frac{[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!_{\mathfrak{a}}}{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{a}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{a}}}\mathsf{t}^T(\eta^{\boldsymbol{bc}}_{\boldsymbol{rt},\boldsymbol{su}})\\ &= \frac{[\boldsymbol{bc},\boldsymbol{rt},\boldsymbol{su}]^!_{\mathfrak{a}}}{[\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}]^!_{\mathfrak{a}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{a}}}\delta_{\boldsymbol{rt},\boldsymbol{su}}\mathsf{t}(b_1)\cdots\mathsf{t}(b_{d_1})\mathsf{t}(c_1)\cdots\mathsf{t}(c_{d_2}).\end{aligned}$$

Since $\mathsf{t}(b)=0$ whenever $b \in \mathfrak{a}$, we have the result. □

Lemma 6.5. *Let* $(b^d, r^d, s^d) \in \mathrm{Tri}^B(n,d)$. *Let* $(\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}) \in \mathrm{Tri}^{B^*}(n,d)$. *Then*

$$\mathsf{t}^T(\eta^{b^d}_{r^d,s^d}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = \begin{cases}\pm 1 & \text{if } \boldsymbol{c} = (b^*)^d,\ \boldsymbol{t} = s^d,\ \boldsymbol{u} = r^d,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Recall the central forms t^M and t^S from Lemmas 6.1 and 6.2. We have

$$\begin{aligned} d!\,\mathsf{t}^T(\eta^{b^d}_{r^d,s^d}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) &= \mathsf{t}^S\big([b^d,r^d,s^d]^!_{\mathfrak{c}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{c}}\,\xi^{b^d}_{r^d,s^d}\xi^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}\big)\\ &= [b^d,r^d,s^d]^!_{\mathfrak{c}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{c}}\,\mathsf{t}^M\Bigg(\xi^{b^d}_{r^d,s^d}\sum_{\sigma\in{}^{\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}}\mathscr{D}}(\xi^{c_1}_{t_1,u_1}\otimes\cdots\otimes\xi^{c_d}_{t_d,u_d})^{\sigma}\Bigg)\\ &= [b^d,r^d,s^d]^!_{\mathfrak{c}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{c}}\,\mathsf{t}^M\Bigg(\sum_{\sigma\in{}^{\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}}\mathscr{D}}\pm\delta_{s^d,\boldsymbol{t}\sigma}(\xi^{bc_{\sigma 1}}_{r,u_{\sigma 1}}\otimes\cdots\otimes\xi^{bc_{\sigma d}}_{r,u_{\sigma d}})\Bigg)\\ &= [b^d,r^d,s^d]^!_{\mathfrak{c}}[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathfrak{c}}\sum_{\sigma\in{}^{\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}}\mathscr{D}}\pm\delta_{s^d,\boldsymbol{t}\sigma}\delta_{r^d,\boldsymbol{u}\sigma}\mathsf{t}(bc_{\sigma_1})\cdots\mathsf{t}(bc_{\sigma d}).\end{aligned}$$

Since $\mathrm{t}(bc) = 0$ for all $c \in B^* \backslash \{b^*\}$, we have that $\mathrm{t}^T(\eta^{b^d}_{r^d,s^d}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = 0$ unless $\boldsymbol{c} = (b^*)^d$, $\boldsymbol{t} = s^d$, and $\boldsymbol{u} = r^d$. Assume now that $\boldsymbol{c} = (b^*)^d$, $\boldsymbol{t} = s^d$, and $\boldsymbol{u} = r^d$. Then ${}^{\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}}\mathscr{D} = \{1\}$, so the above simplifies to

$$d!\,\mathrm{t}^T(\eta^{b^d}_{r^d,s^d}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = \pm[b^d, r^d, s^d]^!_{\mathfrak{c}}[(b^*)^d, s^d, r^d]^!_{\mathfrak{c}}.$$

If $b \in \mathfrak{a}$, then $b^* \in \mathfrak{c}$, so we have

$$[b^d, r^d, s^d]_{\mathfrak{c}} = 0, \quad [(b^*)^d, s^d, r^d]_{\mathfrak{c}} = d.$$

Conversely, if $b \in \mathfrak{c}$, then $b^* \in \mathfrak{a}$, so we have

$$[b^d, r^d, s^d]_{\mathfrak{c}} = d, \quad [(b^*)^d, s^d, r^d]_{\mathfrak{c}} = 0.$$

If $b \in A_{\bar{1}}$, then $b^* \in A_{\bar{1}}$, and $(b^d, r^d, s^d) \in \mathrm{Tri}^B(n, d)$ implies $d = 1$, so we have

$$[b^d, r^d, s^d]_{\mathfrak{c}} = [(b^*)^d, s^d, r^d]_{\mathfrak{c}} = 0.$$

Thus, in any case, we have $d!\,\mathrm{t}^T(\eta^{b^d}_{r^d,s^d}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = \pm d!$, so the result follows since $\Bbbk$ is a characteristic zero domain. ☐

Now we upgrade this lemma. For $\boldsymbol{b} \in B^d$, write $\boldsymbol{b}^* := (b_1^*, \dots, b_d^*) \in (B^*)^d$.

Lemma 6.6. *Let $(\boldsymbol{b}, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)$ and $(\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}) \in \mathrm{Tri}^{B^*}(n, d)$. Then*

$$\mathrm{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = \begin{cases} \pm 1 & \text{if } (\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}) \sim (\boldsymbol{b}^*, \boldsymbol{s}, \boldsymbol{r}), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We go by induction on d. The base case follows from Lemma 6.5. Let $d > 1$. We have by Lemma 4.6 that either $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \eta^{b^d}_{r^d,s^d}$ for some $b \in B$, $r, s \in [1, n]$, or else $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$ may be written as

$$\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = \pm\eta^{\boldsymbol{b}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} * \eta^{\boldsymbol{b}^{(2)}}_{\boldsymbol{r}^{(2)},\boldsymbol{s}^{(2)}},$$

for some δ-separated $(\boldsymbol{b}^{(1)}, \boldsymbol{r}^{(1)}, \boldsymbol{s}^{(1)}) \in \mathrm{Tri}^B(n, d_1)$, $(\boldsymbol{b}^{(2)}, \boldsymbol{r}^{(2)}, \boldsymbol{s}^{(2)}) \in \mathrm{Tri}^B(n, d_2)$, with $d_1, d_2 > 0$ and $d_1 + d_2 = d$. In the former situation, the claim follows from Lemma 6.5, so assume we are in the latter situation.

Recalling the notation of Section 3C, we assume without loss of generality that $(\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}) \in \mathrm{Tri}_0^{B^*}(n, d)$. We have that

$$\begin{aligned}
\mathrm{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) &= \mathrm{t}^T((\pm\eta^{\boldsymbol{b}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}} * \eta^{\boldsymbol{b}^{(2)}}_{\boldsymbol{r}^{(2)},\boldsymbol{s}^{(2)}})\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) \\
&= \mathrm{t}^T\Bigg(\sum_{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}(\boldsymbol{c},\boldsymbol{t},\boldsymbol{u})} \frac{\pm[\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}]^!_{\mathfrak{c}}}{[\mathcal{T}^1]^!_{\mathfrak{c}}[\mathcal{T}^2]^!_{\mathfrak{c}}} (\eta^{\boldsymbol{b}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}}\eta_{\mathcal{T}^1}) * (\eta^{\boldsymbol{b}^{(2)}}_{\boldsymbol{r}^{(2)},\boldsymbol{s}^{(2)}}\eta_{\mathcal{T}^2}) \Bigg) \\
&= \sum_{(\mathcal{T}^1,\mathcal{T}^2)\in\mathrm{Spl}(\boldsymbol{c},\boldsymbol{t},\boldsymbol{u})} \frac{\pm[\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}]^!_{\mathfrak{c}}}{[\mathcal{T}^1]^!_{\mathfrak{c}}[\mathcal{T}^2]^!_{\mathfrak{c}}} \mathrm{t}^T(\eta^{\boldsymbol{b}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}}\eta_{\mathcal{T}^1})\mathrm{t}^T(\eta^{\boldsymbol{b}^{(2)}}_{\boldsymbol{r}^{(2)},\boldsymbol{s}^{(2)}}\eta_{\mathcal{T}^2}),
\end{aligned}$$

applying Corollary 3.24 and Lemma 4.5 for the second equality and Lemma 6.4 for the third equality. If this is nonzero, then by the induction assumption we must have $\mathcal{T}^1 \sim ((\boldsymbol{b}^{(1)})^*, \boldsymbol{s}^{(1)}, \boldsymbol{r}^{(1)})$ and $\mathcal{T}^2 \sim ((\boldsymbol{b}^{(2)})^*, \boldsymbol{s}^{(2)}, \boldsymbol{r}^{(2)})$ for some $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}(\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u})$, which implies $(\boldsymbol{b}^*, \boldsymbol{s}, \boldsymbol{r}) \sim (\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u})$.

On the other hand, assume $(\boldsymbol{b}^*, \boldsymbol{s}, \boldsymbol{r}) \sim (\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u})$. Then there is exactly one $(\mathcal{T}^1, \mathcal{T}^2) \in \mathrm{Spl}(\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u})$ such that $\mathcal{T}^1 \sim ((\boldsymbol{b}^{(1)})^*, \boldsymbol{s}^{(1)}, \boldsymbol{r}^{(1)})$ and $\mathcal{T}^2 \sim ((\boldsymbol{b}^{(2)})^*, \boldsymbol{s}^{(2)}, \boldsymbol{r}^{(2)})$. Then by the above we have

$$\mathtt{t}^T(\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}\eta^{\boldsymbol{c}}_{\boldsymbol{t},\boldsymbol{u}}) = \pm\frac{[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathtt{c}}}{[\mathcal{T}^1]^!_{\mathtt{c}}[\mathcal{T}^2]^!_{\mathtt{c}}}\mathtt{t}^T(\eta^{\boldsymbol{b}^{(1)}}_{\boldsymbol{r}^{(1)},\boldsymbol{s}^{(1)}}\eta_{\mathcal{T}^1})\mathtt{t}^T(\eta^{\boldsymbol{b}^{(2)}}_{\boldsymbol{r}^{(2)},\boldsymbol{s}^{(2)}}\eta_{\mathcal{T}^2}) = \pm\frac{[\boldsymbol{c},\boldsymbol{t},\boldsymbol{u}]^!_{\mathtt{c}}}{[\mathcal{T}^1]^!_{\mathtt{c}}[\mathcal{T}^2]^!_{\mathtt{c}}},$$

using the induction assumption. But note, since $(\boldsymbol{b}^{(1)}, \boldsymbol{r}^{(1)}, \boldsymbol{s}^{(1)})$ and $(\boldsymbol{b}^{(2)}, \boldsymbol{r}^{(2)}, \boldsymbol{s}^{(2)})$ are δ-separated, we have $[\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}]^!_{\mathtt{c}} = [\mathcal{T}^1]^!_{\mathtt{c}}[\mathcal{T}^2]^!_{\mathtt{c}}$, which completes the proof. □

Corollary 6.7. *If* $\mathtt{t}$ *is an* $(A, \mathfrak{a})$*-symmetrizing form on* A, *then the algebra* $T^A_{\mathfrak{a}}(n, d)$ *is symmetric, with symmetrizing form* $\mathtt{t}^T$.

Proof. The form $\mathtt{t}^T$ is central by Lemma 6.3. Moreover, by Lemma 6.6, we have that $(\cdot\,,\,\cdot)_{\mathtt{t}^T}$ is a perfect pairing. □

Remark 6.8. Given a superalgebra C, this result recovers the symmetricity property of the Turner double algebra $D^C(n, d)$ described in Section 5D2. In particular, $E(C) = C \oplus C^*$ has a symmetrizing form given by $\mathtt{t}(a, \alpha) = \alpha(a)$, so the symmetricity of $D^C(n, d) \cong T^{E(C)}_{C_{\bar{0}}}(n, d)$ follows from Corollary 6.7.

7. Double centralizer property

Throughout the section we assume that $\Bbbk$ is a principal ideal domain (as usual of characteristic 0). All modules and algebras are assumed to be free of finite rank as $\Bbbk$-modules. An element v of a $\Bbbk$-module V is called *divisible* if there is $w \in V$ and a nonunit $m \in \Bbbk$ with $v = mw$. Otherwise v is called *indivisible*. We let $\mathbb{K}$ to be the field of fractions of $\Bbbk$.

7A. *Double centralizer idempotents.* Let S be a $\Bbbk$-algebra and $e \in S$ be an idempotent. Considering Se as a right eSe-module, we have an algebra homomorphism

$$\lambda : S \to \mathrm{End}_{eSe}(Se), \quad s \mapsto \lambda_s$$

where $\lambda_s(s'e) = ss'e$ for all $s, s' \in S$. We say that e is a *double centralizer idempotent for* S if λ is an isomorphism. We say that e is a *sound idempotent for* S if λ sends indivisible elements of S to indivisible elements of $\mathrm{End}_{eSe}(Se)$. Clearly, a double centralizer idempotent is sound.

Set $S_{\mathbb{K}} := S \otimes_{\Bbbk} \mathbb{K}$ and use the map $s \mapsto s \otimes 1_{\mathbb{K}}$ to identify S as a subset of $S_{\mathbb{K}}$. Then $Se \otimes_{\Bbbk} \mathbb{K} = S_{\mathbb{K}}e$. Using the universal coefficient theorem, we identify $\mathrm{End}_{eSe}(Se) \otimes_{\Bbbk} \mathbb{K} = \mathrm{End}_{eS_{\mathbb{K}}e}(S_{\mathbb{K}}e)$, so that the $\lambda_{\mathbb{K}} := \lambda \otimes \mathrm{id}_{\mathbb{K}}$ is the map

$$\lambda_{\mathbb{K}} : S_{\mathbb{K}} \to \mathrm{End}_{eS_{\mathbb{K}}e}(S_{\mathbb{K}}e), \quad s \mapsto \lambda_s$$

where λ_s is the left multiplication by s. Clearly, if e is a double centralizer idempotent for S then e is a double centralizer idempotent for $S_{\mathbb{K}}$, but not vice versa in general.

Lemma 7.1. *Let $e \in S$ be an idempotent. Then e is a double centralizer idempotent for S if and only if e is a double centralizer idempotent for $S_{\mathbb{K}}$ and e is sound for S.*

Proof. The "only if" part is immediate. For the "if" part, the assumption that e is a double centralizer idempotent for $S_{\mathbb{K}}$ implies that $\lambda : S \to \mathrm{End}_{eSe}(Se)$ is injective. Moreover,

$$\mathrm{rank}_{\Bbbk} S = \dim_{\mathbb{K}} S_{\mathbb{K}} = \dim \mathrm{End}_{eS_{\mathbb{K}}e}(S_{\mathbb{K}}e) = \mathrm{rank}_{\Bbbk} \mathrm{End}_{eSe}(Se).$$

So λ is a full rank embedding. As e is sound, it now follows that λ is surjective. $\square$

7B. *Double centralizer property for $S^A(n, d)$.* Let $(A, \mathfrak{a})$ be a unital good pair, $e \in \mathfrak{a}$ be an idempotent, and $\xi^e \in T^A_{\mathfrak{a}}(n, d)$ be the idempotent of (5.1). Suppose that e is a double centralizer idempotent for A. Although it is not true in general that ξ^e is then a double centralizer for $T^A_{\mathfrak{a}}(n, d)$, this is true in some interesting situations.

In view of Lemma 7.1, to verify that ξ^e is a double centralizer for $T^A_{\mathfrak{a}}(n, d)$, it suffices to check that ξ^e is a double centralizer idempotent for $S^A(n, d)$ and that ξ^e is sound for $T^A_{\mathfrak{a}}(n, d)_{\mathbb{K}} = S^A(n, d)_{\mathbb{K}}$. It turns out that the first condition is always true provided $d \leq n$. This will follow from the following stronger theorem:

Theorem 7.2. *If $e \in A$ is a double centralizer idempotent for A and $d \leq n$, then ξ^e is a double centralizer idempotent for $S^A(n, d)$.*

Proof. As usual, we write $\bar{A} := eAe$. First we show

$$M_n(A) \cong \mathrm{End}_{M_n(\bar{A})}(M_n(Ae)). \tag{7.3}$$

We have an algebra homomorphism $\lambda : M_n(A) \to \mathrm{End}_{M_n(\bar{A})}(M_n(Ae))$ given by left multiplication. On the other hand, let $\varphi \in \mathrm{End}_{M_n(\bar{A})}(M_n(Ae))$. Since $\varphi(E^{ae}_{i,j}) = \varphi(E^{ae}_{i,t} E^{e}_{t,j}) = \varphi(E^{ae}_{i,t}) E^{e}_{t,j}$ for any $a \in A$ and $i, t, j \in [1, n]$, it follows that there exist functions $\varphi_{k,i} : Ae \to Ae$ such that

$$\varphi(E^{ae}_{i,j}) = \sum_k E^{\varphi_{k,i}(ae)}_{k,j}.$$

Moreover, each $\varphi_{k,i} \in \mathrm{End}_{\bar{A}}(Ae)$, since for all $a \in A$, $b \in \bar{A}$, we have

$$\begin{aligned}\sum_k E^{\varphi_{k,i}(aeb)}_{k,j} &= \varphi(E^{aeb}_{i,j}) = \varphi(E^{ae}_{i,j} E^{b}_{j,j}) = \varphi(E^{ae}_{i,j}) E^{b}_{j,j} \\ &= \sum_k E^{\varphi_{k,i}(ae)}_{k,j} E^{b}_{j,j} = \sum_k E^{\varphi_{k,i}(ae)b}_{k,j},\end{aligned}$$

which implies that $\varphi_{k,i}(aeb) = \varphi_{k,i}(ae)b$. Thus, since $A \cong \mathrm{End}_{\bar{A}}(Ae)$, we have that $\varphi_{k,i}$ is given by left multiplication by a unique $x_{k,i} \in A$. Then, since

$$\left(\sum_{k,i} E^{x_{k,i}}_{k,i}\right) E^{ae}_{r,s} = \sum_k E^{x_{k,r}ae}_{k,s} = \sum_k E^{\varphi_{r,k}(ae)}_{k,s} = \varphi(E^{ae}_{r,s}),$$

we have a well-defined map $\rho : \mathrm{End}_{M_n(\bar{A})}(M_n(Ae)) \to M_n(A)$ given by $\varphi \mapsto \sum_{k,i} E_{k,i}^{x_{k,i}}$. It is clear that λ and ρ are mutual inverses, proving (7.3).

Let us now write

$$\begin{aligned} S^A &:= S^A(n,d), & S^{Ae} &:= (M_n(Ae)^{\otimes d})^{\mathfrak{S}_d}, & S^{\bar{A}} &:= S^{\bar{A}}(n,d), \\ M^A &:= M_n(A)^{\otimes d}, & M^{Ae} &:= M_n(Ae)^{\otimes d}, & M^{\bar{A}} &:= M_n(\bar{A})^{\otimes d}. \end{aligned}$$

For $\boldsymbol{a} = (a_1, \dots, a_d) \in A^d$, $\boldsymbol{r}, \boldsymbol{s} \in [1,n]^d$, we will write

$$E_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{a}} := \xi_{r_1,s_1}^{a_1} \otimes \cdots \otimes \xi_{r_d,s_d}^{a_d}.$$

Our next task is to show

$$\mathrm{End}_{S^{\bar{A}}}(S^{Ae}) \cong \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}. \tag{7.4}$$

The $\mathfrak{S}_d$-action on $\mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}$ is given by $f^\sigma(x) := f(x^{\sigma^{-1}})^\sigma$. Let

$$f \in \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}.$$

Consider the restriction f_{res} of f to the $\Bbbk$-submodule S^{Ae}. For any $\alpha \in S^{Ae}$, we have

$$f_{\mathrm{res}}(\alpha)^\sigma = f(\alpha)^\sigma = f(\alpha^{\sigma^{-1}})^\sigma = f^\sigma(\alpha) = f(\alpha) = f_{\mathrm{res}}(\alpha),$$

so $f_{\mathrm{res}} \in \mathrm{End}_{S^{\bar{A}}}(S^{Ae})$ and $\mathrm{res} : \mathrm{End}_{S^{\bar{A}}}(S^{Ae}) \to \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}$ is an algebra homomorphism.

Now, let $g \in \mathrm{End}_{S^{\bar{A}}}(S^{Ae})$. Write $\boldsymbol{e} := (e, \dots, e) \in A^d$, and $\omega = (1, \dots, d) \in [1,n]^d$. We inflate g to a linear map $g_{\mathrm{infl}} : M^{Ae} \to M^{Ae}$ via

$$g_{\mathrm{infl}}(E_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{x}}) := g(\xi_{\boldsymbol{r},\omega}^{\boldsymbol{x}}) E_{\omega,\boldsymbol{s}}^{\boldsymbol{e}}.$$

Note that for any $\sigma \in \mathfrak{S}_d$, we also have

$$\begin{aligned} g_{\mathrm{infl}}(E_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{x}}) &= g(\xi_{\boldsymbol{r},\omega}^{\boldsymbol{x}}) E_{\omega,\boldsymbol{s}}^{\boldsymbol{e}} = g(\xi_{\boldsymbol{r},\omega^\sigma}^{\boldsymbol{x}} \xi_{\omega^\sigma,\omega}^{\boldsymbol{e}}) E_{\omega,\boldsymbol{s}}^{\boldsymbol{e}} \\ &= g(\xi_{\boldsymbol{r},\omega^\sigma}^{\boldsymbol{x}}) \xi_{\omega^\sigma,\omega}^{\boldsymbol{e}} E_{\omega,\boldsymbol{s}}^{\boldsymbol{e}} = g(\xi_{\boldsymbol{r},\omega^\sigma}^{\boldsymbol{x}}) E_{\omega^\sigma,\boldsymbol{s}}^{\boldsymbol{e}}. \end{aligned}$$

We will show that $g_{\mathrm{infl}} \in \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}$. First we check symmetry:

$$\begin{aligned} g_{\mathrm{infl}}^\sigma(E_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{x}}) &= g_{\mathrm{infl}}((E_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{x}})^{\sigma^{-1}})^\sigma = (-1)^{\langle \boldsymbol{x},\sigma^{-1}\rangle} g_{\mathrm{infl}}(E_{\boldsymbol{r}^{\sigma^{-1}},\boldsymbol{s}^{\sigma^{-1}}}^{\boldsymbol{x}^{\sigma^{-1}}})^\sigma \\ &= (-1)^{\langle \boldsymbol{x},\sigma^{-1}\rangle} [g(\xi_{\boldsymbol{r}^{\sigma^{-1}},\omega}^{\boldsymbol{x}^{\sigma^{-1}}}) E_{\omega,\boldsymbol{s}^{\sigma^{-1}}}^{\boldsymbol{e}}]^\sigma \\ &= (-1)^{\langle \boldsymbol{x},\sigma^{-1}\rangle} g(\xi_{\boldsymbol{r}^{\sigma^{-1}},\omega}^{\boldsymbol{x}^{\sigma^{-1}}})^\sigma (E_{\omega,\boldsymbol{s}^{\sigma^{-1}}}^{\boldsymbol{e}})^\sigma \\ &= (-1)^{\langle \boldsymbol{x},\sigma^{-1}\rangle} g(\xi_{\boldsymbol{r}^{\sigma^{-1}},\omega}^{\boldsymbol{x}^{\sigma^{-1}}}) E_{\omega^\sigma,\boldsymbol{s}}^{\boldsymbol{e}} \\ &= (-1)^{\langle \boldsymbol{x},\sigma^{-1}\rangle} (-1)^{\langle \boldsymbol{x}^{\sigma^{-1}},\sigma\rangle} g(\xi_{\boldsymbol{r},\omega^\sigma}^{\boldsymbol{x}}) E_{w^\sigma,\boldsymbol{s}}^{\boldsymbol{e}} = g_{\mathrm{infl}}(E_{\boldsymbol{r},\boldsymbol{s}}^{\boldsymbol{x}}). \end{aligned}$$

Now we check that g_{infl} is an $M^{\bar{A}}$-homomorphism. Let $\boldsymbol{x} \in (Ae)^d$, $\boldsymbol{b} \in \bar{A}^d$, and $\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u} \in [1, n]^d$.

$$\begin{aligned} g_{\mathrm{infl}}(E^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}} E^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}}) &= \delta_{\boldsymbol{s},\boldsymbol{t}} g_{\mathrm{infl}}(E^{\boldsymbol{xb}}_{\boldsymbol{r},\boldsymbol{u}}) = \delta_{\boldsymbol{s},\boldsymbol{t}} g(\xi^{\boldsymbol{xb}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{e}}_{\omega,\boldsymbol{u}} = \delta_{\boldsymbol{s},\boldsymbol{t}} g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega} \xi^{\boldsymbol{b}}_{\omega,\omega}) E^{\boldsymbol{e}}_{\omega,\boldsymbol{u}} \\ &= \delta_{\boldsymbol{s},\boldsymbol{t}} g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) \xi^{\boldsymbol{b}}_{\omega,\omega} E^{\boldsymbol{e}}_{\omega,\boldsymbol{u}} = \delta_{\boldsymbol{s},\boldsymbol{t}} g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) \xi^{\boldsymbol{b}}_{\omega,\omega} E^{\boldsymbol{e}}_{\omega,\boldsymbol{u}} = \delta_{\boldsymbol{s},\boldsymbol{t}} g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{b}}_{\omega,\boldsymbol{u}} \\ &= g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{e}}_{\omega,\boldsymbol{s}} E^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}} = g_{\mathrm{infl}}(E^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}) E^{\boldsymbol{b}}_{\boldsymbol{t},\boldsymbol{u}}. \end{aligned}$$

Therefore $g_{\mathrm{infl}} \in \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}$. Now we show that res and infl are mutual inverses. Let $\boldsymbol{x} \in (Ae)^d$, and $\boldsymbol{r}, \boldsymbol{s} \in [1, n]^d$. For $f \in \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d}$ we have

$$(f_{\mathrm{res}})_{\mathrm{infl}}(E^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}) = f_{\mathrm{res}}(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{e}}_{\omega,\boldsymbol{s}} = f(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{e}}_{\omega,\boldsymbol{s}} = f(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega} E^{\boldsymbol{e}}_{\omega,\boldsymbol{s}}) = f(E^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}),$$

so $(f_{\mathrm{res}})_{\mathrm{infl}} = f$.

Now, let $g \in \mathrm{End}_{S^{\bar{A}}}(S^{Ae})$, and write $g' := (g_{\mathrm{infl}})_{\mathrm{res}}$. We have

$$\begin{aligned} g'(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) = g_{\mathrm{infl}}(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) &= g_{\mathrm{infl}}\Bigg(\sum_{\sigma \in \mathfrak{S}_d} (E^{\boldsymbol{x}}_{\boldsymbol{r},\omega})^{\sigma} \Bigg) = g_{\mathrm{infl}}\Bigg(\sum_{\sigma \in \mathfrak{S}_d} (-1)^{\langle \boldsymbol{x},\sigma \rangle} E^{\boldsymbol{x}^{\sigma}}_{\boldsymbol{r}^{\sigma},\omega^{\sigma}} \Bigg) \\ &= \sum_{\sigma \in \mathfrak{S}_d} (-1)^{\langle \boldsymbol{x},\sigma \rangle} g_{\mathrm{infl}}(E^{\boldsymbol{x}^{\sigma}}_{\boldsymbol{r}^{\sigma},\omega^{\sigma}}) = \sum_{\sigma \in \mathfrak{S}_d} (-1)^{\langle \boldsymbol{x},\sigma \rangle} g(\xi^{\boldsymbol{x}^{\sigma}}_{\boldsymbol{r}^{\sigma},\omega^{\sigma}}) E^{\boldsymbol{e}}_{\omega^{\sigma},\omega^{\sigma}} \\ &= \sum_{\sigma \in \mathfrak{S}_d} (-1)^{\langle \boldsymbol{x},\sigma \rangle} (-1)^{\langle \boldsymbol{x}^{\sigma},\sigma^{-1} \rangle} g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{e}}_{\omega^{\sigma},\omega^{\sigma}} = \sum_{\sigma \in \mathfrak{S}_d} g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) E^{\boldsymbol{e}}_{\omega^{\sigma},\omega^{\sigma}} \\ &= g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) \sum_{\sigma \in \mathfrak{S}_d} E^{\boldsymbol{e}}_{w^{\sigma},w^{\sigma}} = g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) \xi^{\boldsymbol{e}}_{\omega,\omega} = g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega} \xi^{\boldsymbol{e}}_{\omega,\omega}) = g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}), \end{aligned}$$

so $g'(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) = g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega})$. We also have

$$\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega} \xi^{e^d}_{\omega,\boldsymbol{s}} = \xi^{x_1}_{r_1,s_1} * \cdots * \xi^{x_d}_{r_d,s_d} = |\mathfrak{S}_{\boldsymbol{x},\boldsymbol{r},\boldsymbol{s}}| \cdot \xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}.$$

Thus, writing $m := |\mathfrak{S}_{\boldsymbol{x},\boldsymbol{r},\boldsymbol{s}}| \in \Bbbk$, we have

$$\begin{aligned} mg'(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}) = g'(m\xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}) &= g'(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega} \xi^{e^d}_{\omega,\boldsymbol{s}}) = g'(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) \xi^{e^d}_{\omega,\boldsymbol{s}} \\ &= g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega}) \xi^{e^d}_{\omega,\boldsymbol{s}} = g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\omega} \xi^{e^d}_{\omega,\boldsymbol{s}}) = g(m\xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}) = mg(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}). \end{aligned}$$

As $m \neq 0$ and S^{Ae} is free over $\Bbbk$, this implies that $g'(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}}) = g(\xi^{\boldsymbol{x}}_{\boldsymbol{r},\boldsymbol{s}})$. Thus res and infl are mutual inverses, proving (7.4).

Now, note that the action of $\mathfrak{S}_d$ intertwines the isomorphisms

$$M_n(A)^{\otimes d} \cong \mathrm{End}_{M_n(\bar{A})}(M_n(Ae))^{\otimes d} \cong \mathrm{End}_{M^{\bar{A}}}(M^{Ae}),$$

so we have

$$\mathrm{End}_{\xi^e S^A \xi^e}(S^A \xi^e) = \mathrm{End}_{S^{\bar{A}}}(S^{Ae}) \cong \mathrm{End}_{M^{\bar{A}}}(M^{Ae})^{\mathfrak{S}_d} \cong (M_n(A)^{\otimes d})^{\mathfrak{S}_d} = S^A,$$

as desired. □

Corollary 7.5. *If $e \in A$ is a double centralizer idempotent for A and $d \le n$, then ξ^e is a double centralizer idempotent for*

$$S^A(n,d)_{\mathbb{K}} = T^A_{\mathfrak{a}}(n,d)_{\mathbb{K}}.$$

7C. *Computations in extended zigzag Schur algebras.* Recall the notation of Section 5D3. In particular, we have the extended zigzag algebra Z for a fixed ℓ and the idempotent $e := e_0 + \cdots + e_{\ell-1} \in Z$. We will use the standard basis $B = B_{\mathfrak{z}} \sqcup B_{\mathfrak{c}} \sqcup B_{\bar{1}}$ of Z.

For $\boldsymbol{r} \in [1,n]^d$, set $\mathfrak{S}_{\boldsymbol{r}} := \{\sigma \in \mathfrak{S}_d \mid \boldsymbol{r}\sigma = \boldsymbol{r}\}$, and denote by ${}^{\boldsymbol{r}}\mathscr{D}$ the set of the shortest coset representatives for $\mathfrak{S}_{\boldsymbol{r}} \backslash \mathfrak{S}_d$.

Lemma 7.6. *Let $t \in [1,n]$ and $\boldsymbol{r},\boldsymbol{s},\boldsymbol{u} \in [1,n]^d$. Suppose that $r_a \neq r_b$ and $s_a \neq s_b$ for all $1 \le a \neq b \le d$. Then*

(i) $\eta^{a^d_{\ell-1,\ell}}_{\boldsymbol{r},t^d}\eta^{a^d_{\ell,\ell-1}}_{t^d,\boldsymbol{s}} = \pm \sum_{\sigma\in\mathfrak{S}_d}(\operatorname{sgn}\sigma)\eta^{c^d_{\ell-1}}_{\boldsymbol{r}\sigma,\boldsymbol{s}}$,

(ii) $\eta^{e^d_\ell}_{\boldsymbol{u},t^d}\eta^{a^d_{\ell,\ell-1}}_{t^d,\boldsymbol{s}} = \sum_{\sigma\in{}^{\boldsymbol{u}}\mathscr{D}}\eta^{a^d_{\ell,\ell-1}}_{\boldsymbol{u}\sigma,\boldsymbol{s}}$.

Proof. (i) We have

$$\begin{aligned}
\eta^{a^d_{\ell-1,\ell}}_{\boldsymbol{r},t^d}\eta^{a^d_{\ell,\ell-1}}_{t^d,\boldsymbol{s}} &= \Big(\sum_{\sigma\in\mathfrak{S}_d}(\xi^{a_{\ell-1,\ell}}_{r_1,t}\otimes\cdots\otimes\xi^{a_{\ell-1,\ell}}_{r_d,t})^\sigma\Big)\Big(\sum_{\tau\in\mathfrak{S}_d}(\xi^{a_{\ell,\ell-1}}_{t,s_1}\otimes\cdots\otimes\xi^{a_{\ell,\ell-1}}_{t,s_d})^\tau\Big)\\
&= \pm\sum_{\sigma,\tau\in\mathfrak{S}_d}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)\xi^{c_{\ell-1}}_{r_{\sigma1},s_{\tau1}}\otimes\cdots\otimes\xi^{c_{\ell-1}}_{r_{\sigma d},s_{\tau d}}\\
&= \pm\sum_{\sigma,\tau\in\mathfrak{S}_d}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)\xi^{c_{\ell-1}}_{r_{\sigma\tau^{-1}\tau1},s_{\tau1}}\otimes\cdots\otimes\xi^{c_{\ell-1}}_{r_{\sigma\tau^{-1}\tau d},s_{\tau d}}\\
&= \pm\sum_{\sigma,\tau\in\mathfrak{S}_d}(\operatorname{sgn}\sigma\tau^{-1})(\xi^{c_{\ell-1}}_{r_{\sigma\tau^{-1}1},s_1}\otimes\cdots\otimes\xi^{c_{\ell-1}}_{r_{\sigma\tau^{-1}d},s_d})^\sigma,
\end{aligned}$$

which equals the right-hand side of the equation in (i).

(ii) We have

$$\begin{aligned}
\eta^{e^d_\ell}_{\boldsymbol{u},t^d}\eta^{a^d_{\ell,\ell-1}}_{t^d,\boldsymbol{s}} &= \Big(\sum_{\sigma\in{}^{\boldsymbol{u}}\mathscr{D}}(\xi^{e_\ell}_{u_1,t}\otimes\cdots\otimes\xi^{e_\ell}_{u_d,t})^\sigma\Big)\Big(\sum_{\tau\in\mathfrak{S}_d}(\xi^{a_{\ell,\ell-1}}_{t,s_1}\otimes\cdots\otimes\xi^{a_{\ell,\ell-1}}_{t,s_d})^\tau\Big)\\
&= \sum_{\sigma\in{}^{\boldsymbol{u}}\mathscr{D},\,\tau\in\mathfrak{S}_d}(\operatorname{sgn}\tau)\,\xi^{a_{\ell,\ell-1}}_{u_{\sigma1},s_{\tau1}}\otimes\cdots\otimes\xi^{a_{\ell,\ell-1}}_{u_{\sigma d},s_{\tau d}}\\
&= \sum_{\sigma\in{}^{\boldsymbol{u}}\mathscr{D},\,\tau\in\mathfrak{S}_d}(\operatorname{sgn}\tau)\,\xi^{a_{\ell,\ell-1}}_{u_{\sigma\tau^{-1}\tau1},s_{\tau1}}\otimes\cdots\otimes\xi^{a_{\ell,\ell-1}}_{u_{\sigma\tau^{-1}\tau d},s_{\tau d}}\\
&= \sum_{\sigma\in{}^{\boldsymbol{u}}\mathscr{D},\,\tau\in\mathfrak{S}_d}(\xi^{a_{\ell,\ell-1}}_{u_{\sigma\tau^{-1}1},s_1}\otimes\cdots\otimes\xi^{a_{\ell,\ell-1}}_{u_{\sigma\tau^{-1}d},s_d})^\tau,
\end{aligned}$$

which equals the right-hand side of the equation in (ii). □

We set

$$P_d := [1,n]^d \times [1,n]^d,$$

i.e., elements of P_d are pairs $(\boldsymbol{r}, \boldsymbol{s})$ of words $\boldsymbol{r} = r_1 \cdots r_d$, $\boldsymbol{s} = s_1 \cdots s_d$ in $[1, n]^d$. We also define

$$P'_d := \{(\boldsymbol{r}, \boldsymbol{s}) \in P_d \mid (r_a, s_a) \neq (r_b, s_b) \text{ for all } 1 \leq a \neq b \leq c\}.$$

For $b \in B_{\bar{0}}$, the triple $(b^d, \boldsymbol{r}, \boldsymbol{s})$ belongs to $\mathrm{Tri}^B(n, d)$ for all $(\boldsymbol{r}, \boldsymbol{s}) \in P_d$, while for $b \in B_{\bar{1}}$, we have $(b^d, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)$ if and only if $(\boldsymbol{r}, \boldsymbol{s}) \in P'_d$.

Given $\lambda \in \Lambda(n, d)$, we define

$$\boldsymbol{r}^\lambda := 1^{\lambda_1} \cdots n^{\lambda_n} \in [1, n]^d.$$

We refer to such tuples as *leading tuples*. Then

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\boldsymbol{r}^\lambda} = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}.$$

For $(b^d, \boldsymbol{r}, \boldsymbol{s}) \in \mathrm{Tri}^B(n, d)$, the corresponding $\mathfrak{S}_d$-orbit $[b^d, \boldsymbol{r}, \boldsymbol{s}]$ has a representative of the form $[b^d, \boldsymbol{t}, \boldsymbol{r}^\lambda]$ for some $\lambda \in \Lambda(n, d)$ and a representative of the form $[b^d, \boldsymbol{r}^\mu, \boldsymbol{u}]$ for some $\mu \in \Lambda(n, d)$. So while working with elements of the form $\eta^{b^d}_{\boldsymbol{r},\boldsymbol{s}} \in T^Z_{\mathfrak{z}}(n, d)$ we will often assume that $\boldsymbol{s}$ or $\boldsymbol{r}$ is a leading tuple when convenient.

Lemma 7.7. *Let $\lambda \in \Lambda(n, d)$, and $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{u} \in [1, n]^d$ be such that $(\boldsymbol{s}, \boldsymbol{r}^\lambda), (\boldsymbol{r}^\lambda, \boldsymbol{t}) \in P'_d$. Then*

(i) $\eta^{a^d_{\ell-1,\ell}}_{\boldsymbol{s},\boldsymbol{r}^\lambda} \eta^{a^d_{\ell,\ell-1}}_{\boldsymbol{r}^\lambda,\boldsymbol{t}} = \pm \sum_{\sigma \in \mathfrak{S}_\lambda} (\operatorname{sgn} \sigma) \eta^{c^d_{\ell-1}}_{\boldsymbol{s}\sigma,\boldsymbol{t}}$,

(ii) $\eta^{e^d_\ell}_{\boldsymbol{u},\boldsymbol{r}^\lambda} \eta^{a^d_{\ell,\ell-1}}_{\boldsymbol{r}^\lambda,\boldsymbol{t}} = \sum_{\sigma \in {}^{\boldsymbol{u}}\mathfrak{S}_\lambda} \eta^{a^d_{\ell,\ell-1}}_{\boldsymbol{s}\sigma,\boldsymbol{t}}$, *where ${}^{\boldsymbol{u}}\mathfrak{S}_\lambda$ is the set of the shortest coset representatives for $(\mathfrak{S}_{\boldsymbol{u}} \cap \mathfrak{S}_\lambda)\backslash\mathfrak{S}_\lambda$.*

Proof. For $u = 1, \ldots, n$, there exist words $\boldsymbol{s}^u, \boldsymbol{t}^u \in [1, n]^{\lambda_u}$ such that $\boldsymbol{s} = \boldsymbol{s}^1 \cdots \boldsymbol{s}^n$, $\boldsymbol{t} = \boldsymbol{t}^1 \cdots \boldsymbol{t}^n$. We have

$$\begin{aligned}
\eta^{a^d_{\ell-1,\ell}}_{\boldsymbol{s},\boldsymbol{r}^\lambda} \eta^{a^d_{\ell,\ell-1}}_{\boldsymbol{r}^\lambda,\boldsymbol{t}} &= \pm(\eta^{a^{\lambda_1}_{\ell-1,\ell}}_{\boldsymbol{s}^1,1^{\lambda_1}} \eta^{a^{\lambda_1}_{\ell,\ell-1}}_{1^{\lambda_1},\boldsymbol{t}^1}) * \cdots * (\eta^{a^{\lambda_n}_{\ell-1,\ell}}_{\boldsymbol{s}^n,n^{\lambda_n}} \eta^{a^{\lambda_n}_{\ell,\ell-1}}_{n^{\lambda_n},\boldsymbol{t}^n}) \\
&= \pm\Bigg(\sum_{\sigma^1 \in \mathfrak{S}_{\lambda_1}} (\operatorname{sgn} \sigma^1) \eta^{c^{\lambda_1}_{\ell-1}}_{\boldsymbol{s}^1\sigma^1,\boldsymbol{t}^1} \Bigg) * \cdots * \Bigg(\sum_{\sigma^n \in \mathfrak{S}_{\lambda_n}} (\operatorname{sgn} \sigma^n) \eta^{c^{\lambda_n}_{\ell-1}}_{\boldsymbol{s}^n\sigma^n,\boldsymbol{t}^n} \Bigg) \\
&= \pm \sum_{\sigma \in \mathfrak{S}_\lambda} (\operatorname{sgn} \sigma) \eta^{c^d_{\ell-1}}_{\boldsymbol{s}\sigma,\boldsymbol{t}},
\end{aligned}$$

where we have used Lemma 4.7 for the first equality, Lemma 7.6(i) for the second equality and Lemma 4.2(iii) for the last equality. This proves (i). The proof of (ii) is similar but uses Lemma 7.6(ii) instead of Lemma 7.6(i). □

Let $\lambda \in \Lambda(n, d)$. For $r \in [1, n]$, let $x_r := 1 + \sum_{s=1}^{r-1} \lambda_s$ and $y_r := \sum_{s=1}^{r} \lambda_s$ so that

$$[1, d] = [x_1, y_1] \sqcup \cdots \sqcup [x_n, y_n]$$

and if $\boldsymbol{r}^\lambda = r_1 \cdots r_d$ then $r_s = r$ if and only if $s \in [x_r, y_r]$. Let $0 \leq c \leq d$ and $\mu \in \Lambda(n, c)$, $\nu \in \Lambda(n, d-c)$ satisfy $\mu + \nu = \lambda$. We denote

$$\Omega^{\mu,\nu} := \{U \subseteq [1, d] \mid |U \cap [x_r, y_r]| = \mu_r \text{ for all } r = 1, \ldots, n\}$$

Let $\boldsymbol{t} = t_1 \cdots t_d \in [1,n]^d$. For $U = \{u_1 < \cdots < u_c\} \in \Omega^{\mu,\nu}$, we set

$$U' = \{v_1 < \cdots < v_{d-c}\} := [1,d] \setminus U, \quad \boldsymbol{t}^U := t_{u_1} \cdots t_{u_c}, \quad \boldsymbol{t}^{U'} := t_{v_1} \cdots t_{v_{d-c}}.$$

With this notation, Corollary 3.24 yields:

Lemma 7.8. *Let $\lambda \in \Lambda(n,d)$ and $\boldsymbol{t} \in [1,n]^d$. If $(\boldsymbol{r}^\lambda, \boldsymbol{t}) \in P'_d$ then*

$$\nabla(\eta_{\boldsymbol{r}^\lambda,\boldsymbol{t}}^{a^d_{\ell,\ell-1}}) = \sum_{c=0}^{d} \sum_{\substack{\mu\in\Lambda(n,c),\,\nu\in\Lambda(n,d-c)\\ \mu+\nu=\lambda}} \sum_{U\in\Omega^{\mu,\nu}} \pm\eta_{\boldsymbol{r}^\mu,\boldsymbol{t}^U}^{a^c_{\ell,\ell-1}} \otimes \eta_{\boldsymbol{r}^\nu,\boldsymbol{t}^{U'}}^{a^{d-c}_{\ell,\ell-1}}.$$

Lemma 7.9. *Let $0 \le c \le d$, $\mu \in \Lambda(n,c)$, $\nu \in \Lambda(n,d-c)$, $\boldsymbol{r} \in [1,n]^c$, $\boldsymbol{s} \in [1,n]^{d-c}$ and $\boldsymbol{t} \in [1,n]^d$. Suppose that $(\boldsymbol{r}, \boldsymbol{r}^\mu) \in P'_c$, and $(\boldsymbol{r}^{\mu+\nu}, \boldsymbol{t}) \in P'_d$. Then*

$$(\eta_{\boldsymbol{r},\boldsymbol{r}^\mu}^{a^c_{\ell-1,\ell}} * \eta_{\boldsymbol{s},\boldsymbol{r}^\nu}^{e^{d-c}_\ell})\eta_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}}^{a^d_{\ell,\ell-1}} = \sum_{U\in\Omega^{\mu,\nu}} \sum_{\sigma\in\mathfrak{S}_\mu,\,\tau\in{}^s\mathfrak{S}_\nu} \pm\eta_{(\boldsymbol{r}\sigma)(\boldsymbol{s}\tau),\boldsymbol{t}^U\boldsymbol{t}^{U'}}^{c^c_{\ell-1}a^{d-c}_{\ell,\ell-1}}.$$

Proof. Using Lemmas 4.5, 7.7 and 7.8, we have

$$\begin{aligned}(\eta_{\boldsymbol{r},\boldsymbol{r}^\mu}^{a^c_{\ell-1,\ell}} * \eta_{\boldsymbol{s},\boldsymbol{r}^\nu}^{e^{d-c}_\ell})\eta_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}}^{a^d_{\ell,\ell-1}} &= \sum_{U\in\Omega^{\mu,\nu}} \pm(\eta_{\boldsymbol{r},\boldsymbol{r}^\mu}^{a^c_{\ell-1,\ell}}\eta_{\boldsymbol{r}^\mu,\boldsymbol{t}^U}^{a^c_{\ell,\ell-1}}) * (\eta_{\boldsymbol{s},\boldsymbol{r}^\nu}^{e^{d-c}_\ell}\eta_{\boldsymbol{r}^\nu,\boldsymbol{t}^{U'}}^{a^{d-c}_{\ell,\ell-1}})\\ &= \sum_{U\in\Omega^{\mu,\nu}} \Big(\sum_{\sigma\in\mathfrak{S}_\mu} \pm\eta_{\boldsymbol{r}\sigma,\boldsymbol{t}^U}^{c^c_{\ell-1}}\Big) * \Big(\sum_{\tau\in{}^s\mathfrak{S}_\nu} \eta_{\boldsymbol{s}\tau,\boldsymbol{t}^{U'}}^{a^{d-c}_{\ell,\ell-1}}\Big),\end{aligned}$$

which equals the right-hand side of the equality in the claim by Lemma 4.6. □

Remark 7.10. Let $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}), (\boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}') \in \mathrm{Tri}^B(n,d)$. By (3.9) and Lemmas 3.3, 3.4, we have that $\eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}}$ and $\eta^{\boldsymbol{b}'}_{\boldsymbol{r}',\boldsymbol{s}'}$ are proportional if and only if $(\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}) \sim (\boldsymbol{b}',\boldsymbol{r}',\boldsymbol{s}')$. It will be important later on that all the basis elements appearing in the right-hand side of Lemma 7.9 are linearly independent.

Let

$$K_d := \bigsqcup_{c=0}^{d} \Lambda(4,c) = \{\kappa = (\kappa_1,\kappa_2,\kappa_3,\kappa_4) \in \Lambda(4) \mid |\kappa| \le d\}.$$

For $\kappa \in K_d$, we set

$$P_\kappa := P'_{\kappa_1} \times P_{\kappa_2} \times P'_{\kappa_3} \times P_{\kappa_4}.$$

Let $\big((\boldsymbol{r}^1,\boldsymbol{s}^1), (\boldsymbol{r}^2,\boldsymbol{s}^2), (\boldsymbol{r}^3,\boldsymbol{s}^3), (\boldsymbol{r}^4,\boldsymbol{s}^4)\big) \in P_\kappa$. We often denote

$$\boldsymbol{r} := \boldsymbol{r}^1\boldsymbol{r}^2\boldsymbol{r}^3\boldsymbol{r}^4, \ \ \boldsymbol{s} := \boldsymbol{s}^1\boldsymbol{s}^2\boldsymbol{s}^3\boldsymbol{s}^4 \in [1,n]^{|\kappa|},$$

and, abusing notation, write $(\boldsymbol{r},\boldsymbol{s}) \in P_\kappa$. Given $(\boldsymbol{r},\boldsymbol{s}) \in P_\kappa$, we define the following element of $T^Z_{\mathfrak{z}}(n,|\kappa|)$:

$$\eta^\kappa_{\boldsymbol{r},\boldsymbol{s}} := \eta_{\boldsymbol{r},\boldsymbol{s}}^{a^{\kappa_1}_{\ell-1,\ell}e^{\kappa_2}_\ell a^{\kappa_3}_{\ell,\ell-1}c^{\kappa_4}_{\ell-1}} = \eta_{\boldsymbol{r}^1,\boldsymbol{s}^1}^{a^{\kappa_1}_{\ell-1,\ell}} * \eta_{\boldsymbol{r}^2,\boldsymbol{s}^2}^{e^{\kappa_2}_\ell} * \eta_{\boldsymbol{r}^3,\boldsymbol{s}^3}^{a^{\kappa_3}_{\ell,\ell-1}} * \eta_{\boldsymbol{r}^4,\boldsymbol{s}^4}^{c^{\kappa_4}_{\ell-1}}.$$

We have the equivalence relation $\sim$ on P_κ given by

$$(\boldsymbol{r},\boldsymbol{s}) \sim (\boldsymbol{t},\boldsymbol{u}) \quad \text{if and only if} \quad (\boldsymbol{r}^h,\boldsymbol{s}^h) \sim (\boldsymbol{t}^h,\boldsymbol{u}^h) \text{ for } h = 1,2,3,4.$$

Let $B' := B \setminus \{a_{\ell-1,\ell}, e_\ell, a_{\ell,\ell-1}, c_{\ell-1}\}$. Recall the equivalence relation $\sim$ on $\mathrm{Tri}^{B'}(n, d-|\kappa|)$ from Section 2B. By Lemmas 3.10 and 4.6, we have that

$$\bigl\{\eta^{\kappa}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}} \mid \kappa \in K_d,\ (\boldsymbol{r},\boldsymbol{s}) \in P^{\kappa}/\sim,\ (\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}) \in \mathrm{Tri}^{B'}(n, d-|\kappa|)/\sim\bigr\} \tag{7.11}$$

is a basis of $T^Z_{\mathfrak{z}}(n,d)$.

Lemma 7.12. *Let $\kappa \in K_d$, $(\boldsymbol{r},\boldsymbol{s}) \in P^\kappa$ with $\boldsymbol{s}^1 = \boldsymbol{r}^\mu$, $\boldsymbol{s}^2 = \boldsymbol{r}^\nu$, and $(\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}) \in \mathrm{Tri}^{B'}(n, d-|\kappa|)$. Let $k = \kappa_1+\kappa_2$ and $(\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}) \in P'_k$. Then we have*

$$(\eta^{\kappa}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}})(\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}} * \xi^{e^{d-k}}) = \sum_{U \in \Omega^{\mu,\nu}} \sum_{\sigma \in \mathfrak{S}_\mu,\, \tau \in {}^{r^2}\mathfrak{S}_\nu} \pm \eta^{c^{\kappa_1+\kappa_4}_{\ell-1} a^{\kappa_2+\kappa_3}_{\ell,\ell-1}}_{(\boldsymbol{r}^1\sigma)\boldsymbol{r}^4(\boldsymbol{r}^2\tau)\boldsymbol{r}^3,\,\boldsymbol{t}^U\boldsymbol{s}^4\boldsymbol{t}^{U'}\boldsymbol{s}^3} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}}.$$

Proof. By Lemmas 4.7 and 7.9, the left-hand side equals

$$\begin{aligned}&\bigl((\eta^{a^{\kappa_1}_{\ell-1,\ell}}_{\boldsymbol{r}^1,\boldsymbol{r}^\mu} * \eta^{e^{\kappa_2}_\ell}_{\boldsymbol{r}^2,\boldsymbol{r}^\nu})\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}}\bigr) * \bigl((\eta^{a^{\kappa_3}_{\ell,\ell-1}}_{\boldsymbol{r}^3,\boldsymbol{s}^3} * \eta^{c^{\kappa_4}_{\ell-1}}_{\boldsymbol{r}^4,\boldsymbol{s}^4} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}})\xi^{e^{d-k}}\bigr)\\ &\qquad = \Bigl(\sum_{U \in \Omega^{\mu,\nu}} \sum_{\sigma \in \mathfrak{S}_\mu,\, \tau \in {}^{r^2}\mathfrak{S}_\nu} \pm \eta^{c^{\kappa_1}_{\ell-1}}_{\boldsymbol{r}^1\sigma,\boldsymbol{t}^U} * \eta^{a^{\kappa_2}_{\ell,\ell-1}}_{\boldsymbol{r}^2\tau,\boldsymbol{t}^{U'}}\Bigr) * \bigl(\eta^{a^{\kappa_3}_{\ell,\ell-1}}_{\boldsymbol{r}^3,\boldsymbol{s}^3} * \eta^{c^{\kappa_4}_{\ell-1}}_{\boldsymbol{r}^4,\boldsymbol{s}^4} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}}\bigr),\end{aligned}$$

which equals the right-hand side of the equality in the claim by Lemma 4.2(iii) and Lemma 4.6. ☐

Remark 7.13. Suppose that in the assumption of Lemma 7.12, we have additionally that $\boldsymbol{t} = t_1,\dots,t_k$ satisfies $t_a \neq t_b$ for all $1 \le a \neq b \le k$ and $\boldsymbol{t}$ shares no letters in common with the words $\boldsymbol{s}^3$ and $\boldsymbol{s}^4$. Taking into account Remark 7.10, one can see that all the basis elements appearing in the right-hand side of Lemma 7.12 are linearly independent.

Recall the idempotent $\xi^e \in T^Z_{\mathfrak{z}}(n,d)$. In this subsection we sometimes write $\xi^{e^d} := \xi^e$, so that we also have idempotents $\xi^{e^c} \in T^Z_{\mathfrak{z}}(n,c)$ for all $c \in \mathbb{Z}_{\ge 0}$. Recalling (5.2), we also introduce idempotents

$$\xi^{(c,d-c)} := \bigl((E^{e_\ell})^{\otimes c}\bigr) * \bigl((E^e)^{\otimes d-c}\bigr) \in T^Z_{\mathfrak{z}}(n,d) \qquad (0 \le c \le d).$$

Note that $\xi^{(0,d)} = \xi^e$ and $\xi^{(d,0)} = \xi^{e_\ell}$. Moreover,

$$\xi^{(c,d-c)}\xi^{(b,d-b)} = \delta_{b,c}\xi^{(c,d-c)}. \tag{7.14}$$

The following is easily checked using Lemma 4.5:

Lemma 7.15. *Let $\kappa \in K_d$, $(\boldsymbol{r},\boldsymbol{s}) \in P_\kappa$, $(\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}) \in \mathrm{Tri}^{B'}(n, d-|\kappa|)$, $0 \le k \le d$, $\lambda \in \Lambda(n,k)$, and suppose that $(\boldsymbol{r}^\lambda,\boldsymbol{t}) \in P'_k$. Then*

$$\xi^{(\kappa_2+\kappa_3,d-\kappa_2-\kappa_3)}(\eta^{\kappa}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}})\xi^{(\kappa_1+\kappa_2,d-\kappa_1-\kappa_2)} = \eta^{\kappa}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}},$$

$$\xi^{(k,d-k)}(\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^\lambda,\boldsymbol{t}} * \xi^{e^{d-k}})\xi^e = \eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^\lambda,\boldsymbol{t}} * \xi^{e^{d-k}}.$$

In particular, $(\eta^{\kappa}_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}})(\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^\lambda,\boldsymbol{t}} * \xi^{e^{d-k}}) = 0$, *unless* $\kappa_1 + \kappa_2 = k$ *and* $\boldsymbol{s}^1\boldsymbol{s}^2 \sim \boldsymbol{r}^\lambda$.

7D. ***Double centralizer property for zigzag Schur algebras.*** Recall that $\bar{Z} := eZe$ is the zigzag algebra for a fixed $\ell \geq 1$.

Lemma 7.16. *We have that e is a double centralizer idempotent for Z.*

Proof. As a right $\bar{Z}$-module, Ze decomposes as

$$Ze = e_\ell Ze \oplus e_{\ell-1}Ze \oplus \cdots \oplus e_0 Ze,$$

so it is enough to check that the algebra map $\lambda : Z \to \mathrm{End}_{\bar{Z}}(Ze)$ restricts to an isomorphism $e_j Ze_i \to \mathrm{Hom}_{\bar{Z}}(e_i Ze, e_j Ze)$, for all $i, j \in I$.

Let $i, j \in I$ with $i \neq \ell$. The map $\lambda|_{e_j Ze_i} : e_j Ze_i \to \mathrm{Hom}_{\bar{Z}}(e_i Ze, e_j Ze)$ is injective, since $\lambda_x(e_i) = x$ for all $x \in e_j Ze_i$. Now let $f \in \mathrm{Hom}_{\bar{Z}}(e_i Ze, e_j Ze)$. As f is a right $\bar{Z}$-module homomorphism, f is determined by the image of e_i. Moreover, since $f(e_i) = f(e_i)e_i$, we have $f(e_i) \in e_j Ze_i$, and thus $f = \lambda_{f(e_i)}$, so $\lambda|_{e_j Ze_i}$ is surjective, and thus an isomorphism.

Now let $j \in I$, and consider the map $\lambda|_{e_j Ze_\ell} : e_j Ze_\ell \to \mathrm{Hom}_{\bar{Z}}(e_\ell Ze, e_j Ze)$. Note that $e_\ell Ze = \mathrm{span}(a_{\ell,\ell-1})$. First we show $\lambda|_{e_j Ze_\ell}$ is injective. If $j \neq \ell, \ell-1$ then $e_j Ze_\ell = 0$, and this is trivially true. If $j = \ell - 1$ then $e_j Ze_\ell = \mathrm{span}(a_{\ell-1,\ell})$, and $\lambda_{a_{\ell-1,\ell}}(a_{\ell,\ell-1}) = c_{\ell-1} \neq 0$. If $j = \ell$, then $e_j Ze_\ell = \mathrm{span}(e_\ell)$, and $\lambda_{e_\ell}(a_{\ell,\ell-1}) = a_{\ell,\ell-1} \neq 0$. Thus, in any case $\lambda|_{e_j Ze_\ell}$ is injective.

Now we show that $\lambda|_{e_j Ze_\ell}$ is surjective. Let $f \in \mathrm{Hom}_{\bar{Z}}(e_\ell Ze, e_j Ze)$. Since $f(a_{\ell,\ell-1}) = f(a_{\ell,\ell-1})e_{\ell-1}$, we have that $f(a_{\ell,\ell-1}) \in e_j Ze_{\ell-1}$. Since $e_j Ze_{\ell-1} = 0$ for $j \neq \ell, \ell-1, \ell-2$ we may assume $j \in \{\ell, \ell-1, \ell-2\}$. If $j = \ell-2$, then $f(a_{\ell,\ell-1}) = \alpha a_{\ell-2,\ell-1}$ for some $\alpha \in \Bbbk$. But then $0 = f(a_{\ell,\ell-1}a_{\ell-1,\ell-2}) = f(a_{\ell,\ell-1})a_{\ell-1,\ell-2} = \alpha c_{\ell-2}$, so $f = 0 = \lambda_0$. If $j = \ell - 1$, then $f(a_{\ell,\ell-1}) = \alpha e_{\ell-1} + \beta c_{\ell-1}$ for some $\alpha, \beta \in \Bbbk$. But then $0 = f(a_{\ell,\ell-1}c_{\ell-1}) = f(a_{\ell,\ell-1})c_{\ell-1} = \alpha c_{\ell-1}$ implies that $\alpha = 0$. Thus $f(a_{\ell,\ell-1}) = \beta c_{\ell-1} = \beta a_{\ell-1,\ell}a_{\ell,\ell-1}$ and $f = \lambda_{\beta a_{\ell-1,\ell}}$. Finally, assume $j = \ell$. Then $f(a_{\ell,\ell-1}) = \alpha a_{\ell,\ell-1} = \alpha e_\ell a_{\ell,\ell-1}$ for some $\alpha \in \Bbbk$, so $f = \lambda_{\alpha e_\ell}$. Thus in any case $\lambda|_{e_j Ze_\ell}$ is surjective, and thereby an isomorphism, completing the proof. □

In view of Lemma 5.12, we have $\xi^e T^Z_{\mathfrak{z}}(n,d)\xi^e = T^{\bar{Z}}_{\mathfrak{z}}(n,d)$. The main result of this subsection is:

Theorem 7.17. *Let* $d \leq n$. *Then* ξ^e *is a double centralizer idempotent for* $T^Z_{\mathfrak{z}}(n,d)$. *In particular*, $T^Z_{\mathfrak{z}}(n,d) \cong \mathrm{End}_{T^{\bar{Z}}_{\mathfrak{z}}(n,d)}(T^Z_{\mathfrak{z}}(n,d)\xi^e)$.

Theorem 7.17 follows immediately from Lemma 7.1, Corollary 7.5, and the following proposition:

Proposition 7.18. *Let* $d \leq n$. *Then* ξ^e *is a sound idempotent for* $T^Z_{\mathfrak{z}}(n,d)$.

Proof. Set $T := T^Z_{\mathfrak{z}}(n,d)$. We will use the basis (7.11) of T.

Suppose for a contradiction that there exists an indivisible element

$$x := \sum_{\substack{\kappa \in K_d,\, (\boldsymbol{r},\boldsymbol{s}) \in P_\kappa/\sim,\\ (\boldsymbol{b},\boldsymbol{p},\boldsymbol{q}) \in \mathrm{Tri}^{B'}(n,d-|\kappa|)/\sim}} q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}} (\eta^\kappa_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}}) \in T \qquad (q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}} \in \Bbbk)$$

such that $\lambda_x : T\xi^e \to T\xi^e$ is divisible, i.e., there exists a nonzero nonunit $m \in \Bbbk$ such that $x\eta\xi^e \in mT$ for all $\eta \in T$. By the remarks preceding Lemma 7.7, we may assume that all $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ are leading tuples. If $\boldsymbol{s}^1 = \boldsymbol{r}^\mu$ and $\boldsymbol{s}^2 = \boldsymbol{r}^\nu$, we write $(\boldsymbol{r}, \boldsymbol{s}) \in P_\kappa^{\mu,\nu}$.

We may also assume that among all indivisible elements x as above, our x has the smallest possible number of nonzero coefficients $q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}}$. Then m does not divide $q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}}$ whenever $q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}} \neq 0$.

Let $k \in \mathbb{Z}_{\geq 0}$ be such that some coefficient $q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}}$ with $\kappa_1 + \kappa_2 = k$ is nonzero. We assume that $(\boldsymbol{r}, \boldsymbol{s}) \in P_\kappa^{\mu,\nu}$ for some such nonzero coefficient. We now pick $\boldsymbol{t} = (t_1, \ldots, t_k) \in [1, n]^d$ such that:

(1) $t_r \neq t_s$ for all $1 \leq r \neq s \leq k$.

(2) The words $\boldsymbol{s}^4$ and $\boldsymbol{s}^3$ have no letters of the form t_r for $1 \leq r \leq k$.

Such $\boldsymbol{t}$ exists by the assumption that $d \leq n$.

We have $(\boldsymbol{r}^{\mu+\nu}, \boldsymbol{t}) \in P'_k$. By Lemma 7.15, we have that $\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}} * \xi^{e^{d-k}} \in T\xi^e$, so by the assumptions made, we must have $x(\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}} * \xi^{e^{d-k}}) \in mT$. On the other hand, by Lemmas 7.12 and 7.15, we have that $x(\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}} * \xi^{e^{d-k}})$ equals

$$\sum_{\substack{\kappa\in K_d,\,(\boldsymbol{r},\boldsymbol{s})\in P_\kappa^{\mu,\nu}/\sim,\\ (\boldsymbol{b},\boldsymbol{p},\boldsymbol{q})\in \mathrm{Tri}^{B'}(n,d-|\kappa|)/\sim}} q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}} (\eta^\kappa_{\boldsymbol{r},\boldsymbol{s}} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}})(\eta^{a^k_{\ell,\ell-1}}_{\boldsymbol{r}^{\mu+\nu},\boldsymbol{t}} * \xi^{e^{d-k}})$$

$$= \sum_{\substack{\kappa\in K_d,\,(\boldsymbol{r},\boldsymbol{s})\in P_\kappa^{\mu,\nu}/\sim,\\ (\boldsymbol{b},\boldsymbol{p},\boldsymbol{q})\in \mathrm{Tri}^{B'}(n,d-|\kappa|)/\sim}} q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}} \sum \pm \eta^{c^{\kappa_1+\kappa_4}_{\ell-1} a^{\kappa_2+\kappa_3}_{\ell,\ell-1}}_{(\boldsymbol{r}^1\sigma)\boldsymbol{r}^4(\boldsymbol{r}^2\tau)\boldsymbol{r}^3,\,\boldsymbol{t}^U\boldsymbol{s}^4\boldsymbol{t}^{U'}\boldsymbol{s}^3} * \eta^{\boldsymbol{b}}_{\boldsymbol{p},\boldsymbol{q}},$$

where the second sum is over all $U \in \Omega^{\mu,\nu}$, $\sigma \in \mathfrak{S}_\mu$ and $\tau \in {}^{\boldsymbol{r}^2}\mathfrak{S}_\nu$. By Remark 7.13, all the basis elements appearing in the whole sum above are linearly independent. By our assumptions this implies that all of the nonzero coefficients $q^{\kappa,\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s};\boldsymbol{p},\boldsymbol{q}}$ appearing there are divisible by m, which is a contradiction. □

Remark 7.19. For an arbitrary algebra A with double centralizer idempotent e, it is not the case that ξ^e is in general a double centralizer idempotent for $T^A_{\mathfrak{a}}(n, d)$. For example, take arbitrary $n \in \mathbb{Z}_{>0}$, and consider the case $A = A_{\bar{0}} = M_2(\Bbbk)$, $\mathfrak{a} = \mathrm{span}(E_{11}, E_{22})$, $\mathfrak{c} = \mathrm{span}(E_{12}, E_{21})$, and $e = E_{11}$. Then e is clearly a double centralizer idempotent for A.

The element $\eta^{E_{12},E_{12}}_{11,11} = 2\xi^{E_{12},E_{12}}_{11,11}$ is indivisible in $T^A_{\mathfrak{a}}(n, 2)$, and $\{E_{11}, E_{21}\}$ is a basis for Ae. For $\boldsymbol{b} \in \{E_{11}, E_{21}\}^2$ and $\boldsymbol{r}, \boldsymbol{s} \in [1, n]^2$, we have that $\eta^{E_{12},E_{12}}_{11,11} \cdot \eta^{\boldsymbol{b}}_{\boldsymbol{r},\boldsymbol{s}} = 0$ unless $\boldsymbol{b} = (E_{21}, E_{21})$ and $\boldsymbol{r} = (1, 1)$. Then, if $s_1 = s_2$ we have

$$\eta^{E_{12},E_{12}}_{11,11} \cdot \eta^{E_{21},E_{21}}_{11,\boldsymbol{s}} = (2\xi^{E_{12},E_{12}}_{11,11}) \cdot (2\xi^{E_{21},E_{21}}_{11,\boldsymbol{s}}) = 4\xi^{E_{11},E_{11}}_{11,\boldsymbol{s}} = 4\eta^{E_{11},E_{11}}_{11,\boldsymbol{s}}.$$

If $s_1 \neq s_2$ we have

$$\eta^{E_{12},E_{12}}_{11,11} \cdot \eta^{E_{21},E_{21}}_{11,\boldsymbol{s}} = (2\xi^{E_{12},E_{12}}_{11,11}) \cdot (\xi^{E_{21},E_{21}}_{11,\boldsymbol{s}}) = 2\xi^{E_{11},E_{11}}_{11,\boldsymbol{s}} = 2\eta^{E_{11},E_{11}}_{11,\boldsymbol{s}}.$$

This implies that $\lambda_{\eta^{E_{12},E_{12}}_{11,11}}$ is divisible in $\mathrm{End}_{T^{\bar{A}}_{\bar{\mathfrak{a}}}(n,2)}(T^A_{\mathfrak{a}}(n, 2)\xi^e)$, so ξ^e is not sound, and hence not a double centralizer idempotent by Lemma 7.1.

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References

[Brundan and Kleshchev 1999] J. Brundan and A. Kleshchev, "Modular Littlewood–Richardson coefficients", *Math. Z.* **232**:2 (1999), 287–320. MR Zbl

[Donkin 1986] S. Donkin, "On Schur algebras and related algebras, I", *J. Algebra* **104**:2 (1986), 310–328. MR Zbl

[Donkin 1987] S. Donkin, "On Schur algebras and related algebras, II", *J. Algebra* **111**:2 (1987), 354–364. MR Zbl

[Donkin 1998] S. Donkin, *The q-Schur algebra*, Lond. Math. Soc. Lect. Note Ser. **253**, Cambridge Univ. Press, 1998. MR Zbl

[Doty 1998] S. Doty, "Polynomial representations, algebraic monoids, and Schur algebras of classical type", *J. Pure Appl. Algebra* **123**:1-3 (1998), 165–199. MR Zbl

[Evseev and Kleshchev 2017] A. Evseev and A. Kleshchev, "Turner doubles and generalized Schur algebras", *Adv. Math.* **317** (2017), 665–717. MR Zbl

[Evseev and Kleshchev 2018] A. Evseev and A. Kleshchev, "Blocks of symmetric groups, semicuspidal KLR algebras and zigzag Schur–Weyl duality", *Ann. of Math.* (2) **188**:2 (2018), 453–512. MR Zbl

[Green 2007] J. A. Green, *Polynomial representations of* GL_n, 2nd ed., Lect. Notes in Math. **830**, Springer, 2007. MR Zbl

[Kleshchev and Muth 2018] A. Kleshchev and R. Muth, "Schurifying quasi-hereditary algebras", preprint, 2018. arXiv

[Marko and Zubkov 2006] F. Marko and A. N. Zubkov, "Schur superalgebras in characteristic p, II", *Bull. Lond. Math. Soc.* **38**:1 (2006), 99–112. MR Zbl

[Turner 2008a] W. Turner, "Bialgebras and caterpillars", *Q. J. Math.* **59**:3 (2008), 379–388. MR Zbl

[Turner 2008b] W. Turner, "Tilting equivalences: from hereditary algebras to symmetric groups", *J. Algebra* **319**:10 (2008), 3975–4007. MR Zbl

[Turner 2009] W. Turner, *Rock blocks*, Mem. Amer. Math. Soc. **947**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

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