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Artin–Schreier covers

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Let $\pi : Y \to X$ be a branched $\mathbf{Z}/p\mathbf{Z}$-cover of smooth, projective, geometrically connected curves over a perfect field of characteristic $p > 0$. We investigate the relationship between the a-numbers of Y and X and the ramification of the map π. This is analogous to the relationship between the genus (respectively p-rank) of Y and X given the Riemann–Hurwitz (respectively Deuring–Shafarevich) formula. Except in special situations, the a-number of Y is not determined by the a-number of X and the ramification of the cover, so we instead give bounds on the a-number of Y. We provide examples showing our bounds are sharp. The bounds come from a detailed analysis of the kernel of the Cartier operator.

1. Introduction

Let k be a field and $\pi : Y \to X$ a finite morphism of smooth, projective, and geometrically connected curves over k that is generically Galois with group G. The most fundamental numerical invariant of a curve is its genus, and the famous *Riemann–Hurwitz formula* says that the genus of Y is determined by that of X and the ramification of the cover π: letting $S \subset X(\bar{k})$ denote the branch locus,

$$2g_Y - 2 = |G| \cdot (2g_X - 2) + \sum_{y \in \pi^{-1}(S)} \sum_{i \geq 0} (|G_i(y)| - 1). \tag{1-1}$$

Here $G_i(y) \leqslant G$ is the i-th ramification group (in the lower numbering) at y.

When k is perfect of characteristic $p > 0$, which we will assume henceforth, there are important numerical invariants of curves beyond the genus coming from the existence of the Frobenius morphism. Writing σ for the p-power Frobenius automorphism of k, the *Cartier operator* is a σ^{-1}-semilinear map $V : H^0(X, \Omega^1_{X/k}) \to H^0(X, \Omega^1_{X/k})$ which is dual to the pullback by absolute Frobenius on $H^1(X, \mathcal{O}_X)$ using Grothendieck–Serre duality.

The Cartier operator gives the k-vector space of holomorphic differentials on X the structure of a (left) module of finite length over the (noncommutative in general) polynomial ring $k[V]$. Fitting's lemma provides a canonical direct sum decomposition of $k[V]$-modules

$$H^0(X, \Omega^1_{X/k}) = H^0(X, \Omega^1_{X/k})^{\text{bij}} \oplus H^0(X, \Omega^1_{X/k})^{\text{nil}}$$

with V bijective (respectively nilpotent) on $H^0(X, \Omega^1_{X/k})^\star$ for $\star = \mathrm{bij}$ (respectively $\star = \mathrm{nil}$). Let us write f_X for the k-dimension of $H^0(X, \Omega^1_{X/k})^{\mathrm{bij}}$; this integer is called the *p-rank* of X, or more properly of the Jacobian J_X of X, since one also has the description $f_X = \dim_{\mathbf{F}_p} \mathrm{Hom}(\mu_p, J_X[p])$. When $\pi : Y \to X$ is a branched G-cover with G a p-group, the *Deuring–Shafarevich formula* relates the p-ranks of X and Y:

$$f_Y - 1 = |G| \cdot (f_X - 1) + \sum_{y \in \pi^{-1}(S)} (|G_0(y)| - 1). \tag{1-2}$$

Like the Riemann–Hurwitz formula, (1-2) says that the numerical invariant f_Y of Y is determined by f_X and the ramification of π; unlike the Riemann–Hurwitz formula, it *only* applies when G has p-power order, and requires only limited information about the ramification filtration. As Crew [1984, Remark 1.8.1] pointed out, there can be no version of the Deuring–Shafarevich formula if G is not assumed to be a p-group, since (for example) if $p > 2$ any elliptic curve E over k is a $\mathbf{Z}/2\mathbf{Z}$-cover of the projective line branched at exactly 4 points (necessarily with ramification degree 2), but f_E can be 0 or 1, so that f_E is *not* determined by $f_{\mathbf{P}^1} = 0$ and the ramification of $\pi : E \to \mathbf{P}^1$. Of course, thanks to the solvability of p-groups, the essential case of (1-2) is when $G = \mathbf{Z}/p\mathbf{Z}$.

Since the k-dimension δ_X of the nilpotent part $H^0(X, \Omega^1_{X/k})^{\mathrm{nil}}$ satisfies $\delta_X = g_X - f_X$, together the Riemann–Hurwitz and Deuring–Shafarevich formulae provide a similar formula relating δ_X, δ_Y, and the (wild) ramification of π for any p-group branched cover $\pi : Y \to X$. Beyond this fact, very little seems to be understood about the behavior of the nilpotent part in p-group covers.

In this paper, we will study the behavior of the *a-number* of curves in branched $\mathbf{Z}/p\mathbf{Z}$-covers $\pi : Y \to X$. By definition, the *a*-number of a curve C is

$$a_C := \dim_k \ker\big(V : H^0(C, \Omega^1_{C/k}) \to H^0(C, \Omega^1_{C/k})\big). \tag{1-3}$$

Equivalently, a_C is the number of nonzero cyclic direct summands in the invariant factor decomposition of $H^0(C, \Omega^1_{C/k})^{\mathrm{nil}}$ as a $k[V]$-module. Yet a third interpretation is $a_C = \dim_k \mathrm{Hom}(\alpha_p, J_C[p])$, where α_p denotes the group-scheme $\ker(F : \mathbf{G}_a \to \mathbf{G}_a)$ over k.[1] A curve X is said to be *ordinary* if $a_X = 0$. For elliptic curves, the *a*-number is 0 or 1 depending on whether the curve is ordinary or supersingular in the standard senses.

Although this fundamental numerical invariant of curves in positive characteristic has been extensively studied (e.g., [Washio and Kodama 1986; Kodama and Washio 1988; Re 2001; Elkin and Pries 2007; Johnston 2007; Elkin 2011; Friedlander et al. 2013; Dummigan and Farwa 2014; Montanucci and Speziali 2018; Frei 2018; Zhou 2019]), it remains rather mysterious. When $p = 2$ and X is ordinary, Voloch [1988] established an explicit formula for a_Y in terms of the ramification of π and the genus of X. If in addition $X = \mathbf{P}^1$, Elkin and Pries [2013] showed that this data completely determines the Ekedahl–Oort type of $J_Y[p]$; note that this situation is quite special, as every Artin–Schreier cover of $\mathbf{P}^1$ in characteristic 2 is hyperelliptic. For general p, Farnell and Pries [2013] studied branched $\mathbf{Z}/p\mathbf{Z}$-covers $\pi : Y \to \mathbf{P}^1$,

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[1] The equivalence of this description with the given definition follows from Dieudonné theory and a theorem of Oda [1969, Corollary 5.11], which provides a canonical isomorphism of $k[V]$-modules $H^0(C, \Omega^1_{C/k}) \simeq k \otimes_{k,\sigma^{-1}} D(J_C[F])$, where $D(\cdot)$ is the contravariant Dieudonné module.

and proved that there is an explicit formula for a_Y in terms of the ramification of π whenever the unique break in the ramification filtration at every ramified point is a divisor of $p-1$. Unfortunately, there can be no such "*a*-number formula" in the spirit of (1-2) in general: simple examples with $p > 2$ show that there are $\mathbf{Z}/p\mathbf{Z}$-covers even of $X = \mathbf{P}^1$ branched only at ∞ which have *identical* ramification filtrations, but different *a*-numbers; cf. Example 7.2.

Nonetheless, we will prove that the possibilities for the *a*-number of Y are tightly constrained by the *a*-number of X and the ramification of π:

Theorem 1.1. *Let $\pi : Y \to X$ be a finite morphism of smooth, projective and geometrically connected curves over a perfect field k of characteristic $p > 0$ that is generically Galois with group $\mathbf{Z}/p\mathbf{Z}$. Let $S \subseteq X(\bar{k})$ be the finite set of geometric closed points over which π ramifies, and for $Q \in S$ let d_Q be the unique break in the lower-numbering ramification filtration at the unique point of Y over Q. Then for any $1 \le j \le p-1$,*

$$\sum_{Q \in S} \sum_{i=j}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right)\frac{jd_Q}{p} \right\rfloor \right) \le a_Y \le pa_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - (p-i)\left\lfloor \frac{id_Q}{p^2} \right\rfloor \right).$$

In fact, our main result (Theorem 6.26) features a slightly sharper—if somewhat messier and less explicit in general — upper bound, and Theorem 1.1 is an immediate consequence of this.

Remark 1.2. The lower bound is largest for $j \approx p/2$, and in applications we will take $j = \lceil p/2 \rceil$. Fixing X and $S \subseteq X(\bar{k})$ and writing $T := (p-1)\sum_{Q \in S} d_Q$, elementary estimates show that our lower (respectively upper) bound is asymptotic to $\left(1 - \frac{1}{p^2}\right)\frac{1}{4}T$ $\left(\text{respectively } \left(1 - \frac{1}{2p}\right)\frac{1}{3}T\right)$ as $T \to \infty$; see Corollary 6.27 for more precise estimates. Equivalently, since $g_Y \sim \frac{1}{2}T$ as $T \to \infty$ by Riemann–Hurwitz, our lower and upper bounds are asymptotic to $\left(1 - \frac{1}{p^2}\right)\frac{1}{2}g_Y$ and $\left(1 - \frac{1}{2p}\right)\frac{2}{3}g_Y$, respectively, as $g_Y \to \infty$ with X and S fixed. In contrast, f_Y/g_Y approaches 0 as $T \to \infty$.

Remark 1.3. Using only information about X, elementary arguments give "trivial" bounds

$$\dim_k \ker\left(V : H^0(X, \Omega_X^1(E_0)) \to H^0(X, \Omega_X^1(E_0))\right) \le a_Y \le p \cdot g_X - p \cdot f_X + \sum_{Q \in S} \tfrac{1}{2}(p-1)(d_Q - 1),$$

where $E_0 = \sum_{Q \in S}(d_Q - \lfloor d_Q/p \rfloor)[Q])$. The trivial lower bound comes from the fact that there is an inclusion $\Omega_X^1(E_0) \hookrightarrow \pi_*\Omega_Y^1$ compatible with the Cartier operator; see Lemma 4.1. The trivial upper bound comes from the fact that $a_Y + f_Y \le g_Y$ and from applying the Riemann–Hurwitz formula for the genus and Deuring–Shafarevich formula for the p-rank to $\pi : Y \to X$.

The lower bound is explicit. When $X = \mathbf{P}^1$, for a divisor $D = \sum n_i[P_i]$ with $n_i \ge 0$ we know that

$$\dim_k \ker\left(V : H^0(X, \Omega_X^1(D)) \to H^0(X, \Omega_X^1(D))\right) = \sum_i \left(n_i - \left\lceil \frac{n_i}{p} \right\rceil\right)$$

so the lower bound is explicit. For general X, a theorem of Tango (Fact 6.11, the main theorem of [Tango 1972]) generalizes this to provide information about the kernel of the Cartier operator when the degree of D is sufficiently large.

Theorem 1.1 is substantially better than the trivial bounds; see Example 7.1 for an illustration.

When $p = 2$ and $a_X = 0$, the upper and lower bounds of Theorem 6.26 (with $j = 1$) *coincide*, and we recover Voloch's formula [1988, Theorem 2]; see Remark 6.30. Similarly, when p is odd, all d_Q divide $p - 1$, and $a_X = 0$, we prove in Corollary 6.32 that our (sharpest) upper bound and our lower bound with $j = \lceil p/2 \rceil = (p + 1)/2$ also coincide, thereby establishing the following "a-number formula":

Corollary 1.4. *With hypotheses and notation as in Theorem 1.1 and p odd, assume that $d_Q | (p - 1)$ for all $Q \in S$ and that X is* ordinary *(i.e., $a_X = 0$). Then*

$$a_Y = \sum_{Q \in S} a_Q, \qquad where \quad a_Q := \frac{(p - 1)}{2}(d_Q - 1) - \frac{p - 1}{d_Q} \left\lfloor \frac{(d_Q - 1)^2}{4} \right\rfloor.$$

Specializing Corollary 1.4 to the case of $X = \mathbf{P}^1$ recovers the main result of [Farnell and Pries 2013].

To get a sense of the bounds in Theorem 1.1, in Section 7 we work out a number of examples. We show in particular that for $X = \mathbf{P}^1$ our upper bound of Theorem 6.26 is *sharp*, with the family of covers $y^p - y = t^{-d}$ (for t a choice of coordinate on $\mathbf{P}^1$) achieving the upper bound for all $p > 2$ and all d with $p \nmid d$; see Example 7.5. We similarly find in our specific examples that the lower bound is sharp and that most covers have a-number equal to the lower bound.[2] For various p, we also provide examples using covers of the elliptic curve with affine equation $y^2 = x^3 - x$, which has $a_X = 1$ when $p \equiv 3 \bmod 4$ and $a_X = 0$ when $p \equiv 1 \bmod 4$.

1A. *Outline of the proof.* Without loss of generality, we may assume that k is algebraically closed. A key idea in the proof is that the Cartier operator is not defined only on global differentials, but actually is a map of sheaves. Let X be a smooth projective curve over k. Functorially associated to the finite flat absolute Frobenius map $F : X \to X$ by Grothendieck's theory of the trace [Conrad 2000, 2.7.36] is an $\mathcal{O}_X$-linear map of sheaves

$$V_X : F_* \Omega^1_X \to \Omega^1_X;$$

the Cartier operator considered previously is obtained by taking global sections. (For the remainder of the paper, we include subscripts to clarify which curve/ring we are working with.) The advantage of this perspective is that the Cartier operator admits a simple description on stalks, allowing local arguments. In particular, the Cartier operator on completed stalks at any k-point is given by

$$V \left(\sum_i a_i t^i \frac{dt}{t} \right) = \sum_j a_{pj}^{1/p} t^j \frac{dt}{t}; \tag{1-4}$$

see, for example [Cais 2018, Proposition 2.1]. To relate the kernels of V_X and V_Y on global differentials, we will combine an analysis over the generic point with an analysis at stalks at the points where the cover $\pi : Y \to X$ is ramified. This strategy allows the use of geometric methods, and allows us to work with

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[2]When $p = 3$ or $p = 5$ and X is ordinary, for any branch locus S and choice of d_Q for $Q \in S$ with $p \nmid d_Q$, subsequent work constructs covers of X whose a-number is the lower bound [Abney-McPeek et al. 2020].

general Artin–Schreier covers instead of only covers of $\mathbf{P}^1$: previous work has focused on curves defined by explicit equations or covers Y of $\mathbf{P}^1$ where it is possible to find a nice and explicit basis of $H^0(Y, \Omega_Y^1)$.

For now, we will ignore a few technical issues and sketch the argument. None of these technical issues arise for $X = \mathbf{P}^1$, which is a helpful simplification on a first reading. Writing η for the generic point of X, one has an isomorphism

$$\pi_* \Omega_{Y,\eta}^1 \simeq \bigoplus_{i=0}^{p-1} \Omega_{X,\eta}^1. \tag{1-5}$$

This follows from the fact that the function field $K' = k(Y)$ of Y is an Artin–Schreier extension of the function field $K = k(X)$ given by $y^p - y = f$ for some $f \in K$. This induces an isomorphism

$$(\pi_* \ker V_Y)_\eta \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta. \tag{1-6}$$

This is Proposition 4.4; using (1-5) to write $\omega \in \pi_* \Omega_{Y,\eta}^1$ as $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ with $\omega_i \in \Omega_{X,\eta}^1$, the key observation is that if $V_Y(\omega) = 0$ then for all $0 \le j \le p-1$, $V_X(\omega_j)$ is determined by $V_X(\omega_i)$ for $j < i \le p-1$.

Unfortunately, (1-5)–(1-6) do not generalize to isomorphisms of sheaves. Instead, there are explicit divisors E_i (Definition 3.4) depending on the ramification of π and an isomorphism of $\mathcal{O}_X$-modules

$$\pi_* \Omega_Y^1 \simeq \bigoplus_{i=0}^{p-1} \Omega_X^1(E_i), \tag{1-7}$$

as well as an injection (Definition 4.3)

$$\varphi : \pi_* \ker V_Y \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_* E_i) \tag{1-8}$$

inducing (1-6) at the generic point. Here $\ker V_X(F_* E_i) := (\ker V_X)_\eta \cap F_*(\Omega_X^1(E_i))$. But φ is not surjective as a map of sheaves. The problem is that while (in the generic fiber) $V_X(\omega_j)$ is determined by ω_i for $i > j$, it is not automatic that the resulting form ω_j satisfies $\mathrm{ord}_Q(\omega_j) \ge -\mathrm{ord}_Q(E_j)$ when $\mathrm{ord}_Q(\omega_i) \ge -\mathrm{ord}_Q(E_i)$ for $i > j$ and $Q \in S$. Example 4.6 is a key illustration of this problem.

To deal with this issue, we wish to find relations that describe the image of φ. Fortunately, this is a purely local problem at the points S where π is ramified. We will identify $\pi_* \ker V_Y$ with a certain subsheaf of the target of (1-8) cut out by linear relations on the coefficients of the power series expansions of elements at points $Q \in S$. These relations force the corresponding differentials ω_i on X to be regular. These relations are expressed in Section 5 as maps to skyscraper sheaves supported on S, and the kernels of these maps describe the image of φ; see Theorem 5.20.

The final step is to extract useful information about the a-number of Y (the dimension of the space of global sections of $\pi_* \ker V_Y$) from the short exact sequences resulting from our description of the image of φ. The key to doing so is knowledge about the dimension of the kernel of the Cartier operator on $H^0(X, \Omega_X^1(D))$ for various divisors D, and related questions about the existence of elements of that

space with specified behavior at points in S. In Section 6, we analyze these questions using a theorem of Tango (Fact 6.11, the main theorem of [Tango 1972]) and obtain bounds on the a-number of Y by taking global sections of the exact sequences from Theorem 5.20.

Remark 1.5. Tango's theorem yields precise results only when the degree of the divisor is sufficiently large. By exploiting the flexibility of our local analysis—which in particular allows us to arbitrarily increase the degree of certain auxiliary divisors—we may always work in this case; see the proof of Theorem 6.26. Given additional information about the size of the dimension of the kernel of the Cartier operator on $H^0(X, \Omega^1_X(D))$ for divisors D of small degree, modest improvements are possible. See Section 6E for an example with unramified covers.

Remark 1.6. The trivial lower bound of Remark 1.3 follows from the inclusion $\Omega^1_X(E_0) \hookrightarrow \pi_* \Omega^1_Y$ that is compatible with the Cartier operator. This inclusion is a consequence of (1-7), which is an isomorphism of $\mathcal{O}_X$-modules and is not in general compatible with the Cartier operator. The significant improvements in the bounds come from incorporating information about the Cartier operator to obtain a more refined inclusion (1-8), analyzing the image, and using Tango's theorem.

Remark 1.7. As mentioned previously, there are some technical complications to the strategy outlined above. In the end, these have no effect on the final result. The issues are:

- The short exact sequence

$$0 \to \ker V_X \to F_* \Omega^1_X \to \operatorname{Im} V_X \to 0$$

  is not always split, although it does split when $X = \mathbf{P}^1$. In Section 2, we show that we may produce maps which split the sequence over the generic point and introduce poles in a controlled manner. This is used to define the map φ of 4.3.

- The Artin–Schreier extension of function fields cannot always be described as $y^p - y = f$ where f is regular away from S, and $\operatorname{ord}_Q(f) = -d_Q$ for $Q \in S$. This is possible when $X = \mathbf{P}^1$ by using the theory of partial fractions. In Section 3 we allow f to have a pole at one additional (nonbranched) point Q' to ensure the desired property holds for $Q \in S$, and then keep track of this complication throughout the remainder of the argument.

Remark 1.8. The same arguments, with minor modifications, should yield bounds on the dimension of the kernel of powers of the Cartier operator. We leave that for future work.

2. Producing splittings

Let k be an algebraically closed field of characteristic p, and X a smooth projective and connected curve over k. Writing $V_X : F_* \Omega^1_{X/k} \to \Omega^1_{X/k}$ for the Cartier operator, we are interested in splitting the tautological short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \ker V_X \to F_* \Omega^1_X \to \operatorname{Im} V_X \to 0. \tag{2-1}$$

Lemma 2.1. *When* $X = \mathbf{P}^1_k$, *(2-1) is a split exact sequence of sheaves.*

Proof. Identify the generic fiber of X with $\mathrm{Spec}(k(t))$. From (1-4), we know that $V_X(dt/t) = (dt/t)$ and $V_X(t^i(dt/t)) = 0$ if $p \nmid i$. Thus we see that

$$(\ker V_X)_\eta = \left\{ \sum_{i=1}^{p-1} h_i(t) t^i \frac{dt}{t} : h_i \in k(t^p) \right\} \quad \text{and} \quad (\mathrm{Im}\, V_X)_\eta = \Omega^1_{X,\eta}.$$

An explicit splitting $s : (\mathrm{Im}\, V_X)_\eta \to F_*\Omega^1_{X,\eta}$ of (2-1) over the generic fiber is given by

$$s\left(\sum_i a_i t^i \frac{dt}{t} \right) = \sum_i a_i^p t^{pi} \frac{dt}{t}. \tag{2-2}$$

For $Q \in \mathbf{P}^1_k$, a direct calculation shows that for a section ω to Ω^1_X, if $\mathrm{ord}_Q(\omega) \geq 0$ then $\mathrm{ord}_Q(s(\omega)) \geq 0$. Thus this defines a map of sheaves. $\qquad\square$

Remark 2.2. The corresponding projector $r : F_*\Omega^1_X \to \ker V_X$ is given by

$$r\left(\sum_i a_i t^i \frac{dt}{t} \right) = \sum_{p \nmid i} a_i t^i \frac{dt}{t}. \tag{2-3}$$

In general, (2-1) splits over the generic point for any smooth curve X, although it is not clear the sequence itself splits. However, it *does* split if we allow the splitting to introduce controlled poles.

Let D be a divisor on X. Note that $(F_*\Omega^1_X)(D) = F_*(\Omega^1_X(pD))$, which complicates the relationship between twists and order of vanishing. To more closely connect twists with order of vanishing, we make the following definition.

Definition 2.3. For a subsheaf $\mathscr{F} \subset F_*\Omega^1_X$ and divisor D on X we define a sheaf $\mathscr{F}(F_*D)$ via

$$\mathscr{F}(F_*D)(U) := F_*(\Omega^1_X(D))(U) \cap \mathscr{F}_\eta$$

for open $U \subset X$. In particular,

$$\ker V_X(F_*D)(U) = \{\omega \in \Omega^1_X(D)(U) : V_X(\omega) = 0\}. \tag{2-4}$$

Example 2.4. It is clear upon taking $\mathscr{F} = F_*\Omega^1_X(D)$ that

$$(F_*\Omega^1_X)(F_*D) = F_*(\Omega^1_X(D)).$$

Note that $\ker V_X(F_*D)$ consists of differentials ω that lie in the kernel of V_X and satisfy $\mathrm{ord}_Q(\omega) \geq -\mathrm{ord}_Q(D)$ for all Q. On the other hand, $\ker V_X(D) = \ker V_X \otimes \mathcal{O}_X(D) = \ker V_X(F_*pD)$, which consists of differentials that lie in the kernel of V_X and satisfy $\mathrm{ord}_Q(\omega) \geq -p\,\mathrm{ord}_Q(D)$.

For any divisor $E = \sum_i n_i P_i \geq 0$, define $\bar{E} := \sum_i \lceil n_i/p \rceil P_i$.

Lemma 2.5. *There is an exact sequence of $\mathcal{O}_X$-modules*

$$0 \to \ker V_X(F_*E) \to F_*(\Omega^1_X(E)) \xrightarrow{V_X} \mathrm{Im}\, V_X(\bar{E}) \to 0. \tag{2-5}$$

Furthermore, each term is locally free.

Proof. For a closed point Q of X and a section ω of $F_*(\Omega_X^1(E))$ defined at Q, a local calculation shows that $\mathrm{ord}_Q(V_X(\omega)) \geq \lceil \mathrm{ord}_Q(\omega)/p \rceil$, so the right map is well-defined. The kernel is $\ker V_X(F_*E)$ by definition. As the completion $\mathcal{O}_{X,Q}^\wedge$ is faithfully flat over $\mathcal{O}_{X,Q}$, we may check surjectivity on completed stalks. Since $-n_i \leq p\lceil -n_i/p \rceil$, for t_Q a local uniformizer at Q we have

$$V_X\left(\sum_i a_i^p t_Q^{ip} \frac{dt_Q}{t_Q}\right) = \sum a_i t_Q^i \frac{dt_Q}{t_Q}$$

thanks to (1-4). Over a smooth curve, to check local freeness it suffices to check the sheaves are torsion-free, which is clear as the sheaves are subsheaves of $F_*\Omega_{X,\eta}^1$. $\square$

We will prove the following:

Proposition 2.6. *Let S be a finite set of points on a smooth projective curve X over k and*

$$0 \to \mathscr{F}_1 \xrightarrow{\iota} \mathscr{F}_2 \to \mathscr{F}_3 \to 0$$

an exact sequence of locally free sheaves. There exists a divisor $D = \sum_i [P_i]$ with the P_i distinct points of X not in S and a morphism $r : \mathscr{F}_2 \to \mathscr{F}_1(D)$ such that $r \circ \iota$ is the natural inclusion $\mathscr{F}_1 \to \mathscr{F}_1(D)$.

The following corollary will be useful in Section 4, especially in Definition 4.3.

Corollary 2.7. *Let S be a finite set of points of X. Fix an effective divisor E supported on S. There is a divisor $D = \sum_i [P_i]$ with the P_i distinct points of X not in S and a morphism*

$$r : F_*(\Omega_X^1(E)) \to \ker V_X(F_*(E + pD))$$

such that $r \circ \iota$ is the natural inclusion $\ker V_X(F_*E) \to \ker V_X(F_*(E + pD))$, where ι is the inclusion $\ker V_X(F_*E) \to F_*(\Omega_X^1(E))$.

Proof. Apply Proposition 2.6 to the exact sequence of Lemma 2.5. It is elementary to verify using Definition 2.3 that $(\ker V_X(F_*E))(D) = \ker V_X(F_*(E + pD))$. $\square$

The rest of this section is devoted to proving Proposition 2.6. The key input is:

Lemma 2.8. *Let S be a finite set of points of X, and $\mathscr{F}$ be a locally free sheaf on X. Then for any divisor $D = \sum_i [P_i]$ with the P_i distinct points of X not in S and $\deg D \gg 0$, we have*

$$H^1(X, \mathscr{F} \otimes \mathcal{O}_X(D)) = 0.$$

Proof. Pick an ample line bundle $\mathscr{L} = \mathcal{O}_X(D')$. By Serre's cohomological criterion for ampleness, we know that there is an N such that

$$H^1(X, \mathscr{F} \otimes \mathscr{L}^n) = 0 \tag{2-6}$$

for $n \geq N$. Writing g for the genus of X, the Riemann–Roch theorem gives

$$h^0(X, \mathcal{O}_X(D - ND')) - h^1(X, \mathcal{O}(D - ND')) = \deg(\mathcal{O}(D - ND')) - g + 1,$$

and when $\deg(\mathcal{O}(D - ND')) > 2g - 2$, we have $h^1(X, \mathcal{O}(D - ND')) = 0$ for degree reasons. Thus when $\deg D \gg 0$ we conclude that

$$h^0(X, \mathcal{O}_X(D - ND')) > 0.$$

Using a global section of $\mathcal{O}_X(D - ND')$, we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D' - ND) \to \mathscr{G} \to 0,$$

where $\mathscr{G}$ is a skyscraper sheaf supported on $D' - ND$. Tensoring with $\mathcal{O}_X(ND')$ is exact, as is tensoring with the locally free $\mathscr{F}$, so we obtain a short exact sequence

$$0 \to \mathscr{F} \otimes \mathcal{O}_X(ND') \to \mathscr{F} \otimes \mathcal{O}_X(D) \to \mathscr{G}' \to 0,$$

where $\mathscr{G}'$ is still a skyscraper sheaf supported on $D - ND'$. Part of the long exact sequence of cohomology is

$$H^1(X, \mathscr{F} \otimes \mathcal{O}_X(ND')) \to H^1(X, \mathscr{F} \otimes \mathcal{O}_X(D)) \to H^1(X, \mathscr{G}')$$

The left term vanishes by (2-6), and the right vanishes as $\mathscr{G}'$ is a skyscraper sheaf. Thus $H^1(X, \mathscr{F} \otimes \mathcal{O}_X(D)) = 0$. $\qquad\square$

We now prove Proposition 2.6. By looking at stalks we see that the Hom-sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1)$ is locally free. Also, notice that for a divisor D,

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1(D)) \simeq \mathscr{F}_3^\vee \otimes \mathscr{F}_1 \otimes \mathcal{O}_X(D) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1) \otimes \mathcal{O}_X(D). \tag{2-7}$$

Applying the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\cdot, \mathscr{F}_1(D))$ to the exact sequence in Proposition 2.6 and using the assumption that $\mathscr{F}_3$ is locally free, we obtain an exact sequence

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1(D)) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_2, \mathscr{F}_1(D)) \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_1(D)) \to 0.$$

Passing to global sections, part of the long exact sequence of cohomology is

$$H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_2, \mathscr{F}_1(D))) \xrightarrow{f} H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_1(D)))$$
$$\to H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1(D))).$$

Applying Lemma 2.8 with $\mathscr{F} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1)$ and appealing to (2-7), we choose $D = \sum_i [P_i]$ where the distinct P_i avoid S such that $H^1(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_3, \mathscr{F}_1(D))) = 0$. Thus f is surjective, and the desired morphism r is the preimage of the natural inclusion in $H^0(X, \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_1(D)))$. $\qquad\square$

Corollary 2.9. *A choice of projector* $r : F_*(\Omega_X^1(E)) \to \ker V_X(F_*(E + pD))$ *as in Corollary 2.7 is equivalent to a choice of splitting of* (2-1)

$$s : \mathrm{Im}\, V_X(\bar{E}) \to F_*(\Omega_X^1(E + pD))$$

such that $V_X \circ s$ is the natural inclusion $\mathrm{Im}\, V_X(\bar{E}) \to \mathrm{Im}\, V_X(\bar{E} + D)$. *Furthermore, we may choose s so that for any point $Q \in \sup(E)$, if* $\mathrm{ord}_Q(\omega) \geq d$ *then*

$$\mathrm{ord}_Q(s(\omega)) \geq (d+1)p - 1. \tag{2-8}$$

Proof. Given r, a section of

$$0 \to (\ker V_X)_\eta \xrightarrow{\iota} (F_* \Omega^1_{X,\eta}) \to (\mathrm{Im}\, V_X)_\eta \to 0$$

is given by $s(m) = \widetilde{m} - \iota r(\widetilde{m})$, where $\widetilde{m}$ is any preimage of $m \in (\mathrm{Im}\, V_X)_\eta$. It is independent of the choice of lift, which allows us to make local calculations using (1-4) to check that this recipe defines a map $s : \mathrm{Im}\, V_X(\bar{E}) \to F_*(\Omega^1_X(E + pD))$. Conversely, given s we define $r(x) = x - sV_X(x)$ for any section x of $F_*(\Omega^1_X(E))$. It is straightforward to check that $r \circ \iota$ is the natural inclusion and that the recipes for r given s and s given r are inverse to each other, so that this process really does give an equivalence between a splitting s and a projector r as claimed.

To get the last claim, set

$$E' := \sum_{Q \in \sup(E)} p\left(\left\lceil \frac{\mathrm{ord}_Q(E)}{p} \right\rceil - 1\right) + 1 \tag{2-9}$$

and note that $\bar{E}' = \bar{E}$ and $E' \leq E$. Thanks to Corollary 2.7 and its proof, we obtain a morphism $r' : F_*(\Omega^1_X(E')) \to \ker V_X(F_*(E' + pD))$ with the *same* D as above. Via the correspondence already established, we obtain a section

$$s' : \mathrm{Im}\, V_X(\bar{E}) \to F_*(\Omega^1_X(E' + pD)).$$

Composing with the natural inclusion $F_*(\Omega^1_X(E' + pD)) \hookrightarrow F_*(\Omega^1_X(E + pD))$, we obtain the desired map s: the condition on orders of vanishing follows by twisting. $\qquad\square$

Remark 2.10. Let $r : F_*(\Omega^1_X(E)) \to \ker V_X(F_*(E + pD))$ be the projector corresponding to a splitting $s : \mathrm{Im}\, V_X(\bar{E}) \to F_*(\Omega^1_X(E + pD))$. No matter the choice of E and D, the generic fiber of (2-1) is split by r and s.

Remark 2.11. When $X = \mathbf{P}^1$, the explicit splitting of Lemma 2.1 shows we may take $D = 0$. This is a good simplification for subsequent arguments on a first reading.

3. Artin–Schreier covers and differential forms

Let $\pi : Y \to X$ be a branched Galois cover of smooth projective and connected curves over an algebraically closed field k of characteristic p, with Galois group $G \simeq \mathbf{Z}/p\mathbf{Z}$ and branch locus $S \subset X$. We are mainly interested in the case that S is nonempty, though we do not assume this at the outset. By Artin–Schreier theory, $K' := k(Y)$ is a degree-p Artin–Schreier extension of $K := k(X)$; that is, there exists $f \in K$ such that $K' = K(y)$, where

$$y^p - y = f. \tag{3-1}$$

We may and do assume a fixed choice of generator τ of G sends y to $y + 1$, and we write g_X for the genus of X. We henceforth *fix* a closed point $Q' \in X$ with $\mathrm{ord}_{Q'}(f) = 0$.

The choice of f is far from unique. In particular, by replacing y with $y + h$ for $h \in K$ we may replace f with $f + h^p - h$ but obtain the same extension. Nonetheless, it is possible to normalize the choice of f as follows.

Definition 3.1. We say $\psi \in K$ is *minimal* for $\pi : Y \to X$ (or for K'/K) if

- $\mathrm{ord}_Q(\psi) \geq 0$ or $p \nmid \mathrm{ord}_Q(\psi)$ for all $Q \in X$ with $Q \neq Q'$,
- $p \mid \mathrm{ord}_{Q'}(\psi)$ and $-\mathrm{ord}_{Q'}(\psi) \leq p(2g_X - 2)$.

Lemma 3.2. *The Artin–Schreier extension K'/K may be described as in* (3-1) *with f minimal.*

Proof. Let $f \in K$ be as in (3-1), and suppose f has a pole of order $p \cdot d$ at $Q \neq Q'$. By Riemann–Roch, there exists a function with a pole of order m at Q and a pole of order n at Q' provided $m + n > 2g_X - 2$. We may therefore choose $h \in K$ to have a pole of order d at Q and no poles except possibly at Q', and satisfy $\mathrm{ord}_Q(f + h^p - h) > \mathrm{ord}_Q(f)$ and $\mathrm{ord}_{Q'}(h^p - h) \geq -p(2g_X - 2)$. Replacing f with $f + h^p - h$ and repeating this procedure, we may thereby arrange for f to be minimal. $\square$

We now *fix* a choice of minimal $f \in K$ giving the Artin–Schreier extension K'/K as in (3-1).

Remark 3.3. When $X = \mathbf{P}^1$, it is easy to arrange for f to have poles only at the branch locus since there are functions on $\mathbf{P}^1$ with a single simple pole. In this case, our analysis below at Q' is unnecessary: this is a helpful simplification on a first reading. This reduction is not possible in general; see [Shabat 2001, §7], especially the second example after Proposition 49.

We now fix some notation relating to the cover $\pi : Y \to X$ and our fixed choice of Q' and minimal f giving the corresponding extension of function fields as in (3-1).

Definition 3.4. With notation as above:

- Let $S' := S \cup \{Q'\}$.
- For $Q \in X$ define $d_Q := \max\{0, -\mathrm{ord}_Q(f)\}$.
- For $0 \leq i \leq p - 1$ and $Q \in X$ define

$$
n_{Q,i} := \begin{cases} \left\lceil \dfrac{(p-1-i)d_Q}{p} \right\rceil & \text{if } Q \neq Q', \\ (p-1-i)d_{Q'} & \text{if } Q = Q', \end{cases}
$$

and set

$$
E_i := \sum_{Q \in S'} n_{Q,i}[Q] \quad \text{and} \quad \bar{E}_i := \sum_{Q \in S'} \lceil n_{Q,i}/p \rceil [Q].
$$

Lemma 3.5. *The map $\pi : Y \to X$ is ramified over $Q \in X$ if and only if $d_Q > 0$ and $p \nmid d_Q$, in which case d_Q is the unique break in the lower-numbering ramification filtration of G above Q.*

Proof. This is well known; see for example [Stichtenoth 2009, Proposition 3.7.8]. $\square$

Remark 3.6. It follows from Lemma 3.5 and the fact that the fixed f is minimal that if $Q \notin S'$ then $d_Q = 0$ and hence also $n_{Q,i} = 0$ for all i.

We now investigate when a meromorphic differential on Y is regular.

Lemma 3.7. Let $\omega \in (\pi_* \Omega_Y^1)_\eta$ be a meromorphic differential on Y, and write $\omega = \sum_i \omega_i y^i$ with $\omega_i \in \Omega_{X,\eta}^1$ using the identification (1-5). Let $P \in Y$ with $\pi(P) = Q \neq Q'$. Then ω is regular at P provided $\mathrm{ord}_Q(\omega_i) \geq -n_{Q,i}$ for all $0 \leq i \leq p-1$.

Proof. This is also probably known, but we were unable to locate a reference in the degree of generality we need: for example, Boseck [1958, Satz 15] only treated the case that $X = \mathbf{P}^1$. We provide a proof for the convenience of the reader.

Let $P \in Y$ and set $Q = \pi(P)$. Suppose first that $Q \in S$. Let t_Q be a uniformizer of $\mathcal{O}_{X,Q}$, and note that the fraction field of $\mathcal{O}_{Y,P}$ can be obtained from the fraction field of $\mathcal{O}_{X,Q}$ by adjoining a root of the polynomial $y^p - y = f$ where f has a pole of order d_Q at Q by Lemma 3.5; in particular, $\mathrm{ord}_P(y) = -d_Q$.

As $p \nmid d_Q$, we may choose positive integers a and b with $1 = ap - bd_Q$, so that $u = t_Q^a y^b$ is a uniformizer of $\mathcal{O}_{Y,P}$. A direct computation shows that $du = u^{-(p-1)(d_Q+1)} \beta dt_Q$, where $\beta \in \mathcal{O}_{Y,P}^\times$. Hence $\mathrm{ord}_P(dt_Q) = (p-1)(d_Q+1)$, and $\mathrm{ord}_P(\omega_i) = (p-1)(d_Q+1) + p\,\mathrm{ord}_Q(\omega_i)$. We conclude that

$$\mathrm{ord}_P(\omega_i y^i) = p\,\mathrm{ord}_Q(\omega_i) + (p-1)(d_Q+1) - i d_Q. \tag{3-2}$$

This is nonnegative precisely when $\mathrm{ord}_Q(\omega_i) \geq -n_{Q,i}$. As $p \nmid d_Q$ and $0 \leq i \leq p-1$, the p integers in (3-2) are all distinct modulo p, and hence distinct, so

$$\mathrm{ord}_P(\omega) = \min_{0 \leq i \leq p-1} \mathrm{ord}_P(\omega_i y^i) \tag{3-3}$$

and ω is regular at P precisely when $\mathrm{ord}_Q(\omega_i) \geq -n_{Q,i}$ for all i.

Now suppose $Q \notin S'$, so that π is étale over Q. There are then p points $P = P_0, \ldots, P_{p-1}$ over Q, and $\mathcal{O}_{Y,P_j} \simeq \mathcal{O}_{X,Q}$ (and likewise for differentials) for all j. Under these identifications, the function y corresponds to regular functions $h, h+1, \ldots, h+(p-1)$ in each of the $\mathcal{O}_{Y,P_j} \simeq \mathcal{O}_{X,Q}$. Thus $\omega = \sum_i \omega_i y^i$ is regular at one (and hence every) point of Y over Q provided

$$\mathrm{ord}_Q\left(\sum_i \omega_i(h+j)^i\right) \geq 0$$

for $0 \leq j \leq p-1$. Define $\lambda_j = \sum_i \omega_i(h+j)^i \in \Omega_{X,Q}^1$, depending on ω. The λ_j and ω_i are related by the Vandermonde matrix $M = ((h+j)^i)_{0 \leq i,j \leq p-1}$:

$$\begin{pmatrix} (h+0)^0 & (h+0)^1 & \cdots & (h+0)^{p-1} \\ (h+1)^0 & (h+1)^1 & \cdots & (h+1)^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ (h+(p-1))^0 & (h+(p-1))^1 & \cdots & (h+(p-1))^{p-1} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{p-1} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{p-1} \end{pmatrix}.$$

The determinant of M is well-known to be

$$\prod_{1 \leq i, j \leq p, \, i \neq j} ((h+i) - (h+j)) = \prod_{i \neq j} (i-j) \in \mathbf{F}_p^\times,$$

and thus all ω_i are regular at Q if and only if all λ_j are. This completes the proof. $\qquad \square$

Remark 3.8. If $\pi(P) = Q'$, then the analogous condition that a meromorphic differential on Y be regular at P is considerably more complicated as $\operatorname{ord}_P(y) < 0$, so the function h in the proof of Lemma 3.7 has a pole at Q and the entries of M are no longer in $\mathcal{O}_{X,Q}$. One simple case will be analyzed in the proof of Proposition 3.12, while a more complete analysis is deferred to Lemma 5.16.

We next study a filtration on $\pi_* \Omega_Y^1$ arising from the $G \simeq \mathbf{Z}/p\mathbf{Z}$ action on Y. We fix a generator $\tau \in G$ and obtain a map of sheaves $\Omega_Y^1 \to \tau_* \Omega_Y^1$. After pushing forward using the equivariant π, we obtain a map $\tau : \pi_* \Omega_Y^1 \to \pi_* \Omega_Y^1$. This induces a map at the stalk at the generic point.

For technical reasons, instead of $\pi_* \Omega_Y^1$, we will work with a slightly larger sheaf of meromorphic differentials defined as follows. Recall that $d_{Q'}$ is the order of the pole of f above Q' and that $p | d_{Q'}$. Furthermore, y has a pole of order $d_{Q'}$ at any point above Q'. Recall we defined $n_{Q',i} = (p-1-i)d_{Q'}$.

Definition 3.9. Define the sheaf $\mathscr{F}_0 \subset \pi_* \Omega_{Y,\eta}^1$ on X which on an open set U has sections

$$\left\{ \omega = \sum_i \omega_i y^i : \omega \text{ is regular above all } Q \in U \setminus \{Q'\}, \text{ and } \operatorname{ord}_{Q'} \omega_i \geq -n_{Q',i} \, \forall i \text{ if } Q' \in U \right\}.$$

Lemma 3.10. *The natural G-action on $\pi_* \Omega_{Y,\eta}^1$ preserves $\mathscr{F}_0$, and $\pi_* \Omega_Y^1$ is a subsheaf of $\mathscr{F}_0$.*

Proof. As the first assertion is clear, it suffices to show that any meromorphic differential $\omega = \sum \omega_i y^i$ on Y that is regular above Q' has $\operatorname{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$. We will do so by descending induction on i. Observe that

$$(\tau - 1)^{p-1} \omega = (p-1)! \omega_{p-1}$$

which must be regular above Q': this happens only if $\operatorname{ord}_{Q'}(\omega_{p-1}) \geq 0$. To deal with $i = n$ for the inductive step, note that

$$(\tau - 1)^n \omega = n! \omega_n + f_{n+1}(y) \omega_{n+1} + \cdots + f_{p-1}(y) \omega_{p-1}, \tag{3-4}$$

where $f_j(y)$ is a polynomial of degree at most $j - n$ of y. As in the proof of Lemma 3.7, let $h, h+1, h+2, \ldots, h+(p-1) \in \mathcal{O}_{X,Q'}$ correspond to the image of y in the local rings $\mathcal{O}_{Y,P'}$ under the identifications $\mathcal{O}_{Y,P'} \simeq \mathcal{O}_{X,Q'}$ for all P' lying over Q'. Note that $\operatorname{ord}_{Q'}(h) = -d_{Q'}/p \geq -d_{Q'}$. Together with (3-4), the assumption that ω is regular above Q' gives

$$\operatorname{ord}_{Q'}\big(n! \omega_n + f_{n+1}(h) \omega_{n+1} + \cdots f_{p-1}(h) \omega_{p-1}\big) \geq 0.$$

Furthermore, $\operatorname{ord}_{Q'}(f_j(h)\omega_j) \geq (j-n)(-d_Q) - (p-1-j)d_{Q'} = -(p-1-n)d_{Q'}$ by our inductive hypothesis, and it follows that $\operatorname{ord}_{Q'}(\omega_n) \geq -(p-1-n)d_{Q'}$. $\qquad \square$

Thanks to Lemma 3.10, our fixed generator $\tau \in G$ induces a map $\tau : \mathscr{F}_0 \to \mathscr{F}_0$ that is compatible with the canonical G-action on $\pi_* \Omega_Y^1$. We use this to define a filtration on $\mathscr{F}_0$:

Definition 3.11. For $-1 \le i \le p-1$ let $W_i := \ker((\tau - 1)^{i+1} : \mathscr{F}_0 \to \mathscr{F}_0) \subset \mathscr{F}_0$.

Note that $W_{-1} = 0$ and $W_{p-1} = \mathscr{F}_0$.

Proposition 3.12. *For $0 \le i \le p-1$, we have split exact sequences of $\mathcal{O}_X$-modules*

$$0 \to W_{i-1} \to W_i \to \Omega_X^1(E_i) \to 0 \tag{3-5}$$

with E_i as in Definition 3.4. A splitting is given by sending a section ω of $\Omega_X^1(E_i)$ to ωy^i.

Proof. We compute that $(\tau - 1)$ reduces the degree in y:

$$(\tau - 1)\omega_j y^j = \omega_j((y+1)^j - y^j) = \omega_j(jy^{j-1} + \cdots).$$

Hence $(W_i)_\eta = \bigoplus_{j=0}^{i} (\Omega_X^1)_\eta y^j$. We define a map $\psi_i : W_i \to \Omega_X^1(E_i)$ by the formula

$$\psi_i\left(\sum_{j=0}^{i} \omega_j y^j\right) := \omega_i.$$

To check ψ_i is well-defined, given any element $\omega = \sum_{j=0}^{i} \omega_j y^j$ in the stalk of W_i at $Q \in X$, it suffices to check that $\mathrm{ord}_Q(\omega_i) \ge -\mathrm{ord}_Q(E_i)$. Recall the definition of E_i from Definition 3.4 and that $\mathrm{ord}_Q(E_i) = n_{Q,i}$. For $Q \ne Q'$, the claim follows immediately from Lemma 3.7, while for $Q = Q'$ it follows from the definitions; for the $i = 0$ case use Definition 3.9. One checks easily that the kernel of ψ_i is W_{i-1}, and that the map sending a section ω of $\Omega_X^1(E_i)$ to the section ωy^i of W_i indeed provides a splitting. $\square$

Remark 3.13. The filtration W_i on $\mathscr{F}_0$ is our replacement for the (perhaps more natural) filtration $V_i := \ker((\tau - 1)^i : \pi_* \Omega_Y^1 \to \pi_* \Omega_Y^1)$ on $\pi_* \Omega_Y^1$. Unfortunately, the corresponding exact sequence

$$0 \to V_{i-1} \to V_i \to \Omega_X^1(\widetilde{E}_i) \to 0$$

with $\widetilde{E}_i := \sum_{Q \in S} n_{Q,i}[Q]$ is *not* split: the natural splitting at the generic point given by $\omega \mapsto \omega y^i$ does not extend to a map of sheaves on X as ωy^i is not regular above Q' because y has a pole above Q'. It is precisely for this reason that we work instead with the sheaf $\mathscr{F}_0$ which has modified behavior above Q'.

Let $\mathrm{gr}^\bullet \mathscr{F}_0$ denote the associated graded sheaf for the filtration $\{W_i\}$.

Corollary 3.14. *There is an isomorphism of $\mathcal{O}_X$-modules*

$$\mathscr{F}_0 \simeq \mathrm{gr}^\bullet \mathscr{F}_0 \simeq \bigoplus_{i=0}^{p-1} \Omega_X^1(E_i). \tag{3-6}$$

Proof. The isomorphism is provided by the splittings of the sequences (3-5) of Proposition 3.12. $\square$

Remark 3.15. The isomorphism of Corollary 3.14 is only as $\mathcal{O}_X$-modules: it is not compatible with the Cartier operator.

4. The Cartier operator on stalks

We keep the notation and assumptions of Section 3. In particular, $\pi : Y \to X$ is a branched Galois cover of smooth, projective, and connected curves over $k = \bar{k}$ with group $G \simeq \mathbf{Z}/p\mathbf{Z}$. Recall that we have fixed a point Q' of X over which π is unramified, and that the degree-p Artin–Schreier extension $K' := k(Y)$ of $K = k(X)$ is the splitting field of $y^p - y = f$ with $f \in K$ *minimal* in the sense of Definition 3.1. Writing simply F for the absolute Frobenius morphism, we denote by $V_Y : F_* \Omega^1_Y \to \Omega^1_Y$ and $V_X : F_* \Omega^1_X \to \Omega^1_X$ the Cartier operators on Y and X respectively. In this section, we will analyze these maps at the generic points η' and η of Y and X, respectively, and will construct an isomorphism

$$(\ker V_Y)_{\eta'} \simeq (\pi_* \ker V_Y)_\eta \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta$$

which we will show gives rise to an inclusion of sheaves

$$\varphi : \pi_* \ker V_Y \hookrightarrow \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$$

for certain auxiliary divisors D_i on X that we are able to control. In the next section, we will analyze the image of this map.

Lemma 4.1. *For* $\omega \in \Omega^1_{Y, \eta'}$, *write* $\omega = \sum_{i=0}^{p-1} \omega_i y^i$ *with* $\omega_i \in \Omega^1_{X, \eta}$. *Then*

$$V_Y(\omega) = \sum_{j=0}^{p-1} \left(\sum_{i=j}^{p-1} V_X \left(\binom{i}{j} \omega_i (-f)^{i-j} \right) \right) y^j.$$

Proof. Using the relation $y = y^p - f$ and the fact that the Cartier operator is $1/p$-linear, we compute

$$V_Y(\omega_i y^i) = V_Y(\omega_i (y^p - f)^i)$$

$$= V_Y \left(\sum_{j=0}^i \omega_i \binom{i}{j} y^{pj} (-f)^{i-j} \right)$$

$$= \sum_{j=0}^i V_X \left(\omega_i \binom{i}{j} (-f)^{i-j} \right) y^j,$$

and the result follows by collecting y^j terms. $\square$

Corollary 4.2. *Let* $\omega_0, \ldots, \omega_{p-1} \in \Omega^1_{X, \eta}$ *and set* $\omega := \sum_{i=0}^{p-1} \omega_i y^i$. *Then* $V_Y(\omega) = 0$ *if and only if*

$$V_X(\omega_j) = - \sum_{i=j+1}^{p-1} V_X \left(\binom{i}{j} \omega_i (-f)^{i-j} \right) \tag{4-1}$$

for $0 \le j \le p - 1$. *In particular, if* $V_Y(\omega) = 0$ *then* $V_X(\omega_{p-1}) = 0$ *and* $V_X(\omega_j)$ *is determined via* (4-1) *by* ω_i *for* $i > j$.

Proof. Clear from Lemma 4.1. □

Recall the notation of Definition 3.4. Using Corollaries 2.7 and 2.9 with S enlarged to include Q' and the zeroes of f, for each $0 \le i \le p - 1$ pick a divisor $D_i = \sum_j [P_{i,j}]$ consisting of distinct points $P_{i,j}$ of X where f has neither pole nor zero (so π is unramified over $P_{i,j}$ by Lemma 3.5) and maps

$$r_i : F_*(\Omega^1_X(E_i)) \to \ker V_X(F_*(E_i + pD_i)),$$
$$s_i : \operatorname{Im} V_X(\bar{E}_i) \to F_*(\Omega^1_X(E_i + pD_i)). \tag{4-2}$$

These induce corresponding maps on stalks at the generic point of X, which we again denote simply by r_i and s_i, respectively. Note that, by the very construction of these maps, any $\omega_i \in \Omega^1_{X,\eta}$ is determined by $V_X(\omega_i)$ and $v_i := r_i(\omega_i)$ via

$$\omega_i = v_i + s_i(V_X(\omega_i)). \tag{4-3}$$

Definition 4.3. Given the fixed choices of r_i and D_i above, we define a map of $\mathcal{O}_X$-modules

$$\varphi : \pi_* \ker V_Y \hookrightarrow F_* \mathscr{F}_0 \simeq F_* \operatorname{gr}^\bullet \mathscr{F}_0$$

$$\simeq \bigoplus_{i=0}^{p-1} F_*(\Omega^1_X(E_i)) \xrightarrow{\oplus r_i} \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)) \tag{4-4}$$

with the first inclusion coming from Lemma 3.10 and the middle isomorphisms coming from Corollary 3.14.

Let φ_η denote the induced map on stalks at the generic point of X.

Proposition 4.4. *The map φ_η is an isomorphism of $k(X)$-vector spaces. For $\omega = \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \ker V_Y)_\eta$ we have*

$$\varphi_\eta(\omega) = (r_0(\omega_0), r_1(\omega_1), \ldots, r_{p-1}(\omega_{p-1})).$$

Proof. Let $v := (v_0, \ldots, v_{p-1}) \in \bigoplus_{i=0}^{p-1} (\ker V_X)_\eta$ be arbitrary, and also define $\omega_0, \ldots, \omega_{p-1} \in \Omega^1_{X,\eta}$ as follows. Set $\omega_{p-1} := v_{p-1}$, and if ω_i has been defined for $i > j$, then define

$$\omega_j := v_j + s_j\left(-\sum_{i=j+1}^{p-1} V_X\left(\binom{i}{j}\omega_i(-f)^{i-j}\right)\right). \tag{4-5}$$

By construction, since $V_X \circ s_j$ is the natural inclusion, the differentials ω_j satisfy the relation (4-1) for all j, so that $\omega := \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \Omega^1_Y)_\eta$ lies in $(\pi_* \ker V_Y)_\eta$ thanks to Corollary 4.2. One checks easily that the resulting map $v \mapsto \omega$ is inverse to φ_η. □

Remark 4.5. Notice that the proof of Proposition 4.4 shows that for fixed j, ω_j depends only on v_i for $j \ge i$, and in particular that $\omega_i = 0$ for $i \ge j$ if and only if $v_i = 0$ for $i \ge j$.

Unfortunately, (4-4) is not itself an isomorphism:

Example 4.6. Let $p = 5$, $X = \mathbf{P}^1$, and consider the Artin–Schreier covers $Y_i \to X$ for $i = 1, 2$ given by $y^p - y = f_i$, where $f_1(t) = t^{-3}$, and $f_2(t) = t^{-3} + t^{-2}$; each of these covers is ramified only over $Q := 0$

and has $d_Q = 3$. The *a*-number of Y_1 is 4, while the *a*-number of Y_2 is 3. Using the simple projectors of Remark 2.2, which allow us to take $D_i = 0$ for all $0 \leq i \leq p - 1$, we obtain for $Y = Y_1, Y_2$ a map

$$\varphi : \pi_* \ker V_Y \to \bigoplus_{i=0}^{4} \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q])).$$

The induced map on global sections is injective (as φ_η is an isomorphism), but need not be an isomorphism in general. Indeed, given an element $(v_0, \ldots, v_4) \in \bigoplus_i H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q])))$, let us investigate when it lies in the image of φ.

Recalling Definition 3.4, we calculate

$$n_{Q,i} = \begin{cases} 0, & i = 4, \\ 1, & i = 3, \\ 2, & i = 2, \\ 2, & i = 1, \\ 3, & i = 0, \end{cases} \quad \text{and} \quad \dim_{\mathbf{F}_p} H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q]))) = \begin{cases} 0, & i = 4, \\ 0, & i = 3, \\ 1, & i = 2, \\ 1, & i = 1, \\ 2, & i = 0, \end{cases}$$

so the elements of $\bigoplus_i H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_*(n_{Q,i}[Q])))$ are easy to describe. The space is four dimensional, with basis

$$v_{0,3} = (t^{-3}dt, 0, 0, 0, 0), \quad v_{0,2} = (t^{-2}dt, 0, 0, 0, 0),$$
$$v_{1,2} = (0, t^{-2}dt, 0, 0, 0), \quad v_{2,2} = (0, 0, t^{-2}dt, 0, 0).$$

Using (4-5), we may compute the (unique) preimage under φ_η of these differentials for each of the covers under consideration. For Y_1, we find

$$\varphi_\eta^{-1}(v_{0,3}) = t^{-3}dt,$$
$$\varphi_\eta^{-1}(v_{0,2}) = t^{-2}dt,$$
$$\varphi_\eta^{-1}(v_{1,2}) = t^{-2}dt \cdot y,$$
$$\varphi_\eta^{-1}(v_{2,2}) = t^{-2}dt \cdot y^2.$$

On the other hand, for Y_2 we have

$$\varphi_\eta^{-1}(v_{0,3}) = t^{-3}dt,$$
$$\varphi_\eta^{-1}(v_{0,2}) = t^{-2}dt,$$
$$\varphi_\eta^{-1}(v_{1,2}) = t^{-2}dt \cdot y,$$
$$\varphi_\eta^{-1}(v_{2,2}) = -t^{-6}dt + t^{-2}dt \cdot y^2.$$

The computation of $\varphi_\eta^{-1}(v_{2,2})$ is special, as it depends on the exact choice of Artin–Schreier equation $y^p - y = f$. With $f = f_1 = t^{-3}$, we have

$$\varphi_\eta^{-1}(v_{2,2}) = t^{-2}dty^2$$

because there are no terms of the form $t^i dt$ with $i \equiv -1 \bmod 5$ appearing in $V_{\mathbf{P}^1}(\omega_0)$ when using Corollary 4.2. On the other hand, for $f = f_2 = t^{-3} + t^{-2}$ we have

$$\varphi_\eta^{-1}(v_{2,2}) = -t^{-6}dt + t^{-2}dt\, y^2$$

since $V_{\mathbf{P}^1}(\omega_0) = -V_{\mathbf{P}^1}(t^{-2}(t^{-6} + 2t^{-5} + t^{-4})dt)$. This is not an element of $H^0(Y, \Omega_Y^1)$ because of the $t^{-6}dt$ term.

All of the other differentials showing up are regular. Thus, Y_1 has a-number 4, while Y_2 has a-number 3. This behavior illustrates why the a-number of the cover cannot depend only on the d_Q and must incorporate finer information (in this case, expressed as whether certain coefficients of powers of f are nonzero).

This example shows that to ensure the regularity of $\varphi_\eta^{-1}(v_i)$, the coefficients of the v_i need to satisfy certain relations (in this case, the coefficient of $t^{-2}dt$ in v_2 must be zero). These relations describe the image of φ, and are the subject of the next section. This example will be reinterpreted in that context in Example 5.21.

5. Short exact sequences and the kernel

Using the conventions and notation of Section 3–4, and motivated by Example 4.6, the goal of this section is to describe the image of the map (4-4) by means of a collection of linear relations on the coefficients of local expansions of meromorphic differentials at a fixed set of closed points containing the branch locus S. These linear relations will be encoded via an ascending filtration by subsheaves

$$0 \subset \mathscr{G}_{-1} \subset \mathscr{G}_0 \subset \cdots \subset \mathscr{G}_{p-1} \subset \mathscr{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)), \tag{5-1}$$

with $\mathscr{G}_{-1} = \mathrm{Im}(\varphi)$ and whose successive gradeds are identified with explicit skyscraper sheaves; see Theorem 5.20 for a precise statement.

For most of the argument, we will work with the sheaf $\mathscr{F}_0$ of Definition 3.9 and variants thereof having modified behavior at the auxiliary point Q'. Only at the very end (see Lemma 5.16) will we properly account for the modified behavior at Q' in order to recover the desired $\mathrm{Im}(\varphi) \simeq \pi_* \ker V_Y$, rather than the (generally larger) $\pi_* \ker(V_Y|_{\mathscr{F}_0})$. At a first reading, it could be useful to assume that $X = \mathbf{P}^1$ to remove the need for Q' and the divisors D_i coming from Corollary 2.7.

5A. *Filtration and skyscraper sheaves.* The first step is to define the filtration (5-1). Recall we have fixed projectors r_i as in (4-2) and used them to define φ in Definition 4.3. Recall the notation of Definition 3.4, noting Remark 3.6, and for $Q \in X$ choose a local uniformizer t_Q at Q.

If A is an abelian group, then by a slight abuse of notation we will again write A to denote the constant sheaf associated to A. For any k-valued point $P : \mathrm{Spec}\, k \to X$, we write P_*A for the pushforward of (the constant sheaf) A along P; it is the skyscraper sheaf on X supported at P with stalk $(P_*A)_P = A$. Likewise, if $h : A \to B$ is any homomorphism of abelian groups, we write $P_*h : P_*A \to P_*B$ for the induced morphism of sheaves on X.

Definition 5.1. Set $S'_j := S' \cup \sup(D_j)$. Define $g_{j+1} := \rho_j \circ \iota$, where ι is the inclusion

$$\iota : \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)) \hookrightarrow \bigoplus_{i=0}^{p-1}(\ker V_X)_\eta \xrightarrow[\simeq]{\varphi_\eta^{-1}} (\pi_* \ker V_Y)_\eta \hookrightarrow F_*(\pi_* \Omega_Y^1)_\eta \xrightarrow[\simeq]{(1\text{-}5)} \bigoplus_{i=0}^{p-1} F_* \Omega_{X,\eta}^1$$

and ρ_j is the composite

$$\rho_j : \bigoplus_{i=0}^{p-1} F_* \Omega_{X,\eta}^1 \xrightarrow{\pi_j} F_* \Omega_{X,\eta}^1 \xrightarrow{\sigma_j} F_* \bigoplus_{Q \in S_j'} Q_* \left(\Omega_{X,Q}^1 \left[\frac{1}{t_Q} \right] / t_Q^{-n_{Q,j}} \Omega_{X,Q}^1 \right).$$

Here, the first and third maps in the definition of ι are the natural inclusions, the map π_j is projection onto the j-th summand of the direct sum, and σ_j is induced by the identifications $\Omega_{X,\eta}^1 = \Omega_{X,Q}^1[1/t_Q]$ and the canonical quotient maps.

If $v := (v_0, \ldots, v_{p-1})$ is a section of $\bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$ with $\iota(v) = (\omega_0, \ldots, \omega_{p-1})$ then $r_i(\omega_i) = v_i$ and writing $\omega := \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \ker V_Y)_\eta$ we have $\varphi_\eta(\omega) = v$ and $g_{j+1}(v) = \sigma_j(\omega_j)$.

Definition 5.2. For $0 \le j \le p$, let $\mathscr{G}_j$ be the $\mathcal{O}_X$-submodule

$$\mathscr{G}_j \subset \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$$

whose sections are precisely $v = (v_0, \ldots, v_{p-1})$ with $\iota(v) = (\omega_0, \ldots, \omega_{p-1})$ such that ω_i is a section of $F_*(\Omega_X^1(E_i))$ whenever $j \le i \le p-1$. Define $\mathscr{G}_{-1} = \operatorname{Im}(\varphi)$.

Lemma 5.3. *Suppose that $0 \le j \le p-1$. Then*:

(i) $\mathscr{G}_j = \ker(g_{j+1}|_{\mathscr{G}_{j+1}})$.

(ii) *For $Q \in X - S_j'$, the inclusion $\mathscr{G}_j \hookrightarrow \mathscr{G}_{j+1}$ induces an isomorphism on stalks at Q.*

(iii) *The action of $G = \mathbf{Z}/p\mathbf{Z}$ on $(F_* \pi_* \Omega_Y^1)_\eta \xrightarrow{\varphi_\eta} \bigoplus_i (\ker V_X)_\eta$ preserves $\mathscr{G}_j$.*

(iv) $\mathscr{G}_0 \simeq \pi_* \ker(V_Y|_{\mathscr{F}_0})$.

(v) $\mathscr{G}_{-1} \simeq \pi_* \ker V_Y$.

(vi) $\mathscr{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$.

Proof. Let $v = (v_0, \ldots, v_{p-1})$ be a section of $\bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i))$ over an open set $U \subseteq X$ with $\iota(v) = (\omega_0, \ldots, \omega_{p-1})$. By definition, if $v \in \mathscr{G}_{j+1}(U)$ then $\omega_i \in F_*(\Omega_X^1(E_i))(U)$ for $i > j$. In addition, ω_j lies in $F_*(\Omega_X^1(E_j))(U)$ if and only if

$$\operatorname{ord}_Q(\omega_j) \ge -\operatorname{ord}_Q(E_j) = -n_{Q,j} \tag{5-2}$$

for every point Q of U. Condition (5-2) holds automatically for $Q \notin S_j'$. On the other hand, v lies in the kernel of g_{j+1} if and only if $\omega_j \in t_Q^{-n_{Q,j}} \Omega_{X,Q}^1$ for all $Q \in S_j' \cap U$, which is equivalent to the condition (5-2), and this gives the claimed identification $\mathscr{G}_j = \ker(g_{j+1}|_{\mathscr{G}_{j+1}})$ in (i)enumi. This also shows (ii)enumi.

Since $E_i \ge E_j$ if $i < j$ and the induced action of a generator τ of G on $\mathscr{G}_p$ from Proposition 4.4 is given by $\tau(y) = y + 1$, it is clear that this action preserves $\mathscr{G}_j$ for each j.

It follows from Lemma 3.7 and Definition 5.2 that $v \in \mathscr{G}_0(U)$ if and only if $\omega := \sum_{i=0}^{p-1} \omega_i y^i \in (\pi_* \Omega_Y^1)_\eta$ is regular over every point Q of U except possibly $Q = Q'$. It is then clear from the definition of the map ι, Definition 3.9, and Definition 4.3 that $\omega \in \mathscr{F}_0(U)$ and ω is killed by V_Y; that is, ω is a section of $\pi_* \ker(V_Y|_{\mathscr{F}_0})$, which gives the identification (iv)enumi. Since φ is injective with source $\pi_* \ker V_Y$, it induces an isomorphism $\pi_* \ker V_Y \simeq \mathrm{Im}(\varphi) =: \mathscr{G}_{-1}$ as in (v)enumi, and the description (vi)enumi of $\mathscr{G}_p$ is clear from definitions. $\qquad\square$

The main step is to analyze the quotient $\mathscr{G}_{j+1}/\mathscr{G}_j$: it will be a skyscraper sheaf supported on S_j'. We will do so by studying the image of g_{j+1}. In particular, for $Q \in S_j'$ we will define $\mathcal{O}_{X,Q}$-modules $M_{j,Q}$ and a map of skyscraper sheaves

$$c_j : \mathrm{Im}(g_{j+1}|_{\mathscr{G}_{j+1}}) \to \bigoplus_{Q \in S_j'} Q_*(M_{j,Q}). \tag{5-3}$$

Composing with the restriction of g_{j+1} to $\mathscr{G}_{j+1}$, we obtain maps of sheaves on X

$$g_{j+1}' = c_j \circ (g_{j+1}|_{\mathscr{G}_{j+1}}) : \mathscr{G}_{j+1} \to \bigoplus_{Q \in S_j'} Q_*(M_{j,Q})$$

which we will show via stalk-wise calculations induce isomorphisms

$$g_{j+1}' : \mathscr{G}_{j+1}/\mathscr{G}_j \xrightarrow{\sim} \bigoplus_{Q \in S_j'} Q_*(M_{j,Q}),$$

for $0 \le j \le p - 1$, thereby providing an explicit description of these quotients.

In order to motivate the definition of $M_{j,Q}$, we first record a result about orders of vanishing that will be useful in what follows.

Lemma 5.4. *Let $v = (v_0, \ldots, v_{p-1}) \in H^0(X, \mathscr{G}_p)$ and set $(\omega_0, \ldots, \omega_{p-1}) := \iota(v)$. For $0 \le j \le p - 1$:*

(i) *For fixed j and $Q \in S$, suppose that $v_i = 0$ for $i > j$ and $\mathrm{ord}_Q(v_j) \ge -n$ for some nonnegative integer n. Let $\mu_{Q,i}$ be the largest multiple of p such that $\mu_{Q,i} + 1 \le n + d_Q(j - i)$. Then $\omega_i = 0$ for $i > j$, $\mathrm{ord}_Q(\omega_j) \ge -n$, and for $i < j$*

$$\mathrm{ord}_Q(\omega_i) \ge \min(-\mu_{Q,i} - 1, \mathrm{ord}_Q(v_i)) \ge -n - d_Q(j - i). \tag{5-4}$$

(ii) *If $Q \in D_j$ then $\mathrm{ord}_Q(\omega_j) \ge -p$.*

(iii) *$\mathrm{ord}_{Q'}(\omega_j) \ge -d_{Q'}(p - 1 - j)$.*

Proof. Suppose first that $Q \in S$, fix j with $0 \le j \le p - 1$, and assume that $v_i = 0$ for $i > j$. Then $\omega_i = 0$ for $i > j$ thanks to Remark 4.5, while if $i = j$ we have $\omega_j = v_j$ thanks to (4-5), so our assumption that $\mathrm{ord}_Q(v_j) \ge -n$ gives $\mathrm{ord}_Q(\omega_j) \ge -n$ as well. Thus to establish (i)enumi, it remains to prove (5-4) for each $i < j$. We will establish this by descending induction on i. So suppose that $\ell < j$ and that (5-4)

holds for all i with $\ell < i < j$. Since $\omega_i = 0$ for $i > j$, Corollary 4.2 gives $V_X(\omega_\ell) = V_X(\xi)$ for

$$\xi := - \sum_{i=\ell+1}^{j} \binom{i}{\ell} \omega_i (-f)^{i-\ell}. \tag{5-5}$$

Since $\mathrm{ord}_Q(f) = -d_Q$, our inductive hypothesis and the already established $\mathrm{ord}_Q(\omega_j) \geq -n$ immediately imply that

$$\mathrm{ord}_Q(\xi) \geq \min_{\ell < i \leq j} \left\{ -n - d_Q(j-i) - d_Q(i-\ell) \right\} = -n - d_Q(j-\ell).$$

Using (1-4), we see that $\mathrm{ord}_Q(V_X(\xi)) \geq -\mu_{Q,\ell}/p - 1$. It then follows from Corollary 2.9 that

$$\mathrm{ord}_Q(s_\ell(V_X(\xi))) \geq -\mu_{Q,\ell} - 1 \geq -n - d_Q(j-\ell),$$

so using (4-3) and remembering that $\mathrm{ord}_Q(v_\ell) \geq -n_{Q,\ell} = -\lceil (p-1-\ell)d_Q/p \rceil \geq -d_Q$ yields (5-4) for $i = \ell$, completing the inductive step.

The proof of (ii)enumi proceeds by a similar—but simpler—argument, using descending induction on j and the fact that for $Q \in D_j$ one has $\mathrm{ord}_Q(f) = 0$ and $\mathrm{ord}_Q(v_j) \geq -p$ by definition; we leave the details to the reader. Case (iii)enumi likewise follows from a similar argument, using $\mathrm{ord}_{Q'}(f) = -d_{Q'}$ and $\mathrm{ord}_{Q'}(v_j) \geq -(p-1-j)d_{Q'}$ by definition; see the proof of Lemma 5.14 for a more detailed version of the analysis in this case. $\square$

Corollary 5.5. *Let* $v = (v_0, \ldots, v_{p-1}) \in H^0(X, \mathscr{G}_p)$ *and set* $(\omega_0, \ldots, \omega_{p-1}) := \iota(v)$. *For* $Q \in S$, *we have that*

$$\mathrm{ord}_Q(\omega_i) \geq -d_Q(p-1-i).$$

Proof. Take $j = p-1$ and $n = 0$ in Lemma 5.4(i)enumi. $\square$

For a $k[\![t_Q]\!]$-module M, let $F_* M$ denote the $k[\![t_Q^p]\!]$-module with underlying additive group M and the action of t_Q^p on $F_* M$ given by multiplication by t_Q on M. Recall the definition of $n_{Q,i}$ from Definition 3.4.

Definition 5.6. For $0 \leq j \leq p-1$ and $Q \in S'_j$ let $m_{Q,j} := p(n_{Q,j} - 1)$ and $\beta_{Q,j} := \lfloor (p-1)n_{Q,j}/p \rfloor$, and set

$$M_{j,Q} := \begin{cases} k[\![t_Q^p]\!]/(t_Q^{p\beta_{,j}}) = F_*(k[\![t_Q]\!]/(t_Q^{\beta_{Q,j}})) & \text{if } Q \in S, \\ k[\![t_Q^p]\!]/(t_Q^{p(p-1)}) = F_*(k[\![t_Q]\!]/(t_Q^{p-1})) & \text{if } Q \in \sup(D_j), \\ 0 & \text{if } Q = Q', \end{cases}$$

considered as an $\mathcal{O}_{X,Q}$-module via $\mathcal{O}_{X,Q} \hookrightarrow \mathcal{O}_{X,Q}^{\wedge} \simeq k[\![t_Q]\!]$. Putting these together, we define a skyscraper sheaf on X

$$M_j := \bigoplus_{Q \in S'_j} Q_*(M_{j,Q}).$$

Note that by construction, if $Q \in S'_j$ then the stalk of M_j at Q is precisely $M_{j,Q}$, which justifies the notation.

Remark 5.7. For $Q \in S$, one checks easily that $m_{Q,j}$ is the largest integral multiple of p with $-m_{Q,j} \geq -(p-1-j)d_Q + 1$, and that $\beta_{Q,j}$ is the number of integer multiples of p between $-m_{Q,j}$ and $-n_{Q,j}$ inclusive. In view of Corollary 5.5, the skyscraper sheaves M_j will record all of the possible ways in which ω_j could fail to be a section of $\Omega^1_X(E_j)$ when $(\omega_0, \ldots, \omega_{p-1}) = \iota(v)$ for a section v of $\mathscr{G}_{j+1}$.

In the next subsections, we will define the maps (5-3) stalk-by-stalk and check that they are surjective.

5B. *Local calculations above ramified points.* Fix j, let $Q \in S$ and as before let t_Q be a uniformizer at Q. For $v := (v_0, \ldots, v_{p-1}) \in \mathscr{G}_{p,Q}$, put $(\omega_0, \ldots, \omega_{p-1}) := \iota(v)$ and define $\omega = \sum_{i=0}^{p-1} \omega_i y^i$, so that $\varphi_\eta(\omega) = v$ and $V_X(\omega) = 0$. By (4-5), we have

$$\omega_j = v_j + s_j(V_X(\xi)), \qquad \text{where } \xi := -\sum_{i=j+1}^{p-1} \binom{i}{j}\omega_i(-f)^{i-j}, \tag{5-6}$$

and $\operatorname{ord}_Q(\xi) \geq -(p-1-j)d_Q$ thanks to Corollary 5.5. Working in the completion $\mathcal{O}^\wedge_{X,Q} \simeq k[\![t_Q]\!]$, we may therefore expand

$$\xi = \sum_{\ell=-N+1}^{\infty} a_\ell t_Q^\ell \frac{dt_Q}{t_Q} \tag{5-7}$$

where $N := -(p-1-j)d_Q$. Using (1-4), we then compute

$$s_j(V_X(\xi)) = \sum_{\ell=-m_{Q,j}/p}^{\infty} a_{p\ell} s_j\left(t_Q^\ell \frac{dt_Q}{t_Q}\right), \tag{5-8}$$

where $m_{Q,j}$ is as in Definition 5.6. If $p\ell - 1 \geq -n_{Q,j}$ then $\operatorname{ord}_Q(s_j(t_Q^\ell(dt_Q/t_Q))) \geq -n_{Q,j}$ by (2-8). Since we know that $\operatorname{ord}_Q(v_j) \geq -n_{Q,j}$, it follows from (5-6) that

$$g_{j+1}(v_0, \ldots, v_{p-1}) \in \operatorname{Im}(g_{j+1})_Q \subset F_*\left(\Omega^1_{X,Q}\left[\frac{1}{t_Q}\right]/t_Q^{-n_{Q,j}}\Omega^1_{X,Q}\right).$$

Furthermore, $g_{j+1}(v_0, \ldots, v_{p-1})$ is determined by the $a_{p\ell}$ for $-m_{Q,j} \leq p\ell \leq -n_{Q,j}$. There are $\beta_{Q,j}$ such integers by definition (see Remark 5.7). We thus obtain an injection $c_{j,Q} : \operatorname{Im}(g_{j+1})_Q \hookrightarrow M_{j,Q} = k[\![t_Q^p]\!]/(t_Q^{p\beta_{Q,j}})$ given by

$$c_{j,Q}: \sum_{\ell=-m_{Q,j}/p}^{\infty} a_{p\ell} s_j\left(t_Q^\ell \frac{dt_Q}{t_Q}\right) \bmod t_Q^{-n_{Q,j}} k[\![t_Q]\!]dt_Q \mapsto \sum_{\ell=0}^{\beta_{Q,j}-1} a_{-m_{Q,j}+p\ell} t_Q^{p\ell} \bmod t_Q^{p\beta_{Q,j}} k[\![t]\!]. \tag{5-9}$$

One checks that this is a well-defined and $\mathcal{O}_X$-linear map, where the action of $\mathcal{O}_X$ on the first factor is the natural one coming from localizing the action on $F_*\Omega^1_X$ and the action of the second factor comes from the action of $\mathcal{O}_{X,Q}$ on $F_*k[\![t_Q]\!]$ via the identification $\mathcal{O}^\wedge_{X,Q} \simeq k[\![t_Q]\!]$.

Definition 5.8. For $Q \in S$ and $0 \leq j \leq p-1$ we define $g'_{j+1,Q}$ as the composite map of $\mathcal{O}_{X,Q}$-modules

$$g'_{j+1,Q} : \mathscr{G}_{j+1,Q} \xrightarrow{(g_{j+1}|_{\mathscr{G}_{j+1}})_Q} \operatorname{Im}(g_{j+1}|_{\mathscr{G}_{j+1}})_Q \xrightarrow{c_{j,Q}} M_{j,Q}$$

Proposition 5.9. *For $Q \in S$ and $0 \leq j \leq p-1$, the map $g'_{j+1,Q}$ is surjective.*

To prove this, we will make use of the following technical result. As above, we identify $\mathcal{O}^{\wedge}_{X,Q}$ with $k[\![t_Q]\!]$ and note that the completion of $\Omega^1_{X,Q}$ is isomorphic to $\Omega^1_{X,Q} \otimes_0 \mathcal{O}_{X,Q}\mathcal{O}^{\wedge}_{X,Q} \simeq k[\![t_Q]\!]dt_Q$.

Lemma 5.10. *Given $\omega_0, \ldots, \omega_{p-2} \in k(\!(t_Q)\!)dt_Q$ satisfying $\mathrm{ord}_Q(\omega_i) \geq -(p-1-i)d_Q$ for $0 \leq i \leq p-2$, there exists $\omega_{p-1} \in k[\![t_Q]\!]dt_Q$ such that $\omega := \sum_{i=0}^{p-1} \omega_i y^i$ is an element of $(\pi_* \ker V_Y)_Q \otimes_0 \mathcal{O}_{X,Q}k(\!(t_Q)\!)$.*

Proof of Proposition 5.9. Assuming Lemma 5.10, we can prove Proposition 5.9 easily. As in the proof of Lemma 2.5, it suffices to check surjectivity after tensoring with $\mathcal{O}^{\wedge}_{X,Q}$ for each $Q \in S$. Note that the completion $\mathcal{G}^{\wedge}_{j+1,Q}$ is isomorphic to $\mathcal{G}_{j+1,Q} \otimes_0 \mathcal{O}_{X,Q}\mathcal{O}^{\wedge}_{X,Q}$.

Pick any $\omega_0, \ldots \omega_{j-1}, \omega_{j+1}, \ldots, \omega_{p-2} \in k(\!(t_Q)\!)dt_Q$ with $\mathrm{ord}_Q(\omega_i) \geq -n_{Q,i}$ for $i \neq j, p-1$, and set

$$\omega_j := s_j\left(t_Q^{-m_{Q,j}/p}\frac{dt_Q}{t_Q}\right).$$

By Corollary 2.9 and Definition 5.6, we have $\mathrm{ord}_Q(\omega_j) \geq -m_{Q,j} - 1 \geq -(p-1-j)d_Q$. As we visibly also have $-n_{Q,i} \geq -(p-1-i)d_Q$ for $i \neq j, p-1$, we may apply Lemma 5.10 to find ω_{p-1} such that $\omega := \sum_{i=0}^{p-1} \omega_i y^i$ lies in $(\pi_* \ker V_Y)_Q \otimes_0 \mathcal{O}_{X,Q}k(\!(t_Q)\!)$ and $\mathrm{ord}_Q(\omega_{p-1}) \geq 0$. Put $(v_0, \ldots, v_{p-1}) := \varphi_\eta(\omega)$, and note that as $v_i = r_i(\omega_i)$ for all i, when $i \neq j$ we have $\mathrm{ord}_Q(v_i) \geq -n_{Q,i}$. On the other hand, $v_j = 0$ since $r_j \circ s_j = 0$ (as $r(m) = m - s(V_X(m))$ in Corollary 2.9). Thus $(v_0, \ldots, v_{p-1}) \in \mathcal{G}^{\wedge}_{j+1,Q}$. By the definition of $c_{j,Q}$ in (5-9) and the choice of ω_j, we have arranged that

$$g'_{j+1,Q}(v_0, \ldots v_{p-1}) = 1 \in k[\![t_Q^p]\!]/(t^{p\beta_{Q,j}}) = M_{j,Q} = M_{j,Q} \otimes_{\mathcal{O}_{X,Q}} \mathcal{O}^{\wedge}_{X,Q}.$$

This suffices to show that $g'_{j+1,Q}$ is surjective, as it is a map of $\mathcal{O}_{X,Q}$-modules. $\qquad\square$

Proof of Lemma 5.10. For $h \in k(\!(t_Q)\!)$, let us write $\mathrm{coef}_m(h)$ for the coefficient of t_Q^m in h, and for $\xi \in k(\!(t_Q)\!)dt_Q$, write $\mathrm{coef}_m(\xi)$ for the coefficient of $t_Q^m(dt_Q/t_Q)$ in ξ. Given $\omega_0, \ldots, \omega_{p-2}$ as in the statement of the lemma, it follows from the formula

$$0 = V_Y(\omega) = \sum_{j=0}^{p-1}\left(\sum_{i=j}^{p-1} V_X\left(\binom{i}{j}\omega_i(-f)^{i-j}\right)\right)y^j$$

of Lemma 4.1 that our goal is to construct $\omega_{p-1} \in k[\![t_Q]\!]dt_Q$ satisfying

$$V_X\left(\binom{p-1}{j}\omega_{p-1}(-f)^{p-1-j}\right) = -\sum_{i=j}^{p-2} V_X\left(\binom{i}{j}\omega_i(-f)^{i-j}\right) \tag{5-10}$$

for $0 \leq j < p-1$. By (1-4) it suffices to check that for each j and every $m \equiv 0 \bmod p$, the m-th coefficients of both sides of (5-10) agree. So we need to show that

$$\sum_i \binom{p-1}{j} \mathrm{coef}_i(\omega_{p-1}) \mathrm{coef}_{m-i}((-f)^{p-1-j}) = -\mathrm{coef}_m\left(\sum_{\ell=j}^{p-2}\binom{\ell}{j}\omega_\ell(-f)^{\ell-j}\right). \tag{5-11}$$

The right side is determined by the choice of $\omega_0, \ldots, \omega_{p-2}$. We observe:

(i) $\mathrm{coef}_i((-f)^{p-1-j}) = 0$ for $i < -(p-1-j)d_Q$;

(ii) $\mathrm{coef}_{-(p-1-j)d_Q}((-f)^{p-1-j}) \neq 0$;

(iii) $\mathrm{coef}_m\left(\sum_{\ell=j}^{p-2}\binom{\ell}{j}\omega_\ell(-f)^{\ell-j}\right) = 0$ for $m \leq -(p-1-j)d_Q$.

Notice that (i)enumi and (ii)enumi are immediate consequences of the fact that $\mathrm{ord}_Q(f) = -d_Q$, while (iii)enumi follows from the fact that by hypothesis

$$\mathrm{ord}_Q(\omega_\ell(-f)^{\ell-j}) \geq -(p-1-\ell)d_Q + d_Q(j-\ell) = -(p-1-j)d_Q.$$

We construct $\omega_{p-1} = \sum_i b_i t_Q^i (dt_Q/t_Q)$ by specifying $b_i = \mathrm{coef}_i(\omega_{p-1})$ inductively as follows. For $i \leq 0$, set $b_i = 0$; this choice implies that for each j and any $m \leq -(p-1-j)d_Q$ that is a multiple of p, the left side of (5-11) vanishes, and likewise for the right side by (iii)enumi. To specify b_1, choose $0 \leq j < p$ so $d_Q(p-1-j) \equiv 1 \bmod p$. In light of (i)enumi and the fact that $b_i = 0$ for $i \leq 0$, (5-11) with $m = -d_Q(p-1-j)+1$ specifies that

$$\cdots + 0 + \binom{p-1}{j}b_1 \mathrm{coef}_{-d_Q(p-1-j)}((-f)^{p-1-j}) + 0 + \cdots$$
$$= -\mathrm{coef}_{-d_Q(p-1-j)+1}\left(\sum_{\ell=j}^{p-2}\binom{\ell}{j}\omega_\ell(-f)^{\ell-j}\right),$$

where the right side has already been specified. By (ii)enumi, there is a unique solution b_1.

In general, if N is any positive integer and b_i has been chosen for all $i < N$, first choose $0 \leq j < p$ so that $d_Q(p-1-j) \equiv N \bmod p$. The right side of (5-11) with $m = -d_Q(p-1-j)+N$ is already specified. The finitely many nonzero terms of the left side with $i < N$ are determined by our previous choices, while the terms with $i > N$ are zero by (i)enumi. By (ii)enumi, we may uniquely solve for b_N. Since $p \nmid d_Q$, when considering b_i with i between b and $b+(p-1)$, each j satisfying $0 \leq j < p$ occurs once. Thus the inductive choice of the b_i's makes (5-11) hold for every j and every m that is a multiple of p. This completes the proof. $\qquad\square$

5C. *Local calculations at poles of sections.* We repeat the analysis of Section 5B for points $Q \in \mathrm{sup}(D_j)$. Fix j and $Q \in \mathrm{sup}(D_j)$, let t_Q be a uniformizer of $\mathcal{O}_{X,Q}$, and let $v := (v_0, \ldots, v_{p-1}) \in \mathscr{G}_{p,Q}$. As before, we put $\omega := \sum_{i=0}^{p-1} \omega_i y^i$ where $(\omega_0, \ldots, \omega_{p-1}) = \iota(v)$, and note that $V_Y(\omega) = 0$.

We have $\mathrm{ord}_Q(\omega_j) \geq -p$ thanks to Lemma 5.4(ii)enumi, whence a local expansion $\omega_j = \sum_{i=-p}^{\infty} b_i t_Q^i dt_Q$ in the completed stalk at Q. By definition, we then have $g_{j+1,Q}(v) \equiv \sum_{i=-p}^{-1} b_i t_Q^i dt_Q \bmod k[\![t_Q]\!]dt_Q$, and we define a map $c_{j,Q} : \mathrm{Im}(g_{j+1})_Q \to M_{j,Q} = k[\![t_Q]\!]/(t_Q^{p-1})$ by

$$c_{j,Q} : \sum_{i=-p}^{-1} b_i t_Q^i dt_Q \bmod k[\![t_Q]\!]dt_Q \mapsto \sum_{i=0}^{p-2} b_{i-p} t_Q^i \bmod t_Q^{p-1} k[\![t]\!]. \tag{5-12}$$

This is a well-defined map of $\mathcal{O}_{X,Q}$-modules, using the natural maps $\mathcal{O}_{X,Q} \hookrightarrow \mathcal{O}_{X,Q}^\wedge \simeq k[\![t_Q]\!]$.

Definition 5.11. For $0 \leq j \leq p-1$ and $Q \in \sup(D_j)$ we define $g'_{j+1,Q}$ as the $\mathcal{O}_{X,Q}$-linear composite

$$g'_{j+1,Q} : \mathcal{G}_{j+1,Q} \xrightarrow{(g_{j+1}|_{\mathcal{G}_{j+1}})_Q} \mathrm{Im}(g_{j+1}|_{\mathcal{G}_{j+1}})_Q \xrightarrow{c_{j,Q}} M_{j,Q}.$$

In other words, $g'_{j+1,Q}$ extracts the coefficients of the monomials $t_Q^{-i} dt_Q$ for $2 \leq i \leq p$ in the local expansion of ω_j at Q. This suffices to check regularity at Q in the following sense:

Lemma 5.12. *If* $v \in \mathcal{G}_{j+1,Q}$ *and* $g'_{j+1,Q}(v) = 0$, *then* $\mathrm{ord}_Q(\omega_j) \geq -n_{Q,j} = 0$, *i.e.,* $v \in \mathcal{G}_{j,Q}$.

Proof. Recall (4-3) that $\omega_j = v_j + s_j(V_X(\omega_j))$. By definition of $\mathcal{G}_{j+1}$ (Definition 5.2), we have $\mathrm{ord}_Q(\omega_i) \geq 0$ for $i > j$, and $\mathrm{ord}_Q(f) = 0$ by the very choice of the divisors D_j in Section 4. It follows that the differential

$$\xi := - \sum_{i=j+1}^{p-1} \binom{i}{j} \omega_i (-f)^{i-j}$$

is regular at Q. By Corollary 4.2 and the fact that $V(\omega) = 0$, we have $V_X(\omega_j) = V_X(\xi)$, which must also then be regular at Q, so working in the completed stalk at Q we may expand it locally as $V_X(\omega_j) = (a_1 + a_2 t_Q + \cdots) dt_Q$. But then

$$\omega_j = v_j + s_j(V_X(\omega_j)) = v_j + a_1^p s_j(dt_Q) + a_2^p t_Q^p s_j(dt_Q) + \cdots, \tag{5-13}$$

Now $V_X(v_j) = 0$, so the local expansion of v_j has no $t_Q^{-1} dt_Q$-term, and since $V_X \circ s_j$ is the identity, $s_j(dt_Q)$ has no $t_Q^{-1} dt_Q$-term either. From (5-13) and the fact that $\mathrm{ord}_Q(s_j(dt_Q)) \geq -p$ (by the very construction of s_j), we conclude that the local expansion of ω_j has no $t_Q^{-1} dt_Q$-term. If $g'_{j+1,Q}(v) = 0$ as well, then the local expansion of ω_j has no $t_Q^{-i} dt_Q$-terms for $2 \leq i \leq p$ either, and ω_j is regular at Q. $\square$

Lemma 5.13. *For* $0 \leq j \leq p-1$ *and* $Q \in \sup(D_j)$, *the map* $g'_{j+1,Q}$ *is surjective.*

Proof. Such surjectivity may be checked after passing to completions, where it follows immediately from the proof of Lemma 5.12 as $(\ker V_X)_Q^{\wedge}[1/t_Q]$ is (topologically) generated as a k-vector space by $\{t_Q^{i-1} dt_Q : p \nmid i\}$, and the only restriction on v_j is that $\mathrm{ord}_Q(v_j) \geq -p$. $\square$

5D. *Local calculations at* Q'. Finally, we analyze the behavior at Q'. Because we chose to work with the sheaf $\mathcal{F}_0$ instead of $\pi_* \Omega_Y^1$, the relationship between $\mathcal{G}_j$ and $\mathcal{G}_{j+1}$ at Q' is particularly simple when $0 \leq j \leq p-1$:

Lemma 5.14. *For* $0 \leq j \leq p-1$, *the natural inclusion* $\mathcal{G}_{j,Q'} \to \mathcal{G}_{j+1,Q'}$ *is an isomorphism.*

Proof. We check surjectivity. Consider $v := (v_0, \ldots, v_{p-1}) \in \mathcal{G}_{p,Q'}$ with $\iota(v) = (\omega_0, \ldots, \omega_{p-1})$. By Lemma 5.4 (iii)enumi, we in fact have $\mathrm{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ for *all* i, which implies in particular that $v \in \mathcal{G}_{0,Q'}$, whence the composite map $\mathcal{G}_{0,Q'} \hookrightarrow \mathcal{G}_{j,Q'} \hookrightarrow \mathcal{G}_{j+1,Q'} \hookrightarrow \mathcal{G}_{p,Q'}$ is an isomorphism, which gives the claim. As the details of the proof of Lemma 5.4 (iii)enumi were abridged, we will spell them out here (sans the descending induction, which is unnecessary if we assume that $v \in \mathcal{G}_{j+1,Q'}$). So assume

that $v \in \mathscr{G}_{j+1,Q'}$, or equivalently that $\mathrm{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ for $i \geq j+1$. Since $\mathrm{ord}_{Q'}(f) = -d_{Q'}$, we compute that

$$\mathrm{ord}_{Q'}(\omega_i(-f)^{i-j}) \geq -(p-1-i)d_{Q'} + (j-i)d_{Q'} = -(p-1-j)d_{Q'}$$

for $i \geq j+1$. On the other hand, for *any* meromorphic differential ξ on X, Corollary 2.9 shows that $\mathrm{ord}_{Q'}(s_j(V_X(\xi))) \geq \mathrm{ord}_{Q'}(\xi)$. Then (4-5) shows that $\mathrm{ord}_{Q'}(\omega_j) \geq -(p-1-j)d_{Q'}$, and hence that $v \in \mathscr{G}_{j,Q'}$ as desired.   □

The downside of working with the sheaf $\mathscr{F}_0$ from Definition 3.9 is that we require a separate analysis to relate $\mathscr{G}_0$ with the sheaf we care about, $\mathrm{Im}(\varphi) \simeq \pi_* \ker V_Y$. Recall (Definition 5.2) that we defined $\mathscr{G}_{-1} := \mathrm{Im}(\varphi) \subseteq \mathscr{G}_0$.

Definition 5.15. Define

$$M_{-1,Q'} := \bigoplus_{i=0}^{p-1} \left(k[\![t_{Q'}^p]\!]/(t_{Q'}^{(p-1-i)d_{Q'}})\right)^{\oplus(p-1)}$$

and write $M_{-1} := Q'_*(M_{-1,Q'})$ for the skyscraper sheaf on X supported at Q' with stalk $M_{-1,Q'}$. Put $S'_{-1} := \{Q'\}$.

Lemma 5.16. *The cokernel of the natural inclusion map $\mathscr{G}_{-1} \hookrightarrow \mathscr{G}_0$ is isomorphic to M_{-1}.*

We interpret Lemma 5.16 as giving a short exact sequence

$$0 \to \mathscr{G}_{-1} \to \mathscr{G}_0 \xrightarrow{g'_0} M_{-1} \to 0, \tag{5-14}$$

where g'_0 is the composition of the natural map $\mathscr{G}_0 \to \mathrm{coker}(\mathscr{G}_{-1} \to \mathscr{G}_0)$ with the isomorphism of Lemma 5.16.

Proof. We already know that the map $\mathscr{G}_{-1} \to \mathscr{G}_0$ is an inclusion, and is an isomorphism away from Q', so this is a local question at Q'. We have $\mathscr{G}_{-1} \simeq \pi_* \ker V_Y$ and $\mathscr{G}_0 \simeq \pi_* \ker(V_Y|_{\mathscr{F}_0})$ thanks to Lemma 5.3, and $(\pi_* \ker V_Y)_{Q'}$ consists of meromorphic forms on Y that are regular above Q' and lie in the kernel of V_Y, while $\pi_* \ker(V_Y|_{\mathscr{F}_0})_{Q'}$ consists of meromorphic $\omega = \sum_i \omega_i y^i$ in the kernel of V_Y that satisfy $\mathrm{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ for all i (Definition 3.9). Despite the fact that π is étale over Q', the decomposition $\omega = \sum_i \omega_i y^i$ is tricky to analyze since the element f defining the Artin–Schreier extension of function fields has a pole at Q' (as does y).

By the very choice of f in Section 3, we may find a meromorphic function $g \in K = k(X)$ such that $f' := f + g^p - g$ has $\mathrm{ord}_{Q'}(f') = 0$; necessarily $\mathrm{ord}_{Q'}(g) = -d_{Q'}/p$. Let $y' = y + g$, so that

$$(y')^p - y' = f'$$

and $K' = k(Y)$ is the Artin–Schreier extension of K given by adjoining y'. Observe that y' is regular above Q'. Any meromorphic differential ω on Y may be written as

$$\omega = \sum_{i=0}^{p-1} \omega_i y^i = \sum_{i=0}^{p-1} \omega'_i (y')^i. \tag{5-15}$$

While the condition of ω being regular above Q' is tricky to describe in terms of the ω_i, it is simple to describe in terms of the ω_i'. Indeed, the proof of Lemma 3.7 shows that ω is regular above Q' if and only if $\mathrm{ord}_{Q'}(\omega_i') \geq 0$ for $0 \leq i \leq p-1$. We deduce that the stalk of $\mathscr{G}_{-1}$ at Q' is isomorphic to

$$N_{-1} := \left\{ \omega = \sum_{i=0}^{p-1} \omega_i'(y')^i : \omega \in (\pi_* \ker V_Y)_{Q'} \text{ and } \omega_i' \in \Omega^1_{X,Q'} \right\}.$$

On the other hand, substituting $y' = y + g$ in (5-15) and collecting y^j-terms gives

$$\omega_j = \sum_{i=j}^{p-1} \binom{i}{j} \omega_i' g^{i-j}.$$

A descending induction on i using $\mathrm{ord}_{Q'}(g) = -d_{Q'}/p$ then shows that $\mathrm{ord}_{Q'}(\omega_i) \geq -(p-1-i)d_{Q'}$ if and only if $\mathrm{ord}_{Q'}(\omega_i') \geq -(p-1-i)d_{Q'}$, and we conclude that the stalk of $\mathscr{G}_0$ at Q' is isomorphic to

$$N_0 := \left\{ \omega = \sum_{i=0}^{p-1} \omega_i'(y')^i : \omega \in (\pi_* \ker V_Y)_{Q'} \text{ and } \omega_i' \in t_{Q'}^{-(p-1-i)d_{Q'}} \Omega^1_{X,Q'} \right\}.$$

We view both N_{-1} and N_0 as $k[t_{Q'}^p]$-modules. To complete the proof, it suffices to show the cokernel of the natural inclusion $N_{-1} \hookrightarrow N_0$ is isomorphic to $M_{-1,Q'}$. It suffices to do so after completing.

To analyze the cokernel, we first observe that (2-2)–(2-3) give a section $s : (\mathrm{Im}\, V_X)_{Q'}^\wedge \to F_*\Omega^{1,\wedge}_{X,Q'}$ and projector $r : F_*\Omega^{1,\wedge}_{X,Q'} \to (\ker V_X)_{Q'}^\wedge$ to the completion of the exact sequence (2-1) at Q'. By Corollary 4.2, we have the relations

$$V_X(\omega_j') = - \sum_{i=j+1}^{p-1} V_X\left(\binom{i}{j} \omega_i'(-f')^{i-j} \right) \tag{5-16}$$

for $0 \leq j \leq p-1$, and in particular $V_X(\omega_j')$ is determined by ω_i' for $i > j$. It follows that the map

$$\varphi_{Q'}' : (\pi_* \ker V_Y)_{Q'}^\wedge \to \bigoplus_{i=0}^{p-1} (\ker V_X)_{Q'}^\wedge, \qquad \sum_{i=0}^{p-1} \omega_i'(y')^i \mapsto (r(\omega_0'), \ldots, r(\omega_{p-1}'))$$

is an isomorphism after inverting $t_{Q'}^p$. Now for any differential ξ on X, we have $\mathrm{ord}_{Q'}(s(V_X(\xi))) \geq p\lfloor \mathrm{ord}_{Q'}(\xi)/p \rfloor$. As $\mathrm{ord}_{Q'}(f') = 0$, it follows from this and (5-16) that:

- $s(V_X(\omega_j')) \in \Omega^{1,\wedge}_{X,Q'}$ if $\omega_i' \in \Omega^{1,\wedge}_{X,Q'}$ for $i > j$,
- $s(V_X(\omega_j')) \in t_{Q'}^{-(p-1-j)d_{Q'}} \Omega^{1,\wedge}_{X,Q'}$ if $\omega_i' \in t_{Q'}^{-(p-1-i)d_{Q'}} \Omega^{1,\wedge}_{X,Q'}$ for $i > j$.

As $\omega_j' = r(\omega_j') + s(V_X(\omega_j'))$ for all j, together with (5-16) this shows that $\varphi_{Q'}'$ induces identifications

$$N_{-1}^\wedge \simeq \bigoplus_{i=0}^{p-1} (\ker V_X)_{Q'}^\wedge \quad \text{and} \quad N_0^\wedge \simeq \bigoplus_{i=0}^{p-1} (t_{Q'}^p)^{-(p-1-i)(d_{Q'}/p)} (\ker V_X)_{Q'}^\wedge$$

as submodules of $\bigoplus_i (\ker V_X)^\wedge_{Q'}[1/t^p_{Q'}]$, with the natural inclusion $N^\wedge_{-1} \hookrightarrow N^\wedge_0$ corresponding to the canonical inclusion of direct summands. As $(\ker V_X)^\wedge_{Q'}$ is a free $k[\![t^p_{Q'}]\!]$-module generated by $dt_{Q'}, \ldots, t^{p-1}_{Q'} dt_{Q'}$, this completes the proof. $\qquad\square$

Remark 5.17. Tracing through the proof of Lemma 5.16, we see that the image under $H^0(g'_0)$ of $H^0(X, \ker((\tau-1)^{j+1} : \mathcal{G}_0 \to \mathcal{G}_0))$ has dimension at most

$$\sum_{i=0}^j \frac{(p-1-i)d_{Q'}}{p}(p-1)$$

as $\omega_i = \omega'_i = 0$ for $i > j$ whenever $\omega = \sum_i \omega_i y^i = \sum_i \omega'_i (y')^i$ is killed by $(\tau-1)^{j+1}$.

5E. *Short exact sequences.* Combining the local analyses of Section 5B–5D, we can finally construct the desired exact sequences relating the $\mathcal{G}_j$. Recall the definitions of the skyscraper sheaf M_j in Definition 5.6 (for $0 \le j \le p-1$) and Definition 5.15 (for $j = -1$). From Sections 5B, 5C, and 5D we have maps from the stalks of $\mathcal{G}_j$ to pieces of these skyscraper sheaves. We now put them together.

Definition 5.18. For $0 \le j \le p-1$, put $S_j := S \cup \sup(D_j)$ and define

$$c_j : \mathrm{Im}(g_{j+1}) \to \bigoplus_{Q \in S_j} Q_*(\mathrm{Im}(g_{j+1})_Q) \xrightarrow{\bigoplus_{Q \in S_j} Q_*(c_{j,Q})} \bigoplus_{Q \in S_j} Q_*(M_{j,Q}) =: M_j,$$

where the first map is the canonical one. Let $g'_{j+1} := c_j \circ (g_{j+1}|_{\mathcal{G}_{j+1}})$.

Recall that we also defined g'_0 in (5-14).

Remark 5.19. Notice that the map induced by g'_{j+1} on stalks at Q coincides with the previously defined map $g'_{j+1,Q}$, justifying our notation.

Theorem 5.20. *For $-1 \le j \le p-1$, there are short exact sequences of sheaves on X*

$$0 \to \mathcal{G}_j \to \mathcal{G}_{j+1} \xrightarrow{g'_{j+1}} M_j \to 0.$$

We have

$$\mathcal{G}_{-1} = \mathrm{Im}(\varphi) \simeq \pi_* \ker V_Y \qquad and \qquad \mathcal{G}_p = \bigoplus_{i=0}^{p-1} \ker V_X(F_*(E_i + pD_i)).$$

Furthermore, M_j is a skyscraper sheaf supported on $S_j := S \cup \sup(D_j)$ for $j \ge 0$, and supported at Q' for $j = -1$.

Proof. We first establish the exact sequence. The case $j = -1$ is just Lemma 5.16, so suppose $0 \le j \le p-1$. Left exactness is obvious from the definition of $\mathcal{G}_j$ and $\mathcal{G}_{j+1}$. We can check the rest locally. For points Q not in S'_j, exactness in the middle and on the right is (ii)enumi of Lemma 5.3 plus the fact that $M_{j,Q} = 0$. At Q' such exactness is the content of Lemma 5.14 plus the fact that $M_{j,Q'} = 0$. For $Q \in S$, exactness in the middle is simply the first statement of Lemma 5.3 plus the observation that $c_{j,Q}$ is injective when $Q \in S$, while exactness on the right is Proposition 5.9. For $Q \in \sup(D_j)$, exactness in the middle and on

the right follow from Lemmas 5.12 and 5.13, respectively. Finally, the statements about the support of M_j are clear from the definitions. $\qquad\square$

Example 5.21. We continue the notation of Example 4.6, working with the two covers Y_1, Y_2 of $\mathbf{P}^1$ given by $y^5 - y = f_1$ and $y^5 - y = f_2$, respectively. The fact that $\omega = \sum \omega_i y^i = \varphi_\eta^{-1}(v_{2,2})$ is regular for one cover but not the is captured by the machinery of this section as follows.

Note that the element $v_{2,2}$ naturally lies in $H^0(\mathbf{P}^1, \mathcal{G}_2)$. For both covers, it is moreover in the kernel of $H^0(g_2')$: looking at the definition of g_2' in Section 5B, the computation that $H^0(g_2')(v_{2,2}) = 0$ is equivalent to the computation that $\omega_1 = 0$. Thus, in both cases we also have $v_{2,2} \in H^0(\mathbf{P}^1, \mathcal{G}_1)$.

In Example 4.6, we computed ω_0 using the relations

$$s V_{\mathbf{P}^1}(\omega_0) = -s V_{\mathbf{P}^1}(t^{-2}(t^{-6}dt)) = 0,$$

and

$$s V_{\mathbf{P}^1}(\omega_0) = -s V_{\mathbf{P}^1}(t^{-2}(t^{-6} + 2t^{-5} + t^{-4})dt) = -t^{-6}dt$$

for the covers Y_1 and Y_2, respectively. The definition of g_1' in Section 5B is exactly recording the coefficient of $t^{-6}dt$. Thus $H^0(g_1')(v_{2,2})$ is 0 in the case of Y_1, but nonzero for the cover Y_2, reflecting the fact that $\varphi_\eta^{-1}(v_{2,2})$ is regular for Y_1 but not Y_2. We therefore see that in the case of Y_1 we have $a_{Y_1} = \dim H^0(\mathbf{P}^1, \mathcal{G}_0) = 4$ while $a_{Y_2} = \dim H^0(\mathbf{P}^1, \mathcal{G}_0) = 3$ in the case of Y_2.

6. Bounds

We continue with the notation and conventions of the previous sections, so $\pi : Y \to X$ is a degree-p Artin–Schreier cover of smooth projective curves over k with branch locus $S \subseteq X$. In this section we will use the short exact sequences of Theorem 5.20 to extract bounds on the a-number of Y in terms of the a-number of X and the breaks in the ramification filtrations at points above $Q \in S$. Our starting point is the equality

$$a_Y = \dim_k \ker\bigl(V_Y : H^0(Y, \Omega_Y^1) \to H^0(Y, \Omega_Y^1)\bigr)$$
$$= \dim_k H^0(Y, \ker V_Y) = \dim_k H^0(X, \mathcal{G}_{-1}),$$

which follows immediately from Lemma 5.3.

6A. *Abstract bounds.* We will first obtain a relatively abstract upper bound on the a-number using the short exact sequences of Theorem 5.20. To streamline the analysis, we first encode the information contained in these exact sequences in the study of a single linear transformation.

Definition 6.1. For $0 \le j \le p - 1$ define

$$\tilde{g}_{j+1} : H^0(X, \mathcal{G}_p) \xrightarrow{H^0(c_j \circ g_{j+1})} H^0(X, M_j) = \bigoplus_{Q \in S_j'} M_{j,Q} \tag{6-1}$$

to be the map on global sections induced by $c_j \circ g_{j+1} : \mathscr{G}_p \to M_j$, where c_j and g_{j+1} are as in Definitions 5.18 and 5.1, respectively. Define

$$\tilde{g} : H^0(X, \mathscr{G}_p) \to \bigoplus_{j=0}^{p-1} H^0(X, M_j) \quad \text{by} \quad \tilde{g}(\nu) := (\tilde{g}_1(\nu), \dots, \tilde{g}_p(\nu)) \tag{6-2}$$

Finally define

$$\tilde{g}_0 : H^0(X, \mathscr{G}_0) \xrightarrow{H^0(g_0')} H^0(X, M_{-1}) = M_{-1,Q'}, \tag{6-3}$$

where g_0' is as in (5-14) and $M_{-1,Q'}$ is as in Definition 5.15.

Remark 6.2. By construction, the restriction of $\tilde{g}_{j+1}$ to $H^0(X, \mathscr{G}_{j+1}) \subseteq H^0(X, \mathscr{G}_p)$ coincides with the map $H^0(g_{j+1}') : H^0(X, \mathscr{G}_{j+1}) \to H^0(X, M_j)$, where g_{j+1}' is as in Definition 5.18 for $j \geq 0$ and as in (5-14) for $j = -1$. While the use of the maps g_{j+1}' is essential for Theorem 5.20, our analysis below is much simpler when phrased in terms of $\tilde{g}$ and $\tilde{g}_0$ as the following key Lemma indicates.

Lemma 6.3. *The kernel of $\tilde{g}$ is $H^0(X, \mathscr{G}_0)$.*

Proof. Let $\nu \in H^0(X, \mathscr{G}_p)$ with $\tilde{g}(\nu) = 0$, i.e., $\tilde{g}_{j+1}(\nu) = 0$ for $0 \leq j \leq p - 1$. Supposing that $\nu \in H^0(X, \mathscr{G}_{j+1})$ for *some* $j \leq p - 1$, we have $H^0(g_{j+1}')(\nu) = \tilde{g}_{j+1}(\nu) = 0$ by assumption and the compatibility between g_{j+1}' and $\tilde{g}_{j+1}$ noted in Remark 6.2. Thus $\nu \in H^0(X, \mathscr{G}_j)$ thanks to Theorem 5.20 and left exactness of H^0. We conclude by descending induction on j that $\nu \in H^0(X, \mathscr{G}_0)$, whence $\ker(\tilde{g}) \subseteq H^0(X, \mathscr{G}_0)$, and the reverse containment is clear. $\square$

Definition 6.4. Set $N_1(X, \pi) := \dim_k \operatorname{Im}(\tilde{g})$ and $N_2(X, \pi) := \dim_k \operatorname{Im}(\tilde{g}_0)$, and let

$$N(X, \pi) = N_1(X, \pi) + N_2(X, \pi).$$

Define

$$U(X, \pi) := \sum_{i=0}^{p-1} \dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))) - N(X, \pi).$$

Lemma 6.5. *We have that $a_Y = U(X, \pi)$.*

Proof. Lemma 6.3 and the rank-nullity theorem give

$$\dim_k H^0(X, \mathscr{G}_0) = \dim_k H^0(X, \mathscr{G}_p) - N_1(X, \pi).$$

Likewise, the $j = -1$ case of Theorem 5.20, left exactness of H^0 and the rank-nullity theorem yield

$$a_Y = \dim_k H^0(Y, \ker V_Y) = \dim_k H^0(X, \mathscr{G}_{-1}) = \dim_k H^0(X, \mathscr{G}_0) - N_2(X, \pi).$$

Combining these equations with the computation

$$\dim_k H^0(X, \mathscr{G}_p) = \sum_{i=0}^{p-1} \dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))),$$

which follows at once from Lemma 5.3 (vi)enumi, gives the claimed equality. $\square$

The lower bound is a more elaborate application of linear algebra. The idea is to find a family of subspaces $U_j \subset H^0(X, \mathcal{G}_p)$ and choose j so that $\dim_k U_j$ is "large" while it is easy to see that $\dim_k \tilde{g}(U_j)$ is "small". Then the rank-nullity theorem gives a lower bound on the kernel of $\tilde{g}$. Recall that $G = \mathbf{Z}/p\mathbf{Z}$ acts on $\pi_* \Omega^1_Y$ by having a generator $\tau \in G$ send $y \mapsto y + 1$. By Lemma 5.3, this action preserves $\mathcal{G}_j$.

Definition 6.6. Let i and j be integers with $0 \le i \le j \le p - 1$.

- Define $U_j := \ker\big((\tau - 1)^{j+1} : H^0(X, \mathcal{G}_{j+1}) \to H^0(X, \mathcal{G}_{j+1})\big)$.

- For $Q \in S$, define $c(i, j, Q)$ to be the number of integers n congruent to -1 modulo p such that $-n_{Q,j} - d_Q(j - i) \le n < -n_{Q,i}$.

- For $Q \in \sup(D_i)$, define $c(i, j, Q) = p - 1$.

- Define $c(i, j, Q') = \dfrac{d_{Q'}(p-1)(p-1-i)}{p}$.

- Define $L(X, \pi)$ to be

$$L(X, \pi) := \max_{0 \le j \le p-1} \left(\dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S_i'} c(i, j, Q) \right).$$

Lemma 6.7. *There is an isomorphism*

$$U_j \simeq \bigoplus_{i=0}^{j} H^0(X, \ker V_X(F_*(E_i + pD_i))).$$

Proof. Both sides are isomorphic to

$$\left\{ \omega = \sum_{i=0}^{j} \omega_i y^i \in (\pi_* \ker V_Y)_\eta : \varphi_\eta(\omega) \in H^0(X, \mathcal{G}_p) \right\}.$$

This follows from the formulas in Lemma 4.1, the fact that $\tau - 1$ reduces the maximum power of y appearing in a differential by one, and the definition of φ. $\qquad\square$

Lemma 6.8. *We have that $a_Y \ge L(X, \pi)$.*

Proof. Fix j. The key idea is to use descending induction on i and the rank-nullity theorem to give a lower bound on $\dim_k(U_j \cap H^0(X, \mathcal{G}_i))$ for $-1 \le i \le j + 1$. Taking $i = -1$ gives a lower bound (depending on j) on $\dim_k(U_j \cap H^0(Y, \ker V_Y)) \le a_Y$, which will establish the result.

For $i = j + 1$, we see that

$$\dim_k(U_j \cap H^0(X, \mathcal{G}_{j+1})) = \dim_k U_j. \tag{6-4}$$

To ease the notational burden, for $-1 \le i \le j$ let us write ψ_{i+1}^j for the map

$$\psi_{i+1}^j : U_j \cap H^0(X, \mathcal{G}_{i+1}) \to H^0(M_i),$$
$$\text{given by } \psi_{i+1}^j := H^0(g_{i+1}')|_{U_j \cap H^0(X, \mathcal{G}_{i+1})}. \tag{6-5}$$

Theorem 5.20 then gives exact sequences

$$0 \to U_j \cap H^0(X, \mathcal{G}_i) \to U_j \cap H^0(X, \mathcal{G}_{i+1}) \xrightarrow{\psi_{i+1}^j} H^0(X, M_i) = \bigoplus_{Q \in S_i'} M_{i,Q} \qquad (6\text{-}6)$$

for $-1 \leq i \leq j$, which need not be right exact. Nonetheless, we may work stalk-by-stalk to obtain information about the image of ψ_{i+1}^j as follows. For $Q \in S_i'$, let $\psi_{i+1,Q}^j$ denote the composition of ψ_{i+1}^j with projection onto the factor of the direct sum indexed by Q; by construction $\psi_{i+1,Q}^j$ coincides with the composition

$$\psi_{i+1,Q}^j : U_j \cap H^0(X, \mathcal{G}_{i+1}) \hookrightarrow H^0(X, \mathcal{G}_{i+1}) \to \mathcal{G}_{i+1,Q} \xrightarrow{g_{i+1,Q}'} M_{i,Q} \qquad (6\text{-}7)$$

wherein the first two maps are the canonical ones. We have the obvious inclusion

$$\operatorname{Im}(\psi_{i+1}^j) \subseteq \bigoplus_{Q \in S_i'} \operatorname{Im}(\psi_{i+1,Q}^j) \qquad (6\text{-}8)$$

so to bound $\dim_k \operatorname{Im}(\psi_{i+1}^j)$ from above it suffices to do so for each $\dim_k \operatorname{Im}(\psi_{i+1,Q}^j)$. First suppose that $i \geq 0$. Lemma 6.9 below gives an upper bound of $c(i, j, Q)$ on $\dim_k \operatorname{Im}(\psi_{i+1,Q}^j)$ for $Q \in S$. For $Q \in \sup(D_i)$, the dimension of $\operatorname{Im}(\psi_{i+1,Q}^j)$ is at most $p - 1$ by the very definition of $g_{i+1,Q}'$ in Section 5C, and we defined $c(i, j, Q) := p - 1$ in this case. Applying the rank-nullity theorem to the exact sequence (6-6) and using (6-8) then gives

$$\dim_k(U_j \cap H^0(X, \mathcal{G}_i)) \geq \dim_k(U_j \cap H^0(X, \mathcal{G}_{i+1})) - \sum_{Q \in S_i'} c(i, j, Q).$$

By descending induction on i and (6-4), we deduce

$$\dim_k(U_j \cap H^0(X, \mathcal{G}_0)) \geq \dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S_i'} c(i, j, Q).$$

Finally, using Remark 5.17 to analyze (6-6) when $i = -1$, we conclude

$$\dim_k(U_j \cap H^0(X, \mathcal{G}_{-1})) \geq \dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S_i'} c(i, j, Q). \qquad \square$$

Lemma 6.9. *If $Q \in S$ and $0 \leq i \leq j < p$, then $\dim_k \operatorname{Im}(\psi_{i+1,Q}^j) \leq c(i, j, Q)$.*

Proof. We continue the notation of Section 5B. Fixing $0 \leq i \leq j < p$ and recalling Lemma 5.4, for $v \in U_j$ we have

$$v = (v_0, \ldots, v_j, 0, \ldots, 0) \text{ and } (\omega_0, \ldots, \omega_j, 0, \ldots, 0) := \iota(v).$$

From (4-5), we have $\omega_i = v_i + s_i(V_X(\xi))$, where

$$\xi := -\sum_{\ell=i+1}^{j} \binom{\ell}{j} \omega_\ell (-f)^{\ell-i} \qquad (6\text{-}9)$$

since ν_ℓ and ω_ℓ are 0 for $\ell > j$. From Lemma 5.4 (i)enumi with $n = n_{Q,j}$ we have

$$\operatorname{ord}_Q(\xi) \geq -n_{Q,j} - (j-i)d_Q. \tag{6-10}$$

By the very definitions (see (6-7) and Definitions 5.8 and 5.1), to compute $\psi^j_{i+1,Q}(\nu)$, we first expand ξ locally at Q as a power series $\xi = \sum_n a_n t_Q^n (dt_Q/t_Q)$ and evaluate

$$s_i(V_X(\xi)) = \sum_{n=-B}^{\infty} a_{pn} s_i\left(t_Q^n \frac{dt_Q}{t_Q}\right) \tag{6-11}$$

for some integer B. Then $\psi^j_{i+1,Q}(\nu)$ records only those coefficients on the right side of (6-11) where $n \geq -B$ and $pn - 1 < -n_{Q,i}$. In our situation we have *fewer* coefficients to record than in the general analysis of Section 5B, precisely because $\nu_\ell = 0 = \omega_\ell$ for $\ell > j$. Indeed, based on (6-10) we may take B to be the smallest integer greater than or equal to $(-n_{Q,j} - (j-i)d_Q + 1)/p$. In particular, the number of potentially nonzero coefficients a_{pn} is the number of multiples of p between $-n_{Q,j} - (j-i)d_Q + 1$ and $-n_{Q,i}$ inclusive, which by definition is the integer $c(i, j, Q)$. This integer therefore gives an upper bound on the dimension of $\operatorname{Im}(\psi^j_{i+1,Q})$ $\qquad\square$

Proposition 6.10. *The a-number of Y satisfies*

$$L(X, \pi) \leq a_Y = U(X, \pi).$$

Proof. Combine Lemmas 6.5 and 6.8. $\qquad\square$

6B. *Tools*. The quantities $L(X, \pi)$ and $U(X, \pi)$ are quite abstract. We now explain how to bound $L(X, \pi)$ and $U(X, \pi)$ in terms of the ramification of $\pi : Y \to X$ and the genus and *a*-number of X.

The key is a theorem of Tango which allows us to compute the dimension of the kernel of the Cartier operator on certain spaces of (global) meromorphic differential forms on X whose poles are "sufficiently bad". Let $\sigma : k \to k$ denote the p-power Frobenius automorphism of k, and let g_X be the genus of X. Attached to X is its *Tango number*:

$$\mathrm{n}(X) := \max\left\{ \sum_{x \in X(k)} \left\lfloor \frac{\operatorname{ord}_x(df)}{p} \right\rfloor : f \in k(X) - k(X)^p \right\}. \tag{6-12}$$

In Lemma 10 and Proposition 14 of [Tango 1972], one sees that $\mathrm{n}(X)$ is well-defined and is an integer satisfying $-1 \leq \mathrm{n}(X) \leq \lfloor (2g_X - 2)/p \rfloor$, with the lower bound an equality if and only if $g_X = 0$.

Fact 6.11 (Tango's theorem). *Let $\mathscr{L}$ be a line bundle on X. If $\deg \mathscr{L} > \mathrm{n}(X)$ then the natural σ-linear map*

$$F_X^* : H^1(X, \mathscr{L}^{-1}) \to H^1(X, \mathscr{L}^{-p}) \tag{6-13}$$

induced by pullback by the absolute Frobenius of X is injective, and the σ^{-1}-linear Cartier operator

$$V_X : H^0(X, \Omega^1_{X/k} \otimes \mathscr{L}^p) \to H^0(X, \Omega^1_{X/k} \otimes \mathscr{L}) \tag{6-14}$$

is surjective.

Remark 6.12. This is [Tango 1972, Theorem 15]; strictly speaking, Tango requires $g_X > 0$; however, by tracing through Tango's argument—or by direct calculation—one sees easily that the result holds when $g_X = 0$ as well.

To simplify notation, let $\delta(H^0(X, \Omega_X^1(E)))$ denote the dimension of the kernel of V_X on that space of differentials. Tango's theorem tells us the following:

Corollary 6.13. *Let D, R be divisors on X with $R = \sum r_i P_i$ where $0 \leq r_i < p$. If $\deg(D) > \max(n(X), 0)$, then*

$$\delta(H^0(X, \Omega_X^1(pD + R))) = (p - 1)\deg(D) + \sum_i \left(r_i - \left\lceil \frac{r_i}{p} \right\rceil\right).$$

For arbitrary D, we have the weaker statement

$$0 \leq \delta(H^0(X, \Omega_X^1(pD + R))) - \left((p - 1)\deg(D) + \sum_i \left(r_i - \left\lceil \frac{r_i}{p} \right\rceil\right)\right) \leq a_X.$$

Proof. When $R = 0$, the first case follows from the surjectivity of the Cartier operator in Fact 6.11 taking $\mathcal{L} = \mathcal{O}_X(D)$, plus the fact that

$$\dim_k H^0(X, \Omega_X^1(D)) = g_X - 1 + \deg(D),$$
$$\dim_k H^0(X, \Omega_X^1(pD)) = (g_X - 1) + p\deg(D)$$

from the Riemann–Roch theorem.

We can build on this to prove the remaining cases of the first statement and establish the inequalities of the second statement. We know that for any divisor E with $\deg(E) \geq 0$ and any closed point P of X,

$$\dim_k H^0(X, \Omega_X^1(E + [P])) \leq \dim_k H^0(X, \Omega_X^1(E)) + 1.$$

with equality whenever $\deg(E) > 0$. Thus we know that

$$0 \leq \delta(H^0(X, \Omega_X^1(E + [P]))) - \delta(H^0(X, \Omega_X^1(E))) \leq 1.$$

Further, if $p \mid \mathrm{ord}_P(E)$, this difference is 0, as a differential in $H^0(X, \Omega_X^1(E + [P]))$ not in $H^0(X, \Omega_X^1(E))$ must have a nonzero $t_P^{-\mathrm{ord}_P(E)}(dt_P/t_p)$-term in the completed stalk at P, which forces the differential to not lie in the kernel of the Cartier operator.

When $\deg(D) > \max(n(X), 0)$ and $R \neq 0$, we prove the equality by induction on the number of points in the support of R. Assume the equality holds for some fixed R and all D with $\deg(D) > \max(n(X), 0)$. Pick another point P (not in the support of R) and $0 < r < p$. Then

$$\delta(H^0(X, \Omega_X^1(pD + R))) \leq \delta(H^0(X, \Omega_X^1(pD + R + [P])))$$
$$\leq \cdots \leq \delta(H^0(X, \Omega_X^1(p(D + [P]) + R)))$$

and by the inductive hypothesis the last is $p - 1$ more than the first. This means that at each step after the first, the dimension of the kernel must increase by one. This shows that

$$\delta(H^0(X, \Omega_X^1(pD + R + r[P]))) = \delta(H^0(X, \Omega_X^1(pD + R))) + (r - 1),$$

which completes the induction.

The second statement follows from similar reasoning. When passing from

$$a_X = \delta(H^0(X, \Omega_X^1)) \quad \text{to} \quad \delta(H^0(X, \Omega_X^1(pD + R))),$$

there are $(p-1)\deg(D) + \sum(r_i - \lceil r_i/p \rceil)$ times the dimension might increase by one. When passing from

$$\delta(H^0(X, \Omega_X^1(pD + R))) \quad \text{to} \quad \delta(H^0(X, \Omega_X^1(pD'))) = (p-1)\deg(D')$$

with D' chosen so that $pD' \geq pD + R$ and $D' > \mathrm{n}(X)$, there are

$$(p-1)\deg(D' - D) + \sum(p - 1 - r_i)$$

chances for the dimension to increase by one. This completes the proof.   □

Remark 6.14. Choosing $D' \geq D$ with $\deg(D') > \mathrm{n}(X)$ and $pD' \geq pD + R$, we also obtain the bound $\delta(H^0(X, \Omega_X^1(pD + R))) \leq (p-1)\deg D'$ from the inclusion $H^0(X, \Omega_X^1(pD + R)) \subset H^0(X, \Omega_X^1(pD'))$ and Tango's theorem.

Fix a divisor $E = pD + R = \sum_i a_i Q_i$ with $R = \sum r_i Q_i$ and $0 \leq r_i < p$. For fixed j and $1 \leq n \leq a_j$,

$$\delta\left(H^0\left(X, \Omega_X^1\left(\sum_{i<j} a_i Q_i + (n-1)Q_j\right)\right)\right) \leq \delta\left(H^0\left(X, \Omega_X^1\left(\sum_{i<j} a_i Q_i + nQ_j\right)\right)\right).$$

If the dimension increases by one, let $\xi_{Q_j,n}$ be a differential in the larger space not in the smaller space. This differential satisfies the following properties:

(i) $V_X(\xi_{Q_j,n}) = 0$;

(ii) $\mathrm{ord}_{Q_i}(\xi_{Q_j,n}) \geq -a_i$ for $i < j$;

(iii) $\mathrm{ord}_{Q_j}(\xi_{Q_j,n}) = -n$; and

(iv) $\mathrm{ord}_{Q_i}(\xi_{Q_j,n}) \geq 0$ for $i > j$.

Note that such a differential never exists if $n \equiv 1 \bmod p$.

Example 6.15. Suppose $X = \mathbf{P}^1$. For any positive integer n with $n \not\equiv 1 \bmod p$ and closed point $Q = [\alpha]$ of X, we may take $\xi_{Q,n}$ to be $(t - \alpha)^{-n} dt$.

Corollary 6.16. *The differentials $\xi_{Q_j,n}$ are linearly independent. There are at least $(p-1)\deg D + \sum_i (r_i - \lceil r_i/p \rceil) - a_X$ of them.*

Proof. This follows from the proof of Corollary 6.13.   □

We can use Corollary 6.13 to relatively easily give a formula for $\dim_k U_j$ using Lemma 6.7 provided that the auxiliary divisors D_i are sufficiently large. This will allow us to bound $L(X, \pi)$; we do so in the proof of Theorem 6.26. We can also use Corollary 6.13 to obtain a bound on the number $N(X, \pi)$ of Definition 6.4; this is more involved, and is the subject of the next section.

6C. *Estimating $N(X, \pi)$.* We will use Corollary 6.16 to construct differential forms to give a lower bound on $N(X, \pi)$, and hence an upper bound on $U(X, \pi)$.

Definition 6.17. Define T to be the set of triples (Q, n, j), where

- $0 \leq j \leq p - 1$ and $Q \in S'_j$;
- $0 < n \leq \text{ord}_Q(E_j + pD_j)$;
- $n \not\equiv 1 \mod p$;
- if $Q \in S$, an integer m $(0 \leq m \leq j)$ exists such that $m \equiv j + (n-1)d_Q^{-1} \mod p$.

We fix an ordering on $\bigcup_j S'_j$ with Q' the smallest element. For $(Q, n, j) \in T$, suppose the element $\xi_{Q,j}$ from Corollary 6.16 exists $\big($taking $E = E_j + pD_j$ and $a_i = n_{Q_i,j}$ in the notation there, and using the ordering on $\bigcup_j S'_j\big)$. We set

$$\nu_{Q,n,j} := (0, \ldots, 0, \xi_{Q,j}, 0, \ldots, 0) \in H^0(X, \mathscr{G}_{j+1}),$$

where $\xi_{Q,j}$ occurs in the j-th component. In that case, we say that $\nu_{Q,n,j}$ *exists*. As Q' is smallest in the chosen ordering, we know that $\nu_{Q',n,j} \in H^0(X, \mathscr{G}_0)$. Our aim is to prove:

Proposition 6.18. *Let $N(X, \pi)$ be as in Definition 6.4. Then $N(X, \pi) \geq \#T - B$, where B is the number of triples $(Q, n, j) \in T$ such that $\nu_{Q,n,j}$ does not exist.*

The idea is to show that the images under $\tilde{g}$ of the $\nu_{Q,n,j}$ for $(Q, n, j) \in T$ with $Q \neq Q'$ are linearly independent, and similarly for the images under $\tilde{g}_0$ of the $\nu_{Q',n,j}$ with $(Q', n, j) \in T$.

Definition 6.19. We define an ordering on T by setting $(P, a, j) < (P', a', j')$ provided that

(i) $j < j'$; or

(ii) $j = j'$ and $P < P'$; or

(iii) $j = j'$ and $P = P'$ and $a < a'$.

For $(Q, n, j) \in T$, we then define

$$U_{Q,n,j} := \text{span}_k\big\{\tilde{g}(\nu_{P,a,i}) : \nu_{P,a,i} \text{ exists and } (P, a, i) < (Q, n, j)\big\}.$$

Lemma 6.20. *Let $(P, a, i), (Q, n, j) \in T$ with $(P, a, i) < (Q, n, j)$ and suppose that the differentials $\nu_{P,a,i}$ and $\nu_{Q,n,j}$ both exist. Writing $(\omega_0, \ldots, \omega_{p-1}) = \iota(\nu_{Q,n,j})$ and $(\omega'_0, \ldots, \omega'_{p-1}) = \iota(\nu_{P,a,i})$, we have $\text{ord}_Q(\omega'_j) > \text{ord}_Q(\omega_j) = -n$.*

Proof. Suppose first that $i < j$. Since the ℓ-th component of $\nu_{P,a,i}$ is zero for $\ell > i$, we have $\omega'_\ell = 0$ for $\ell > i$ by Lemma 5.4 (i)enumi, and in particular $\omega'_j = 0$ which immediately gives the claim. So suppose that $i = j$ and $P < Q$. In this case, by very construction of the differentials in Corollary 6.16 and our choice of ordering, we have $\text{ord}_Q(\omega'_j) = \text{ord}_Q(\xi_{P,a}) \geq 0$ while $\text{ord}_Q(\omega_j) = \text{ord}_Q(\xi_{Q,n}) = -n$. Finally, suppose that $i = j$, $P = Q$, and $a < n$. Then,

$$\text{ord}_Q(\omega'_j) = \text{ord}_Q(\xi_{Q,a}) = -a > -n = \text{ord}_Q(\xi_{P,n}) = \text{ord}_Q(\omega_j)$$

as claimed. □

Lemma 6.21. *Suppose that $v_{Q,n,j}$ exists with $(Q, n, j) \in T$ and $Q \in S$. Then $\tilde{g}(v_{Q,n,j}) \notin U_{Q,n,j}$.*

Proof. In this proof, let $\tilde{g}_{m+1,Q}$ denote the composition of $\tilde{g}_{m+1}$ with the projection onto $M_{j,Q}$. Write $(\omega_0, \ldots, \omega_j, 0, \ldots, 0) = \iota(v_{Q,n,j})$. As $Q \in S$, by Definition 6.17 there exists an integer $0 \le m \le j$ such that $m \equiv j + (n-1)d_Q^{-1} \bmod p$. Looking at the definition of $g'_{m+1,Q}$ in Section 5B, to compute $\tilde{g}_{m+1,Q}(v_{Q,n,j})$ we work in the completed stalk at Q and write

$$\xi := -\sum_{i=m+1}^{j} \binom{i}{m} \omega_i (-f)^{i-m} = \sum_{i=-d_Q(p-1-m)+1}^{\infty} a_i t_Q^i \frac{dt_Q}{t_Q}. \tag{6-15}$$

Now $\tilde{g}_{m+1,Q}$ records the a_i for which $p \mid i$ and $-d_Q(p-1-m) \le i < -n_{Q,m}$.

Using Lemma 5.4 (i)enumi we find for $i \le j$

$$\mathrm{ord}_Q(\omega_i(-f)^{i-m}) \ge -n - (j-m)d_Q \tag{6-16}$$

In fact, we know more: for $i \ne j$, the i-th component of $v_{Q,n,j}$ is *zero*, whence Lemma 5.4 (i)enumi shows that for $m < i < j$

$$\begin{aligned}
\mathrm{ord}_Q(\omega_i(-f)^{i-m}) &= \mathrm{ord}_Q(\omega_i) - (i-m)d_Q \\
&\ge -p \left\lfloor \frac{n + (j-i)d_Q - 1}{p} \right\rfloor - 1 - (i-m)d_Q \\
&> -(n + (j-i)d_Q - 1) - 1 - (i-m)d_Q \\
&= -n - (j-m)d_Q, \tag{6-17}
\end{aligned}$$

where the strict inequality comes from the fact that $n + (j-i)d_Q \not\equiv 1 \bmod p$ for $m < i \le j$, as m is by definition the *unique* solution modulo p to

$$n + (j-x)d_Q \equiv 1 \bmod p.$$

We adopt the notation from the proof of Lemma 5.10 for coefficients of differentials and functions. As $\omega_j = \xi_{Q,n}$, we have $\mathrm{coef}_{1-n}(\omega_j) \ne 0$. As $\mathrm{ord}_Q(f) = -d_Q$, we have $\mathrm{coef}_{-d_Q}(f) \ne 0$. By (6-17), we know that $\mathrm{coef}_{1-n-(j-m)d_Q}(\omega_i(-f)^{i-m}) = 0$ for $m < i < j$. Then looking at coefficients in (6-15) we see that

$$\begin{aligned}
a_{1-n-(j-m)d_Q} &= \mathrm{coef}_{1-n-(j-m)d_Q}\left(-\sum_{i=m+1}^{j} \binom{i}{m} \omega_i(-f)^{i-m}\right) \\
&= \mathrm{coef}_{1-n-(j-m)d_Q}\left(-\binom{j}{m} \omega_j(-f)^{j-m}\right) \\
&= -\binom{j}{m} \mathrm{coef}_{1-n}(\omega_j) \cdot \mathrm{coef}_{-(j-m)d_Q}((-f)^{j-m}) \ne 0. \tag{6-18}
\end{aligned}$$

But $1 - n - (j-m)d_Q \equiv 0 \bmod p$, so $a_{1-n-(j-m)d_Q}$ is one of the *nonzero* terms that $\tilde{g}_{m+1,Q}$ records. This is enough to show that $\tilde{g}(v_{Q,n,j})$ does not lie in $U_{Q,n,j}$. Indeed, suppose $(P, a, i) \in T$ with $(P, a, i) <$

(Q, n, j) and that $\nu_{P,a,i}$ exists. Set $(\omega'_0, \ldots, \omega'_{p-1}) = \iota(\nu_{P,a,i})$. By Lemma 6.20, we have $\operatorname{ord}_Q(\omega'_j) > -n$, so thanks to Lemma 5.4 (i)enumi we see that $\operatorname{ord}_Q(\omega'_\ell) > -n - d_Q(j - \ell)$ for all $\ell \leq j$. It follows that

$$\operatorname{coef}_{1-n-(j-m)d_Q}(\omega'_m) = 0.$$

As $U_{Q,n,j}$ is spanned by $\tilde{g}(\nu_{P,a,i})$ for such triples (P, a, i), this shows $\tilde{g}(\nu_{Q,n,j})$ does not lie in $U_{Q,n,j}$ as claimed. $\square$

Lemma 6.22. *Suppose that $\nu_{Q,n,j}$ exists with $(Q, n, j) \in T$ and $Q \in D_j$. Then $\tilde{g}(\nu_{Q,n,j}) \notin U_{Q,n,j}$.*

Proof. In this proof, let $\tilde{g}_{m+1,Q}$ denote the composition of $\tilde{g}_{m+1}$ with the projection onto $M_{j,Q}$. As $\operatorname{ord}_Q(E_j + pD_j) = p$ and we assumed $n \neq 1$, we know $1 < n \leq p$. Set $(\omega_0, \ldots, \omega_{p-1}) = \iota(\nu_{Q,n,j})$ and notice that $\omega_j = \xi_{Q,n}$. Looking at the definition of $g'_{j+1,Q}$ in Section 5C, we see that $g'_{j+1,Q}$ extracts the coefficients of $t_Q^{-p} dt_Q$ through $t_Q^{-2} dt_Q$ in a local expansion of ω_j at Q. In particular, since $\operatorname{ord}_Q(\omega_j) = -n$, we see that the component of $\tilde{g}_{j+1,Q}(\nu_{Q,n,j})$ corresponding to the coefficient of $t_Q^{-n} dt_Q$ in the local expansion of ω_j at Q is *nonzero*. On the other hand, for any $(P, a, i) \in T$ such that $\nu_{P,a,i}$ exists and $(P, a, i) < (Q, n, j)$, consider $(\omega'_0, \ldots, \omega'_{p-1}) = \iota(\nu_{P,a,i})$. By Lemma 6.20 we have $\operatorname{ord}_Q(\omega'_j) > -n$ so that the component of $\tilde{g}_{j+1,Q}(\nu_{P,a,i})$ corresponding to the coefficient of $t_Q^{-n} dt_Q$ in the local expansion of ω'_j at Q is by contrast *zero*. As the $\tilde{g}(\nu_{P,a,i})$ for $(P, a, i) < (Q, n, j)$ span $U_{Q,n,j}$, this shows $\tilde{g}(\nu_{Q,n,j}) \notin U_{Q,n,j}$ as desired. $\square$

Lemma 6.23. *The elements*

$$\{\tilde{g}_0(\nu_{Q',n,j}) : (Q', n, j) \in T \text{ and } \nu_{Q',n,j} \text{ exists}\}$$

are linearly independent.

Proof. For notational convenience, let $T' \subseteq T$ be the subset of triples of the form (Q', n, j) such that $\nu_{Q',n,j}$ exists, and note that $\nu_{Q',n,j} \in H^0(X, \mathscr{G}_0)$ as $n \leq (p-1-j)d_{Q'} = \operatorname{ord}_{Q'}(E_j)$. Let $\omega_{Q',n,j} = \varphi_\eta^{-1}(\nu_{Q',n,j})$; we have that $(\tau - 1)^j \omega_{Q',n,j} = j! \xi_{Q',n}$ and $(\tau - 1)^{j+1} \omega_{Q',n,j} = 0$.

Suppose that there exist scalars $c_{Q',n,j} \in k$ for $(Q', n, j) \in T'$ such that

$$\sum_{(Q',n,j) \in T'} c_{Q',n,j} \omega_{Q',n,j} = \omega' \in H^0(X, \mathscr{G}_{-1}) = H^0(Y, \ker V_Y).$$

We will show that $c_{Q',n,j} = 0$ for all $(Q', n, j) \in T'$ by descending induction on j. First observe that there are no triples of the form $(Q', n, p-1)$ in T (so a fortiori there are none in T') as $\operatorname{ord}_{Q'}(E_{p-1} + pD_{p-1}) = 0$. Now fix j and suppose $c_{Q',n,i} = 0$ for all n whenever $i > j$. We compute

$$(\tau - 1)^j \omega' = j! \sum_{\{n \, : \, (Q',n,j) \in T'\}} c_{Q',n,j} \xi_{Q',n} \tag{6-19}$$

since $(\tau - 1)^j \omega_{Q',n,i} \neq 0$ implies that $i \geq j$. As $\operatorname{ord}_{Q'}(\xi_{Q',n}) = -n$, each of the nonzero terms on the right of (6-19) has a different, negative valuation at Q', while the left side is *regular* at Q' as $(\tau - 1)^j \omega'$ is regular at the points above Q' (in fact, everywhere) and $\pi : Y \to X$ is unramified over Q'. This forces $c_{Q',n,j} = 0$ for all n in the sum, as desired.

We conclude that the images of the $\omega_{Q',n,j}$ are linearly independent in the cokernel of the inclusion $H^0(Y, \pi_* \ker V_Y) \simeq H^0(X, \mathscr{G}_{-1}) \hookrightarrow H^0(X, \mathscr{G}_0)$ and the result follows. $\qquad \square$

Proof of Proposition 6.18. By Definition 6.4, $N(X, \pi) := N_1(X, \pi) + N_2(X, \pi)$, where $N_1(X, \pi) := \dim_k \mathrm{Im}(\tilde{g})$ and $N_2(X, \pi) := \dim_k \mathrm{Im}(\tilde{g}_0)$. Let $T_1 \subseteq T$ be the set of triples $(Q, n, j) \in T$ with $Q \in S_j$ for some $0 \le j \le p - 1$ such that $\nu_{Q,n,j}$ exists and $T_2 \subseteq T$ the set of triples $(Q', n, j) \in T$ such that $\nu_{Q',n,j}$ exists. Thanks to Lemmas 6.21 and 6.22 we have $\#T_1 \le N_1(X, \pi)$, while $\#T_2 \le N_2(X, \pi)$ due to Lemma 6.23. It follows that

$$N(X, \pi) = N_1(X, \pi) + N_2(X, \pi) \ge \#T_1 + \#T_2 = \#T - B,$$

where B is the number of triples $(Q, n, j) \in T$ for which $\nu_{Q,n,j}$ does not exist. $\qquad \square$

Remark 6.24. Using Corollary 6.13 and the observation that at most a_X of the $\xi_{Q,n}$ do not exist, we will use Proposition 6.18 to bound $U(X, \pi)$, thereby obtaining bounds on a_Y. A precise statement will be given in Theorem 6.26.

Remark 6.25. Let $Q \in S$. This argument uses very limited information about the coefficients of a local expansion of f at Q. In particular, it only uses that f has a leading term (a nonzero $t_Q^{-d_Q}$ term). If there were additional nonzero coefficients, the image of $\tilde{g}$ could very well be larger, and $N(X, \pi)$ could be strictly larger than $\#T - B$. However, Example 7.5 shows it is not possible to do better in general. This is the main place the argument loses information.

6D. *Explicit bounds.* We continue to use the notation from Section 3; in particular, $\pi : Y \to X$ is a fixed degree-p Artin–Schreier cover of smooth, projective, and connected curves over k with branch locus $S \subseteq X(\bar{k})$, and d_Q is the unique break in the ramification filtration at the unique point of Y above $Q \in S$. For nonnegative integers d, i with $p \nmid d$ let $\tau_p(d, i)$ be the number of positive integers $n \le \lfloor id/p \rfloor$ with the property that $-n \equiv md \mod p$ for some m with $0 < m \le p - 1 - i$.

Theorem 6.26. *With notation as above,*

$$a_Y \le p a_X + \sum_{Q \in S} \sum_{i=1}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor - \tau_p(d_Q, i) \right) \tag{6-20}$$

and for any j with $1 \le j \le p - 1$

$$a_Y \ge \sum_{Q \in S} \sum_{i=j}^{p-1} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right) \frac{jd_Q}{p} \right\rfloor \right). \tag{6-21}$$

Moreover, for any nonnegative integers d, i with $p \nmid d$ and $i \le p - 1$,

$$\tau_p(d, i) \ge (p - 1 - i) \left\lfloor \frac{1}{p} \left\lceil \frac{id_Q}{p} \right\rceil \right\rfloor \ge (p - 1 - i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor. \tag{6-22}$$

Proof. Let $\mathrm{n}(X)$ be the Tango number of X, and for $0 \le i \le p - 1$ let D_i be as above Definition 4.3 (see also Corollary 2.7). Adding in more points to each D_i as needed, we may without loss of generality

assume that $\deg(D_i) > \mathrm{n}(X)$ for all i. Using this assumption, we will analyze the lower and upper bounds from Proposition 6.10 separately. Recall that we have fixed an Artin–Schreier equation $y^p - y = f$ giving the extension of function fields $k(Y)/k(X)$ with f minimal in the sense of Definition 3.1; in particular f has a pole of order d_Q at each $Q \in S$, as well as a pole of some order $d_{Q'}$ that is divisible by p at the fixed point $Q' \in X(k) - S$. As in Definition 3.4, for $Q \in S' = S \cup \{Q'\}$ and $0 \le i \le p - 1$ we set $n_{Q,i} := \lceil (p-1-i)d_Q/p \rceil$ if $Q \ne Q'$ and $n_{Q',i} := (p-1-i)d_{Q'}$, and we put $E_i := \sum_{Q \in S'} n_{Q,i}[Q]$. We now define

$$A_i := \sum_{Q \in S'} \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor [Q], \qquad R_i = \sum_{Q \in S} r(Q,i)[Q] := E_i - pA_i.$$

Note that R_i is supported on S, as $n_{Q',i}$ is a multiple of p. Let $s(Q,i) = r(Q,i) - 1$ if $r(Q,i) > 0$, and 0 otherwise.

We will first establish (6-20). By Proposition 6.10 we have

$$a_Y \le \sum_{i=0}^{p-1} \dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))) - N(X, \pi). \tag{6-23}$$

To understand the first sum we will use Corollary 6.13, and to understand $N(X, \pi)$ we will use Proposition 6.18.

Fix i. As $\deg(D_i) > n(X)$, the first part of Corollary 6.13 implies that

$\dim_k H^0(X, \ker V_X(F_*(E_i + pD_i)))$

$$= \dim_k H^0(X, \ker V_X(F_*(p(A_i + D_i) + R_i))) = (p-1)\deg(A_i + D_i) + \sum_{Q \in S} s(Q,i)$$

$$= (p-1)\left(\sum_{Q \in S} \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor + \frac{d_{Q'}}{p}(p-1-i) + \#\sup(D_i) \right) + \sum_{Q \in S} s(Q,i). \tag{6-24}$$

We also want to count elements of the form (Q, n, i) in the set T of Proposition 6.18 for fixed i and $Q \in S_i' = S' \cup \sup(D_i)$. If $Q \in \sup(D_i)$, then there are $p - 1$ such elements by definition since $\mathrm{ord}_Q(E_i + pD_i) = p$. If $Q = Q'$, there are $(p-1)(p-1-i)d_{Q'}/p$ such elements since $\mathrm{ord}_{Q'}(E_i) = (p-1-i)d_{Q'}$. When $Q \in S$, the number of elements of the form (Q, n, i) in T is precisely $\tau_p(d_Q, p-1-i)$ since the set of positive integers $n \le n_{Q,i}$ with the property that $i + (n-1)d_Q^{-1} \equiv m \bmod p$ for some m with $0 \le m < i$ is in bijection with the set of positive integers $n' \le \lfloor (p-1-i)d_Q/p \rfloor$ with $-n' \equiv m'd_Q \bmod p$ for some m' with $0 < m' \le i$ via $n' := n - 1$ and $m' = i - m$. Putting this together, we find that there are exactly

$$\sum_{Q \in S} \tau_p(d_Q, p-1-i) + (p-1)\#\sup(D_i) + (p-1)(p-1-i)\frac{d_{Q'}}{p} \tag{6-25}$$

elements of T of the form (Q, n, i).

Putting (6-23) together with (6-24) and the bound on $N(X, \pi)$ coming from (6-25), Proposition 6.18, and the observation that at most pa_X of the $\nu_{Q,n,j}$ do not exist (since at most a_X of the $\xi_{Q,n}$ do not exist),

we conclude that

$$a_Y \leq p \cdot a_X + \sum_{i=0}^{p-2} \sum_{Q \in S} (p-1) \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor + s(Q,i) - \tau_p(d_Q, p-1-i). \tag{6-26}$$

Using the very definition of $s(Q,i)$, one finds the formula

$$s(Q,i) = n_{Q,i} - p \cdot \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor - \left(\left\lceil \frac{n_{Q,i}}{p} \right\rceil - \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor \right)$$

$$= n_{Q,i} - \left\lceil \frac{n_{Q,i}}{p} \right\rceil - (p-1) \cdot \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor \tag{6-27}$$

Substituting (6-27) into (6-26), reindexing the sum $i \mapsto p-1-i$ and using the equality $\lceil x \rceil = \lfloor x \rfloor + 1$ for $x \notin \mathbf{Z}$ gives (6-20).

To make this bound more explicit, we must bound $\tau_p(d,i)$ from below. To do this, simply note that for any $0 < m \leq p-1-i$ and any interval of length p, there is a unique n in that interval such that

$$md \equiv -n \mod p$$

and that necessarily $n \not\equiv 0 \mod p$ as $p \nmid d$. Thus,

$$\tau_p(d_Q, i) \geq (p-1-i) \cdot \left\lfloor \frac{1}{p} \left(\left\lfloor \frac{id_Q}{p} \right\rfloor + 1 \right) \right\rfloor = (p-1-i) \cdot \left\lfloor \frac{1}{p} \left\lceil \frac{id_Q}{p} \right\rceil \right\rfloor$$

$$\geq (p-1-i) \left\lfloor \frac{id_Q}{p^2} \right\rfloor, \tag{6-28}$$

as $0 \leq n \leq \lfloor id_Q/p \rfloor = \lceil id_Q/p \rceil - 1$.

We now turn to the lower bound (6-21). Proposition 6.10 and Definition 6.6 give that for each $0 \leq j \leq p-1$,

$$\dim_k U_j - \sum_{i=0}^{j} \sum_{Q \in S'_j} c(i,j,Q) \leq a_Y. \tag{6-29}$$

Using Lemma 6.7, Corollary 6.13 and the assumption that $\deg D_i > n(X)$ for all i, we calculate

$$\dim_k U_j = \sum_{i=0}^{j} \dim_k H^0(X, \Omega_X^1(F_*(E_i + pD_i)))$$

$$= \sum_{i=0}^{j} \left((p-1)\deg(D_i) + (p-1)(p-1-i)\frac{d_{Q'}}{p} + \sum_{Q \in S} (p-1) \left\lfloor \frac{n_{Q,i}}{p} \right\rfloor + s(Q,i) \right).$$

Now for $Q \in \sup(D_i)$ we have $c(i,j,Q) = p-1$, while

$$c(i,j,Q') = (p-1)(p-1-i)d_{Q'}/p.$$

If $Q \in S$, then $c(i, j, Q)$ is the number of integers $n \equiv -1 \mod p$ satisfying $-n_{Q,j} - d_Q(j-i) \le n < -n_{Q,i}$. That is,

$$c(i, j, Q) = \left\lceil \frac{n_{Q,j} + d_Q(j-i)}{p} \right\rceil - \left\lceil \frac{n_{Q,i}}{p} \right\rceil.$$

Thus, the contributions from D_i and Q' in (6-29) cancel, so the formula (6-27) for $s(Q, i)$ yields

$$\sum_{i=0}^{j} \sum_{Q \in S} \left(n_{Q,i} - \left\lceil \frac{n_{Q,j} + d_Q(j-i)}{p} \right\rceil \right) \le a_Y$$

for all $0 \le j < p - 1$. Using the equality

$$\left\lceil \frac{n_{Q,j} + (j-i)d_Q}{p} \right\rceil = \left\lceil \frac{(p-1-i)d_Q}{p} - \left(1 - \frac{1}{p}\right)\frac{(p-1-j)d_Q}{p} \right\rceil,$$

changing variables $i \mapsto p - 1 - i$ and $j \mapsto p - 1 - j$, and employing again the equality $\lceil x \rceil = \lfloor x \rfloor + 1$ for $x \notin \mathbf{Z}$ then gives (6-21). $\qquad\square$

Corollary 6.27. *Suppose p is odd. With the hypotheses of Theorem 6.26 we have*

$$a_Y \le p \cdot a_X + \left(\frac{(p-1)(p-2)}{2} + \left(1 - \frac{1}{p}\right)^2 \right) \cdot \#S + \left(1 - \frac{1}{p}\right) \sum_{Q \in S} \frac{(2p-1)}{6} d_Q$$

and

$$a_Y \ge \left(1 - \frac{1}{p}\right)^2 \left(\sum_{Q \in S} \frac{(p+1)}{4} d_Q - \#S \cdot \frac{p}{2} \right).$$

Proof. The upper bound follows easily from (6-20) and the first inequality in (6-22) by basic properties of the floor function and the well-known equality

$$\sum_{i=1}^{n-1} \left\lfloor \frac{id}{n} \right\rfloor = \frac{(n-1)(d-1)}{2} \tag{6-30}$$

for any positive and coprime integers d, n. For the lower bound, we have

$$\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right)\frac{jd_Q}{p} \right\rfloor \ge \frac{id_Q - (p-1)}{p} - \left(\frac{id_Q}{p} - \left(1 - \frac{1}{p}\right)\frac{jd_Q}{p} \right)$$

$$= \left(1 - \frac{1}{p}\right)\left(\frac{jd_Q}{p} - 1 \right).$$

Summing over i with $j \le i \le p - 1$ and then Q and using (6-21) gives the lower bound

$$\left(1 - \frac{1}{p}\right) \sum_{Q \in S} \left(\frac{d_Q}{p}(p-j)j - (p-j) \right) \tag{6-31}$$

which holds for all $1 \le j \le p - 1$; we then take $j = (p+1)/2$. Remark 6.28 motivates this choice. $\qquad\square$

Remark 6.28. The choice of $j = (p+1)/2$ in the proof of Corollary 6.27 is optimized for $\sum d_Q$ large relative to #S. Indeed, (6-31) is a quadratic function in j which attains its maximum when

$$j \approx \frac{p}{2}\left(1 - \frac{\#S}{\sum_{Q \in S} d_Q}\right). \tag{6-32}$$

Any nearby value of j will give a similar bound. Supposing that $\sum d_Q$ is large relative to #S gives optimal choice $j = \lceil p/2 \rceil = (p+1)/2$. When all d_Q are small and p is large, one can get a better explicit lower bound by choosing a value of j in accordance with (6-32); cf. Example 7.1.

Remark 6.29. For fixed X, p and S with all d_Q becoming large, the dominant terms of the lower and upper bounds in Corollary 6.27 are respectively

$$\sum_{Q \in S} d_Q \frac{p}{4} \quad \text{and} \quad \sum_{Q \in S} d_Q \frac{p}{3}.$$

On the other hand, the dominant term in the Riemann–Hurwitz formula for the genus of Y is $\sum_{Q \in S} d_Q p/2$, so that, for large d_Q, the a-number is approximately between $1/2$ and $2/3$ of the genus of Y.

Remark 6.30. When $p = 2$, the statement and proof of the Corollary do not work as written. If we take $j = 1$ in Theorem 6.26 we obtain

$$\sum_{Q \in S}\left\lfloor \frac{d_Q}{2} \right\rfloor - \left\lfloor \frac{d_Q}{4} \right\rfloor \le a_Y \le 2a_X + \sum_{Q \in S}\left\lfloor \frac{d_Q}{2} \right\rfloor - \left\lfloor \frac{d_Q}{4} \right\rfloor.$$

In particular, when X is ordinary (i.e., $a_X = 0$) we obtain an exact formula for a_Y. This recovers [Voloch 1988, Theorem 2] (note that the formula there is for the rank of the Cartier operator, and that for $Q_i \in S$, our d_{Q_i} is $2n_i - 1$).

Similarly, Corollary 6.32 will give an exact formula for a_Y when p is odd, X is ordinary, and $d_Q | (p-1)$. To derive it, we will need to investigate situations when it is possible to derive an exact formula for the quantity

$$\tau_p(d) := \sum_{i=0}^{p-1} \tau_p(d, i) \tag{6-33}$$

occurring in the upper bound (6-20):

Proposition 6.31. *Let $p > 2$ and suppose that $d \equiv d'$ mod p^2. Then*

$$\tau_p(d) = \tau_p(d') + (d - d')\frac{(p-1)(p-2)}{6p}. \tag{6-34}$$

Moreover, $\tau_p(1) = 0$ and for $1 < d < p$ we have

$$\tau_p(d) = u_p(d) \cdot \frac{p}{d} + v_p(d), \tag{6-35}$$

where $u_p(d)$ and $v_p(d)$ are the integers (depending only on p mod d) given by

$$u_p(d) := \sum_{j=0}^{b-1} \sum_{r \in T_j} \big((j+1)d - (b+1)r\big),$$

$$(6\text{-}36)$$

$$v_p(d) := \sum_{j=0}^{b-1} \sum_{r \in T_j} \left(\frac{r}{d} - \left\{\frac{a(d-r)}{d}\right\}\right),$$

where $a := p$ mod d and $b := a^{-1}$ mod d with $0 < a, b < d$ and

$$T_j := \{r \in \mathbf{Z} \ : \ jd/b < r < (j+1)d/(b+1)\}.$$

In particular, for $1 < d < p$ the quantity $\tau_p(d)$ depends only on d and p mod d.

Before proving Proposition 6.31, let us give some indication of its utility. For example, if $p \equiv 1$ mod d, we have $a = b = 1$, whence

$$u_p(d) = \sum_{r=1}^{\lfloor d/2 \rfloor} (d - 2r) = \left\lfloor \frac{d}{2} \right\rfloor \cdot \left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{(d-1)^2}{4} \right\rfloor$$

and

$$v_p(d) = \sum_{r=1}^{\lfloor d/2 \rfloor} \frac{r}{d} - \frac{d-r}{d} = \sum_{r=1}^{\lfloor d/2 \rfloor} \frac{2r-d}{d} = -\left\lfloor \frac{(d-1)^2}{4} \right\rfloor.$$

Thus

$$\tau_p(d) = \left\lfloor \frac{(d-1)^2}{4} \right\rfloor \cdot \frac{p-1}{d}$$

$$(6\text{-}37)$$

whenever $d < p$ and $p \equiv 1$ mod d. Another simple example is when $p \equiv -1$ mod d, so that $a = b = d-1$. In this case, $T_j := \{r \in \mathbf{Z} \ : \ jd/(d-1) < r < (j+1)\}$ is the empty set for all j, whence $\tau_p(d) = 0$.

Proof. The first assertion follows easily from the fact that, as observed immediately prior to (6-28), for any $0 < m \leq p - 1 - i$ and any interval of length p, there is a unique n in that interval such that

$$md \equiv -n \bmod p$$

and necessarily $n \not\equiv 0$ mod p. So suppose that $d < p$, and for integers m, i, define

$$\chi(m, i) := \begin{cases} 1 & \text{if } m \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

We use the convention that "x mod y" denotes the unique integer x_0 with $0 \leq x_0 < y$ and $x \equiv x_0$ mod y, and we set $a := p$ mod d and $b := a^{-1}$ mod d. By definition,

$$\tau_p(d) = \sum_{i=0}^{p-1} \sum_{\substack{0 < j \leq n_i(d) \\ j \not\equiv 0 \bmod p}} \chi(-jd^{-1} \bmod p, \, p - 1 - i),$$

$$(6\text{-}38)$$

where $n_i(d) := \lfloor id/p \rfloor$. For an integer r, observe that $r \le n_i(d) < d$ if and only if $i > rp/d$, and that $-jd^{-1} \equiv j(bp-1)/d \bmod p$ (note that $d|(bp-1)$). Taking this into account, we swap the order of summation in (6-38), collecting all terms with $j = r$ together to obtain

$$\sum_{r=1}^{d-1} \sum_{i=\lceil rp/d \rceil}^{p-1} \chi\left(r\frac{bp-1}{d} \bmod p, \, p-1-i\right). \tag{6-39}$$

Since $r < d < p$ and $rb \equiv rp^{-1} \not\equiv 0 \bmod d$, we have

$$r\frac{bp-1}{d} \bmod p = \frac{(rb \bmod d) \cdot p - r}{d}$$

and moreover $rb \bmod d = rb - \ell d$ for $\ell d/b \le r < (\ell+1)d/b$. Breaking the sum over r up into these regions converts (6-39) into

$$\sum_{\ell=0}^{b-1} \sum_{\substack{\ell d/b < r \\ r < (\ell+1)d/b}} \sum_{i=\lceil rp/d \rceil}^{p-1} \chi\left(\frac{(rb-\ell d)p-r}{d}, \, p-1-i\right). \tag{6-40}$$

Indeed, the quantity $\ell d/b$ is either 0 or *nonintegral* since $\gcd(b, d) = 1$ and $0 \le \ell < b$; thus $\ell d/b \le r$ is equivalent to $\ell d/b < r$ since r ranges over all positive integers less than d. The innermost sum in (6-40) has value the cardinality of the subset of positive integers

$$T(r, \ell) := \left\{i \in \mathbf{N} \; : \; \frac{(rb-\ell d)p-r}{d} \le p-1-i \le p-1-\left\lceil\frac{rp}{d}\right\rceil = \left\lfloor\frac{(d-r)p}{d}\right\rfloor - 1\right\}.$$

Now $T(r, \ell)$ is empty unless

$$\frac{(d-r)p}{d} - 1 > \frac{(rb-\ell d)p-r}{d}, \tag{6-41}$$

in which case

$$\begin{aligned}
\#T(r, \ell) &= \left\lfloor\frac{(d-r)p}{d}\right\rfloor - \frac{(rb-\ell d)p-r}{d} \\
&= \frac{(d-r)(p-a)}{d} + \left\lfloor\frac{a(d-r)}{d}\right\rfloor - \frac{(rb-\ell d)p-r}{d} \\
&= ((\ell+1)d - (b+1)r)\frac{p}{d} + \left(\frac{r}{d} - \left\{\frac{a(d-r)}{d}\right\}\right).
\end{aligned} \tag{6-42}$$

The inequality (6-41) is equivalent to $((\ell+1)d - (b+1)r)p > (d-r)$ which, as $p > d > d-r$ is equivalent to $(\ell+1)d > (b+1)r$, or what is the same thing, $r < (\ell+1)d/(b+1)$. In other words, the contribution from the innermost sum (6-39) for r in the ranges $(\ell+1)d/(b+1) < r < (\ell+1)d/b$ is zero so that these values of r in the middle sum may be omitted, and the expression (6-42) then substituted for the innermost sum, which yields the claimed formula for $\tau_p(d)$ when $d < p$. $\qquad\square$

Using the bounds of Theorem 6.26, we are able to generalize the main result of [Farnell and Pries 2013], which provides an *a*-number *formula* for branched $\mathbf{Z}/p\mathbf{Z}$-covers of $X = \mathbf{P}^1$ with all ramification breaks d dividing $p-1$, to the case of *arbitrary* ordinary base curves X:

Corollary 6.32. *Let* $\pi : Y \to X$ *be a branched* $\mathbf{Z}/p\mathbf{Z}$*-cover with* $a_X = 0$, *and suppose* p *is odd. If* d_Q *divides* $p - 1$ *for every branch point* Q, *then*

$$a_Y = \sum_Q a_Q, \qquad \text{where } a_Q := \frac{(p-1)}{2}(d_Q - 1) - \frac{p-1}{d_Q}\left\lfloor \frac{(d_Q - 1)^2}{4} \right\rfloor. \tag{6-43}$$

Proof. We will compute the upper and lower bounds for a_Y given by (6-20) and (6-21), and show that these bounds coincide when $d_Q | (p-1)$ for all Q and $a_X = 0$, and agree with the stated formula. Using the hypothesis $a_X = 0$ together with the explicit formula for $\tau_p(d)$ when $d \equiv 1 \mod p$ provided by (6-37), the upper bound (6-20) becomes

$$a_Y \leq \sum_{Q \in S} \sum_{i=1}^{p-1}\left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor\right) - \sum_{Q \in S} \frac{p-1}{d_Q}\left\lfloor \frac{(d_Q - 1)^2}{4} \right\rfloor. \tag{6-44}$$

Now $d_Q < p$ for all Q, so $\lfloor id_Q/p^2 \rfloor = 0$ for $1 \leq i \leq p - 1$. Using (6-30), we conclude that

$$\sum_{Q \in S} \sum_{i=1}^{p-1}\left(\left\lfloor \frac{id_Q}{p} \right\rfloor - \left\lfloor \frac{id_Q}{p^2} \right\rfloor\right) = \frac{(p-1)}{2}(d_Q - 1). \tag{6-45}$$

Combining (6-44) and (6-45) gives the formula (6-43) as the upper bound of a_Y. It remains to prove that a_Y is also bounded below by the same quantity.

Set $j := (p+1)/2$, and write $d_Q \cdot f_Q = p - 1$. For $i < p$, we see that

$$\left\lfloor \frac{id_Q}{p} \right\rfloor = \begin{cases} \lfloor i/f_Q \rfloor & \text{when } f_Q \nmid i, \\ \lfloor i/f_Q \rfloor - 1 & \text{when } f_Q | i. \end{cases}$$

Likewise we can check that

$$\left\lfloor \frac{id_Q}{p} - \left(1 - \frac{1}{p}\right)\frac{jd_Q}{p} \right\rfloor = \left\lfloor \frac{i-j}{f_Q} \right\rfloor.$$

The inner sum from (6-21) becomes

$$\sum_{i=(p+1)/2}^{p-1}\left\lfloor \frac{i}{f_Q} \right\rfloor - \sum_{i=0}^{(p-3)/2}\left\lfloor \frac{i}{f_Q} \right\rfloor - \#\{(p+1)/2 \leq i \leq p - 1 : f_Q | i\}.$$

The term in the first sum can be rewritten as $\lfloor d_Q/2 + (i - (p-1)/2) \rfloor$. When d_Q is even, $d_Q/2$ can be removed from the floor function, allowing cancellation with the second sum giving that the inner sum from (6-21) is

$$\frac{(p-1)d_Q}{4}.$$

This equals a_Q from (6-43) when d_Q is even. When d_Q is odd, a similar argument removing $\lfloor d_Q/2 \rfloor = \frac{1}{2}(d_Q - 1)$ from the first sum shows that the lower bound is

$$\lfloor d_Q/2 \rfloor \frac{p-1}{2} + \frac{f_Q \cdot \lfloor d_Q/2 \rfloor}{2} - f_Q/2 = f_Q \cdot \frac{(d_Q + 1)(d_Q - 1)}{4}.$$

Again, this matches the formula for a_Q. Summing over Q completes the proof. $\qquad\square$

Remark 6.33. Note that the proof shows that one has the alternative expression

$$a_Q = \begin{cases} \dfrac{(p-1)d_Q}{4} & d_Q \text{ is even,} \\[2mm] \dfrac{(p-1)(d_Q+1)(d_Q-1)}{4d_Q} & d_Q \text{ is odd.} \end{cases}$$

for a_Q as in (6-43); cf. [Farnell and Pries 2013, Theorem 1.1].

6E. *Unramified covers.* Now suppose that $\pi : Y \to X$ is an unramified degree-p Artin–Schreier cover. (No such covers exist when $X = \mathbf{P}^1$.) Theorem 6.26 applies with $S = \varnothing$, but the analysis is not optimized for this situation. In particular, the argument uses the dimension of $H^0(X, \ker V_X)$ (the a-number) but no extra information about the dimension of $H^0(X, \ker V_X(F_*D))$ for other divisors D. This is not an issue when π has a substantial amount of ramification, but would be essential to obtaining sharper bounds for unramified covers. We now illustrate this.

In this setting, the bounds of Theorem 6.26 become

$$0 \le a_Y \le p \cdot a_X.$$

On the other hand, since $E_0 = 0$ the trivial bounds are

$$a_X \le a_Y \le p \cdot (g_X - f_X).$$

Note that $p \cdot (g_X - f_X) \ge p \cdot a_X$, so our upper bound is an improvement, but the trivial lower bound is better!

Any of these bounds are enough to rederive the following fact (which is typically deduced from the Deuring-Shafarevich formula):

Corollary 6.34. *Let $\pi : Y \to X$ be an unramified degree-p Artin–Schreier cover. Then Y is ordinary if and only if X is ordinary.*

The trivial lower bound is better because we lost some information in our applications of Tango's theorem. In particular, in the proof of Theorem 6.26 the first step is to add more points to the auxiliary divisors D_i (if needed) so that $\deg(D_i) > \mathrm{n}(X)$ for all i and we can apply Corollary 6.13 to exactly calculate

$$\dim_k H^0(X, \ker V_X(F_*(E_i + pD_i))) = \dim_k H^0(X, \ker V_X(F_*(pD_i)))$$
$$= (p-1)\deg(D_i).$$

The trivial bounds comes from the inclusion $\ker V_X \hookrightarrow \pi_* \ker V_Y$ given by $\omega \mapsto \omega y^0$, which would correspond to taking $j = 0$ in Definition 6.6 and being allowed to take $D_0 = 0$. If that were the case, S is empty and the lower bound is

$$\dim U_0 - \sum_{Q \in S_0} c(0, 0, Q) = \dim H^0(X, \ker V_X(F_*0)) = a_X$$

which matches the trivial lower bound. However, our analysis uses Corollary 6.13, which requires a sufficiently large D_0. In that case,

$$\dim U_0 - \sum_{Q \in S_0} c(0, 0, Q) = \dim H^0(X, \ker V_X(F_* D_0))$$

$$= (p-1)\deg(D_0) - \sum_{Q \in D_0}(p-1) = 0$$

as $c(0, 0, Q) = (p-1)$ for $Q \in D_0$. Thus the lower bound of Theorem 6.26 is zero. (In Theorem 6.26, we reindexed so there we are taking $j = p - 1$.)

The technical requirements of using sufficiently large divisors D_i in order to apply Tango's theorem are not an issue when π is "highly ramified" in the sense that we do not need to increase the degree of the D_i in order to apply Corollary 6.13 to compute the dimension of $H^0(X, \ker V_X(F_*(E_i + pD_i)))$. Once we have reached that point, increasing D_i further does not change the bound: the dimension of this space increases, but the increase is canceled by additional $c(i, j, Q)$ terms for $Q \in D_i$. This is why the divisor D_i makes no appearance in Theorem 6.26.

7. Examples

As always, let $\pi : Y \to X$ be a degree-p Artin–Schreier cover of curves. We give some examples of the bounds given by Theorem 6.26 and the trivial bounds

$$\dim_k H^0(X, \ker V_X(F_* E_0)) \le a_Y \le p \cdot g_X - p \cdot f_X + \sum_{Q \in S} \frac{p-1}{2}(d_Q - 1)$$

discussed in Remark 1.3. When $p = 3$, one checks that the lower bound $L(X, \pi)$ *coincides* with the trivial lower bound. Outside of this special case, the bounds in Theorem 6.26 are *always* better than the trivial bounds, and often sharp in the sense that there are degree-p Artin–Schreier covers $\pi : Y \to X$ with a_Y realizing our bounds in many cases. We will give a number of examples illustrating these features. Magma programs which do the calculations in the following examples are available on the authors' websites.

7A. *The projective line.* Suppose $X = \mathbf{P}^1$. Then $n(X) = -1$, $g_X = 0$, $a_X = 0$, and $f_X = 0$. Remark 3.3 shows that we may choose $f \in k(X)$ so that the extension of function fields $k(Y)/k(X)$ is given by adjoining the roots of $y^p - y = f$ with $f \in k(X)$ regular outside the branch locus S of π, and we may further assume that f has a pole of order d_Q at each $Q \in S$, where d_Q is the unique break in the lower-numbering ramification filtration above Q.

For $Q \in S$ put $n_{Q,i} = \lceil (p - 1 - i)d_Q/p \rceil$; note that $n_{Q,0} = d_Q - \lfloor d_Q/p \rfloor$. The trivial bounds are

$$\sum_{Q \in S}\left(n_{Q,0} - \left\lceil \frac{n_{Q,0}}{p}\right\rceil\right) \le a_Y \le \sum_{Q \in S}\frac{p-1}{2}(d_Q - 1).$$

Example 7.1. Let $p = 13$, and suppose f has a single pole. Table 1 shows the trivial upper and lower bounds, as well as the bounds from Theorem 6.26 for various values of d_Q. When $d_Q > 4$, an optimum

value of j to use in the lower bound turns out to be $\frac{1}{2}(p+1) = 7$. Notice that our bounds are substantially better than the trivial bounds.

Example 7.2. Using Magma [Bosma et al. 1997] or a MAPLE program from Shawn Farnell's thesis [2010], we can compute the *a*-number for covers of $\mathbf{P}^1$. For example, let $p = 13$ and suppose f has a single pole of order 7. Our results show that the *a*-number of the cover is between 21 and 36. Table 2 lists the *a*-numbers for some choices of f, and shows that our bounds are sharp in this instance.

Example 7.3. Let us generalize Examples 4.6 and 5.21. As before, let $p = 5$ and $X = \mathbf{P}^1$, but now consider covers $Y_{A,B}$ of X given by $y^5 - y = f_{A,B} = t^{-3} + At^{-2} + Bt^{-1}$. Then $Y_{A,B} \to X$ is branched only over $Q := 0$ and has $d_Q = 3$, so our bounds are $3 \le a_{Y_{A,B}} \le 4$.

Recall the maps g_i' depend on the cover via the choice $f_{A,B}$. It is clear that the elements $v_{0,2}$, $v_{0,3}$, and $v_{1,2}$ lie in $H^0(\mathbf{P}^1, \mathscr{G}_1)$. In fact, so does $v_{2,2}$ as $g_2'(v_{2,2}) = 0$: indeed, recalling Section 5B, we see that $g_2'(v_{2,2})$ records the coefficient of $t^{-6}dt$ in

$$-2 \cdot (t^{-2}dt) \cdot -(t^{-3} + At^{-2} + Bt^{-1})$$

which is visibly zero.

To compute the *a*-number (equal to $\dim H^0(\mathbf{P}^1, \mathscr{G}_0)$ in this case), it therefore suffices to understand $\ker H^0(g_1') = H^0(\mathbf{P}^1, \mathscr{G}_0)$. It is clear that $v_{0,2}$ and $v_{0,3}$ lie in $H^0(\mathbf{P}^1, \mathscr{G}_0)$. An identical calculation to the one above shows that $g_1'(v_{1,2}) = 0$ as well. Finally, we see that $g_1'(v_{2,2})$ records the coefficient of $t^{-6}dt$ in

$$-t^{-2}(t^{-3} + At^{-2} + Bt^{-1})^2 dt = -(t^{-8} + 2At^{-7} + (A^2 + 2B)t^{-6} + \cdots)dt.$$

Thus $g_1'(v_{2,2}) = 0$ when $A^2 + 2B = 0$, and is nonzero otherwise. In particular, $a_{Y_{A,B}} = 3$ when $A^2 + 2B \ne 0$ and $a_{Y_{A,B}} = 4$ when $A^2 + 2B = 0$. This generalizes Example 5.21, again showing why the *a*-number of the cover cannot depend only on the d_Q and must incorporate finer information.

The set T in Proposition 6.18 is an attempt to produce differentials not in the kernel of some g_i'. It only uses the leading terms of powers of $f_{A,B}$ (since those are the only terms guaranteed to be nonzero). For the cover $Y_{0,0}$, the leading term is the only term, and there are no differentials not in the kernel of some g_i'. This is reflected in the fact that T is empty in this case. When $A^2 + 2B \ne 0$, the *a*-number was smaller because there was a differential not in $\ker H^0(g_1')$; this relied on the nonleading terms of $f_{A,B}$.

Example 7.4. Let us now consider an example with multiple poles. Let $p = 5$, and suppose that f has two poles of order 7 (at ∞ and -1). Then our bounds say that $14 \le a_Y \le 16$. Computing the *a*-number

$d_Q =$	1	2	3	4	5	6	7	8	9	10	11	12	14	15	32	128	1024
trv. lower	0	1	2	3	4	5	6	7	8	9	10	11	12	12	27	109	873
$L(\mathbf{P}^1, \pi)$	0	6	8	12	15	18	21	24	26	30	33	36	42	45	96	382	3054
$U(\mathbf{P}^1, \pi)$	0	6	8	12	16	18	36	30	34	36	38	36	78	60	120	488	3936
trv. upper	0	6	12	18	24	30	36	42	48	54	60	66	78	84	186	762	6138

Table 1. Bounds for a single pole with $p = 13$. (trivial abbreviated as trv.)

polynomial	a_Y
$t^{-7} + 2t^{-6} + 7t^{-5}$	21
$t^{-7} + t^{-2} + t^{-1}$	23
$t^{-7} + 8t^{-2}$	24
$t^{-7} + t^{-1}$	27
t^{-7}	36

Table 2. *a*-numbers for some select Artin–Schreier curves.

for a thousand random choices of f defined over $\mathbf{F}_5$ subject to the constraint on the poles, 942 of them had *a*-number 14, 41 had *a*-number 15, and 1 had *a*-number 16. The f giving *a*-number 16 was

$$t^7 + 3t^6 + t^4 + t^3 + 2t^2 + t + 3(t+1)^{-1} + 3(t+1)^{-2} + 4(t+1)^{-3} + 4(t+1)^{-4} + 4(t+1)^{-6} + 3(t+1)^{-7}.$$

It would be interesting to understand the geometry of the relevant moduli space of Artin–Schreier covers with prescribed ramification. In this situation, these numerical examples seem to suggest that the locus of curves with *a*-numbers greater than or equal to 15 (respectively 16) is of codimension two in the locus with *a*-number greater than or equal to 14 (respectively 15).

Example 7.5. For general p, take $f(t) = t^{-d}$ with $p \nmid d$. We will show that this family achieves our upper bound. As discussed in [Farnell and Pries 2013, Remark 2.1] (which extracts the results from [Pries 2005]), the resulting *a*-number is

$$\sum_{b=0}^{d-2} \min\left(h_b, \left\lfloor \frac{pd - bp - p - 1}{d} \right\rfloor \right)$$

where h_b is the unique integer in $[0, p-1]$ such that $h_b \equiv (-1-b)d^{-1} \bmod p$. Note that if $b \equiv -1 \bmod p$ then $h_b = 0$. This counts the number of elements in the set

$$T' := \left\{ (b, j) : 0 \le b \le d-2, \ 0 \le jd \le p(d-b-1) - d - 1, \ j < h_b \right\}.$$

On the other hand, our upper bound is $\bigoplus_{i=0}^{p-1} \dim_{\mathbf{F}_p} H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_* E_i)) - \#T$. For $0 \le i \le p-1$, the differentials $t^{-n} dt$ with $0 < n \le n_{Q,i} = \lceil (p-1-i)d/p \rceil$ and $n \not\equiv 1 \bmod p$ form a basis for $H^0(\mathbf{P}^1, \ker V_{\mathbf{P}^1}(F_* E_i))$. The condition $n \le n_{Q,i}$ can be expressed as $pn \le (p-1-i)d + (p-1)$, or equivalently

$$i \cdot d \le (p-1)d + (p-1) - pn = p(d-n+1) - 1 - d.$$

Assigning to each such basis element the triple (Q, n, i), note that (Q, n, j) *does not* lie in T if there does not exist an integer $m \in [0, j]$ such that $m \equiv j - (n-1)d_Q^{-1} \bmod p$: this can be rephrased as $j < h_{n-2}$. Thus our upper bound is the size of the set

$$T'' := \left\{ (n, j) : 2 \le n \le d, \ 0 \le j \cdot d \le p(d-n+1) - 1 - d, \ j < h_{n-2} \right\}.$$

(The condition $n \not\equiv 1 \bmod p$ is implicit, as in that case $h_{n-2} = 0$.) But T' and T'' have the same size: there is a bijection given by taking $b = n - 2$. Thus the covers given by $f(t) = t^{-d}$ for $p \nmid d$ realize the upper bound for a cover ramified at a single point.

7B. Elliptic curves. We now suppose that E is the elliptic curve over $\mathbf{F}_p$ with affine equation $y^2 = x^3 - x$ (recall that $p > 2$). Of course, $g_E = 1$ and it is not hard to compute that the Tango number of E is $n(E) = 0$ and that E is ordinary (so $a_E = 0$ and $f_E = 1$) when $p \equiv 1 \bmod 4$ and supersingular ($a_E = 1$ and $f_E = 0$) when $p \equiv 3 \bmod 4$. In this simple case, we can say more than what Tango's theorem tells us.

Lemma 7.6. *Let $D = \sum n_Q \cdot [Q] > 0$ be a divisor on E with $n_Q \geq 2$ for some Q, and set $D' := \sum \lceil n_Q/p \rceil \cdot [Q]$. Then*

$$V_E : H^0(E, \Omega_E^1(D)) \to H^0(E, \Omega_E^1(D'))$$

is surjective.

Proof. This lemma is an immediate consequence of Corollary 6.13 when $n_Q \geq p$ for some Q or when $a_E = 0$, so we suppose that $n_Q < p$ for all Q and $p \equiv 3 \bmod 4$. A straightforward induction on the size of the support of D reduces us to the case that $D = n \cdot [Q]$ with $n \geq 2$, and then as $D' = [Q]$ it suffices to treat the case $n = 2$. Suppose that Q is not the point P at infinity on E. Consider the commutative diagram with exact rows

$$0 \longrightarrow H^0(E, \Omega_E^1(2[P])) \longrightarrow H^0(E, \Omega_E^1(2[P] + 2[Q])) \longrightarrow (k[x_Q]/x_Q^2)x_Q^{-2}dx_Q \longrightarrow 0$$

$$\Big\downarrow V \qquad\qquad\qquad \Big\downarrow V \qquad\qquad\qquad \Big\downarrow V$$

$$0 \longrightarrow H^0(E, \Omega_E^1([P])) \longrightarrow H^0(E, \Omega_E^1([P] + [Q])) \longrightarrow (k[x_Q]/x_Q)x_Q^{-1}dx_Q \longrightarrow 0$$

The local description of V shows that the right vertical map is surjective, and it therefore suffices to prove the lemma in the special case $D = 2[P]$. Now $\{dx/y, x\,dx/y\}$ is a basis of $H^0(E, \Omega_E^1(2[P]))$ and one calculates

$$V\left(x\frac{dx}{y}\right) = \frac{1}{y}V(x(x^3 - x)^{(p-1)/2}dx) = (-1)^{(p+1)/4}\binom{\frac{1}{2}(p-1)}{\frac{1}{4}(p-3)}\frac{dx}{y} \neq 0.$$

Since $V(dx/y) = 0$, the image of V is 1-dimensional. But $H^0(E, \Omega_E^1([P])) = H^0(E, \Omega_E^1)$ because the sum of the residues of a meromorphic differential on a smooth projective curve is zero, and this space therefore has dimension one as well; the lemma follows. $\square$

We first consider Artin–Schreier covers $Y \to E$ with defining equation of the form

$$z^p - z = f$$

with $f \in H^0(E, \mathcal{O}_E(d_P \cdot [P]))$ having a pole of exact order d_P at P, for P the point at infinity on E, where $d_P \geq 2$ is prime to p. Such covers are branched only over P with unique break in the ramification filtration d_P. Conversely, every $\mathbf{Z}/p\mathbf{Z}$-cover of E that is defined over $\mathbf{F}_p$ and is ramified only above P with unique ramification break d_P occurs this way by Lemma 3.2.

$d_P =$	2	3	4	6	7	8	9	11	12	13	14	16	32	128	1024
trv. lower	1	2	3	4	4	5	6	7	8	8	9	10	20	82	656
lower	2	3	4	6	7	8	9	10	12	12	14	15	31	123	984
upper	2	4	4	10	8	10	10	14	16	14	16	20	38	154	1230
trv. upper	2	4	6	10	12	14	16	20	22	24	26	30	62	254	2046

Table 3. Bounds for a single pole at the point at infinity with $p = 5$. (trivial abbreviated as trv.)

$d_P =$	2	3	4	5	6	8	9	10	11	12	13	15	16	32	128	1024
trv. lower	1	2	3	4	5	6	6	7	8	9	10	11	12	24	94	752
lower	3	4	6	7	9	12	13	15	16	18	20	21	24	47	188	1505
upper	10	11	16	15	16	28	23	24	27	30	29	37	40	68	244	1910
trv. upper	10	13	16	19	22	28	31	34	37	40	43	49	52	100	388	3076

Table 4. Bounds for a single pole at the point at infinity with $p = 7$. (trivial abbreviated as trv.)

	$d_P = 3$	$d_P = 6$	$d_P = 8$	$d_P = 11$
a_Y			f	
3	$-2y - x$			
4	$2y$			
6		$x^3 - 2x^2$		
7		$x^3 + x$		
8		$2x^3 - xy$	x^4	
9		$-x^3 + x$	$2x^4 - x^2 + x$	
10			$-x^4 - 2xy - 2x^2$	$(x^4 - x)y - 2x^5 - x^4$
11				$(-2x^4 + 2x + 1)y + 2x^5 + x^2$
12				$(-2x^4 - x^2 - x)y - 2x^5 - 1$
13				$(2x^4 - x^2 + 2x)y - 2x^5 - x^3 - x$

Table 5. a-numbers of $\mathbf{Z}/p\mathbf{Z}$-covers of E branched only over P with $p = 5$.

Example 7.7. For $p = 5, 7$ we compare the bounds given by Theorem 6.26 with the trivial upper and lower bounds, which are

$$\left\lceil \frac{p-1}{p} d_P \right\rceil - \left\lceil \frac{p-1}{p^2} d_P \right\rceil \leq a_Y \leq p \cdot a_E + \frac{p-1}{2}(d_P - 1)$$

thanks to Lemma 7.6 and the fact that $\lceil (p-1)d_P/p \rceil = d_P - \lfloor d_P/p \rfloor \geq 2$. We summarize these computations in Tables 3 and 4.

Example 7.8. Again for $p = 5, 7$ and select d_P in the tables above, we have computed a_Y for several thousand randomly selected $f \in H^0(E, \mathcal{O}_E(d_P \cdot [P]))$ (working over $\mathbf{F}_p$). In Tables 5 and 6, we record

	$d_P = 6$	$d_P = 8$	$d_P = 10$
a_Y		f	
10	$x^3 + xy$		
11	$x^3 + y$		
12	x^3	$3xy + 2x^4 - 2x^3$	
13		$2xy - 3x^4 - 1$	
14	$2x^3 + x$	$2xy - 2x^4 + x - 3$	
15		$-x^4 + 2x^3 + 2x^2 - 3x - 3$	$-3x^5 - 3x^4 - x^2 + 3x$
16		$(x^2 + x)y + 3x^4 - 2x^3 - 3x^2 - x + 1$	$2x^5 - 3x^3 + x^2 + 3x$
17			$-x^5 + x^3 + 2x^2 - 1$
18			$2xy - x^5 + x^3 + 2x^2 - x - 1$

Table 6. *a*-numbers of $\mathbf{Z}/p\mathbf{Z}$-covers of E branched only over P with $p = 7$.

the values of a_Y that we found, as well as a function f that produced it. These values of a_Y should be compared to the bounds in Tables 3 and 4.

Remark 7.9. Larger values of a_Y are more rare. For example, when $p = 7$, $S = \{P\}$ and $d_P = 6$, among *all* $100842 = 6 \cdot 7^5$ functions $f \in H^0(E, \mathcal{O}_E(d_P \cdot [P]))$ with a pole of exact order 6 at P, there are $86436 (= 85.71\%)$ with $a_Y = 10$, $11760 (= 11.66\%)$ with $a_Y = 11$, $2562 (= 2.54\%)$ with $a_Y = 12$ and $84 (= 0.08\%)$ with $a_Y = 14$. Curiously, none had $a_Y = 13$. In the spirit of [Cais et al. 2013], it would be interesting to investigate the limiting distribution of *a*-numbers in branched $\mathbf{Z}/p\mathbf{Z}$-covers of a fixed base curve with fixed branch locus, as the sum of the ramification breaks tends to infinity. Soumya Sankar [2019] investigated the limiting distribution of nonordinary (*a*-number greater than 0) covers of the projective line.

Although $d_P = 6$ divides $p - 1$ when $p = 7$, the *a*-number of Y can be 10, 11, 12 or 14; in particular, the ordinarity hypothesis in Corollary 6.32 is necessary.

Example 7.10. We now work out some examples with $\pi : Y \to E$ branched at exactly two points. As before, let P be the point at infinity on E and Q be the point $(0, 0)$. For $p = 5$ we considered Artin–Schreier covers Y of E branched only over P, Q with $d_P = 6$ and $d_Q = 4$. Our bounds are $10 \leq a_Y \leq 14$, and among a sample of 10001 functions $f \in H^0(E, \mathcal{O}_E(d_P \cdot [P] + d_Q \cdot [Q]))$ with a pole of exact order $d_\star$ at $\star = P, Q$, we found 8021 with $a_Y = 10$, 1818 with $a_Y = 11$, 149 with $a_Y = 12$, and 13 with $a_Y = 13$. One of the 13 functions we found giving *a*-number 13 was

$$f = -\frac{1}{xy} + \frac{x^5 - x^3 + 2x^2 - 2x + 1}{x^2}.$$

Similarly, with $p = 7$ and $d_P = 6$, $d_Q = 8$ our bounds are $21 \leq a_Y \leq 37$. Among a sample of 5001 random functions satisfying the required constraints we found 4318 with $a_Y = 21$, 668 with $a_Y = 22$, 14 with $a_Y = 23$, and 1 with $a_Y = 24$. The function producing a cover Y with *a*-number 24 was

$$f = \frac{-x^4 - 3x^3 + 2x^2 - x + 1}{x^4 y} + \frac{-2x^7 + 3x^5 - 3x^4 + x^3 + x^2 + x + 3}{x^4}.$$

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