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The algebraic de Rham realization of the elliptic polylogarithm via the Poincaré bundle

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We describe the algebraic de Rham realization of the elliptic polylogarithm for arbitrary families of elliptic curves in terms of the Poincaré bundle. Our work builds on previous work of Scheider and generalizes results of Bannai, Kobayashi and Tsuji, and Scheider. As an application, we compute the de Rham–Eisenstein classes explicitly in terms of certain algebraic Eisenstein series.

1. Introduction

In his groundbreaking paper, Beilinson [1984] stated his important conjectures expressing special values of L-functions up to a rational factor in terms of motivic cohomology classes under the regulator map to Deligne cohomology. For finer integrality questions one has to consider additionally regulator maps to other cohomology theories. In order to study particular cases of these conjectures, one needs to construct such cohomology classes and understand their realizations. An important source of such cohomology classes are polylogarithmic cohomology classes and their associated Eisenstein classes.

The elliptic polylogarithm has been defined by Beilinson and Levin [1994] in their seminal paper. It is a motivic cohomology class living on the complement of certain torsion sections of an elliptic curve $E \to S$. By specializing the elliptic polylogarithm along torsion sections one obtains the associated *Eisenstein classes*. Eisenstein classes have been fruitfully applied for studying special values of L-functions of imaginary quadratic fields, e.g., in [Deninger 1989; Kings 2001]. They also proved to be an important tool for gaining a better understanding of L-functions of modular forms. They appear in the construction of Kato's celebrated Euler system and more recently in the important works of Bertolini, Darmon and Rotger and Kings, Loeffler and Zerbes.

The aim of this work is to describe the algebraic de Rham realization of the elliptic polylogarithm for arbitrary families of elliptic curves. In the case of a single elliptic curve with complex multiplication, such a description has been given by Bannai, Kobayashi and Tsuji [Bannai et al. 2010]. Building on this, Scheider [2014] has generalized this to arbitrary families of complex elliptic curves in his PhD thesis. Even more importantly, he has given an explicit description of the de Rham logarithm sheaves in terms of the Poincaré bundle on the universal vectorial extension of the dual elliptic curve.

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Theorem ([Scheider 2014], more precisely stated as Theorem 4.7). *For an elliptic curve E/S let $E^\dagger$ be the universal vectorial extension of the dual elliptic curve and* $(\mathcal{P}^\dagger, \nabla_{\mathcal{P}^\dagger})$ *the Poincaré bundle with connection on* $E \times_S E^\dagger$. *Then*

$$\mathcal{L}_n^\dagger := (\mathrm{pr}_E)_*(\mathcal{P}^\dagger|_{\mathrm{Inf}_e^n E^\dagger})$$

provides an explicit model for the (*abstractly defined*) *n-th de Rham logarithm sheaf* $\mathrm{Log}_{\mathrm{dR}}^n$.

For a positive integer D, the de Rham polylogarithm is a prosystem of cohomology classes with values in the logarithm sheaves, more explicitly

$$\mathrm{pol}_{D,\mathrm{dR}} \in \varprojlim_n H^1_{\mathrm{dR}}(E \setminus E[D], \mathrm{Log}^n_{\mathrm{dR}}).$$

Building on this, Scheider used certain theta functions of the Poincaré bundle to construct *analytic* differential forms representing the de Rham polylogarithm class for families of complex elliptic curves. Here, he followed the approach of Bannai, Kobayashi and Tsuji, who gave a similar construction for elliptic curves with complex multiplication. For all arithmetic applications it is indispensable to have an explicit *algebraic* representative of the polylogarithm class. In the CM case studied by Bannai, Kobayashi and Tsuji, it is possible to prove the algebraicity of the coefficient functions of the involved theta function using the algebraicity of the Hodge decomposition. This leads to a purely algebraic description of the de Rham polylogarithm for CM elliptic curves. Unfortunately, this approach fails for arbitrary families of elliptic curves and thus it does not apply to the situation studied by Scheider.

In this work, we address this problem and construct *algebraic* differential forms with values in the logarithm sheaves representing the de Rham polylogarithm for families of elliptic curves. For an elliptic curve E/S and a positive integer D, we have defined a certain 1-form with values in the Poincaré bundle $s^D_{\mathrm{can}} \in \mathcal{P}^\dagger \otimes \Omega^1_{E/S}$, see [Sprang 2018]. This section s^D_{can} is called *the Kronecker section* and serves as a substitute for the analytic theta functions appearing in the work of Bannai, Kobayashi and Tsuji and Scheider. Scheider's theorem allows us to view

$$l_n^D := (\mathrm{pr}_E)_*(s^D_{\mathrm{can}}|_{\mathrm{Inf}_e^n E^\dagger}) \in \Gamma(E \setminus E[D], \mathcal{L}_n^\dagger \otimes \Omega^1_{E/S})$$

as a 1-form with values in the n-th de Rham logarithm sheaf. In a second step, we lift these relative 1-forms to absolute 1-forms

$$L_n^D \in \Gamma(E \setminus E[D], \mathcal{L}_n^\dagger \otimes \Omega^1_E).$$

Now, our main result states that the prosystem $(L_n^D)_n$ gives explicit algebraic representatives of the de Rham polylogarithm class.

Theorem (more precisely stated as Theorem 5.8). *Let E/S be a family of elliptic curves over a smooth scheme S over a field of characteristic zero. The* 1*-forms L_n^D form explicit algebraic representatives of the de Rham polylogarithm class*, *i.e.*,

$$([L_n^D])_n = \mathrm{pol}_{D,\mathrm{dR}} \in \varprojlim_n H^1_{\mathrm{dR}}(E \setminus E[D], \mathrm{Log}^n_{\mathrm{dR}}).$$

As a byproduct, we deduce explicit formulas for the de Rham–Eisenstein classes in terms of certain holomorphic Eisenstein series.

The results of this paper depend heavily on the results of the unpublished PhD thesis of Scheider [2014]. In particular, Scheider's explicit description of the de Rham logarithm sheaves in terms of the Poincaré bundle will play a fundamental role in this paper. Scheider's original proof of this result is long and involved. A substantial part of this paper is devoted to making the results of Scheider available to the mathematical community. At the same time, we will simplify the proof of Scheider's theorem considerably.

Our main motivation for this work comes from the wish of gaining a better understanding of the syntomic realization of the elliptic polylogarithm. Syntomic cohomology can be seen as a p-adic analogue of Deligne cohomology and replaces Deligne cohomology in the formulation of the p-adic Beilinson conjectures. Till now, we only understand the syntomic realization in the case of a single elliptic curve with complex multiplication [Bannai et al. 2010] as well as the specializations of the syntomic polylogarithm along torsion sections, i.e., the syntomic Eisenstein classes [Bannai and Kings 2010]. Building on the results of this paper, we generalize the results of [Bannai and Kings 2010; Bannai et al. 2010] and provide an explicit description of the syntomic realization over the ordinary locus of the modular curve in [Sprang 2019]. Here, it is indispensable to have explicit *algebraic* representatives for the de Rham polylogarithm class.

The polylogarithm can also be defined for higher dimensional Abelian schemes and is expected to have interesting arithmetic applications. While we have a good understanding of the elliptic polylogarithm, not much is known in the higher dimensional case. Combining the results of this paper with Scheider's results gives a conceptional understanding of the elliptic polylogarithm in terms of the Poincaré bundle. We expect that the general structure of the argument should allow the generalization to higher dimensional Abelian schemes. A good understanding of the de Rham realization is an essential step towards the realization in Deligne and syntomic cohomology.

2. The de Rham logarithm sheaves

The aim of this section is to define the prosystem of the de Rham logarithm sheaves. The de Rham logarithm sheaves satisfy a universal property among all unipotent vector bundles with integrable connections. In this section we will present the basic properties of the de Rham logarithm sheaves, these have been worked out by Scheider [2014] in his PhD thesis.

Vector bundles with integrable connections. The coefficients for algebraic de Rham cohomology are vector bundles with integrable connections. Let us start by with recalling some basic definitions on vector bundles with integrable connections. For details we refer to [Katz 1970, (1.0) and (1.1)]. For a smooth morphism $\pi : S \to T$ between smooth separated schemes of finite type over a field K of characteristic 0 let us denote by $\mathrm{VIC}(S/T)$ the category of vector bundles on S with integrable T-connection and horizontal maps as morphisms. Since every coherent $\mathcal{O}_S$-module with integrable K-connection is a vector bundle, the category $\mathrm{VIC}(S/K)$ is Abelian (see [Berthelot and Ogus 1978, Section 2, Note 2.17]). The pullback along a smooth map $\pi : S \to T$ of smooth K-schemes induces an exact functor

$$\pi^* : \mathrm{VIC}(T/K) \to \mathrm{VIC}(S/K).$$

By restricting the connection we get a forgetful map

$$\mathrm{res}_T\colon\ \mathrm{VIC}(S/K) \to \mathrm{VIC}(S/T).$$

To an object $\mathcal{F}$ in $\mathrm{VIC}(S/T)$ we can associate a complex of $\pi^{-1}\mathcal{O}_T$-modules

$$\Omega^\bullet_{S/T}(\mathcal{F})\colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_S} \Omega^1_{S/T} \to \cdots$$

called the *algebraic de Rham complex*. The differentials in this complex are induced by the integrable connection. The relative algebraic de Rham cohomology for $\pi\colon S \to T$ is defined as

$$\underline{H}^i_{\mathrm{dR}}(S/T, \mathcal{F}) := R^i\pi_*(\Omega^\bullet_{S/T}(\mathcal{F})).$$

For $T = \operatorname{Spec} K$ we obtain the absolute algebraic de Rham cohomology $H^i_{\mathrm{dR}}(S, \mathcal{F})$. For $i = 0, 1$ the i-th de Rham cohomology can be seen as an extension group in the category $\mathrm{VIC}(S/K)$, i.e.,

$$H^i_{\mathrm{dR}}(S, \mathcal{F}) = \mathrm{Ext}^i_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{F}), \quad \text{for } i = 0, 1.$$

For $\mathcal{F} \in \mathrm{VIC}(X/T)$ and smooth morphisms $f : X \to S$ and $S \to T$ the relative de Rham cohomology

$$\underline{H}^i_{\mathrm{dR}}(X/S, \mathcal{F})$$

is canonically equipped with an integrable T-connection called *Gauss–Manin* connection.

Definition of the de Rham logarithm sheaves. Let $\pi : E \to S$ be an elliptic curve over a smooth separated K-scheme of finite type. Let us write $\mathcal{H} := \underline{H}^1_{\mathrm{dR}}(E/S)^\vee$ for the dual of the relative de Rham cohomology and $\mathcal{H}_E := \pi^*\mathcal{H}$ for its pullback to the elliptic curve. The Gauss–Manin connection equips $\mathcal{H}_E$ with an integrable K-connection. The group $\mathrm{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E)$ classifies isomorphism classes $[\mathcal{F}]$ of extensions

$$0 \to \mathcal{H}_E \to \mathcal{F} \to \mathcal{O}_E \to 0 \tag{1}$$

in the category $\mathrm{VIC}(E/K)$. In general such an extension will have nontrivial automorphisms. In the case where the extension (1) splits horizontally after pullback along the unit section $e\colon S \to E$ we can rigidify the situation by fixing a splitting, i.e., an isomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{H} \oplus \mathcal{O}_S & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\ & & \| & & \downarrow{\cong} & & \| & & \\ 0 & \longrightarrow & e^*\mathcal{H}_E & \longrightarrow & e^*\mathcal{F} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \end{array}$$

in the category $\mathrm{VIC}(S/K)$. Such a horizontal splitting is uniquely determined by the image of $1 \in \Gamma(S, \mathcal{O}_S)$ under the splitting. In other words, the group

$$\ker(e^*\colon \operatorname{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E) \to \operatorname{Ext}^1_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H}))$$

classifies isomorphism classes of pairs $[\mathcal{F}, s]$ consisting of an extension of $\mathcal{O}_E$ by $\mathcal{H}_E$ in the category $\mathrm{VIC}(E/K)$ together with a horizontal section $s \in \Gamma(S, e^*\mathcal{F})$ mapping to 1 under $e^*\mathcal{F} \to \mathcal{O}_S$. A pair $[F, s]$ is uniquely determined by its extension class up to unique isomorphism.

The Leray spectral sequence in de Rham cohomology

$$E_2^{p,q} = H^p_{\mathrm{dR}}(S/K, \underline{H}^q_{\mathrm{dR}}(E/S, \mathcal{H}_E)) \Longrightarrow E^{p+q} = H^{p+q}_{\mathrm{dR}}(E/K, \mathcal{H}_E)$$

gives a split short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H}) \underset{e^*}{\overset{\pi^*}{\rightleftarrows}} \operatorname{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E) \longrightarrow \operatorname{Hom}_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H} \otimes_{\mathcal{O}_S} \mathcal{H}^\vee) \longrightarrow 0.$$

Definition 2.1. Let $[\mathrm{Log}^1_{\mathrm{dR}}, \mathbb{1}^{(1)}]$ be the unique extension class corresponding to $\mathrm{id}_{\mathcal{H}}$ under the isomorphism

$$\ker(e^*\colon \operatorname{Ext}^1_{\mathrm{VIC}(E/K)}(\mathcal{O}_E, \mathcal{H}_E) \to \operatorname{Ext}^1_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H})) \cong \operatorname{Hom}_{\mathrm{VIC}(S/K)}(\mathcal{O}_S, \mathcal{H} \otimes_{\mathcal{O}_S} \mathcal{H}^\vee).$$

This pair $(\mathrm{Log}^1_{\mathrm{dR}}, \mathbb{1}^{(1)})$ is uniquely determined up to unique isomorphism. This pair is called the *first de Rham logarithm sheaf.*

The tensor product in the category $\mathrm{VIC}(E/K)$ allows us to define for $n \geq 0$ an integrable connection on the n-th tensor symmetric powers

$$\mathrm{Log}^n_{\mathrm{dR}} := \underline{\mathrm{TSym}}^n \, \mathrm{Log}^1_{\mathrm{dR}}.$$

For details on symmetric tensors, let us for example refer to [Bourbaki 1990, Chapter IV, Section 5]. Recall that the n-th tensor symmetric power is the sheaf of invariants of the n-th tensor power under the action of the symmetric groups S_n. The shuffle product defines a ring structure on the tensor symmetric powers and we obtain a graded ring

$$\bigoplus_{k \geq 0} \underline{\mathrm{TSym}}^k \, \mathrm{Log}^1_{\mathrm{dR}}.$$

The map

$$\mathrm{Log}^1_{\mathrm{dR}} \to \underline{\mathrm{TSym}}^k \, \mathrm{Log}^1_{\mathrm{dR}}, \quad x \mapsto x^{[k]} := \underbrace{x \otimes \cdots \otimes x}_{k}$$

defines divided powers on the graded algebra of symmetric tensors. In particular, the horizontal section $\mathbb{1}^{(1)}$ induces a horizontal section

$$\mathbb{1}^{(n)} := (\mathbb{1}^{(1)})^{[n]} \in \Gamma(S, e^* \mathrm{Log}^n_{\mathrm{dR}}).$$

This allows us to define the *n-th de Rham logarithm sheaves* as the pair $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^{(n)})$. Later we will see that the *n-th de Rham logarithm sheaf* is uniquely determined by a universal property, see Proposition 2.4. The horizontal epimorphism $\mathrm{Log}^1_{\mathrm{dR}} \twoheadrightarrow \mathcal{O}_E$ induces horizontal transition maps

$$\mathrm{Log}^n_{\mathrm{dR}} \twoheadrightarrow \mathrm{Log}^{n-1}_{\mathrm{dR}}\,.$$

The transition maps allow us to define a descending filtration

$$A^0\,\mathrm{Log}^n_{\mathrm{dR}} = \mathrm{Log}^n_{\mathrm{dR}} \supseteq A^1\,\mathrm{Log}^n_{\mathrm{dR}} \supseteq \cdots \supseteq A^{n+1}\,\mathrm{Log}^n_{\mathrm{dR}} = 0$$

by subobjects

$$A^i\,\mathrm{Log}^n_{\mathrm{dR}} := \ker(\mathrm{Log}^n_{\mathrm{dR}} \twoheadrightarrow \mathrm{Log}^{n-i}_{\mathrm{dR}}), \quad i = 1, \dots, n.$$

in the category $\mathrm{VIC}(E/K)$. The graded pieces of this filtration are given by

$$\mathrm{gr}^i_A\,\mathrm{Log}^n_{\mathrm{dR}} = A^i\,\mathrm{Log}^n_{\mathrm{dR}}\,/A^{i+1}\,\mathrm{Log}^n_{\mathrm{dR}} = \underline{\mathrm{TSym}}^i\,\mathcal{H}_E.$$

The cohomology of the de Rham logarithm sheaves. For the definition of the de Rham polylogarithm we will need a good understanding of the cohomology of the logarithm sheaves. The necessary computations are the same as for other realizations. In the de Rham realization, they can also be found in Scheider's PhD thesis [2014, Section 1.2]. Let us briefly recall the arguments for the convenience of the reader.

Proposition 2.2. *For $i = 0, 1$ the transition map $\mathrm{Log}^n_{\mathrm{dR}} \twoheadrightarrow \mathrm{Log}^{n-1}_{\mathrm{dR}}$ induces the zero morphism*

$$\underline{H}^i_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}}) \to \underline{H}^i_{\mathrm{dR}}(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}}).$$

For $i = 2$ we have isomorphisms

$$\underline{H}^2_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}}) \xrightarrow{\sim} \underline{H}^2_{\mathrm{dR}}(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{O}_S.$$

Proof. This is a classical result due to Beilinson and Levin [1994]. For the de Rham realization see also [Scheider 2014, Theorem 1.2.1]. For the convenience of the reader let us include a proof. For $n \geq 1$ consider the long exact sequence

$$\begin{array}{ccccccc}
0 \to & \underline{H}^0_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \to & \underline{H}^0_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}}) & \to & \underline{H}^0_{\mathrm{dR}}(E/S) & \xrightarrow{\delta^0} \\
 & \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \to & \underline{H}^1_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}}) & \to & \underline{H}^1_{\mathrm{dR}}(E/S) & \xrightarrow{\delta^1} \\
 & \underline{H}^2_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \to & \underline{H}^2_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}}) & \to & \underline{H}^2_{\mathrm{dR}}(E/S) & \to 0
\end{array}$$

associated to

$$0 \to \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}} \to \mathrm{Log}^n_{\mathrm{dR}} \to \mathcal{O}_E \to 0.$$

Let us first consider the case $n = 1$. It follows from the defining property of the extension class of $\mathrm{Log}^1_{\mathrm{dR}}$ that the map $\delta^0\colon \mathcal{O}_S = \underline{H}^0_{\mathrm{dR}}(E/S) \to \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E) = \mathcal{H} \otimes \mathcal{H}^\vee$ maps $1 \in \Gamma(S, \mathcal{O}_S)$ to $\mathrm{id}_{\mathcal{H}} \in \Gamma(S, \mathcal{H} \otimes \mathcal{H}^\vee)$, i.e.,

$$\delta^0(1) = \mathrm{id}_{\mathcal{H}} \tag{2}$$

In particular, we deduce that δ^0 is injective. Again by (2), we obtain that

$$\underline{H}^1_{\mathrm{dR}}(E/S) \cong \underline{H}^0_{\mathrm{dR}}(E/S) \otimes \underline{H}^1_{\mathrm{dR}}(E/S) \xrightarrow{\delta^0 \otimes \mathrm{id}} \underline{H}^1_{\mathrm{dR}}(\mathcal{H}_E) \otimes \underline{H}^1_{\mathrm{dR}}(E/S) \xrightarrow{\cup} \underline{H}^2_{\mathrm{dR}}(\mathcal{H}_E) \tag{3}$$

coincides with the map

$$\underline{H}^1_{\mathrm{dR}}(E/S) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{H}^1_{\mathrm{dR}}(E/S), \mathcal{O}_S) \cong \underline{H}^2_{\mathrm{dR}}(\mathcal{H}_E), \quad x \mapsto (y \mapsto x \cup y).$$

Since the cup product

$$\cup\colon \underline{H}^1_{\mathrm{dR}}(E/S) \otimes \underline{H}^1_{\mathrm{dR}}(E/S) \to \underline{H}^2_{\mathrm{dR}}(E/S)$$

defines a perfect pairing, we deduce that (3) is an isomorphism. By the compatibility of the cup product with the connecting homomorphism

$$\begin{array}{ccc} \underline{H}^0_{\mathrm{dR}}(E/S) \otimes \underline{H}^1_{\mathrm{dR}}(E/S) & \xrightarrow{\delta^0 \otimes \mathrm{id}} & \underline{H}^1_{\mathrm{dR}}(\mathcal{H}_E) \otimes \underline{H}^1_{\mathrm{dR}}(E/S) \\ \downarrow \cup & & \downarrow \cup \\ \underline{H}^1_{\mathrm{dR}}(E/S) & \xrightarrow{\delta^1} & \underline{H}^2_{\mathrm{dR}}(\mathcal{H}_E) \end{array}$$

we deduce that δ^1 is an isomorphism. The fact that δ^1 is an isomorphism implies that

$$\underline{H}^2_{\mathrm{dR}}(E/S, \mathrm{Log}^1_{\mathrm{dR}}) \to \underline{H}^2_{\mathrm{dR}}(E/S)$$

is an isomorphism and that

$$\underline{H}^1_{\mathrm{dR}}(E/S, \mathrm{Log}^1_{\mathrm{dR}}) \to \underline{H}^1_{\mathrm{dR}}(E/S)$$

is zero. The injectivity of δ^0 implies that

$$\underline{H}^0_{\mathrm{dR}}(E/S, \mathrm{Log}^1_{\mathrm{dR}}) \to \underline{H}^0_{\mathrm{dR}}(E/S)$$

is zero. This settles the case $n = 1$. Let us now proceed by induction. For $n \geq 2$ let us assume that the claim has been proven for the transition map $\mathrm{Log}^{n-1}_{\mathrm{dR}} \to \mathrm{Log}^{n-2}_{\mathrm{dR}}$. The morphism of exact complexes associated to the map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}} & \longrightarrow & \mathrm{Log}^{n}_{\mathrm{dR}} & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{H}_E \otimes \mathrm{Log}^{n-2}_{\mathrm{dR}} & \longrightarrow & \mathrm{Log}^{n-1}_{\mathrm{dR}} & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \end{array}$$

splits up into pieces by the induction hypothesis. Indeed, note that $\mathrm{Log}^n_{\mathrm{dR}} \to \mathcal{O}_E$ factors through $\mathrm{Log}^n_{\mathrm{dR}} \to \mathrm{Log}^{n-1}_{\mathrm{dR}} \to \cdots \to \mathcal{O}_E$ and thus induces the zero map in cohomological degree 0 and 1 and an isomorphism in degree 2. Using this, the above diagram of short exact sequences induces the following diagrams with exact rows:

$$
\begin{array}{ccccc}
0 & \longrightarrow & \underline{H}^0_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \xrightarrow{\cong} & \underline{H}^0_{\mathrm{dR}}(E/S, \mathrm{Log}^{n}_{\mathrm{dR}}) \\
 & & \downarrow{\scriptstyle 0\text{ (by IH)}} & & \downarrow{\scriptstyle (1)} \\
0 & \longrightarrow & \underline{H}^0_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-2}_{\mathrm{dR}}) & \xrightarrow{\cong} & \underline{H}^0_{\mathrm{dR}}(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}})
\end{array}
$$

$$
\begin{array}{ccccc}
\underline{H}^0_{\mathrm{dR}}(E/S) & \hookrightarrow & \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \twoheadrightarrow & \underline{H}^1_{\mathrm{dR}}(E/S, \mathrm{Log}^{n}_{\mathrm{dR}}) \\
\| & & \downarrow{\scriptstyle 0\text{ (by IH)}} & & \downarrow{\scriptstyle (2)} \\
\underline{H}^0_{\mathrm{dR}}(E/S) & \hookrightarrow & \underline{H}^1_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-2}_{\mathrm{dR}}) & \twoheadrightarrow & \underline{H}^1_{\mathrm{dR}}(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}})
\end{array}
$$

$$
\begin{array}{ccccccc}
\underline{H}^1_{\mathrm{dR}}(E/S) & \hookrightarrow & \underline{H}^2_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \longrightarrow & \underline{H}^2_{\mathrm{dR}}(E/S, \mathrm{Log}^{n}_{\mathrm{dR}}) & \twoheadrightarrow & \underline{H}^2_{\mathrm{dR}}(E/S) \\
\| & & \downarrow{\scriptstyle \cong\text{ (by IH)}} & & \downarrow{\scriptstyle (3)} & & \| \\
\underline{H}^1_{\mathrm{dR}}(E/S) & \hookrightarrow & \underline{H}^2_{\mathrm{dR}}(E/S, \mathcal{H}_E \otimes \mathrm{Log}^{n-2}_{\mathrm{dR}}) & \longrightarrow & \underline{H}^2_{\mathrm{dR}}(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}}) & \twoheadrightarrow & \underline{H}^2_{\mathrm{dR}}(E/S)
\end{array}
$$

The maps denoted by (IH) are zero and isomorphisms, respectively, by the induction hypothesis. From the commutativity of the diagrams we deduce that the transition maps in (1) and (2) are zero maps while (3) is an isomorphism, as desired. □

The universal property of the de Rham logarithm sheaves. In this subsection we will prove the universal property of the de Rham logarithm sheaves among all unipotent vector bundles with integrable connection. Recall that we have a descending filtration $A^\bullet \mathrm{Log}^n_{\mathrm{dR}}$ on the n-th logarithm sheaves satisfying

$$\mathrm{gr}^i_A \mathrm{Log}^n_{\mathrm{dR}} = \underline{\mathrm{TSym}}^i \mathcal{H}_E = \pi^* \underline{\mathrm{TSym}}^i \mathcal{H}.$$

In particular, all subquotients are given by pullback. Motivated by this property we state the following definition:

Definition 2.3. Let $S \to T$ be a smooth morphism of smooth separated K-schemes and E/S an elliptic curve:

(a) An object $\mathcal{U}$ in $\mathrm{VIC}(E/T)$ is called unipotent of length n for $E/S/T$ if there exists a descending filtration in the category $\mathrm{VIC}(E/T)$

$$\mathcal{U} = A^0\mathcal{U} \supseteq A^1\mathcal{U} \supseteq \cdots \supseteq A^{n+1}\mathcal{U} = 0$$

such that for all $0 \le i \le n$, $\mathrm{gr}^i_A \mathcal{U} = A^i\mathcal{U}/A^{i+1}\mathcal{U} = \pi^* Y_i$ for some $Y_i \in \mathrm{VIC}(S/T)$.

(b) Let $U^\dagger_n(E/S/T)$ be the full subcategory of objects of $\mathrm{VIC}(E/T)$ which are unipotent of length n for $E/S/T$. For the case $S = T$ let us write $U^\dagger(E/S) := U^\dagger(E/S)$ for simplicity.

For the moment we will consider the absolute case, i.e., $T = \operatorname{Spec} K$. Later, we will naturally by concerned with the relative case, i.e., the case $T = S$. For $F, G \in \mathrm{VIC}(E/K)$ there is a natural connection on the sheaf of homomorphisms of the underlying modules. Let us write $\underline{\mathrm{Hom}}(F, G)$ for the internal-Hom in the category $\mathrm{VIC}(S/K)$. Let us also introduce the notation $\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(F, G)$ for the sheaf of horizontal morphisms relative S. By the definition of the connection on the internal-Hom sheaves, we see that $\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(F, G)$ is the subsheaf of S-horizontal sections of $\underline{\mathrm{Hom}}(F, G)$. We can rephrase this as follows:

$$\pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(F, G) = \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(F, G)).$$

In particular, the Gauss–Manin connection gives a K-connection on $\pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(F, G)$. With this observation we can finally characterize the logarithm sheaves through a universal property:

Proposition 2.4 [Scheider 2014, Theorem 1.3.6]. *The pair* $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^{(n)})$ *is the unique pair, consisting of a unipotent object of* $\mathrm{U}^\dagger_n(E/S/K)$ *together with a horizontal section along* e, *such that for all* $\mathcal{U} \in \mathrm{U}^\dagger_n(E/S/K)$ *the map*

$$\pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathrm{Log}^n_{\mathrm{dR}}, \mathcal{U}) \to e^*\mathcal{U}, \quad f \mapsto (e^*f)(\mathbb{1}^{(n)})$$

is an isomorphism in $\mathrm{VIC}(S/K)$.

Proof. For the convenience of the reader let us sketch the proof. We have for $0 \le i \le 2$ a canonical horizontal isomorphism

$$\underline{H}^{2-i}_{\mathrm{dR}}(E/S, (\mathrm{Log}^n_{\mathrm{dR}})^\vee) \xrightarrow{\sim} \underline{H}^i_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}})^\vee \tag{4}$$

induced by the perfect cup product pairing

$$\underline{H}^{2-i}_{\mathrm{dR}}(E/S, (\mathrm{Log}^n_{\mathrm{dR}})^\vee) \otimes_{\mathcal{O}_S} \underline{H}^i_{\mathrm{dR}}(E/S, \mathrm{Log}^n_{\mathrm{dR}}) \to \underline{H}^2_{\mathrm{dR}}(E/S) \xrightarrow{\sim} \mathcal{O}_S.$$

We prove the result by a double induction over $0 \le k \le n$, where k is the length of the shortest unipotent filtration of the object $\mathcal{U}$. For $k = n = 0$, we have $\mathcal{U} = \pi^*Z$ for some object Z of $\mathrm{VIC}(S/K)$. Indeed, we have

$$\pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathrm{Log}^0_{\mathrm{dR}}, \pi^*Z) \cong \pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathcal{O}_E, \pi^*Z) \cong \underline{\mathrm{Hom}}(\mathcal{O}_S, Z) \cong Z$$

and the map is given by $f \mapsto e^*f(1) = e^*f(\mathbb{1}^{(0)})$. For the case $0 = k < n$ observe, that the transition maps

$$\underline{H}^0_{\mathrm{dR}}(E/S, (\mathrm{Log}^n_{\mathrm{dR}})^\vee) \xrightarrow{\sim} \underline{H}^0_{\mathrm{dR}}(E/S, (\mathrm{Log}^{n-1}_{\mathrm{dR}})^\vee)$$

are isomorphisms by the above perfect pairing and Proposition 2.2. We deduce the claim for $0 = k < n$ by the commutative diagram

$$\begin{array}{ccccc}
\underline{H}^0_{\mathrm{dR}}(E/S, (\mathrm{Log}^n_{\mathrm{dR}})^\vee) & \xrightarrow{\simeq} & \pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathrm{Log}^n_{\mathrm{dR}}, \pi^*Z) & \longrightarrow & Z \\
\downarrow{\simeq} & & \downarrow & & \| \\
\underline{H}^0_{\mathrm{dR}}(E/S, (\mathrm{Log}^0_{\mathrm{dR}})^\vee) & \xrightarrow{\simeq} & \pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathrm{Log}^0_{\mathrm{dR}}, \pi^*Z) & \xrightarrow{\simeq} & Z.
\end{array}$$

Here, let us observe, that $\mathbb{1}^{(n)}$ maps to $\mathbb{1}^{(0)}$ under the transition map. Let us now consider the case $0 < k \leq n$ and assume that the case $k-1 \leq n$ has already been settled. For an unipotent object $\mathcal{U}$ of length k we have an horizontal exact sequence

$$0 \to \pi^* Z \to \mathcal{U} \to \mathcal{U}/\pi^* Z \to 0$$

with $\mathcal{U}/\pi^* Z$ in $U^{\dagger}_{k-1}(E/S/K)$. This sequence induces a long exact sequences in $\mathrm{VIC}(S/K)$. Let us first show that the connecting homomorphism

$$\delta^0_{(n)}\colon \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^n_{\mathrm{dR}}, \mathcal{U}/\pi^* Z)) \to \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^n_{\mathrm{dR}}, \pi^* Z))$$

is trivial for all $n \geq k$. Let us consider the following commutative diagram, where the right vertical map is induced by the transition maps of the logarithm sheaves:

$$\begin{array}{ccc} \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n-1}_{dR}, \mathcal{U}/\pi^* Z)) & \longrightarrow & e^*(\mathcal{U}/\pi^* Z) \\ \downarrow & & \| \\ \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n}_{\mathrm{dR}}, \mathcal{U}/\pi^* Z)) & \longrightarrow & e^*(\mathcal{U}/\pi^* Z) \end{array}$$

By the induction hypothesis, the horizontal maps in this diagram are isomorphisms. We deduce that the map

$$\underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n-1}_{\mathrm{dR}}, \mathcal{U}/\pi^* Z)) \to \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n}_{\mathrm{dR}}, \mathcal{U}/\pi^* Z))$$

is an isomorphism, too. On the other hand, the transition maps

$$\underline{H}^1_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n-1}_{\mathrm{dR}}, \pi^* Z)) \to \underline{H}^1_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n}_{\mathrm{dR}}, \pi^* Z)) \tag{5}$$

are identified with the dual of the transition maps

$$\underline{H}^1_{\mathrm{dR}}(E/S, \mathrm{Log}^{n-1}_{\mathrm{dR}})^\vee \otimes Z \to \underline{H}^1_{\mathrm{dR}}(E/S, \mathrm{Log}^{n}_{\mathrm{dR}})^\vee \otimes Z$$

under the perfect cup product pairing. Thus, Section 2 implies that (5) is the zero morphism. The connecting homomorphisms $\delta^0_{(n)}$ and $\delta^0_{(n-1)}$ fit in the following commutative diagram, where the vertical maps are induced by the transition maps of the logarithm sheaves:

$$\begin{array}{ccc} \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n-1}_{\mathrm{dR}}, \mathcal{U}/\pi^* Z)) & \xrightarrow{\delta^0_{(n-1)}} & \underline{H}^1_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n-1}_{\mathrm{dR}}, \pi^* Z)) \\ \downarrow \cong & & \downarrow 0 \\ \underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n}_{\mathrm{dR}}, \mathcal{U}/\pi^* Z)) & \xrightarrow{\delta^0_{(n)}} & \underline{H}^1_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^{n}_{\mathrm{dR}}, \pi^* Z)) \end{array}$$

Above, we have already shown that the left vertical map is an isomorphism, while the right vertical map is zero. This shows the vanishing of the connecting homomorphism $\delta^0_{(n)}$. Now, the claim for $0 < k \leq n$ is easily deduced from the induction hypothesis. Indeed, we get the following commutative diagram with vertical exact sequences:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^n_{\mathrm{dR}}, \pi^* Z)) & \xrightarrow{\simeq} & Z \\
\downarrow & & \downarrow \\
\underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^n_{\mathrm{dR}}, \mathcal{U})) & \longrightarrow & e^*\mathcal{U} \\
\downarrow & & \downarrow \\
\underline{H}^0_{\mathrm{dR}}(E/S, \underline{\mathrm{Hom}}(\mathrm{Log}^n_{\mathrm{dR}}, \mathcal{U}/\pi^* Z)) & \xrightarrow{\simeq} & e^*(\mathcal{U}/\pi^* Z) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Here, the first and the last horizontal maps are isomorphisms by induction. The left-exactness of the first column follows from the vanishing of the connecting homomorphism $\delta^0_{(n)}$. We deduce the desired isomorphism in the middle. □

Remark 2.5. Another way to formulate the universal property is as follows. Consider the category consisting of pairs $(\mathcal{U}, s)$ with $\mathcal{U} \in \mathrm{U}^\dagger_n(E/S/K)$ and a fixed horizontal section $s \in \Gamma(S, e^*\mathcal{U})$. Morphisms are supposed to be horizontal and respect the fixed section after pullback along e. Then, the universal property reformulates as the fact that this category has an initial object. This initial object is $(\mathrm{Log}^n_{\mathrm{dR}}, \mathbb{1}^{(n)})$.

3. The definition of the polylogarithm in de Rham cohomology

Let us briefly recall the definition of the de Rham cohomology class of the polylogarithm following [Scheider 2014, Chapter 1.5]. Let us fix a positive integer D. Let us define the sections $1_e, 1_{E[D]} \in \Gamma(E[D], \mathcal{O}_{E[D]})$ as follows: let $1_{E[D]}$ correspond to $1 \in \mathcal{O}_{E[D]}$ and 1_e correspond to the section which is zero on $E[D] \setminus \{e\}$ and 1 on $\{e\}$. The localization sequence in de Rham cohomology for the situation

$$\begin{array}{ccccc}
U_D := E \setminus E[D] & \overset{j_D}{\hookrightarrow} & E & \overset{i_D}{\hookleftarrow} & E[D] \\
& {\scriptstyle \pi_{U_D}} \searrow & \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \pi_{E[D]}} & \\
& & S & &
\end{array} \tag{6}$$

combined with the vanishing results (see Section 2) gives the following:

Lemma 3.1 [Scheider 2014, Section 1.5.2, Lemma 1.5.4]. *Let us write $\mathcal{H}_{E[D]} := \pi^*_{E[D]}\mathcal{H}$ and $\mathcal{H}_{U_D} := \pi^*_{U_D}\mathcal{H}$. The localization sequence in de Rham cohomology for* (6) *induces an exact sequence*

$$0 \to \varprojlim_n H^1_{\mathrm{dR}}(U_D/K, \mathrm{Log}^n_{\mathrm{dR}}) \xrightarrow{\mathrm{Res}} \prod_{k=0}^{\infty} H^0_{\mathrm{dR}}(E[D]/K, \underline{\mathrm{Sym}}^k\, \mathcal{H}_{E[D]}) \xrightarrow{\sigma} K \to 0.$$

If we view the horizontal section $D^2 \cdot 1_e - 1_{E[D]} \in \Gamma(S, \mathcal{O}_{E[D]})$ *as sitting in degree zero of*

$$\prod_{k=0}^{\infty} \underline{H}^0_{\mathrm{dR}}(E[D]/K, \underline{\mathrm{Sym}}^k \mathcal{H}_{E[D]}),$$

it is contained in the kernel of the augmentation map σ.

Proof. For the convenience of the reader let us recall the construction of the short exact sequence. The localization sequence and the vanishing of $\varprojlim_n H^1_{\mathrm{dR}}(E, \mathrm{Log}^n_{\mathrm{dR}}) = 0$ gives

$$0 \to \varprojlim_n H^1_{\mathrm{dR}}(U_D, \mathrm{Log}^n_{\mathrm{dR}}) \xrightarrow{\mathrm{Res}} \varprojlim_n H^0_{\mathrm{dR}}(E[D], i_D^* \mathrm{Log}^n_{\mathrm{dR}}) \to \varprojlim_n H^2_{\mathrm{dR}}(E, \mathrm{Log}^n_{\mathrm{dR}}) \to 0$$

Now, the exact sequence in the claim follows by Section 2 and the isomorphism

$$i_D^* \mathrm{Log}^n_{\mathrm{dR}} \xrightarrow{\sim} i_D^*[D]^* \mathrm{Log}^n_{\mathrm{dR}} = \pi^*_{E[D]} e^* \mathrm{Log}^n_{\mathrm{dR}} \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\mathrm{Sym}}^k \mathcal{H}_{E[D]}. \qquad \square$$

Definition 3.2. Let $\mathrm{pol}_{D,\mathrm{dR}} = (\mathrm{pol}^n_{D,\mathrm{dR}})_{n \geq 0} \in \varprojlim_n H^1_{\mathrm{dR}}(U_D/K, \mathrm{Log}^n_{\mathrm{dR}})$ be the unique prosystem mapping to $D^2 1_e - 1_{E[D]}$ under the residue map. We call $\mathrm{pol}_{D,\mathrm{dR}}$ *the (D-variant) of the elliptic polylogarithm.*

Remark 3.3. Let us write $U := E \setminus \{e\}$. The *classical polylogarithm* in de Rham cohomology

$$(\mathrm{pol}^n_{\mathrm{dR}})_{n \geq 0} \in \varprojlim_n H^1_{\mathrm{dR}}(U/K, \mathcal{H}_E^\vee \otimes_{\mathcal{O}_E} \mathrm{Log}^n_{\mathrm{dR}})$$

is defined as the unique element mapping to $\mathrm{id}_{\mathcal{H}}$ under the isomorphism

$$\varprojlim_n H^1_{\mathrm{dR}}(U/K, \mathcal{H}_E^\vee \otimes_{\mathcal{O}_E} \mathrm{Log}^n_{\mathrm{dR}}) \xrightarrow{\sim} \prod_{k=1}^{\infty} H^0_{\mathrm{dR}}(S/\mathbb{Q}, \mathcal{H}^\vee \otimes_{\mathcal{O}_S} \underline{\mathrm{Sym}}^k \mathcal{H}).$$

This isomorphism comes from the localization sequence for $U := E \setminus \{e\} \hookrightarrow E$. For details we refer to [Scheider 2014, Section 1.5.1]. Indeed, there is not much difference between the classical polylogarithm and its D-variant. For a comparison of both we refer to [loc. cit., Section 1.5.3].

4. The de Rham logarithm sheaves via the Poincaré bundle

In his PhD thesis Scheider [2014, Theorem 2.3.1] gave an explicit model for the de Rham logarithm sheaves constructed out of the Poincaré bundle. Since the material has never been published, we will recall his approach to the de Rham logarithm sheaves via the Poincaré bundle. The main result of this section is due to Scheider, but we provide a considerably shorter proof of this theorem.

The geometric logarithm sheaves. Let $\pi : E \to S$ be an elliptic curve over a separated locally Noetherian base scheme S. Let us recall the definition of the Poincaré bundle and thereby fix some notation. A *rigidification* of a line bundle $\mathcal{L}$ on E over S is an isomorphism

$$r : e^* \mathcal{L} \xrightarrow{\sim} \mathcal{O}_S.$$

A morphism of rigidified line bundles is a morphism of line bundles respecting the rigidification. The dual elliptic curve $E^\vee$ represents the functor

$$T \mapsto \mathrm{Pic}^0(E_T/T) := \{\text{isomorphism classes of rigidified line bundles } (\mathcal{L}, r) \text{ of degree 0 on } E_T/T\}$$

on the category of S-schemes. Because a rigidified line bundle does not have any nontrivial automorphisms, there is a universal rigidified line bundle $(\mathcal{P}, r_0)$ called the *Poincaré bundle* over $E \times_S E^\vee$. By interchanging the roles of E and $E^\vee$, we get a unique trivialization $s_0\colon (e \times \mathrm{id})^*\mathcal{P} \xrightarrow{\sim} \mathcal{O}_{E^\vee}$ and we call $(\mathcal{P}, r_0, s_0)$ the *birigidified Poincaré bundle*. Similarly, let us consider the group valued functor

$$T \mapsto \mathrm{Pic}^\dagger(E_T/T) := \left\{ \begin{array}{l} \text{isomorphism classes of rigidified line bundles } (\mathcal{L}, r\nabla) \text{ of degree 0 on } E_T/T \\ \text{with an integrable } T\text{-connection } \nabla\colon \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_{E_T}} \Omega^1_{E_T/T} \text{ on } \mathcal{L} \end{array} \right\}.$$

This functor is representable by an S-group scheme $E^\dagger$. By forgetting the connection, we obtain an epimorphism $q^\dagger\colon E^\dagger \to E^\vee$ of group schemes over S. The pullback $\mathcal{P}^\dagger := (q^\dagger)^*\mathcal{P}$ is equipped with a unique integrable $E^\dagger$-connection

$$\nabla_{\mathcal{P}^\dagger} : \mathcal{P}^\dagger \to \mathcal{P}^\dagger \otimes \Omega^1_{E \times_S E^\dagger / E^\dagger}$$

making $(\mathcal{P}^\dagger, \nabla_{\mathcal{P}^\dagger}, r_0)$ universal among all rigidified line bundles with integrable connection. Let us mention that the group scheme $E^\dagger$ satisfies another universal property: it sits in a short exact sequence

$$0 \to V(\omega_{E/S}) \to E^\dagger \to E^\vee \to 0$$

with $V(\omega_{E/S})$ the vector group over S associated with $\omega_{E/S}$ and every other such vectorial extension of $E^\vee$ is a pushout of this extension. This explains why $E^\dagger$ is called *universal vectorial extension* of $E^\vee$. For a more detailed discussion on the universal vectorial extension and its properties, let us refer to the first chapter of the book of Mazur and Messing [1974].

Let us denote the inclusions of the infinitesimal thickenings of e in $E^\dagger$ and $E^\vee$, respectively, by

$$\iota_n^\dagger : E_n^\dagger := \mathrm{Inf}_e^n E^\dagger \hookrightarrow E^\dagger, \quad \iota_n : E_n^\vee := \mathrm{Inf}_e^n E^\vee \hookrightarrow E^\vee.$$

Definition 4.1. For $n \geq 0$ define

$$\mathcal{L}_n^\dagger = (\mathrm{pr}_E)_*(\mathrm{id}_E \times \iota_n^\dagger)^*\mathcal{P}^\dagger, \quad \mathcal{L}_n := (\mathrm{pr}_E)_*(\mathrm{id}_E \times \iota_n)^*\mathcal{P}.$$

Both $\mathcal{L}_n$ and $\mathcal{L}_n^\dagger$ are locally free $\mathcal{O}_E$-modules of finite rank equipped with canonical isomorphisms

$$\mathrm{triv}_e : e^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \mathcal{O}_{E_n^\dagger}, \quad \mathrm{triv}_e : e^*\mathcal{L}_n \xrightarrow{\sim} \mathcal{O}_{E_n^\vee}$$

induced by the rigidifications of the Poincaré bundle. Furthermore, $\nabla_{\mathcal{P}^\dagger}$ induces an integrable S-connection $\nabla_{\mathcal{L}_n^\dagger}$ on $\mathcal{L}_n^\dagger$. We call $\mathcal{L}_n^\dagger$ the *n-th geometric logarithm sheaf*. Sometimes we will write $\mathcal{L}_{n,E}^\dagger$ to emphasize the dependence on the elliptic curve E/S.

Let us discuss some immediate properties of the geometric logarithm sheaves. The reader who is familiar with the formal properties of the abstract logarithm sheaves will immediately recognize many of the following properties: the compatibility of the Poincaré bundle with base change along $f: T \to S$ shows immediately that the geometric logarithm sheaves are *compatible with base change*, i.e.,

$$\mathrm{pr}_E^* \mathcal{L}_{n,E/S} \xrightarrow{\sim} \mathcal{L}_{n,E_T/T}, \quad \mathrm{pr}_E^* \mathcal{L}^\dagger_{n,E/S} \xrightarrow{\sim} \mathcal{L}^\dagger_{n,E_T/T}$$

where $\mathrm{pr}_E : E_T = E \times_S T \to E$ is the projection. By restricting form the n-th infinitesimal thickening to the $(n-1)$-th, we obtain *transition maps*

$$\mathcal{L}^\dagger_n \twoheadrightarrow \mathcal{L}^\dagger_{n-1}, \quad \mathcal{L}_n \twoheadrightarrow \mathcal{L}_{n-1}.$$

The decompositions $\mathcal{O}_{\mathrm{Inf}^1_e E^\vee} = \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S}$ and $\mathcal{O}_{\mathrm{Inf}^1_e E^\dagger} = \mathcal{O}_S \oplus \mathcal{H}$ show that the transition maps $\mathcal{L}_1 \to \mathcal{O}_E$ and $\mathcal{L}^\dagger_1 \to \mathcal{O}_E$ sit in *short exact sequences*

$$0 \to \pi^* \underline{\omega}_{E^\vee/S} \to \mathcal{L}_1 \to \mathcal{O}_E \to 0 \quad \text{and} \quad 0 \to \mathcal{H}_E \to \mathcal{L}^\dagger_1 \to \mathcal{O}_E \to 0.$$

Since $\mathcal{P}^\dagger$ is the pullback of $\mathcal{P}$, we obtain natural inclusions

$$\mathcal{L}_n \hookrightarrow \mathcal{L}^\dagger_n.$$

These inclusions can be interpreted as the first nontrivial step of the *Hodge filtration* of the geometric logarithm sheaf $\mathcal{L}^\dagger_n$: the Hodge filtration on $\mathcal{H}$ induces a descending filtration of $\mathcal{O}_E$-modules on $\mathcal{L}^\dagger_1$ such that all morphisms in

$$0 \to \mathcal{H}_E \to \mathcal{L}^\dagger_1 \to \mathcal{O}_E \to 0$$

are strictly compatible with the filtration. Here, $\mathcal{O}_E$ is considered to be concentrated in filtration step 0. Explicitly this filtration is given as

$$F^{-1}\mathcal{L}^\dagger_1 = \mathcal{L}^\dagger_1 \supseteq F^0 \mathcal{L}^\dagger_1 = \mathcal{L}_1 \supseteq F^1 \mathcal{L}^\dagger_1 = 0.$$

Let us write $[D]: E \to E$ for the isogeny given by D-multiplication. The dual isogeny of $[D]$ is D-multiplication on $E^\vee$. By the universal property of the Poincaré bundle over $E \times_S E^\vee$, there is a unique isomorphism

$$\gamma_{\mathrm{id},[D]} \colon (\mathrm{id} \times [D])^* \mathcal{P} \xrightarrow{\sim} ([D] \times \mathrm{id})^* \mathcal{P}.$$

Since we are working over a field of characteristic zero, the D-multiplication induces an isomorphism on $\mathrm{Inf}^n_e E^\vee$:

$$\begin{array}{ccc} \mathrm{Inf}^n_e E^\vee & \hookrightarrow & E^\vee \\ \downarrow \cong & & \downarrow [D] \\ \mathrm{Inf}^n_e E^\vee & \hookrightarrow & E^\vee \end{array}$$

Restricting $\gamma_{\mathrm{id},[D]}$ along $E \times_S E^\vee$ and using the above commutative diagram gives

$$(\mathrm{pr}_E)_*(\mathcal{P}^\dagger|_{E \times \mathrm{Inf}^n_e E^\vee}) \cong (\mathrm{pr}_E)_*((\mathrm{id} \times [D])^* \mathcal{P}^\dagger|_{E \times \mathrm{Inf}^n_e E^\vee}) \xrightarrow{\sim} (\mathrm{pr}_E)_*(([D] \times \mathrm{id})^* \mathcal{P}^\dagger|_{E \times \mathrm{Inf}^n_e E^\vee}).$$

Recalling $\mathcal{L}_n^\dagger = (\mathrm{pr}_E)_*(\mathcal{P}^\dagger|_{E\times \mathrm{Inf}_e^n E^\vee})$ gives an *invariance under isogenies* isomorphism:

$$\mathcal{L}_n^\dagger \xrightarrow{\sim} [D]^*\mathcal{L}_n^\dagger. \tag{7}$$

The Fourier–Mukai transform of Laumon. For a smooth morphism $X \to S$ let us denote by $\mathcal{D}_{X/S}$ the sheaf of differential operators of X/S. Furthermore, let us denote by $D^b_{qc}(\mathcal{D}_{X/S})$ (resp. $D^b_{qc}(\mathcal{O}_X)$) the derived category of bounded complexes of quasicoherent $\mathcal{D}_{X/S}$-modules (resp. $\mathcal{O}_X$-modules). As usually, let E/S be an elliptic curve over a smooth base S over a field of characteristic zero. The Poincaré bundle with connection $(\mathcal{P}^\dagger, \nabla_{\mathcal{P}^\dagger})$ on $E\times_S E^\dagger$ serves as kernel for Laumon's Fourier–Mukai equivalence.

Theorem 4.2 [Laumon 1996, (3.2)]. *The functor*

$$\Phi_{\mathcal{P}^\dagger}: D^b_{qc}(\mathcal{O}_{E^\dagger}) \to D^b_{qc}(\mathcal{D}_{E/S}), \quad \mathcal{F}^\bullet \mapsto R\,\mathrm{pr}_{E,*}((\mathcal{P}^\dagger, \nabla_{\mathcal{P}^\dagger}) \otimes_{\mathcal{O}_{E\times E^\dagger}} \mathrm{pr}^*_{E^\dagger}\mathcal{F}^\bullet)$$

establishes an equivalence of triangulated categories.

We will restrict this derived Fourier–Mukai transform to certain complexes of unipotent objects which are concentrated in a single cohomological degree. Let us first introduce the following category:

Definition 4.3. Let $\mathcal{J}$ be the ideal sheaf of $\mathcal{O}_{E^\dagger}$ defined by the unit section. Let $\mathrm{U}_n(\mathcal{O}_{E^\dagger})$ be the full subcategory of the category of quasicoherent $\mathcal{O}_{E^\dagger}$-modules $\mathcal{F}$, such that $\mathcal{J}^{n+1}\mathcal{F}=0$ and $\mathcal{J}^i\mathcal{F}/\mathcal{J}^{i+1}\mathcal{F}$ is a locally free $\mathcal{O}_S = \mathcal{O}_{E^\dagger}/\mathcal{J}$-module of finite rank for $i = 0, \dots, n$.

The following results appear in the work of Scheider:

Lemma 4.4 [Scheider 2014, Proposition 2.2.6, Theorem 2.2.12(i)]. *Let us write $e_{E^\dagger}: S \to E^\dagger$ for the unit section of the universal vectorial extension of $E^\vee$:*

(a) *For a locally free $\mathcal{O}_S$-module $\mathcal{G}$, we have the formula*

$$\Phi_{\mathcal{P}^\dagger}((e_{E^\dagger})_*\mathcal{G}) = \pi^*\mathcal{G}.$$

(b) *The Fourier–Mukai transform of $\mathcal{F} \in \mathrm{U}_n(\mathcal{O}_{E^\dagger})$ is concentrated in cohomological degree zero, i.e.,*

$$H^i(\Phi_{\mathcal{P}^\dagger}(\mathcal{F})) = 0 \quad \textit{for } i \neq 0.$$

Proof. We follow closely the argument of Scheider [2014, Proposition 2.2.6, Theorem 2.2.12(i)]:

(a) Base change along the Cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}\times e_{E^\dagger}} & E\times_S E^\dagger \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{pr}_{E^\dagger}} \\ S & \xrightarrow{e_{E^\dagger}} & E^\dagger \end{array}$$

gives a canonical isomorphism of $\mathcal{D}_{E\times_S E^\dagger/E^\dagger}$-modules

$$\mathrm{pr}^*_{E^\dagger}(e_{E^\dagger})_*\mathcal{G} \simeq (\mathrm{id}\times e_{E^\dagger})_*\pi^*\mathcal{G}.$$

The trivialization of the Poincaré bundle gives a $\mathcal{D}_{E/S}$-linear isomorphism

$$(\mathrm{id}\times e_{E^\dagger})^*(\mathcal{P}^\dagger,\nabla_\dagger)\simeq(\mathcal{O}_E,\mathrm{d}).$$

Thus, we get an isomorphism of $\mathcal{D}_{E\times_S E^\dagger/E^\dagger}$-modules

$$\mathcal{P}^\dagger\otimes_{\mathcal{O}_{E\times_S E^\dagger}}\mathrm{pr}_{E^\dagger}^*(e_{E^\dagger})_*\mathcal{G}\cong(\mathrm{id}\times e_{E^\dagger})_*\pi^*\mathcal{G}.$$

Since $\mathrm{id}\times e_{E^\dagger}$ is affine, it is exact and we get

$$\begin{aligned}\Phi_{\mathcal{P}^\dagger}((e_{E^\dagger})_*\mathcal{G})&=R(\mathrm{pr}_E)_*(\mathcal{P}^\dagger\otimes_{\mathcal{O}_{E\times_S E^\dagger}}\mathrm{pr}_{E^\dagger}^*(e_{E^\dagger})_*\mathcal{G})\\&\cong R(\mathrm{pr}_E)_*R(\mathrm{id}\times e_{E^\dagger})_*\pi^*\mathcal{G}\\&=R(\mathrm{id}_E)_*\pi^*\mathcal{G}\\&=\pi^*\mathcal{G}.\end{aligned}$$

(b) Let us prove the claim by induction on n. For $n=0$ we have $\mathcal{F}=(e_{E^\dagger})_*\mathcal{G}$ for some locally free $\mathcal{O}_S$-module $\mathcal{G}$. Thus, the case $n=0$ follows from (a). For $n>0$ the claim follows by induction: indeed, the exact sequence

$$0\to\mathcal{J}^n\mathcal{F}\to\mathcal{F}\to\mathcal{F}/\mathcal{J}^n\mathcal{F}\to 0$$

induces a triangle

$$\Phi_{\mathcal{P}^\dagger}(\mathcal{J}^n\mathcal{F})\to\Phi_{\mathcal{P}^\dagger}(\mathcal{F})\to\Phi_{\mathcal{P}^\dagger}(\mathcal{F}/\mathcal{J}^n\mathcal{F})\to\Phi_{\mathcal{P}^\dagger}(\mathcal{J}^n\mathcal{F})[1]$$

in $D^b_{qc}(\mathcal{D}_{E/S})$. Now we conclude since $\Phi_{\mathcal{P}^\dagger}(\mathcal{J}^n\mathcal{F})$ and $\Phi_{\mathcal{P}^\dagger}(\mathcal{F}/\mathcal{J}^n\mathcal{F})$ are concentrated in degree zero by the induction hypothesis. □

Proposition 4.5 [Scheider 2014, Theorem 2.2.12]. *The functor*

$$\mathrm{U}_n(\mathcal{O}_{E^\dagger})\xrightarrow{\sim}\mathrm{U}_n^\dagger(E/S),\quad \mathcal{F}\mapsto\hat{\mathcal{F}}^\dagger:=H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}))$$

is a well-defined equivalence of categories.

Proof. Again, we follow Scheider's proof [2014, Theorem 2.2.12]. Let us first prove the well-definedness by induction over n. For $n=0$, Lemma 4.4(a) shows that the functor is well-defined. For $n>0$ an object $\mathcal{F}\in\mathrm{U}_n(\mathcal{O}_{E^\dagger})$ fits in an exact sequence

$$0\to\mathcal{J}^n\mathcal{F}\to\mathcal{F}\to\mathcal{F}/\mathcal{J}^n\mathcal{F}\to 0$$

with $\mathcal{J}^n\mathcal{F}\in\mathrm{U}_0(\mathcal{O}_{E^\dagger})$ and $\mathcal{F}/\mathcal{J}^n\mathcal{F}\in\mathrm{U}_{n-1}(\mathcal{O}_{E^\dagger})$. We have seen in Lemma 4.4(b) that the Fourier–Mukai transform of each term in the above exact sequence is concentrated in degree 0, thus we get the short exact sequence

$$0\to H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{J}^n\mathcal{F}))\to H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}))\to H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}/\mathcal{J}^n\mathcal{F}))\to 0.$$

By the induction hypothesis, $H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{J}^n\mathcal{F}))\in\mathrm{U}_0^\dagger(E/S)$, $H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}/\mathcal{J}^n\mathcal{F}))\in\mathrm{U}_{n-1}^\dagger(E/S)$ and we deduce $H^0(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}))\in\mathrm{U}_n^\dagger(E/S)$. This proves the well-definedness.

Since $\Phi_{\mathcal{P}^\dagger}$ is an equivalence and all considered sheaves are concentrated in cohomological degree zero, we deduce that $\mathrm{U}_n(\mathcal{O}_{E^\dagger}) \to \mathrm{U}_n^\dagger(E/S)$ is fully faithful. The proof that $\widehat{(\cdot)}^\dagger$ is essentially surjective proceeds again by induction over n. For $n = 0$, we have $\mathcal{F} = (e_{E^\dagger})_* \mathcal{G}$ for some locally free $\mathcal{O}_S$-module and the explicit formula $\hat{\mathcal{F}}^\dagger = \pi^* \mathcal{G}$ establishes the case $n = 0$. Let $n > 0$ and assume that we already know that $\widehat{(\cdot)}^\dagger : \mathrm{U}_m(\mathcal{O}_{E^\dagger}) \to \mathrm{U}_m^\dagger(E/S)$ is essentially surjective for $m < n$. Every unipotent sheaf $\mathcal{U} \in \mathrm{U}_n^\dagger(E/S)$ sits in an exact sequence

$$0 \to A^1\mathcal{U} \to \mathcal{U} \to \pi^*\mathcal{Y} \to 0$$

for some locally free $\mathcal{O}_S$-module $\mathcal{Y}$. By the induction hypothesis there exists $\mathcal{F}' \in \mathrm{U}_{n-1}(\mathcal{O}_{E^\dagger})$ and $\mathcal{F}'' \in \mathrm{U}_0(\mathcal{O}_{E^\dagger})$ with $\widehat{(\mathcal{F}')}^\dagger \cong A^1\mathcal{U}$ and $\widehat{(\mathcal{F}'')}^\dagger \cong \pi^*\mathcal{Y}$, i.e.,

$$0 \to \widehat{(\mathcal{F}')}^\dagger \to \mathcal{U} \to \widehat{(\mathcal{F}'')}^\dagger \to 0.$$

This gives us a distinguished triangle

$$\Phi_{\mathcal{P}^\dagger}(\mathcal{F}') \to \mathcal{U} \to \Phi_{\mathcal{P}^\dagger}(\mathcal{F}') \to \Phi_{\mathcal{P}^\dagger}(\mathcal{F}')[1] \tag{8}$$

in $D^b_{qc}(\mathcal{D}_{E/S})$. Let us denote by $\Phi^{-1}_{\mathcal{P}^\dagger}$ an quasiinverse of the equivalence $\Phi_{\mathcal{P}^\dagger}$. Applied to (8) this yields a triangle in $D^b_{qc}(\mathcal{O}_{E^\dagger})$:

$$\Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}')) \to \Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U}) \to \Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}')) \to \Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}'))[1].$$

Since $\Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}')) \cong \mathcal{F}'$ and $\Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}'')) \cong \mathcal{F}''$ we may replace $\Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}'))$ and $\Phi^{-1}_{\mathcal{P}^\dagger}(\Phi_{\mathcal{P}^\dagger}(\mathcal{F}''))$ by $\mathcal{F}'$ and $\mathcal{F}''$, respectively, in the above triangle:

$$\mathcal{F}' \to \Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U}) \to \mathcal{F}'' \to \mathcal{F}'[1].$$

Because $\mathcal{F}'$ and $\mathcal{F}''$ are concentrated in degree zero we deduce that $\Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U})$ is concentrated in degree zero. Applying H^0 gives

$$0 \to \mathcal{F}' \to H^0(\Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U})) \to \mathcal{F}'' \to 0$$

As $\mathcal{F}' \in \mathrm{U}_{n-1}(\mathcal{O}_{E^\dagger})$ and $\mathcal{F}'' \in \mathrm{U}_0(\mathcal{O}_{E^\dagger})$ we get $\mathcal{F} := H^0(\Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U})) \in \mathrm{U}_n^\dagger(E/S)$. Applying $\widehat{(\cdot)}^\dagger$ gives

$$0 \to \widehat{(\mathcal{F}')}^\dagger \to \widehat{(\mathcal{F})}^\dagger \to \widehat{(\mathcal{F}'')}^\dagger \to 0.$$

Using once again, that $\Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U})$ is concentrated in degree zero, we deduce

$$\widehat{\mathcal{F}}^\dagger = H^0(\Phi_{E^\dagger}(H^0(\Phi^{-1}_{\mathcal{P}^\dagger}(\mathcal{U})))) \cong \mathcal{U}.$$

This proves that $\mathcal{U}$ is in the essential image of $\widehat{(\cdot)}^\dagger$ and concludes the induction step of the essential surjectivity of $\widehat{(\cdot)}^\dagger$. □

Finally, let us state the following result:

Proposition 4.6 [Scheider 2014, Prop. 2.2.16]. *For $\mathcal{F} \in \mathrm{U}_n(\mathcal{O}_{E^\dagger})$ there is a canonical isomorphism*

$$e^*\widehat{\mathcal{F}}^\dagger \xrightarrow{\sim} (\pi_n)_*\mathcal{F}.$$

Here, $\pi_n : \mathrm{Inf}^n_e E^\dagger \to S$ *is the structure morphism of* $\mathrm{Inf}^n_e E^\dagger$.

Proof. For the convenience of the reader, let us recall Scheider's proof. Let us view $\mathcal{F} \in \mathrm{U}_n(\mathcal{O}_{E^\dagger})$ as an $\mathcal{O}_{\mathrm{Inf}^n_e E^\dagger}$-module. The base change isomorphism associated to the diagram

$$\begin{array}{ccc} E \times_S \mathrm{Inf}^n E^\dagger & \xrightarrow{\mathrm{id} \times \iota_n} & E \times_S E^\dagger \\ \big\downarrow{\scriptstyle \mathrm{pr}_{\mathrm{Inf}^n_e E^\dagger}} & & \big\downarrow{\scriptstyle \mathrm{pr}_E} \\ \mathrm{Inf}^n E^\dagger & \xrightarrow{\iota_n} & E \end{array}$$

gives an horizontal isomorphism

$$\widehat{\mathcal{F}}^\dagger \cong (\mathrm{pr}_E)_*(\mathrm{pr}^*_{\mathrm{Inf}^n E^\dagger} \mathcal{F} \otimes_{\mathcal{O}_{E \times_S \mathrm{Inf}^n E^\dagger}} (\mathrm{id} \times \iota_n)^* \mathcal{P}^\dagger).$$

Again, by base change along the diagram

$$\begin{array}{ccc} \mathrm{Inf}^n E^\dagger & \xrightarrow{\pi_n} & S \\ \big\downarrow{\scriptstyle e \times \mathrm{id}} & & \big\downarrow{\scriptstyle e} \\ E \times_S \mathrm{Inf}^n E^\dagger & \xrightarrow{\mathrm{pr}_E} & E \end{array}$$

we obtain the desired $\mathcal{O}_S$-linear isomorphism

$$\begin{aligned} e^*\widehat{\mathcal{F}}^\dagger &\cong e^*(\mathrm{pr}_E)_*(\mathrm{pr}^*_{\mathrm{Inf}^n E^\dagger} \mathcal{F} \otimes_{\mathcal{O}_{E \times_S E^\dagger}} (\mathrm{id} \times \iota_n)^* \mathcal{P}^\dagger) \\ &\cong (\pi_n)_*(e \times \mathrm{id})^*(\mathrm{pr}^*_{\mathrm{Inf}^n E^\dagger} \mathcal{F} \otimes_{\mathcal{O}_{E \times_S E^\dagger}} (\mathrm{id} \times \iota_n)^* \mathcal{P}^\dagger) \\ &\cong (\pi_n)_*\mathcal{F}. \end{aligned}$$

□

The geometric logarithm sheaves as de Rham logarithm sheaves. The aim of this section is to show that the geometric logarithm sheaves $\mathcal{L}^\dagger_n$ give us a concrete geometric realization of the abstractly defined de Rham logarithm sheaves. This is one of the main results of Scheider [2014, Theorem 2.3.1]. By working with the universal property of the logarithm sheaves instead of its extension class, we can give a much simpler proof then the original one. The idea of our proof is quite simple: the pair $(\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}, 1)$ is initial in the category consisting of pairs $(\mathcal{F}, s)$ of unipotent $\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}$ modules $\mathcal{F} \in \mathrm{U}_n(\mathcal{O}_{E^\dagger})$ with a marked section $s \in \Gamma(S, (\pi_n)_*\mathcal{F})$. Thus, the Fourier–Mukai transform of $(\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}, 1)$ should be initial in the corresponding category of unipotent vector bundles (with a marked section) through the equivalence of categories:

$$\mathrm{U}_n(\mathcal{O}_{E^\dagger}) \xrightarrow{\sim} \mathrm{U}^\dagger_n(E/S).$$

Now recall, that being initial in the category of unipotent vector bundles with a marked section is exactly the universal property of the logarithm sheaves. Finally, the formula

$$(\widehat{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}})^\dagger = (\mathrm{pr}_E)_*((\mathrm{pr}_{E^\dagger})^*\mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} \otimes_{\mathcal{O}_{E\times E^\dagger}} \mathcal{P}^\dagger) = (\mathrm{pr}_E)_*(\mathrm{id}\times\iota_n^\dagger)^*\mathcal{P}^\dagger \overset{\mathrm{Def.}}{=} \mathcal{L}_n^\dagger,$$

which has already been observed by Scheider, allows us to conclude that the geometric logarithm sheaves satisfy the universal property of the abstractly defined logarithm sheaves.

In order to make this argument work, we have to be more precise: First, let us observe that

$$e^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger} = \mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}$$

gives us a canonical section $1 \in \Gamma(S, \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger}) \cong \Gamma(S, e^*\mathcal{L}_n^\dagger)$ of $e^*\mathcal{L}_n^\dagger$. Furthermore, let us observe that the connection $\nabla_{\mathcal{L}_n^\dagger}$ is a connection relative S, while the connection $\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}$ is an absolute connection. In order to compare both objects, we have to restrict the connection $\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}$ relative S. Let us define $\nabla_{\mathrm{Log}^n_{\mathrm{dR}},E/S} := \mathrm{res}_S(\nabla_{\mathrm{Log}^n_{\mathrm{dR}}})$. Now, we can state one of the main results of Scheider's PhD thesis:

Theorem 4.7 [Scheider 2014, Theorem 2.3.1]. *There is a unique horizontal isomorphism*

$$(\mathrm{Log}^n_{\mathrm{dR}}, \nabla_{\mathrm{Log}^n_{\mathrm{dR}},E/S}) \to (\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger})$$

mapping $\mathbb{1}^{(n)}$ *to* 1 *after pullback along* e.

The proof of is long and involved. Scheider compares the extension classes of both tuples. It is much simpler to deduce this isomorphism by proving that both objects satisfy the same universal property, or stated differently that they are initial in the same category. Since we already know the universal property of the abstractly defined logarithm sheaves, it remains to show that the geometric logarithm sheaves satisfy the same universal property.

Theorem 4.8. *The tuple* $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger}, 1)$ *is the unique tuple, up to unique isomorphism, consisting of an object* $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger}) \in \mathrm{U}_n^\dagger(E/S)$ *and a section*

$$1 \in \Gamma(S, e^*\mathcal{L}_n) = \Gamma(S, \mathcal{O}_{\mathrm{Inf}^n_e E^\dagger})$$

such that the following universal property holds: For all $\mathcal{G} \in \mathrm{U}_n^\dagger(E/S)$ *the map*

$$\pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathcal{L}_n^\dagger, \mathcal{G}) \to e^*\mathcal{G}, \quad f \mapsto (e^*f)(1)$$

is an isomorphism of $\mathcal{O}_S$*-modules.*

Proof. Let $\mathcal{G} \in \mathrm{U}_n^\dagger(E/S)$. By the equivalence

$$\widehat{(\cdot)}^\dagger : \mathrm{U}_n(\mathcal{O}_{E^\dagger}) \xrightarrow{\sim} \mathrm{U}_n^\dagger(E/S),$$

we may assume $\mathcal{G} = \hat{\mathcal{F}}^\dagger$ for some $\mathcal{F} \in \mathrm{U}_n(\mathcal{O}_{E^\dagger})$. Then, we have the following chain of isomorphisms

$$\begin{aligned}\pi_*\underline{\mathrm{Hom}}_{\mathrm{U}_n^\dagger(E/S)}(\widehat{\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}}^\dagger, \mathcal{G}) &\overset{(A)}{\cong} (\pi_n)_*\underline{\mathrm{Hom}}_{\mathrm{U}_n(\mathcal{O}_{E^\dagger})}(\mathcal{O}_{E^\dagger}/\mathcal{J}^{n+1}, \mathcal{F}) \\ &= (\pi_n)_*\mathcal{F} \\ &\overset{(B)}{\cong} e^*\hat{\mathcal{F}}^\dagger \\ &= e^*\mathcal{G}\end{aligned}$$

where (A) is induced by the Fourier–Mukai type equivalence Proposition 4.5 and (B) is Proposition 4.6. It is straightforward to check that this chain of isomorphisms sends f to $(e^*f)(1)$. This proves the universal property of $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger}, 1)$. □

As a corollary, we can deduce the following result about absolute connections:

Corollary 4.9 [Scheider 2014, Corollary 2.3.2]. *There exists a unique K-connection $\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}$ on $\mathcal{L}_n^\dagger$, such that*

(a) $\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}$ *restricts to the S-connection $\nabla_{\mathcal{L}_n^\dagger}$, i.e.,* $\mathrm{res}_S(\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}) = \nabla_{\mathcal{L}_n^\dagger}$ *and*
(b) $(\mathcal{L}_n^\dagger, \nabla^{abs}_{\mathcal{L}_n^\dagger}, 1)$ *satisfies the universal property of the absolute n-th de Rham logarithm sheaf stated in Proposition 2.4.*

Proof. Uniqueness follows immediately from the universal property in (b). For the existence of $\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}$ let us consider n-th (absolute) de Rham logarithm sheaves $((\mathrm{Log}^n_{\mathrm{dR}}, \nabla_{\mathrm{Log}^n_{\mathrm{dR}}}), \mathbb{1}^{(n)})$. The forgetful functor

$$\mathrm{res}_S \colon \mathrm{U}_n^\dagger(E/S/K) \to \mathrm{U}_n^\dagger(E/S)$$

gives an object $(\mathrm{Log}^n_{\mathrm{dR}}, \mathrm{res}_S(\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}), \mathbb{1}^{(n)})$ in the category $\mathrm{U}_n^\dagger(E/S)$ satisfying the following universal property (by Proposition 2.4): for all $\mathcal{G} \in \mathrm{U}_n^\dagger(E/S)$ the map

$$\pi_*\underline{\mathrm{Hom}}^{\mathrm{hor}}_{E/S}(\mathrm{Log}^n_{\mathrm{dR}}, \mathcal{G}) \to e^*\mathcal{G}, \quad f \mapsto (e^*f)(\mathbb{1}^{(n)})$$

is an isomorphism of $\mathcal{O}_S$-modules. Since $((\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger}), 1)$ satisfies the same universal property (see Theorem 4.8), there is a unique isomorphism

$$\alpha : \mathcal{L}_n^\dagger \xrightarrow{\sim} \mathrm{Log}^n_{\mathrm{dR}}$$

which is S-horizontal, i.e., $\alpha^*\,\mathrm{res}_S(\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}) = \nabla_{\mathcal{L}_n^\dagger}$ and satisfies $(e^*\alpha)(1) = \mathbb{1}^{(n)}$. Let us define $\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger} := \alpha^*\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}$. With this definition α provides an isomorphism

$$(\mathcal{L}_n^\dagger, \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}, 1) \xrightarrow{\sim} ((\mathrm{Log}^n_{\mathrm{dR}}, \nabla_{\mathrm{Log}^n_{\mathrm{dR}}}), \mathbb{1}^{(n)})$$

in the category $\mathrm{U}_n^\dagger(E/S/K)$. In particular, the tuple $(\mathcal{L}_n^\dagger, \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}, 1)$ satisfies the universal property of the n-th de Rham logarithm sheaf, i.e., (b) holds. (a) follows from the formula

$$\mathrm{res}_S(\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}) = \mathrm{res}_S(\alpha^*\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}) = \alpha^*\,\mathrm{res}_S(\nabla_{\mathrm{Log}^n_{\mathrm{dR}}}) = \nabla_{\mathcal{L}_n^\dagger}.$$ □

Extension classes and the Kodaira–Spencer map. Simultaneously with the geometric logarithm sheaves $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger})$, we introduced a variant $\mathcal{L}_n$ of geometric logarithm sheaves without a connection. One might ask about a similar universal property for $\mathcal{L}_n$. Indeed, let us define $U_n(E/S)$ as the full subcategory of the category of vector bundles consisting of unipotent objects of length n for E/S, i.e., there is a n-step filtration with graded pieces of the form $\pi^*\mathcal{G}$ for some vector bundle $\mathcal{G}$ on S. The pullback $e^*\mathcal{L}_n = \mathcal{O}_{\mathrm{Inf}_e^n E^\vee}$ is equipped with a distinguished section $1 \in \Gamma(S, \mathcal{O}_{\mathrm{Inf}_e^n E^\vee})$.

Theorem 4.10. *The pair $(\mathcal{L}_n, 1)$ is the unique pair, up to unique isomorphism, consisting of a unipotent vector bundle of length n for E/S and a section*

$$1 \in \Gamma(S, e^*\mathcal{L}_n) = \Gamma(S, \mathcal{O}_{\mathrm{Inf}_e^n E^\vee})$$

such that the following universal property holds: For all $\mathcal{U} \in \mathrm{U}_n(E/S)$ the map

$$\pi_*\underline{\mathrm{Hom}}_{\mathcal{O}_E}(\mathcal{L}_n, \mathcal{G}) \to e^*\mathcal{G}, \quad f \mapsto (e^*f)(1)$$

is an isomorphism of $\mathcal{O}_S$-modules.

Proof. The same proof as in Theorem 4.8 works if one replaces the Fourier–Mukai transform of Laumon by the classical Fourier–Mukai transform. □

Let us recall that $\mathcal{L}^1$ sits in a short exact sequence

$$0 \to \pi^*\underline{\omega}_{E^\vee/S} \to \mathcal{L}^1 \to \mathcal{O}_E \to 0$$

and that the rigidification of the Poincaré bundle induces a trivializing isomorphism along the zero section $\mathrm{triv}_e \colon e^*\mathcal{L}_1 \xrightarrow{\sim} \mathcal{O}_S \oplus \underline{\omega}_{E^\vee/S}$. An immediate reformulation of the above universal property of $\mathcal{L}^1$ is the following:

Corollary 4.11. *Let E/S be an elliptic curve, M a locally free $\mathcal{O}_S$-module of finite rank and let $(\mathcal{F}, \sigma)$ be a pair consisting of an extension*

$$\mathcal{F} : 0 \to \pi^*M \to F \to \mathcal{O}_E \to 0$$

together with a splitting of $e^*\mathcal{F}$, i.e., σ is an isomorphism $e^*F \xrightarrow{\sim} \mathcal{O}_S \oplus M$ which is compatible with the extension structure. Then, there is a unique morphism

$$\varphi : \underline{\omega}_{E^\vee/S} \to M$$

such that the pair $(\mathcal{F}, \sigma)$ is the pushout of the pair $(\mathcal{L}_1, \mathrm{triv}_e)$ along φ.

Proof. The pair $(\mathcal{F}, \sigma)$ induces a pair (F, s) with $F \in U_1(E/S)$ and $s := \sigma^{-1}(1, 0) \in \Gamma(S, e^*F)$. By the universal property of $\mathcal{L}^1$, there is a unique morphism $f : \mathcal{L}_1 \to F$ which identifies $1 \in \Gamma(S, e^*\mathcal{L}_1)$ with $s \in \Gamma(S, e^*F)$. The pushout of

$$\mathcal{F} : 0 \to \pi^*M \to F \to \mathcal{O}_E \to 0$$

along $\varphi := (e^* f)|_{\underline{\omega}_{E^\vee/S}} : \underline{\omega}_{E^\vee/S} \to M$ is isomorphic to $(\mathcal{F}, \sigma)$. Uniqueness follows from the rigidity of extensions with a fixed splitting along the pullback e^*. □

An interesting application of this corollary relates $\mathcal{L}_1$ to the absolute Kähler differentials of the universal elliptic curve. Let $E \to S$ be an elliptic curve and $S \to T$ be a smooth morphism. We have the following fundamental short exact sequences of Kähler differentials:

$$0 \to \pi^*\Omega^1_{S/T} \to \Omega^1_{E/T} \to \Omega^1_{E/S} \to 0 \tag{9}$$

and

$$0 \to \mathcal{I}/\mathcal{I}^2 \to e^*\Omega^1_{E/T} \to \Omega^1_{S/T} \to 0$$

with $\mathcal{I}$ the ideal sheaf defining the zero section $e : S \to E$. The second short exact sequence induces a canonical splitting

$$e^*\Omega^1_{E/T} \xrightarrow{\sim} \Omega^1_{S/T} \oplus e^*\Omega^1_{E/S}$$

of the pullback of (9) along e. The pullback of the Kodaira–Spencer map

$$\underline{\omega}_{E/S} \otimes \underline{\omega}_{E^\vee/S} \to \Omega^1_{S/T}$$

along $\pi : E \to S$ induces a map

$$\mathrm{ks} : \Omega^1_{E/S} \otimes_{\mathcal{O}_E} \pi^*\underline{\omega}_{E^\vee/S} \to \pi^*\Omega^1_{S/T}.$$

Corollary 4.12. *The short exact sequence of Kähler differentials*

$$0 \to \pi^*\Omega^1_{S/T} \to \Omega^1_{E/T} \to \Omega^1_{E/S} \to 0$$

is the pushout of $\mathcal{L}^1 \otimes_{\mathcal{O}_E} \Omega^1_{E/S}$ *along the Kodaira–Spencer map* ks.

Proof. Assume, we have a short exact sequence

$$\mathcal{F} : 0 \to \pi^*M \to F \to \mathcal{O}_E \to 0$$

together with a splitting σ of $e^*\mathcal{F}$ as in Corollary 4.11. Then we can describe the unique map

$$\varphi : \underline{\omega}_{E^\vee/S} \to M$$

whose existence is guaranteed by Corollary 4.11, as the image of 1 under the connecting morphism

$$\delta : \mathcal{O}_E \to R^1\pi_*(\pi^*M) = \mathrm{Hom}_{\mathcal{O}_S}(\underline{\omega}_{E^\vee/S}, M).$$

Now the result follows from the definition of the Kodaira–Spencer map [Faltings and Chai 1990, page 80]: the Kodaira–Spencer map is the image of 1 under the connecting homomorphism of the short exact sequence

$$0 \to \pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_E} (\Omega^1_{E/S})^\vee \to \Omega^1_{E/T} \otimes_{\mathcal{O}_E} (\Omega^1_{E/S})^\vee \to \mathcal{O}_E \to 0 \tag{10}$$

obtained by tensoring the short exact sequence of Kähler differentials with $\otimes_{\mathcal{O}_E}(\Omega^1_{E/S})^\vee$. □

In particular, the pushout along the Kodaira–Spencer map induces a map

$$\mathrm{KS}\colon \mathcal{L}_1 \otimes_{\mathcal{O}_E} \Omega^1_{E/S} \to \Omega^1_{E/T}. \tag{11}$$

If E/S is the universal elliptic curve over the modular curve (with some level structure) then KS is an isomorphism. In particular, $\mathcal{L}_1$ is essentially given by the absolute Kähler differentials.

5. The de Rham realization of the elliptic polylogarithm

The aim of this section is to give an explicit algebraic description of the de Rham realization of the elliptic polylogarithm in terms of the Kronecker section of the Poincaré bundle for arbitrary families of elliptic curves E/S over a smooth separated K-scheme S of finite type. From now on, we will use Scheider's explicit description of the de Rham logarithm sheaves and fix $(\mathcal{L}_n^\dagger, \nabla^{abs}_{\mathcal{L}_n^\dagger}, 1)$ as an explicit model for the de Rham logarithm sheaves (see Corollary 4.9).

The Kronecker section of the geometric logarithm sheaves. Recall that the dual elliptic curve $E^\vee$ represents the connected component of the functor

$$T \mapsto \mathrm{Pic}(E_T/T) := \{\text{isomorphism classes of rigidified line bundles } (\mathcal{L}, r) \text{ on } E_T/T\}$$

on the category of S-schemes. The polarization associated to the ample line bundle $\mathcal{O}_E([e])$ gives us an explicit autoduality isomorphism

$$\begin{aligned} \lambda : E &\to \underline{\mathrm{Pic}}^0_{\mathrm{E/S}} =: E^\vee \\ P &\mapsto (\mathcal{O}_E([-P]-[e]) \otimes_{\mathcal{O}_E} \pi^* e^* \mathcal{O}_E([-P]-[e])^{-1}, \mathrm{can}) \end{aligned} \tag{12}$$

Here, can is the canonical rigidification given by the canonical isomorphism

$$e^*\mathcal{O}_E([-P]-[e]) \otimes_{\mathcal{O}_S} e^*\mathcal{O}_E([-P]-[e])^{-1} \xrightarrow{\sim} \mathcal{O}_S.$$

With this chosen identification of E and $E^\vee$ we can describe the birigidified Poincaré bundle as follows

$$(\mathcal{P}, r_0, s_0) := (\mathrm{pr}_1^* \mathcal{O}_E([e])^{\otimes -1} \otimes \mathrm{pr}_2^* \mathcal{O}_E([e])^{\otimes -1} \otimes \mu^* \mathcal{O}_E([e]) \otimes \pi^*_{E\times E} \underline{\omega}^{\otimes -1}_{E/S}, r_0, s_0).$$

Here, $\Delta = \ker(\mu : E \times E^\vee \to E)$ is the antidiagonal and r_0, s_0 are the rigidifications induced by the canonical isomorphism

$$e^*\mathcal{O}_E(-[e]) \xrightarrow{\sim} \underline{\omega}_{E/S}.$$

This description of the Poincaré bundle gives the following isomorphisms of locally free $\mathcal{O}_{E\times E}$-modules, i.e., all tensor products over $\mathcal{O}_{E\times E}$:

$$\begin{aligned} \mathcal{P} \otimes \mathcal{P}^{\otimes -1} &= \mathcal{P} \otimes (\mathcal{O}_{E\times E}(-[e\times E] - [E\times e] + \Delta) \otimes \pi^*_{E\times E} \omega^{\otimes -1}_{E/S})^{\otimes -1} \\ &\cong \mathcal{P} \otimes \Omega^1_{E\times E/E}([e\times E] + [E\times e]) \otimes \mathcal{O}_{E\times E}(-\Delta) \end{aligned} \tag{13}$$

The line bundle $\mathcal{O}_{E\times E}(-\Delta)$ can be identified with the ideal sheaf $\mathcal{J}_\Delta$ of the antidiagonal Δ in $E\times_S E$ in a canonical way. If we combine the inclusion

$$\mathcal{O}_{E\times E}(-\Delta)\cong\mathcal{J}_\Delta\hookrightarrow\mathcal{O}_{E\times E}$$

with (13), we get a morphism of $\mathcal{O}_{E\times E}$-modules

$$\mathcal{P}\otimes\mathcal{P}^{\otimes -1}\hookrightarrow\mathcal{P}\otimes\Omega^1_{E\times E/E}([e\times E]+[E\times e]). \tag{14}$$

Recall, that we agreed to identify E with its dual with the canonical polarization associated to the line bundle $\mathcal{O}_E([e])$. With this identification, the *Kronecker section*

$$s_{\mathrm{can}}\in\Gamma(E\times_S E^\vee,\mathcal{P}\otimes_{\mathcal{O}_{E\times E^\vee}}\Omega^1_{E\times E^\vee/E^\vee}([e\times E^\vee]+[E\times e]))$$

is then defined as the image of the identity section $\mathrm{id}_{\mathcal{P}}\in\Gamma(E\times E,\mathcal{P}\otimes\mathcal{P}^{\otimes -1})$ under (14). The universal property of the Poincaré bundle gives us a canonical isomorphisms for $D>1$:

$$\gamma_{1,D}\colon(\mathrm{id}\times[D])^*\mathcal{P}\xrightarrow{\sim}([D]\times\mathrm{id})^*\mathcal{P}$$

Let us define the *D-variant of the Kronecker section* by

$$s^D_{\mathrm{can}}:=D^2\cdot\gamma_{1,D}((\mathrm{id}\times[D])^*(s_{\mathrm{can}}))-([D]\times\mathrm{id})^*(s_{\mathrm{can}}).$$

This is a priori an element in

$$\Gamma(E\times_S E^\vee,([D]\times\mathrm{id})^*[\mathcal{P}\otimes\Omega^1_{E\times E^\vee/E^\vee}([E\times E^\vee[D]]+[E[D]\times E^\vee])]),$$

but below, we will see that it is actually contained in

$$\Gamma(E\times_S E^\vee,([D]\times\mathrm{id})^*[\mathcal{P}\otimes\Omega^1_{E\times E^\vee/E^\vee}([E\times(E^\vee[D]\setminus\{e\})]+[E[D]\times E^\vee])]).$$

In other words, passing from the Kronecker section to its D-variant removes a pole along the divisor $E\times e$. For the proof, we will need the distribution relation of the Kronecker section: For a torsion section $t\in E[D](S)$, let us define the translation operator

$$\mathcal{U}^D_t:=\gamma_{1,D}\circ(\mathrm{id}\times T_t)^*\gamma^{-1}_{1,D}\colon([D]\times T_t)^*\mathcal{P}\to([D]\times\mathrm{id})^*\mathcal{P}.$$

For a section $f\in\Gamma(E\times E^\vee,\mathcal{P}\otimes_{\mathcal{O}_{E\times E^\vee/E^\vee}}\Omega^1_{E\times E^\vee/E^\vee}(E\times e+e\times E]))$ let us introduce the notation

$$U^D_t(f):=(\mathcal{U}^D_t\otimes\mathrm{id}_{\Omega^1})(([D]\times T_t)^*(f)).$$

With this notation, the distribution relation of the Kronecker section reads in the convenient form:

Proposition 5.1 (distribution relation). *For an elliptic curve E/S with D invertible on S and $|E[D](S)|=D^2$ we have*

$$\sum_{e\neq t\in E[D](S)}U^D_t(s_{\mathrm{can}})=s^D_{\mathrm{can}}.$$

Proof. Let us refer to [Sprang 2018, Corollary A.3] for the proof of the distribution relation. ☐

The distribution relation shows that passing to the D-variant of the Kronecker section removes a pole along the divisor $E \times e$.

Corollary 5.2. *The D-variant of the Kronecker section is contained in*

$$\Gamma(E \times_S E^\vee, ([D] \times \mathrm{id})^*[\mathcal{P} \otimes \Omega^1_{E \times E^\vee / E^\vee}([E \times (E^\vee[D] \setminus \{e\})] + [E[D] \times E^\vee])]).$$

Proof. By definition of the translation operators U_t^D, we get

$$U_t^D(s_{\mathrm{can}}) \in \Gamma(E \times_S E^\vee, ([D] \times \mathrm{id})^*(\mathcal{P} \otimes \Omega^1_{E \times_S E^\vee / E^\vee}([E \times (-t)] + [e \times E^\vee]))).$$

Now the result follows from the distribution relation. □

The rigidification $(\mathrm{id} \times e)^*\mathcal{P} \cong \mathcal{O}_E$ of the Poincaré bundle gives us the identification

$$([D] \times e)^*(\mathcal{P} \otimes \Omega^1_{E \times_S E^\vee / E^\vee}([E \times (-t)] + [e \times E^\vee])) \cong [D]^*\Omega^1_{E/S}([e]) \cong \Omega^1_{E/S}(E[D]).$$

Using this identification, we get $(\mathrm{id} \times e)^* s_{\mathrm{can}} \in \Gamma(E, \Omega^1_{E/S}(E[D]))$. This allows us to identify the 1-form $(\mathrm{id} \times e)^* s_{\mathrm{can}}$ with the logarithmic derivative of the Kato–Siegel function, which has been introduced by Kato [2004, Proposition 1.3]. Let us briefly recall the definition of the Kato–Siegel function.

Proposition 5.3 [Kato 2004, Proposition 1.3]. *Let E be an elliptic curve over a base scheme S and D be an integer which is prime to* 6. *There exists a unique section ${}_D\theta \in \Gamma(E \setminus E[D], \mathcal{O}_E^\times)$ satisfying the following conditions*:

(a) *${}_D\theta$ has divisor $D^2[e] - [E[D]]$.*

(b) *$\mathrm{N}_M({}_D\theta) = {}_D\theta$ for every integer M which is prime to D. Here $\mathrm{N}_M \colon \Gamma(E \setminus E[MD], \mathcal{O}_E) \to \Gamma(E \setminus E[D], \mathcal{O}_E)$ denotes the norm map along M.*

The function ${}_D\theta$ is called Kato–Siegel *function.*

The Kato–Siegel functions play an important role in modern number theory. While their values at torsion points are elliptic units, the logarithmic derivatives $\mathrm{d}\log_D \theta \in \Gamma(E, \mathcal{O}_E(E[D]))$ are closely related to algebraic Eisenstein series. These two properties make them an important tool for proving explicit reciprocity laws for twists of Tate modules of elliptic curves, see for example [Kato 1993; Tsuji 2004]. Let us remark, that the logarithmic derivatives of the Kato–Siegel functions appear as specializations of the Kronecker section.

Proposition 5.4 [Sprang 2018, Corollary 5.7]. *The section*

$$(\mathrm{id} \times e)^* s^D_{\mathrm{can}} \in \Gamma(E, \Omega^1_{E/S}(E[D]))$$

coincides with the logarithmic derivative of the Kato–Siegel function, *i.e.*,

$$(\mathrm{id} \times e)^* s^D_{\mathrm{can}} = \mathrm{d}\log_D \theta.$$

Proof. In [Sprang 2018, Corollary 5.7] we proved

$$\mathrm{d}\log_D \theta = \sum_{e \neq t \in E[D](S)} (\mathrm{id} \times e)^*(U_t^D(s_{\mathrm{can}})).$$

Now, the result follows from the distribution relation. □

By restricting s_{can}^D along $E \times \mathrm{Inf}_e^n E^\vee$ we obtain a compatible system of 1-forms with values in the logarithm sheaves, more precisely the canonical isomorphism $[D]^*\Omega^1_{E/S}([e]) \xrightarrow{\sim} \Omega^1_{E/S}(E[D])$ tensored with the invariance under isogenies isomorphism (see (7)) gives

$$[D]^*(\mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}([e])) \xrightarrow{\sim} \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D]). \tag{15}$$

Definition 5.5. Define the *n-th infinitesimal Kronecker section*

$$l_n^D \in \Gamma(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D]))$$

as the image of $(\mathrm{pr}_E)_*(\mathrm{id} \times \iota_n)^*(s_{\mathrm{can}}^D) \in \Gamma(E, [D]^*[\mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}([e])])$ under (15).

By construction the l_n^D form a compatible system of sections with respect to the transition maps of the geometric logarithm sheaves. By abuse of notation, we will denote the image of l_n^D under the canonical inclusion

$$\mathcal{L}_n \hookrightarrow \mathcal{L}_n^\dagger$$

again by l_n^D.

Lifting the infinitesimal Kronecker sections. By the results of Scheider, we already have a good explicit model of the de Rham logarithm sheaves in terms of the Poincaré bundle. Our next aim is to give an explicit description of the elliptic de Rham polylogarithm in terms of the Poincaré bundle. Therefore, we would like to lift the infinitesimal Kronecker sections

$$l_n^D \in \Gamma(E, \mathcal{L}_n \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D])) \subseteq \Gamma(E, \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^1_{E/S}(E[D]))$$

to absolute 1-forms. In (11) we used the Kodaira–Spence map to construct a map

$$\mathrm{KS}\colon \mathcal{L}^1 \otimes_{\mathcal{O}_E} \Omega^1_{E/S} \to \Omega^1_{E/K}.$$

Further, by the universal property of $\mathcal{L}_n$ we have comultiplication maps $\mathcal{L}_{k+l} \to \mathcal{L}_k \otimes \mathcal{L}_l$ mapping 1 to $1 \otimes 1$ after e^*. If we combine the Kodaira–Spencer map with the comultiplication of the geometric logarithm sheaves, we obtain a map

$$\mathcal{L}_{n+1} \otimes \Omega^1_{E/S} \to \mathcal{L}_n \otimes \mathcal{L}_1 \otimes \Omega^1_{E/S} \to \mathcal{L}_n \otimes \Omega^1_{E/K} \tag{16}$$

lifting relative 1-forms to absolute 1-forms.

Definition 5.6. Let us define the n-th *absolute infinitesimal Kronecker section*

$$L_n^D \in \Gamma(E, \mathcal{L}_n \otimes \Omega^1_{E/S}(E[D]))$$

as the image of l_{n+1}^D under the lifting map (16).

A first step in proving that the absolute infinitesimal Kronecker sections represent the polylogarithm is the following:

Proposition 5.7. *Let us view L_n^D as section of $\mathcal{L}_n^\dagger|_{U_D} \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K}$ via the inclusion $\mathcal{L}_n \hookrightarrow \mathcal{L}_n^\dagger$. Then*

$$L_n^D \in \Gamma(U_D, \ker(\mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^1_{E/K} \xrightarrow{\mathrm{d}^{(1)}} \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_E} \Omega^2_{E/K}))$$

where $\mathrm{d}^{(1)}$ is the second differential in the absolute de Rham complex of $(\mathcal{L}_n^\dagger, \nabla_{\mathcal{L}_n^\dagger, abs})$.

Proof. The question is étale locally on the base. Indeed, for a Cartesian diagram

$$\begin{array}{ccc} E_T & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

with f finite étale we have an isomorphism

$$\tilde{f}^*(\mathcal{L}_n \otimes \Omega^1_{E/K}) \xrightarrow{\sim} \mathcal{L}_{n,E_T} \otimes \Omega^1_{E_T/K}$$

which identifies $(\tilde{f}^* L_n^D)_{n\geq 0}$ with $(L_n^D)_{n\geq 0}$. Thus, we may prove the claim after a finite étale base change. Now, choose an arbitrary $N > 3$. Since we are working over a scheme of characteristic zero, the integer N is invertible and there exists étale locally a $\Gamma_1(N)$-level structure. Again, by compatibility with base change it is enough to prove the claim for the universal elliptic curve $\mathcal{E}$ with $\Gamma_1(N)$-level structure over the modular curve $\mathcal{M}$ over K. The vanishing of $\mathrm{d}^{(1)}(L_n^D)$ can be checked after analytification. The necessary analytic computation is shifted to page 573. □

The polylogarithm class via the Poincaré bundle. The edge morphism $E_2^{1,0} \to E^1$ in the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(U_D, \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^p_{U_D/K}) \Rightarrow E^{p+q} = H^{p+q}_{\mathrm{dR}}(U_D/K, \mathcal{L}_n^\dagger)$$

induces a morphism

$$\Gamma(U_D, \ker(\mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^1_{U_D/K} \xrightarrow{\mathrm{d}^{(1)}} \mathcal{L}_n^\dagger \otimes_{\mathcal{O}_{U_D}} \Omega^2_{U_D/K})) \xrightarrow{[\cdot]} H^1_{\mathrm{dR}}(U_D/K, \mathcal{L}_n^\dagger). \quad (17)$$

We show that $\mathrm{pol}_{D,\mathrm{dR}}$ is represented by the compatible system $(L_n^D)_{n\geq 0}$ under (17).

Theorem 5.8. *The D-variant of the elliptic polylogarithm in de Rham cohomology is explicitly given by*

$$\mathrm{pol}_{D,\mathrm{dR}} = ([L_n^D])_{n\geq 0}$$

where $[L_n^D]$ is the de Rham cohomology class associated with L_n^D via (17). Here, L_n^D are the (absolute) infinitesimal Kronecker sections associated to the Kronecker section.

Proof. First let us recall that L_n^D is contained in the kernel of the first differential of the de Rham complex $\Omega^\bullet_{E/K} \otimes \mathcal{L}_n^\dagger$, see Proposition 5.7. Thus, $[L_n^D]$ is well-defined. Further, the question is étale locally on the base. Indeed, for a Cartesian diagram

$$\begin{array}{ccc} E_T & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

with f finite étale we have an isomorphism

$$\tilde{f}^*(\mathcal{L}_n^\dagger \otimes \Omega^1_{E/K}) \xrightarrow{\sim} \mathcal{L}^\dagger_{n,E_T} \otimes \Omega^1_{E_T/K}$$

which identifies $(\tilde{f}^* L_n^D)_{n\geq 0}$ with $(L_n^D)_{n\geq 0}$. Furthermore, the canonical map

$$\underline{H}^1_{\mathrm{dR}}(U_D, \mathcal{L}_n^\dagger) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(U_D \times_S T, \mathcal{L}^\dagger_{n,E_T})$$

is an isomorphism and identifies the polylogarithm classes. Thus, we may prove the claim after a finite étale base change. Now, choose an arbitrary $N > 3$. Since we are working over a scheme of characteristic zero, the integer N is invertible and there exists étale locally a $\Gamma_1(N)$-level structure. Again, by compatibility with base change it is enough to prove the claim for the universal elliptic curve $\mathcal{E}$ with $\Gamma_1(N)$-level structure over the modular curve $\mathcal{M}$. By the defining property of the polylogarithm we have to show

$$\mathrm{Res}(([L_n^D])_{n\geq 0}) = D^2 1_e - 1_{\mathcal{E}[D]}.$$

We split this into two parts:

(A) $\mathrm{Res}\, L_0^D = D^2 1_e - 1_{\mathcal{E}[D]}$.

(B) The image of $\mathrm{Res}(([L_n^D])_{n\geq 0})$ under

$$\prod_{n=0}^{\infty} H^0_{\mathrm{dR}}(\mathcal{E}[D], \underline{\mathrm{Sym}}^k\, \mathcal{H}_{E[D]}) \twoheadrightarrow \prod_{n=1}^{\infty} H^0_{\mathrm{dR}}(\mathcal{E}[D], \underline{\mathrm{Sym}}^k\, \mathcal{H}_{E[D]})$$

is zero.

(A) Since $\mathcal{M}$ is affine, the Leray spectral sequence for de Rham cohomology shows that we obtain the localization sequence for $n = 1$ by applying $H^0_{\mathrm{dR}}(\mathcal{M}, \cdot)$ to

$$0 \to \underline{H}^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{M}) \to \underline{H}^1_{\mathrm{dR}}(U_D/\mathcal{M}) \xrightarrow{\mathrm{Res}} \underline{H}^0_{\mathrm{dR}}(\mathcal{E}/\mathcal{M}).$$

This exact sequence can be obtained by applying $R\pi_*$ to the short exact sequence

$$0 \to \Omega^\bullet_{\mathcal{E}/\mathcal{M}} \to \Omega^\bullet_{\mathcal{E}/\mathcal{M}}(\mathcal{E}[D]) \xrightarrow{\mathrm{Res}} (i_{\mathcal{E}[D]})_* \mathcal{O}_{\mathcal{E}[D]}[-1] \to 0$$

of complexes. Thus, it is enough to show $\mathrm{Res}(l_0^D) = D^2 1_e - 1_{\mathcal{E}[D]}$. But we have already seen that l_0^D coincides with the logarithmic derivative of the Kato–Siegel function, see Proposition 5.4. The residue condition $\mathrm{Res}(l_0^D) = D^2 1_e - 1_{\mathcal{E}[D]}$ follows immediately by one of the defining properties of the Kato–Siegel function. This proves (A).

(B) The following lemma implies the vanishing of

$$H^0_{\mathrm{dR}}(\mathcal{E}[D], \underline{\mathrm{Sym}}^k\, \mathcal{H}_{\mathcal{E}[D]}) = H^0_{\mathrm{dR}}(\mathcal{M}, \mathrm{Sym}^k\, \mathcal{H}) = 0, \quad \text{for } k > 0.$$

So (B) holds in the universal case for trivial reasons. □

Lemma 5.9. *Let $N > 3$ and $\mathcal{E}/\mathcal{M}$ be the universal elliptic curve with $\Gamma_1(N)$-level structure over K. Then*

$$H^0_{\mathrm{dR}}(\mathcal{M}, \underline{\mathrm{Sym}}^k\, \mathcal{H}) = 0$$

for all $k \geq 1$.

Proof. One can show the vanishing of $H^0_{\mathrm{dR}}(\mathcal{M}, \underline{\mathrm{Sym}}^k\, \mathcal{H})$ after analytification. Then, the statement boils down, using the Riemann–Hilbert correspondence, to the obvious vanishing result

$$H^0(\Gamma_1(N), \underline{\mathrm{Sym}}^k\, \mathbb{Z}^2) = 0, \quad k \geq 1$$

in group cohomology. Here, $\mathbb{Z}^2$ is the regular representation of $\Gamma_1(N) \subseteq \mathrm{Sl}_2(\mathbb{Z})$ on $\mathbb{Z}^2$. □

Remark 5.10. The above Theorem gives an explicit representative $(L_n^D)_{n\geq 0}$ of the de Rham polylogarithm class. One could ask about uniqueness of this representative. We have already seen that the compatible system of sections $(L_n^D)_{n\geq 0}$ is contained in the first nontrivial step

$$F^0\mathcal{L}_n^\dagger \otimes \Omega^1_{E/K} = \mathcal{L}_n \otimes \Omega^1_{E/K}$$

of the Hodge filtration. With some more effort one can actually prove that the system $(L_n^D)_{n\geq 0}$ is the unique system in $F^0\mathcal{L}_n^\dagger \otimes \Omega^1_{E/K}$ representing the de Rham polylogarithm. Let us refer the interested reader to our PhD thesis [Sprang 2017, Proposition 5.2.12].

Proof of Proposition 5.7 via the mixed heat equation for theta functions. In this subsection we will pass to the universal elliptic curve and deduce Proposition 5.7 from the mixed heat equation

$$2\pi i \cdot \partial_\tau J(z, w, \tau) = \partial_z \partial_w J(z, w, \tau)$$

of the Jacobi theta function. Let $\mathcal{E}/\mathcal{M}$ be the universal elliptic curve over $\mathbb{Q}$ with $\Gamma_1(N)$-level structure. The complex manifolds $\mathcal{E}(\mathbb{C})$ and $\mathcal{M}(\mathbb{C})$ can be explicitly described as

$$\mathcal{E}(\mathbb{C}) = \mathbb{C} \times \mathbb{H}/\mathbb{Z}^2 \rtimes \Gamma_1(N) \twoheadrightarrow \mathcal{M}(\mathbb{C}) = \mathbb{H}/\Gamma_1(N).$$

Recall that we fixed the polarization associated with $\mathcal{O}([e])$ as autoduality isomorphism. The above explicit analytification together with this autoduality isomorphism gives

$$\mathbb{C} \times \mathbb{C} \times \mathbb{H} \twoheadrightarrow \mathcal{E}(\mathbb{C}) \times_{\mathcal{M}(\mathbb{C})} \mathcal{E}^\vee(\mathbb{C})$$

as universal covering. Let us write (z, w, τ) for the coordinates on the universal covering. Using the autoduality isomorphism from (12) we can write the rigidified Poincaré bundle on $\mathcal{E}\times_{\mathcal{M}}\mathcal{E}$ as

$$\mathcal{P}=\mathrm{pr}_1^*\,\mathcal{O}_{\mathcal{E}}([e])^{\otimes -1}\otimes \mathrm{pr}_2^*\,\mathcal{O}_{\mathcal{E}}([e])^{\otimes -1}\otimes \mu^*\mathcal{O}_{\mathcal{E}}([e])\otimes \pi^*_{\mathcal{E}\times\mathcal{E}}\underline{\omega}^{\otimes -1}_{\mathcal{E}/\mathcal{M}}.$$

Let us write $\tilde{\mathcal{P}}$ for the pullback of the analytified Poincaré bundle to the above universal covering. Then $\tilde{\mathcal{P}}$ is a trivial line bundle on the complex manifold $\mathbb{C}\times\mathbb{C}\times\mathbb{H}$ and by the above explicit description of $\mathcal{P}$ in terms of $\mathcal{O}_{\mathcal{E}}([e])$ it can be trivialized in terms of a suitable theta function. Let us choose the Jacobi theta function

$$J(z, w, \tau):=\frac{\vartheta(z+w)}{\vartheta(z)\vartheta(w)}$$

with

$$\vartheta(z):=\exp\left(\frac{z^2}{2}\eta_1(\tau)\right)\sigma(z,\tau)$$

as trivializing section of the pullback $\tilde{\mathcal{P}}$ of the Poincaré bundle to the universal covering, i.e.,

$$\mathcal{O}_{\mathbb{C}\times\mathbb{C}\times\mathbb{H}}\xrightarrow{\sim}\tilde{\mathcal{P}},\quad 1\mapsto\tilde{\mathfrak{t}}:=\frac{1}{J(z,w,\tau)}\otimes(\mathrm{d}z)^\vee.$$

The analytification of the universal vectorial extension $\mathcal{E}^\dagger$ of $\mathcal{E}^\vee$ sits in a short exact sequence (see [Mazur and Messing 1974, Chapter I, 4.4])

$$0\to R^1(\pi_{\mathcal{E}}^{an})_*(2\pi i\mathbb{Z})\to \underline{H}^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{M})\to\mathcal{E}^\dagger(\mathbb{C})\to 0$$

In particular, the pullback of the geometric vector bundle $\underline{H}^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{M})$ to the universal covering $\mathbb{H}\to\mathcal{M}(\mathbb{C})$ serves as a universal covering of $\mathcal{E}^\dagger(\mathbb{C})$. Choosing coordinates on this universal covering is tantamount to choosing a basis of

$$\underline{H}^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{M})^\vee\xrightarrow{\sim}\underline{H}^1_{\mathrm{dR}}(\mathcal{E}^\vee/\mathcal{M}). \tag{18}$$

The isomorphism (18) is canonically induced by Deligne's pairing. Let us choose the differentials of the first and second kind $\omega=[\mathrm{d}w]$ and $\eta=[\wp(w,\tau)\mathrm{d}w]$ as generators of $\underline{H}^1_{\mathrm{dR}}(\mathcal{E}^\vee/\mathcal{M})$ and denote the resulting coordinates by (w, v). We can summarize the resulting covering spaces in the following commutative diagram:

$$\begin{array}{ccc}\mathbb{C}^2\times\mathbb{H} & \xrightarrow{\mathrm{pr}_1} & \mathbb{C}\times\mathbb{H}\\ \downarrow & & \downarrow\\ \mathcal{E}^\dagger(\mathbb{C}) & \longrightarrow & \mathcal{E}^\vee(\mathbb{C})\end{array}$$

The pullback of the Poincaré bundle $\mathcal{P}^\dagger$ to $\mathcal{E}\times\mathcal{E}^\dagger$ is equipped with a canonical integrable $\mathcal{E}^\dagger$-connection

$$\nabla_\dagger:\mathcal{P}^\dagger\to\mathcal{P}^\dagger\otimes_{\mathcal{O}_{\mathcal{E}\times\mathcal{E}^\dagger}}\Omega^1_{\mathcal{E}\times\mathcal{E}^\dagger/\mathcal{E}^\dagger}.$$

Let us write $\tilde{\mathcal{P}}^\dagger$ for the pullback of the Poincaré bundle to the universal covering. The trivializing section $\tilde{\mathfrak{t}}^\dagger$ of $\tilde{\mathcal{P}}$ induces a trivializing section $\tilde{\mathfrak{t}}^\dagger$ of $\tilde{\mathcal{P}}^\dagger$ via pullback

$$\mathcal{O}_{\mathbb{C}\times\mathbb{C}^2\times\mathbb{H}} \xrightarrow{\sim} \tilde{\mathcal{P}}^\dagger, \quad 1 \mapsto \tilde{\mathfrak{t}}^\dagger. \tag{19}$$

Let us write $\tilde{\mathcal{L}}_1^\dagger$ for the pullback of $\mathcal{L}_1^\dagger := \pi_*(\mathcal{P}^\dagger|_{\mathcal{E}\times_{\mathcal{M}}\mathrm{Inf}_e^1\mathcal{E}^\dagger})$ along the universal covering

$$\tilde{\pi} : \tilde{\mathcal{E}} = \mathbb{C}\times\mathbb{H} \twoheadrightarrow \mathcal{E}(\mathbb{C}).$$

Using $\mathrm{Inf}_e^1\mathcal{E}^\dagger = \mathcal{O}_{\mathcal{M}} \oplus \underline{H}^1_{\mathrm{dR}}(\mathcal{E}^\vee/\mathcal{M})$ the trivialization (19) induces

$$\mathrm{triv} : \tilde{\mathcal{L}}_1^\dagger \xrightarrow{\sim} \mathcal{O}_{\tilde{\mathcal{E}}} \oplus \tilde{\pi}^*\underline{H}^1_{\mathrm{dR}}(\mathcal{E}^\vee/\mathcal{M}).$$

Recall, that the higher de Rham logarithm sheaves were defined by tensor symmetric powers of the first de Rham logarithm sheaf. The generators $\omega = [\mathrm{d}w]$, $\eta = [\wp(w,\tau)\mathrm{d}w]$ of $\tilde{\mathcal{H}} := \tilde{\pi}^*\underline{H}^1_{\mathrm{dR}}(\mathcal{E}^\vee/\mathcal{M})$ together with the isomorphism

$$\mathrm{triv}\colon \tilde{\mathcal{L}}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n\,\tilde{\mathcal{L}}_1^\dagger \cong \underline{\mathrm{TSym}}^n(\mathcal{O}_{\tilde{\mathcal{E}}}\oplus\tilde{\mathcal{H}}) = \bigoplus_{i=0}^n \underline{\mathrm{TSym}}^i\,\tilde{\mathcal{H}}$$

induce a decomposition

$$\mathrm{triv}\colon \tilde{\mathcal{L}}_n^\dagger \xrightarrow{\sim} \bigoplus_{i+j\le n} \tilde{\omega}^{[i,j]}\cdot\mathcal{O}_{\tilde{\mathcal{E}}}$$

with $\tilde{\omega}^{[i,j]} := \mathrm{triv}^{-1}(\omega^{[i]}\cdot\eta^{[j]})$. Here, $(\cdot)^{[i]}$ denotes the canonical divided power structure on the algebra of tensor symmetric powers.

Lemma 5.11 [Scheider 2014, (3.4.16)]. *In terms of this decomposition the connection $\nabla_{\mathcal{L}_n^\dagger}$ is given by*

$$\nabla_{\mathcal{L}_n^\dagger}(\tilde{\omega}^{[i,j]}) = -(i+1)\eta_1(\tau)\cdot\tilde{\omega}^{[i+1,j]}\otimes\mathrm{d}z + (j+1)\tilde{\omega}^{[i,j+1]}\otimes\mathrm{d}z$$

with the convention that $\tilde{\omega}^{[i,j]} = 0$ if $i+j>n$. Here, $\eta_1(\tau)$ is the "period of the second kind" $\eta_1(\tau) = \zeta(z,\tau)-\zeta(z+1,\tau)$.

Proof. The horizontality of the isomorphism

$$\mathcal{L}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n\,\mathcal{L}_1^\dagger$$

reduces us to prove the statement in the case $n=1$. Since the restriction of $\nabla_{\mathcal{L}_1^\dagger}$ to $\mathcal{H}_E$ is trivial, we get

$$\nabla_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[1,0]}) = \nabla_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[0,1]}) = 0$$

and it remains to prove

$$\nabla_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[0,0]}) = -\eta_1(\tau)\cdot\tilde{\omega}^{[1,0]}\otimes\mathrm{d}z + \tilde{\omega}^{[0,1]}\otimes\mathrm{d}z.$$

The connection $\nabla_{\mathcal{L}_1^\dagger}$ is induced from the connection $\nabla_\dagger$ on $\mathcal{P}^\dagger$. The explicit description of the connection in [Katz 1977, Theorem C.6 (1)] yields immediately the formula

$$\nabla_\dagger(\tilde{\mathfrak{t}}^\dagger) = \left[-\frac{\partial_z J(z,w)}{J(z,w)} + (\zeta(z+w) - \zeta(z) + v) \right] \tilde{\mathfrak{t}}^\dagger \otimes \mathrm{d}z$$

Using

$$\frac{\partial_z J(z,w)}{J(z,w)} = \partial_z \log J(z,w) = w \cdot \eta_1(\tau) + \zeta(z+w) - \zeta(z)$$

we get

$$\nabla_\dagger(\tilde{\mathfrak{t}}^\dagger) = (v - w \cdot \eta_1(\tau)) \tilde{\mathfrak{t}}^\dagger \otimes \mathrm{d}z.$$

Restricting this to the first infinitesimal neighborhood gives

$$\nabla_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[0,0]}) = -\eta_1(\tau) \cdot \tilde{\omega}^{[1,0]} \otimes \mathrm{d}z + \tilde{\omega}^{[0,1]} \otimes \mathrm{d}z. \qquad \square$$

Corollary 5.12. *The absolute connection on $\mathcal{L}_n^\dagger$ is given by the formula*

$$\begin{aligned}\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}(\tilde{\omega}^{[k,j]}) = (-(k+1)\eta_1(\tau) \cdot \tilde{\omega}^{[k+1,j]} + (j+1)\tilde{\omega}^{[k,j+1]}) \otimes \mathrm{d}z \\ + \left(-\frac{\eta_1(\tau)}{2\pi i} k \tilde{\omega}^{[k,j]} + \frac{1}{2\pi i}(j+1)\tilde{\omega}^{[k-1,j+1]} \right) \otimes \mathrm{d}\tau \\ + \left(\left(\partial_\tau \eta_1(\tau) - \frac{\eta_1(\tau)^2}{2\pi i} \right)(k+1)\tilde{\omega}^{[k+1,j-1]} + \frac{\eta_1(\tau)}{2\pi i} j \tilde{\omega}^{[k,j]} \right) \otimes \mathrm{d}\tau\end{aligned}$$

with the convention that $\tilde{\omega}^{[i,j]} = 0$ if $i + j > n$ or $i, j < 0$.

Proof. By the horizontality of

$$\mathcal{L}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n \mathcal{L}_1^\dagger$$

it is enough to prove the statement in the case $n = 1$. In the following, let us write $\tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}$ for the connection defined by the formulas in the statement. So, in the case $n = 1$ these formulas reduce to

$$\begin{aligned}\tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[0,0]}) &= (-\eta_1(\tau) \cdot \tilde{\omega}^{[1,0]} + \tilde{\omega}^{[0,1]}) \otimes \mathrm{d}z \\ \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[0,1]}) &= \left(\partial_\tau \eta_1(\tau) - \frac{\eta_1(\tau)^2}{2\pi i} \right) \tilde{\omega}^{[1,0]} \otimes \mathrm{d}\tau + \frac{\eta_1}{2\pi i} \tilde{\omega}^{[0,1]} \otimes \mathrm{d}\tau \\ \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}(\tilde{\omega}^{[1,0]}) &= -\frac{\eta_1(\tau)}{2\pi i} \tilde{\omega}^{[1,0]} \otimes \mathrm{d}\tau + \frac{1}{2\pi i} \tilde{\omega}^{[0,1]} \otimes \mathrm{d}\tau.\end{aligned}$$

A straightforward calculation shows that these formulas define an integrable holomorphic connection on $\mathcal{L}_1^\dagger$. Let us now verify, that $(\mathcal{L}_1^\dagger, \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger})$ represents the extension class of the first logarithm sheaf. The logarithm sheaf splits after pullback along e:

$$e^* \mathcal{L}_1^\dagger \xrightarrow{\sim} \mathcal{O}_{\mathcal{M}} \oplus \mathcal{H}.$$

The section $e^* \tilde{\omega}^{[0,0]}$ is a generator of $\mathcal{O}_{\mathcal{M}}$ while $e^* \tilde{\omega}^{[0,1]} = \eta$ and $e^* \tilde{\omega}^{[1,0]} = \omega$ form a basis of $\mathcal{H}$. Let us first check, that this splitting is horizontal if we equip the left-hand side with $e^* \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}$, $\mathcal{O}_{\mathcal{M}}$ with the derivation $\mathrm{d}\colon \mathcal{O}_{\mathcal{M}} \to \Omega^1_{\mathcal{M}}$ and $\mathcal{H}$ with the Gauss–Manin connection ∇_{GM}. The Gauss–Manin connection is given by the formulas (see for example [Katz 1973, A1.3.8])

$$\nabla_{\mathrm{GM}}(\eta) = \left(\partial_\tau \eta_1(\tau) - \frac{\eta_1(\tau)^2}{2\pi i}\right)\omega \otimes \mathrm{d}\tau + \frac{\eta_1}{2\pi i}\eta \otimes \mathrm{d}\tau \quad \text{and} \quad \nabla_{\mathrm{GM}}(\omega) = -\frac{\eta_1(\tau)}{2\pi i}\omega \otimes \mathrm{d}\tau + \frac{1}{2\pi i}\eta \otimes \mathrm{d}\tau.$$

Comparing this to the defining formulas for $\tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}$ shows immediately the horizontality of the splitting along e. Stated differently, we have shown that the extension class $[(\mathcal{L}_1^\dagger, \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger})]$ maps to zero under the map

$$\mathrm{Ext}^1_{\mathrm{VIC}(\mathcal{E}/\mathbb{C})}(\mathcal{O}_{\mathcal{E}}, \mathcal{H}_{\mathcal{E}}) \xrightarrow{e^*} \mathrm{Ext}^1_{\mathrm{VIC}(\mathcal{M}/\mathbb{C})}(\mathcal{O}_{\mathcal{M}}, \mathcal{H}).$$

It remains to show, that $[(\mathcal{L}_1^\dagger, \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}))]$ maps to $\mathrm{id}_{\mathcal{H}}$ under the map

$$\mathrm{Ext}^1_{\mathrm{VIC}(\mathcal{E}/\mathbb{C})}(\mathcal{O}_{\mathcal{E}}, \mathcal{H}_{\mathcal{E}}) \to \mathrm{Hom}_{\mathrm{VIC}(\mathcal{M}/\mathbb{C})}(\mathcal{O}_{\mathcal{M}}, \mathcal{H} \otimes \mathcal{H}^\vee)$$

appearing in the defining property of the first logarithm sheaf in Definition 2.1. Restricting the absolute connection relative $\mathcal{M}$ gives us vertical maps making the diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathrm{VIC}(\mathcal{E}/\mathbb{C})}(\mathcal{O}_{\mathcal{E}}, \mathcal{H}_{\mathcal{E}}) & \longrightarrow & \mathrm{Hom}_{\mathrm{VIC}(\mathcal{M}/\mathbb{C})}(\mathcal{O}_{\mathcal{M}}, \mathcal{H} \otimes \mathcal{H}^\vee) \\
\Big\downarrow \mathrm{res}_{\mathcal{M}} & & \Big\downarrow \\
\mathrm{Ext}^1_{\mathrm{VIC}(\mathcal{E}/\mathcal{M})}(\mathcal{O}_{\mathcal{E}}, \mathcal{H}_{\mathcal{E}}) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{O}_{\mathcal{M}}, \mathcal{H} \otimes \mathcal{H}^\vee)
\end{array}$$

commute. The defining formulas for $\tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}$ together with Lemma 5.11 yield $\mathrm{res}_{\mathcal{M}}(\tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger}) = \nabla_{\mathcal{L}_1^\dagger}$. But we already know that $\nabla_{\mathcal{L}_1^\dagger} = \mathrm{res}_{\mathcal{M}}(\nabla^{\mathrm{abs}}_{\mathcal{L}_1^\dagger})$, hence $(\mathcal{L}_1^\dagger, \nabla_{\mathcal{L}_1^\dagger})$ maps to $\mathrm{id}_{\mathcal{H}}$ under the lower horizontal morphism in the above diagram. Since the left vertical map is injective, we deduce that $(\mathcal{L}_1^\dagger, \tilde{\nabla}^{\mathrm{abs}}_{\mathcal{L}_1^\dagger})$ maps to $\mathrm{id}_{\mathcal{H}}$ under the upper horizontal map. □

In terms of our trivializing section $\tilde{\mathfrak{t}}^\dagger$ the Kronecker section s_{can} expresses as follows:

$$s_{\mathrm{can}} = J(z, w, \tau) \cdot \tilde{\mathfrak{t}}^\dagger \otimes \mathrm{d}z.$$

This implies the following formula for the D-variant of the Kronecker section:

$$s^D_{\mathrm{can}} = (D^2 J(z, Dw, \tau) - DJ(Dz, w, \tau)) \cdot \tilde{\mathfrak{t}}^\dagger \otimes \mathrm{d}z.$$

The expansion coefficients

$$D^2 J(z, w, \tau) - DJ\left(Dz, \frac{w}{D}, \tau\right) = \sum_{k=0}^{\infty} s^D_k(z, \tau) w^k$$

allow us to describe the restriction of s^D_{can} to the n-th infinitesimal neighborhood along $\mathcal{E}$ as

$$l^D_n = \sum_{k=0}^{n} k! s^D_k(z, \tau) \tilde{\omega}^{[k,0]} \otimes \mathrm{d}z.$$

The Kodaira–Spencer isomorphism identifies $\mathrm{d}z \otimes \mathrm{d}w$ with $\frac{1}{2\pi i}\mathrm{d}\tau$ so we get

$$L^D_n = \sum_{k=0}^{n} \left(k! s^D_k(z, \tau) \tilde{\omega}^{[k,0]} \otimes \mathrm{d}z + \frac{1}{2\pi i}(k+1)! s^D_{k+1}(z, \tau) \tilde{\omega}^{[k,0]} \otimes \mathrm{d}\tau \right). \tag{20}$$

In particular, we deduce that the analytification of the 1-forms L_n^D coincide with the analytic 1-forms used by Scheider to describe the de Rham realization of the elliptic polylogarithm on the universal elliptic curve. The analytic expression (20) is exactly the analytic section of the de Rham logarithm sheaves which was used by Scheider to describe the de Rham realization of the elliptic polylogarithm analytically. We have reduced the purely algebraic statement of Proposition 5.7 to the analytification of the modular curve and identified the objects with the analytic description of Scheider. Thus from here on we can follow the argument in [Scheider 2014, Theorem 3.6.2]. For the convenience of the reader let us nevertheless finish the proof. Indeed, it will be the mixed heat equation

$$2\pi i \cdot \partial_\tau J(z, w, \tau) = \partial_z \partial_w J(z, w, \tau).$$

which will be responsible for the vanishing of L_n^D under the differential in the de Rham complex. The mixed heat equation implies the formula

$$\partial_\tau s_k^D = \frac{1}{2\pi i}(k+1)\partial_z s_{k+1}^D \tag{21}$$

and we compute

$$\begin{aligned}
&(\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger} \wedge \mathrm{id} + \mathrm{id} \otimes \mathrm{d})(L_n^D) \\
&\quad = (\nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger} \wedge \mathrm{id} + \mathrm{id} \otimes \mathrm{d})\left(\sum_{k=0}^{n}\left(k! s_k^D \tilde{\omega}^{[k,0]} \otimes \mathrm{d}z + \frac{1}{2\pi i}(k+1)! s_{k+1}^D \tilde{\omega}^{[k,0]} \otimes \mathrm{d}\tau\right)\right) \\
&\quad = \sum_{k=0}^{n}\left(k! \partial_\tau s_k^D \tilde{\omega}^{[k,0]} \otimes \mathrm{d}\tau \wedge \mathrm{d}z + \frac{1}{2\pi i}(k+1)! \partial_z s_{k+1}^D \tilde{\omega}^{[k,0]} \otimes \mathrm{d}z \wedge \mathrm{d}\tau\right) \\
&\qquad\qquad + \sum_{k=0}^{n} k! s_k^D \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}(\tilde{\omega}^{[k,0]}) \otimes \mathrm{d}z + \sum_{k=0}^{n} \frac{1}{2\pi i}(k+1)! s_{k+1}^D \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}(\tilde{\omega}^{[k+1,0]}) \otimes \mathrm{d}\tau \\
&\quad \overset{(21)}{=} \sum_{k=0}^{n} k! s_k^D \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}(\tilde{\omega}^{[k,0]}) \otimes \mathrm{d}z + \sum_{k=0}^{n} \frac{1}{2\pi i}(k+1)! s_{k+1}^D \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger}(\tilde{\omega}^{[k+1,0]}) \otimes \mathrm{d}\tau \\
&\quad \overset{5.12}{=} \sum_{k=0}^{n} \frac{k!}{2\pi i} s_k^D \cdot (-\eta_1(\tau) \cdot k \cdot \tilde{\omega}^{[k,0]} + \tilde{\omega}^{[k-1,1]}) \otimes \mathrm{d}\tau \wedge \mathrm{d}z \\
&\qquad\qquad + \sum_{k=0}^{n} \frac{(k+1)!}{2\pi i} s_{k+1}^D \cdot (-\eta_1(\tau) \cdot (k+1) \cdot \tilde{\omega}^{[k+1,0]} + \tilde{\omega}^{[k,1]}) \otimes \mathrm{d}z \wedge \mathrm{d}\tau \\
&\quad = \sum_{k=1}^{n} \frac{k!}{2\pi i} s_k^D \cdot (-\eta_1(\tau) \cdot k \cdot \tilde{\omega}^{[k,0]} + \tilde{\omega}^{[k-1,1]}) \otimes \mathrm{d}\tau \wedge \mathrm{d}z \\
&\qquad\qquad - \sum_{k=1}^{n} \frac{k!}{2\pi i} s_k^D \cdot (-\eta_1(\tau) \cdot k \cdot \tilde{\omega}^{[k,0]} + \tilde{\omega}^{[k-1,1]}) \otimes \mathrm{d}\tau \wedge \mathrm{d}z = 0
\end{aligned}$$

Thus L_n^D is a closed form with respect to the differential of the de Rham complex of $\mathcal{L}_n^\dagger$ and the proof of the proposition is finished.

6. The de Rham–Eisenstein classes

The aim of this section is to describe the de Rham–Eisenstein classes explicitly. We will identify them with cohomology classes associated to certain Eisenstein series. In the following we will use $(\mathcal{L}_n^\dagger, \nabla^{\mathrm{abs}}_{\mathcal{L}_n^\dagger})$ as an explicit model for the de Rham logarithm sheaves. The canonical horizontal isomorphism

$$\mathcal{L}_n^\dagger \xrightarrow{\sim} \underline{\mathrm{TSym}}^n\, \mathcal{L}_1^\dagger$$

together with the horizontal isomorphism

$$e^*\mathcal{L}_1^\dagger \xrightarrow{\sim} \mathcal{O}_S \oplus \mathcal{H}$$

induces a splitting isomorphism

$$e^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \prod_{k=0}^{n} \underline{\mathrm{TSym}}^k_{\mathcal{O}_S}\, \mathcal{H}.$$

It might be more common to work with $\underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}}(E/S)$ instead of $\underline{\mathrm{TSym}}^k\, \underline{H}^1_{\mathrm{dR}}(E^\vee/S)$. Thus, let us make the following identifications: By the universal property of the symmetric algebra, we have a canonical ring homomorphism

$$\underline{\mathrm{Sym}}^\bullet\, \mathcal{H} \to \underline{\mathrm{TSym}}^\bullet\, \mathcal{H},$$

which is an isomorphism since we are working over a field of characteristic zero. Further, let us use the polarization $E \xrightarrow{\sim} E^\vee$ associated with the ample line bundle $\mathcal{O}_E([e])$ to identify

$$\mathcal{H} = \underline{H}^1_{\mathrm{dR}}(E^\vee/S) \xrightarrow{\sim} \underline{H}^1_{\mathrm{dR}}(E/S).$$

With these identifications, we can write the above splitting isomorphism as follows:

$$e^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \prod_{k=0}^{n} \underline{\mathrm{Sym}}^k_{\mathcal{O}_S}\, \underline{H}^1_{\mathrm{dR}}(E/S).$$

Similarly, we have

$$e^*\mathcal{L}_n \xrightarrow{\sim} \prod_{k=0}^{n} \underline{\omega}_{E/S}^{\otimes k}.$$

Further, by invariance under isogenies we have an isomorphism

$$\mathcal{L}_n^\dagger \xrightarrow{\sim} [N]^*\mathcal{L}_n^\dagger.$$

For a torsion section $s \in E[N](S)$ we get a canonical horizontal isomorphism

$$T_s^*\mathcal{L}_n^\dagger \xrightarrow{\sim} T_s^*[N]^*\mathcal{L}_n^\dagger = [N]^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \mathcal{L}_n^\dagger.$$

where $T_s : E \to E$ is the translation by s. Together with the splitting isomorphism we obtain a horizontal isomorphism

$$\mathrm{triv}_s\colon s^*\mathcal{L}_n^\dagger = e^*T_s^*\mathcal{L}_n^\dagger \xrightarrow{\sim} e^*\mathcal{L}_n^\dagger \xrightarrow{\sim} \prod_{k=0}^{n} \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}}(E/S).$$

The trivialization map is compatible with the Hodge filtration, i.e., we have

$$\mathrm{triv}_s\colon s^*\mathcal{L}_n = e^*T_s^*\mathcal{L}_n \xrightarrow{\sim} e^*\mathcal{L}_n \xrightarrow{\sim} \prod_{k=0}^{n} \underline{\omega}_{E/S}^{\otimes k}.$$

The map triv_s induces the *specialization map*

$$s^*\colon H^1_{\mathrm{dR}}(U_D, \mathcal{L}_n^\dagger) \to \prod_{k=0}^{n} H^1_{\mathrm{dR}}(S, \underline{\mathrm{Sym}}^k\, \underline{H}^1_{\mathrm{dR}}(E/S)).$$

The aim of this section is to identify $s^*\,\mathrm{pol}^n_{D,\mathrm{dR}}$ with cohomology classes of certain Eisenstein series; let us consider the following analytic Eisenstein series

$$F^{(k)}_{(a,b)}(\tau) = (-1)^{k+1}(k-1)! \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(m\tau+n)^k}\zeta_N^{mb-na}, \quad \zeta_N := \exp\!\left(\frac{2\pi i}{N}\right).$$

and define

$${}_D\mathrm{F}^{(k)}_{(a,b)}(\tau) = D^2F^{(k)}_{(a,b)}(\tau) - D^{1-k}F^{(k)}_{(Da,Db)}(\tau).$$

These are exactly the Eisenstein series appearing in Kato's Euler system; see [Kato 2004, Section 3.6]. Let $\mathcal{E}/\mathcal{M}$ be the universal elliptic curve over $\mathbb{Q}$ with $\Gamma(N)$-level structure. Recall that modular forms of level $\Gamma(N)$ and weight k are exactly the sections of $\Gamma(\mathcal{M}, \underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes k})$ which are finite at the cusps. The Kodaira–Spencer map

$$\underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathcal{M}}$$

allows us to associate de Rham cohomology classes to modular forms of weight $k \geq 2$ via

$$\Gamma(\mathcal{M}, \underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes k}) \xrightarrow{\sim} \Gamma(\mathcal{M}, \underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes(k-2)} \otimes \Omega^1_{\mathcal{M}}) \xrightarrow{[\cdot]} H^1_{\mathrm{dR}}(\mathcal{M}, \underline{\mathrm{Sym}}^{k-2}\, \underline{H}^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{M})).$$

For a modular form f of weight k let us write $[f] \in H^1_{\mathrm{dR}}(\mathcal{M}, \underline{\mathrm{Sym}}^{k-2}\, \underline{H}^1_{\mathrm{dR}}(\mathcal{E}/\mathcal{M}))$ for its associated cohomology class. The explicit description of the de Rham polylogarithm via the Kronecker section allows us to deduce an explicit formula for the de Rham–Eisenstein classes. The de Rham–Eisenstein classes have been known previously, see for example [Bannai and Kings 2010, Proposition 3.8; Scheider 2014, Theorem 3.8.15].

Theorem 6.1. *Let $\mathcal{E}/\mathcal{M}$ be the universal elliptic curve over $\mathbb{Q}$ with $\Gamma(N)$-level structure. Let $(0,0) \neq (a,b) \in (\mathbb{Z}/N\mathbb{Z})^2$ and $s = s_{(a,b)}$ be the associated N-torsion section. The D-variant of the polylogarithm specializes to the following cohomology classes of Eisenstein series*:

$$s^* \operatorname{pol}^n_{D,\mathrm{dR}} = \left(\left[\frac{{}_D\mathrm{F}^{(k+2)}_{(a,b)}}{k!} \right] \right)^n_{k=0}$$

Proof. By Theorem 5.8 it suffices to prove

$$s^*[L^D_n] = \left(\left[\frac{{}_D\mathrm{F}^{(k+2)}_{(a,b)}}{k!} \right] \right)^n_{k=0}.$$

Since L^D_n is obtained by applying

$$\mathcal{L}_{n+1} \otimes \Omega^1_{\mathcal{E}/\mathcal{M}} \to \mathcal{L}_n \otimes \mathcal{L}_1 \otimes \Omega^1_{\mathcal{E}/\mathcal{M}} \to \mathcal{L}_n \otimes \Omega^1_{\mathcal{E}/\mathbb{Q}}. \tag{22}$$

to l^D_{n+1}, we are reduced to prove

$$\operatorname{triv}_s(s^* l^D_{n+1}) = \left(\frac{{}_D\mathrm{F}^{(k+1)}_{(a,b)}}{k!} \right)^{n+1}_{k=0}.$$

Our aim is to reduce this claim to the construction of Eisenstein–Kronecker series via the Poincaré bundle. As a first step we have to compare the translation operators of the Poincaré bundle to the translation operators of the logarithm sheaves. Let us recall the definitions: The definition of the translation isomorphism

$$T_s^* \mathcal{L}^\dagger_n \xrightarrow{\sim} T_s^*[N]^* \mathcal{L}^\dagger_n = [N]^* \mathcal{L}^\dagger_n \xrightarrow{\sim} \mathcal{L}^\dagger_n.$$

involves twice the invariance under isogenies isomorphism $[N]^*\mathcal{L}^\dagger_n$ which in turn is induced by restricting $\gamma_{\mathrm{id},[N]} \colon (\mathrm{id} \times [N])^* \mathcal{P}^\dagger \xrightarrow{\sim} ([N] \times \mathrm{id})^* \mathcal{P}^\dagger$ along $E \times \operatorname{Inf}^n E^\dagger$. More generally, one can define $\gamma_{[N],[D]}$ as the diagonal in the commutative diagram

$$\begin{array}{ccc} ([N] \times [D])^* \mathcal{P} & \xrightarrow{([N]\times \mathrm{id})^* \gamma_{\mathrm{id},[D]}} & ([ND] \times \mathrm{id})^* \mathcal{P} \\ \Big\downarrow {\scriptstyle (\mathrm{id} \times [D])^* \gamma_{[N],\mathrm{id}}} & & \Big\downarrow {\scriptstyle ([D]\times \mathrm{id})^* \gamma_{[N],\mathrm{id}}} \\ (\mathrm{id} \times [DN])^* \mathcal{P} & \xrightarrow{(\mathrm{id} \times [N])^* \gamma_{\mathrm{id},[D]}} & ([D] \times [N])^* \mathcal{P}. \end{array}$$

Using this, we have defined in [Sprang 2018, Section 3.3] translation isomorphisms $\mathcal{U}^{N,D}_{s,t}$ for $s \in \mathcal{E}[N](\mathcal{M})$ and $t \in \mathcal{E}^\vee[D](\mathcal{M})$ on the Poincaré bundle $\mathcal{P}$:

$$\mathcal{U}^{N,D}_{s,t} := \gamma_{[N],[D]} \circ (T_s \times T_t)^* \gamma_{[D],[N]} : (T_s \times T_t)^*([D] \times [N])^* \mathcal{P} \to ([D] \times [N])^* \mathcal{P}.$$

For sections $\sigma \in \Gamma(U, \mathcal{P})$ it is convenient to introduce the notation

$$U^{N,D}_{s,t}(\sigma) := \mathcal{U}^{N,D}_{s,t}((T_s \times T_t)^*([D] \times [N])^* \sigma).$$

Let $\sigma \in \Gamma(U, \mathcal{P})$ be a section of the Poincaré bundle and $s \in E[N](S)$ and $t \in E^\vee[D](S)$. In particular, we get the section

$$\sigma_t := (\mathrm{pr}_E)_*(U^{1,D}_{e,t}(\sigma)|_{E\times_S \mathrm{Inf}^n_e E^\vee})$$

of the geometric logarithm sheaf $[D]^*\mathcal{L}_n$. Similarly, by taking translates by s and t we get the section

$$\sigma_{s,t} := (\mathrm{pr}_E)_*((\mathrm{id}\times[N]^\sharp)^{-1}[U^{N,D}_{s,t}(\sigma)|_{E\times_S \mathrm{Inf}^n_e E^\vee}])$$

of $[D]^*\mathcal{L}_n$. Here, we wrote $[N]^\sharp$ for the isomorphism of structure sheaves $\mathcal{O}_{\mathrm{Inf}^n_e E^\vee} \xrightarrow{\sim} [N]^*\mathcal{O}_{\mathrm{Inf}^n_e E^\vee}$ induced by N-multiplication. Let us write

$$\mathrm{inv}_{[D]}\colon \mathcal{L}_n \xrightarrow{\sim} [D]^*\mathcal{L}_n$$

for the invariance under isogenies isomorphism. Unwinding the definitions it is straightforward to check that the translation operators of the Poincaré bundle and the translation operators of the geometric logarithm sheaves are compatible in the following precise sense:

$$\mathrm{trans}_s(T_s^*\,\mathrm{inv}_{[D]}^{-1}(\sigma_t)) = \mathrm{inv}_{[D]}^{-1}(\sigma_{s,t}).$$

Applying this to the Kronecker section $\sigma = s_{\mathrm{can}}$ gives us sections

$$\sigma_t \in \Gamma(\mathcal{E}, [D]^*\mathcal{L}_n \otimes \Omega^1_{E/S}(E[D])) \quad \text{and} \quad \sigma_{s,t} \in \Gamma(\mathcal{E}, [D]^*\mathcal{L}_n \otimes \Omega^1_{E/S}(T_s^*E[D])).$$

By the distribution relation [Sprang 2018, Corollary A.3]

$$\sum_{e\neq t\in E^\vee[D](S)} U^D_t(s_{\mathrm{can}}) = (D)^2\cdot\gamma_{1,D}((\mathrm{id}\times[D])^*(s_{\mathrm{can}})) - ([D]\times\mathrm{id})^*(s_{\mathrm{can}}).$$

and the definition of l^D_n we obtain

$$l^D_n = \sum_{e\neq t\in E^\vee[D]} (\mathrm{inv}_{[D]}^{-1}\otimes\mathrm{can})(\sigma_t)$$

where

$$\mathrm{can}\colon [D]^*\Omega^1_{\mathcal{E}/\mathcal{M}}([e]) \xrightarrow{\sim} \Omega^1_{\mathcal{E}/\mathcal{M}}(\mathcal{E}[D])$$

is the canonical isomorphism. Now, the above formula gives

$$(\mathrm{trans}_s\otimes\mathrm{id}_{\Omega^1_{\mathcal{E}/\mathcal{M}}})(T_s^* l^D_n) = (\mathrm{inv}_{[D]}\otimes\mathrm{can})^{-1}\Bigg(\sum_{e\neq t\in\mathcal{E}^\vee[D]}\sigma_{s,t}\Bigg). \tag{23}$$

In particular, we obtain the formula

$$\begin{aligned}(\mathrm{triv}_s \otimes \mathrm{id}_{\omega_{\mathcal{E}/\mathcal{M}}})(s^* l_n^D) &\overset{\text{Def.}}{=} (e^* \mathrm{triv}_e \otimes \mathrm{id}_{\omega_{\mathcal{E}/\mathcal{M}}}) \circ (e^* \mathrm{trans}_s \otimes \mathrm{id}_{\omega_{\mathcal{E}/\mathcal{M}}})(e^* T_s^* l_n^D) \\ &\overset{(23)}{=} (e^* \mathrm{triv}_e \otimes \mathrm{id}_{\omega_{\mathcal{E}/\mathcal{M}}}) \circ (e^* \mathrm{inv}_{[D]} \otimes e^* \mathrm{can})^{-1} \Bigg(\sum_{e \neq t \in \mathcal{E}^\vee[D]} e^* \sigma_{s,t} \Bigg) \\ &\overset{(A)}{=} \Bigg(\bigoplus_{k=0}^{n} (\cdot D^{-k-1}) \Bigg) \circ (e^* \mathrm{triv}_e) \Bigg(\sum_{e \neq t \in \mathcal{E}^\vee[D]} e^* \sigma_{s,t} \Bigg) \\ &\overset{(B)}{=} \Bigg(\frac{D^{-k+1}}{k!} \cdot \sum_{e \neq t \in \mathcal{E}^\vee[D]} (e \times e)^* [([D] \times [N])^* \nabla_\sharp^{\circ k} U_{s,t}^{N,D}(s_{\mathrm{can}})] \Bigg)_{k=0}^{n} \\ &\overset{(C)}{=} \Bigg(\frac{D^{-k+1}}{k!} \cdot \sum_{e \neq t \in \mathcal{E}^\vee[D]} E_{s,t}^{k,1} \Bigg)_{k=0}^{n}\end{aligned}$$

Here, (A) is induced by the commutativity of

$$\begin{array}{ccc} e^* \mathcal{L}_n \otimes e^* \Omega^1_{\mathcal{E}/\mathcal{M}} & \xrightarrow{e^* \mathrm{inv}_{[D]} \otimes e^* \mathrm{can}^{-1}} & e^* [D]^* \mathcal{L}_n \otimes e^* [D]^* \Omega^1_{\mathcal{E}/\mathcal{M}} \\ \big\downarrow \mathrm{triv}_e & & \big\downarrow \mathrm{triv}_e \\ \bigoplus_{k=0}^n \underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes k} \otimes \underline{\omega}_{\mathcal{E}/\mathcal{M}} & \xrightarrow{([D]^*)^{\otimes k} \otimes ([D]^*)^{-1}} & \bigoplus_{k=0}^n \underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes k} \otimes \underline{\omega}_{\mathcal{E}/\mathcal{M}} \end{array}$$

and the fact that pullback along $[D]$ induces multiplication by D on the cotangent space $\underline{\omega}_{\mathcal{E}/\mathcal{M}}$. For the equality (B), let us recall that $\nabla_\sharp$ is the universal integrable $\mathcal{E}^\sharp$-connection on the pullback $\mathcal{P}^\sharp$ of the Poincaré bundle to $\mathcal{E}^\sharp \times_\mathcal{M} \mathcal{E}^\vee$, where $\mathcal{E}^\sharp$ denotes the universal vectorial extension of $\mathcal{E}$. The $\mathcal{O}_\mathcal{M}$-linear map

$$(e \times \mathrm{id}_{\mathcal{E}^\vee})^* \nabla_\sharp \colon \mathcal{O}_{\mathcal{E}^\vee} \to \mathcal{O}_{\mathcal{E}^\vee} \otimes_{\mathcal{O}_{\mathcal{E}^\vee}} \Omega^1_{\mathcal{E}^\vee/\mathcal{M}} = \mathcal{O}_{\mathcal{E}^\vee} \otimes_{\mathcal{O}_\mathcal{M}} \underline{\omega}_{\mathcal{E}/\mathcal{M}}$$

is nothing than the invariant derivation on $\mathcal{E}^\vee$. On the other hand, the map

$$\mathrm{triv}_e \colon \mathcal{O}_{\mathrm{Inf}_e^n \mathcal{E}^\vee} = (e \times \mathrm{id}_{\mathrm{Inf}_e^n \mathcal{E}^\vee})^* (\mathcal{P}|_{\mathcal{E} \times \mathrm{Inf}_e^n \mathcal{E}^\vee}) \xrightarrow{\sim} \bigoplus_{k=0}^{n} \underline{\omega}_{\mathcal{E}/\mathcal{M}}^{\otimes k}$$

coincides with $f \mapsto (e^*(\partial^{\circ k} f)/k!)_{k=0}^n$, i.e., it is given by iteratively applying the invariant derivation

$$\partial \colon \mathcal{O}_{\mathrm{Inf}_e^n \mathcal{E}^\vee} \to \mathcal{O}_{\mathrm{Inf}_e^{n-1} \mathcal{E}^\vee} \otimes_{\mathcal{O}_\mathcal{M}} \underline{\omega}_{\mathcal{E}/\mathcal{M}}$$

to sections of $\mathcal{O}_{\mathrm{Inf}_e^n \mathcal{E}^\vee}$. Combining these two facts with the definition of $\sigma_{s,t}$ gives (B). The equality (C) is the definition of the geometric modular forms $E_{s,t}^{k,1}$ given in [Sprang 2018, Definition 4.1].

So far, we have proven that the specialization $s^* \mathrm{pol}_{D,\mathrm{dR}}^n$ is represented by the cohomology classes associated to the geometric modular forms $D^{-k-1} \cdot E_{s,t}^{1,k}$. It remains to relate $D^{-k-1} \cdot E_{s,t}^{1,k}$ to Kato's Eisenstein series ${}_D\mathrm{F}_{(a,b)}^{(k)}$. Let us compare $D^{-k-1} \cdot \sum_{e \neq t \in E^\vee[D]} E_{s,t}^{1,k}$ to ${}_D\mathrm{F}_{(a,b)}^{(k)}$ on the analytification of the universal elliptic curve.

Let $\mathcal{E}/\mathcal{M}$ be the universal elliptic curve of level $\Gamma_1(N)$. Let us choose the following explicit model for the analytification

$$\mathcal{E}(\mathbb{C}) = ((\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{C} \times \mathbb{H})/(\mathbb{Z}^2 \rtimes \Gamma_1(N))$$

with coordinates (j, z, τ) on $(\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{C} \times \mathbb{H}$. We will use the trivializing section $\mathrm{d}z$ of $\underline{\omega}_{\mathcal{E}(\mathbb{C})/\mathcal{M}(\mathbb{C})}$ to identify classical modular forms with sections of $\underline{\omega}^{\otimes k}_{\mathcal{E}(\mathbb{C})/\mathcal{M}(\mathbb{C})}$. According to [Sprang 2018, Theorem 4.2] and the functional equation of Eisenstein–Kronecker series the geometric modular form $D^{-k-1} \cdot \sum_{e \neq t \in E^\vee[D]} E^{1,k}_{s,t}$ corresponds to the classical modular form

$$\begin{aligned}(-1)^k k! \cdot D^{-k+1} \sum_{(0,0)\neq(c,d)\in(\mathbb{Z}/D\mathbb{Z})^2} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(m\tau+n+c\tau/D+d/D)^{k+1}} \zeta_N^{(mb-na)} \\ = (-1)^k k! \cdot D^{-k+1} \sum_{(m,n)\in\mathbb{Z}^2\setminus(D\mathbb{Z})^2} D^{k+1} \frac{1}{(m\tau+n)^{k+1}} \zeta_N^{(mb-na)} \\ = D^2 F^{(k+1)}_{(a,b)}(\tau) - D^{2-(k+1)} F^{(k+1)}_{(Da,Db)}(\tau) =_D \mathrm{F}^{(k+1)}_{(a,b)}(\tau)\end{aligned}$$

This proves the desired formula

$$s^* \operatorname{pol}^n_{D,\mathrm{dR}} = \left(\left[\frac{{}_D\mathrm{F}^{(k+2)}_{(a,b)}}{k!} \right] \right)^n_{k=0}.$$

□

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a-numbers of curves in Artin–Schreier covers

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Let $\pi : Y \to X$ be a branched $\mathbf{Z}/p\mathbf{Z}$-cover of smooth, projective, geometrically connected curves over a perfect field of characteristic $p > 0$. We investigate the relationship between the a-numbers of Y and X and the ramification of the map π. This is analogous to the relationship between the genus (respectively p-rank) of Y and X given the Riemann–Hurwitz (respectively Deuring–Shafarevich) formula. Except in special situations, the a-number of Y is not determined by the a-number of X and the ramification of the cover, so we instead give bounds on the a-number of Y. We provide examples showing our bounds are sharp. The bounds come from a detailed analysis of the kernel of the Cartier operator.

1. Introduction

Let k be a field and $\pi : Y \to X$ a finite morphism of smooth, projective, and geometrically connected curves over k that is generically Galois with group G. The most fundamental numerical invariant of a curve is its genus, and the famous *Riemann–Hurwitz formula* says that the genus of Y is determined by that of X and the ramification of the cover π: letting $S \subset X(\bar{k})$ denote the branch locus,

$$2g_Y - 2 = |G| \cdot (2g_X - 2) + \sum_{y \in \pi^{-1}(S)} \sum_{i \geq 0} (|G_i(y)| - 1). \tag{1-1}$$

Here $G_i(y) \leqslant G$ is the i-th ramification group (in the lower numbering) at y.

When k is perfect of characteristic $p > 0$, which we will assume henceforth, there are important numerical invariants of curves beyond the genus coming from the existence of the Frobenius morphism. Writing σ for the p-power Frobenius automorphism of k, the *Cartier operator* is a σ^{-1}-semilinear map $V : H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^1_{X/k})$ which is dual to the pullback by absolute Frobenius on $H^1(X, \mathcal{O}_X)$ using Grothendieck–Serre duality.

The Cartier operator gives the k-vector space of holomorphic differentials on X the structure of a (left) module of finite length over the (noncommutative in general) polynomial ring $k[V]$. Fitting's lemma provides a canonical direct sum decomposition of $k[V]$-modules

$$H^0(X, \Omega^1_{X/k}) = H^0(X, \Omega^1_{X/k})^{\mathrm{bij}} \oplus H^0(X, \Omega^1_{X/k})^{\mathrm{nil}}$$

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with V bijective (respectively nilpotent) on $H^0(X, \Omega^1_{X/k})^\star$ for $\star = \text{bij}$ (respectively $\star = \text{nil}$). Let us write f_X for the k-dimension of $H^0(X, \Omega^1_{X/k})^{\text{bij}}$; this integer is called the *p-rank* of X, or more properly of the Jacobian J_X of X, since one also has the description $f_X = \dim_{\mathbf{F}_p} \operatorname{Hom}(\mu_p, J_X[p])$. When $\pi : Y \to X$ is a branched G-cover with G a p-group, the *Deuring–Shafarevich formula* relates the p-ranks of X and Y:

$$f_Y - 1 = |G| \cdot (f_X - 1) + \sum_{y \in \pi^{-1}(S)} (|G_0(y)| - 1). \tag{1-2}$$

Like the Riemann–Hurwitz formula, (1-2) says that the numerical invariant f_Y of Y is determined by f_X and the ramification of π; unlike the Riemann–Hurwitz formula, it *only* applies when G has p-power order, and requires only limited information about the ramification filtration. As Crew [1984, Remark 1.8.1] pointed out, there can be no version of the Deuring–Shafarevich formula if G is not assumed to be a p-group, since (for example) if $p > 2$ any elliptic curve E over k is a $\mathbf{Z}/2\mathbf{Z}$-cover of the projective line branched at exactly 4 points (necessarily with ramification degree 2), but f_E can be 0 or 1, so that f_E is *not* determined by $f_{\mathbf{P}^1} = 0$ and the ramification of $\pi : E \to \mathbf{P}^1$. Of course, thanks to the solvability of p-groups, the essential case of (1-2) is when $G = \mathbf{Z}/p\mathbf{Z}$.

Since the k-dimension δ_X of the nilpotent part $H^0(X, \Omega^1_{X/k})^{\text{nil}}$ satisfies $\delta_X = g_X - f_X$, together the Riemann–Hurwitz and Deuring–Shafarevich formulae provide a similar formula relating δ_X, δ_Y, and the (wild) ramification of π for any p-group branched cover $\pi : Y \to X$. Beyond this fact, very little seems to be understood about the behavior of the nilpotent part in p-group covers.

In this paper, we will study the behavior of the *a-number* of curves in branched $\mathbf{Z}/p\mathbf{Z}$-covers $\pi : Y \to X$. By definition, the a-number of a curve C is

$$a_C := \dim_k \ker\bigl(V : H^0(C, \Omega^1_{C/k}) \to H^0(C, \Omega^1_{C/k})\bigr). \tag{1-3}$$

Equivalently, a_C is the number of nonzero cyclic direct summands in the invariant factor decomposition of $H^0(C, \Omega^1_{C/k})^{\text{nil}}$ as a $k[V]$-module. Yet a third interpretation is $a_C = \dim_k \operatorname{Hom}(\alpha_p, J_C[p])$, where α_p denotes the group-scheme $\ker(F : \mathbf{G}_a \to \mathbf{G}_a)$ over k.[1] A curve X is said to be *ordinary* if $a_X = 0$. For elliptic curves, the a-number is 0 or 1 depending on whether the curve is ordinary or supersingular in the standard senses.

Although this fundamental numerical invariant of curves in positive characteristic has been extensively studied (e.g., [Washio and Kodama 1986; Kodama and Washio 1988; Re 2001; Elkin and Pries 2007; Johnston 2007; Elkin 2011; Friedlander et al. 2013; Dummigan and Farwa 2014; Montanucci and Speziali 2018; Frei 2018; Zhou 2019]), it remains rather mysterious. When $p = 2$ and X is ordinary, Voloch [1988] established an explicit formula for a_Y in terms of the ramification of π and the genus of X. If in addition $X = \mathbf{P}^1$, Elkin and Pries [2013] showed that this data completely determines the Ekedahl–Oort type of $J_Y[p]$; note that this situation is quite special, as every Artin–Schreier cover of $\mathbf{P}^1$ in characteristic 2 is hyperelliptic. For general p, Farnell and Pries [2013] studied branched $\mathbf{Z}/p\mathbf{Z}$-covers $\pi : Y \to \mathbf{P}^1$,

[1] The equivalence of this description with the given definition follows from Dieudonné theory and a theorem of Oda [1969, Corollary 5.11], which provides a canonical isomorphism of $k[V]$-modules $H^0(C, \Omega^1_{C/k}) \simeq k \otimes_{k,\sigma^{-1}} \boldsymbol{D}(J_C[F])$, where $\boldsymbol{D}(\cdot)$ is the contravariant Dieudonné module.

and proved that there is an explicit formula for a_Y in terms of the ramification of π whenever the unique break in the ramification filtration at every ramified point is a divisor of $p-1$. Unfortunately, there can be no such "a-number formula" in the spirit of (1-2) in general: simple examples with $p>2$ show that there are $\mathbf{Z}/p\mathbf{Z}$-covers even of $X=\mathbf{P}^1$ branched only at ∞ which have *identical* ramification filtrations, but different a-numbers; cf. Example 7.2.

Nonetheless, we will prove that the possibilities for the a-number of Y are tightly constrained by the a-number of X and the ramification of π:

Theorem 1.1. *Let $\pi : Y \to X$ be a finite morphism of smooth, projective and geometrically connected curves over a perfect field k of characteristic $p>0$ that is generically Galois with group $\mathbf{Z}/p\mathbf{Z}$. Let $S \subseteq X(\bar{k})$ be the finite set of geometric closed points over which π ramifies, and for $Q \in S$ let d_Q be the unique break in the lower-numbering ramification filtration at the unique point of Y over Q. Then for any $1 \le j \le p-1$,*

$$\sum_{Q\in S}\sum_{i=j}^{p-1}\left(\left\lfloor \frac{id_Q}{p}\right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1-\frac{1}{p}\right)\frac{jd_Q}{p}\right\rfloor\right) \le a_Y \le pa_X + \sum_{Q\in S}\sum_{i=1}^{p-1}\left(\left\lfloor \frac{id_Q}{p}\right\rfloor - (p-i)\left\lfloor \frac{id_Q}{p^2}\right\rfloor\right).$$

In fact, our main result (Theorem 6.26) features a slightly sharper—if somewhat messier and less explicit in general — upper bound, and Theorem 1.1 is an immediate consequence of this.

Remark 1.2. The lower bound is largest for $j \approx p/2$, and in applications we will take $j = \lceil p/2 \rceil$. Fixing X and $S \subseteq X(\bar{k})$ and writing $T := (p-1)\sum_{Q\in S} d_Q$, elementary estimates show that our lower (respectively upper) bound is asymptotic to $\bigl(1-\frac{1}{p^2}\bigr)\frac{1}{4}T$ $\bigl($respectively $\bigl(1-\frac{1}{2p}\bigr)\frac{1}{3}T\bigr)$ as $T\to\infty$; see Corollary 6.27 for more precise estimates. Equivalently, since $g_Y \sim \frac{1}{2}T$ as $T\to\infty$ by Riemann–Hurwitz, our lower and upper bounds are asymptotic to $\bigl(1-\frac{1}{p^2}\bigr)\frac{1}{2}g_Y$ and $\bigl(1-\frac{1}{2p}\bigr)\frac{2}{3}g_Y$, respectively, as $g_Y\to\infty$ with X and S fixed. In contrast, f_Y/g_Y approaches 0 as $T\to\infty$.

Remark 1.3. Using only information about X, elementary arguments give "trivial" bounds

$$\dim_k \ker\bigl(V : H^0(X,\Omega^1_X(E_0)) \to H^0(X,\Omega^1_X(E_0))\bigr) \le a_Y \le p\cdot g_X - p\cdot f_X + \sum_{Q\in S}\tfrac{1}{2}(p-1)(d_Q-1),$$

where $E_0 = \sum_{Q\in S}(d_Q - \lfloor d_Q/p\rfloor)[Q])$. The trivial lower bound comes from the fact that there is an inclusion $\Omega^1_X(E_0) \hookrightarrow \pi_*\Omega^1_Y$ compatible with the Cartier operator; see Lemma 4.1. The trivial upper bound comes from the fact that $a_Y + f_Y \le g_Y$ and from applying the Riemann–Hurwitz formula for the genus and Deuring–Shafarevich formula for the p-rank to $\pi : Y \to X$.

The lower bound is explicit. When $X = \mathbf{P}^1$, for a divisor $D = \sum n_i[P_i]$ with $n_i \ge 0$ we know that

$$\dim_k \ker\bigl(V : H^0(X,\Omega^1_X(D)) \to H^0(X,\Omega^1_X(D))\bigr) = \sum_i\left(n_i - \left\lceil \frac{n_i}{p}\right\rceil\right)$$

so the lower bound is explicit. For general X, a theorem of Tango (Fact 6.11, the main theorem of [Tango 1972]) generalizes this to provide information about the kernel of the Cartier operator when the degree of D is sufficiently large.

Theorem 1.1 is substantially better than the trivial bounds; see Example 7.1 for an illustration.

When $p = 2$ and $a_X = 0$, the upper and lower bounds of Theorem 6.26 (with $j = 1$) *coincide*, and we recover Voloch's formula [1988, Theorem 2]; see Remark 6.30. Similarly, when p is odd, all d_Q divide $p-1$, and $a_X = 0$, we prove in Corollary 6.32 that our (sharpest) upper bound and our lower bound with $j = \lceil p/2 \rceil = (p+1)/2$ also coincide, thereby establishing the following "a-number formula":

Corollary 1.4. *With hypotheses and notation as in Theorem 1.1 and p odd, assume that $d_Q | (p-1)$ for all $Q \in S$ and that X is* ordinary (*i.e.*, $a_X = 0$). *Then*

$$a_Y = \sum_{Q \in S} a_Q, \qquad \textit{where} \quad a_Q := \frac{(p-1)}{2}(d_Q - 1) - \frac{p-1}{d_Q}\left\lfloor \frac{(d_Q-1)^2}{4} \right\rfloor.$$

Specializing Corollary 1.4 to the case of $X = \mathbf{P}^1$ recovers the main result of [Farnell and Pries 2013].

To get a sense of the bounds in Theorem 1.1, in Section 7 we work out a number of examples. We show in particular that for $X = \mathbf{P}^1$ our upper bound of Theorem 6.26 is *sharp*, with the family of covers $y^p - y = t^{-d}$ (for t a choice of coordinate on $\mathbf{P}^1$) achieving the upper bound for all $p > 2$ and all d with $p \nmid d$; see Example 7.5. We similarly find in our specific examples that the lower bound is sharp and that most covers have a-number equal to the lower bound.[2] For various p, we also provide examples using covers of the elliptic curve with affine equation $y^2 = x^3 - x$, which has $a_X = 1$ when $p \equiv 3 \bmod 4$ and $a_X = 0$ when $p \equiv 1 \bmod 4$.

1A. ***Outline of the proof.*** Without loss of generality, we may assume that k is algebraically closed. A key idea in the proof is that the Cartier operator is not defined only on global differentials, but actually is a map of sheaves. Let X be a smooth projective curve over k. Functorially associated to the finite flat absolute Frobenius map $F : X \to X$ by Grothendieck's theory of the trace [Conrad 2000, 2.7.36] is an $\mathcal{O}_X$-linear map of sheaves

$$V_X : F_* \Omega^1_X \to \Omega^1_X;$$

the Cartier operator considered previously is obtained by taking global sections. (For the remainder of the paper, we include subscripts to clarify which curve/ring we are working with.) The advantage of this perspective is that the Cartier operator admits a simple description on stalks, allowing local arguments. In particular, the Cartier operator on completed stalks at any k-point is given by

$$V\left(\sum_i a_i t^i \frac{dt}{t}\right) = \sum_j a_{pj}^{1/p} t^j \frac{dt}{t}; \tag{1-4}$$

see, for example [Cais 2018, Proposition 2.1]. To relate the kernels of V_X and V_Y on global differentials, we will combine an analysis over the generic point with an analysis at stalks at the points where the cover $\pi : Y \to X$ is ramified. This strategy allows the use of geometric methods, and allows us to work with

[2] When $p = 3$ or $p = 5$ and X is ordinary, for any branch locus S and choice of d_Q for $Q \in S$ with $p \nmid d_Q$, subsequent work constructs covers of X whose a-number is the lower bound [Abney-McPeek et al. 2020].

general Artin–Schreier covers instead of only covers of $\mathbf{P}^1$: previous work has focused on curves defined by explicit equations or covers Y of $\mathbf{P}^1$ where it is possible to find a nice and explicit basis of $H^0(Y, \Omega^1_Y)$.

For now, we will ignore a few technical issues and sketch the argument. None of these technical issues arise for $X = \mathbf{P}^1$, which is a helpful simplification on a first reading. Writing η for the generic point of X, one has an isomorphism

$$\pi_* \Omega^1_{Y,\eta} \simeq \bigoplus_{i=0}^{p-1} \Omega^1_{X,\eta}. \tag{1-5}$$

This follows from the fact that the function field $K' = k(Y)$ of Y is an Artin–Schreier extension of the function field $K = k(X)$ given by $y^p - y = f$ for some $f \in K$. This induces an isomorphism

$$(\pi_* \ker V_Y)_\eta \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta. \tag{1-6}$$

This is Proposition 4.4; using (1-5) to write $\omega \in \pi_* \Omega^1_{Y,\eta}$ as $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ with $\omega_i \in \Omega^1_{X,\eta}$, the key observation is that if $V_Y(\omega) = 0$ then for all $0 \leq j \leq p-1$, $V_X(\omega_j)$ is determined by $V_X(\omega_i)$ for $j < i \leq p-1$.

Unfortunately, (1-5)–(1-6) do not generalize to isomorphisms of sheaves. Instead, there are explicit divisors E_i (Definition 3.4) depending on the ramification of π and an isomorphism of $\mathcal{O}_X$-modules

$$\pi_* \Omega^1_Y \simeq \bigoplus_{i=0}^{p-1} \Omega^1_X(E_i), \tag{1-7}$$

as well as an injection (Definition 4.3)

$$\varphi : \pi_* \ker V_Y \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_* E_i) \tag{1-8}$$

inducing (1-6) at the generic point. Here $\ker V_X(F_* E_i) := (\ker V_X)_\eta \cap F_*(\Omega^1_X(E_i))$. But φ is not surjective as a map of sheaves. The problem is that while (in the generic fiber) $V_X(\omega_j)$ is determined by ω_i for $i > j$, it is not automatic that the resulting form ω_j satisfies $\operatorname{ord}_Q(\omega_j) \geq -\operatorname{ord}_Q(E_j)$ when $\operatorname{ord}_Q(\omega_i) \geq -\operatorname{ord}_Q(E_i)$ for $i > j$ and $Q \in S$. Example 4.6 is a key illustration of this problem.

To deal with this issue, we wish to find relations that describe the image of φ. Fortunately, this is a purely local problem at the points S where π is ramified. We will identify $\pi_* \ker V_Y$ with a certain subsheaf of the target of (1-8) cut out by linear relations on the coefficients of the power series expansions of elements at points $Q \in S$. These relations force the corresponding differentials ω_i on X to be regular. These relations are expressed in Section 5 as maps to skyscraper sheaves supported on S, and the kernels of these maps describe the image of φ; see Theorem 5.20.

The final step is to extract useful information about the a-number of Y (the dimension of the space of global sections of $\pi_* \ker V_Y$) from the short exact sequences resulting from our description of the image of φ. The key to doing so is knowledge about the dimension of the kernel of the Cartier operator on $H^0(X, \Omega^1_X(D))$ for various divisors D, and related questions about the existence of elements of that space with specified behavior at points in S. In Section 6, we analyze these questions using a theorem of Tango (Fact 6.11, the main theorem of [Tango 1972]) and obtain bounds on the a-number of Y by taking global sections of the exact sequences from Theorem 5.20.

Remark 1.5. Tango's theorem yields precise results only when the degree of the divisor is sufficiently large. By exploiting the flexibility of our local analysis—which in particular allows us to arbitrarily increase the degree of certain auxiliary divisors—we may always work in this case; see the proof of Theorem 6.26. Given additional information about the size of the dimension of the kernel of the Cartier operator on $H^0(X, \Omega^1_X(D))$ for divisors D of small degree, modest improvements are possible. See Section 6E for an example with unramified covers.

Remark 1.6. The trivial lower bound of Remark 1.3 follows from the inclusion $\Omega^1_X(E_0) \hookrightarrow \pi_*\Omega^1_Y$ that is compatible with the Cartier operator. This inclusion is a consequence of (1-7), which is an isomorphism of $\mathcal{O}_X$-modules and is not in general compatible with the Cartier operator. The significant improvements in the bounds come from incorporating information about the Cartier operator to obtain a more refined inclusion (1-8), analyzing the image, and using Tango's theorem.

Remark 1.7. As mentioned previously, there are some technical complications to the strategy outlined above. In the end, these have no effect on the final result. The issues are:

- The short exact sequence

$$0 \to \ker V_X \to F_*\Omega^1_X \to \operatorname{Im} V_X \to 0$$

  is not always split, although it does split when $X = \mathbf{P}^1$. In Section 2, we show that we may produce maps which split the sequence over the generic point and introduce poles in a controlled manner. This is used to define the map φ of 4.3.

- The Artin–Schreier extension of function fields cannot always be described as $y^p - y = f$ where f is regular away from S, and $\operatorname{ord}_Q(f) = -d_Q$ for $Q \in S$. This is possible when $X = \mathbf{P}^1$ by using the theory of partial fractions. In Section 3 we allow f to have a pole at one additional (nonbranched) point Q' to ensure the desired property holds for $Q \in S$, and then keep track of this complication throughout the remainder of the argument.

Remark 1.8. The same arguments, with minor modifications, should yield bounds on the dimension of the kernel of powers of the Cartier operator. We leave that for future work.

2. Producing splittings

Let k be an algebraically closed field of characteristic p, and X a smooth projective and connected curve over k. Writing $V_X : F_*\Omega^1_{X/k} \to \Omega^1_{X/k}$ for the Cartier operator, we are interested in splitting the tautological short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \ker V_X \to F_*\Omega^1_X \to \operatorname{Im} V_X \to 0. \tag{2-1}$$

Lemma 2.1. *When* $X = \mathbf{P}^1_k$, (2-1) *is a split exact sequence of sheaves.*

Proof. Identify the generic fiber of X with $\operatorname{Spec}(k(t))$. From (1-4), we know that $V_X(dt/t) = (dt/t)$ and $V_X(t^i(dt/t)) = 0$ if $p \nmid i$. Thus we see that

$$(\ker V_X)_\eta = \left\{ \sum_{i=1}^{p-1} h_i(t) t^i \frac{dt}{t} : h_i \in k(t^p) \right\} \quad \text{and} \quad (\operatorname{Im} V_X)_\eta = \Omega^1_{X,\eta}.$$

An explicit splitting $s : (\operatorname{Im} V_X)_\eta \to F_* \Omega^1_{X,\eta}$ of (2-1) over the generic fiber is given by

$$s\left(\sum_i a_i t^i \frac{dt}{t} \right) = \sum_i a_i^p t^{pi} \frac{dt}{t}. \tag{2-2}$$

For $Q \in \mathbf{P}^1_k$, a direct calculation shows that for a section ω to Ω^1_X, if $\operatorname{ord}_Q(\omega) \geq 0$ then $\operatorname{ord}_Q(s(\omega)) \geq 0$. Thus this defines a map of sheaves. □

Remark 2.2. The corresponding projector $r : F_* \Omega^1_X \to \ker V_X$ is given by

$$r\left(\sum_i a_i t^i \frac{dt}{t} \right) = \sum_{p \nmid i} a_i t^i \frac{dt}{t}. \tag{2-3}$$

In general, (2-1) splits over the generic point for any smooth curve X, although it is not clear the sequence itself splits. However, it *does* split if we allow the splitting to introduce controlled poles.

Let D be a divisor on X. Note that $(F_* \Omega^1_X)(D) = F_*(\Omega^1_X(pD))$, which complicates the relationship between twists and order of vanishing. To more closely connect twists with order of vanishing, we make the following definition.

Definition 2.3. For a subsheaf $\mathscr{F} \subset F_* \Omega^1_X$ and divisor D on X we define a sheaf $\mathscr{F}(F_* D)$ via

$$\mathscr{F}(F_* D)(U) := F_*(\Omega^1_X(D))(U) \cap \mathscr{F}_\eta$$

for open $U \subset X$. In particular,

$$\ker V_X(F_* D)(U) = \{\omega \in \Omega^1_X(D)(U) : V_X(\omega) = 0\}. \tag{2-4}$$

Example 2.4. It is clear upon taking $\mathscr{F} = F_* \Omega^1_X(D)$ that

$$(F_* \Omega^1_X)(F_* D) = F_*(\Omega^1_X(D)).$$

Note that $\ker V_X(F_* D)$ consists of differentials ω that lie in the kernel of V_X and satisfy $\operatorname{ord}_Q(\omega) \geq -\operatorname{ord}_Q(D)$ for all Q. On the other hand, $\ker V_X(D) = \ker V_X \otimes \mathcal{O}_X(D) = \ker V_X(F_* pD)$, which consists of differentials that lie in the kernel of V_X and satisfy $\operatorname{ord}_Q(\omega) \geq -p \operatorname{ord}_Q(D)$.

For any divisor $E = \sum_i n_i P_i \geq 0$, define $\bar{E} := \sum_i \lceil n_i/p \rceil P_i$.

Lemma 2.5. *There is an exact sequence of* $\mathcal{O}_X$*-modules*

$$0 \to \ker V_X(F_* E) \to F_*(\Omega^1_X(E)) \xrightarrow{V_X} \operatorname{Im} V_X(\bar{E}) \to 0. \tag{2-5}$$

Furthermore, each term is locally free.

Proof. For a closed point Q of X and a section ω of $F_*(\Omega^1_X(E))$ defined at Q, a local calculation shows that $\mathrm{ord}_Q(V_X(\omega)) \geq \lceil \mathrm{ord}_Q(\omega)/p \rceil$, so the right map is well-defined. The kernel is $\ker V_X(F_*E)$ by definition. As the completion $\mathcal{O}^{\wedge}_{X,Q}$ is faithfully flat over $\mathcal{O}_{X,Q}$, we may check surjectivity on completed stalks. Since $-n_i \leq p\lceil -n_i/p \rceil$, for t_Q a local uniformizer at Q we have

$$V_X\left(\sum_i a_i^p t_Q^{ip} \frac{dt_Q}{t_Q}\right) = \sum a_i t_Q^i \frac{dt_Q}{t_Q}$$

thanks to (1-4). Over a smooth curve, to check local freeness it suffices to check the sheaves are torsion-free, which is clear as the sheaves are subsheaves of $F_*\Omega^1_{X,\eta}$. □

We will prove the following:

Proposition 2.6. *Let S be a finite set of points on a smooth projective curve X over k and*

$$0 \to \mathscr{F}_1 \xrightarrow{\iota} \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

an exact sequence of locally free sheaves. There exists a divisor $D = \sum_i [P_i]$ with the P_i distinct points of X not in S and a morphism $r : \mathscr{F}_2 \to \mathscr{F}_1(D)$ such that $r \circ \iota$ is the natural inclusion $\mathscr{F}_1 \to \mathscr{F}_1(D)$.

The following corollary will be useful in Section 4, especially in Definition 4.3.

Corollary 2.7. *Let S be a finite set of points of X. Fix an effective divisor E supported on S. There is a divisor $D = \sum_i [P_i]$ with the P_i distinct points of X not in S and a morphism*

$$r : F_*(\Omega^1_X(E)) \to \ker V_X(F_*(E + pD))$$

such that $r \circ \iota$ is the natural inclusion $\ker V_X(F_*E) \to \ker V_X(F_*(E + pD))$, *where ι is the inclusion* $\ker V_X(F_*E) \to F_*(\Omega^1_X(E))$.

Proof. Apply Proposition 2.6 to the exact sequence of Lemma 2.5. It is elementary to verify using Definition 2.3 that $(\ker V_X(F_*E))(D) = \ker V_X(F_*(E + pD))$. □

The rest of this section is devoted to proving Proposition 2.6. The key input is:

Lemma 2.8. *Let S be a finite set of points of X, and $\mathscr{F}$ be a locally free sheaf on X. Then for any divisor $D = \sum_i [P_i]$ with the P_i distinct points of X not in S and* $\deg D \gg 0$, *we have*

$$H^1(X, \mathscr{F} \otimes \mathcal{O}_X(D)) = 0.$$

Proof. Pick an ample line bundle $\mathscr{L} = \mathcal{O}_X(D')$. By Serre's cohomological criterion for ampleness, we know that there is an N such that

$$H^1(X, \mathscr{F} \otimes \mathscr{L}^n) = 0 \tag{2-6}$$

for $n \geq N$. Writing g for the genus of X, the Riemann–Roch theorem gives

$$h^0(X, \mathcal{O}_X(D - ND')) - h^1(X, \mathcal{O}(D - ND')) = \deg(\mathcal{O}(D - ND')) - g + 1,$$

and when $\deg(\mathcal{O}(D-ND')) > 2g-2$, we have $h^1(X, \mathcal{O}(D-ND')) = 0$ for degree reasons. Thus when $\deg D \gg 0$ we conclude that

$$h^0(X, \mathcal{O}_X(D-ND')) > 0.$$

Using a global section of $\mathcal{O}_X(D-ND')$, we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D'-ND) \to \mathcal{G} \to 0,$$

where $\mathcal{G}$ is a skyscraper sheaf supported on $D'-ND$. Tensoring with $\mathcal{O}_X(ND')$ is exact, as is tensoring with the locally free $\mathcal{F}$, so we obtain a short exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(ND') \to \mathcal{F} \otimes \mathcal{O}_X(D) \to \mathcal{G}' \to 0,$$

where $\mathcal{G}'$ is still a skyscraper sheaf supported on $D-ND'$. Part of the long exact sequence of cohomology is

$$H^1(X, \mathcal{F} \otimes \mathcal{O}_X(ND')) \to H^1(X, \mathcal{F} \otimes \mathcal{O}_X(D)) \to H^1(X, \mathcal{G}')$$

The left term vanishes by (2-6), and the right vanishes as $\mathcal{G}'$ is a skyscraper sheaf. Thus $H^1(X, \mathcal{F} \otimes \mathcal{O}_X(D)) = 0$. □

We now prove Proposition 2.6. By looking at stalks we see that the Hom-sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1)$ is locally free. Also, notice that for a divisor D,

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D)) \simeq \mathcal{F}_3^\vee \otimes \mathcal{F}_1 \otimes \mathcal{O}_X(D) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1) \otimes \mathcal{O}_X(D). \tag{2-7}$$

Applying the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\cdot, \mathcal{F}_1(D))$ to the exact sequence in Proposition 2.6 and using the assumption that $\mathcal{F}_3$ is locally free, we obtain an exact sequence

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D)) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{F}_1(D)) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_1(D)) \to 0.$$

Passing to global sections, part of the long exact sequence of cohomology is

$$\begin{aligned} H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{F}_1(D))) \xrightarrow{f} {} & H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_1(D))) \\ \to {} & H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D))). \end{aligned}$$

Applying Lemma 2.8 with $\mathcal{F} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1)$ and appealing to (2-7), we choose $D = \sum_i [P_i]$ where the distinct P_i avoid S such that $H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_3, \mathcal{F}_1(D))) = 0$. Thus f is surjective, and the desired morphism r is the preimage of the natural inclusion in $H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_1(D)))$. □

Corollary 2.9. *A choice of projector $r : F_*(\Omega^1_X(E)) \to \ker V_X(F_*(E+pD))$ as in Corollary 2.7 is equivalent to a choice of splitting of* (2-1)

$$s : \mathrm{Im}\, V_X(\bar{E}) \to F_*(\Omega^1_X(E+pD))$$

such that $V_X \circ s$ *is the natural inclusion* $\operatorname{Im} V_X(\bar{E}) \to \operatorname{Im} V_X(\bar{E}+D)$. *Furthermore, we may choose* s *so that for any point* $Q \in \sup(E)$, *if* $\operatorname{ord}_Q(\omega) \geq d$ *then*

$$\operatorname{ord}_Q(s(\omega)) \geq (d+1)p - 1. \tag{2-8}$$

Proof. Given r, a section of

$$0 \to (\ker V_X)_\eta \xrightarrow{\iota} (F_*\Omega^1_{X,\eta}) \to (\operatorname{Im} V_X)_\eta \to 0$$

is given by $s(m) = \widetilde{m} - \iota r(\widetilde{m})$, where $\widetilde{m}$ is any preimage of $m \in (\operatorname{Im} V_X)_\eta$. It is independent of the choice of lift, which allows us to make local calculations using (1-4) to check that this recipe defines a map $s : \operatorname{Im} V_X(\bar{E}) \to F_*(\Omega^1_X(E+pD))$. Conversely, given s we define $r(x) = x - sV_X(x)$ for any section x of $F_*(\Omega^1_X(E))$. It is straightforward to check that $r \circ \iota$ is the natural inclusion and that the recipes for r given s and s given r are inverse to each other, so that this process really does give an equivalence between a splitting s and a projector r as claimed.

To get the last claim, set

$$E' := \sum_{Q \in \sup(E)} p\left(\left\lceil \frac{\operatorname{ord}_Q(E)}{p} \right\rceil - 1\right) + 1 \tag{2-9}$$

and note that $\bar{E}' = \bar{E}$ and $E' \leq E$. Thanks to Corollary 2.7 and its proof, we obtain a morphism $r' : F_*(\Omega^1_X(E')) \to \ker V_X(F_*(E'+pD))$ with the *same* D as above. Via the correspondence already established, we obtain a section

$$s' : \operatorname{Im} V_X(\bar{E}) \to F_*(\Omega^1_X(E'+pD)).$$

Composing with the natural inclusion $F_*(\Omega^1_X(E'+pD)) \hookrightarrow F_*(\Omega^1_X(E+pD))$, we obtain the desired map s: the condition on orders of vanishing follows by twisting. □

Remark 2.10. Let $r : F_*(\Omega^1_X(E)) \to \ker V_X(F_*(E+pD))$ be the projector corresponding to a splitting $s : \operatorname{Im} V_X(\bar{E}) \to F_*(\Omega^1_X(E+pD))$. No matter the choice of E and D, the generic fiber of (2-1) is split by r and s.

Remark 2.11. When $X = \mathbf{P}^1$, the explicit splitting of Lemma 2.1 shows we may take $D = 0$. This is a good simplification for subsequent arguments on a first reading.

3. Artin–Schreier covers and differential forms

Let $\pi : Y \to X$ be a branched Galois cover of smooth projective and connected curves over an algebraically closed field k of characteristic p, with Galois group $G \simeq \mathbf{Z}/p\mathbf{Z}$ and branch locus $S \subset X$. We are mainly interested in the case that S is nonempty, though we do not assume this at the outset. By Artin–Schreier theory, $K' := k(Y)$ is a degree-p Artin–Schreier extension of $K := k(X)$; that is, there exists $f \in K$ such that $K' = K(y)$, where

$$y^p - y = f. \tag{3-1}$$

We may and do assume a fixed choice of generator τ of G sends y to $y+1$, and we write g_X for the genus of X. We henceforth *fix* a closed point $Q' \in X$ with $\mathrm{ord}_{Q'}(f) = 0$.

The choice of f is far from unique. In particular, by replacing y with $y+h$ for $h \in K$ we may replace f with $f + h^p - h$ but obtain the same extension. Nonetheless, it is possible to normalize the choice of f as follows.

Definition 3.1. We say $\psi \in K$ is *minimal* for $\pi : Y \to X$ (or for K'/K) if

- $\mathrm{ord}_Q(\psi) \geq 0$ or $p \nmid \mathrm{ord}_Q(\psi)$ for all $Q \in X$ with $Q \neq Q'$,
- $p \mid \mathrm{ord}_{Q'}(\psi)$ and $-\mathrm{ord}_{Q'}(\psi) \leq p(2g_X - 2)$.

Lemma 3.2. *The Artin–Schreier extension K'/K may be described as in* (3-1) *with f minimal.*

Proof. Let $f \in K$ be as in (3-1), and suppose f has a pole of order $p \cdot d$ at $Q \neq Q'$. By Riemann–Roch, there exists a function with a pole of order m at Q and a pole of order n at Q' provided $m + n > 2g_X - 2$. We may therefore choose $h \in K$ to have a pole of order d at Q and no poles except possibly at Q', and satisfy $\mathrm{ord}_Q(f + h^p - h) > \mathrm{ord}_Q(f)$ and $\mathrm{ord}_{Q'}(h^p - h) \geq -p(2g_X - 2)$. Replacing f with $f + h^p - h$ and repeating this procedure, we may thereby arrange for f to be minimal. □

We now *fix* a choice of minimal $f \in K$ giving the Artin–Schreier extension K'/K as in (3-1).

Remark 3.3. When $X = \mathbf{P}^1$, it is easy to arrange for f to have poles only at the branch locus since there are functions on $\mathbf{P}^1$ with a single simple pole. In this case, our analysis below at Q' is unnecessary: this is a helpful simplification on a first reading. This reduction is not possible in general; see [Shabat 2001, §7], especially the second example after Proposition 49.

We now fix some notation relating to the cover $\pi : Y \to X$ and our fixed choice of Q' and minimal f giving the corresponding extension of function fields as in (3-1).

Definition 3.4. With notation as above:

- Let $S' := S \cup \{Q'\}$.
- For $Q \in X$ define $d_Q := \max\{0, -\mathrm{ord}_Q(f)\}$.
- For $0 \leq i \leq p-1$ and $Q \in X$ define
$$n_{Q,i} := \begin{cases} \left\lceil \dfrac{(p-1-i)d_Q}{p} \right\rceil & \text{if } Q \neq Q', \\ (p-1-i)d_{Q'} & \text{if } Q = Q', \end{cases}$$
and set
$$E_i := \sum_{Q \in S'} n_{Q,i}[Q] \quad \text{and} \quad \bar{E}_i := \sum_{Q \in S'} \lceil n_{Q,i}/p \rceil [Q].$$

Lemma 3.5. *The map $\pi : Y \to X$ is ramified over $Q \in X$ if and only if $d_Q > 0$ and $p \nmid d_Q$, in which case d_Q is the unique break in the lower-numbering ramification filtration of G above Q.*

Proof. This is well known; see for example [Stichtenoth 2009, Proposition 3.7.8]. □

Remark 3.6. It follows from Lemma 3.5 and the fact that the fixed f is minimal that if $Q \notin S'$ then $d_Q = 0$ and hence also $n_{Q,i} = 0$ for all i.

We now investigate when a meromorphic differential on Y is regular.

Lemma 3.7. *Let $\omega \in (\pi_* \Omega^1_Y)_\eta$ be a meromorphic differential on Y, and write $\omega = \sum_i \omega_i y^i$ with $\omega_i \in \Omega^1_{X,\eta}$ using the identification* (1-5). *Let $P \in Y$ with $\pi(P) = Q \neq Q'$. Then ω is regular at P provided* $\operatorname{ord}_Q(\omega_i) \geq -n_{Q,i}$ *for all* $0 \leq i \leq p-1$.

Proof. This is also probably known, but we were unable to locate a reference in the degree of generality we need: for example, Boseck [1958, Satz 15] only treated the case that $X = \mathbf{P}^1$. We provide a proof for the convenience of the reader.

Let $P \in Y$ and set $Q = \pi(P)$. Suppose first that $Q \in S$. Let t_Q be a uniformizer of $\mathcal{O}_{X,Q}$, and note that the fraction field of $\mathcal{O}_{Y,P}$ can be obtained from the fraction field of $\mathcal{O}_{X,Q}$ by adjoining a root of the polynomial $y^p - y = f$ where f has a pole of order d_Q at Q by Lemma 3.5; in particular, $\operatorname{ord}_P(y) = -d_Q$.

As $p \nmid d_Q$, we may choose positive integers a and b with $1 = ap - bd_Q$, so that $u = t_Q^a y^b$ is a uniformizer of $\mathcal{O}_{Y,P}$. A direct computation shows that $du = u^{-(p-1)(d_Q+1)} \beta \, dt_Q$, where $\beta \in \mathcal{O}^\times_{Y,P}$. Hence $\operatorname{ord}_P(dt_Q) = (p-1)(d_Q+1)$, and $\operatorname{ord}_P(\omega_i) = (p-1)(d_Q+1) + p \operatorname{ord}_Q(\omega_i)$. We conclude that

$$\operatorname{ord}_P(\omega_i y^i) = p \operatorname{ord}_Q(\omega_i) + (p-1)(d_Q+1) - i d_Q. \tag{3-2}$$

This is nonnegative precisely when $\operatorname{ord}_Q(\omega_i) \geq -n_{Q,i}$. As $p \nmid d_Q$ and $0 \leq i \leq p-1$, the p integers in (3-2) are all distinct modulo p, and hence distinct, so

$$\operatorname{ord}_P(\omega) = \min_{0 \leq i \leq p-1} \operatorname{ord}_P(\omega_i y^i) \tag{3-3}$$

and ω is regular at P precisely when $\operatorname{ord}_Q(\omega_i) \geq -n_{Q,i}$ for all i.

Now suppose $Q \notin S'$, so that π is étale over Q. There are then p points $P = P_0, \ldots, P_{p-1}$ over Q, and $\mathcal{O}_{Y,P_j} \simeq \mathcal{O}_{X,Q}$ (and likewise for differentials) for all j. Under these identifications, the function y corresponds to regular functions $h, h+1, \ldots, h+(p-1)$ in each of the $\mathcal{O}_{Y,P_j} \simeq \mathcal{O}_{X,Q}$. Thus $\omega = \sum_i \omega_i y^i$ is regular at one (and hence every) point of Y over Q provided

$$\operatorname{ord}_Q\Bigl(\sum_i \omega_i (h+j)^i\Bigr) \geq 0$$

for $0 \leq j \leq p-1$. Define $\lambda_j = \sum_i \omega_i (h+j)^i \in \Omega^1_{X,Q}$, depending on ω. The λ_j and ω_i are related by the Vandermonde matrix $M = ((h+j)^i)_{0 \leq i,j \leq p-1}$:

$$\begin{pmatrix} (h+0)^0 & (h+0)^1 & \cdots & (h+0)^{p-1} \\ (h+1)^0 & (h+1)^1 & \cdots & (h+1)^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ (h+(p-1))^0 & (h+(p-1))^1 & \cdots & (h+(p-1))^{p-1} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{p-1} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{p-1} \end{pmatrix}.$$

The determinant of M is well-known to be

$$\prod_{1\le i,j\le p,\, i\ne j} ((h+i)-(h+j)) = \prod_{i\ne j}(i-j) \in \mathbf{F}_p^{\times},$$

and thus all ω_i are regular at Q if and only if all λ_j are. This completes the proof. □

Remark 3.8. If $\pi(P) = Q'$, then the analogous condition that a meromorphic differential on Y be regular at P is considerably more complicated as $\mathrm{ord}_P(y) < 0$, so the function h in the proof of Lemma 3.7 has a pole at Q and the entries of M are no longer in $\mathcal{O}_{X,Q}$. One simple case will be analyzed in the proof of Proposition 3.12, while a more complete analysis is deferred to Lemma 5.16.

We next study a filtration on $\pi_*\Omega^1_Y$ arising from the $G \simeq \mathbf{Z}/p\mathbf{Z}$ action on Y. We fix a generator $\tau \in G$ and obtain a map of sheaves $\Omega^1_Y \to \tau_*\Omega^1_Y$. After pushing forward using the equivariant π, we obtain a map $\tau : \pi_*\Omega^1_Y \to \pi_*\Omega^1_Y$. This induces a map at the stalk at the generic point.

For technical reasons, instead of $\pi_*\Omega^1_Y$, we will work with a slightly larger sheaf of meromorphic differentials defined as follows. Recall that $d_{Q'}$ is the order of the pole of f above Q' and that $p | d_{Q'}$. Furthermore, y has a pole of order $d_{Q'}$ at any point above Q'. Recall we defined $n_{Q',i} = (p-1-i)d_{Q'}$.

Definition 3.9. Define the sheaf $\mathscr{F}_0 \subset \pi_*\Omega^1_{Y,\eta}$ on X which on an open set U has sections

$$\left\{\omega = \sum_i \omega_i y^i : \omega \text{ is regular above all } Q \in U \setminus \{Q'\}, \text{ and } \mathrm{ord}_{Q'}\,\omega_i \ge -n_{Q',i}\ \forall\, i \text{ if } Q' \in U\right\}.$$

Lemma 3.10. *The natural G-action on $\pi_*\Omega^1_{Y,\eta}$ preserves $\mathscr{F}_0$, and $\pi_*\Omega^1_Y$ is a subsheaf of $\mathscr{F}_0$.*

Proof. As the first assertion is clear, it suffices to show that any meromorphic differential $\omega = \sum \omega_i y^i$ on Y that is regular above Q' has $\mathrm{ord}_{Q'}(\omega_i) \ge -(p-1-i)d_{Q'}$. We will do so by descending induction on i. Observe that

$$(\tau-1)^{p-1}\omega = (p-1)!\,\omega_{p-1}$$

which must be regular above Q': this happens only if $\mathrm{ord}_{Q'}(\omega_{p-1}) \ge 0$. To deal with $i = n$ for the inductive step, note that

$$(\tau-1)^n\omega = n!\,\omega_n + f_{n+1}(y)\omega_{n+1} + \cdots + f_{p-1}(y)\omega_{p-1}, \tag{3-4}$$

where $f_j(y)$ is a polynomial of degree at most $j-n$ of y. As in the proof of Lemma 3.7, let $h, h+1, h+2, \ldots, h+(p-1) \in \mathcal{O}_{X,Q'}$ correspond to the image of y in the local rings $\mathcal{O}_{Y,P'}$ under the identifications $\mathcal{O}_{Y,P'} \simeq \mathcal{O}_{X,Q'}$ for all P' lying over Q'. Note that $\mathrm{ord}_{Q'}(h) = -d_{Q'}/p \ge -d_{Q'}$. Together with (3-4), the assumption that ω is regular above Q' gives

$$\mathrm{ord}_{Q'}\big(n!\,\omega_n + f_{n+1}(h)\omega_{n+1} + \cdots f_{p-1}(h)\omega_{p-1}\big) \ge 0.$$

Furthermore, $\mathrm{ord}_{Q'}(f_j(h)\omega_j) \ge (j-n)(-d_Q) - (p-1-j)d_{Q'} = -(p-1-n)d_{Q'}$ by our inductive hypothesis, and it follows that $\mathrm{ord}_{Q'}(\omega_n) \ge -(p-1-n)d_{Q'}$. □

Thanks to Lemma 3.10, our fixed generator $\tau \in G$ induces a map $\tau : \mathscr{F}_0 \to \mathscr{F}_0$ that is compatible with the canonical G-action on $\pi_*\Omega^1_Y$. We use this to define a filtration on $\mathscr{F}_0$:

Definition 3.11. For $-1 \le i \le p-1$ let $W_i := \ker((\tau-1)^{i+1} : \mathscr{F}_0 \to \mathscr{F}_0) \subset \mathscr{F}_0$.

Note that $W_{-1} = 0$ and $W_{p-1} = \mathscr{F}_0$.

Proposition 3.12. *For $0 \le i \le p-1$, we have split exact sequences of $\mathcal{O}_X$-modules*

$$0 \to W_{i-1} \to W_i \to \Omega^1_X(E_i) \to 0 \tag{3-5}$$

with E_i as in Definition 3.4. A splitting is given by sending a section ω of $\Omega^1_X(E_i)$ to ωy^i.

Proof. We compute that $(\tau-1)$ reduces the degree in y:

$$(\tau-1)\omega_j y^j = \omega_j((y+1)^j - y^j) = \omega_j(jy^{j-1} + \cdots).$$

Hence $(W_i)_\eta = \bigoplus_{j=0}^{i} (\Omega^1_X)_\eta y^j$. We define a map $\psi_i : W_i \to \Omega^1_X(E_i)$ by the formula

$$\psi_i\left(\sum_{j=0}^{i} \omega_j y^j\right) := \omega_i.$$

To check ψ_i is well-defined, given any element $\omega = \sum_{j=0}^{i} \omega_j y^j$ in the stalk of W_i at $Q \in X$, it suffices to check that $\operatorname{ord}_Q(\omega_i) \ge -\operatorname{ord}_Q(E_i)$. Recall the definition of E_i from Definition 3.4 and that $\operatorname{ord}_Q(E_i) = n_{Q,i}$. For $Q \ne Q'$, the claim follows immediately from Lemma 3.7, while for $Q = Q'$ it follows from the definitions; for the $i = 0$ case use Definition 3.9. One checks easily that the kernel of ψ_i is W_{i-1}, and that the map sending a section ω of $\Omega^1_X(E_i)$ to the section ωy^i of W_i indeed provides a splitting. □

Remark 3.13. The filtration W_i on $\mathscr{F}_0$ is our replacement for the (perhaps more natural) filtration $V_i := \ker((\tau-1)^i : \pi_*\Omega^1_Y \to \pi_*\Omega^1_Y)$ on $\pi_*\Omega^1_Y$. Unfortunately, the corresponding exact sequence

$$0 \to V_{i-1} \to V_i \to \Omega^1_X(\widetilde{E}_i) \to 0$$

with $\widetilde{E}_i := \sum_{Q \in S} n_{Q,i}[Q]$ is *not* split: the natural splitting at the generic point given by $\omega \mapsto \omega y^i$ does not extend to a map of sheaves on X as ωy^i is not regular above Q' because y has a pole above Q'. It is precisely for this reason that we work instead with the sheaf $\mathscr{F}_0$ which has modified behavior above Q'.

Let $\operatorname{gr}^\bullet \mathscr{F}_0$ denote the associated graded sheaf for the filtration $\{W_i\}$.

Corollary 3.14. *There is an isomorphism of $\mathcal{O}_X$-modules*

$$\mathscr{F}_0 \simeq \operatorname{gr}^\bullet \mathscr{F}_0 \simeq \bigoplus_{i=0}^{p-1} \Omega^1_X(E_i). \tag{3-6}$$

Proof. The isomorphism is provided by the splittings of the sequences (3-5) of Proposition 3.12. □

Remark 3.15. The isomorphism of Corollary 3.14 is only as $\mathcal{O}_X$-modules: it is not compatible with the Cartier operator.

4. The Cartier operator on stalks

We keep the notation and assumptions of Section 3. In particular, $\pi : Y \to X$ is a branched Galois cover of smooth, projective, and connected curves over $k = \bar{k}$ with group $G \simeq \mathbf{Z}/p\mathbf{Z}$. Recall that we have fixed a point Q' of X over which π is unramified, and that the degree-p Artin–Schreier extension $K' := k(Y)$ of $K = k(X)$ is the splitting field of $y^p - y = f$ with $f \in K$ *minimal* in the sense of Definition 3.1. Writing simply F for the absolute Frobenius morphism, we denote by $V_Y : F_*\Omega^1_Y \to \Omega^1_Y$ and $V_X : F_*\Omega^1_X \to \Omega^1_X$ the Cartier operators on Y and X respectively. In this section, we will analyze these maps at the generic points η' and η of Y and X, respectively, and will construct an isomorphism

$$(\ker V_Y)_{\eta'} \simeq (\pi_* \ker V_Y)_\eta \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta$$

which we will show gives rise to an inclusion of sheaves

$$\varphi : \pi_* \ker V_Y \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$$

for certain auxiliary divisors D_i on X that we are able to control. In the next section, we will analyze the image of this map.

Lemma 4.1. *For $\omega \in \Omega^1_{Y,\eta'}$, write $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ with $\omega_i \in \Omega^1_{X,\eta}$. Then*

$$V_Y(\omega) = \sum_{j=0}^{p-1} \biggl(\sum_{i=j}^{p-1} V_X\Bigl(\binom{i}{j}\omega_i(-f)^{i-j}\Bigr)\biggr) y^j.$$

Proof. Using the relation $y = y^p - f$ and the fact that the Cartier operator is $1/p$-linear, we compute

$$\begin{aligned} V_Y(\omega_i y^i) &= V_Y(\omega_i (y^p - f)^i) \\ &= V_Y\Bigl(\sum_{j=0}^{i} \omega_i \binom{i}{j} y^{pj}(-f)^{i-j}\Bigr) \\ &= \sum_{j=0}^{i} V_X\Bigl(\omega_i \binom{i}{j}(-f)^{i-j}\Bigr) y^j, \end{aligned}$$

and the result follows by collecting y^j terms. ☐

Corollary 4.2. *Let $\omega_0, \dots, \omega_{p-1} \in \Omega^1_{X,\eta}$ and set $\omega := \sum_{i=0}^{p-1} \omega_i y^i$. Then $V_Y(\omega) = 0$ if and only if*

$$V_X(\omega_j) = -\sum_{i=j+1}^{p-1} V_X\Bigl(\binom{i}{j}\omega_i(-f)^{i-j}\Bigr) \tag{4-1}$$

for $0 \le j \le p-1$. In particular, if $V_Y(\omega) = 0$ then $V_X(\omega_{p-1}) = 0$ and $V_X(\omega_j)$ is determined via (4-1) *by ω_i for $i > j$.*

Proof. Clear from Lemma 4.1. □

Recall the notation of Definition 3.4. Using Corollaries 2.7 and 2.9 with S enlarged to include Q' and the zeroes of f, for each $0 \le i \le p-1$ pick a divisor $D_i = \sum_j [P_{i,j}]$ consisting of distinct points $P_{i,j}$ of X where f has neither pole nor zero (so π is unramified over $P_{i,j}$ by Lemma 3.5) and maps

$$\begin{aligned} r_i &: F_*(\Omega^1_X(E_i)) \to \ker V_X(F_*(E_i + pD_i)), \\ s_i &: \operatorname{Im} V_X(\bar{E}_i) \to F_*(\Omega^1_X(E_i + pD_i)). \end{aligned} \tag{4-2}$$

These induce corresponding maps on stalks at the generic point of X, which we again denote simply by r_i and s_i, respectively. Note that, by the very construction of these maps, any $\omega_i \in \Omega^1_{X,\eta}$ is determined by $V_X(\omega_i)$ and $\nu_i := r_i(\omega_i)$ via

$$\omega_i = \nu_i + s_i(V_X(\omega_i)). \tag{4-3}$$

Definition 4.3. Given the fixed choices of r_i and D_i above, we define a map of $\mathcal{O}_X$-modules

$$\begin{aligned} \varphi : \pi_* \ker V_Y \hookrightarrow F_*\mathscr{F}_0 &\simeq F_* \operatorname{gr}^\bullet \mathscr{F}_0 \\ &\simeq \bigoplus_{i=0}^{p-1} F_*(\Omega^1_X(E_i)) \xrightarrow{\oplus r_i} \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)) \end{aligned} \tag{4-4}$$

with the first inclusion coming from Lemma 3.10 and the middle isomorphisms coming from Corollary 3.14.

Let φ_η denote the induced map on stalks at the generic point of X.

Proposition 4.4. *The map φ_η is an isomorphism of $k(X)$-vector spaces. For $\omega = \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \ker V_Y)_\eta$ we have*

$$\varphi_\eta(\omega) = (r_0(\omega_0), r_1(\omega_1), \dots, r_{p-1}(\omega_{p-1})).$$

Proof. Let $\nu := (\nu_0, \dots, \nu_{p-1}) \in \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta$ be arbitrary, and also define $\omega_0, \dots, \omega_{p-1} \in \Omega^1_{X,\eta}$ as follows. Set $\omega_{p-1} := \nu_{p-1}$, and if ω_i has been defined for $i > j$, then define

$$\omega_j := \nu_j + s_j\biggl(-\sum_{i=j+1}^{p-1} V_X\biggl(\binom{i}{j}\omega_i(-f)^{i-j}\biggr)\biggr). \tag{4-5}$$

By construction, since $V_X \circ s_j$ is the natural inclusion, the differentials ω_j satisfy the relation (4-1) for all j, so that $\omega := \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \Omega^1_Y)_\eta$ lies in $(\pi_* \ker V_Y)_\eta$ thanks to Corollary 4.2. One checks easily that the resulting map $\nu \mapsto \omega$ is inverse to φ_η. □

Remark 4.5. Notice that the proof of Proposition 4.4 shows that for fixed j, ω_j depends only on ν_i for $j \ge i$, and in particular that $\omega_i = 0$ for $i \ge j$ if and only if $\nu_i = 0$ for $i \ge j$.

Unfortunately, (4-4) is not itself an isomorphism:

Example 4.6. Let $p = 5$, $X = \mathbf{P}^1$, and consider the Artin–Schreier covers $Y_i \to X$ for $i = 1, 2$ given by $y^p - y = f_i$, where $f_1(t) = t^{-3}$, and $f_2(t) = t^{-3} + t^{-2}$; each of these covers is ramified only over $Q := 0$

and has $d_Q = 3$. The a-number of Y_1 is 4, while the a-number of Y_2 is 3. Using the simple projectors of Remark 2.2, which allow us to take $D_i = 0$ for all $0 \leq i \leq p-1$, we obtain for $Y = Y_1, Y_2$ a map

$$\varphi : \pi_* \ker V_Y \to \bigoplus_{i=0}^{4} \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q])).$$

The induced map on global sections is injective (as φ_η is an isomorphism), but need not be an isomorphism in general. Indeed, given an element $(\nu_0, \dots, \nu_4) \in \bigoplus_i H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q])))$, let us investigate when it lies in the image of φ.

Recalling Definition 3.4, we calculate

$$n_{Q,i} = \begin{cases} 0, & i=4, \\ 1, & i=3, \\ 2, & i=2, \\ 2, & i=1, \\ 3, & i=0, \end{cases} \quad \text{and} \quad \dim_{\mathbf{F}_p} H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q]))) = \begin{cases} 0, & i=4, \\ 0, & i=3, \\ 1, & i=2, \\ 1, & i=1, \\ 2, & i=0, \end{cases}$$

so the elements of $\bigoplus_i H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q])))$ are easy to describe. The space is four dimensional, with basis

$$\nu_{0,3} = (t^{-3}dt, 0, 0, 0, 0), \quad \nu_{0,2} = (t^{-2}dt, 0, 0, 0, 0),$$
$$\nu_{1,2} = (0, t^{-2}dt, 0, 0, 0), \quad \nu_{2,2} = (0, 0, t^{-2}dt, 0, 0).$$

Using (4-5), we may compute the (unique) preimage under φ_η of these differentials for each of the covers under consideration. For Y_1, we find

$$\begin{aligned} \varphi_\eta^{-1}(\nu_{0,3}) &= t^{-3}dt, \\ \varphi_\eta^{-1}(\nu_{0,2}) &= t^{-2}dt, \\ \varphi_\eta^{-1}(\nu_{1,2}) &= t^{-2}dt \cdot y, \\ \varphi_\eta^{-1}(\nu_{2,2}) &= t^{-2}dt \cdot y^2. \end{aligned}$$

On the other hand, for Y_2 we have

$$\begin{aligned} \varphi_\eta^{-1}(\nu_{0,3}) &= t^{-3}dt, \\ \varphi_\eta^{-1}(\nu_{0,2}) &= t^{-2}dt, \\ \varphi_\eta^{-1}(\nu_{1,2}) &= t^{-2}dt \cdot y, \\ \varphi_\eta^{-1}(\nu_{2,2}) &= -t^{-6}dt + t^{-2}dt \cdot y^2. \end{aligned}$$

The computation of $\varphi_\eta^{-1}(\nu_{2,2})$ is special, as it depends on the exact choice of Artin–Schreier equation $y^p - y = f$. With $f = f_1 = t^{-3}$, we have

$$\varphi_\eta^{-1}(\nu_{2,2}) = t^{-2}dty^2$$

because there are no terms of the form $t^i dt$ with $i \equiv -1 \bmod 5$ appearing in $V_{\mathbf{P}^1}(\omega_0)$ when using Corollary 4.2. On the other hand, for $f = f_2 = t^{-3} + t^{-2}$ we have

$$\varphi_\eta^{-1}(\nu_{2,2}) = -t^{-6}dt + t^{-2}dty^2$$

since $V_{\mathbf{P}^1}(\omega_0) = -V_{\mathbf{P}^1}(t^{-2}(t^{-6} + 2t^{-5} + t^{-4})dt)$. This is not an element of $H^0(Y, \Omega^1_Y)$ because of the $t^{-6}dt$ term.

All of the other differentials showing up are regular. Thus, Y_1 has a-number 4, while Y_2 has a-number 3. This behavior illustrates why the a-number of the cover cannot depend only on the d_Q and must incorporate finer information (in this case, expressed as whether certain coefficients of powers of f are nonzero).

This example shows that to ensure the regularity of $\varphi_\eta^{-1}(\nu_i)$, the coefficients of the ν_i need to satisfy certain relations (in this case, the coefficient of $t^{-2}dt$ in ν_2 must be zero). These relations describe the image of φ, and are the subject of the next section. This example will be reinterpreted in that context in Example 5.21.

5. Short exact sequences and the kernel

Using the conventions and notation of Section 3–4, and motivated by Example 4.6, the goal of this section is to describe the image of the map (4-4) by means of a collection of linear relations on the coefficients of local expansions of meromorphic differentials at a fixed set of closed points containing the branch locus S. These linear relations will be encoded via an ascending filtration by subsheaves

$$0 \subset \mathscr{G}_{-1} \subset \mathscr{G}_0 \subset \cdots \subset \mathscr{G}_{p-1} \subset \mathscr{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)), \tag{5-1}$$

with $\mathscr{G}_{-1} = \operatorname{Im}(\varphi)$ and whose successive gradeds are identified with explicit skyscraper sheaves; see Theorem 5.20 for a precise statement.

For most of the argument, we will work with the sheaf $\mathscr{F}_0$ of Definition 3.9 and variants thereof having modified behavior at the auxiliary point Q'. Only at the very end (see Lemma 5.16) will we properly account for the modified behavior at Q' in order to recover the desired $\operatorname{Im}(\varphi) \simeq \pi_* \ker V_Y$, rather than the (generally larger) $\pi_* \ker(V_Y|_{\mathscr{F}_0})$. At a first reading, it could be useful to assume that $X = \mathbf{P}^1$ to remove the need for Q' and the divisors D_i coming from Corollary 2.7.

5A. ***Filtration and skyscraper sheaves.*** The first step is to define the filtration (5-1). Recall we have fixed projectors r_i as in (4-2) and used them to define φ in Definition 4.3. Recall the notation of Definition 3.4, noting Remark 3.6, and for $Q \in X$ choose a local uniformizer t_Q at Q.

If A is an abelian group, then by a slight abuse of notation we will again write A to denote the constant sheaf associated to A. For any k-valued point $P : \operatorname{Spec} k \to X$, we write P_*A for the pushforward of (the constant sheaf) A along P; it is the skyscraper sheaf on X supported at P with stalk $(P_*A)_P = A$. Likewise, if $h : A \to B$ is any homomorphism of abelian groups, we write $P_*h : P_*A \to P_*B$ for the induced morphism of sheaves on X.

Definition 5.1. Set $S'_j := S' \cup \sup(D_j)$. Define $g_{j+1} := \rho_j \circ \iota$, where ι is the inclusion

$$\iota : \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)) \hookrightarrow \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta \xrightarrow[\simeq]{\varphi_\eta^{-1}} (\pi_* \ker V_Y)_\eta \hookrightarrow F_*(\pi_*\Omega^1_Y)_\eta \xrightarrow[\simeq]{(1\text{-}5)} \bigoplus_{i=0}^{p-1} F_*\Omega^1_{X,\eta}$$

and ρ_j is the composite

$$\rho_j : \bigoplus_{i=0}^{p-1} F_*\Omega^1_{X,\eta} \xrightarrow{\pi_j} F_*\Omega^1_{X,\eta} \xrightarrow{\sigma_j} F_* \bigoplus_{Q\in S'_j} Q_*\left(\Omega^1_{X,Q}\left[\frac{1}{t_Q}\right]/t_Q^{-n_{Q,j}}\Omega^1_{X,Q}\right).$$

Here, the first and third maps in the definition of ι are the natural inclusions, the map π_j is projection onto the j-th summand of the direct sum, and σ_j is induced by the identifications $\Omega^1_{X,\eta} = \Omega^1_{X,Q}[1/t_Q]$ and the canonical quotient maps.

If $\nu := (\nu_0, \dots, \nu_{p-1})$ is a section of $\bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$ with $\iota(\nu) = (\omega_0, \dots, \omega_{p-1})$ then $r_i(\omega_i) = \nu_i$ and writing $\omega := \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \ker V_Y)_\eta$ we have $\varphi_\eta(\omega) = \nu$ and $g_{j+1}(\nu) = \sigma_j(\omega_j)$.

Definition 5.2. For $0 \le j \le p$, let $\mathscr{G}_j$ be the $\mathcal{O}_X$-submodule

$$\mathscr{G}_j \subset \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$$

whose sections are precisely $\nu = (\nu_0, \dots, \nu_{p-1})$ with $\iota(\nu) = (\omega_0, \dots, \omega_{p-1})$ such that ω_i is a section of $F_*(\Omega^1_X(E_i))$ whenever $j \le i \le p-1$. Define $\mathscr{G}_{-1} = \mathrm{Im}(\varphi)$.

Lemma 5.3. *Suppose that* $0 \le j \le p-1$. *Then*:

(i) $\mathscr{G}_j = \ker(g_{j+1}|_{\mathscr{G}_{j+1}})$.

(ii) *For* $Q \in X - S'_j$, *the inclusion* $\mathscr{G}_j \hookrightarrow \mathscr{G}_{j+1}$ *induces an isomorphism on stalks at* Q.

(iii) *The action of* $G = \mathbf{Z}/p\mathbf{Z}$ *on* $(F_*\pi_*\Omega^1_Y)_\eta \overset{\varphi_\eta}{\simeq} \bigoplus_i (\ker V_X)_\eta$ *preserves* $\mathscr{G}_j$.

(iv) $\mathscr{G}_0 \simeq \pi_* \ker(V_Y|_{\mathscr{F}_0})$.

(v) $\mathscr{G}_{-1} \simeq \pi_* \ker V_Y$.

(vi) $\mathscr{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$.

Proof. Let $\nu = (\nu_0, \dots, \nu_{p-1})$ be a section of $\bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$ over an open set $U \subseteq X$ with $\iota(\nu) = (\omega_0, \dots, \omega_{p-1})$. By definition, if $\nu \in \mathscr{G}_{j+1}(U)$ then $\omega_i \in F_*(\Omega^1_X(E_i))(U)$ for $i > j$. In addition, ω_j lies in $F_*(\Omega^1_X(E_j))(U)$ if and only if

$$\mathrm{ord}_Q(\omega_j) \ge -\,\mathrm{ord}_Q(E_j) = -n_{Q,j} \tag{5-2}$$

for every point Q of U. Condition (5-2) holds automatically for $Q \notin S'_j$. On the other hand, ν lies in the kernel of g_{j+1} if and only if $\omega_j \in t_Q^{-n_{Q,j}}\Omega^1_{X,Q}$ for all $Q \in S'_j \cap U$, which is equivalent to the condition (5-2), and this gives the claimed identification $\mathscr{G}_j = \ker(g_{j+1}|_{\mathscr{G}_{j+1}})$ in (i)enumi. This also shows (ii)enumi.

Since $E_i \ge E_j$ if $i < j$ and the induced action of a generator τ of G on $\mathscr{G}_p$ from Proposition 4.4 is given by $\tau(y) = y + 1$, it is clear that this action preserves $\mathscr{G}_j$ for each j.

It follows from Lemma 3.7 and Definition 5.2 that $\nu \in \mathscr{G}_0(U)$ if and only if $\omega := \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_*\Omega^1_Y)_\eta$ is regular over every point Q of U except possibly $Q = Q'$. It is then clear from the definition of the map ι, Definition 3.9, and Definition 4.3 that $\omega \in \mathscr{F}_0(U)$ and ω is killed by V_Y; that is, ω is a section of $\pi_* \ker(V_Y|_{\mathscr{F}_0})$, which gives the identification (iv)enumi. Since φ is injective with source $\pi_* \ker V_Y$, it induces an isomorphism $\pi_* \ker V_Y \simeq \operatorname{Im}(\varphi) =: \mathscr{G}_{-1}$ as in (v)enumi, and the description (vi)enumi of $\mathscr{G}_p$ is clear from definitions. □

The main step is to analyze the quotient $\mathscr{G}_{j+1}/\mathscr{G}_j$: it will be a skyscraper sheaf supported on S'_j. We will do so by studying the image of g_{j+1}. In particular, for $Q \in S'_j$ we will define $\mathcal{O}_{X,Q}$-modules $M_{j,Q}$ and a map of skyscraper sheaves

$$c_j : \operatorname{Im}(g_{j+1}|_{\mathscr{G}_{j+1}}) \to \bigoplus_{Q \in S'_j} Q_*(M_{j,Q}). \tag{5-3}$$

Composing with the restriction of g_{j+1} to $\mathscr{G}_{j+1}$, we obtain maps of sheaves on X

$$g'_{j+1} = c_j \circ (g_{j+1}|_{\mathscr{G}_{j+1}}) : \mathscr{G}_{j+1} \to \bigoplus_{Q \in S'_j} Q_*(M_{j,Q})$$

which we will show via stalk-wise calculations induce isomorphisms

$$g'_{j+1} : \mathscr{G}_{j+1}/\mathscr{G}_j \xrightarrow{\sim} \bigoplus_{Q \in S'_j} Q_*(M_{j,Q}),$$

for $0 \le j \le p-1$, thereby providing an explicit description of these quotients.

In order to motivate the definition of $M_{j,Q}$, we first record a result about orders of vanishing that will be useful in what follows.

Lemma 5.4. *Let $\nu = (\nu_0, \dots, \nu_{p-1}) \in H^0(X, \mathscr{G}_p)$ and set $(\omega_0, \dots, \omega_{p-1}) := \iota(\nu)$. For $0 \le j \le p-1$:*

(i) *For fixed j and $Q \in S$, suppose that $\nu_i = 0$ for $i > j$ and $\operatorname{ord}_Q(\nu_j) \ge -n$ for some nonnegative integer n. Let $\mu_{Q,i}$ be the largest multiple of p such that $\mu_{Q,i} + 1 \le n + d_Q(j-i)$. Then $\omega_i = 0$ for $i > j$, $\operatorname{ord}_Q(\omega_j) \ge -n$, and for $i < j$*

$$\operatorname{ord}_Q(\omega_i) \ge \min(-\mu_{Q,i} - 1, \operatorname{ord}_Q(\nu_i)) \ge -n - d_Q(j-i). \tag{5-4}$$

(ii) *If $Q \in D_j$ then $\operatorname{ord}_Q(\omega_j) \ge -p$.*

(iii) $\operatorname{ord}_{Q'}(\omega_j) \ge -d_{Q'}(p-1-j)$.

Proof. Suppose first that $Q \in S$, fix j with $0 \le j \le p-1$, and assume that $\nu_i = 0$ for $i > j$. Then $\omega_i = 0$ for $i > j$ thanks to Remark 4.5, while if $i = j$ we have $\omega_j = \nu_j$ thanks to (4-5), so our assumption that $\operatorname{ord}_Q(\nu_j) \ge -n$ gives $\operatorname{ord}_Q(\omega_j) \ge -n$ as well. Thus to establish (i)enumi, it remains to prove (5-4) for each $i < j$. We will establish this by descending induction on i. So suppose that $\ell < j$ and that (5-4) holds for all i with $\ell < i < j$. Since $\omega_i = 0$ for $i > j$, Corollary 4.2 gives $V_X(\omega_\ell) = V_X(\xi)$ for

$$\xi := -\sum_{i=\ell+1}^{j} \binom{i}{\ell} \omega_i (-f)^{i-\ell}. \tag{5-5}$$

Since $\mathrm{ord}_Q(f) = -d_Q$, our inductive hypothesis and the already established $\mathrm{ord}_Q(\omega_j) \geq -n$ immediately imply that

$$\mathrm{ord}_Q(\xi) \geq \min_{\ell < i \leq j} \big\{ -n - d_Q(j-i) - d_Q(i-\ell) \big\} = -n - d_Q(j-\ell).$$

Using (1-4), we see that $\mathrm{ord}_Q(V_X(\xi)) \geq -\mu_{Q,\ell}/p - 1$. It then follows from Corollary 2.9 that

$$\mathrm{ord}_Q(s_\ell(V_X(\xi))) \geq -\mu_{Q,\ell} - 1 \geq -n - d_Q(j-\ell),$$

so using (4-3) and remembering that $\mathrm{ord}_Q(\nu_\ell) \geq -n_{Q,\ell} = -\lceil (p-1-\ell) d_Q / p \rceil \geq -d_Q$ yields (5-4) for $i = \ell$, completing the inductive step.

The proof of (ii)enumi proceeds by a similar—but simpler—argument, using descending induction on j and the fact that for $Q \in D_j$ one has $\mathrm{ord}_Q(f) = 0$ and $\mathrm{ord}_Q(\nu_j) \geq -p$ by definition; we leave the details to the reader. Case (iii)enumi likewise follows from a similar argument, using $\mathrm{ord}_{Q'}(f) = -d_{Q'}$ and $\mathrm{ord}_{Q'}(\nu_j) \geq -(p-1-j) d_{Q'}$ by definition; see the proof of Lemma 5.14 for a more detailed version of the analysis in this case. ☐

Corollary 5.5. *Let $\nu = (\nu_0, \ldots, \nu_{p-1}) \in H^0(X, \mathscr{G}_p)$ and set $(\omega_0, \ldots, \omega_{p-1}) := \iota(\nu)$. For $Q \in S$, we have that*

$$\mathrm{ord}_Q(\omega_i) \geq -d_Q(p-1-i).$$

Proof. Take $j = p-1$ and $n = 0$ in Lemma 5.4(i)enumi. ☐

For a $k[\![t_Q]\!]$-module M, let $F_* M$ denote the $k[\![t_Q^p]\!]$-module with underlying additive group M and the action of t_Q^p on $F_* M$ given by multiplication by t_Q on M. Recall the definition of $n_{Q,i}$ from Definition 3.4.

Definition 5.6. For $0 \leq j \leq p-1$ and $Q \in S'_j$ let $m_{Q,j} := p(n_{Q,j} - 1)$ and $\beta_{Q,j} := \lfloor (p-1) n_{Q,j} / p \rfloor$, and set

$$M_{j,Q} := \begin{cases} k[\![t_Q^p]\!]/(t_Q^{p\beta_{Q,j}}) = F_*(k[\![t_Q]\!]/(t_Q^{\beta_{Q,j}})) & \text{if } Q \in S, \\ k[\![t_Q^p]\!]/(t_Q^{p(p-1)}) = F_*(k[\![t_Q]\!]/(t_Q^{p-1})) & \text{if } Q \in \sup(D_j), \\ 0 & \text{if } Q = Q', \end{cases}$$

considered as an $\mathcal{O}_{X,Q}$-module via $\mathcal{O}_{X,Q} \hookrightarrow \mathcal{O}^\wedge_{X,Q} \simeq k[\![t_Q]\!]$. Putting these together, we define a skyscraper sheaf on X

$$M_j := \bigoplus_{Q \in S'_j} Q_*(M_{j,Q}).$$

Note that by construction, if $Q \in S'_j$ then the stalk of M_j at Q is precisely $M_{j,Q}$, which justifies the notation.

Remark 5.7. For $Q \in S$, one checks easily that $m_{Q,j}$ is the largest integral multiple of p with $-m_{Q,j} \geq -(p-1-j)d_Q + 1$, and that $\beta_{Q,j}$ is the number of integer multiples of p between $-m_{Q,j}$ and $-n_{Q,j}$ inclusive. In view of Corollary 5.5, the skyscraper sheaves M_j will record all of the possible ways in which ω_j could fail to be a section of $\Omega^1_X(E_j)$ when $(\omega_0, \ldots, \omega_{p-1}) = \iota(\nu)$ for a section ν of $\mathscr{G}_{j+1}$.

In the next subsections, we will define the maps (5-3) stalk-by-stalk and check that they are surjective.

5B. ***Local calculations above ramified points.*** Fix j, let $Q \in S$ and as before let t_Q be a uniformizer at Q. For $\nu := (\nu_0, \ldots, \nu_{p-1}) \in \mathscr{G}_{p,Q}$, put $(\omega_0, \ldots, \omega_{p-1}) := \iota(\nu)$ and define $\omega = \sum_{i=0}^{p-1} \omega_i y^i$, so that $\varphi_\eta(\omega) = \nu$ and $V_X(\omega) = 0$. By (4-5), we have

$$\omega_j = \nu_j + s_j(V_X(\xi)), \qquad \text{where } \xi := -\sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j}, \tag{5-6}$$

and $\mathrm{ord}_Q(\xi) \geq -(p-1-j)d_Q$ thanks to Corollary 5.5. Working in the completion $\mathscr{O}^\wedge_{X,Q} \simeq k[\![t_Q]\!]$, we may therefore expand

$$\xi = \sum_{\ell=-N+1}^{\infty} a_\ell t_Q^\ell \frac{dt_Q}{t_Q} \tag{5-7}$$

where $N := -(p-1-j)d_Q$. Using (1-4), we then compute

$$s_j(V_X(\xi)) = \sum_{\ell=-m_{Q,j}/p}^{\infty} a_{p\ell} s_j\Big(t_Q^\ell \frac{dt_Q}{t_Q}\Big), \tag{5-8}$$

where $m_{Q,j}$ is as in Definition 5.6. If $p\ell - 1 \geq -n_{Q,j}$ then $\mathrm{ord}_Q(s_j(t_Q^\ell(dt_Q/t_Q))) \geq -n_{Q,j}$ by (2-8). Since we know that $\mathrm{ord}_Q(\nu_j) \geq -n_{Q,j}$, it follows from (5-6) that

$$g_{j+1}(\nu_0, \ldots, \nu_{p-1}) \in \mathrm{Im}(g_{j+1})_Q \subset F_*\Big(\Omega^1_{X,Q}\Big[\frac{1}{t_Q}\Big] / t_Q^{-n_{Q,j}} \Omega^1_{X,Q}\Big).$$

Furthermore, $g_{j+1}(\nu_0, \ldots, \nu_{p-1})$ is determined by the $a_{p\ell}$ for $-m_{Q,j} \leq p\ell \leq -n_{Q,j}$. There are $\beta_{Q,j}$ such integers by definition (see Remark 5.7). We thus obtain an injection $c_{j,Q} : \mathrm{Im}(g_{j+1})_Q \hookrightarrow M_{j,Q} = k[\![t_Q^p]\!]/(t_Q^{p\beta_{Q,j}})$ given by

$$c_{j,Q} : \sum_{\ell=-m_{Q,j}/p}^{\infty} a_{p\ell} s_j\Big(t_Q^\ell \frac{dt_Q}{t_Q}\Big) \bmod t^{-n_{Q,j}} k[\![t_Q]\!] dt_Q \mapsto \sum_{\ell=0}^{\beta_{Q,j}-1} a_{-m_{Q,j}+p\ell} t_Q^{p\ell} \bmod t_Q^{p\beta_{Q,j}} k[\![t]\!]. \tag{5-9}$$

One checks that this is a well-defined and $\mathscr{O}_X$-linear map, where the action of $\mathscr{O}_X$ on the first factor is the natural one coming from localizing the action on $F_*\Omega^1_X$ and the action of the second factor comes from the action of $\mathscr{O}_{X,Q}$ on $F_*k[\![t_Q]\!]$ via the identification $\mathscr{O}^\wedge_{X,Q} \simeq k[\![t_Q]\!]$.

Definition 5.8. For $Q \in S$ and $0 \leq j \leq p-1$ we define $g'_{j+1,Q}$ as the composite map of $\mathscr{O}_{X,Q}$-modules

$$g'_{j+1,Q} : \mathscr{G}_{j+1,Q} \xrightarrow{(g_{j+1}|_{\mathscr{G}_{j+1}})_Q} \mathrm{Im}(g_{j+1}|_{\mathscr{G}_{j+1}})_Q \xrightarrow{c_{j,Q}} M_{j,Q}$$

Proposition 5.9. *For $Q \in S$ and $0 \le j \le p-1$, the map $g'_{j+1,Q}$ is surjective.*

To prove this, we will make use of the following technical result. As above, we identify $\mathcal{O}^{\wedge}_{X,Q}$ with $k[\![t_Q]\!]$ and note that the completion of $\Omega^1_{X,Q}$ is isomorphic to $\Omega^1_{X,Q} \otimes_0 \mathcal{O}_{X,Q}\mathcal{O}^{\wedge}_{X,Q} \simeq k[\![t_Q]\!]dt_Q$.

Lemma 5.10. *Given $\omega_0, \dots, \omega_{p-2} \in k(\!(t_Q)\!)dt_Q$ satisfying $\mathrm{ord}_Q(\omega_i) \ge -(p-1-i)d_Q$ for $0 \le i \le p-2$, there exists $\omega_{p-1} \in k[\![t_Q]\!]dt_Q$ such that $\omega := \sum_{i=0}^{p-1} \omega_i y^i$ is an element of $(\pi_* \ker V_Y)_Q \otimes_0 \mathcal{O}_{X,Q} k(\!(t_Q)\!)$.*

Proof of Proposition 5.9. Assuming Lemma 5.10, we can prove Proposition 5.9 easily. As in the proof of Lemma 2.5, it suffices to check surjectivity after tensoring with $\mathcal{O}^{\wedge}_{X,Q}$ for each $Q \in S$. Note that the completion $\mathscr{G}^{\wedge}_{j+1,Q}$ is isomorphic to $\mathscr{G}_{j+1,Q} \otimes_0 \mathcal{O}_{X,Q}\mathcal{O}^{\wedge}_{X,Q}$.

Pick any $\omega_0, \dots \omega_{j-1}, \omega_{j+1}, \dots, \omega_{p-2} \in k(\!(t_Q)\!)dt_Q$ with $\mathrm{ord}_Q(\omega_i) \ge -n_{Q,i}$ for $i \ne j$, $p-1$, and set

$$\omega_j := s_j\left(t_Q^{-m_{Q,j}/p}\frac{dt_Q}{t_Q}\right).$$

By Corollary 2.9 and Definition 5.6, we have $\mathrm{ord}_Q(\omega_j) \ge -m_{Q,j} - 1 \ge -(p-1-j)d_Q$. As we visibly also have $-n_{Q,i} \ge -(p-1-i)d_Q$ for $i \ne j$, $p-1$, we may apply Lemma 5.10 to find ω_{p-1} such that $\omega := \sum_{i=0}^{p-1} \omega_i y^i$ lies in $(\pi_* \ker V_Y)_Q \otimes_0 \mathcal{O}_{X,Q} k(\!(t_Q)\!)$ and $\mathrm{ord}_Q(\omega_{p-1}) \ge 0$. Put $(\nu_0, \dots, \nu_{p-1}) := \varphi_\eta(\omega)$, and note that as $\nu_i = r_i(\omega_i)$ for all i, when $i \ne j$ we have $\mathrm{ord}_Q(\nu_i) \ge -n_{Q,i}$. On the other hand, $\nu_j = 0$ since $r_j \circ s_j = 0$ (as $r(m) = m - s(V_X(m))$ in Corollary 2.9). Thus $(\nu_0, \dots, \nu_{p-1}) \in \mathscr{G}^{\wedge}_{j+1,Q}$. By the definition of $c_{j,Q}$ in (5-9) and the choice of ω_j, we have arranged that

$$g'_{j+1,Q}(\nu_0, \dots \nu_{p-1}) = 1 \in k[\![t_Q^p]\!]/(t^{p\beta_{Q,j}}) = M_{j,Q} = M_{j,Q} \otimes_{\mathcal{O}_{X,Q}} \mathcal{O}^{\wedge}_{X,Q}.$$

This suffices to show that $g'_{j+1,Q}$ is surjective, as it is a map of $\mathcal{O}_{X,Q}$-modules. ☐

Proof of Lemma 5.10. For $h \in k(\!(t_Q)\!)$, let us write $\mathrm{coef}_m(h)$ for the coefficient of t_Q^m in h, and for $\xi \in k(\!(t_Q)\!)dt_Q$, write $\mathrm{coef}_m(\xi)$ for the coefficient of $t_Q^m(dt_Q/t_Q)$ in ξ. Given $\omega_0, \dots, \omega_{p-2}$ as in the statement of the lemma, it follows from the formula

$$0 = V_Y(\omega) = \sum_{j=0}^{p-1}\left(\sum_{i=j}^{p-1} V_X\left(\binom{i}{j}\omega_i(-f)^{i-j}\right)\right)y^j$$

of Lemma 4.1 that our goal is to construct $\omega_{p-1} \in k[\![t_Q]\!]dt_Q$ satisfying

$$V_X\left(\binom{p-1}{j}\omega_{p-1}(-f)^{p-1-j}\right) = -\sum_{i=j}^{p-2} V_X\left(\binom{i}{j}\omega_i(-f)^{i-j}\right) \tag{5-10}$$

for $0 \le j < p-1$. By (1-4) it suffices to check that for each j and every $m \equiv 0 \bmod p$, the m-th coefficients of both sides of (5-10) agree. So we need to show that

$$\sum_i \binom{p-1}{j} \mathrm{coef}_i(\omega_{p-1})\, \mathrm{coef}_{m-i}((-f)^{p-1-j}) = -\,\mathrm{coef}_m\left(\sum_{\ell=j}^{p-2}\binom{\ell}{j}\omega_\ell(-f)^{\ell-j}\right). \tag{5-11}$$

The right side is determined by the choice of $\omega_0, \dots, \omega_{p-2}$. We observe:

(i) $\operatorname{coef}_i((-f)^{p-1-j}) = 0$ for $i < -(p-1-j)d_Q$;

(ii) $\operatorname{coef}_{-(p-1-j)d_Q}((-f)^{p-1-j}) \neq 0$;

(iii) $\operatorname{coef}_m\left(\sum_{\ell=j}^{p-2}\binom{\ell}{j}\omega_\ell(-f)^{\ell-j}\right) = 0$ for $m \leq -(p-1-j)d_Q$.

Notice that (i)enumi and (ii)enumi are immediate consequences of the fact that $\operatorname{ord}_Q(f) = -d_Q$, while (iii)enumi follows from the fact that by hypothesis

$$\operatorname{ord}_Q(\omega_\ell(-f)^{\ell-j}) \geq -(p-1-\ell)d_Q + d_Q(j-\ell) = -(p-1-j)d_Q.$$

We construct $\omega_{p-1} = \sum_i b_i t_Q^i (dt_Q/t_Q)$ by specifying $b_i = \operatorname{coef}_i(\omega_{p-1})$ inductively as follows. For $i \leq 0$, set $b_i = 0$; this choice implies that for each j and any $m \leq -(p-1-j)d_Q$ that is a multiple of p, the left side of (5-11) vanishes, and likewise for the right side by (iii)enumi. To specify b_1, choose $0 \leq j < p$ so $d_Q(p-1-j) \equiv 1 \bmod p$. In light of (i)enumi and the fact that $b_i = 0$ for $i \leq 0$, (5-11) with $m = -d_Q(p-1-j)+1$ specifies that

$$\cdots + 0 + \binom{p-1}{j} b_1 \operatorname{coef}_{-d_Q(p-1-j)}((-f)^{p-1-j}) + 0 + \cdots = -\operatorname{coef}_{-d_Q(p-1-j)+1}\left(\sum_{\ell=j}^{p-2}\binom{\ell}{j}\omega_\ell(-f)^{\ell-j}\right),$$

where the right side has already been specified. By (ii)enumi, there is a unique solution b_1.

In general, if N is any positive integer and b_i has been chosen for all $i < N$, first choose $0 \leq j < p$ so that $d_Q(p-1-j) \equiv N \bmod p$. The right side of (5-11) with $m = -d_Q(p-1-j)+N$ is already specified. The finitely many nonzero terms of the left side with $i < N$ are determined by our previous choices, while the terms with $i > N$ are zero by (i)enumi. By (ii)enumi, we may uniquely solve for b_N. Since $p \nmid d_Q$, when considering b_i with i between b and $b+(p-1)$, each j satisfying $0 \leq j < p$ occurs once. Thus the inductive choice of the b_i's makes (5-11) hold for every j and every m that is a multiple of p. This completes the proof. □

5C. ***Local calculations at poles of sections.*** We repeat the analysis of Section 5B for points $Q \in \sup(D_j)$. Fix j and $Q \in \sup(D_j)$, let t_Q be a uniformizer of $\mathcal{O}_{X,Q}$, and let $\nu := (\nu_0, \ldots, \nu_{p-1}) \in \mathcal{G}_{p,Q}$. As before, we put $\omega := \sum_{i=0}^{p-1}\omega_i y^i$ where $(\omega_0, \ldots, \omega_{p-1}) = \iota(\nu)$, and note that $V_Y(\omega) = 0$.

We have $\operatorname{ord}_Q(\omega_j) \geq -p$ thanks to Lemma 5.4(ii)enumi, whence a local expansion $\omega_j = \sum_{i=-p}^{\infty} b_i t_Q^i dt_Q$ in the completed stalk at Q. By definition, we then have $g_{j+1,Q}(\nu) \equiv \sum_{i=-p}^{-1} b_i t_Q^i dt_Q \bmod k[[t_Q]]dt_Q$, and we define a map $c_{j,Q} : \operatorname{Im}(g_{j+1})_Q \to M_{j,Q} = k[[t_Q]]/(t_Q^{p-1})$ by

$$c_{j,Q} : \sum_{i=-p}^{-1} b_i t_Q^i dt_Q \bmod k[[t_Q]]dt_Q \mapsto \sum_{i=0}^{p-2} b_{i-p} t_Q^i \bmod t_Q^{p-1}k[[t]]. \tag{5-12}$$

This is a well-defined map of $\mathcal{O}_{X,Q}$-modules, using the natural maps $\mathcal{O}_{X,Q} \hookrightarrow \mathcal{O}_{X,Q}^{\wedge} \simeq k[[t_Q]]$.

Definition 5.11. For $0 \le j \le p-1$ and $Q \in \sup(D_j)$ we define $g'_{j+1,Q}$ as the $\mathcal{O}_{X,Q}$-linear composite

$$g'_{j+1,Q} : \mathscr{G}_{j+1,Q} \xrightarrow{(g_{j+1}|_{\mathscr{G}_{j+1}})_Q} \operatorname{Im}(g_{j+1}|_{\mathscr{G}_{j+1}})_Q \xrightarrow{c_{j,Q}} M_{j,Q}.$$

In other words, $g'_{j+1,Q}$ extracts the coefficients of the monomials $t_Q^{-i} dt_Q$ for $2 \le i \le p$ in the local expansion of ω_j at Q. This suffices to check regularity at Q in the following sense:

Lemma 5.12. *If* $\nu \in \mathscr{G}_{j+1,Q}$ *and* $g'_{j+1,Q}(\nu) = 0$, *then* $\operatorname{ord}_Q(\omega_j) \ge -n_{Q,j} = 0$, *i.e.*, $\nu \in \mathscr{G}_{j,Q}$.

Proof. Recall (4-3) that $\omega_j = \nu_j + s_j(V_X(\omega_j))$. By definition of $\mathscr{G}_{j+1}$ (Definition 5.2), we have $\operatorname{ord}_Q(\omega_i) \ge 0$ for $i > j$, and $\operatorname{ord}_Q(f) = 0$ by the very choice of the divisors D_j in Section 4. It follows that the differential

$$\xi := -\sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j}$$

is regular at Q. By Corollary 4.2 and the fact that $V(\omega) = 0$, we have $V_X(\omega_j) = V_X(\xi)$, which must also then be regular at Q, so working in the completed stalk at Q we may expand it locally as $V_X(\omega_j) = (a_1 + a_2 t_Q + \cdots) dt_Q$. But then

$$\omega_j = \nu_j + s_j(V_X(\omega_j)) = \nu_j + a_1^p s_j(dt_Q) + a_2^p t_Q^p s_j(dt_Q) + \cdots, \tag{5-13}$$

Now $V_X(\nu_j) = 0$, so the local expansion of ν_j has no $t_Q^{-1} dt_Q$-term, and since $V_X \circ s_j$ is the identity, $s_j(dt_Q)$ has no $t_Q^{-1} dt_Q$-term either. From (5-13) and the fact that $\operatorname{ord}_Q(s_j(dt_Q)) \ge -p$ (by the very construction of s_j), we conclude that the local expansion of ω_j has no $t_Q^{-1} dt_Q$-term. If $g'_{j+1,Q}(\nu) = 0$ as well, then the local expansion of ω_j has no $t_Q^{-i} dt_Q$-terms for $2 \le i \le p$ either, and ω_j is regular at Q. □

Lemma 5.13. *For* $0 \le j \le p-1$ *and* $Q \in \sup(D_j)$, *the map* $g'_{j+1,Q}$ *is surjective.*

Proof. Such surjectivity may be checked after passing to completions, where it follows immediately from the proof of Lemma 5.12 as $(\ker V_X)^{\wedge}_Q[1/t_Q]$ is (topologically) generated as a k-vector space by $\{t_Q^{i-1} dt_Q \ : \ p \nmid i\}$, and the only restriction on ν_j is that $\operatorname{ord}_Q(\nu_j) \ge -p$. □

5D. *Local calculations at Q'.* Finally, we analyze the behavior at Q'. Because we chose to work with the sheaf $\mathscr{F}_0$ instead of $\pi_* \Omega^1_Y$, the relationship between $\mathscr{G}_j$ and $\mathscr{G}_{j+1}$ at Q' is particularly simple when $0 \le j \le p-1$:

Lemma 5.14. *For* $0 \le j \le p-1$, *the natural inclusion* $\mathscr{G}_{j,Q'} \to \mathscr{G}_{j+1,Q'}$ *is an isomorphism.*

Proof. We check surjectivity. Consider $\nu := (\nu_0, \ldots, \nu_{p-1}) \in \mathscr{G}_{p,Q'}$ with $\iota(\nu) = (\omega_0, \ldots, \omega_{p-1})$. By Lemma 5.4 (iii)enumi, we in fact have $\operatorname{ord}_{Q'}(\omega_i) \ge -(p-1-i) d_{Q'}$ for *all* i, which implies in particular that $\nu \in \mathscr{G}_{0,Q'}$, whence the composite map $\mathscr{G}_{0,Q'} \hookrightarrow \mathscr{G}_{j,Q'} \hookrightarrow \mathscr{G}_{j+1,Q'} \hookrightarrow \mathscr{G}_{p,Q'}$ is an isomorphism, which gives the claim. As the details of the proof of Lemma 5.4 (iii)enumi were abridged, we will spell them out here (sans the descending induction, which is unnecessary if we assume that $\nu \in \mathscr{G}_{j+1,Q'}$). So assume that $\nu \in \mathscr{G}_{j+1,Q'}$, or equivalently that $\mathrm{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ for $i \geq j+1$. Since $\mathrm{ord}_{Q'}(f) = -d_{Q'}$, we compute that

$$\mathrm{ord}_{Q'}(\omega_i(-f)^{i-j}) \geq -(p-1-i)d_{Q'} + (j-i)d_{Q'} = -(p-1-j)d_{Q'}$$

for $i \geq j+1$. On the other hand, for *any* meromorphic differential ξ on X, Corollary 2.9 shows that $\mathrm{ord}_{Q'}(s_j(V_X(\xi))) \geq \mathrm{ord}_{Q'}(\xi)$. Then (4-5) shows that $\mathrm{ord}_{Q'}(\omega_j) \geq -(p-1-j)d_{Q'}$, and hence that $\nu \in \mathscr{G}_{j,Q'}$ as desired. □

The downside of working with the sheaf $\mathscr{F}_0$ from Definition 3.9 is that we require a separate analysis to relate $\mathscr{G}_0$ with the sheaf we care about, $\mathrm{Im}(\varphi) \simeq \pi_* \ker V_Y$. Recall (Definition 5.2) that we defined $\mathscr{G}_{-1} := \mathrm{Im}(\varphi) \subseteq \mathscr{G}_0$.

Definition 5.15. Define

$$M_{-1,Q'} := \bigoplus_{i=0}^{p-1} \bigl(k[\![t_{Q'}^p]\!]/(t_{Q'}^{(p-1-i)d_{Q'}})\bigr)^{\oplus(p-1)}$$

and write $M_{-1} := Q'_*(M_{-1,Q'})$ for the skyscraper sheaf on X supported at Q' with stalk $M_{-1,Q'}$. Put $S'_{-1} := \{Q'\}$.

Lemma 5.16. *The cokernel of the natural inclusion map $\mathscr{G}_{-1} \hookrightarrow \mathscr{G}_0$ is isomorphic to M_{-1}.*

We interpret Lemma 5.16 as giving a short exact sequence

$$0 \to \mathscr{G}_{-1} \to \mathscr{G}_0 \xrightarrow{g_0'} M_{-1} \to 0, \tag{5-14}$$

where g_0' is the composition of the natural map $\mathscr{G}_0 \to \mathrm{coker}(\mathscr{G}_{-1} \to \mathscr{G}_0)$ with the isomorphism of Lemma 5.16.

Proof. We already know that the map $\mathscr{G}_{-1} \to \mathscr{G}_0$ is an inclusion, and is an isomorphism away from Q', so this is a local question at Q'. We have $\mathscr{G}_{-1} \simeq \pi_* \ker V_Y$ and $\mathscr{G}_0 \simeq \pi_* \ker(V_Y|_{\mathscr{F}_0})$ thanks to Lemma 5.3, and $(\pi_* \ker V_Y)_{Q'}$ consists of meromorphic forms on Y that are regular above Q' and lie in the kernel of V_Y, while $\pi_* \ker(V_Y|_{\mathscr{F}_0})_{Q'}$ consists of meromorphic $\omega = \sum_i \omega_i y^i$ in the kernel of V_Y that satisfy $\mathrm{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ for all i (Definition 3.9). Despite the fact that π is étale over Q', the decomposition $\omega = \sum_i \omega_i y^i$ is tricky to analyze since the element f defining the Artin–Schreier extension of function fields has a pole at Q' (as does y).

By the very choice of f in Section 3, we may find a meromorphic function $g \in K = k(X)$ such that $f' := f + g^p - g$ has $\mathrm{ord}_{Q'}(f') = 0$; necessarily $\mathrm{ord}_{Q'}(g) = -d_{Q'}/p$. Let $y' = y + g$, so that

$$(y')^p - y' = f'$$

and $K' = k(Y)$ is the Artin–Schreier extension of K given by adjoining y'. Observe that y' is regular above Q'. Any meromorphic differential ω on Y may be written as

$$\omega = \sum_{i=0}^{p-1} \omega_i y^i = \sum_{i=0}^{p-1} \omega_i' (y')^i. \tag{5-15}$$

While the condition of ω being regular above Q' is tricky to describe in terms of the ω_i, it is simple to describe in terms of the ω'_i. Indeed, the proof of Lemma 3.7 shows that ω is regular above Q' if and only if $\operatorname{ord}_{Q'}(\omega'_i) \geq 0$ for $0 \leq i \leq p-1$. We deduce that the stalk of $\mathscr{G}_{-1}$ at Q' is isomorphic to

$$N_{-1} := \left\{ \omega = \sum_{i=0}^{p-1} \omega'_i (y')^i : \omega \in (\pi_* \ker V_Y)_{Q'} \text{ and } \omega'_i \in \Omega^1_{X,Q'} \right\}.$$

On the other hand, substituting $y' = y + g$ in (5-15) and collecting y^j-terms gives

$$\omega_j = \sum_{i=j}^{p-1} \binom{i}{j} \omega'_i g^{i-j}.$$

A descending induction on i using $\operatorname{ord}_{Q'}(g) = -d_{Q'}/p$ then shows that $\operatorname{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ if and only if $\operatorname{ord}_{Q'}(\omega'_i) \geq -(p-1-i)d_{Q'}$, and we conclude that the stalk of $\mathscr{G}_0$ at Q' is isomorphic to

$$N_0 := \left\{ \omega = \sum_{i=0}^{p-1} \omega'_i (y')^i : \omega \in (\pi_* \ker V_Y)_{Q'} \text{ and } \omega'_i \in t_{Q'}^{-(p-1-i)d_{Q'}} \Omega^1_{X,Q'} \right\}.$$

We view both N_{-1} and N_0 as $k[t^p_{Q'}]$-modules. To complete the proof, it suffices to show the cokernel of the natural inclusion $N_{-1} \hookrightarrow N_0$ is isomorphic to $M_{-1,Q'}$. It suffices to do so after completing.

To analyze the cokernel, we first observe that (2-2)–(2-3) give a section $s : (\operatorname{Im} V_X)^\wedge_{Q'} \to F_*\Omega^{1,\wedge}_{X,Q'}$ and projector $r : F_*\Omega^{1,\wedge}_{X,Q'} \to (\ker V_X)^\wedge_{Q'}$ to the completion of the exact sequence (2-1) at Q'. By Corollary 4.2, we have the relations

$$V_X(\omega'_j) = -\sum_{i=j+1}^{p-1} V_X\left(\binom{i}{j} \omega'_i (-f')^{i-j}\right) \tag{5-16}$$

for $0 \leq j \leq p-1$, and in particular $V_X(\omega'_j)$ is determined by ω'_i for $i > j$. It follows that the map

$$\varphi'_{Q'} : (\pi_* \ker V_Y)^\wedge_{Q'} \to \bigoplus_{i=0}^{p-1} (\ker V_X)^\wedge_{Q'}, \qquad \sum_{i=0}^{p-1} \omega'_i (y')^i \mapsto (r(\omega'_0), \ldots, r(\omega'_{p-1}))$$

is an isomorphism after inverting $t^p_{Q'}$. Now for any differential ξ on X, we have $\operatorname{ord}_{Q'}(s(V_X(\xi))) \geq p\lfloor \operatorname{ord}_{Q'}(\xi)/p \rfloor$. As $\operatorname{ord}_{Q'}(f') = 0$, it follows from this and (5-16) that:

- $s(V_X(\omega'_j)) \in \Omega^{1,\wedge}_{X,Q'}$ if $\omega'_i \in \Omega^{1,\wedge}_{X,Q'}$ for $i > j$,
- $s(V_X(\omega'_j)) \in t_{Q'}^{-(p-1-j)d_{Q'}} \Omega^{1,\wedge}_{X,Q'}$ if $\omega'_i \in t_{Q'}^{-(p-1-i)d_{Q'}} \Omega^{1,\wedge}_{X,Q'}$ for $i > j$.

As $\omega'_j = r(\omega'_j) + s(V_X(\omega'_j))$ for all j, together with (5-16) this shows that $\varphi'_{Q'}$ induces identifications

$$N^\wedge_{-1} \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)^\wedge_{Q'} \quad \text{and} \quad N^\wedge_0 \simeq \bigoplus_{i=0}^{p-1} (t^p_{Q'})^{-(p-1-i)(d_{Q'}/p)} (\ker V_X)^\wedge_{Q'}$$

as submodules of $\bigoplus_i (\ker V_X)^{\wedge}_{Q'}[1/t^p_{Q'}]$, with the natural inclusion $N^{\wedge}_{-1} \hookrightarrow N^{\wedge}_0$ corresponding to the canonical inclusion of direct summands. As $(\ker V_X)^{\wedge}_{Q'}$ is a free $k[\![t^p_{Q'}]\!]$-module generated by $dt_{Q'}, \dots, t^{p-1}_{Q'} dt_{Q'}$, this completes the proof. □

Remark 5.17. Tracing through the proof of Lemma 5.16, we see that the image under $H^0(g'_0)$ of $H^0(X, \ker((\tau-1)^{j+1} : \mathscr{G}_0 \to \mathscr{G}_0))$ has dimension at most

$$\sum_{i=0}^{j} \frac{(p-1-i)d_{Q'}}{p}(p-1)$$

as $\omega_i = \omega'_i = 0$ for $i > j$ whenever $\omega = \sum_i \omega_i y^i = \sum_i \omega'_i (y')^i$ is killed by $(\tau-1)^{j+1}$.

5E. ***Short exact sequences.*** Combining the local analyses of Section 5B–5D, we can finally construct the desired exact sequences relating the $\mathscr{G}_j$. Recall the definitions of the skyscraper sheaf M_j in Definition 5.6 (for $0 \le j \le p-1$) and Definition 5.15 (for $j = -1$). From Sections 5B, 5C, and 5D we have maps from the stalks of $\mathscr{G}_j$ to pieces of these skyscraper sheaves. We now put them together.

Definition 5.18. For $0 \le j \le p-1$, put $S_j := S \cup \sup(D_j)$ and define

$$c_j : \operatorname{Im}(g_{j+1}) \to \bigoplus_{Q \in S_j} Q_*(\operatorname{Im}(g_{j+1})_Q) \xrightarrow{\bigoplus_{Q \in S_j} Q_*(c_{j,Q})} \bigoplus_{Q \in S_j} Q_*(M_{j,Q}) =: M_j,$$

where the first map is the canonical one. Let $g'_{j+1} := c_j \circ (g_{j+1}|_{\mathscr{G}_{j+1}})$.

Recall that we also defined g'_0 in (5-14).

Remark 5.19. Notice that the map induced by g'_{j+1} on stalks at Q coincides with the previously defined map $g'_{j+1,Q}$, justifying our notation.

Theorem 5.20. *For $-1 \le j \le p-1$, there are short exact sequences of sheaves on X*

$$0 \to \mathscr{G}_j \to \mathscr{G}_{j+1} \xrightarrow{g'_{j+1}} M_j \to 0.$$

We have

$$\mathscr{G}_{-1} = \operatorname{Im}(\varphi) \simeq \pi_* \ker V_Y \qquad \text{and} \qquad \mathscr{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)).$$

Furthermore, M_j is a skyscraper sheaf supported on $S_j := S \cup \sup(D_j)$ for $j \ge 0$, and supported at Q' for $j = -1$.

Proof. We first establish the exact sequence. The case $j = -1$ is just Lemma 5.16, so suppose $0 \le j \le p-1$. Left exactness is obvious from the definition of $\mathscr{G}_j$ and $\mathscr{G}_{j+1}$. We can check the rest locally. For points Q not in S'_j, exactness in the middle and on the right is (ii)enumi of Lemma 5.3 plus the fact that $M_{j,Q} = 0$. At Q' such exactness is the content of Lemma 5.14 plus the fact that $M_{j,Q'} = 0$. For $Q \in S$, exactness in the middle is simply the first statement of Lemma 5.3 plus the observation that $c_{j,Q}$ is injective when $Q \in S$, while exactness on the right is Proposition 5.9. For $Q \in \sup(D_j)$, exactness in the middle and on the right follow from Lemmas 5.12 and 5.13, respectively. Finally, the statements about the support of M_j are clear from the definitions. □

Example 5.21. We continue the notation of Example 4.6, working with the two covers Y_1, Y_2 of $\mathbf{P}^1$ given by $y^5 - y = f_1$ and $y^5 - y = f_2$, respectively. The fact that $\omega = \sum \omega_i y^i = \varphi_\eta^{-1}(\nu_{2,2})$ is regular for one cover but not the is captured by the machinery of this section as follows.

Note that the element $\nu_{2,2}$ naturally lies in $H^0(\mathbf{P}^1, \mathscr{G}_2)$. For both covers, it is moreover in the kernel of $H^0(g_2')$: looking at the definition of g_2' in Section 5B, the computation that $H^0(g_2')(\nu_{2,2}) = 0$ is equivalent to the computation that $\omega_1 = 0$. Thus, in both cases we also have $\nu_{2,2} \in H^0(\mathbf{P}^1, \mathscr{G}_1)$.

In Example 4.6, we computed ω_0 using the relations

$$sV_{\mathbf{P}^1}(\omega_0) = -sV_{\mathbf{P}^1}(t^{-2}(t^{-6}dt)) = 0,$$

and

$$sV_{\mathbf{P}^1}(\omega_0) = -sV_{\mathbf{P}^1}(t^{-2}(t^{-6} + 2t^{-5} + t^{-4})dt) = -t^{-6}dt$$

for the covers Y_1 and Y_2, respectively. The definition of g_1' in Section 5B is exactly recording the coefficient of $t^{-6}dt$. Thus $H^0(g_1')(\nu_{2,2})$ is 0 in the case of Y_1, but nonzero for the cover Y_2, reflecting the fact that $\varphi_\eta^{-1}(\nu_{2,2})$ is regular for Y_1 but not Y_2. We therefore see that in the case of Y_1 we have $a_{Y_1} = \dim H^0(\mathbf{P}^1, \mathscr{G}_0) = 4$ while $a_{Y_2} = \dim H^0(\mathbf{P}^1, \mathscr{G}_0) = 3$ in the case of Y_2.

6. Bounds

We continue with the notation and conventions of the previous sections, so $\pi : Y \to X$ is a degree-p Artin–Schreier cover of smooth projective curves over k with branch locus $S \subseteq X$. In this section we will use the short exact sequences of Theorem 5.20 to extract bounds on the a-number of Y in terms of the a-number of X and the breaks in the ramification filtrations at points above $Q \in S$. Our starting point is the equality

$$\begin{aligned} a_Y &= \dim_k \ker\big(V_Y : H^0(Y, \Omega^1_Y) \to H^0(Y, \Omega^1_Y)\big) \\ &= \dim_k H^0(Y, \ker V_Y) = \dim_k H^0(X, \mathscr{G}_{-1}), \end{aligned}$$

which follows immediately from Lemma 5.3.

6A. *Abstract bounds.* We will first obtain a relatively abstract upper bound on the a-number using the short exact sequences of Theorem 5.20. To streamline the analysis, we first encode the information contained in these exact sequences in the study of a single linear transformation.

Definition 6.1. For $0 \leq j \leq p-1$ define

$$\tilde{g}_{j+1} : H^0(X, \mathscr{G}_p) \xrightarrow{H^0(c_j \circ g_{j+1})} H^0(X, M_j) = \bigoplus_{Q \in S_j'} M_{j,Q} \tag{6-1}$$

to be the map on global sections induced by $c_j \circ g_{j+1} : \mathscr{G}_p \to M_j$, where c_j and g_{j+1} are as in Definitions 5.18 and 5.1, respectively. Define

$$\tilde{g} : H^0(X, \mathscr{G}_p) \to \bigoplus_{j=0}^{p-1} H^0(X, M_j) \quad \text{by} \quad \tilde{g}(\nu) := (\tilde{g}_1(\nu), \ldots, \tilde{g}_p(\nu)) \tag{6-2}$$

Finally define

$$\tilde{g}_0 : H^0(X, \mathscr{G}_0) \xrightarrow{H^0(g_0')} H^0(X, M_{-1}) = M_{-1,Q'}, \tag{6-3}$$

where g_0' is as in (5-14) and $M_{-1,Q'}$ is as in Definition 5.15.

Remark 6.2. By construction, the restriction of $\tilde{g}_{j+1}$ to $H^0(X, \mathscr{G}_{j+1}) \subseteq H^0(X, \mathscr{G}_p)$ coincides with the map $H^0(g'_{j+1}) : H^0(X, \mathscr{G}_{j+1}) \to H^0(X, M_j)$, where g'_{j+1} is as in Definition 5.18 for $j \geq 0$ and as in (5-14) for $j = -1$. While the use of the maps g'_{j+1} is essential for Theorem 5.20, our analysis below is much simpler when phrased in terms of $\tilde{g}$ and $\tilde{g}_0$ as the following key Lemma indicates.

Lemma 6.3. *The kernel of $\tilde{g}$ is $H^0(X, \mathscr{G}_0)$.*

Proof. Let $\nu \in H^0(X, \mathscr{G}_p)$ with $\tilde{g}(\nu) = 0$, i.e., $\tilde{g}_{j+1}(\nu) = 0$ for $0 \leq j \leq p-1$. Supposing that $\nu \in H^0(X, \mathscr{G}_{j+1})$ for *some* $j \leq p-1$, we have $H^0(g'_{j+1})(\nu) = \tilde{g}_{j+1}(\nu) = 0$ by assumption and the compatibility between g'_{j+1} and $\tilde{g}_{j+1}$ noted in Remark 6.2. Thus $\nu \in H^0(X, \mathscr{G}_j)$ thanks to Theorem 5.20 and left exactness of H^0. We conclude by descending induction on j that $\nu \in H^0(X, \mathscr{G}_0)$, whence $\ker(\tilde{g}) \subseteq H^0(X, \mathscr{G}_0)$, and the reverse containment is clear. □

Definition 6.4. Set $N_1(X, \pi) := \dim_k \operatorname{Im}(\tilde{g})$ and $N_2(X, \pi) := \dim_k \operatorname{Im}(\tilde{g}_0)$, and let

$$N(X, \pi) = N_1(X, \pi) + N_2(X, \pi).$$

Define

$$U(X, \pi) := \sum_{i=0}^{p-1} \dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))) - N(X, \pi).$$

Lemma 6.5. *We have that $a_Y = U(X, \pi)$.*

Proof. Lemma 6.3 and the rank-nullity theorem give

$$\dim_k H^0(X, \mathscr{G}_0) = \dim_k H^0(X, \mathscr{G}_p) - N_1(X, \pi).$$

Likewise, the $j = -1$ case of Theorem 5.20, left exactness of H^0 and the rank-nullity theorem yield

$$a_Y = \dim_k H^0(Y, \ker V_Y) = \dim_k H^0(X, \mathscr{G}_{-1}) = \dim_k H^0(X, \mathscr{G}_0) - N_2(X, \pi).$$

Combining these equations with the computation

$$\dim_k H^0(X, \mathscr{G}_p) = \sum_{i=0}^{p-1} \dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))),$$

which follows at once from Lemma 5.3 (vi)enumi, gives the claimed equality. □

The lower bound is a more elaborate application of linear algebra. The idea is to find a family of subspaces $U_j \subset H^0(X, \mathscr{G}_p)$ and choose j so that $\dim_k U_j$ is "large" while it is easy to see that $\dim_k \tilde{g}(U_j)$ is "small". Then the rank-nullity theorem gives a lower bound on the kernel of $\tilde{g}$. Recall that $G = \mathbf{Z}/p\mathbf{Z}$ acts on $\pi_* \Omega^1_Y$ by having a generator $\tau \in G$ send $y \mapsto y + 1$. By Lemma 5.3, this action preserves $\mathscr{G}_j$.

Definition 6.6. Let i and j be integers with $0 \le i \le j \le p-1$.

- Define $U_j := \ker\bigl((\tau - 1)^{j+1} : H^0(X, \mathscr{G}_{j+1}) \to H^0(X, \mathscr{G}_{j+1})\bigr)$.
- For $Q \in S$, define $c(i, j, Q)$ to be the number of integers n congruent to -1 modulo p such that $-n_{Q,j} - d_Q(j-i) \le n < -n_{Q,i}$.
- For $Q \in \sup(D_i)$, define $c(i, j, Q) = p - 1$.
- Define $c(i, j, Q') = \dfrac{d_{Q'}(p-1)(p-1-i)}{p}$.
- Define $L(X, \pi)$ to be

$$L(X, \pi) := \max_{0 \le j \le p-1} \left(\dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S'_i} c(i, j, Q) \right).$$

Lemma 6.7. *There is an isomorphism*

$$U_j \simeq \bigoplus_{i=0}^{j} H^0(X, \ker V_X(F_*(E_i + pD_i))).$$

Proof. Both sides are isomorphic to

$$\left\{ \omega = \sum_{i=0}^{j} \omega_i y^i \in (\pi_* \ker V_Y)_\eta : \varphi_\eta(\omega) \in H^0(X, \mathscr{G}_p) \right\}.$$

This follows from the formulas in Lemma 4.1, the fact that $\tau - 1$ reduces the maximum power of y appearing in a differential by one, and the definition of φ. □

Lemma 6.8. *We have that $a_Y \ge L(X, \pi)$.*

Proof. Fix j. The key idea is to use descending induction on i and the rank-nullity theorem to give a lower bound on $\dim_k(U_j \cap H^0(X, \mathscr{G}_i))$ for $-1 \le i \le j+1$. Taking $i = -1$ gives a lower bound (depending on j) on $\dim_k(U_j \cap H^0(Y, \ker V_Y)) \le a_Y$, which will establish the result.

For $i = j + 1$, we see that

$$\dim_k(U_j \cap H^0(X, \mathscr{G}_{j+1})) = \dim_k U_j. \tag{6-4}$$

To ease the notational burden, for $-1 \le i \le j$ let us write ψ^j_{i+1} for the map

$$\psi^j_{i+1} : U_j \cap H^0(X, \mathscr{G}_{i+1}) \to H^0(M_i),$$
$$\text{given by } \psi^j_{i+1} := H^0(g'_{i+1})|_{U_j \cap H^0(X, \mathscr{G}_{i+1})}. \tag{6-5}$$

Theorem 5.20 then gives exact sequences

$$0 \to U_j \cap H^0(X, \mathscr{G}_i) \to U_j \cap H^0(X, \mathscr{G}_{i+1}) \xrightarrow{\psi^j_{i+1}} H^0(X, M_i) = \bigoplus_{Q \in S'_i} M_{i,Q} \tag{6-6}$$

for $-1 \le i \le j$, which need not be right exact. Nonetheless, we may work stalk-by-stalk to obtain information about the image of ψ^j_{i+1} as follows. For $Q \in S'_i$, let $\psi^j_{i+1,Q}$ denote the composition of ψ^j_{i+1} with projection onto the factor of the direct sum indexed by Q; by construction $\psi^j_{i+1,Q}$ coincides with the composition

$$\psi^j_{i+1,Q} : U_j \cap H^0(X, \mathscr{G}_{i+1}) \hookrightarrow H^0(X, \mathscr{G}_{i+1}) \to \mathscr{G}_{i+1,Q} \xrightarrow{g'_{i+1,Q}} M_{i,Q} \tag{6-7}$$

wherein the first two maps are the canonical ones. We have the obvious inclusion

$$\operatorname{Im}(\psi^j_{i+1}) \subseteq \bigoplus_{Q \in S'_i} \operatorname{Im}(\psi^j_{i+1,Q}) \tag{6-8}$$

so to bound $\dim_k \operatorname{Im}(\psi^j_{i+1})$ from above it suffices to do so for each $\dim_k \operatorname{Im}(\psi^j_{i+1,Q})$. First suppose that $i \ge 0$. Lemma 6.9 below gives an upper bound of $c(i, j, Q)$ on $\dim_k \operatorname{Im}(\psi^j_{i+1,Q})$ for $Q \in S$. For $Q \in \sup(D_i)$, the dimension of $\operatorname{Im}(\psi^j_{i+1,Q})$ is at most $p-1$ by the very definition of $g'_{i+1,Q}$ in Section 5C, and we defined $c(i, j, Q) := p - 1$ in this case. Applying the rank-nullity theorem to the exact sequence (6-6) and using (6-8) then gives

$$\dim_k(U_j \cap H^0(X, \mathscr{G}_i)) \ge \dim_k(U_j \cap H^0(X, \mathscr{G}_{i+1})) - \sum_{Q \in S'_i} c(i, j, Q).$$

By descending induction on i and (6-4), we deduce

$$\dim_k(U_j \cap H^0(X, \mathscr{G}_0)) \ge \dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S'_i} c(i, j, Q).$$

Finally, using Remark 5.17 to analyze (6-6) when $i = -1$, we conclude

$$\dim_k(U_j \cap H^0(X, \mathscr{G}_{-1})) \ge \dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S'_i} c(i, j, Q). \qquad \square$$

Lemma 6.9. *If $Q \in S$ and $0 \le i \le j < p$, then* $\dim_k \operatorname{Im}(\psi^j_{i+1,Q}) \le c(i, j, Q)$.

Proof. We continue the notation of Section 5B. Fixing $0 \le i \le j < p$ and recalling Lemma 5.4, for $\nu \in U_j$ we have

$$\nu = (\nu_0, \dots, \nu_j, 0, \dots, 0) \text{ and } (\omega_0, \dots, \omega_j, 0, \dots, 0) := \iota(\nu).$$

From (4-5), we have $\omega_i = \nu_i + s_i(V_X(\xi))$, where

$$\xi := -\sum_{\ell=i+1}^{j} \binom{\ell}{j} \omega_\ell (-f)^{\ell-i} \tag{6-9}$$

since ν_ℓ and ω_ℓ are 0 for $\ell > j$. From Lemma 5.4 (i)enumi with $n = n_{Q,j}$ we have

$$\mathrm{ord}_Q(\xi) \ge -n_{Q,j} - (j-i)d_Q. \tag{6-10}$$

By the very definitions (see (6-7) and Definitions 5.8 and 5.1), to compute $\psi^j_{i+1,Q}(\nu)$, we first expand ξ locally at Q as a power series $\xi = \sum_n a_n t_Q^n (dt_Q/t_Q)$ and evaluate

$$s_i(V_X(\xi)) = \sum_{n=-B}^{\infty} a_{pn} s_i\left(t_Q^n \frac{dt_Q}{t_Q}\right) \tag{6-11}$$

for some integer B. Then $\psi^j_{i+1,Q}(\nu)$ records only those coefficients on the right side of (6-11) where $n \ge -B$ and $pn - 1 < -n_{Q,i}$. In our situation we have *fewer* coefficients to record than in the general analysis of Section 5B, precisely because $\nu_\ell = 0 = \omega_\ell$ for $\ell > j$. Indeed, based on (6-10) we may take B to be the smallest integer greater than or equal to $(-n_{Q,j} - (j-i)d_Q + 1)/p$. In particular, the number of potentially nonzero coefficients a_{pn} is the number of multiples of p between $-n_{Q,j} - (j-i)d_Q + 1$ and $-n_{Q,i}$ inclusive, which by definition is the integer $c(i, j, Q)$. This integer therefore gives an upper bound on the dimension of $\mathrm{Im}(\psi^j_{i+1,Q})$ □

Proposition 6.10. *The a-number of Y satisfies*

$$L(X, \pi) \le a_Y = U(X, \pi).$$

Proof. Combine Lemmas 6.5 and 6.8. □

6B. *Tools.* The quantities $L(X, \pi)$ and $U(X, \pi)$ are quite abstract. We now explain how to bound $L(X, \pi)$ and $U(X, \pi)$ in terms of the ramification of $\pi : Y \to X$ and the genus and a-number of X.

The key is a theorem of Tango which allows us to compute the dimension of the kernel of the Cartier operator on certain spaces of (global) meromorphic differential forms on X whose poles are "sufficiently bad". Let $\sigma : k \to k$ denote the p-power Frobenius automorphism of k, and let g_X be the genus of X. Attached to X is its *Tango number*:

$$\mathrm{n}(X) := \max\left\{ \sum_{x \in X(k)} \left\lfloor \frac{\mathrm{ord}_x(df)}{p} \right\rfloor : f \in k(X) - k(X)^p \right\}. \tag{6-12}$$

In Lemma 10 and Proposition 14 of [Tango 1972], one sees that $\mathrm{n}(X)$ is well-defined and is an integer satisfying $-1 \le \mathrm{n}(X) \le \lfloor (2g_X - 2)/p \rfloor$, with the lower bound an equality if and only if $g_X = 0$.

Fact 6.11 (Tango's theorem). *Let $\mathscr{L}$ be a line bundle on X. If* $\deg \mathscr{L} > \mathrm{n}(X)$ *then the natural σ-linear map*

$$F_X^* : H^1(X, \mathscr{L}^{-1}) \to H^1(X, \mathscr{L}^{-p}) \tag{6-13}$$

induced by pullback by the absolute Frobenius of X is injective, and the σ^{-1}-linear Cartier operator

$$V_X : H^0(X, \Omega^1_{X/k} \otimes \mathscr{L}^p) \to H^0(X, \Omega^1_{X/k} \otimes \mathscr{L}) \tag{6-14}$$

is surjective.

Remark 6.12. This is [Tango 1972, Theorem 15]; strictly speaking, Tango requires $g_X > 0$; however, by tracing through Tango's argument—or by direct calculation—one sees easily that the result holds when $g_X = 0$ as well.

To simplify notation, let $\delta(H^0(X, \Omega^1_X(E)))$ denote the dimension of the kernel of V_X on that space of differentials. Tango's theorem tells us the following:

Corollary 6.13. *Let D, R be divisors on X with $R = \sum r_i P_i$ where $0 \le r_i < p$. If $\deg(D) > \max(\mathrm{n}(X), 0)$, then*

$$\delta(H^0(X, \Omega^1_X(pD + R))) = (p-1)\deg(D) + \sum_i \left(r_i - \left\lceil \frac{r_i}{p} \right\rceil\right).$$

For arbitrary D, we have the weaker statement

$$0 \le \delta(H^0(X, \Omega^1_X(pD + R))) - \left((p-1)\deg(D) + \sum_i \left(r_i - \left\lceil \frac{r_i}{p} \right\rceil\right)\right) \le a_X.$$

Proof. When $R = 0$, the first case follows from the surjectivity of the Cartier operator in Fact 6.11 taking $\mathscr{L} = \mathcal{O}_X(D)$, plus the fact that

$$\dim_k H^0(X, \Omega^1_X(D)) = g_X - 1 + \deg(D),$$
$$\dim_k H^0(X, \Omega^1_X(pD)) = (g_X - 1) + p\deg(D)$$

from the Riemann–Roch theorem.

We can build on this to prove the remaining cases of the first statement and establish the inequalities of the second statement. We know that for any divisor E with $\deg(E) \ge 0$ and any closed point P of X,

$$\dim_k H^0(X, \Omega^1_X(E + [P])) \le \dim_k H^0(X, \Omega^1_X(E)) + 1.$$

with equality whenever $\deg(E) > 0$. Thus we know that

$$0 \le \delta(H^0(X, \Omega^1_X(E + [P]))) - \delta(H^0(X, \Omega^1_X(E))) \le 1.$$

Further, if $p \mid \mathrm{ord}_P(E)$, this difference is 0, as a differential in $H^0(X, \Omega^1_X(E+[P]))$ not in $H^0(X, \Omega^1_X(E))$ must have a nonzero $t_P^{-\mathrm{ord}_P(E)}(dt_P/t_p)$-term in the completed stalk at P, which forces the differential to not lie in the kernel of the Cartier operator.

When $\deg(D) > \max(\mathrm{n}(X), 0)$ and $R \ne 0$, we prove the equality by induction on the number of points in the support of R. Assume the equality holds for some fixed R and all D with $\deg(D) > \max(\mathrm{n}(X), 0)$. Pick another point P (not in the support of R) and $0 < r < p$. Then

$$\begin{aligned}\delta(H^0(X, \Omega^1_X(pD + R))) &\le \delta(H^0(X, \Omega^1_X(pD + R + [P])))\\ &\le \cdots \le \delta(H^0(X, \Omega^1_X(p(D + [P]) + R)))\end{aligned}$$

and by the inductive hypothesis the last is $p-1$ more than the first. This means that at each step after the first, the dimension of the kernel must increase by one. This shows that

$$\delta(H^0(X, \Omega^1_X(pD + R + r[P]))) = \delta(H^0(X, \Omega^1_X(pD + R))) + (r-1),$$

which completes the induction.

The second statement follows from similar reasoning. When passing from

$$a_X = \delta(H^0(X, \Omega^1_X)) \quad \text{to} \quad \delta(H^0(X, \Omega^1_X(pD + R))),$$

there are $(p-1)\deg(D) + \sum(r_i - \lceil r_i/p \rceil)$ times the dimension might increase by one. When passing from

$$\delta(H^0(X, \Omega^1_X(pD + R))) \quad \text{to} \quad \delta(H^0(X, \Omega^1_X(pD'))) = (p-1)\deg(D')$$

with D' chosen so that $pD' \geq pD + R$ and $D' > \mathrm{n}(X)$, there are

$$(p-1)\deg(D' - D) + \sum(p - 1 - r_i)$$

chances for the dimension to increase by one. This completes the proof. ☐

Remark 6.14. Choosing $D' \geq D$ with $\deg(D') > \mathrm{n}(X)$ and $pD' \geq pD + R$, we also obtain the bound $\delta(H^0(X, \Omega^1_X(pD + R))) \leq (p-1)\deg D'$ from the inclusion $H^0(X, \Omega^1_X(pD + R)) \subset H^0(X, \Omega^1_X(pD'))$ and Tango's theorem.

Fix a divisor $E = pD + R = \sum_i a_i Q_i$ with $R = \sum r_i Q_i$ and $0 \leq r_i < p$. For fixed j and $1 \leq n \leq a_j$,

$$\delta\Bigl(H^0\Bigl(X, \Omega^1_X\Bigl(\sum_{i<j} a_i Q_i + (n-1)Q_j\Bigr)\Bigr)\Bigr) \leq \delta\Bigl(H^0\Bigl(X, \Omega^1_X\Bigl(\sum_{i<j} a_i Q_i + nQ_j\Bigr)\Bigr)\Bigr).$$

If the dimension increases by one, let $\xi_{Q_j,n}$ be a differential in the larger space not in the smaller space. This differential satisfies the following properties:

(i) $V_X(\xi_{Q_j,n}) = 0$;

(ii) $\mathrm{ord}_{Q_i}(\xi_{Q_j,n}) \geq -a_i$ for $i < j$;

(iii) $\mathrm{ord}_{Q_j}(\xi_{Q_j,n}) = -n$; and

(iv) $\mathrm{ord}_{Q_i}(\xi_{Q_j,n}) \geq 0$ for $i > j$.

Note that such a differential never exists if $n \equiv 1 \bmod p$.

Example 6.15. Suppose $X = \mathbf{P}^1$. For any positive integer n with $n \not\equiv 1 \bmod p$ and closed point $Q = [\alpha]$ of X, we may take $\xi_{Q,n}$ to be $(t-\alpha)^{-n}\,dt$.

Corollary 6.16. *The differentials $\xi_{Q_j,n}$ are linearly independent. There are at least $(p-1)\deg D + \sum_i (r_i - \lceil r_i/p \rceil) - a_X$ of them.*

Proof. This follows from the proof of Corollary 6.13. ☐

We can use Corollary 6.13 to relatively easily give a formula for $\dim_k U_j$ using Lemma 6.7 provided that the auxiliary divisors D_i are sufficiently large. This will allow us to bound $L(X, \pi)$; we do so in the proof of Theorem 6.26. We can also use Corollary 6.13 to obtain a bound on the number $N(X, \pi)$ of Definition 6.4; this is more involved, and is the subject of the next section.

6C. *Estimating $N(X,\pi)$.* We will use Corollary 6.16 to construct differential forms to give a lower bound on $N(X,\pi)$, and hence an upper bound on $U(X,\pi)$.

Definition 6.17. Define T to be the set of triples (Q,n,j), where

- $0 \le j \le p-1$ and $Q \in S'_j$;
- $0 < n \le \mathrm{ord}_Q(E_j + pD_j)$;
- $n \not\equiv 1 \bmod p$;
- if $Q \in S$, an integer m $(0 \le m \le j)$ exists such that $m \equiv j + (n-1)d_Q^{-1} \bmod p$.

We fix an ordering on $\bigcup_j S'_j$ with Q' the smallest element. For $(Q,n,j) \in T$, suppose the element $\xi_{Q,j}$ from Corollary 6.16 exists (taking $E = E_j + pD_j$ and $a_i = n_{Q_i,j}$ in the notation there, and using the ordering on $\bigcup_j S'_j$). We set

$$\nu_{Q,n,j} := (0,\dots,0,\xi_{Q,j},0,\dots,0) \in H^0(X,\mathscr{G}_{j+1}),$$

where $\xi_{Q,j}$ occurs in the j-th component. In that case, we say that $\nu_{Q,n,j}$ *exists*. As Q' is smallest in the chosen ordering, we know that $\nu_{Q',n,j} \in H^0(X,\mathscr{G}_0)$. Our aim is to prove:

Proposition 6.18. *Let $N(X,\pi)$ be as in Definition 6.4. Then $N(X,\pi) \ge \#T - B$, where B is the number of triples $(Q,n,j) \in T$ such that $\nu_{Q,n,j}$ does not exist.*

The idea is to show that the images under $\tilde{g}$ of the $\nu_{Q,n,j}$ for $(Q,n,j) \in T$ with $Q \ne Q'$ are linearly independent, and similarly for the images under $\tilde{g}_0$ of the $\nu_{Q',n,j}$ with $(Q',n,j) \in T$.

Definition 6.19. We define an ordering on T by setting $(P,a,j) < (P',a',j')$ provided that

(i) $j < j'$; or

(ii) $j = j'$ and $P < P'$; or

(iii) $j = j'$ and $P = P'$ and $a < a'$.

For $(Q,n,j) \in T$, we then define

$$U_{Q,n,j} := \mathrm{span}_k\big\{\tilde{g}(\nu_{P,a,i}) : \nu_{P,a,i} \text{ exists and } (P,a,i) < (Q,n,j)\big\}.$$

Lemma 6.20. *Let $(P,a,i),(Q,n,j) \in T$ with $(P,a,i) < (Q,n,j)$ and suppose that the differentials $\nu_{P,a,i}$ and $\nu_{Q,n,j}$ both exist. Writing $(\omega_0,\dots,\omega_{p-1}) = \iota(\nu_{Q,n,j})$ and $(\omega'_0,\dots,\omega'_{p-1}) = \iota(\nu_{P,a,i})$, we have $\mathrm{ord}_Q(\omega'_j) > \mathrm{ord}_Q(\omega_j) = -n$.*

Proof. Suppose first that $i < j$. Since the ℓ-th component of $\nu_{P,a,i}$ is zero for $\ell > i$, we have $\omega'_\ell = 0$ for $\ell > i$ by Lemma 5.4 (i)enumi, and in particular $\omega'_j = 0$ which immediately gives the claim. So suppose that $i = j$ and $P < Q$. In this case, by very construction of the differentials in Corollary 6.16 and our choice of ordering, we have $\mathrm{ord}_Q(\omega'_j) = \mathrm{ord}_Q(\xi_{P,a}) \ge 0$ while $\mathrm{ord}_Q(\omega_j) = \mathrm{ord}_Q(\xi_{Q,n}) = -n$. Finally, suppose that $i = j$, $P = Q$, and $a < n$. Then,

$$\mathrm{ord}_Q(\omega'_j) = \mathrm{ord}_Q(\xi_{Q,a}) = -a > -n = \mathrm{ord}_Q(\xi_{P,n}) = \mathrm{ord}_Q(\omega_j)$$

as claimed. □

Lemma 6.21. *Suppose that $\nu_{Q,n,j}$ exists with $(Q, n, j) \in T$ and $Q \in S$. Then $\tilde{g}(\nu_{Q,n,j}) \notin U_{Q,n,j}$.*

Proof. In this proof, let $\tilde{g}_{m+1,Q}$ denote the composition of $\tilde{g}_{m+1}$ with the projection onto $M_{j,Q}$. Write $(\omega_0, \dots, \omega_j, 0, \dots, 0) = \iota(\nu_{Q,n,j})$. As $Q \in S$, by Definition 6.17 there exists an integer $0 \le m \le j$ such that $m \equiv j + (n-1)d_Q^{-1} \bmod p$. Looking at the definition of $g'_{m+1,Q}$ in Section 5B, to compute $\tilde{g}_{m+1,Q}(\nu_{Q,n,j})$ we work in the completed stalk at Q and write

$$\xi := -\sum_{i=m+1}^{j} \binom{i}{m} \omega_i (-f)^{i-m} = \sum_{i=-d_Q(p-1-m)+1}^{\infty} a_i t_Q^i \frac{dt_Q}{t_Q}. \tag{6-15}$$

Now $\tilde{g}_{m+1,Q}$ records the a_i for which $p|i$ and $-d_Q(p-1-m) \le i < -n_{Q,m}$.

Using Lemma 5.4 (i)enumi we find for $i \le j$

$$\mathrm{ord}_Q(\omega_i(-f)^{i-m}) \ge -n - (j-m)d_Q \tag{6-16}$$

In fact, we know more: for $i \ne j$, the i-th component of $\nu_{Q,n,j}$ is *zero*, whence Lemma 5.4 (i)enumi shows that for $m < i < j$

$$\begin{aligned}
\mathrm{ord}_Q(\omega_i(-f)^{i-m}) &= \mathrm{ord}_Q(\omega_i) - (i-m)d_Q \\
&\ge -p\left\lfloor \frac{n + (j-i)d_Q - 1}{p} \right\rfloor - 1 - (i-m)d_Q \\
&> -(n + (j-i)d_Q - 1) - 1 - (i-m)d_Q \\
&= -n - (j-m)d_Q,
\end{aligned} \tag{6-17}$$

where the strict inequality comes from the fact that $n + (j-i)d_Q \not\equiv 1 \bmod p$ for $m < i \le j$, as m is by definition the *unique* solution modulo p to

$$n + (j-x)d_Q \equiv 1 \bmod p.$$

We adopt the notation from the proof of Lemma 5.10 for coefficients of differentials and functions. As $\omega_j = \xi_{Q,n}$, we have $\mathrm{coef}_{1-n}(\omega_j) \ne 0$. As $\mathrm{ord}_Q(f) = -d_Q$, we have $\mathrm{coef}_{-d_Q}(f) \ne 0$. By (6-17), we know that $\mathrm{coef}_{1-n-(j-m)d_Q}(\omega_i(-f)^{i-m}) = 0$ for $m < i < j$. Then looking at coefficients in (6-15) we see that

$$\begin{aligned}
a_{1-n-(j-m)d_Q} &= \mathrm{coef}_{1-n-(j-m)d_Q}\left(-\sum_{i=m+1}^{j} \binom{i}{m} \omega_i (-f)^{i-m} \right) \\
&= \mathrm{coef}_{1-n-(j-m)d_Q}\left(-\binom{j}{m} \omega_j (-f)^{j-m} \right) \\
&= -\binom{j}{m} \mathrm{coef}_{1-n}(\omega_j) \cdot \mathrm{coef}_{-(j-m)d_Q}((-f)^{j-m}) \ne 0.
\end{aligned} \tag{6-18}$$

But $1 - n - (j-m)d_Q \equiv 0 \bmod p$, so $a_{1-n-(j-m)d_Q}$ is one of the *nonzero* terms that $\tilde{g}_{m+1,Q}$ records. This is enough to show that $\tilde{g}(\nu_{Q,n,j})$ does not lie in $U_{Q,n,j}$. Indeed, suppose $(P, a, i) \in T$ with $(P, a, i) <$

(Q, n, j) and that $\nu_{P,a,i}$ exists. Set $(\omega'_0, \dots, \omega'_{p-1}) = \iota(\nu_{P,a,i})$. By Lemma 6.20, we have $\mathrm{ord}_Q(\omega'_j) > -n$, so thanks to Lemma 5.4 (i)enumi we see that $\mathrm{ord}_Q(\omega'_\ell) > -n - d_Q(j-\ell)$ for all $\ell \le j$. It follows that

$$\mathrm{coef}_{1-n-(j-m)d_Q}(\omega'_m) = 0.$$

As $U_{Q,n,j}$ is spanned by $\tilde{g}(\nu_{P,a,i})$ for such triples (P, a, i), this shows $\tilde{g}(\nu_{Q,n,j})$ does not lie in $U_{Q,n,j}$ as claimed. □

Lemma 6.22. *Suppose that $\nu_{Q,n,j}$ exists with $(Q, n, j) \in T$ and $Q \in D_j$. Then $\tilde{g}(\nu_{Q,n,j}) \notin U_{Q,n,j}$.*

Proof. In this proof, let $\tilde{g}_{m+1,Q}$ denote the composition of $\tilde{g}_{m+1}$ with the projection onto $M_{j,Q}$. As $\mathrm{ord}_Q(E_j + pD_j) = p$ and we assumed $n \ne 1$, we know $1 < n \le p$. Set $(\omega_0, \dots, \omega_{p-1}) = \iota(\nu_{Q,n,j})$ and notice that $\omega_j = \xi_{Q,n}$. Looking at the definition of $g'_{j+1,Q}$ in Section 5C, we see that $g'_{j+1,Q}$ extracts the coefficients of $t_Q^{-p}dt_Q$ through $t_Q^{-2}dt_Q$ in a local expansion of ω_j at Q. In particular, since $\mathrm{ord}_Q(\omega_j) = -n$, we see that the component of $\tilde{g}_{j+1,Q}(\nu_{Q,n,j})$ corresponding to the coefficient of $t_Q^{-n}dt_Q$ in the local expansion of ω_j at Q is *nonzero*. On the other hand, for any $(P, a, i) \in T$ such that $\nu_{P,a,i}$ exists and $(P, a, i) < (Q, n, j)$, consider $(\omega'_0, \dots, \omega'_{p-1}) = \iota(\nu_{P,a,i})$. By Lemma 6.20 we have $\mathrm{ord}_Q(\omega'_j) > -n$ so that the component of $\tilde{g}_{j+1,Q}(\nu_{P,a,i})$ corresponding to the coefficient of $t_Q^{-n}dt_Q$ in the local expansion of ω'_j at Q is by contrast *zero*. As the $\tilde{g}(\nu_{P,a,i})$ for $(P, a, i) < (Q, n, j)$ span $U_{Q,n,j}$, this shows $\tilde{g}(\nu_{Q,n,j}) \notin U_{Q,n,j}$ as desired. □

Lemma 6.23. *The elements*

$$\{\tilde{g}_0(\nu_{Q',n,j}) : (Q', n, j) \in T \text{ and } \nu_{Q',n,j} \text{ exists}\}$$

are linearly independent.

Proof. For notational convenience, let $T' \subseteq T$ be the subset of triples of the form (Q', n, j) such that $\nu_{Q',n,j}$ exists, and note that $\nu_{Q',n,j} \in H^0(X, \mathscr{G}_0)$ as $n \le (p-1-j)d_{Q'} = \mathrm{ord}_{Q'}(E_j)$. Let $\omega_{Q',n,j} = \varphi_\eta^{-1}(\nu_{Q',n,j})$; we have that $(\tau-1)^j\omega_{Q',n,j} = j!\xi_{Q',n}$ and $(\tau-1)^{j+1}\omega_{Q',n,j} = 0$.

Suppose that there exist scalars $c_{Q',n,j} \in k$ for $(Q', n, j) \in T'$ such that

$$\sum_{(Q',n,j)\in T'} c_{Q',n,j}\omega_{Q',n,j} = \omega' \in H^0(X, \mathscr{G}_{-1}) = H^0(Y, \ker V_Y).$$

We will show that $c_{Q',n,j} = 0$ for all $(Q', n, j) \in T'$ by descending induction on j. First observe that there are no triples of the form $(Q', n, p-1)$ in T (so a fortioti there are none in T') as $\mathrm{ord}_{Q'}(E_{p-1} + pD_{p-1}) = 0$. Now fix j and suppose $c_{Q',n,i} = 0$ for all n whenever $i > j$. We compute

$$(\tau-1)^j\omega' = j! \sum_{\{n \,:\, (Q',n,j)\in T'\}} c_{Q',n,j}\xi_{Q',n} \tag{6-19}$$

since $(\tau-1)^j\omega_{Q',n,i} \ne 0$ implies that $i \ge j$. As $\mathrm{ord}_{Q'}(\xi_{Q',n}) = -n$, each of the nonzero terms on the right of (6-19) has a different, negative valuation at Q', while the left side is *regular* at Q' as $(\tau-1)^j\omega'$ is regular at the points above Q' (in fact, everywhere) and $\pi : Y \to X$ is unramified over Q'. This forces $c_{Q',n,j} = 0$ for all n in the sum, as desired.

We conclude that the images of the $\omega_{Q',n,j}$ are linearly independent in the cokernel of the inclusion $H^0(Y, \pi_* \ker V_Y) \simeq H^0(X, \mathscr{G}_{-1}) \hookrightarrow H^0(X, \mathscr{G}_0)$ and the result follows. □

Proof of Proposition 6.18. By Definition 6.4, $N(X,\pi) := N_1(X,\pi) + N_2(X,\pi)$, where $N_1(X,\pi) := \dim_k \operatorname{Im}(\tilde{g})$ and $N_2(X,\pi) := \dim_k \operatorname{Im}(\tilde{g}_0)$. Let $T_1 \subseteq T$ be the set of triples $(Q, n, j) \in T$ with $Q \in S_j$ for some $0 \le j \le p-1$ such that $\nu_{Q,n,j}$ exists and $T_2 \subseteq T$ the set of triples $(Q', n, j) \in T$ such that $\nu_{Q',n,j}$ exists. Thanks to Lemmas 6.21 and 6.22 we have $\#T_1 \le N_1(X,\pi)$, while $\#T_2 \le N_2(X,\pi)$ due to Lemma 6.23. It follows that

$$N(X,\pi) = N_1(X,\pi) + N_2(X,\pi) \ge \#T_1 + \#T_2 = \#T - B,$$

where B is the number of triples $(Q, n, j) \in T$ for which $\nu_{Q,n,j}$ does not exist. □

Remark 6.24. Using Corollary 6.13 and the observation that at most a_X of the $\xi_{Q,n}$ do not exist, we will use Proposition 6.18 to bound $U(X,\pi)$, thereby obtaining bounds on a_Y. A precise statement will be given in Theorem 6.26.

Remark 6.25. Let $Q \in S$. This argument uses very limited information about the coefficients of a local expansion of f at Q. In particular, it only uses that f has a leading term (a nonzero $t_Q^{-d_Q}$ term). If there were additional nonzero coefficients, the image of $\tilde{g}$ could very well be larger, and $N(X,\pi)$ could be strictly larger than $\#T - B$. However, Example 7.5 shows it is not possible to do better in general. This is the main place the argument loses information.

6D. ***Explicit bounds.*** We continue to use the notation from Section 3; in particular, $\pi : Y \to X$ is a fixed degree-p Artin–Schreier cover of smooth, projective, and connected curves over k with branch locus $S \subseteq X(\bar{k})$, and d_Q is the unique break in the ramification filtration at the unique point of Y above $Q \in S$. For nonnegative integers d, i with $p \nmid d$ let $\tau_p(d,i)$ be the number of positive integers $n \le \lfloor id/p \rfloor$ with the property that $-n \equiv md \bmod p$ for some m with $0 < m \le p-1-i$.

Theorem 6.26. *With notation as above,*

$$a_Y \le p a_X + \sum_{Q\in S} \sum_{i=1}^{p-1} \left(\left\lfloor \frac{i d_Q}{p} \right\rfloor - \left\lfloor \frac{i d_Q}{p^2} \right\rfloor - \tau_p(d_Q, i) \right) \tag{6-20}$$

and for any j with $1 \le j \le p-1$

$$a_Y \ge \sum_{Q\in S} \sum_{i=j}^{p-1} \left(\left\lfloor \frac{i d_Q}{p} \right\rfloor - \left\lfloor \frac{i d_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{j d_Q}{p} \right\rfloor \right). \tag{6-21}$$

Moreover, for any nonnegative integers d, i with $p \nmid d$ and $i \le p-1$,

$$\tau_p(d,i) \ge (p-1-i) \left\lfloor \frac{1}{p} \left\lceil \frac{i d_Q}{p} \right\rceil \right\rfloor \ge (p-1-i) \left\lfloor \frac{i d_Q}{p^2} \right\rfloor. \tag{6-22}$$

Proof. Let $\mathrm{n}(X)$ be the Tango number of X, and for $0 \le i \le p-1$ let D_i be as above Definition 4.3 (see also Corollary 2.7). Adding in more points to each D_i as needed, we may without loss of generality assume that $\deg(D_i) > \mathrm{n}(X)$ for all i. Using this assumption, we will analyze the lower and upper bounds from Proposition 6.10 separately. Recall that we have fixed an Artin–Schreier equation $y^p - y = f$ giving the extension of function fields $k(Y)/k(X)$ with f minimal in the sense of Definition 3.1; in particular f has a pole of order d_Q at each $Q \in S$, as well as a pole of some order $d_{Q'}$ that is divisible by p at the fixed point $Q' \in X(k) - S$. As in Definition 3.4, for $Q \in S' = S \cup \{Q'\}$ and $0 \le i \le p-1$ we set $n_{Q,i} := \lceil (p-1-i)d_Q/p \rceil$ if $Q \neq Q'$ and $n_{Q',i} := (p-1-i)d_{Q'}$, and we put $E_i := \sum_{Q \in S'} n_{Q,i}[Q]$. We now define

$$A_i := \sum_{Q \in S'} \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor [Q], \qquad R_i = \sum_{Q \in S} r(Q,i)[Q] := E_i - pA_i.$$

Note that R_i is supported on S, as $n_{Q',i}$ is a multiple of p. Let $s(Q,i) = r(Q,i) - 1$ if $r(Q,i) > 0$, and 0 otherwise.

We will first establish (6-20). By Proposition 6.10 we have

$$a_Y \le \sum_{i=0}^{p-1} \dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))) - N(X,\pi). \tag{6-23}$$

To understand the first sum we will use Corollary 6.13, and to understand $N(X,\pi)$ we will use Proposition 6.18.

Fix i. As $\deg(D_i) > n(X)$, the first part of Corollary 6.13 implies that

$$\begin{aligned}\dim_k &H^0(X, \ker V_X(F_*(E_i + pD_i))) \\ &= \dim_k H^0(X, \ker V_X(F_*(p(A_i + D_i) + R_i)) = (p-1)\deg(A_i + D_i) + \sum_{Q \in S} s(Q,i) \\ &= (p-1)\left(\sum_{Q \in S} \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor + \frac{d_{Q'}}{p}(p-1-i) + \#\sup(D_i) \right) + \sum_{Q \in S} s(Q,i).\end{aligned} \tag{6-24}$$

We also want to count elements of the form (Q,n,i) in the set T of Proposition 6.18 for fixed i and $Q \in S'_i = S' \cup \sup(D_i)$. If $Q \in \sup(D_i)$, then there are $p-1$ such elements by definition since $\mathrm{ord}_Q(E_i + pD_i) = p$. If $Q = Q'$, there are $(p-1)(p-1-i)d_{Q'}/p$ such elements since $\mathrm{ord}_{Q'}(E_i) = (p-1-i)d_{Q'}$. When $Q \in S$, the number of elements of the form (Q,n,i) in T is precisely $\tau_p(d_Q, p-1-i)$ since the set of positive integers $n \le n_{Q,i}$ with the property that $i + (n-1)d_Q^{-1} \equiv m \bmod p$ for some m with $0 \le m < i$ is in bijection with the set of positive integers $n' \le \lfloor (p-1-i)d_Q/p \rfloor$ with $-n' \equiv m' d_Q \bmod p$ for some m' with $0 < m' \le i$ via $n' := n-1$ and $m' = i - m$. Putting this together, we find that there are exactly

$$\sum_{Q \in S} \tau_p(d_Q, p-1-i) + (p-1)\#\sup(D_i) + (p-1)(p-1-i)\frac{d_{Q'}}{p} \tag{6-25}$$

elements of T of the form (Q,n,i).

Putting (6-23) together with (6-24) and the bound on $N(X,\pi)$ coming from (6-25), Proposition 6.18, and the observation that at most pa_X of the $\nu_{Q,n,j}$ do not exist (since at most a_X of the $\xi_{Q,n}$ do not exist), we conclude that

$$a_Y \le p \cdot a_X + \sum_{i=0}^{p-2} \sum_{Q\in S} (p-1)\left\lfloor \frac{n_{Q,i}}{p} \right\rfloor + s(Q,i) - \tau_p(d_Q, p-1-i). \tag{6-26}$$

Using the very definition of $s(Q,i)$, one finds the formula

$$\begin{aligned} s(Q,i) &= n_{Q,i} - p\cdot\left\lfloor \frac{n_{Q,i}}{p}\right\rfloor - \left(\left\lceil \frac{n_{Q,i}}{p}\right\rceil - \left\lfloor \frac{n_{Q,i}}{p}\right\rfloor\right) \\ &= n_{Q,i} - \left\lceil \frac{n_{Q,i}}{p}\right\rceil - (p-1)\cdot\left\lfloor \frac{n_{Q,i}}{p}\right\rfloor \end{aligned} \tag{6-27}$$

Substituting (6-27) into (6-26), reindexing the sum $i \mapsto p-1-i$ and using the equality $\lceil x\rceil = \lfloor x\rfloor + 1$ for $x \notin \mathbf{Z}$ gives (6-20).

To make this bound more explicit, we must bound $\tau_p(d,i)$ from below. To do this, simply note that for any $0 < m \le p-1-i$ and any interval of length p, there is a unique n in that interval such that

$$md \equiv -n \bmod p$$

and that necessarily $n \not\equiv 0 \bmod p$ as $p \nmid d$. Thus,

$$\begin{aligned} \tau_p(d_Q,i) &\ge (p-1-i)\cdot\left\lfloor \frac{1}{p}\left(\left\lfloor \frac{id_Q}{p}\right\rfloor + 1\right)\right\rfloor = (p-1-i)\cdot\left\lfloor \frac{1}{p}\left\lceil \frac{id_Q}{p}\right\rceil\right\rfloor \\ &\ge (p-1-i)\left\lfloor \frac{id_Q}{p^2}\right\rfloor, \end{aligned} \tag{6-28}$$

as $0 \le n \le \lfloor id_Q/p\rfloor = \lceil id_Q/p\rceil - 1$.

We now turn to the lower bound (6-21). Proposition 6.10 and Definition 6.6 give that for each $0 \le j \le p-1$,

$$\dim_k U_j - \sum_{i=0}^{j} \sum_{Q\in S'_j} c(i,j,Q) \le a_Y. \tag{6-29}$$

Using Lemma 6.7, Corollary 6.13 and the assumption that $\deg D_i > n(X)$ for all i, we calculate

$$\begin{aligned} \dim_k U_j &= \sum_{i=0}^{j} \dim_k H^0(X, \Omega^1_X(F_*(E_i + pD_i))) \\ &= \sum_{i=0}^{j}\left((p-1)\deg(D_i) + (p-1)(p-1-i)\frac{d_{Q'}}{p} + \sum_{Q\in S}(p-1)\left\lfloor \frac{n_{Q,i}}{p}\right\rfloor + s(Q,i)\right). \end{aligned}$$

Now for $Q \in \sup(D_i)$ we have $c(i,j,Q) = p-1$, while

$$c(i,j,Q') = (p-1)(p-1-i)d_{Q'}/p.$$

If $Q\in S$, then $c(i,j,Q)$ is the number of integers $n\equiv -1 \mod p$ satisfying $-n_{Q,j}-d_Q(j-i)\le n<-n_{Q,i}$. That is,

$$c(i,j,Q)=\left\lceil\frac{n_{Q,j}+d_Q(j-i)}{p}\right\rceil-\left\lceil\frac{n_{Q,i}}{p}\right\rceil.$$

Thus, the contributions from D_i and Q' in (6-29) cancel, so the formula (6-27) for $s(Q,i)$ yields

$$\sum_{i=0}^{j}\sum_{Q\in S}\left(n_{Q,i}-\left\lceil\frac{n_{Q,j}+d_Q(j-i)}{p}\right\rceil\right)\le a_Y$$

for all $0\le j<p-1$. Using the equality

$$\left\lceil\frac{n_{Q,j}+(j-i)d_Q}{p}\right\rceil=\left\lceil\frac{(p-1-i)d_Q}{p}-\left(1-\frac{1}{p}\right)\frac{(p-1-j)d_Q}{p}\right\rceil,$$

changing variables $i\mapsto p-1-i$ and $j\mapsto p-1-j$, and employing again the equality $\lceil x\rceil=\lfloor x\rfloor+1$ for $x\notin \mathbf{Z}$ then gives (6-21). □

Corollary 6.27. *Suppose p is odd. With the hypotheses of Theorem 6.26 we have*

$$a_Y\le p\cdot a_X+\left(\frac{(p-1)(p-2)}{2}+\left(1-\frac{1}{p}\right)^2\right)\cdot \#S+\left(1-\frac{1}{p}\right)\sum_{Q\in S}\frac{(2p-1)}{6}d_Q$$

and

$$a_Y\ge\left(1-\frac{1}{p}\right)^2\left(\sum_{Q\in S}\frac{(p+1)}{4}d_Q-\#S\cdot\frac{p}{2}\right).$$

Proof. The upper bound follows easily from (6-20) and the first inequality in (6-22) by basic properties of the floor function and the well-known equality

$$\sum_{i=1}^{n-1}\left\lfloor\frac{id}{n}\right\rfloor=\frac{(n-1)(d-1)}{2}\tag{6-30}$$

for any positive and coprime integers d,n. For the lower bound, we have

$$\begin{aligned}\left\lfloor\frac{id_Q}{p}\right\rfloor-\left\lfloor\frac{id_Q}{p}-\left(1-\frac{1}{p}\right)\frac{jd_Q}{p}\right\rfloor&\ge\frac{id_Q-(p-1)}{p}-\left(\frac{id_Q}{p}-\left(1-\frac{1}{p}\right)\frac{jd_Q}{p}\right)\\&=\left(1-\frac{1}{p}\right)\left(\frac{jd_Q}{p}-1\right).\end{aligned}$$

Summing over i with $j\le i\le p-1$ and then Q and using (6-21) gives the lower bound

$$\left(1-\frac{1}{p}\right)\sum_{Q\in S}\left(\frac{d_Q}{p}(p-j)j-(p-j)\right)\tag{6-31}$$

which holds for all $1\le j\le p-1$; we then take $j=(p+1)/2$. Remark 6.28 motivates this choice. □

Remark 6.28. The choice of $j=(p+1)/2$ in the proof of Corollary 6.27 is optimized for $\sum d_Q$ large relative to $\#S$. Indeed, (6-31) is a quadratic function in j which attains its maximum when

$$j \approx \frac{p}{2}\left(1-\frac{\#S}{\sum_{Q\in S} d_Q}\right). \tag{6-32}$$

Any nearby value of j will give a similar bound. Supposing that $\sum d_Q$ is large relative to $\#S$ gives optimal choice $j=\lceil p/2\rceil=(p+1)/2$. When all d_Q are small and p is large, one can get a better explicit lower bound by choosing a value of j in accordance with (6-32); cf. Example 7.1.

Remark 6.29. For fixed X, p and S with all d_Q becoming large, the dominant terms of the lower and upper bounds in Corollary 6.27 are respectively

$$\sum_{Q\in S} d_Q\frac{p}{4} \quad \text{and} \quad \sum_{Q\in S} d_Q\frac{p}{3}.$$

On the other hand, the dominant term in the Riemann–Hurwitz formula for the genus of Y is $\sum_{Q\in S} d_Q p/2$, so that, for large d_Q, the a-number is approximately between $1/2$ and $2/3$ of the genus of Y.

Remark 6.30. When $p=2$, the statement and proof of the Corollary do not work as written. If we take $j=1$ in Theorem 6.26 we obtain

$$\sum_{Q\in S}\left\lfloor\frac{d_Q}{2}\right\rfloor-\left\lfloor\frac{d_Q}{4}\right\rfloor \le a_Y \le 2a_X+\sum_{Q\in S}\left\lfloor\frac{d_Q}{2}\right\rfloor-\left\lfloor\frac{d_Q}{4}\right\rfloor.$$

In particular, when X is ordinary (i.e., $a_X=0$) we obtain an exact formula for a_Y. This recovers [Voloch 1988, Theorem 2] (note that the formula there is for the rank of the Cartier operator, and that for $Q_i\in S$, our d_{Q_i} is $2n_i-1$).

Similarly, Corollary 6.32 will give an exact formula for a_Y when p is odd, X is ordinary, and $d_Q|(p-1)$. To derive it, we will need to investigate situations when it is possible to derive an exact formula for the quantity

$$\tau_p(d) := \sum_{i=0}^{p-1}\tau_p(d,i) \tag{6-33}$$

occurring in the upper bound (6-20):

Proposition 6.31. *Let $p>2$ and suppose that $d\equiv d' \bmod p^2$. Then*

$$\tau_p(d)=\tau_p(d')+(d-d')\frac{(p-1)(p-2)}{6p}. \tag{6-34}$$

Moreover, $\tau_p(1)=0$ and for $1<d<p$ we have

$$\tau_p(d)=u_p(d)\cdot\frac{p}{d}+v_p(d), \tag{6-35}$$

where $u_p(d)$ and $v_p(d)$ are the integers (depending only on $p \bmod d$) given by

$$u_p(d) := \sum_{j=0}^{b-1} \sum_{r\in T_j} \bigl((j+1)d-(b+1)r\bigr),$$
$$v_p(d) := \sum_{j=0}^{b-1} \sum_{r\in T_j} \left(\frac{r}{d} - \left\{\frac{a(d-r)}{d}\right\}\right), \tag{6-36}$$

where $a := p \bmod d$ and $b := a^{-1} \bmod d$ with $0 < a, b < d$ and

$$T_j := \{r \in \mathbf{Z} \ :\ jd/b < r < (j+1)d/(b+1)\}.$$

In particular, for $1 < d < p$ the quantity $\tau_p(d)$ depends only on d and $p \bmod d$.

Before proving Proposition 6.31, let us give some indication of its utility. For example, if $p \equiv 1 \bmod d$, we have $a = b = 1$, whence

$$u_p(d) = \sum_{r=1}^{\lfloor d/2\rfloor} (d-2r) = \left\lfloor \frac{d}{2}\right\rfloor \cdot \left\lfloor \frac{d-1}{2}\right\rfloor = \left\lfloor \frac{(d-1)^2}{4}\right\rfloor$$

and

$$v_p(d) = \sum_{r=1}^{\lfloor d/2\rfloor} \frac{r}{d} - \frac{d-r}{d} = \sum_{r=1}^{\lfloor d/2\rfloor} \frac{2r-d}{d} = -\left\lfloor \frac{(d-1)^2}{4}\right\rfloor.$$

Thus

$$\tau_p(d) = \left\lfloor \frac{(d-1)^2}{4}\right\rfloor \cdot \frac{p-1}{d} \tag{6-37}$$

whenever $d < p$ and $p \equiv 1 \bmod d$. Another simple example is when $p \equiv -1 \bmod d$, so that $a = b = d-1$. In this case, $T_j := \{r \in \mathbf{Z} \ :\ jd/(d-1) < r < (j+1)\}$ is the empty set for all j, whence $\tau_p(d) = 0$.

Proof. The first assertion follows easily from the fact that, as observed immediately prior to (6-28), for any $0 < m \le p-1-i$ and any interval of length p, there is a unique n in that interval such that

$$md \equiv -n \bmod p$$

and necessarily $n \not\equiv 0 \bmod p$. So suppose that $d < p$, and for integers m, i, define

$$\chi(m,i) := \begin{cases} 1 & \text{if } m \le i, \\ 0 & \text{otherwise.} \end{cases}$$

We use the convention that "$x \bmod y$" denotes the unique integer x_0 with $0 \le x_0 < y$ and $x \equiv x_0 \bmod y$, and we set $a := p \bmod d$ and $b := a^{-1} \bmod d$. By definition,

$$\tau_p(d) = \sum_{i=0}^{p-1} \sum_{\substack{0<j\le n_i(d) \\ j\not\equiv 0 \bmod p}} \chi(-jd^{-1} \bmod p,\ p-1-i), \tag{6-38}$$

where $n_i(d) := \lfloor id/p \rfloor$. For an integer r, observe that $r \le n_i(d) < d$ if and only if $i > rp/d$, and that $-jd^{-1} \equiv j(bp-1)/d \bmod p$ (note that $d|(bp-1)$). Taking this into account, we swap the order of summation in (6-38), collecting all terms with $j = r$ together to obtain

$$\sum_{r=1}^{d-1} \sum_{i=\lceil rp/d \rceil}^{p-1} \chi\left(r\frac{bp-1}{d} \bmod p,\, p-1-i \right). \tag{6-39}$$

Since $r < d < p$ and $rb \equiv rp^{-1} \not\equiv 0 \bmod d$, we have

$$r\frac{bp-1}{d} \bmod p = \frac{(rb \bmod d)\cdot p - r}{d}$$

and moreover $rb \bmod d = rb - \ell d$ for $\ell d/b \le r < (\ell+1)d/b$. Breaking the sum over r up into these regions converts (6-39) into

$$\sum_{\ell=0}^{b-1} \sum_{\substack{\ell d/b < r \\ r < (\ell+1)d/b}} \sum_{i=\lceil rp/d \rceil}^{p-1} \chi\left(\frac{(rb-\ell d)p - r}{d},\, p-1-i \right). \tag{6-40}$$

Indeed, the quantity $\ell d/b$ is either 0 or *nonintegral* since $\gcd(b, d) = 1$ and $0 \le \ell < b$; thus $\ell d/b \le r$ is equivalent to $\ell d/b < r$ since r ranges over all positive integers less than d. The innermost sum in (6-40) has value the cardinality of the subset of positive integers

$$T(r,\ell) := \left\{ i \in \mathbf{N} \,:\, \frac{(rb-\ell d)p - r}{d} \le p-1-i \le p-1-\left\lceil \frac{rp}{d} \right\rceil = \left\lfloor \frac{(d-r)p}{d} \right\rfloor - 1 \right\}.$$

Now $T(r,\ell)$ is empty unless

$$\frac{(d-r)p}{d} - 1 > \frac{(rb-\ell d)p - r}{d}, \tag{6-41}$$

in which case

$$\begin{aligned} \#T(r,\ell) &= \left\lfloor \frac{(d-r)p}{d} \right\rfloor - \frac{(rb-\ell d)p - r}{d} \\ &= \frac{(d-r)(p-a)}{d} + \left\lfloor \frac{a(d-r)}{d} \right\rfloor - \frac{(rb-\ell d)p - r}{d} \\ &= \big((\ell+1)d - (b+1)r\big)\frac{p}{d} + \left(\frac{r}{d} - \left\{ \frac{a(d-r)}{d} \right\} \right). \end{aligned} \tag{6-42}$$

The inequality (6-41) is equivalent to $((\ell+1)d - (b+1)r)p > (d-r)$ which, as $p > d > d-r$ is equivalent to $(\ell+1)d > (b+1)r$, or what is the same thing, $r < (\ell+1)d/(b+1)$. In other words, the contribution from the innermost sum (6-39) for r in the ranges $(\ell+1)d/(b+1) < r < (\ell+1)d/b$ is zero so that these values of r in the middle sum may be omitted, and the expression (6-42) then substituted for the innermost sum, which yields the claimed formula for $\tau_p(d)$ when $d < p$. □

Using the bounds of Theorem 6.26, we are able to generalize the main result of [Farnell and Pries 2013], which provides an a-number *formula* for branched $\mathbf{Z}/p\mathbf{Z}$-covers of $X = \mathbf{P}^1$ with all ramification breaks d dividing $p-1$, to the case of *arbitrary* ordinary base curves X:

Corollary 6.32. *Let $\pi : Y \to X$ be a branched $\mathbf{Z}/p\mathbf{Z}$-cover with $a_X = 0$, and suppose p is odd. If d_Q divides $p-1$ for every branch point Q, then*

$$a_Y = \sum_Q a_Q, \qquad \text{where } a_Q := \frac{(p-1)}{2}(d_Q - 1) - \frac{p-1}{d_Q}\left\lfloor \frac{(d_Q-1)^2}{4} \right\rfloor. \tag{6-43}$$

Proof. We will compute the upper and lower bounds for a_Y given by (6-20) and (6-21), and show that these bounds coincide when $d_Q|(p-1)$ for all Q and $a_X = 0$, and agree with the stated formula. Using the hypothesis $a_X = 0$ together with the explicit formula for $\tau_p(d)$ when $d \equiv 1 \bmod p$ provided by (6-37), the upper bound (6-20) becomes

$$a_Y \le \sum_{Q\in S} \sum_{i=1}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right) - \sum_{Q\in S} \frac{p-1}{d_Q} \left\lfloor \frac{(d_Q-1)^2}{4} \right\rfloor. \tag{6-44}$$

Now $d_Q < p$ for all Q, so $\lfloor id_Q/p^2 \rfloor = 0$ for $1 \le i \le p-1$. Using (6-30), we conclude that

$$\sum_{Q\in S} \sum_{i=1}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor \right) = \frac{(p-1)}{2}(d_Q - 1). \tag{6-45}$$

Combining (6-44) and (6-45) gives the formula (6-43) as the upper bound of a_Y. It remains to prove that a_Y is also bounded below by the same quantity.

Set $j := (p+1)/2$, and write $d_Q \cdot f_Q = p - 1$. For $i < p$, we see that

$$\left\lfloor \frac{id_Q}{p} \right\rfloor = \begin{cases} \lfloor i/f_Q \rfloor & \text{when } f_Q \nmid i, \\ \lfloor i/f_Q \rfloor - 1 & \text{when } f_Q | i. \end{cases}$$

Likewise we can check that

$$\left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{jd_Q}{p} \right\rfloor = \left\lfloor \frac{i-j}{f_Q} \right\rfloor.$$

The inner sum from (6-21) becomes

$$\sum_{i=(p+1)/2}^{p-1} \left\lfloor \frac{i}{f_Q} \right\rfloor - \sum_{i=0}^{(p-3)/2} \left\lfloor \frac{i}{f_Q} \right\rfloor - \#\{(p+1)/2 \le i \le p-1 : f_Q | i\}.$$

The term in the first sum can be rewritten as $\lfloor d_Q/2 + (i - (p-1)/2) \rfloor$. When d_Q is even, $d_Q/2$ can be removed from the floor function, allowing cancellation with the second sum giving that the inner sum from (6-21) is

$$\frac{(p-1)d_Q}{4}.$$

This equals a_Q from (6-43) when d_Q is even. When d_Q is odd, a similar argument removing $\lfloor d_Q/2 \rfloor = \frac{1}{2}(d_Q - 1)$ from the first sum shows that the lower bound is

$$\lfloor d_Q/2 \rfloor \frac{p-1}{2} + \frac{f_Q \cdot \lfloor d_Q/2 \rfloor}{2} - f_Q/2 = f_Q \cdot \frac{(d_Q+1)(d_Q-1)}{4}.$$

Again, this matches the formula for a_Q. Summing over Q completes the proof. □

Remark 6.33. Note that the proof shows that one has the alternative expression

$$a_Q = \begin{cases} \dfrac{(p-1)d_Q}{4} & d_Q \text{ is even,} \\ \dfrac{(p-1)(d_Q+1)(d_Q-1)}{4d_Q} & d_Q \text{ is odd.} \end{cases}$$

for a_Q as in (6-43); cf. [Farnell and Pries 2013, Theorem 1.1].

6E. ***Unramified covers.*** Now suppose that $\pi : Y \to X$ is an unramified degree-p Artin–Schreier cover. (No such covers exist when $X = \mathbf{P}^1$.) Theorem 6.26 applies with $S = \varnothing$, but the analysis is not optimized for this situation. In particular, the argument uses the dimension of $H^0(X, \ker V_X)$ (the a-number) but no extra information about the dimension of $H^0(X, \ker V_X(F_*D))$ for other divisors D. This is not an issue when π has a substantial amount of ramification, but would be essential to obtaining sharper bounds for unramified covers. We now illustrate this.

In this setting, the bounds of Theorem 6.26 become

$$0 \le a_Y \le p \cdot a_X.$$

On the other hand, since $E_0 = 0$ the trivial bounds are

$$a_X \le a_Y \le p \cdot (g_X - f_X).$$

Note that $p \cdot (g_X - f_X) \ge p \cdot a_X$, so our upper bound is an improvement, but the trivial lower bound is better!

Any of these bounds are enough to rederive the following fact (which is typically deduced from the Deuring-Shafarevich formula):

Corollary 6.34. *Let $\pi : Y \to X$ be an unramified degree-p Artin–Schreier cover. Then Y is ordinary if and only if X is ordinary.*

The trivial lower bound is better because we lost some information in our applications of Tango's theorem. In particular, in the proof of Theorem 6.26 the first step is to add more points to the auxiliary divisors D_i (if needed) so that $\deg(D_i) > \mathrm{n}(X)$ for all i and we can apply Corollary 6.13 to exactly calculate

$$\begin{aligned}\dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))) &= \dim_k H^0(X, \ker V_X(F_*(pD_i)) \\ &= (p-1)\deg(D_i).\end{aligned}$$

The trivial bounds comes from the inclusion $\ker V_X \hookrightarrow \pi_* \ker V_Y$ given by $\omega \mapsto \omega y^0$, which would correspond to taking $j = 0$ in Definition 6.6 and being allowed to take $D_0 = 0$. If that were the case, S is empty and the lower bound is

$$\dim U_0 - \sum_{Q \in S_0} c(0, 0, Q) = \dim H^0(X, \ker V_X(F_*0)) = a_X$$

which matches the trivial lower bound. However, our analysis uses Corollary 6.13, which requires a sufficiently large D_0. In that case,

$$\dim U_0 - \sum_{Q\in S_0} c(0,0,Q) = \dim H^0(X, \ker V_X(F_* D_0)) = (p-1)\deg(D_0) - \sum_{Q\in D_0}(p-1) = 0$$

as $c(0,0,Q) = (p-1)$ for $Q \in D_0$. Thus the lower bound of Theorem 6.26 is zero. (In Theorem 6.26, we reindexed so there we are taking $j = p-1$.)

The technical requirements of using sufficiently large divisors D_i in order to apply Tango's theorem are not an issue when π is "highly ramified" in the sense that we do not need to increase the degree of the D_i in order to apply Corollary 6.13 to compute the dimension of $H^0(X, \ker V_X(F_*(E_i + pD_i)))$. Once we have reached that point, increasing D_i further does not change the bound: the dimension of this space increases, but the increase is canceled by additional $c(i, j, Q)$ terms for $Q \in D_i$. This is why the divisor D_i makes no appearance in Theorem 6.26.

7. Examples

As always, let $\pi : Y \to X$ be a degree-p Artin–Schreier cover of curves. We give some examples of the bounds given by Theorem 6.26 and the trivial bounds

$$\dim_k H^0(X, \ker V_X(F_* E_0)) \le a_Y \le p\cdot g_X - p\cdot f_X + \sum_{Q\in S}\frac{p-1}{2}(d_Q - 1)$$

discussed in Remark 1.3. When $p = 3$, one checks that the lower bound $L(X, \pi)$ *coincides* with the trivial lower bound. Outside of this special case, the bounds in Theorem 6.26 are *always* better than the trivial bounds, and often sharp in the sense that there are degree-p Artin–Schreier covers $\pi : Y \to X$ with a_Y realizing our bounds in many cases. We will give a number of examples illustrating these features. Magma programs which do the calculations in the following examples are available on the authors' websites.

7A. ***The projective line.*** Suppose $X = \mathbf{P}^1$. Then $\mathrm{n}(X) = -1$, $g_X = 0$, $a_X = 0$, and $f_X = 0$. Remark 3.3 shows that we may choose $f \in k(X)$ so that the extension of function fields $k(Y)/k(X)$ is given by adjoining the roots of $y^p - y = f$ with $f \in k(X)$ regular outside the branch locus S of π, and we may further assume that f has a pole of order d_Q at each $Q \in S$, where d_Q is the unique break in the lower-numbering ramification filtration above Q.

For $Q \in S$ put $n_{Q,i} = \lceil (p-1-i)d_Q/p\rceil$; note that $n_{Q,0} = d_Q - \lfloor d_Q/p\rfloor$. The trivial bounds are

$$\sum_{Q\in S}\left(n_{Q,0} - \left\lceil\frac{n_{Q,0}}{p}\right\rceil\right) \le a_Y \le \sum_{Q\in S}\frac{p-1}{2}(d_Q - 1).$$

Example 7.1. Let $p = 13$, and suppose f has a single pole. Table 1 shows the trivial upper and lower bounds, as well as the bounds from Theorem 6.26 for various values of d_Q. When $d_Q > 4$, an optimum value of j to use in the lower bound turns out to be $\frac{1}{2}(p+1)=7$. Notice that our bounds are substantially better than the trivial bounds.

Example 7.2. Using Magma [Bosma et al. 1997] or a MAPLE program from Shawn Farnell's thesis [2010], we can compute the a-number for covers of $\mathbf{P}^1$. For example, let $p=13$ and suppose f has a single pole of order 7. Our results show that the a-number of the cover is between 21 and 36. Table 2 lists the a-numbers for some choices of f, and shows that our bounds are sharp in this instance.

Example 7.3. Let us generalize Examples 4.6 and 5.21. As before, let $p=5$ and $X=\mathbf{P}^1$, but now consider covers $Y_{A,B}$ of X given by $y^5-y=f_{A,B}=t^{-3}+At^{-2}+Bt^{-1}$. Then $Y_{A,B}\to X$ is branched only over $Q:=0$ and has $d_Q=3$, so our bounds are $3\le a_{Y_{A,B}}\le 4$.

Recall the maps g_i' depend on the cover via the choice $f_{A,B}$. It is clear that the elements $\nu_{0,2}$, $\nu_{0,3}$, and $\nu_{1,2}$ lie in $H^0(\mathbf{P}^1,\mathscr{G}_1)$. In fact, so does $\nu_{2,2}$ as $g_2'(\nu_{2,2})=0$: indeed, recalling Section 5B, we see that $g_2'(\nu_{2,2})$ records the coefficient of $t^{-6}dt$ in

$$-2\cdot(t^{-2}dt)\cdot-(t^{-3}+At^{-2}+Bt^{-1})$$

which is visibly zero.

To compute the a-number (equal to $\dim H^0(\mathbf{P}^1,\mathscr{G}_0)$ in this case), it therefore suffices to understand $\ker H^0(g_1')=H^0(\mathbf{P}^1,\mathscr{G}_0)$. It is clear that $\nu_{0,2}$ and $\nu_{0,3}$ lie in $H^0(\mathbf{P}^1,\mathscr{G}_0)$. An identical calculation to the one above shows that $g_1'(\nu_{1,2})=0$ as well. Finally, we see that $g_1'(\nu_{2,2})$ records the coefficient of $t^{-6}dt$ in

$$-t^{-2}(t^{-3}+At^{-2}+Bt^{-1})^2dt=-(t^{-8}+2At^{-7}+(A^2+2B)t^{-6}+\cdots)dt.$$

Thus $g_1'(\nu_{2,2})=0$ when $A^2+2B=0$, and is nonzero otherwise. In particular, $a_{Y_{A,B}}=3$ when $A^2+2B\neq 0$ and $a_{Y_{A,B}}=4$ when $A^2+2B=0$. This generalizes Example 5.21, again showing why the a-number of the cover cannot depend only on the d_Q and must incorporate finer information.

The set T in Proposition 6.18 is an attempt to produce differentials not in the kernel of some g_i'. It only uses the leading terms of powers of $f_{A,B}$ (since those are the only terms guaranteed to be nonzero). For the cover $Y_{0,0}$, the leading term is the only term, and there are no differentials not in the kernel of some g_i'. This is reflected in the fact that T is empty in this case. When $A^2+2B\neq 0$, the a-number was smaller because there was a differential not in $\ker H^0(g_1')$; this relied on the nonleading terms of $f_{A,B}$.

Example 7.4. Let us now consider an example with multiple poles. Let $p=5$, and suppose that f has two poles of order 7 (at ∞ and -1). Then our bounds say that $14\le a_Y\le 16$. Computing the a-number

$d_Q=$	1	2	3	4	5	6	7	8	9	10	11	12	14	15	32	128	1024
trv. lower	0	1	2	3	4	5	6	7	8	9	10	11	12	12	27	109	873
$L(\mathbf{P}^1,\pi)$	0	6	8	12	15	18	21	24	26	30	33	36	42	45	96	382	3054
$U(\mathbf{P}^1,\pi)$	0	6	8	12	16	18	36	30	34	36	38	36	78	60	120	488	3936
trv. upper	0	6	12	18	24	30	36	42	48	54	60	66	78	84	186	762	6138

Table 1. Bounds for a single pole with $p=13$. (trivial abbreviated as trv.)

polynomial	a_Y
$t^{-7}+2t^{-6}+7t^{-5}$	21
$t^{-7}+t^{-2}+t^{-1}$	23
$t^{-7}+8t^{-2}$	24
$t^{-7}+t^{-1}$	27
t^{-7}	36

Table 2. a-numbers for some select Artin–Schreier curves.

for a thousand random choices of f defined over $\mathbf{F}_5$ subject to the constraint on the poles, 942 of them had a-number 14, 41 had a-number 15, and 1 had a-number 16. The f giving a-number 16 was

$$t^7+3t^6+t^4+t^3+2t^2+t+3(t+1)^{-1}+3(t+1)^{-2}+4(t+1)^{-3}+4(t+1)^{-4}+4(t+1)^{-6}+3(t+1)^{-7}.$$

It would be interesting to understand the geometry of the relevant moduli space of Artin–Schreier covers with prescribed ramification. In this situation, these numerical examples seem to suggest that the locus of curves with a-numbers greater than or equal to 15 (respectively 16) is of codimension two in the locus with a-number greater than or equal to 14 (respectively 15).

Example 7.5. For general p, take $f(t)=t^{-d}$ with $p\nmid d$. We will show that this family achieves our upper bound. As discussed in [Farnell and Pries 2013, Remark 2.1] (which extracts the results from [Pries 2005]), the resulting a-number is

$$\sum_{b=0}^{d-2}\min\left(h_b,\left\lfloor\frac{pd-bp-p-1}{d}\right\rfloor\right)$$

where h_b is the unique integer in $[0,\, p-1]$ such that $h_b\equiv(-1-b)d^{-1} \bmod p$. Note that if $b\equiv -1 \bmod p$ then $h_b=0$. This counts the number of elements in the set

$$T' := \left\{(b,\, j) : 0\le b\le d-2,\ 0\le jd\le p(d-b-1)-d-1,\ \ j<h_b\right\}.$$

On the other hand, our upper bound is $\bigoplus_{i=0}^{p-1}\dim_{\mathbf{F}_p} H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*E_i)) - \#T$. For $0\le i\le p-1$, the differentials $t^{-n}dt$ with $0<n\le n_{Q,i}=\lceil(p-1-i)d/p\rceil$ and $n\not\equiv 1 \bmod p$ form a basis for $H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*E_i))$. The condition $n\le n_{Q,i}$ can be expressed as $pn\le(p-1-i)d+(p-1)$, or equivalently

$$i\cdot d\le(p-1)d+(p-1)-pn=p(d-n+1)-1-d.$$

Assigning to each such basis element the triple (Q,n,i), note that $(Q,n,\,j)$ *does not* lie in T if there does not exist an integer $m\in[0,\,j]$ such that $m\equiv j-(n-1)d_Q^{-1} \bmod p$: this can be rephrased as $j<h_{n-2}$. Thus our upper bound is the size of the set

$$T'' := \left\{(n,\, j) : 2\le n\le d,\ 0\le j\cdot d\le p(d-n+1)-1-d,\ j<h_{n-2}\right\}.$$

(The condition $n \not\equiv 1 \bmod p$ is implicit, as in that case $h_{n-2} = 0$.) But T' and T'' have the same size: there is a bijection given by taking $b = n-2$. Thus the covers given by $f(t) = t^{-d}$ for $p \nmid d$ realize the upper bound for a cover ramified at a single point.

7B. *Elliptic curves.* We now suppose that E is the elliptic curve over $\mathbf{F}_p$ with affine equation $y^2 = x^3 - x$ (recall that $p > 2$). Of course, $g_E = 1$ and it is not hard to compute that the Tango number of E is $n(E) = 0$ and that E is ordinary (so $a_E = 0$ and $f_E = 1$) when $p \equiv 1 \bmod 4$ and supersingular ($a_E = 1$ and $f_E = 0$) when $p \equiv 3 \bmod 4$. In this simple case, we can say more than what Tango's theorem tells us.

Lemma 7.6. *Let $D = \sum n_Q \cdot [Q] > 0$ be a divisor on E with $n_Q \geq 2$ for some Q, and set $D' := \sum \lceil n_Q/p \rceil \cdot [Q]$. Then*

$$V_E : H^0(E, \Omega^1_E(D)) \to H^0(E, \Omega^1_E(D'))$$

is surjective.

Proof. This lemma is an immediate consequence of Corollary 6.13 when $n_Q \geq p$ for some Q or when $a_E = 0$, so we suppose that $n_Q < p$ for all Q and $p \equiv 3 \bmod 4$. A straightforward induction on the size of the support of D reduces us to the case that $D = n \cdot [Q]$ with $n \geq 2$, and then as $D' = [Q]$ it suffices to treat the case $n = 2$. Suppose that Q is not the point P at infinity on E. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(E, \Omega^1_E(2[P])) & \longrightarrow & H^0(E, \Omega^1_E(2[P]+2[Q])) & \longrightarrow & (k[x_Q]/x_Q^2)x_Q^{-2}dx_Q & \longrightarrow 0 \\ & \downarrow V & & \downarrow V & & \downarrow V & \\ 0 \longrightarrow & H^0(E, \Omega^1_E([P])) & \longrightarrow & H^0(E, \Omega^1_E([P]+[Q])) & \longrightarrow & (k[x_Q]/x_Q)x_Q^{-1}dx_Q & \longrightarrow 0 \end{array}$$

The local description of V shows that the right vertical map is surjective, and it therefore suffices to prove the lemma in the special case $D = 2[P]$. Now $\{dx/y, x\,dx/y\}$ is a basis of $H^0(E, \Omega^1_E(2[P]))$ and one calculates

$$V\left(x\frac{dx}{y}\right) = \frac{1}{y} V(x(x^3-x)^{(p-1)/2}dx) = (-1)^{(p+1)/4} \binom{\frac{1}{2}(p-1)}{\frac{1}{4}(p-3)} \frac{dx}{y} \neq 0.$$

Since $V(dx/y) = 0$, the image of V is 1-dimensional. But $H^0(E, \Omega^1_E([P])) = H^0(E, \Omega^1_E)$ because the sum of the residues of a meromorphic differential on a smooth projective curve is zero, and this space therefore has dimension one as well; the lemma follows. □

We first consider Artin–Schreier covers $Y \to E$ with defining equation of the form

$$z^p - z = f$$

with $f \in H^0(E, \mathcal{O}_E(d_P \cdot [P]))$ having a pole of exact order d_P at P, for P the point at infinity on E, where $d_P \geq 2$ is prime to p. Such covers are branched only over P with unique break in the ramification filtration d_P. Conversely, every $\mathbf{Z}/p\mathbf{Z}$-cover of E that is defined over $\mathbf{F}_p$ and is ramified only above P with unique ramification break d_P occurs this way by Lemma 3.2.

$d_P =$	2	3	4	6	7	8	9	11	12	13	14	16	32	128	1024
trv. lower	1	2	3	4	4	5	6	7	8	8	9	10	20	82	656
lower	2	3	4	6	7	8	9	10	12	12	14	15	31	123	984
upper	2	4	4	10	8	10	10	14	16	14	16	20	38	154	1230
trv. upper	2	4	6	10	12	14	16	20	22	24	26	30	62	254	2046

Table 3. Bounds for a single pole at the point at infinity with $p = 5$. (trivial abbreviated as trv.)

$d_P =$	2	3	4	5	6	8	9	10	11	12	13	15	16	32	128	1024
trv. lower	1	2	3	4	5	6	6	7	8	9	10	11	12	24	94	752
lower	3	4	6	7	9	12	13	15	16	18	20	21	24	47	188	1505
upper	10	11	16	15	16	28	23	24	27	30	29	37	40	68	244	1910
trv. upper	10	13	16	19	22	28	31	34	37	40	43	49	52	100	388	3076

Table 4. Bounds for a single pole at the point at infinity with $p = 7$. (trivial abbreviated as trv.)

	$d_P = 3$	$d_P = 6$	$d_P = 8$	$d_P = 11$
a_Y	f			
3	$-2y - x$			
4	$2y$			
6		$x^3 - 2x^2$		
7		$x^3 + x$		
8		$2x^3 - xy$	x^4	
9		$-x^3 + x$	$2x^4 - x^2 + x$	
10			$-x^4 - 2xy - 2x^2$	$(x^4 - x)y - 2x^5 - x^4$
11				$(-2x^4 + 2x + 1)y + 2x^5 + x^2$
12				$(-2x^4 - x^2 - x)y - 2x^5 - 1$
13				$(2x^4 - x^2 + 2x)y - 2x^5 - x^3 - x$

Table 5. a-numbers of $\mathbf{Z}/p\mathbf{Z}$-covers of E branched only over P with $p = 5$.

Example 7.7. For $p = 5, 7$ we compare the bounds given by Theorem 6.26 with the trivial upper and lower bounds, which are

$$\left\lceil \frac{p-1}{p} d_P \right\rceil - \left\lceil \frac{p-1}{p^2} d_P \right\rceil \leq a_Y \leq p \cdot a_E + \frac{p-1}{2}(d_P - 1)$$

thanks to Lemma 7.6 and the fact that $\lceil (p-1)d_P/p \rceil = d_P - \lfloor d_P/p \rfloor \geq 2$. We summarize these computations in Tables 3 and 4.

Example 7.8. Again for $p = 5, 7$ and select d_P in the tables above, we have computed a_Y for several thousand randomly selected $f \in H^0(E, \mathcal{O}_E(d_P \cdot [P]))$ (working over $\mathbf{F}_p$). In Tables 5 and 6, we record

	$d_P=6$	$d_P=8$	$d_P=10$
a_Y	f		
10	x^3+xy		
11	x^3+y		
12	x^3	$3xy+2x^4-2x^3$	
13		$2xy-3x^4-1$	
14	$2x^3+x$	$2xy-2x^4+x-3$	
15		$-x^4+2x^3+2x^2-3x-3$	$-3x^5-3x^4-x^2+3x$
16		$(x^2+x)y+3x^4-2x^3-3x^2-x+1$	$2x^5-3x^3+x^2+3x$
17			$-x^5+x^3+2x^2-1$
18			$2xy-x^5+x^3+2x^2-x-1$

Table 6. a-numbers of $\mathbf{Z}/p\mathbf{Z}$-covers of E branched only over P with $p=7$.

the values of a_Y that we found, as well as a function f that produced it. These values of a_Y should be compared to the bounds in Tables 3 and 4.

Remark 7.9. Larger values of a_Y are more rare. For example, when $p=7$, $S=\{P\}$ and $d_P=6$, among *all* $100842=6\cdot 7^5$ functions $f\in H^0(E,\mathcal{O}_E(d_P\cdot[P]))$ with a pole of exact order 6 at P, there are 86436 (= 85.71%) with $a_Y=10$, 11760 (= 11.66%) with $a_Y=11$, 2562 (= 2.54%) with $a_Y=12$ and 84 (= 0.08%) with $a_Y=14$. Curiously, none had $a_Y=13$. In the spirit of [Cais et al. 2013], it would be interesting to investigate the limiting distribution of a-numbers in branched $\mathbf{Z}/p\mathbf{Z}$-covers of a fixed base curve with fixed branch locus, as the sum of the ramification breaks tends to infinity. Soumya Sankar [2019] investigated the limiting distribution of nonordinary (a-number greater than 0) covers of the projective line.

Although $d_P=6$ divides $p-1$ when $p=7$, the a-number of Y can be 10, 11, 12 or 14; in particular, the ordinarity hypothesis in Corollary 6.32 is necessary.

Example 7.10. We now work out some examples with $\pi: Y\to E$ branched at exactly two points. As before, let P be the point at infinity on E and Q be the point $(0,0)$. For $p=5$ we considered Artin–Schreier covers Y of E branched only over P, Q with $d_P=6$ and $d_Q=4$. Our bounds are $10\le a_Y\le 14$, and among a sample of 10001 functions $f\in H^0(E,\mathcal{O}_E(d_P\cdot[P]+d_Q\cdot[Q]))$ with a pole of exact order $d_\star$ at $\star=P,Q$, we found 8021 with $a_Y=10$, 1818 with $a_Y=11$, 149 with $a_Y=12$, and 13 with $a_Y=13$. One of the 13 functions we found giving a-number 13 was

$$f=-\frac{1}{xy}+\frac{x^5-x^3+2x^2-2x+1}{x^2}.$$

Similarly, with $p=7$ and $d_P=6$, $d_Q=8$ our bounds are $21\le a_Y\le 37$. Among a sample of 5001 random functions satisfying the required constraints we found 4318 with $a_Y=21$, 668 with $a_Y=22$, 14 with $a_Y=23$, and 1 with $a_Y=24$. The function producing a cover Y with a-number 24 was

$$f = \frac{-x^4 - 3x^3 + 2x^2 - x + 1}{x^4 y} + \frac{-2x^7 + 3x^5 - 3x^4 + x^3 + x^2 + x + 3}{x^4}.$$

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On the locus of 2-dimensional crystalline representations with a given reduction modulo p

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We consider the family of irreducible crystalline representations of dimension 2 of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ given by the V_{k,a_p} for a fixed weight $k \geq 2$. We study the locus of the parameter a_p where these representations have a given reduction modulo p. We give qualitative results on this locus and show that for a fixed p and k it can be computed by determining the reduction modulo p of V_{k,a_p} for a finite number of values of the parameter a_p. We also generalize these results to other Galois types.

Contents

Introduction

Let p be a prime number. Fix a continuous representation $\bar{\rho}$ of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with values in $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Kisin [2008] defined local rings $R^{\psi}(k, \bar{\rho})$ that parametrize the deformations of $\bar{\rho}$ to characteristic 0 representations that are crystalline with Hodge–Tate weights $(0, k-1)$ and determinant ψ. These rings are very hard to compute, even for relatively small values of k. We are interested in this paper in the rings $R^{\psi}(k, \bar{\rho})[1/p]$. These rings lose some information from $R^{\psi}(k, \bar{\rho})$, but still retain all the information about the parametrization of deformations of $\bar{\rho}$ in characteristic 0.

We can relate the study of the rings $R^{\psi}(k, \bar{\rho})[1/p]$ to another problem: When we fix an integer $k \geq 2$ and set the character ψ to be $\chi_{\mathrm{cycl}}^{k-1}$, the set of isomorphism classes of irreducible crystalline representations of dimension 2, determinant ψ and Hodge–Tate weights $(0, k-1)$ is in bijection with the set $D = \{x \in \overline{\mathbb{Q}}_p, v_p(x) > 0\}$ via a parameter a_p, and we call V_{k,a_p} the representation corresponding

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to a_p. So given a residual representation $\bar{\rho}$ we can consider the set $X(k, \bar{\rho})$ of $a_p \in D$ such that the semisimplified reduction modulo p of V_{k,a_p} is equal to $\bar{\rho}^{ss}$.

It turns out that $X(k, \bar{\rho})$ has a special form. We say that a subset of $\overline{\mathbb{Q}}_p$ is a standard subset if it is a finite union of rational open disks from which we have removed a finite union of rational closed disks. Then we show that (when $\bar{\rho}$ has trivial endomorphisms, so that the rings $R^{\psi}(k, \bar{\rho})$ are well-defined):

Theorem A (Theorem 5.3.3 and Proposition 5.3.5). *The set $X(k, \bar{\rho})$ is a standard subset of $\overline{\mathbb{Q}}_p$, and $R^{\psi}(k, \bar{\rho})[1/p]$ is the ring of bounded analytic functions on $X(k, \bar{\rho})$.*

This tells us that we can recover $R^{\psi}(k, \bar{\rho})[1/p]$ from $X(k, \bar{\rho})$. But we need to be able to understand $X(k, \bar{\rho})$ better.

We can define a notion of complexity for a standard subset X which is invariant under the absolute Galois group of E for some finite extension E of $\mathbb{Q}_p$. This complexity is a positive integer $c_E(X)$, which mostly counts the number of disks involved in the definition of X, but with some arithmetic multiplicity that measures how hard it is to define the disk on the field E. A consequence of this definition is that if an upper bound for $c_E(X)$ is given, then X can be recovered from the sets $X \cap F$ for some finitely many finite extensions F of E, and even from the intersection of X with some finite set of points under an additional hypothesis (Theorems 4.5.1 and 4.5.2).

A key point is that this complexity, which is defined in a combinatorial way, is actually related to the Hilbert–Samuel multiplicity of the special fiber of the rings of analytic functions bounded by 1 on the set X (Theorem 4.4.1). This is especially interesting in the case where the set X is $X(k, \bar{\rho})$ as in this case this Hilbert–Samuel multiplicity can be bounded explicitly using the Breuil–Mézard conjecture. So, when $\bar{\rho}$ has trivial endomorphisms and under some conditions that ensure that the Breuil–Mézard conjecture is known in this case:

Theorem B (Theorem 5.3.3). *There is an explicit upper bound for the complexity of $X(k, \bar{\rho})$.*

As a consequence we get (with some additional conditions on $\bar{\rho}$, that are satisfied for example when $\bar{\rho}$ is irreducible):

Theorem C (Corollary 5.4.11). *The set $X(k, \bar{\rho})$ can be determined by computing the reduction modulo p of V_{k,a_p} for a_p in some finite set.*

In particular, it is possible to compute the set $X(k, \bar{\rho})$, and the ring $R^{\psi}(k, \bar{\rho})[1/p]$, by a finite number of numerical computations. We give some examples of this in Section 6. One interesting outcome of these computations is that when $\bar{\rho}$ is irreducible, in every example that we computed we observed that the upper bound for the complexity given by Theorem B is actually an equality. It would be interesting to have an interpretation for this fact and to know if it is true in general.

Finally, we could ask the same questions about more general rings parametrizing potentially semistable deformations of a given Galois type, instead of only rings parametrizing crystalline deformations. Our method relies on the fact that we work with rings that have relative dimension 1 over $\mathbb{Z}_p$, so we cannot use it beyond the case of 2-dimensional representations of $G_{\mathbb{Q}_p}$. But in this case we can actually generalize our results to all Galois types. In order to do this, we need to introduce a parameter classifying the representations that plays a role similar to the role the function a_p plays for crystalline representations, and to show that it defines an analytic function on the rigid space attached to the deformation ring. This is the result of Theorem 5.3.1. Once we have this parameter, we show that an analogue of Theorem A holds, and an analogue of Theorem B (Theorem 5.3.3). However we get only a weaker analogue of Theorem C (Theorem 5.3.6). The main ingredient of this theorem that is known in the crystalline case, but missing in the case of more general Galois types, is the fact that the reduction of the representation is locally constant with respect to the parameter a_p, with an explicit radius for local constancy.

Plan of the article. The first three sections contain some preliminaries. In Section 1 we prove some results on the smallest degree of an extension generated by a point of a disk in $\overline{\mathbb{Q}}_p$. These results may be of independent interest. In Section 2 we prove some results on Hilbert–Samuel multiplicities and how to compute them for some special rings of dimension 1. In Section 3 we introduce the notion of a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ and prove some results about some special rigid subspaces of the affine line.

Section 4 contains the main technical results. This is where we introduce the complexity of so-called standard subsets of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, and show that it can be defined in either a combinatorial or an algebraic way.

We apply these results in Section 5 to the locus of points parametrizing potentially semistable representations of a fixed Galois type with a given reduction. We also explain some particularities of the case of parameter rings for crystalline representations.

In Section 6 we report on some numerical computations that were made using the results of Section 5 in the case of crystalline representations, and mention some questions inspired by these computations.

Finally in Section 7 we explain the construction of a parameter classifying the representations on the potentially semistable deformation rings.

Notation. If E is a finite extension of $\mathbb{Q}_p$, we denote its ring of integers by $\mathcal{O}_E$, with maximal ideal $\mathfrak{m}_E$, and its residue field by k_E. We write π_E for a uniformizer of E, and v_E for the valuation on E normalized so that $v_E(\pi_E) = 1$ and its extension to $\overline{\mathbb{Q}}_p$. We also write v_p for $v_{\mathbb{Q}_p}$. Finally, G_E denotes the absolute Galois group of E.

If R is a ring and n a positive integer, we denote by $R[X]_{<n}$ the subspace of $R[X]$ of polynomials of degree at most $n-1$.

If $a \in \overline{\mathbb{Q}}_p$ and $r \in \mathbb{R}$, we write $D(a,r)^+$ for the set $\{x \in \overline{\mathbb{Q}}_p, |x-a| \leq r\}$ (closed disk) and $D(a,r)^-$ for the set $\{x \in \overline{\mathbb{Q}}_p, |x-a| < r\}$ (open disk).

We denote by χ_{cycl} the p-adic cyclotomic character, and ω its reduction modulo p. We denote by $\mathrm{unr}(x)$ the unramified character that sends a geometric Frobenius to x.

1. Points in disks in extensions of the base field

Definition 1.0.1. Let X be a subset of $\overline{\mathbb{Q}}_p$, and let E be an algebraic extension of $\mathbb{Q}_p$. We say that X is defined over E if it is invariant by the action of G_E.

Let $D \subset \overline{\mathbb{Q}}_p$ be a disk (open or closed). It can happen that D is defined over a finite extension E of $\mathbb{Q}_p$, but $E \cap D$ is empty. For example, let π be a p-th root of p and let D be the disk $\{x,\ v_p(x-\pi) > 1/p\}$. Then D is defined over $\mathbb{Q}_p$, as it contains all the conjugates of π, that is, the $\zeta_p^i \pi$ for a primitive p-th root ζ_p of 1. On the other hand, D does not contain any element of $\mathbb{Q}_p$. The goal of this section is to understand the relationship between the smallest ramification degree over E of an extension field F such that $F \cap D \neq \varnothing$, and the smallest degree over E of such a field F.

In this section a disk will mean either a closed or an open disk.

The results of this section are used in the proofs of Propositions 4.5.6 and 4.5.8.

1.1. *Statements.*

Theorem 1.1.1. *Let E be a finite extension of $\mathbb{Q}_p$. Let D be a disk defined over E. Let e be the smallest integer such that there exists a finite extension F of E with $e_{F/E} = e$ and $F \cap D \neq \varnothing$. Then $e = p^s$ for some s, and there exists an extension F of E with $[F : E] \leq \max(1, p^{2s-1})$ such that $F \cap D \neq \varnothing$. For $s \leq 1$ any such F/E is totally ramified.*

We can in fact do better in the case where $p = 2$. Note that this result proves Conjecture 2 of [Benedetto 2015] in this case.

Theorem 1.1.2. *Let $p = 2$. Let E be a finite extension of $\mathbb{Q}_p$. Let D be a disk defined over E. Let e be the smallest integer such that there exists a finite extension F of E with $e_{F/E} = e$ and $F \cap D \neq \varnothing$. Then $e = p^s$ for some s, and there exists a totally ramified extension F of E with $[F : E] = p^s$ such that $F \cap D \neq \varnothing$.*

1.2. *Preliminaries.* We recall the following result, which is [Benedetto 2015, Lemma 3.6] (it is stated only for closed disks, but applies also to open disks).

Lemma 1.2.1. *Let K be an algebraic extension of $\mathbb{Q}_p$. Let D be a disk defined over K. Suppose that D contains an $a \in \overline{\mathbb{Q}}_p$ of degree n over K. Then D contains an element $b \in \overline{\mathbb{Q}}_p$ of degree $\leq p^s$ over K where $s = v_p(n)$.*

Corollary 1.2.2. *Let K be an algebraic extension of $\mathbb{Q}_p$. Let D be a disk defined over K. Suppose that D contains an element a such that $[K(a) : K] = n$. Then the minimal degree over K of an element of D is of the form p^t for some $t \leq v_p(n)$.*

Proof. It follows from Lemma 1.2.1 that the minimal degree over K of an element of D is a power of p. On the other hand, applying Lemma 1.2.1 to a, we get an element of degree at most p^s for $s = v_p(n)$. Hence the minimal degree is of the form p^t for some $t \leq s$. ☐

Corollary 1.2.3. *Let E be a finite extension of $\mathbb{Q}_p$. Let D be a disk defined over E. Then the minimal ramification degree over E of an element of D is a power of p, and it can be reached for an element a such that $[E(a) : E]$ is a power of p.*

Proof. We first apply Corollary 1.2.2 with $K = E^{\mathrm{unr}}$ to see that the minimal ramification degree is a power of p. Let $b \in D$ be such that $e_{E(b)/E} = p^t$ is the minimal ramification degree.

Let $E(b)_0 = E^{\mathrm{unr}} \cap E(b)$, and let F be the maximal extension of E contained in $E(b)_0$ such that $[F : E]$ is a power of p. Note that $v_p([E(b) : F]) = t$, as $[E(b)_0 : F]$ is prime to p. We apply Corollary 1.2.2 to $K = F$, and we get an element $a \in D$ of degree at most p^t over F. By minimality of t, we get that in fact $[F(a) : F] = p^t$, and $F(a)/F$ is totally ramified. As $[F(a) : E]$ is a power of p, so is $[E(a) : E]$, and $e_{E(a)/E} = p^t$. □

Let π_E be a uniformizer of E, and let F be a finite unramified extension of E. For $x \in F$, we define the E-part of x, which we denote by x^0, as follows: we write x as $x = \sum_{n \geq N} a_n \pi_E^n$, where the a_n are Teichmüller lifts of elements of the residue field of F. Let $x^0 = \sum_{n=N}^{m} a_n \pi_E^n$ with $a_n \in E$ for all $n \leq m$ and $a_{m+1} \notin E$ (or $m = \infty$ if $a \in E$) so that $x^0 \in E$. We have that $v_E(x - x^0) = m + 1$. This definition depends on the choice of π_E.

Proposition 1.2.4. *Let D be a disk defined over E, and suppose that $F \cap D \neq \varnothing$ for some unramified extension F of E. Then $E \cap D \neq \varnothing$.*

Proof. Let $a \in F \cap D$. We fix π_E a uniformizer of E, and let a^0 be the E-part of a. Let σ be the Frobenius of $\mathrm{Gal}(F/E)$. Then $v_E(a - \sigma(a)) = v_E(a - a^0)$. So any disk containing a and $\sigma(a)$ also contains a^0. □

We also recall the well-known result:

Lemma 1.2.5. *Let $f \in \overline{\mathbb{Q}}_p(X)$ be a rational fraction with indeterminate X. Then for any disk D, if f does not have a pole in D, then $f(D)$ is also a disk. Moreover, if D is defined over E and $f \in E(X)$, then $f(D)$ is defined over E.*

1.3. *Proofs.* The part that states that e is a power of p in Theorems 1.1.1 and 1.1.2 is a consequence of Corollary 1.2.3.

We start with the rest of the proof of Theorem 1.1.2 which is actually easier.

Proof of Theorem 1.1.2. By applying Corollary 1.2.3, we get an element $a \in D$ that generates a totally ramified extension F of K of degree $e = p^s$, where K is an unramified extension of E of degree a power of p, and we take $[K : E]$ minimal. If $K \neq E$, let $K' \subset K$ with $[K : K'] = p$. We will show that we can find $b \in D$ of degree e over K', which gives a contradiction by minimality of K so in fact $K = E$.

Let μ be the minimal polynomial of a over K, so $\mu \in K[X]$ is monic of degree e. Now we use that $p = 2$: let $(1, u)$ be a basis of K over K', and write $\mu = \mu_0 + u\mu_1$ with μ_0, μ_1 in $K'[X]$. If μ_0 has a root in D we are finished, so we can assume that μ_0 has no zero in D, and let $f = \mu_1/\mu_0 \in K'(X)$. Let $D' = f(D)$. It is a disk defined over K', containing $-u \in K$, so by Proposition 1.2.4, D' contains an element $c \in K'$. This means that $\mu_0 - c\mu_1$ has a root b in D.

Then b is of degree at most e over K'. By minimality of e, it means that b is of degree exactly e over K', and $K'(b)/K'$ is totally ramified. So this gives the contradiction we were looking for. □

Now we turn to the proof of Theorem 1.1.1. We first prove the result when we assume an additional condition.

Proposition 1.3.1. *Let D be a disk defined over E and $a \in D$. Suppose that $v_E(a) = n/e$, where $e = e_{E(a)/E}$ and n is prime to e. Then there exists an extension F of E of degree at most e such that $F \cap D \neq \varnothing$.*

Proof. Let $K = E(a) \cap E^{\mathrm{unr}}$. Let μ be the minimal polynomial of a over K, so that μ has degree e. We write $\mu = \sum b_i X^i$, $b_i \in K$. Define $\mu^0 = \sum b_i^0 X^i$, where $b_i^0 \in E$ is the E-part of b_i. Let $x_1, \dots, x_e$ be the roots of μ^0. Then $v_E(\mu^0(a)) = \sum_{i=1}^{e} v_E(a - x_i)$. On the other hand, $\mu^0(a) = \mu^0(a) - \mu(a) = \sum_{i=0}^{e-1}(b_i^0 - b_i)a^i$. By the condition on $v_E(a)$, we get that $v_E(\mu^0(a)) = \min_{0 \le i < e}(v_E(b_i^0 - b_i) + in/e)$. Let σ be an element of G_E that induces the Frobenius on K. Let $y_1, \dots, y_e$ be the roots of $\sigma(\mu) = \sum \sigma(b_i) X^i$. Then as before, $v_E(\sigma(\mu)(a)) = \sum v_E(a - y_i)$, and $v_E(\sigma(\mu)(a)) = \min_{0 \le i < e}(v_E(\sigma(b_i) - b_i) + in/e)$. As $v_E(b_i^0 - b_i) = v_E(\sigma(b_i) - b_i)$ for all i, we get that $v_E(\mu^0(a)) = v_E(\sigma(\mu)(a))$.

Suppose first that D is closed. Write D as the set $\{z,\, v_E(z - a) \ge \lambda\}$ for some λ, then we get that $v_E(\sigma(\mu)(a)) \ge e\lambda$ as the y_i are among the conjugates of a over E and hence are in D, so $v_E(\mu^0(a)) \ge e\lambda$ and so there exists an i with $x_i \in D$. Let $F = E(x_i)$; then F is an extension of E of degree at most e. The case of an open disk is similar. □

Note that if we take e to be minimal, then necessarily F/E is totally ramified and of degree e.

Proof of Theorem 1.1.1. The case $e = 1$ is a consequence of Proposition 1.2.4.

Assume now that $e > 1$. Let $a \in D$, $F = E(a)$ with $e_{F/E} = e$. If a is a uniformizer of F, the result follows from Proposition 1.3.1. Otherwise, let $f \in E[X]_{<e}$ be a polynomial such that $f(a)$ is a uniformizer of F.

Assume first that such an f exists. Let $D' = f(D)$. Then D' is a disk defined over E by Lemma 1.2.5, containing an element $\varpi = f(a)$ with $e_{E(\varpi)/E} = e$ and $v_E(\varpi) = 1/e$, so it satisfies the hypotheses of Proposition 1.3.1. Hence there exists some $c \in D'$ with $[E(c) : E] \le e$. Let $b \in D$ such that $f(b) = c$, then $[E(b) : E] \le e(e-1)$ as b is a root of $f(X) - c$, which is a polynomial of degree at most $e - 1$ with coefficients in an extension of degree e of E. Moreover, by minimality of e, we get that $e_{E(b)/E} \ge e$, and so $[E(b) \cap E^{\mathrm{unr}} : E] \le e - 1$. Let K be the maximal extension of E contained in $E(b) \cap E^{\mathrm{unr}}$ such that $[K : E]$ is a power of p. Then $[K : E] \le p^{s-1}$ where $e = p^s$, because $[K : E] \le e - 1$. Now we apply again Lemma 1.2.1, to the field K: D contains a point a' with $[K(a') : K] \le p^{v_p([E(b):K])}$, that is, $[K(a') : K] \le p^s$. So finally $a' \in D$ and $[E(a') : E] \le p^{2s-1}$.

We prove now the existence of such a polynomial f. Fix a uniformizer π_F of F, and let $K = E(a) \cap E^{\mathrm{unr}}$. Let $\mathcal{E}$ be the set of pairs of e-tuples (α, P), where $\alpha = \alpha_1, \dots, \alpha_e$ are elements of K, $P = P_1, \dots, P_e$ are elements of $E[X]_{<e}$, and $\sum_i \alpha_i P_i(a) = \pi_F$. Then $\mathcal{E}$ is not empty: we can write $\pi_F = Q(a)$ for some $Q \in K[X]_{<e}$; now let $\alpha_1, \dots, \alpha_e$ be a basis of K over E, and write $Q = \sum \alpha_i P_i$ with $P_i \in E[X]_{<e}$. For each $(\alpha, P) \in \mathcal{E}$ let $m_{(\alpha,P)} = \inf_i v_E(\alpha_i P_i(a))$, so $m_{(\alpha,P)} \le 1/e$. It is enough to show that there is an (α, P) with $m_{(\alpha,P)} = 1/e$. Indeed, if $v_E(\alpha_i P_i(a)) = 1/e$, let $\beta_i \in E$ with $v_E(\alpha_i) = v_E(\beta_i)$ then $\beta_i P_i$ is the f we are looking for.

So choose an $(\alpha, P) \in \mathcal{E}$ with $m = m_{(\alpha,P)}$ minimal, and with minimal number of indices i such that $v_E(\alpha_i P_i(a)) = m$. Suppose that $m < 1/e$. Then there are at least two indices i with $v_E(\alpha_i P_i(a)) = m$. Say for simplicity that $v_E(\alpha_1 P_1(a)) = v_E(\alpha_2 P_2(a)) = m$. By minimality of e, P_1 and P_2 have no root in D. Let $f = P_1/P_2$, and $D' = f(D)$. Then D' is defined over E, and contains an element $f(a)$ of valuation $r = v_E(P_1(a)/P_2(a)) \in \mathbb{Z}$, as $r = v_E(\alpha_2/\alpha_1)$. Consider $\pi_E^{-r} D'$. It contains an element of valuation 0 and it does not contain 0, so it is contained in a disk $\{z, v_E(z - c) > 0\}$ for some element c that is the Teichmüller lift of an element of $\bar{\mathbb{F}}_p^\times$. So $v_E(\pi_E^{-r} P_1(a)/P_2(a) - c) > 0$. As $\pi_E^{-r} D'$ is defined over E, we have that $c \in E$. Let $x = c\pi_E^r$, then $v_E(P_1(a) - x P_2(a)) > r + v_E(P_2(a)) = v_E(P_1(a))$. We define an element (α', P') of $\mathcal{E}$ by setting $P_1' = P_1 - x P_2$ and $\alpha_2' = \alpha_2 + x\alpha_1$, and $\alpha_i' = \alpha_i$ and $P_i' = P_i$ for all other indices. We observe that $v_E(\alpha_1' P_1'(a)) > m$, $v_E(\alpha_2' P_2'(a)) \geq m$, and all other valuations are unchanged. This contradicts the choice we made for (α, P) at the beginning. So in fact $m = 1/e$. □

2. Some results on Hilbert–Samuel multiplicities

2.1. ***Hilbert–Samuel multiplicity.*** Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, and d be the dimension of A. Let M be a finitely generated module over A. We recall the definition of the Hilbert–Samuel multiplicity $e(A, M)$; see [Matsumura 1986, Chapter 13]. For n large enough, $\operatorname{len}_A(M/\mathfrak{m}^n M)$ is a polynomial in n of degree at most d. We can write its term of degree d as $e(A, M)n^d/d!$ for an integer $e(A, M)$, which is the Hilbert–Samuel multiplicity of M (relative to $(A, \mathfrak{m})$). We also write $e(A)$ for $e(A, A)$.

If $\dim A = 1$, it follows from the definition that

$$e(A, M) = \operatorname{len}_A(M/\mathfrak{m}^{n+1} M) - \operatorname{len}_A(M/\mathfrak{m}^n M) = \operatorname{len}_A(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M)$$

for n large enough.

We give some results that will enable us to compute $e(A)$ for some special cases of rings A of dimension 1.

Lemma 2.1.1. *Let k be a field, and $(A, \mathfrak{m})$ be a local noetherian k-algebra of dimension* 1, *with $A/\mathfrak{m} = k$. Suppose that there exists an element $z \in \mathfrak{m}$ such that A has no z-torsion and for all n large enough, $z\mathfrak{m}^n = \mathfrak{m}^{n+1}$. Then $e(A) = \dim_k A/(z)$.*

Proof. For n large enough, we have $\mathfrak{m}^{n+1} \subset (z)$. So the surjective map $A \to A/(z)$ factors through $A/\mathfrak{m}^{n+1}$ (and in particular $\operatorname{len}_A(A/(z))$ is finite). We have an exact sequence

$$A \xrightarrow{z} A/\mathfrak{m}^{n+1} \to A/(z) \to 0$$

For n large enough, the kernel of the first map is $\mathfrak{m}^n$ by the assumptions on z: it contains $\mathfrak{m}^n$, and as multiplication by z is injective, it is exactly equal to $\mathfrak{m}^n$. So we have an exact sequence

$$0 \to A/\mathfrak{m}^n \xrightarrow{z} A/\mathfrak{m}^{n+1} \to A/(z) \to 0$$

This gives $\operatorname{len}_A(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = \operatorname{len}_A(A/(z)) = \dim_k A/(z)$ as stated. □

Corollary 2.1.2. *Let k be a field, and $(A, \mathfrak{m})$ be a local noetherian k-algebra of dimension* 1, *with $A/\mathfrak{m} = k$. Suppose that there exist an element $z \in \mathfrak{m}$ such that A has no z-torsion and a nilpotent ideal I such that $\mathfrak{m} = (z, I)$. Then $e(A) = \dim_k A/(z)$.*

Proof. We need only show that $z\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for all n large enough, as we can then apply Lemma 2.1.1. Let m be an integer such that $I^m = 0$. Then for $n > m$ we have $\mathfrak{m}^n = \sum_{i=0}^m I^i z^{n-i}$, which gives the result. □

Definition 2.1.3. Let k be a field. Let $A_1, \dots, A_s$ be a family of local noetherian complete k-algebras of dimension 1 with maximal ideals M_i and residue field k. Let A be a local noetherian complete k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. We say that A is nearly the sum of the family (A_i) if there are injective k-algebra maps $u_i : A_i \to A$ such that $A = k \oplus \big(\bigoplus_{i=1}^s u_i(M_i)\big)$ as a k-vector space and $\mathfrak{m} = \bigoplus_{i=1}^s u_i(M_i)$.

In this case, we write V_i for $u_i(M_i)$, and for all $n > 0$, V_i^n is defined as $u_i(M_i^n)$, and V_i^0 is defined as $\{1\}$. For $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}_{\geq 0}^s$, we denote by V^α the vector space generated by elements of the form $x_1 \cdots x_s$, where x_i is an element of $V_i^{\alpha_i}$. Note that this is not in general an ideal of A. We also denote by $V^\alpha V_i^n$ the set V^β where $\beta_j = \alpha_j$, except for $\beta_i = \alpha_i + n$.

Example 2.1.4. Let $A = k[[z_1, x_2, z_2]]/(x_2^2, z_1 x_2, z_1 z_2 - x_2)$, $A_1 = k[[z_1]]$, and $A_2 = k[[x_2, z_2]]/(x_2^2)$. Then A is nearly the sum of A_1 and A_2, for the natural maps $A_i \to A$.

Lemma 2.1.5. *Let k be a field. Let $A_1, \dots, A_s$ be a family of local noetherian complete k-algebras of dimension* 1 *with maximal ideals M_i and residue field k. Suppose that for all i, there is an element $z_i \in A_i$ such that A_i has no z_i-torsion and that for all n large enough, $z_i M_i^n = M_i^{n+1}$.*

Let A be a k-algebra with maximal ideal $\mathfrak{m}$ that is nearly the sum of the family (A_i), and let $V_i = u_i(M_i)$ as in Definition 2.1.3. Moreover, suppose that there exist integers N_0 and t_0 with $t_0 < N_0$ such that for all i and j, $V_j V_i^n \subset V_i^{n-t_0}$ for all $n \geq N_0$.

Then $e(A) = \sum_{i=1}^s e(A_i)$.

Note that if we had the stronger property that $V_i V_j = 0$ for all $i \neq j$ the result would be trivial, as then $\mathfrak{m}^n = \bigoplus_{i=1}^s V_i^n$ and $\mathfrak{m}^n/\mathfrak{m}^{n+1} = \bigoplus_{i=1}^s V_i^n / V_i^{n+1}$.

Proof. Observe first that there exist integers N and t with $N > t$ such that for all α, for all i, $V^\alpha V_i^n \subset V_i^{n-t}$ for all $n \geq N$. Indeed, $V^\alpha \subset V_{j_1} \cdots V_{j_r}$ where $\{j_1, \dots, j_r\} \subset \{1, \dots, s\}$ is the set of indices with $\alpha_j > 0$. Then if $n \geq r N_0$, then $V_{j_1} \cdots V_{j_r} V_i^n \subset V_i^{n - r t_0}$. So we can take $N = s N_0$ and $t = s t_0$.

If $\alpha_j > N$, then $V^\alpha \subset V_j^{\alpha_j - t} \subset V_j$. So if there are two different indices i, j with $\alpha_i > N$ and $\alpha_j > N$, then $V^\alpha = 0$ as it is contained in $V_i \cap V_j$. If $|\alpha| > sN$, then there exists at least one i with $\alpha_i > N$ so $V^\alpha = \sum (V^\alpha \cap V_j)$.

Fix some index i. Let $n > 0$. Then $\mathfrak{m}^n = \sum_{|\alpha| = n} V^\alpha$. So if $n > Ns$, then $(\mathfrak{m}^n \cap V_i) = \sum_\alpha (V^\alpha \cap V_i)$ and the only contributing terms are those with $\alpha_j \leq N$ for all $j \neq i$, and $\alpha_i > N$. For such an α, we have $V^\alpha \subset V_i^{n - sN}$ as $\alpha_i \geq n - (s-1)N$. Let $r = sN$, so that $V_i^{n-r} \subset V_i$ for all $n > r$. So for all $n > r$ and all such α we have $V^\alpha \subset V_i$, so finally for $n > r$ we have

$$(\mathfrak{m}^n \cap V_i) = \sum_{|\alpha|=n, \alpha_j \le N \text{ if } j \ne i} V^\alpha. \tag{1}$$

We see that $V_i^n \subset (\mathfrak{m}^n \cap V_i) \subset V_i^{n-r}$ for all $n > r$.

Note that $(\mathfrak{m}^n \cap V_i)$ is an ideal of A_i, which we denote by $W_{i,n}$. We know that $z_i V_i^n = V_i^{n+1}$ for all n large enough, so by the formula (1) for $W_{i,n}$ we see that $z_i W_{i,n} = W_{i,n+1}$ for all n large enough. In A_i, multiplication by z_i induces an isomorphism from V_i^n to V_i^{n+1} and from $W_{i,n}$ to $W_{i,n+1}$, so it also induces an isomorphism from $V_i^{n-r}/W_{i,n}$ to $V_i^{n+1-r}/W_{i,n+1}$ for all n large enough. Note that these vector spaces are finite-dimensional, so they have the same dimension, as $\dim_k V_i^{n-r}/V_i^n$ is finite for all n.

We consider the inclusions

$$V_i^n \subset W_{i,n} \subset V_i^{n-r} \subset W_{i,n-r} \subset V_i^{n-2r}.$$

For all $n \gg 0$, we know that $\dim_k V_i^{n-r}/V_i^n = \dim_k V_i^{n-2r}/V_i^{n-r} = re(A_i)$ and $\dim_k V_i^{n-r}/W_{i,n} = \dim_k V_i^{n-2r}/W_{i,n-r}$, which gives $\dim_k W_{i,n-r}/W_{i,n} = re(A_i)$.

We now go back to A. For all $n \gg 0$ we have that $\dim_k(\mathfrak{m}^{n-r}/\mathfrak{m}^n) = re(A)$. On the other hand, for all $n \gg 0$, we have seen that $\mathfrak{m}^n = \bigoplus_i (\mathfrak{m}^n \cap V_i)$, so $\mathfrak{m}^{n-r}/\mathfrak{m}^n$ is isomorphic to $\bigoplus_i (\mathfrak{m}^{n-r} \cap V_i)/(\mathfrak{m}^n \cap V_i) = \bigoplus_i (W_{i,n-r}/W_{i,n})$. So $re(A) = \sum_{i=1}^s re(A_i)$, and so $e(A) = \sum_i e(A_i)$. □

Example 2.1.6. Let us take again A_1, A_2, A as in Example 2.1.4. For all $n > 0$, $\mathfrak{m}^n$ is the set of terms of the form $\sum_{i \ge n} a_i z_1^i + \sum_{i \ge n} b_i z_2^i + \sum_{i \ge n-2} c_i x_2 z_2^i$. Indeed, we have that $x_2 \in V_1 V_2 \subset \mathfrak{m}^2$, although in A_2 we have $x_2 \in M_2 \setminus M_2^2$. So we do not have $\mathfrak{m}^n \cap V_2 = V_2^n$, but $\mathfrak{m}^n \cap V_2 = V_2^n + (V_1 V_2^{n-1} \cap \mathfrak{m}^n)$, where $V_1 V_2^{n-1}$ is the part that contains the term $x_2 z_2^{n-2}$.

2.2. *Hilbert–Samuel multiplicity of the special fiber.* Let R be a discrete valuation ring with uniformizer π and residue field k.

Let A be a local R-algebra with maximal ideal $\mathfrak{m}$, and let M be an A-module of finite type. We denote by $\bar{e}_R(A, M)$ the Hilbert–Samuel multiplicity of $M \otimes_R k$ as an $A \otimes_R k$-module, with respect to the ideal $\mathfrak{m} \otimes_R k$. When $M = A$ we just write $\bar{e}_R(A)$ instead of $\bar{e}_R(A, A)$, and we omit the subscript R when the choice of the ring is clear from the context.

Lemma 2.2.1. *Let $(T, \mathfrak{m}_T) \to (S, \mathfrak{m}_S)$ be a local morphism of local noetherian rings of the same dimension, with residue fields k_T and k_S respectively. Then $e(T, S) \ge [k_S : k_T] e(S)$.*

Proof. Let $n \ge 0$ be an integer. Then $S/\mathfrak{m}_S^n$ is a quotient of $S/(\mathfrak{m}_T S)^n$, so $\operatorname{len}_T(S/\mathfrak{m}_S^n) \le \operatorname{len}_T(S/(\mathfrak{m}_T S)^n)$. Moreover,

$$\operatorname{len}_T(S/\mathfrak{m}_S^n) = \sum_{i=0}^{n-1} \dim_{k_T} \mathfrak{m}_S^i/\mathfrak{m}_S^{i+1} = [k_S : k_T] \sum_{i=0}^{n-1} \dim_{k_S} \mathfrak{m}_S^i/\mathfrak{m}_S^{i+1} = [k_S : k_T] \operatorname{len}_S(S/\mathfrak{m}_S^n),$$

so finally $[k_S : k_T] \operatorname{len}_S(S/\mathfrak{m}_S^n) \le \operatorname{len}_T(S/(\mathfrak{m}_T S)^n)$ which gives the result. □

Proposition 2.2.2. *Let A be a complete noetherian local R-algebra which is a domain. Let $B \subset A[1/\pi]$ be a finite A-algebra. Let k_A and k_B be the residue fields of A and B respectively. Then $\bar{e}(A) \ge [k_B : k_A] \bar{e}(B)$.*

Proof. Note that B is also a complete noetherian local R-algebra which is a domain. Indeed, A is henselian and B is a finite A-algebra, so B is a finite product of local rings, and so it is a local ring as it is a domain.

It is enough to prove the result when $\pi B \subset A$, as B is generated over A by a finite number of elements of the form x/π^n for $x \in A$.

We have an exact sequence of R-modules,

$$0 \to A \to B \to B/A \to 0.$$

After tensoring by k over R we get the exact sequence

$$0 \to B/A \to A \otimes_R k \to B \otimes_R k \to B/A \to 0.$$

Indeed, $(B/A) \otimes_R k = B/A$, and $(B/A)[\pi] = B/A$ and B is π-torsion free so $B[\pi] = 0$.

Hence we get that $\bar{e}(A, B) = \bar{e}(A, A)$. So we only need to show that $\bar{e}(A, B) \geq [k_B : k_A]\bar{e}(B)$, which follows from Lemma 2.2.1 applied to $T = A \otimes_R k$ and $S = B \otimes_R k$. □

Remark 2.2.3. We give some examples: Let $R = \mathbb{Z}_p$, $C = R[[X]]$, and $A_n = R[[pX, X^n]] \subset C$ for $n \geq 1$, and $B_n = R[[pX, pX^2, \dots, pX^{n-1}, X^n]] \subset C$ for $n \geq 1$. We check easily that $A_n \subset B_n \subset C$ and that C is finite over A_n, and A_n is not equal to B_n if $n > 2$. We compute that $\bar{e}(A_n) = \bar{e}(B_n) = n$, and $\bar{e}(C) = 1$. So we see that in Proposition 2.2.2, both possibilities $\bar{e}(B) < \bar{e}(A)$ and $\bar{e}(B) = \bar{e}(A)$ can happen for $A \neq B$. See the end of Section 4.1 for more examples.

2.3. *Change of ring.* We suppose now that R is the ring of integers of a finite extension K of $\mathbb{Q}_p$. If K' is a finite extension of K, we denote by R' its ring of integers.

Proposition 2.3.1. *Let K' be a finite extension of K, with ramification degree $e_{K'/K}$. Let A be a local noetherian R'-algebra. Then $\bar{e}_R(A) = e_{K'/K}\bar{e}_{R'}(A)$.*

Proof. Suppose first that K' is an unramified extension of K, and let k and k' be the residue fields of K and K' respectively, and let π be a uniformizer of R and R'. Then $A \otimes_{R'} k' = A \otimes_R k = A/\pi A$. So $\bar{e}_R(A) = e(A/\pi A) = \bar{e}_{R'}(A)$.

Suppose now that K' is a totally ramified extension of K. Let u be an Eisenstein polynomial defining the extension, so that $R' = R[X]/u(X)$, and $\bar{u}(X) = X^s$, where $s = [K' : K]$. Then $A \otimes_R k = A \otimes_{R'} (R' \otimes_R k) = A \otimes_{R'} (k[X]/(X^s)) = (A \otimes_{R'} k) \otimes_k k[X]/(X^s)$. So $\bar{e}_R(A) = s\bar{e}_{R'}(A) = [K' : K]\bar{e}_{R'}(A)$.

For the general case, let R_0 be the ring of integers of the maximal unramified extension K_0 of K in K'. Then $e_R(A) = e_{R_0}(A)$ and $e_{R_0}(A) = [K' : K_0]e_{R'}(A)$ which gives the result. □

We recall the following result, which is [Breuil and Mézard 2002, Lemme 2.2.2.6]:

Lemma 2.3.2. *Let A be a local noetherian R-algebra, with the same residue field as R and A is complete and topologically of finite type over R. Let K' be a finite extension of K, and $A' = R' \otimes_R A$. Suppose that A' is still a local ring. Then $\bar{e}_R(A) = \bar{e}_{R'}(A')$.*

3. Rigid geometry and standard subsets of the affine line

3.1. *Quasiaffinoid algebras and rigid spaces.*

Quasiaffinoid algebras. Let F be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_F$. We denote by $R_{n,m}$, or $R_{n,m,F}$, the F-algebra $\mathcal{O}_F\langle x_1, \dots, x_n\rangle[[y_1, \dots, y_m]] \otimes_{\mathcal{O}_F} F$. Following [Lipshitz and Robinson 2000], we say that an F-algebra is a quasiaffinoid algebra (or an F-quasiaffinoid algebra) if it is a quotient of $R_{n,m}$ for some n, m. For example, an F-affinoid algebra is F-quasiaffinoid, as it is a quotient of the Tate algebra $R_{n,0,F}$ for some n. The theory of quasiaffinoid algebras has also been studied by other authors under the name "semiaffinoid algebras"; see for example [Kappen 2012].

Let A be an F-quasiaffinoid algebra. Following [Kappen 2012, Definition 2.2], we say that an $\mathcal{O}_F$-subalgebra $\underline{A}$ of A is an $\mathcal{O}_F$-model of A if the canonical morphism $\underline{A} \otimes_{\mathcal{O}_F} F \to A$ is an isomorphism. Note that an $\mathcal{O}_F$-model is automatically $\mathcal{O}_F$-flat. Assume that A is normal. Let $\underline{A}$ be an $\mathcal{O}_F$-model of A, and let A^0 be the integral closure of $\underline{A}$ in A. Then A^0 is normal, and is an $\mathcal{O}_F$-model of A.

A quasiaffinoid algebra is said to be of open type if it has an $\mathcal{O}_F$-model that is local, or equivalently, if it is a quotient of $R_{0,m}$ for some m. For example, let $\mathcal{R}$ be one of the potentially semistable deformation rings defined by Kisin (as recalled in Section 5.1), then $\mathcal{R}[1/p]$ is a quasiaffinoid algebra of open type. These will be our main focus of interest, but we need to use quasiaffinoid algebras that are not necessarily of open type in order to study them.

Quasiaffinoid algebras have some properties that are similar to affinoid algebras: for example they are noetherian and they are Jacobson rings, and the Nullstellensatz holds for them.

Rigid spaces attached to quasiaffinoid algebras. Let A be an F-quasiaffinoid algebra. Using Berthelot's construction, as described in [de Jong 1995, Section 7], we can attach canonically to it a rigid space $\mathcal{X} = \mathcal{X}_A$ defined over F. We say that such a rigid space is the quasiaffinoid space attached to A. We say that a quasiaffinoid space is of open type if it is attached to a quasiaffinoid algebra of open type.

We give some properties of this construction. We denote by $\underline{A}$ an $\mathcal{O}_F$-model of A.

Proposition 3.1.1. (1) *We have a natural map $A \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ which induces a map $\underline{A} \to \Gamma(\mathcal{X}, \mathcal{O}^0_{\mathcal{X}})$ (where $\mathcal{O}^0_{\mathcal{X}}$ is the sheaf of functions bounded by* 1).

(2) *The map $\underline{A} \to \Gamma(\mathcal{X}, \mathcal{O}^0_{\mathcal{X}})$ is an isomorphism as soon as $\underline{A}$ is normal. In particular, in this case A is isomorphic to the subring of $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of functions that are bounded.*

(3) *There is a functorial bijection between* $\mathrm{Max}(A)$ *and the points of $\mathcal{X}$.*

(4) *This construction is compatible with base change by a finite extension $F \to F'$.*

Proof. Property (1) is [de Jong 1995, 7.1.8], (2) is [de Jong 1995, 7.4.1], using the fact that an $\mathcal{O}_F$-model is $\mathcal{O}_F$-flat.

Property (3) is [de Jong 1995, 7.1.9] and (4) is [de Jong 1995, 7.2.6]. □

If $\mathcal{X}$ is a rigid space over F, we write $\mathcal{A}^0_F(\mathcal{X})$ for $\Gamma(\mathcal{X}, \mathcal{O}^0_{\mathcal{X}})$ and $\mathcal{A}_F(\mathcal{X})$ for the subring of $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ of functions that are bounded. If $\mathcal{X}$ is the rigid space attached to an F-quasiaffinoid algebra A that is normal, then A has a normal $\mathcal{O}_F$-model $\underline{A}$, and we have $A = \mathcal{A}_F(\mathcal{X})$ and $\underline{A} = \mathcal{A}_F^0(\mathcal{X})$ (in particular, there is actually only one $\mathcal{O}_F$-model of A that is normal, and it contains all other $\mathcal{O}_F$-models).

A map $f : \mathcal{X} \to \mathcal{Y}$ between F-quasiaffinoid rigid spaces is quasiaffinoid if it is induced by an F-algebra map $f^\# : \mathcal{A}_F(\mathcal{Y}) \to \mathcal{A}_F(\mathcal{X})$. By Proposition 3.1.1, it is easy to see that any rigid analytic map $f : \mathcal{X} \to \mathcal{Y}$ between F-quasiaffinoid spaces is in fact quasiaffinoid as soon as $\mathcal{X}$ is normal.

R-subdomains. As in the case of affinoid algebras and rigid spaces, we define some special subsets of quasiaffinoid spaces.

Let $\mathcal{X}$ be a quasiaffinoid space. Let $h, f_1, \dots, f_n, g_1, \dots, g_m$ be elements of $\mathcal{A}_F(\mathcal{X})$ that generate the unit ideal of $\mathcal{A}_F(\mathcal{X})$. A quasirational subdomain of $\mathcal{X}$ is a subset U of the form $\{x, |f_i(x)| \leq |h(x)| \text{ for all } i \text{ and } |g_i(x)| < |h(x)| \text{ for all } i\}$; see [Lipshitz and Robinson 2000, Definition 5.3.3]. This generalizes the notion of an affinoid subdomain: let $\mathcal{X}$ be an affinoid rigid space, then an affinoid subdomain of $\mathcal{X}$ is a subset defined by equations of the form $\{x, |f_i(x)| \leq |h(x)| \text{ for all } i\}$, where $f_1, \dots, f_n$ and h generate the unit ideal.

In contrast to the case of affinoid subdomains, it is not necessarily true that a quasirational subdomain of a quasirational subdomain of $\mathcal{X}$ is itself a quasirational subdomain of $\mathcal{X}$; see [Lipshitz and Robinson 2000, Example 5.3.7]. We recall the definition of a R-subdomain of $\mathcal{X}$ [Lipshitz and Robinson 2000, Definition 5.3.3]: the set of R-subdomains of $\mathcal{X}$ is defined as the smallest set of subsets of $\mathcal{X}$ that contains $\mathcal{X}$ and is closed by the operation of taking a quasirational subdomain of an element of this set. In particular, any finite intersection of R-subdomains of $\mathcal{X}$ is also an R-subdomain.

Any R-subdomain of a quasiaffinoid space $\mathcal{X}$ is itself a quasiaffinoid space in a canonical way, attached to the quasiaffinoid algebra constructed as in [Lipshitz and Robinson 2000, Definition 5.3.3].

3.2. ***R-subdomains of the unit disk.*** Our goal now is to understand the subsets of $\overline{\mathbb{Q}}_p$ that are the set of points of some quasiaffinoid space. For simplicity, we consider for the moment only subsets of the unit disk. Let $\mathcal{D}$ be the rigid closed unit disk, seen as a quasiaffinoid space defined over $\mathbb{Q}_p$, or over any finite extension of $\mathbb{Q}_p$, so that $\mathcal{D}(\overline{\mathbb{Q}}_p) = D(0, 1)^+$. We say that $X \subset D(0, 1)^+$ is an R-subdomain if it is of the form $\mathcal{X}(\overline{\mathbb{Q}}_p)$ for some R-subdomain $\mathcal{X}$ of $\mathcal{D}$, and that it is a quasirational subdomain if it is of the form $\mathcal{X}(\overline{\mathbb{Q}}_p)$ for some quasirational subdomain $\mathcal{X}$ of $\mathcal{D}$.

Definition 3.2.1. We say that a subset of $\overline{\mathbb{Q}}_p$ is a rational disk if it is a set of the form $\{x, |x - a| < r\}$ with $a \in \overline{\mathbb{Q}}_p$, $r \in |\overline{\mathbb{Q}}_p^\times|$ (open disk), or of the form $\{x, |x - a| \leq r\}$ with $a \in \overline{\mathbb{Q}}_p$, $r \in |\overline{\mathbb{Q}}_p^\times|$ (closed disk).

Let F be a finite extension of $\mathbb{Q}_p$. We say that a disk is well-defined over F if it can be written as $\{x, |x - a| < r\}$ or as $\{x, |x - a| \leq r\}$ for some $a \in F$ and $r \in |F^\times|$.

Recall (see Definition 1.0.1) that a disk is defined over F if it is fixed by G_F, so a disk that is well-defined over F is defined over F, although the converse is not necessarily true.

From now on, when we write "disk" we always mean "rational disk". It is clear that a rational disk is a quasirational subdomain of the affine line.

Following [Lipshitz and Robinson 1996, Definition 4.1], we define:

Definition 3.2.2. A special subset of $\overline{\mathbb{Q}}_p$ is a subset of one of the following forms:

(1) $\{x, r < |x-a| < r'\}$ for some $a \in \overline{\mathbb{Q}}_p$ and r, r' in $|\overline{\mathbb{Q}}_p^\times|$.

(2) $\{x, |x-a| \leq r$ and for all $i \in \{1, \dots, N\}, |x-\alpha_i| \geq r_i\}$ for some $a, (\alpha_i)_{1\leq i\leq N}, \in \overline{\mathbb{Q}}_p$ and $r, (r_i)_{1\leq i\leq N}$ in $|\overline{\mathbb{Q}}_p^\times|$.

Special subsets are R-subdomains, as they are finite intersections of quasirational subdomains. We have the following result:

Lemma 3.2.3 [Lipshitz and Robinson 1996, Theorem 4.5]. *An R-subdomain of $D(0, 1)^+$ is a finite union of special sets.*

Definition 3.2.4. We say that a subset X of $D(0, 1)^+$ is a connected R-subset if it is of the following form: $D_0 \setminus \bigcup_{i=1}^n D_i$, where the D_i are rational disks contained in $D(0, 1)^+$, $D_0 \neq D_i$ for all $i > 0$, $D_i \subset D_0$, and D_i and D_j are disjoint if $i \neq j$ and $i, j > 0$.

We say that a subset X of $D(0, 1)^+$ is an R-subset if it is a finite disjoint union of connected R-subsets.

We say that a connected R-subset is of closed type if D_0 is closed and the D_i, $i > 0$ are open. We say that it is of open type if D_0 is open and the D_i, $i > 0$ are closed. We say that an R-subset is of closed type (resp. open type) if it is a finite union of connected R-subset of closed type (resp. open type). We say that a connected R-subset is well-defined over some extension F of $\mathbb{Q}_p$ if each disk involved in its description is well-defined over F.

We check easily the following result:

Lemma 3.2.5. *Let X and Y be two connected R-subsets of closed* (*resp. open*) *type. If $X \cap Y \neq \varnothing$ then $X \cap Y$ and $X \cup Y$ are connected R-subsets of closed* (*resp. open*) *type. As a consequence*, *any finite union of connected R-subsets of closed* (*resp. open*) *type is an R-subset of closed* (*resp. open*) *type.*

From Lemma 3.2.3, we get the following property of R-subdomains of the unit disk:

Proposition 3.2.6. *Any R-subdomain of the unit disk is an R-subset.*

On the other hand, we can ask whether any R-subset is an R-subdomain.

Proposition 3.2.7. *Let X be a connected R-subset. Let F be a finite extension of $\mathbb{Q}_p$ such that X is well-defined over F. Then X is a quasiaffinoid subdomain of $D(0, 1)^+$, and it is the set of points of a quasiaffinoid space defined over F which is uniquely defined as a quasirational subdomain of $\mathcal{D}$.*

Proof. From Definition 3.2.4, we see that X can be defined by a finite number of equations of the form $|x-a| < |b|$ or $|x-a| \leq |b|$ or $|x-a| > |b|$ or $|x-a| \geq |b|$ for a, b in F and $b \neq 0$. □

In particular, if X is a connected R-subset of closed type, then it is the set of points of an affinoid subdomain of the unit disk, and any affinoid subdomain of the unit disk is of this form by [Bosch et al. 1984, Theorem 9.7.2/2].

Definition 3.2.8. Let $\mathcal{X}$ be an R-subdomain of $\mathcal{D}$, defined over F as a quasiaffinoid space, and let $X = \mathcal{X}(\overline{\mathbb{Q}}_p)$. Then we write $\mathcal{A}_F(X)$ for $\mathcal{A}_F(\mathcal{X})$ and $A_F^0(X)$ for $\mathcal{A}_F^0(\mathcal{X})$.

For example, let X be the disk defined by $|x-a| < |b|$ for some a, b in F, $b \neq 0$. Then $\mathcal{A}_F^0(X) = \mathcal{O}_F[[(x-a)/b]]$ is isomorphic to the power series ring $\mathcal{O}_F[[t]]$. Let Y be the annulus defined by $|c| < |x-a| < |b|$ for some a, b, c in F with $c \neq 0$ and $|c| < |b|$. Then $\mathcal{A}_F^0(Y) = \mathcal{O}_F[[(x-a)/b, c/(x-a)]]$, which is isomorphic to $\mathcal{O}_F[[t, u]]/(tu - c/b)$. In general the ring $\mathcal{A}_F(X)$ can be entirely described using [Lipshitz and Robinson 2000, Definition 5.3.3]. We will not give a formula, but we see easily that for a connected R-subset X, $\mathcal{A}_F^0(X)$ is local if and only if X is of open type. So we have:

Proposition 3.2.9. *Let X be a connected R-subset. Assume that we know that X is the set of points of a quasiaffinoid space of open type. Then X is an R-subset of open type.*

3.3. ***Rings of functions on standard subsets.*** We continue studying subsets of $\overline{\mathbb{Q}}_p$, or more generally of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, coming from quasiaffinoid spaces. From now on, we will be only interested in R-subsets that are of open type, but we will not necessarily assume that the subsets are contained in the unit disk anymore.

Standard subsets. We make the following definitions:

Definition 3.3.1. We say that a subset X of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ is a connected standard subset if it is of one of the following forms:

(1) $D_0 \setminus \bigcup_{i=1}^n D_i$, where the D_i are rational disks, D_0 is open and each D_i is closed for $i > 0$, $\infty \notin D_0$, $D_0 \neq D_i$ for all $i > 0$, $D_i \subset D_0$, and D_i and D_j are disjoint if $i \neq j$ and $i, j > 0$ (bounded connected standard subset).

(2) $\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus \bigcup_{i=1}^n D_i$, where the D_i are rational disks, each D_i is closed, and D_i and D_j are disjoint if $i \neq j$ (unbounded connected standard subset).

The disks (D_i) are called the defining disks of X.

So a bounded standard subset contained in the unit disk is the same thing as a connected R-subset of open type.

Definition 3.3.2. A standard subset is a finite disjoint union of connected standard subsets of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. The connected standard subsets that appear are called the connected components of the standard subset. The defining disks of a standard subsets are the defining disks of each of its connected components.

It is clear that a standard subset can be written in a unique way as a finite disjoint union of connected standard subsets so the notion of connected component is well-defined.

Let F be a finite extension of $\mathbb{Q}_p$. We say that a standard subset is well-defined over F if each defining disk of X is well-defined over F.

Definition of the rings of functions of standard subsets. Let $X \subset \overline{\mathbb{Q}}_p$ a connected standard subset, which is well-defined over some finite extension F of $\mathbb{Q}_p$. Although it is not necessarily contained in the unit disk, it is contained in some closed disk, and so all the results of Section 3.2 apply to X. In particular we can define $\mathcal{A}_F(X)$ and $\mathcal{A}_F^0(X)$ as in Definition 3.2.8.

Let $X \subset \mathbb{P}^1(\overline{\mathbb{Q}}_p)$ be an unbounded connected standard subset not equal to all of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. Let f be a rational function with $\overline{\mathbb{Q}}_p$-coefficients defining a bijection of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, and such that its pole is outside of

X, then $Y = f(X)$ is a bounded connected standard subset of $\overline{\mathbb{Q}}_p$. Let F be a finite extension of $\mathbb{Q}_p$ such that X is well-defined over F, and such that the rational function f has coefficients in F. Then $\mathcal{A}_F(Y)$ and $\mathcal{A}_F^0(Y)$ are well-defined. We define $\mathcal{A}_F(X)$ and $\mathcal{A}_F^0(X)$ to be the functions of X of the form $u \circ f$ for $u \in \mathcal{A}_F(Y)$ and $\mathcal{A}_F^0(Y)$ respectively. It is clear that this does not depend on the choice of f, as different choices of f give rise to bounded connected standard subsets coming from isomorphic quasiaffinoids.

We give now a general formula for these functions rings, which can be obtained using [Lipshitz and Robinson 2000, Definition 5.3.3]:

Proposition 3.3.3. *Let X be a connected standard subset that is well-defined over some finite extension E of $\mathbb{Q}_p$. Write $X = D(a_0, r_0)^- \setminus \bigcup_{j=1}^n D(a_j, r_j)^+$ or $X = \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus \bigcup_{j=1}^n D(a_j, r_j)^+$ with $a_j \in E$ for all j, and the sets $D(a_j, r_j)^+$ are pairwise disjoint for $j > 0$. For each j, let $t_j \in E$ be such that $|t_j| = r_j$. Then for any finite extension F/E, we have*

$$\mathcal{A}_F(X) = \left\{ f,\ f(x) = \sum_{i \geq 0} c_{i,0}\left(\frac{x - a_0}{t_0}\right)^i + \sum_{j=1}^n \sum_{i>0} c_{i,j}\left(\frac{t_j}{x - a_j}\right)^i, \right.$$
$$\left. \text{with } c_{i,j} \in F \text{ for all } i, j \text{ and } \{c_{i,j}, 0 \leq j \leq n, i \geq 0\} \text{ bounded} \right\}$$

if X is bounded and

$$\mathcal{A}_F(X) = \left\{ f,\ f(x) = c_0 + \sum_{j=1}^n \sum_{i>0} c_{i,j}\left(\frac{t_j}{x - a_j}\right)^i \right.$$
$$\left. \text{with } c_{i,j} \in F \text{ for all } i, j \text{ and } \{c_{i,j}, 1 \leq j \leq n, i \geq 0\} \text{ bounded} \right\}$$

if X is unbounded.

Also, $\|f\|_X = \sup_{i,j} |c_{i,j}|$ if f is as above. If we write $f_0 = \sum_{i \geq 0} c_{i,0}((x - a_0)/t_0)^i$ (or $f_0 = c_0$ in the unbounded case), and $f_j = \sum_{i>0} c_{i,j}(t_j/(x - a_j))^i$ for $j > 0$ so that $f = \sum_{i=0}^n f_i$, then $\|f\|_X = \max_{0 \leq i \leq n} \|f_i\|_X$.

In particular, $f \in \mathcal{A}_F^0(X)$ if and only if $c_{i,j} \in \mathcal{O}_F$ for all i, j.

Remark 3.3.4. Assume that X is unbounded. Then the value of the constant term c_0 is independent from the choice of the $a_i \in D(a_i, r_i)^+$ used to write the decomposition, as it is the value of the function at ∞.

Let now X be a standard subset. It can be written uniquely as $X = \bigcup_{i=1}^n X_i$, where the X_i are disjoint connected standard subsets. Then we set $\mathcal{A}_F(X) = \bigoplus_{i=1}^n \mathcal{A}_F(X_i)$ and $\mathcal{A}_F^0(X) = \bigoplus_{i=1}^n \mathcal{A}_F^0(X_i)$, where F is a finite extension of $\mathbb{Q}_p$ such that X is well-defined over F.

Field of definition and change of field. Let F be a finite extension of $\mathbb{Q}_p$. The field of definition of $X \subset \mathbb{P}^1(\overline{\mathbb{Q}}_p)$ over F is the fixed field of $\{\sigma \in G_F, \sigma(X) = X\}$. The field of definition of X is the field of definition of X over $\mathbb{Q}_p$. Then X is defined over F (as in Definition 1.0.1) if and only if F contains the field of definition of X.

Let X be a standard subset defined over some finite extension E of $\mathbb{Q}_p$. Let F be a finite Galois extension of E such that X is well-defined over F. In this case $\mathrm{Gal}(F/E)$ acts on $\mathcal{A}_F(X)$ and $\mathcal{A}_F^0(X)$

by $(\sigma f)(x)=\sigma(f(\sigma^{-1}x))$. We write $\mathcal{A}_E(X)$ and $\mathcal{A}_E^0(X)$ for $\mathcal{A}_F(X)^{\mathrm{Gal}(F/E)}$ and $\mathcal{A}_F^0(X)^{\mathrm{Gal}(F/E)}$. So for example, if $X=D(0,1)^-$, then X is defined over $\mathbb{Q}_p$, and $\mathcal{A}_E^0(X)$ is $\mathcal{O}_E[[x]]$ for any finite extension E of $\mathbb{Q}_p$. It is clear that the definition of $\mathcal{A}_E(X)$ and $\mathcal{A}_E^0(X)$ does not depend on the choice of the extension F over which X is well-defined.

Proposition 3.3.5. *Let X be a standard subset defined over E. Let F be a finite extension of E. Then $\mathcal{A}_F(X)=F\otimes_E\mathcal{A}_E(X)$, and $\mathcal{O}_F\otimes_{\mathcal{O}_E}\mathcal{A}_E^0(X)\subset\mathcal{A}_F^0(X)$, with $\mathcal{A}_F^0(X)$ finite over $\mathcal{O}_F\otimes_{\mathcal{O}_E}\mathcal{A}_E^0(X)$. If F/E is unramified, then this inclusion is an isomorphism.*

Proof. When X is well-defined over E, the formulas of Proposition 3.3.3 give the isomorphisms $\mathcal{A}_F(X)=F\otimes_E\mathcal{A}_E(X)$ and $\mathcal{O}_F\otimes_{\mathcal{O}_E}\mathcal{A}_E^0(X)=\mathcal{A}_F^0(X)$ (even if F/E is ramified).

We now treat the general case. We define a map $\phi: F\otimes_E\mathcal{A}_E(X)\to\mathcal{A}_F(X)$ by $\phi(a\otimes f)=af$. Let us describe the inverse ψ of ϕ. Let $Q=G_E/G_F$. If a is in F and $f\in\mathcal{A}_F(X)$, $\sigma(a)$ and $\sigma(f)$ are well-defined for $\sigma\in Q$ as a and f are invariant by G_F. Moreover, for $a\in F$, we have that $\mathrm{tr}_{F/E}(a)=\sum_{\sigma\in Q}\sigma(a)$.

Let $(e_1,\dots,e_n)$ be a basis of F over E, and $(u_1,\dots,u_n)\in F^n$ be the dual basis with respect to $\mathrm{tr}_{F/E}$, that is, $\mathrm{tr}_{F/E}(e_iu_j)=\delta_{i,j}$. We see that for $\sigma\in G_E$, we have $\sum_{i=1}^n e_i\sigma(u_i)=1$ if $\sigma\in G_F$, and 0 otherwise.

For $f\in\mathcal{A}_F(X)$, we set $t_i(f)=\sum_{\sigma\in Q}\sigma(u_if)$. Let $\psi(f)=\sum_{i=1}^n e_i\otimes t_i(f)$. Let us check that ψ is the inverse of ϕ. Let $f\in\mathcal{A}_F(X)$, and $f'=\phi(\psi(f))$. Then $f'=\sum_i e_i\sum_Q\sigma(u_i)\sigma(f)=\sum_Q\sigma(f)\big(\sum_i e_i\sigma(u_i)\big)$, so $f'=f$. Let $f\in\mathcal{A}_E(X)$, and $a\in F$. Let $g=\phi(a\otimes f)$. Then $t_i(g)=\mathrm{tr}_{F/E}(au_i)f$, as $\sigma(f)=f$ for all $\sigma\in Q$. So $\psi(g)=\sum_i e_i\otimes\mathrm{tr}_{F/E}(au_i)f=\big(\sum_i e_i\,\mathrm{tr}_{F/E}(au_i)\big)\otimes f$ as $\mathrm{tr}_{F/E}(au_i)\in E$. Then we check that $\sum_i e_i\,\mathrm{tr}_{F/E}(au_i)=a$, so $\psi(\phi(a\otimes f))=a\otimes f$. So we see that ψ is the inverse map of ϕ, so ϕ is an isomorphism.

We see that ϕ induces a map ϕ^0 from $\mathcal{O}_F\otimes_{\mathcal{O}_E}\mathcal{A}_E^0(X)$ to $\mathcal{A}_F^0(X)$. When F/E is unramified, we can choose (e_i) and (u_i) to be in $\mathcal{O}_F$, and in this case the restriction ψ^0 of ψ to $\mathcal{A}_F^0(X)$ maps into $\mathcal{O}_F\otimes_{\mathcal{O}_E}\mathcal{A}_E^0(X)$, and so ψ^0 is the inverse map of ϕ^0, and so ϕ^0 is an isomorphism. □

Some algebraic results. Let X be a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ that is defined over E for some finite extension E of $\mathbb{Q}_p$. Let F be a finite extension of E. We say that X is irreducible over F if it cannot be written as a finite disjoint union of standard subsets of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ that are defined over F. There exists a unique decomposition of X as a finite disjoint union of standard subsets of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ that are irreducible over F. A standard subset is connected if and only if it is irreducible over any field of definition.

Lemma 3.3.6. *Let X be a connected standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ defined over E. Then $\mathcal{A}_E(X)$ is a domain, and $\mathcal{A}_E^0(X)$ is a local ring which has the same residue field as E.*

Proof. Let F be a finite Galois extension of E such that X is well-defined over F. The result for $\mathcal{A}_F^0(X)$ holds from the description given in Proposition 3.3.3, and the result for $\mathcal{A}_E^0(X)$ follows from the fact that it is equal to $\mathcal{A}_F^0(X)^{\mathrm{Gal}(F/E)}$ and the results of Proposition 3.3.5. Note that the maximal ideal is the set of functions f such that $|f(x)|<1$ for all x in X, that is, the functions f that are topologically nilpotent. □

Lemma 3.3.7. *Let X be a standard subset that is defined and irreducible over E, and let $X=\bigcup_{i=1}^r X_i$ be its decomposition in a finite union of connected standard subsets. Let F be the field of definition of*

X_1 over E. Then the restriction map $\mathcal{A}^0(X) \to \mathcal{A}^0(X_1)$ induces an $\mathcal{O}_E$-linear isomorphism $\mathcal{A}_E^0(X) \to \mathcal{A}_F^0(X_1)$.

Note in particular that $[F : E]$ is the number of connected components of X, and the isomorphism class of $\mathcal{A}_F^0(X_1)$ as an $\mathcal{O}_E$-algebra does not depend on the choice of X_1.

Proof. The group G_E acts transitively on the set of the (X_i) as X is irreducible, and G_F is the stabilizer of X_1. We fix a system (σ_i) of representatives of G_E/G_F, numbered so that $\sigma_i(X_1) = X_i$ for all i.

Let f be an element of $\mathcal{A}_E^0(X)$. First note that f is invariant under the action of G_F, as it is by definition invariant under the action of G_E, so $f_{|X_1}$ is in $\mathcal{A}_F^0(X_1)$. Moreover, we have that for all $x \in X_i$,

$$f(x) = \sigma_i((\sigma_i^{-1} f)(\sigma_i^{-1}(x)) = \sigma_i(f_{|X_1}(\sigma_i^{-1} x)).$$

So $f_{|X_i}$ is entirely determined by $f_{|X_1}$, so the restriction map is injective, and moreover for any $f \in \mathcal{A}_F^0(X_1)$ the formula above defines an element of $\mathcal{A}_E^0(X)$, so the restriction map is bijective. □

Corollary 3.3.8. *If X is defined and irreducible over E, then $\mathcal{A}_E(X)$ is a domain and $\mathcal{A}_E^0(X)$ is a local ring.*

Proof. We apply Lemma 3.3.7: $\mathcal{A}_E^0(X)$ is isomorphic as a ring to $\mathcal{A}_F^0(X_1)$, which is local. □

Definition 3.3.9. If X is defined and irreducible over E, we denote by $k_{X,E}$ the residue field of $\mathcal{A}_E^0(X)$.

By construction, $k_{X,E}$ is a finite extension of k_E. In the notation of Lemma 3.3.7, we have $k_{X,E} = k_{X_1,F}$, and by Lemma 3.3.6, $k_{X_1,F} = k_F$ as X_1 is connected.

Example 3.3.10. Let $a \in \mathbb{Q}_{p^2}$ such that $v_p(a) = 0$, and $\bar{a}$ is not in $\mathbb{F}_p$. Let a' be its Galois conjugate, so that the disks $D = D(a, 1)^-$ and $D' = D(a', 1)^-$ are disjoint. Let X be the union of D and D'. Then X is defined and irreducible over $\mathbb{Q}_p$, although it is not connected. Moreover, $\mathcal{A}^0_{\mathbb{Q}_p}(X) = \mathcal{A}^0_{\mathbb{Q}_{p^2}}(D)$ is isomorphic to $\mathbb{Z}_{p^2}[[w]]$ (where w corresponds to $x - a$), so $k_{X,\mathbb{Q}_p} = \mathbb{F}_{p^2}$.

3.4. *Some maps from quasiaffinoid spaces to the unit disk.*

Theorem 3.4.1. *Let $\mathcal{X}$ be a normal, Zariski geometrically connected quasiaffinoid space over some finite extension of $\mathbb{Q}_p$, $\mathcal{D}$ be the closed unit disk, and $f : \mathcal{X} \to \mathcal{D}$ be a rigid analytic map that is an open immersion. Then the image $f(\mathcal{X})$ of $\mathcal{X}$ is a connected R-subset of $\mathcal{D}$, and f is an isomorphism from $\mathcal{X}$ to its image.*

Let us first recall what is known in the affinoid case. Let $\mathcal{X}$ and $\mathcal{Y}$ be affinoid spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a rigid analytic map which is an open immersion. Then by [Bosch et al. 1984, Corollary 8.2/4], the image of f in $\mathcal{Y}$ is an affinoid subdomain of $\mathcal{Y}$ and f is an isomorphism from $\mathcal{X}$ to its image.

But this does not hold in the quasiaffinoid case without extra hypotheses, as illustrated by the following example: let $\mathcal{X}$ be the disjoint union of the open unit disk and the unit circle, and i the natural map from $\mathcal{X}$ to the closed unit disk $\mathcal{D}$. Then i is an open immersion and is bijective, but is not an isomorphism (as $\mathcal{X}$ is not connected, whereas the closed unit disk is).

We need some lemmas in order to prove Theorem 3.4.1.

Lemma 3.4.2. *Let $\mathcal{X}$ be a quasiaffinoid space and $\mathcal{Y}$ be a rigid space, both defined over some finite extension of $\mathbb{Q}_p$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a rigid analytic map which is a surjective open immersion.*

Assume that there exists a covering $(\mathcal{Y}_i)$ of $\mathcal{Y}$ by affinoid subdomains, such that each $f^{-1}(\mathcal{Y}_i)$ is an affinoid subdomain of $\mathcal{X}$, and $(\mathcal{Y}_i)$ is an admissible covering of $\mathcal{Y}$ (that is, any affinoid subdomain $\mathcal{Y}'$ of $\mathcal{Y}$ can be covered by a finite number of $\mathcal{Y}_i$). Then f is an isomorphism from $\mathcal{X}$ to $\mathcal{Y}$.

Proof. We need to construct the inverse $g : \mathcal{Y} \to \mathcal{X}$. It is enough to construct $g' : \mathcal{Y}' \to \mathcal{X}$ satisfying $f \circ g' = \mathrm{id}$ for each affinoid subdomain $\mathcal{Y}'$ of $\mathcal{Y}$ (as these are necessarily compatible and glue to form g). Fix such a $\mathcal{Y}'$. Then it is covered by some $\mathcal{Y}_i$ for i in some finite set I. Set $\mathcal{X}_i = f^{-1}(\mathcal{Y}_i)$, so that $\mathcal{X}_i$ is affinoid. As $\mathcal{X}$ is quasiaffinoid, there exists some affinoid subdomain $\mathcal{X}'$ of $\mathcal{X}$ containing all the $\mathcal{X}_i$ for $i \in I$. Note that we have, for each $i \in I$, a map $g_i : \mathcal{Y}_i \to \mathcal{X}_i$ which is the inverse of the restriction of f to $\mathcal{X}_i$, and these are compatible. So they glue to form the function $g' : \mathcal{Y}' \to \mathcal{X}'$ by the admissibility condition and Tate's acyclicity theorem; see [Bosch et al. 1984, Corollary 8.2/3]. □

Looking back at the inclusion $i : \mathcal{X} \to \mathcal{D}$ as above, we see that i is a surjective open immersion, but there does not exist a covering of $\mathcal{D}$ satisfying the conditions of Lemma 3.4.2. Indeed, let $\mathcal{Y}$ be a connected affinoid subdomain of $\mathcal{D}$, then $i^{-1}(\mathcal{Y})$ is affinoid if and only if $\mathcal{Y}$ is either contained in the unit circle or in the open unit disk. But it is not possible to have an admissible covering of $\mathcal{D}$ by affinoids satisfying this condition.

Corollary 3.4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ quasiaffinoid spaces, both defined over some finite extension of $\mathbb{Q}_p$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a rigid analytic map which is finite and an open immersion. Assume that $\mathcal{Y}$ is connected and $\mathcal{X}$ is nonempty. Then f is an isomorphism.*

Proof. Assume first that $\mathcal{Y}$ is affinoid. Then so is $\mathcal{X}$ as f is finite, and so f is an isomorphism by [Bosch et al. 1984, Corollary 8.2/4]. In general, $\mathcal{Y}$ has an admissible covering $(\mathcal{Y}_i)$ by connected affinoid subdomains. Let $\mathcal{X}_i = f^{-1}(\mathcal{Y}_i)$; then each $\mathcal{X}_i$ is affinoid as f is finite. Moreover, for each i, either $\mathcal{X}_i$ is empty or f induces an isomorphism between $\mathcal{X}_i$ and $\mathcal{Y}_i$. By connectedness of $\mathcal{Y}$ and the fact that $\mathcal{X}$ is nonempty, we get that $f(\mathcal{X}_i) = \mathcal{Y}_i$ for all i and in particular f is surjective. So the conditions of Lemma 3.4.2 are satisfied. □

Lemma 3.4.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be quasiaffinoid rigid spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a quasiaffinoid map. Assume that f is an open immersion. There exists a finite covering $(\mathcal{Y}_i)$ of $\mathcal{Y}$ by connected R-subdomains such that for each i, either $f^{-1}(\mathcal{Y}_i)$ is empty, or f induces an isomorphism from $f^{-1}(\mathcal{Y}_i)$ to $\mathcal{Y}_i$.*

Proof. As f is an open immersion, it is in particular quasifinite. So we can apply [Lipshitz and Robinson 2000, Theorem 6.1.2]: there exists a finite covering $(\mathcal{Y}_i)$ of $\mathcal{Y}$ by R-subdomains such that f induces a finite map f_i from $\mathcal{X}_i = f^{-1}(\mathcal{Y}_i)$ to $\mathcal{Y}_i$. We can assume that each $\mathcal{Y}_i$ is connected. By Corollary 3.4.3, for each i we have that either $\mathcal{X}_i$ is empty or f_i is an isomorphism. □

Proof of Theorem 3.4.1. Let $f : \mathcal{X} \to \mathcal{D}$ be as in the statement of the theorem. First observe that f is a quasiaffinoid map from $\mathcal{X}$ to $\mathcal{D}$, as it is a bounded analytic function on $\mathcal{X}$ and $\mathcal{X}$ is normal. By Lemma 3.4.4, there exists a finite covering $(\mathcal{Y}_i)$ of $\mathcal{D}$ by R-subdomains of D such that for all i, either

$f^{-1}(\mathcal{Y}_i)$ is empty or f induces an isomorphism from $f^{-1}(\mathcal{Y}_i)$ to $\mathcal{Y}_i$. By Lemma 3.2.3, we can assume that each $Y_i = \mathcal{Y}_i(\overline{\mathbb{Q}}_p)$ is a special subset of $\overline{\mathbb{Q}}_p$. We see $\mathcal{X}$ and f as defined on some finite extension F of $\mathbb{Q}_p$ that is large enough so that each of the Y_i are well-defined over F. Let Y be the union of the Y_i for those i such that $f^{-1}(\mathcal{Y}_i)$ is not empty. We see that Y is a finite union of R-subsets of $\overline{\mathbb{Q}}_p$, and is equal to $f(\mathcal{X}(\overline{\mathbb{Q}}_p))$. As $\mathcal{X}$ is connected, so is Y, and so Y is in fact a connected R-subset of $\overline{\mathbb{Q}}_p$ and so is the set of points of a quasiaffinoid subdomain $\mathcal{Y}$ of $\mathcal{D}$.

We now want to prove that f induces an isomorphism between $\mathcal{X}$ and $\mathcal{Y}$. We want to apply Lemma 3.4.2, and so we want to construct an appropriate covering of $\mathcal{Y}$. It is the same to work with quasiaffinoid subdomains or their sets of points, so from now on we work with subsets of $\overline{\mathbb{Q}}_p$.

We write the family (Y_i) as $(S_i) \cup (A_i)$, where S_i are subsets of the form (2) of Definition 3.2.2 (and hence affinoid), and A_i are subsets of the form (1). We can cover each $A_i = \{x, r < |x-a| < r'\}$ by a family of affinoid subsets $A_{i,\eta} = \{x, r/\eta \leq |x-a| \leq r'\eta\}$ for $\eta > 1$, $\eta \in \sqrt{|F^\times|}$, η close enough to 1. So we get a covering of Y by affinoid subsets, that is, the S_i and the $A_{i,\eta}$. Their inverse images in $\mathcal{X}$ are affinoid, as each of them is contained in one of the Y_i.

This covering is not necessarily admissible, so we add some other affinoid subsets of Y in order to get an admissible covering. We know that if $\mathcal{Z}$ is an affinoid subset of $\mathcal{X}$, then $f(\mathcal{Z})$ is an affinoid subdomain of $\mathcal{D}$ and f induces an isomorphism between $\mathcal{Z}$ and $f(\mathcal{Z})$ by [Bosch et al. 1984, Corollary 8.2/4]. Let $\mathcal{C}$ be the covering of Y by the union of families of elements (S_i), $(A_{i,\eta})$, and all the sets $f(\mathcal{Z})(\overline{\mathbb{Q}}_p)$ for $\mathcal{Z}$ an affinoid subset of X. We want to show that $\mathcal{C}$ is an admissible covering of Y. Then it will satisfy the conditions of Lemma 3.4.2 and so the conclusion will follow.

Write Y as $D(a_0, r_0) \setminus \bigcup_{i=1}^m D(a_i, r_i)$, where each of the disks is rational and either open or closed and $a_0 \in Y$. Let $\eta > 1$, $\eta \in \sqrt{|F^\times|}$. We set $r_{0,\eta} = r_0$ if $D(a_0, r_0)$ is closed and r_0/η otherwise, and for $i > 0$ let $r_{i,\eta} = r_i$ if $D(a_i, r_i)$ is open and $r_{i,\eta} = r_i\eta$ otherwise. Let $Y_\eta = D(a_0, r_{0,\eta})^+ \setminus \bigcup_{i=1}^m D(a_i, r_{i,\eta})^-$, so that Y_η is an affinoid contained in Y (for η close enough to 1), and the family (Y_η) forms an admissible covering of Y. So it is enough to show that each Y_η can be covered by a finite number of elements of $\mathcal{C}$.

For each $0 \leq i \leq m$, let $b_i \in Y$ be such that $|a_i - b_i| = r_{i,\eta}$. Let c_i be an element of A_i for each i. Writing A_i as $\{r < |x-a| < r'\}$, we choose some c'_i in $Y \cap D(a, r)^+$ if it is not empty. By [Liu 1987], as $\mathcal{X}$ is connected, there is a connected subset Z of $\mathcal{X}$ that is a finite union of affinoid subdomains of $\mathcal{X}$, such that $f(Z)$ contains a_0 and each of the b_i, c_i and c'_i. Let $Z' = f(Z)$. Then it is a finite union of elements of $\mathcal{C}$, and a finite union of connected closed R-subsets, as it is a finite union of images of affinoid subsets of $\mathcal{X}$. As Z is connected, so is Z', so it is a connected closed R-subset by Lemma 3.2.5. By construction, there is a finite number of open disks (D_i) that do not meet Z' such that $D_i \subset Y$ and Y_η is contained in $Z' \cup (\cup_i D_i)$.

So it suffices to show that each D_i can be covered by a finite number of elements of $\mathcal{C}$. If D_i does not meet any A_j, then it is covered by the elements of $\mathcal{C}$ of the form S_j. If D_i meets A_j, then as D_i does not contain c_j (nor c'_j), then $D_i \subset A_j$ and so D_i is covered by $A_{j,t}$ for some $t > 0$. □

Corollary 3.4.5. *Let $\mathcal{X}$ be a normal rigid space that is quasiaffinoid space of open type over some finite extension E of $\mathbb{Q}_p$. Let $\mathcal{D}$ be the rigid closed unit disk. Let $f : \mathcal{X} \to \mathcal{D}$ be a rigid analytic map over E*

that is an open immersion. Let $Y = f(\mathcal{X})(\overline{\mathbb{Q}}_p)$. Then Y is an R-subset of open type defined over E, and if $\mathcal{X}$ is geometrically Zariski connected then Y is a connected R-subset. Moreover, f induces an E-algebra isomorphism between $\mathcal{A}_E(Y)$ and $\mathcal{A}_E(\mathcal{X})$, and between $\mathcal{A}_E^0(Y)$ and $\mathcal{A}_E^0(\mathcal{X})$.

Proof. Let F be an finite extension of E that is large enough so that each geometric Zariski connected component is defined over F, and F/E is Galois.

Write $\mathcal{X}$ as a disjoint union of $\mathcal{X}_i$ where each $\mathcal{X}_i$ is geometrically Zariski connected. Let f_i be the restriction of f to $\mathcal{X}_i$; it is still an open immersion, and is defined over F. We apply Theorem 3.4.1 to f_i: f_i induces an isomorphism between $\mathcal{X}_i$ and its image $f(\mathcal{X}_i) = \mathcal{Y}_i$. In particular, $\mathcal{A}_F(\mathcal{Y}_i)$ and $\mathcal{A}_F(\mathcal{X}_i)$ are isomorphic by the map $f_i^{\#}$. As $\mathcal{X}$ is of open type, so is $\mathcal{X}_i$ and hence so is $\mathcal{Y}_i$. By Proposition 3.2.9, this implies that $Y_i = \mathcal{Y}_i(\overline{\mathbb{Q}}_p)$ is a connected R-subset of open type. Moreover, the Y_i are disjoint as f is injective. Let Y be the disjoint union of the Y_i.

So we get an F-algebra isomorphism $f^{\#}$ between $\mathcal{A}_F(Y) = \bigoplus_{i=1}^n \mathcal{A}_F(Y_i)$ and $\mathcal{A}_F(\mathcal{X})$, which is equal to $\bigoplus_{i=1}^n \mathcal{A}_F(\mathcal{X}_i)$. As $\mathcal{X}$ is defined over E and f is an E-morphism, we see that $\mathcal{Y}$ is defined over E. We have an action of $\mathrm{Gal}(F/E)$ on both sides, and $f^{\#}$ is $\mathrm{Gal}(F/E)$-equivariant. So $f^{\#}$ induces an isomorphism between the $\mathrm{Gal}(F/E)$ invariants on both sides; hence the result. □

4. Complexity of standard subsets

4.1. *Algebraic complexity of a standard subset over a field of definition.*

Definition. Recall that we defined $\bar{e}$ in Section 2.2.

Definition 4.1.1. Let X be a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ that is defined over E. If X is irreducible over E, we define the complexity of X over E to be

$$c_E(X) = [k_{X,E} : k_E]\bar{e}_{\mathcal{O}_E}(\mathcal{A}_E^0(X)).$$

In general, let $X = \bigcup_{i=1}^r X_i$ be the decomposition of X as a disjoint union of standard subsets that are defined and irreducible over E. We define the complexity of X over E to be $c_E(X) = \sum_{i=1}^r c_E(X_i)$.

The above definition makes sense as $\mathcal{A}_E^0(X)$ is a complete noetherian local $\mathcal{O}_E$-algebra if X is irreducible over E by Corollary 3.3.8.

Note that in particular if X is connected then $c_E(X) = \bar{e}_{\mathcal{O}_E}(\mathcal{A}_E^0(X))$ as $k_{X,E} = k_E$ in this case.

Some general results on algebraic complexity. We now give explicit formulas for the complexity. It is enough to give such formulas for subsets X that are irreducible over E.

Proposition 4.1.2. *In the situation of Lemma 3.3.7, we have $c_E(X) = [F : E]c_F(X_1)$.*

Note that $c_F(X_1)$ does not depend on the choice of X_1 among the connected components.

Proof. Let $e_{F/E}$ be the ramification degree of F/E. We have that $\mathcal{A}_F^0(X_1) = \mathcal{A}_E^0(X)$ as $\mathcal{O}_E$-algebras, and $k_{X,E} = k_{X_1,F} = k_F$. So $c_E(X) = [k_F : k_E]\bar{e}_{\mathcal{O}_E}(\mathcal{A}_E^0(X)) = [k_F : k_E]\bar{e}_{\mathcal{O}_E}(\mathcal{A}_F^0(X_1))$ which equals $[k_F : k_E]e_{F/E}\bar{e}_{\mathcal{O}_F}(\mathcal{A}_F^0(X_1)) = [F : E]c_F(X_1)$ by Proposition 2.3.1. □

Proposition 4.1.3. *Let X be a connected standard subset defined over E, and F a finite extension of E. Then $c_E(X) \geq c_F(X)$ with equality when F/E is unramified.*

Proof. From Lemma 2.3.2 we see that $\bar{e}(\mathcal{O}_F \otimes_{\mathcal{O}_E} \mathcal{A}_E^0(X)) = \bar{e}(\mathcal{A}_E^0(X)) = c_E(X)$, and from Propositions 3.3.5 and 2.2.2 we see that $\bar{e}(\mathcal{O}_F \otimes_{\mathcal{O}_E} \mathcal{A}_E^0(X)) \geq \bar{e}(\mathcal{A}_F^0(X))$ with equality when F/E is unramified. □

Proposition 4.1.4. *Let X be a standard subset defined over E, and F a finite extension of E. Then $c_E(X) \geq c_F(X)$ with equality when F/E is unramified.*

Proof. By additivity of the complexity we can assume that X is irreducible over E. Write $X = \bigcup_{i=1}^n X_i$, where each X_i is connected. Let E_i be the field of definition of X_i over E, so that $c_E(X) = nc_{E_i}(X_i)$ for all i. Then FE_i is the field of definition of X_i over F. Suppose that the action of G_F on the set of the irreducible components of X has r orbits, with representatives say $X_1, \dots, X_r$. Then $c_F(X) = \sum_{j=1}^r [FE_j : F] c_{FE_j}(X_j)$. We have that $c_{FE_j}(X_j) \leq c_{E_j}(X_j)$ by Proposition 4.1.3, and $c_{E_j}(X_j)$ is independent of j, and equal to $(1/n)c_E(X)$. Moreover, $[FE_j : F]$ is the cardinality of the orbit of X_j, so $\sum_{j=1}^r [FE_j : F] = n$. Finally we get that $c_F(X) \leq c_E(X)$, with equality if and only if $c_{FE_j}(X_j) = c_{E_j}(X_j)$ for all j, which happens in particular if F/E is unramified. □

Does $c_E(X)$ characterize $\mathcal{A}_E^0(X)$? We ask the following question: let X be defined and irreducible over E. Let $R \subset \mathcal{A}_E^0(X)$ be a local, noetherian, complete, $\mathcal{O}_E$-flat $\mathcal{O}_E$-subalgebra of $\mathcal{A}_E^0(X)$, such that $R[1/p] = \mathcal{A}_E(X)$. Suppose moreover that R and $\mathcal{A}_E^0(X)$ both have residue field k_E, and $\bar{e}(R) = \bar{e}(\mathcal{A}_E^0(X))$, that is $\bar{e}(R) = c_E(X)$. Do we have $R = \mathcal{A}_E^0(X)$?

It follows from [Breuil and Mézard 2002, Lemme 5.1.8] that the equality holds if $c_E(X) = 1$, and in this case both rings are isomorphic to $\mathcal{O}_E[[x]]$, and X is a disk of the form $\{x, |x-a| < |b|\}$ for some $a, b \in E$.

But as soon as $c_E(X) > 1$ there are counterexamples. We give a few, with $E = \mathbb{Q}_p$.

(1) Let $X = \{x, 0 < v_p(x) < 1\}$. Then $\mathcal{A}_{\mathbb{Q}_p}^0(X)$ is isomorphic to $\mathbb{Z}_p[[x, y]]/(xy - p)$. Let R be the closure of the subring generated by px, py and $x - y$. Here $\bar{e}(R) = c_{\mathbb{Q}_p}(X) = 2$.

(2) Let $X = \left\{x, v_p(x) > \frac{1}{2}\right\}$. Then $\mathcal{A}_{\mathbb{Q}_p}^0(X)$ is isomorphic to $\mathbb{Z}_p[[x, y]]/(x^2 - py)$. Let R be the closure of the subring generated by y and px. Here $\bar{e}(R) = c_{\mathbb{Q}_p}(X) = 2$.

(3) Let $X = \{x, |x-\pi| < |\pi|\}$, where $\pi^p = p$. Then $\mathcal{A}_{\mathbb{Q}_p}^0(X)$ is isomorphic to $\mathbb{Z}_p[[x, y]]/(x^p - p(y+1))$. Let R be the closure of the subring generated by y and px. Here $\bar{e}(R) = c_{\mathbb{Q}_p}(X) = p$.

4.2. *Computations of the algebraic complexity in some special cases.*

Preliminaries. If $P \in E[x]$, and $a \in \overline{\mathbb{Q}}_p$, let $P_a(x) = P(x+a) \in \overline{\mathbb{Q}}_p[x]$.

Lemma 4.2.1. *Let D be an open disk defined over E, let s be the smallest degree over E of an element in D. Let a be an element of D of degree s over E. Let $\lambda \in \mathbb{R}$ be such that $D = \{x, v_E(x-a) > \lambda\}$.*

Let $P \in E[x]_{<s}$, and write $P_a(x) = \sum_{i=0}^{s-1} b_i x^i$. Then $v_E(b_i) \geq v_E(b_0) - i\lambda$ for all i. In particular, if $v_E(b_0) \geq 0$, then $v_E(b_i) \geq -i\lambda$ for all $i > 0$, and if $v_E(b_0) > 0$, then $v_E(b_i) > -i\lambda$ for all $i > 0$.

Proof. Consider the Newton polygon of P_a: if the conclusion of the lemma is not satisfied, then it has at least one slope μ which is $< -\lambda$. So P_a has a root y of valuation $-\mu > \lambda$. Let $b = a + y$, then b is a root of P, so of degree $< s$ over E. On the other hand, $v_E(b-a) = v_E(y) > \lambda$ so b is in D, which contradicts the definition of s. □

A similar proof shows:

Lemma 4.2.2. *Let D be a closed disk defined over E, let s be the smallest degree over E of an element in D. Let a be in D of degree s over E. Let $\lambda \in \mathbb{R}$ be such that $D = \{x, v_E(x-a) \geq \lambda\}$.*

Let $P \in E[x]_{<s}$, and write $P_a(x) = \sum_{i=0}^{s-1} b_i x^i$. Then $v_E(b_i) > v_E(b_0) - i\lambda$ for all $i > 0$. In particular, if $v_E(b_0) \geq 0$, then $v_E(b_i) > -i\lambda$ for all $i > 0$.

Definition 4.2.3. Let $L/\mathbb{Q}_p$ be a finite extension. Let $f \in \mathcal{O}_L[[w]]$, $f = \sum_{i \geq 0} f_i w^i$. We say that f is regular of degree n if $f_n \in \mathcal{O}_L^\times$ and $f_m \in \mathfrak{m}_L$ for all $m < n$.

Definition 4.2.4. Let $L/\mathbb{Q}_p$ be a finite extension. Let $f \in \mathcal{O}_L[[w]]$, $f = \sum_{i \geq 0} f_i w^i$. We define the valuation of f as $v_E(f) = \min_i v_E(f_i)$, and the leading term of f as w^i for the smallest i such that $v_E(f) = v_E(f_i)$. In particular, f is regular of degree n if and only if $v_E(f) = 0$ and the leading term of f is w^n.

We recall the following result (see, e.g., [Washington 1997, Proposition 7.2]):

Lemma 4.2.5 (Weierstrass division theorem). *Let $f \in \mathcal{O}_L[[w]]$ that is regular of degree n, and $g \in \mathcal{O}_L[[w]]$. Then there exists a unique pair (q, r) with $q \in \mathcal{O}_L[[w]]$, $r \in \mathcal{O}_L[w]_{<n}$ and $g = qf + r$.*

Let X be a connected standard subset defined over E. Then we have the easy but useful result:

Lemma 4.2.6. *Let $f \in \mathcal{A}_E^0(X)$. Then f reduces to 0 in $\mathcal{A}_E^0(X)/(\pi_E)$ if and only if $\|f\|_X \leq |\pi_E|$. The image of f in $\mathcal{A}_E^0(X)/(\pi_E)$ is nilpotent if and only if $\|f\|_X < 1$.*

Open disks. We want to give the general formula for the complexity of a disk. We start with some examples. We see that there are two kinds of difficulties: one from the radius that is not necessarily the norm of an element of E, and one from the fact that the disk does not necessarily contain an element in E.

Example 4.2.7. Let a, b be in E with $b \neq 0$. Let D be the disk $\{x, |x-a| < |b|\}$. Then $c_E(D) = 1$. Indeed, $\mathcal{A}_E^0(D)$ is isomorphic to $\mathcal{O}_E[[w]]$, where w corresponds to the function $(x-a)/b$, so $\mathcal{A}_E^0(D)/(\pi_E) = k_E[[w]]$.

Example 4.2.8. Let D be the disk $\left\{x, v_p(x) > \frac{1}{2}\right\}$. Then $c_{\mathbb{Q}_p}(D) = 2$. Indeed, $\mathcal{A}_{\mathbb{Q}_p}^0(D)$ is isomorphic to $\mathbb{Z}_p[[w, t]]/(t^2 - pw)$, where w corresponds to the function x^2/p and t to the function x. So $\mathcal{A}_{\mathbb{Q}_p}^0(D)/(p)$ is isomorphic to $\mathbb{F}_p[[w, t]]/(t^2)$.

Example 4.2.9. Let $D = \{x, |x - \pi| < |\pi|\}$, where $\pi^p = p$. Then $c_{\mathbb{Q}_p}(D) = p$. Indeed $\mathcal{A}_{\mathbb{Q}_p}^0(D)$ is isomorphic to $\mathbb{Z}_p[[t, w]]/(t^p - p(w+1))$, where t is the function x and w is the function $(x^p - p)/p$.

Proposition 4.2.10. *Let D be an open disc of radius $r \in p^{\mathbb{Q}}$ defined over E. Let s be the smallest ramification degree of $E(a)/E$ for $a \in D$. Let t be the smallest positive integer such that $r^{st} \in |E(a)^\times|$. Then $c_E(D) = st$.*

Proof. There are two steps in the proof: the first is to find a description of $\mathcal{A}_E^0(D)$, and the second to use this description to show that $\mathcal{A}_E^0(D)/(\pi_E)$ satisfies the conditions of Corollary 2.1.2 and apply this to compute $c_E(D)$.

Step 1: Let $a \in D$ be as in the statement. As the complexity does not change by unramified extensions by Proposition 4.1.4, we can enlarge E so that $E(a)/E$ is totally ramified. Let μ be the minimal polynomial of a over E, so that μ has degree s. Write $F = E(a)$. For $\nu \in \mathbb{Q}$, let F_ν be the set $\{x \in F, v_E(x) \geq \nu\}$ (so that $F_0 = \mathcal{O}_F$).

Let λ be such that $D = \{x, v_E(x-a) > \lambda\}$. Let also $\rho \in F$ such that $v_E(\rho) = st\lambda$, which is possible by the definition of t. When $s > 1$, we see that $v_E(a) \leq \lambda$ (otherwise $0 \in D$), and if a' is another root of μ then $v_E(a-a') > \lambda$ as D is defined over E.

For $n \in \mathbb{Z}$, let $\mathcal{E}_n$ be the subset of $E[x]_{<s}$ of polynomials that can be written as $\sum_{i=0}^{s-1} b_i(x-a)^i$ with $v_E(b_i) \geq -(i+ns)\lambda$. Note that by Lemma 4.2.1, $\mathcal{E}_n$ is the set of polynomials in $E[x]_{<s}$ with $v_E(b_0) \geq -ns\lambda$. In fact $\mathcal{E}_n$ is in bijection with the set $F_{-ns\lambda}$ by $P \mapsto P(a)$, as any element of F can be written uniquely as $P(a)$ for some $P \in E[x]_{<s}$. Note that $\rho^{-1} \in F_{-st\lambda}$. We fix $R \in \mathcal{E}_t$ the unique polynomial such that $R(a) = \rho^{-1}$. We set $\alpha = R\mu^t$.

Let L be a Galois extension of E containing F and an element ξ such that $v_E(\xi) = \lambda$. Then $\mathcal{A}_L^0(D)$ is isomorphic to $\mathcal{O}_L[[w]]$, with w corresponding to $(x-a)/\xi$. We consider now α as a polynomial in $w = (x-a)/\xi$. Then an easy computation shows that $\alpha \in \mathcal{O}_L[w]$, and it is a polynomial of degree at most $st+s-1$ which is regular of degree st in the sense of Definition 4.2.3 when seen as an element of $\mathcal{A}_L^0(D) = \mathcal{O}_L[[w]]$.

Let $\mathcal{E}'$ be the subset of $E[x]_{<st}$ of polynomials that can be written as

$$\sum_{i=0}^{st-1} b_i(x-a)^i$$

with $v_E(b_i) \geq -i\lambda$. Then

$$\mathcal{A}_E^0(D) = \left\{ \sum_{n\geq 0} P_n\alpha^n,\, P_n \in \mathcal{E}' \right\}$$

and any element of $\mathcal{A}_E^0(D)$ can be written uniquely in such a way. Indeed, let $f \in \mathcal{A}_E^0(D)$, which we see as an element of $\mathcal{A}_L^0(D) = \mathcal{O}_L[[w]]$. Applying repeatedly the Weierstrass division theorem, f can be written uniquely as $\sum_{n\geq 0} P_n\alpha^n$ with $P_n \in \mathcal{O}_L[w]_{<st}$. The fact that f is in $\mathcal{A}_E^0(D)$ means that f is invariant under $\mathrm{Gal}(L/E)$. As α itself is invariant under this group, this means that each P_n is invariant, and so $P_n \in \mathcal{E}'$ (where we see $\mathcal{E}' \subset \mathcal{O}_L[w]_{<st}$ by $w = (x-a)/\xi$).

Step 2: We now want to check to conditions of Corollary 2.1.2. We see that $\mathcal{E}' = \bigoplus_{j=0}^{t-1} \mu^j \mathcal{E}_j$. For $0 \leq i < t$, let $(u_{i,j})_{1\leq i\leq s}$ be a basis of $\mathcal{E}_j$ as an $\mathcal{O}_E$-module, taking $u_{1,0} = 1$, and $v_E(u_{i,0}(a)) > 0$ for $i > 1$. We can satisfy this condition as taking a basis of $\mathcal{E}_0$ is the same as taking a basis of $\mathcal{O}_F$ over $\mathcal{O}_E$, and F is totally ramified over E. We also observe that for $j > 0$, we have $v_E(u_{i,j}(a)) > -js\lambda$ by definition of t.

Write $y_{i,j} = u_{i,j}\mu^j$ and $z = \alpha$ (note that $y_{1,0} = 1$). Then $\mathcal{A}_E^0(D)$ is a quotient of $\mathcal{O}_E[[y_{i,j}, z]]$; hence the ring $A = \mathcal{A}_E^0(D)/(\pi_E)$ is a quotient of $k_E[[y_{i,j}, z]]$. Let $\bar{y}_{i,j}$, $\bar{z}$ be the images of $y_{i,j}$, z in A. Let I be the ideal generated by the $\bar{y}_{i,j}$ for $(i, j) \neq (1, 0)$. Then the maximal ideal $\mathfrak{m}$ of A is generated by I and $\bar{z}$.

We show first that I is nilpotent. We see $y_{i,j}$ as an element of $\mathcal{A}_L^0(D) = \mathcal{O}_L[[w]]$, then $\|y_{i,j}\|_X = \max_n a_n$, where $y_{i,j} = \sum_{n \geq 0} a_n w^n$. So we see that for $(i, j) \neq (1, 0)$, we have that $\|y_{i,j}\|_X < 1$, and so $y_{i,j}$ is nilpotent by Lemma 4.2.6.

Let us see now that A has no $\bar{z}$-torsion. As before, we see $\mathcal{A}_E^0(X)$ as a subalgebra of $\mathcal{O}_L[[w]]$. From the existence of this inclusion, we see that the norm on $\mathcal{A}_E^0(X)$ is actually multiplicative. As $\|z\|_X = 1$, we deduce that $\|zf\|_X = \|f\|_X$ for all $f \in \mathcal{A}_E^0(X)$, and so A has no $\bar{z}$-torsion.

We deduce that the conditions of Corollary 2.1.2 are satisfied. So $e(A) = \dim_k A/(\bar{z})$, and we see easily that 1 and the $\bar{y}_{i,j}$, $1 \leq i \leq s$ and $0 \leq i < t$, $(i, j) \neq (1, 0)$, form a k-basis of $A/(\bar{z})$. □

Holes.

Proposition 4.2.11. *Let $X = \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus T$, where $T = \bigcup_{i=1}^N D_i$ is a G_E-orbit of closed disks of positive radius $r \in p^{\mathbb{Q}}$, with each disk defined over a totally ramified extension of E. Let K be the field of definition of D_1. Let s be the smallest ramification degree of $K(a)/K$ for $a \in D_1$. Let t be the smallest positive integer such that $r^{st} \in |E(a)^\times|$. Assume that $K(a)/E$ is totally ramified. Then $c_E(X) = Nst$.*

When $N = 1$, that is, when T is a disk, then the formula and the proof are similar to what happens in Proposition 4.2.10. But there are additional difficulties when there is not only one hole, but a whole Galois orbit of them, that is, when $N > 1$.

Proof of Proposition 4.2.11. We divide the proof into several steps.

Step 1: We first give a description of the ring of functions in the case where $N = 1$, that is, when there is only one hole. Write $X' = \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D_1$, so that X' is defined over K. We will give a description of the ring $\mathcal{A}_K^0(X')$, forgetting T and E for the moment. This computation is similar to the computation in Proposition 4.2.10, although complicated by the fact that we work with rational fractions and not only with polynomials.

Let $a \in D_1$ as in the statement of the proposition. Note that $[K : E] = N$. Let $F = E(a)$. Note that $K \subset F$ so $E(a) = K(a)$. By hypothesis, F/E is totally ramified. We write $[F : K] = s$. Write D_1 as the set $\{x, v_E(x - a) \geq \lambda\}$ for some $\lambda \in \mathbb{Q}$. Let μ be the minimal polynomial of a over K, so that μ has degree s. Let also $\rho \in F$ be such that $v_E(\rho) = st\lambda$, which is possible by the definition of t.

Let R be the unique element of $K[x]_{<s}$ such that $R(a) = \rho$. Note that when we write $R(x) = \sum b_i (x - a)^i$, we have $v_E(b_i) > (st - i)\lambda$ for all $i > 0$ by Lemma 4.2.2. Set $v = R/\mu^t$, and for $n \geq 1$, set $\alpha_n = \rho\mu^{-t} v^{n-1}$.

Let L be an extension of E containing a and an element ξ such that $v_E(\xi) = \lambda$, and which is Galois over E. Note that $\mathcal{A}_L^0(X')$ is isomorphic to $\mathcal{O}_L[[w]]$, with w corresponding to the function $\xi/(x - a)$. In this isomorphism, observe that α_n is regular of degree nst and is divisible by w^{st}, and $v = R\mu^{-t} = \rho^{-1} R\alpha_1$ is regular of degree st, and divisible by w^{st-s+1}.

Let $f = wg \in w\mathcal{A}_L^0(X')$. Then by applying Lemma 4.2.5 repeatedly we can write $w^{st-1}f = w^{st}g$ as $\sum_{n\geq 1} P_n(w)\alpha_n$ for $P_n \in \mathcal{O}_L[w]_{<st}$ (there is no remainder as w^{st} and α_1 differ by a unit). So $f = \sum_{n\geq 1} w^{1-st}P_n(w)\alpha_n$. So any element of $\mathcal{A}_L^0(X')$ can be written uniquely as $f = a_0 + \sum_{n\geq 1} \rho w^{1-st}P_n(w)\mu^{-t}v^{n-1}$, where $a_0 \in \mathcal{O}_L$ and $P_n(w) \in \mathcal{O}_L[w]_{<st}$.

We want to know when such an element is in $\mathcal{A}_K(X')$. As v and μ^{-t} are in $\mathcal{A}_K(X')$, we see that it is the case if and only if $a_0 \in \mathcal{O}_K$ and $\rho w^{1-st}P_n(w)$ is invariant under the action of $\mathrm{Gal}(L/K)$. Note that $\rho w^{1-st}P_n(w)$ in actually in $L[x]_{<st}$, so it is invariant by $\mathrm{Gal}(L/K)$ if and only if it is in $K[x]$. Let $\mathcal{E}'$ the set of elements $Q \in K[x]_{<st}$ such that when we write $Q(x) = \sum_{i\geq 0} b_i(x-a)^i$, we have $v_E(b_i) \geq (st-i)\lambda$. Then we see that $\mathcal{E}'$ is exactly the set of elements of $K[x]_{<st}$ that are of the form $\rho w^{1-st}P(w)$ for some $P \in \mathcal{O}_L[w]_{<st}$.

Then we have shown that

$$\mathcal{A}_K^0(X') = \left\{ a_0 + \sum_{n\geq 0} \frac{Q_n(x)}{\mu(x)^t} v^n,\ a_0 \in \mathcal{O}_K,\ Q_n \in \mathcal{E}' \right\},$$

where $v = R/\mu^t$.

Note that if $f \in \mathcal{A}_K^0(X')$ is written as $a_0 + \sum_{n\geq 0}(Q_n(x)/\mu(x)^t)v^n$ with $a_0 \in \mathcal{O}_K$, $Q_n \in \mathcal{E}'$, then

$$\|f\|_{X'} = \max(|a_0|, \max_n \|Q_n\mu^{-t}\|_{X'}). \tag{2}$$

We can see f as being in $\mathcal{O}_L[[w]]$ and reason in terms of $v_E(f)$. We have that $v_E(v) = 0$ as v is regular of degree st. Moreover, writing $Q_n\mu^{-t}$ as $\rho^{-1}Q_n\alpha_1$, we see that the leading term of $Q_n\mu^{-t}$ is w^{st}. Using this, we see easily that $v_E(f) = \min(v_E(a_0), \min_n(v_E(Q_n\mu^{-t})))$ which gives the result.

Step 2: We introduce the tools that allow us to go from the description of $\mathcal{A}_K^0(X')$ to the description of $\mathcal{A}_E^0(X)$.

Let $Q = \{\sigma_1, \dots, \sigma_N\}$ be a system of representatives in G_E of G_E/G_K, numbered so that $\sigma_i D_1 = D_i$ (so we take $\sigma_1 = \mathrm{id}$). Recall that $w = \xi/(x-a)$. Let ξ_i, a_i be conjugates of ξ and a_i by σ_i, and let $w_i = \xi_i/(x-a_i)$ (so that $w_i = w$). If $f \in \mathcal{O}_L[[w]] = \mathcal{A}_L^0(X') \subset \mathcal{A}_L^0(X)$, with $f = \sum f_n w^n$, we denote by $\mathrm{tr}\, f \in \mathcal{A}_L^0(X)$ the element $\sum_{i=1}^N \sum_n \sigma_i(f_n)w_i^n$. Note that if $f \in \mathcal{A}_K(X')$, then $\mathrm{tr}\, f \in \mathcal{A}_E(X)$ and $\mathcal{A}_E^0(X) = \{a + \mathrm{tr}\, f, a \in \mathcal{O}_E, f \in \mathcal{A}_K^0(X')\}$. We can actually make this more precise: let $\mathcal{A}_E^0(X)_0$ and $\mathcal{A}_K^0(X')_0$ the subspaces of $\mathcal{A}_E^0(X)$ and $\mathcal{A}_K^0(X')$ of functions with no constant term (see Remark 3.3.4). Then tr induces a bijection between $\mathcal{A}_K^0(X')_0$ and $\mathcal{A}_E^0(X)_0$. Moreover if $f \in \mathcal{A}_K^0(X')_0$ then $\|f\|_{X'} = \|\mathrm{tr}\, f\|_X$, as can be seen from Proposition 3.3.3.

Note that $\mathcal{A}_K^0(X') \subset \mathcal{O}_L[[w]]$, and this injection is multiplicative and preserves the norm. So we also get an injection $\mathcal{A}_E^0(X)_0 \to \mathcal{O}_L[[w]]$, which preserves the norm as noted earlier. But this is not multiplicative, as in general $\mathrm{tr}(fg) \neq (\mathrm{tr}\, f)(\mathrm{tr}\, g)$. However, we have for all f, g in $w\mathcal{O}_L[[w]]$:

$$\|\mathrm{tr}(fg) - (\mathrm{tr}\, f)(\mathrm{tr}\, g)\|_X < \|fg\|_{X'}. \tag{3}$$

In particular, we have that $\|\mathrm{tr}(fg)\|_X = \|(\mathrm{tr}\, f)(\mathrm{tr}\, g)\|_X$.

Let us prove (3). Write $f=\sum_n f_n w^n$ and $g=\sum_n g_n w^n$. Then $(\operatorname{tr} f)(\operatorname{tr} g)$ is of the form $\operatorname{tr}(u)$ for some u in $w\mathcal{O}_L[[w]]$. We can assume that $\|f\|_{X'}=\|g\|_{X'}=1$, so that $\|fg\|_{X'}=1$. Let w' be one of the w_i for $i>1$. Consider $w^n w'^m$ for some integers $n, m>0$. It can be written as a sum of an element of $w\mathcal{O}_L[[w]]$ and an element of $w'\mathcal{O}_L[[w']]$, and we want to understand the term in $w\mathcal{O}_L[[w]]$. Note that for $n>0$, $m>0$, we can write $w^n w'^m=\sum_{i=1}^n \alpha_i w^i+\sum_{i=1}^m \beta_i w'^i$, with $\alpha_i\in\mathcal{O}_L$ and $\beta_i\in\mathcal{O}_L$ and $|\alpha_i|<1$ and $|\beta_i|<1$ for all i as $|\xi|<|a-a'|$. So we see that all the terms contributing to $fg-u$ have norm <1. This proves (3).

Step 3: We give a description of $\mathcal{A}_E^0(X)$. Combining the description of $\mathcal{A}_K^0(X')$ in Step 1 and using the trace we get

$$\mathcal{A}_E^0(X)=\left\{a_0+\sum_{n\geq 0}\operatorname{tr}\Big(\frac{Q_n}{\mu^t}v^n\Big),\ a_0\in\mathcal{O}_E,\ Q_n\in\mathcal{E}'\right\}\tag{4}$$

and elements of $\mathcal{A}_E^0(X)$ can be written uniquely in such a way. We want to change this description to something more convenient. Let $z=\operatorname{tr} v$. Then

$$\mathcal{A}_E^0(X)=\left\{a_0+\sum_{n\geq 0}\operatorname{tr}\Big(\frac{Q_n}{\mu^t}\Big)z^n,\ a_0\in\mathcal{O}_E,\ Q_n\in\mathcal{E}'\right\}.\tag{5}$$

In order to prove this, we transform an element written as in (4) into an element written as in (5) by successive approximation, using the inequality (3) and the formula (2) for the norm.

Step 4: We now give a set of generators of $\mathcal{A}_E^0(X)$ as a complete $\mathcal{O}_E$-algebra, which will be useful in the next steps.

We start by giving a basis of the $\mathcal{O}_K$-module $\mathcal{E}'$. For $0\leq j<t$, let $\mathcal{E}_j$ be the subset of $K[x]_{<s}$ of polynomials that can be written as $\sum_{i=0}^{s-1} b_i(x-a)^i$ with $b_i\in F$, $v_E(b_i)\geq(s(t-j)-i)\lambda$. Note that by Lemma 4.2.2, $\mathcal{E}_j$ is the subset of elements of $K[x]_{<s}$ with $v_E(b_0)\geq s(t-j)\lambda$, and if $P\in\mathcal{E}_j$, then $v_E(b_i)>(s(t-j)-i)\lambda$ for all $i>0$. Moreover, $\mathcal{E}_j$ is in bijection with the set

$$F_{s(t-j)\lambda}=\{b\in F,\ v_E(b)\geq s(t-j)\lambda\}$$

by $P\mapsto P(a)$. Indeed, if $b\in F$, it can be written uniquely as $b=P(a)$ for some $P\in K[x]_{<s}$ as $F=K(a)$. By definition, $F_{s(t-j)\lambda}$ does not contain an element of valuation $s(t-j)\lambda$ for $0<j<t$. We note that $\mathcal{E}'=\bigoplus_{j=0}^{t-1}\mu^j\mathcal{E}_j$. We define bases for the $\mathcal{E}_j$ as $\mathcal{O}_K$-modules as follows: fix δ_j in $F_{s(t-j)\lambda}$ of minimal valuation (take $\delta_0=1$, and note that $v_E(\delta_j)>s(t-j)\lambda$ if $j\neq 0$). Let ϖ be a uniformizer of F, so that $(1,\varpi,\dots,\varpi^{s-1})$ is a basis of $\mathcal{O}_F$ as an $\mathcal{O}_K$-module. Then let $Q_{i,j}\in\mathcal{E}_j$ be the polynomial such that $Q_{i,j}(a)=\delta_j\varpi^{i-1}$ for $1\leq i\leq s$. So we deduce a basis $(P_{i,j})_{0\leq j<t,1\leq i\leq s}$ of $\mathcal{E}'$ as an $\mathcal{O}_K$-module by taking $P_{i,j}=Q_{i,j}\mu^j$.

Finally let $u_{i,j}=P_{i,j}/\mu^t\in\mathcal{A}_K^0(X')$, so that $v=R/\mu^t=u_{1,0}$. Let α be a uniformizer of K, so that $\mathcal{O}_K=\mathcal{O}_E[\alpha]$ (recall that K is a totally ramified extension of degree N of E). Let $y_{i,j,\ell}=\operatorname{tr}(\alpha^\ell u_{i,j})$. Then

$\mathcal{A}_E^0(X)$ is generated by z and the $y_{i,j,\ell}$, and more precisely

$$\mathcal{A}_E^0(X)=\left\{a_0+\sum_{n\geq 0}\left(\sum_{i=1}^{s}\sum_{j=0}^{t-1}\sum_{\ell=0}^{N-1}a_{i,j,\ell,n}y_{i,j,\ell}\right)z^n,\ \ a_0,a_{i,j,\ell,n}\in\mathcal{O}_E\ \forall i,j,\ell\right\}. \tag{6}$$

Step 5: Let $A=\mathcal{A}_E^0(X)/(\pi_E)$. We now show that the hypotheses of Corollary 2.1.2 are satisfied by A. Denote by $\bar{y}_{i,j,\ell}$ the image of $y_{i,j,\ell}$ in A, and by $\bar{z}$ the image of z (observe that $y_{1,0,0}=z$). Let I be the ideal of A generated by the $\bar{y}_{i,j,\ell}$ for $(i,j,\ell)\neq(1,0,0)$. Then it is clear from (6) that the maximal ideal of A is generated by I and $\bar{z}$.

Then I is a nilpotent ideal. Indeed, consider f one of the elements $y_{i,j,\ell}$, that is, $f=\operatorname{tr}\alpha^\ell u_{i,j}$. We see f as an element of $\mathcal{A}_L^0(X)$. When we write $\alpha^\ell u_{i,j}$ as an element of $\mathcal{O}_L[[w]]$, with $w=\xi/(x-a)$ as before, we see that in fact it is in $\pi_L\mathcal{O}_L[[w]]$, as either $\ell>0$ or $(i,j)\neq(1,0)$. So f is in $\pi_L\mathcal{A}_L^0(X)$. By Lemma 4.2.6, this means that the image of f in A is nilpotent. So I is nilpotent.

Let us show that A has no $\bar{z}$-torsion. Let $f\in A$ which is not a unit. Then $f=\bar{g}$ for some $g\in\mathcal{A}_E^0(X)$ that can be written as $\operatorname{tr}(h)$ for some $h\in\mathcal{A}_K^0(X')$. We compute $\|zg\|_X=\|(\operatorname{tr}v)(\operatorname{tr}h)\|_X=\|\operatorname{tr}(vh)\|_X$ (by (3)), so finally $\|zg\|_X=\|vh\|_{X'}=\|v\|_{X'}\|h\|_{X'}$. Moreover, $\|v\|_{X'}=1$, so $\|zg\|_X=\|h\|_{X'}=\|g\|_X$. By Lemma 4.2.6, this means that $\bar{z}f\neq 0$ if $f\neq 0$.

So finally we are in the conditions of Corollary 2.1.2.

Step 6 We compute the dimension of $A/(\bar{z})$, which is the complexity we are looking for by Corollary 2.1.2. It is clear from (6) that $\dim_k A/(\bar{z})\leq Nst$, as $A/(\bar{z})$ is generated as a k-vector space by 1 and the $\bar{y}_{i,j,\ell}$ for $(i,j,\ell)\neq(1,0,0)$. Let us show that it is in fact an equality.

Let $x=\mu+\sum_{(i,j,\ell)\neq(1,0,0)}\lambda_{i,j,\ell}\bar{y}_{i,j,\ell}$ in A that reduces to 0 in $A/(\bar{z})$, and let us show that all the coefficients are in fact 0. First, $\mu=0$ otherwise x in a unit in A. Lift each $\lambda_{i,j,\ell}$ to some $a_{i,j,\ell}\in\mathcal{O}_E$. Let $f=\sum_{(i,j,\ell)\neq(1,0,0)}a_{i,j,\ell}\alpha^\ell u_{i,j}$, so that $x=\overline{\operatorname{tr}f}$, and assume that $f\neq 0$.

The fact that x reduces to 0 in $A/(\bar{z})$ means that there exists some $g\in\mathcal{A}_E^0(X)$ such that $\|\operatorname{tr}f-zg\|_X\leq|\pi_E|$. Then g is in the maximal ideal of $\mathcal{A}_E^0(X)$ (it cannot be a unit as $\overline{\operatorname{tr}f}$ is nilpotent in A but $\bar{z}$ is not). So we can take g to be of the form $\operatorname{tr}h$ for some $h\in\mathcal{A}_K^0(X')_0$. Let us compare $\operatorname{tr}f$ and $(\operatorname{tr}v)(\operatorname{tr}h)$: they are both in $\mathcal{A}_E^0(X)_0$ so we see them in $\mathcal{O}_L[[w]]=\mathcal{A}_K^0(X')$.

We compute easily that the valuation of $\alpha^\ell u_{i,j}$ is $\ell v_E(\alpha)+(i-1)v_E(\varpi)+v_E(\delta_j)$ and the leading term is $w^{s(t-j)}$. So we can determine j from the leading term. Note also that $v_E(\alpha)=1/N$, $v_E(\varpi)=1/sN$. As $0\leq\ell<N$ and $0\leq i-1<s$, we see that for a given j, the valuations of $\alpha^\ell u_{i,j}$ and $\alpha^{\ell'}u_{i',j}$ are not equal modulo $\mathbb{Z}$ except if $i=i'$ and $\ell=\ell'$. This means that in f there are no cancellations, and in particular the leading term of f is $w^{s(t-j)}$ for some $j<t$. On the other hand, the leading term of vh is w^n for some $n>st$. This contradicts the fact that $\|\operatorname{tr}f-(\operatorname{tr}v)(\operatorname{tr}h)\|\leq|\pi_E|$.

So finally $e(A)=\dim_k A/(\bar{z})=Nst$. □

Additivity formula. We know want to compute the complexity of any connected standard subset defined over E. Using the fact that the complexity is invariant under unramified extension of the definition field, we see that Proposition 4.2.12, combined with Propositions 4.2.10 and 4.2.11, gives us a way do this computation.

Proposition 4.2.12. *Let X be a connected standard subset defined over E. Assume that X is of the form $Y \setminus T$, where Y is either $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ or a disk defined over E, $T = \bigcup_{i=1}^m T_i$, where each T_i is a disjoint union of closed disks $D_{i,j}$ such that the T_i are pairwise disjoint, with each defined and irreducible over E, contained in Y, and each $D_{i,j}$ is well-defined over an extension of E that is totally ramified over E. Then $c_E(X) = c_E(Y) + \sum_{i=1}^m c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus T_i)$ if Y is a disk, and $c_E(X) = \sum_{i=1}^m c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus T_i)$ if $Y = \mathbb{P}^1(\overline{\mathbb{Q}}_p)$.*

Proof. Let $X_i = \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus T_i$ for $1 \le i \le m$, and set $X_0 = Y$ if Y is a disk. Then $X = \bigcap_i X_i$, and each X_i is defined over E, and if $i \ge 1$ then X_i is of the form of the subsets studied in Proposition 4.2.11.

Using the description of the ring of functions in Proposition 3.3.3, we see that for each i, we can write $\mathcal{A}_E^0(X_i) = \mathcal{O}_E \oplus \mathcal{M}_i$ for some submodule $\mathcal{M}_i$, where $\mathcal{O}_E$ is the subring of constant functions (note that if $i \ge 1$ we can choose $\mathcal{M}_i$ canonically by taking the functions that are zero at infinity). Then we have a natural injection $\mathcal{A}_E^0(X_i) \to \mathcal{A}_E^0(X)$ for all i, such that $\mathcal{A}_E^0(X) = \mathcal{O}_E \oplus \left(\bigoplus_i \mathcal{M}_i\right)$ by Proposition 3.3.3. Let $f \in \mathcal{A}_E^0(X)$, and write f in this decomposition. Then f has a nonzero component on $\mathcal{M}_i$ if and only f has a pole in T_i. Let $A_i = \mathcal{A}_E^0(X_i)/(\pi_E)$, and $M_i = \mathcal{M}_i/(\pi_E)$. Then each A_i contains an element z_i as in Lemma 2.1.5: it is the element called $\bar{z}$ in Propositions 4.2.10 (for $i = 0$, if X is bounded) and 4.2.11 (for $i \ge 1$).

Let $A = \mathcal{A}_E^0(X)/(\pi_E)$. Then $A = k \oplus \left(\bigoplus_i V_i\right)$, where V_i is the image of M_i, and A is nearly the sum of the A_i's as in Definition 2.1.3. In order to compute the multiplicity of A, we want to apply Lemma 2.1.5. So we need to prove: for all $i \ne j$, there exist some integers N and t, with $t < N$, such that $V_i^n V_j \subset V_i^{n-t}$ for all $n > N$. It is clear that $V_i V_j \subset V_i + V_j$. So we can assume without loss of generality that $m \le 2$.

We will treat only the case where $i = 1$, $j = 2$. The case where i or j is equal to 0 (which can occur only when X is bounded) is similar. For simplicity, we will assume from now on that T_1 and T_2 are actually connected, that is, each is a single closed disk D_i defined over E. The general case needs no new ideas but requires more complicated notation.

We first describe a little the ring $\mathcal{A}_E^0(X)$. We fix a finite Galois extension L of E such that X is well-defined over L. Let $t_0 = e_{L/E}$. So for $i = 1, 2$ we write $D_i = D(a_i, |\xi_i|)^+$, with a_i and ξ_i in L. Note that $|\xi_i/(a_i - a_j)| < 1$ if $\{i, j\} = \{1, 2\}$, so $v_L(\xi_i/(a_i - a_j)) \ge 1$. Let $y_i = \xi_i/(x - a_i)$ for $i = 1, 2$. Then $\mathcal{A}_E^0(X_i) \subset \mathcal{O}_L[[y_i]] = \mathcal{A}_L^0(X_i)$. If $h \in \mathcal{A}_E^0(X_i) \cap \pi_L^{t_0}\mathcal{O}_L[[y_i]]$, then h is in $\pi_E \mathcal{A}_E^0(X_i)$.

We have a decomposition $\mathcal{A}_E^0(X) = \mathcal{O}_E \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$ as before. We denote by α_i the projection to $\mathcal{M}_i$ in this decomposition. We also have a decomposition of $A = \mathcal{A}_E^0(X)/(\pi_E)$ as $k \oplus V_1 \oplus V_2$, and we denote by $\bar{\alpha}_i$ the map to V_i which is the composition of reduction modulo π_E and projection to V_i. The maps α_i extend to the decomposition $\mathcal{A}_L^0(X) = \mathcal{O}_L \oplus \mathcal{M}_{1,L} \oplus \mathcal{M}_{2,L}$.

Denote by z_1 the element that was called z in the proof of Proposition 4.2.11 applied to X_1 (which is also the element called v, as we are in the case where $N = 1$), and denote by τ the integer that was denoted by st. Then in $\mathcal{O}_L[[y_1]]$, z_1 is equal to $\pi_L h + y_1^\tau u$ for some $h \in \mathcal{O}_L[y_1]_{<\tau}$ and $u \in \mathcal{O}_L[[y_1]]^\times$.

For $m \geq 0$, write $z_1^m = \sum_{j \geq 0} c_{m,j} y_1^j$ with $c_{m,j} \in \mathcal{O}_L$. Then we have that $v_L(c_{m,j}) \geq m - j/\tau$. On the other hand, we can write $y_1^{m\tau} = \sum_{i \geq 0} q_i z_1^i$ with $q_i \in \pi_L^{\max(0,m-i)} \mathcal{O}_L[[t_1]]$.

Let $\bar{z}_1$ be the image of z_1 in A_1. Then as in the proof of Proposition 4.2.11, V_1 is generated by $\bar{z}_1$ and a nilpotent ideal I of A_1. Let t_1 be an integer such that $I^{t_1} = 0$. Then any element of V_1^n for n large enough is a multiple of $z_1^{n-t_1}$.

Fix some $f \in \mathcal{M}_1$ such that its image in V_1 is in V_1^n, and $g \in \mathcal{M}_2$. As we are interested only in working in A, we can assume that f is divisible by $z_1^{n-t_1}$. So when we write f (seen as an element of $\mathcal{A}_L^0(X_1)$) as $\sum_j f_j y_1^j$, we have $v_L(f_j) \geq n - t_1 - j/\tau$.

We have that $fg \in \mathcal{M}_1 \oplus \mathcal{M}_2$, so its image in A is in $V_1 \oplus V_2$. We want to show that in this decomposition, the projection $\bar{\alpha}_2(fg)$ of fg to V_2 is zero, and the projection $\bar{\alpha}_1(fg)$ to V_1 is contained in V_1^{n-t} (for some t independent of n to be determined).

Clearly, for all integers a, b, we can write $y_1^a y_2^b = \sum_{i=1}^a \lambda_{a,b,i} {y_1}^i + \sum_{i=1}^b \mu_{a,b,i} y_2^i$ with $\lambda_{a,b,i}$ and $\mu_{a,b,i}$ in $\mathcal{O}_L$, and $v_L(\lambda_{a,b,i}) \geq a + b - i$ and $v_L(\mu_{a,b,i}) \geq a + b - i$.

We study first $\alpha_1(fg)$ in $\mathcal{A}_L^0(X)$. We have $\alpha_1(fg) = \sum_{j \geq 0} f_j \alpha_1(y_1^j g)$. As $v_L(f_j) \geq n - t_1 - j/\tau$, all terms $f_j \alpha_1(y_1^j g)$ for $j \leq (n - t_0 - t_1)\tau$ contribute elements that are in $\pi_L^{t_0} \mathcal{O}_L[[y_1]]$. Consider $\alpha_1(y_1^j g)$ for $j > (n - t_0 - t_1)\tau$. It contributes to y_1^i with a coefficient of valuation $\geq j - i$. So all terms in y_1^i with $i \leq (n - t_0 - t_1)\tau - t_0$ are in $\pi_L^{t_0} \mathcal{O}_L[[y_1]]$. Thus $\alpha_1(fg)$ is in $(\pi_L^{t_0} \mathcal{O}_L[[y_1]] + y_1^{(n-t_2)\tau} \mathcal{O}_L[[y_1]]) \cap \mathcal{A}_E^0(X_1)$ for $t_2 = t_1 + 2t_0$. We have that $y_1^{(n-t_2)\tau} = \sum_i q_i z_1^i$ with $q_i \in \pi_L^{\max(0,(n-t_2-i)\tau)} \mathcal{O}_L[[y_1]]$. So finally, $\alpha_1(fg) \in (\pi_L^{t_0} \mathcal{O}_L[[y_1]] + z_1^{(n-t_3)} \mathcal{O}_L[[y_1]]) \cap \mathcal{A}_E^0(X_1)$ for $t_3 = t_2 + t_0$. From this we deduce that $\bar{\alpha}_1(fg)$ is a multiple of $\bar{z}_1^{n-t_3}$, and so is in $V_1^{n-t_3}$.

We see also that if $n \geq 2t_0 + t_1$, then $\alpha_2(fg)$ goes to 0 in V_2.

So we get the result we wanted by taking $t = t_3$ and any $N > \max(t_3, 2t_0 + t_1)$. □

4.3. *Combinatorial complexity of a standard subset with respect to a field.* We give another definition of complexity of a standard subset. It is defined in more cases than the algebraic complexity, as we do not require X to be defined over E to define the complexity of X with respect to E.

Definition. Let X be a standard subset, and E be a finite extension of $\mathbb{Q}_p$. We define an integer $\gamma_E(X)$ which we call combinatorial complexity of X.

Let D be a disk (open or closed). Let F be the field of definition of D over E. Let s be the smallest integer such that there exists an extension K of F, with $e_{K/F} = s$, and $K \cap D \neq \varnothing$. Let t be the smallest positive integer such that D can be written as $\{x, stv_E(x-a) \geq v_E(b)\}$ or as $\{x, stv_E(x-a) > v_E(b)\}$ for elements a, b in K. Then we set $\gamma_E(D) = st$. We also set $\gamma_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p)) = 0$.

Then if X is a standard subset, we define $\gamma_E(X)$ to be the sum of the combinatorial complexities of its defining disks. That is, if X is a connected standard subset, it can be written uniquely as $D_0 \setminus \bigcup_{j=1}^n D_j$ with D_0 an open disk or $D_0 = \mathbb{P}^1(\overline{\mathbb{Q}}_p)$, D_j a closed disk for $j > 0$, and the D_j are disjoint for $j > 0$. We set $\gamma_E(X) = \sum_{j=0}^n \gamma_E(D_j)$. If X be a standard subset, we can write uniquely $X = \bigcup_{i=1}^s X_i$, where X_i is a connected standard subset and the X_i are disjoint. Then we set $\gamma_E(X) = \sum_{i=1}^s \gamma_E(X_i)$.

Some properties of the combinatorial complexity.

Lemma 4.3.1. *Let X be a standard subset. Let F/E be a finite extension. Then $\gamma_E(X) \geq \gamma_F(X)$, with equality when F/E is unramified, or when F is contained in the field of definition of X.*

Proof. It suffices to show that $\gamma_E(D) \geq \gamma_F(D)$, with equality when F/E is unramified, for any disk D (open or closed), and then it is clear from the definition. □

Proposition 4.3.2. *Let X be a standard subset defined and irreducible over E, and write $X = \bigcup_{i=1}^s X_i$ its decomposition in connected standard subsets. Let E_1 be the field of definition of X_1 over E. Then $\gamma_E(X) = [E_1 : E]\gamma_{E_1}(X_1)$.*

Proof. We have $\gamma_E(X) = \sum_{i=1}^s \gamma_E(X_i) = \sum_{i=1}^s \gamma_{E_i}(X_i)$. Observe first that $\gamma_{E_i}(X_i)$ does not depend on i. Indeed, for all i there exists $\sigma \in G_E$ such that $\sigma(X_1) = X_i$ and $\sigma(E_1) = E_i$. Such a σ transforms an equation $\{x, v_E(x-a) \geq v_E(b)\}$ (or $\{x, v_E(x-a) > v_E(b)\}$) of a defining disk of X_1 to an equation defining the corresponding disk in X_i. Moreover, $s = [E_1 : E]$, as G_E acts transitively on the set of X_i because we have assumed X to be irreducible over E. □

4.4. *Comparison of complexities.* The important result is that the two definitions of complexity actually coincide when both are defined.

Theorem 4.4.1. *Let X be a standard subset defined over E. Then $c_E(X) = \gamma_E(X)$.*

Proof. We can assume that X is irreducible over E, as both multiplicities are additive with respect to irreducible standard subsets.

Write now $X = \bigcup X_i$, where the X_i are connected standard subsets, and let E_i be the field of definition of X_i. Then $c_E(X) = [E : E_1]c_{E_1}(X_1)$ by Proposition 4.1.2, and $\gamma_E(X) = [E : E_1]\gamma_{E_1}(X_1)$ by Proposition 4.3.2.

So we can assume that X is a connected standard subset defined over E. Note that $c_E(X) = c_{E'}(X)$ and $\gamma_E(X) = \gamma_{E'}(X)$ for any finite unramified extension E'/E by Propositions 4.1.4 and 4.3.1. So we can enlarge E if needed to an unramified extension, and we can assume that we have written $X = D \setminus \bigcup Y_i$ satisfying the hypotheses of Proposition 4.2.12. So we have $c_E(X) = c_E(D) + \sum_i c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus Y_i)$ by Proposition 4.2.12, and the analogous result for γ_E follows from the definition. So we need only prove the equality for these standard subsets.

Let D be a disk defined over E, of the form $\{x, v_E(x-a) > \lambda\}$. Let s be the minimal ramification degree of an extension F of E such that $F \cap D \neq \varnothing$, and $t > 0$ be the smallest integer such that $st\lambda \in (1/s)\mathbb{Z}$. Then $c_E(D) = \gamma_E(D) = st$. For $c_E(D)$ it follows from Proposition 4.2.10, and for $\gamma_E(D)$ it is the definition. So we get that $c_E(D) = \gamma_E(D)$.

Let now $X = \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus T$, where T is defined and irreducible over E, and $T = \bigcup_{i=1}^N D_i$, where the D_i are disjoint closed disks defined over a totally ramified extension of E. We have $\gamma_E(X) = \sum \gamma_E(D_i) = N\gamma_E(D_1)$ as the D_i are G_E-conjugates. Let F be the field of definition of D_1. Then $\gamma_E(X) = N\gamma_F(D_1) = N\gamma_F(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D_1)$. On the other hand, it follows from Proposition 4.2.11 that $c_E(X) = Nc_F(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D_1)$. The proof that $\gamma_F(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D_1) = c_F(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D_1)$ is now the same as in the case of a disk. So finally $c_E(X) = \gamma_E(X)$. □

From now on we only write c_E to denote either c_E or γ_E (so we can consider $c_E(X)$ even for X that is not defined over E, or for X a disjoint union of closed disks).

Corollary 4.4.2. *The complexity of X is at least equal to the number defining disks of X. It is at least equal to the number of connected components of X.*

4.5. *Finding a standard subset from a finite set of points.*

Approximations of a standard subset. Let $X = \bigcup_{n=1}^{N} \big(D_{n,0} \setminus \bigcup_{i=1}^{m_n} D_{n,i}\big)$ be a bounded standard subset, where the $D_{n,0} \setminus \bigcup_{i=1}^{m_n} D_{n,i}$ form the decomposition of X as a disjoint union of connected standard subsets, so that the disks $D_{i,j}$ are the defining disks of X.

For $J \subset \{1, \dots, N\}$ and $I_n \subset \{1, \dots, m_n\}$ for $n \in J$, we set $Y_{J,I} = \bigcup_{n\in J} \big(D_{n,0} \setminus \bigcup_{i\in I_n} D_{n,i}\big)$. This is a standard subset with $c_E(Y_{J,I}) \leq c_E(X)$ and equality if and only if $X = Y_{J,I}$. Such standard subsets are called approximations of X.

For a bounded connected standard subset X, written as $D(a, r)^- \setminus \Delta$ for some finite union of closed disks Δ, we define its outer part as $D(a, r)^-$. If X is any bounded standard subset, we define its outer part as the union of the outer parts of its connected components. Note that if X is defined over a field E, then so is its outer part X', and X' is an approximation of X, and it contains X.

We make similar definitions for unbounded standard subsets. If X is an unbounded standard subset, then we define its outer part to be $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$.

Main results.

Theorem 4.5.1. *Let X be a standard subset defined over E. Let m be an integer such that $c_E(X) \leq m$. Then there exists a finite set $\mathcal{E}$ of finite extensions of E, depending only on E and m, such that X is entirely determined by the sets $X \cap F$ for all extensions $F \in \mathcal{E}$.*

We can actually take the set $\mathcal{E}$ to be the set of all extensions of E of degree at most N for some N depending only on E and m. This theorem will be proved later, after we establish some preliminary results.

Corollary 4.5.2. *Let X be a standard subset of $D(0, 1)^-$ defined over E. Let m be an integer such that $c_E(X) \leq m$. Let $\varepsilon > 0$ be such that $D(x, \varepsilon)^- \subset X$ for all $x \in X$, and $D(x, \varepsilon)^- \cap X = \varnothing$ for all $x \notin X$. Then there exists a finite subset $\mathcal{P}$ of $D(0, 1)^-$, depending only on E, m, and ε, such that X is entirely determined by $X \cap \mathcal{P}$.*

Proof. Let $\mathcal{E}$ be the set of extensions of E given by Theorem 4.5.1. For each extension F of E which is in $\mathcal{E}$, the set $F \cap D(0, 1)^-$ can be covered by a finite number of open disks of radius ε, and we define a finite set $\mathcal{P}_F$ by taking an element in each of these disks. Then $X \cap F$ can be entirely determined from $X \cap \mathcal{P}_F$. We set $\mathcal{P}$ to be the union of the sets $\mathcal{P}_F$ for the extensions F of E that are in $\mathcal{E}$. This is a finite set, as $\mathcal{E}$ is finite, and X is determined by $X \cap \mathcal{P}$, as it is determined by the intersections $X \cap F$ for $F \in \mathcal{E}$ by Theorem 4.5.1. □

Remark 4.5.3. As is clear from the proof, the set $\mathcal{P}$ can be huge. However in practice for a given X we need only test points in a very small proportion of this subset.

Notation. Let $c \in \overline{\mathbb{Q}}_p$. If $a < b$ are rational numbers, denote by $A_c(a, b)$ the annulus $\{x, b < v_E(x-c) < a\}$. If a is a rational number, denote by $C_c(a)$ the circle $\{x, v_E(x-c) = a\}$.

If $t \in \mathbb{Q}$, let $\operatorname{denom}(t)$ be the denominator of t, that is, the smallest integer d such that $t \in (1/d)\mathbb{Z}$. Note that $[E(x) : E] \geq \operatorname{denom}(v_E(x))$.

Preliminaries.

Lemma 4.5.4. *Let $x, z \in \overline{\mathbb{Q}}_p$, with $\operatorname{denom}(v_E(x-z)) > te_{E(x)/E}$ for some integer t. Let D be a rational disk (open or closed) containing z but not x. Then $c_E(D) > t$.*

Proof. It is enough to prove that $c_{E(x)}(D) \geq t$, as $c_E(D) \geq c_{E(x)}(D)$. Let D be such a disk. As D does not contain x, we have $D \subset C_x(\lambda)$ where $\lambda = v_E(x-z)$. So for all $y \in D$, $\operatorname{denom}(v_{E(x)}(y-z)) = \operatorname{denom}(v_{E(x)}(x-z)) > t$, which implies that $e_{E(x,y)/E(x)} > t$. By the definition of combinatorial complexity, we deduce that $c_{E(x)}(D) > t$. □

Fix an integer B. We say that $\lambda \in \mathbb{Q}$ has a large denominator (with respect to B) if $\operatorname{denom}(\lambda) > B$, and a small denominator otherwise. The set of elements of $\mathbb{Q}$ with small denominator can be enumerated as a strictly increasing sequence $(t_i)_{i\in\mathbb{Z}}$.

Corollary 4.5.5. *Let $x \in \overline{\mathbb{Q}}_p$, $m \in \mathbb{Z}$, and X be a standard subset defined over E with $c_E(X) \leq m$. Let $B \geq me_{E(x)/E}$, and define the sequence (t_i) of rationals that have a small denominator with respect to B. Let $i \in \mathbb{Z}$, and let D be a defining disk of X (open or closed). Then either $A_x(t_i, t_{i+1}) \cap D = \varnothing$, or $A_x(t_i, t_{i+1}) \subset D$ (and then $x \in D$).*

Proof. Assume that $A_x(t_i, t_{i+1}) \cap D$ is not empty, and let $z \in \overline{\mathbb{Q}}_p$ be an element of this set. Then $\operatorname{denom}(v_E(x-z)) > B$, so in particular $\operatorname{denom}(v_E(x-z)) > me_{E(x)/E}$. By Lemma 4.5.4, either $c_E(D) > m$ or $x \in D$. As the first is impossible because $c_E(X) \leq m$, we get that $x \in D$. Assume that D is a closed disk (the case of an open disk being similar). So D is a set of the form $\{y, v_E(x-y) \geq t\}$ for some $t \in \mathbb{Q}$, or equivalently of the form $\{y, v_{E(x)}(x-y) \geq t'\}$ for $t' = te_{E(x)/E}$. We have $c_{E(x)}(D) = \operatorname{denom}(t')$ and $c_{E(x)}(D) \leq c_E(D) \leq c_E(X) \leq m$, so $\operatorname{denom}(t') \leq m$, and so t has a small denominator and hence is one of the t_j. As D contains an element of $A_x(t_i, t_{i+1})$, we see that $t_j \leq t_i$ and so $A_x(t_i, t_{i+1}) \subset D$. □

Proposition 4.5.6. *Let E be a finite extension of $\mathbb{Q}_p$. There exists a function ψ_E such that for any bounded standard subset X defined over E, if $c_E(X) \leq m$, then there exists an extension F of E with $[F : E] \leq \psi_E(m)$ and $X \cap F \neq \varnothing$.*

Lemma 4.5.7. *Let E be a finite extension of $\mathbb{Q}_p$. There exists a function ψ_E^0 such that for any open or closed disk D of $\overline{\mathbb{Q}}_p$ defined over E, if $c_E(D) \leq m$ then there exists an extension F of E with $[F : E] \leq \psi_E^0(m)$ and $D \cap F \neq \varnothing$ and the radius of D is in $|F^\times|$. For $m < p^2$ or $p = 2$ we can take $\psi_E^0(m) = m$ and consider only extensions F/E that are totally ramified.*

Proof. We write the proof for D open, the proof for D closed being nearly identical.

Let s be the minimal ramification degree of an extension K of E with $K \cap D \neq \varnothing$, and fix $a \in K \cap D$. Let t be the smallest positive integer such that D can be written as $\{x, stv_E(x-a) > v_E(b)\}$ for an element $b \in K$. So by definition $c_E(D) = st$. By Theorem 1.1.1, there exists an extension K of E with $e_{K/E} = s$ and $[K : E] \leq s^2$ and $K \cap D \neq \varnothing$. Then if F is a totally ramified extension of degree t of K, then F satisfies the conditions, and we have $[F : E] \leq s^2 t$. As $st \leq m$, this means that we can take $\psi_E^0(m) = m^2$.

Note that s is a power of p by Theorem 1.1.1, and $s \leq m$. So if $m < p^2$ then $s = 1$ or $s = p$, so we can take $[K : E] \leq s$ and K/E totally ramified instead of $[K : E] \leq s^2$, and so we can take $[F : E] \leq m$.

When $p = 2$ the result comes from applying Theorem 1.1.2 instead of 1.1.1. □

Proof of Proposition 4.5.6. We show first that there exists a function ψ_E^1 such that for all X a standard connected subset defined over E with $c_E(X) \leq m$, there exists an extension L of E with $[L : E] \leq \psi_E^1(m)$ and $X \cap L \neq \varnothing$.

Consider first the case where X is of the form $D(0, 1)^- \setminus Y$, with Y a disjoint union of closed disks. Then either $0 \notin Y$, in which case $0 \in X$, so $E \cap X \neq \varnothing$ and there is nothing more to do, or $0 \in Y$. We assume from now on that $0 \in Y$. Then $m > 1$ and we can write Y as $D(0, |a|)^+ \cup Z$, with Z a union of disjoint closed disks. By the additivity formula for complexity, we have that $c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D(0, |a|)^+) + c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus Z) \leq m - 1$. Let $\lambda \in \mathbb{Q}^\times$ with $\operatorname{denom}(\lambda) \geq m$. Then $Z \cap C_0(\lambda) = \varnothing$, by Lemma 4.5.4 and the fact that $c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus Z) < m$. Let $s = \operatorname{denom}(v_E(|a|))$. Then by definition of the combinatorial complexity, we have that $c_E(\mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus D(0, |a|)^+) = s$, so we know that $s < m$, and in particular $v_E(a) > 1/m$. So we see that $C_0(1/2m) \subset X$ by the two previous remarks. Let L be a totally ramified extension of E of degree $2m$, then $L \cap C_0(1/2m) \neq \varnothing$, and so $L \cap X \neq \varnothing$. So there exists an extension of E of degree at most $2m$ such that X has points in this extension.

Consider now the case where X is of the form $D \setminus Y$, but D is not necessarily $D(0, 1)^-$ anymore. By Lemma 4.5.7, there exists an extension F of E of degree at most $\psi_E^0(m)$ such that D contains a point in F and has a radius in $|F^\times|$. Moreover, $c_F(X) \leq c_E(X) \leq m$. By doing some affine transformation defined over F, we can reduce to the case where $D = D(0, 1)^-$, so we see that X contains a point in some extension L of F with $[L : F] \leq 2m$, and so X contains a point in L with $[L : E] \leq \psi_E^1(m)$, where $\psi_E^1(m) = 2m\psi_E^0(m)$.

Now we go back to the general case, where X is not necessarily connected. Write X as a disjoint union of irreducible components over E. Each of them has complexity at most m, and it is enough to find a point in one of them. So we can assume that X is irreducible over E.

Suppose now that X is irreducible over E: write $X = \bigcup_{i=1}^s X_i$, where the X_i form a G_E-orbit. Let F be the field of definition of X_1, and $s = [F : E]$. Then $c_E(X) = sc_F(X_1)$, so $c_F(X_1) \leq m' = \lfloor m/s \rfloor$. There exists an extension L of F of degree at most $\psi_F^1(m')$ such that $K \cap X_1 \neq \varnothing$. As L is an extension of E of degree at most $s\psi_F^1(m')$, we see that we can take $\psi_E(m) = \sup_{1 \leq s \leq m} \sup_{[F:E]=s} s\psi_F^1(\lfloor m/s \rfloor)$, which is finite as E has only a finite number of extensions of a given degree. □

By inverting the role of closed and open disks, we obtain the following statement:

Proposition 4.5.8. *Let E be a finite extension of $\mathbb{Q}_p$. There exists a function ϕ_E such that for any standard subset X defined over E and different from $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, if $c_E(X) \leq m$, then there exists an extension F of E with $[F : E] \leq \phi_E(m)$ and there exists an element of F that does not belong to X.*

Following the proofs above, we see that we can actually take $\psi_E^0(m) = m^2$, $\psi_E^1(m) = 2m^3$, and $\psi_E(m) = \phi_E(m) = 2m^3$.

Proof of Theorem 4.5.1. To help with the understanding of the method, we will explain the steps on the following example (for $p > 2$): let

$$X = (D_{1,0} \setminus D_{1,1}) \cup (D_{2,0} \setminus D_{2,1}) \cup D_{3,0} \cup D_{4,0},$$

where $D_{1,0} = D(0, 1)^-$, $D_{1,1} = D(0, |p|)^+$, $D_{2,0} = D(0, |p|)^-$, $D_{2,1} = D(0, |p^2|)^+$, $D_{3,0} = D(1/\sqrt{p}, 1)^-$, $D_{3,0} = D(-1/\sqrt{p}, 1)^-$. Here X is defined over $\mathbb{Q}_p$ and $c_{\mathbb{Q}_p}(X) = 6$ so we can take any $m \geq 6$.

We assume first that we know that X is bounded.

Write X as $\bigcup_{n=1}^{N} \bigl(D_{n,0} \setminus \bigcup_{i=1}^{m_n} D_{n,i}\bigr)$, where the $D_{n,0} \setminus \bigcup_{i=1}^{m_n} D_{n,i}$ form the decomposition of X as a disjoint union of connected standard subsets (in particular $D_{n,0}$ is open and $D_{n,i}$ is closed for $i > 0$). We number the disks so that the $D_{n,0}$ for $0 \leq n \leq M$ are maximal, that is, they are not included in any other defining disk of X, and the $D_{n,0}$ for $M < n \leq N$ are all included in another defining disk of X. In this way, the outer part of X is $\bigcup_{n=1}^{M} D_{n,0}$. In the example the outer part of X is $D(0, 1)^- \cup D(1/\sqrt{p}, 1)^- \cup D(-1/\sqrt{p}, 1)^-$.

Let $\mathcal{E}$ be the set of extensions of E of degree at most $2m^2 \max(\psi_E(m), \phi_E(m))^3$, where $c_E(X) \leq m$. Let $\mathcal{P} = \cup_{F \in \mathcal{E}} F$. We have to show that X can be recovered from the knowledge of $X \cap \mathcal{P}$.

We work by constructing a sequence (X_i) of approximations of X, such that each X_i is defined over E and is an approximation of X_{i+1} and $c_E(X_{i+1}) > c_E(X_i)$, so that at some point $X_i = X$ and we stop.

We first describe how to solve the following problem: given some fixed $x \in X$, with $[E(x) : E] \leq \psi_E(m)$, find the largest defining disk D of X containing x (note that D is necessarily open).

Let the sequence (t_i) be as before Corollary 4.5.5, with $B = m\psi_E(m)$. For each $i \in \mathbb{Z}$, let $\lambda_i = (t_i + t_{i+1})/2$. By construction λ_i has a large denominator, but $\operatorname{denom}(\lambda_i) \leq 2m^2\psi_E(m)^2$. Choose some $z_i \in \mathcal{P}$ such that $v_E(x - z_i) = \lambda_i$. This is possible as we can choose z_i in a totally ramified extension of $E(x)$ of degree at most $2m^2\psi_E(m)^2$. Then:

Lemma 4.5.9. *Let $i \in \mathbb{Z}$ be the smallest element such that $z_i \in X$. The largest defining disk of X containing x is the disk $D = \{z, v_E(x - z) > t_i\}$.*

Proof. Let D be the largest defining disk of X containing x. Then $z_i \in D$. Otherwise, z_i is contained in some (open) defining disk of X that does not contain x, which contradicts Corollary 4.5.5.

We can write D as $\{z, v_E(x - z) > t\}$ for some $t \in \mathbb{Q}$. Moreover $t = t_j$ for some j, as $c_E(D) \leq m$. Then $t \leq t_i$ as D contains z_i so $j \leq i$.

Let us show now that $j = i$. If $j < i$ then by definition of i, $z_j \notin X$. As $z_j \in D$, it means that $z_j \in D'$ for some closed defining set of X contained in D. But then $A_x(t_j, t_{j+1}) \subset D'$ by Corollary 4.5.5, so D' is of the form $\{y, v_E(y - x) \geq s\}$ for some $s \leq t_j$, which contradicts the fact that $D' \subset D$. $\square$

Next we describe how to solve the following problem: given some fixed $x \notin X$ but x in the outer part of X, with $[E(x) : E] \leq \phi_E(m)$, find the largest defining disk D of X containing x (note that D is necessarily closed).

Let the sequence (t_i) be as before Corollary 4.5.5, with $B = m\phi_E(m)$. Then D is of the form $\{z, v_E(x-z) \geq t_i\}$ for some i, as $c_{E(x)}(D) \leq m$. As x is in the outer part of X, there exists a largest defining disk D' of X containing x. Let $i_0 \in \mathbb{Z}$ be the integer such that $D' = \{z, v_E(x-z) > t_{i_0}\}$. For each i, we can find an element z_i in $C_x(t_i)$, with $[E(x, z_i) : E(x)] \leq m^2\phi_E(m)$, such that the $G_{E(x)}$-orbit of z_i contains at least m elements $z_i^{(1)} = z_i, \dots, z_i^{(m)}$ satisfying $v_E(z_i^{(j)} - z_i^{(\ell)}) = t_i$ for all $j \neq \ell$. We can find such a z_i as follows: first find y_i in a totally ramified extension of $E(x)$ of degree at most $m\phi_E(m)$ such that $v_E(x-y_i) = t_i$. Next, find u_i generating the unramified extension of $E(x)$ of degree m and such that $|u_i| = 1$ and the $G_{E(x)}$-conjugates of u_i have distinct reductions modulo p. Let $z_i = u_i y_i$. Then z_i satisfies the property we want, as it has m $G_{E(x,y_i)}$-conjugates $(z_i^{(\ell)})_{1\leq\ell\leq m}$ and they satisfy the property about $v_E(z_i^{(j)} - z_i^{(\ell)})$.

Lemma 4.5.10. *Let $i \in \mathbb{Z}$ be the smallest element $> i_0$ such that $z_i \notin X$. Then $D = \{z, v_E(x-z) \geq t_i\}$.*

Proof. Let $j \in \mathbb{Z}$, $j > i_0$ be such that $z_j \notin X$. As X is G_E-stable, this means that $z_j^{(\ell)}$ is not in X for all $1 \leq \ell \leq m$. As each $z_j^{(\ell)}$ is in the outer part of X (in fact in D'), it means that each $z_j^{(\ell)}$ is contained in some closed defining D_ℓ disk of X contained in D'. Then in fact there exists a closed defining disk of X containing x and all the $z_j^{(\ell)}$ for $1 \leq \ell \leq m$. Indeed, if a disk contains two of the $z_j^{(\ell)}$ it contains all of them and also x, due to the condition on the $v_E(z_j^{(\ell)} - z_j^{(\ell')})$. So if there is not a closed defining disk containing all the $z_j^{(\ell)}$, then the disks D_ℓ are all distinct, which gives that $c_E(X) > m$. So, in particular, z_j is contained in D.

On the other hand, assume that $D = \{z, v_E(x-z) \geq t_j\}$ for some j. Necessarily $j > i_0$ as $D \subset D'$. Then $z_j \notin X$: if z_j is in X, then so is $z_j^{(\ell)}$ for all $1 \leq \ell \leq m$. So each $z_j^{(\ell)}$ is contained in an open defining disk of X contained in D, and so as before there exists some open defining disk of X containing z_j and x and contained in D. But this is impossible as $v_E(x - z_j) = t_j$. □

We show now how to find the outer part X_1 of X, that is, $X_1 = \bigcup_{n=0}^{M} D_{n,0}$ in the notation of the beginning of the proof. Start with $X_1 = \varnothing$.

(1) Find some $x \in X \cap \mathcal{P}$ that is not in X_1, if there is one. If there is not, then X_1 is the outer part of X.

(2) Find the largest defining (open) disk D of X that contains x, using Lemma 4.5.9. Add to X_1 the G_E-orbit of D. Go back to the first step.

For the first step, Proposition 4.5.6 ensures that if X is not contained in X_1, we can find some element of $X \setminus X_1$ that is also in $\mathcal{P}$. For the second step, note that by construction the G_E-orbit of D is disjoint from X_1, and during the construction the set X_1 is always defined over G_E.

In the example: note that $X \cap \mathbb{Q}_p = \varnothing$, so we find points in $X \cap \mathbb{Q}_p(\sqrt{p})$. The fact that $1/\sqrt{p} \in X$ gives us the defining disk $D_{3,0} = D(1/\sqrt{p}, 1)^-$, and the fact that X is defined over $\mathbb{Q}_p$ gives the other defining disk $D_{4,0} = D(-1/\sqrt{p}, 1)^-$. The fact that $\sqrt{p} \in X$ gives us the defining disk $D_{1,0} = D(0, 1)^-$. At this point we have the outer part $X_1 = D_{1,0} \cup D_{3,0} \cup D_{4,0}$.

We now want to find $X_2 = \bigcup_{n=0}^{M} \big(D_{n,0} \setminus \bigcup_{i=0}^{m_n} D_{n,i}\big)$. Note that X_2 is defined over E.

The method as it is very similar to the method to find X_1. For each $D_{n,0}$, $n \leq M$, find if there is an element x that is in X_1 but not in X. If such an element exists, we can take it with $[E(x) : E] \leq \phi_E(m)$ by Proposition 4.5.8. Then we find the largest (closed) defining disk of X containing x and contained in $D_{n,0}$ using Lemma 4.5.10.

In the example $X_2 = (D_{1,0} \setminus D_{1,1}) \cup D_{3,0} \cup D_{4,0}$. The fact that p is in $D_{1,0}$ but not in X gives us the defining disk $D_{1,1}$.

Once X_2 is found, we have the decomposition $X = X_2 \cup X'$, where X_2 and X' are both approximations of X, and $X' = \bigcup_{n=M+1}^{N} \big(D_{n,0} \setminus \bigcup_{i=1}^{m_n} D_{n,i}\big)$. In particular, we have that $c_E(X') = c_E(X) - c_E(X_2)$. Let $m' = m - c_E(X_2)$. If $m' = 0$ then $X_2 = X$. Otherwise, we must find X', given that X' is defined over E and $c_E(X') \leq m'$. Moreover, as we know X_2 entirely, if we know $X \cap \mathcal{P}$ then we also know $X' \cap \mathcal{P}$. Applying the same steps as before, we can find an approximation of X' and so work recursively.

In the example, $X' = D_{2,0} \setminus D_{2,1}$, and as $c_{\mathbb{Q}_p}(X_2) = 4$ we get $m' = m - 4$, and we need to find X'.

Finally, we want to remove the hypothesis that we know that X is bounded. First, we can determine whether X is bounded by considering only $X \cap \mathbb{Q}_p$. If X is bounded apply the algorithm described. If X is not bounded, then set $X_1 = X$, and then apply the algorithm starting at the step where we determine X_2.

5. Application to potentially semistable deformation rings

5.1. ***Definition of the potentially semistable deformation rings.*** We recall the definition and some properties of the rings defined in [Kisin 2008]; see also [Kisin 2010].

Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a potentially semistable representation. Then we know from [Fontaine 1994] that we can attach to ρ a Weil-Deligne representation $\mathrm{WD}(\rho)$, that is, a smooth representation σ of the Weil group $W_{\mathbb{Q}_p}$ with values in $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, and a $\overline{\mathbb{Q}}_p$-linear, nilpotent endomorphism N of $\overline{\mathbb{Q}}_p^2$ such that $N\sigma(x) = p^{\deg x}\sigma(x)N$ for all $x \in W_{\mathbb{Q}_p}$. We say that σ is the extended type of ρ, and $\sigma_{|I_{\mathbb{Q}_p}}$ the inertial type of ρ, where $I_{\mathbb{Q}_p}$ is the inertia subgroup of $W_{\mathbb{Q}_p}$.

Kisin defines deformation rings that parametrize potentially semistable representations with fixed (distinct) Hodge–Tate weights and a fixed inertial type. However, this is not entirely adapted to our purposes: we would like each of these families of representations to be classified by one parameter (see Theorem 5.3.1). This is not the case for the rings defined by Kisin: for example, if we take the trivial inertial type, the deformation ring classifies a family of crystalline representations, and a family of semistable, noncrystalline representations, and we cannot classify all of these with a single parameter. So we introduce a refinement of Kisin's rings, where in some cases we will consider deformations with a fixed extended type instead, and use a refinement of Kisin's rings defined in [Rozensztajn 2015].

Definition of the Galois types. We make the following definition:

Definition 5.1.1. A Galois type of dimension 2 is one of the following representations with values in $\mathrm{GL}_2(\overline{\mathbb{Q}}_p)$:

(1) A scalar smooth representation $\tau = \chi \oplus \chi$ of $I_{\mathbb{Q}_p}$, such that χ extends to a character of $W_{\mathbb{Q}_p}$.

(2) A smooth representation $\tau = \chi_1 \oplus \chi_2$ of $I_{\mathbb{Q}_p}$, where both χ_1 and χ_2 extend to characters of $W_{\mathbb{Q}_p}$, and $\chi_1 \neq \chi_2$.

(3) If $p > 2$, a smooth representation $\tau = \chi_1 \oplus \chi_2$ of $W_{\mathbb{Q}_p}$, such that χ_1 and χ_2 have the same restriction to inertia, and $\chi_1(F) = p\chi_2(F)$ for any Frobenius element F in $W_{\mathbb{Q}_p}$.

(4) If $p > 2$, a smooth irreducible representation τ of $W_{\mathbb{Q}_p}$.

We call Galois types of the form (1) and (2) inertial types, and those of the forms (3) and (4) discrete series extended types. If ρ is a potentially semistable representation of $G_{\mathbb{Q}_p}$ of dimension 2 and $p > 2$, then we know from the classification of 2-dimension smooth representations of $W_{\mathbb{Q}_p}$ that either its inertial type is isomorphic to a Galois type of the form (1) or (2), or its extended type is isomorphic to a Galois type of the form (3) or (4) (if $p = 2$ there are other possibilities). Of course the possibilities are not mutually exclusive, as a representation that has its extended type of the form (3) also has its inertial type of the form (1), but we will define different deformation rings using these Galois types. Note that if the Galois type of ρ is of the form (2) or (4) then it is potentially crystalline (that is, the endomorphism N of the Weil–Deligne representation is zero), and that if ρ is potentially semistable but not potentially crystalline (that is, $N \neq 0$) then its Galois type is of the form (3).

Definition of the deformation rings.

Definition 5.1.2. A deformation data $(k, \tau, \bar{\rho}, \psi)$ is the data of

(1) an integer $k \geq 2$,

(2) a Galois type τ,

(3) a continuous representation $\bar{\rho}$ of $G_{\mathbb{Q}_p}$ of dimension 2, with trivial endomorphisms, over some finite extension $\mathbb{F}$ of $\mathbb{F}_p$,

(4) a continuous character $\psi : G_{\mathbb{Q}_p} \to \overline{\mathbb{Q}}_p^\times$ lifting $\det \bar{\rho}$ such that ψ and $\chi_{\mathrm{cycl}}^{k-1} \det \tau$ coincide.

If the type τ is a discrete series extended type, we will assume that $p > 2$.

Let $(k, \tau, \bar{\rho}, \psi)$ be a deformation data, and let E be a finite extension of $\mathbb{Q}_p$ over which τ and ψ are defined, and such that its residue field contains $\mathbb{F}$. Let $R(\bar{\rho})$ be the universal deformation ring of $\bar{\rho}$ over $\mathcal{O}_E$, it is a local noetherian complete $\mathcal{O}_E$-algebra. Let $R^\psi(\bar{\rho})$ the quotient of $R(\bar{\rho})$ that parametrizes deformations of determinant ψ. Kisin [2008] defined deformation rings $R^\psi(k, \tau, \bar{\rho})$ that are quotients of $R^\psi(\bar{\rho})$. We will also use a refinement of these rings introduced in [Rozensztajn 2015], which are better for our purposes in view of Theorem 5.3.1.

If the Galois type τ is an inertial type, we denote by $R^\psi(k, \tau, \bar{\rho})$ the ring classifying potentially crystalline representations with Hodge–Tate weights $(0, k-1)$, inertial type τ, determinant ψ with reduction isomorphic to $\bar{\rho}$, as defined by Kisin [2008].

If the Galois type τ is a discrete series extended type, we denote by $R^{\psi}(k,\tau,\bar{\rho})$ the $\mathcal{O}_E$-algebra which is a quotient of $R^{\psi}(\bar{\rho})$, classifying potentially semistable representations with Hodge–Tate weights $(0,k-1)$, extended type τ, determinant ψ with reduction isomorphic to $\bar{\rho}$ defined in [Rozensztajn 2015, 2.3.3].

We know that $R^{\psi}(k,\tau,\bar{\rho})$ is a complete flat local $\mathcal{O}_E$-algebra (in particular it has no p-torsion), such that Spec $R^{\psi}(k,\tau,\bar{\rho})[1/p]$ is formally smooth of dimension 1.

The characterizing property of these potentially semistable deformation rings is the following: There is a bijection between the maximal ideals of $R^{\psi}(k,\tau,\bar{\rho})[1/p]$ and the set of isomorphism classes of lifts ρ of $\bar{\rho}$ of determinant ψ, potentially crystalline of inertial type τ (resp. potentially semistable of extended type τ), and Hodge–Tate weights 0 and $k-1$. In this bijection, a maximal ideal x, corresponding to a finite extension E_x of E, corresponds to a representation $\rho_x : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(E_x)$ such that there exists a lattice giving the reduction $\bar{\rho}$ (as we consider only representations $\bar{\rho}$ that have trivial endomorphisms, the lattice is unique up to homothety if it exists, so there is no need to specify it).

The Breuil–Mézard conjecture [2002] gives us some information about these rings (proved in [Kisin 2009; Paškūnas 2015; 2016]; see also Rozensztajn 2015 for the cases of discrete series extended type):

Theorem 5.1.3. *Let $\bar{\rho}$ be a continuous representation of $G_{\mathbb{Q}_p}$ of dimension* 2, *with trivial endomorphisms. If $p=3$, assume that $\bar{\rho}$ is not a twist of an extension of* 1 *by ω, and let $(k,\tau,\bar{\rho},\psi)$ be a deformation data. Then there is an explicit integer $\mu_{\mathrm{aut}}(k,\tau,\bar{\rho})$ such that $e(R^{\psi}(k,\tau,\bar{\rho})/(\pi_E)) = \mu_{\mathrm{aut}}(k,\tau,\bar{\rho})$.*

For our purposes, what is important to know about $\mu_{\mathrm{aut}}(k,\tau,\bar{\rho})$ is that it can be easily computed in a combinatorial way, in terms of $\bar{\rho}$, k and τ. For more details on the formula for this integer see the introduction of [Breuil and Mézard 2002].

Definition 5.1.4. We will say that a representation $\bar{\rho}$ with trivial endomorphisms is good if it satisfies the hypothesis of Theorem 5.1.3, that is, if $p=3$ then $\bar{\rho}$ is not a twist of an extension of 1 by ω.

Note that the condition of trivial endomorphisms implies that $\bar{\rho}$ is not reducible with scalar semisimplification.

5.2. ***Rigid spaces attached to deformation rings.*** As $R^{\psi}(k,\tau,\bar{\rho})$ is a complete noetherian $\mathcal{O}_E$-algebra, the E-algebra $R^{\psi}(k,\tau,\bar{\rho})[1/p]$ is an E-quasiaffinoid algebra of open type as described in Section 3.1, and $R^{\psi}(k,\tau,\bar{\rho})$ is an $\mathcal{O}_E$-model of it. We denote by $\mathcal{X}^{\psi}(k,\tau,\bar{\rho})$ the rigid space attached to $R^{\psi}(k,\tau,\bar{\rho})[1/p]$ by the construction of Berthelot as recalled at the end of Section 3.1.

Let $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ be the minimal prime ideals of $R^{\psi}(k,\tau,\bar{\rho})$, and let $R_i = R^{\psi}(k,\tau,\bar{\rho})/\mathfrak{p}_i$. As $R^{\psi}(k,\tau,\bar{\rho})$ has no p-torsion by construction, the set of ideals $(\mathfrak{p}_i)$ is in bijection with the set of minimal prime ideals $(\mathfrak{p}_i')$ of $R^{\psi}(k,\tau,\bar{\rho})[1/p]$, with $R_i[1/p] = R^{\psi}(k,\tau,\bar{\rho})[1/p]/\mathfrak{p}_i'$. Let $\mathcal{X}_i$ be the rigid space attached to $R_i[1/p]$, then $\mathcal{X}^{\psi}(k,\tau,\bar{\rho}) = \cup_{i=1}^{n}\mathcal{X}_i$, and each $\mathcal{X}_i$ is an E-quasiaffinoid space of open type.

Let R_i^0 be the integral closure of R_i in $R_i[1/p]$, so that $R_i \subset R_i^0 \subset R_i[1/p]$ and R_i^0 is finite over R_i. As $R_i[1/p]$ is formally smooth, it is normal; hence so is R_i^0. Hence we see that R_i^0 is equal to the ring $\Gamma(\mathcal{X}_i,\mathcal{O}^0_{X_i})$ of analytic functions on $\mathcal{X}_i$ that are bounded by 1, that $R_i[1/p]$ is equal to the ring of bounded analytic functions on $\mathcal{X}_i$. We deduce that $R^{\psi}(k,\tau,\bar\rho)[1/p]$ is equal to the ring $\mathcal{A}_E(\mathcal{X}^{\psi}(k,\tau,\bar\rho))$ and $\bigoplus_i R_i^0$ is equal to its subring $\mathcal{A}_E^0(\mathcal{X}^{\psi}(k,\tau,\bar\rho))$.

5.3. *Results.*

Parameters on deformation spaces.

Theorem 5.3.1. *For all deformation data $(k,\tau,\bar\rho,\psi)$, there exist a finite extension $E=E(k,\tau,\bar\rho,\psi)$ of $\mathbb{Q}_p$ such that $\mathcal{X}^{\psi}(k,\tau,\bar\rho)$ is defined over E, and an analytic function $\lambda_{(k,\tau,\bar\rho,\psi)}:\mathcal{X}^{\psi}(k,\tau,\bar\rho)\to\mathbb{P}_E^{1,\mathrm{rig}}$ defined over E, satisfying the following condition: for all $\bar\rho$ and $\bar\rho'$, and (k,τ,ψ) such that $(k,\tau,\bar\rho,\psi)$ and $(k,\tau,\bar\rho',\psi)$ are deformation data, then $\lambda_{(k,\tau,\bar\rho,\psi)}(x)=\lambda_{(k,\tau,\bar\rho',\psi)}(x')$ if and only if x and x' correspond to isomorphic representations.*

In particular, each $\lambda_{(k,\tau,\bar\rho,\psi)}$ is injective on $\mathcal{X}^{\psi}(k,\tau,\bar\rho)(\overline{\mathbb{Q}}_p)$, and if there exist x and x' such that $\lambda_{(k,\tau,\bar\rho,\psi)}(x)=\lambda_{(k,\tau,\bar\rho',\psi)}(x')$, then $\bar\rho$ and $\bar\rho'$ have the same semisimplification.

The existence of the functions $\lambda_{(k,\tau,\bar\rho,\psi)}$ will be proved as Propositions 7.4.1, 7.5.3, 7.6.1, and 7.7.4, with an explanation of the choice of the field $E(k,\tau,\bar\rho,\psi)$.

Corollary 5.3.2. *In the conditions of Theorem 5.3.1, the map $\lambda_{(k,\tau,\bar\rho,\psi)}$ defines an open immersion of analytic spaces. The image of $\mathcal{X}^{\psi}(k,\tau,\bar\rho)(\overline{\mathbb{Q}}_p)$ by $\lambda_{(k,\tau,\bar\rho,\psi)}$ is a standard subset $X^{\psi}(k,\tau,\bar\rho)$ of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ that is defined over $E(k,\tau,\bar\rho,\psi)$. Moreover we have that $\mathcal{A}^0_{E(k,\tau,\bar\rho,\psi)}(\mathcal{X}^{\psi}(k,\tau,\bar\rho))=\mathcal{A}^0_{E(k,\tau,\bar\rho,\psi)}(X^{\psi}(k,\tau,\bar\rho))$.*

Proof. Let $\mathcal{X}$ be a rigid analytic space that is smooth of dimension 1, and let $f:\mathcal{X}\to\mathbb{P}^{1,\mathrm{rig}}$ be a rigid map that induces an injective map $\mathcal{X}(\overline{\mathbb{Q}}_p)\to\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. Then f is an open immersion. Indeed, this follows from the well-known fact that an analytic function f from some open disk D to $\overline{\mathbb{Q}}_p$ that is injective on $\overline{\mathbb{Q}}_p$-points satisfies $f'(x)\neq 0$ for all $x\in D$. Now we apply this to $\mathcal{X}=\mathcal{X}^{\psi}(k,\tau,\bar\rho)$ and $f=\lambda_{(k,\tau,\bar\rho,\psi)}$. We write λ for $\lambda_{(k,\tau,\bar\rho,\psi)}$. Let $X=X^{\psi}(k,\tau,\bar\rho)$ be the image of $\mathcal{X}(\overline{\mathbb{Q}}_p)$ by λ. It is clear that X is defined over E.

Assume first that X is contained in some bounded subset of $\overline{\mathbb{Q}}_p$ (this is automatic when τ is an inertial type; see Sections 7.4 and 7.5). Then λ is an analytic open immersion from the quasiaffinoid space $\mathcal{X}$ to some quasiaffinoid space $\mathcal{D}$ attached to an open disk in $\mathbb{A}^{1,\mathrm{rig}}$. By Corollary 3.4.5, X is a bounded standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, and λ induces an isomorphism between $\mathcal{A}_E(X)$ and $\mathcal{A}_E(\mathcal{X})$, and between $\mathcal{A}_E^0(X)$ and $\mathcal{A}_E^0(\mathcal{X})$.

We do not assume anymore that X is contained in some bounded subset of $\overline{\mathbb{Q}}_p$. By the Breuil–Mézard conjecture, there is an infinite number of $\bar\rho'$ with trivial endomorphisms such that $X'=X^{\psi}(k,\tau,\bar\rho')$ is nonempty. For such a $\bar\rho'$, X' contains a disk $D(a,r)^-$ for some $r>0$ as it is open. For any $\bar\rho'$ with trivial endomorphisms such that its semisimplification is not the same as the semisimplification of $\bar\rho$, we have that the intersection of X and X' is empty. So there exists some $a\in\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ and $r>0$ such that $D(a,r)^-\cap X=\varnothing$. Let u be the rational function $u(x)=1/(x-a)$, so that it sends a to ∞; then $u(X)$ is a bounded subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. This means that $u\circ\lambda$ is a bounded analytic function on $\mathcal{X}$. So we can apply the same reasoning as before to show that $u(X)$ is a bounded standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, and so X is a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. □

Complexity bounds. Recall that $X^\psi(k,\tau,\bar\rho)$ denotes the subset $\lambda_{(k,\tau,\bar\rho,\psi)}(\mathcal{X}^\psi(k,\tau,\bar\rho)(\overline{\mathbb{Q}}_p))$ of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$. Now we give more information on the sets $X^\psi(k,\tau,\bar\rho)$.

Theorem 5.3.3. *Let $(k,\tau,\bar\rho,\psi)$ be a deformation data. Then $X^\psi(k,\tau,\bar\rho)$ is a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$, defined over $E = E(k,\tau,\bar\rho,\psi)$, with $c_E(X^\psi(k,\tau,\bar\rho)) \le e(R^\psi(k,\tau,\bar\rho)/(\pi_E))$. In particular, $c_E(X^\psi(k,\tau,\bar\rho)) \le \mu_{\text{aut}}(k,\tau,\bar\rho)$ if $\bar\rho$ is good.*

Remark 5.3.4. Note that the right-hand side of the inequality does not depend on the choice of E, whereas the left-hand side can get smaller when E has more ramification. In particular, to get a statement as strong as possible we want to take E with as little ramification as possible.

Proof. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ be the minimal prime ideals of $R^\psi(k,\tau,\bar\rho)$, let $R_i = R^\psi(k,\tau,\bar\rho)/\mathfrak{p}_i$ and let R_i^0 be the integral closure of R_i in $R_i[1/p]$ as in Section 5.2. Let $\mathcal{X}_i$ be the rigid space attached to $R_i[1/p]$, then $X^\psi(k,\tau,\bar\rho)$ is the disjoint union of the $X_i = \lambda(\mathcal{X}_i(\overline{\mathbb{Q}}_p))$, and each of the X_i is a standard subset of $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ which is defined over E. Then $\mathcal{A}_E^0(X_i) = R_i^0$, so $c_E(X_i) = [k_{X_i,E} : k_E]\bar e(R_i^0)$ by definition. Note that $k_{X_i,E}$ is the residue field of R_i^0, while k_E is the residue field of R_i. So by Proposition 2.2.2, we have $c_E(X_i) \le \bar e(R_i)$. So we get $c_E(X^\psi(k,\tau,\bar\rho)) \le \sum_{i=1}^n \bar e(R_i)$. Finally, $\sum_{i=1}^n \bar e(R_i) = \bar e(R^\psi(k,\tau,\bar\rho))$ by [Breuil and Mézard 2002, Lemme 5.1.6]. □

Note that in the proof above, the decomposition $X^\psi(k,\tau,\bar\rho) = \bigcup_i X_i$ is the decomposition of $X^\psi(k,\tau,\bar\rho)$ in standard subsets that are defined and irreducible over E. So we also have the following result:

Proposition 5.3.5. *Let $X^\psi(k,\tau,\bar\rho) = \bigcup_i X_i$ be the decomposition of $X^\psi(k,\tau,\bar\rho)$ in standard subsets that are defined and irreducible over E. Then $R^\psi(k,\tau,\bar\rho)[1/p] = \bigoplus_i \mathcal{A}_E(X_i)$.*

Finally, we have the following result:

Theorem 5.3.6. *Let $(k,\tau,\bar\rho,\psi)$ be a deformation data, and assume that $\bar\rho$ is good. There exists a finite set $\mathcal{E}$ of finite extensions of $E = E(k,\tau,\bar\rho,\psi)$, depending only on $\mu_{\text{aut}}(k,\tau,\bar\rho)$, such that $X^\psi(k,\tau,\bar\rho)$ is determined by the sets $X^\psi(k,\tau,\bar\rho)\cap F$ for $F \in \mathcal{E}$.*

Proof. This is a consequence of Theorem 5.1.3 and Theorem 4.5.1, where we take $m = \mu_{\text{aut}}(k,\tau,\bar\rho)$. □

5.4. ***The case of crystalline deformation rings.*** We are interested here in the case of the deformation ring of crystalline representations, that is, we take τ to be the trivial representation. This case is of particular interest as we are able to deduce additional information.

In this case $R^\psi(k,\text{triv},\bar\rho)$ is zero unless ψ is a twist of χ_{cycl}^{k-1} by an unramified character. Note that $R^\psi(k,\text{triv},\bar\rho)$ and $R^{\psi'}(k,\text{triv},\bar\rho)$ are isomorphic as long as ψ/ψ' is an unramified character with trivial reduction modulo p. So without loss of generality we will assume from now on that $\psi = \chi_{\text{cycl}}^{k-1}$ and $\det\bar\rho = \omega^{k-1}$.

We denote by $R(k,\bar\rho)$ the ring $R^{\chi_{\text{cycl}}^{k-1}}(k,\text{triv},\bar\rho)$. It parametrizes the set of crystalline lifts of $\bar\rho$ with determinant χ_{cycl}^{k-1} and Hodge–Tate weights 0 and $k-1$. We also write $X(k,\bar\rho)$ for $X^{\chi_{\text{cycl}}^{k-1}}(k,\text{triv},\bar\rho)$ and $\mu_{\text{aut}}(k,\bar\rho)$ for $\mu_{\text{aut}}(k,\text{triv},\bar\rho)$.

Let $\mathbb{F}$ be the extension of $\mathbb{F}_p$ over which $\bar{\rho}$ is defined (so $\mathbb{F} = \mathbb{F}_p$ when $\bar{\rho}$ is irreducible), and $E(\bar{\rho})$ the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$ (so $E(\bar{\rho}) = \mathbb{Q}_p$ when $\bar{\rho}$ is irreducible). Then $R(k, \bar{\rho})$ is an $\mathcal{O}_{E(\bar{\rho})}$-algebra with residue field $\mathbb{F}$.

Classification of filtered ϕ-modules. For $a_p \in \overline{\mathbb{Z}}_p$ and F a finite extension of $\mathbb{Q}_p$ containing a_p, we define a filtered ϕ-module D_{k,a_p} as follows:

$$D_{k,a_p} = Fe_1 \oplus Fe_2, \qquad \phi(e_1) = p^{k-1}e_2, \qquad \phi(e_2) = -e_1 + a_p e_2,$$

$$\operatorname{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0, \\ Fe_1 & \text{if } 1 \leq i \leq k-1, \\ 0 & \text{if } i \geq k. \end{cases}$$

Denote by V_{k,a_p} the crystalline representation such that $\mathrm{D}_{\mathrm{crys}}(V_{k,a_p}^*) = D_{k,a_p}$. Then V_{k,a_p} has Hodge–Tate weights $(0, k-1)$ and determinant $\chi_{\mathrm{cycl}}^{k-1}$. Moreover, V_{k,a_p} is irreducible if $v_p(a_p) > 0$, and a reducible nonsplit extension of an unramified character by the product of an unramified character by $\chi_{\mathrm{cycl}}^{k-1}$ if $v_p(a_p) = 0$. We have the following well-known result:

Lemma 5.4.1. *Let V be a crystalline representation with Hodge–Tate weights* $(0, k-1)$ *and determinant $\chi_{\mathrm{cycl}}^{k-1}$. If V is irreducible there exists a unique $a_p \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$ such that V is isomorphic to V_{k,a_p}. If V is reducible nonsplit there exists a unique $a_p \in \overline{\mathbb{Z}}_p^\times$ such that V is isomorphic to V_{k,a_p}.*

The parameter a_p. We show in Proposition 7.4.1 that the parameter a_p actually defines a rigid analytic function. This is the function that plays the role of λ of Theorem 5.3.1 for crystalline representations.

From Theorem 5.3.1 we can already deduce some results. It is a well-known conjecture (see [Buzzard and Gee 2016, Conjecture 4.1.1]) that if $p > 2$, k is even, and $v(a_p) \notin \mathbb{Z}$, then $\overline{V}_{k,a_p}^{ss}$ is irreducible. From this we get:

Proposition 5.4.2. *Let $p > 2$, k even, $n \in \mathbb{Z}_{\geq 0}$. If the conjecture above is true, then there is an irreducible representation $\bar{\rho}$ (depending on n, k) such that the set $\{x, n < v_p(x) < n+1\}$ is contained in $X(k, \bar{\rho})$.*

Proof. If the conjecture holds, then the set $C = \{x, n < v_p(x) < n+1\}$ is the union of the $C \cap X(k, \bar{\rho})$ for $\bar{\rho}$ irreducible. So we have written C as a finite disjoint union of standard subsets, which means that one of these subsets is equal to C. □

Reduction and semisimplification. We now want to show that the case of crystalline deformation rings is accessible to numerical computations. However we must change slightly our setting. Indeed, we can compute numerically only the semisimplified reduction of V_{k,a_p}. So we need to express the result of Theorem 5.3.3 in terms of semisimple representations instead of in terms of representations with trivial endomorphisms.

Let $\bar{r}$ be a semisimple representation of $G_{\mathbb{Q}_p}$ with values in $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$. We define $Y(k, \bar{r})$ to be the set $\{a_p \in D(0,1)^-, \overline{V}_{k,a_p}^{ss} = \bar{r}\}$. Let $\bar{\rho}$ be a representation of $G_{\mathbb{Q}_p}$ with trivial endomorphisms with semisimplification isomorphic to $\bar{r}$. Let $X'(k, \bar{\rho}) = X(k, \bar{\rho}) \cap D(0,1)^-$. This means we are only interested in elements in $X(k, \bar{\rho})$ that correspond to irreducible representations $V_{k,x}$. Then we have that $X'(k, \bar{\rho}) \subset Y(k, \bar{r})$. We want to know when this is an equality.

Definition 5.4.3. We say that a representation $\bar{\rho}$ with trivial endomorphisms is nice if either $\bar{\rho}$ is irreducible, or $\bar{\rho}$ is a nonsplit extension of α by β where $\beta/\alpha \notin \{1, \omega\}$.

We say that a semisimple representation $\bar{r}$ is nice if $\bar{r}$ is not scalar, and in addition when $p = 3$ if $\bar{r}$ is not of the form $\alpha \oplus \beta$ with $\alpha/\beta \neq \omega$.

Note that any $\bar{\rho}$ with trivial endomorphisms that is nice is also good (in the sense of Definition 5.1.4); hence satisfies the hypotheses of Theorem 5.1.3. Moreover, its semisimplification is a nice semisimple representation. If $\bar{r}$ is semisimple and nice, then there exists a nice $\bar{\rho}$ with trivial endomorphisms such that $\bar{\rho}^{ss} = \bar{r}$, so we have $Y(k, \bar{r}) = X'(k, \bar{\rho})$. We can choose such a $\bar{\rho}$ so that in addition, $E(\bar{\rho}) = E(\bar{r})$.

Proposition 5.4.4. *Let $\bar{\rho}$ be a nice representation with trivial endomorphisms. Then $X'(k, \bar{\rho}) = Y(k, \bar{\rho}^{ss})$.*

Proof. The result is clear when $\bar{\rho}$ is irreducible. Recall that $\dim \mathrm{Ext}^1(\alpha, \beta) > 1$ if and only if $\beta/\alpha \in \{1, \omega\}$. Suppose that $\bar{\rho}$ is an extension of α by β where $\beta/\alpha \notin \{1, \omega\}$. Let $x \in Y(k, \bar{\rho}^{ss})$. By Ribet's lemma, there exists a $G_{\mathbb{Q}_p}$-invariant lattice $T \subset V_{k,x}$ such that $\overline{T}$ is a nonsplit extension of α by β, and so is isomorphic to $\bar{\rho}$. This means that $x \in X'(k, \bar{\rho})$. □

We know some information about the difference between $X(k, \bar{\rho})$ and $X'(k, \bar{\rho})$:

Proposition 5.4.5. *Let $\bar{\rho}$ be a representation of $G_{\mathbb{Q}_p}$ with trivial endomorphisms. If $\bar{\rho}$ is not an extension of* $\mathrm{unr}(u)$ *by* $\mathrm{unr}(u^{-1})\omega^n$ *for some n which is equal to $k-1$ modulo $p-1$, and $u \in \overline{\mathbb{F}}_p^\times$, then $X(k, \bar{\rho}) \subset D(0, 1)^-$. If $\bar{\rho}$ is an extension of* $\mathrm{unr}(u)$ *by* $\mathrm{unr}(u^{-1})\omega^n$ *for some $u \in \overline{\mathbb{F}}_p^\times$ and $0 \leq n < p-1$, and $n = k-1$ modulo $p-1$, and $u \notin \{\pm 1\}$ if $n = 0$ or $n = 1$, then $X(k, \bar{\rho}) \cap \{x, |x| = 1\}$ is the disk $\{x, \bar{x} = u\}$.*

Proof. For $a_p \in \overline{\mathbb{Z}}_p^\times$, the representation V_{k,a_p} is the unique crystalline nonsplit extension of $\mathrm{unr}(u)$ by $\mathrm{unr}(u^{-1})\chi_{\mathrm{cycl}}^{k-1}$, where $u \in \overline{\mathbb{Z}}_p^\times$ and u and $u^{-1}p^{k-1}$ are the roots of $X^2 - a_p X + p^{k-1}$. In particular, for any invariant lattice $T \subset V_{k,a_p}$ such that $\overline{T}$ is nonsplit, we get that $\overline{T}$ is an extension of $\mathrm{unr}(\bar{u})$ by $\mathrm{unr}(\bar{u}^{-1})\omega^{k-1}$. So $X(k, \bar{\rho})$ does not meet $\{x, |x| = 1\}$ unless $\bar{\rho}$ has the specific form given. Moreover, $\bar{u} = \overline{a_p}$. So $X(k, \bar{\rho}) \cap \{x, |x| = 1\} \subset \{x, \bar{x} = u\}$. If $\bar{\rho}$ is an extension of $\mathrm{unr}(u)$ by $\mathrm{unr}(u^{-1})\omega^n$ for some $u \in \overline{\mathbb{F}}_p$ and $0 \leq n < p-1$, the conditions on (n, u) imply there is a unique nonsplit extension of $\mathrm{unr}(u)$ by $\mathrm{unr}(u^{-1})\omega^n$, and so $X(k, \bar{\rho}) \cap \{x, |x| = 1\} = \{x, \bar{x} = u\}$ □

Corollary 5.4.6. *Let $\bar{\rho}$ be a representation with trivial endomorphisms. Let $X'(k, \bar{\rho}) = X(k, \bar{\rho}) \cap D(0, 1)^-$. If $\bar{\rho}$ is not an extension* $\mathrm{unr}(u)$ *by* $\mathrm{unr}(u^{-1})\omega^n$ *for some n which is equal to $k-1$ modulo $p-1$, then $X'(k, \bar{\rho}) = X(k, \bar{\rho})$ and $c_E(X'(k, \bar{\rho})) \leq \bar{e}(R(k, \bar{\rho}))$. If $\bar{\rho}$ is an extension* $\mathrm{unr}(u)$ *by* $\mathrm{unr}(u^{-1})\omega^n$ *for some n which is equal to $k-1$ modulo $p-1$, and $u \notin \{\pm 1\}$ if $n = 0$ or $n = 1$, then $c_E(X'(k, \bar{\rho})) \leq \bar{e}(R(k, \bar{\rho})) - 1$.*

Proof. The first part is clear by Proposition 5.4.5.

For the second part, we can write $X(k, \bar{\rho})$ as a disjoint union of $X'(k, \bar{\rho})$ and $X^+(k, \bar{\rho}) = X(k, \bar{\rho}) \cap \{x, |x| = 1\}$, and both are standard subsets defined over E, so $c_E(X(k, \bar{\rho})) = c_E(X'(k, \bar{\rho})) + c_E(X^+(k, \bar{\rho}))$. By Proposition 5.4.5, $c_E(X^+(k, \bar{\rho})) = 1$ under the hypotheses, hence the result. □

Local constancy results. We recall the following results:

Proposition 5.4.7. *Let $a_p \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. If $a_p \neq 0$, then*

$$\overline{V}^{ss}_{k,a_p} \simeq \overline{V}^{ss}_{k,a'_p}$$

for all a'_p such that $v_p(a_p - a'_p) > 2v_p(a_p) + \lfloor p(k-1)/(p-1)^2 \rfloor$. Further, $\overline{V}^{ss}_{k,a_p} \simeq \overline{V}^{ss}_{k,0}$ for all a_p with $v_p(a_p) > \lfloor (k-2)/(p-1) \rfloor$.

Proof. The result for $a_p \neq 0$ is Theorem A of [Berger 2012]. The result for $a_p = 0$ is the main result of [Berger et al. 2004]. □

Computation of $Y(k, \bar{r})$. We explain now how we can compute numerically the sets $Y(k, \bar{r})$ for $\bar{r}$ semisimple and nice (and hence the sets $X(k, \bar{\rho})$ for $\bar{\rho}$ with nice semisimplification).

From Corollary 5.4.6 we deduce (using the fact that a nice representation with trivial endomorphisms is good and so satisfies the hypotheses of Theorem 5.1.3):

Proposition 5.4.8. *Suppose that $\bar{r}$ is a nice semisimple representation, and let $\bar{\rho}$ be a nice representation with trivial endomorphisms with $\bar{\rho}^{ss} = \bar{r}$. Then $Y(k, \bar{r})$ is a standard subset of $D(0,1)^-$ defined over $E = E(\bar{r})$, with $c_E(Y(k, \bar{r})) \leq \mu_{\mathrm{aut}}(k, \bar{\rho})$. Moreover if $\bar{\rho}$ is an extension of an unramified character by another character then $c_E(Y(k, \bar{r})) \leq \mu_{\mathrm{aut}}(k, \bar{\rho}) - 1$.*

Theorem 5.3.6 specializes here to:

Theorem 5.4.9. *Let $\bar{r}$ be a nice semisimple representation. Then there exists a finite set $\mathcal{E}$ of finite extensions of $E = E(\bar{r})$, depending only on k and $\bar{r}$, such that $Y(k, \bar{r})$ is determined by the sets $Y(k, \bar{r}) \cap F$ for $F \in \mathcal{E}$.*

Proof. This is Theorem 4.5.1, where we take for E the field $E(\bar{r})$, and for m the bound given by Proposition 5.4.8, that is $m = \mu_{\mathrm{aut}}(k, \bar{\rho})$ or $\mu_{\mathrm{aut}}(k, \bar{\rho}) - 1$, where $\bar{\rho}$ is some nice representation with $\bar{\rho}^{ss} = \bar{r}$. □

Theorem 5.4.10. *Let $\bar{r}$ be a nice semisimple representation. Then there exists a finite set of points $\mathcal{P} \subset D(0,1)^-$, depending only on k and $\bar{r}$, such that $Y(k, \bar{r})$ is determined by $Y(k, \bar{r}) \cap \mathcal{P}$.*

Proof. This is Corollary 4.5.2, where we take for E the field $E(\bar{r})$, for m the bound given by Proposition 5.4.8, and for ε we can take the norm of an element of valuation $\lfloor 3p(k-1)/(p-1)^2 \rfloor$ by Proposition 5.4.7. □

Corollary 5.4.11. *Let $\bar{\rho}$ be a nice representation with trivial endomorphisms. Then there exists a finite set of points $\mathcal{P} \subset D(0,1)^-$, depending only on k and $\bar{\rho}^{ss}$, such that $X(k, \bar{\rho})$ is determined by $X(k, \bar{\rho}) \cap \mathcal{P}$.*

Proof. Let $\bar{r} = \bar{\rho}^{ss}$. Then $\bar{r}$ is a nice semisimple representation, so we can apply Theorem 5.4.10 to compute $Y(k, \bar{r}) = X(k, \bar{\rho}) \cap D(0,1)^-$, and Proposition 5.4.5 to determine the rest of $X(k, \bar{\rho})$. □

As a consequence, if we are able to compute $\overline{V}^{ss}_{k,a_p}$ for given p, k, a_p, then we can compute $Y(k,\bar{r})$ for $\bar{r}$ nice in a finite number of such computations, bounded in terms of $E(\bar{r})$ and k. We give some examples of such computations in Section 6.

We give a last application of these results: It follows from the formula giving $\mu_{\mathrm{aut}}(k,\bar{\rho})$ that there exists an integer $m(k)$, depending only on k, such that $\mu_{\mathrm{aut}}(k,\bar{\rho}) \le m(k)$ for all $\bar{\rho}$. The optimal value for $m(k)$ is of the order of $4k/p^2$ when k is large.

In general, the value of $\overline{V}^{ss}_{k,a_p}$ depends on more information than just the valuation of a_p. But there are some cases where it depends only on $v_p(a_p)$:

Corollary 5.4.12. *Fix k, and let m be an integer such that $m \ge \bar{e}(R(k,\bar{\rho}))$ for all nice $\bar{\rho}$ with trivial endomorphisms. Let a and b be rational numbers such that for all rational c between a and b, the denominator of c is strictly larger than m. Then, for all a_p with $a < v_p(a_p) < b$, either $\overline{V}^{ss}_{k,a_p}$ is not nice, or $\overline{V}^{ss}_{k,a_p}$ is constant on the annulus $A_0(a,b)$.*

In particular, let $c \in \mathbb{Q}$ with denominator strictly larger than m. Then, for all a_p with $v_p(a_p) = c$, either $\overline{V}^{ss}_{k,a_p}$ is not nice, or $\overline{V}^{ss}_{k,a_p}$ is constant on the circle $C_0(c)$.

Note that if $p > 3$ and k is even, $\overline{V}^{ss}_{k,a_p}$ is always nice.

Proof. Suppose that there exists at least an a_p in $A_0(a,b)$ such that $\bar{r} = \overline{V}^{ss}_{k,a_p}$ is nice. Then $c_E(Y(k,\bar{r})) \le m$ for $E = E(\bar{\rho})$ which is an unramified extension of $\mathbb{Q}_p$. So we can apply Corollary 4.5.5: the annulus $A_0(a,b)$ is a subset of $Y(k,\bar{r})$. □

6. Numerical examples

We give some numerical examples for the deformations rings of crystalline representations. We have computed some examples of $X(k,\bar{\rho})$ using Theorem 5.4.10 and a computer program written in [SAGE] that implements the algorithm described in [Rozensztajn 2018]. We also used the fact that $\overline{V}^{ss}_{k,a_p}$ is known for $v_p(a_p) < 2$ in all cases for $p \ge 5$, by the results of [Buzzard and Gee 2009; 2013; Ganguli and Ghate 2015; Bhattacharya and Ghate 2015; Bhattacharya et al. 2018; Ghate and Rai 2019] which reduces the number of computations that are necessary to determine $X(k,\bar{\rho})$.

We make the following remark: let $\bar{\rho}$ be a representation such that $\bar{\rho} \otimes \mathrm{unr}(-1)$ is isomorphic to $\bar{\rho}$. Then $X(k,\bar{\rho})$ is invariant by $x \mapsto -x$. Indeed, $V_{k,-a_p}$ is isomorphic to $V_{k,a_p} \otimes \mathrm{unr}(-1)$. This applies in particular when $\bar{\rho}$ is irreducible.

6.1. ***Observations for $p = 5$.*** We have computed $X(k,\bar{\rho})$ for $p = 5$, k even, $k \le 102$, or k odd and $k \le 47$, and $\bar{\rho}$ irreducible (so in this case we have $E(\bar{\rho}) = \mathbb{Q}_p$).

We summarize here some observations from these computations:

(1) In each case, we have $\overline{V}^{ss}_{k,a_p} = \overline{V}^{ss}_{k,0}$ for all a_p with $v_p(a_p) > \lfloor (k-2)/(p+1) \rfloor$, and not only $v_p(a_p) > \lfloor (k-2)/(p-1) \rfloor$ which is the value predicted by [Berger et al. 2004].

(2) In each case, we have $c_{\mathbb{Q}_p}(X(k,\bar{\rho})) = \bar{e}(R(k,\bar{\rho}))$, that is, the inequality of Proposition 5.4.8 is an equality.

(3) Each defining disk D of a $X(k, \bar{\rho})$ has $\gamma_{\mathbb{Q}_p}(D) = 1$.

(4) Each defining disk D of a $X(k, \bar{\rho})$ is defined over an extension of $\mathbb{Q}_p$ of degree at most 2, which is unramified if k is even and totally ramified if k is odd.

(5) For each defining disk D of a $X(k, \bar{\rho})$, either $0 \in D$, or D is included in the set $\{x, v_p(x) = n\}$ for some $n \in \mathbb{Z}_{\geq 0}$ if k is even, and in the set $\left\{x, v_p(x) = n + \frac{1}{2}\right\}$ for some $n \in \mathbb{Z}_{\geq 0}$ if k is odd.

It would be interesting to know which of these properties hold in general. Property (1) is expected to be in fact true for all p and k, but nothing is known about the other properties. We comment further on Property (2) in Section 6.4.

6.2. ***Some detailed examples.*** Let $p = 5$. Let $\bar{r}_0 = \operatorname{ind} \omega_2$ and $\bar{r}_1 = \operatorname{ind} \omega_2^3$, and $\bar{r}(n) = \bar{r} \otimes \omega^n$ for all n. We describe a few examples of sets $X(k, \bar{r})$. In each case, the sets given contain all the values of a_p for which $\overline{V}_{k,a_p}^{ss}$ is irreducible. We also give the generic fibers of the deformation rings.

(a) *The case* $k = 26$.

- $$X(26, \bar{r}_0) = \{x, v_p(x) < 2\} \cup \{x, v_p(x) > 2\},$$

with $c_{\mathbb{Q}_p}(X(26, \bar{r}_0)) = 3$ and $R(26, \bar{r}_0)[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p) \times (\mathbb{Z}_p[[X, Y]]\,/\,(XY - p) \otimes \mathbb{Q}_p)$..

- $$X(26, \bar{r}_0(2)) = \{x, v_p(x - a) > 3\} \cup \{x, v_p(x + a) > 3\},$$

where $a = 4{\cdot}5^2$, with $c_{\mathbb{Q}_p}(X(26, \bar{r}_0(2))) = 2$ and $R(26, \bar{r}_0(2))[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p)^2$..

- $$X(26, \bar{r}_1(1)) = \{x, 2 < v_p(x - a) < 3\} \cup \{x, 2 < v_p(x + a) < 3\},$$

with $c_{\mathbb{Q}_p}(X(26, \bar{r}_1(1))) = 4$ and $R(26, \bar{r}_1(1))[1/p] = (\mathbb{Z}_p[[X, Y]]/(XY - p) \otimes \mathbb{Q}_p)^2$.

Here we see an example where the geometry begins to be a little complicated, with annuli that do not have 0 as a center.

(b) *The case* $k = 28$.

- $$X(28, \bar{r}_1) = \{x, v_p(x) > 2,\ v_p(x-a) < 4,\ v_p(x+a) < 4\} \cup \{x, 0 < v_p(x) < 1\},$$

where $a = 4 \cdot 5^3 + 5^4$, with $c_{\mathbb{Q}_p}(X(28, \bar{r}_1)) = 5$ and

$$R(28, \bar{r}_1)[1/p] = \big(\mathbb{Z}_p[[X, Y]]/(XY - p) \otimes \mathbb{Q}_p\big) \\ \times \big(\mathbb{Z}_p[[X, Y, Z]]/(XY - p^2 - (a/p^2)Y,\ XZ - p^2 + (a/p^2)Z,\ YZ - (p^4/2a)(Y - Z)) \otimes \mathbb{Q}_p\big). \quad (7)$$

- $$X(28, \bar{r}_0(1)) = \{x, 1 < v_p(x) < 2\},$$

with $c_{\mathbb{Q}_p}(X(28, \bar{r}_0(1))) = 2$ and $R(28, \bar{r}_0(1))[1/p] = (\mathbb{Z}_p[[X, Y]]/(XY - p) \otimes \mathbb{Q}_p)$.

- $$X(28, \bar{r}_0(3)) = \{x, v_p(x - a) > 4\} \cup \{x, v_p(x + a) > 4\}$$

with $c_{\mathbb{Q}_p}(X(28, \bar{r}_0(3))) = 2$ and $R(28, \bar{r}_0(3))[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p)^2$.

Here we see an example with an irreducible component that has complexity 3.

(c) *The case* $k = 30$.

- $$X(30, \bar{r}_0) = \{x, 0 < v_p(x) < 1\} \cup \{x, v_p(x) > 4\}$$

with $c_{\mathbb{Q}_p}(X(30, \bar{r}_0)) = 3$ and $R(30, \bar{r}_0)[1/p] = (\mathbb{Z}_p[[X]] \otimes \mathbb{Q}_p) \times (\mathbb{Z}_p[[X, Y]]/(XY - p) \otimes \mathbb{Q}_p)$.

- $$X(30, \bar{r}_0(2)) = \{x, v_p(x - a) > 3\} \cup \{x, v_p(x + a) > 3\},$$

where $a = 5^3 \cdot \sqrt{3}$, with $c_{\mathbb{Q}_p}(X(30, \bar{r}_0(2))) = 2$ and $R(30, \bar{r}_0(2))[1/p] = (\mathbb{Z}_{p^2}[[X]] \otimes \mathbb{Q}_{p^2})$

- $$X(30, \bar{r}_1(1)) = \{x, 1 < v_p(x) < 3\} \cup \{x, 3 < v_p(x) < 4\}$$

with $c_{\mathbb{Q}_p}(X(30, \bar{r}_1(1))) = 4$ and

$$R(30, \bar{r}_1(1))[1/p] = \big(\mathbb{Z}_p[[X, Y]]/(XY - p) \otimes \mathbb{Q}_p\big) \times \big(\mathbb{Z}_p[[X, Y]]/(XY - p^2) \otimes \mathbb{Q}_p\big).$$

The interesting part here is $X(30, \bar{r}_0(2))$: we see that $\mathcal{A}^0_{\mathbb{Q}_p}(X(30, \bar{r}_0(2)))$, which is a domain, has residue field $\mathbb{F}_{p^2}$, whereas $R(30, \bar{r}_0(2))$ has residue field $\mathbb{F}_p$. So $R(30, \bar{r}_0(2)) \neq \mathcal{A}^0_{\mathbb{Q}_p}(X(30, \bar{r}_0(2)))$.

6.3. ***Criteria for nonnormality.*** Recall the notation of Section 5.2. Then we see, by Proposition 5.3.5, that if we know $X(k, \bar{\rho})$ then we know $R(k, \bar{\rho})[1/p] = \bigoplus_i R_i[1/p] = \bigoplus_i \mathcal{A}_E(X_i)$. We can ask whether we can recover each R_i, that is, if $R_i = \mathcal{A}^0_E(X_i)$, or equivalently if $R_i = R_i^0$ for all i (the description of $X(k, \bar{\rho})$ gives no indication about how the R_i glue together so we cannot hope for complete information on $R(k, \bar{\rho})$ anyway if it is not irreducible). We do not expect this to hold, as this would mean that each of the R_i is a normal ring. So we can ask instead, how can we recognize when R_i is not R_i^0?

A first criterion is when they have different residue fields, as in the example of $R(30, \bar{r}_0(2))$ in Section 6.2(c). Another criterion is when R_i and R_i^0 have the same residue field (a situation that we can always obtain by replacing E by an unramified extension, which does not change the complexities), but $\bar{e}(R_i^0) < \bar{e}(R_i)$. This is a situation that does not seem to arise often; see Section 6.4.

We give a last, more subtle criterion. Let X_i be one of the components of $X(k, \bar{\rho})$, and assume that each of the disks that appears in the description of X_i is defined over $\mathbb{Q}_p$, and has complexity 1. In this case, a closer look at the proof of Proposition 4.2.12 shows that $\mathrm{Spec}(\mathcal{A}^0_{\mathbb{Q}_p}(X_i)/p)$ has exactly $c_{\mathbb{Q}_p}(X_i)$ distinct irreducible components. On the other hand, the geometric version of the Breuil-Mézard conjecture, proved in [Emerton and Gee 2014], shows that if $\bar{\rho}$ is irreducible then $\mathrm{Spec}(R(k, \bar{\rho})/p)$ has at most two irreducible components (which can have large multiplicity), and so $\mathrm{Spec}(R_i/p)$ also has at most two irreducible components. So if $c_{\mathbb{Q}_p}(X_i) > 2$ then we certainly have that $R_i \neq R_i^0$. This happens for example for the second irreducible component of $X(28, \bar{r}_1)$. It would be interesting in this case to understand how the irreducible components of $\mathrm{Spec}(R_i^0/p)$ map to the irreducible components of $\mathrm{Spec}(R_i/p)$.

6.4. ***Complexity and multiplicity.*** An interesting result coming from our computations is the following: for $p = 5$, for all irreducible representation $\bar{\rho}$, for all $k \leq 47$ and all even $k \leq 102$, we have that $c_{\mathbb{Q}_p}(X(k, \bar{\rho})) = \bar{e}(R(k, \bar{\rho}))$, instead of simply the inequality $c_{\mathbb{Q}_p}(X(k, \bar{\rho})) \leq \bar{e}(R(k, \bar{\rho}))$. Given this, it is tempting to make the following conjecture: For all $p > 2$, for all $k \geq 2$ and for all irreducible $\bar{\rho}$, we have that $c_{\mathbb{Q}_p}(X(k, \bar{\rho})) = \bar{e}(R(k, \bar{\rho}))$.

Note that this equality between complexity and multiplicity does not necessarily hold when $\bar{\rho}$ is reducible. Consider the following example: let $p = 5$ and $k = 16$. Then we can compute that $X(16, \bar{r}_1)$

is the set $\{x, v(x) > 0, v(x) \neq 1\}$. So the set of a_p for which the reduction is reducible is contained in the set of a_p with $v(a_p) = 0$ or $v(a_p) = 1$. For $v(a_p) = 0$, the reduction is of the form $\omega^3 \oplus 1$ when restricted to inertia. The reduction for the values of a_p with $v(a_p) = 1$ is entirely computed in [Bhattacharya et al. 2018], from which we get that for $\lambda \in \bar{\mathbb{F}}_p^\times$, the (semisimplified) reduction is $\mathrm{unr}(\lambda)\omega^2 \oplus \mathrm{unr}(\lambda^{-1})\omega$ for exactly the values of a_p of valuation 1 for which $\lambda = 2(\overline{a_p/p} - \overline{p/a_p})$. So for each λ, we get that $X(16, \mathrm{unr}(\lambda)\omega^2 \oplus \mathrm{unr}(\lambda^{-1})\omega)$ is the union of two disks, and has complexity 2, except for $\lambda = \pm 1$, where this is just one disk, and has complexity 1. On the other hand, for any $\lambda \in \bar{\mathbb{F}}_p^\times$ we have $\bar{e}(R(16, \mathrm{unr}(\lambda)\omega^2 \oplus \mathrm{unr}(\lambda^{-1})\omega)) = 2$. So we do not always have the equality of multiplicity and complexity in the case where $\bar{\rho}$ is reducible. However, it may be true that for all $p > 2$, for all $k \geq 2$, there is only a finite number of reducible (nice) representations $\bar{\rho}$ for which the equality does not hold.

We can also reformulate this equality in a different way: recall the notation of Section 5.2. So $R(k, \bar{\rho})$ has a family of quotients R_i that are integral domains, and $\bar{e}(R(k, \bar{\rho})) = \sum_i \bar{e}(R_i)$. On the other hand, $c_{\mathbb{Q}_p}(X(k, \bar{\rho})) = \sum_i [k_{R_i^0} : \mathbb{F}_p]\bar{e}(R_i^0)$, where $k_{R_i^0}$ is the residue field of R_i^0. The equality between complexity and multiplicity can be reformulated as saying that for all i, $\bar{e}(R_i) = [k_{R_i^0} : \mathbb{F}_p]\bar{e}(R_i^0)$. Written in this way without any reference to the sets $X(k, \bar{\rho})$, the equality can be generalized to any potentially semistable deformation ring, including those that are of dimension larger than 1, such as the deformation rings classifying representations of dimension larger than 2 or representations of G_K for some finite extension $K/\mathbb{Q}_p$.

7. Parameters classifying potentially semistable representations

This section is devoted to the proof of Theorem 5.3.1. We start with some preliminaries, and then give the proof for the various cases starting in Section 7.4.

7.1. *Results on Weil representations.*

Field of definition. Let $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$. A Weil representation is a representation of $W_{\mathbb{Q}_p}$ with coefficients in $\bar{\mathbb{Q}}_p$ that is trivial on an open subgroup of $I_{\mathbb{Q}_p}$.

Let τ be a Weil representation. The field of definition of τ, denoted by $E(\tau)$, is the subfield of $\bar{\mathbb{Q}}_p$ generated by the $\operatorname{tr}\tau(x)$, $x \in W_{\mathbb{Q}_p}$. This is a finite extension of $\mathbb{Q}_p$, as a Weil representation factors through a finitely generated group.

Let E be a finite extension of $\mathbb{Q}_p$. We say that τ is realizable over E if there is a representation $\tau' : W_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ that is isomorphic to τ. Then we have:

Lemma 7.1.1. *Let τ be an irreducible Weil representation. Then there exists a finite unramified extension E of $E(\tau)$ such that τ is realizable over E.*

Proof. From the results of [Kratzer 1983, 1.4], we see that the obstruction to realizing τ over $E(\tau)$ is in the Brauer group of $E(\tau)$. An element of the Brauer group can be killed by taking a finite unramified extension, hence the result. □

$(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$*-modules.* We fix a finite Galois extension F of $\mathbb{Q}_p$, and denote by F_0 the maximal subextension of F that is unramified over $\mathbb{Q}_p$.

Let A be a $\mathbb{Q}_p$-algebra. Then a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module M over $F_0 \otimes_{\mathbb{Q}_p} A$ is a free $F_0 \otimes_{\mathbb{Q}_p} A$-module of finite rank, endowed with commuting actions of an automorphism ϕ and the group $\mathrm{Gal}(F/\mathbb{Q}_p)$. The action of ϕ is A-linear and F_0-semilinear (with respect to the Frobenius automorphism of F_0), and the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ is F_0-semilinear (with respect to the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on F_0) and A-linear.

Proposition 7.1.2. *Let A be an F_0-algebra. There is an equivalence of categories between $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules over $F_0 \otimes_{\mathbb{Q}_p} A$ and Weil representations over a free A-module that are trivial on I_F, and this equivalence preserves rank. Moreover this construction is functorial in A (in the category of F_0-algebras).*

Proof. The construction of the Weil representation from the $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module is explained in [Breuil and Mézard 2002] for a given A, and the converse construction is immediate. □

We will make use of this equivalence as some things are more naturally expressed in terms of $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules, whereas others are more easily proved in terms of representations of the Weil group (for example Theorem 7.3.2).

In the same situation, we define a $(\phi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} A$ to be a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} A$ that is additionally endowed with a $F_0 \otimes_{\mathbb{Q}_p} A$-linear endomorphism N satisfying $N\phi = p\phi N$ that commutes with the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$.

7.2. *Universal (filtered) (ϕ, N)-modules with descent data.* We recall a few definitions concerning objects attached to p-adic representations of $G_{\mathbb{Q}_p}$. If $F/\mathbb{Q}_p$ is a finite extension, we denote by F_0 the maximal unramified extension of $\mathbb{Q}_p$ contained in F.

Let V be a continuous representation of $G_{\mathbb{Q}_p}$ over an E-vector space for some finite $E/\mathbb{Q}_p$. Let F be a finite Galois extension of $\mathbb{Q}_p$. We denote by $\mathrm{D}^F_{\mathrm{crys}}(V)$ the $F_0 \otimes_{\mathbb{Q}_p} E$-module $(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V)^{G_F}$. It is a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} E$. If V becomes crystalline over F then $\mathrm{D}^F_{\mathrm{crys}}(V)$ is a free $F_0 \otimes_{\mathbb{Q}_p} E$-module of rank $\dim_E(V)$. We denote by $\mathrm{D}^F_{\mathrm{st}}(V)$ the $F_0 \otimes_{\mathbb{Q}_p} E$-module $(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_F}$. It is endowed with a structure of $(\phi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} E$. If V becomes semistable over F then is it a free $F_0 \otimes_{\mathbb{Q}_p} E$-module of rank $\dim_E(V)$. If V becomes crystalline over F then $\mathrm{D}^F_{\mathrm{st}}(V)$ and $\mathrm{D}^F_{\mathrm{crys}}(V)$ coincide as $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules, and $N = 0$. We denote by $\mathrm{D}^F_{\mathrm{dR}}(V)$ the $F \otimes_{\mathbb{Q}_p} E$-module $(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_F}$. It is an $F \otimes_{\mathbb{Q}_p} E$-module with a semilinear action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, and is endowed with a separated exhaustive decreasing filtration by sub-$F \otimes_{\mathbb{Q}_p} E$-modules that is stable under the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, and satisfies an additional condition called admissibility. If V is potentially semistable, then $\mathrm{D}^{\mathbb{Q}_p}_{\mathrm{dR}}(V)$ is an E-vector space of dimension $\dim_E(V)$. Moreover, we have that $\mathrm{D}^F_{\mathrm{dR}}(V) = F \otimes_{F_0} \mathrm{D}^F_{\mathrm{st}}(V)$ as an $F \otimes_{\mathbb{Q}_p} E$-module, so this endows $F \otimes_{F_0} \mathrm{D}^F_{\mathrm{st}}(V)$ with a filtration as above, that is, a structure of filtered $(\phi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module.

Theorem 7.2.1. *Let F be a finite Galois extension of $\mathbb{Q}_p$. Let X be a reduced rigid analytic space, let $\mathcal{V}$ be a locally free $\mathcal{O}_X$-module of rank n with a continuous action of $G_{\mathbb{Q}_p}$. For all $x \in X$, assume that $\mathcal{V}_x$ is potentially semistable with weights independent of x, and becomes semistable over F. Then there exists a* *projective $F_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X$-module D of rank n, endowed with a structure of $(\phi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} \mathcal{O}_X$, such that for all x, D_x is isomorphic, as a $(\phi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module, to $\mathrm{D}_{\mathrm{st}}^F(\mathcal{V}_x)$.*

Proof. This follows immediately from [Bellovin 2015, Theorem 5.1.2]: we take the module D to be the module called $\mathcal{D}_{B_{\mathrm{st}}}(\mathcal{V})$ there, considering $\mathcal{V}$ as a representation of G_F; see also [Berger and Colmez 2008, Théorème C]. □

Theorem 7.2.2. *Let F be a finite Galois extension of $\mathbb{Q}_p$. Let X be a reduced rigid analytic space, let $\mathcal{V}$ be a locally free $\mathcal{O}_X$-module of rank n with a continuous action of $G_{\mathbb{Q}_p}$. For all $x \in X$, assume that $\mathcal{V}_x$ is potentially semistable with weights independent of x, and becomes semistable over F. Let D be as in the conclusion of Theorem 7.2.1. Then $F \otimes_{F_0} D$ is endowed of a filtration by locally free sub-$F \otimes_{\mathbb{Q}_p} \mathcal{O}_X$-modules, such that the graded parts are also locally free, such that for all x, $(F \otimes_{F_0} D)_x$ is isomorphic, as a filtered $(\phi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module, to $\mathrm{D}_{\mathrm{dR}}^F(\mathcal{V}_x)$.*

Proof. This follows from [Bellovin 2015, Theorem 5.1.7], as $F \otimes_{F_0} D$ is the $F \otimes_{\mathbb{Q}_p} \mathcal{O}_X$-module that is called $\mathcal{D}_{B_{\mathrm{dR}}}(\mathcal{V})$ there, considering $\mathcal{V}$ as a representation of G_F. Indeed the filtration, and the graded parts, are given by the modules called $\mathcal{D}_{B_{\mathrm{dR}}}^{[a,b]}(\mathcal{V})$. The point that we need to check is that for all $[a, b]$, the $F \otimes_{\mathbb{Q}_p} E_x$-modules $\mathcal{D}_{B_{\mathrm{dR}}}^{[a,b]}(\mathcal{V}_x)$ are actually free (then their rank is independent of x by the condition on the weights). This comes from [Savitt 2005, Lemma 2.1], and here we use the fact that we start from a representation of $G_{\mathbb{Q}_p}$. □

Let now $(k, \tau, \bar{\rho}, \psi)$ be a deformation data, as defined in Definition 5.1.2. Let E be a finite extension of $\mathbb{Q}_p$ satisfying the following conditions:

(1) The residual representation $\bar{\rho}$ can be realized on the residue field of E.

(2) The type τ can be realized on E.

(3) The character ψ takes its values in $E^\times$.

Let $R^\psi(k, \tau, \bar{\rho})[1/p]$ be the ring defined by Kisin attached to this data, as recalled in Section 5.1. It is an $\mathcal{O}_E$-algebra. We can apply Theorems 7.2.1 and 7.2.2 to the rigid analytic space $X = \mathcal{X}^\psi(k, \tau, \bar{\rho})$ attached to the Kisin ring $R^\psi(k, \tau, \bar{\rho})[1/p]$. Indeed, we know that these rings are reduced, and the hypotheses come from the definition of the rings.

7.3. *Working in families.*

Reduction of an endomorphism.

Proposition 7.3.1. *Let K be a field and A be a K-algebra. Let ϕ be an A-linear endomorphism of A^2, and assume that the characteristic polynomial of ϕ is in fact in $K[X]$, and that it is split over K with distinct eigenvalues. Then, Zariski-locally on A, ϕ is diagonalizable.*

Proof. Let λ and μ be the roots of the characteristic polynomial of ϕ, and let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the matrix of ϕ in the canonical basis of A^2 (so that $a + d = \lambda + \mu$ and $ad - bc = \lambda\mu$).

We are looking for a basis (f_1, f_2) of A^2, with $f_1 = xe_1 + ye_2$, $f_2 = e_2$, such that the matrix of ϕ in this basis is upper triangular. The new basis is as wanted if x, y satisfy one of the following systems of equations:

$$(a-\lambda)x + by = 0 \quad \text{and} \quad cx + (d-\lambda)y = 0$$

or

$$(a-\mu)x + by = 0 \quad \text{and} \quad cx + (d-\mu)y = 0.$$

Assume that $u = d-\lambda$ is invertible. We solve the first system by setting $x = 1$, $y = -c/(d-\lambda)$. In the first case, in our new basis ϕ has a matrix of the form $\left(\begin{smallmatrix}\lambda & b\\ 0 & d\end{smallmatrix}\right)$, and actually $d = \mu$ by the trace condition. As $\lambda - \mu$ is invertible, we can change the basis again so that in the new basis, ϕ has matrix $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)$.

Assume now that $v = a - \lambda$ is invertible. Then so is $d - \mu = -v$. We solve the second system by setting $x = 1$, $y = -c/(d-\mu)$. In this case we do the same thing after exchanging λ and μ.

Note that $u + v = \mu - \lambda$ is invertible by assumption. We set $f = (d-\lambda)/(\mu-\lambda)$, $A_1 = A[f^{-1}]$, $A_2 = A[(1-f)^{-1}]$. Then as we just saw in A_1 and A_2 there is a basis in which the matrix of ϕ is $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \mu\end{smallmatrix}\right)$, which gives the result. □

Isomorphism of group representations.

Theorem 7.3.2. *Let K be a field of characteristic zero, and A a K-algebra.*

Let G be a group. Let $\rho : G \to \mathrm{GL}_n(K)$ be a representation that is absolutely irreducible. Let $\rho' : G \to \mathrm{GL}_n(A)$ be a representation. Assume that for all g in G, we have $\operatorname{tr}\rho(g) = \operatorname{tr}\rho'(g)$.

Then, Zariski-locally on A, there is an $M \in \mathrm{GL}_n(A)$ such that $\rho'(g) = M\rho(g)M^{-1}$ for all $g \in G$.

Proof. By [Rouquier 1996, Théorème 5.1], there is an A-algebra automorphism τ of $M_n(A)$ such that for all $g \in G$, $\rho'(g) = \tau\rho(g)$. By [Knus and Ojanguren 1974, IV. Proposition 1.3], there is a family (f_i) in A generating the unit ideal such that for all i, the automorphism of $M_n(A[1/f_i])$ induced by τ is inner. Hence the result. □

Variations on Hilbert 90.

Proposition 7.3.3. *Let K be an infinite field, and L/K be a finite Galois extension of fields.*

(1) *Let M be a finite K-algebra. Then $H^1(\mathrm{Gal}(L/K), (L \otimes_K M)^\times) = 0$.*

(2) *Let A be a K-algebra. Assume that for every maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is a finite extension of K. Let $c \in H^1(\mathrm{Gal}(L/K), (L \otimes_K A)^\times)$. There exists a family of elements (f_i) in A that generate the unit ideal such that the image of c in $H^1(\mathrm{Gal}(L/K), (L \otimes_K A[f_i^{-1}])^\times)$ is zero for all i.*

Proof. Let M be a K-algebra, and $c \in H^1(\mathrm{Gal}(L/K), (L \otimes_K M)^\times)$. Let $x \in L$. We set $\phi(c, x) = \sum_{\gamma \in \mathrm{Gal}(L/K)} \gamma(x)c(\gamma) \in L \otimes_K M$. We have $c(g)g(\phi(c, x)) = \phi(c, x)$ for all $g \in \mathrm{Gal}(L/K)$, so $c = 0$ as soon as we can find an x such that $\phi(c, x)$ is invertible in $L \otimes_K M$. Point (1) is well-known, and is proved by showing that if M is finite over K then such an x exists, with a proof similar to the case where $M = M_n(K)$ (here we do not need M to be commutative).

For any commutative K-algebra M, the M-algebra $L \otimes_K M$ is finite. We denote by N_M the norm map $L \otimes_K M \to M$, so that for all $x \in L \otimes_K M$, we have $x \in (L \otimes_K M)^\times$ if and only if $N_M(x) \in M^\times$.

Moreover the norm map commutes with base change: let $u : M \to M'$ be a map of K-algebras, then $N_{M'}(1 \otimes u)(x) = u(N_M(x))$ for all $x \in L \otimes_K M$.

Let now A be as in point (2) and let $c \in H^1(\mathrm{Gal}(L/K), (L \otimes_K A)^\times)$. For an extension A' of A, denote by $c_{A'}$ the image of c in $H^1(\mathrm{Gal}(L/K), (L \otimes_K A')^\times)$.

Let $\mathfrak{m}$ be a maximal ideal of A, and $K_\mathfrak{m} = A/\mathfrak{m}$. Then $K_\mathfrak{m}$ is a finite extension of K. So there exists an $x \in L$ such that $\phi(c_{K_\mathfrak{m}}, x)$ is invertible in $L \otimes_K K_\mathfrak{m}$. Let $f = N_A(\phi(c, x)) \in A$. Then D_f is a neighborhood of $\mathfrak{m}$ in Spec A. Moreover the image of $\phi(c, x)$ in $L \otimes_K A[f^{-1}]$ is invertible, so $c_{A[f^{-1}]} = 0$.

So we see that there is a covering of Spec A by open subsets of the form D_f with $c_{A[f^{-1}]} = 0$, which is what we wanted. □

7.4. ***The crystalline case.*** We want to prove Theorem 5.3.1 for the case where the Galois type is of the form (1), that is, $\tau = \chi \oplus \chi$ for some smooth character χ of $I_{\mathbb{Q}_p}$ that extends to $W_{\mathbb{Q}_p}$. By twisting by the character χ, we can reduce to the case where τ is the trivial representation of $I_{\mathbb{Q}_p}$, that is, the case of crystalline deformation rings. Recall from Section 5.4 the definition of the parameter a_p.

Proposition 7.4.1. *There is an element $a_p \in R(k, \bar{\rho})[1/p]$ such that for any finite extension E_x of E and $x : R(k, \bar{\rho})[1/p] \to E_x$ corresponding to a representation ρ_x, $a_p(x)$ is the value of a_p corresponding to ρ_x by the classification of Lemma 5.4.1.*

In particular, we can see a_p as an analytic map from $\mathcal{X}(k, \bar{\rho})$ to $\mathbb{A}^{1,\mathrm{rig}}$. Moreover, a_p induces an injective map from $\mathcal{X}(k, \bar{\rho})(\overline{\mathbb{Q}}_p)$ to $D(0, 1)^+$.

Proof. Consider the ϕ-module D which is obtained from applying Theorem 7.2.1 to the rigid space $\mathcal{X}(k, \bar{\rho})$ attached to the ring $R(k, \bar{\rho})[1/p]$. It is a projective module of rank 2 over $R(k, \bar{\rho})[1/p]$ and is such that for all $x : R(k, \bar{\rho})[1/p] \to E_x$ corresponding to a representation ρ_x, $D \otimes_{R(k,\bar{\rho})[1/p]} E_x$ is the ϕ-module D_x attached to ρ_x (forgetting the filtration). Now observe that a_p, as defined in Lemma 5.4.1, is the trace of ϕ on the dual of D, so it is an element of $R(k, \bar{\rho})[1/p]$, and $a_p(x)$ is the evaluation at x of the trace of ϕ on the dual of D. □

7.5. ***The crystabelline case.*** We suppose here that $\tau = \chi_1 \oplus \chi_2$, where χ_1 and χ_2 are distinct characters of $I_{\mathbb{Q}_p}$ with finite image that extend to characters of $W_{\mathbb{Q}_p}$, so that the representations classified by $R^\psi(k, \tau, \bar{\rho})$ become crystalline on an abelian extension of $\mathbb{Q}_p$. In this case we show the existence of a function λ as in Theorem 5.3.1 when $\chi_1 \neq \chi_2$. We make use of the results of [Ghate and Mézard 2009], which classifies the filtered ϕ-modules with descent data that give rise to a Galois representation of inertial type τ and Hodge–Tate weights $(0, k-1)$. We summarize their results for such a τ.

The characters χ_i factor through $F = \mathbb{Q}_p(\zeta_{p^m})$ for some $m \geq 1$, so the Galois representations we are interested in become crystalline on F, and so are given by filtered $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules. Note that here $F_0 = \mathbb{Q}_p$.

Let E be a finite extension of $\mathbb{Q}_p$ containing the values of χ_1 and χ_2. Let α, β be in $\mathcal{O}_E$ with $v_p(\alpha)+v_p(\beta)=k-1$. We define a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module $\Delta_{\alpha,\beta}$ as follows: let $\Delta_{\alpha,\beta}=Ee_1\oplus Ee_2$, with $g(e_1)=\chi_1(g)e_1$ and $g(e_2)=\chi_2(g)e_2$ for all $g\in\mathrm{Gal}(F/\mathbb{Q}_p)$. The action of ϕ is given by: $\phi(e_1)=\alpha^{-1}e_1$ and $\phi(e_2)=\beta^{-1}e_2$. We are looking at filtrations on $\Delta_{\alpha,\beta,F}=F\otimes_{\mathbb{Q}_p}\Delta_{\alpha,\beta}$ satisfying $\mathrm{Fil}^i\,\Delta_{\alpha,\beta,F}=0$ if $i\le 1-k$, $\mathrm{Fil}^i\,\Delta_{\alpha,\beta,F}=\Delta_{\alpha,\beta}$ if $i>0$, and $\mathrm{Fil}^i\,\Delta_{\alpha,\beta,F}=\mathrm{Fil}^0\,\Delta_{\alpha,\beta,F}$ for $1-k<i\le 0$ is a $F\otimes_{\mathbb{Q}_p}E$-line.

We summarize now the results given in [Ghate and Mézard 2009, Section 3].

Proposition 7.5.1. *Fix α, β in $\mathcal{O}_E$ with $v_p(\alpha)+v_p(\beta)=k-1$. Then there exists a way to choose $\mathrm{Fil}^0(\Delta_{\alpha,\beta,F})\subset\Delta_{\alpha,\beta,F}=\Delta_{\alpha,\beta}\otimes F$ that makes it an admissible filtered $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module.*

If neither α nor β is a unit, then all such choices give rise to isomorphic filtered $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules, which are irreducible.

If α or β is a unit, the choices give rise to two isomorphism classes of filtered $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules, one reducible split and the other reducible nonsplit.

We denote by $D_{\alpha,\beta}$ the isomorphism class of admissible filtered $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module given by a choice of filtration that makes it into either an irreducible module (if neither α nor β is a unit) or a reducible nonsplit module (if α or β is a unit).

Then it follows from the computations of [Ghate and Mézard 2009, Section 3] that:

Proposition 7.5.2. *Let V be a potentially crystalline representation with coefficients in E, of inertial type τ and Hodge–Tate weights $(0, k-1)$ that is not reducible split. Then there exists a unique pair $(\alpha,\beta)\in\mathcal{O}_E$ with $v_p(\alpha)+v_p(\beta)=k-1$ such that $\mathrm{D}^F_{\mathrm{crys}}(V)$ is isomorphic to $D_{\alpha,\beta}$ as a filtered $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module.*

Let $E=E(k,\tau,\bar\rho,\psi)$ be a finite extension of $\mathbb{Q}_p$ such that $\bar\rho$ can be defined over the residue field of E, E contains the images of χ_1 and χ_2 and of the character ψ. Then the ring $R^\psi(k,\tau,\bar\rho)$ can be defined over E. Moreover:

Proposition 7.5.3. *Let $\bar\rho$ be a representation with trivial endomorphisms. There are elements $\alpha,\beta\in R^\psi(k,\tau,\bar\rho)[1/p]$ such that $\mathrm{D}^F_{\mathrm{crys}}(\rho_x)$ is isomorphic to $\Delta_{\alpha(x),\beta(x)}$ as a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module for each closed point x of $\mathrm{Spec}\,R^\psi(k,\tau,\bar\rho)[1/p]$ corresponding to a representation ρ_x.*

Proof. By Theorem 7.2.1 applied to the rigid analytic space $\mathcal{X}^\psi(k,\tau,\bar\rho)$ attached to $R^\psi(k,\tau,\bar\rho)[1/p]$, there exists a ϕ-module D with descent data by $\mathrm{Gal}(F/\mathbb{Q}_p)$, where D is a projective module of rank 2 over $R^\psi(k,\tau,\bar\rho)[1/p]$, such that for each closed point x of $\mathrm{Spec}\,R^\psi(k,\tau,\bar\rho)[1/p]$, $\mathrm{D}^F_{\mathrm{crys}}(\rho_x)$ is isomorphic to $D\otimes_R E_x$ (where E_x is the field of coefficients of ρ_x) as a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module.

Applying Proposition 7.3.1, we observe that the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on D is given as the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on each $\Delta_{\alpha,\beta}$: that is, Zariski-locally on $\mathrm{Spec}\,R^\psi(k,\tau,\bar\rho)[1/p]$, we can write $D=Re_1\oplus Re_2$, with $g(e_1)=\chi_1(g)e_1$ and $g(e_2)=\chi_2(g)e_2$.

As the action of ϕ on D commutes with the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, this shows that the eigenvalues of ϕ acting on D are in fact in $R^\psi(k,\tau,\bar\rho)[1/p]$, that is, α and β are elements of $R^\psi(k,\tau,\bar\rho)[1/p]$. □

Moreover, if we fix the determinant of the Galois representation corresponding to $D_{\alpha,\beta}$ then we fix $\alpha\beta$. So the function α is injective on points, so it can play the role of the function λ of Theorem 5.3.1.

Let $X^{\psi}(k, \tau, \bar{\rho})$ be the image of $\mathcal{X}^{\psi}(k, \tau, \bar{\rho})(\overline{\mathbb{Q}}_p)$ in $\overline{\mathbb{Q}}_p$, then we see that $X^{\psi}(k, \tau, \bar{\rho})$ is contained in the set $\{x, 0 \leq v_p(x) \leq k-1\}$, with the irreducible representations corresponding to the subset of elements that are in $\{x, 0 < v_p(x) < k-1\}$.

7.6. ***Semistable representations.*** We now assume $p > 2$ and we study the case of the deformation rings attached to a discrete series extended type of the form $\tau = \chi_1 \oplus \chi_2$, where χ_1 and χ_2 are characters of $W_{\mathbb{Q}_p}$ that have the same restriction to inertia, and such that $\chi_1(F) = p\chi_2(F)$ for any Frobenius element F. As in the case of crystalline representations, we can twist by a smooth character of $W_{\mathbb{Q}_p}$ and reduce to the case where χ_1 and χ_2 are trivial on inertia. Then the deformation rings $R^{\psi}(k, \tau, \bar{\rho})$ classify representations that are semistable, and only a finite number of the representations that appear can be crystalline.

Let ρ be a semistable, noncrystalline representation of dimension 2 of $G_{\mathbb{Q}_p}$, with Hodge–Tate weights $(0, k-1)$ for some $k \geq 2$. Then we know (see, for example, [Ghate and Mézard 2009, Section 3.1]), that the filtered (ϕ, N)-module $\mathrm{D}_{\mathrm{st}}(\rho)$ is isomorphic to exactly one $D_{\alpha,\mathcal{L}}$ for some α with $v(\alpha) = k/2$, some $\mathcal{L} \in \overline{\mathbb{Q}}_p$ and some finite extension E containing α and $\mathcal{L}$, for (ϕ, N)-modules $D_{\alpha,\mathcal{L}}$ defined as follows: $D_{\alpha,\mathcal{L}} = Ee_1 \oplus Ee_2$, $\phi(e_1) = p\alpha^{-1}e_1$, $\phi(e_2) = \alpha^{-1}e_2$, $Ne_1 = e_2$, $\mathrm{Fil}^0 D_{\alpha,\mathcal{L}} = E(e_1 - \mathcal{L}e_2)$. Then $\mathcal{L}$ is the $\mathcal{L}$-invariant of Fontaine, as defined in [Mazur 1994, Section 9]. Let ρ be a crystalline representation of dimension 2 of $G_{\mathbb{Q}_p}$, we set its $\mathcal{L}$-invariant to be ∞.

Proposition 7.6.1. *Let $\mathcal{X}$ be a rigid analytic space defined over some finite extension E of $\mathbb{Q}_p$. Assume that $\mathcal{X}$ is endowed with a* 2*-dimensional representation ρ of $G_{\mathbb{Q}_p}$ such that for all $x \in \mathcal{X}$, ρ_x is semistable with Hodge–Tate weights* $(0, k-1)$, *the Weil representation attached to ρ_x is independent of x, there exists at least one x such that ρ_x is not crystalline, and none of the ρ_x are reducible split. Then there exists a rigid analytic map $\mathcal{L} : \mathcal{X} \to \mathbb{P}^1_E$, defined over E, such that for all x, $\mathcal{L}(x)$ is the $\mathcal{L}$-invariant of ρ_x.*

Note that under these conditions, the α of $D_{\alpha,\mathcal{L}}$ is independent of x, and is in E.

This proposition applies in the following situation: let $p > 2$, let $\mathcal{X} = \mathcal{X}^{\psi}(k, \tau, \bar{\rho})$ be the deformation space for the extended type τ, and $\bar{\rho}$ is not reducible split. Then the function $\mathcal{L}$ can play the role of λ of Theorem 5.3.1.

Proof. In order to prove this result, it is enough to prove it for an admissible covering of $\mathcal{X}$. Indeed, the condition that $\mathcal{L}(x)$ is the $\mathcal{L}$-invariant of ρ_x ensures that the functions defined on each subset of the covering will glue. In particular, we can assume that $\mathcal{X}$ is affinoid, coming from a Tate algebra A over E.

By Theorems 7.2.1 and 7.2.2, there is a projective A-module D of rank 2 over A, endowed with a structure of filtered (ϕ, N)-module, such that D_x is $\mathrm{D}_{\mathrm{st}}(\rho_x)$ for all $x \in \mathrm{Max}(A)$. Consider the action of ϕ on D: it has eigenvalues $p\alpha^{-1}$ and α^{-1}. By Proposition 7.3.1, we can assume, after replacing A by a Zariski covering, that D is free over A, with a basis e_1, e_2 such that $\phi(e_1) = p\alpha^{-1}e_1$ and $\phi(e_2) = \alpha^{-1}e_2$. By the commutation relations between ϕ and N, there is a $\lambda \in A$ such that $Ne_1 = \lambda e_2$. Moreover, we can assume that there is a free A-module L of rank 1 in D, with quotient that is also free of rank 1, that gives the nontrivial step of the filtration. We fix a basis f of L.

Let $h = \det(f, \phi(f))$. Let us show that N and h do not vanish simultaneously. If this is the case, let x be a point where they both vanish. Then ρ_x is crystalline, as $N_x = 0$, and the filtration of the associated filtered ϕ-module is generated by an eigenvector of ϕ, as $h_x = 0$. Then the representation ρ_x is necessarily split reducible. But by hypothesis this cannot happen. So by replacing $\mathrm{Max}(A)$ by a Zariski cover, we can assume that either N never vanishes, or h in a unit in A.

Assume first that N never vanishes, that is, ρ_x is never crystalline. Then the λ as defined above is actually a unit in A, so we can modify the basis (e_1, e_2) so that $\lambda = 1$. Write f in this basis as $ae_1 + be_2$, with $a, b \in A$. By specializing at each $x \in \mathrm{Max}(A)$, we see that $a(x) \neq 0$ for all x, as this would contradict the admissibility condition of the filtered module. So $a \in A^\times$. Then by definition of the $\mathcal{L}$-invariant, we have $\mathcal{L}(x) = -(b/a)(x)$ for all $x \in \mathrm{Max}(A)$. So the function $\mathcal{L}$ is indeed an analytic function on $\mathrm{Max}(A)$.

Assume now that h is a unit in A. Let (e_1, e_2) be the basis of D defined above such that each e_i is an eigenvector for ϕ. We can write $f = ae_1 + be_2$ for some $a, b \in A$. Then the condition on h implies that a and b are in $A^\times$, that is, (ae_1, be_2) is also a basis of D over A. So we can modify the basis so that we have moreover $f = e_1 + e_2$. After specializing at $x \in \mathrm{Max}(A)$ an easy computation shows that $\lambda(x) = -1/\mathcal{L}(x)$ (and in particular the condition on h implies that $\mathcal{L}$ does not take the value 0). So we have defined an analytic function $\mathrm{Max}(A) \to \mathbb{P}^1$ by taking $\mathcal{L} = 1/\lambda$. □

7.7. *Supercuspidal types.* In this subsection, assume that $p > 2$. We consider now the case where the type is supercuspidal, that is, the Weil representation is (absolutely) irreducible.

Defining the generalized $\mathcal{L}$-invariant. We fix once and for all a supercuspidal extended type τ, that is, a smooth absolutely irreducible representation $\tau : W_{\mathbb{Q}_p} \to \mathrm{GL}_2(E_0)$ for some finite extension E_0 of $\mathbb{Q}_p$. This corresponds to cases (2) and (3) of the classification of types of [Ghate and Mézard 2009, Lemma 2.1]. Note that we can take E_0 to be an unramified extension of the definition field of τ by Lemma 7.1.1.

Let F be a finite Galois extension of $\mathbb{Q}_p$ such that τ is trivial on I_F, and let F_0 be the maximal unramified extension of $\mathbb{Q}_p$ contained in F. We assume, after taking an unramified extension of E_0 if necessary, that $F_0 \subset E_0$.

Let $\mathrm{D}_{\mathrm{crys},0}$ be the $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module associated to τ via the correspondence of Proposition 7.1.2. Let $\mathrm{D}_{\mathrm{dR},0} = F \otimes_{F_0} \mathrm{D}_{\mathrm{crys},0}$. It is endowed with an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ coming from the one on $\mathrm{D}_{\mathrm{crys},0}$.

Lemma 7.7.1. *Assume that there exists at least one potentially crystalline representation ρ with coefficients in E for some finite extension E of E_0, such that $\mathrm{D}^F_{\mathrm{dR}}(\rho)$ is isomorphic to $\mathrm{D}_{\mathrm{dR},0}^{\mathrm{Gal}(F/\mathbb{Q}_p)} \otimes_{E_0} E$ as a $F \otimes_{\mathbb{Q}_p} E$-module with an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$. Then $\mathrm{D}_{\mathrm{dR},0}^{\mathrm{Gal}(F/\mathbb{Q}_p)}$ is an E_0-vector space of dimension 2.*

Proof. Let $D = \mathrm{D}_{\mathrm{dR},0} \otimes_{E_0} E$, with its action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, which is isomorphic to the ϕ-module $\mathrm{D}^F_{\mathrm{dR}}(\rho)$ with its action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ for some potentially crystalline representation ρ. Then we have $\mathrm{D}^F_{\mathrm{dR}}(\rho)^{\mathrm{Gal}(F/\mathbb{Q}_p)} = \mathrm{D}^{\mathbb{Q}_p}_{\mathrm{dR}}(\rho)$ is an E-vector space of dimension 2, as ρ is de Rham as a $G_{\mathbb{Q}_p}$-representation. The action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $\mathrm{D}_{\mathrm{dR},0}$ is E_0-linear. So the dimension of its subspace of fixed elements is invariant by extension of scalars. Hence the result. □

Remark 7.7.2. We could also make use of the results of [Ghate and Mézard 2009], which give an explicit basis of the E-vector space $(\mathrm{D}_{\mathrm{dR},0} \otimes_{E_0} E)^{\mathrm{Gal}(F/\mathbb{Q}_p)}$ for some extension E of E_0.

We denote by V_τ the E_0-vector space of dimension 2 given by Lemma 7.7.1.

Any potentially semistable representation of extended type τ becomes crystalline when restricted to G_F. For any such representation ρ, with coefficients in an extension E of E_0, $D^F_{crys}(\rho)$ is a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} E$. We have that $D^F_{dR}(\rho)$ is canonically isomorphic to $F \otimes_{F_0} D^F_{crys}(\rho)$, and is endowed with an admissible filtration. Furthermore, we have $D^F_{dR}(\rho)^{\mathrm{Gal}(F/\mathbb{Q}_p)} = D^{\mathbb{Q}_p}_{dR}(\rho)$ is an E-vector space of dimension 2.

We also fix an integer $k \geq 2$, a continuous character $\psi : G_{\mathbb{Q}_p} \to E_0^\times$. Note that there is no loss of generality in considering only characters with values in E_0, as the compatibility condition between type and determinant shows that if $R^\psi(k, \tau, \bar{\rho})$ is nonzero then ψ takes its values in E_0.

Let $\mathcal{E}_\tau$ be the set of Galois representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ that are potentially crystalline of extended type τ, Hodge–Tate weights $(0, k-1)$, and determinant ψ. Then:

Theorem 7.7.3. *There exists a map $\mathcal{L}_\tau : \mathcal{E}_\tau \to \mathbb{P}(V_\tau \otimes_{E_0} \overline{\mathbb{Q}}_p)$ such that two elements ρ, ρ' of $\mathcal{E}_\tau$ are isomorphic if and only if $\mathcal{L}_\tau(\rho) = \mathcal{L}_\tau(\rho')$.*

Proof. We can assume that $\mathcal{E}_\tau$ is not empty, otherwise the statement is trivially true. Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be an element of $\mathcal{E}_\tau$. Then $\mathrm{WD}(\rho)$, the Weil–Deligne representation attached to ρ, is actually a Weil representation as ρ is potentially crystalline. By definition, $\mathrm{WD}(\rho)$ is isomorphic to $\tau \otimes_{E_0} \overline{\mathbb{Q}}_p$ as a representation of $W_{\mathbb{Q}_p}$. We fix such an isomorphism u, it is unique up to a scalar by the irreducibility of τ. Then u gives us an isomorphism between $D^F_{crys}(\rho)$ and $D_{crys,0} \otimes_{E_0} \overline{\mathbb{Q}}_p$ as ϕ-modules with an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$, by Proposition 7.1.2. This also gives us an isomorphism, that we still call u, between $D^F_{dR}(\rho)$ and $D_{dR,0} \otimes_{E_0} \overline{\mathbb{Q}}_p$.

The isomorphism class of ρ is entirely determined by the filtration on $D^F_{dR}(\rho)$. As the Hodge–Tate weights of ρ are known, the only necessary information is the $F \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-line corresponding to the nontrivial steps of the filtration. This line is invariant by the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$. By the isomorphism u, this gives rise to a $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariant $F \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$-line in $D_{dR,0} \otimes_{E_0} \overline{\mathbb{Q}}_p$. This line is generated by an element of $D_{dR,0} \otimes_{E_0} \overline{\mathbb{Q}}_p$ that is invariant by $\mathrm{Gal}(F/\mathbb{Q}_p)$ by (1) of Proposition 7.3.3, hence by an element of $D^{\mathrm{Gal}(F/\mathbb{Q}_p)}_{dR,0} \otimes_{E_0} \overline{\mathbb{Q}}_p$.

We define $\mathcal{L}_\tau(\rho) \in \mathbb{P}(D^{\mathrm{Gal}(F/\mathbb{Q}_p)}_{dR,0} \otimes_{E_0} \overline{\mathbb{Q}}_p)$ to be the line generated by this element in $D^{\mathrm{Gal}(F/\mathbb{Q}_p)}_{dR,0} \otimes_{E_0} \overline{\mathbb{Q}}_p$. This does not depend on the choices made, as u is unique up to multiplication by a scalar, and the invariant element generating the line is well-defined up to multiplication by a scalar. □

Making it into an analytic function. Let $\mathcal{X}$ be the rigid analytic space corresponding to the deformation ring $R^\psi(k, \tau, \bar{\rho})$ for some representation $\bar{\rho}$ with trivial endomorphisms and some supercuspidal extended type τ. Let $E = E(k, \tau, \bar{\rho}, \psi)$ be the field E_0 defined above.

Proposition 7.7.4. *There exists a rigid analytic map $\mathcal{L}_\tau : \mathcal{X} \to \mathbb{P}(V_\tau)$, defined over E, such that for all x, $\mathcal{L}_\tau(x)$ is the $\mathcal{L}_\tau$-invariant of ρ_x as defined in Theorem 7.7.3.*

By fixing a basis of the 2-dimensional E-vector space V_τ, we then get a map $\mathcal{L}_\tau : \mathcal{X} \to \mathbb{P}^1_E$, which plays the role of λ in Theorem 5.3.1.

Proof. It is enough to do this on an admissible covering of $\mathcal{X}$ by affinoid subspaces. So we can assume that $\mathcal{X} = \mathrm{Max}(A)$ for some affinoid algebra A, and replace $\mathcal{X}$ by an admissible covering by affinoid subspaces as needed.

Let $\mathrm{D}^F_{\mathrm{crys}}(A)$ be the $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module corresponding to the representation ρ. We can assume that $\mathrm{D}_{\mathrm{crys}}(A)$ is a free A-module of rank 2. Using the correspondence between $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules and representations of the Weil group as in Proposition 7.1.2, and Theorem 7.3.2, we can assume that $\mathrm{D}^F_{\mathrm{crys}}(A) = \mathrm{D}^F_{\mathrm{crys},0} \otimes_E A$ as a $(\phi, \mathrm{Gal}(F/\mathbb{Q}_p))$-module over $F_0 \otimes_{\mathbb{Q}_p} A$.

Consider now $\mathrm{D}^F_{\mathrm{dR}}(A)$. It is isomorphic to $F \otimes_{F_0} \mathrm{D}^F_{\mathrm{crys}}(A)$, so to $\mathrm{D}^F_{\mathrm{dR},0} \otimes_E A$ as a ϕ-module with action of $\mathrm{Gal}(F/\mathbb{Q}_p)$. In particular, it is trivial as an $F \otimes_{\mathbb{Q}_p} A$-module with an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$. Also, it has a basis as an A-module given by the chosen basis of $\mathrm{D}^F_{\mathrm{dR},0}$. The module $\mathrm{D}^F_{\mathrm{dR}}(A)$ contains a locally free sub-$F \otimes_{Q_p} A$-module $\mathcal{F}$ of rank 1, such that $\mathrm{D}^F_{\mathrm{dR}}(A)/\mathcal{F}$ is also locally free of rank 1, that gives at each point x the filtration on $\mathrm{D}^F_{\mathrm{dR}}(\rho_x)$. We can assume that $\mathcal{F}$ and $\mathrm{D}^F_{\mathrm{dR}}(A)$ are free of rank 1 over $F \otimes_{\mathbb{Q}_p} A$. Moreover, this submodule is invariant by the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$. Consider a basis f of $\mathcal{F}$. Then the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on f gives rise to an element $c \in H^1(\mathrm{Gal}(F/\mathbb{Q}_p), (F \otimes_{\mathbb{Q}_p} A)^\times)$. Using Proposition 7.3.3 and replacing $\mathrm{Max}(A)$ by an admissible covering if necessary, we can assume that f itself is fixed by the action of $\mathrm{Gal}(F/\mathbb{Q}_p)$.

So we get that f is in $\mathrm{D}^F_{\mathrm{dR}}(A)^{\mathrm{Gal}(F/\mathbb{Q}_p)}$, which is canonically isomorphic to $\mathrm{D}^{\mathbb{Q}_p}_{\mathrm{dR},0} \otimes_E A$. Thus f defines an analytic map over $\mathrm{Max}(A)$ with values in $\mathbb{P}(\mathrm{D}^{\mathbb{Q}_p}_{\mathrm{dR},0}) = \mathbb{P}(V_\tau)$, which is what we wanted. □

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Third Galois cohomology group of function fields of curves over number fields

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Let K be a number field or a p-adic field and F the function field of a curve over K. Let ℓ be a prime. Suppose that K contains a primitive ℓ-th root of unity. If $\ell = 2$ and K is a number field, then assume that K is totally imaginary. In this article we show that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol. This leads to the finite generation of the Chow group of zero-cycles on a quadric fibration of a curve over a totally imaginary number field.

1. Introduction

Let F be a field and ℓ a prime not equal to the characteristic of F. For $n \geq 1$, let $H^n(F, \mu_\ell^{\otimes n})$ be the n-th Galois cohomology group with coefficients in $\mu_\ell^{\otimes n}$. We have $F^*/F^{*\ell} \simeq H^1(F, \mu_\ell)$. For $a \in F^*$, let $(a) \in H^1(F, \mu_\ell)$ denote the image of the class of a in $F^*/F^{*\ell}$. Let $a_1, \ldots, a_n \in F^*$. The cup product $(a_1) \cdots (a_n) \in H^n(F, \mu_\ell^{\otimes n})$ is called a *symbol*. A theorem of Voevodsky [2003] asserts that every element in $H^n(F, \mu_\ell^{\otimes n})$ is a sum of symbols. Let $\alpha \in H^n(F, \mu_\ell^{\otimes n})$. The *symbol length* of α is defined as the smallest m such that α is a sum of m symbols in $H^n(F, \mu_\ell^{\otimes n})$.

Let K be a p-adic field. Then it is well-known that every element in $H^2(K, \mu_\ell^{\otimes 2})$ is a symbol and $H^n(K, \mu_\ell^{\otimes n}) = 0$ for all $n \geq 3$. Let F be the function of a curve over K. Suppose that K contains a primitive ℓ-th root of unity. If $\ell \neq p$, then it was proved in [Suresh 2010] (see [Brussel and Tengan 2014]) that the symbol length of every element in $H^2(F, \mu_\ell^{\otimes 2})$ is at most 2. If $p \neq \ell$, then it was proved in [Parimala and Suresh 2010] (see [Parimala and Suresh 2016]) that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol. If $\ell = p$, then it was proved in [Parimala and Suresh 2014] that for every central simple algebra A over F, the index of A divides the square of the period of A. In particular if $p = 2$, then the symbol length of every element in $H^2(F, \mu_2^{\otimes 2})$ is at most 2. Since $u(F) = 8$ [Heath-Brown 2010; Leep 2013] (see [Parimala and Suresh 2014]), it follows that every element in $H^3(F, \mu_2^{\otimes 3})$ is a symbol.

If F is the function field of a curve over a global field of positive characteristic p, $\ell \neq p$ and F contains a primitive ℓ-th root of unity, then it was proved in [Parimala and Suresh 2016] that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol.

Let K be a number field. A consequence of class field theory is that every element in $H^n(K, \mu_\ell^{\otimes n})$ is a symbol. A classical lemma of Tate states that given finitely many elements $\alpha_1, \ldots, \alpha_r \in H^2(K, \mu_\ell^{\otimes 2})$, there

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exist $a, b_i \in K^*$ such that $\alpha_i = (a)\cdot(b_i)$. Let F be the function field of a curve over K. Suresh [2004] proved a higher dimensional version of this lemma over F: given finitely any elements $\alpha_1, \dots, \alpha_r \in H^3(F, \mu_2^{\otimes 3})$, there exists $f \in F^*$ such that $\alpha_i = (f)\cdot\beta_i$ for some $\beta_i \in H^2(F, \mu_2^{\otimes 2})$. In particular if there exists an integer N such that the symbol length of every element in $H^2(F, \mu_2^{\otimes 2})$ is bounded by N, then the symbol length of every element in $H^3(F, \mu_2^{\otimes 3})$ is bounded by N. In [Lieblich et al. 2014], it was proved that such an integer N exists under the hypothesis that a conjecture of Colliot-Thélène on the Hasse principle for the existence of 0-cycles of degree 1 holds. However, unconditionally the existence of such N is still open.

In this paper we prove the following (see Corollary 7.8):

Theorem 1.1. *Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to* char(K). *Suppose that K contains a primitive ℓ-th root of unity and one of the following holds*:

(i) $\ell \neq 2$.

(ii) *K is a local field.*

(iii) *K is a totally imaginary number field.*

Then every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol.

The above theorem for K a p-adic field and $\ell \neq p$ is proved in [Parimala and Suresh 2010] (see [Parimala and Suresh 2016]). Our method in this paper is uniform, it covers both global and local fields at the same time and we do not exclude the case $\ell = p$.

We have the following (see Corollary 8.3):

Corollary 1.2. *Let K be a totally imaginary number field and F the function field of a curve over K. Let q be a quadratic form over F and $\lambda \in F^*$. If the dimension of q is at least* 5, *then $q \otimes \langle 1, -\lambda\rangle$ is isotropic.*

Let L be a field of characteristic not equal to 2 and $u(L)$ be the u-invariant of L. By a theorem of Pfister if $u(L) \leq 2^n$ for some n, then every element in $H^n(L, \mu_2^{\otimes n})$ is a symbol. Let K be a totally imaginary number field. Then it is well-known that $u(K)$ is 4. Let F be a function field over K of transcendence degree n. It is a wide open question whether $u(F) = 2^{n+2}$. The finiteness of $u(F)$ is not known even for $n = 1$. In the perspective of Pfister's theorem, the conclusion from (iii) of Theorem 1.1 strengthens the expectations that $u(F)$ is 8 for function fields of curves over totally imaginary number fields

In a related direction Colliot-Thélène raised the question whether every element of $H^{n+2}(F, \mu_\ell^{\otimes(n+2)})$ is a symbol if F is a function field of transcendence degree n over a totally imaginary number field. Our main theorem gives an affirmative answer to this question for function fields of curves.

For a smooth integral variety X over a field k, let $\mathrm{CH}_0(X)$ be the Chow group of 0-cycles modulo rational equivalence. If k is a number field and X a smooth projective geometrically integral curve, the Mordell–Weil theorem implies that $\mathrm{CH}_0(X)$ is finitely generated.

Let C be a smooth projective geometrically integral curve over a field k. Let $X \to C$ be an (admissible) quadric fibration (see [Colliot-Thélène and Skorobogatov 1993]). Let $\mathrm{CH}_0(X/C)$ be the kernel of the natural homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(C)$. If char$(k) \neq 2$, Colliot-Thélène and Skorobogatov identified

$\mathrm{CH}_0(X/C)$ with a certain subquotient of $k(C)^*$ [Colliot-Thélène and Skorobogatov 1993]. From this identification it follows that $\mathrm{CH}_0(X/C)$ is a 2-torsion group. Thus $\mathrm{CH}_0(X/C)$ is finitely generated if and only if it is finite. Suppose that k is a number field. If $\dim(X) \leq 2$, then the finiteness of $\mathrm{CH}_0(X/C)$ is a result of Gros [1987]. If $\dim(X) = 3$, then it was proved in [Colliot-Thélène and Skorobogatov 1993; Parimala and Suresh 1995] that $\mathrm{CH}_0(X/C)$ is finite. Thus for $\dim(X) \leq 3$, $\mathrm{CH}_0(X)$ is finitely generated. As a consequence of Corollary 1.2, we prove the following conjecture of Colliot-Thélène and Skorobogatov (see Theorem 8.4).

Theorem 1.3. *Let K be a totally imaginary number field, C a smooth projective geometrically integral curve over K. Let $X \to C$ be an admissible quadric fibration. If $\dim(X) \geq 4$, then $\mathrm{CH}_0(X/C) = 0$. In particular $\mathrm{CH}_0(X)$ is finitely generated.*

Let K be a global field of positive characteristic p or a local field with the characteristic of the residue field p. Let F be the function field of a curve over K and ℓ a prime not equal to p. Let us recall that the main ingredient in the proof of the fact that every element in $H^3(F, \mu_\ell^{\otimes 3})$ is a symbol [Parimala and Suresh 2010], is a certain local-global principle for divisibility of an element of $H^3(F, \mu_\ell^{\otimes 3})$ by a symbol in $H^2(F, \mu_\ell^{\otimes 2})$ [Parimala and Suresh 2010; 2016]. In fact it was proved that for a given $\zeta \in H^3(F, \mu_\ell^{\otimes 3})$ and a symbol $\alpha \in H^2(F, \mu_\ell^{\otimes 2})$ if for every discrete valuation ν of F there exists $f_\nu \in F^*$ such that $\zeta - \alpha \cdot (f_\nu)$ is unramified at ν, then there exists $f \in F^*$ such that $\zeta = \alpha \cdot (f)$. In the proof of this local-global principle, the existence of residue homomorphisms on $H^2(F, \mu_\ell^{\otimes 2})$ and $H^3(F, \mu_\ell^{\otimes 3})$ is used. However note that if K is a global field or a p-adic field with $\ell = p$, then there is no "residue homomorphism" on $H^2(F, \mu_\ell^{\otimes 2})$ which can be used to describe the unramified Brauer group.

We now briefly explain the main ingredients of our result. Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to characteristic of K. Suppose that K contains a primitive ℓ-th root of unity. Let ν be a discrete valuation on F and $\kappa(\nu)$ the residue field at ν. Then Kato [1986, Section 1] defined a residue homomorphism $H^3(F, \mu_\ell^{\otimes 3}) \to {}_\ell\mathrm{Br}(\kappa(\nu))$. Let $\zeta \in H^3(F, \mu_\ell^{\otimes 3})$ and $\alpha = [a, b) \in H^2(F, \mu_\ell^{\otimes 2})$. First we show that if there is a regular proper model $\mathscr{X}$ of F such that the triple $(\zeta, \alpha, \mathscr{X})$ satisfies certain assumptions, then there is a local global principle for the divisibility of ζ by α (see Theorem 6.5). One of the key assumptions is that $a \in F^*$ has some "nice" properties at closed points of $\mathscr{X}$ which are on the support of the prime ℓ and in the ramification of ζ or α (see Assumptions 5.1 and 6.3). These assumptions on a enable us to work in spite of the absence of a residue homomorphisms on $H^2(F, \mu_\ell^{\otimes 2})$ for discrete valuations with residue fields of characteristic ℓ and also enable us to blow up the given model so that there are no chilly loops (as defined by Saltman).

Let $\zeta \in H^3(F, \mu_\ell^{\otimes 3})$. First we choose a regular proper model $\mathscr{X}$ of F where the ramification of ζ and the support of ℓ is a union of regular curves with normal crossings on $\mathscr{X}$. For each irreducible curve C on $\mathscr{X}$ which is in the union of the ramification of ζ and support of ℓ, let β_C be the residue of ζ at C. Since the residue field $\kappa(C)$ at C is either a global field or a local field, β_C is a cyclic algebra. Using the class field theory and weak approximation, we write $\beta_C = [a_C, b_C)$ with some conditions on a_C and b_C at finitely many closed points of the model. Then we lift these a_C and b_C to $a, b \in F^*$ which satisfy some “nice” conditions and let $\alpha = [a, b)$. By the choice of a and b, α is unramified at all irreducible curves in the support of ℓ and also unramified at some predetermined finitely many closed points of the model. Suppose that $\ell \neq 2$ or K is a local field or K is a global field without real places. Then we show that there exists a sequence of blow-ups $\mathscr{Y}$ of $\mathscr{X}$ such that $\alpha = [a, b) \in H^2(F, \mu_\ell^{\otimes 2})$ and $\mathscr{Y}$ satisfies the assumption of Section 6. Thus, by the local global principle for the divisibility, there exists $f \in F^*$ such that $\zeta - \alpha \cdot (f)$ is unramified on $\mathscr{X}$. Then, using a result of Kato [1986], we arrive at the proof of Theorem 7.7.

2. Preliminaries

Lemma 2.1 [Colliot-Thélène 1999, Proposition 4.1.2(i)]. *Let K be a field with a discrete valuation ν and κ the residue field at ν. Let m be the maximal ideal of the valuation ring R at ν. Suppose that* $\text{char}(K) = 0$ *and* $\text{char}(\kappa) = \ell > 0$. *Suppose that K contains a primitive ℓ-th root of unity ρ. Then $\ell = x(\rho - 1)^{\ell-1}$ for some unit x at ν with $x \equiv -1$ modulo m. In particular $\nu(\rho - 1) = \nu(\ell)/(\ell - 1)$.*

Proof. The congruence $x \equiv -1$ modulo m holds according to the proof of [Colliot-Thélène 1999, Proposition 4.1.2(i)]. □

Lemma 2.2. *Suppose R is a discrete valuation ring with field of fractions K and residue field κ. Suppose that* $\text{char}(K) = 0$, $\text{char}(\kappa) = \ell > 0$ *and K contains a primitive ℓ-th root of unity ρ. Let $u \in R$ and $\bar{u} \in \kappa$ the image of u. If $1 - u(\rho - 1)^\ell \in R^\ell$, then $X^\ell - X + \bar{u}$ has a root in κ. The converse is true if R is complete.*

Proof. Let m be the maximal ideal of R. Suppose that $u \in m$. Then $\bar{u} = 0$ and $X^\ell - X$ has a root in κ.

Suppose that $u \in R$ is a unit. Suppose $1 - u(\rho - 1)^\ell \in R^\ell$. Let $z \in R$ with $z^\ell = 1 - u(\rho - 1)^\ell \in R$. Since $\rho - 1 \in m$, $1 - u(\rho - 1)^\ell$ is a unit in R and hence z is a unit in R with $z^\ell \equiv 1$ modulo m. Since $\text{char}(\kappa) = \ell$, $z \equiv 1$ modulo m. Thus $z = 1 + d$ for some $d \in m$. Since $z^\ell = (1 + d)^\ell = 1 + \ell d + \cdots + d^\ell$, all the nontrivial binomial coefficients are divisible by ℓ and $d \in m$, we have $z^\ell = 1 + \ell d y + d^\ell$ for some unit $y \in R$ with $y \equiv 1$ modulo m. Since $z^\ell = 1 - u(\rho - 1)^\ell$, we have $\ell d y + d^\ell = -u(\rho - 1)^\ell$.

We claim that $\nu(d) = \nu(\rho - 1)$. Suppose that $\nu(\ell d) = \nu(d^\ell)$. Then $\nu(\ell) + \nu(d) = \ell\nu(d)$ and hence $\nu(d) = \nu(\ell)/(\ell - 1) = \nu(\rho - 1)$ (Lemma 2.1). Suppose that $\nu(\ell d) < \nu(d^\ell)$. Then $\nu(\ell d y + d^\ell) = \nu(\ell d) = \nu(\ell) + \nu(d)$. Since $\ell d y + d^\ell = -u(\rho - 1)^\ell$, $\nu(\ell) + \nu(d) = \ell\nu(\rho - 1)$ and hence $\nu(d) = \ell\nu(\rho - 1) - \nu(\ell) = \ell\nu(\ell)/(\ell - 1) - \nu(\ell) = \nu(\ell)/(\ell - 1) = \nu(\rho - 1)$. Suppose that $\nu(\ell d y) > \nu(d^\ell)$. Then $\ell\nu(\rho - 1) = \nu(d^\ell) = \ell\nu(d)$ and hence $\nu(d) = \nu(\rho - 1)$.

Since $\nu(d) = \nu(\rho - 1)$, we have $d = w(\rho - 1)$ for some unit $w \in R$. By Lemma 2.1, we have $\ell = x(\rho - 1)^{\ell-1}$ with $x \equiv -1$ modulo m. Thus

$$-u(\rho - 1)^\ell = \ell d y + d^\ell = x y w(\rho - 1)^\ell + w^\ell(\rho - 1)^\ell$$

and hence

$$-u = w^\ell + x y w.$$

Since $x \equiv -1$ modulo m and $y \equiv 1$ modulo m, we have $\overline{w}^\ell - \overline{w} + \bar{u} = 0$. In particular $X^\ell - X + \bar{u}$ has a root in κ.

Suppose R is complete and $X^\ell - X + \bar{u}$ has a root in κ. Since $\mathrm{char}(\kappa) = \ell$, $X^\ell - X + \bar{u}$ has ℓ distinct roots in κ. Since R is complete, $X^\ell - X + u$ has a root w in R. Let $d = w(\rho - 1) \in R$. Then, as above, we have $(1 + d)^\ell = 1 + \ell dy + d^\ell$ for some $y \in R$ with $y \equiv 1$ modulo m_R. By Lemma 2.1, we have $\ell = x(\rho - 1)^{\ell - 1}$ for some $x \in R$ with $x \equiv -1$ modulo m_R. Since $w^\ell = w - u$ and $d = w(\rho - 1)$, we have

$$\begin{aligned}(1 + d)^\ell = 1 + \ell dy + d^\ell &= 1 + \ell w(\rho - 1)y + w^\ell(\rho - 1)^\ell \\ &= 1 + \ell w(\rho - 1)y + w(\rho - 1)^\ell - u(\rho - 1)^\ell \\ &= 1 + xyw(\rho - 1)^\ell + w(\rho - 1)^\ell - u(\rho - 1)^\ell \\ &= 1 + w(\rho - 1)^\ell(xy + 1) - u(\rho - 1)^\ell.\end{aligned}$$

Since $xy+1 \equiv 0$ modulo m, we have $(1+d)^\ell = 1-u(\rho-1)^\ell$ modulo $(\rho-1)^\ell m$ and hence $1-u(\rho-1)^\ell \in R^{*\ell}$ (see [Epp 1973, Section 0.3]). □

Let R be a regular domain with field of fractions K and let L/K be a finite separable extension. Let S be the integral closure of R in L. We say that L/K is *unramified* at a prime ideal P of R, if S_P/PS_P is a separable algebra over the field R_P/PR_P, where $S_P = S \otimes_R R_P$ is the same as the integral closure of the local ring R_P in L. We say that L/K is *unramified* on R if it is unramified at every prime ideal of R. If L/K is unramified at a prime ideal P of R, the separable R_P/PR_P-algebra S_P/PS_P is called the *residue field* of L at P. Note that S_P/PS_P is a product of separable field extensions of R_P/PR_P. If R is a regular local ring, then L/K is unramified at R if and only if the discriminant of L/K is a unit in R (see [Milne 1980, Exercise 3.9, page 24]). Thus in particular, L/K is unramified on R if and only is L/K is unramified at all height one prime ideals of R. If L is a product of fields L_i with $K \subset L_i$, then we say that L/K is *unramified* on R if each L_i/K is unramified on R.

We have the following (see [Epp 1973, Proposition 1.4]):

Proposition 2.3. *Suppose R is a discrete valuation ring with field of fractions K and residue field κ. Suppose that* $\mathrm{char}(K) = 0$, $\mathrm{char}(\kappa) = \ell > 0$ *and K contains a primitive ℓ-th root of unity ρ. Let $u \in R$ and $L = K[X]/(X^\ell - (1 - u(\rho - 1)^\ell))$. Let S be the integral closure of R in L. Then L/K is unramified on R and*:

- *If $X^\ell - X + \bar{u}$ is irreducible in $\kappa[X]$, then S has a unique maximal ideal, it is generated by the maximal ideal m_R of R, and $S/m_RS \simeq \kappa[X]/(X^\ell - X + \bar{u})$, where $\bar{u}$ is the image of u in κ.*
- *If $X^\ell - X + \bar{u}$ is reducible in $\kappa[X]$, then m_RS is the product of ℓ distinct maximal ideals of S and again $S/m_RS \simeq \kappa[X]/(X^\ell - X + \bar{u})$.*

Proof. Without loss of generality we assume that R is complete. If L is not a field, which happens if and only if $X^\ell - X - \bar{u}$ is reducible in $\kappa[X]$ by Lemma 2.2, then the result is clearly true. So we further assume that L is a field and $X^\ell - X - \bar{u}$ is irreducible in $\kappa[X]$. Then S is a complete discrete valuation ring. Let m_R be the maximal ideal of R and m_S the maximal ideal of S. Since $1 - u(\rho - 1)^\ell \in S^\ell$, by

Lemma 2.2, $X^\ell - X - \bar{u}$ has a root in S/m_S. Since $[S/m_S : \kappa] \le \ell$, $S/m_S \simeq \kappa[X]/(X^\ell - x + \bar{u})$ and hence the ramification index of S over R is 1 and $m_S = m_R S$. It follows that L/K unramified on R. □

Corollary 2.4. *Suppose that A is a regular local ring of dimension two with field of fractions F, maximal ideal m and residue field κ. Suppose that* $\operatorname{char}(F) = 0$, $\operatorname{char}(\kappa) = \ell > 0$ *and F contains a primitive ℓ-th root of unity ρ. Let $u \in A$ and $L = F[X]/(X^\ell - (1 - u(\rho - 1)^\ell))$. Suppose that L is a field. Let S be the integral closure of A in L. Then L/F is unramified on A and $S/mS \simeq \kappa[X]/(X^\ell - X + \bar{u})$, where $\bar{u}$ is the image of u in κ.*

Proof. Since $\operatorname{char}(\kappa) = \ell$ and $\rho^\ell = 1$, $1 - \rho$ is in the maximal ideal of A and hence $1 - u(\rho - 1)^\ell$ is a unit in A. Let P be a prime ideal of A of height one. Suppose $\operatorname{char}(A/P) \neq \ell$. Since $1 - u(\rho - 1)^\ell$ is a unit in A, L/F is unramified at P. If $\operatorname{char}(A/P) = \ell$, then by Proposition 2.3, L/F is unramified at P. Thus L/F is unramified on A.

Let $m = (\pi, \delta)$ be the maximal ideal of A. Since L/F is unramified on A, $S/\pi S$ is a regular semilocal ring (see [Milne 1980, Proposition 3.17, page 27]). Suppose that $\operatorname{char}(A/(\pi)) \neq \ell$. Since $1 - u(\rho - 1)^\ell$ is a unit at π, L/F is unramified at π and $S \otimes_A A_{(\pi)}/(\pi) \simeq (A_{(\pi)}/(\pi))[X]/(X^\ell - (1 - \bar{u}(\bar{\rho} - 1)^\ell))$, where $^-$ denotes the image modulo (π). Hence by Proposition 2.3, $S/(\pi, \delta)S = \kappa[X]/(X^\ell - X + \bar{u})$. Suppose that $\operatorname{char}(A/(\pi)) = \ell$. Then, by Proposition 2.3, the field of fractions of $S/\pi S$ is the field of fractions of $(A/(\pi))[X]/(X^\ell - X + \bar{u})$. Since u is a unit in $A/(\pi)$, $A/(\pi)[X]/(X^\ell - X + \bar{u})$ is a regular local ring and hence $S/\pi S \simeq A/(\pi)[X]/(X^\ell - X + \bar{u})$. Hence $S/(\pi, \delta)S = \kappa[X]/(X^\ell - X + \bar{u})$. □

Let K be a field and ℓ a prime. Then every nontrivial element in $H^1(K, \mathbb{Z}/\ell)$ is represented by a pair (L, σ), where L/K is a cyclic field extension of degree ℓ and σ a generator of $\operatorname{Gal}(L/K)$.

Suppose $\ell \neq \operatorname{char}(K)$ and K contains a primitive ℓ-th root of unity. Fix a primitive ℓ-th root of unity $\rho \in K$. Let L/K be a cyclic extension of degree ℓ. Then, by Kummer theory, we have $L = K(\sqrt[\ell]{a})$ for some $a \in K^*$ and $\sigma \in \operatorname{Gal}(L/K)$ given by $\sigma(\sqrt[\ell]{a}) = \rho\sqrt[\ell]{a}$ is a generator of $\operatorname{Gal}(L/K)$. Thus we have an isomorphism $K^*/K^{*\ell} \to H^1(K, \mathbb{Z}/\ell\mathbb{Z})$ given by sending the class of a in $K^*/K^{*\ell}$ to the pair (L, σ), where $L = K[X]/(X^\ell - a)$ and $\sigma(\sqrt[\ell]{a}) = \rho\sqrt[\ell]{a}$. Let $a \in K^*$. If the image of the class of a in $H^1(F, \mathbb{Z}/\ell\mathbb{Z})$ is (L, σ) and i is coprime to ℓ, then the image of a^i is (L, σ^i). In particular $(L, \sigma)^i = (L, \sigma^i)$ for all i coprime to ℓ.

Suppose $\operatorname{char}(K) = \ell$ and L/K is a cyclic extension of degree ℓ. Then, by Artin–Schreier theory, $L = K[X]/(X^\ell - X + a)$ for some $a \in K$. The element $\sigma \in \operatorname{Gal}(L/K)$ given by $\sigma(x) = x + 1$, where $x \in L$ is the image of X in L, is a generator of $\operatorname{Gal}(L/K)$. Let $\wp : K \to K$ be the Artin–Schreier map $\wp(b) = b^\ell - b$. We have an isomorphism $K/\wp(K) \to H^1(K, \mathbb{Z}/\ell\mathbb{Z})$ given by sending the class of a to the pair (L, σ), where $L = K[X]/(X^\ell - X + a)$ and $\sigma(x) = x + 1$. We note that if the image the class of a is (L, σ), then the image of the class of ia is (L, σ^i) for all $1 \le i \le \ell - 1$.

In either case ($\operatorname{char}(K) \neq \ell$ or $\operatorname{char}(K) = \ell$), for $a \in K^*$ (or K), the pair (L, σ) is denoted by $[a)$. Sometimes, by abuse of notation, we also denote the cyclic extension L by $[a)$.

Let R be a regular ring of dimension at most 2 with field of fractions K and ℓ a prime. If ℓ is not equal to $\operatorname{char}(K)$, then assume that K contains a primitive ℓ-th root of unity ρ. Suppose $L = [a)$ is a cyclic extension of K of degree ℓ. Let P be a prime ideal of R, $\kappa(P) = R_P/PR_P$ and S_P the integral closure of R_P in L. Suppose $\mathrm{char}(\kappa(P)) \neq \ell$. Then $L = K[X]/(X^\ell - a)$ and hence $S_P/PS_P \simeq \kappa(P)[X]/(X^\ell - \bar{a})$ where $\bar{a}$ is the image of a in $\kappa(P)$. Suppose $\mathrm{char}(\kappa(P)) = \ell$, $\mathrm{char}(K) \neq \ell$ and $a = 1 - u(\rho - 1)^\ell$ for some $u \in R_P$. Then, by (Proposition 2.3 and Corollary 2.4), $S_P/PS_P \simeq \kappa(P)[X]/(X^\ell - X + \bar{u})$. Suppose $\mathrm{char}(\kappa(P)) = \mathrm{char}(K) = \ell$ and $a \in R_P$. Then $L = K[X]/(X^\ell - X + a)$ and hence $S_P/PS_P \simeq \kappa(P)[X]/(X^\ell - X + \bar{a})$. Thus, in either case, S_P/PS_P is either a cyclic field extension of degree ℓ over $\kappa(P)$ or the split extension of degree ℓ over $\kappa(P)$ and we denote these S_P/PS_P by $[a(P))$. If $P = (\pi)$ for some $\pi \in R$, then we also denote $[a(P))$ by $[a(\pi))$. If P induces a discrete valuation ν on K, then we also denote $[a(P))$ by $[a(\nu))$. For an element $b \in R$, we also denote the image of b in R/P by $b(P)$. If $b \in R$ and $c \in R/P$, we write $b = c \in R/P$ for $b \equiv c$ modulo P.

Lemma 2.5. *Let A be a semilocal regular ring of dimension at most two with field of fractions F. Let ℓ be a prime not equal to the characteristic of F. Suppose that F contains a primitive ℓ-th root of unity. For each maximal ideal m of A, let $[u_m)$ be a cyclic extension of A/m of degree ℓ. Then there exists $a \in A$ such that*:

- *$[a)$ is unramified on A with residue field $[u_m)$ at each maximal ideal m of A.*
- *If $\ell = 2$ and A/m is finite for all maximal ideals m of A, then a can be chosen to be a sum of two squares in A.*

Proof. Let $\rho \in F$ be a primitive ℓ-th root of unity. Let m be a maximal ideal of A. If $\mathrm{char}(A/m) \neq \ell$, then let $b_m = (1 - u_m/(\rho - 1)^\ell) \in A/m$. If $\mathrm{char}(A/m) = \ell$, then let $b_m = u_m \in A/m$. Choose $b \in A$ with $b = b_m \in A/m$ for all maximal ideals m of A and $a = 1 - b(\rho - 1)^\ell$. Let m be a maximal ideal of A. Suppose that $\mathrm{char}(A/m) \neq \ell$. Then, by the choice of a and b, we have $a = 1 - b_m(\rho - 1)^\ell = u_m \in A/m$. Thus $[a)$ is unramified on A_m with the residue field $[u_m)$ at m. Suppose that $\mathrm{char}(A/m) = \ell$. Then, by (Proposition 2.3 and Corollary 2.4), $[a)$ is unramified on A_m with the residue field $[\bar{b})$. Since $b = b_m = u_m \in A/m$, the residue field of $[a)$ at m is $[u_m)$.

Suppose $\ell = 2$ and A/m is a finite field for all maximal ideals m of A. Let m be a maximal ideal of A. Suppose that $\mathrm{char}(A/m) \neq 2$. Since every element of A/m is a sum of two squares in A/m [Scharlau 1985, page 39, 3.7], there exist $x_m, y_m \in A/m$ such that $x_m^2 + y_m^2 = 1 - 4u_m$. Suppose that $\mathrm{char}(A/m) = 2$. Since A/m is a finite field, every element in A/m is a square. Let $y_m \in A/m$ be such that $y_m^2 = u_m$. Let $x, y \in A$ be such that for every maximal ideal m of A:

- If $\mathrm{char}(A/m) \neq 2$, then $x = \frac{1}{4}(x_m - 1) \in A/m$ and $y = \frac{1}{2}y_m \in A/m$.
- If $\mathrm{char}(A/m) = 2$, then $x = 0 \in A/m$ and $y = y_m \in A/m$.

Let $a = (1 + 4x)^2 + (2y)^2 \in A$. Let m be a maximal ideal of A. Suppose $\mathrm{char}(A/m) \neq 2$. Then $a = x_m^2 + y_m^2 = u_m \in A/m$ and hence $[a)$ is unramified on A_m with residue field at m equal to $[u_m)$. Suppose that $\mathrm{char}(A/m) = 2$. Then $\frac{1}{4}(1 - a) = u_m \in A/m$ and hence $[a)$ is unramified on A_m with residue field $[u_m)$ (Proposition 2.3 and Corollary 2.4). □

Lemma 2.6. *Let R be a semilocal regular domain of dimension 1 and K its field of fractions. Let ℓ be a prime not equal to* $\operatorname{char}(K)$. *Suppose that K contains a primitive ℓ-th root of unity ρ. Let $L = K(\sqrt[\ell]{u})$ for some $u \in R$. Let $m_1, \ldots, m_r, m_{r+1}, \ldots, m_n$ be the maximal ideals of R. Suppose that* $\operatorname{char}(\kappa(m_j)) = \ell$ *and L/K is unramified at m_j for all $r+1 \leq j \leq n$. Then there exists $v \in R$ such that $L = K(\sqrt[\ell]{v})$, $v \equiv u$ modulo m_i for all $1 \leq i \leq r$ and $(1-v)/(\rho-1)^\ell \in R_{m_j}$ for all $r+1 \leq j \leq n$.*

Proof. For a maximal ideal m of R, let K_m denote the field of factions of the completion of R at m.

Let $r+1 \leq j \leq n$. Since $\operatorname{char}(\kappa(m_j)) = \ell$ and L/K unramified at m_j, the residue field of L at m_j is $\kappa(m_j)[X]/(X^\ell - X + \overline{w}_j)$ for some $w_j \in R_{m_j}$. Since the residue field of $K[X]/(X^\ell - (1 - w_j(\rho-1)^\ell))$ is isomorphic to $\kappa(m_j)[X]/(X^\ell - X + \overline{w}_j)$ (Proposition 2.3 and Corollary 2.4),

$$L \otimes K_{m_j} \simeq K_{m_j}[X]/(X^\ell - (1 - w_j(\rho-1)^\ell)).$$

Since $\operatorname{char}(K) \neq \ell$ and $L = K(\sqrt[\ell]{u})$, there exists $\theta_j \in K_{m_j}$ such that $u\theta_j^\ell = 1 - w_j(\rho-1)^\ell$. Let N be an integer larger than the sum of the valuations of u and $(\rho-1)^\ell$ at all m_i. By the weak approximation, there exists $\theta \in K$ such that $\theta \equiv 1$ modulo m_i for $1 \leq i \leq r$ and $\theta\theta_j^{-1} \equiv 1$ modulo m_j^{N+1} for $r+1 \leq j \leq n$.

Let $v = u\theta^\ell$. Let $1 \leq i \leq r$. Since $\theta \equiv 1$ modulo m_i, $v \equiv u$ modulo m_i. Let $r+1 \leq j \leq n$. Let $\pi_j \in R$ be a generator of the ideal m_j. Then $\theta^\ell\theta_j^{-\ell} = 1 + a_j\pi_j^{N+1}$ for some $a_j \in \hat{R}_{m_j}$. Since $u\theta_j^\ell = 1 - w_j(\rho-1)^\ell \in R_{m_j}$ is a unit and N is bigger than the sum of the valuations of u and $(\rho-1)^\ell$, we have $\theta_j^\ell a_j \pi_j^{N+1} = b_j(\rho-1)^\ell$ for some $b_j \in \hat{R}_{m_j}$. Hence

$$v = u\theta^\ell = u\theta_j^\ell + ub_j(\rho-1)^\ell = 1 - w_j(\rho-1)^\ell + ub_j(\rho-1)^\ell = 1 - c_j(\rho-1)^\ell$$

for some $c_j \in \hat{R}_{m_j}$ Since $c_j = (1-v)/(\rho-1)^\ell \in K \cap \hat{R}_{m_j} = R_{m_j}$, v has the required properties. □

The following is a generalization of a result of Saltman [2008, Proposition 0.3].

Lemma 2.7. *Let A be a UFD. For $1 \leq i \leq n$, let $I_i = (a_i) \subset A$ with $gcd(a_i, a_j) = 1$ for all $i \neq j$. For each $i < j$, let $I_{ij} = I_i + I_j$. Suppose that the ideals I_{ij} are comaximal. Then*

$$A \to \bigoplus_i A/I_i \to \bigoplus_{i<j} A/I_{ij}$$

is exact, where for $i < j$, the map from $A/I_i \oplus A/I_j \to A/I_{ij}$ is given by $(x, y) \mapsto x - y$.

Proof. Proof by induction on n. The case $n = 2$ is in [Saltman 2008, Lemma 0.2]. Assume that $n \geq 3$. Suppose $(x_i) \in A/I_i$ maps to zero in $\oplus A/I_{ij}$. By induction, there exists $b \in A$ such that $b = x_i \in A/I_i$ for $1 \leq i \leq n-1$. We claim that $I_1 \cap \cdots \cap I_{n-1} + I_n = (I_1 + I_n) \cap \cdots \cap (I_{n-1} + I_n)$. Since both sides contain I_n, it is enough to prove the equality modulo I_n. Since $\gcd(a_i, a_j) = 1$ for all $i \neq j$, we have $I_1 \cap \cdots \cap I_{n-1} = Aa_1 \cdots a_{n-1}$ and hence $I_1 \cap \cdots \cap I_{n-1} + I_n/I_n = (A/I_n)\bar{a}_1 \cdots \bar{a}_{n-1}$. Since I_{ij} are comaximal, $I_{in}/I_n = (A/I_n)\bar{a}_i$ are comaximal for $1 \leq i \leq n-1$ and hence $(A/I_n)\bar{a}_1 \cdots \bar{a}_{n-1} = (A/I_n)\bar{a}_1 \cap \cdots \cap (A/I_n)\bar{a}_{n-1}$. Let $b_1 \in A/(I_1 \cap \cdots \cap I_{n-1})$ be the image of b. Then, by the case $n = 2$, there exists $a \in A$ such that $a = b_1 \in A/I_1 \cap \cdots \cap I_{n-1}$ and $a = x_n \in A/I_n$.Thus a has the required properties. □

3. Central simple algebras

Let K be a field, L/K a cyclic extension of degree n with $\sigma \in \mathrm{Gal}(L/K)$ a generator and $b \in K^*$. Let (L, σ, b) denote the cyclic algebra $L \oplus Lx \oplus \cdots \oplus Lx^{n-1}$ with relations $x^n = b$, $x\lambda = \sigma(\lambda)x$ for all $\lambda \in L$. Then (L, σ, b) is a central simple algebra over K and represents an element in the n-torsion subgroup ${}_n\mathrm{Br}(K)$ of the Brauer group $\mathrm{Br}(K)$ [Albert 1939, Theorem 18, page 98]. Suppose that n is coprime to $\mathrm{char}(K)$ and K contains a primitive n-th root of unity. Then $L = K(\sqrt[n]{a})$ for some $a \in K^*$. Fix a primitive n-th root of unity ρ in K. Let σ be the generator of $\mathrm{Gal}(L/K)$ given by $\sigma(\sqrt[n]{a}) = \rho\sqrt[n]{a}$. Then, the cyclic algebra (L, σ, b) is denoted by $[a, b)$. Suppose that n is prime and equal to $\mathrm{char}(K)$. Then, $L = K[X]/(X^n - X + a)$ for some $a \in K$. If σ is the generator of $\mathrm{Gal}(L/K)$ given by $\sigma(x) = x + 1$, then the cyclic algebra (L, σ, b) is also denoted by $[a, b)$.

For any Galois module M over K, let $H^n(K, M)$ denote the Galois cohomology of K with coefficients in M. Let ℓ be a prime. Let $\mathbb{Z}/\ell(i)$ be the Galois modules over K as in [Kato 1986, Section 0]. We have canonical isomorphisms $H^1(K, \mathbb{Z}/\ell) \simeq \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{\mathrm{ab}}/K), \mathbb{Z}/\ell)$ and ${}_\ell\mathrm{Br}(K) \simeq H^2(K, \mathbb{Z}/\ell(1))$, where K^{ab} is the maximal abelian extension of K [Kato 1986, Section 0].

Suppose A is a regular domain with field of fractions F. We say that an element $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$ is *unramified* on A if α is represented by a central simple algebra over F which comes from an Azumaya algebra over A. If it is not unramified, then we say that α is *ramified* on A. Suppose P is a prime ideal of A and $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$. We say that α is *unramified* at P if α is unramified on A_P. If α is not unramified at P, then we say that α is *ramified* at P. Suppose that α is unramified at P. Let $\mathscr{A}$ be an Azumaya algebra over A_P with the class of $\mathscr{A} \otimes_{A_P} F$ equal to α. The algebra $\bar{\alpha} = \mathscr{A} \otimes_{A_P} (A_P/PA_P)$ is called the *specialization* of α at P. Since A_P is a regular local ring, the class of $\bar{\alpha}$ is independent of the choice of $\mathscr{A}$. Let $a, b \in F$ and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. If the cyclic extension $[a)$ is unramified at P and b is a unit at P, then α is unramified at P and the specialization of α at P is $[a(P), b(P))$, where $[a(P))$ is the residue field of $[a)$ at P and $b(P)$ is the image of b in A_P/PA_P.

Suppose that R is a discrete valuation ring with field of fractions K and residue field κ. Let ℓ be a prime not equal to $\mathrm{char}(K)$. Suppose that $\mathrm{char}(\kappa) \neq \ell$ or $\mathrm{char}(\kappa) = \ell$ with $\kappa = \kappa^\ell$. Then there is a *residue homomorphism* $\partial : H^2(K, \mathbb{Z}/\ell(1)) \to H^1(\kappa, \mathbb{Z}/\ell)$ [Kato 1986, Section 1]. Further a class $\alpha \in H^2(K, \mathbb{Z}/\ell(1))$ is unramified at R if and only if $\partial(\alpha) = 0$. Let $a, b \in K^*$. If $[a)$ is unramified at R, then $\partial([a, b)) = [a(\nu))^{\nu(b)}$, where ν is the discrete valuation on K. In particular if $[a)$ is unramified on R and ℓ divides $\nu(b)$, then $[a, b)$ is unramified on R.

Lemma 3.1 ([Auslander and Goldman 1960, Proposition 7.4], see [Lieblich et al. 2014, Lemma 3.1]). *Let A be a regular ring of dimension 2 and F its field of fractions. Let ℓ be a prime not equal to* $\mathrm{char}(F)$ *and $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$. If α is unramified at all height one prime ideals of A, then α is unramified on A.*

Lemma 3.2. *Let R be a complete discrete valuation ring with field of fractions K and residue field κ. Let ℓ be a prime not equal to* $\mathrm{char}(\kappa)$. *Let D be a central simple algebra of index ℓ over K. Suppose that D is ramified at R. If L/K is the unramified extension of K with residue field equal to the residue of D at R, then $D \otimes L$ is a split algebra.*

Proof. We have $D = D_0 \otimes (L, \sigma, \pi)$ for some generator of $\mathrm{Gal}(L/K)$, π a parameter in R and D_0 unramified at R (see [Parimala et al. 2018, Lemma 4.1]). Further $\ell = \mathrm{ind}(D) = \mathrm{ind}(D_0 \otimes L)[L : K]$ (see [loc. cit., Lemma 4.2]). Since D is ramified at R, $[L : K] = \ell$ and hence $D_0 \otimes L = 0$. Hence $D_0 = (L, \sigma, u)$ for some $u \in K$ and $D = (L, \sigma, u\pi)$. Thus $D \otimes L$ is a split algebra. □

Lemma 3.3. *Let A be a complete regular local ring of dimension 2 with field of fractions F and residue field κ. Suppose that κ is a finite field. Let $m = (\pi, \delta)$ be the maximal ideal of A. Let ℓ be a prime not equal to* $\mathrm{char}(F)$ *and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$ for some $a, b \in F^*$. Suppose that*:

- *If* $\mathrm{char}(\kappa) = \ell$, *then the cyclic extension $[a)$ is unramified on A.*
- *α is unramified on A except possibly at δ.*
- *The specialization of α at π is unramified on $A/(\pi)$.*

Then $\alpha = 0$.

Proof. Suppose that $\mathrm{char}(\kappa) \neq \ell$. Then, it follows from [Reddy and Suresh 2013, Proposition 3.4] that $\alpha = 0$ (see [Parimala et al. 2018, Corollary 5.5]).

Suppose that $\mathrm{char}(\kappa) = \ell$. Since F is the field of fractions of A, without loss of generality, we assume that $b \in A$ and not divisible by θ^ℓ for any prime $\theta \in A$. Write $b = v\delta^n\theta_1^{n_1} \cdots \theta_r^{n_r}$ for some distinct primes $\theta_i \in A$ with $(\delta) \neq (\theta_i)$ for all i, $1 \le n_i \le \ell - 1$, $0 \le n \le \ell - 1$ and $v \in A$ a unit. Since κ is a finite field, A is complete and $[a)$ is unramified on A, we have $[a, v) = 0$ and hence $\alpha = [a, b) = [a, \delta^n\theta_1^{n_1} \cdots \theta_r^{n_r})$.

Since $[a)$ is unramified on A, for any prime $\theta \in A$, $[a, \theta)$ is unramified on A except possibly at θ. Let $1 \le j \le r$. Since $\alpha = [a, b) = [a, \delta^n) \prod [a, \theta_i^{n_i})$, $[a, \delta^n)$ and $[a, \theta_i^{n_i})$ are unramified at θ_j for all $i \neq j$, $[a, \theta_j^{n_j})$ is unramified at θ_j and hence $[a, \theta_j^{n_j})$ is unramified on A (see Lemma 3.1). Since κ is a finite field and A is complete, $[a, \theta_j^{n_j}) = 0$. Thus, we have $\alpha = [a, \delta^n)$.

If $n = 0$, then $\alpha = 0$. Suppose $1 \le n \le \ell - 1$. Let $\bar{\alpha}$ be the specialization of α at π. Since $\alpha = [a, \delta^n)$ and $[a)$ is unramified at π, we have $\bar{\alpha} = [a(\pi), \bar{\delta}^n)$, where $[a(\pi))$ is the residue field of $[a)$ at π and $\bar{\delta}$ is the image of δ in $A_P/(\pi)$. Since $\bar{\alpha}$ is unramified on $A/(\pi)$, A is complete and κ is a finite field, $\bar{\alpha} = [a(\pi), \bar{\delta}^n) = 0$. Since $\partial(\bar{\alpha}) = [a(m))^n = 1$ and n is coprime to ℓ, $[a(m)) = 0$. Since A is complete, $[a)$ is trivial and hence $\alpha = 0$. □

We now recall the chilly, cool, hot and cold points and the chilly loops associated to a central simple algebra, due to Saltman [2007; 2008]. Let $\mathscr{X}$ be a regular integral excellent scheme of dimension 2 and F its field of fractions. Let ℓ be a prime which is not equal to $\mathrm{char}(F)$. Suppose that F contains a primitive ℓ-th root of unity. Let $\alpha \in H^2(F, \mathbb{Z}/\ell(1))$. Suppose that $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_n\}$ for some regular irreducible curves D_i on $\mathscr{X}$ with normal crossings. Suppose $P \in D_i \cap D_j$ is a closed point. Let A_P be the local ring at P. Let $\pi_i, \pi_j \in A_P$ be primes defining D_i and D_j at P respectively. Suppose that $\mathrm{char}(\kappa(P)) \neq \ell$. Suppose that $\alpha = \alpha_0 + (u, \pi_i) + (v, \pi_j)$ for some α_0 unramified at P, u, v units at P. We say that P is a *chilly point* of α if $u(P)$ and $v(P)$ generate the same nontrivial subgroup of $\kappa(P)^*/\kappa(P)^{*\ell}$, a *cool point* of α if $u(P), v(P) \in \kappa(P)^{*\ell}$, a *hot point* of α if $u(P)$ and $v(P)$ generate different subgroup of $\kappa(P)^*/\kappa(P)^{*\ell}$. We say that P is a *cold point* of α If $\alpha = \alpha_0 + (u\pi_i, v\pi_j^s)$ for some α_0 unramified at P, u, v units at P and s coprime to ℓ.

Let Γ be a graph with vertices D_i's and edges as chilly points, i.e., two distinct vertices D_i and D_j have an edge between them if there is a chilly point in $D_i \cap D_j$. A loop in this graph is called a *chilly loop* on $\mathscr{X}$. Let $\mathscr{X}\left[\frac{1}{\ell}\right]$ be the open subscheme of $\mathscr{X}$ obtained by inverting ℓ. Since, by the definition of chilly point, $\mathrm{char}(\kappa(P)) \neq \ell$ for any chilly point P, we have the following

Proposition 3.4 [Saltman 2007, Corollary 2.9]. *There exists a sequence of blow-ups $\mathscr{X}' \to \mathscr{X}$ centered at closed points $P \in \mathscr{X}\left[\frac{1}{\ell}\right]$ such that α has no chilly loops on $\mathscr{X}'$.*

Let K be a global field and ℓ a prime. Let $\beta \in {}_{\ell}\mathrm{Br}(K)$. Let ν be a discrete vacation of K, K_ν the completion of K at ν and $\kappa(\nu)$ the residue field at ν. Since K_ν is a local field, the invariant map gives an isomorphism $\partial_\nu : {}_{\ell}\mathrm{Br}(K_\nu) = H^2(K_\nu, \mathbb{Z}/\ell(1)) \to H^1(\kappa(\nu), \mathbb{Z}/\ell)$.

Proposition 3.5. *Let K be a global field and ℓ a prime. If ℓ is not equal to* $\mathrm{char}(K)$, *then assume that K contains a primitive ℓ-th root of unity ρ. Let $\beta \in {}_{\ell}\mathrm{Br}(K)$. Let S be a finite set of discrete valuations of K containing all the discrete valuations ν of K with $\partial_\nu(\beta) \neq 0$. Let S' be a finite set of discrete valuations of K with $S \cap S' = \varnothing$. Let $a \in K^*$ and for each $\nu \in S'$, let $n_\nu \geq 2$ be an integer. Suppose that for every $\nu \in S$, $[a)$ is unramified at ν with $\partial_\nu(\beta) = [a(\nu))$. Further assume that if $\ell = 2$, then $\beta \otimes K_\nu(\sqrt{a}) = 0$ for all real places ν of K. Then there exists $b \in K^*$ such that*:

- $\beta = [a, b)$.
- *If $\nu \in S$, then $\nu(b) = 1$.*
- *If $\nu \in S'$, then $\nu(b-1) \geq n_\nu$.*

Proof. Let $L = [a)$. Let $\nu \in S$. If $\partial_\nu(\beta) = 0$, then $\beta \otimes K_\nu = 0$ [Cassels and Fröhlich 1967, page 131]. Suppose that $\partial_\nu(\beta) \neq 0$. Then $[a(\nu))$ is a field extension of $\kappa(\nu)$ of degree ℓ and hence $L \otimes_K K_\nu$ is a degree ℓ field extension of K_ν. Thus $\beta \otimes_K (L \otimes K_\nu) = 0$ [loc. cit., page 131]. Suppose ν is a real place of K. Then, by the assumption on a, $\beta \otimes_K (L \otimes_K K_\nu) = 0$. Thus $\beta \otimes L = 0$ [loc. cit., page 187] and hence there exists $c \in K^*$ such that $\beta = [a, c)$ [Albert 1939, page 94].

Let R be the semilocal ring at the discrete valuations in $S \cup S'$. Replacing c by $c\theta^\ell$ for some $\theta \in K^*$, we assume that $c \in R$. For $\nu \in S \cup S'$, let $\pi_\nu \in R$ be a parameter at ν. Let $\nu \in S$. Since $[a)$ is unramified at ν, $\partial_\nu(\beta) = \partial_\nu([a, c)) = [a(\nu))^{\nu(c)}$. Suppose $[a(\nu))$ is nontrivial. Since, by the hypothesis, $\partial_\nu(\beta) = [a(\nu))$, $\nu(c) - 1$ is divisible by ℓ. Since $[L : K] = \ell$, $\pi_\nu^{\nu(c)-1}$ is a norm from $L \otimes_K K_\nu/K_\nu$. Suppose that $[a(\nu))$ is trivial. Then $L \otimes_K K_\nu$ is the split extension and hence every element of K_ν is a norm from $L \otimes_K K_\nu/K_\nu$. Thus for each $\nu \in S$, there exists $x_\nu \in L \otimes_K K_\nu$ with norm $\pi_\nu^{\nu(c)-1}$. Let $\nu \in S'$. Then $\partial_\nu(\beta) = 0$ and we have $\beta \otimes K_\nu = [a, c) \otimes K_\nu = 0$ [Cassels and Fröhlich 1967, page 131]. Hence c is a norm from $L \otimes_K K_\nu$. For each $\nu \in S'$, $x_\nu \in L \otimes_K K_\nu$ with norm c. Let $z \in L$ be sufficiently close to x_ν such that $\nu(N_{L\otimes_K K_\nu}(z) - \pi_\nu^{\nu(c)-1}) \geq \nu(c)$ for all $\nu \in S$ and $\nu(N_{L\otimes_K K_\nu}(z) - c) \geq \nu(c) + n_\nu$ for all $\nu \in S'$.

Let d be the norm of z and $b=cd^{-1}$. Then $\beta=[a, cd^{-1})=[a, b)$. Let $\nu\in S$. Since $\nu(d-\pi_\nu^{\nu(c)-1})\geq\nu(c)$, we have $\nu(d)=\nu(c)-1$ and hence $\nu(b)=\nu(cd^{-1})=1$. Let $\nu\in S'$. Since $\nu(d-c)\geq\nu(c)+n_\nu\geq 2$, $\nu(d)=\nu(c)$ and $\nu(b-1)=\nu(cd^{-1}-1)\geq n_\nu$. □

4. A complex of Kato

Let K be a complete discrete valued field with residue field κ. Let ℓ be a prime not equal to characteristic of K. If $\ell=\mathrm{char}(\kappa)$, then assume that $[\kappa:\kappa^\ell]\leq\ell$. Then, there is a residue homomorphism $\partial: H^3(K,\mathbb{Z}/\ell(2))\to H^2(\kappa,\mathbb{Z}/\ell(1))$ [Kato 1986, Section 1]. We say that an element $\zeta\in H^3(K,\mathbb{Z}/\ell(2))$ is *unramified* at the discrete valuation of F if $\partial(\zeta)=0$.

Let $\mathscr{X}$ be a two-dimensional regular integral excellent Noetherian scheme quasiprojective over some affine scheme and F the function field of $\mathscr{X}$. For $x\in\mathscr{X}$, let F_x be the field of fractions of the completion $\hat{A}_x$ of the local ring A_x at x on $\mathscr{X}$ and $\kappa(x)$ the residue field at x. Let $x\in\mathscr{X}$ and C be the closure of $\{x\}$ in $\mathscr{X}$. Then, we also denote F_x by F_C. If the dimension of C is one, then C defines a discrete valuation ν_C (or ν_x) on F. Let $\mathscr{X}_{(i)}$ be the set of points of $\mathscr{X}$ with the dimension of the closure of $\{x\}$ equal to i. Let ℓ be a prime not equal to char(F). Suppose that F contains a primitive ℓ-th root of unity. If $P\in\mathscr{X}_{(0)}$ is a closed point of $\mathscr{X}$ with $\mathrm{char}(\kappa(P))=\ell$, then we assume $\kappa(P)=\kappa(P)^\ell$. Let $x\in\mathscr{X}_{(1)}$. We have a *residue homomorphism*

$$\partial_x: H^3(F,\mathbb{Z}/\ell(2))\to H^2(\kappa(x),\mathbb{Z}/\ell(1))$$

[Kato 1986, Section 1]. We say that an element $\zeta\in H^3(F,\mathbb{Z}/\ell(2))$ is *unramified* at x (or C) if ζ is unramified at ν_x. Further if $P\in\mathscr{X}_{(0)}$ is in the closure of $\{x\}$, then we have a *residue homomorphism*

$$\partial_P: H^2(\kappa(x),\mathbb{Z}/\ell(1))\to H^1(\kappa(P),\mathbb{Z}/\ell)$$

[Kato 1986, Section 1]. For $x\in\mathscr{X}_{(1)}$, if C is the closure of $\{x\}$, we also denote ∂_x by ∂_C. An element $\alpha\in H^2(\kappa(x),\mathbb{Z}/\ell(1))\simeq{}_\ell\mathrm{Br}(\kappa(x))$ is unramified at P if and only if $\partial_P(\alpha)=0$. We use the additive notation for the group operations on $H^2(F,\mathbb{Z}/\ell(1))$ and $H^3(F,\mathbb{Z}/\ell(2))$ and multiplicative notation for the group operation on $H^1(F,\mathbb{Z}/\ell)$.

Proposition 4.1 [Kato 1986, Proposition 1.7]. *Then*

$$H^3(F,\mathbb{Z}/\ell(2))\xrightarrow{\partial}\oplus_{x\in\mathscr{X}_{(1)}}H^2(\kappa(x),\mathbb{Z}/\ell(1))\xrightarrow{\partial}\oplus_{P\in\mathscr{X}_{(0)}}H^1(\kappa(P),\mathbb{Z}/\ell).$$

is a complex, where the maps are given by the residue homomorphism.

Lemma 4.2 [Kato 1980, Section 3.2, Lemma 3; 1986, Lemma 1.4(3). *Let $x\in\mathscr{X}_{(1)}$ and ν_x be the discrete valuation on F at x. Then $\partial_x: H^3(F_x,\mathbb{Z}/\ell(2))\to H^2(\kappa(x),\mathbb{Z}/\ell(1))$ is an isomorphism. Further if $\alpha\in H^2(F,\mathbb{Z}/\ell(1))$ is unramified at x and $f\in F^*$, then $\partial_x(\alpha\cdot(f))=\bar{\alpha}^{\nu_x(f)}$.*

The following is a consequence of Proposition 4.1.

Corollary 4.3. *Let C_1 and C_2 be two irreducible regular curves in $\mathscr{X}$ intersecting at a closed point P. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Suppose that ζ is unramified at all codimension one points of $\mathscr{X}$ passing through P except possibly at C_1 and C_2. Then*

$$\partial_P(\partial_{C_1}(\zeta)) = \partial_P(\partial_{C_2}(\zeta))^{-1}.$$

Corollary 4.4. *Let C be an irreducible curve on $\mathscr{X}$ and $P \in C$ with C regular at P. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Suppose that ζ is unramified at all codimension one points of $\mathscr{X}$ passing through P except possibly at C. If $\kappa(P)$ is finite, then $\zeta \otimes F_P = 0$. In particular if $\kappa(P)$ is finite, then ζ is unramified at every discrete valuation of F centered at P.*

Proof. Since C is regular at P, there exists an irreducible curve C' passing through P and intersecting C transversely at P. Then, by Corollary 4.3, we have $\partial_P(\partial_C(\zeta)) = \partial_P(\partial_{C'}(\zeta))^{\ell-1}$. Since, by assumption, $\partial_{C'}(\zeta) = 0$, we have $\partial_P(\partial_C(\zeta)) = 1$.

Let $\pi \in A_P$ be a prime defining C at P. Since C is regular at P, $A_P/(\pi)$ is a discrete valued ring with residue field $\kappa(P)$ and $\kappa(C)$ is the field of fractions of $A_P/(\pi)$. Further π remains a regular prime in $\hat{A}_P$ and $\hat{A}_P/(\pi)$ is the completion of $A_P/(\pi)$. In particular the field of fractions of $\hat{A}_P/(\pi)$ is the completion $\kappa(C)_P$ of the field $\kappa(C)$ at the discrete valuation given by the discrete valuation ring $A_P/(\pi)$. Let $\tilde{\nu}$ be the discrete valuation on F_P given by the height one prime ideal (π) of $\hat{A}$ and ν the discrete valuation of F given by the height one prime ideal (π) of A. Then the restriction of $\tilde{\nu}$ to F is ν and the residue field $\kappa(\tilde{\nu})$ at $\tilde{\nu}$ is $\kappa(C)_P$.

Since $\partial_P(\partial_C(\zeta)) = 1$, we have $\partial_C(\zeta) \otimes \kappa(C)_P = 0$ [Kato 1986, Lemma 1.4(3)]. Hence

$$\partial_{\tilde{\nu}}(\zeta \otimes F_P) = \partial_C(\zeta) \otimes \kappa(C)_P = 0.$$

Let $F_{P,\tilde{\nu}}$ be the completion of F_P at $\tilde{\nu}$. Since $\partial_{\tilde{\nu}} : H^3(F_{P,\tilde{\nu}}, \mathbb{Z}/\ell(2)) \to H^2(\kappa(C)_P, \mathbb{Z}/\ell(2))$ is an isomorphism [loc. cit., Lemma 1.4(3)], $\zeta \otimes F_{P,\tilde{\nu}} = 0$.

Let ν' be a discrete valuation of F_P given by a height one prime ideal of $\hat{A}$ not equal to (π). Then, by the assumption on ζ, $\partial_{\nu'}(\zeta \otimes F_P) = 0$ and hence $\zeta \otimes F_{P,\nu'} = 0$ [loc. cit., Lemma 1.4(3)], where $F_{P,\nu'}$ is the completion of F_P at ν'. Hence, by [Saito 1987, Theorem 5.3], $\zeta \otimes F_P = 0$. □

5. A local global principle

Let $\mathscr{X}$, F and ℓ be as in Section 4. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Let $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. In this section we show that under some additional assumptions on $\mathscr{X}$, ζ and α, there exists $f \in F^*$ such that $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at all the discrete valuations of $\kappa(x)$ centered at closed points of $\overline{\{x\}}$ for all $x \in \mathscr{X}_{(1)}$ (see Theorem 5.7).

For the rest of this section, we assume the following.

Assumptions 5.1. Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the following conditions:

(A1) $\text{ram}_{\mathscr{X}}(\zeta) = \{C_1, \ldots, C_r\}$, the C_i are regular irreducible curves with normal crossings.

(A2) $\text{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_n\}$, the D_j are regular curves with normal crossings and $C_i \neq D_j$ for all i, j.

By reindexing, we have $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \dots, D_m, \dots, D_n\}$, with $\mathrm{char}(\kappa(D_i)) = \ell$ for $1 \le i \le m$ and $\mathrm{char}(\kappa(D_j)) \ne \ell$ for $m+1 \le j \le n$:

(A3) $D_i \cap D_j = \varnothing$ for all $1 \le i \le m$ and $m+1 \le j \le n$.

(A4) If $P \in D_i \cap D_j$ for some $m+1 \le i < j \le n$, then $\mathrm{char}(\kappa(P)) \ne \ell$.

(A5) There are no chilly loops (see Section 3) for α on $\mathscr{X}$.

(A6) $\partial_{C_i}(\zeta)$ is the specialization of α at C_i for all i.

(A7) $C_i \cap D_j = \varnothing$ for all i and $1 \le j \le m$.

(A8) If $P \in C_i \cap D_s$ for some i and s, then $P \in C_i \cap C_j$ for some $i \ne j$.

(A9) For every $i \ne j$, through any point of $C_i \cap C_j$ there is at most one D_t.

(A10) In the representation $\alpha = [a, b)$ the element a can be chosen such that if $P \in \mathscr{X}_{(0)}$ with $\mathrm{char}(\kappa(P)) = \ell$ and $P \in D_i$ for some i, then $(1-a)/(\rho-1)^\ell \in A_P$.

(A11) If $P \in C_i \cap C_j \cap D_t$ for some $i < j$ and for some t, then D_t is given by a regular prime $u\pi_i^{\ell-1} + v\pi_j$ at P, for some prime π_i (resp. π_j) defining C_i (resp. C_j) at P and units u, v at P.

Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{X}$ containing $C_i \cap C_j$, $D_i \cap D_j$ for all $i \ne j$, $C_i \cap D_j$ for all i, j and at least one point from each C_i and D_j. Let A be the regular semilocal ring at $\mathscr{P}$ on $\mathscr{X}$. For every $P \in \mathscr{P}$, let M_P be the maximal ideal of A at P. For $1 \le i \le r$ and $1 \le j \le n$, let $\pi_i \in A$ be a prime defining C_i on A and $\delta_j \in A$ a prime defining D_j on A.

Lemma 5.2. *For $1 \le j \le n$, let $n_j = \ell\nu_{D_j}(\ell) + 1$. Then there exists a unit $u \in A$ such that $u \prod \pi_i$ is an ℓ-th power modulo $\delta_j^{n_j}$ for all $1 \le j \le n$. In particular $u \prod \pi_i \in F_{D_j}^{\ell}$ for all j.*

Proof. Let $\pi = \prod_1^r \pi_i$ and $\delta = \prod_1^m \delta_j^{n_j}$. Since, by the assumption (A7), $C_i \cap D_j = \varnothing$ for all i and $1 \le j \le m$, the ideals $A\pi$ and $A\delta$ are comaximal in A. In particular the image of π in $A/(\delta)$ is a unit. Let $P \in \mathscr{P} \setminus \big(\big(\bigcup_1^r C_i\big) \cup \big(\bigcup_1^m D_j\big)\big)$. Then π is a unit at P and the ideals (π), (δ), m_P are comaximal. By the Chinese remainder theorem, there exists $u_1 \in A$ be such that $u_1 = \pi \in A/(\delta)$, $u_1 = 1 \in A/(\pi)$ and $u_1 = \pi \in A/M_P$ for all $P \in \mathscr{P} \setminus \big(\big(\bigcup_1^r C_i\big) \cup \big(\bigcup_1^m D_j\big)\big)$. Since the image of π in $A/(\delta)$ is a unit, u_1 is a unit in A. Let $\pi' = u_1^{-1}\pi$.

Let $m+1 \le s \le n$ and a_s be the image of π' in $A/(\delta_s)$. We claim that $a_s = w_s b_s^\ell$ for some $w_s, b_s \in A/(\delta_s)$ with w_s a unit in $A/(\delta_s)$ and $w_s(P) = 1$ for all $P \in D_s \cap D_{s'}$, $s \ne s'$. Let M be a maximal ideal of $A/(\delta_s)$. Then $M = M_P/(\delta_s)$ for some $P \in D_s \cap \mathscr{P}$. Suppose $P \notin C_i$ for all i. Then π' is a unit at P and hence a_s is a unit at M. Suppose $P \in C_i$ for some i. Then $P \in C_i \cap D_s$. Thus, by the assumption (A8), there exists $j \ne i$ such that $P \in C_i \cap C_j$. Suppose $i < j$. Then, by the assumption (A11), $\delta_s = v_i\pi_i^{\ell-1} + v_j\pi_j$ for some units v_i and v_j at P. Hence

$$a_s \equiv u_1^{-1}\Big(\prod_{t \ne i,j} \pi_t\Big)\pi_i\pi_j = u_1^{-1}\Big(\prod_{t \ne i,j} \pi_t\Big)\pi_i\Big(-\frac{v_i}{v_j}\pi_i^{\ell-1}\Big) = u_1^{-1}\Big(\prod_{t \ne i,j} \pi_t\Big)\Big(-\frac{v_i}{v_j}\Big)\pi_i^\ell \quad \text{modulo } \delta_s.$$

Since π_t, $t \neq i, j$, is a unit at P (assumption (A1)), $a_s \equiv w_P \pi_j^\ell$ modulo δ_s, for some $w_P \in A/(\delta_s)$ a unit at P. Suppose $i > j$. Then $\delta_s = v_j \pi_j + v_i \pi_i^{\ell-1}$ for some units v_i and v_j at P. Hence, as above, $a_s \equiv w_P \pi_i^\ell$ modulo δ_s, for some $w_P \in A/(\delta_s)$ a unit at P. Hence at every maximal ideal of $A/(\delta_s)$, a_s is a product of a unit and an ℓ-th power. Since D_s is a regular curve on $\mathscr{X}$, $A/(\delta_s)$ is a semilocal regular ring and hence $A/(\delta_s)$ is an UFD. In particular $a_s = w_s b_s^\ell$ for some $w_s, b_s \in A/(\delta_s)$ with w_s a unit.

Let $P \in D_s \cap D_{s'}$ for some $s' \neq s$. Since $m+1 \leq s \leq n$, by the assumption (A3), $P \notin D_i$ for all $1 \leq i \leq m$. By the assumptions (A8) and (A9), $P \notin C_i$ for all i. Thus, by the choice of u_1, $\pi'(P) = 1$. In particular $a_s(P) = 1$ and hence $w_s(P) = b_s(P)^{-\ell}$. Let $\tilde{w}_s \in A/(\delta_s)$ be a unit such that $\tilde{w}_s(P) = b_s(P)$ for all $P \in D_s \cap D_{s'}$, $s \neq s'$. Since $a_s = w_s \tilde{w}_s^\ell (\tilde{w}_s^{-1} b_s)^\ell$ and $w_s \tilde{w}_s^\ell(P) = 1$, replacing w_s by $w_s \tilde{w}_s^\ell$ and b_s by $\tilde{w}_s^{-1} b_s$, we assume that $a_s = w_s b_s^\ell$ with $w_s(P) = 1$ for all $P \in D_s \cap D_{s'}$, $s \neq s'$. Since $m+1 \leq s \leq n$, by the assumption (A3), $(\delta_s, \delta) = A$. Hence, by Lemma 2.7, there exists $w \in A$ such that $w = 1 \in \kappa(P)$ for all $P \in \mathscr{P} \setminus \left(\bigcup_1^n D_i\right)$, $w = 1 \in A/(\delta)$ and $w = w_s \in A/(\delta_s)$. Since $w_s \in A/(\delta_s)$ is a unit, w is a unit in A.

Let $u = w^{-1} u_1^{-1}$. Since u_1 and w are units in A, $u \in A$ is a unit. We have $u \prod \pi_i = w^{-1}\pi' \equiv w_s^{-1} a_s = b_s^\ell$ modulo δ_s for $m+1 \leq s \leq n$ and $u \prod \pi_i = w^{-1}\pi' = w_\delta^{-\ell} \in A/(\delta)$. Since $\nu_{D_j}(\ell) = 0$ for $m+1 \leq j \leq n$ (assumption (A2)), $u \prod \pi_i$ is an ℓ-th power in $A/(\delta_j^{n_j})$ for $1 \leq j \leq n$. Since $n_j = \ell \nu_{D_j}(\delta_j) + 1$, $u \prod \pi_i \in F_{D_j}^\ell$ for all j (see [Epp 1973, Section 0.3]). ☐

Let $u \in A$ be a unit as in Lemma 5.2 and $\pi = u \prod_1^r \pi_i \in A$. Then $\operatorname{div}_{\mathscr{X}}(\pi) = \sum C_i + \sum_1^d t_s E_s$ for some irreducible curves E_s with $E_s \cap \mathscr{P} = \varnothing$. In particular $C_i \neq E_s$, $D_j \neq E_s$ for all i, j and s. Let $\mathscr{P}'$ be a finite set of points of $\mathscr{X}$ containing $\mathscr{P}$, $C_i \cap E_s$, $D_j \cap E_s$ for all i, j and s and at least one point from each E_s. Let A' be the semilocal ring at $\mathscr{P}'$. For $1 \leq i \leq n$, let $\delta_i' \in A'$ be a prime defining D_i on A'. Note that $\delta_i A \cap A' = \delta_i' A'$ for all i.

Lemma 5.3. *There exists $v \in A'$ such that*:

- *v is a unit and $F(\sqrt[\ell]{v})/F$ is unramified at all the points $P \in \mathscr{P}'$ except possible at the points P in $D_i \cap D_j$ for all $i \neq j$ with* $\operatorname{char}(\kappa(P)) \neq \ell$.
- *If* $\operatorname{char}(\kappa(D_j)) \neq \ell$, *then the extension $F(\sqrt[\ell]{v})/F$ is unramified at D_j with the residue field of $F(\sqrt[\ell]{v})$ at D_j equal to $\partial_{D_j}(\alpha)$.*
- *If* $\operatorname{char}(\kappa(D_j)) = \ell$, *then $F_{D_j}(\sqrt[\ell]{v}) \simeq F_{D_j}(\sqrt[\ell]{a})$. In particular $\alpha \otimes F_{D_j}(\sqrt[\ell]{v})$ is trivial.*

Proof. For $1 \leq i \leq n$, we show that there exists $u_i \in A'/(\delta_i') \subset \kappa(D_i)$ which patch to get an element in A' having the required properties.

Let $1 \leq i \leq m$. Then $\operatorname{char}(\kappa(D_i)) = \ell$. By the assumption (A10), $(a-1)/(\rho-1)^\ell \in A_P$ for all $P \in D_i$. In particular $(a-1)/(\rho-1)^\ell$ is regular at D_i and the image of $(a-1)/(\rho-1)^\ell$ in $\kappa(D_i)$ is in $A'/(\delta_i')$. Let u_i be the image of $(1-a)/(\rho-1)^\ell$ in $A'/(\delta_i')$.

Let $m+1 \leq i \leq n$. Then $\operatorname{char}(\kappa(D_i)) \neq \ell$. If $\operatorname{char}(\kappa(P)) = \ell$ for all $P \in D_i$, then let $w_i \in \kappa(D_i)$ be such that $\partial_{D_i}(\alpha) = [w_i)$.

Suppose there exists $P \in D_i$ with $\mathrm{char}(\kappa(P)) \neq \ell$. By [Saltman 2008, Proposition 7.10], there exists $w_i \in \kappa(D_i)^*$ such that:

- $\partial_{D_i}(\alpha) = \kappa(D_i)(\sqrt[\ell]{w_i})$.
- w_i is defined at all $P \in \mathscr{P}' \cap D_i$ with $\mathrm{char}(\kappa(P)) \neq \ell$.
- w_i is a unit at all $P \in (\mathscr{P}' \cap D_i) \setminus \bigl(\bigcup_{j \neq i} D_j\bigr)$ with $\mathrm{char}(\kappa(P)) \neq \ell$.
- $w_i(P) = w_j(P)$ for all $P \in D_i \cap D_j$, $i \neq j$ with P a chilly point or a cold point.

Let $P \in D_i \cap D_j$ for some $i \neq j$. Then, by assumptions (A3) and (A4), $\mathrm{char}(\kappa(P)) \neq \ell$. Suppose P is neither a chilly point nor a cold point. Since α is a symbol, there are no hot points [Saltman 2007, Theorem 2.5]. Hence P is a cool point. Since $\partial_{D_i}(\alpha) = \kappa(D_i)(\sqrt[\ell]{w_i})$, by the definition of a cool point, it follows that $w_i \in \kappa(D_i)_P^{*\ell}$. Write $w_i = w'^{\ell}_{iP}$ for some $w'_{iP} \in \kappa(D_i)_P^*$. Let $w'_i \in \kappa(D_i)^*$ be such that w'_i is close to w'_{iP} for all cool points $P \in D_i$ and w'_i is close to 1 for all other $P \in D_i \cap \mathscr{P}'$. Then, replacing w_i by $w_i w'^{-\ell}_i$, we assume that $w_i(P) = w_j(P)$ at all $P \in D_i \cap D_j$ with $\mathrm{char}(\kappa(P) \neq \ell$.

Let $P \in \mathscr{P}' \cap D_i$. Suppose $\mathrm{char}(\kappa(P)) = \ell$. Then, by the assumptions (A10), $[a)$ is unramified at P (see Proposition 2.3). Since $\alpha = [a, b)$, $\partial_{D_i}(\alpha) = [a(D_i))^{\nu_{D_i}(b)}$. In particular $\partial_{D_i}(\alpha) = \kappa(D_i)(\sqrt[\ell]{w_i})$ is unramified at P. Thus, by Lemma 2.6, we assume that $(1 - w_i)/(\rho - 1)^\ell$ is regular at all $P \in \mathscr{P}' \cap D_i \setminus \bigl(\bigcap_{j \neq i} D_j\bigr)$ with $\mathrm{char}(\kappa(P)) = \ell$. Since $\mathrm{char}(\kappa(D_i)) \neq \ell$, by assumptions (A3) and (A4), if $P \in D_i \cap D_j$ for some $j \neq i$, then $\mathrm{char}(\kappa(P)) \neq \ell$. Thus $(1 - w_i)/(\rho - 1)^\ell \in A'/(\delta_i)$. Let $u_i = (1 - w_i)/(\rho - 1)^\ell \in A'/(\delta'_i)$.

Let $P \in D_i \cap D_j$ for some $i \neq j$. Suppose $\mathrm{char}(\kappa(P)) = \ell$. Then, by the assumption (A3) and (A4), $1 \leq i, j \leq m$ and hence by the choice of u_i, we have $u_i(P) = u_j(P) \in \kappa(P)$. Suppose $\mathrm{char}(\kappa(P)) \neq \ell$. Then, $m + 1 \leq i, j \leq n$ and hence by the choice of w_i, we have $u_i(P) = u_j(P)$. Thus, by Lemma 2.7, there exists $u' \in A'$ such that $u' = u_i$ modulo (δ'_i) for all i. By the Chinese remainder theorem, we get $v' \in A'$ such that $v' = u' \in A'/\bigl(\prod \delta'_i\bigr)$ and $v' = 0 \in \kappa(P)$ for all $P \in \mathscr{P}'$ with $P \notin D_i$ for all i.

We now show that $v = 1 - (\rho - 1)^\ell v'$ has all the required properties.

Let $P \in \mathscr{P}'$. Suppose $\mathrm{char}(\kappa(P)) = \ell$. Then $\rho - 1 \in M_P$. Since $v' \in A'$, v is a unit at P and $F(\sqrt[\ell]{v})$ is unramified at P (Corollary 2.4). Suppose $\mathrm{char}(\kappa(P)) \neq \ell$. Suppose that $P \notin D_i$ for all i. Then, by the choice of v', $v' \in M_P$ and hence v is a unit at P and $F(\sqrt[\ell]{v})/F$ is unramified at P. Suppose that $P \in D_i$ for some i. Since $\mathrm{char}(\kappa(P)) \neq \ell$, $\mathrm{char}(\kappa(D_i)) \neq \ell$. Thus, by the choice of v', we have $v' = u' = u_i = (1 - w_i)/(\rho - 1)^\ell \in A'/(\delta'_i)$. Hence $v = w_i \in A'/(\delta'_i)$. Suppose $P \notin D_j$ for all $j \neq i$. Then, by the choice w_i is a unit at P and hence v is a unit at P. In particular $F(\sqrt[\ell]{v})/F$ is unramified at P. Thus v is a unit and $F(\sqrt[\ell]{v})/F$ is unramified at all $P \in \mathscr{P}'$ except possibly at $P \in D_i \cap D_j$ with $\mathrm{char}(\kappa(P)) \neq \ell$.

Suppose $\mathrm{char}(\kappa(D_i)) \neq \ell$. Then, by the choice of v, we have $v = 1 - (\rho - 1)^\ell v' = 1 - (\rho - 1)^\ell u_i = w_i \in A'/(\delta'_i) \subset \kappa(D_i)$. Since $w_i \neq 0$, v is a unit at δ_i and $F(\sqrt[\ell]{v})$ is unramified at D_i with residue field $\kappa(D_i)(\sqrt[\ell]{w_i}) = \partial_{D_i}(\alpha)$.

Suppose that $\mathrm{char}(\kappa(D_i)) = \ell$. Since $v = 1 - (\rho - 1)^\ell v'$ and $v' = u_i = w_i \in A'/(\delta'_i)$, $F(\sqrt[\ell]{v})$ is unramified at D_i with residue field equal to $\kappa(D_i)[X]/(X^\ell - X + w_i)$ (Proposition 2.3). Since w_i is the image of $(1-a)/(\rho-1)^\ell$ in $A'/(\delta_i')$, the residue field of $F(\sqrt[\ell]{a})$ at δ_i' is $\kappa(D_i)[X]/(X^\ell - X + w_i)$ (Proposition 2.3). Hence $F_{D_i}(\sqrt[\ell]{v}) \simeq F_{D_i}(\sqrt[\ell]{a})$. Since $\alpha = [a, b)$, $\alpha \otimes F_{\delta_i'}(\sqrt{v})$ is trivial. □

Remark 5.4. If ℓ is a unit in A', then the extension $F(\sqrt[\ell]{v})/F$ given in the above lemma is the lift of the residues of α which is in the sense of [Saltman 2008, Proposition 7.11].

Let $v \in A'$ be as in Lemma 5.3. Let $V_1, \dots, V_q$ be the irreducible curves in $\mathscr{X}$ where $F(\sqrt[\ell]{v\pi})$ is ramified. Since $\pi \in F_{D_j}^\ell$ Lemma 5.2 and $F(\sqrt[\ell]{v})$ is unramified at D_j Lemma 5.3 for all j, $V_i \neq D_j$ for all i and j. Let $\mathscr{P}'' = \mathscr{P} \cup (\cup(D_i \cap E_s)) \cup (\cup(D_i \cap V_j)$. After reindexing E_s, we assume that there exists $d_1 \le d$ such that $E_s \cap \mathscr{P}'' \neq \varnothing$ for $1 \le s \le d_1$ and $E_s \cap \mathscr{P}'' = \varnothing$ for $d_1 + 1 \le s \le d$.

Lemma 5.5. *There exists $h \in F^*$ which is a norm from the extension $F(\sqrt[\ell]{v\pi})$ such that*

$$\operatorname{div}_{\mathscr{X}}(h) = -\sum_1^{d_1} t_i E_i + \sum r_i E_i',$$

where $E_j' \cap \mathscr{P}'' = \varnothing$ for all j.

Proof. Let A'' be the regular semilocal ring at $\mathscr{P}''$. Let $L = F(\sqrt[\ell]{v\pi})$ and T be the integral closure of A'' in L.

Let $1 \le s \le d_1$ and $P \in \mathscr{P}'' \cap E_s$. Since $E_s \cap \mathscr{P} = \varnothing$, $P \in D_i \cap E_s$ for some i. Since v is a unit at all $P \in (\mathscr{P}' \setminus \mathscr{P})$ Lemma 5.3 and $D_i \cap E_s \subset \mathscr{P}'$, v is a unit at P and hence v is a unit at E_s.

Let e_s and f_s be the ramification index and the residue degree of L/F at E_s respectively. Suppose that $e_s = \ell$. Then there is a unique curve $\tilde{E}_s$ in T lying over E_s and let $t_s' = t_s$. Suppose that $e_s = 1$. Since $\operatorname{div}_{\mathscr{X}}(\pi) = \sum C_i + \sum_1^d t_s E_s$ and v is a unit at E_s, ℓ divides t_s. Suppose that $f_s = 1$. Let $t_s' = t_s/\ell$ and $\tilde{E}_s = t_s' \sum E_{s,i}$, where $E_{s,i}$ are the irreducible divisors in T which lie over E_s. Suppose that $f_s = \ell$. Then there is a unique curve $\tilde{E}_s$ in T lying over E_s and let $t_s' = t_s$.

Let $\tilde{E} = -\sum t_s' \tilde{E}_s$. Then the pushforward of $\tilde{E}$ from T to A'' is $-\sum_1^d t_s E_s$. We claim that $\tilde{E}$ is a principal divisor on T. Since T is normal it is enough to check this at every maximal ideal of T. Let M be a maximal ideal of T. Then $M \cap A'' = M_P$ for some $P \in \mathscr{P}''$. Suppose $P \notin E_s$ for all $1 \le s \le d_1$. Then $\tilde{E}$ is trivial at M. Suppose that $P \in E_s$ for some s with $1 \le s \le d_1$. Then, as we have seen above, $P \in D_i \cap E_s$ for some i. Since $D_i \cap C_j \in \mathscr{P}$ for all i and j and $\mathscr{P} \cap E_s = \varnothing$, $P \notin C_i$ for all i. Hence $\operatorname{div}_{A_P}(\pi) = \sum_{P \in E_i} t_i E_i$. Since v is a unit at P Lemma 5.3, $\operatorname{div}_{A_P}(v\pi) = \operatorname{div}_{A_P}(\pi)$ and hence $\tilde{E} = -\operatorname{div}(\sqrt[\ell]{v\pi})$ at M. In particular $\tilde{E}$ is principal at M. Hence $\tilde{E} = div_T(g)$ for some $g \in L$. Let $h = N_{L/F}(g)$. Since the pushforward of $\tilde{E}$ from T to A'' is $-\sum_1^d t_s E_s$, $\operatorname{div}_{A''}(h) = -\sum_1^{d_1} t_i E_i$ and hence h has the required properties. □

Lemma 5.6. *Let $h \in F^*$ be as in Lemma 5.5 with $\operatorname{div}_{\mathscr{X}}(h) = -\sum_1^{d_1} t_i E_i + \sum r_j E_j'$. Then α is unramified at E_j'. Further, if r_j is coprime to ℓ for some j, then the specialization of α at E_j' is unramified at every discrete valuation of $\kappa(E_j')$ which is centered on E_j'.*

Proof. Since $E_j' \cap \mathscr{P}'' = \varnothing$ and $D_i \cap \mathscr{P}'' \neq \varnothing$ for all i, $E_j' \neq D_i$ for all i. Hence, by the assumption (A2), α is unramified at E_j'.

Let P be a closed point of E'_j for some j with r_j coprime to ℓ. Let $L = F(\sqrt[\ell]{v\pi})$ and B_P be the integral closure of A_P in L. We first show that there exists an Azumaya algebra $\mathscr{A}_P$ over B_P such that $\alpha \otimes_F L$ is the class of $\mathscr{A}_P \otimes_{B_P} L$.

Suppose $P \notin D_i$ for all i. Then α is unramified at P (assumption (A2)). Hence there exists an Azumaya algebra $\mathscr{A}'_P$ over A_P such that α is the class of $\mathscr{A}'_P \otimes_{A_P} F$ (see Lemma 3.1). Let $\mathscr{A}_P = \mathscr{A}'_P \otimes_{A_P} B_Q$. Then $\alpha \otimes_F L$ is the class of $\mathscr{A}_P \otimes_{B_P} L$.

Suppose $P \in D_i$ for some i. Since $E'_j \cap \mathscr{P}'' = \varnothing$ Lemma 5.5, $P \notin \mathscr{P}''$. Since $\cup(V_{i'} \cap D_i) \subset \mathscr{P}''$, $P \notin \cup V_{i'}$ for all i' and hence L is unramified at P. Hence B_P is a regular semilocal domain. Let $Q \subset B_P$ be a height one prime ideal and $Q_0 = Q \cap A_P$. Then Q is a height one prime ideal of A_P. If α is unramified at Q_0, then $\alpha \otimes_F L$ is unramified at Q. Suppose that α is ramified at Q_0. Since $P \notin D_j$ for $j \neq i$, Q_0 is the prime ideal corresponding to D_i. Since $\pi \in F^{\ell}_{D_i}$ (Lemma 5.2), $F_{D_i}(\sqrt[\ell]{v\pi}) = F_{D_i}(\sqrt[\ell]{v})$. Suppose that $\operatorname{char}(\kappa(D_i)) \neq \ell$. Since L/F is unramified at D_i with residue field equal to $\partial_{D_i}(\alpha)$ (Lemma 5.3), $\alpha \otimes_F L$ is unramified at Q (see [Parimala et al. 2018, Lemma 4.1]). Suppose that $\operatorname{char}(\kappa(D_i)) = \ell$. Since $\alpha \otimes F_{D_i}(\sqrt[\ell]{v})$ is trivial (Lemma 5.3), $\alpha \otimes_F L$ is unramified at Q. Since B_P is a regular semilocal ring of dimension two, $\alpha \otimes F(\sqrt[\ell]{v\pi})$ is unramified at B_P (see Lemma 3.1). Hence there exists an Azumaya algebra $\mathscr{A}_P$ over B_P such that $\alpha \otimes_F L$ is the class of $\mathscr{A}_P \otimes_{B_P} L$.

Let $\beta \in H^2(\kappa(E'_j), \mathbb{Z}/\ell(1))$ be the specialization of α at E'_j. Suppose that r_j is coprime to ℓ. Let ν be a discrete valuation of $\kappa(E'_j)$ centered on a closed point P of E'_j. Let $Q_0 \subset A_P$ be the prime ideal defining E_j at P. Let $Q \subset B_P$ be a height one prime ideal of B_P lying over Q_0. Since E'_j is in the support of h, r_j is coprime to ℓ and h is a norm from L, the valuation on F given by Q_0 is either ramified or splits in L. Hence $A_P/Q_0 \subseteq B_P/Q \subset \kappa(E'_j)$. Thus β is the class of $\mathscr{A}_P \otimes_{B_P/Q} \kappa(E'_j)$. Since B_P/Q is integral over A_P/Q_0, the ring of integers at ν contains B_P/Q. In particular β is unramified at ν. □

Theorem 5.7. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 5.1. Then there exists $f \in K^*$ such that for every $x \in \mathscr{X}_{(1)}$, $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$ centered on the closure of $\{x\}$.*

Proof. We use the same notation as above and let $h \in F^*$ be as in Lemma 5.5. We claim that $f = h\pi$ has the required properties, i.e., $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$ for all $x \in \mathscr{X}_{(1)}$.

Let $x \in \mathscr{X}_{(1)}$ and D be the closure of $\{x\}$. Suppose $D = C_i$ for some i. Then h is a unit at C_i (Lemma 5.5), α is unramified at C_i (assumption (A2)) and π is a parameter at C_i, we have $\partial_{C_i}(\alpha \cdot (f))$ is the specialization of α at C_i (Lemma 4.2). Hence, by the assumption (A6), $\partial_{C_i}(\zeta - \alpha \cdot (f)) = 0$.

Suppose that $D = D_j$ for some j. By the assumption (A2), $\partial_{D_j}(\zeta) = 0$ and α is ramified at D_j. If $\operatorname{char}(\kappa(D_j)) = \ell$, then by the choice $\alpha \otimes F_{D_j}(\sqrt[\ell]{v}) = 0$ (Lemma 5.3). Suppose that $\operatorname{char}(\kappa(D_j)) \neq \ell$. Since $F_{D_j}(\sqrt[\ell]{v})$ is unramified with residue field equal to $\partial_{D_j}(\alpha)$ (Lemma 5.3), we have $\alpha \otimes F_{D_j}(\sqrt[\ell]{v}) = 0$ (Lemma 3.2). In particular, in either case, $\alpha \cdot (g) = 0 \in H^3(F_{D_j}(\sqrt{v}), \mathbb{Z}/\ell(2))$. Since $\pi \in F^{\ell}_{D_i}$ (Lemma 5.2), $L \otimes F_{D_j} = F_{D_j}(\sqrt[\ell]{v})$ and $\alpha \cdot (\pi) = 0 \in H^3(F_{D_j}, \mathbb{Z}/\ell(2))$. Thus $\alpha \cdot (h) = \operatorname{cor}_{L/F}(\alpha \cdot (g)) = 0 \in H^3(F_{D_j}, \mathbb{Z}/\ell(2))$ and $\partial_{D_j}(\alpha \cdot (h)) = 0$. Hence $\partial_{D_j}(\zeta - \alpha \cdot (f)) = 0$.

Suppose $D \neq C_i$ and D_j for all i and j. Then $\partial_D(\zeta) = 0$ and α is unramified at D. If $\nu_D(f)$ is a multiple of ℓ, then $\partial_D(\alpha \cdot (f)) = 0$. Suppose that $\nu_D(f)$ is coprime to ℓ. Since $\operatorname{div}_{\mathscr{X}}(\pi) =$

$\sum C_i + \sum_1^d t_i E_i$ (Lemma 5.2), $\mathrm{div}_{\mathscr{X}}(h) = -\sum_1^{d_1} t_s E_s + \sum r_i E'_i$ (Lemma 5.5) and $f = h\pi$, we have $\mathrm{div}_{\mathscr{X}}(f) = \sum C_i + \sum_{d_1+1}^{d} t_s E_s + \sum r_i E'_i$. Since $\nu_D(f)$ is coprime to ℓ and $D \neq C_i$ for all i, $D = E_s$ for some $d_1 + 1 \leq s \leq d$ or $D = E'_i$ for some i.

If $D = E'_i$, then by Lemma 5.6, the specialization $\bar{\alpha}$ of α at D is unramified at every discrete valuation of $\kappa(D)$ centered on D. Suppose $D = E_s$ for some $d_1+1 \leq s \leq d$. Then by the choice of d_1, $E_s \cap \mathscr{P}'' = \varnothing$ and hence $E_s \cap D_j = \varnothing$ for all j. Let $P \in E_s$. Then α is unramified at P (assumption (A2)) and hence $\bar{\alpha}$ is unramified at P. In particular $\bar{\alpha}$ is unramified at every discrete valuation of $\kappa(E_s)$ centered at P. Since α is unramified at E_s, $\partial_{E_s}(\alpha \cdot (f)) = \bar{\alpha}^{\nu_{E_s}(f)}$ (Lemma 4.2). Since $\bar{\alpha}$ is unramified at every discrete valuation of $\kappa(E_s)$ centered on E_s, $\partial_{E_s}(\alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(E_s)$ centered on E_s. Hence f has the required property. □

6. Divisibility of elements in H^3 by symbols in H^2

Let K be a global field or a local field and F the function field of a curve over K. If K is a number field or a local field, let R be the ring of integers in K. If K is a global field of positive characteristic, let R be the field of constants of K. Let $\mathscr{X}$ be a regular proper model of F over $\mathrm{Spec}(R)$. Let ℓ be a prime not equal to $\mathrm{char}(K)$. Suppose that K contains a primitive ℓ-th root of unity ρ. Then for any $P \in \mathscr{X}_{(0)}$, $\kappa(P)$ is a finite field. Hence if $\mathrm{char}(\kappa(P)) = \ell$, then $\kappa(P) = \kappa(P)^\ell$.

Thus we have a complex (see Proposition 4.1)

$$0 \to H^3(F, \mathbb{Z}/\ell(2)) \xrightarrow{\partial} \oplus_{x \in \mathscr{X}_{(1)}} H^2(\kappa(x), \mathbb{Z}/\ell(1)) \xrightarrow{\partial} \oplus_{P \in \mathscr{X}_{(0)}} H^1(\kappa(P), \mathbb{Z}/\ell).$$

Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. In this section we prove (see Theorem 6.5) a certain local global principle for divisibility of ζ by α if $(\mathscr{X}, \zeta, \alpha)$ satisfies certain assumptions (see Assumptions 6.3).

For a sequence of blow-ups $\eta : \mathscr{Y} \to \mathscr{X}$ and for an irreducible curve C in $\mathscr{X}$, we denote the strict transform of C in $\mathscr{Y}$ by C itself.

We begin with the following:

Lemma 6.1. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption* (A1) *of Assumptions 5.1. Let $\mathscr{Y} \to \mathscr{X}$ be a sequence of blow-ups centered on closed points of $\mathscr{X}$ which are not in $C_i \cap C_j$ for all $i \neq j$. Let $1 \leq I \leq 11$ with $I \neq 3, 5, 7$. If $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption* (AI) *of Assumptions 5.1, then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption* (AI).

Proof. Let Q be a closed point of $\mathscr{X}$ which is not in $C_i \cap C_j$ for $i \neq j$ and $\eta : \mathscr{Y} \to \mathscr{X}$ a simple blow-up at Q. It is enough to prove the lemma for $(\mathscr{Y}, \zeta, \alpha)$.

Let E be the exceptional curve in $\mathscr{Y}$. Since $Q \notin C_i \cap C_j$ for $i \neq j$ and $(\mathscr{X}, \zeta, \alpha)$ satisfies (A1) of Assumptions 5.1, by Corollary 4.4, ζ is unramified at E.

Let $1 \leq I \leq 11$ with $I \neq 3, 5, 7$. Suppose further $I \neq 4, 10$. Since the exceptional curve E is not in $\mathrm{ram}_{\mathscr{Y}}(\zeta)$, if $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (AI) of Assumptions 5.1, then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the same assumption.

Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (A4) of Assumptions 5.1. Suppose $\mathrm{char}(\kappa(Q)) = \ell$. Then $\mathrm{char}(\kappa(E)) = \ell$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A4) of Assumptions 5.1. Suppose $\mathrm{char}(\kappa(Q)) \neq \ell$. Then $\mathrm{char}(\kappa(P)) \neq \ell$ for all $P \in E$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A4) of Assumptions 5.1.

Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (A10) of Assumptions 5.1. If $\mathrm{char}(\kappa(Q)) \neq \ell$, then $\mathrm{char}(\kappa(P)) \neq \ell$ for all $P \in E$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A10) of Assumptions 5.1. Suppose that $\mathrm{char}(\kappa(Q)) = \ell$. If $Q \notin D_i$ for any i, then α is unramified at Q and hence α is unramified at E. In particular $E \notin \mathrm{ram}_{\mathscr{Y}}(\alpha)$ and hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A10) of Assumptions 5.1. Suppose $Q \in D_i$ for some i. Since $(\mathscr{X}, \zeta, \alpha)$ satisfies (A10) of Assumptions 5.1, $(1-a)/(\rho-1)^\ell \in A_Q$. Let $P \in E$. Since $A_Q \subset A_P$, $(1-a)/(\rho-1)^\ell \in A_P$. Hence $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the assumption (A10) of Assumptions 5.1. □

Lemma 6.2. *Let $\mathscr{Y} \to \mathscr{X}$ be a sequence of blow-ups centered on closed points Q of $\mathscr{X}$ with* $\mathrm{char}(\kappa(Q)) \neq \ell$. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfy the assumptions* (A1) *and* (A2). *If $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption* (A3) *or* (A7) *of Assumptions 5.1, then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies the same assumption.*

Proof. Let Q be a closed point of $\mathscr{X}$ with $\mathrm{char}(\kappa(Q)) \neq \ell$ and E the exceptional curve in $\mathscr{Y}$. Since $\mathrm{char}(\kappa(E)) \neq \ell$ and for any closed point P of E $\mathrm{char}(\kappa(P)) \neq \ell$, the lemma follows. □

Assumptions 6.3. Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies the following:

(B1) $\mathrm{ram}_{\mathscr{X}}(\zeta) = \{C_1, \dots, C_r\}$, the C_i are irreducible regular curves with normal crossings.

(B2) $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \dots, D_n\}$ with the D_j irreducible curves such that $C_i \neq D_j$ for all i and j.

(B3) If $D_s \cap C_i \cap C_j \neq \varnothing$ for some s, $i \neq j$, then $\mathrm{char}(\kappa(D_s)) \neq \ell$.

(B4) If $P \in D_j$ for some $1 \leq j \leq n$ with $\mathrm{char}(\kappa(P)) = \ell$, then $(1-a)/(\rho-1)^\ell \in A_P$.

(B5) $\partial_{C_i}(\zeta)$ is the specialization of α at C_i for all i.

(B6) If $\ell = 2$, then $\zeta \otimes F \otimes K_\nu$ is trivial for all real places ν of K.

(B7) If $\ell = 2$, then a is a sum of two squares in F.

(B8) For $1 \leq i < j \leq r$, through any point of $C_i \cap C_j$ there passes at most one D_s and if $P \in D_s \cap C_i \cap C_j$, then D_s is defined by $u\pi_i^{\ell-1} + v\pi_j$ at P for some units u and v at P and π_i, π_j primes defining C_i and C_j at P.

Lemma 6.4. *Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3. Let $\mathscr{Y} \to \mathscr{X}$ be a sequence of blow-ups centered on closed points of $\mathscr{X}$ which are not in $C_i \cap C_j$ for $i \neq j$. Then $(\mathscr{Y}, \zeta, \alpha)$ also satisfies Assumptions 6.3.*

Proof. Let Q be a closed point of $\mathscr{X}$ which is not in $C_i \cap C_j$ for $i \neq j$ and $\eta : \mathscr{Y} \to \mathscr{X}$ a simple blow-up at Q. It is enough to show that $(\mathscr{Y}, \zeta, \alpha)$ satisfies Assumptions 6.3.

Since (B1), (B4), (B5) and (B8) are restatements of (A1), (A10), (A6) and (A9), (A11), by Lemma 6.1, $(\mathscr{Y}, \zeta, \alpha)$ satisfies (B1), (B4), (B5) and (B8). Let E be the exceptional curve in $\mathscr{Y}$. Since $Q \notin C_i \cap C_j$

for $i \neq j$, by Corollary 4.4, ζ is unramified at E. Hence $\mathrm{ram}_{\mathscr{Y}}(\zeta) = \{C_1, \ldots, C_r\}$. Since $\mathrm{ram}_{\mathscr{Y}}(\alpha) \subset \{D_1, \ldots, D_n, E\}$, $(\mathscr{Y}, \zeta, \alpha)$ satisfies (B2). Since $E \cap C_i \cap C_j = \varnothing$ for all $i \neq j$, $(\mathscr{Y}, \zeta, \alpha)$ satisfies (B3).

Since (B6) and (B7) do not depend on the model, $(\mathscr{Y}, \zeta, \alpha)$ satisfies all Assumptions 6.3. □

Theorem 6.5. *Let K, F and $\mathscr{X}$ be as above. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ and $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$. Suppose that F contains a primitive ℓ-th root of unity. If $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3, then there exists $f \in F^*$ such that $\zeta = \alpha \cdot (f)$.*

Proof. Suppose $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3. First we show that there exists a sequence of blow-ups $\eta : \mathscr{Y} \to \mathscr{X}$ such that $(\mathscr{Y}, \zeta, \alpha)$ satisfies Assumptions 5.1.

Let $P \in \mathscr{X}_{(0)}$. Suppose $P \in D_s$ for some s and D_s is not regular at P or $P \in D_s \cap D_t$ for some $s \neq t$. Then, by the assumption (B8), $P \notin C_i \cap C_j$ for all $i \neq j$. Thus, there exists a sequence of blow-ups $\mathscr{X}' \to \mathscr{X}$ at closed points which are not in $C_i \cap C_j$ for all $i \neq j$ such that $\mathrm{ram}_{\mathscr{X}'}(\alpha)$ is a union of regular with normal crossings. By Lemma 6.4, $\mathscr{X}'$ also satisfies Assumptions 6.3. Thus, replacing $\mathscr{X}$ by $\mathscr{X}'$ we assume that $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3, D_i's are regular with normal crossings and D_s, C_i have normal crossings at all $P \notin C_j$ for all $j \neq i$. In particular $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumptions (A1) and (A2) of Assumptions 5.1.

Suppose there exists $i \neq j$ and $P \in D_i \cap D_j$ such that $\mathrm{char}(\kappa(D_i)) \neq \ell$, $\mathrm{char}(\kappa(D)_j) \neq \ell$ and $\mathrm{char}(\kappa(P)) = \ell$. Let $\mathscr{X}' \to \mathscr{X}$ be the blow-up at P and E the exceptional curve in $\mathscr{X}'$. Then $\mathrm{char}(\kappa(E)) = \mathrm{char}(\kappa(P)) = \ell$ and $D_i \cap D_j \cap E = \varnothing$ in $\mathscr{X}'$. By the assumption (B8), $P \notin C_{i'} \cap C_{j'}$ for all $i' \neq j'$ and hence $\mathscr{X}'$ satisfies Assumptions 6.3 (see Lemma 6.4) and assumptions (A1) and (A2) of Assumptions 5.1 (see Lemma 6.1). Thus replacing $\mathscr{X}$ by a sequence of blow-ups at closed points in $D_i \cap D_j$ for $i \neq j$, we assume that $\mathscr{X}$ satisfies Assumptions 6.3 and assumptions (A1), (A2) and (A4) of Assumptions 5.1.

Since $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumptions (B4), (B5) and (B8) of Assumptions 6.3, $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumptions (A6), (A9), (A10) and (A11) of Assumptions 5.1.

Suppose $P \in C_i \cap D_s$ for some i, s and $P \notin C_j$ for all $j \neq i$. Since ζ is unramified at P except at C_i, $\partial_{C_i}(\zeta)$ is zero over $\kappa(C_i)_P$ (Corollary 4.4). By the assumption (B5), we have $\partial_{C_i}(\zeta) = \bar{\alpha}$. Since $P \notin C_j$ for all $j \neq i$, C_i and D_s have normal crossings at P and $P \notin D_{s'}$ for all $s' \neq s$. Thus, by Lemma 3.3, $\alpha \otimes F_P = 0$. Let $\mathscr{X}' \to \mathscr{X}$ be the blow-up at P and E the exceptional curve in $\mathscr{X}'$. Since $\alpha \otimes F_P = 0$ and $F_P \subset F_E$, α is unramified at E and hence $\mathrm{ram}_{\mathscr{X}'}(\alpha) = \{D_1, \ldots, D_n\}$. Since $\zeta \otimes F_P = 0$, $\mathrm{ram}_{\mathscr{X}'}(\zeta) = \{C_1, \ldots, C_r\}$. Note that $C_i \cap D_s = \varnothing$ in $\mathscr{X}'$. Hence $(\mathscr{X}', \zeta, \alpha)$ satisfies assumption (A8) of Assumptions 5.1. Since $P \notin C_j$ for all $j \neq i$, $(\mathscr{X}', \zeta, \alpha)$ satisfies Assumptions 6.3, 6.4, 5.1, except possibly (A3), (A5) and (A7), and 6.1. Thus, replacing $\mathscr{X}$ by $\mathscr{X}'$ we assume that $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3 and 5.1 except possibly (A3), (A5) and (A7).

Let $\mathrm{ram}_{\mathscr{X}}(\alpha) = \{D_1, \ldots, D_m, D_{m+1}, \ldots, D_n\}$ with $\mathrm{char}(\kappa(D_s)) = \ell$ for $1 \leq s \leq m$ and $\mathrm{char}(\kappa(D_t)) \neq \ell$ for $m+1 \leq t \leq n$. Suppose $D_s \cap D_t \neq \varnothing$ for some $1 \leq s \leq m$ and $m+1 \leq t \leq n$. Let $P \in D_s \cap D_t$. Then $\mathrm{char}(\kappa(P)) = \ell$ and hence $(a-1)/(\rho-1)^\ell \in A_P$ (assumption (B4)). In particular $[a)$ is unramified at P (see Proposition 2.3). Since α is ramified at D_t, $\nu_{D_t}(b)$ is coprime to ℓ and hence there exists i such that $\nu_{D_s}(b) + i\nu_{D_t}(b)$ is divisible by ℓ. Let $\mathscr{X}_1 \to \mathscr{X}$ be the blow-up at P and E_1 the exceptional curve in $\mathscr{X}_1$.

We have $\nu_{E_1}(b) = \nu_{D_s}(b) + \nu_{D_t}(b)$. Let Q_1 be the point in $E_1 \cap D_t$ and $\mathscr{X}_2 \to \mathscr{X}_1$ be the blow-up at Q_1. Let E_2 be the exceptional curve in $\mathscr{X}_2$. We have $\nu_{E_2}(b) = \nu_{E_1}(b) + \nu_{D_t}(b) = \nu_{D_s}(b) + 2\nu_{D_t}(b)$. Continue this process i times and get $\mathscr{X}_i \to \mathscr{X}_{i-1}$ and E_i the exceptional curve in $\mathscr{X}_i$. Then $\nu_{E_i}(b) = \nu_{D_t}(b) + i\nu_D(b)$ is divisible by ℓ. Since $[a)$ is unramified at P, α is unramified at E_i. Since $\mathrm{char}(\kappa(E_j)) = \ell$ for all j, $E_{i-1} \cap D_t = \varnothing$ in $\mathscr{X}_i$ and E_i is in not in $\mathrm{ram}_{\mathscr{X}_i}(\alpha)$. Since $P \notin C_i \cap C_j$ for all $i \neq j$ (assumption (B4)), $\mathscr{X}_i$ satisfies Assumptions 6.3 (see Lemma 6.4). Thus, replacing $\mathscr{X}$ by $\mathscr{X}_i$, we assume that $D_s \cap D_t = \varnothing$ for all $1 \leq s \leq m$ and $m+1 \leq t \leq n$ and $\mathscr{X}$ satisfies Assumptions 6.3. Thus $\mathscr{X}$ satisfies all the assumptions of Assumptions 5.1 expect possibly (A5) and (A7) (see Lemma 6.1).

Suppose $C_i \cap D_t \neq \varnothing$ for some i and t. Since $(\mathscr{X}, \zeta, \alpha)$ satisfies (A8) and (A9) of Assumptions 5.1, there exists $j \neq i$ such that $C_i \cap C_j \cap D_t \neq \varnothing$. Since $(\mathscr{X}, \zeta, \alpha)$ satisfies the assumption (B3) of Assumptions 6.3, $\mathrm{char}(\kappa(D_t)) \neq \ell$. Hence $C_i \cap D_t = \varnothing$ for all i and $1 \leq t \leq m$. In particular $(\mathscr{X}, \zeta, \alpha)$ satisfies (A7) of Assumptions 5.1 and hence $(\mathscr{X}, \zeta, \alpha)$ satisfies all the assumptions of Assumptions 5.1 except possibly (A5).

Let $P \in \mathscr{X}_{(0)}$. Suppose that P is a chilly point for α. Then $P \in D_s \cap D_t$ for some $D_s, D_t \in \mathrm{ram}_{\mathscr{X}}(\alpha)$ with $D_s \neq D_t$ with $\mathrm{char}(\kappa(P)) \neq \ell$. In particular $P \notin C_i \cap C_j$ for all $i \neq j$ (assumption (B8)). Since there is a sequence of blow-ups $\mathscr{Y} \to \mathscr{X}$ centered on chilly points of α on $\mathscr{X}$ with no chilly loops on $\mathscr{Y}$ (Proposition 3.4), by Lemmas 6.1 and 6.2, replacing $\mathscr{X}$ by $\mathscr{Y}$ we assume that $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3 and 5.1.

Thus, by Theorem 5.7, there exists $f \in F^*$ such that for every $x \in \mathscr{X}_{(1)}$, $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$ centered at a closed point of the closure $\overline{\{x\}}$ of $\{x\}$. Since $\kappa(x)$ is a global field or a local field, every discrete valuation of $\kappa(x)$ is centered on a closed point of $\overline{\{x\}}$. Hence $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$.

For place ν of K, let K_ν be the completion of K at ν and $F_\nu = F \otimes_K K_\nu$.

Let ν be a real place of K. Since a is a sum of two squares in F, a is a norm from the extension $F_\nu(\sqrt{-1})$. Let $\tilde{a} \in F_\nu(\sqrt{-1})$ with norm equal to a. Since $H^2(F_\nu(\sqrt{-1}), \mathbb{Z}/2(1)) = 0$ [Serre 1997, page 80] and $\mathrm{cor}_{F_\nu(\sqrt{-1})/F_\nu}[\tilde{a}, b) = [a, b) \otimes F_\nu$, $\alpha = [a, b) = 0 \in H^2(F_\nu, \mathbb{Z}/2(1))$. Since, by assumption $\zeta \otimes F_\nu = 0$,

$$\zeta - \alpha \cdot (f) = 0 \in H^3(F_\nu, \mathbb{Z}/2(2)).$$

Let $x \in \mathscr{X}_{(1)}$. Since $\zeta - \alpha \cdot (f) = 0 \in H^3(F_\nu, \mathbb{Z}/2(2))$ for all real places ν of K, it follows that $\partial_x(\zeta - \alpha \cdot (f)) = 0 \in H^2(\kappa(x)_{\nu'}, \mathbb{Z}/2(1))$ for all real places ν' of $\kappa(x)$. Since $\partial_x(\zeta - \alpha \cdot (f))$ is unramified at every discrete valuation of $\kappa(x)$, $\partial_x(\zeta - \alpha \cdot (f)) = 0$ [Cassels and Fröhlich 1967, page 130]. Hence $\zeta - \alpha \cdot (f)$ is unramified on $\mathscr{X}$.

Let ν be a finite place of K. Since $\zeta - \alpha \cdot (f)$ is unramified on $\mathscr{X}$,

$$(\zeta - \alpha \cdot (f)) \otimes_F F_\nu) = 0 \in H^3(F_\nu, \mathbb{Z}/\ell(2))$$

[Kato 1986, Corollary page 145]. Hence $\zeta = \alpha \cdot (f)$ [loc. cit., Theorem 0.8(2)]. □

7. Main theorem

In this section we prove our main result Theorem 7.7. Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to $\mathrm{char}(K)$. Suppose that F contains a primitive ℓ-th root of unity ρ. If K is a number field or a local field, let R be the ring of integers in K. If K is a global field of positive characteristic, let R be the field of constants of K.

To prove our main result Theorem 7.7, we first show Proposition 7.6 that given $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ with $\zeta \otimes_F (F \otimes_K K_\nu) = 0$ for all real places ν of K, there exist $\alpha = [a, b) \in H^2(F, \mathbb{Z}/\ell(1))$ and a regular proper model $\mathscr{X}$ of F over R such that the triple $(\mathscr{X}, \zeta, \alpha)$ satisfies Assumptions 6.3.

Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$ be such that $\zeta \otimes_F (F \otimes_K K_\nu) = 0$ for all real places ν of K. Choose a regular proper model $\mathscr{X}$ of F over R [Saltman 1997, page 38] such that:

- $\mathrm{ram}_{\mathscr{X}}(\zeta) \cup \mathrm{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \dots, C_{r_1}, \dots, C_r\}$, where the C_i are irreducible regular curves with normal crossings.
- For $i \neq j$, C_i and C_j intersect at most at one closed point.
- $C_i \cap C_j = \varnothing$ if $i, j \leq r_1$ or $i, j > r_1$.

For $x \in \mathscr{X}_{(1)}$, let $\beta_x = \partial_x(\zeta)$. Let $\mathscr{P}_0 \subset \cup C_i$ be a finite set of closed points of $\mathscr{X}$ containing $C_i \cap C_j$ for $1 \leq i < j \leq r$, and at least one closed point from each C_i. Let A be the regular semilocal ring at the points of $\mathscr{P}_0$. Let $Q \in C_i$ be a closed point. Since C_i is regular on $\mathscr{X}$, Q gives a discrete valuation ν_Q^i on $\kappa(C_i)$.

Lemma 7.1. *There exists $a \in A$ such that*:

- *$(a-1)/(\rho-1)^\ell \in A$ and $[a)$ is unramified on A.*
- *For $1 \leq i \leq r_1$ and $P \in C_i \cap \mathscr{P}_0$, $\partial_P(\beta_{x_i}) = [a(P))$.*
- *For $r_1 + 1 \leq i \leq r$ and $P \in C_i \cap \mathscr{P}_0$, $\partial_P(\beta_{x_i}) = [a(P))^{-1}$.*
- *If $P \in \mathscr{P}_0$ and $P \notin C_i \cap C_j$ for all $i \neq j$, then $[a(P))$ is the trivial extension.*
- *If $\ell = 2$, then a is a sum of two squares in A.*

Proof. Let $P \in \mathscr{P}_0$. Suppose $P \in C_i \cap C_j$ for some $i < j$. Then, by the choice of $\mathscr{X}$, the pair (i, j) is uniquely determined by P. Let $u_P \in \kappa(P)$ be such that $\partial_P(\partial_{x_i}(\zeta)) = [u_P)$. If $P \notin C_i \cap C_j$ for all $i \neq j$, let $u_P \in \kappa(P)$ with $[u_P)$ the trivial extension.

Then, by Lemma 2.5, there exists $a \in A$ such that for every $P \in \mathscr{P}_0$, the cyclic extension $[a)$ over F is unramified on A with the residue field $[a(P))$ of $[a)$ at P is $[u_P)$. Further if $\ell = 2$, choose a to be a sum of two squares in A (Lemma 2.5). From the proof of Lemma 2.5, we have $(a-1)/(\rho-1)^\ell \in A$.

Let $P \in \mathscr{P}_0$. Suppose that $P \in C_i$ for some i and $P \notin C_j$ for all $i \neq j$. Then $\partial_P(\partial_{x_i}(\zeta)) = 1$ (Corollary 4.3) and by the choice of a and u_P, we have $[a(P)) = [u_P) = 1$. Suppose that $P \in C_i \cap C_j$ for some $i \neq j$. Suppose $i < j$. Then by the choice of a and u_P we have $\partial_P(\partial_{x_i}(\zeta)) = [u_P) = [a(P))$. Suppose $i > j$. Then by the choice of a and u_P we have $\partial_P(\partial_{x_j}(\zeta)) = [u_P) = [a(P))$. Since $\partial_P(\partial_{x_i}(\zeta)) = \partial_P(\partial_{x_j}(\zeta))^{-1}$ (Corollary 4.3), we have $\partial_P(\partial_{x_i}(\zeta)) = [a(P))^{-1}$. Thus a has the required properties. □

Let $a \in A$ be as in Lemma 7.1. Let $L_1, \dots, L_d$ be the irreducible curves in $\mathscr{X}$ which are in the ramification of $[a)$ or $\nu_{L_i}((a-1)/(\rho-1)^\ell) < 0$.

Lemma 7.2. *Then $L_i \cap \mathscr{P}_0 = \varnothing$ for all i. In particular $L_i \neq C_j$ for all i, j and* $\mathrm{char}(\kappa(L_i)) \neq \ell$.

Proof. By the choice of a, $[a)$ is unramified on A and $(a-1)/(\rho-1)^\ell \in A$ (Lemma 7.1). Hence $\mathscr{P}_0 \cap L_i = \varnothing$ for all i. Since $\mathscr{P}_0$ contains at least one point from each C_j, $L_i \neq C_j$ for all i and j. Since $\mathrm{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \dots, C_r\}$, $\mathrm{char}(\kappa(L_i)) \neq \ell$ for all i. □

Let $\mathscr{P}_1 \subset \bigcup_j L_j$ be a finite set of closed points of $\mathscr{X}$ consisting of $L_i \cap L_j$ for $i \neq j$, $L_i \cap C_j$, one point from each L_i. Since $L_i \cap \mathscr{P}_0 = \varnothing$ for all i (Lemma 7.2), $\mathscr{P}_0 \cap \mathscr{P}_1 = \varnothing$.

Let $\mathscr{P} = \mathscr{P}_0 \cup \mathscr{P}_1$ and B be the semilocal ring at $\mathscr{P}$ on $\mathscr{X}$. For each i and j, let $\pi_i \in B$ be a prime defining C_i and $\delta_j \in B$ a prime defining L_j.

Lemma 7.3. *For each $P \in C_i \cap \mathscr{P}_1$, let n_i^P be a positive integer. Then for each i, $1 \le i \le r$, there exists $b_i \in B/(\pi_i) \subset \kappa(C_i)$ such that*:

- $\partial_{C_i}(\zeta) = [a(C_i), b_i)$.
- $\nu_P^i(b_i) = 1$ *for all* $P \in C_i \cap \mathscr{P}_0$, $1 \le i \le r_1$.
- $\nu_P^i(b_i) = \ell - 1$ *for all* $P \in C_i \cap \mathscr{P}_0$, $r_1 + 1 \le i \le r$.
- $\nu_P^i(b_i - 1) \ge n_i^P$ *for all* $P \in \mathscr{P}_1 \cap C_i$ *for all* i.

Proof. Let $1 \le i \le r$. Let $\beta_{x_i} = \partial_{x_i}(\zeta) \in H^2(\kappa(C_i), \mathbb{Z}/\ell(1))$ and $a_i = a(C_i)$.

Suppose $1 \le i \le r_1$. By Lemma 7.1, $\partial_P(\beta_{x_i}) = [a_i(P))$ for all $P \in C_i \cap \mathscr{P}_0$. If $P \notin \mathscr{P}_0$, then $\partial_P(\beta_{x_i}) = 0$ for all i (Corollary 4.3). By the assumption, $\beta_{x_i} \otimes \kappa(C_i)_\nu = 0$ for all real places ν of $\kappa(C_i)$. Thus, by Proposition 3.5, there exists $b_i \in \kappa(C_i)^*$ such that $\beta_{x_i} = [a_i, b_i)$, with $\nu_P^i(b_i) = 1$ for all $P \in C_i \cap \mathscr{P}_0$ and $\nu_P^i(b_i - 1) \ge n_i^P$ for all $P \in C_i \cap \mathscr{P}_1$. In particular b_i is regular at all $P \in C_i \cap \mathscr{P}$ and hence $b_i \in B/(\pi_i)$.

Suppose $r_1 + 1 \le i \le r$. Let $P \in C_i \cap \mathscr{P}_0$. Since $\partial_P(\beta_{x_i}) = [a(P))^{-1}$ for all $P \in C_i \cap \mathscr{P}_0$ (Lemma 7.1), $\partial_P(\beta_{x_i}^{-1}) = [a(P))$. Thus, as above, by Proposition 3.5, there exists $c_i \in B/(\pi_i)$ such that $\beta_{x_i}^{-1} = [a_i, c_i)$, with $\nu_P^i(c_i) = 1$ for all $P \in C_i \cap \mathscr{P}_0$ and $\nu_P^i(c_i - 1) \ge n_i^P$ for all $P \in C_i \cap \mathscr{P}_1$. Let $b_i = c_i^{\ell-1} \in B/(\pi_i)$. Then $\beta_{x_i} = [a_i, b_i)$. Let $P \in C_i \cap \mathscr{P}_1$. Since $c_i \in B/(\pi_i)$ and $\nu_P^i(c_i - 1) \ge n_i^P$, it follows tat $\nu_P^i(b_i - 1) \ge n_i^P$. Thus b_i has the required properties. □

Let $\delta = \prod \delta_j \in B$. For $1 \le i \le r$, let $\bar{\delta}(i) \in B/(\pi_i)$ be the image of δ. Let d be an integer greater than $\nu_P^i(\bar{\delta}(i)) + 1$ for all i and $P \in C_i \cap \mathscr{P}$.

Lemma 7.4. *Let $b_i \in B/(\pi_i)$ be as in Lemma 7.3 for $n_i^P = d$ for all $P \in C_i \cap \mathscr{P}$. Then there exists $b \in B$ such that*:

- $b = b_i$ *modulo* π_i *for all* i.
- $b = 1$ *modulo* δ_j *for all* j.
- b *is a unit at all* $P \in \mathscr{P}_1$.

Proof. For $1 \le i \le r$, let $I_i = (\pi_i) \subset B$ and $I_{r+1} = (\delta) \subset B$. Clearly the $\gcd(\pi_i, \pi_j) = 1$ and $\gcd(\pi_i, \delta) = 1$ for all $1 \le i < j \le r$. For $1 \le i < j \le r$, $I_{ij} = I_i + I_j$ is either maximal ideal or equal to B. For $1 \le i \le r$, we have $I_{i(r+1)} = (\pi_i, \delta)$. Since $L_s \cap \mathscr{P}_0 = \varnothing$ for all s, $(\delta_s, \pi_i, \pi_j) = A$ for all $1 \le i < j \le r$ and for all s. Thus the ideals I_{ij}, $1 \le i < j \le r+1$, are coprime. Let $b_{r+1} = 1 \in B/(I_{r+1})$.

Let $1 \le i < j \le r$. Suppose $(\pi_i, \pi_j) \ne B$. Then (π_i, π_j) is a maximal ideal of B corresponding to a point $P \in C_i \cap C_j$. Since $P \in \mathscr{P}_0$, by the choice of b_i and b_j (see Lemma 7.4), we have $\nu^i_P(b_i) = 1$, $\nu^i_P(b_j) = \ell - 1$ and hence $b_i = b_j = 0 \in B/(\pi_i, \pi_j) = B/I_{ij}$.

Suppose $I_{i(r+1)} \ne B$ for some $1 \le i \le r$. Then we claim that $b_i = 1 \in B/I_{i(r+1)}$. For each $P \in L_j \cap C_i$, let M_P be the maximal ideal of B at P. Since $\mathscr{X}$ is regular and C_i is regular on $\mathscr{X}$, we have $M_P = (\pi_i, \pi_{i,P})$ for some $\pi_{i,P} \in M_P$ and the image of $\pi_{i,P}$ in $B/(\pi_i)$ is a parameter at the discrete valuation ν^i_P. Since $d > \nu^i_P(\bar\delta(i))$, we have $\big(\pi_i, \prod \pi^d_{i,P}\big) \subset (\pi_i, \delta) = I_{i(r+1)}$. Since $B/\big(\pi_i, \prod \pi^d_{i,P}\big) \simeq \prod_P B/(\pi_i, \pi^d_{i,P})$ and $\nu^i_P(b_i - 1) \ge d$, we have $b_i = 1 \in B/\big(\pi_i, \prod \pi^d_{i,P}\big)$. Since $B/I_i + I_{r+1}$ is a quotient of $B/I_i + \big(\prod_P \pi_{i,P}\big)^d$, it follows that $b_i = b_{r+1} = 1 \in B/I_i + I_{r+1} = B/I_{i(r+1)}$.

Thus, by Lemma 2.7, there exists $b \in B$ such that $b = b_i \in B/(\pi_i)$ for all i and $b = 1 \in B/I_{r+1}$. Since $I_{r+1} = (\delta) \subset (\delta_j)$ and $b = 1 \in B/(\delta)$, we have $b = 1 \in B/(\delta_j)$ for all j. Let $P \in \mathscr{P}_1$. Then $P \in L_j$ for some j. Since $b = 1 \in B/(\delta_j)$, b is a unit at P. Thus b has all the required properties. □

Lemma 7.5. *Let a be as in Lemma 7.1 and b as in Lemma 7.4 and $\alpha = [a, b)$. Then α is unramified at all C_i, L_j and at all $Q \in \mathscr{P}_1$. Further $\partial_{C_i}(\zeta)$ is the specialization of α at C_i for all $1 \le i \le r$.*

Proof. Since $[a)$ is unramified at C_i (Lemma 7.1) and b is a unit at C_i for all i (Lemma 7.4), α is unramified at C_i and the specialization of α at C_i is $[a(C_i), b_i) = \partial_{C_i}(\zeta)$ (Lemmas 7.3 and 7.4). Since $\operatorname{char}(\kappa(L_j)) \ne \ell$ (Lemma 7.2) and $b = 1$ modulo δ_j (Lemma 7.4), b is an ℓ-th power in F_{L_j} and hence $\alpha \otimes F_{L_j} = 0$. In particular α is unramified at L_j.

Let $Q \in \mathscr{P}_1$. Then b is a unit at Q (Lemma 7.4). Let x be a dimension one point of $\operatorname{Spec}(B_Q)$. Then b is a unit at x. If $[a)$ is unramified at x, then α is unramified at x. Suppose $[a)$ is ramified at x. Then, by the choice of the L_j, x is the generic point of L_j for some j and hence α is unramified at x. Thus α is unramified at Q (see Lemma 3.1). □

Proposition 7.6. *The triple $(\mathscr{X}, \zeta, [a, b))$ satisfies Assumptions 6.3.*

Proof. By the choice of $\mathscr{X}$, (B1) of Assumptions 6.3 is satisfied. Let $\operatorname{ram}_{\mathscr{X}}(\alpha) = \{D_1, \dots, D_n\}$. Since α is unramified at all C_i (Lemma 7.5), (B2) of Assumptions 6.3 is satisfied. Since $\operatorname{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \dots, C_r\}$ and $D_i \ne C_j$ for all i and j, $\operatorname{char}(\kappa(D_i)) \ne \ell$ for all i and hence (B3) of Assumptions 6.3 is satisfied.

Let $P \in D_j$ some j with $\operatorname{char}(\kappa(P)) = \ell$. Since $\operatorname{supp}_{\mathscr{X}}(\ell) \subset \{C_1, \dots, C_r\}$, $P \in C_i$ for some i. Since α is unramified at all $Q \in \mathscr{P}_1$ (Lemma 7.5), $P \notin \mathscr{P}_1$. Since $C_i \cap L_s \subset \mathscr{P}_1$ for all s, $P \notin L_s$ for all s and hence $(a-1)/(\rho-1)^\ell \in A_P$. Thus (B4) of Assumptions 6.3 is satisfied.

Since $\partial_{C_i}(\zeta)$ is the specialization of α at C_i (Lemma 7.5), (B5) of Assumptions 6.3 is satisfied.

By the assumption on ζ, (B6) of Assumptions 6.3 is satisfied. If $\ell = 2$, then, by the choice of a (Lemma 7.1), (B7) of Assumptions 6.3 is satisfied.

Let $P \in C_i \cap C_j$ for some $i < j$. Then, by the choice of b_i and b_j (Lemma 7.3), we have $b_i = \bar{u}_j \bar{\pi}_j$ for some unit u_j at P and $b_j = \bar{u}_i \bar{\pi}_i^{\ell-1}$ for some unit u_i at P. Since $b = b_i$ modulo π_i and $b = b_j$ modulo π_j, we have $b = v_i \pi_i^{\ell-1} + v_j \pi_j$ for some units v_i, v_j at P. In particular b is a regular prime at P. Since $[a)$ is unramified at P (Lemma 7.1) and b being a prime at P, α is unramified at P except possibly at b. Thus there is at most one D_s with $P \in D_s$ and such a D_s is defined by $b = v_i \pi_i^{\ell-1} + v_j \pi_j$ for some units v_i, v_j at P. In particular (B8) of Assumptions 6.3 is satisfied. □

Theorem 7.7. *Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to the characteristic of K. Suppose that K contains a primitive ℓ-th root of unity. Let $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$. Suppose that $\zeta \otimes_F (F \otimes_K K_\nu)$ is trivial for all real places ν of K. Then there exist $a, b, f \in F^*$ such that $\zeta = [a, b) \cdot (f)$.*

Proof. By Proposition 7.6, there exist $a, b \in F^*$ and regular proper model $\mathscr{X}$ of F such that the triple $(\mathscr{X}, \zeta, \alpha)$ satisfy the Assumptions 6.3. Thus, by Theorem 6.5, there exists $f \in F^*$ such that $\zeta = \alpha \cdot (f) = [a, b) \cdot (f)$. □

Corollary 7.8. *Let K be a global field or a local field and F the function field of a curve over K. Let ℓ be a prime not equal to the characteristic of K. Suppose that K contains a primitive ℓ-th root of unity. Suppose that either $\ell \neq 2$ or K has no real places. Then for every element $\zeta \in H^3(F, \mathbb{Z}/\ell(2))$, there exist $a, b, c \in F^*$ such that $\zeta = [a, b) \cdot (c)$.*

8. Applications

In this section we given some applications of our main result to quadratic forms and Chow group of zero-cycles.

Let K be a field of characteristic not equal to 2. Let $W(K)$ denote the Witt group of quadratic forms over K and $I(K)$ the fundamental ideal of $W(K)$ consisting of classes of even dimensional forms [Scharlau 1985, Chapter 2]. For $n \geq 1$, let $I^n(K)$ denote the n-th power of $I(K)$. For $a_1, \ldots, a_n \in F^*$, let $\langle\langle a_1, \ldots, a_n \rangle\rangle$ denote the n-fold Pfister form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ [loc. cit., Chapter 4].

Theorem 8.1. *Let k be a totally imaginary number field and F the function field of a curve over k. Then every element in $I^3(F)$ is represented by a 3-fold Pfister form. In particular if the class of a quadratic form q is in $I^3(F)$ and dimension of q is at least 9, then q is isotropic.*

Proof. Since every element in $H^3(F, \mathbb{Z}/2(3))$ is a symbol (Corollary 7.8) and $\mathrm{cd}_2(F) \leq 3$, it follows from [Arason et al. 1986, Theorem 2] that every element in $I^3(F)$ is represented by a 3-fold Pfister form (see the proof of [Parimala and Suresh 1998, Theorem 4.1]). □

Proposition 8.2. *Let F be a field of characteristic not equal to 2 with $cd_2(F) \leq 3$ Suppose that every element in $H^3(F, \mathbb{Z}/2(3))$ is a symbol. If q is a quadratic form over F of dimension at least 5 and $\lambda \in F^*$, then $q \otimes \langle 1, -\lambda \rangle$ is isotropic.*

Proof. Without loss of generality we assume that dimension of q is 5. By scaling we also assume that $q = \langle -a, -b, ab, c, d \rangle$ for some $a, b, c, d \in F^*$. Let $q' = \langle -a, -b, ab, c, d, -cd \rangle \otimes \langle 1, -\lambda \rangle$. Since

$\langle -a, -b, ab, c, d, -cd\rangle \in I^2(K)$ [Scharlau 1985, page 82], $q' \in I^3(F)$. Hence, by Theorem 8.1, q' is represented by 3-fold Pfister form. Since $q' \otimes F(\sqrt{\lambda}) = 0$, $q' = \langle 1, -\lambda\rangle \otimes \langle 1, \mu\rangle \otimes \langle 1, \mu'\rangle$ for some $\mu, \mu' \in F^*$ (see [Scharlau 1985, Theorem 5.2 on page 45, Corollary 1.5 on page 143 and Theorem 1.4 on page 144]). Since $H^4(F, \mathbb{Z}/2(4)) = 0$, $I^4(F) = 0$ [Arason et al. 1986, Corollary 2], we have $q' = -cd\langle 1, -\lambda\rangle \otimes \langle 1, \mu\rangle \otimes \langle 1, \mu'\rangle$.

Thus we have

$$\begin{aligned}\langle -a, -b, ab, c, d\rangle \otimes \langle 1, -\lambda\rangle &= -cd\langle 1, -\lambda\rangle \otimes \langle 1, \mu\rangle \otimes \langle 1, \mu'\rangle + cd\langle 1, -\lambda\rangle \\ &= -cd\langle 1 - \lambda\rangle \otimes \langle \mu, \mu', \mu\mu'\rangle.\end{aligned}$$

In particular $\langle -a, -b, ab, c, d\rangle \otimes \langle 1, -\lambda\rangle$ is isotropic [Scharlau 1985, page 34]. □

Corollary 8.3. *Let K be a totally imaginary number field and F the function field a curve over K. Let q be a quadratic forms over F of dimension at least* 5. *Let $\lambda \in F^*$. Then the quadratic form $q \otimes \langle 1, -\lambda\rangle$ is isotropic.*

Proof. Since K is a totally imaginary number field and F is a function field of a curve over k, we have $H^4(F, \mathbb{Z}/2(4)) = 0$. Since every element in $H^3(F, \mathbb{Z}/2(3))$ is a symbol (Corollary 7.8), $q \otimes \langle 1, -\lambda\rangle$ is isotropic (Proposition 8.2). □

The following was conjectured by Colliot-Thélène and Skorobogatov [1993].

Theorem 8.4. *Let k be a totally imaginary number field and C a smooth projective geometrically integral curve over K. Let $\eta : X \to C$ be an admissible quadric fibration. If* $\dim(X) \geq 4$, *then* $\mathrm{CH}_0(X)$ *is a finitely generated abelian group.*

Proof. Let q be a quadratic form over $k(C)$ defining the generic fiber of $\eta : X \to C$. Let $N_q(k(C))$ be the subgroup of $k(C)^*$ generated by fg with $f, g \in k(C)^*$ represented by q. Let $\lambda \in k(C)^*$. Since $\dim(X) \geq 4$, the dimension of q is at least 5. Thus, by Corollary 8.3, $q \otimes \langle 1, -\lambda\rangle$ is isotropic. Hence λ is a product of two values of q. In particular $\lambda \in N_q(k(C))$ and $k(C)^* = N_q(k(C))$.

Let $\mathrm{CH}_0(X/C)$ be the kernel of the induced homomorphism $\mathrm{CH}_0(X) \to \mathrm{CH}_0(C)$. Then, by [Colliot-Thélène and Skorobogatov 1993], $\mathrm{CH}_0(X/C)$ is a subquotient of the group $k(C)^*/N_q(k(C))$ and hence $\mathrm{CH}_0(X/C) = 0$. In particular $\mathrm{CH}_0(X)$ is isomorphic to a subgroup of $\mathrm{CH}_0(C)$. Since, by a theorem of Mordell–Weil, $\mathrm{CH}_0(C)$ is finitely generated, $\mathrm{CH}_0(X)$ is finitely generated. □

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On upper bounds of Manin type

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We introduce a certain birational invariant of a polarized algebraic variety and use that to obtain upper bounds for the counting functions of rational points on algebraic varieties. Using our theorem, we obtain new upper bounds of Manin type for 28 deformation types of smooth Fano 3-folds of Picard rank ≥ 2 following the Mori–Mukai classification. We also find new upper bounds for polarized K3 surfaces S of Picard rank 1 using Bayer and Macrì's result on the nef cone of the Hilbert scheme of two points on S.

1. Introduction

A driving question in diophantine geometry is to prove asymptotic formulae for the counting function of rational points on a projective variety. Manin's conjecture, originally formulated in [Batyrev and Manin 1990], predicts a precise asymptotic formula when the underlying variety is smooth Fano, or more generally smooth and rationally connected. This asymptotic formula has a description in terms of the geometric invariants of the underlying variety.

In this paper we consider questions related to the following weaker version of the conjecture which is called the weak Manin's conjecture: let X be a geometrically uniruled smooth projective variety defined over a number field k and let L be a big and nef divisor on X. One can associate a height function

$$H_L : X(k) \to \mathbb{R}_{>0}$$

to (X, L), and we consider the counting function

$$N(U, L, T) = \#\{P \in U(k) | H_L(P) \leq T\}$$

for an appropriate Zariski open subset $U \subset X$. The weak Manin's conjecture predicts that this function is governed by the following geometric invariant of (X, L):

$$a(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\},$$

where $\overline{\mathrm{Eff}}^1(X)$ is the cone of pseudoeffective divisors on X. Here is the statement of the weak Manin's conjecture:

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Conjecture 1.1 (the weak Manin's conjecture/linear growth conjecture). Let X be a geometrically uniruled smooth projective variety defined over a number field k and let L be a big and nef divisor on X. Then there exists a nonempty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$,

$$N(U, L, T) = O_\epsilon(T^{a(X,L)+\epsilon}).$$

Note that for $L = -K_X$, we have $a(X, L) = 1$, backing up the phrase "linear growth".

Remark 1.2. There are counterexamples to a version of this conjecture where one assumes L to be only big but not nef; see [Lehmann et al. 2018a, Section 5.1].

The starting point of the current research is [McKinnon 2011], where it is shown that Vojta's conjecture implies the weak Manin's conjecture for K3 surfaces and more generally varieties with Kodaira dimension 0 assuming the nonvanishing conjecture in the minimal model program. (For such varieties, the a-invariant is 0.) While McKinnon's result is conditional on Vojta's conjecture, our results are unconditional: they do not rely on Vojta's conjecture. In our approach, instead of appealing to Vojta's conjecture, we use the positivity of divisors by introducing the following invariant measuring the local positivity of big divisors:

Definition 1.3. Let X be a normal projective variety defined over an algebraically closed field of characteristic 0 and H be a big $\mathbb{Q}$-Cartier divisor on X. We consider $W = X \times X$ and denote each projection by $\pi_i : W \to X_i$. Let $\alpha : W' \to W$ be the blow-up of the diagonal and denote its exceptional divisor by E, i.e., the pullback of the diagonal. For any $\mathbb{Q}$-Cartier divisor L on X we denote $\alpha^*\pi_1^*L + \alpha^*\pi_2^*L$ by $L[2]$. We define the invariant

$$\delta(X, H) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{for any component } V \subset \mathrm{SB}(sH[2]-E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i. \end{array} \right\},$$

where $\mathrm{SB}(sH[2]-E)$ is the stable base locus of a $\mathbb{R}$-divisor $sH[2]-E$. We call this invariant the δ-invariant.

Inspired by [McKinnon 2011], we obtain the following general result on the counting functions of rational points on algebraic varieties:

Theorem 1.4. *Let X be a normal projective variety of dimension n defined over a number field k and L be a big $\mathbb{Q}$-Cartier divisor on X. Then for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that we have*

$$N(U, L, T) = O_\epsilon(T^{2n\delta(X,L)+\epsilon}).$$

Previous results applying to general projective varieties are results related to dimension growth conjecture obtained by Browning, Heath-Brown, and Salberger [Browning et al. 2006], and Salberger [2007]. Recall that the dimension growth conjecture of Heath-Brown, which is proved by Salberger, states that for any subvariety $X \subset \mathbb{P}^n$, the hyperplane class H, and any $\epsilon > 0$ we have

$$N(X, H, T) = O(T^{\dim X+\epsilon}).$$

For many examples of projective varieties X of dimension ≤ 3 we proved that $\delta(X, -K_X) \leq \frac{1}{2}$. (For example, we compute $\delta(X, -K_X)$ for most of 3-dimensional Fano conic bundles with rational sections in Section 9 following the Mori–Mukai classification, and we confirm that $\delta(X, -K_X) = \frac{1}{2}$ for 40 deformation types out of 53 deformation types.) Thus this theorem recovers some statements of [Browning et al. 2006; Salberger 2007] on the dimension growth conjecture for varieties with $\delta(X, -K_X) \leq \frac{1}{2}$. We conjecture that for a large portion of the class of Fano manifolds, we have $\delta(X, -K_X) \leq \frac{1}{2}$, so our theorem should lead to an alternative proof of the dimension growth conjecture for certain Fano varieties.

However, there is no direct comparison between our result and those in [Browning et al. 2006; Salberger 2007], which are certainly better in the sense that they obtain a bound for $N(X, L, T)$ and their constants only depend on the dimension of X, ϵ, and the dimension of the ambient projective space where X is embedded into. On the other hand, our method has the advantage in the sense that our theorem applies to arbitrary big divisors and in many cases where $\delta(X, L)$ is the minimum, e.g., the 3-dimensional Fano conic bundles mentioned above, one does not need to introduce $\epsilon > 0$ in the above theorem. For most smooth projective varieties with nonnegative Kodaira dimension, we conjecture that Theorem 1.4 gives better upper bounds than [Browning et al. 2006; Salberger 2007]. For example, we have the following application of Theorem 1.4 to K3 surfaces:

Theorem 1.5. *Let S be a K3 surface defined over a number field k with a polarization H of degree $2d$ such that* $\mathrm{Pic}(\bar{S}) = \mathbb{Z}H$. *Then for any $\epsilon > 0$, we have*

$$N(S, H, T) = O_\epsilon(T^{4\sqrt{(4/d)+(5/d^2)}+\epsilon}).$$

The existence of K3 surfaces satisfying the assumptions of Theorem 1.5 is justified by [Terasoma 1985; Ellenberg 2004; van Luijk 2007]. Our proof relies on the work of Bayer and Macrì [2014] on the nef cone of the Hilbert scheme of 2 points $\mathrm{Hilb}^{[2]}(S)$. Indeed, $\delta(S, H)$ is bounded by the s-invariant of H, and the computation of the s-invariant can be done using the description of the nef cone of $\mathrm{Hilb}^{[2]}(S)$. A certain bound is also obtained for Enriques surfaces by using [Nuer 2016]; see Theorem 7.2. It would be interesting to compute these invariants for surfaces of general type. Bounds of this type are obtained for hypersurfaces in $\mathbb{P}^n$ by Heath-Brown [2002].

For some 3-dimensional Fano conic bundles we are able to improve bounds of Theorem 1.4 using conic bundle structures.

Theorem 1.6. *Let $f : X \to S$ be a conic bundle defined over a number field k with a rational section. We assume that X and S are smooth Fano. Let $W = X \times X$ and W' be the blow-up of W along the diagonal with the exceptional divisor E. We denote each projection $W' \to X_i$ by π_i. Let α, β be positive real numbers such that $2\alpha - 2\beta = 1$. We further make the following assumptions*:

(1) *The weak Manin's conjecture for $(S, -K_S)$ holds.*

(2) *For any component V of the stable locus of the divisor*

$$-\alpha K_{X/S}[2] - \beta f^* K_S[2] - E,$$

such that V is not contained in E, one of projections $\pi_i|_V$ is not dominant.

Then there exists a nonempty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$, there exists $C = C_\epsilon > 0$ such that

$$N(U, -K_X, T) < CT^{2\alpha+\epsilon}.$$

Using this theorem, new upper bounds for 28 deformation types of Fano 3-folds are obtained and these bounds are better than the dimension growth conjecture in [Browning et al. 2006; Salberger 2007]. These examples are discussed in Section 9. Here are some examples of Fano 3-folds which our theorem applies to. Note that for examples below we have $\delta(X, -K_X) = \frac{1}{2}$.

Example 1.7 (Example 9.2). Let X be the blow-up of a quadric threefold Q defined over a number field k with center a line defined over the same ground field. Let H be the pullback of hyperplane class from Q and we denote the exceptional divisor by D. Then the linear system $|H - D|$ defines a $\mathbb{P}^1$-fibration over $\mathbb{P}^2$. We prove that $\alpha = \frac{5}{6}$ satisfies the assumptions of Theorem 1.6: thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have

$$N(U, -K_X, T) = O_\epsilon(T^{5/3+\epsilon}).$$

Example 1.8 (Example 9.3). Let V_7 be the blow-up of $\mathbb{P}^3$ at a point P. This is isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$. Let X be the blow-up of V_7 with center the strict transform of a conic passing through P. Then X is a Fano conic bundle with singular fibers. We prove that $\alpha = \frac{5}{6}$ satisfies the assumptions of Theorem 1.6: thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have

$$N(U, -K_X, T) = O_\epsilon(T^{5/3+\epsilon}).$$

Example 1.9 (Example 9.5). Let X be the blow-up of $\mathbb{P}^3$ with center a disjoint union of three lines. Then X is a Fano conic bundle with singular fibers. We prove that $\alpha = 1$ satisfies the assumptions of Theorem 1.6: thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have

$$N(U, -K_X, T) = O_\epsilon(T^{2+\epsilon}).$$

It is natural to wonder whether $2\alpha < 2\delta(X, -K_X) \dim X$ holds in general. While we do not have a proof of this inequality, we do not have any counterexample either. Finally note that for del Pezzo surfaces, there are many better results on bounds of the counting functions; see, e.g., [Heath-Brown 1997; Broberg 2001; Browning and Swarbrick Jones 2014; Frei et al. 2018; Browning and Sofos 2019].

The method of proofs. McKinnon proves the weak Manin's conjecture for K3 surfaces using a certain repulsion principle which he proves assuming Vojta's conjecture. We instead prove a different repulsion principle using the δ-invariant and this proof does not rely on Vojta's conjecture. Here is our theorem:

Theorem 1.10 (repulsion principle). *Let X be a normal projective variety defined over a number field k. We fix a place v of k. Let A be a big $\mathbb{Q}$-Cartier divisor on X. Then for any $\epsilon > 0$ there exists a constant*

$C = C_\epsilon > 0$ and a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that

$$\operatorname{dist}_v(P, Q) > C(H_A(P)H_A(Q))^{-(\delta(X,A)+\epsilon)},$$

for any $P, Q \in U(k)$ with $P \neq Q$, where $\operatorname{dist}_v(P, Q)$ is the v-adic distant function on X.

Combining this theorem and counting arguments in [McKinnon 2011], we prove Theorem 1.4.

This paper is organized as follows. In Section 2, we recall the constructions of height functions and their basic properties. In Section 3 we recall basic properties and results of the a-invariants. In Section 4, we discuss some basic properties of the δ-invariants and compute them for some examples, e.g., del Pezzo surfaces. In Section 5, we prove the Repulsion principle for projective varieties (Theorem 1.10). In Section 6 we establish Theorem 1.4. In Section 7, we study K3 surfaces and Enriques surfaces and prove Theorem 1.5. In Section 8, we prove Theorem 1.6. In Section 9, we study 3-dimensional Fano conic bundles using Theorem 1.6.

2. Height functions

Here we recall the constructions of height functions and their basic properties which will be needed for the rest of the paper. The main references are [Hindry and Silverman 2000; Chambert-Loir and Tschinkel 2010]. Let k be a number field and M_k denote the set of places of k. For each place $v \in M_k$, k_v denotes its completion with respect to v, and we fix a Haar measure μ_v on k_v. We normalize our absolute value $|\cdot|_v$ on k_v by the following property: for any $a \in k_v$ and a measurable set $\Omega \subset k_v$ we have

$$\mu_v(a\Omega) = |a|_v \mu_v(\Omega).$$

When $k_v = \mathbb{R}$, $|\cdot|_v$ is the usual absolute value. When $k_v = \mathbb{Q}_p$ we have $|p|_v = 1/p$. For a finite extension $k_v/\mathbb{Q}_w$, we have $|a|_v = |N_{k_v/\mathbb{Q}_w}(a)|_w$ for any $a \in k_v$. Due to the normalizations, we have the product formula, i.e., for any $a \in k^\times$ we have

$$\prod_{v \in M_k} |a|_v = 1. \tag{2-1}$$

A variety X defined over k is a geometrically integral separated scheme of finite type over k. For any place $v \in M_k$ the topological space $X(k_v)$ is endowed with a natural structure as an analytic space over k_v. For any invertible sheaf L on X, we consider the underlying line bundle $\pi : \underline{L} \to X$.

Definition 2.1 (height functions). Let X be a projective variety defined over k and L be an invertible sheaf on X. Let $\{\|\cdot\|_v\}_{v \in M_k}$ be an adelic metric for L, i.e., for each $v \in M_k$, $\|\cdot\|_v$ is a v-adic metric on the analytic line bundle $\pi : \underline{L}(k_v) \to X(k_v)$ and they satisfy a certain integral condition; see [Chambert-Loir and Tschinkel 2010, Section 2.2.3] for more details. For each $P \in X(k)$, we pick a nonzero element ℓ of $\underline{L}_P(k)$ where $\underline{L}_P$ is a fiber of $\pi : \underline{L} \to X$ at $P \in X$. We define the multiplicative height function associated to $\mathcal{L} = (L, \|\cdot\|_v)$ by

$$H_{\mathcal{L}}(P) = \prod_{v \in M_k} \|\ell\|_v^{-1}.$$

Note that this does not depend on the choice of ℓ because of the product formula (2-1). We also define the logarithmic height function by

$$h_{\mathcal{L}}(P) = \log H_{\mathcal{L}}(P).$$

Remark 2.2. There is another construction of height functions using the framework of Weil height machines [Hindry and Silverman 2000, Theorem B.3.2]. One can show that two constructions are equivalent in the sense that two height functions associated to the same line bundle are equal up to a bounded function; see [Hindry and Silverman 2000, Theorem B.10.7]. Also note that the construction of height functions in this paper uses normalizations which differ from ones in [Hindry and Silverman 2000]. This is because in this paper we only consider height functions defined on the set of rational points while [Hindry and Silverman 2000] considers the height functions defined over the set of algebraic points. Thus in [Hindry and Silverman 2000], one needs to normalize each height function by the degree of the definition field of an algebraic point.

Remark 2.3. In the later discussions, we frequently omit the discussion of metrics and we consider a height function h_L associated to a line bundle L. In this situation, we implicitly make a choice of an adelic metric, but we will not make this dependence explicit as this does not matter for our discussion.

There are two important properties of height functions we frequently use:

Theorem 2.4 [Hindry and Silverman 2000, Theorem B.3.2]. *Let X be a projective variety defined over k and L be an invertible sheaf on X.*

(1) ***Positivity***: *Let B be the stable base locus of L. Then we have*

$$h_L(P) \geq O(1)$$

for any $P \in (X \setminus B)(k)$.

(2) ***Northcott property***: *Suppose that L is ample. Then for any $T > 0$ the set*

$$\{P \in X(k) \in H_L(P) \leq T\}$$

is finite.

Finally we recall the construction of local height functions:

Definition 2.5. Let X be a projective variety defined over k and L be a Cartier divisor on X. For a place $v \in M_k$ we fix a v-adic metric $\| \cdot \|_v$ for L, i.e., a v-adic metric on the analytic line bundle $\pi : \underline{\mathcal{O}(L)}(k_v) \to X(k_v)$. Let D be an effective divisor linearly equivalent to L. Let s_D be a k-section associated to D. Then the multiplicative local height function associated to D is given by

$$H_{D,v}(P) = \|\mathsf{s}_D(P)\|_v^{-1}$$

for any $P \in (X \setminus D)(k_v)$. We also define the logarithmic local height function by

$$h_{D,v}(P) := \log H_{D,v}(P).$$

Suppose we fix an adelic metrized line bundle $\mathcal{L}$. Then the height function is the Euler product of local height functions, i.e., we have

$$H_{\mathcal{L}}(P) = \prod_{v \in M_k} H_{D,v}(P)$$

for any $P \in (X \setminus D)(k)$.

3. The Fujita invariant in Manin's conjecture

Here we assume that our ground field k is a field of characteristic zero, but not necessarily algebraically closed. Recently the geometric study of Fujita invariants has been conducted in [Hassett et al. 2015; Lehmann et al. 2018b; Hacon and Jiang 2017; Lehmann and Tanimoto 2017; 2018; 2019a; 2019b; Sengupta 2017; Lehmann et al. 2018a.] We recall its definition here.

Definition 3.1. Let X be a smooth projective variety defined over k. Let L be a big and nef $\mathbb{Q}$-divisor on X. We define the *Fujita invariant* (or a-invariant) by

$$a(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\},$$

where $\overline{\mathrm{Eff}}^1(X)$ is the cone of pseudoeffective divisors on X. By [Boucksom et al. 2013] $a(X, L) > 0$ if and only if X is geometrically uniruled. When L is not big, we simply set $a(X, L) = +\infty$. When X is singular, we take a resolution $\beta : X' \to X$ and we define the Fujita invariant by

$$a(X, L) := a(X', \beta^* L).$$

This is well-defined because the Fujita invariant is a birational invariant [Hassett et al. 2015, Proposition 2.7].

This invariant plays a central role in Manin's conjecture. For example, one can predict the exceptional set of Manin's conjecture by studying this invariant and the following result is a consequence of Birkar's celebrated papers [2016; 2019]:

Theorem 3.2 [Lehmann et al. 2018b; Hacon and Jiang 2017; Lehmann and Tanimoto 2019a]. *Assume that our ground field is algebraically closed. Let X be a smooth projective uniruled variety and let L be a big and nef $\mathbb{Q}$-divisor on X. Let V be the union of subvarieties Y with $a(Y, L) > a(X, L)$. Then V is a proper closed subset of X.*

For computations of this exceptional set V for some examples, see [Lehmann et al. 2018b; Lehmann and Tanimoto 2019b].

4. The invariant $\delta(X, H)$

Here we assume that our ground field k is an algebraically closed field of characteristic 0. Let X be a normal projective variety and H be a big $\mathbb{Q}$-Cartier divisor on X. We consider $W = X \times X$ and denote each projection by $\pi_i : W \to X_i$. Let $\alpha : W' \to W$ be the blow-up of the diagonal and we denote its exceptional divisor by E. For any $\mathbb{Q}$-Cartier divisor L on X we denote $\alpha^*\pi_1^* L + \alpha^*\pi_2^* L$ by $L[2]$.

Definition 4.1. Let X, W', E as above. We define the invariant

$$\delta(X, H) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{for any component } V \subset \mathrm{SB}(sH[2]-E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i. \end{array} \right\},$$

where $\mathrm{SB}(sH[2]-E)$ is the stable base locus of a $\mathbb{R}$-divisor $sH[2]-E$.

Remark 4.2. It follows from the definition that when the rational map $\Phi_{|H|}$ associated to $|H|$ is birational, we have $\delta(X, H) \leq 1$. Indeed, let Z be a closed subset such that on $X \setminus Z$, $\Phi_{|H|}$ is well-defined and an isomorphism onto the image. Then one can conclude that

$$\mathrm{Bs}(H[2]-E) \subset E \cup \pi_1^{-1}(Z) \cup \pi_2^{-1}(Z),$$

where $\mathrm{Bs}(H[2]-E)$ is the base locus of $H[2]-E$. Thus our assertion follows. Also it follows from the definition that $\delta(X, H)H[2]-E$ is pseudoeffective.

Lemma 4.3. *Let s_0 be a positive real number. Let F_P be a fiber of the first projection $\pi_1 \circ \alpha : W' \to X_1$ at $P \in X_1$. Suppose that there exists a family of irreducible curves $C_t \subset F_{P(t)}$ such that*

(i) *$P(t)$ covers a Zariski open subset of X_1 as t varies,*

(ii) *the image of C_t in X_2 is an irreducible curve containing $P(t)$, and*

(iii) $(s_0H[2]-E).C_t = 0$.

Then we have $\delta(X, H) \geq s_0$.

Proof. Let $V = \overline{\bigcup C_t}$ in W' which is irreducible. Then V is not contained in E and each projection $\pi_i|_V$ is dominant. Let $s < s_0$. Then since $(sH[2]-E).C_t < 0$, $\mathrm{SB}(sH[2]-E)$ contains V. Thus we must have $s \leq \delta(X, H)$. Since this is true for any $s < s_0$, our assertion follows. □

Example 4.4. Let $X = \mathbb{P}^n$ and H be the hyperplane class. Then $\delta(X, H) = 1$. Indeed, it follows from Remark 4.2 that $\delta(X, H) \leq 1$. On the other hand, let F_1 be a general fiber of the first projection $\pi_1 \circ \alpha : W' \to X_1 = \mathbb{P}^n$ at $P \in X_1$ and ℓ be the strict transform of a line passing through P in F_1. Then we have $(H[2]-E).\ell = 0$. Thus our assertion follows from Lemma 4.3.

Example 4.5. Let $X \subset \mathbb{P}^n$ be a normal projective variety and H be the hyperplane class. Suppose that X is covered by lines. Then the same proof of the above example shows that $\delta(X, H) = 1$.

Next we show that the invariant $\delta(X, H)$ is a birational invariant.

Lemma 4.6. *Let X be a normal projective variety and H be a big $\mathbb{Q}$-Cartier divisor on X. Let $\beta : X' \to X$ be a birational morphism between normal projective varieties. Then we have*

$$\delta(X', \beta^*H) = \delta(X, H).$$

Proof. Let W_X be the blow-up of $X \times X$ along the diagonal and $W_{X'}$ be the blow-up of $X' \times X'$ along the diagonal. We denote their exceptional divisors by E_X and $E_{X'}$ respectively. Then we have a birational map

$$\phi : W_{X'} \dashrightarrow W_X$$

which is a birational contraction and the indeterminacy of this map is not dominant to both X_i'. Also for a component V of the nonisomorphic loci of this map such that V is not contained in $E_{X'}$, one of projections is not dominant.

Fix $\epsilon > 0$. Suppose that the stable locus of $(\delta(X, H)+\epsilon)\beta^* H[2] - E_{X'}$ contains a subvariety $Y \subset W_{X'}$ such that $Y \not\subset E_{X'}$ and Y maps dominantly to both X_i'. By the definition, $(\delta(X, H)+\epsilon)H[2] - E_X$ does not contain $\phi(Y)$ in the stable locus so that there exists $0 \leq D \sim_{\mathbb{R}} (\delta(X, H)+\epsilon)H[2] - E_X$ such that $\phi(Y) \not\subset \mathrm{Supp}(D)$. Then we have $\phi^* D \sim_{\mathbb{R}} (\delta(X, H)+\epsilon)\beta^* H[2] - E_{X'}$ because $\phi^* E_X = E_{X'}$. Furthermore we have $Y \not\subset \mathrm{Supp}(\phi^* D)$. This contradicts with our assumption. Thus we conclude

$$\delta(X', \beta^* H) \leq \delta(X, H).$$

Suppose that the stable locus of $(\delta(X', H)+\epsilon)H[2] - E_X$ contains a subvariety $Y \subset W_X$ such that $Y \not\subset E_X$ and Y maps dominantly to both X_i. We take the strict transform $Y' \subset W_{X'}$ of Y. By the definition, $(\delta(X', H)+\epsilon)\beta^* H[2] - E_{X'}$ does not contain Y' in the stable locus so there exists $0 \leq D \sim_{\mathbb{R}} (\delta(X', H)+\epsilon)\beta^* H[2] - E_{X'}$ such that $Y' \not\subset \mathrm{Supp}(D)$. Then $\phi_* D \sim_{\mathbb{R}} (\delta(X', H)+\epsilon)H[2] - E_X$. Furthermore we have $Y \not\subset \mathrm{Supp}(\phi_* D)$. This contradicts with our assumption. Thus we conclude

$$\delta(X', \beta^* H) \geq \delta(X, H).$$

Thus our assertion follows. □

Here is a relation between $\delta(X, H)$ and $a(X, H)$.

Proposition 4.7. *Let X be a smooth weak Fano variety, i.e., $-K_X$ is big and nef, and let H be a big and nef divisor on X. Then we have*

$$\delta(X, H) \leq a(X, H)\delta(X, -K_X).$$

Proof. We write $a(X, H)H + K_X \sim_{\mathbb{Q}} D \geq 0$. Fix $\epsilon > 0$. Then we have

$$a(X, H)(\delta(X, -K_X)+\epsilon)H[2] - E \sim_{\mathbb{Q}} -(\delta(X, -K_X)+\epsilon)K_X[2] + (\delta(X, -K_X)+\epsilon)D[2] - E.$$

Thus we see that the stable locus of $|a(X, H)(\delta(X, -K_X)+\epsilon)H[2] - E|$ does not contain any dominant component possibly other than subvarieties in E. Thus our assertion follows. □

Next we consider the s-invariants and its relation to the δ-invariants:

Definition 4.8. Let X be a smooth projective variety and H be an ample divisor on X. Let W be the blow-up of $X \times X$ along the diagonal and we denote its exceptional divisor by E. The s-invariant of H is defined by

$$s(X, H) = \inf\{s \in \mathbb{R} \mid sH[2] - E \text{ is nef}\}.$$

This is a positive real number in general; see [Lazarsfeld 2004, Section 5.4] for many properties of this invariant.

Proposition 4.9. *Let X be a smooth projective variety and H be an ample divisor on X. Then we have*

$$\delta(X, H) \leq s(X, H).$$

Proof. For every $\epsilon > 0$, $(s(X, H) + \epsilon)H[2] - E$ is ample so its stable base locus is empty. Thus our assertion follows. □

Del Pezzo surfaces. Next we discuss del Pezzo surfaces. Let S be a smooth del Pezzo surface. We consider $W = S \times S$ and we denote each projection by $\pi_i : W \to S_i$. Let $\alpha : W' \to W$ be the blow-up of the diagonal and we denote its exceptional divisor by E. First we record a lower bound for the δ-invariant:

Lemma 4.10. *Let S be a smooth del Pezzo surface. Then we have*

$$\delta(S, -K_S) \geq \frac{1}{\epsilon(-K_S, P)}$$

for any general point $P \in X$ where $\epsilon(-K_S, P)$ is the Seshadri constant of $-K_S$ at P.

Proof. The Seshadri constant for the anticanonical divisor on a smooth del Pezzo surface is computed in [Broustet 2006]. According to this paper, $\epsilon(-K_S, P)$ is constant for a general point $P \in S$ and for such a P we have

$$\epsilon(-K_S, P) = \min_{P \in C \subset S} \frac{-K_S.C}{\mathrm{mult}_P(C)}.$$

Moreover, curves achieving the minimum are completely described for del Pezzo surfaces of degree ≥ 2 in [Broustet 2006] and they are members of one family from the Hilbert scheme. For a del Pezzo surface of degree 1, this minimum is achieved by members of the anticanonical system.

Let $P \in S_1$ be a general point and let C_t be the strict transform of a curve in $\{P\} \times S_2$ achieving the minimum $\epsilon(-K_S, P)$. Then we have

$$(-K_S[2] - \epsilon(-K_S, P)E).C_t = -K_X.C_t - \epsilon(-K_S, P)\mathrm{mult}_P(C_t) = 0.$$

Thus our assertion follows from Lemma 4.3. □

Now we compute $\delta(S, -K_S)$ for a del Pezzo surface S.

Proposition 4.11. *Let S be a del Pezzo surface of degree d where $4 \leq d \leq 8$. Then we have $\delta(S, -K_S) = \frac{1}{2}$.*

Proof. We only discuss the case of degree 4 del Pezzo surfaces. Other cases are easier.

Suppose that S is a del Pezzo surface of degree 4. Let F_1 be a $-K_S$-conic on S and F_2 be another $-K_S$-conic on S such that $-K_S \sim F_1 + F_2$. Indeed, one may find such a pair of $-K_S$-conics in the following way: let $\phi : S \to \mathbb{P}^2$ be a blow-down to $\mathbb{P}^2$ and we may assume that ϕ is the blow-up at $P_1, \ldots, P_5 \in \mathbb{P}^2$. Then one can find a general plane conic C_1 and a general line C_2 such that C_1 contains $P_1, \ldots, P_4$ and C_2 contains P_5. Then their strict transforms satisfy the desired property.

Now the linear system $|F_i|$ defines a conic fibration $p_i : S \to \mathbb{P}^1$. It induces a morphism $p_i[2] : W' \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\Delta_{\mathbb{P}^1}$ be the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $F_i[2] - E$ is linearly equivalent to a unique effective divisor $p_i[2]^*\Delta_{\mathbb{P}^1} - E$. We denote it by Δ_{F_i}. Let D_1 be a third conic such that $D_1.F_1 = D_1.F_2 = 1$.

Indeed, one may take D_1 as the strict transform of a conic passing through P_1, P_2, P_3, P_5. Let D_2 be the class of conics such that $-K_S \sim D_1 + D_2$. Then we have $D_2.F_1 = D_2.F_2 = 1$. We also consider Δ_{D_i}. Then we have

$$\mathrm{SB}(-K_S[2] - 2E) \subset (\Delta_{F_1} \cup \Delta_{F_2}) \cap (\Delta_{D_1} \cup \Delta_{D_2})$$

but the only possible dominant component of $\Delta_{F_i} \cap \Delta_{D_i}$ is contained in E. Thus we conclude that $\delta(X, H) \leq \frac{1}{2}$. The opposite inequality follows from Lemma 4.10 and [Broustet 2006]. □

Proposition 4.12. *Let S be a del Pezzo surface of degree 3. Then $\delta(S, -K_S) = \frac{2}{3}$.*

Proof. Let F_1 be a $-K_S$-conic. One can find a (-1)-curve E such that $-K_S \sim F_1 + E$. Indeed, let $\phi : S \to \mathbb{P}^2$ be the blow-up at $P_1, \dots, P_6 \in \mathbb{P}^2$. We let F_1 to be the strict transform of a general conic passing through $P_1, \dots, P_4$ and E be the strict transform of a line passing through P_5, P_6. Then they satisfy the desired property. Now let $D = -2K_S - F_1 \sim -K_S + E$. Then D is the pullback of the anticanonical class from a degree 4 del Pezzo surface. The upshot is that we have

$$-2K_S[2] - 3E = D[2] - 2E + F_1[2] - E.$$

Therefore it follows from the proof of Proposition 4.11 that the stable locus of $D[2] - 2E$ minus E is not dominant. Thus the stable locus of $-2K_S[2] - 3E$ is contained in Δ_{F_1}. By considering another conic and applying the same discussion, we conclude that $\delta(S, -K_S) \leq \frac{2}{3}$. The opposite inequality follows from Lemma 4.10 and [Broustet 2006]. □

Proposition 4.13. *Let S be a del Pezzo surface of degree 2. Then $\delta(S, -K_S) = 1$.*

Proof. We may write $-K_S \sim E_1 + E_2$ where E_i is a (-1)-curve. Indeed, let $\phi : S \to \mathbb{P}^2$ be the blow-up at $P_1, \dots, P_7$. Then we may define E_1 as the strict transform of a conic passing through $P_1, \dots, P_5$ and E_2 be the strict transform of a line passing through P_6, P_7. Let $f_i : S \to S_i$ be the blow-down of E_i to a cubic surface. Then $-3K_S$ can be expressed as

$$-3K_S \sim -f_1^* K_{S_1} - f_2^* K_{S_2}.$$

Thus arguing as in Proposition 4.12 we prove that the stable locus of $-K_S[2] - E$ does not contain any dominant component except E. This shows that $\delta(S, -K_S) \leq 1$.

On the other hand, let $\phi : S \to \mathbb{P}^2$ be the anticanonical double cover. We denote the involution associated to ϕ by ι and we consider the image S^ι of the map

$$S \to S \times S, \quad P \mapsto (P, \iota(P)).$$

Then one can show that for any curve C in S^ι and any $\epsilon > 0$ we have

$$(-K_S[2] - (1 + \epsilon)E).C < 0$$

Thus C is contained in the stable locus of $-K_S[2] - (1 + \epsilon)E$, proving the claim. □

Proposition 4.14. *Let S be a del Pezzo surface of degree 1. Then $\frac{3}{2} \leq \delta(S, -K_S) \leq 2$.*

Proof. Let $\phi : S \to Q \subset \mathbb{P}^3$ be the double cover associated to $|-2K_S|$. Let E_1 be a (-1)-curve on S. Then $\phi|_{E_1} : E_1 \to \phi(E_1)$ is one-to-one and its pullback consists of two (-1)-curves including E_1. Thus we may write as $-2K_S \sim E_1 + E_2$.

Let $f_i : S \to S_i$ be the blow-down of E_i to a degree 2 del Pezzo surface. Then $-4K_S$ can be expressed as

$$-4K_S \sim -f_1^* K_{S_1} - f_2^* K_{S_2}$$

Thus by Proposition 4.13, one may conclude that $\delta(S, -K_S) \leq 2$.

Another inequality follows from the discussion of Proposition 4.13 using the double cover $\phi : S \to Q \subset \mathbb{P}^3$. □

Remark 4.15. In the proofs of Propositions 4.11, 4.12 and 4.13 we show that for any component $V \subset \mathrm{SB}(-\delta(S, -K_S)K_S[2] - E)$ not contained in E, the projection $\pi_i \circ \alpha|_V$ is not dominant. In particular we conclude that

$$\delta(S, -K_S) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c}\text{for any component } V \subset \mathrm{SB}(-sK_S[2]-E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } S_i\end{array} \right\}$$

is the minimum.

5. Repulsion principle for projective varieties

Assuming Vojta's conjecture and the nonvanishing conjecture in the minimal model program, McKinnon [2011] showed a repulsion principle for varieties of nonnegative Kodaira dimension. In this paper we develop a weaker repulsion principle for projective varieties in general. We introduce some notations. We refer readers to [Silverman 1987] for the definitions and their basic properties.

Let k be a number field. Suppose that we have a projective variety X defined over k and a big $\mathbb{Q}$-divisor L on X. Let D be a closed subscheme on X. Let $h_{D,v}$ be a local height function for D with respect to v. Note that in this paper, we use unnormalized heights, i.e., we do not normalize heights by the degree of k. Let Δ be the diagonal of $X \times X$. We define the v-adic distant function by

$$h_{\Delta,v}(P, Q) = -\log \mathrm{dist}_v(P, Q);$$

see [Silverman 1987] for basic properties of this function.

Let X be a normal projective variety defined over a number field k and L be a big $\mathbb{Q}$-Cartier divisor on X. We set

$$\delta(X, L) = \delta(\overline{X}, \bar{L}),$$

where $\overline{X}$, $\bar{L}$ are the base change of X, L to an algebraic closure. Our main theorem is:

Theorem 5.1 (the repulsion principle). *Let X be a normal projective variety defined over a number field k. Let v be a place of k. Let A be a big Cartier divisor on X. Then for any $\epsilon > 0$ there exists a constant*

$C = C_\epsilon > 0$ and a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that we have

$$\mathrm{dist}_v(P, Q) > C(H_A(P)H_A(Q))^{-(\delta(X,A)+\epsilon)},$$

for any $P, Q \in U(k)$ with $P \neq Q$.

Proof. We let $W = X \times X$ with projections $\pi_i : W \to X_i$ and we let $L = \pi_1^* A + \pi_2^* A$. We denote the blow-up of the diagonal by $\alpha : W' \to W$ and its exceptional divisor by E. Fix $\epsilon > 0$. Let $\overline{B} \subset \overline{W}'$ be the stable locus $\mathrm{SB}((\delta(X, A) + \epsilon)\alpha^* L - E)$ where $\overline{W}'$ is the base change to an algebraic closure. One can express $\overline{B}$ as the intersection of supports of finitely many effective $\mathbb{R}$-divisors which are $\mathbb{R}$-linearly equivalent to $(\delta(X, A) + \epsilon)\alpha^* L - E$. After taking some finite extension k' of k, we may assume that these divisors are defined over k' and so is $\overline{B}$. We denote the union of the Galois orbits of $\overline{B}$ by B'. Then it is a property of height functions that for any $(P, Q) \in W'(k) \setminus B'(k)$, we have

$$0 \leq h_{((\delta(X,A)+\epsilon)\alpha^* L - E)}(P, Q) + O(1).$$

From this, we may conclude that

$$h_{E,v}(P, Q) \leq h_E(P, Q) + O(1) \leq h_{((\delta(X,A)+\epsilon)\alpha^* L)}(P, Q) + O(1).$$

Let $V \subset B'$ be a component not contained in E. Then one of projections $\pi_i \circ \alpha|_V$ is not dominant, and we denote its image by F_V. Now we define U by $X \setminus \bigcup_V F_V$. Our assertion follows for this U. □

Remark 5.2. Note that $\delta(X, A)$ is defined as

$$\delta(X, A) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{for any component } V \subset \mathrm{SB}(s\alpha^* L - E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i. \end{array} \right\}.$$

If this is the minimum, then in the above proof, one does not need to introduce $\epsilon > 0$.

Remark 5.3. When A is ample, we may replace $\delta(X, A)$ by $s(X, A)$ in Theorem 5.1. In this situation, one can take our exceptional set to be empty because of the emptiness of the base locus in the proof of Proposition 4.9.

6. Counting problems: general cases

In this section, we discuss some applications of Theorem 5.1 to the counting problems of rational points on algebraic varieties.

Local Tamagawa measures. Here we record some auxiliary results for local Tamagawa measures. Let X be a smooth projective variety defined over a number field k. Let v be a place of k. We fix a v-adic metrization on $\mathcal{O}(K_X)$ and it induces the Tamagawa measure $\tau_{X,v}$ on $X(k_v)$. We refer readers to [Chambert-Loir and Tschinkel 2010, Section 2.1.8] for its definition.

Lemma 6.1. *Let $n = \dim X$. There exists $C > 0$ such that for sufficiently small T and $P \in X(k_v)$, we have*

$$CT^n < \tau_{X,v}(\{Q \in X(k_v) \mid \mathrm{dist}_v(P, Q) < T\}).$$

Proof. Let $Y = X \times X$. We take a finite open cover $\{U_i\}$ of Y such that on U_i, we have that Δ is the scheme-theoretic intersection of $D_{i,1}, \dots, D_{i,n}$, where $D_i = \sum D_{i,j}$ is a strict normal crossings divisor on U_i. On $U_i(k_v)$, there exists $C > 0$ such that

$$\mathrm{dist}_v(P, Q) < C \max_j \{H_{D_{i,j}}^{-1}(P, Q)\}$$

for all $(P, Q) \in U_i(k_v)$. For each $P \in X_i(k_v)$, there exists a v-adic open neighborhood $V_P \subset X(k_v)$ such that $\overline{V}_P \times \overline{V}_P \subset U_i(k_v)$ for some i and $D_{i,j}$ induces local coordinates $x_{i,j}$ on V_P. Since $X(k_v)$ is compact, finitely many V_P cover $X(k_v)$. We denote them by V_l. For each l let

$$\omega_l = \mathrm{d}x_{l,1} \wedge \cdots \wedge \mathrm{d}x_{l,l}.$$

Then on V_l we have a uniform upper bound $C' > 0$ such that

$$\|\omega_l\|_v < C'.$$

Also let $d_l(P) = \min\{\mathrm{dist}_v(P, Q) \mid Q \in V_l^c\}$ and define $d(P) = \max_l\{d_l(P)\}$. Then $d(P) > 0$ for any $P \in X(k_v)$ so there is the minimum $d_m = \min\{d(P)\} > 0$. Now by the definition of the Tamagawa measure, for $0 < T < d_m$, we have

$$\tau_{X,v}(\{Q \in X(k_v) \mid \mathrm{dist}_v(P, Q) < T\}) > \int_{\{\max\{|x_{l,i} - x_{l,i}(P)|_v\} < C^{-1}T\}} \|\omega_l\|_v^{-1} |\omega_l|$$

Thus our assertion follows. □

The local Tamagawa number is defined by

$$\tau_v(X) = \tau_{X,v}(X(k_v)).$$

General estimates. Let X be a projective variety defined over a number field k and let L be a big $\mathbb{Q}$-divisor on X. We fix an adelic metrization on $\mathcal{O}(L)$ and consider the induced height:

$$H_L : X(F) \to \mathbb{R}_{>0}.$$

For each Zariski open subset $U \subset X$ we define the counting function

$$N(U, L, T) = \#\{P \in U(k) \mid H_L(P) \leq T\}.$$

Here is a general result using the repulsion principle:

Theorem 6.2. *Let X be a normal projective variety of dimension n defined over a number field k and let L be a big $\mathbb{Q}$-Cartier divisor on X. We fix an adelic metrization on $\mathcal{O}(L)$. Then for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that we have*

$$N(U, L, T) = O(T^{2n\delta(X,L)+\epsilon}).$$

Proof. We may assume that X is smooth after applying a resolution. This does not affect the invariant $\delta(X, L)$ because of Lemma 4.6. Let v be a place of k. By Theorem 5.1, for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U(\epsilon) \subset X$ such that there exists $C = C_\epsilon > 0$ such that

$$\mathrm{dist}_v(P, Q) > C(H_L(P)H_L(Q))^{-(\delta(X,L)+\epsilon)},$$

for any $P, Q \in U(k)$ with $P \neq Q$. We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$

For $P \in A_T$, we define the v-adic ball by

$$B_T(P) = \left\{R \in U(k_v) \mid \mathrm{dist}_v(P, R) < \tfrac{1}{2} C T^{-2(\delta(X,L)+\epsilon)}\right\}$$

Then $\bigcup_{P\in A_T} B_T(P)$ is disjoint because of the triangle inequality. Hence we have

$$\tau_v(X) > \sum_{P\in A_T} \tau_{X,v}(B_T(P)) \gg N(U, L, T)T^{-2n(\delta(X,L)+\epsilon)}$$

by Lemma 6.1. Thus our assertion follows. □

Remark 6.3. In the case that $\delta(X, L)$ is the minimum, one does not need to introduce ϵ in Theorem 6.2 because of Remark 5.2.

Remark 6.4. In Theorem 6.2, assuming L is ample and X is smooth we may replace $\delta(X, L)$ by $s(X, L)$ because of Remark 5.3. In this case, one can take $U = X$.

In view of Manin's conjecture, we expect the following is true:

Conjecture 6.5. Let X be a geometrically rationally connected smooth projective variety of dimension n and L be a big and nef $\mathbb{Q}$-divisor on X. Then we have

$$a(X, L) \leq 2n\delta(X, L).$$

7. K3 surfaces and Enriques surfaces

In this section we discuss applications of Theorem 6.2 to surfaces of Kodaira dimension 0. Let S be a K3 surface or an Enriques surface with a polarization H of degree $2d$. In this section, we obtain an upper bound for $s(X, H)$ using [Bayer and Macrì 2014; Nuer 2016]. Let W be the blow-up of $S \times S$ along the diagonal and we denote the exceptional divisor by E. We also consider the Hilbert scheme of two points on S, i.e., $\mathrm{Hilb}^{[2]}(S)$. The variety $\mathrm{Hilb}^{[2]}(S)$ comes with the divisor $H(2)$ induced by H and a divisor class B such that $2B$ is the class of the exceptional divisor of the Hilbert–Chow morphism. The variety W admits a degree 2 finite morphism $f : W \to \mathrm{Hilb}^{[2]}(S)$ and we have

$$f^*H(2) = H[2], \quad f^*B = E.$$

We then have

$$sH[2] - E \text{ is nef} \iff sH(2) - B \text{ is nef},$$

because of $f^*(sH(2)-B)=sH[2]-E$. Thus one needs to study the nef cone of $\mathrm{Hilb}^{[2]}(S)$ and this is studied in [Hassett and Tschinkel 2001; 2009; Bayer and Macrì 2014] for K3 surfaces. We use results from [Bayer and Macrì 2014] for the nef cone of $\mathrm{Hilb}^{[2]}(S)$. Here is the theorem:

Theorem 7.1 [Bayer and Macrì 2014]. *Let S be a $K3$ surface with a polarization H of degree $2d$ such that* $\mathrm{Pic}(S)=\mathbb{Z}H$. *Then we have*

$$s(S,H)\le\sqrt{\frac{4}{d}+\frac{5}{d^2}}.$$

Proof. We recall a result on the nef boundary of $sH(2)-B$ based on properties of certain Pell's equation. First we consider

$$X^2-4dY^2=5.$$

Suppose that there is a nontrivial solution (x_1,y_1) with $x_1>0$ minimal and $y_1>0$ even. Then it follows from [Bayer and Macrì 2014, Lemma 13.3] that

$$s(S,H)=\frac{x_1}{dy_1}\le\sqrt{\frac{4}{d}+\frac{5}{d^2}}.$$

Next suppose that there is no nontrivial solution to the above Pell's equation. Then for $\mathrm{Hilb}^{[2]}(S)$, the nef cone and the movable cone coincides by [Bayer and Macrì 2014, Lemma 13.3]. Suppose that d is a square. Then it follows from [Bayer and Macrì 2014, Proposition 13.1] that

$$s(S,H)=\frac{1}{\sqrt{d}}.$$

Next suppose that d is not a square. We consider the Pell's equation

$$X^2-dY^2=1.$$

This has a solution. Let $x_1,y_1>0$ be the solution with x_1 minimal. Then by [Bayer and Macrì 2014, Proposition 13.1], we have

$$s(S,H)=\frac{x_1}{dy_1}\le\sqrt{\frac{1}{d}+\frac{1}{d^2}}.$$

Thus our assertion follows. □

Now Theorem 1.5 follows from Theorem 7.1 and Remark 6.4. We also obtain bounds for Enriques surfaces:

Theorem 7.2 [Nuer 2016]. *Let Y be an unnodal Enriques surface*, *i.e.*, *Y contains no curve of negative self-intersection. Let H be a k-very ample divisor. Then*

$$s(Y,H)\le\frac{2}{k+2}.$$

Proof. It follows from [Nuer 2016, Theorem 12.3] that

$$s(Y,H)=\frac{2}{\phi(H)},$$

where $\phi(H)$ is the Cossec–Dolgachev function. Then it follows from [Szemberg 2001, Theorem 2.4] that $\phi(H) \geq k+2$. Thus our assertion follows. □

For a necessary and sufficient condition for k-very ampleness, see [Szemberg 2001, Proposition 2.3]

8. Manin type upper bounds for Fano conic bundles

In this section, we study the counting problems of rational points on conic bundles.

Definition 8.1. Let $f : X \to S$ be a flat projective morphism between smooth projective varieties. The fibration f is a conic bundle if there exist a rank 3 vector bundle $\mathcal{E}$ on S and an embedding $X \to \mathbb{P}_S(\mathcal{E})$ over S such that every fiber X_s is isomorphic to a conic in $\mathbb{P}^2_s$, where X_s and $\mathbb{P}^2_s$ are fibers of X and $\mathbb{P}(\mathcal{E})$ at $s \in S$.

Suppose that X is 3-dimensional. Then if every fiber X_s is isomorphic to a conic in $\mathbb{P}^2$, then $f_*\omega_X^{-1}$ is a rank 3 vector bundle on S and a natural map $X \to \mathbb{P}_S(f_*\omega_X^{-1})$ is an embedding by [Mori and Mukai 1983, Proposition 6.2]. In this way $f : X \to S$ is a conic bundle in the above sense.

Lemma 8.2. *Let $f : X \to S$ be a conic bundle and H be a big $\mathbb{Q}$-divisor on X. Suppose that for a fiber X_s of f, we have $H.X_s = 2$, i.e., X_s is a H-conic. Then*

$$\delta(X, H) \geq \tfrac{1}{2}.$$

Proof. Let $W = X \times X$ and $\alpha : W' \to W$ be the blow-up of the diagonal. We denote its exceptional divisor by E. Let C_P be a conic in the fiber at $P \in X_1$ passing through P. Then we have

$$(H[2] - 2E).C_P = 0.$$

As P varies over X_1 C_P forms a subvariety D in W' which is dominant to both X_1 and X_2. Thus our assertion follows from Lemma 4.3. □

Proposition 4.12 shows that in general, $\delta(X, H)$ may not be $\frac{1}{2}$.

Local Tamagawa measures of conics in families. We study the behavior of local Tamagawa measures of conics in a family. Let $f : X \to S$ be a conic bundle defined over a number field k. Let S° be the complement of the discriminant locus Δ_f of f.

Let v be a place of k. We fix a v-adic metrization on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. This induces a v-adic metrization on $\mathcal{O}(K_{X/S})$. For each local 1-form $\mathrm{d}t \in \Omega^1_{X/S}$, one can define the local Tamagawa measure $\tau_{X_s,v}$ on a conic X_s for any $s \in S^\circ(k_v)$ by

$$\tau_{X_s,v}(U) = \int_U \frac{|\mathrm{d}t|}{\|\mathrm{d}t\|_v}$$

which is independent of a choice of $\mathrm{d}t$.

Lemma 8.3. *Suppose that $f : X \to S$ admits a rational section. Then there exists $C > 0$ such that for sufficiently small T, any $s \in S^\circ(k_v)$, $P \in X_s(k_v)$, we have*

$$C\,\mathrm{dist}_v(\Delta_f, f(P))T < \tau_{X_s,v}(\{Q \in X_s(k_v) \mid \mathrm{dist}_v(P, Q) < T\}).$$

Proof. We fix a rational section S_0 and an ample divisor A on S. Let $S_m = S_0 + mf^*A$. Now $f_*(\mathcal{O}(S_0)) \otimes \mathcal{O}(mA)$ is globally generated for $m \gg 0$. Using this for each point $p \in S$, one may find a rational section $S_p \sim S_m$ such that S_p is a local section in a neighborhood of the point $p \in S$. By the definition of conic bundles one can embed $f : X \to S$ into a projective bundle $\mathbb{P}(\mathcal{E})$. Take a finite open affine covering $\{U_i\}$ of S so that over U_i, we have that $\mathbb{P}(\mathcal{E})|_{U_i}$ is trivialized, i.e., isomorphic to $U_i \times \mathbb{P}^2$. Taking a finer finite open covering, we may assume that f admits a local section S_i over U_i. By taking a finer finite open covering and applying a change of coordinates, one can assume that the local section S_i corresponds to $(1:0:0)$ in $\mathbb{P}^2$. Moreover we may assume that the tangent line of X_s at $(1:0:0)$ is given by $x_1 = 0$. Let A_j $(j = 0, 1, 2)$ be the standard affine charts of $\mathbb{P}^2$ and we define $V_{i,j} = f^{-1}(U_i) \cap (U_i \times A_j)$ which is affine.

Now we take a finite v-adic open covering B_l of $X(k_v)$ such that $\bar{B}_l$ is contained in some $V_{i,j}(k_v)$. Then on B_l there exists a positive constant C_1 such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) \leq C_1 \max\{|x_j(P) - x_j(Q)|_v, |y_j(P) - y_j(Q)|_v\}, \tag{8-1}$$

where x_j, y_j is the coordinates of A_j.

Now we are going to parametrize conics in the family. By our construction, $f^{-1}(U_i) \subset U_i \times \mathbb{P}^2$ is defined by the equation

$$d(s)y^2 + f(s)z^2 + 2xy + 2e(s)yz = 0,$$

where d, f, e are functions on U_i. Note that the discriminant locus Δ_f is defined by $f = 0$ and it is a smooth divisor by our assumption. After further simplifications, we may assume that the equation is given by

$$f(s)z^2 + 2xy = 0.$$

Lines $uy - vz = 0$ passing through $(1:0:0)$ are parametrized by $(u : v) \in \mathbb{P}^1$. Then the rational parametrization of conics is given by

$$(fu^2 : -2v^2 : -2uv).$$

In particular, any smooth conic X_s over $s \in U_i(k_v)$ is covered by $V_{i,0}$ and $V_{i,1}$. Also note that while this rational parametrization is not valid along singular fibers, a rational map mapping $(s, P) \in f^{-1}(U_i)$ to $(u(P) : v(P)) \in \mathbb{P}^1$ is a well-defined morphism.

Suppose that B_l is contained in $V_{i,0}$. The inequality (8-1) shows that there exists $C_2 > 0$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) < C_2 \,\mathrm{dist}_v(\Delta_f, f(P))^{-1} |t(P) - t(Q)|_v,$$

where $t = v/u$ and $\mathrm{dist}_v(\Delta_f, f(P))$ is the distant function of Δ_f.

Suppose that B_l is contained in $V_{i,1}$. The inequality (8-1) shows that there exists $C_3 > 0$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) < C_3 |t(P) - t(Q)|_v,$$

where $t = u/v$.

Suppose that B_l is contained in $V_{i,2}$. The inequality (8-1) shows that there exists $C_4 > 0$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) < C_4 |t(P) - t(Q)|_v,$$

where $t = v/u$. Now by arguing as in Lemma 6.1 our assertion follows. □

Lemma 8.4. *Let $f : X \to S$ be a conic bundle defined over a number field k with a rational section. Let v be an archimedean place of k. We fix a v-adic metrization on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Then for any sufficiently small $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that for any $s \in S^\circ(k_v)$ we have*

$$\tau_v(X_s) < C_\epsilon \, \mathrm{dist}_v(\Delta_f, s)^{1-\epsilon}$$

Proof. This follows from the descriptions in the proof of Lemma 8.3 and an explicit computations of local Tamagawa numbers using the naive metrization. □

Fano conic bundles: the anticanonical height. In this section, we discuss upper bounds of Manin type for the anticanonical height of Fano conic bundles.

Theorem 8.5. *Let $f : X \to S$ be a conic bundle defined over a number field k with a rational section. We assume that X and S are Fano. Let $W = X \times X$ and W' be the blow-up of W along the diagonal with the exceptional divisor E. We denote each projection $W' \to X_i$ by π_i. Let α, β be positive real numbers such that $2\alpha - 2\beta = 1$. We further make the following assumptions*:

(1) *The weak Manin's conjecture for $(S, -K_S)$ holds.*

(2) *For any component V of the stable locus of*

$$|-\alpha K_{X/S}[2] - \beta f^* K_S[2] - E|$$

such that V is not contained in E, one of projections $\pi_i|_V$ is not dominant.

Then there exists a nonempty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$ there exists $C = C_\epsilon > 0$ such that

$$N(U, -K_X, T) < C T^{2\alpha+\epsilon}.$$

Proof. First of all note that the assumption (2) implies that $-2\alpha K_X + f^* K_S$ is big. Indeed, it implies that $-\alpha K_{X/S} - \beta f^* K_S$ is big; otherwise the linear system of this divisor defines a nontrivial fibration up to a birational modification and the assumption (2) cannot be true. Then note that we have $-K_{X/S} = -K_X + f^* K_S$ and $2\alpha - 2\beta = 1$ so that $-\alpha K_{X/S} - \beta f^* K_S = -\alpha K_X + \frac{1}{2} f^* K_S$ so our claim follows.

Let v be a place of k and fix v-adic metrizations on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Fix $\epsilon > 0$. Arguing as Theorem 5.1, the assumption (2) implies that there exists $U \subset X$ and C such that for any $P, Q \in U(k)$ with $P \neq Q$ and $f(P) = f(Q) = s$ we have

$$\mathrm{dist}_v(P, Q) > C (H_{-K_S}(s))^{-2\beta} (H_{-K_{X/S}}(P) H_{-K_{X/S}}(Q))^{-\alpha}. \tag{8-2}$$

We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$

For $P \in A_T \cap X_s$, we define the v-adic ball by

$$B_T(P) = \left\{R \in X_s \cap U(k_v) \mid \mathrm{dist}_v(P, R) < \tfrac{1}{2}CT^{-2\alpha}H_{-K_S}(s)\right\}$$

Then $\bigcup B_T(P)$ is disjoint because of (8-2) and the triangle inequality. Note that after shrinking U, $T^{-2\alpha}H_{-K_S}(s)$ uniformly goes to 0 as $T \to \infty$ for any $s \in S$ with $A_T \cap X_s \neq \varnothing$ because of the Northcott property of the height function associated to $-2\alpha K_X + f^*K_S$ which is big. Thus we have

$$\begin{aligned}\mathrm{dist}_v(\Delta_f, s)^{1-\epsilon} \gg_\epsilon \tau_v(X_s) &> \sum_{P \in A_T \cap X_s(k)} \tau_{X_s,v}(B_T(P)) \\ &\gg_\epsilon N(U \cap X_s, L, T)\,\mathrm{dist}_v(\Delta_f, s)T^{-2\alpha}H_{-K_S}(s)\end{aligned}$$

by Lemmas 8.3 and 8.4. Let m be a positive integer such that $mL - f^*\Delta_f$ is ample. We conclude that

$$N(U \cap X_s, L, T) \ll_\epsilon T^{2\alpha}H_{-K_S}(s)^{-1}H_{\Delta_f,v}(s)^\epsilon \ll T^{2\alpha+m\epsilon}H_{-K_S}(s)^{-1}.$$

Since $-kK_X \geq -f^*K_S$ for some k, our assertion follows from the fact that S satisfies the weak Manin's conjecture and Tauberian theorem. Indeed, the weak Manin's conjecture for S and Tauberian argument implies that

$$\sum_{s \in S^\circ(k)} H_{-K_S}(s)^{-(1+\epsilon)} < +\infty,$$

where $S^\circ \subset S$ is a some open subset of S. Let $U^\circ = U \cap f^{-1}(S^\circ)$. Then one can conclude that

$$\begin{aligned}N(U^\circ, -K_X, T) &= \sum_{s \in S^\circ(k)} N(U \cap X_s, L, T) \\ &\leq T^{2\alpha+m\epsilon+\epsilon k} \sum_{s \in S^\circ(k)} H_{-K_S}(s)^{-1-\epsilon} \ll T^{2\alpha+m\epsilon+\epsilon k}.\end{aligned}$$

Thus our assertion follows. □

Fano conic bundles: the non-anticanonical heights. In this section, we discuss the weak Manin's conjecture for non-anticanonical height functions in some cases:

Theorem 8.6. *Let $f : X \to S$ be a conic bundle defined over a number field k with a rational section. We assume that X and S are Fano. Let $L = -K_X - tf^*K_S$. We make the following assumptions*:

(1) *The weak Manin's conjecture conjecture for $(S, -K_S)$ holds.*

(2) $t \geq \dfrac{1}{2\delta(X, -K_X)}$.

Then for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U = U(\epsilon) \subset X$ and $C = C_\epsilon > 0$ such that

$$N(U, L, T) < CT^{2\delta(X,-K_X)+\epsilon}.$$

In particular when $\delta(X, -K_X) = \frac{1}{2}$, Conjecture 1.1 holds for (X, L) except independence of U on ϵ.

For such a height function, [Frei and Loughran 2019] establishes Manin's conjecture when the base is the projective space using conic bundle structures. Our theorem is flexible in the sense that S can be other Fano manifold other than the projective space. For example one may find a smooth Fano threefold with a conic bundle structure over the Hirzebruch surface $\mathbb{F}_1$ in Section 9.

Proof. Let v be a place and fix v-adic metrizations on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Fix $\epsilon > 0$. Theorem 5.1 implies that there exists $U \subset X$ and C such that for any $P, Q \in U(k)$ with $P \neq Q$ we have

$$\mathrm{dist}_v(P, Q) > C(H_{-K_X}(P)H_{-K_X}(Q))^{-(\delta(X,-K_X)+\epsilon)}. \tag{8-3}$$

We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$

For $P \in A_T \cap X_s$, we define the v-adic ball by

$$B_T(P) = \left\{R \in X_s \cap U(k_v) \mid \mathrm{dist}_v(P, R) < \tfrac{1}{2}CT^{-2(\delta(X,-K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X,-K_X)+\epsilon)}\right\}.$$

Then $\bigcup B_T(P)$ is disjoint because of (8-3) and the triangle inequality. Note that

$$T^{-2(\delta(X,-K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X,-K_X)+\epsilon)}$$

uniformly goes to 0 as $T \to \infty$ for any $s \in S$ with $A_T \cap X_s \neq \varnothing$ after shrinking U. Thus we have

$$\begin{aligned}\mathrm{dist}_v(\Delta_f, s)^{1-\epsilon} \gg_\epsilon \tau_v(X_s) &> \sum_{P \in A_T \cap X_s(k)} \tau_{X_s,v}(B_T(P)) \\ &\gg_\epsilon N(U \cap X_s, L, T)\, \mathrm{dist}_v(\Delta_f, s) T^{-2(\delta(X,-K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X,-K_X)+\epsilon)}\end{aligned}$$

by Lemmas 8.3 and 8.4. Let m be a positive integer such that $mL - f^*\Delta_f$ is ample. We conclude that

$$N(U \cap X_s, L, T) \ll_\epsilon T^{2(\delta(X,-K_X)+\epsilon)+m\epsilon} H_{-K_S}(s)^{-2t(\delta(X,-K_X)+\epsilon)}.$$

Since $L \geq -tf^*K_S$, our assertion follows by arguing as in Theorem 8.5. □

Remark 8.7. If $\delta(X, -K_X)$ is the minimum, one can take U to be independent of ϵ.

9. 3-dimensional Fano conic bundles

In this section we list smooth 3-dimensional Fano conic bundles and compute $\delta(X, -K_X)$ and the smallest 2α satisfying the conditions of Theorem 8.5. Fano 3-folds with Picard rank ≥ 2 were classified by Mori and Mukai [1981; 1983]. We follow their classification. We assume that our ground field is an algebraically closed field of characteristic 0. In our computations of $\delta(X, -K_X)$ and the minimum 2α satisfying the conditions of Theorem 8.5, it is important to know a description of the nef cone of divisors of X. Such a description was obtained in [Matsuki 1995]. We freely use the results in this article.

Fano threefolds with Picard rank 2. According to [Mori and Mukai 1981], there are 36 deformation types of smooth Fano 3-folds with Picard rank 2. Among them there are 16 deformation types of smooth Fano 3-folds which come with conic bundle structures. Since Fano 3-folds have Picard rank 2, these conic bundle structures are extremal contractions. Thus in these cases, a conic bundle structure comes with a rational section if and only if there is no singular fiber. Thus there are 7 deformation types of smooth Fano 3-folds which come with a conic bundle structure with a rational section. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 2] is given in Table 1.

Here $Q \subset \mathbb{P}^4$ is a smooth quadric 3-fold. Note that numbers 34–36 are toric; thus Manin's conjecture is known for these cases by [Batyrev and Tschinkel 1996; 1998]. In [Blomer et al. 2018], Manin's conjecture for an example of Fano 3-folds of number 24 is proven.

Let us illustrate the computation of $\delta(X, -K_X)$ and $\alpha > 0$ in some cases:

Example 9.1 (number 32). Let W be a smooth divisor of $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$. We denote each projection by $\pi_i : W \to \mathbb{P}^2$ and let H_i be the pullback of the hyperplane class via π_i. Then we have

$$-K_X = 2H_1 + 2H_2.$$

Since $H_1 + H_2$ is very ample, it follows that $\delta(X, -K_X) = \frac{1}{2}$ by Lemma 8.2.

Next we consider

$$\alpha(2H_1 + 2H_2 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = \frac{4\alpha - 3}{2} H_1 + 2\alpha H_2.$$

Then when $\frac{1}{2}(4\alpha - 3) \geq 1$, i.e., $\alpha \geq \frac{5}{4}$, the above divisor satisfies the assumptions of Theorem 8.5. On the other hand, for each $P \in X$, let C_P be a fiber of π_2 meeting with P. Then we have $C_P.H_1 = 1$ and $C_P.H_2 = 0$. Thus we have

$$\left(\frac{4\alpha - 3}{2} H_1 + 2\alpha H_2\right).C_P = \frac{4\alpha - 3}{2}.$$

Thus by Lemma 4.3, we conclude that $\alpha = \frac{5}{4}$ is the minimum value satisfying the assumptions of Theorem 8.5.

no.	$(-K_X)^3$	X	$6\delta(X, -K_X)$	2α
24	30	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 2)$	≤ 6	≤ 5
27	38	blow-up of $\mathbb{P}^3$ with center a twisted cubic	3	2
31	46	blow-up of $Q \subset \mathbb{P}^4$ with center a line on it	3	$\frac{5}{3}$
32	48	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$	3	$\frac{5}{2}$
34	54	$\mathbb{P}^1 \times \mathbb{P}^2$	3	$\frac{5}{3}$
35	56	$V_7 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$	3	$\frac{5}{4}$
36	62	$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbb{P}^2$	3	$\frac{5}{3}$

Table 1. Fano 3-folds with Picard rank 2.

Example 9.2 (number 31). Let X be the blow-up of Q along a line. Then X has Picard rank 2 so it comes with two extremal contractions, one is a $\mathbb{P}^1$-bundle $\pi_1 : X \to \mathbb{P}^2$, and the other is a divisorial contraction $\pi_2 : X \to Q$. Let H_i be the pullback of the hyperplane class via π_i. Then it follows from [Mori and Mukai 1983, Theorem 5.1] that

$$-K_X = H_1 + 2H_2.$$

Since $\delta(X, H_2) = \delta(Q, H) = 1$ and π_2 is birational, it follows that $\delta(X, -K_X) \leq \frac{1}{2}$. Thus by Lemma 8.2, $\delta(X, -K_X) = \frac{1}{2}$ is proved.

Next we have

$$\alpha(H_1 + 2H_2 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = 2\alpha H_2 + \frac{2\alpha - 3}{2} H_1.$$

Let D be the exceptional divisor of π_2. Then we have $H_1 = H_2 - D$. Thus the above divisor becomes

$$\frac{6\alpha - 3}{2} H_2 - \frac{2\alpha - 3}{2} D.$$

Thus when $\frac{1}{2}(6\alpha - 3) \geq 1$ and $2\alpha - 3 \leq 0$, i.e, $\frac{5}{6} \leq \alpha \leq \frac{3}{2}$ the assumption of Theorem 8.5 holds. On the other hand let $\ell \subset X$ be the strict transform of a line on Q not meeting with center of π_2. Then we have

$$\left(\frac{6\alpha - 3}{2} H_2 - \frac{2\alpha - 3}{2} D\right).\ell = \frac{6\alpha - 3}{2}.$$

Thus since such ℓ deforms to cover X, by arguing as in Lemma 4.3, we conclude that $\alpha = \frac{5}{6}$ is the minimum value satisfying the assumptions of Theorem 8.5.

***Fano threefolds with Picard rank* 3.** According to [Mori and Mukai 1981], there are 31 deformation types of smooth Fano 3-folds with Picard rank 3. It follows from [Mori and Mukai 1983, p. 125, (9.1)] that all such Fano 3-folds come with a conic bundle structure except the blow-up of $\mathbb{P}^3$ along a disjoint union of a line and a conic. Again a conic bundle structure with singular fibers which is extremal never comes with a rational section. Note that if X is a Fano conic bundle which does not admit a divisorial contraction to a Fano conic bundle of Picard rank 2 with a rational section, then its extremal conic bundle structure admits singular fibers. For such a 3-fold, one can conclude that it does not admit a rational section. This implies that there are 25 deformation types of 3-dimensional Fano conic bundles with a rational section. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 3] is given in 2.

Note that numbers 24–31 are toric; thus Manin's conjecture is known for these cases by [Batyrev and Tschinkel 1996; 1998]. Let us demonstrate the computation of $\delta(X, -K_X)$ and α in some cases:

Example 9.3 (number 23). Let X be a Fano 3-fold of number 23. Then X admits a divisorial contraction $\beta : X \to V_7$ with the exceptional divisor D_1. The Fano 3-fold V_7 admits two extremal contractions: one is a $\mathbb{P}^1$-bundle $\pi_1 : V_7 \to \mathbb{P}^2$ and the other is the blow-down $V_7 \to \mathbb{P}^3$. We denote the pullback of the hyperplane class via π_i by H_i. One can conclude that the only conic bundle structure on X is $\pi_1 \circ \beta$. It follows from [Mori and Mukai 1983, Theorem 5.1] that

$$-K_{V_7} = 2H_1 + 2H_2.$$

no.	X	6δ	2α
3	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 2)$	≤ 6	≤ 5
5	blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a curve C of bidegree $(5, 2)$ such that the projection $C \to \mathbb{P}^2$ is an embedding	≤ 6	≤ 5
7	blow-up of W (number 32) with center an intersection of two members of $\left\|-\frac{1}{2}K_W\right\|$	≤ 4	≤ 3
8	a member of the linear system $\|p_1^* g^* \mathcal{O}(1) \otimes p_2 \mathcal{O}(2)\|$ on $\mathbb{F}_1 \times \mathbb{P}^2$, where p_i is the projection to each factor and $g : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowing up	≤ 6	≤ 5
9	blow-up of the cone $W_4 \subset \mathbb{P}^6$ over the Veronese surface $R_4 \subset \mathbb{P}^5$ with center a disjoint union of the vertex and quartic in $R_4 = \mathbb{P}^2$	3	$\leq \frac{7}{5}$
11	blow-up of V_7 (number 35) with center an intersection of two members of $\left\|-\frac{1}{2}K_{V_7}\right\|$	3	$\frac{5}{2}$
12	blow-up of $\mathbb{P}^3$ with center a disjoint union of a line and a twisted cubic	3	$\frac{8}{3}$
13	blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ with center a curve C of bidegree $(2, 2)$ on it such that each projection from C to $\mathbb{P}^2$ is an embedding	3	$\leq \frac{5}{2}$
14	blow-up of $\mathbb{P}^3$ with center a disjoint union of a point and a plane cubic	≤ 6	$\leq \frac{9}{5}$
15	blow-up of $Q \subset \mathbb{P}^4$ with center a disjoint union of a line and a conic	3	$\frac{5}{2}$
16	blow-up of V_7 with center the strict transform of a twisted cubic passing through the center of the blow-up $V_7 \to \mathbb{P}^3$	3	$\frac{5}{2}$
17	a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 1)$	3	$\frac{5}{2}$
19	blow-up of $Q \subset \mathbb{P}^4$ with center two points which are not colinear	3	$\frac{5}{3}$
20	blow-up of $Q \subset \mathbb{P}^4$ with center a disjoint union of two lines	3	$\frac{5}{2}$
21	blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a curve of bidegree $(2, 1)$	3	$\frac{5}{2}$
22	blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a conic in $\{t\} \times \mathbb{P}^2$	3	$\frac{5}{3}$
23	blow-up of V_7 with center the strict transform of a conic passing through the center of the blow-up $V_7 \to \mathbb{P}^3$	3	$\frac{5}{3}$
24	fiber product $W \times_{\mathbb{P}^2} \mathbb{F}_1$, where W is number 32, $W \to \mathbb{P}^2$ is the $\mathbb{P}^1$-bundle and $\mathbb{F}_1 \to \mathbb{P}^2$ is the blowing up	3	$\frac{5}{2}$
25	$\mathbb{P}(\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$	3	$\frac{3}{2}$
26	blow-up of $\mathbb{P}^3$ with center a disjoint union of a point and a line	3	$\frac{5}{3}$
27	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	3	2
28	$\mathbb{P}^1 \times \mathbb{F}_1$	3	2
29	blow-up of V_7 with center a line on the exceptional set $D = \mathbb{P}^2$ of the blow-up $V_7 \to \mathbb{P}^3$	3	$\leq \frac{7}{5}$
30	blow-up of V_7 with center the strict transform of a line passing through the center of the blow-up $V_7 \to \mathbb{P}^3$	3	$\frac{4}{3}$
31	$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1, 1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$	3	$\leq \frac{4}{3}$

Table 2. Fano 3-folds with Picard rank 3.

Thus we have

$$-K_X = 2\beta^* H_1 + 2\beta^* H_2 - D_1.$$

Since $\beta^* H_1 - D_1$ is effective and the morphism associated to $|H_2|$ is birational, one can conclude that $\delta(X, -K_X) = \frac{1}{2}$ by Lemma 8.2.

Let D_2 be the strict transform of the exceptional divisor of π_2. Then we have

$$\alpha(2\beta^* H_1 + 2\beta^* H_2 - D_1 - 3\beta^* H_1) + \frac{2\alpha - 1}{2} \cdot 3\beta^* H_1 = 2\alpha\beta^* H_2 - \alpha D_1 + \frac{4\alpha - 3}{2}\beta^* H_1$$

Since we have $\beta^* H_2 = \beta^* H_1 + D_2$, the above divisor becomes

$$\beta^* H_2 + (2\alpha - 1)D_2 + \frac{8\alpha - 5}{2}\beta^* H_1 - \alpha D_1$$

Since $\beta^* H_1 - D_1 \geq 0$, in the case of $\frac{1}{2}(8\alpha - 5) \geq \alpha$, i.e., $\alpha \geq \frac{5}{6}$, the above divisor satisfies the assumption of Theorem 8.5. On the other hand let ℓ be the strict transform of a line meeting with the center of D_1. When $\alpha = \frac{5}{6}$, we have

$$\left(\beta^* H_2 + (2\alpha - 1)D_2 + \frac{8\alpha - 5}{2}\beta^* H_1 - \alpha D_1\right).\ell = 1$$

Thus since such ℓ deforms to cover X, by arguing as in Lemma 4.3, we conclude that $\alpha = \frac{5}{6}$ is the minimum value satisfying the assumption of Theorem 8.5.

Example 9.4 (number 12). Let X be a Fano 3-fold of number 12. Then it admits a conic bundle structure $\pi_1 : X \to \mathbb{P}^2$, a birational morphism $\pi_2 : X \to \mathbb{P}^3$, and a del Pezzo fibration $\pi_3 : X \to \mathbb{P}^1$. Let H_i be the pullback of the hyperplane class via π_i. Then we have

$$-K_X = H_1 + H_2 + H_3.$$

Since $|H_2|$ defines a birational morphism to $\mathbb{P}^3$ and $|H_1 + H_3|$ defines a birational morphism to $\mathbb{P}^2 \times \mathbb{P}^1$ we can conclude that $\delta(X, -K_X) \leq \frac{1}{2}$. Thus by Lemma 8.2, we have $\delta(X, -K_X) = \frac{1}{2}$. Next we consider the divisor

$$\alpha(H_1 + H_2 + H_3 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = (3\alpha - 3)H_2 + \alpha H_3 - \frac{2\alpha - 3}{2}D_1,$$

where D_1 is the exceptional divisor of π_2 whose center is a twisted cubic. When $3\alpha - 3 \geq 1$ and $2\alpha - 3 \leq 0$, i.e., $\frac{4}{3} \leq \alpha \leq \frac{3}{2}$, the assumptions of Theorem 8.5 hold. On the other hand let ℓ be the strict transform of a general line meeting with the line which is the center of π_2. Then we have

$$\left((3\alpha - 3)H_2 + \alpha H_3 - \frac{2\alpha - 3}{2}D_1\right).\ell = 3\alpha - 3.$$

Thus since such ℓ deforms to cover X, by arguing as in Lemma 4.3, we conclude that $\alpha = \frac{4}{3}$ is the minimum value satisfying the assumption of Theorem 8.5.

no.	X	$6\delta(X, -K_X)$	2α
1	a smooth divisor on $(\mathbb{P}^1)^4$ of multidegree $(1, 1, 1, 1)$	3	3
2	blow-up of the cone over a quadric surface $S \subset \mathbb{P}^3$ with center a disjoint union of the vertex and an elliptic curve on S	≤ 6	≤ 2
3	blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve of tridegree $(1, 1, 2)$	3	≤ 2
4	blow-up of Y (no. 19, Table 2) with center the strict transform of a conic passing through p and q	≤ 6	≤ 2
5	blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center two disjoint curves of bidegree $(2, 1)$ and $(1, 0)$	≤ 6	2
6	blow-up of $\mathbb{P}^3$ with center three disjoint lines	3	2
7	blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ with center two disjoint curves of bidegree $(0, 1)$ and $(1, 0)$	3	$\frac{5}{2}$
8	blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve of tridegree $(0, 1, 1)$	3	2
9	blow-up of Y (no. 25, Table 2) with center an exceptional line of the blowing up $Y \to \mathbb{P}^3$	3	2
10	$\mathbb{P}^1 \times S_7$	3	2
11	blow-up of $\mathbb{P}^1 \times \mathbb{F}_1$ with center $t \times e$, where $t \in \mathbb{P}^1$ and e is an exceptional curve on $\mathbb{F}_1$	3	2
12	blow-up of Y (no. 33, [Mori and Mukai 1981, Table 2]) with center two exceptional lines of the blowing up $Y \to \mathbb{P}^3$	3	2
13	blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve of tridegree $(1, 1, 3)$	≤ 6	≤ 3

Table 3. Fano 3-folds with Picard rank 4.

***Fano threefolds with Picard rank* 4 *or* 5.** For Fano 3-folds in this range, all of them admit conic bundle structures with a rational section. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 4] is given in Table 3.

Here S_7 is a smooth del Pezzo surface of degree 7. Notice numbers 9–12 are toric, so Manin's conjecture is known for these cases by [Batyrev and Tschinkel 1996; 1998]. Again let us illustrate the computation of $\delta(X, -K_X)$ and α in some cases:

Example 9.5 (number 6). Let X be a Fano 3-fold of number 6. Then X admits three del Pezzo fibrations $\pi_i : X \to \mathbb{P}^1$. It also admits a birational morphism $\pi : X \to \mathbb{P}^3$. Let H_i be the pullback of the hyperplane class via π_i. Let H be the pullback of the hyperplane class via π. Then we have

$$-K_X = H + H_1 + H_2 + H_3.$$

Since both $|H|$, $|H_1 + H_2 + H_3|$ are birational, we conclude that $\delta(X, -K_X) = \frac{1}{2}$.

Next any conic bundle structure on X is given by $|H_i + H_j|$, where $i \neq j$. So we look at the conic bundle structure defined by $|H_1 + H_2|$. We consider

$$\alpha(H + H_3 - H_1 - H_2) + (2\alpha - 1)(H_1 + H_2) = \alpha H + \alpha H_3 + (\alpha - 1)H_1 + (\alpha - 1)H_2.$$

From this description we may conclude that $\alpha = 1$ satisfies the assumptions of Theorem 8.5. On the other hand, by looking at the strict transform C of a line such that $H_3.C = 0$, we may conclude that $\alpha = 1$ is the minimum value satisfying the assumptions of Theorem 8.5.

Example 9.6 (number 4). Let V_7 be the blow-up of $\mathbb{P}^3$ at a point p. It admits two extremal rays and we denote each contraction morphism by $\pi_1 : V_7 \to \mathbb{P}^2$ and $\pi_2 : X \to \mathbb{P}^3$. Let H_i be the pullback of the hyperplane via π_i. Let X' be the blow-up of a fiber of π_1 which is the strict transform of a line ℓ passing through p. It admits a conic bundle structure over $\mathbb{F}_1$. Let X be the blow-up of X' along the strict transform of a conic C_0 not meeting with ℓ. Then X is a smooth Fano 3-fold of number 4. We denote the strict transform of the exceptional divisor of $X' \to V_7$ by D_1 and the exceptional divisor of $X \to X'$ by D_2. Then we have

$$-K_X = 2H_1 + 2H_2 - D_1 - D_2.$$

Since $2H_1 + H_2 - D_1 - D_2$ is linearly equivalent to an effective divisor, it follows that $\delta(X, -K_X) \leq 1$.

Next we consider

$$\alpha(-K_X - (3H_1 - D_1)) + \frac{2\alpha - 1}{2}(3H_1 - D_1) = 2\alpha H_2 - \alpha D_2 + \frac{4\alpha - 3}{2}H_1 - \frac{2\alpha - 1}{2}D_1.$$

Since $H_2 - D_2$ and $H_1 - D_1$ are effective, it follows that $\alpha = 1$ satisfies the assumptions of Theorem 8.5.

Finally we discuss the case of Picard rank 5. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 5] is given in Table 4.

no.	X	$6\delta(X, -K_X)$	2α
1	blow-up of Y (no. 29, [Mori and Mukai 1981, Table 2]) with center three exceptional lines of the blowing up $Y \to Q$	3	≤ 2
2	blow-up of Y (no. 25, Table 2) with center two exceptional lines ℓ and ℓ' of the blowing up $\phi : Y \to \mathbb{P}^3$ such that ℓ and ℓ' lie on the same irreducible component of the exceptional set for ϕ	3	≤ 2
3	$\mathbb{P}^1 \times S_6$	3	2

Table 4. Fano 3-folds with Picard rank 5.

Here S_6 is a smooth del Pezzo surface of degree 6. Note that numbers 2 and 3 are toric, so Manin's conjecture is known by [Batyrev and Tschinkel 1996; 1998].

Fano threefolds with Picard rank ≥ 6.

Theorem 9.7 [Mori and Mukai 1983, Theorem 1.2]. *Let X be a smooth Fano 3-fold and we denote its Picard rank by $\rho(X)$. Suppose that $\rho(X) \geq 6$. Then X is isomorphic to $\mathbb{P}^1 \times S_{11-\rho(X)}$, where S_d is a smooth del Pezzo surface of degree d.*

Thus in this section, we study the product of a smooth del Pezzo surface S_d with $\mathbb{P}^1$. Note that the weak Manin's conjecture for the product follows as soon as the weak Manin's conjecture is known for S_d by [Franke et al. 1989], so we omit the discussion of α in this section.

Proposition 9.8. *Let $X = \mathbb{P}^1 \times S$, where S is a smooth del Pezzo surface of degree d with $1 \leq d \leq 8$. Then we have $\delta(X, -K_X) = \delta(S, -K_S)$.*

Proof. Let $W_X = X \times X$ and $\alpha : W'_X \to W_X$ be the blow-up of the diagonal. We use the same notation for S as well. Let H_1 be the pullback of the ample generator via $p_1 : X \to \mathbb{P}^1$ and let H_2 be the pullback of the anticanonical divisor via $p_2 : X \to S$. Then the anticanonical divisor of X is

$$-K_X = 2H_1 + H_2.$$

Fix $\epsilon > 0$ and consider

$$-(\delta(S, -K_S) + \epsilon)K_X[2] - E = 2(\delta(S, -K_S) + \epsilon)H_1[2] + \epsilon H_2[2] + \delta(S, -K_S)H_2[2] - E.$$

Since $2(\delta(S, -K_S) + \epsilon)H_1[2] + \epsilon H_2[2]$ is semi-ample, we know the stable locus of $-(\delta(S, -K_S) + \epsilon)K_X[2] - E$ is contained in the stable locus of $\delta(S, -K_S)H_2[2] - E$. The possible dominant components of the stable locus of $\delta(S, -K_S)H_2[2] - E$ are E and the strict transform of

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \Delta_S,$$

where Δ_S is the diagonal of W_S. Next we consider

$$-(\delta(S, -K_S) + \epsilon)K_X[2] - E = 2\epsilon H_1[2] + (\delta(S, -K_S) + \epsilon)H_2[2] + 2\delta(S, -K_S)H_1[2] - E.$$

Since $2\epsilon H_1[2] + (\delta(S, -K_S) + \epsilon)H_2[2]$ is again semi-ample, the stable locus of $-(\delta(S, -K_S) + \epsilon)K_X[2] - E$ is therefore contained in the stable locus of $2\delta(S, -K_S)H_1[2] - E$. Since $\delta(S, -K_S) \geq \frac{1}{2}$, it follows that the stable locus of $2\delta(S, -K_S)H_1[2] - E$ is contained in the strict transform Z of

$$\Delta_{\mathbb{P}^1} \times W_S \subset W_X,$$

where $\Delta_{\mathbb{P}^1}$ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. The variety Z is isomorphic to $\mathbb{P}^1 \times W'_S$. From this one may conclude that $\delta(X, -K_X) \leq \delta(S, -K_S)$ by taking the intersection of two loci.

On the other hand the discussion after Proposition 4.9 shows that in each case there are curves C on W'_S such that C deforms to dominate both S_i and also that $(-\delta(S, -K_S)K_S[2] - E).C = 0$. Thus we conclude that $\delta(X, -K_X) \geq \delta(S, -K_S)$. Thus our assertion follows. □

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Tubular approaches to Baker's method for curves and varieties

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Baker's method, relying on estimates on linear forms in logarithms of algebraic numbers, allows one to prove in several situations the effective finiteness of integral points on varieties. In this article, we generalize results of Levin regarding Baker's method for varieties, and explain how, quite surprisingly, it mixes (under additional hypotheses) with Runge's method to improve some known estimates in the case of curves by bypassing (or more generally reducing) the need for linear forms in p-adic logarithms. We then use these ideas to improve known estimates on solutions of S-unit equations. Finally, we explain how a finer analysis and formalism can improve upon the conditions given, and give some applications to the Siegel modular variety $A_2(2)$.

1. Introduction

One of the main concerns of number theory is solving polynomial equations in integers, which amounts to determining the integral points on the variety defined by those equations. For a smooth projective curve over a number field, Siegel's theorem says that there are generally only finitely many integral points on this curve, but this result is in general deeply ineffective in that it does not provide us with any way to actually determine this set of integral points.

We focus here on Baker's method (and to a lesser extent Runge's method), which are both *effective*: when applicable, they give a bound on the height of the integral points considered. Our work is based on Bilu's conceptual approach [1995] for curves, and its generalization to higher-dimensional varieties by Levin [2014]. It is also heavily inspired (sometimes implicitly) by a previous article [Le Fourn 2019], dealing with Runge's method. Before stating the main results, let us give some notations and motivations.

K is a fixed number field, L is a finite extension of K with set of places M_L divided into its archimedean places M_L^∞ and its finite places M_L^f, and S a finite set of places of L containing M_L^∞ (the pair (L, S) will be allowed to change). The ring of S-integers of L is denoted by $\mathcal{O}_{L,S}$ and the regulator of $\mathcal{O}_{L,S}^*$ by R_S. We also denote by P_S the largest norm of an ideal coming from a finite place of S (equal to 1 if $S = M_L^\infty$).

The notion of integral point on a projective variety X will be precisely defined (model-theoretically) in Section 2A, but the result is compatible with all reasonable definitions, e.g., the one in [Vojta 1987, Section I.4] and the intuition we give below. For Z a closed algebraic subvariety of X, the set $(X \setminus Z)(\mathcal{O}_{L,S})$ will thus be the set of S-integral points of $X \setminus Z$. It can be interpreted in the following way. Every place

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v of M_L defines a v-adic topology on $X(K_v)$ where K_v is the completion of K at v, and in this topology we can say if a point $P \in X(K_v)$ is "v-close" to the subvariety Z or not. The point P is then S-integral with respect to Z if for every place v of M_L except maybe the ones of S, P is v-far from Z.

Furthermore, as $X(K_v)$ is compact, given two algebraic (hence closed) subvarieties Z, Z' of X, a point P is v-close to both Z and Z' if and only if it is v-close to $Z \cap Z'$, in particular if this intersection is empty, P can be v-close to only one of them. This simple fact is very useful in our arguments.

The basic context (C) (which will be made more complex later) of our arguments will be the following:

- X is a smooth projective variety over a number field K.
- $D_1, \ldots, D_n$ are effective divisors on X and $D = \bigcup_{i=1}^n D_i$.
- The point P belongs to $(X \setminus D)(\mathcal{O}_{L,S})$.

One of the first ideas to prove finiteness of integral points is the following: if a point P is v-far from D_i for *every* place v of M_L, the global height $h_{D_i}(P)$ relative to D_i can be bounded, so under some geometric property called *ampleness* of D_i, P must belong to a finite set that can be in principle effectively determined. The goal of many finiteness methods is thus to find conditions which ensure automatically that such a situation as above happens for P and one of the D_i, which would then entail finiteness. Notice that if $P \in (X \setminus D)(\mathcal{O}_{L,S})$, this is automatic for places not in S, so we only need to prove it for the (finitely many) places of S.

On the other hand, Baker's method proceeds quite differently: it relies on the fact (based on linear forms in logarithms) that a unit $x \in \mathcal{O}_{L,S}^* \setminus \{1\}$ cannot be too v-close to 1 (this depending on the global height of x itself), so if we force it somehow to be v-close, we obtain in return a bound on the global height of x, which in turn gives effective finiteness. For $P \in (X \setminus D)(\mathcal{O}_{L,S})$ as before, the game is then to ensure that there is some place $v \in S$ and some point P_0 in an explicit finite set (independent of P) such that P is very v-close to P_0, and there is a rational function ϕ sending P_0 to 1 and P in $\mathcal{O}_{L,S}^*$. This approach looks quite orthogonal to the previous paragraph but it turns out they can complement each other in diverse situations, and this is the topic of the present paper.

Let us start from (C), and fix $m_Y \geq 1$ an integer and Y the union of intersections of any $m_Y + 1$ divisors amongst $D_1, \ldots, D_n$ (the notation might seem backwards, but it is because one can also first fix a closed subvariety Y and define m_Y from it, which is a natural way to proceed in practical cases). We also define $m_{\mathcal{B}} \geq 1$ such that any intersection of $m_{\mathcal{B}}$ distinct divisors amongst $D_1, \ldots, D_n$ is a finite set. The main point of Theorem 1.1 is proving that, for any finite sets of places $M_L^\infty \subset S' \subset S$ the set of points of $(X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$ is effectively finite (up to an explicit proper closed subset) when

$$(m_{\mathcal{B}} - 1)|S'| + m_Y |S \setminus S'| < n.$$

The basic idea (to which the reduction is nevertheless quite technical, and we forget the exceptional cases for simplicity here) is the following: given a point $P \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$, for $v \in S \setminus S'$, if P is v-close to more than m_Y divisors D_i, it is v-close to Y, which is ruled out by S'-integrality with respect to Y. On another hand, if $v \in S'$ and P is v-close to $m_{\mathcal{B}}$ divisors, it is close to some point in their intersection which is finite, and we can then apply Baker's method. Now, the inequality above guarantees that if none of these situations happens, then by the pigeonhole principle there is some D_i such that P is v-far away from D_i for every place $v \in M_L$, which proves effective finiteness by the arguments above.

We now state the complete version of this result.

Theorem 1.1. *Let X be a smooth projective variety over K and $D_1, \dots, D_n$ be ample effective divisors on X, $D = \bigcup_{i=1}^n D_i$, and h_D a choice of absolute logarithmic height relative to D.*

The number $m_{\mathcal{B}}$ (assumed to exist) is the smallest integer such that for any set $I \subset \{1, \dots, n\}$ with $|I| = m_{\mathcal{B}}$, the intersection $T_I = \bigcap_{i \in I} \operatorname{Supp}(D_i)(\overline{K})$ is finite. For any point P in the finite set

$$T := \bigcup_{|I|=m_{\mathcal{B}}} T_I,$$

assume $(H)_P$:

> *There exists $\phi_P \in K(X)$ nonconstant whose support is included in $(\operatorname{Supp} D) \setminus P$ (and $\phi_P(P) = 1$ for simplicity).*

Such a function will be fixed in the following.

Let $m_Y \geq 1$ be an integer and

$$Y = \bigcup_{|I|>m_Y} \bigcap_{i \in I} \operatorname{Supp}(D_i)(\overline{K}).$$

Then, there exists an effectively computable constant $C > 0$ and an explicit function $C_1(d, s)$ such that for any triple (L, S, S') with $M_L^\infty \subset S' \subset S$ finite such that $[L : \mathbb{Q}] = d$, $|S| = s$ and

$$(m_{\mathcal{B}} - 1)|S'| + m_Y |S \setminus S'| < n, \tag{1-1}$$

for every $Q \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$,

$$h_D(Q) \leq C \cdot C_1(d, s) h_L R_S P_{S'} \log^*(h_L R_S), \tag{1-2}$$

where $\log^*(x) = \max(\log x, 1)$, unless

$$Q \in Z := \bigcup_{P \in T} Z_{\phi_P}, \quad Z_{\phi_P} := \overline{\{Q \in (X \setminus \operatorname{Supp} \phi_P)(\overline{K}),\ \phi_P(Q) = 1\}},$$

this set Z being an effective strict closed subset of X independent of (L, S, S') and the bar denoting the Zariski closure.

A remark developed below is that the dependence on S' in (1-2) is the factor $P_{S'}$, so an ideal choice for S' is for it to be as large as possible to satisfy (1-1) but with its finite places as small as possible. This leeway in the choice of S' allows for improved bounds even on the well-trodden ground of curves, for which one obtains the following corollary.

Corollary 1.2. *Let C be a smooth projective curve over K and $\phi \in K(C)$ nonconstant.*

Assume that every pole P of ϕ satisfies $(H)_P$ (and to simplify, is defined over K). Then, there are a constant $C > 0$ and an explicit function $C_1(d, s)$ such that for any triple (L, S, S') satisfying

$$|S \setminus S'| < n, \tag{1-3}$$

any point $Q \in C(L)$ such that $\phi(Q) \in \mathcal{O}_{L,S}$ satisfies

$$h(\phi(Q)) \leq C \cdot C_1(d, s) h_L R_S \log^*(h_L R_S) P_{S'}.$$

Remarks 1.3. Let us make some comments about these results:

- For $S' = M_L^\infty$, under the assumption (1-1) that $|S \setminus M_L^\infty|$ is small (exactly translated by (1-3) for curves), one obtains a bound on the height which only grows in $R_S^{1+\varepsilon}$. In particular, there is no linear dependence on P_S (which would come from estimates of linear forms in p-adic logarithms in a straightforward application of Baker's method), but rather in $\log P_S$ (implicitly contained in R_S), which might prove useful for some applications.

- The set Z of Theorem 1.1 can actually be made smaller: as done in [Levin 2014], we can replace each Z_{ϕ_P} by the intersection of *all* Z_{ϕ_P} where ϕ_P runs through all functions satisfying the hypotheses of $(H)_P$.

- Theorem 1.1 applied for $m_Y = 0$ and $S = S'$ retrieves Levin's result [2014, Theorem 1], and for smaller S' (hence more hypotheses) improves upon the quantitative estimates it implicitly gave.

- For general m_Y, Theorem 1.1 improves qualitatively (when the hypotheses $(H)_P$ hold) upon a previous result based on Runge's method [Le Fourn 2019, Theorem 5.1 and Remark 5.2(b)], as condition (1-1) is generally weaker than the tubular Runge condition defined there (the choice of set $(X \setminus Y)(\mathcal{O}_{L,S'})$ is inspired by the notion of tubular neighborhood defined in that article, see section 3 there). Indeed, the $m = m_\varnothing$ in the statement of [Le Fourn 2019, Remark 5.2] is not $m_\mathcal{B}$, and in general one only knows that $m_\mathcal{B} \leq m + 1$. If $m_\mathcal{B} = m + 1$, the original Runge's method can be applied as in [Levin 2008] and gives better (and uniform) estimates than [Levin 2014] (and the same holds for the tubular variants we propose), but if $m_\mathcal{B} \leq m$ which is the most likely situation, the condition (1-1) is indeed more easily satisfied than for Runge's method.

- The words "effectively computable" for the constant C deserve to be made more precise. One requires to know an embedding of X in a projective space $\mathbb{P}_K^N$, explicit equations and formulas for X, the D_i, D, h_D, the points of T and expressions of ϕ_P relative to this embedding. With this data, the effectivity boils down to an effective Nullstellensatz such as e.g., [Masser and Wüstholz 1983, Theorem IV]. Now, the functions $C_1(d, s)$ (as well as $R_S \log^*(R_S)$) are coming from the theory of linear forms in logarithms, and are as such completely explicit. The addition of h_L is a technicality due to the necessity of slightly increasing the ring of units $\mathcal{O}_{L,S}^*$ upon which to apply Baker's estimates, and can often be removed in special cases.

• Unless we are in the case of curves, $m_{\mathcal{B}} > 1$ and then (1-1) bounds d and s in terms of n, which allows us to replace $C_1(d, s)$ by an explicit function $C_1(n)$.

• As will be discussed in Section 5, in some situations one can apply the same methods as in the proof of Theorem 1.1 without having (1-1), and one can also devise some more uniform variants of this result in intermediary cases.

As an illustration of the effectivity of the method, we prove the following result on the S-unit equation: fix L to be a number field of degree d, $S \supset M_L^\infty$ a set of places of L of cardinality s and $\alpha, \beta \in L^*$. We consider the S-unit equation

$$\alpha x + \beta y = 1, \quad x, y \in \mathcal{O}_{L,S}^*. \tag{1-4}$$

Theorem 1.4. *Let L, S, α, β be as above*:

- *If S contains at most two finite places, all solutions of* (1-4) *satisfy*

$$\max(h(x), h(y)) \leq 2c(d, s) R_S \log^*(R_S) H,$$

where $H = \max(h(\alpha), h(\beta), 1, \pi/d)$ and $c(d, s)$ is the constant defined as $c_{26}(s, d)$ in formula (30) *of [Győry and Yu 2006].*

- *For any set of places S, all solutions of* (1-4) *satisfy*

$$\max(h(x), h(y)) \leq 2c'(d, s) P'_S R_S (1 + \log^*(R_S)/\log^* P'_S) H,$$

where $c'(d, s) = c_1(s, d)$ from Theorem 1 *of [Győry and Yu 2006], and P'_S the third largest value of the norms of ideals coming from finite places of S.*

This result provides an improvement on known bounds for solutions of the S-unit equations. More precisely, its dependence on P'_S (instead of P_S, the largest norm of an ideal coming from a place of S) becomes particularly interesting when there are at most two places of S of large relative norm, and by construction it improves Theorem 1 of [Győry and Yu 2006]. One can also remark that such an estimate is likely to be close to optimal in terms of dependence on the primes in S, as replacing $R_S \log^*(R_S)$ by $o(R_S)$ in the first bound would imply that there are only finitely many Mersenne primes. Notice that there is an additional factor 2 in the inequalities of Theorem 1.4 when compared to the reference [Győry and Yu 2006], which is due to a special case in the proof.

On another hand, Theorem 4.1.7 of [Evertse and Győry 2015], based on slightly different Baker-type estimates, has a better dependence on s and d. It is possible to combine the strategy of proof of the latter theorem with our own to obtain an improvement of both results, essentially replacing again P_S by P'_S. This is achieved in a recent preprint of Győry [2019] (which also takes into account and deals with the factor 2 discussed above).

After proving Theorem 1.1 and Corollary 1.2 in Section 3 (Section 2 gathering the necessary reminders and tools for the proof), we prove Theorem 1.4 in Section 4. This application is heavily based on computations undertaken in [Győry and Yu 2006], hence we have chosen to refer to it whenever possible, and focus on pointing out where the improvements come from our approach.

In the last part of this paper (Section 5), inspired by comments from the referees, we discuss a rewording of the elementary ideas behind Runge and Baker's method in terms of a graph defined by the divisors D_i, which leads in some situations to hypotheses of application weaker than e.g., (1-1). In the spirit of [Le Fourn 2019] and as an example of the potential for improvement it reveals, we apply these ideas to the Siegel modular variety $A_2(2)^S$ and obtain finiteness of abelian surfaces over quadratic fields with full 2-torsion and satisfying conditions on their places of bad reduction (Propositions 5.5 and 5.7).

2. Reminders on Baker's theory and local heights

For any place w of L, the norm $|\cdot|_w$ associated to w is normalized to extend the norm on $\mathbb{Q}$ defined by v_0 below w, where $|\cdot|_\infty$ is the usual norm on $\mathbb{Q}$ and for every prime p and nonzero fraction a/b,

$$\left|\frac{a}{b}\right|_p = p^{\operatorname{ord}_p b - \operatorname{ord}_p a}.$$

We also define $n_w = [L_w : \mathbb{Q}_{v_0}]$ the local degree of L at w.

In all discussions below, X is a fixed projective smooth algebraic variety over the number field K and closed subset of X will mean a closed algebraic K-subvariety of X.

Regarding the integrality, we choose a model-theoretic definition as follows. Assume $\mathcal{X}$ is a proper model of X over $\mathcal{O}_K$, fixed until the end of this article. For every closed subset Y of X, denote by $\mathcal{Y}$ the Zariski closure of Y in $\mathcal{X}$. The set of integral points $(X \setminus Y)(\mathcal{O}_{L,S})$ will then implicitly denote the set of points $P \in X(L)$ whose reduction in $\mathcal{X}_v(\kappa(w))$ for a place w of $M_L \setminus S$ above $v \in M_K$ (well-defined by the valuative criterion of properness) never belongs to $\mathcal{Y}$.

2A. *M_K-constants and M_K-bounded functions.*

The arguments below will be much simpler to present with the formalism of M_K-constants and M_K-functions briefly recalled here.

Definition 2.1. • An *M_K-constant* is a family $(c_v)_{v \in M_K}$ of nonnegative real numbers, all but finitely many of them being zero.

• An *M_K-function* f (on X) is a function defined on a subset E of $X(\overline{K}) \times M_{\overline{K}}$ with real values (typically, a local height function as below). Equivalently, it is defined as a function on a subset of $\bigsqcup_{K \subset L \subset \overline{K}} X(L) \times M_L$, consistently in the sense that if f is defined at (P, w) with $P \in X(L)$ and $w \in M_L$, then it is defined at (P, w') with $w' \mid w \in M_{L'}$ for any extension L' of L, and $f(P, w) = f(P, w')$.

• An M_K-function $f : E \to \mathbb{R}$ is *M_K-bounded* if there exists an M_K-constant $(c_v)_{v \in M_K}$ for which for all $(P, w) \in E$,

$$|f(P, w)| \leq c_v \quad (w \mid v).$$

The notation $O_{M_K}(1)$ will be used for an M_K-bounded function depending on the context (in particular, its domain E will often be implicit but obvious).

• Two M_K-functions $f, g : E \to \mathbb{R}$ are *M_K-proportional* when there is an absolute constant $C > 0$ and a M_K-constant $(c_v)_{v\in M_K}$ for which for all $(P, w) \in E$,

$$\tfrac{1}{C}|f(P, w)| - c_v \le |g(P, w)| \le C|f(P, w)| + c_v \quad (w \mid v).$$

• Two functions f, g defined on an open subset O of $X(\overline{K})$ (typically, global height functions) are *proportional* if there are absolute constants $C_1, C_2 > 0$ such that for every $P \in O$

$$\tfrac{1}{C_1} f(P) - C_2 \le g(P) \le C_1 f(P) + C_2.$$

2B. *Local heights associated to closed subsets.* We will now define explicitly local height functions relative to closed subsets of a projective variety X:

• For any point $P \in \mathbb{P}^N(L)$, one denotes by $x_P = (x_{P,0}, \dots, x_{P,n}) \in L^{n+1}$ a choice of coordinates representing P and $\|x_P\|_w = \max_i |x_{P,i}|_w$.

• For a polynomial $g \in L[X_0, \dots, X_N]$ and $w \in M_L$, the norm $\|g\|_w$ is the maximum norm of its coefficients for $|\cdot|_w$.

• Given a closed subset Y of $\mathbb{P}^N_K$ and homogeneous polynomials $g_1, \dots, g_m \in K[X_0, \dots, X_N]$ generating the ideal of definition of Y, for any $w \in M_L$ and any $P \in (\mathbb{P}^N \setminus Y)(L)$, one defines explicitly a choice of local height of P at Y for w by

$$h_{Y,w}(P) := -\min_i \log \frac{|g_i(x_P)|_w}{\|g_i\|_w \|x_P\|_w^{\deg g_i}}, \tag{2-1}$$

and the global height by

$$h_Y(P) := \frac{1}{[L : \mathbb{P}]} \sum_{w\in M_L} n_w \cdot h_{Y,w}(P).$$

With this normalization, for any $w \in M_L^f$ and $P \in (\mathbb{P}^N \setminus Y)(L)$, $h_{Y,w}(P) \ge 0$ and it is positive if and only if P reduces in Y modulo w.

Let us now sum up the main properties of those functions that we will need.

Proposition 2.2 (local heights). *Let X be a smooth projective variety over K, with an implicit embedding in a $\mathbb{P}^n_K$ and fixed choices of local heights as in* (2-1) *for all closed subsets considered below*:

(a) *For any closed subsets Y, Y' of X the functions $h_{Y\cap Y',w}$ and* $\min(h_{Y,w}, h_{Y,w'})$ *are M_K-proportional on $(X \setminus (Y \cup Y'))(\overline{K}) \times M_{\overline{K}}$.*

(b) *For a disjoint union $Y \sqcup Y'$ of closed subsets of X, one has*

$$h_{Y,w}(P) + h_{Y',w}(P) = h_{Y\sqcup Y',w}(P) + O_{M_K}(1)$$

on $(X \setminus (Y \cup Y'))(\overline{K}) \times M_{\overline{K}}$.

(c) *For $Y \subset Y'$ closed subsets, one has*

$$h_{Y,w}(P) \le h_{Y',w}(P) + O_{M_K}(1)$$

on $(X \setminus Y')(\overline{K}) \times M_{\overline{K}}$.

(d) *If* $\phi : X' \to X$ *is a morphism of projective varieties, the functions* $(P, w) \mapsto h_{Y,w}(\phi(P))$ *(resp.* $h_{\phi^{-1}(Y),w}(P)$*) are* M_K*-proportional on* $(X' \setminus \phi^{-1}((Y))(\overline{K}) \times M_{\overline{K}}$.

(e) *For any closed subset* Y *of* X, *the function* $(P, w) \mapsto h_{Y,w}(P)$ *is* M_K*-bounded on the set of pairs satisfying* $P \in (X \setminus Y)(\mathcal{O}_{L,w})$ *(independently of the number field* L*).*

(f) *For any effective divisor* D *on* X *and any function* $\phi \in K(X)$ *with support of poles included in* Supp D, *the function* $(P, w) \mapsto |\phi(P)|_w$ *is* M_K*-bounded on the set of pairs* $(P, w) \in X(L) \times M_L^f$ *satisfying* $P \in (X \setminus D)(\mathcal{O}_{L,w})$ *(independently of the number field* L*).*

(g) *If* D *and* D' *are two ample divisors on* X, *for any two choices of global heights* h_D *and* $h_{D'}$, *they are proportional on* $(X \setminus \mathrm{Supp}(D \cup D'))(\overline{K})$.

Furthermore, all the implied M_K*-constants and constants are effective.*

Proof. This proposition is mostly a reformulation of results of [Silverman 1987] already quoted in [Levin 2014]. First, (2-1) indeed defines local heights associated to closed subsets by [Silverman 1987, Proposition 2.4] so most of the proposition is contained in [loc. cit., Theorem 2.1]. Let us point out the slight differences and explain how it is effective. In that article, local height functions are more precisely defined by their ideal sheaves, whereas we consider closed subsets hence reduced closed subschemes. Now, if two ideal sheaves $\mathcal{I}$ and $\mathcal{I}'$ have the same support, their local height functions are M_K-proportional. More concretely, let us fix $Y \subset Y'$ closed subsets of $X \subset \mathbb{P}^N_K$ and two systems of homogeneous generators $g_1, \dots, g_m \in K[X_0, \dots, X_N]$ and $h_1, \dots, h_p$ of ideal sheaves with respective supports Y and Y' in $\mathbb{P}^N_K$. After multiplying by a suitable $n \geq 1$, one can assume all those polynomials' coefficients belong to $\mathcal{O}_K$, and such an n can be made effective in terms of the $\|g_i\|_v$ and $\|h_j\|_v$ for $v \in M_K^f$. Now, an effective Nullstellensatz (e.g., [Masser and Wüstholz 1983] applied to multiples of those generators), translated in the projective case, will give relations

$$a g_i^k = \sum_{j=1}^{p} f_{i,j} h_j$$

with $a \in \mathcal{O}_K$ nonzero, all the $f_{i,j}$ with coefficients in $\mathcal{O}_K$ and bounded $\|f_{i,j}\|_v$ and $|a|_v$ in terms of the norms of the polynomials for all $v \in M_K^\infty$. Furthermore, the power k is effectively bounded in terms of $[K : \mathbb{Q}]$,N and the degrees of the polynomials. This will clearly give an effective inequality

$$h_{Y,w}(P) \leq k \cdot h_{Y',w}(P) + O_{M_K}(1).$$

for all $(P, w) \in (X \setminus Y')(\overline{K}) \times M_{\overline{K}}$, and this argument works for parts (a) to (d) of the Proposition (the inequality in (c) without a factor k coming from the fact that we can extend generators of an ideal sheaf for Y' to an ideal sheaf for Y).

The only parts remaining to be proven are now (e), (f) and (g). Part (e), essentially saying that local height functions detect integral points up to some M_K-bounded error, is classical (see [Vojta 1987, Proposition 1.4.7]) and in fact automatic for the exact definition given in (2-1). Part (f) then comes from

(e) and Lemma 11 of [Levin 2014]. Finally, part (g) is a classical result on heights (e.g., [Lang 1983, Chapter IV, Proposition 5.4], and there also, the constants implied can be made effective. □

2C. *Baker's theory of linear forms in logarithms.* Let us now give our second main tool: estimates from Baker's theory in a special form sufficient for our purposes.

For any place w of M_L, $N(w)$ is defined to be 2 if w is archimedean and the norm of the associated prime ideal otherwise.

Proposition 2.3. *Define* $\log^*(x) = \max(\log(x), 1)$ *for* $x > 0$.

Let $d = [L : \mathbb{Q}]$ *and* $s = |S|$. *There is an effectively computable function* $C(d, s)$ *such that for any pair* (L, S), *any* $\alpha \in \mathcal{O}_{L,S}^* \setminus \{1\}$ *and any* $w \in M_L$,

$$\log|\alpha - 1|_w \geq -C(d, s) \frac{N(w)}{\log N(w)} R_S \log^*(N(w)h(\alpha)). \tag{2-2}$$

In terms of local heights, one can choose the local height $h_{1,w}(\alpha)$ *to be* $\max(-\log|\alpha - 1|_w, 0)$, *which gives us*

$$h_{1,w}(\alpha) \leq C(d, s) \frac{N(w)}{\log N(w)} R_S \log^*(N(w)h(\alpha)).$$

Proof. This result, although natural when one knows estimates for linear forms in logarithms, is not often presented in this form, so the following proof will explain how one can get to such an expression with known results. First, let us assume $s \geq 2$ (for $s = 1$, the result is trivial). By Lemma 1 of [Bugeaud and Győry 1996] ($\log h$ there is our logarithmic height here), one can choose a family of fundamental units $\varepsilon_1, \dots, \varepsilon_{s-1}$ of $\mathcal{O}_{L,S}^*$ such that

$$\prod_{i=1}^{s-1} h(\varepsilon_i) \leq c_1(s) R_S,$$

where $c_1(s) = ((s-1)!)^2/(2^{s-1} d^{s-2})$.

Now, by Theorem 4.2.1 of [Evertse and Győry 2015] applied to $\Gamma = \mathcal{O}_{L,S}^*$ gives us the bound (taking into account our normalization of $|\cdot|_w$) with $C(d, s) = c_1(s) c_8$ where c_8 is defined as in the reference. □

3. Proof of the main theorem

We now have all the tools to prove the theorem. We keep the notations from its statement, and assume that we have an embedding $X \subset \mathbb{P}_K^N$ from which all local heights considered below are defined. Recall that $s = |S|$, $d = [L : \mathbb{Q}]$ and n_w is the local degree of L at w. The constants c_i below are absolute and can be made effective.

First, let us notice that for every point $P \in T$, as the support of ϕ_P is in D, by Proposition 2.2(f) applied to ϕ_P and ϕ_P^{-1}, there is an absolute positive integer m (independent on the choice of (L, S) and $P \in T$) such that for every $Q \in (X \setminus D)(\mathcal{O}_{L,S})$, one has $m\phi_P(Q) \in \mathcal{O}_{L,S}$ and $m\phi_P(Q)^{-1} \in \mathcal{O}_{L,S}$. Defining S_m the set of primes of L dividing m, one thus has $\phi_P(Q) \in \mathcal{O}_{L,S \cup S_m}^*$ for all $Q \in (X \setminus D)(\mathcal{O}_{L,S})$.

By Proposition 2.2(e), the map $(Q,w)\mapsto h_{D_i,w}(Q)$ is M_K-bounded on pairs (Q,w) with $w\in M_L\setminus S$ and $Q\in(X\setminus D)(\mathcal{O}_{L,S})$, and $(Q,w)\mapsto h_{Y,w}(Q)$ is M_K-bounded on pairs (Q,w) with $w\in M_L\setminus S'$ and $Q\in(X\setminus Y)(\mathcal{O}_{L,S'})$.

Let us assume now that $Q\in(X\setminus D)(\mathcal{O}_{L,S})\cap(X\setminus Y)(\mathcal{O}_{L,S'})$. The previous paragraphs imply that for every $i\in\{1,\dots,n\}$

$$h_{D_i}(Q)=\frac{1}{[L:\mathbb{Q}]}\sum_{w\in S}n_w h_{D_i,w}(Q)+O(1)$$

where $O(1)$ is absolutely (and effectively bounded) on the set of such points Q (even if (L,S) is allowed to change).

Thus, for all $i\in\{1,\dots,n\}$, there is $w\in S$ such that

$$h_{D_i,w}(Q)\geq\frac{n_w}{[L:\mathbb{Q}]}h_{D_i,w}(Q)\geq\frac{1}{s}h_{D_i}(Q)+O(1).$$

After choosing for every $i\in\{1,\dots,n\}$ such a $w\in S$, we obtain a function $\{1,\dots,n\}\to S$. Now, if the fiber above a place $w\in S\setminus S'$ was a set J with $|J|>m_Y$, by Proposition 2.2(a), one would obtain an absolute effective (computable) upper bound on the minimum of such $h_{D_j,w}(Q)$, therefore on $h_{D_i}(Q)/s$ and $h_D(Q)/s$ by Proposition 2.2(g).

We can thus assume from now on that this is not the case. Therefore, the fibers of this function are of cardinality at most m_Y above $S\setminus S'$. Consequently, by hypothesis (1-1), one of the fibers above S', defined as I, has to be of cardinality at least $m_{\mathcal{B}}$ (if it's more, we extract a subset of cardinality $m_{\mathcal{B}}$), which gives $w\in S'$ such that

$$\min_{i\in I}h_{D_i,w}(Q)\geq\frac{1}{s}\min_{i\in I}h_{D_i}(Q)\geq\frac{c_1}{s}h_D(Q)+O(1).$$

Now, by Proposition 2.2(a) again and construction of T (if $T=\varnothing$, we directly obtain an absolute M_K-constant bound), there exists $P\in T$ such that

$$h_{P,w}(Q)\geq\frac{c_2}{s}h_D(Q)+O_{M_K}(1)\geq\frac{c_3}{s}h(\phi_P(Q))+O_{M_K}(1),\tag{3-1}$$

using Proposition 2.2(g), with absolute effective constants $c_1,c_2,c_3>0$. Moreover, by Proposition 2.2(d), as $\phi_P(P)=1$, if ϕ_P can be evaluated at Q and $\phi_P(Q)\neq1$,

$$h_{1,w}(\phi_P(Q))\geq c_4\cdot h_{P,w}(Q)+O_{M_K}(1)\geq\frac{c_3c_4}{s}h(\phi_P(Q))+O_{M_K}(1),\tag{3-2}$$

for $c_4>0$ absolute effective and $O_{M_K}(1)$ computable in terms of the initial data of embeddings and equations (but bounding it crudely by an absolute constant would suffice in the following argument).

On another hand, applying Proposition 2.3 to $\mathcal{O}^*_{L,S\cup S_m}$, we get

$$h_{1,w}(\phi_P(Q))\leq C(d,s+|S_m|)\cdot R_{S\cup S_m}\frac{N(w)}{\log N(w)}\cdot\log^*(N(w)h(\phi_P(Q))).\tag{3-3}$$

By formula (1.8.3) of [Evertse and Győry 2015], one has

$$R_{S\cup S_m} \le h_L R_S \prod_{\mathfrak{P}\in S_m\setminus S} \log N(\mathfrak{P})) \le h_L R_S \prod_{p\,|\,m} e^{d/e \log p} \le h_L R_S m^{d/e} \tag{3-4}$$

after optimizing the products of logarithms.

Combining (3-2), (3-3) and (3-4) and with some care about the logarithmic terms, we obtain an affine bound of the shape (1-2) for $h(\phi_P(Q))$, hence on $h_{P,v}(Q)$ by Proposition 2.2(d) applied the other way, which finally gives a bound on $h_D(Q)$ by (3-1) (there is a constant term which we can absorb in the linear one as it is effectively boundable).

4. Applications to the S-unit equation in the case of curves

In this section, we realize our method in the practical case of the S-unit equation (1-4), to prove Theorem 1.4.

This problem is related to finding the integral points of $(\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathcal{O}_{L,S})$ (up to taking into account the factors α, β), and this is the interpretation we will follow below to illustrate the main theorem. We follow closely the definitions and lemmas of [Győry and Yu 2006] (except their normalizations of norms), as our improvements intervene only at the beginning of the proof. As in that article, define

$$d = [L : \mathbb{Q}], \quad H = \max(h(\alpha), h(\beta), 1, \pi/d), \quad s = |S|.$$

For any $t \in L$, define $h_w(t) := h_{0,w}(t) = \log^+(1/|t|_w)$.

For the sake of symmetry of the exposition, we will do most computations with αx and βy, before coming back to H. This means we deal with $h_w(P)$ for

$$P \in E = \left\{\alpha x, \beta y, \frac{1}{\alpha x}\right\}.$$

Lemma 4.1. *For any $x, y \in L$ with $\alpha x + \beta y = 1$:*

- *For any place $w \in M_L$, at most one value of $h_w(P)$ for $P \in E$ can exceed $\delta_w \log 2$, where $\delta_w = 1$ if w is infinite, 0 otherwise.*
- *The maximum modulus of the difference of logarithmic heights of any two of them amongst $h(x)$, $h(y)$, $h(\alpha x)$, $h(\beta y)$ is at most $3H$, and even $2H$ except for $|h(x) - h(y)|$.*
- *If $x, y \in \mathcal{O}_{L,S}^*$ and $h = \max(h(x), h(y))$, we always have, for $P \in E$,*

$$\sum_{w\in S} \frac{n_w}{[L:\mathbb{Q}]} h_w(P) \ge h(P) - H \ge h - 3H.$$

Proof. The first item is the translation of the fact that if $z + z' = 1$ one of z, z' has to have norm at least 1 if the norm is ultrametric, and at least $\frac{1}{2}$ if it is archimedean.

The second item uses that for any nonzero algebraic numbers z, z', $h(zz') \leq h(z)+h(z')$ and $h(z+z') \leq h(z)+h(z')+\log 2$. For example, we obtain $|h(x)-h(\alpha x)| \leq H$ and $|h(\alpha x)-h(\beta y)| \leq \log 2$, and by symmetric role this leads to all other bounds on difference of heights, as $\log 2 \leq H$.

For the third item, in each of the three cases,

$$\sum_{w \in S} \frac{n_w}{[L:\mathbb{Q}]} h_w(P) = h(P) - \sum_{w \notin S} \frac{n_w}{[L:\mathbb{Q}]} \log^+(1/|P|_w)$$

but for $w \notin S$ and each of our three P's, the contribution of x or y to $1/|P|_w$ is by a factor 1 so this sum is bounded by H. The second inequality follows directly from the second item. □

Proof of Theorem 1.4. First, notice that if $s \leq 2$, Lemma 4.1 alone gives immediately that there is $P \in E$ such that $h_w(P) \leq \delta_w \log 2$ for all $w \in S$, and elsewhere we have $h_w(P) = h_w(\alpha)$, $h_w(\beta)$ or $h(\beta/\alpha)$ depending on the value of P, because x and y are S-units. Consequently, $h(P) \leq 2H + \log 2$, hence $h \leq 4H + \log 2$ in this case. We can now assume that $s \geq 3$.

For the first part of Theorem 1.4, the assumption amounts to saying that (1-1) holds in this case for $S' = M_L^\infty$. By Lemma 4.1, for any choice of $P \in E$, there is $w \in S$ such that

$$\frac{n_w}{[L:\mathbb{Q}]} h_w(P) \geq \frac{1}{|S|}(\max(h(x), h(y)) - 3H). \tag{4-1}$$

We want to fall back on a case where for one of the three choices of P, one can impose that $w \in M_L^\infty$. If that is not possible, by the pigeonhole principle and our hypothesis on S, there is a finite place w and two points $P, Q \in \{E\}$ distinct with

$$\frac{n_w}{[L:\mathbb{Q}]} \min(h_w(P), h_w(Q)) \geq \frac{1}{|S|}(\max(h(x), h(y)) - 3H).$$

By the same lemma, we get $\max(h(x), h(y)) \leq 3H$. This bound will be readily checked to be smaller than the other case.

One can thus assume from now on that for some $w \in M_L^\infty$ and $P \in E$, (4-1) holds.

The only thing to do is then to get back to the situation of [Győry and Yu 2006, page 24] in all three cases, after which we will obtain the exact same bounds. We fix a fundamental system $\varepsilon_1, \ldots, \varepsilon_{s-1}$ of units of $\mathcal{O}_{L,S}^*$ with the properties of [Győry and Yu 2006, Lemma 2].

• Assume first $P = \alpha x$, and write

$$y = \zeta \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}}, \tag{4-2}$$

with ζ a root of unity in L and $b_i \in \mathbb{Z}$ for all i. By the arguments of [Bugeaud and Győry 1996, page 76], we obtain that

$$B = \max(|b_1|, \ldots, |b_{s-1}|) \leq c_1(d, s) h(y)$$

with

$$c_1(d,s) = \begin{cases} ((s-1)!)^2/(2^{s-3}\log 2) & \text{if } d = 1, \\ ((s-1)!)^2/2^{s-2})\log(3d)^3 & \text{if } d \geq 2. \end{cases}$$

We set $\alpha_s = \zeta\beta$ and $b_s = 1$ so that

$$|\alpha x|_w = |1 - \varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}} \alpha_s^{b_s}|_w.$$

We set the A_i and A_s as in [Győry and Yu 2006, (31)] and can make the same assumption (otherwise, we obtain a smaller bound). By [loc. cit., Proposition 4 and Lemma 5], we thus obtain

$$h_w(P) = -\log|\alpha x|_w < c_2(d,s)c_3(d,s)R_S H \log\left(\frac{c_1(d,s)h(y)}{\sqrt{2}H}\right) \tag{4-3}$$

(as we always have $s \geq 3$ here), with

$$\begin{aligned} c_2(d,s) &= d^3\log(ed)\min(1.451(30\sqrt{2})^{s+4}(s+1)^{5.5}, \pi 2^{6.5s+27}),\\ c_3(d,s) &= e\sqrt{s-2}(((s-1)!)^2/(2^{s-2}))\pi^{s-2} \cdot \begin{cases} 8.5 & \text{if } d = 1,\\ 29d\log d & \text{if } d \geq 2. \end{cases} \end{aligned}$$

Let us now define $h = \max(h(x), h(y))$. We use inequality (4-1), and replace $h(y)$ by h in the right-hand side of (4-3) to obtain

$$\frac{h}{s} - H \leq \frac{h-3H}{s} \leq \frac{n_w}{[L:\mathbb{Q}]} c_2(d,s)c_3(d,s)R_S H \log\left(\frac{c_1(d,s)h}{\sqrt{2}H}\right) \tag{4-4}$$

and these are equivalent to the two inequalities used on page 24 of [Győry and Yu 2006] to obtain the result.

• Assume $P = \beta y$. We apply the same argument by symmetry, replacing α by β and x by y everywhere, to finally obtain the same bound.

• Assume $P = \frac{1}{\alpha x}$. We thus write

$$h_w(P) = -\log\left|\frac{1}{\alpha x}\right|_w = -\log\left|1 - \frac{\beta y}{\alpha x}\right|_w.$$

Let us fix then

$$\frac{y}{x} = \zeta\varepsilon_1^{b_1} \cdots \varepsilon_{s-1}^{b_{s-1}}, \quad \alpha_s = \frac{\beta}{\alpha},$$

and proceed in the same fashion as before, with a loss of precision because $h(\alpha_s) \leq 2H$ and not H. This is the reason for the factor 2 in the final result of Theorem 1.4, and has been taken into account in [Győry 2019, end of proof of Theorem 1].

For the second part of Theorem 1.4, we can play the same game, by defining S' the set of places S deprived of its two prime ideals with largest norm. The same elimination work as before will then give $w \in S'$ and $P \in E$ satisfying (4-1), and from there we can apply for $P = \alpha x$ the exact method of [Győry and Yu 2006, page 25] with a prime ideal $\mathfrak{p}$ from S'. This finally leads to the same estimates with P'_S instead of P_S, using again Lemma 4.1. □

5. A general framework for integrality methods

This section is motivated by comments from the referees asking for comparisons with other results, which inspired the author to present an attempt at conceptualizing more closely the current approaches.

The formalism presented here does not bring anything completely new in this regard but allows to understand many versions of Runge's or Baker's method simultaneously, and sheds some light on how they can possibly be combined. There is also some degree of equivalence with preexisting statements in the literature, which we will try to emphasize.

5A. *Graph-theoretic definitions.* The context (C) is still the same as before:

- X is a (smooth, to simplify) projective variety over the number field K.
- $D_1, \dots, D_n$ are reduced effective pairwise distinct divisors on X.

We will interpret everything in terms of a directed graph, which motivates the following definitions.

Definition 5.1.

- A *descending directed graph* is a directed graph $\mathcal{G}$ with finitely many vertices and edges such that:
  - Every vertex $\mathfrak{v} \in V$ is given a *depth* $d_{\mathfrak{v}} \in \mathbb{N}$.
  - If there is an edge $\mathfrak{v}' \mapsto \mathfrak{v}$, $d_{\mathfrak{v}'} < d_{\mathfrak{v}}$ (and we then say that $\mathfrak{v}$ is a child of $\mathfrak{v}'$). If $\mathfrak{v}'$ has no children, it is called *extremal*.
- For a given vertex $\mathfrak{v}$, the *cone of ancestors* of (we will also say *originating in*) $\mathfrak{v}$ is defined as

$$\mathcal{C}_{\mathfrak{v}} = \{\mathfrak{v}\} \cup \{\mathfrak{v}', \ \exists \text{ path } \mathfrak{v}' \mapsto \cdots \mapsto \mathfrak{v}\},$$

  and its *depth* is also defined as $d_{\mathfrak{v}}$.
- A family of cones of ancestors $(\mathcal{C}_{\mathfrak{v}})_{\mathfrak{v}}$ *spans the graph in depth 1* when the union of those cones contains all vertices of depth 1. If it does not, its *remainder* $\mathcal{R}$ is the set of unspanned vertices of depth 1.

Such a graph is particularly easy to draw and arrange by rows with fixed depth, hence its name.

Definition 5.2 (intersection graph). The *intersection graph* $\mathcal{G}$ of $X, D_1, \dots, D_n$ is the descending directed graph defined as such:

- For every subset $I \subset \{1, \dots, n\}$, we define

$$Z_I := \bigcap_{i \in I} \operatorname{Supp}(D_i)$$

and say that I is *optimal* if $Z_I \neq \varnothing$ and there is no $I' \supsetneq I$ such that $Z_{I'} = Z_I$, in other words no $i \notin I$ such that $Z_I \subset \operatorname{Supp}(D_i)$. In this way, every nonempty set-theoretic intersection of the divisors corresponds to a unique optimal set of indices

• The vertices $\mathfrak{v}$ of $\mathcal{G}$ are indexed by the optimal sets of indices. We thus can associate to each $\mathfrak{v}$ its optimal set $I_{\mathfrak{v}}$ and $Z_{\mathfrak{v}} := Z_{I_{\mathfrak{v}}}$. The depth of $\mathfrak{v}$ is defined as $|I_{\mathfrak{v}}|$, in other words the maximal number of divisors whose intersection defines $Z_{\mathfrak{v}}$. In particular, the unique vertex of depth 0 corresponds to X and the vertices of depth 1 correspond bijectively to the divisors $D_1, \ldots, D_n$ (unless one divisor contains another which we can assume to never hold).

• There is a directed arrow from $\mathfrak{v}'$ to $\mathfrak{v}$ if and only if $Z_{\mathfrak{v}'} \supsetneq Z_{\mathfrak{v}}$ with no intermediary set $Z_{\mathfrak{v}''}$, which by our construction is equivalent to saying that $I_{\mathfrak{v}'} \subsetneq I_{\mathfrak{v}}$ with no intermediary optimal set of indices (this is mainly to reduce the number of arrows in the graph). Considering a vertex $\mathfrak{v}$, the ancestors of depth 1 of $\mathfrak{v}$ correspond to the divisors D_i containing $Z_{\mathfrak{v}}$.

• Extremal vertices correspond to minimal nonempty intersections of the divisors (and optimal sets of indices which are maximal for the inclusion).

Remark 5.3. The number m in Runge's method [Levin 2008, Theorem 4] is exactly the maximal depth of a vertex in $\mathcal{G}$. When the divisors D_i are ample and in general position and X is of dimension d, the graph is particularly simple: for every $i \leq d$, it has $\binom{n}{i}$ edges of depth i, and the maximal depth is d. Notice also that the finite sets are exactly the ones of depth d in this case. This ideal situation would not need the formalism of the graph above, but our purpose is precisely to deal with the situations where m is larger than it should be.

5B. ***Runge and Baker methods for integrality and their variations.*** To proceed with integrality, the intuition is that for a point $P \in (X \setminus D)(\mathcal{O}_{L,S})$, for every place $v \in S$, one considers the set of closed subsets Z_I which are v-close to S. If the notion of closeness is defined rigorously and consistently (which we postpone until later), the following property $(\mathcal{P})$ holds:

> *For every place v and every point $P \in X(K_v)$, the set of vertices $\mathfrak{v}$ such that P is v-close to $Z_{\mathfrak{v}}$ is the cone of ancestors of a vertex $\mathfrak{v}_o$.*

Notice that we can also do the same with a point P in $X(K_v)$ and look at the $\mathfrak{v}$ for which $Z_{\mathfrak{v}}$ contains P (instead of merely being close to it), and we again get cones of ancestors. More precisely, approximating P v-adically by points $P_v \in X(K_v)$, we can obtain by compactness (when P_v is close enough) this same cone as the set of vertices $\mathfrak{v}$ such that $Z_{\mathfrak{v}}$ is v-close to P_v.

Let us now, as before, fix (L, S) and $P \in (X \setminus D)(\mathcal{O}_{L,S})$. To each $v \in S$ we thus associate a cone of ancestors $\mathcal{C}(P, v)$ corresponding to P seen in $X(K_v)$.

It now turns out that Runge and Baker's method can each be applied under conditions on those cones of ancestors (and a geometric condition on their remainder) in a very straightforward way, which we call terminal conditions (because if they apply, we can finish the analysis via computations quite independent from the graph then).

For each case, we provide an hypothesis which would make termination conditions automatically realized.

We start with the classical versions of Runge and Baker's method:

(1) (a) Termination for Runge. The cones of ancestors $\mathcal{C}(P, v)$, $v \in S$ do not span the intersection graph in depth 1 and the remainder $\mathcal{R}$ satisfies the geometric condition $(G)_R$:

The sum $D_{\mathcal{R}}$ of the divisors $D_{\mathfrak{v}}$, $\mathfrak{v} \in \mathcal{R}$ is ample (resp. big, of positive Kodaira–Iitaka dimension).

By construction, every divisor $D_{\mathfrak{v}}$ with $\mathfrak{v} \in \mathcal{R}$ is v-far from P for every place $v \in S$, and the global height $h_{D_{\mathcal{R}}}(P)$ is thus sufficiently controlled to ensure finiteness (resp. finiteness outside of an explicit proper algebraic subset independent of (L, S), non-Zariski density) by $(G)_R$.

(b) Automatic guarantee for Runge. $|S| = s$, and a family of s cones of ancestors (of extremal vertices) cannot span the graph in depth 1, and every divisor D_i is ample (resp. big, of positive dimension)

It is sufficient here (and more convenient for computations) to prove this for cones of extremal vertices because otherwise one might take a strictly larger cone, which would thus span at least the same depth 1 vertices.

Remark 5.4. Let us first compare it with the original higher-dimensional version of Runge's method, found in Theorem 4 of [Levin 2008]. In this context, recall that the number m is the maximal depth of the graph, and by definition of the intersection graph, a vertex $\mathfrak{v}$ of the intersection graph has exactly $d_{\mathfrak{v}}$ ancestors of depth 1. If $ms < n$, then any given s cones of ancestors cannot span the graph in depth 1, which means that $ms < n$ implies the automatic guarantee for Runge (in the three cases ample, big, positive Kodaira–Iitaka dimension of that theorem).

Now, this statement (for the case of positive Kodaira–Iitaka dimension) is in fact equivalent to Proposition 4.2 of [Corvaja et al. 2015], which formulates it in terms of the maximal number s of points $P_1, \ldots, P_s$ such that the sum of divisors D_i avoiding all of them make up a positive-dimensional divisor. This is exactly what $D_{\mathcal{R}}$ is meant to be above, hence the equivalence. One can also see with this graph-theoretic interpretation that the worst-case scenario is when each P_i belongs to a $Z_{\mathfrak{v}}$ where $\mathfrak{v}$ is an extremal vertex. One difficulty regarding the divisors dealt with in [Corvaja et al. 2015] is that they are not one by one of positive Kodaira dimension, and there is no obvious way to choose positive-dimensional sums of those divisors for which a classical Runge condition $ms < n$ can be applied, which is another reason why the improvement brought there by Proposition 4.2 is needed for the end result of that paper.

We give below a concrete example of application of this weaker Runge condition for a Siegel modular variety, following [Le Fourn 2019].

Proposition 5.5. *There is an absolute effectively computable constant $C > 0$ such that the following holds for any pair (K, S) with K quadratic and $S \supset M_K^\infty$ with $|S| = 2$.*

Let A be a principally polarized abelian surface over K with $A[2] \subset A(K)$ and such that the semistable reduction of A is always the jacobian of a smooth curve of genus 2 except at the finite places of S. Then, the stable Faltings height of A satisfies $h_{\mathcal{F}}(A) \leq C$. In particular, the set of all such abelian surfaces A up to isomorphism is effectively finite.

Proof. This amounts to a problem of $\mathcal{O}_{K,S}$-integral points on $A_2(2)^{\mathrm{Sa}} \setminus D$ when $|S| \leq 2$, where $A_2(2)^{\mathrm{Sa}}$ is the Satake compactification of the Siegel modular variety $A_2(2)$ of degree 2 and level 2 and D is the union of the ten divisors of moduli of products of elliptic curves. For all details, we refer to the author's previous work in [Le Fourn 2019, Section 8] with only some reminders here (every claimed fact with no reference can be found there). On this variety, the ten even theta constants define (as divisors of zeroes) ten divisors $D_1, \dots, D_{10}$ whose union D is exactly the boundary $\partial A_2(2) := A_2(2)^{\mathrm{Sa}} \setminus A_2(2)$ of the compactification together with the locus of products of elliptic curves. Furthermore, the fourth powers of these theta constants define an embedding

$$\psi : A_2(2)^{\mathrm{Sa}} \to \mathbb{P}^9,$$

for which the equations of the image are, with canonical ordering of the coordinates in $\mathbb{P}^9$:

$$x_1 - x_2 - x_6 - x_9 = 0 \tag{5-1}$$

$$x_1 - x_4 - x_5 - x_8 = 0 \tag{5-2}$$

$$x_2 - x_3 + x_5 - x_7 = 0 \tag{5-3}$$

$$x_3 - x_4 - x_6 + x_{10} = 0 \tag{5-4}$$

$$x_7 - x_8 - x_9 + x_{10} = 0 \tag{5-5}$$

$$\left(\sum_{i=1}^{10} x_i^2\right)^2 - 4\sum_{i=1}^{10} x_i^4 = 0, \tag{5-6}$$

which can also be seen as a quartic in $\mathbb{P}^4$ [Igusa 1964a, page 397]. The inverse image of the coordinate hyperplanes in $A_2(2)$ is thus the locus of moduli of products of elliptic curves. This description also holds for the semistable reduction modulo a prime $\mathfrak{P}$ of $\mathcal{O}_K$ of a point $P \in A_2(2)(K)$, in the sense that the reduction of the image by ψ of the modulus of a point given by A lands inside a coordinate hyperplane if and only if the semistable reduction of A is multiplicative, or a product of elliptic curves. This is true only up to some $M_{\mathbb{Q}}$-constant error, in particular the case of primes above 2 adds a lot of technical subtleties (see [Le Fourn 2019, paragraphs 8B to 8D]) so we do not try to obtain the constant C explicitly here as new computations would be needed (depending on the type of reduction of a curve in Théorème 1 of [Liu 1993]). By moduli arguments and the given equations for $\operatorname{Im}\psi$, one sees that a point Q in $\operatorname{Im}\psi$ has at most one zero coordinate if $\psi^{-1}(Q) \in A_2(2)$, and at most 6 if $\psi^{-1}(Q) \in \partial A_2(2)$. This gave the strong Runge condition $6|S| < 10$, hence in turn S had to be reduced to the unique archimedean place.

The termination condition for Runge written above will allow us to do a bit better; indeed, every D_i is ample and thus it is enough to ensure that the remainder is nonempty. To do this, it is possible to describe completely the intersection graph of the divisors D_i by first looking for the optimal sets of indices and then study the graph itself. Every time we "compute with equations" below, it means we proceed manually with the equations for $\operatorname{Im}\psi$ given above (and it will always be straightforward).

We recall from [Igusa 1964b, page 227] or [Streng 2010] that the canonical action of $\mathrm{Sp}_4(\mathbb{Z})$ on $A_2(2)^{\mathrm{Sa}}$ is transitive on their divisors D_i because the action on the ten fourth powers of theta constants is up to multiplication factors. Even better, this action is actually 2-transitive, which allows us to say that all

45 sets of indices $I \subset \{1, \ldots, 10\}$ with $|I| = 2$ are optimal (it is enough to check it for one of them). Notice that as for all arguments below, it amounts to say that assuming that if two of the x_i are zero, the equations above do not imply that another one is. Now, the 120 sets of indices with $|I| = 3$ are of two different types: they can be *syzygous* or *azygous* [Igusa 1964a, page 403], each case happening 60 times, and the action of $\mathrm{Sp}_4(\mathbb{Z})$ is transitive on syzygous (resp. azygous) unordered triples so again it is enough to see what happens for one of each type. The syzygous triples are optimal by computing the equations (and define a nonirreducible curve), whereas for any azygous triple I, there is a unique $j \notin I$ such that $Z_I \subset D_j$ and then $\tilde{I} = I \cup \{j\}$ completes I and is optimal, and $Z_{\tilde{I}}$ is irreducible.

Now, for each syzygous triple I, there is a unique $j \notin I$ completing it into a "Göpel quadruple" (quadruple such that every triple in it is syzygous), and it defines an empty intersection (again using the equations and transitivity). For any other j, the set of indices is not optimal and one needs to add two more indices. In other words, the vertex associated to a syzygous triple $\mathfrak{v}'$ is the starting point of two arrows corresponding to a unique partition in triples of the remaining 6 indices, and for each child $\mathfrak{v}$, the set $Z_{\mathfrak{v}}$ is reduced to a point.

For each completion of an azygous triple, the 6 remaining indices split into three pairs who each define an extremal vertex, and a point again.

To sum up the situation for the intersection graph, there are:

- 10 vertices of depth 1, defining divisors, each with 9 children of depth 2.
- 45 vertices of depth 2, defining nonirreducible curves in the boundary of $A_2(2)$, each with 4 children of depth 3 and 2 children of depth 4.
- 60 vertices of depth 3, corresponding to the syzygous triples, defining curves with two components in the boundary, and each having 2 children of depth 6.
- 15 vertices of depth 4 corresponding to azygous quadruples [van der Geer 1982, page 337], defining irreducible curves in the boundary, and each having 3 children of depth 6.
- 15 vertices of depth 6, the extremal ones, corresponding to complements of Göpel quadruples, each defining a unique point in the boundary.

Now, given the explicit list of the Göpel quadruples, it is easy to see that two of them always have nonempty intersection. This means that the cones of ancestors of any two extremal vertices cannot span the graph in depth 1. Therefore, if we fix $|S| = 2$, there is a remaining divisor D_i which is v-far away from our integral point P at every place by our hypotheses, hence giving an absolute bound on the height. $\square$

(2) (a) <u>Termination for Baker</u>. Either the condition of termination for Runge holds, or one of the cones of ancestors $\mathcal{C}(P, v)$ comes from a $Z_{\mathfrak{v}}$ which is finite *and* its remainder $\mathcal{R}_{\mathfrak{v}}$ (with respect to only this cone) satisfies the following condition $(G)_B$:

> *There is a nonconstant rational function ϕ on $\overline{K}(X)$ with support in $\mathcal{R}_{\mathfrak{v}}$ (which is thus disjoint from $Z_{\mathfrak{v}}$).*

We then obtain a bound on the height of points of $(X \setminus D)(\mathcal{O}_{L,S})$ depending on (L, S) and outside a fixed effectively computable proper subvariety of X.

(b) Automatic guarantee for Baker. With the same notations, s cones of ancestors (none of them giving a finite $Z_\mathfrak{v}$) cannot span the graph in depth 1, the D_i are all ample *and* $(B)_{\text{rank}}$ the rank of the group of principal divisors with support in $\bigcup_{i=1}^n D_i$ is larger than the depth of any vertex $\mathfrak{v}$ with $Z_\mathfrak{v}$ finite.

Remark 5.6. Again, the condition $(m_\mathcal{B} - 1)s < n$ in [Levin 2014, Theorem 1] together with $(G)_B$ imply the automatic guarantee; indeed, for every edge $\mathfrak{v}$ of depth at least $m_\mathcal{B}$, the set $Z_\mathfrak{v}$ is finite by definition, so if we restrict to the other cones of ancestors, each vertex is below at most $m_\mathcal{B} - 1$ divisors and a pigeonhole principle applies again.

Another remark that might deserve to be pointed out is that $(G)_B$ is made to be able to send any point of $Z_\mathfrak{v}$ to 1 via one of finitely many rational functions ϕ, and thus go back to the 1-dimension situation of Baker's method. It turns out, following the proof of Levin (especially [Levin 2014, Lemma 10]) that one can do the same process with nonfinite $Z_\mathfrak{v}$ assuming that *for each of the irreducible components of* $Z_\mathfrak{v}$*, there is nonconstant* ϕ *with support in* $\mathcal{R}_v$ *sending it to a single point*. This hypothesis is of course very strong (even more so than $(G)_B$ itself), but it can sometimes be satisfied. Indeed, if X is embedded in $\mathbb{P}^n$ in such a way that the D_i become the coordinate hyperplanes and $\mathcal{I}_X$ is the homogeneous ideal of definition of X in $\mathbb{P}^n$, this hypothesis is true for a given $I_\mathfrak{v}$ if there are disjoint sets of indices I, J (disjoint with $I_\mathfrak{v}$) such that there is a nonzero homogenous polynomial

$$P_I - P_J \in (\mathcal{I}_X, x_i \ (i \in I_\mathfrak{v}))$$

where P_I (resp. P_J) is a monomial in the x_i, $i \in I$ (resp. $i \in J$). If we go back to the case of $A_2(2)$ as in Proposition 5.5, using again the equations, we prove easily that for any optimal set $I_\mathfrak{v}$ with $|I_\mathfrak{v}| \geq 2$, one can find such $j, k \notin I_\mathfrak{v}$ such that $x_j - x_k$ or $x_j + x_k$ belong to $(\mathcal{I}_X, x_i \ (i \in I_\mathfrak{v}))$: for example, taking $I_\mathfrak{v} = \{1, 2\}$, the first equation gives $x_6 - x_9$. For the syzygous triple $\{1, 2, 3\}$, the difference $x_6 - x_8$ works, and for the azygous quadruple $\{1, 2, 6, 9\}$, the sum $x_4 + x_8$ works. All this implies that one can in fact apply the termination condition for Baker's method as soon as one cone of ancestors is of depth at least 2! This allows us to modify this condition which in practice amounts to considering that $m_\mathcal{B} = 2$, to obtain the following statement:

Proposition 5.7. *For any pair* (L, S), *with* $M_L^\infty \subset S$ *and* $|S| < 10$, *there is an effectively computable bound* $C(L, S)$ *such that the following holds*:

For a principally polarized abelian surface A *defined over* L *with* $A[2] \subset A(L)$, *if the semistable reduction of* A *is the jacobian of a smooth curve at every prime except maybe the ones in* S, *then* $h_\mathcal{F}(A) \leq C(L, S)$ *unless some two theta coordinates of a point of* $A_2(2)$ (*seen in* $\mathbb{P}^9$) *representing* A *are equal or opposite.*

The equality up to sign of those theta coordinates comes from the need to take out an exceptional subset (where the auxiliary rational functions, which are exactly the functions $(x_j/x_i) \circ \psi$ here, can be equal to 1) already appearing in [Levin 2014] for the same reasons.

We will now consider variants of those methods, the tubular ones devised by the author and then the "reductions" devised by Levin [2018], and compare them.

Let us add to the context (C) an integer $m_Y \geq 1$ and the corresponding closed subset Y as in Theorem 1.1: notice that it amounts to drawing an horizontal line between depths m_Y and $m_Y + 1$. We denote by $\mathcal{G}_Y$ the part of the graph below this line, and consider triples (L, S, S') with $M_L^\infty \subset S' \subset S$ finite, $s = |S|$, $s' = |S'|$, and points $P \in (X \setminus D)(\mathcal{O}_{L,S}) \cap (X \setminus Y)(\mathcal{O}_{L,S'})$. Of course, when $Y = \varnothing$, this intersection is the set $(X \setminus D)(\mathcal{O}_{L,S})$ again.

(3) (a) Termination for tubular Runge [Le Fourn 2019, Theorem 5.1]. Either one of the $s - s'$ cones of ancestors $\mathcal{C}(P, v)$, $v \in S \setminus S'$ originates in $\mathcal{G}_Y$ or the s cones do not span $\mathcal{G}$ in depth 1 and the termination condition for Runge holds.

(b) Automatic guarantee for tubular Runge. When we consider s cones of ancestors with at most s' of them of depth $> m_Y$, they do not span the graph in depth 1 and each divisor is ample (resp. big, of positive dimension).

Remark 5.8. Following the now usual ideas, this guarantee holds when $m \cdot s' + m_Y(s - s') < n$, which is exactly equation (4) of [Le Fourn 2019]. Notice also that in this context, as a byproduct of Runge's strategy and the tools used here, the bound on the height is still uniform.

Notice that one cannot improve upon Proposition 5.5 in this tubular context, because when $s = s' = 2$, there can be only one divisor in the remainder (see the proof) and so no room for increasing s.

(4) (a) Termination for tubular Baker. Either the termination condition for tubular Runge with m_Y and $\mathcal{G}_Y$ holds, or one of the s' cones of ancestors coming from S' originates in a finite $Z_{\mathfrak{v}}$ and the remainder of this cone satisfies $(G)_B$.

(b) Automatic guarantee for tubular Baker. One cannot span the graph in depth 1 with s cones of ancestors when $s - s'$ of them originate outside $\mathcal{G}_Y$ and the s' other ones do not originate at a finite $Z_{\mathfrak{v}}$, and the condition $(B)_{\text{rank}}$ holds.

Remark 5.9. Again, this guarantee is ensured whenever $(m_{\mathcal{B}} - 1)s' + m_Y(s - s') < n$ under the hypothesis $(H)_P$ of Theorem 1.1, where we recall that $m_{\mathcal{B}}$ is the depth starting from which every $Z_{\mathfrak{v}}$ is finite. As a byproduct of the proof made here, each case of the termination condition is uniform in the choice of (L, S, S') except the one where one cone from S' originates from a finite $Z_{\mathfrak{v}}$.

In the case $Y = \varnothing$, which amounts to $m_Y = m$ as in Runge's method, Baker's method applies when s, s' are such that $s - s'$ cones of ancestors (of any possible depth) together with s' ones not originating from finite $Z_{\mathfrak{v}}$'s cannot span the graph in depth 1. If we define for $s - s'$, $n_{s-s'}$ the minimal cardinality of the remainder of $s - s'$ cones of ancestors, this automatically holds then when $(m_{\mathcal{B}} - 1)s' < n_{s-s'}$.

For results from [Levin 2018, especially Theorem 5.15], we are yet in another context (we modified slightly the notations there for consistency here): S_0 is a fixed set of places of K, $S_{L,0}$ the set of places above S_0 in L of cardinality s' and we consider (L, S) and s such that $s = |S|$ this time.

(5) (a) Levin's reduction. The $s - s'$ cones of ancestors coming from $S \setminus S_{L,0}$ do not span the graph in depth 1, and the remainder $\mathcal{R}$ is such that a further method could be applied to its intersection graph, i.e., to a set of $S_{L,0}$-integral points on $(X \setminus D_{\mathcal{R}})$.

(b) Automatic guarantee for Levin's reduction. Any union of $s - s'$ cones of ancestors leave a remainder large enough to apply another method to its graph.

Remark 5.10. The big gain in this reduction is uniformity in the choice of $S \setminus S_{L,0}$. Of course, when S_0 is empty, one retrieves exactly Runge's method in higher dimension.

Notice the link with the case $Y = \varnothing$ of tubular Baker above: on the graph-theoretic side, the condition of reduction of Levin (and then the ability to apply Baker's condition) are very close, in particular

$$(m_{\mathcal{B}} - 1)s' + m(s - s')$$

implies both. The statements are not completely equivalent though: when Levin's reduction applies, it provides, as stated, uniformity in the choice of $S \setminus S_{L,0}$, which in practical estimates will give a bound only depending on the primes of S_0, $h_L R_L$ and $[L : \mathbb{Q}]$. By comparison, one can see in (1-2) that our tubular Baker estimates do depend on the primes of S and of the regulator of S in particular.

On the other hand, there are cases where Levin's reduction cannot be applied for effective results, starting with the S-unit equation: one cannot eliminate even one point and still satisfy the hypothesis $(G)_B$.

It thus seems that the best way to proceed (in the case $Y = \varnothing$, i.e., when one does not want to add another hypothesis of integrality) is to figure out the maximal number of cones of ancestors one can choose so that their remainder contains the support of a nonconstant rational function (which is, as we repeat, a *uniform* process of reduction) and then apply our tubular Baker method.

Notice that in the case of curves, Levin [2018, Theorems 7.19 and 7.20] uses reduction to respectively Siegel's theorem for $g \geq 1$ (thus ineffective) and Thue–Mahler equations for $\mathbb{P}^1$. As for Theorem 7.22 of [loc. cit.], the number s is exactly the number $s - s'$ mentioned above for which one can take this many cones of ancestors and still ensure there is a rational function with support in $D_{\mathcal{R}}$.

Finally, Levin's reduction and our tubular Baker approach can naturally be combined, making full use of the notion of tubular neighborhoods in [Le Fourn 2019]. In other words, start with $P \in (X \setminus D)(\mathcal{O}_{L,S})$ which is v-far away from Y at every $v \notin S'$. We then reproduce Levin's reduction for the places $v \in S \setminus S'$, which generate cone of ancestors of depth at most m_Y by hypothesis on P, and we are allowed to consider $|S \setminus S'|$ of them as long as for the remainder of their union, the associated graph satisfies $(G)_B$ with s' cones of ancestors. The end result will then be, as Levin's reduction, only depending on s, S' and L.

An example of application, not undertaken here, would be our pet example $A_2(2)^{\mathrm{Sa}}$ with Y being the boundary. We would obtain more uniformity than in Proposition 5.7 but at the cost of an hypothesis of potentially good reduction of the abelian surface at every prime except a bounded number of them.

5C. *Formalizing the closeness condition.* To prove rigorously all previous statements, one needs, as explained in the previous subsection, a rigorous definition of v-closeness that satisfies $\mathcal{P}$. Let us start with the divisors (i.e., depth 1 vertices). A first look gives us two rather different possibilities:

(1) For Runge's method, one wants to end up with one of the divisors D_i such that $h_{D_i,v}(P) \geq c_v$ for all $v \in M_K$, for an explicit M_K-constant $(c_v)_v$.

It is thus natural to define the v-closeness of P to $Z_{\mathfrak{v}}$ as $h_{Z_{\mathfrak{v}},v}(P) \geq c_{\mathfrak{v},v}$ for an M_K-constant deduced from the one above, such that the v-closeness is stable by intersection. Such M_K-constants exists by Proposition 2.2(a), and all arguments hold in this case.

(2) For Baker's method, given that we want a bound of the type $h \ll \log^* h$ for some global height at the end (see Section 3), we want to define v-closeness rather as something of the shape

$$h_{D_i,v}(P) > \lambda_i \cdot h_{D_i}(P).$$

(in fact any function of the height against which the logarithm would be dominated would suffice, but we always choose it linear here).

Again, we can make this v-closeness property inheritable thanks to Proposition 2.2(a), and v-closeness is then given by

$$h_{Z_{\mathfrak{v}},v}(P) > \lambda_{\mathfrak{v}} \cdot \min_{i \in \mathcal{C}(\mathfrak{v})} h_{D_i}(P),$$

where i goes through all ancestors of $\mathfrak{v}$ of depth 1 and $\lambda_{\mathfrak{v}} > 0$ is a well-chosen value in terms of the λ_i. Notice also that to apply Baker's method (and not fall back to the Runge case), one needs to ensure that every v does create a cone of ancestors, hopefully deep enough. The best way to guarantee that is fixing $\lambda_i < 1/|S|$ because the sum of $h_{D_i,v}(P)$ for $v \in S$ is almost $h_{D_i}(P)$ for our integral points.

To combine the principles behind the proofs above, one needs a consistent definition of closeness that fits both definitions. The most natural way to do this is

$$h_{D_i,v}(P) > \lambda_i \cdot h_{D_i}(P) + c_v$$

and deduced estimates for the $h_{Z_{\mathfrak{v}},v}(P)$. In passing, one can remark that the classical Runge condition of closeness is in fact too strong: a linear condition as above would also be amenable to the (classical) Runge method as long as $|S|\lambda_i = 1 - \varepsilon$, $\varepsilon > 0$, and the obtained bounds would then depend on $1/\varepsilon$.

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Fano 4-folds with rational fibrations

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We study (smooth, complex) Fano 4-folds X having a rational contraction of fiber type, that is, a rational map $X \dashrightarrow Y$ that factors as a sequence of flips followed by a contraction of fiber type. The existence of such a map is equivalent to the existence of a nonzero, nonbig movable divisor on X. Our main result is that if Y is not $\mathbb{P}^1$ or $\mathbb{P}^2$, then the Picard number ρ_X of X is at most 18, with equality only if X is a product of surfaces. We also show that if a Fano 4-fold X has a dominant rational map $X \dashrightarrow Z$, regular and proper on an open subset of X, with $\dim(Z) = 3$, then either X is a product of surfaces, or ρ_X is at most 12. These results are part of a program to study Fano 4-folds with large Picard number via birational geometry.

1. Introduction

Smooth, complex Fano varieties play an important role in projective geometry, both from the classical and modern point of view, in the framework of the minimal model program. There are finitely many families of Fano varieties of any given dimension, which are classified up to dimension 3 — the classification of Fano 3-folds was achieved more than 30 years ago, see [Iskovskikh and Prokhorov 1999] and references therein. In dimensions 4 and higher there is no classification apart from some special classes, and we still lack a good understanding of the geometry of Fano 4-folds.

This paper is part of a program to study Fano 4-folds X with large Picard number ρ_X, by means of birational geometry, more precisely via the study of contractions and flips of Fano 4-folds. Our goal is to get a sharp bound on ρ_X, and possibly to classify Fano 4-folds X with "large" Picard number. Let us notice that, among the known examples of Fano 4-folds, products of del Pezzo surfaces have $\rho_X \leq 18$, and the others have $\rho_X \leq 9$ (see [Casagrande et al. 2019] for the case $\rho_X = 9$).

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In this paper we focus on Fano 4-folds X having a rational contraction of fiber type. Here a *contraction* is a morphism $f\colon X\to Y$ with connected fibers onto a normal projective variety. More generally, a *rational contraction* is a rational map $f\colon X \dashrightarrow Y$ that can be factored as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} Y$, where X' is a normal and $\mathbb{Q}$-factorial projective variety, φ is birational and an isomorphism in codimension 1, and f' is a contraction. As usual, f is of fiber type if $\dim Y < \dim X$. Note that X has a nonconstant rational contraction of fiber type if and only if there is a nonzero, nonbig movable divisor. Our main results are the following.

Theorem 1.1. *Let X be a smooth Fano 4-fold with a rational contraction of fiber type $f\colon X \dashrightarrow Y$, where* $\dim Y > 0$. *If $Y \ncong \mathbb{P}^1$ and $Y \ncong \mathbb{P}^2$, then $\rho_X \le 18$, with equality only if X is a product of surfaces.*

Theorem 1.2. *Let X be a smooth Fano 4-fold. Suppose that there exists a dominant rational map $f\colon X \dashrightarrow Y$, regular and proper on an open subset of X, with* $\dim Y = 3$. *Then either X is a product of surfaces, or $\rho_X \le 12$.*

Let us say something on the techniques and strategy used in the paper. We consider the following classes of rational contractions of fiber type:

$$\{\text{“quasielementary”}\} \subset \{\text{“special”}\} \subset \{\text{general}\}.$$

Quasielementary rational contractions of fiber type have been introduced in [Casagrande 2008; 2013a] (see Section 2A for more details); when f is quasielementary Theorem 1.1 is already known [loc. cit.], and one can even allow $Y \cong \mathbb{P}^1$ and $Y \cong \mathbb{P}^2$.

In this paper we introduce a more general notion, that of “special” rational contraction of fiber type, which plays a key role in the proof of Theorem 1.1. We define special (regular and rational) contractions in Section 2B; then we show that every rational contraction of fiber type of a Mori dream space can be factored as a special rational contraction, followed by a birational map (Proposition 2.13). In particular, if a Fano variety has a rational contraction of fiber type, then it also has a special rational contraction of fiber type, so that we can reduce to prove Theorem 1.1 when f is special.

Secondly, we show that up to flips, every special rational contraction of a Mori dream space can be factored as a sequence of elementary divisorial contractions, followed by a quasielementary contraction (Theorem 2.15). This allows to relate the study of special rational contractions of Fano 4-folds X to our previous study of elementary divisorial contractions and quasielementary contractions of 4-folds obtained from X with a sequence of flips, in [Casagrande 2013a; 2017].

Another key ingredient used in the paper is the Lefschetz defect δ_X, an invariant of X which basically allows to bound ρ_X in terms of the Picard number of prime divisors in X (see Section 3A for an account).

After developing the necessary techniques and preliminary results in Sections 2–4, we prove Theorem 1.1 first in the case where $\dim Y = 2$ in Section 5, and then in the case where $\dim Y = 3$ in Section 6. Theorem 1.2 is then an easy consequence of the case where $\dim Y = 3$.

1A. ***Notation and terminology.*** If $\mathcal{N}$ is a finite-dimensional real vector space and $a_1, \dots, a_r \in \mathcal{N}$, $\langle a_1, \dots, a_r \rangle$ denotes the convex cone in $\mathcal{N}$ generated by $a_1, \dots, a_r$. Moreover, for every $a \neq 0$, $a^\perp$ is the hyperplane orthogonal to a in the dual vector space $\mathcal{N}^*$.

We refer the reader to [Hu and Keel 2000] for the notion of Mori dream space; *we always assume that a Mori dream space is projective, normal and $\mathbb{Q}$-factorial.* We recall that Fano varieties are Mori dream spaces by [Birkar et al. 2010, Corollary 1.3.2]. We also refer to [Kollár and Mori 1998] for the standard notions in birational geometry, in particular the definition of flip [loc. cit., Definition 6.5]

Let X be a normal and $\mathbb{Q}$-factorial projective variety.

A small $\mathbb{Q}$-factorial modification (SQM) is a birational map $\varphi\colon X \dashrightarrow X'$ which is an isomorphism in codimension one, where X' is a normal and $\mathbb{Q}$-factorial projective variety. If X is a Mori dream space, every SQM can be factored as a finite sequence of flips.

Let $f\colon X \to Y$ be an elementary contraction, namely a contraction with $\rho_X - \rho_Y = 1$. We say that f is of type (a, b) if

$$\dim \operatorname{Exc}(f) = a \quad \text{and} \quad \dim f(\operatorname{Exc}(f)) = b.$$

We say that f is of type $(\dim X - 1, b)^{\mathrm{sm}}$ if it is the blow-up of a smooth b-dimensional subvariety of Y, contained in Y_{reg}. If X is a smooth 4-fold, we say that f is of type $(3,0)^Q$ if f is of type $(3,0)$, $\operatorname{Exc}(f)$ is isomorphic to an irreducible quadric Q, and $\mathcal{N}_{\operatorname{Exc}(f)/X} \cong \mathcal{O}_Q(-1)$.

Let D be a divisor. A contraction $f\colon X \to Y$ is D-negative (respectively, D-positive) if there exists $m \in \mathbb{Z}_{>0}$ such that $-mD$ (respectively, mD) is Cartier and f-ample. A D-negative flip is the flip of a small, D-negative elementary contraction, and similarly for D-positive. *We do not assume that contractions or flips are K-negative, unless specified.*

When X is a Mori dream space, given a contraction $f\colon X \to Y$ and a divisor D in X, one can run an MMP for D relative to f. This means that there exists a birational map $\psi\colon X \dashrightarrow X'$, given by a composition of D-negative flips and elementary divisorial contractions, such that $f' := f \circ \psi^{-1}\colon X' \to Y$ is regular, and if D' is the transform of D in X', then either D' is f'-nef, or f' factors through a D'-negative elementary contraction of fiber type of X'.

A movable divisor is an effective divisor D such that the stable base locus of the linear system $|D|$ has codimension ≥ 2. A fixed prime divisor is a prime divisor D which is the stable base locus of $|D|$. We will consider the usual cones of divisors and of curves

$$\operatorname{Nef}(X) \subseteq \operatorname{Mov}(X) \subseteq \operatorname{Eff}(X) \subset \mathcal{N}^1(X), \quad \operatorname{mov}(X) \subseteq \operatorname{NE}(X) \subset \mathcal{N}_1(X),$$

where all the notations are standard except $\operatorname{mov}(X)$, which is the convex cone generated by classes of curves moving in a family covering X. When X is a Mori dream space, all these cones are closed, rational and polyhedral. If D is a divisor and C is a curve in X, we denote by $[D] \in \mathcal{N}^1(X)$ and $[C] \in \mathcal{N}_1(X)$ their numerical equivalence classes.

For every closed subset $Z \subset X$, we denote by $\mathcal{N}_1(Z, X)$ the linear subspace of $\mathcal{N}_1(X)$ spanned by classes of curves contained in Z. We will use the following simple property.

Remark 1.3. Let D be a prime divisor. If $Z \cap D = \varnothing$, then $\mathcal{N}_1(Z, X) \subseteq D^{\perp}$, in particular $\mathcal{N}_1(Z, X) \subsetneq \mathcal{N}_1(X)$. This is because $D \cdot C = 0$ for every curve $C \subset Z$.

Let X be a smooth 4-fold. An *exceptional plane* is a closed subset $L \subset X$ such that $L \cong \mathbb{P}^2$ and $\mathcal{N}_{L/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$; an *exceptional line* is a closed subset $\ell \subset X$ such that $\ell \cong \mathbb{P}^1$ and $\mathcal{N}_{\ell/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$.

2. Special contractions of fiber type

When studying Fano varieties, or more generally Mori dream spaces, one often needs to consider contractions of fiber type $f : X \to Y$ which are not elementary. In full generality, such contractions are hard to deal with, in particular Y may be very singular and/or non-$\mathbb{Q}$-factorial. For this reason, it is useful to introduce some classes of contractions of fiber type with good properties, which should include the elementary case. A first notion of this type is that of "quasielementary" contraction; we briefly recall this definition and some properties in Section 2A.

Here we introduce a more general notion, that of "special" contraction of fiber type. In Section 2B we define special contractions, in the regular and rational case; the target is automatically $\mathbb{Q}$-factorial.

In Section 2C we show two factorization results for rational contractions of fiber type of Mori dream spaces. More precisely, we show that every rational contraction of fiber type of a Mori dream space can be factored as a special rational contraction, followed by a birational map (Proposition 2.13). Moreover, up to flips, every special rational contraction of a Mori dream space can be factored as a sequence of elementary divisorial contractions, followed by a quasielementary contraction (Theorem 2.15).

Finally, in Section 2D we consider special contractions of fiber type $f : X \to Y$ which are also $(K+\Delta)$-negative for a suitable boundary Δ on X, and we show that if X has good singularities, then Y has good singularities too.

2A. ***Quasielementary contractions.*** We refer the reader to [Casagrande 2013a, Section 2.2; 2008] for the notion of quasielementary contraction of fiber type; here we just recall the definition.

Definition 2.1 (quasielementary contraction). Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f : X \to Y$ a contraction of fiber type. We say that f is quasielementary if for every fiber F of f we have $\mathcal{N}_1(F, X) = \ker f_*$, where $f_* : \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ is the push-forward of one-cycles (see Section 1A for $\mathcal{N}_1(F, X)$).

Let us give an equivalent characterization, for Mori dream spaces.

Proposition 2.2. *Let X be a Mori dream space and $f : X \to Y$ a contraction of fiber type. The following are equivalent*:

(i) *f is quasielementary.*

(ii) *For every prime divisor D in X, either $f(D) = Y$, or $D = \lambda f^* B$ for some $\mathbb{Q}$-Cartier prime divisor B in Y and $\lambda \in \mathbb{Q}_{>0}$.*

(iii) *Y is $\mathbb{Q}$-factorial and for every prime divisor B in Y, the pull-back $f^* B$ is irreducible* (*but possibly nonreduced*).

Proof. Let $F \subset X$ be a general fiber of f.

(i)⇒(iii) The target Y is $\mathbb{Q}$-factorial by [Casagrande 2013a, proof of Remark 2.26]. Let B be a prime divisor in Y, and let D be an irreducible component of f^*B. Then $D \cap F = \varnothing$, so that $\mathcal{N}_1(F, X) \subseteq D^{\perp}$ by Remark 1.3. Since f is quasielementary, we have $\mathcal{N}_1(F, X) = \ker f_*$, hence $\ker f_* \subseteq D^{\perp}$, and D is the pull-back of a $\mathbb{Q}$-divisor in Y (see [loc. cit., Remark 2.9]). Since $B = f(D)$, we must have $D = \lambda f^*B$ with $\lambda \in \mathbb{Q}_{>0}$, so f^*B is irreducible.

(ii)⇒(i) Let σ be the minimal face of $\mathrm{Eff}(X)$ containing $f^*(\mathrm{Nef}(Y))$; by [Casagrande 2013a, Lemma 2.21 and Proposition 2.22] we have $\sigma = \mathrm{Eff}(X) \cap \mathcal{N}_1(F, X)^{\perp}$, and f is quasielementary if and only if $\dim \sigma = \rho_Y$.

Suppose that f is not quasielementary. Then $\dim \sigma > \rho_Y$, so that $\sigma \not\subseteq f^*\mathcal{N}^1(Y)$, and there exists a one-dimensional face τ of σ such that $\tau \not\subseteq f^*\mathcal{N}^1(Y)$. Let $D \subset X$ be a prime divisor with $[D] \in \tau$. Then D is not the pull-back of a $\mathbb{Q}$-Cartier prime divisor in Y. On the other hand, we also have $[D] \in \mathcal{N}_1(F, X)^{\perp}$, so that $D \cdot C = 0$ for every curve $C \subset F$. Since $F \not\subset D$, we must have $F \cap D = \varnothing$, hence $f(D) \subsetneq Y$.

(iii)⇒(ii) Let $D \subset X$ be a prime divisor which does not dominate Y. Let $B \subset Y$ be a prime divisor containing $f(D)$. Then B is $\mathbb{Q}$-Cartier, and D is an irreducible component of f^*B, hence $f^*B = \mu D$ with $\mu \in \mathbb{Q}_{>0}$. □

2B. *Special contractions.*

Definition 2.3 (special contraction). Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f : X \to Y$ a contraction of fiber type. We say that f is special if for every prime divisor $D \subset X$ we have that either $f(D) = Y$, or $f(D)$ is a $\mathbb{Q}$-Cartier prime divisor in Y.

Remark 2.4. Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f : X \to Y$ a contraction of fiber type. Then f is special if and only if the following conditions hold:

(1) $\operatorname{codim} f(D) \leq 1$ for every prime divisor $D \subset X$.

(2) Y is $\mathbb{Q}$-factorial.

Condition (1) above is not enough to ensure that Y is $\mathbb{Q}$-factorial, as the following simple example shows.

Example 2.5. Set $Z := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$, $X := Z \times \mathbb{P}^1$, and let $\pi : X \to Z$ be the projection. Then Z has a small elementary contraction $g : Z \to Y$, and $f := g \circ \pi : X \to Y$ satisfies (1) but not (2), in particular it is not special. Note that X is Fano and f is K-negative.

Remark 2.6. Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f : X \to Y$ a contraction of fiber type:

(a) If X is a Mori dream space and f is elementary, or quasielementary, then f is special by Proposition 2.2.

(b) If f is special, then the locus where f is not equidimensional has codimension at least 3 in Y.

(c) Let f be special, and $\varphi : X \dashrightarrow X'$ a SQM such that $f' := f \circ \varphi^{-1}$ is regular. Then f' is special.

The following is a consequence of [Druel 2018, Lemma 2.6].

Lemma 2.7. *Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \to Y$ a contraction of fiber type. If f is equidimensional, then Y is $\mathbb{Q}$-factorial and f is special.*

Definition 2.8 (special rational contraction). Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \dashrightarrow Y$ a rational contraction of fiber type. We say that f is special if there exists a SQM $\varphi\colon X \dashrightarrow X'$ such that $f' := f \circ \varphi^{-1}$ is regular and special.

Remark 2.9. If $f\colon X \dashrightarrow Y$ is special, then:

- Y is $\mathbb{Q}$-factorial, by Remark 2.4.
- For every SQM $\varphi\colon X \dashrightarrow X'$ such that $f' := f \circ \varphi^{-1}$ is regular, we have that f' is special, by Remark 2.6(c)enumi.

In the next subsection we will prove the following characterization of special rational contractions of Mori dream spaces.

Proposition 2.10. *Let X be a Mori dream space and $f\colon X \dashrightarrow Y$ a rational contraction of fiber type. Then f is special if and only if f cannot be factored as*

$$X \overset{g}{\dashrightarrow} Z \overset{h}{\dashrightarrow} Y, \qquad f = h \circ g$$

where g is a rational contraction, h is birational, and $\rho_Z > \rho_Y$.

2C. *Factorizations.* We start this subsection with a construction that will be used in the proofs of two factorization results, Proposition 2.13 and Theorem 2.15.

Construction 2.11. Let X be a Mori dream space, $f\colon X \to Y$ a contraction, and $D \subset X$ a prime divisor such that $f(D) \subsetneq Y$. Let us run a MMP for $-D$, relative to f (see Section 1A). We get a commutative diagram:

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & W \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f_W} & \downarrow{\scriptstyle j} \\ Y & \underset{k}{\longleftarrow} & T \end{array} \tag{2.12}$$

where W is $\mathbb{Q}$-factorial, ψ is a composition of D-positive flips and divisorial contractions (in particular D cannot be exceptional for ψ, so it has a proper transform D_W in W), and $f_W := f \circ \psi^{-1}$ is regular. Since $f(D) \subsetneq Y$, the MMP cannot end with a fiber type contraction, and $-D_W$ is f_W-nef. Let $j\colon W \to T$ be the contraction given by $\mathrm{NE}(f_W) \cap D_W^{\perp}$, so that f_W factors as in (2.12); there exists a $\mathbb{Q}$-Cartier prime divisor D_T in T such that $D_W = \lambda j^* D_T$ for some $\lambda \in \mathbb{Q}_{>0}$, and $-D_T$ is k-ample. We have the following properties:

(a) k is birational, $\mathrm{Exc}(k) \subseteq D_T$, $f(D) = k(D_T)$.

(b) f, f_W, and j coincide in the open subset $X \smallsetminus f^{-1}(f(D))$.

(c) The divisorial irreducible components of $f^{-1}(f(D))$ are exactly D and the prime exceptional divisors of ψ.

Proof. By construction ψ is a composition of D-positive flips and divisorial contractions (relative to f), hence the images under f of the exceptional divisors of ψ are all contained in $f(D)$, so these divisors must be divisorial irreducible components of $f^{-1}(f(D))$. On the other hand $k^{-1}(k(D_T)) = D_T$, so $f_W^{-1}(f(D)) = j^{-1}(D_T) = D_W$ is irreducible. □

(d) $f^{-1}(f(D))$ has $\rho_X - \rho_W + 1$ divisorial irreducible components.

(e) k is an isomorphism if and only if $f(D)$ is a $\mathbb{Q}$-Cartier prime divisor in Y.

Proof. The "only if" direction is clear, because D_T is $\mathbb{Q}$-Cartier and $f(D) = k(D_T)$. For the other, suppose that $f(D)$ is a $\mathbb{Q}$-Cartier prime divisor in Y. Since $k^{-1}(f(D)) = k^{-1}(k(D_T)) = D_T$, we must have $k^*(f(D)) = \mu D_T$, with $\mu \in \mathbb{Q}_{>0}$. Then $-D_T$ is both k-trivial and k-ample, so that k must be an isomorphism. □

(f) $\operatorname{Exc}(k)$ is a prime divisor if and only if $\operatorname{codim} f(D) > 1$.

(g) k is not an isomorphism and $\operatorname{codim}\operatorname{Exc}(k) > 1$ if and only if $f(D)$ is a non-$\mathbb{Q}$-Cartier prime divisor.

Proposition 2.13. *Let X be a Mori dream space and $f\colon X \dashrightarrow Y$ a rational contraction of fiber type. Then f can be factored as follows*:

$$\begin{array}{ccccc} & & \overset{f}{\overbrace{}} & & \\ X & \overset{}{\underset{g}{\dashrightarrow}} & Z & \underset{h}{\longrightarrow} & Y \end{array}$$

where g is a special rational contraction, and h is birational. Moreover, such a factorization is unique up to composition with a SQM of Z.

Proof. To show existence of the factorization, we proceed by induction on $\rho_X - \rho_Y$.

If $\rho_X - \rho_Y = 1$, then f is elementary and hence special, so the statement holds with $g = f$ and $h = \mathrm{Id}_Y$.

For the general case, up to composing with a SQM of X, we can assume that f is regular. If f is special, then as before the statement holds with $g = f$. Otherwise, there exists a prime divisor D in X such that $f(D) \subsetneq Y$ and $f(D)$ is not a $\mathbb{Q}$-Cartier divisor in Y.

We apply Construction 2.11 to f and D. We get a diagram as (2.12), where k is not an isomorphism by (e), because $f(D)$ is not a $\mathbb{Q}$-Cartier divisor in Y; in particular $\rho_T > \rho_Y$.

The composition $\tilde{f} := j \circ \psi\colon X \dashrightarrow T$ is a rational contraction of fiber type with $\rho_X - \rho_T < \rho_X - \rho_Y$; by the induction assumption, $\tilde{f}$ can be factored as follows:

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & Z \\ {\scriptstyle f}\downarrow & \overset{\tilde{f}}{\searrow} & \downarrow{\scriptstyle \tilde{h}} \\ Y & \underset{k}{\longleftarrow} & T \end{array}$$

where g is a special rational contraction of fiber type, and $\tilde{h}$ is birational. Then $h := k \circ \tilde{h}\colon Z \to Y$ is birational, so we have a factorization as in the statement.

To show uniqueness, suppose that f has another factorization $X \overset{g'}{\dashrightarrow} Z' \overset{h'}{\longrightarrow} Y$ with g' special and h' birational; notice that both Z and Z' are $\mathbb{Q}$-factorial by Remark 2.9. We show that the birational map $\varphi := (h')^{-1} \circ h \colon Z \dashrightarrow Z'$ is a SQM.

Let $B \subset Z$ be a prime divisor. Up to composing g and g' with a SQM of X, we can assume that $g' \colon X \to Z'$ is regular. Let $D \subset X$ be a prime divisor dominating B under g; then $g'(D) \subsetneq Z'$, and since g' is special, $B' := g'(D)$ is a prime divisor in Z'. This means that φ does not contract B. Similarly, we see that φ^{-1} does not contract divisors, hence φ is a SQM. □

Proof of Proposition 2.10. Suppose that f is not special, and consider the factorization of f given by Proposition 2.13. Then h cannot be an isomorphism, thus $\rho_Z > \rho_Y$.

Conversely, suppose that f has a factorization as in the statement. By applying Proposition 2.13 to g, we get a factorization of f as follows:

$$X \overset{g'}{\dashrightarrow} Z' \overset{h'}{\longrightarrow} Z \overset{h}{\dashrightarrow} Y, \qquad f \colon X \dashrightarrow Y$$

where g' is special and h' is birational. Thus $h \circ h'$ is birational with $\rho_{Z'} > \rho_Y$; by the uniqueness part of Proposition 2.13, f is not special. □

Notation 2.14. Let X be a Mori dream space and $f \colon X \to Y$ a special contraction; recall that Y is $\mathbb{Q}$-factorial by Remark 2.4. If B is a prime divisor in Y, then every irreducible component of f^*B must dominate B. As the general fiber of f is irreducible, there are at most finitely many prime divisors in Y whose pullback to X is reducible. We fix the notation $B_1, \dots, B_m$ for these divisors in Y, where $m \in \mathbb{Z}_{\geq 0}$, and we denote by $r_i \in \mathbb{Z}_{\geq 2}$ the number of irreducible components of f^*B_i, for $i = 1, \dots, m$ (we ignore the multiplicities of these components, and ignore the possible prime divisors B such that f^*B is irreducible but nonreduced). Note that by Proposition 2.2, f is quasielementary if and only if $m = 0$.

Given a special rational contraction $f \colon X \dashrightarrow Y$, we will use the same notation $B_1, \dots, B_m$ and $r_1, \dots, r_m$, with the obvious meaning.

Theorem 2.15. *Let X be a Mori dream space and $f \colon X \to Y$ a special contraction; we use Notation 2.14. Let E be the union of (arbitrarily chosen) $r_i - 1$ components of f^*B_i, for $i = 1, \dots, m$. Then there is a factorization*

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X' \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f'} & \\ Y & & \end{array}$$

where X' is projective, normal, and $\mathbb{Q}$-factorial, g is birational with $\operatorname{Exc}(g) = E$,[1] *the general fiber of f is contained in the open subset where g is an isomorphism, and f' is quasielementary.*

[1] We denote by $\operatorname{Exc}(g)$ the closure in X of the exceptional locus of g in its domain.

Proof. We proceed by induction on $\rho_X - \rho_Y$. If f is elementary, then it is quasielementary, so $E = \varnothing$ and the statement holds with $X' = X$ and $f' = f$.

Let us consider the general case. If f is quasielementary, then again the statement holds with $f' = f$.

Suppose that f is not quasielementary, so that $m \geq 1$ by Proposition 2.2, and consider the divisor $B_1 \subset Y$. Let D be the irreducible component of f^*B_1 not contained in E; we have $f(D) = B_1$ because f is special. We apply Construction 2.11 to f and D, and get a diagram:

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & W \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f_W} & \\ Y & & \end{array}$$

where W is $\mathbb{Q}$-factorial, ψ is a sequence of D-positive flips and divisorial contractions, relative to f, and the general fiber of f is contained in the open subset where ψ is an isomorphism (by (b)). Moreover $f_W^*B_1$ is irreducible (by (e)), and the exceptional divisors of ψ are all the components of f^*B_1 except D (by (c)). In particular, $r_1 - 1 \geq 1$ elementary divisorial contractions occur in ψ, so $\rho_W < \rho_X$. Clearly f_W is still special, and we conclude by applying the induction assumption to f_W. □

In particular, given a special contraction $f : X \to Y$ with general fiber F, one can bound ρ_X in terms of ρ_Y, ρ_F, and the number of irreducible components of f^*B_i, $i = 1, \dots, m$.

Corollary 2.16. *Let X be a Mori dream space, $f : X \to Y$ a special contraction, and $F \subset X$ a general fiber of f. We use Notation 2.14. Then*

$$\rho_X = \rho_Y + \dim \mathcal{N}_1(F, X) + \sum_{i=1}^{m} (r_i - 1) \leq \rho_Y + \rho_F + \sum_{i=1}^{m} (r_i - 1).$$

For the proof of Corollary 2.16 we need the following simple property.

Lemma 2.17. *Let $\varphi : X \dashrightarrow X'$ be a birational map between normal and $\mathbb{Q}$-factorial projective varieties. Let $T \subset X$ be a closed subset contained in the open subset where φ is an isomorphism, and set $T' := \varphi(T) \subset X'$. Then* $\dim \mathcal{N}_1(T, X) = \dim \mathcal{N}_1(T', X')$.

Proof. We note that $\mathcal{N}_1(T, X)$ is the quotient of the vector space of real 1-cycles in T by the subspace of 1-cycles γ such that $\gamma \cdot D = 0$ for every divisor D in X, so it is determined by the image of the restriction map $\mathcal{N}^1(X) \to \mathcal{N}^1(T)$, and similarly for $\mathcal{N}_1(T', X')$. Since X and X' are $\mathbb{Q}$-factorial, and T is contained in the open subset where φ is an isomorphism, it is easy to see that the images of the maps $\mathcal{N}^1(X) \to \mathcal{N}^1(T)$ and $\mathcal{N}^1(X') \to \mathcal{N}^1(T')$ are the same, under the natural isomorphism $\mathcal{N}^1(T) \cong \mathcal{N}^1(T')$. □

Proof of Corollary 2.16. Let us consider the factorization of f given by Theorem 2.15. The difference $\rho_X - \rho_{X'}$ is the number of prime exceptional divisors of g, namely $\sum_{i=1}^{m} (r_i - 1)$. Moreover F is contained in the open subset where g is an isomorphism, $g(F) \subset X'$ is a general fiber of f', and

$\dim \mathcal{N}_1(F, X) = \dim \mathcal{N}_1(g(F), X')$ by Lemma 2.17. Finally, since f' is quasielementary, we have $\rho_{X'} = \rho_Y + \dim \mathcal{N}_1(g(F), X')$. This yields the statement. □

Corollary 2.18. *Let X be a Mori dream space and $f\colon X \to Y$ a special contraction; we use Notation 2.14. Then every prime divisor in f^*B_i is a fixed divisor, for $i = 1, \dots, m$.*

Moreover, let E be the union of (arbitrarily chosen) $r_i - 1$ components of f^*B_i, for $i = 1, \dots, m$. Then the classes of the components of E in $\mathcal{N}^1(X)$ generate a simplicial face σ of $\mathrm{Eff}(X)$, and $\sigma \cap \mathrm{Mov}(X) = \{0\}$.

Proof. Theorem 2.15 implies the existence of a contracting birational map $g\colon X \dashrightarrow X'$, with X' $\mathbb{Q}$-factorial, whose prime exceptional divisors are precisely the components of E. This gives the statement (see for instance [Okawa 2016, Lemma 2.7]). □

We will also need the following technical property.

Lemma 2.19. *Let X be a Mori dream space and $f\colon X \dashrightarrow Y$ a special rational contraction; we use Notation 2.14. Let E_0 be an irreducible component of f^*B_i for some $i \in \{1, \dots, m\}$. Then there is a factorization of f:*

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & \hat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma} \\ Y & \longleftarrow & Z \end{array}$$

where φ is a SQM, σ is an elementary divisorial contraction, $\mathrm{Exc}(\sigma)$ is the transform of E_0, and $\dim \sigma(\mathrm{Exc}\,\sigma) \geq \dim Y - 1$.

Proof. Let us choose a SQM $\psi\colon X \dashrightarrow X'$ such that $f' := f \circ \psi^{-1}\colon X' \to Y$ is regular.

We still denote by E_0 the transform of E_0 in X'; by Corollary 2.18, E_0 is a fixed divisor, and it is easy to see that it cannot be f'-nef. We run a MMP in X' for E_0, relative to f', and get a diagram:

$$\begin{array}{ccccc} X & \overset{\psi}{\dashrightarrow} & X' & \overset{\xi}{\dashrightarrow} & \hat{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle \sigma} \\ & & Y & \overset{h}{\longleftarrow} & Z \end{array}$$

where ξ is a sequence of E_0-negative flips, and σ is an elementary divisorial contraction with exceptional divisor (the transform of) E_0.

Now $h \circ \sigma\colon \hat{X} \to Y$ is a special contraction, therefore $h(\sigma(\mathrm{Exc}(\sigma)))$ is a divisor in Y, and $\dim \sigma(\mathrm{Exc}(\sigma)) \geq \dim Y - 1$. □

2D. *Singularities of the target.*

The goal of this subsection is to prove the following result.

Proposition 2.20. *Let X be a smooth projective variety, and Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $f\colon X \to Y$ be a $(K+\Delta)$-negative special contraction of fiber type. Then Y has locally factorial, canonical singularities, and is nonsingular in codimension* 2.

Proposition 2.20 will follow from some technical lemmas.

Lemma 2.21. *Let X be a projective variety with locally factorial, canonical singularities, and Δ a boundary such that (X, Δ) is klt. Let $f\colon X \to Y$ be a $(K+\Delta)$-negative special contraction of fiber type. Then Y has locally factorial, canonical singularities.*

Proof. It follows from [Fujino 1999, Corollary 4.5] that Y has rational singularities, so it is enough to show that it is locally factorial [Kollár and Mori 1998, Corollary 5.24].

Let B be a prime divisor in Y. Since Y is $\mathbb{Q}$-factorial, there exists $m \in \mathbb{Z}_{>0}$ such that mB is Cartier. Set $U := f^{-1}(Y_{\mathrm{reg}})$; since Y is normal and f is special, we have

$$\operatorname{codim} \operatorname{Sing}(Y) \geq 2 \quad \text{and} \quad \operatorname{codim}(X \setminus U) \geq 2.$$

Then $B \cap Y_{\mathrm{reg}}$ is a Cartier divisor on Y_{reg}, and $f_{|U}^*(B \cap Y_{\mathrm{reg}})$ is a Cartier divisor on U. Since X is locally factorial, there exists a Cartier divisor D in X such that $D_{|U} = f_{|U}^*(B \cap Y_{\mathrm{reg}})$. Then $(mD)_{|U} = f_{|U}^*((mB)_{|Y_{\mathrm{reg}}}) = f^*(mB)_{|U}$, and hence $mD = f^*(mB)$.

We deduce that $D \cdot C = 0$ for every curve $C \subset X$ contracted by f. Since f is $(K+\Delta)$-negative, this implies that there exists a Cartier divisor B' on Y such that $D = f^*B'$ [Kollár and Mori 1998, Theorem 3.7(4)]. Thus we have $B'_{|Y_{\mathrm{reg}}} = B \cap Y_{\mathrm{reg}}$, and hence $B = B'$ is Cartier. □

The following two lemmas are basically [Andreatta et al. 1992, Proposition 1.4 and 1.4.1], where they are attributed to Fujita.

Lemma 2.22. *Let X be a smooth projective variety, and Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $f\colon X \to Y$ be an equidimensional, $(K+\Delta)$-negative contraction of fiber type. If Y has at most finite quotient singularities, then Y is smooth.*

Proof. Let $F \subset X$ be a general fiber of f. Then F is smooth and $(F, \Delta_{|F})$ is klt [Kollár and Mori 1998, Lemma 5.17]; moreover $-(K_F + \Delta_{|F}) \equiv -(K_X + \Delta)_{|F}$ is ample, so that $(F, \Delta_{|F})$ is log Fano. By Kawamata–Viehweg vanishing, $h^i(F, \mathcal{O}_F) = 0$ for every $i > 0$, hence $\chi(F, \mathcal{O}_F) = 1$. Then the same proof as [Andreatta et al. 1992, Proposition 1.4] applies. □

Lemma 2.23. *Let X be a smooth projective variety with* $\dim X \geq 3$, *and Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $f\colon X \to S$ be an equidimensional, $(K+\Delta)$-negative contraction onto a surface. Then S is smooth.*

Proof. Notice first of all that S is $\mathbb{Q}$-factorial by Lemma 2.7. Moreover, by [Fujino 1999, Corollary 4.5], there exists $\mathbb{Q}$-divisor Δ' on S such that (S, Δ') is klt; in particular S has log terminal singularities, and hence finite quotient singularities [Kollár and Mori 1998, Proposition 4.18]. Then S is smooth by Lemma 2.22. □

Lemma 2.24. *Let X be a smooth projective variety, Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt, and $f\colon X \to Y$ a $(K+\Delta)$-negative contraction of fiber type.*

Suppose that the locus where f is not equidimensional has codimension at least 3 *in Y, equivalently that there is no prime divisor $D \subset X$ such that* $\operatorname{codim} f(D) = 2$.

Then Y is smooth in codimension 2.

Proof. Set $m = \dim Y$ and let $H_1, \ldots, H_{m-2}$ be general very ample divisors in Y. Consider $S := H_1 \cap \cdots \cap H_{m-2}$ and $Z := f^{-1}(S) = f^*H_1 \cap \cdots f^*H_{m-2}$. Then S is a normal projective surface, Z is smooth, and f is equidimensional over S, so that $f_Z := f_{|Z} \colon Z \to S$ is an equidimensional contraction. Moreover $(Z, \Delta_{|Z})$ is klt [Kollár and Mori 1998, Lemma 5.17].

Let $C \subset Z$ be a curve contracted by f; then $f^*H_i \cdot C = 0$ for every i, so that by adjunction

$$(K_Z + \Delta_{|Z}) \cdot C = (K_X + \Delta) \cdot C < 0,$$

and f_Z is $(K_Z + \Delta_{|Z})$-negative. Thus S is smooth by Lemma 2.23, so $S \subseteq Y_{\text{reg}}$ and hence $\operatorname{codim} \operatorname{Sing} Y \geq 3$. □

Proposition 2.20 follows from Lemma 2.21, Remark 2.6(b)enumi, and Lemma 2.24.

3. Special contractions of Fano varieties of relative dimension 1

3A. ***Preliminaries on the Lefschetz defect.*** Let X be a normal and $\mathbb{Q}$-factorial Fano variety. The *Lefschetz defect* δ_X is an invariant of X, introduced in [Casagrande 2012], and defined as follows:

$$\delta_X = \max\{\operatorname{codim} \mathcal{N}_1(D, X) \mid D \text{ a prime divisor in } X\}$$

(see Section 1A for $\mathcal{N}_1(D, X)$). The main properties of δ_X are the following.

Theorem 3.1 [Casagrande 2012; Della Noce 2014]. *Let X be a $\mathbb{Q}$-factorial, Gorenstein Fano variety, with canonical singularities and at most finitely many nonterminal points. Then $\delta_X \leq 8$.*

If moreover X is smooth and $\delta_X \geq 4$, then $X \cong S \times Y$, where S is a surface.

Theorem 3.2 [Casagrande 2012, Corollary 1.3; 2013b, Theorem 1.2]. *Let X be a smooth Fano 4-fold. Then one of the following holds*:

(i) *X is a product of surfaces.*

(ii) $\delta_X = 3$ *and* $\rho_X \leq 6$.

(iii) $\delta_X = 2$ *and* $\rho_X \leq 12$.

(iv) $\delta_X \leq 1$.

3B. ***The case of relative dimension one.*** In this subsection we show that if X is a Fano variety and $f \colon X \to Y$ is a special contraction with $\dim Y = \dim X - 1$, then $\rho_X - \rho_Y \leq 9$; this is a generalization of an analogous result in [Romano 2019] in the case where f is a conic bundle. The strategy of proof is the same: we use f to produce $\rho_X - \rho_Y - 1$ pairwise disjoint divisors in X, and then we use them to show that if $\rho_X - \rho_Y \geq 3$, then $\delta_X \geq \rho_X - \rho_Y - 1$; finally we apply Theorem 3.1.

Proposition 3.3. *Let X be a $\mathbb{Q}$-factorial, Gorenstein Fano variety, with canonical singularities and at most finitely many nonterminal points. Let $f\colon X \to Y$ be a special contraction with* $\dim Y = \dim X - 1$. *Then the following hold*:

(a) $\rho_X - \rho_Y \leq 9$.

(b) *If* $\rho_X - \rho_Y \geq 3$, *then* $\delta_X \geq \rho_X - \rho_Y - 1$.

If moreover X is smooth and $\rho_X - \rho_Y \geq 5$, then there exists a surface S such that $X \cong S \times Z$, $Y \cong \mathbb{P}^1 \times Z$, and f is induced by a conic bundle $S \to \mathbb{P}^1$.

For the proof of Proposition 3.3 we need some technical lemmas, that will be used also in Section 6.

Lemma 3.4. *Let X be a Mori dream space, and suppose that K_X is Cartier in codimension* 2, *namely that there exists a closed subset $T \subset X$ such that* $\operatorname{codim} T \geq 3$ *and $K_{X\smallsetminus T}$ is Cartier.*

Let $f\colon X \to Y$ be a K-negative special contraction with $\dim Y = \dim X - 1$; *we use Notation 2.14. Then $\rho_X = \rho_Y + 1 + m$ and $r_i = 2$ for every $i = 1, \dots, m$.*

Let moreover E_i, $\hat{E}_i$ be the irreducible components of f^*B_i. Then the general fiber of f over B_i is $e_i + \hat{e}_i$, where e_i and $\hat{e}_i$ are integral curves with $E_i \cdot e_i < 0$, $\hat{E}_i \cdot \hat{e}_i < 0$, and $-K_X \cdot e_i = -K_X \cdot \hat{e}_1 = 1$.

Proof. Fix $i \in \{1, \dots, m\}$. The closed subset T cannot dominate B_i, hence the general fiber of f over B_i is a curve F_i contained in $X \setminus T$ where K_X is Cartier. Since $-K_X \cdot F_i = 2$, and f is K-negative, F_i has at most two irreducible components. This implies that $r_i = 2$ and $F_i = e_i + \hat{e}_i$, with $e_i \subset E_i$, $\hat{e}_i \subset \hat{E}_i$, and conversely $e_i \not\subset \hat{E}_i$, $\hat{e}_i \not\subset E_i$. The fiber F_i is connected, hence we have $E_i \cap \hat{e}_i \neq \emptyset$, and therefore $E_i \cdot \hat{e}_i > 0$. Since $E_i \cdot F_i = 0$, we get $E_i \cdot e_i < 0$; similarly for $\hat{E}_i$. Finally $\rho_X = \rho_Y + 1 + m$ by Corollary 2.16. □

Lemma 3.5. *In the setting of Lemma 3.4, if moreover* $\operatorname{codim} T \geq 4$, *then $B_1, \dots, B_m$ are pairwise disjoint.*

Proof. By contradiction, suppose that $B_1 \cap B_2 \neq \emptyset$. Then $B_1 \cap B_2$ has pure dimension $\dim X - 3$, because Y is $\mathbb{Q}$-factorial (see Remark 2.4); let W be an irreducible component. Since f is special, the general fiber F_W of f over W is a curve. Moreover, F_W is contained in the open subset where K_X is Cartier, so that $F_W = C + C'$ with C and C' integral curves of anticanonical degree 1.

By Lemma 3.4, for $i = 1, 2$ the general fiber F_i of f over B_i is $e_i + \hat{e}_i$, with $-K_X \cdot e_i = 1$, and F_i degenerates to F_W. Thus, up to switching the components, we can assume that both e_1 and e_2 are numerically equivalent to C, which implies that $e_1 \equiv e_2$. This is impossible, because $E_1 \neq E_2$, $E_i \cdot e_i < 0$, and e_i moves in a family of curves dominating E_i, for $i = 1, 2$. □

Proof of Proposition 3.3. This the same as the proof of [Romano 2019, Theorem 1.1 and 1.3], so we give only a sketch. We have $\rho_X = \rho_Y + 1 + m$ by Lemma 3.4. As in [loc. cit., Lemmas 3.9 and 3.10], using Lemma 3.5, one sees that if $m \geq 2$, then $\delta_X \geq m$. Hence the statement follows from Theorem 3.1. □

4. Preliminary results on Fano 4-folds

From now on, we focus on smooth Fano 4-folds. After giving in Section 4A some preliminary results on rational contractions of Fano 4-folds, in Section 4B we recall the classification of fixed prime divisors in a Fano 4-fold X with $\rho_X \geq 7$, and report some properties that will be crucial in the sequel. Then in Section 4C we apply the previous results to study special rational contractions of fiber type of X, when $\rho_X \geq 7$.

4A. *Rational contractions of Fano 4-folds.*

Lemma 4.1 [Casagrande 2013a, Remark 3.6 and its proof]. *Let X be a smooth Fano 4-fold and $\varphi\colon X \dashrightarrow \tilde{X}$ an SQM*:

(a) *$\tilde{X}$ is smooth, the indeterminacy locus of φ is a disjoint union of exceptional planes (see Section 1A), and the indeterminacy locus of φ^{-1} is a disjoint union of exceptional lines.*

(b) *An exceptional line in $\tilde{X}$ cannot meet any integral curve of anticanonical degree 1, in particular it cannot meet an exceptional plane.*

(c) *Let $\psi : \tilde{X} \dashrightarrow \hat{X}$ be a SQM that factors as a sequence of K-negative flips. Then the indeterminacy locus of ψ (respectively, ψ^{-1}) is a disjoint union of exceptional planes (respectively, lines).*

Lemma 4.2 [Casagrande 2013a, Remark 3.7]. *Let X be a smooth Fano 4-fold and $f : X \dashrightarrow Y$ a rational contraction. Then one can factor f as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} Y$, where φ is a SQM, X' is smooth, and f' is a K-negative contraction.*

These results allow to conclude that the target of a special rational contraction of a Fano 4-fold has mild singularities.

Lemma 4.3. *Let X be a smooth Fano 4-fold and $f : X \dashrightarrow Y$ a special rational contraction. If* $\dim Y = 2$, *then Y is smooth. If* $\dim Y = 3$, *then Y has isolated locally factorial, canonical singularities.*

Proof. By Lemma 4.2 we can factor f as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} Y$ where φ is a SQM, X' is smooth, and f' is regular, K-negative, and special. Then the statement follows from Proposition 2.20. □

4B. *Fixed prime divisors in Fano 4-folds with ρ ≥ 7.*

Let X be a Fano 4-fold with $\rho_X \geq 7$. Fixed prime divisors in X have been classified in [Casagrande 2013a; 2017] in four types, and have many properties; this explicit information on the geometry of fixed divisors is a key ingredient in the proof of Theorem 1.1. In this subsection we recall this classification, and show some properties that will be used in the sequel.

Theorem–Definition 4.4 [Casagrande 2017, Theorem 5.1, Definition 5.3, Corollary 5.26, Definition 5.27]. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, and D a fixed prime divisor in X. The following hold*:

(a) *Given a SQM $X \dashrightarrow X'$ and an elementary divisorial contraction $k\colon X' \to Y$ with* $\mathrm{Exc}(k)$ *the transform of D, then k is of type* $(3,0)^{\mathrm{sm}}$, $(3,0)^{Q}$, $(3,1)^{\mathrm{sm}}$, *or* $(3,2)$.

(b) *The type of k depends only on D, so we define D to be of type* $(3,0)^{\mathrm{sm}}$, $(3,0)^{Q}$, $(3,1)^{\mathrm{sm}}$, *or* $(3,2)$, *respectively.*

(c) *If D is of type* $(3,2)$, *then D is the exceptional divisor of an elementary divisorial contraction of X, of type* $(3,2)$.

(d) *We define $C_D \subset D \subset X$ to be the transform of a general irreducible curve $\Gamma \subset X'$ contracted by k, of minimal anticanonical degree; the curve C_D depends only on D.*

(e) *$C_D \cong \mathbb{P}^1$, $D \cdot C_D = -1$, C_D is contained in the open subset where the birational map $X \dashrightarrow X'$ is an isomorphism, and C_D moves in a family of curves dominating D.*

(f) *Let $\varphi\colon X \dashrightarrow \tilde{X}$ be a SQM, and E a fixed prime divisor in $\tilde{X}$. We define the type of E to be the type of its transform in X.*

We will frequently use the notation $C_D \subset D$ introduced in the Theorem–Definition above.

The next property of fixed divisors of type $(3,2)$ will be crucial in the sequel.

Lemma 4.5. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, $X \dashrightarrow \tilde{X}$ a SQM, and $D \subset \tilde{X}$ a fixed divisor of type* $(3,2)$. *If $\mathcal{N}_1(D, \tilde{X}) \subsetneq \mathcal{N}_1(\tilde{X})$, then either $\rho_X \leq 12$, or X is a product of surfaces.*

Proof. If $\delta_X \geq 2$, we have the statement by Theorem 3.2, so let us assume that $\delta_X \leq 1$. Let D_X be the transform of D in X, so that D_X is the exceptional divisor of an elementary divisorial contraction of X, of type $(3,2)$. By [Casagrande 2017, Remark 2.17(2)], D_X cannot contain exceptional planes, hence $\dim \mathcal{N}_1(D_X, X) = \dim \mathcal{N}_1(D, \tilde{X})$ by [Casagrande 2013a, Corollary 3.14]. Then $\rho_X \leq 12$ by [Casagrande 2017, Proposition 5.32]. □

Lemma 4.6. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, and let $D_1, D_2 \subset X$ be two distinct fixed prime divisors. We have the following:*

(a) *First*

$$\dim\langle [D_1], [D_2]\rangle \cap \mathrm{Mov}(X) = \dim\langle [C_{D_1}], [C_{D_2}]\rangle \cap \mathrm{mov}(X) = \begin{cases} 0 & \text{if } D_1 \cdot C_{D_2} = 0 \text{ or } D_2 \cdot C_{D_1} = 0, \\ 1 & \text{if } D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1, \\ 2 & \text{if } (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) \geq 2. \end{cases}$$

(b) *If $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1$, then*

$$\langle [D_1], [D_2]\rangle \cap \mathrm{Mov}(X) = \langle [D_1 + D_2]\rangle \quad \text{and} \quad \langle [C_{D_1}], [C_{D_2}]\rangle \cap \mathrm{mov}(X) = \langle [C_{D_1} + C_{D_2}]\rangle.$$

Moreover $(D_1 + D_2) \cdot (C_{D_1} + C_{D_2}) = 0$ and $D_1 + D_2$ is not big.

(c) *If $D_1 \cdot C_{D_2} = 0$ or $D_2 \cdot C_{D_1} = 0$, then $\langle [D_1], [D_2]\rangle$ is a face of* $\mathrm{Eff}(X)$, *and $\langle [C_{D_1}], [C_{D_2}]\rangle$ is a face of* $\mathrm{Mov}(X)^{\vee}$.

For the proof, we need the following elementary property in convex geometry.

Lemma 4.7. *Let σ be a convex polyhedral cone, of maximal dimension, in a finite dimensional real vector space $\mathcal{N}$. Let τ_1 be a one-dimensional face of σ, and let $\alpha \in \mathcal{N}^*$ (the dual vector space) be such that $\alpha \cdot \tau_1 < 0$ and $\alpha \cdot \eta \geq 0$ for every one-dimensional face $\eta \neq \tau_1$ of σ.*

If τ_2 is a one-dimensional face of σ such that $\alpha \cdot \tau_2 = 0$, then $\tau_1 + \tau_2$ is a face of σ.

Proof. Since τ_2 is a face of σ, there exists $\beta \in \mathcal{N}^*$ such that $\beta \cdot x \geq 0$ for every $x \in \sigma$, and $\beta^\perp \cap \sigma = \tau_2$. Let $y \in \tau_1$ be a nonzero element, and set $a := \alpha \cdot y$ and $b := \beta \cdot y$. Then $a, b \in \mathbb{R}$, $a < 0$, and $b > 0$ (because $\tau_2 \neq \tau_1$ by our assumptions). Let us consider $\gamma := b\alpha + |a|\beta \in \mathcal{N}^*$.

We have $\alpha \cdot \tau_2 = \beta \cdot \tau_2 = 0$, hence $\gamma \cdot \tau_2 = 0$. Moreover $\gamma \cdot y = b\alpha \cdot y + |a|\beta \cdot y = 0$, namely $\gamma \cdot \tau_1 = 0$. Finally if η is a one-dimensional face of σ, different from τ_1 and τ_2, we have $\alpha \cdot \eta \geq 0$, $\beta \cdot \eta > 0$, and hence $\gamma \cdot \eta > 0$.

Therefore $\gamma \cdot x \geq 0$ for every $x \in \sigma$, and $\gamma^\perp \cap \sigma = \tau_1 + \tau_2$. This shows the statement. □

Proof of Lemma 4.6. We compute $\langle [D_1], [D_2] \rangle \cap \mathrm{Mov}(X)$. Set $B := \lambda_1 D_1 + \lambda_2 D_2$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for $i = 1, 2$. By [Casagrande 2017, Lemma 5.29(2)], B is movable if and only if $B \cdot C_D \geq 0$ for every fixed prime divisor $D \subset X$, and this is equivalent to $B \cdot C_{D_i} \geq 0$ for $i = 1, 2$, namely to

$$\begin{cases} -\lambda_1 + \lambda_2 D_2 \cdot C_{D_1} \geq 0 \\ \lambda_1 D_1 \cdot C_{D_2} - \lambda_2 \geq 0. \end{cases} \tag{4.8}$$

Let $\mathcal{S} \subseteq (\mathbb{R}_{\geq 0})^2$ be the set of nonnegative solutions (λ_1, λ_2) of (4.8), so that $\mathcal{S}$ determines the intersection $\langle [D_1], [D_2] \rangle \cap \mathrm{Mov}(X)$. Notice that $(D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1})$ is always nonnegative, because $D_1 \neq D_2$. It is elementary to check that:

- $\mathcal{S} = \{(0, 0)\} \Leftrightarrow 1 - (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) > 0 \Leftrightarrow D_1 \cdot C_{D_2} = 0$ or $D_2 \cdot C_{D_1} = 0$.
- $\mathcal{S}$ is a half-line $\Leftrightarrow 1 - (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) = 0 \Leftrightarrow D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1$, moreover in this case $\mathcal{S} = \{(\lambda, \lambda) \mid \lambda \geq 0\}$.
- $\mathcal{S}$ is a 2-dimensional cone $\Leftrightarrow 1 - (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) < 0 \Leftrightarrow (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) \geq 2$.

Similarly, we compute $\langle [C_{D_1}], [C_{D_2}] \rangle \cap \mathrm{mov}(X)$. We have

$$\mathrm{mov}(X)^\vee = \mathrm{Eff}(X) = \langle [D] \rangle_{D \text{ fixed}} + \mathrm{Mov}(X).$$

Set $\gamma := \lambda_1 C_{D_1} + \lambda_2 C_{D_2}$ with $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. We have $\gamma \cdot M \geq 0$ for every movable divisor M in X (see [Casagrande 2017, Lemma 5.29(2)]). Hence $\gamma \in \mathrm{mov}(X)$ if and only if $\gamma \cdot D \geq 0$ for every fixed prime divisor $D \subset X$, and this is equivalent to $\gamma \cdot D_i \geq 0$ for $i = 1, 2$, namely to

$$\begin{cases} -\lambda_1 + \lambda_2 D_1 \cdot C_{D_2} 0 \\ \lambda_1 D_2 \cdot C_{D_1} - \lambda_2 0, \end{cases}$$

which is the same system as (4.8), but with λ_1 and λ_2 interchanged. Thus the previous discussion yields (a) and (b).

We show (c). Suppose for instance that $D_1 \cdot C_{D_2} = 0$. To see that $\langle[D_1], [D_2]\rangle$ is a face of $\mathrm{Eff}(X)$, we apply Lemma 4.7 with $\sigma = \mathrm{Eff}(X)$, $\tau_1 = \langle[D_2]\rangle$, $\alpha = [C_{D_2}]$, and $\tau_2 = \langle[D_1]\rangle$. It is enough to remark that $D \cdot C_{D_2} \geq 0$ for every prime divisor $D \neq D_2$.

Similarly, to see that $\langle[C_{D_1}], [C_{D_2}]\rangle$ is a face of $\mathrm{Mov}(X)^\vee$, we apply Lemma 4.7 with $\sigma = \mathrm{Mov}(X)^\vee$, $\tau_1 = \langle[C_{D_1}]\rangle$, $\alpha = [D_1]$, and $\tau_2 = \langle[C_{D_2}]\rangle$. Indeed $\langle[C_{D_1}]\rangle$ and $\langle[C_{D_2}]\rangle$ are one-dimensional faces of $\mathrm{Mov}(X)^\vee$ by [Casagrande 2017, Lemma 5.29(1)]. Moreover $D_1 \cdot \gamma \geq 0$ for every $\gamma \in \mathrm{mov}(X)$, and $D_1 \cdot C_D \geq 0$ for every fixed prime divisor $D \neq D_1$. By [loc. cit., Lemma 5.29(2)] we have

$$\mathrm{Mov}(X)^\vee = \langle[C_D]\rangle_{D\,\mathrm{fixed}} + \mathrm{mov}(X),$$

therefore $D_1 \cdot \eta \geq 0$ for every one-dimensional face η of $\mathrm{Mov}(X)^\vee$ different from $\langle[C_{D_1}]\rangle$. Thus the hypotheses of Lemma 4.7 are satisfied, and we get (c). ☐

Lemma 4.9. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, and let $D_1, D_2 \subset X$ be two distinct fixed prime divisors such that $\langle[D_1], [D_2]\rangle \cap \mathrm{Mov}(X) = \{0\}$. Then, up to exchanging D_1 and D_2, one of the following holds*:

(a) $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 0$ *and* $D_1 \cap D_2 = \varnothing$.

(b) $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 0$ *and $D_1 \cap D_2$ is a disjoint union of exceptional planes.*

(c) $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 0$, *$D_1$ is of type $(3, 2)$, and D_2 is not of type $(3, 0)^{\mathrm{sm}}$.*

(d) $D_1 \cdot C_{D_2} > 0$, $D_2 \cdot C_{D_1} = 0$, *D_1 is of type $(3, 2)$, and D_2 is of type $(3, 1)^{\mathrm{sm}}$ or $(3, 0)^Q$.*

Proof. By [Casagrande 2017, Theorem 5.1] there is a diagram

$$X \dashrightarrow \tilde{X} \xrightarrow{f} Y$$

where the first map is a SQM and f is an elementary divisorial contraction with exceptional divisor the transform $\tilde{D}_2 \subset \tilde{X}$ of D_2. Let $\tilde{D}_1 \subset \tilde{X}$ be the transform of D_1. By [loc. cit., Lemma 2.21], D_1 is the transform of a fixed prime divisor $B_1 \subset Y$.

If $\tilde{D}_1 \cap \tilde{D}_2 = \varnothing$, then $D_1 \cap D_2$ is contained in the indeterminacy locus of the map $X \dashrightarrow \tilde{X}$, which is a disjoint union of exceptional planes by Lemma 4.1(a). Therefore either $D_1 \cap D_2 = \varnothing$ and we get (a), or $D_1 \cap D_2$ has pure dimension 2 and we get (b).

We assume from now on that $\tilde{D}_1 \cap \tilde{D}_2 \neq \varnothing$.

Suppose that D_2 is of type $(3, 1)^{\mathrm{sm}}$. Then Y is a smooth Fano 4-fold by [Casagrande 2017, Theorem 5.1], f is the blow-up of a smooth curve $C \subset Y$, and $B_1 \cap C \neq \varnothing$. Then [loc. cit., Lemma 5.11] yields that B_1 is the exceptional divisor of an elementary divisorial contraction of type $(3, 2)$, and either $B_1 \cdot C > 0$, or $B_1 \cdot C < 0$. Thus B_1 is generically a $\mathbb{P}^1$-bundle over a surface, and the general fiber F of this $\mathbb{P}^1$-bundle satisfies $B_1 \cdot F = K_Y \cdot F = -1$. Using Lemma 4.1(a) and [loc. cit., Lemma 2.18], one sees that D_1 must be of type $(3, 2)$. Moreover $C \cap F = \varnothing$ implies that $\tilde{D}_2$ is disjoint from the transform $\tilde{F}$ of F in $\tilde{X}$, and $\tilde{D}_1$ is still generically a $\mathbb{P}^1$-bundle with fiber $\tilde{F}$. The indeterminacy locus of the map $\tilde{X} \dashrightarrow X$ has dimension at most one (see Lemma 4.1(a)), hence $\tilde{F}$ is contained in the open subset where this map is an isomorphism, and in X we get $D_2 \cdot C_{D_1} = \tilde{D}_2 \cdot \tilde{F} = 0$. Finally it is easy to check that $D_1 \cdot C_{D_2} = 0$ if $B_1 \cdot C > 0$ (and we have (c)), while $D_1 \cdot C_{D_2} > 0$ if $B_1 \cdot C < 0$ (and we have (d)). So we get the statement.

We can assume now that neither D_1 nor D_2 are of type $(3,1)^{\mathrm{sm}}$. Suppose that D_2 is of type $(3,0)^{\mathrm{sm}}$ or $(3,0)^Q$. Then $\tilde{D}_2$ is isomorphic to $\mathbb{P}^3$ or to an irreducible quadric; let $\Gamma \subset \tilde{D}_2$ be a curve corresponding to a line. We have $\tilde{D}_1 \cdot \Gamma > 0$, and since Γ is contained in the open subset where the map $\tilde{X} \dashrightarrow X$ is an isomorphism (see Theorem–Definition 4.4(e)), we also have $D_1 \cdot C_{D_2} > 0$. This yields $D_2 \cdot C_{D_1} = 0$ by Lemma 4.6. Therefore D_1 cannot be of type $(3,0)^{\mathrm{sm}}$ nor $(3,0)^Q$, and the only possibility is that D_1 is of type $(3,2)$. Moreover, since $f(\tilde{D}_2)$ is contained in B_1, [Casagrande 2017, Lemma 5.41] yields that D_2 cannot be of type $(3,0)^{\mathrm{sm}}$, so we get again (d).

We are left with the case where both D_1 and D_2 are of type $(3,2)$, and we can assume that $D_1 \cdot C_{D_2} = 0$ by Lemma 4.6. If $\delta_X \geq 3$, then Theorem 3.2 implies that X is a product of surfaces; in this case it is easy to check directly that $D_2 \cdot C_{D_1} = 0$. If $\delta_X \leq 2$, then we get $D_2 \cdot C_{D_1} = 0$ by [Casagrande 2013b, Lemma 2.2(b)]. So we have (c). □

4C. *Special rational contractions of Fano 4-folds with $\rho_X \geq 7$.* Given a Fano 4-fold X with $\rho_X \geq 7$, and a special rational contraction of fiber type $f\colon X \dashrightarrow Y$, in this subsection we show that, for every prime divisor B of Y, f^*B has at most two irreducible components. Moreover we give conditions on the type of the fixed prime divisors in f^*B, when f^*B is reducible.

Lemma 4.10. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, and $f\colon X \dashrightarrow Y$ a special rational contraction; we use Notation 2.14. Let $i \in \{1, \dots, m\}$:*

- *If* $\dim Y = 3$, *then every fixed divisor in f^*B_i is of type* $(3,2)$.
- *If* $\dim Y = 2$, *then every fixed divisor in f^*B_i is of type* $(3,2)$ *or* $(3,1)^{\mathrm{sm}}$.

Proof. Let E_0 be an irreducible component of f^*B_i. By Lemma 2.19 there are a SQM $X \dashrightarrow \tilde{X}$ and an elementary divisorial contraction $\sigma\colon \tilde{X} \to Z$ such that $\mathrm{Exc}(\sigma)$ is the transform of E_0, and $\dim \sigma(\mathrm{Exc}(\sigma)) \geq \dim Y - 1$. Theorem–Definition 4.4 yields the statement. □

Lemma 4.11. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, and $f\colon X \dashrightarrow Y$ a special rational contraction; we use Notation 2.14. Then $r_i = 2$ for every $i = 1, \dots, m$.*

Proof. We consider for simplicity $i = 1$.

Claim. *For every irreducible component D of f^*B_1, there exists another component E of f^*B_1 such that $E \cdot C_D > 0$.*

Let us first show that the Claim implies the statement. Assume by contradiction that $r_1 > 2$, and let us consider a component D_1 of f^*B_1. By the Claim, there exists a second component D_2 with $D_2 \cdot C_{D_1} > 0$, and since $r_1 \geq 3$, we have $\langle [D_1], [D_2] \rangle \cap \mathrm{Mov}(X) = \{0\}$ by Corollary 2.18. Applying Lemma 4.9, we conclude that D_1 is not of type $(3,2)$, and D_2 is of type $(3,2)$.

Now we restart with D_2, and we deduce that D_2 is not of type $(3,2)$, a contradiction. Hence $r_1 = 2$.

We prove the Claim. By Lemma 2.19, there exists a diagram:

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & \tilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma} \\ Y & \underset{g}{\longleftarrow} & Z \end{array}$$

where φ is a SQM and σ is an elementary divisorial contraction with $\mathrm{Exc}(\sigma) = \tilde{D}$, the transform of D in $\tilde{X}$.

Since $g \circ \sigma$ is special, we have $g(\sigma(\tilde{D})) = B_1$ and hence $\sigma(\tilde{D}) \subset g^{-1}(B_1)$; let $E_Z \subset Z$ be an irreducible component of $g^{-1}(B_1)$ containing $\sigma(\tilde{D})$. Let $\tilde{E} \subset \tilde{X}$ and $E \subset X$ be the transforms of E_Z, so that E is an irreducible component of f^*B_1. Note that $\tilde{E} \cdot \mathrm{NE}(\sigma) > 0$ by construction.

Now let $\Gamma \subset \tilde{D}$ be a general minimal irreducible curve contracted by σ; by Theorem–Definition 4.4(d) and (e), the transform of Γ in X is the curve C_D, and Γ is contained in the open subset where $\varphi^{-1} \colon \tilde{X} \dashrightarrow X$ is an isomorphism. Therefore $E \cdot C_D = \tilde{E} \cdot \Gamma > 0$. □

5. Fano 4-folds to surfaces

In this section we study rational contractions from a Fano 4-fold to a surface, and show the following.

Theorem 5.1. *Let X be a smooth Fano 4-fold having a rational contraction $f \colon X \dashrightarrow S$ with $\dim S = 2$. Then one of the following holds*:

(i) *X is a product of surfaces.*

(ii) $\rho_X \leq 12$.

(iii) $13 \leq \rho_X \leq 17$, *S is a smooth del Pezzo surface, the general fiber F of f is a smooth del Pezzo surface with* $4 \leq \dim \mathcal{N}_1(F, X) \leq \rho_F \leq 8$, *and* $\rho_X \leq 9 + \dim \mathcal{N}_1(F, X)$.

(iv) $S \cong \mathbb{P}^2$ *and f is special.*

Lemma 5.2. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, and $f \colon X \dashrightarrow S$ a special rational contraction with* $\dim S = 2$; *we use Notation 2.14. Then for every $i = 1, \dots, m$ the divisor f^*B_i has two irreducible components*, *one a fixed divisor of type* $(3, 2)$, *and the other one of type* $(3, 2)$ *or* $(3, 1)^{\mathrm{sm}}$.

Proof. We consider for simplicity $i = 1$. By Lemma 4.11 f^*B_1 has two irreducible components, and by Lemma 4.10 they are of type $(3, 2)$ or $(3, 1)^{\mathrm{sm}}$. We have to show that they cannot be both of type $(3, 1)^{\mathrm{sm}}$.

Let us choose a SQM $\varphi \colon X \dashrightarrow \tilde{X}$ such that $\tilde{f} := f \circ \varphi^{-1} \colon \tilde{X} \to S$ is regular, K-negative, and special (see Lemma 4.2). Let $E, \hat{E} \subset \tilde{X}$ be the irreducible components of $\tilde{f}^*(B_1)$, and $F \subset \tilde{X}$ a general fiber of $\tilde{f}$ over the curve B_1.

Suppose that E is of type $(3, 1)^{\mathrm{sm}}$. By Theorems 2.19 and 4.4, we have a diagram:

$$\begin{array}{ccccccc} X & \stackrel{\varphi}{\dashrightarrow} & \tilde{X} & \stackrel{\psi}{\dashrightarrow} & \hat{X} & \stackrel{k}{\longrightarrow} & \tilde{X}_1 \\ & {\scriptstyle f}\searrow & & {\scriptstyle \tilde{f}}\searrow & \downarrow{\scriptstyle \hat{f}} & \swarrow{\scriptstyle f_1} & \\ & & & & S & & \end{array}$$

where ψ is SQM and k is the blow-up of a smooth irreducible curve $C \subset \tilde{X}_1$, with exceptional divisor the transform of $E \subset \tilde{X}$, and $f_1(C) = B_1$.

Recall from the proof of Lemma 2.19 that ψ arises from a MMP for E, relative to $\tilde{f}$. Since $\tilde{f}$ is K-negative, one can use a MMP with scaling of $-K_{\tilde{X}}$ (see [Birkar et al. 2010, Section 3.10], and for this specific case [Casagrande 2012, Proposition 2.4] which can be adapted to the relative setting), so that ψ factors as a sequence of K-negative flips, relative to $\tilde{f}$. Then by Lemma 4.1(b) and (c), the indeterminacy locus of ψ is a disjoint union of exceptional planes, and is disjoint from the indeterminacy locus of φ^{-1}.

In particular, the indeterminacy locus of ψ is contracted to points by $\tilde{f}$. Since F is a general fiber of $\tilde{f}$ over B_1, it must be contained in the open subset where ψ is an isomorphism, and $\hat{F} := \psi(F) \subset \hat{X}$ is a general fiber of $\hat{f}$ over B_1. We also note that F is contained in the open subset where φ^{-1} is an isomorphism; otherwise there should be an exceptional line contained in E, and this would give an exceptional line contained in $\mathrm{Exc}(k)$, contradicting [Casagrande 2017, Remark 5.6].

Every irreducible component of $\mathrm{Exc}(k) \cap \hat{F}$ is a fiber of k over C. We deduce that the transform in X of any curve in $E \cap F$ has class in $\mathbb{R}_{\geq 0}[C_E]$.

We have $\dim F \cap E \cap \hat{E} \geq 1$, let Γ be an irreducible curve in $F \cap E \cap \hat{E}$. If $\hat{E}$ were of type $(3, 1)^{\mathrm{sm}}$ too, the transform of Γ in X should have class in both $\mathbb{R}_{\geq 0}[C_E]$ and $\mathbb{R}_{\geq 0}[C_{\hat{E}}]$. This would imply that the classes of C_E and $C_{\hat{E}}$ are proportional, and this is impossible by Theorem–Definition 4.4(e). Therefore E and $\hat{E}$ cannot be both of type $(3, 1)^{\mathrm{sm}}$. □

Proof of Theorem 5.1. We can assume that $\rho_X \geq 13$, otherwise we have (ii).

By Proposition 2.13 f factors as a special rational contraction $g\colon X \dashrightarrow T$ followed by a birational map $T \to S$. There exists a SQM $\varphi\colon X \dashrightarrow \tilde{X}$ such that $\tilde{X}$ is smooth and the composition $\tilde{g} := g \circ \varphi^{-1}\colon \tilde{X} \to T$ is regular, K-negative and special (see Lemma 4.2); in particular T is a smooth surface by Lemma 4.3.

$$\begin{array}{ccc} X & \stackrel{\varphi}{\dashrightarrow} & \tilde{X} \\ {\scriptstyle f}\,\vdots & \stackrel{g}{\searrow} & \downarrow{\scriptstyle \tilde{g}} \\ S & \longleftarrow & T \end{array}$$

Finally g has $r_i = 2$ for every $i = 1, \dots, m$ (we use Notation 2.14) by Lemma 4.11.

Suppose that $m = 0$, equivalently that $\tilde{g}$ is quasielementary. If g is regular, then [Casagrande 2008, Theorem 1.1(i)] together with $\rho_X \geq 13$ yield that X is a product of surfaces, so we have (i).

Assume instead that g is not regular, and let $F \subset X$ be a general fiber of f, which is also a general fiber of g. Since the indeterminacy locus of φ^{-1} has dimension 1 (see Lemma 4.1(a)), it does not meet a general fiber of $\tilde{g}$. This means that F is contained in the open subset where φ is an isomorphism, and $\varphi(F)$ is a general fiber of $\tilde{g}$. By Lemma 2.17 and [Casagrande 2013a, Corollary 3.9 and its proof] we have that F is a smooth del Pezzo surface with $\rho_F \leq 8$ and

$$\rho_X = \dim \mathcal{N}_1(F, X) + \rho_T \leq \rho_F + \rho_T \leq 8 + \rho_T.$$

In particular $\rho_T \geq 13 - 8 = 5$. Then [loc. cit., Proposition 4.1 and its proof] imply that g is not elementary and that T is a del Pezzo surface. Therefore $\rho_X \leq 17$, $\dim \mathcal{N}_1(F, X) = \rho_X - \rho_T \geq 13 - 9 = 4$, and S is a smooth del Pezzo surface too. So we have (iii).

Suppose now that $m \geq 1$. By Lemma 5.2, $(\tilde{g})^* B_1$ has an irreducible component E which is a fixed divisor of type $(3, 2)$. We have $(\tilde{g})_* \mathcal{N}_1(E, \tilde{X}) = \mathbb{R}[B_1]$, so that $\operatorname{codim} \mathcal{N}_1(E, \tilde{X}) \geq \rho_T - 1$. If $\rho_T > 1$, then we get (i) by Lemma 4.5.

Let us assume that $\rho_T = 1$. Then $T \cong \mathbb{P}^2$, because T is a smooth rational surface. Moreover the birational map $T \to S$ must be an isomorphism, hence $S \cong \mathbb{P}^2$ and f is special, and we get (iv). ☐

6. Fano 4-folds to 3-folds

In this section we study rational contractions from a Fano 4-fold to a 3-dimensional target, and show the following.

Theorem 6.1. *Let X be a smooth Fano 4-fold. If there exists a rational contraction $X \dashrightarrow Y$ with $\dim Y = 3$, then either X is a product of surfaces, or $\rho_X \leq 12$.*

Proof. If $\delta_X \geq 3$ the statement follows from Theorem 3.2, so we can assume that $\delta_X \leq 2$; we also assume that $\rho_X \geq 7$. By Proposition 2.13, we can suppose that the map $X \dashrightarrow Y$ is special. Moreover by Lemma 4.2 we can factor it as

$$X \overset{\varphi}{\dashrightarrow} \tilde{X} \overset{f}{\longrightarrow} Y,$$

where φ is a SQM, $\tilde{X}$ is smooth, and f is regular, K-negative and special.

By Lemmas 3.4 and 3.5 we have $\rho_X = \rho_Y + m + 1$, $r_1 = \cdots = r_m = 2$, and the divisors $B_1, \ldots, B_m$ are pairwise disjoint in Y (we use Notation 2.14). For $i = 1, \ldots, m$ the irreducible components of $f^* B_i$ are fixed divisors of type $(3, 2)$ by Lemma 4.10.

If $\rho_X - \rho_Y \geq 3$, then $m \geq 2$. Let E_1, E_2 be the irreducible components of $f^* B_1$, and W an irreducible component of $f^* B_2$. Since $B_1 \cap B_2 = \varnothing$, we have $E_1 \cap W = \varnothing$, so that $\mathcal{N}_1(E_1, \tilde{X}) \subsetneq \mathcal{N}_1(\tilde{X})$ by Remark 1.3, and this implies the statement by Lemma 4.5.

If instead $\rho_X - \rho_Y = 1$, then f is elementary, and $\rho_X \leq 11$ by [Casagrande 2013a, Theorem 1.1].

We are left with the case where $\rho_X - \rho_Y = 2$ and $m = 1$, which we assume from now on. We will adapt the proof of [loc. cit., Theorem 1.1] of the elementary case to the case $\rho_X - \rho_Y = 2$, and divide the proof in several steps. Since $m = 1$, we set for simplicity $B := B_1$.

6.2. If $\mathcal{N}_1(E_1, \tilde{X}) \subsetneq \mathcal{N}_1(\tilde{X})$ we conclude as before, so we can assume that $\mathcal{N}_1(E_1, \tilde{X}) = \mathcal{N}_1(\tilde{X})$; this implies that $\mathcal{N}_1(B, Y) = \mathcal{N}_1(Y)$.

By Lemma 3.4, $E_1 \cup E_2$ is covered by curves of anticanonical degree 1. Since an exceptional line cannot meet such curves (see Lemma 4.1(b)), we deduce that $\ell \cap (E_1 \cup E_2) = \varnothing$ for every exceptional line $\ell \subset \tilde{X}$.

Notice that even if f is not elementary, by specialty it does not have fibers of dimension 3, and has at most isolated fibers of dimension 2. Moreover Y is locally factorial and has (at most) isolated canonical singularities, by Lemma 4.3. More precisely, $\mathrm{Sing}(Y)$ is contained in the images of the 2-dimensional fibers of f (this is due to Ando, see [Andreatta and Wiśniewski 1997, Theorem 4.1 and references therein]).

Since $\tilde{X}$ is smooth and Y is locally factorial, it is easy to see that $f^*B = E_1 + E_2$.

Finally, since X is Fano, by [Prokhorov and Shokurov 2009, Lemma 2.8] there exists a $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is a klt log Fano, so that $-K_Y$ is big.

6.3. Let $g\colon Y \to Y_0$ be a small elementary contraction. Then $\mathrm{Exc}(g)$ is the disjoint union of smooth rational curves lying in the smooth locus of Y, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$; in particular $K_Y \cdot \mathrm{NE}(g) = 0$.

Proof. Exactly the same proof as the one of [Casagrande 2013a, Lemma 4.5] applies, with the only difference that, in the notation of [loc. cit., Lemma 4.5], $\dim \mathcal{N}_1(\tilde{U}/U)$ could be bigger than 2. We take τ to be any extremal ray of $\mathrm{NE}(\tilde{U}/U)$ not contained $\mathrm{NE}(g_{|\tilde{U}})$. □

6.4. Let $g\colon Y \to Y_0$ be an elementary divisorial contraction. Then g is the blow-up of a smooth point of Y_0; in particular $-K_Y \cdot \mathrm{NE}(g) > 0$.

Proof. Set $G := \mathrm{Exc}(g) \subset Y$. Since g is elementary and $\dim g(G) \le 1$, we have $\dim \mathcal{N}_1(G, Y) \le 2$; on the other hand $\dim \mathcal{N}_1(B, Y) = \rho_Y = \rho_X - 2 \ge 5$ (see 6.2), so $G \ne B$, and $D := f^*G$ is a prime divisor in $\tilde{X}$, different from E_1 and E_2, with $\dim \mathcal{N}_1(D, \tilde{X}) \le \dim \ker f_* + \dim \mathcal{N}_1(G, Y) \le 2 + 2 = 4$.

Since G is fixed, also D is a fixed divisor in $\tilde{X}$; let $D_X \subset X$ be the transform of D.

6.4.1. We show that D is not of type $(3, 2)$. Otherwise, as in the proof of Lemma 4.5 we see that $\dim \mathcal{N}_1(D_X, X) = \dim \mathcal{N}_1(D, \tilde{X}) \le 4$. On the other hand we have $\delta_X \le 2$ and $\rho_X \ge 7$, a contradiction.

6.4.2. We show that g is of type $(2, 0)$. By contradiction, suppose that g is of type $(2, 1)$. As in [Casagrande 2013a, proof of Lemma 4.6], we show that there is an open subset $\tilde{U} \subseteq \tilde{X}$ such that $D \cap \tilde{U}$ is covered by curves of anticanonical degree 1. By [Casagrande 2017, Lemma 2.8(3)], D_X still has a nonempty open subset covered by curves of anticanonical degree 1; this implies that D_X and D are of type $(3, 2)$ by [loc. cit., Lemma 2.18], a contradiction to 6.4.1.

6.4.3. Thus g is of type $(2, 0)$; set $p := g(G) \in Y_0$.

Since $\mathcal{N}_1(B, Y) = \mathcal{N}_1(Y)$ by 6.2, we must have $G \cap B \ne \varnothing$ by Remark 1.3. Therefore $p \in g(B)$, hence $g^*(g(B)) = B + aG$ with $a > 0$, and $(g \circ f)^*(g(B)) = E_1 + E_2 + aD$ (see again 6.2).

As in [Casagrande 2013a, proof of Lemma 4.6], we get a diagram:

$$\begin{array}{ccccc} \tilde{X} & \overset{\psi}{\dashrightarrow} & \hat{X} & \overset{k}{\longrightarrow} & \tilde{X}_1 \\ \Big\downarrow f & & & \swarrow f_1 & \\ Y & \overset{g}{\longrightarrow} & Y_0 & & \end{array}$$

where ψ is a sequence of D-negative flips relative to $g \circ f$, k is an elementary divisorial contraction with exceptional divisor the transform $\hat{D} \subset \hat{X}$ of D, and f_1 is a contraction of fiber type with $\dim \ker(f_1)_* = 2$. By 6.4.1 and Theorem–Definition 4.4, k is of type $(3,0)^{\text{sm}}$, $(3,0)^{Q}$, or $(3,1)^{\text{sm}}$; in particular $\tilde{X}_1$ has at most one isolated locally factorial and terminal singularity. Moreover f_1 is special, so that Y_0 has locally factorial, canonical singularities by Lemma 2.21.

6.4.4. Let us consider the factorization of ψ as a sequence of D-negative flips relative to $g \circ f$:

$$\begin{array}{ccccccc} \tilde{X} = Z_0 & \overset{\sigma_1}{\dashrightarrow} \cdots \dashrightarrow & Z_{i-1} & \overset{\sigma_i}{\dashrightarrow} & Z_i & \dashrightarrow \cdots \overset{\sigma_n}{\dashrightarrow} & Z_n = \hat{X} \\ & \searrow^{g\circ f} & \downarrow{\scriptstyle \zeta_{i-1}} & \swarrow_{\zeta_i} & & \swarrow_{f_1\circ k} & \\ & & Y_0 & & & & \end{array}$$

With a slight abuse of notation, we still denote by D, E_1, E_2 the transforms of these divisors in Z_i, for $i = 0, \dots, n$.

We show by induction on $i = 0, \dots, n$ that σ_i is K-negative and that $(E_1 + E_2) \cdot \ell \leq 0$ for every exceptional line $\ell \subset Z_i$. For $i = 0$, this holds by 6.2.

Suppose that the statement is true for $i - 1$. Let R and R' be the small extremal rays of $\mathrm{NE}(Z_{i-1})$ and $\mathrm{NE}(Z_i)$ respectively corresponding to the flip σ_i. By the commutativity of the diagram above and by 6.4.3, we have $E_1 + E_2 + aD = \zeta_{i-1}^*(g(B))$, hence $(E_1 + E_2 + aD) \cdot R = 0$, where $a > 0$. On the other hand $D \cdot R < 0$, thus $(E_1 + E_2) \cdot R > 0$ and $(E_1 + E_2) \cdot R' < 0$.

If $-K_{Z_{i-1}} \cdot R \leq 0$, then by [Casagrande 2013a, Remark 3.6(2)] there exists an exceptional line $\ell_0 \subset Z_{i-1}$ such that $[\ell_0] \in R$, therefore $(E_1 + E_2) \cdot \ell_0 > 0$, contradicting the induction assumption. Hence $-K_{Z_{i-1}} \cdot R > 0$ and σ_i is K-negative.

Finally if $\ell \subset Z_i$ is an exceptional line, by [loc. cit., Remark 4.2] we have either $\ell \subset \operatorname{dom} \sigma_i^{-1}$, or $\ell \cap \operatorname{dom} \sigma_i^{-1} = \varnothing$. In the first case $\sigma_i^{-1}(\ell)$ is an exceptional line in Z_{i-1}, and we deduce that $(E_1 + E_2) \cdot \ell \leq 0$. In the second case, we must have $[\ell] \in R'$ and hence $(E_1 + E_2) \cdot \ell < 0$.

6.4.5. By 6.4.4, ψ factors as a sequence of K-negative flips, and Lemma 4.1(c) yields that the indeterminacy locus of ψ^{-1} is a disjoint union of exceptional lines $\ell_1, \dots, \ell_s$.

6.4.6. Set $F_p := f_1^{-1}(p)$. We show that $\dim F_p = 1$.

Note that $\tilde{X}$ and $\hat{X}$ are isomorphic outside the fibers of $g \circ f$ and $f_1 \circ k$ over p, respectively. In $\tilde{X}$ we have $(g \circ f)^{-1}(p) = D$, and the indeterminacy locus of ψ must be contained in D. In $\hat{X}$ we have $(f_1 \circ k)^{-1}(p) = k^{-1}(F_p) = \hat{D} \cup \overline{F}_p$, where $\overline{F}_p$ is the transform of the components of F_p not contained in $k(\hat{D})$. On the other hand, by 6.4.5 we also have $k^{-1}(F_p) = \hat{D} \cup \ell_1 \cup \cdots \cup \ell_s$. This shows that $\overline{F}_p \subseteq \ell_1 \cup \cdots \cup \ell_s$, in particular $\dim \overline{F}_p \leq 1$, and since $\dim k(\hat{D}) \leq 1$ (see 6.4.3), we conclude that $\dim F_p = 1$.

We have also shown that the transform in $\hat{X}$ of any irreducible component of F_p not contained in $k(\hat{D})$ must be one of the ℓ_i.

6.4.7. We show that f_1 is K-negative. Since f is K-negative and $f_{|\tilde{X}\smallsetminus D} \cong (f_1)_{|\tilde{X}_1\smallsetminus F_p}$, we only have to check the fiber F_p. Let Γ be an irreducible component of F_p.

If $\Gamma \not\subseteq k(\hat{D})$, then by 6.4.6 we can assume that the transform of Γ in $\hat{X}$ is ℓ_1. Since $k^{-1}(F_p)$ is connected and $\ell_1, \dots, \ell_s$ are pairwise disjoint, we have $\hat{D} \cdot \ell_1 > 0$; notice also that $K_{\hat{X}} \cdot \ell_1 = 1$. Thus $-K_{\tilde{X}_1} \cdot \Gamma > 0$ because $k^*(-K_{\tilde{X}_1}) = -K_{\hat{X}} + b\hat{D}$ with $b \in \{2, 3\}$ (see 6.4.3).

If instead $\Gamma \subseteq k(\hat{D})$, then by 6.4.3 k must be of type $(3, 1)^{\mathrm{sm}}$ and $\Gamma = k(\hat{D})$. By [Casagrande 2017, Lemma 5.25] there is a SQM $\varphi_1 \colon \tilde{X}_1 \dashrightarrow X_1$ where X_1 is a Fano 4-fold, and Γ is contained in the open subset where φ_1 is an isomorphism, so that $-K_{\tilde{X}_1} \cdot \Gamma = -K_{X_1} \cdot \varphi_1(\Gamma) > 0$.

6.4.8. By 6.4.3, 6.4.6, and 6.4.7, $\tilde{X}_1$ has isolated locally factorial and terminal singularities, Y_0 has locally factorial canonical singularities, f_1 is K-negative, and $\dim F_p = 1$. Then [Ou 2018, Lemma 5.5] yields that p is a smooth point of Y_0 (note that in [loc. cit.] the contraction is supposed to be elementary, but this is used only to conclude that Y_0 is locally factorial, which here we already know).

In particular p is a terminal singularity, hence g is K-negative. The possibilities for $(G, -K_{\tilde{X}_1|G})$ are given in [Andreatta and Wiśniewski 1997, Theorem 1.19]; moreover we know that G is Gorenstein, and by adjunction that $-K_G \cdot C \geq 2$ for every curve $C \subset G$. Going through the list, it is easy to see that the possibilities for G are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and the quadric cone. In the first two cases, $G \subset Y_{\mathrm{reg}}$, and it follows from [Mori 1982, Corollary 3.4] that $G \cong \mathbb{P}^2$ and g is the blow-up of p.

Suppose instead that G is isomorphic to a quadric cone Q. Then the normal bundle of G has to be $\mathcal{O}_Q(-1)$, and as in [Mori 1982, page 164] and [Cutkosky 1988, proof of Theorem 5] one sees that $\mathcal{I}_p\mathcal{O}_Y = \mathcal{O}_Y(-G)$ where $\mathcal{I}_p$ is the ideal sheaf of p in Y_0, so that $g^{-1}(p) = G$ scheme-theoretically. Then g factors through the blow-up of p, and being g elementary, it must be the blow-up of p, which yields $G \cong \mathbb{P}^2$ and hence a contradiction. □

6.5. If Y has an elementary rational contraction of fiber type $Y \dashrightarrow Z$, then $\rho_Z = \rho_X - 3 \geq 4$, in particular Z is a surface. The composition $X \dashrightarrow Z$ is a rational contraction with $\rho_X - \rho_Z = 3$, and we can apply Theorem 5.1. If (i) or (ii) hold, we have the statement. If (iii) holds, then $\rho_X \geq 13$ and Z is a del Pezzo surface, so that $\rho_Z \leq 9$, which is impossible. Finally (iv) cannot hold because $\rho_Z > 1$.

Therefore we can assume that Y does not have elementary rational contractions of fiber type.

6.6. Let R be an extremal ray of $\mathrm{NE}(Y)$. By 6.5 the associated contraction cannot be of fiber type, thus it is birational, either small of divisorial. By 6.3 and 6.4, $-K_Y \cdot R \geq 0$. Since Y is log Fano, $\mathrm{NE}(Y)$ is closed and polyhedral, and we conclude that $-K_Y$ is nef and Y is a weak Fano variety (see 6.2).

6.7. Let $Y \dashrightarrow \tilde{Y}$ be a SQM. Then the composition $X \dashrightarrow \tilde{Y}$ is again a special rational contraction with $\rho_X - \rho_{\tilde{Y}} = 2$, so all the previous steps apply to $\tilde{Y}$ as well. As in [Casagrande 2013a, page 622], using 6.3 and 6.4 one shows that if $E \subset Y$ is a fixed prime divisor, then E can contain at most finitely many curves of anticanonical degree zero.

6.8. Let us consider all the contracting birational maps $Y \dashrightarrow Y_1$ with $\mathbb{Q}$-factorial target, and choose one with ρ_{Y_1} minimal.

Suppose that $\rho_{Y_1} \geq 3$. By minimality, Y_1 has an elementary rational contraction of fiber type $Y_1 \dashrightarrow Z$, and Z must be a surface with $\rho_Z = \rho_{Y_1} - 1 \geq 2$. The composition $X \dashrightarrow Z$ is a rational contraction, let $F \subset X$ be a general fiber. The general fiber of $Y \dashrightarrow Z$ is a smooth rational curve $\Gamma \subset Y$, and $\dim \mathcal{N}_1(F, X) \leq \dim \mathcal{N}_1(\Gamma, Y) + (\rho_X - \rho_Y) = 3$. Thus we get the statement by Theorem 5.1.

Therefore we can assume that $\rho_{Y_1} \leq 2$.

6.9. By [Casagrande 2017, Lemma 4.18], we can factor the map $Y \dashrightarrow Y_1$ as $Y \dashrightarrow Y' \to Y_1$, where $Y \dashrightarrow Y'$ is a SQM, and $Y' \to Y_1$ is a sequence of elementary divisorial contractions. Now notice that the composition $X \dashrightarrow Y'$ is again a special rational contraction with $\rho_X - \rho_{Y'} = 2$, so up to replacing Y with Y', we can assume that the map $a\colon Y \dashrightarrow Y_1$ is regular and is a sequence of $r := \rho_Y - \rho_{Y_1}$ elementary divisorial contractions:

$$Y = W_0 \xrightarrow{a_1} W_1 \xrightarrow{a_2} W_2 \to \cdots \to W_r = Y_1.$$

Let us show that the exceptional loci of these maps are all disjoint, so that a is just the blow-up of r distinct smooth points of Y_1.

We know by 6.4 that a_1 is the blow-up of a smooth point $w_1 \in W_1$, and since $-K_Y$ is nef, it is easy to see that if $C \subset W_1$ is an irreducible curve containing w_1, then $-K_{W_1} \cdot C \geq 2$.

Suppose that $\mathrm{Exc}(a_2)$ contains w_1. Then a_2 is K-negative, and $\mathrm{Exc}(a_2)$ cannot be covered by curves of anticanonical degree one. By [Andreatta and Wiśniewski 1997, Theorem 1.19] this implies that $\mathrm{Exc}(a_2) \cong \mathbb{P}^2$ and $(-K_{W_1})_{|\mathrm{Exc}(a_2)} \cong \mathcal{O}_{\mathbb{P}^2}(2)$. Then the transform of $\mathrm{Exc}(a_2)$ would be a fixed prime divisor covered by curves of anticanonical degree zero, which is impossible by 6.7. Proceeding in the same way, we conclude that the exceptional loci of the maps a_i are all disjoint.

Now Y_1 is weak Fano with isolated locally factorial, canonical singularities, and we have $(-K_{Y_1})^3 \leq 72$ by [Prokhorov 2005]. Therefore

$$0 < (-K_Y)^3 = (-K_{Y_1})^3 - 8r,$$

which yields $r \leq 8$ and $\rho_X = \rho_{Y_1} + r + 2 \leq 12$. □

Theorem 1.1 is a straightforward consequence of Theorems 5.1 and 6.1.

Proof of Theorem 1.2. Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be open subsets such that $f_0 := f_{|X_0}\colon X_0 \to Y_0$ is a projective morphism. Up to taking the Stein factorization, we can assume that f_0 is a contraction. Let $A \in \mathrm{Pic}(Y)$ be ample and consider $H := f^*A \in \mathrm{Pic}(X)$. Then H is a movable divisor, hence it yields a rational contraction $f'\colon X \dashrightarrow Y'$. It is easy to see that $f'_{|X_0} = f_0$, in particular $\dim Y' = 3$. Then the statement follows from Theorem 6.1. □

7. Fano 4-folds to $\mathbb{P}^1$

Let X be a Fano 4-fold and $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational contraction; notice that f is always special. In the following proposition we collect the information that we can give on f.

Proposition 7.1. *Let X be a smooth Fano 4-fold and $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational contraction. Let $F_1, \dots, F_m$ be the reducible fibers of f. Then one of the following hold*:

(i) $\rho_X \leq 12$.

(ii) *X is a product of surfaces.*

(iii) $\rho_X \leq m+10$, *f is not regular, and every F_i has two irreducible components, which are fixed divisors of type* $(3,1)^{\mathrm{sm}}$ *or* $(3,0)^Q$.

Proof. We can assume that $\rho_X \geq 7$, so that $r_i = 2$ for $i = 1, \dots, m$ by Lemma 4.11. By Lemma 4.2 we can factor f as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} \mathbb{P}^1$ where φ is a SQM, X' is smooth, and f' is regular and K-negative.

If some F_i has a component of type $(3,0)^{\mathrm{sm}}$, then we get (i) by [Casagrande 2017, Theorem 5.40].

If some F_i has a component of type $(3,2)$, let $E \subset X'$ be its transform. Then $\mathcal{N}_1(E, X') \subseteq \ker(f')_* \subsetneq \mathcal{N}_1(X')$, so we get (i) or (ii) by Lemma 4.5.

We are left with the case where every component of every F_i is of type $(3,1)^{\mathrm{sm}}$ or $(3,0)^Q$. The general fiber F of f' is a smooth Fano 3-fold, so that $\rho_F \leq 10$ by Mori and Mukai's classification (see [Iskovskikh and Prokhorov 1999, Corollary 7.1.2]). If f is regular, then φ is an isomorphism, and $\rho_X \leq \rho_F + \delta_X$, so we get (i) or (ii) by Theorem 3.2.

If instead f is not regular, then as in [Casagrande 2013a, proof of Corollary 3.9] one shows that in fact $\rho_F \leq 9$. Therefore Corollary 2.16 yields $\rho_X \leq m+10$, and we have (iii). □

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