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Sho Tanimoto

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# On upper bounds of Manin type

Sho Tanimoto

We introduce a certain birational invariant of a polarized algebraic variety and use that to obtain upper bounds for the counting functions of rational points on algebraic varieties. Using our theorem, we obtain new upper bounds of Manin type for 28 deformation types of smooth Fano 3-folds of Picard rank $\geq 2$ following the Mori–Mukai classification. We also find new upper bounds for polarized K3 surfaces $S$ of Picard rank 1 using Bayer and Macrì's result on the nef cone of the Hilbert scheme of two points on $S$.

## 1. Introduction

A driving question in diophantine geometry is to prove asymptotic formulae for the counting function of rational points on a projective variety. Manin's conjecture, originally formulated in [Batyrev and Manin 1990], predicts a precise asymptotic formula when the underlying variety is smooth Fano, or more generally smooth and rationally connected. This asymptotic formula has a description in terms of the geometric invariants of the underlying variety.

In this paper we consider questions related to the following weaker version of the conjecture which is called the weak Manin's conjecture: let $X$ be a geometrically uniruled smooth projective variety defined over a number field $k$ and let $L$ be a big and nef divisor on $X$. One can associate a height function

$$H_L : X(k) \to \mathbb{R}_{>0}$$

to $(X, L)$, and we consider the counting function

$$N(U, L, T) = \#\{P \in U(k) | H_L(P) \leq T\}$$

for an appropriate Zariski open subset $U \subset X$. The weak Manin's conjecture predicts that this function is governed by the following geometric invariant of $(X, L)$:

$$a(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\},$$

where $\overline{\mathrm{Eff}}^1(X)$ is the cone of pseudoeffective divisors on $X$. Here is the statement of the weak Manin's conjecture:

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*Keywords:* heights, counting rational points, weak Manin conjecture.

**Conjecture 1.1** (the weak Manin's conjecture/linear growth conjecture). Let $X$ be a geometrically uniruled smooth projective variety defined over a number field $k$ and let $L$ be a big and nef divisor on $X$. Then there exists a nonempty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$,

$$N(U, L, T) = O_\epsilon(T^{a(X,L)+\epsilon}).$$

Note that for $L = -K_X$, we have $a(X, L) = 1$, backing up the phrase "linear growth".

**Remark 1.2.** There are counterexamples to a version of this conjecture where one assumes $L$ to be only big but not nef; see [Lehmann et al. 2018a, Section 5.1].

The starting point of the current research is [McKinnon 2011], where it is shown that Vojta's conjecture implies the weak Manin's conjecture for K3 surfaces and more generally varieties with Kodaira dimension 0 assuming the nonvanishing conjecture in the minimal model program. (For such varieties, the $a$-invariant is 0.) While McKinnon's result is conditional on Vojta's conjecture, our results are unconditional: they do not rely on Vojta's conjecture. In our approach, instead of appealing to Vojta's conjecture, we use the positivity of divisors by introducing the following invariant measuring the local positivity of big divisors:

**Definition 1.3.** Let $X$ be a normal projective variety defined over an algebraically closed field of characteristic 0 and $H$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We consider $W = X \times X$ and denote each projection by $\pi_i : W \to X_i$. Let $\alpha : W' \to W$ be the blow-up of the diagonal and denote its exceptional divisor by $E$, i.e., the pullback of the diagonal. For any $\mathbb{Q}$-Cartier divisor $L$ on $X$ we denote $\alpha^*\pi_1^* L + \alpha^*\pi_2^* L$ by $L[2]$. We define the invariant

$$\delta(X, H) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c}\text{for any component } V \subset \mathrm{SB}(sH[2]-E) \text{ not contained}\\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i.\end{array} \right\},$$

where $\mathrm{SB}(sH[2] - E)$ is the stable base locus of a $\mathbb{R}$-divisor $sH[2] - E$. We call this invariant the $\delta$-invariant.

Inspired by [McKinnon 2011], we obtain the following general result on the counting functions of rational points on algebraic varieties:

**Theorem 1.4.** *Let $X$ be a normal projective variety of dimension $n$ defined over a number field $k$ and $L$ be a big $\mathbb{Q}$-Cartier divisor on $X$. Then for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that we have*

$$N(U, L, T) = O_\epsilon(T^{2n\delta(X,L)+\epsilon}).$$

Previous results applying to general projective varieties are results related to dimension growth conjecture obtained by Browning, Heath-Brown, and Salberger [Browning et al. 2006], and Salberger [2007]. Recall that the dimension growth conjecture of Heath-Brown, which is proved by Salberger, states that for any subvariety $X \subset \mathbb{P}^n$, the hyperplane class $H$, and any $\epsilon > 0$ we have

$$N(X, H, T) = O(T^{\dim X+\epsilon}).$$

For many examples of projective varieties $X$ of dimension $\leq 3$ we proved that $\delta(X, -K_X) \leq \frac{1}{2}$. (For example, we compute $\delta(X, -K_X)$ for most of 3-dimensional Fano conic bundles with rational sections in Section 9 following the Mori–Mukai classification, and we confirm that $\delta(X, -K_X) = \frac{1}{2}$ for 40 deformation types out of 53 deformation types.) Thus this theorem recovers some statements of [Browning et al. 2006; Salberger 2007] on the dimension growth conjecture for varieties with $\delta(X, -K_X) \leq \frac{1}{2}$. We conjecture that for a large portion of the class of Fano manifolds, we have $\delta(X, -K_X) \leq \frac{1}{2}$, so our theorem should lead to an alternative proof of the dimension growth conjecture for certain Fano varieties.

However, there is no direct comparison between our result and those in [Browning et al. 2006; Salberger 2007], which are certainly better in the sense that they obtain a bound for $N(X, L, T)$ and their constants only depend on the dimension of $X$, $\epsilon$, and the dimension of the ambient projective space where $X$ is embedded into. On the other hand, our method has the advantage in the sense that our theorem applies to arbitrary big divisors and in many cases where $\delta(X, L)$ is the minimum, e.g., the 3-dimensional Fano conic bundles mentioned above, one does not need to introduce $\epsilon > 0$ in the above theorem. For most smooth projective varieties with nonnegative Kodaira dimension, we conjecture that Theorem 1.4 gives better upper bounds than [Browning et al. 2006; Salberger 2007]. For example, we have the following application of Theorem 1.4 to K3 surfaces:

**Theorem 1.5.** *Let $S$ be a K3 surface defined over a number field $k$ with a polarization $H$ of degree $2d$ such that $\mathrm{Pic}(\bar{S}) = \mathbb{Z}H$. Then for any $\epsilon > 0$, we have*

$$N(S, H, T) = O_\epsilon(T^{4\sqrt{(4/d)+(5/d^2)}+\epsilon}).$$

The existence of K3 surfaces satisfying the assumptions of Theorem 1.5 is justified by [Terasoma 1985; Ellenberg 2004; van Luijk 2007]. Our proof relies on the work of Bayer and Macrì [2014] on the nef cone of the Hilbert scheme of 2 points $\mathrm{Hilb}^{[2]}(S)$. Indeed, $\delta(S, H)$ is bounded by the $s$-invariant of $H$, and the computation of the $s$-invariant can be done using the description of the nef cone of $\mathrm{Hilb}^{[2]}(S)$. A certain bound is also obtained for Enriques surfaces by using [Nuer 2016]; see Theorem 7.2. It would be interesting to compute these invariants for surfaces of general type. Bounds of this type are obtained for hypersurfaces in $\mathbb{P}^n$ by Heath-Brown [2002].

For some 3-dimensional Fano conic bundles we are able to improve bounds of Theorem 1.4 using conic bundle structures.

**Theorem 1.6.** *Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are smooth Fano. Let $W = X \times X$ and $W'$ be the blow-up of $W$ along the diagonal with the exceptional divisor $E$. We denote each projection $W' \to X_i$ by $\pi_i$. Let $\alpha, \beta$ be positive real numbers such that $2\alpha - 2\beta = 1$. We further make the following assumptions*:

(1) *The weak Manin's conjecture for $(S, -K_S)$ holds.*

(2) *For any component $V$ of the stable locus of the divisor*

$$-\alpha K_{X/S}[2] - \beta f^* K_S[2] - E,$$

*such that $V$ is not contained in $E$, one of projections $\pi_i|_V$ is not dominant.*

*Then there exists a nonempty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$, there exists $C = C_\epsilon > 0$ such that*

$$N(U, -K_X, T) < CT^{2\alpha+\epsilon}.$$

Using this theorem, new upper bounds for 28 deformation types of Fano 3-folds are obtained and these bounds are better than the dimension growth conjecture in [Browning et al. 2006; Salberger 2007]. These examples are discussed in Section 9. Here are some examples of Fano 3-folds which our theorem applies to. Note that for examples below we have $\delta(X, -K_X) = \frac{1}{2}$.

**Example 1.7** (Example 9.2). Let $X$ be the blow-up of a quadric threefold $Q$ defined over a number field $k$ with center a line defined over the same ground field. Let $H$ be the pullback of hyperplane class from $Q$ and we denote the exceptional divisor by $D$. Then the linear system $|H - D|$ defines a $\mathbb{P}^1$-fibration over $\mathbb{P}^2$. We prove that $\alpha = \frac{5}{6}$ satisfies the assumptions of Theorem 1.6: thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have

$$N(U, -K_X, T) = O_\epsilon(T^{5/3+\epsilon}).$$

**Example 1.8** (Example 9.3). Let $V_7$ be the blow-up of $\mathbb{P}^3$ at a point $P$. This is isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$. Let $X$ be the blow-up of $V_7$ with center the strict transform of a conic passing through $P$. Then $X$ is a Fano conic bundle with singular fibers. We prove that $\alpha = \frac{5}{6}$ satisfies the assumptions of Theorem 1.6: thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have

$$N(U, -K_X, T) = O_\epsilon(T^{5/3+\epsilon}).$$

**Example 1.9** (Example 9.5). Let $X$ be the blow-up of $\mathbb{P}^3$ with center a disjoint union of three lines. Then $X$ is a Fano conic bundle with singular fibers. We prove that $\alpha = 1$ satisfies the assumptions of Theorem 1.6: thus we may conclude there exists some open subset $U \subset X$ such that for any $\epsilon > 0$, we have

$$N(U, -K_X, T) = O_\epsilon(T^{2+\epsilon}).$$

It is natural to wonder whether $2\alpha < 2\delta(X, -K_X) \dim X$ holds in general. While we do not have a proof of this inequality, we do not have any counterexample either. Finally note that for del Pezzo surfaces, there are many better results on bounds of the counting functions; see, e.g., [Heath-Brown 1997; Broberg 2001; Browning and Swarbrick Jones 2014; Frei et al. 2018; Browning and Sofos 2019].

***The method of proofs.*** McKinnon proves the weak Manin's conjecture for K3 surfaces using a certain repulsion principle which he proves assuming Vojta's conjecture. We instead prove a different repulsion principle using the $\delta$-invariant and this proof does not rely on Vojta's conjecture. Here is our theorem:

**Theorem 1.10** (repulsion principle). *Let $X$ be a normal projective variety defined over a number field $k$. We fix a place $v$ of $k$. Let $A$ be a big $\mathbb{Q}$-Cartier divisor on $X$. Then for any $\epsilon > 0$ there exists a constant*

*$C = C_\epsilon > 0$ and a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that*

$$\mathrm{dist}_v(P, Q) > C(H_A(P)H_A(Q))^{-(\delta(X,A)+\epsilon)},$$

*for any $P, Q \in U(k)$ with $P \neq Q$, where $\mathrm{dist}_v(P, Q)$ is the $v$-adic distant function on $X$.*

Combining this theorem and counting arguments in [McKinnon 2011], we prove Theorem 1.4.

This paper is organized as follows. In Section 2, we recall the constructions of height functions and their basic properties. In Section 3 we recall basic properties and results of the $a$-invariants. In Section 4, we discuss some basic properties of the $\delta$-invariants and compute them for some examples, e.g., del Pezzo surfaces. In Section 5, we prove the Repulsion principle for projective varieties (Theorem 1.10). In Section 6 we establish Theorem 1.4. In Section 7, we study K3 surfaces and Enriques surfaces and prove Theorem 1.5. In Section 8, we prove Theorem 1.6. In Section 9, we study 3-dimensional Fano conic bundles using Theorem 1.6.

## 2. Height functions

Here we recall the constructions of height functions and their basic properties which will be needed for the rest of the paper. The main references are [Hindry and Silverman 2000; Chambert-Loir and Tschinkel 2010]. Let $k$ be a number field and $M_k$ denote the set of places of $k$. For each place $v \in M_k$, $k_v$ denotes its completion with respect to $v$, and we fix a Haar measure $\mu_v$ on $k_v$. We normalize our absolute value $|\cdot|_v$ on $k_v$ by the following property: for any $a \in k_v$ and a measurable set $\Omega \subset k_v$ we have

$$\mu_v(a\Omega) = |a|_v \mu_v(\Omega).$$

When $k_v = \mathbb{R}$, $|\cdot|_v$ is the usual absolute value. When $k_v = \mathbb{Q}_p$ we have $|p|_v = 1/p$. For a finite extension $k_v/\mathbb{Q}_w$, we have $|a|_v = |N_{k_v/\mathbb{Q}_w}(a)|_w$ for any $a \in k_v$. Due to the normalizations, we have the product formula, i.e., for any $a \in k^\times$ we have

$$\prod_{v \in M_k} |a|_v = 1. \tag{2-1}$$

A variety $X$ defined over $k$ is a geometrically integral separated scheme of finite type over $k$. For any place $v \in M_k$ the topological space $X(k_v)$ is endowed with a natural structure as an analytic space over $k_v$. For any invertible sheaf $L$ on $X$, we consider the underlying line bundle $\pi : \underline{L} \to X$.

**Definition 2.1** (height functions). Let $X$ be a projective variety defined over $k$ and $L$ be an invertible sheaf on $X$. Let $\{\|\cdot\|_v\}_{v \in M_k}$ be an adelic metric for $L$, i.e., for each $v \in M_k$, $\|\cdot\|_v$ is a $v$-adic metric on the analytic line bundle $\pi : \underline{L}(k_v) \to X(k_v)$ and they satisfy a certain integral condition; see [Chambert-Loir and Tschinkel 2010, Section 2.2.3] for more details. For each $P \in X(k)$, we pick a nonzero element $\ell$ of $\underline{L}_P(k)$ where $\underline{L}_P$ is a fiber of $\pi : \underline{L} \to X$ at $P \in X$. We define the multiplicative height function associated to $\mathcal{L} = (L, \|\cdot\|_v)$ by

$$H_{\mathcal{L}}(P) = \prod_{v \in M_k} \|\ell\|_v^{-1}.$$

Note that this does not depend on the choice of $\ell$ because of the product formula (2-1). We also define the logarithmic height function by

$$h_{\mathcal{L}}(P) = \log H_{\mathcal{L}}(P).$$

**Remark 2.2.** There is another construction of height functions using the framework of Weil height machines [Hindry and Silverman 2000, Theorem B.3.2]. One can show that two constructions are equivalent in the sense that two height functions associated to the same line bundle are equal up to a bounded function; see [Hindry and Silverman 2000, Theorem B.10.7]. Also note that the construction of height functions in this paper uses normalizations which differ from ones in [Hindry and Silverman 2000]. This is because in this paper we only consider height functions defined on the set of rational points while [Hindry and Silverman 2000] considers the height functions defined over the set of algebraic points. Thus in [Hindry and Silverman 2000], one needs to normalize each height function by the degree of the definition field of an algebraic point.

**Remark 2.3.** In the later discussions, we frequently omit the discussion of metrics and we consider a height function $h_L$ associated to a line bundle $L$. In this situation, we implicitly make a choice of an adelic metric, but we will not make this dependence explicit as this does not matter for our discussion.

There are two important properties of height functions we frequently use:

**Theorem 2.4** [Hindry and Silverman 2000, Theorem B.3.2]. *Let $X$ be a projective variety defined over $k$ and $L$ be an invertible sheaf on $X$.*

(1) ***Positivity***: *Let $B$ be the stable base locus of $L$. Then we have*

$$h_L(P) \geq O(1)$$

*for any $P \in (X \setminus B)(k)$.*

(2) ***Northcott property***: *Suppose that $L$ is ample. Then for any $T > 0$ the set*

$$\{P \in X(k) \in H_L(P) \leq T\}$$

*is finite.*

Finally we recall the construction of local height functions:

**Definition 2.5.** Let $X$ be a projective variety defined over $k$ and $L$ be a Cartier divisor on $X$. For a place $v \in M_k$ we fix a $v$-adic metric $\|\cdot\|_v$ for $L$, i.e., a $v$-adic metric on the analytic line bundle $\pi : \underline{\mathcal{O}(L)}(k_v) \to X(k_v)$. Let $D$ be an effective divisor linearly equivalent to $L$. Let $\mathsf{s}_D$ be a $k$-section associated to $D$. Then the multiplicative local height function associated to $D$ is given by

$$H_{D,v}(P) = \|\mathsf{s}_D(P)\|_v^{-1}$$

for any $P \in (X \setminus D)(k_v)$. We also define the logarithmic local height function by

$$h_{D,v}(P) := \log H_{D,v}(P).$$

Suppose we fix an adelic metrized line bundle $\mathcal{L}$. Then the height function is the Euler product of local height functions, i.e., we have

$$H_{\mathcal{L}}(P) = \prod_{v \in M_k} H_{D,v}(P)$$

for any $P \in (X \setminus D)(k)$.

## 3. The Fujita invariant in Manin's conjecture

Here we assume that our ground field $k$ is a field of characteristic zero, but not necessarily algebraically closed. Recently the geometric study of Fujita invariants has been conducted in [Hassett et al. 2015; Lehmann et al. 2018b; Hacon and Jiang 2017; Lehmann and Tanimoto 2017; 2018; 2019a; 2019b; Sengupta 2017; Lehmann et al. 2018a.] We recall its definition here.

**Definition 3.1.** Let $X$ be a smooth projective variety defined over $k$. Let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. We define the *Fujita invariant* (or $a$-invariant) by

$$a(X, L) = \inf\{t \in \mathbb{R} \mid K_X + tL \in \overline{\mathrm{Eff}}^1(X)\},$$

where $\overline{\mathrm{Eff}}^1(X)$ is the cone of pseudoeffective divisors on $X$. By [Boucksom et al. 2013] $a(X, L) > 0$ if and only if $X$ is geometrically uniruled. When $L$ is not big, we simply set $a(X, L) = +\infty$. When $X$ is singular, we take a resolution $\beta : X' \to X$ and we define the Fujita invariant by

$$a(X, L) := a(X', \beta^* L).$$

This is well-defined because the Fujita invariant is a birational invariant [Hassett et al. 2015, Proposition 2.7].

This invariant plays a central role in Manin's conjecture. For example, one can predict the exceptional set of Manin's conjecture by studying this invariant and the following result is a consequence of Birkar's celebrated papers [2016; 2019]:

**Theorem 3.2** [Lehmann et al. 2018b; Hacon and Jiang 2017; Lehmann and Tanimoto 2019a]. *Assume that our ground field is algebraically closed. Let $X$ be a smooth projective uniruled variety and let $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Let $V$ be the union of subvarieties $Y$ with $a(Y, L) > a(X, L)$. Then $V$ is a proper closed subset of $X$.*

For computations of this exceptional set $V$ for some examples, see [Lehmann et al. 2018b; Lehmann and Tanimoto 2019b].

## 4. The invariant $\delta(X, H)$

Here we assume that our ground field $k$ is an algebraically closed field of characteristic 0. Let $X$ be a normal projective variety and $H$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We consider $W = X \times X$ and denote each projection by $\pi_i : W \to X_i$. Let $\alpha : W' \to W$ be the blow-up of the diagonal and we denote its exceptional divisor by $E$. For any $\mathbb{Q}$-Cartier divisor $L$ on $X$ we denote $\alpha^*\pi_1^* L + \alpha^*\pi_2^* L$ by $L[2]$.

**Definition 4.1.** Let $X$, $W'$, $E$ as above. We define the invariant

$$\delta(X, H) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{for any component } V \subset \mathrm{SB}(sH[2]-E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i. \end{array} \right\},$$

where $\mathrm{SB}(sH[2]-E)$ is the stable base locus of a $\mathbb{R}$-divisor $sH[2]-E$.

**Remark 4.2.** It follows from the definition that when the rational map $\Phi_{|H|}$ associated to $|H|$ is birational, we have $\delta(X, H) \leq 1$. Indeed, let $Z$ be a closed subset such that on $X \setminus Z$, $\Phi_{|H|}$ is well-defined and an isomorphism onto the image. Then one can conclude that

$$\mathrm{Bs}(H[2]-E) \subset E \cup \pi_1^{-1}(Z) \cup \pi_2^{-1}(Z),$$

where $\mathrm{Bs}(H[2]-E)$ is the base locus of $H[2]-E$. Thus our assertion follows. Also it follows from the definition that $\delta(X, H)H[2]-E$ is pseudoeffective.

**Lemma 4.3.** *Let $s_0$ be a positive real number. Let $F_P$ be a fiber of the first projection $\pi_1 \circ \alpha : W' \to X_1$ at $P \in X_1$. Suppose that there exists a family of irreducible curves $C_t \subset F_{P(t)}$ such that*

(i) *$P(t)$ covers a Zariski open subset of $X_1$ as $t$ varies,*

(ii) *the image of $C_t$ in $X_2$ is an irreducible curve containing $P(t)$, and*

(iii) $(s_0 H[2]-E).C_t = 0$.

*Then we have $\delta(X, H) \geq s_0$.*

*Proof.* Let $V = \overline{\bigcup C_t}$ in $W'$ which is irreducible. Then $V$ is not contained in $E$ and each projection $\pi_i|_V$ is dominant. Let $s < s_0$. Then since $(sH[2]-E).C_t < 0$, $\mathrm{SB}(sH[2]-E)$ contains $V$. Thus we must have $s \leq \delta(X, H)$. Since this is true for any $s < s_0$, our assertion follows. □

**Example 4.4.** Let $X = \mathbb{P}^n$ and $H$ be the hyperplane class. Then $\delta(X, H) = 1$. Indeed, it follows from Remark 4.2 that $\delta(X, H) \leq 1$. On the other hand, let $F_1$ be a general fiber of the first projection $\pi_1 \circ \alpha : W' \to X_1 = \mathbb{P}^n$ at $P \in X_1$ and $\ell$ be the strict transform of a line passing through $P$ in $F_1$. Then we have $(H[2]-E).\ell = 0$. Thus our assertion follows from Lemma 4.3.

**Example 4.5.** Let $X \subset \mathbb{P}^n$ be a normal projective variety and $H$ be the hyperplane class. Suppose that $X$ is covered by lines. Then the same proof of the above example shows that $\delta(X, H) = 1$.

Next we show that the invariant $\delta(X, H)$ is a birational invariant.

**Lemma 4.6.** *Let $X$ be a normal projective variety and $H$ be a big $\mathbb{Q}$-Cartier divisor on $X$. Let $\beta : X' \to X$ be a birational morphism between normal projective varieties. Then we have*

$$\delta(X', \beta^* H) = \delta(X, H).$$

*Proof.* Let $W_X$ be the blow-up of $X \times X$ along the diagonal and $W_{X'}$ be the blow-up of $X' \times X'$ along the diagonal. We denote their exceptional divisors by $E_X$ and $E_{X'}$ respectively. Then we have a birational map

$$\phi : W_{X'} \dashrightarrow W_X$$

which is a birational contraction and the indeterminacy of this map is not dominant to both $X'_i$. Also for a component $V$ of the nonisomorphic loci of this map such that $V$ is not contained in $E_{X'}$, one of projections is not dominant.

Fix $\epsilon > 0$. Suppose that the stable locus of $(\delta(X, H)+\epsilon)\beta^* H[2]-E_{X'}$ contains a subvariety $Y \subset W_{X'}$ such that $Y \not\subset E_{X'}$ and $Y$ maps dominantly to both $X'_i$. By the definition, $(\delta(X, H)+\epsilon)H[2]-E_X$ does not contain $\phi(Y)$ in the stable locus so that there exists $0 \leq D \sim_{\mathbb{R}} (\delta(X, H)+\epsilon)H[2]-E_X$ such that $\phi(Y) \not\subset \operatorname{Supp}(D)$. Then we have $\phi^* D \sim_{\mathbb{R}} (\delta(X, H)+\epsilon)\beta^* H[2]-E_{X'}$ because $\phi^* E_X = E_{X'}$. Furthermore we have $Y \not\subset \operatorname{Supp}(\phi^* D)$. This contradicts with our assumption. Thus we conclude

$$\delta(X', \beta^* H) \leq \delta(X, H).$$

Suppose that the stable locus of $(\delta(X', H)+\epsilon)H[2]-E_X$ contains a subvariety $Y \subset W_X$ such that $Y \not\subset E_X$ and $Y$ maps dominantly to both $X_i$. We take the strict transform $Y' \subset W_{X'}$ of $Y$. By the definition, $(\delta(X', H)+\epsilon)\beta^* H[2]-E_{X'}$ does not contain $Y'$ in the stable locus so there exists $0 \leq D \sim_{\mathbb{R}} (\delta(X', H)+\epsilon)\beta^* H[2]-E_{X'}$ such that $Y' \not\subset \operatorname{Supp}(D)$. Then $\phi_* D \sim_{\mathbb{R}} (\delta(X', H)+\epsilon)H[2]-E_X$. Furthermore we have $Y \not\subset \operatorname{Supp}(\phi_* D)$. This contradicts with our assumption. Thus we conclude

$$\delta(X', \beta^* H) \geq \delta(X, H).$$

Thus our assertion follows. □

Here is a relation between $\delta(X, H)$ and $a(X, H)$.

**Proposition 4.7.** *Let $X$ be a smooth weak Fano variety, i.e., $-K_X$ is big and nef, and let $H$ be a big and nef divisor on $X$. Then we have*

$$\delta(X, H) \leq a(X, H)\delta(X, -K_X).$$

*Proof.* We write $a(X, H)H + K_X \sim_{\mathbb{Q}} D \geq 0$. Fix $\epsilon > 0$. Then we have

$$a(X, H)(\delta(X, -K_X)+\epsilon)H[2]-E \sim_{\mathbb{Q}} -(\delta(X, -K_X)+\epsilon)K_X[2]+(\delta(X, -K_X)+\epsilon)D[2]-E.$$

Thus we see that the stable locus of $|a(X, H)(\delta(X, -K_X)+\epsilon)H[2]-E|$ does not contain any dominant component possibly other than subvarieties in $E$. Thus our assertion follows. □

Next we consider the $s$-invariants and its relation to the $\delta$-invariants:

**Definition 4.8.** Let $X$ be a smooth projective variety and $H$ be an ample divisor on $X$. Let $W$ be the blow-up of $X \times X$ along the diagonal and we denote its exceptional divisor by $E$. The $s$-invariant of $H$ is defined by

$$s(X, H) = \inf\{s \in \mathbb{R} \mid sH[2]-E \text{ is nef}\}.$$

This is a positive real number in general; see [Lazarsfeld 2004, Section 5.4] for many properties of this invariant.

**Proposition 4.9.** *Let $X$ be a smooth projective variety and $H$ be an ample divisor on $X$. Then we have*

$$\delta(X, H) \leq s(X, H).$$

*Proof.* For every $\epsilon > 0$, $(s(X, H) + \epsilon)H[2] - E$ is ample so its stable base locus is empty. Thus our assertion follows. □

***Del Pezzo surfaces.*** Next we discuss del Pezzo surfaces. Let $S$ be a smooth del Pezzo surface. We consider $W = S \times S$ and we denote each projection by $\pi_i : W \to S_i$. Let $\alpha : W' \to W$ be the blow-up of the diagonal and we denote its exceptional divisor by $E$. First we record a lower bound for the $\delta$-invariant:

**Lemma 4.10.** *Let $S$ be a smooth del Pezzo surface. Then we have*

$$\delta(S, -K_S) \geq \frac{1}{\epsilon(-K_S, P)}$$

*for any general point $P \in X$ where $\epsilon(-K_S, P)$ is the Seshadri constant of $-K_S$ at $P$.*

*Proof.* The Seshadri constant for the anticanonical divisor on a smooth del Pezzo surface is computed in [Broustet 2006]. According to this paper, $\epsilon(-K_S, P)$ is constant for a general point $P \in S$ and for such a $P$ we have

$$\epsilon(-K_S, P) = \min_{P \in C \subset S} \frac{-K_S.C}{\mathrm{mult}_P(C)}.$$

Moreover, curves achieving the minimum are completely described for del Pezzo surfaces of degree $\geq 2$ in [Broustet 2006] and they are members of one family from the Hilbert scheme. For a del Pezzo surface of degree 1, this minimum is achieved by members of the anticanonical system.

Let $P \in S_1$ be a general point and let $C_t$ be the strict transform of a curve in $\{P\} \times S_2$ achieving the minimum $\epsilon(-K_S, P)$. Then we have

$$(-K_S[2] - \epsilon(-K_S, P)E).C_t = -K_X.C_t - \epsilon(-K_S, P)\mathrm{mult}_P(C_t) = 0.$$

Thus our assertion follows from Lemma 4.3. □

Now we compute $\delta(S, -K_S)$ for a del Pezzo surface $S$.

**Proposition 4.11.** *Let $S$ be a del Pezzo surface of degree $d$ where $4 \leq d \leq 8$. Then we have $\delta(S, -K_S) = \frac{1}{2}$.*

*Proof.* We only discuss the case of degree 4 del Pezzo surfaces. Other cases are easier.

Suppose that $S$ is a del Pezzo surface of degree 4. Let $F_1$ be a $-K_S$-conic on $S$ and $F_2$ be another $-K_S$-conic on $S$ such that $-K_S \sim F_1 + F_2$. Indeed, one may find such a pair of $-K_S$-conics in the following way: let $\phi : S \to \mathbb{P}^2$ be a blow-down to $\mathbb{P}^2$ and we may assume that $\phi$ is the blow-up at $P_1, \dots, P_5 \in \mathbb{P}^2$. Then one can find a general plane conic $C_1$ and a general line $C_2$ such that $C_1$ contains $P_1, \dots, P_4$ and $C_2$ contains $P_5$. Then their strict transforms satisfy the desired property.

Now the linear system $|F_i|$ defines a conic fibration $p_i : S \to \mathbb{P}^1$. It induces a morphism $p_i[2] : W' \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\Delta_{\mathbb{P}^1}$ be the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. Then $F_i[2] - E$ is linearly equivalent to a unique effective divisor $p_i[2]^*\Delta_{\mathbb{P}^1} - E$. We denote it by $\Delta_{F_i}$. Let $D_1$ be a third conic such that $D_1.F_1 = D_1.F_2 = 1$.

Indeed, one may take $D_1$ as the strict transform of a conic passing through $P_1, P_2, P_3, P_5$. Let $D_2$ be the class of conics such that $-K_S \sim D_1 + D_2$. Then we have $D_2.F_1 = D_2.F_2 = 1$. We also consider $\Delta_{D_i}$. Then we have

$$\mathrm{SB}(-K_S[2]-2E) \subset (\Delta_{F_1} \cup \Delta_{F_2}) \cap (\Delta_{D_1} \cup \Delta_{D_2})$$

but the only possible dominant component of $\Delta_{F_i} \cap \Delta_{D_i}$ is contained in $E$. Thus we conclude that $\delta(X, H) \leq \frac{1}{2}$. The opposite inequality follows from Lemma 4.10 and [Broustet 2006]. □

**Proposition 4.12.** *Let $S$ be a del Pezzo surface of degree 3. Then $\delta(S, -K_S) = \frac{2}{3}$.*

*Proof.* Let $F_1$ be a $-K_S$-conic. One can find a $(-1)$-curve $E$ such that $-K_S \sim F_1 + E$. Indeed, let $\phi : S \to \mathbb{P}^2$ be the blow-up at $P_1, \dots, P_6 \in \mathbb{P}^2$. We let $F_1$ to be the strict transform of a general conic passing through $P_1, \dots, P_4$ and $E$ be the strict transform of a line passing through $P_5, P_6$. Then they satisfy the desired property. Now let $D = -2K_S - F_1 \sim -K_S + E$. Then $D$ is the pullback of the anticanonical class from a degree 4 del Pezzo surface. The upshot is that we have

$$-2K_S[2] - 3E = D[2] - 2E + F_1[2] - E.$$

Therefore it follows from the proof of Proposition 4.11 that the stable locus of $D[2]-2E$ minus $E$ is not dominant. Thus the stable locus of $-2K_S[2]-3E$ is contained in $\Delta_{F_1}$. By considering another conic and applying the same discussion, we conclude that $\delta(S, -K_S) \leq \frac{2}{3}$. The opposite inequality follows from Lemma 4.10 and [Broustet 2006]. □

**Proposition 4.13.** *Let $S$ be a del Pezzo surface of degree 2. Then $\delta(S, -K_S) = 1$.*

*Proof.* We may write $-K_S \sim E_1 + E_2$ where $E_i$ is a $(-1)$-curve. Indeed, let $\phi : S \to \mathbb{P}^2$ be the blow-up at $P_1, \dots, P_7$. Then we may define $E_1$ as the strict transform of a conic passing through $P_1, \dots, P_5$ and $E_2$ be the strict transform of a line passing through $P_6, P_7$. Let $f_i : S \to S_i$ be the blow-down of $E_i$ to a cubic surface. Then $-3K_S$ can be expressed as

$$-3K_S \sim -f_1^* K_{S_1} - f_2^* K_{S_2}.$$

Thus arguing as in Proposition 4.12 we prove that the stable locus of $-K_S[2]-E$ does not contain any dominant component except $E$. This shows that $\delta(S, -K_S) \leq 1$.

On the other hand, let $\phi : S \to \mathbb{P}^2$ be the anticanonical double cover. We denote the involution associated to $\phi$ by $\iota$ and we consider the image $S^\iota$ of the map

$$S \to S \times S, \quad P \mapsto (P, \iota(P)).$$

Then one can show that for any curve $C$ in $S^\iota$ and any $\epsilon > 0$ we have

$$(-K_S[2] - (1+\epsilon)E).C < 0$$

Thus $C$ is contained in the stable locus of $-K_S[2]-(1+\epsilon)E$, proving the claim. □

**Proposition 4.14.** *Let $S$ be a del Pezzo surface of degree 1. Then $\frac{3}{2} \leq \delta(S, -K_S) \leq 2$.*

*Proof.* Let $\phi : S \to Q \subset \mathbb{P}^3$ be the double cover associated to $|-2K_S|$. Let $E_1$ be a $(-1)$-curve on $S$. Then $\phi|_{E_1} : E_1 \to \phi(E_1)$ is one-to-one and its pullback consists of two $(-1)$-curves including $E_1$. Thus we may write as $-2K_S \sim E_1 + E_2$.

Let $f_i : S \to S_i$ be the blow-down of $E_i$ to a degree 2 del Pezzo surface. Then $-4K_S$ can be expressed as

$$-4K_S \sim -f_1^* K_{S_1} - f_2^* K_{S_2}$$

Thus by Proposition 4.13, one may conclude that $\delta(S, -K_S) \leq 2$.

Another inequality follows from the discussion of Proposition 4.13 using the double cover $\phi : S \to Q \subset \mathbb{P}^3$. □

**Remark 4.15.** In the proofs of Propositions 4.11, 4.12 and 4.13 we show that for any component $V \subset \mathrm{SB}(-\delta(S, -K_S)K_S[2] - E)$ not contained in $E$, the projection $\pi_i \circ \alpha|_V$ is not dominant. In particular we conclude that

$$\delta(S, -K_S) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c}\text{for any component } V \subset \mathrm{SB}(-sK_S[2]-E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } S_i\end{array} \right\}$$

is the minimum.

## 5. Repulsion principle for projective varieties

Assuming Vojta's conjecture and the nonvanishing conjecture in the minimal model program, McKinnon [2011] showed a repulsion principle for varieties of nonnegative Kodaira dimension. In this paper we develop a weaker repulsion principle for projective varieties in general. We introduce some notations. We refer readers to [Silverman 1987] for the definitions and their basic properties.

Let $k$ be a number field. Suppose that we have a projective variety $X$ defined over $k$ and a big $\mathbb{Q}$-divisor $L$ on $X$. Let $D$ be a closed subscheme on $X$. Let $h_{D,v}$ be a local height function for $D$ with respect to $v$. Note that in this paper, we use unnormalized heights, i.e., we do not normalize heights by the degree of $k$. Let $\Delta$ be the diagonal of $X \times X$. We define the $v$-adic distant function by

$$h_{\Delta,v}(P, Q) = -\log \mathrm{dist}_v(P, Q);$$

see [Silverman 1987] for basic properties of this function.

Let $X$ be a normal projective variety defined over a number field $k$ and $L$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We set

$$\delta(X, L) = \delta(\overline{X}, \bar{L}),$$

where $\overline{X}$, $\bar{L}$ are the base change of $X$, $L$ to an algebraic closure. Our main theorem is:

**Theorem 5.1** (the repulsion principle). *Let $X$ be a normal projective variety defined over a number field $k$. Let $v$ be a place of $k$. Let $A$ be a big Cartier divisor on $X$. Then for any $\epsilon > 0$ there exists a constant*

*$C = C_\epsilon > 0$ and a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that we have*

$$\mathrm{dist}_v(P, Q) > C(H_A(P)H_A(Q))^{-(\delta(X,A)+\epsilon)},$$

*for any $P, Q \in U(k)$ with $P \neq Q$.*

*Proof.* We let $W = X \times X$ with projections $\pi_i : W \to X_i$ and we let $L = \pi_1^* A + \pi_2^* A$. We denote the blow-up of the diagonal by $\alpha : W' \to W$ and its exceptional divisor by $E$. Fix $\epsilon > 0$. Let $\bar{B} \subset \overline{W}'$ be the stable locus $\mathrm{SB}((\delta(X, A)+\epsilon)\alpha^* L - E)$ where $\overline{W}'$ is the base change to an algebraic closure. One can express $\bar{B}$ as the intersection of supports of finitely many effective $\mathbb{R}$-divisors which are $\mathbb{R}$-linearly equivalent to $(\delta(X, A)+\epsilon)\alpha^* L - E$. After taking some finite extension $k'$ of $k$, we may assume that these divisors are defined over $k'$ and so is $\bar{B}$. We denote the union of the Galois orbits of $\bar{B}$ by $B'$. Then it is a property of height functions that for any $(P, Q) \in W'(k) \setminus B'(k)$, we have

$$0 \leq h_{((\delta(X,A)+\epsilon)\alpha^* L - E)}(P, Q) + O(1).$$

From this, we may conclude that

$$h_{E,v}(P, Q) \leq h_E(P, Q) + O(1) \leq h_{((\delta(X,A)+\epsilon)\alpha^* L)}(P, Q) + O(1).$$

Let $V \subset B'$ be a component not contained in $E$. Then one of projections $\pi_i \circ \alpha|_V$ is not dominant, and we denote its image by $F_V$. Now we define $U$ by $X \setminus \bigcup_V F_V$. Our assertion follows for this $U$. □

**Remark 5.2.** Note that $\delta(X, A)$ is defined as

$$\delta(X, A) = \inf\left\{ s \in \mathbb{R} \;\middle|\; \begin{array}{c} \text{for any component } V \subset \mathrm{SB}(s\alpha^* L - E) \text{ not contained} \\ \text{in } E, \text{ one of } \pi_i \circ \alpha|_V \text{ is not dominant to } X_i. \end{array} \right\}.$$

If this is the minimum, then in the above proof, one does not need to introduce $\epsilon > 0$.

**Remark 5.3.** When $A$ is ample, we may replace $\delta(X, A)$ by $s(X, A)$ in Theorem 5.1. In this situation, one can take our exceptional set to be empty because of the emptiness of the base locus in the proof of Proposition 4.9.

## 6. Counting problems: general cases

In this section, we discuss some applications of Theorem 5.1 to the counting problems of rational points on algebraic varieties.

***Local Tamagawa measures.*** Here we record some auxiliary results for local Tamagawa measures. Let $X$ be a smooth projective variety defined over a number field $k$. Let $v$ be a place of $k$. We fix a $v$-adic metrization on $\mathcal{O}(K_X)$ and it induces the Tamagawa measure $\tau_{X,v}$ on $X(k_v)$. We refer readers to [Chambert-Loir and Tschinkel 2010, Section 2.1.8] for its definition.

**Lemma 6.1.** *Let $n = \dim X$. There exists $C > 0$ such that for sufficiently small $T$ and $P \in X(k_v)$, we have*

$$CT^n < \tau_{X,v}(\{Q \in X(k_v) \mid \mathrm{dist}_v(P, Q) < T\}).$$

*Proof.* Let $Y = X \times X$. We take a finite open cover $\{U_i\}$ of $Y$ such that on $U_i$, we have that $\Delta$ is the scheme-theoretic intersection of $D_{i,1}, \dots, D_{i,n}$, where $D_i = \sum D_{i,j}$ is a strict normal crossings divisor on $U_i$. On $U_i(k_v)$, there exists $C > 0$ such that

$$\mathrm{dist}_v(P, Q) < C \max_j \{H_{D_{i,j}}^{-1}(P, Q)\}$$

for all $(P, Q) \in U_i(k_v)$. For each $P \in X_i(k_v)$, there exists a $v$-adic open neighborhood $V_P \subset X(k_v)$ such that $\overline{V}_P \times \overline{V}_P \subset U_i(k_v)$ for some $i$ and $D_{i,j}$ induces local coordinates $x_{i,j}$ on $V_P$. Since $X(k_v)$ is compact, finitely many $V_P$ cover $X(k_v)$. We denote them by $V_l$. For each $l$ let

$$\omega_l = \mathrm{d}x_{l,1} \wedge \cdots \wedge \mathrm{d}x_{l,l}.$$

Then on $V_l$ we have a uniform upper bound $C' > 0$ such that

$$\|\omega_l\|_v < C'.$$

Also let $d_l(P) = \min\{\mathrm{dist}_v(P, Q) \mid Q \in V_l^c\}$ and define $d(P) = \max_l\{d_l(P)\}$. Then $d(P) > 0$ for any $P \in X(k_v)$ so there is the minimum $d_m = \min\{d(P)\} > 0$. Now by the definition of the Tamagawa measure, for $0 < T < d_m$, we have

$$\tau_{X,v}(\{Q \in X(k_v) \mid \mathrm{dist}_v(P, Q) < T\}) > \int_{\{\max\{|x_{l,i} - x_{l,i}(P)|_v\} < C^{-1}T\}} \|\omega_l\|_v^{-1} |\omega_l|$$

Thus our assertion follows. ☐

The local Tamagawa number is defined by

$$\tau_v(X) = \tau_{X,v}(X(k_v)).$$

***General estimates.*** Let $X$ be a projective variety defined over a number field $k$ and let $L$ be a big $\mathbb{Q}$-divisor on $X$. We fix an adelic metrization on $\mathcal{O}(L)$ and consider the induced height:

$$H_L : X(F) \to \mathbb{R}_{>0}.$$

For each Zariski open subset $U \subset X$ we define the counting function

$$N(U, L, T) = \#\{P \in U(k) \mid H_L(P) \leq T\}.$$

Here is a general result using the repulsion principle:

**Theorem 6.2.** *Let $X$ be a normal projective variety of dimension $n$ defined over a number field $k$ and let $L$ be a big $\mathbb{Q}$-Cartier divisor on $X$. We fix an adelic metrization on $\mathcal{O}(L)$. Then for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U = U(\epsilon) \subset X$ such that we have*

$$N(U, L, T) = O(T^{2n\delta(X,L)+\epsilon}).$$

*Proof.* We may assume that $X$ is smooth after applying a resolution. This does not affect the invariant $\delta(X, L)$ because of Lemma 4.6. Let $v$ be a place of $k$. By Theorem 5.1, for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U(\epsilon) \subset X$ such that there exists $C = C_\epsilon > 0$ such that

$$\mathrm{dist}_v(P, Q) > C(H_L(P)H_L(Q))^{-(\delta(X,L)+\epsilon)},$$

for any $P, Q \in U(k)$ with $P \neq Q$. We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$

For $P \in A_T$, we define the $v$-adic ball by

$$B_T(P) = \left\{R \in U(k_v) \mid \mathrm{dist}_v(P, R) < \tfrac{1}{2}CT^{-2(\delta(X,L)+\epsilon)}\right\}$$

Then $\bigcup_{P \in A_T} B_T(P)$ is disjoint because of the triangle inequality. Hence we have

$$\tau_v(X) > \sum_{P \in A_T} \tau_{X,v}(B_T(P)) \gg N(U, L, T)T^{-2n(\delta(X,L)+\epsilon)}$$

by Lemma 6.1. Thus our assertion follows. □

**Remark 6.3.** In the case that $\delta(X, L)$ is the minimum, one does not need to introduce $\epsilon$ in Theorem 6.2 because of Remark 5.2.

**Remark 6.4.** In Theorem 6.2, assuming $L$ is ample and $X$ is smooth we may replace $\delta(X, L)$ by $s(X, L)$ because of Remark 5.3. In this case, one can take $U = X$.

In view of Manin's conjecture, we expect the following is true:

**Conjecture 6.5.** Let $X$ be a geometrically rationally connected smooth projective variety of dimension $n$ and $L$ be a big and nef $\mathbb{Q}$-divisor on $X$. Then we have

$$a(X, L) \leq 2n\delta(X, L).$$

## 7. K3 surfaces and Enriques surfaces

In this section we discuss applications of Theorem 6.2 to surfaces of Kodaira dimension 0. Let $S$ be a K3 surface or an Enriques surface with a polarization $H$ of degree $2d$. In this section, we obtain an upper bound for $s(X, H)$ using [Bayer and Macrì 2014; Nuer 2016]. Let $W$ be the blow-up of $S \times S$ along the diagonal and we denote the exceptional divisor by $E$. We also consider the Hilbert scheme of two points on $S$, i.e., $\mathrm{Hilb}^{[2]}(S)$. The variety $\mathrm{Hilb}^{[2]}(S)$ comes with the divisor $H(2)$ induced by $H$ and a divisor class $B$ such that $2B$ is the class of the exceptional divisor of the Hilbert–Chow morphism. The variety $W$ admits a degree 2 finite morphism $f : W \to \mathrm{Hilb}^{[2]}(S)$ and we have

$$f^*H(2) = H[2], \quad f^*B = E.$$

We then have

$$sH[2] - E \text{ is nef} \iff sH(2) - B \text{ is nef},$$

because of $f^*(sH(2)-B)=sH[2]-E$. Thus one needs to study the nef cone of $\mathrm{Hilb}^{[2]}(S)$ and this is studied in [Hassett and Tschinkel 2001; 2009; Bayer and Macrì 2014] for K3 surfaces. We use results from [Bayer and Macrì 2014] for the nef cone of $\mathrm{Hilb}^{[2]}(S)$. Here is the theorem:

**Theorem 7.1** [Bayer and Macrì 2014]. *Let $S$ be a K3 surface with a polarization $H$ of degree $2d$ such that* $\mathrm{Pic}(S)=\mathbb{Z}H$. *Then we have*

$$s(S,H)\le\sqrt{\frac{4}{d}+\frac{5}{d^2}}.$$

*Proof.* We recall a result on the nef boundary of $sH(2)-B$ based on properties of certain Pell's equation. First we consider

$$X^2-4dY^2=5.$$

Suppose that there is a nontrivial solution $(x_1,y_1)$ with $x_1>0$ minimal and $y_1>0$ even. Then it follows from [Bayer and Macrì 2014, Lemma 13.3] that

$$s(S,H)=\frac{x_1}{dy_1}\le\sqrt{\frac{4}{d}+\frac{5}{d^2}}.$$

Next suppose that there is no nontrivial solution to the above Pell's equation. Then for $\mathrm{Hilb}^{[2]}(S)$, the nef cone and the movable cone coincides by [Bayer and Macrì 2014, Lemma 13.3]. Suppose that $d$ is a square. Then it follows from [Bayer and Macrì 2014, Proposition 13.1] that

$$s(S,H)=\frac{1}{\sqrt{d}}.$$

Next suppose that $d$ is not a square. We consider the Pell's equation

$$X^2-dY^2=1.$$

This has a solution. Let $x_1,y_1>0$ be the solution with $x_1$ minimal. Then by [Bayer and Macrì 2014, Proposition 13.1], we have

$$s(S,H)=\frac{x_1}{dy_1}\le\sqrt{\frac{1}{d}+\frac{1}{d^2}}.$$

Thus our assertion follows. □

Now Theorem 1.5 follows from Theorem 7.1 and Remark 6.4. We also obtain bounds for Enriques surfaces:

**Theorem 7.2** [Nuer 2016]. *Let $Y$ be an unnodal Enriques surface, i.e., $Y$ contains no curve of negative self-intersection. Let $H$ be a $k$-very ample divisor. Then*

$$s(Y,H)\le\frac{2}{k+2}.$$

*Proof.* It follows from [Nuer 2016, Theorem 12.3] that

$$s(Y,H)=\frac{2}{\phi(H)},$$

where $\phi(H)$ is the Cossec–Dolgachev function. Then it follows from [Szemberg 2001, Theorem 2.4] that $\phi(H) \geq k+2$. Thus our assertion follows. □

For a necessary and sufficient condition for $k$-very ampleness, see [Szemberg 2001, Proposition 2.3]

## 8. Manin type upper bounds for Fano conic bundles

In this section, we study the counting problems of rational points on conic bundles.

**Definition 8.1.** Let $f : X \to S$ be a flat projective morphism between smooth projective varieties. The fibration $f$ is a conic bundle if there exist a rank 3 vector bundle $\mathcal{E}$ on $S$ and an embedding $X \to \mathbb{P}_S(\mathcal{E})$ over $S$ such that every fiber $X_s$ is isomorphic to a conic in $\mathbb{P}^2_s$, where $X_s$ and $\mathbb{P}^2_s$ are fibers of $X$ and $\mathbb{P}(\mathcal{E})$ at $s \in S$.

Suppose that $X$ is 3-dimensional. Then if every fiber $X_s$ is isomorphic to a conic in $\mathbb{P}^2$, then $f_*\omega_X^{-1}$ is a rank 3 vector bundle on $S$ and a natural map $X \to \mathbb{P}_S(f_*\omega_X^{-1})$ is an embedding by [Mori and Mukai 1983, Proposition 6.2]. In this way $f : X \to S$ is a conic bundle in the above sense.

**Lemma 8.2.** *Let $f : X \to S$ be a conic bundle and $H$ be a big $\mathbb{Q}$-divisor on $X$. Suppose that for a fiber $X_s$ of $f$, we have $H.X_s = 2$, i.e., $X_s$ is a $H$-conic. Then*

$$\delta(X, H) \geq \tfrac{1}{2}.$$

*Proof.* Let $W = X \times X$ and $\alpha : W' \to W$ be the blow-up of the diagonal. We denote its exceptional divisor by $E$. Let $C_P$ be a conic in the fiber at $P \in X_1$ passing through $P$. Then we have

$$(H[2] - 2E).C_P = 0.$$

As $P$ varies over $X_1$ $C_P$ forms a subvariety $D$ in $W'$ which is dominant to both $X_1$ and $X_2$. Thus our assertion follows from Lemma 4.3. □

Proposition 4.12 shows that in general, $\delta(X, H)$ may not be $\frac{1}{2}$.

***Local Tamagawa measures of conics in families.*** We study the behavior of local Tamagawa measures of conics in a family. Let $f : X \to S$ be a conic bundle defined over a number field $k$. Let $S^\circ$ be the complement of the discriminant locus $\Delta_f$ of $f$.

Let $v$ be a place of $k$. We fix a $v$-adic metrization on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. This induces a $v$-adic metrization on $\mathcal{O}(K_{X/S})$. For each local 1-form $\mathrm{d}t \in \Omega^1_{X/S}$, one can define the local Tamagawa measure $\tau_{X_s,v}$ on a conic $X_s$ for any $s \in S^\circ(k_v)$ by

$$\tau_{X_s,v}(U) = \int_U \frac{|\mathrm{d}t|}{\|\mathrm{d}t\|_v}$$

which is independent of a choice of $\mathrm{d}t$.

**Lemma 8.3.** *Suppose that $f : X \to S$ admits a rational section. Then there exists $C > 0$ such that for sufficiently small $T$, any $s \in S^\circ(k_v)$, $P \in X_s(k_v)$, we have*

$$C\,\mathrm{dist}_v(\Delta_f, f(P))T < \tau_{X_s,v}(\{Q \in X_s(k_v) \mid \mathrm{dist}_v(P, Q) < T\}).$$

*Proof.* We fix a rational section $S_0$ and an ample divisor $A$ on $S$. Let $S_m = S_0 + mf^*A$. Now $f_*(\mathcal{O}(S_0)) \otimes \mathcal{O}(mA)$ is globally generated for $m \gg 0$. Using this for each point $p \in S$, one may find a rational section $S_p \sim S_m$ such that $S_p$ is a local section in a neighborhood of the point $p \in S$. By the definition of conic bundles one can embed $f : X \to S$ into a projective bundle $\mathbb{P}(\mathcal{E})$. Take a finite open affine covering $\{U_i\}$ of $S$ so that over $U_i$, we have that $\mathbb{P}(\mathcal{E})|_{U_i}$ is trivialized, i.e., isomorphic to $U_i \times \mathbb{P}^2$. Taking a finer finite open covering, we may assume that $f$ admits a local section $S_i$ over $U_i$. By taking a finer finite open covering and applying a change of coordinates, one can assume that the local section $S_i$ corresponds to $(1:0:0)$ in $\mathbb{P}^2$. Moreover we may assume that the tangent line of $X_s$ at $(1:0:0)$ is given by $x_1 = 0$. Let $A_j$ $(j = 0, 1, 2)$ be the standard affine charts of $\mathbb{P}^2$ and we define $V_{i,j} = f^{-1}(U_i) \cap (U_i \times A_j)$ which is affine.

Now we take a finite $v$-adic open covering $B_l$ of $X(k_v)$ such that $\bar{B}_l$ is contained in some $V_{i,j}(k_v)$. Then on $B_l$ there exists a positive constant $C_1$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) \leq C_1 \max\{|x_j(P) - x_j(Q)|_v, |y_j(P) - y_j(Q)|_v\}, \tag{8-1}$$

where $x_j, y_j$ is the coordinates of $A_j$.

Now we are going to parametrize conics in the family. By our construction, $f^{-1}(U_i) \subset U_i \times \mathbb{P}^2$ is defined by the equation

$$d(s)y^2 + f(s)z^2 + 2xy + 2e(s)yz = 0,$$

where $d, f, e$ are functions on $U_i$. Note that the discriminant locus $\Delta_f$ is defined by $f = 0$ and it is a smooth divisor by our assumption. After further simplifications, we may assume that the equation is given by

$$f(s)z^2 + 2xy = 0.$$

Lines $uy - vz = 0$ passing through $(1 : 0 : 0)$ are parametrized by $(u : v) \in \mathbb{P}^1$. Then the rational parametrization of conics is given by

$$(fu^2 : -2v^2 : -2uv).$$

In particular, any smooth conic $X_s$ over $s \in U_i(k_v)$ is covered by $V_{i,0}$ and $V_{i,1}$. Also note that while this rational parametrization is not valid along singular fibers, a rational map mapping $(s, P) \in f^{-1}(U_i)$ to $(u(P) : v(P)) \in \mathbb{P}^1$ is a well-defined morphism.

Suppose that $B_l$ is contained in $V_{i,0}$. The inequality (8-1) shows that there exists $C_2 > 0$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) < C_2 \, \mathrm{dist}_v(\Delta_f, f(P))^{-1} |t(P) - t(Q)|_v,$$

where $t = v/u$ and $\mathrm{dist}_v(\Delta_f, f(P))$ is the distant function of $\Delta_f$.

Suppose that $B_l$ is contained in $V_{i,1}$. The inequality (8-1) shows that there exists $C_3 > 0$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) < C_3 |t(P) - t(Q)|_v,$$

where $t = u/v$.

Suppose that $B_l$ is contained in $V_{i,2}$. The inequality (8-1) shows that there exists $C_4 > 0$ such that for any $(s, P), (s, Q) \in B_l \subset U_i(k_v) \times A_j(k_v)$,

$$\mathrm{dist}_v(P, Q) < C_4|t(P) - t(Q)|_v,$$

where $t = v/u$. Now by arguing as in Lemma 6.1 our assertion follows. □

**Lemma 8.4.** *Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. Let $v$ be an archimedean place of $k$. We fix a $v$-adic metrization on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Then for any sufficiently small $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that for any $s \in S^\circ(k_v)$ we have*

$$\tau_v(X_s) < C_\epsilon\, \mathrm{dist}_v(\Delta_f, s)^{1-\epsilon}$$

*Proof.* This follows from the descriptions in the proof of Lemma 8.3 and an explicit computations of local Tamagawa numbers using the naive metrization. □

***Fano conic bundles: the anticanonical height.*** In this section, we discuss upper bounds of Manin type for the anticanonical height of Fano conic bundles.

**Theorem 8.5.** *Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are Fano. Let $W = X \times X$ and $W'$ be the blow-up of $W$ along the diagonal with the exceptional divisor $E$. We denote each projection $W' \to X_i$ by $\pi_i$. Let $\alpha, \beta$ be positive real numbers such that $2\alpha - 2\beta = 1$. We further make the following assumptions*:

(1) *The weak Manin's conjecture for $(S, -K_S)$ holds.*

(2) *For any component $V$ of the stable locus of*

$$|-\alpha K_{X/S}[2] - \beta f^* K_S[2] - E|$$

*such that $V$ is not contained in $E$, one of projections $\pi_i|_V$ is not dominant.*

*Then there exists a nonempty Zariski open subset $U \subset X$ such that for any $\epsilon > 0$ there exists $C = C_\epsilon > 0$ such that*

$$N(U, -K_X, T) < CT^{2\alpha+\epsilon}.$$

*Proof.* First of all note that the assumption (2) implies that $-2\alpha K_X + f^*K_S$ is big. Indeed, it implies that $-\alpha K_{X/S} - \beta f^* K_S$ is big; otherwise the linear system of this divisor defines a nontrivial fibration up to a birational modification and the assumption (2) cannot be true. Then note that we have $-K_{X/S} = -K_X + f^*K_S$ and $2\alpha - 2\beta = 1$ so that $-\alpha K_{X/S} - \beta f^*K_S = -\alpha K_X + \frac{1}{2} f^*K_S$ so our claim follows.

Let $v$ be a place of $k$ and fix $v$-adic metrizations on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Fix $\epsilon > 0$. Arguing as Theorem 5.1, the assumption (2) implies that there exists $U \subset X$ and $C$ such that for any $P, Q \in U(k)$ with $P \neq Q$ and $f(P) = f(Q) = s$ we have

$$\mathrm{dist}_v(P, Q) > C(H_{-K_S}(s))^{-2\beta}(H_{-K_{X/S}}(P)H_{-K_{X/S}}(Q))^{-\alpha}. \tag{8-2}$$

We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$

For $P \in A_T \cap X_s$, we define the $v$-adic ball by

$$B_T(P) = \left\{R \in X_s \cap U(k_v) \mid \mathrm{dist}_v(P, R) < \tfrac{1}{2} C T^{-2\alpha} H_{-K_S}(s)\right\}$$

Then $\bigcup B_T(P)$ is disjoint because of (8-2) and the triangle inequality. Note that after shrinking $U$, $T^{-2\alpha} H_{-K_S}(s)$ uniformly goes to 0 as $T \to \infty$ for any $s \in S$ with $A_T \cap X_s \neq \varnothing$ because of the Northcott property of the height function associated to $-2\alpha K_X + f^* K_S$ which is big. Thus we have

$$\begin{aligned} \mathrm{dist}_v(\Delta_f, s)^{1-\epsilon} \gg_\epsilon \tau_v(X_s) &> \sum_{P \in A_T \cap X_s(k)} \tau_{X_s, v}(B_T(P)) \\ &\gg_\epsilon N(U \cap X_s, L, T)\, \mathrm{dist}_v(\Delta_f, s) T^{-2\alpha} H_{-K_S}(s) \end{aligned}$$

by Lemmas 8.3 and 8.4. Let $m$ be a positive integer such that $mL - f^*\Delta_f$ is ample. We conclude that

$$N(U \cap X_s, L, T) \ll_\epsilon T^{2\alpha} H_{-K_S}(s)^{-1} H_{\Delta_f, v}(s)^\epsilon \ll T^{2\alpha + m\epsilon} H_{-K_S}(s)^{-1}.$$

Since $-kK_X \geq -f^*K_S$ for some $k$, our assertion follows from the fact that $S$ satisfies the weak Manin's conjecture and Tauberian theorem. Indeed, the weak Manin's conjecture for $S$ and Tauberian argument implies that

$$\sum_{s \in S^\circ(k)} H_{-K_S}(s)^{-(1+\epsilon)} < +\infty,$$

where $S^\circ \subset S$ is a some open subset of $S$. Let $U^\circ = U \cap f^{-1}(S^\circ)$. Then one can conclude that

$$\begin{aligned} N(U^\circ, -K_X, T) &= \sum_{s \in S^\circ(k)} N(U \cap X_s, L, T) \\ &\leq T^{2\alpha + m\epsilon + \epsilon k} \sum_{s \in S^\circ(k)} H_{-K_S}(s)^{-1-\epsilon} \ll T^{2\alpha + m\epsilon + \epsilon k}. \end{aligned}$$

Thus our assertion follows. □

***Fano conic bundles: the non-anticanonical heights.*** In this section, we discuss the weak Manin's conjecture for non-anticanonical height functions in some cases:

**Theorem 8.6.** *Let $f : X \to S$ be a conic bundle defined over a number field $k$ with a rational section. We assume that $X$ and $S$ are Fano. Let $L = -K_X - t f^* K_S$. We make the following assumptions*:

(1) *The weak Manin's conjecture conjecture for $(S, -K_S)$ holds.*

(2) $t \geq \dfrac{1}{2\delta(X, -K_X)}$.

*Then for any $\epsilon > 0$ there exists a nonempty Zariski open subset $U = U(\epsilon) \subset X$ and $C = C_\epsilon > 0$ such that*

$$N(U, L, T) < C T^{2\delta(X, -K_X) + \epsilon}.$$

*In particular when $\delta(X, -K_X) = \frac{1}{2}$, Conjecture 1.1 holds for $(X, L)$ except independence of $U$ on $\epsilon$.*

For such a height function, [Frei and Loughran 2019] establishes Manin's conjecture when the base is the projective space using conic bundle structures. Our theorem is flexible in the sense that $S$ can be other Fano manifold other than the projective space. For example one may find a smooth Fano threefold with a conic bundle structure over the Hirzebruch surface $\mathbb{F}_1$ in Section 9.

*Proof.* Let $v$ be a place and fix $v$-adic metrizations on $\mathcal{O}(K_X)$ and $\mathcal{O}(K_S)$. Fix $\epsilon > 0$. Theorem 5.1 implies that there exists $U \subset X$ and $C$ such that for any $P, Q \in U(k)$ with $P \neq Q$ we have

$$\mathrm{dist}_v(P, Q) > C(H_{-K_X}(P)H_{-K_X}(Q))^{-(\delta(X,-K_X)+\epsilon)}. \tag{8-3}$$

We define

$$A_T = \{P \in U(k) \mid H_L(P) \leq T\}.$$

For $P \in A_T \cap X_s$, we define the $v$-adic ball by

$$B_T(P) = \left\{R \in X_s \cap U(k_v) \mid \mathrm{dist}_v(P, R) < \tfrac{1}{2}CT^{-2(\delta(X,-K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X,-K_X)+\epsilon)}\right\}.$$

Then $\bigcup B_T(P)$ is disjoint because of (8-3) and the triangle inequality. Note that

$$T^{-2(\delta(X,-K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X,-K_X)+\epsilon)}$$

uniformly goes to 0 as $T \to \infty$ for any $s \in S$ with $A_T \cap X_s \neq \varnothing$ after shrinking $U$. Thus we have

$$\begin{aligned}\mathrm{dist}_v(\Delta_f, s)^{1-\epsilon} \gg_\epsilon \tau_v(X_s) > \sum_{P \in A_T \cap X_s(k)} \tau_{X_s,v}(B_T(P)) \\ \gg_\epsilon N(U \cap X_s, L, T)\,\mathrm{dist}_v(\Delta_f, s)T^{-2(\delta(X,-K_X)+\epsilon)}(H_{-K_S}(s))^{2t(\delta(X,-K_X)+\epsilon)}\end{aligned}$$

by Lemmas 8.3 and 8.4. Let $m$ be a positive integer such that $mL - f^*\Delta_f$ is ample. We conclude that

$$N(U \cap X_s, L, T) \ll_\epsilon T^{2(\delta(X,-K_X)+\epsilon)+m\epsilon} H_{-K_S}(s)^{-2t(\delta(X,-K_X)+\epsilon)}.$$

Since $L \geq -tf^*K_S$, our assertion follows by arguing as in Theorem 8.5. □

**Remark 8.7.** If $\delta(X, -K_X)$ is the minimum, one can take $U$ to be independent of $\epsilon$.

## 9. 3-dimensional Fano conic bundles

In this section we list smooth 3-dimensional Fano conic bundles and compute $\delta(X, -K_X)$ and the smallest $2\alpha$ satisfying the conditions of Theorem 8.5. Fano 3-folds with Picard rank $\geq 2$ were classified by Mori and Mukai [1981; 1983]. We follow their classification. We assume that our ground field is an algebraically closed field of characteristic 0. In our computations of $\delta(X, -K_X)$ and the minimum $2\alpha$ satisfying the conditions of Theorem 8.5, it is important to know a description of the nef cone of divisors of $X$. Such a description was obtained in [Matsuki 1995]. We freely use the results in this article.

***Fano threefolds with Picard rank 2.*** According to [Mori and Mukai 1981], there are 36 deformation types of smooth Fano 3-folds with Picard rank 2. Among them there are 16 deformation types of smooth Fano 3-folds which come with conic bundle structures. Since Fano 3-folds have Picard rank 2, these conic bundle structures are extremal contractions. Thus in these cases, a conic bundle structure comes with a rational section if and only if there is no singular fiber. Thus there are 7 deformation types of smooth Fano 3-folds which come with a conic bundle structure with a rational section. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 2] is given in Table 1.

Here $Q \subset \mathbb{P}^4$ is a smooth quadric 3-fold. Note that numbers 34–36 are toric; thus Manin's conjecture is known for these cases by [Batyrev and Tschinkel 1996; 1998]. In [Blomer et al. 2018], Manin's conjecture for an example of Fano 3-folds of number 24 is proven.

Let us illustrate the computation of $\delta(X, -K_X)$ and $\alpha > 0$ in some cases:

**Example 9.1** (number 32). Let $W$ be a smooth divisor of $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$. We denote each projection by $\pi_i : W \to \mathbb{P}^2$ and let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Then we have

$$-K_X = 2H_1 + 2H_2.$$

Since $H_1 + H_2$ is very ample, it follows that $\delta(X, -K_X) = \frac{1}{2}$ by Lemma 8.2.

Next we consider

$$\alpha(2H_1 + 2H_2 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = \frac{4\alpha - 3}{2} H_1 + 2\alpha H_2.$$

Then when $\frac{1}{2}(4\alpha - 3) \geq 1$, i.e., $\alpha \geq \frac{5}{4}$, the above divisor satisfies the assumptions of Theorem 8.5. On the other hand, for each $P \in X$, let $C_P$ be a fiber of $\pi_2$ meeting with $P$. Then we have $C_P.H_1 = 1$ and $C_P.H_2 = 0$. Thus we have

$$\left(\frac{4\alpha - 3}{2} H_1 + 2\alpha H_2\right).C_P = \frac{4\alpha - 3}{2}.$$

Thus by Lemma 4.3, we conclude that $\alpha = \frac{5}{4}$ is the minimum value satisfying the assumptions of Theorem 8.5.

| no. | $(-K_X)^3$ | $X$ | $6\delta(X, -K_X)$ | $2\alpha$ |
|---|---|---|---|---|
| 24 | 30 | a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 2)$ | $\leq 6$ | $\leq 5$ |
| 27 | 38 | blow-up of $\mathbb{P}^3$ with center a twisted cubic | 3 | 2 |
| 31 | 46 | blow-up of $Q \subset \mathbb{P}^4$ with center a line on it | 3 | $\frac{5}{3}$ |
| 32 | 48 | a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$ | 3 | $\frac{5}{2}$ |
| 34 | 54 | $\mathbb{P}^1 \times \mathbb{P}^2$ | 3 | $\frac{5}{3}$ |
| 35 | 56 | $V_7 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$ | 3 | $\frac{5}{4}$ |
| 36 | 62 | $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ over $\mathbb{P}^2$ | 3 | $\frac{5}{3}$ |

**Table 1.** Fano 3-folds with Picard rank 2.

**Example 9.2** (number 31). Let $X$ be the blow-up of $Q$ along a line. Then $X$ has Picard rank 2 so it comes with two extremal contractions, one is a $\mathbb{P}^1$-bundle $\pi_1 : X \to \mathbb{P}^2$, and the other is a divisorial contraction $\pi_2 : X \to Q$. Let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Then it follows from [Mori and Mukai 1983, Theorem 5.1] that

$$-K_X = H_1 + 2H_2.$$

Since $\delta(X, H_2) = \delta(Q, H) = 1$ and $\pi_2$ is birational, it follows that $\delta(X, -K_X) \le \frac{1}{2}$. Thus by Lemma 8.2, $\delta(X, -K_X) = \frac{1}{2}$ is proved.

Next we have

$$\alpha(H_1 + 2H_2 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = 2\alpha H_2 + \frac{2\alpha - 3}{2} H_1.$$

Let $D$ be the exceptional divisor of $\pi_2$. Then we have $H_1 = H_2 - D$. Thus the above divisor becomes

$$\frac{6\alpha - 3}{2} H_2 - \frac{2\alpha - 3}{2} D.$$

Thus when $\frac{1}{2}(6\alpha - 3) \ge 1$ and $2\alpha - 3 \le 0$, i.e, $\frac{5}{6} \le \alpha \le \frac{3}{2}$ the assumption of Theorem 8.5 holds. On the other hand let $\ell \subset X$ be the strict transform of a line on $Q$ not meeting with center of $\pi_2$. Then we have

$$\left(\frac{6\alpha - 3}{2} H_2 - \frac{2\alpha - 3}{2} D\right).\ell = \frac{6\alpha - 3}{2}.$$

Thus since such $\ell$ deforms to cover $X$, by arguing as in Lemma 4.3, we conclude that $\alpha = \frac{5}{6}$ is the minimum value satisfying the assumptions of Theorem 8.5.

***Fano threefolds with Picard rank* 3.** According to [Mori and Mukai 1981], there are 31 deformation types of smooth Fano 3-folds with Picard rank 3. It follows from [Mori and Mukai 1983, p. 125, (9.1)] that all such Fano 3-folds come with a conic bundle structure except the blow-up of $\mathbb{P}^3$ along a disjoint union of a line and a conic. Again a conic bundle structure with singular fibers which is extremal never comes with a rational section. Note that if $X$ is a Fano conic bundle which does not admit a divisorial contraction to a Fano conic bundle of Picard rank 2 with a rational section, then its extremal conic bundle structure admits singular fibers. For such a 3-fold, one can conclude that it does not admit a rational section. This implies that there are 25 deformation types of 3-dimensional Fano conic bundles with a rational section. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 3] is given in 2.

Note that numbers 24–31 are toric; thus Manin's conjecture is known for these cases by [Batyrev and Tschinkel 1996; 1998]. Let us demonstrate the computation of $\delta(X, -K_X)$ and $\alpha$ in some cases:

**Example 9.3** (number 23). Let $X$ be a Fano 3-fold of number 23. Then $X$ admits a divisorial contraction $\beta : X \to V_7$ with the exceptional divisor $D_1$. The Fano 3-fold $V_7$ admits two extremal contractions: one is a $\mathbb{P}^1$-bundle $\pi_1 : V_7 \to \mathbb{P}^2$ and the other is the blow-down $V_7 \to \mathbb{P}^3$. We denote the pullback of the hyperplane class via $\pi_i$ by $H_i$. One can conclude that the only conic bundle structure on $X$ is $\pi_1 \circ \beta$. It follows from [Mori and Mukai 1983, Theorem 5.1] that

$$-K_{V_7} = 2H_1 + 2H_2.$$

| no. | $X$ | $6\delta$ | $2\alpha$ |
|---|---|---|---|
| 3 | a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 2)$ | $\leq 6$ | $\leq 5$ |
| 5 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a curve $C$ of bidegree $(5, 2)$ such that the projection $C \to \mathbb{P}^2$ is an embedding | $\leq 6$ | $\leq 5$ |
| 7 | blow-up of $W$ (number 32) with center an intersection of two members of $\lvert -\frac{1}{2}K_W \rvert$ | $\leq 4$ | $\leq 3$ |
| 8 | a member of the linear system $\lvert p_1^* g^* \mathcal{O}(1) \otimes p_2 \mathcal{O}(2) \rvert$ on $\mathbb{F}_1 \times \mathbb{P}^2$, where $p_i$ is the projection to each factor and $g : \mathbb{F}_1 \to \mathbb{P}^2$ is the blowing up | $\leq 6$ | $\leq 5$ |
| 9 | blow-up of the cone $W_4 \subset \mathbb{P}^6$ over the Veronese surface $R_4 \subset \mathbb{P}^5$ with center a disjoint union of the vertex and quartic in $R_4 = \mathbb{P}^2$ | 3 | $\leq \frac{7}{5}$ |
| 11 | blow-up of $V_7$ (number 35) with center an intersection of two members of $\lvert -\frac{1}{2}K_{V_7} \rvert$ | 3 | $\frac{5}{2}$ |
| 12 | blow-up of $\mathbb{P}^3$ with center a disjoint union of a line and a twisted cubic | 3 | $\frac{8}{3}$ |
| 13 | blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ with center a curve $C$ of bidegree $(2, 2)$ on it such that each projection from $C$ to $\mathbb{P}^2$ is an embedding | 3 | $\leq \frac{5}{2}$ |
| 14 | blow-up of $\mathbb{P}^3$ with center a disjoint union of a point and a plane cubic | $\leq 6$ | $\leq \frac{9}{5}$ |
| 15 | blow-up of $Q \subset \mathbb{P}^4$ with center a disjoint union of a line and a conic | 3 | $\frac{5}{2}$ |
| 16 | blow-up of $V_7$ with center the strict transform of a twisted cubic passing through the center of the blow-up $V_7 \to \mathbb{P}^3$ | 3 | $\frac{5}{2}$ |
| 17 | a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 1)$ | 3 | $\frac{5}{2}$ |
| 19 | blow-up of $Q \subset \mathbb{P}^4$ with center two points which are not colinear | 3 | $\frac{5}{3}$ |
| 20 | blow-up of $Q \subset \mathbb{P}^4$ with center a disjoint union of two lines | 3 | $\frac{5}{2}$ |
| 21 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a curve of bidegree $(2, 1)$ | 3 | $\frac{5}{2}$ |
| 22 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center a conic in $\{t\} \times \mathbb{P}^2$ | 3 | $\frac{5}{3}$ |
| 23 | blow-up of $V_7$ with center the strict transform of a conic passing through the center of the blow-up $V_7 \to \mathbb{P}^3$ | 3 | $\frac{5}{3}$ |
| 24 | fiber product $W \times_{\mathbb{P}^2} \mathbb{F}_1$, where $W$ is number 32, $W \to \mathbb{P}^2$ is the $\mathbb{P}^1$-bundle and $\mathbb{F}_1 \to \mathbb{P}^2$ is the blowing up | 3 | $\frac{5}{2}$ |
| 25 | $\mathbb{P}(\mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$ | 3 | $\frac{3}{2}$ |
| 26 | blow-up of $\mathbb{P}^3$ with center a disjoint union of a point and a line | 3 | $\frac{5}{3}$ |
| 27 | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 3 | 2 |
| 28 | $\mathbb{P}^1 \times \mathbb{F}_1$ | 3 | 2 |
| 29 | blow-up of $V_7$ with center a line on the exceptional set $D = \mathbb{P}^2$ of the blow-up $V_7 \to \mathbb{P}^3$ | 3 | $\leq \frac{7}{5}$ |
| 30 | blow-up of $V_7$ with center the strict transform of a line passing through the center of the blow-up $V_7 \to \mathbb{P}^3$ | 3 | $\frac{4}{3}$ |
| 31 | $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1, 1))$ over $\mathbb{P}^1 \times \mathbb{P}^1$ | 3 | $\leq \frac{4}{3}$ |

**Table 2.** Fano 3-folds with Picard rank 3.

Thus we have

$$-K_X = 2\beta^* H_1 + 2\beta^* H_2 - D_1.$$

Since $\beta^* H_1 - D_1$ is effective and the morphism associated to $|H_2|$ is birational, one can conclude that $\delta(X, -K_X) = \frac{1}{2}$ by Lemma 8.2.

Let $D_2$ be the strict transform of the exceptional divisor of $\pi_2$. Then we have

$$\alpha(2\beta^* H_1 + 2\beta^* H_2 - D_1 - 3\beta^* H_1) + \frac{2\alpha - 1}{2} \cdot 3\beta^* H_1 = 2\alpha\beta^* H_2 - \alpha D_1 + \frac{4\alpha - 3}{2}\beta^* H_1$$

Since we have $\beta^* H_2 = \beta^* H_1 + D_2$, the above divisor becomes

$$\beta^* H_2 + (2\alpha - 1)D_2 + \frac{8\alpha - 5}{2}\beta^* H_1 - \alpha D_1$$

Since $\beta^* H_1 - D_1 \geq 0$, in the case of $\frac{1}{2}(8\alpha - 5) \geq \alpha$, i.e., $\alpha \geq \frac{5}{6}$, the above divisor satisfies the assumption of Theorem 8.5. On the other hand let $\ell$ be the strict transform of a line meeting with the center of $D_1$. When $\alpha = \frac{5}{6}$, we have

$$\left(\beta^* H_2 + (2\alpha - 1)D_2 + \frac{8\alpha - 5}{2}\beta^* H_1 - \alpha D_1\right).\ell = 1$$

Thus since such $\ell$ deforms to cover $X$, by arguing as in Lemma 4.3, we conclude that $\alpha = \frac{5}{6}$ is the minimum value satisfying the assumption of Theorem 8.5.

**Example 9.4** (number 12). Let $X$ be a Fano 3-fold of number 12. Then it admits a conic bundle structure $\pi_1 : X \to \mathbb{P}^2$, a birational morphism $\pi_2 : X \to \mathbb{P}^3$, and a del Pezzo fibration $\pi_3 : X \to \mathbb{P}^1$. Let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Then we have

$$-K_X = H_1 + H_2 + H_3.$$

Since $|H_2|$ defines a birational morphism to $\mathbb{P}^3$ and $|H_1 + H_3|$ defines a birational morphism to $\mathbb{P}^2 \times \mathbb{P}^1$ we can conclude that $\delta(X, -K_X) \leq \frac{1}{2}$. Thus by Lemma 8.2, we have $\delta(X, -K_X) = \frac{1}{2}$. Next we consider the divisor

$$\alpha(H_1 + H_2 + H_3 - 3H_1) + \frac{2\alpha - 1}{2} \cdot 3H_1 = (3\alpha - 3)H_2 + \alpha H_3 - \frac{2\alpha - 3}{2}D_1,$$

where $D_1$ is the exceptional divisor of $\pi_2$ whose center is a twisted cubic. When $3\alpha - 3 \geq 1$ and $2\alpha - 3 \leq 0$, i.e., $\frac{4}{3} \leq \alpha \leq \frac{3}{2}$, the assumptions of Theorem 8.5 hold. On the other hand let $\ell$ be the strict transform of a general line meeting with the line which is the center of $\pi_2$. Then we have

$$\left((3\alpha - 3)H_2 + \alpha H_3 - \frac{2\alpha - 3}{2}D_1\right).\ell = 3\alpha - 3.$$

Thus since such $\ell$ deforms to cover $X$, by arguing as in Lemma 4.3, we conclude that $\alpha = \frac{4}{3}$ is the minimum value satisfying the assumption of Theorem 8.5.

| no. | $X$ | $6\delta(X, -K_X)$ | $2\alpha$ |
|---|---|---|---|
| 1 | a smooth divisor on $(\mathbb{P}^1)^4$ of multidegree $(1, 1, 1, 1)$ | 3 | 3 |
| 2 | blow-up of the cone over a quadric surface $S \subset \mathbb{P}^3$ with center a disjoint union of the vertex and an elliptic curve on $S$ | $\leq 6$ | $\leq 2$ |
| 3 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve of tridegree $(1, 1, 2)$ | 3 | $\leq 2$ |
| 4 | blow-up of $Y$ (no. 19, Table 2) with center the strict transform of a conic passing through $p$ and $q$ | $\leq 6$ | $\leq 2$ |
| 5 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ with center two disjoint curves of bidegree $(2, 1)$ and $(1, 0)$ | $\leq 6$ | 2 |
| 6 | blow-up of $\mathbb{P}^3$ with center three disjoint lines | 3 | 2 |
| 7 | blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ with center two disjoint curves of bidegree $(0, 1)$ and $(1, 0)$ | 3 | $\frac{5}{2}$ |
| 8 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve of tridegree $(0, 1, 1)$ | 3 | 2 |
| 9 | blow-up of $Y$ (no. 25, Table 2) with center an exceptional line of the blowing up $Y \to \mathbb{P}^3$ | 3 | 2 |
| 10 | $\mathbb{P}^1 \times S_7$ | 3 | 2 |
| 11 | blow-up of $\mathbb{P}^1 \times \mathbb{F}_1$ with center $t \times e$, where $t \in \mathbb{P}^1$ and $e$ is an exceptional curve on $\mathbb{F}_1$ | 3 | 2 |
| 12 | blow-up of $Y$ (no. 33, [Mori and Mukai 1981, Table 2]) with center two exceptional lines of the blowing up $Y \to \mathbb{P}^3$ | 3 | 2 |
| 13 | blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with center a curve of tridegree $(1, 1, 3)$ | $\leq 6$ | $\leq 3$ |

**Table 3.** Fano 3-folds with Picard rank 4.

***Fano threefolds with Picard rank* 4 *or* 5.** For Fano 3-folds in this range, all of them admit conic bundle structures with a rational section. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 4] is given in Table 3.

Here $S_7$ is a smooth del Pezzo surface of degree 7. Notice numbers 9–12 are toric, so Manin's conjecture is known for these cases by [Batyrev and Tschinkel 1996; 1998]. Again let us illustrate the computation of $\delta(X, -K_X)$ and $\alpha$ in some cases:

**Example 9.5** (number 6). Let $X$ be a Fano 3-fold of number 6. Then $X$ admits three del Pezzo fibrations $\pi_i : X \to \mathbb{P}^1$. It also admits a birational morphism $\pi : X \to \mathbb{P}^3$. Let $H_i$ be the pullback of the hyperplane class via $\pi_i$. Let $H$ be the pullback of the hyperplane class via $\pi$. Then we have

$$-K_X = H + H_1 + H_2 + H_3.$$

Since both $|H|$, $|H_1 + H_2 + H_3|$ are birational, we conclude that $\delta(X, -K_X) = \frac{1}{2}$.

Next any conic bundle structure on $X$ is given by $|H_i + H_j|$, where $i \neq j$. So we look at the conic bundle structure defined by $|H_1 + H_2|$. We consider

$$\alpha(H + H_3 - H_1 - H_2) + (2\alpha - 1)(H_1 + H_2) = \alpha H + \alpha H_3 + (\alpha - 1)H_1 + (\alpha - 1)H_2.$$

From this description we may conclude that $\alpha = 1$ satisfies the assumptions of Theorem 8.5. On the other hand, by looking at the strict transform $C$ of a line such that $H_3.C = 0$, we may conclude that $\alpha = 1$ is the minimum value satisfying the assumptions of Theorem 8.5.

**Example 9.6** (number 4). Let $V_7$ be the blow-up of $\mathbb{P}^3$ at a point $p$. It admits two extremal rays and we denote each contraction morphism by $\pi_1 : V_7 \to \mathbb{P}^2$ and $\pi_2 : X \to \mathbb{P}^3$. Let $H_i$ be the pullback of the hyperplane via $\pi_i$. Let $X'$ be the blow-up of a fiber of $\pi_1$ which is the strict transform of a line $\ell$ passing through $p$. It admits a conic bundle structure over $\mathbb{F}_1$. Let $X$ be the blow-up of $X'$ along the strict transform of a conic $C_0$ not meeting with $\ell$. Then $X$ is a smooth Fano 3-fold of number 4. We denote the strict transform of the exceptional divisor of $X' \to V_7$ by $D_1$ and the exceptional divisor of $X \to X'$ by $D_2$. Then we have

$$-K_X = 2H_1 + 2H_2 - D_1 - D_2.$$

Since $2H_1 + H_2 - D_1 - D_2$ is linearly equivalent to an effective divisor, it follows that $\delta(X, -K_X) \leq 1$.

Next we consider

$$\alpha(-K_X - (3H_1 - D_1)) + \frac{2\alpha - 1}{2}(3H_1 - D_1) = 2\alpha H_2 - \alpha D_2 + \frac{4\alpha - 3}{2}H_1 - \frac{2\alpha - 1}{2}D_1.$$

Since $H_2 - D_2$ and $H_1 - D_1$ are effective, it follows that $\alpha = 1$ satisfies the assumptions of Theorem 8.5.

Finally we discuss the case of Picard rank 5. The list of these Fano 3-folds from [Mori and Mukai 1981, Table 5] is given in Table 4.

| no. | $X$ | $6\delta(X, -K_X)$ | $2\alpha$ |
|---|---|---|---|
| 1 | blow-up of $Y$ (no. 29, [Mori and Mukai 1981, Table 2]) with center three exceptional lines of the blowing up $Y \to Q$ | 3 | $\leq 2$ |
| 2 | blow-up of $Y$ (no. 25, Table 2) with center two exceptional lines $\ell$ and $\ell'$ of the blowing up $\phi : Y \to \mathbb{P}^3$ such that $\ell$ and $\ell'$ lie on the same irreducible component of the exceptional set for $\phi$ | 3 | $\leq 2$ |
| 3 | $\mathbb{P}^1 \times S_6$ | 3 | 2 |

**Table 4.** Fano 3-folds with Picard rank 5.

Here $S_6$ is a smooth del Pezzo surface of degree 6. Note that numbers 2 and 3 are toric, so Manin's conjecture is known by [Batyrev and Tschinkel 1996; 1998].

***Fano threefolds with Picard rank ≥ 6.***

**Theorem 9.7** [Mori and Mukai 1983, Theorem 1.2]. *Let $X$ be a smooth Fano 3-fold and we denote its Picard rank by $\rho(X)$. Suppose that $\rho(X) \geq 6$. Then $X$ is isomorphic to $\mathbb{P}^1 \times S_{11-\rho(X)}$, where $S_d$ is a smooth del Pezzo surface of degree $d$.*

Thus in this section, we study the product of a smooth del Pezzo surface $S_d$ with $\mathbb{P}^1$. Note that the weak Manin's conjecture for the product follows as soon as the weak Manin's conjecture is known for $S_d$ by [Franke et al. 1989], so we omit the discussion of $\alpha$ in this section.

**Proposition 9.8.** *Let $X = \mathbb{P}^1 \times S$, where $S$ is a smooth del Pezzo surface of degree $d$ with $1 \leq d \leq 8$. Then we have $\delta(X, -K_X) = \delta(S, -K_S)$.*

*Proof.* Let $W_X = X \times X$ and $\alpha : W'_X \to W_X$ be the blow-up of the diagonal. We use the same notation for $S$ as well. Let $H_1$ be the pullback of the ample generator via $p_1 : X \to \mathbb{P}^1$ and let $H_2$ be the pullback of the anticanonical divisor via $p_2 : X \to S$. Then the anticanonical divisor of $X$ is

$$-K_X = 2H_1 + H_2.$$

Fix $\epsilon > 0$ and consider

$$-(\delta(S, -K_S) + \epsilon)K_X[2] - E = 2(\delta(S, -K_S) + \epsilon)H_1[2] + \epsilon H_2[2] + \delta(S, -K_S)H_2[2] - E.$$

Since $2(\delta(S, -K_S) + \epsilon)H_1[2] + \epsilon H_2[2]$ is semi-ample, we know the stable locus of $-(\delta(S, -K_S) + \epsilon)K_X[2] - E$ is contained in the stable locus of $\delta(S, -K_S)H_2[2] - E$. The possible dominant components of the stable locus of $\delta(S, -K_S)H_2[2] - E$ are $E$ and the strict transform of

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \Delta_S,$$

where $\Delta_S$ is the diagonal of $W_S$. Next we consider

$$-(\delta(S, -K_S) + \epsilon)K_X[2] - E = 2\epsilon H_1[2] + (\delta(S, -K_S) + \epsilon)H_2[2] + 2\delta(S, -K_S)H_1[2] - E.$$

Since $2\epsilon H_1[2] + (\delta(S, -K_S) + \epsilon)H_2[2]$ is again semi-ample, the stable locus of $-(\delta(S, -K_S) + \epsilon)K_X[2] - E$ is therefore contained in the stable locus of $2\delta(S, -K_S)H_1[2] - E$. Since $\delta(S, -K_S) \geq \frac{1}{2}$, it follows that the stable locus of $2\delta(S, -K_S)H_1[2] - E$ is contained in the strict transform $Z$ of

$$\Delta_{\mathbb{P}^1} \times W_S \subset W_X,$$

where $\Delta_{\mathbb{P}^1}$ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. The variety $Z$ is isomorphic to $\mathbb{P}^1 \times W'_S$. From this one may conclude that $\delta(X, -K_X) \leq \delta(S, -K_S)$ by taking the intersection of two loci.

On the other hand the discussion after Proposition 4.9 shows that in each case there are curves $C$ on $W'_S$ such that $C$ deforms to dominate both $S_i$ and also that $(-\delta(S, -K_S)K_S[2] - E).C = 0$. Thus we conclude that $\delta(X, -K_X) \geq \delta(S, -K_S)$. Thus our assertion follows. □

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stanimoto@kumamoto-u.ac.jp *Department of Mathematics, Faculty of Science, Kumamoto University, Kurokami, Kumamoto, Japan*

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