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Fano 4-folds with rational fibrations

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We study (smooth, complex) Fano 4-folds X having a rational contraction of fiber type, that is, a rational map $X \dashrightarrow Y$ that factors as a sequence of flips followed by a contraction of fiber type. The existence of such a map is equivalent to the existence of a nonzero, nonbig movable divisor on X. Our main result is that if Y is not $\mathbb{P}^1$ or $\mathbb{P}^2$, then the Picard number ρ_X of X is at most 18, with equality only if X is a product of surfaces. We also show that if a Fano 4-fold X has a dominant rational map $X \dashrightarrow Z$, regular and proper on an open subset of X, with $\dim(Z) = 3$, then either X is a product of surfaces, or ρ_X is at most 12. These results are part of a program to study Fano 4-folds with large Picard number via birational geometry.

1. Introduction

Smooth, complex Fano varieties play an important role in projective geometry, both from the classical and modern point of view, in the framework of the minimal model program. There are finitely many families of Fano varieties of any given dimension, which are classified up to dimension 3 — the classification of Fano 3-folds was achieved more than 30 years ago, see [Iskovskikh and Prokhorov 1999] and references therein. In dimensions 4 and higher there is no classification apart from some special classes, and we still lack a good understanding of the geometry of Fano 4-folds.

This paper is part of a program to study Fano 4-folds X with large Picard number ρ_X, by means of birational geometry, more precisely via the study of contractions and flips of Fano 4-folds. Our goal is to get a sharp bound on ρ_X, and possibly to classify Fano 4-folds X with "large" Picard number. Let us notice that, among the known examples of Fano 4-folds, products of del Pezzo surfaces have $\rho_X \leq 18$, and the others have $\rho_X \leq 9$ (see [Casagrande et al. 2019] for the case $\rho_X = 9$).

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In this paper we focus on Fano 4-folds X having a rational contraction of fiber type. Here a *contraction* is a morphism $f\colon X \to Y$ with connected fibers onto a normal projective variety. More generally, a *rational contraction* is a rational map $f\colon X \dashrightarrow Y$ that can be factored as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} Y$, where X' is a normal and $\mathbb{Q}$-factorial projective variety, φ is birational and an isomorphism in codimension 1, and f' is a contraction. As usual, f is of fiber type if $\dim Y < \dim X$. Note that X has a nonconstant rational contraction of fiber type if and only if there is a nonzero, nonbig movable divisor. Our main results are the following.

Theorem 1.1. *Let X be a smooth Fano 4-fold with a rational contraction of fiber type $f\colon X \dashrightarrow Y$, where $\dim Y > 0$. If $Y \ncong \mathbb{P}^1$ and $Y \ncong \mathbb{P}^2$, then $\rho_X \leq 18$, with equality only if X is a product of surfaces.*

Theorem 1.2. *Let X be a smooth Fano 4-fold. Suppose that there exists a dominant rational map $f\colon X \dashrightarrow Y$, regular and proper on an open subset of X, with $\dim Y = 3$. Then either X is a product of surfaces, or $\rho_X \leq 12$.*

Let us say something on the techniques and strategy used in the paper. We consider the following classes of rational contractions of fiber type:

$$\{\text{“quasielementary”}\} \subset \{\text{“special”}\} \subset \{\text{general}\}.$$

Quasielementary rational contractions of fiber type have been introduced in [Casagrande 2008; 2013a] (see Section 2A for more details); when f is quasielementary Theorem 1.1 is already known [loc. cit.], and one can even allow $Y \cong \mathbb{P}^1$ and $Y \cong \mathbb{P}^2$.

In this paper we introduce a more general notion, that of “special” rational contraction of fiber type, which plays a key role in the proof of Theorem 1.1. We define special (regular and rational) contractions in Section 2B; then we show that every rational contraction of fiber type of a Mori dream space can be factored as a special rational contraction, followed by a birational map (Proposition 2.13). In particular, if a Fano variety has a rational contraction of fiber type, then it also has a special rational contraction of fiber type, so that we can reduce to prove Theorem 1.1 when f is special.

Secondly, we show that up to flips, every special rational contraction of a Mori dream space can be factored as a sequence of elementary divisorial contractions, followed by a quasielementary contraction (Theorem 2.15). This allows to relate the study of special rational contractions of Fano 4-folds X to our previous study of elementary divisorial contractions and quasielementary contractions of 4-folds obtained from X with a sequence of flips, in [Casagrande 2013a; 2017].

Another key ingredient used in the paper is the Lefschetz defect δ_X, an invariant of X which basically allows to bound ρ_X in terms of the Picard number of prime divisors in X (see Section 3A for an account).

After developing the necessary techniques and preliminary results in Sections 2–4, we prove Theorem 1.1 first in the case where $\dim Y = 2$ in Section 5, and then in the case where $\dim Y = 3$ in Section 6. Theorem 1.2 is then an easy consequence of the case where $\dim Y = 3$.

1A. ***Notation and terminology.*** If $\mathcal{N}$ is a finite-dimensional real vector space and $a_1, \dots, a_r \in \mathcal{N}$, $\langle a_1, \dots, a_r \rangle$ denotes the convex cone in $\mathcal{N}$ generated by $a_1, \dots, a_r$. Moreover, for every $a \neq 0$, $a^{\perp}$ is the hyperplane orthogonal to a in the dual vector space $\mathcal{N}^*$.

We refer the reader to [Hu and Keel 2000] for the notion of Mori dream space; *we always assume that a Mori dream space is projective, normal and* $\mathbb{Q}$*-factorial.* We recall that Fano varieties are Mori dream spaces by [Birkar et al. 2010, Corollary 1.3.2]. We also refer to [Kollár and Mori 1998] for the standard notions in birational geometry, in particular the definition of flip [loc. cit., Definition 6.5]

Let X be a normal and $\mathbb{Q}$-factorial projective variety.

A small $\mathbb{Q}$-factorial modification (SQM) is a birational map $\varphi\colon X \dashrightarrow X'$ which is an isomorphism in codimension one, where X' is a normal and $\mathbb{Q}$-factorial projective variety. If X is a Mori dream space, every SQM can be factored as a finite sequence of flips.

Let $f\colon X \to Y$ be an elementary contraction, namely a contraction with $\rho_X - \rho_Y = 1$. We say that f is of type (a, b) if

$$\dim \operatorname{Exc}(f) = a \quad \text{and} \quad \dim f(\operatorname{Exc}(f)) = b.$$

We say that f is of type $(\dim X - 1, b)^{\mathrm{sm}}$ if it is the blow-up of a smooth b-dimensional subvariety of Y, contained in Y_{reg}. If X is a smooth 4-fold, we say that f is of type $(3, 0)^Q$ if f is of type $(3, 0)$, $\operatorname{Exc}(f)$ is isomorphic to an irreducible quadric Q, and $\mathcal{N}_{\operatorname{Exc}(f)/X} \cong \mathcal{O}_Q(-1)$.

Let D be a divisor. A contraction $f\colon X \to Y$ is D-negative (respectively, D-positive) if there exists $m \in \mathbb{Z}_{>0}$ such that $-mD$ (respectively, mD) is Cartier and f-ample. A D-negative flip is the flip of a small, D-negative elementary contraction, and similarly for D-positive. *We do not assume that contractions or flips are* K*-negative, unless specified.*

When X is a Mori dream space, given a contraction $f\colon X \to Y$ and a divisor D in X, one can run an MMP for D relative to f. This means that there exists a birational map $\psi\colon X \dashrightarrow X'$, given by a composition of D-negative flips and elementary divisorial contractions, such that $f' := f \circ \psi^{-1}\colon X' \to Y$ is regular, and if D' is the transform of D in X', then either D' is f'-nef, or f' factors through a D'-negative elementary contraction of fiber type of X'.

A movable divisor is an effective divisor D such that the stable base locus of the linear system $|D|$ has codimension ≥ 2. A fixed prime divisor is a prime divisor D which is the stable base locus of $|D|$. We will consider the usual cones of divisors and of curves

$$\operatorname{Nef}(X) \subseteq \operatorname{Mov}(X) \subseteq \operatorname{Eff}(X) \subset \mathcal{N}^1(X), \quad \operatorname{mov}(X) \subseteq \operatorname{NE}(X) \subset \mathcal{N}_1(X),$$

where all the notations are standard except $\operatorname{mov}(X)$, which is the convex cone generated by classes of curves moving in a family covering X. When X is a Mori dream space, all these cones are closed, rational and polyhedral. If D is a divisor and C is a curve in X, we denote by $[D] \in \mathcal{N}^1(X)$ and $[C] \in \mathcal{N}_1(X)$ their numerical equivalence classes.

For every closed subset $Z \subset X$, we denote by $\mathcal{N}_1(Z, X)$ the linear subspace of $\mathcal{N}_1(X)$ spanned by classes of curves contained in Z. We will use the following simple property.

Remark 1.3. Let D be a prime divisor. If $Z \cap D = \emptyset$, then $\mathcal{N}_1(Z, X) \subseteq D^\perp$, in particular $\mathcal{N}_1(Z, X) \subsetneq \mathcal{N}_1(X)$. This is because $D \cdot C = 0$ for every curve $C \subset Z$.

Let X be a smooth 4-fold. An *exceptional plane* is a closed subset $L \subset X$ such that $L \cong \mathbb{P}^2$ and $\mathcal{N}_{L/X} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$; an *exceptional line* is a closed subset $\ell \subset X$ such that $\ell \cong \mathbb{P}^1$ and $\mathcal{N}_{\ell/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$.

2. Special contractions of fiber type

When studying Fano varieties, or more generally Mori dream spaces, one often needs to consider contractions of fiber type $f \colon X \to Y$ which are not elementary. In full generality, such contractions are hard to deal with, in particular Y may be very singular and/or non-$\mathbb{Q}$-factorial. For this reason, it is useful to introduce some classes of contractions of fiber type with good properties, which should include the elementary case. A first notion of this type is that of "quasielementary" contraction; we briefly recall this definition and some properties in Section 2A.

Here we introduce a more general notion, that of "special" contraction of fiber type. In Section 2B we define special contractions, in the regular and rational case; the target is automatically $\mathbb{Q}$-factorial.

In Section 2C we show two factorization results for rational contractions of fiber type of Mori dream spaces. More precisely, we show that every rational contraction of fiber type of a Mori dream space can be factored as a special rational contraction, followed by a birational map (Proposition 2.13). Moreover, up to flips, every special rational contraction of a Mori dream space can be factored as a sequence of elementary divisorial contractions, followed by a quasielementary contraction (Theorem 2.15).

Finally, in Section 2D we consider special contractions of fiber type $f \colon X \to Y$ which are also $(K+\Delta)$-negative for a suitable boundary Δ on X, and we show that if X has good singularities, then Y has good singularities too.

2A. *Quasielementary contractions.*

We refer the reader to [Casagrande 2013a, Section 2.2; 2008] for the notion of quasielementary contraction of fiber type; here we just recall the definition.

Definition 2.1 (quasielementary contraction). Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f \colon X \to Y$ a contraction of fiber type. We say that f is quasielementary if for every fiber F of f we have $\mathcal{N}_1(F, X) = \ker f_*$, where $f_* \colon \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ is the push-forward of one-cycles (see Section 1A for $\mathcal{N}_1(F, X)$).

Let us give an equivalent characterization, for Mori dream spaces.

Proposition 2.2. *Let X be a Mori dream space and $f \colon X \to Y$ a contraction of fiber type. The following are equivalent*:

(i) *f is quasielementary.*

(ii) *For every prime divisor D in X, either $f(D) = Y$, or $D = \lambda f^* B$ for some $\mathbb{Q}$-Cartier prime divisor B in Y and $\lambda \in \mathbb{Q}_{>0}$.*

(iii) *Y is $\mathbb{Q}$-factorial and for every prime divisor B in Y, the pull-back $f^* B$ is irreducible* (*but possibly nonreduced*).

Proof. Let $F \subset X$ be a general fiber of f.

(i)⇒(iii) The target Y is $\mathbb{Q}$-factorial by [Casagrande 2013a, proof of Remark 2.26]. Let B be a prime divisor in Y, and let D be an irreducible component of f^*B. Then $D \cap F = \emptyset$, so that $\mathcal{N}_1(F, X) \subseteq D^{\perp}$ by Remark 1.3. Since f is quasielementary, we have $\mathcal{N}_1(F, X) = \ker f_*$, hence $\ker f_* \subseteq D^{\perp}$, and D is the pull-back of a $\mathbb{Q}$-divisor in Y (see [loc. cit., Remark 2.9]). Since $B = f(D)$, we must have $D = \lambda f^*B$ with $\lambda \in \mathbb{Q}_{>0}$, so f^*B is irreducible.

(ii)⇒(i) Let σ be the minimal face of $\operatorname{Eff}(X)$ containing $f^*(\operatorname{Nef}(Y))$; by [Casagrande 2013a, Lemma 2.21 and Proposition 2.22] we have $\sigma = \operatorname{Eff}(X) \cap \mathcal{N}_1(F, X)^{\perp}$, and f is quasielementary if and only if $\dim \sigma = \rho_Y$.

Suppose that f is not quasielementary. Then $\dim \sigma > \rho_Y$, so that $\sigma \not\subseteq f^*\mathcal{N}^1(Y)$, and there exists a one-dimensional face τ of σ such that $\tau \not\subseteq f^*\mathcal{N}^1(Y)$. Let $D \subset X$ be a prime divisor with $[D] \in \tau$. Then D is not the pull-back of a $\mathbb{Q}$-Cartier prime divisor in Y. On the other hand, we also have $[D] \in \mathcal{N}_1(F, X)^{\perp}$, so that $D \cdot C = 0$ for every curve $C \subset F$. Since $F \not\subset D$, we must have $F \cap D = \emptyset$, hence $f(D) \subsetneq Y$.

(iii)⇒(ii) Let $D \subset X$ be a prime divisor which does not dominate Y. Let $B \subset Y$ be a prime divisor containing $f(D)$. Then B is $\mathbb{Q}$-Cartier, and D is an irreducible component of f^*B, hence $f^*B = \mu D$ with $\mu \in \mathbb{Q}_{>0}$. □

2B. *Special contractions.*

Definition 2.3 (special contraction). Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \to Y$ a contraction of fiber type. We say that f is special if for every prime divisor $D \subset X$ we have that either $f(D) = Y$, or $f(D)$ is a $\mathbb{Q}$-Cartier prime divisor in Y.

Remark 2.4. Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \to Y$ a contraction of fiber type. Then f is special if and only if the following conditions hold:

(1) $\operatorname{codim} f(D) \leq 1$ for every prime divisor $D \subset X$.

(2) Y is $\mathbb{Q}$-factorial.

Condition (1) above is not enough to ensure that Y is $\mathbb{Q}$-factorial, as the following simple example shows.

Example 2.5. Set $Z := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1))$, $X := Z \times \mathbb{P}^1$, and let $\pi\colon X \to Z$ be the projection. Then Z has a small elementary contraction $g\colon Z \to Y$, and $f := g \circ \pi\colon X \to Y$ satisfies (1) but not (2), in particular it is not special. Note that X is Fano and f is K-negative.

Remark 2.6. Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \to Y$ a contraction of fiber type:

(a) If X is a Mori dream space and f is elementary, or quasielementary, then f is special by Proposition 2.2.

(b) If f is special, then the locus where f is not equidimensional has codimension at least 3 in Y.

(c) Let f be special, and $\varphi\colon X \dashrightarrow X'$ a SQM such that $f' := f \circ \varphi^{-1}$ is regular. Then f' is special.

The following is a consequence of [Druel 2018, Lemma 2.6].

Lemma 2.7. *Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \to Y$ a contraction of fiber type. If f is equidimensional, then Y is $\mathbb{Q}$-factorial and f is special.*

Definition 2.8 (special rational contraction). Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \dashrightarrow Y$ a rational contraction of fiber type. We say that f is special if there exists a SQM $\varphi\colon X \dashrightarrow X'$ such that $f' := f \circ \varphi^{-1}$ is regular and special.

Remark 2.9. If $f\colon X \dashrightarrow Y$ is special, then:

- Y is $\mathbb{Q}$-factorial, by Remark 2.4.
- For every SQM $\varphi\colon X \dashrightarrow X'$ such that $f' := f \circ \varphi^{-1}$ is regular, we have that f' is special, by Remark 2.6(c)enumi.

In the next subsection we will prove the following characterization of special rational contractions of Mori dream spaces.

Proposition 2.10. *Let X be a Mori dream space and $f\colon X \dashrightarrow Y$ a rational contraction of fiber type. Then f is special if and only if f cannot be factored as*

$$\begin{tikzcd}
X \arrow[r, dashed, "g"'] \arrow[rr, dashed, bend left, "f"] & Z \arrow[r, dashed, "h"'] & Y
\end{tikzcd}$$

where g is a rational contraction, h is birational, and $\rho_Z > \rho_Y$.

2C. ***Factorizations.*** We start this subsection with a construction that will be used in the proofs of two factorization results, Proposition 2.13 and Theorem 2.15.

Construction 2.11. Let X be a Mori dream space, $f\colon X \to Y$ a contraction, and $D \subset X$ a prime divisor such that $f(D) \subsetneq Y$. Let us run a MMP for $-D$, relative to f (see Section 1A). We get a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, dashed, "\psi"] \arrow[d, "f"'] & W \arrow[d, "j"] \arrow[dl, "f_W"'] \\
Y & T \arrow[l, "k"]
\end{tikzcd} \tag{2.12}$$

where W is $\mathbb{Q}$-factorial, ψ is a composition of D-positive flips and divisorial contractions (in particular D cannot be exceptional for ψ, so it has a proper transform D_W in W), and $f_W := f \circ \psi^{-1}$ is regular. Since $f(D) \subsetneq Y$, the MMP cannot end with a fiber type contraction, and $-D_W$ is f_W-nef. Let $j\colon W \to T$ be the contraction given by $\mathrm{NE}(f_W) \cap D_W^{\perp}$, so that f_W factors as in (2.12); there exists a $\mathbb{Q}$-Cartier prime divisor D_T in T such that $D_W = \lambda j^* D_T$ for some $\lambda \in \mathbb{Q}_{>0}$, and $-D_T$ is k-ample. We have the following properties:

(a) k is birational, $\mathrm{Exc}(k) \subseteq D_T$, $f(D) = k(D_T)$.

(b) f, f_W, and j coincide in the open subset $X \smallsetminus f^{-1}(f(D))$.

(c) The divisorial irreducible components of $f^{-1}(f(D))$ are exactly D and the prime exceptional divisors of ψ.

Proof. By construction ψ is a composition of D-positive flips and divisorial contractions (relative to f), hence the images under f of the exceptional divisors of ψ are all contained in $f(D)$, so these divisors must be divisorial irreducible components of $f^{-1}(f(D))$. On the other hand $k^{-1}(k(D_T)) = D_T$, so $f_W^{-1}(f(D)) = j^{-1}(D_T) = D_W$ is irreducible. □

(d) $f^{-1}(f(D))$ has $\rho_X - \rho_W + 1$ divisorial irreducible components.

(e) k is an isomorphism if and only if $f(D)$ is a $\mathbb{Q}$-Cartier prime divisor in Y.

Proof. The "only if" direction is clear, because D_T is $\mathbb{Q}$-Cartier and $f(D) = k(D_T)$. For the other, suppose that $f(D)$ is a $\mathbb{Q}$-Cartier prime divisor in Y. Since $k^{-1}(f(D)) = k^{-1}(k(D_T)) = D_T$, we must have $k^*(f(D)) = \mu D_T$, with $\mu \in \mathbb{Q}_{>0}$. Then $-D_T$ is both k-trivial and k-ample, so that k must be an isomorphism. □

(f) $\operatorname{Exc}(k)$ is a prime divisor if and only if $\operatorname{codim} f(D) > 1$.

(g) k is not an isomorphism and $\operatorname{codim}\operatorname{Exc}(k) > 1$ if and only if $f(D)$ is a non-$\mathbb{Q}$-Cartier prime divisor.

Proposition 2.13. *Let X be a Mori dream space and $f\colon X \dashrightarrow Y$ a rational contraction of fiber type. Then f can be factored as follows*:

$$\begin{tikzcd}
X \arrow[r, dashed, "g"'] \arrow[rr, dashed, bend left, "f"] & Z \arrow[r, "h"'] & Y
\end{tikzcd}$$

where g is a special rational contraction, and h is birational. Moreover, such a factorization is unique up to composition with a SQM of Z.

Proof. To show existence of the factorization, we proceed by induction on $\rho_X - \rho_Y$.

If $\rho_X - \rho_Y = 1$, then f is elementary and hence special, so the statement holds with $g = f$ and $h = \mathrm{Id}_Y$.

For the general case, up to composing with a SQM of X, we can assume that f is regular. If f is special, then as before the statement holds with $g = f$. Otherwise, there exists a prime divisor D in X such that $f(D) \subsetneq Y$ and $f(D)$ is not a $\mathbb{Q}$-Cartier divisor in Y.

We apply Construction 2.11 to f and D. We get a diagram as (2.12), where k is not an isomorphism by (e), because $f(D)$ is not a $\mathbb{Q}$-Cartier divisor in Y; in particular $\rho_T > \rho_Y$.

The composition $\tilde{f} := j \circ \psi\colon X \dashrightarrow T$ is a rational contraction of fiber type with $\rho_X - \rho_T < \rho_X - \rho_Y$; by the induction assumption, $\tilde{f}$ can be factored as follows:

$$\begin{tikzcd}
X \arrow[r, dashed, "g"] \arrow[d, "f"'] \arrow[dr, dashed, "\tilde{f}"'] & Z \arrow[d, "\tilde{h}"] \\
Y & T \arrow[l, "k"]
\end{tikzcd}$$

where g is a special rational contraction of fiber type, and $\tilde{h}$ is birational. Then $h := k \circ \tilde{h}\colon Z \to Y$ is birational, so we have a factorization as in the statement.

To show uniqueness, suppose that f has another factorization $X \overset{g'}{\dashrightarrow} Z' \xrightarrow{h'} Y$ with g' special and h' birational; notice that both Z and Z' are $\mathbb{Q}$-factorial by Remark 2.9. We show that the birational map $\varphi := (h')^{-1} \circ h \colon Z \dashrightarrow Z'$ is a SQM.

Let $B \subset Z$ be a prime divisor. Up to composing g and g' with a SQM of X, we can assume that $g' \colon X \to Z'$ is regular. Let $D \subset X$ be a prime divisor dominating B under g; then $g'(D) \subsetneq Z'$, and since g' is special, $B' := g'(D)$ is a prime divisor in Z'. This means that φ does not contract B. Similarly, we see that φ^{-1} does not contract divisors, hence φ is a SQM. ☐

Proof of Proposition 2.10. Suppose that f is not special, and consider the factorization of f given by Proposition 2.13. Then h cannot be an isomorphism, thus $\rho_Z > \rho_Y$.

Conversely, suppose that f has a factorization as in the statement. By applying Proposition 2.13 to g, we get a factorization of f as follows:

$$\begin{tikzcd}
X \arrow[r, dashed, "g'"'] \arrow[rrr, dashed, bend left, "f"] & Z' \arrow[r, "h'"'] & Z \arrow[r, dashed, "h"'] & Y
\end{tikzcd}$$

where g' is special and h' is birational. Thus $h \circ h'$ is birational with $\rho_{Z'} > \rho_Y$; by the uniqueness part of Proposition 2.13, f is not special. ☐

Notation 2.14. Let X be a Mori dream space and $f \colon X \to Y$ a special contraction; recall that Y is $\mathbb{Q}$-factorial by Remark 2.4. If B is a prime divisor in Y, then every irreducible component of f^*B must dominate B. As the general fiber of f is irreducible, there are at most finitely many prime divisors in Y whose pullback to X is reducible. We fix the notation $B_1, \dots, B_m$ for these divisors in Y, where $m \in \mathbb{Z}_{\geq 0}$, and we denote by $r_i \in \mathbb{Z}_{\geq 2}$ the number of irreducible components of f^*B_i, for $i = 1, \dots, m$ (we ignore the multiplicities of these components, and ignore the possible prime divisors B such that f^*B is irreducible but nonreduced). Note that by Proposition 2.2, f is quasielementary if and only if $m = 0$.

Given a special rational contraction $f \colon X \dashrightarrow Y$, we will use the same notation $B_1, \dots, B_m$ and $r_1, \dots, r_m$, with the obvious meaning.

Theorem 2.15. *Let X be a Mori dream space and $f \colon X \to Y$ a special contraction; we use Notation 2.14. Let E be the union of (arbitrarily chosen) $r_i - 1$ components of f^*B_i, for $i = 1, \dots, m$. Then there is a factorization*

$$\begin{tikzcd}
X \arrow[r, dashed, "g"] \arrow[d, "f"'] & X' \arrow[dl, "f'"] \\
Y &
\end{tikzcd}$$

where X' is projective, normal, and $\mathbb{Q}$-factorial, g is birational with $\operatorname{Exc}(g) = E$,[1] *the general fiber of f is contained in the open subset where g is an isomorphism, and f' is quasielementary.*

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[1] We denote by $\operatorname{Exc}(g)$ the closure in X of the exceptional locus of g in its domain.

Proof. We proceed by induction on $\rho_X - \rho_Y$. If f is elementary, then it is quasielementary, so $E = \emptyset$ and the statement holds with $X' = X$ and $f' = f$.

Let us consider the general case. If f is quasielementary, then again the statement holds with $f' = f$.

Suppose that f is not quasielementary, so that $m \geq 1$ by Proposition 2.2, and consider the divisor $B_1 \subset Y$. Let D be the irreducible component of f^*B_1 not contained in E; we have $f(D) = B_1$ because f is special. We apply Construction 2.11 to f and D, and get a diagram:

$$\begin{tikzcd}
X \arrow[r, dashed, "\psi"] \arrow[d, "f"'] & W \arrow[dl, "f_W"] \\
Y &
\end{tikzcd}$$

where W is $\mathbb{Q}$-factorial, ψ is a sequence of D-positive flips and divisorial contractions, relative to f, and the general fiber of f is contained in the open subset where ψ is an isomorphism (by (b)). Moreover $f_W^*B_1$ is irreducible (by (e)), and the exceptional divisors of ψ are all the components of f^*B_1 except D (by (c)). In particular, $r_1 - 1 \geq 1$ elementary divisorial contractions occur in ψ, so $\rho_W < \rho_X$. Clearly f_W is still special, and we conclude by applying the induction assumption to f_W. □

In particular, given a special contraction $f: X \to Y$ with general fiber F, one can bound ρ_X in terms of ρ_Y, ρ_F, and the number of irreducible components of f^*B_i, $i = 1, \dots, m$.

Corollary 2.16. *Let X be a Mori dream space, $f: X \to Y$ a special contraction, and $F \subset X$ a general fiber of f. We use Notation 2.14. Then*

$$\rho_X = \rho_Y + \dim \mathcal{N}_1(F, X) + \sum_{i=1}^{m}(r_i - 1) \leq \rho_Y + \rho_F + \sum_{i=1}^{m}(r_i - 1).$$

For the proof of Corollary 2.16 we need the following simple property.

Lemma 2.17. *Let $\varphi: X \dashrightarrow X'$ be a birational map between normal and $\mathbb{Q}$-factorial projective varieties. Let $T \subset X$ be a closed subset contained in the open subset where φ is an isomorphism, and set $T' := \varphi(T) \subset X'$. Then* $\dim \mathcal{N}_1(T, X) = \dim \mathcal{N}_1(T', X')$.

Proof. We note that $\mathcal{N}_1(T, X)$ is the quotient of the vector space of real 1-cycles in T by the subspace of 1-cycles γ such that $\gamma \cdot D = 0$ for every divisor D in X, so it is determined by the image of the restriction map $\mathcal{N}^1(X) \to \mathcal{N}^1(T)$, and similarly for $\mathcal{N}_1(T', X')$. Since X and X' are $\mathbb{Q}$-factorial, and T is contained in the open subset where φ is an isomorphism, it is easy to see that the images of the maps $\mathcal{N}^1(X) \to \mathcal{N}^1(T)$ and $\mathcal{N}^1(X') \to \mathcal{N}^1(T')$ are the same, under the natural isomorphism $\mathcal{N}^1(T) \cong \mathcal{N}^1(T')$. □

Proof of Corollary 2.16. Let us consider the factorization of f given by Theorem 2.15. The difference $\rho_X - \rho_{X'}$ is the number of prime exceptional divisors of g, namely $\sum_{i=1}^{m}(r_i - 1)$. Moreover F is contained in the open subset where g is an isomorphism, $g(F) \subset X'$ is a general fiber of f', and

$\dim \mathcal{N}_1(F, X) = \dim \mathcal{N}_1(g(F), X')$ by Lemma 2.17. Finally, since f' is quasielementary, we have $\rho_{X'} = \rho_Y + \dim \mathcal{N}_1(g(F), X')$. This yields the statement. ☐

Corollary 2.18. *Let X be a Mori dream space and $f\colon X \to Y$ a special contraction; we use Notation 2.14. Then every prime divisor in f^*B_i is a fixed divisor, for $i = 1, \dots, m$.*

Moreover, let E be the union of (arbitrarily chosen) $r_i - 1$ components of f^*B_i, for $i = 1, \dots, m$. Then the classes of the components of E in $\mathcal{N}^1(X)$ generate a simplicial face σ of $\operatorname{Eff}(X)$, and $\sigma \cap \operatorname{Mov}(X) = \{0\}$.

Proof. Theorem 2.15 implies the existence of a contracting birational map $g\colon X \dashrightarrow X'$, with X' $\mathbb{Q}$-factorial, whose prime exceptional divisors are precisely the components of E. This gives the statement (see for instance [Okawa 2016, Lemma 2.7]). ☐

We will also need the following technical property.

Lemma 2.19. *Let X be a Mori dream space and $f\colon X \dashrightarrow Y$ a special rational contraction; we use Notation 2.14. Let E_0 be an irreducible component of f^*B_i for some $i \in \{1, \dots, m\}$. Then there is a factorization of f:*

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "f"'] & \hat{X} \arrow[d, "\sigma"] \\
Y & Z \arrow[l]
\end{tikzcd}$$

where φ is a SQM, σ is an elementary divisorial contraction, $\operatorname{Exc}(\sigma)$ *is the transform of E_0, and* $\dim \sigma(\operatorname{Exc} \sigma) \geq \dim Y - 1$.

Proof. Let us choose a SQM $\psi\colon X \dashrightarrow X'$ such that $f' := f \circ \psi^{-1}\colon X' \to Y$ is regular.

We still denote by E_0 the transform of E_0 in X'; by Corollary 2.18, E_0 is a fixed divisor, and it is easy to see that it cannot be f'-nef. We run a MMP in X' for E_0, relative to f', and get a diagram:

$$\begin{tikzcd}
X \arrow[r, dashed, "\psi"] \arrow[dr, dashed, "f"'] & X' \arrow[r, dashed, "\xi"] \arrow[d, "f'"] & \hat{X} \arrow[d, "\sigma"] \\
& Y & Z \arrow[l, "h"']
\end{tikzcd}$$

where ξ is a sequence of E_0-negative flips, and σ is an elementary divisorial contraction with exceptional divisor (the transform of) E_0.

Now $h \circ \sigma\colon \hat{X} \to Y$ is a special contraction, therefore $h(\sigma(\operatorname{Exc}(\sigma)))$ is a divisor in Y, and $\dim \sigma(\operatorname{Exc}(\sigma)) \geq \dim Y - 1$. ☐

2D. *Singularities of the target.*

The goal of this subsection is to prove the following result.

Proposition 2.20. *Let X be a smooth projective variety, and Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $f\colon X \to Y$ be a $(K+\Delta)$-negative special contraction of fiber type. Then Y has locally factorial, canonical singularities, and is nonsingular in codimension* 2.

Proposition 2.20 will follow from some technical lemmas.

Lemma 2.21. *Let X be a projective variety with locally factorial, canonical singularities, and Δ a boundary such that (X, Δ) is klt. Let $f\colon X \to Y$ be a $(K+\Delta)$-negative special contraction of fiber type. Then Y has locally factorial, canonical singularities.*

Proof. It follows from [Fujino 1999, Corollary 4.5] that Y has rational singularities, so it is enough to show that it is locally factorial [Kollár and Mori 1998, Corollary 5.24].

Let B be a prime divisor in Y. Since Y is $\mathbb{Q}$-factorial, there exists $m \in \mathbb{Z}_{>0}$ such that mB is Cartier.

Set $U := f^{-1}(Y_{\mathrm{reg}})$; since Y is normal and f is special, we have

$$\operatorname{codim} \operatorname{Sing}(Y) \geq 2 \quad \text{and} \quad \operatorname{codim}(X \setminus U) \geq 2.$$

Then $B \cap Y_{\mathrm{reg}}$ is a Cartier divisor on Y_{reg}, and $f_{|U}^{*}(B \cap Y_{\mathrm{reg}})$ is a Cartier divisor on U. Since X is locally factorial, there exists a Cartier divisor D in X such that $D_{|U} = f_{|U}^{*}(B \cap Y_{\mathrm{reg}})$. Then $(mD)_{|U} = f_{|U}^{*}((mB)_{|Y_{\mathrm{reg}}}) = f^{*}(mB)_{|U}$, and hence $mD = f^{*}(mB)$.

We deduce that $D \cdot C = 0$ for every curve $C \subset X$ contracted by f. Since f is $(K+\Delta)$-negative, this implies that there exists a Cartier divisor B' on Y such that $D = f^{*}B'$ [Kollár and Mori 1998, Theorem 3.7(4)]. Thus we have $B'_{|Y_{\mathrm{reg}}} = B \cap Y_{\mathrm{reg}}$, and hence $B = B'$ is Cartier. ☐

The following two lemmas are basically [Andreatta et al. 1992, Proposition 1.4 and 1.4.1], where they are attributed to Fujita.

Lemma 2.22. *Let X be a smooth projective variety, and Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $f\colon X \to Y$ be an equidimensional, $(K+\Delta)$-negative contraction of fiber type. If Y has at most finite quotient singularities, then Y is smooth.*

Proof. Let $F \subset X$ be a general fiber of f. Then F is smooth and $(F, \Delta_{|F})$ is klt [Kollár and Mori 1998, Lemma 5.17]; moreover $-(K_F + \Delta_{|F}) \equiv -(K_X + \Delta)_{|F}$ is ample, so that $(F, \Delta_{|F})$ is log Fano. By Kawamata–Viehweg vanishing, $h^i(F, \mathcal{O}_F) = 0$ for every $i > 0$, hence $\chi(F, \mathcal{O}_F) = 1$. Then the same proof as [Andreatta et al. 1992, Proposition 1.4] applies. ☐

Lemma 2.23. *Let X be a smooth projective variety with* $\dim X \geq 3$, *and Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt. Let $f\colon X \to S$ be an equidimensional, $(K+\Delta)$-negative contraction onto a surface. Then S is smooth.*

Proof. Notice first of all that S is $\mathbb{Q}$-factorial by Lemma 2.7. Moreover, by [Fujino 1999, Corollary 4.5], there exists $\mathbb{Q}$-divisor Δ' on S such that (S, Δ') is klt; in particular S has log terminal singularities, and hence finite quotient singularities [Kollár and Mori 1998, Proposition 4.18]. Then S is smooth by Lemma 2.22. ☐

Lemma 2.24. *Let X be a smooth projective variety, Δ a $\mathbb{Q}$-divisor on X such that (X, Δ) is klt, and $f\colon X \to Y$ a $(K+\Delta)$-negative contraction of fiber type.*

Suppose that the locus where f is not equidimensional has codimension at least 3 *in Y, equivalently that there is no prime divisor $D \subset X$ such that* $\operatorname{codim} f(D) = 2$.

Then Y is smooth in codimension 2.

Proof. Set $m = \dim Y$ and let $H_1, \dots, H_{m-2}$ be general very ample divisors in Y. Consider $S := H_1 \cap \cdots \cap H_{m-2}$ and $Z := f^{-1}(S) = f^*H_1 \cap \cdots f^*H_{m-2}$. Then S is a normal projective surface, Z is smooth, and f is equidimensional over S, so that $f_Z := f_{|Z} \colon Z \to S$ is an equidimensional contraction. Moreover $(Z, \Delta_{|Z})$ is klt [Kollár and Mori 1998, Lemma 5.17].

Let $C \subset Z$ be a curve contracted by f; then $f^*H_i \cdot C = 0$ for every i, so that by adjunction

$$(K_Z + \Delta_{|Z}) \cdot C = (K_X + \Delta) \cdot C < 0,$$

and f_Z is $(K_Z + \Delta_{|Z})$-negative. Thus S is smooth by Lemma 2.23, so $S \subseteq Y_{\mathrm{reg}}$ and hence $\operatorname{codim} \operatorname{Sing} Y \geq 3$. □

Proposition 2.20 follows from Lemma 2.21, Remark 2.6(b)enumi, and Lemma 2.24.

3. Special contractions of Fano varieties of relative dimension 1

3A. ***Preliminaries on the Lefschetz defect.*** Let X be a normal and $\mathbb{Q}$-factorial Fano variety. The *Lefschetz defect* δ_X is an invariant of X, introduced in [Casagrande 2012], and defined as follows:

$$\delta_X = \max\{\operatorname{codim} \mathcal{N}_1(D, X) \mid D \text{ a prime divisor in } X\}$$

(see Section 1A for $\mathcal{N}_1(D, X)$). The main properties of δ_X are the following.

Theorem 3.1 [Casagrande 2012; Della Noce 2014]. *Let X be a $\mathbb{Q}$-factorial, Gorenstein Fano variety, with canonical singularities and at most finitely many nonterminal points. Then $\delta_X \leq 8$.*

If moreover X is smooth and $\delta_X \geq 4$, then $X \cong S \times Y$, where S is a surface.

Theorem 3.2 [Casagrande 2012, Corollary 1.3; 2013b, Theorem 1.2]. *Let X be a smooth Fano 4-fold. Then one of the following holds*:

(i) *X is a product of surfaces.*

(ii) $\delta_X = 3$ *and* $\rho_X \leq 6$.

(iii) $\delta_X = 2$ *and* $\rho_X \leq 12$.

(iv) $\delta_X \leq 1$.

3B. ***The case of relative dimension one.*** In this subsection we show that if X is a Fano variety and $f \colon X \to Y$ is a special contraction with $\dim Y = \dim X - 1$, then $\rho_X - \rho_Y \leq 9$; this is a generalization of an analogous result in [Romano 2019] in the case where f is a conic bundle. The strategy of proof is the same: we use f to produce $\rho_X - \rho_Y - 1$ pairwise disjoint divisors in X, and then we use them to show that if $\rho_X - \rho_Y \geq 3$, then $\delta_X \geq \rho_X - \rho_Y - 1$; finally we apply Theorem 3.1.

Proposition 3.3. *Let X be a $\mathbb{Q}$-factorial, Gorenstein Fano variety, with canonical singularities and at most finitely many nonterminal points. Let $f\colon X \to Y$ be a special contraction with* $\dim Y = \dim X - 1$. *Then the following hold*:

(a) $\rho_X - \rho_Y \leq 9$.

(b) *If* $\rho_X - \rho_Y \geq 3$, *then* $\delta_X \geq \rho_X - \rho_Y - 1$.

If moreover X is smooth and $\rho_X - \rho_Y \geq 5$, then there exists a surface S such that $X \cong S \times Z$, $Y \cong \mathbb{P}^1 \times Z$, and f is induced by a conic bundle $S \to \mathbb{P}^1$.

For the proof of Proposition 3.3 we need some technical lemmas, that will be used also in Section 6.

Lemma 3.4. *Let X be a Mori dream space, and suppose that K_X is Cartier in codimension* 2, *namely that there exists a closed subset $T \subset X$ such that* $\operatorname{codim} T \geq 3$ *and $K_{X \smallsetminus T}$ is Cartier.*

Let $f\colon X \to Y$ be a K-negative special contraction with $\dim Y = \dim X - 1$; *we use Notation 2.14. Then $\rho_X = \rho_Y + 1 + m$ and $r_i = 2$ for every $i = 1, \dots, m$.*

Let moreover E_i, $\hat{E}_i$ be the irreducible components of f^*B_i. Then the general fiber of f over B_i is $e_i + \hat{e}_i$, where e_i and $\hat{e}_i$ are integral curves with $E_i \cdot e_i < 0$, $\hat{E}_i \cdot \hat{e}_i < 0$, and $-K_X \cdot e_i = -K_X \cdot \hat{e}_1 = 1$.

Proof. Fix $i \in \{1, \dots, m\}$. The closed subset T cannot dominate B_i, hence the general fiber of f over B_i is a curve F_i contained in $X \setminus T$ where K_X is Cartier. Since $-K_X \cdot F_i = 2$, and f is K-negative, F_i has at most two irreducible components. This implies that $r_i = 2$ and $F_i = e_i + \hat{e}_i$, with $e_i \subset E_i$, $\hat{e}_i \subset \hat{E}_i$, and conversely $e_i \not\subset \hat{E}_i$, $\hat{e}_i \not\subset E_i$. The fiber F_i is connected, hence we have $E_i \cap \hat{e}_i \neq \varnothing$, and therefore $E_i \cdot \hat{e}_i > 0$. Since $E_i \cdot F_i = 0$, we get $E_i \cdot e_i < 0$; similarly for $\hat{E}_i$. Finally $\rho_X = \rho_Y + 1 + m$ by Corollary 2.16. ☐

Lemma 3.5. *In the setting of Lemma 3.4, if moreover* $\operatorname{codim} T \geq 4$, *then $B_1, \dots, B_m$ are pairwise disjoint.*

Proof. By contradiction, suppose that $B_1 \cap B_2 \neq \varnothing$. Then $B_1 \cap B_2$ has pure dimension $\dim X - 3$, because Y is $\mathbb{Q}$-factorial (see Remark 2.4); let W be an irreducible component. Since f is special, the general fiber F_W of f over W is a curve. Moreover, F_W is contained in the open subset where K_X is Cartier, so that $F_W = C + C'$ with C and C' integral curves of anticanonical degree 1.

By Lemma 3.4, for $i = 1, 2$ the general fiber F_i of f over B_i is $e_i + \hat{e}_i$, with $-K_X \cdot e_i = 1$, and F_i degenerates to F_W. Thus, up to switching the components, we can assume that both e_1 and e_2 are numerically equivalent to C, which implies that $e_1 \equiv e_2$. This is impossible, because $E_1 \neq E_2$, $E_i \cdot e_i < 0$, and e_i moves in a family of curves dominating E_i, for $i = 1, 2$. ☐

Proof of Proposition 3.3. This the same as the proof of [Romano 2019, Theorem 1.1 and 1.3], so we give only a sketch. We have $\rho_X = \rho_Y + 1 + m$ by Lemma 3.4. As in [loc. cit., Lemmas 3.9 and 3.10], using Lemma 3.5, one sees that if $m \geq 2$, then $\delta_X \geq m$. Hence the statement follows from Theorem 3.1. ☐

4. Preliminary results on Fano 4-folds

From now on, we focus on smooth Fano 4-folds. After giving in Section 4A some preliminary results on rational contractions of Fano 4-folds, in Section 4B we recall the classification of fixed prime divisors in a Fano 4-fold X with $\rho_X \geq 7$, and report some properties that will be crucial in the sequel. Then in Section 4C we apply the previous results to study special rational contractions of fiber type of X, when $\rho_X \geq 7$.

4A. *Rational contractions of Fano 4-folds.*

Lemma 4.1 [Casagrande 2013a, Remark 3.6 and its proof]. *Let X be a smooth Fano 4-fold and $\varphi\colon X \dashrightarrow \tilde{X}$ an SQM*:

(a) *$\tilde{X}$ is smooth, the indeterminacy locus of φ is a disjoint union of exceptional planes (see Section 1A), and the indeterminacy locus of φ^{-1} is a disjoint union of exceptional lines.*

(b) *An exceptional line in $\tilde{X}$ cannot meet any integral curve of anticanonical degree* 1, *in particular it cannot meet an exceptional plane.*

(c) *Let $\psi\colon \tilde{X} \dashrightarrow \hat{X}$ be a SQM that factors as a sequence of K-negative flips. Then the indeterminacy locus of ψ (respectively, ψ^{-1}) is a disjoint union of exceptional planes (respectively, lines).*

Lemma 4.2 [Casagrande 2013a, Remark 3.7]. *Let X be a smooth Fano 4-fold and $f\colon X \dashrightarrow Y$ a rational contraction. Then one can factor f as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} Y$, where φ is a SQM, X' is smooth, and f' is a K-negative contraction.*

These results allow to conclude that the target of a special rational contraction of a Fano 4-fold has mild singularities.

Lemma 4.3. *Let X be a smooth Fano 4-fold and $f\colon X \dashrightarrow Y$ a special rational contraction. If* $\dim Y = 2$, *then Y is smooth. If* $\dim Y = 3$, *then Y has isolated locally factorial, canonical singularities.*

Proof. By Lemma 4.2 we can factor f as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} Y$ where φ is a SQM, X' is smooth, and f' is regular, K-negative, and special. Then the statement follows from Proposition 2.20. ☐

4B. *Fixed prime divisors in Fano 4-folds with $\rho \geq 7$.* Let X be a Fano 4-fold with $\rho_X \geq 7$. Fixed prime divisors in X have been classified in [Casagrande 2013a; 2017] in four types, and have many properties; this explicit information on the geometry of fixed divisors is a key ingredient in the proof of Theorem 1.1. In this subsection we recall this classification, and show some properties that will be used in the sequel.

Theorem–Definition 4.4 [Casagrande 2017, Theorem 5.1, Definition 5.3, Corollary 5.26, Definition 5.27]. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, and D a fixed prime divisor in X. The following hold*:

(a) *Given a SQM $X \dashrightarrow X'$ and an elementary divisorial contraction $k\colon X' \to Y$ with* $\mathrm{Exc}(k)$ *the transform of D, then k is of type* $(3,0)^{\mathrm{sm}}$, $(3,0)^{Q}$, $(3,1)^{\mathrm{sm}}$, *or* $(3,2)$.

(b) *The type of k depends only on D, so we define D to be of type* $(3,0)^{\mathrm{sm}}$, $(3,0)^{Q}$, $(3,1)^{\mathrm{sm}}$, *or* $(3,2)$, *respectively.*

(c) *If D is of type* $(3,2)$, *then D is the exceptional divisor of an elementary divisorial contraction of X, of type* $(3,2)$.

(d) *We define $C_D \subset D \subset X$ to be the transform of a general irreducible curve $\Gamma \subset X'$ contracted by k, of minimal anticanonical degree*; *the curve C_D depends only on D.*

(e) $C_D \cong \mathbb{P}^1$, $D \cdot C_D = -1$, *C_D is contained in the open subset where the birational map $X \dashrightarrow X'$ is an isomorphism, and C_D moves in a family of curves dominating D.*

(f) *Let $\varphi\colon X \dashrightarrow \tilde{X}$ be a SQM, and E a fixed prime divisor in $\tilde{X}$. We define the type of E to be the type of its transform in X.*

We will frequently use the notation $C_D \subset D$ introduced in the Theorem–Definition above.

The next property of fixed divisors of type $(3,2)$ will be crucial in the sequel.

Lemma 4.5. *Let X be a smooth Fano* 4*-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, $X \dashrightarrow \tilde{X}$ a SQM, and $D \subset \tilde{X}$ a fixed divisor of type* $(3,2)$. *If $\mathcal{N}_1(D,\tilde{X}) \subsetneq \mathcal{N}_1(\tilde{X})$, then either $\rho_X \leq 12$, or X is a product of surfaces.*

Proof. If $\delta_X \geq 2$, we have the statement by Theorem 3.2, so let us assume that $\delta_X \leq 1$. Let D_X be the transform of D in X, so that D_X is the exceptional divisor of an elementary divisorial contraction of X, of type $(3,2)$. By [Casagrande 2017, Remark 2.17(2)], D_X cannot contain exceptional planes, hence $\dim \mathcal{N}_1(D_X, X) = \dim \mathcal{N}_1(D,\tilde{X})$ by [Casagrande 2013a, Corollary 3.14]. Then $\rho_X \leq 12$ by [Casagrande 2017, Proposition 5.32]. ☐

Lemma 4.6. *Let X be a smooth Fano* 4*-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, and let $D_1, D_2 \subset X$ be two distinct fixed prime divisors. We have the following*:

(a) *First*

$$\dim\langle[D_1],[D_2]\rangle \cap \mathrm{Mov}(X) = \dim\langle[C_{D_1}],[C_{D_2}]\rangle \cap \mathrm{mov}(X) = \begin{cases} 0 & \text{if } D_1 \cdot C_{D_2} = 0 \text{ or } D_2 \cdot C_{D_1} = 0, \\ 1 & \text{if } D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1, \\ 2 & \text{if } (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) \geq 2. \end{cases}$$

(b) *If $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1$, then*

$$\langle[D_1],[D_2]\rangle \cap \mathrm{Mov}(X) = \langle[D_1 + D_2]\rangle \quad \text{and} \quad \langle[C_{D_1}],[C_{D_2}]\rangle \cap \mathrm{mov}(X) = \langle[C_{D_1} + C_{D_2}]\rangle.$$

Moreover $(D_1 + D_2) \cdot (C_{D_1} + C_{D_2}) = 0$ and $D_1 + D_2$ is not big.

(c) *If $D_1 \cdot C_{D_2} = 0$ or $D_2 \cdot C_{D_1} = 0$, then $\langle[D_1],[D_2]\rangle$ is a face of* $\mathrm{Eff}(X)$, *and $\langle[C_{D_1}],[C_{D_2}]\rangle$ is a face of* $\mathrm{Mov}(X)^{\vee}$.

For the proof, we need the following elementary property in convex geometry.

Lemma 4.7. *Let σ be a convex polyhedral cone, of maximal dimension, in a finite dimensional real vector space $\mathcal{N}$. Let τ_1 be a one-dimensional face of σ, and let $\alpha \in \mathcal{N}^*$ (the dual vector space) be such that $\alpha \cdot \tau_1 < 0$ and $\alpha \cdot \eta \geq 0$ for every one-dimensional face $\eta \neq \tau_1$ of σ.*

If τ_2 is a one-dimensional face of σ such that $\alpha \cdot \tau_2 = 0$, then $\tau_1 + \tau_2$ is a face of σ.

Proof. Since τ_2 is a face of σ, there exists $\beta \in \mathcal{N}^*$ such that $\beta \cdot x \geq 0$ for every $x \in \sigma$, and $\beta^{\perp} \cap \sigma = \tau_2$. Let $y \in \tau_1$ be a nonzero element, and set $a := \alpha \cdot y$ and $b := \beta \cdot y$. Then $a, b \in \mathbb{R}$, $a < 0$, and $b > 0$ (because $\tau_2 \neq \tau_1$ by our assumptions). Let us consider $\gamma := b\alpha + |a|\beta \in \mathcal{N}^*$.

We have $\alpha \cdot \tau_2 = \beta \cdot \tau_2 = 0$, hence $\gamma \cdot \tau_2 = 0$. Moreover $\gamma \cdot y = b\alpha \cdot y + |a|\beta \cdot y = 0$, namely $\gamma \cdot \tau_1 = 0$. Finally if η is a one-dimensional face of σ, different from τ_1 and τ_2, we have $\alpha \cdot \eta \geq 0$, $\beta \cdot \eta > 0$, and hence $\gamma \cdot \eta > 0$.

Therefore $\gamma \cdot x \geq 0$ for every $x \in \sigma$, and $\gamma^{\perp} \cap \sigma = \tau_1 + \tau_2$. This shows the statement. □

Proof of Lemma 4.6. We compute $\langle [D_1], [D_2] \rangle \cap \operatorname{Mov}(X)$. Set $B := \lambda_1 D_1 + \lambda_2 D_2$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ for $i = 1, 2$. By [Casagrande 2017, Lemma 5.29(2)], B is movable if and only if $B \cdot C_D \geq 0$ for every fixed prime divisor $D \subset X$, and this is equivalent to $B \cdot C_{D_i} \geq 0$ for $i = 1, 2$, namely to

$$\begin{cases} -\lambda_1 + \lambda_2 D_2 \cdot C_{D_1} \geq 0 \\ \lambda_1 D_1 \cdot C_{D_2} - \lambda_2 \geq 0. \end{cases} \tag{4.8}$$

Let $\mathcal{S} \subseteq (\mathbb{R}_{\geq 0})^2$ be the set of nonnegative solutions (λ_1, λ_2) of (4.8), so that $\mathcal{S}$ determines the intersection $\langle [D_1], [D_2] \rangle \cap \operatorname{Mov}(X)$. Notice that $(D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1})$ is always nonnegative, because $D_1 \neq D_2$. It is elementary to check that:

- $\mathcal{S} = \{(0, 0)\} \Leftrightarrow 1 - (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) > 0 \Leftrightarrow D_1 \cdot C_{D_2} = 0$ or $D_2 \cdot C_{D_1} = 0$.
- $\mathcal{S}$ is a half-line $\Leftrightarrow 1 - (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) = 0 \Leftrightarrow D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 1$, moreover in this case $\mathcal{S} = \{(\lambda, \lambda) \mid \lambda \geq 0\}$.
- $\mathcal{S}$ is a 2-dimensional cone $\Leftrightarrow 1 - (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) < 0 \Leftrightarrow (D_1 \cdot C_{D_2})(D_2 \cdot C_{D_1}) \geq 2$.

Similarly, we compute $\langle [C_{D_1}], [C_{D_2}] \rangle \cap \operatorname{mov}(X)$. We have

$$\operatorname{mov}(X)^{\vee} = \operatorname{Eff}(X) = \langle [D] \rangle_{D \text{ fixed}} + \operatorname{Mov}(X).$$

Set $\gamma := \lambda_1 C_{D_1} + \lambda_2 C_{D_2}$ with $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. We have $\gamma \cdot M \geq 0$ for every movable divisor M in X (see [Casagrande 2017, Lemma 5.29(2)]). Hence $\gamma \in \operatorname{mov}(X)$ if and only if $\gamma \cdot D \geq 0$ for every fixed prime divisor $D \subset X$, and this is equivalent to $\gamma \cdot D_i \geq 0$ for $i = 1, 2$, namely to

$$\begin{cases} -\lambda_1 + \lambda_2 D_1 \cdot C_{D_2} 0 \\ \lambda_1 D_2 \cdot C_{D_1} - \lambda_2 0, \end{cases}$$

which is the same system as (4.8), but with λ_1 and λ_2 interchanged. Thus the previous discussion yields (a) and (b).

We show (c). Suppose for instance that $D_1 \cdot C_{D_2} = 0$. To see that $\langle [D_1], [D_2] \rangle$ is a face of $\mathrm{Eff}(X)$, we apply Lemma 4.7 with $\sigma = \mathrm{Eff}(X)$, $\tau_1 = \langle [D_2] \rangle$, $\alpha = [C_{D_2}]$, and $\tau_2 = \langle [D_1] \rangle$. It is enough to remark that $D \cdot C_{D_2} \geq 0$ for every prime divisor $D \neq D_2$.

Similarly, to see that $\langle [C_{D_1}], [C_{D_2}] \rangle$ is a face of $\mathrm{Mov}(X)^\vee$, we apply Lemma 4.7 with $\sigma = \mathrm{Mov}(X)^\vee$, $\tau_1 = \langle [C_{D_1}] \rangle$, $\alpha = [D_1]$, and $\tau_2 = \langle [C_{D_2}] \rangle$. Indeed $\langle [C_{D_1}] \rangle$ and $\langle [C_{D_2}] \rangle$ are one-dimensional faces of $\mathrm{Mov}(X)^\vee$ by [Casagrande 2017, Lemma 5.29(1)]. Moreover $D_1 \cdot \gamma \geq 0$ for every $\gamma \in \mathrm{mov}(X)$, and $D_1 \cdot C_D \geq 0$ for every fixed prime divisor $D \neq D_1$. By [loc. cit., Lemma 5.29(2)] we have

$$\mathrm{Mov}(X)^\vee = \langle [C_D] \rangle_{D \,\mathrm{fixed}} + \mathrm{mov}(X),$$

therefore $D_1 \cdot \eta \geq 0$ for every one-dimensional face η of $\mathrm{Mov}(X)^\vee$ different from $\langle [C_{D_1}] \rangle$. Thus the hypotheses of Lemma 4.7 are satisfied, and we get (c). □

Lemma 4.9. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, and let $D_1, D_2 \subset X$ be two distinct fixed prime divisors such that $\langle [D_1], [D_2] \rangle \cap \mathrm{Mov}(X) = \{0\}$. Then, up to exchanging D_1 and D_2, one of the following holds*:

(a) $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 0$ *and* $D_1 \cap D_2 = \varnothing$.

(b) $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 0$ *and* $D_1 \cap D_2$ *is a disjoint union of exceptional planes.*

(c) $D_1 \cdot C_{D_2} = D_2 \cdot C_{D_1} = 0$, D_1 *is of type* $(3, 2)$, *and* D_2 *is not of type* $(3, 0)^{\mathrm{sm}}$.

(d) $D_1 \cdot C_{D_2} > 0$, $D_2 \cdot C_{D_1} = 0$, D_1 *is of type* $(3, 2)$, *and* D_2 *is of type* $(3, 1)^{\mathrm{sm}}$ *or* $(3, 0)^Q$.

Proof. By [Casagrande 2017, Theorem 5.1] there is a diagram

$$X \dashrightarrow \tilde{X} \xrightarrow{f} Y$$

where the first map is a SQM and f is an elementary divisorial contraction with exceptional divisor the transform $\tilde{D}_2 \subset \tilde{X}$ of D_2. Let $\tilde{D}_1 \subset \tilde{X}$ be the transform of D_1. By [loc. cit., Lemma 2.21], D_1 is the transform of a fixed prime divisor $B_1 \subset Y$.

If $\tilde{D}_1 \cap \tilde{D}_2 = \varnothing$, then $D_1 \cap D_2$ is contained in the indeterminacy locus of the map $X \dashrightarrow \tilde{X}$, which is a disjoint union of exceptional planes by Lemma 4.1(a). Therefore either $D_1 \cap D_2 = \varnothing$ and we get (a), or $D_1 \cap D_2$ has pure dimension 2 and we get (b).

We assume from now on that $\tilde{D}_1 \cap \tilde{D}_2 \neq \varnothing$.

Suppose that D_2 is of type $(3, 1)^{\mathrm{sm}}$. Then Y is a smooth Fano 4-fold by [Casagrande 2017, Theorem 5.1], f is the blow-up of a smooth curve $C \subset Y$, and $B_1 \cap C \neq \varnothing$. Then [loc. cit., Lemma 5.11] yields that B_1 is the exceptional divisor of an elementary divisorial contraction of type $(3, 2)$, and either $B_1 \cdot C > 0$, or $B_1 \cdot C < 0$. Thus B_1 is generically a $\mathbb{P}^1$-bundle over a surface, and the general fiber F of this $\mathbb{P}^1$-bundle satisfies $B_1 \cdot F = K_Y \cdot F = -1$. Using Lemma 4.1(a) and [loc. cit., Lemma 2.18], one sees that D_1 must be of type $(3, 2)$. Moreover $C \cap F = \varnothing$ implies that $\tilde{D}_2$ is disjoint from the transform $\tilde{F}$ of F in $\tilde{X}$, and $\tilde{D}_1$ is still generically a $\mathbb{P}^1$-bundle with fiber $\tilde{F}$. The indeterminacy locus of the map $\tilde{X} \dashrightarrow X$ has dimension at most one (see Lemma 4.1(a)), hence $\tilde{F}$ is contained in the open subset where this map is an isomorphism, and in X we get $D_2 \cdot C_{D_1} = \tilde{D}_2 \cdot \tilde{F} = 0$. Finally it is easy to check that $D_1 \cdot C_{D_2} = 0$ if $B_1 \cdot C > 0$ (and we have (c)), while $D_1 \cdot C_{D_2} > 0$ if $B_1 \cdot C < 0$ (and we have (d)). So we get the statement.

We can assume now that neither D_1 nor D_2 are of type $(3, 1)^{\mathrm{sm}}$. Suppose that D_2 is of type $(3, 0)^{\mathrm{sm}}$ or $(3, 0)^{Q}$. Then $\tilde{D}_2$ is isomorphic to $\mathbb{P}^3$ or to an irreducible quadric; let $\Gamma \subset \tilde{D}_2$ be a curve corresponding to a line. We have $\tilde{D}_1 \cdot \Gamma > 0$, and since Γ is contained in the open subset where the map $\tilde{X} \dashrightarrow X$ is an isomorphism (see Theorem–Definition 4.4(e)), we also have $D_1 \cdot C_{D_2} > 0$. This yields $D_2 \cdot C_{D_1} = 0$ by Lemma 4.6. Therefore D_1 cannot be of type $(3, 0)^{\mathrm{sm}}$ nor $(3, 0)^{Q}$, and the only possibility is that D_1 is of type $(3, 2)$. Moreover, since $f(\tilde{D}_2)$ is contained in B_1, [Casagrande 2017, Lemma 5.41] yields that D_2 cannot be of type $(3, 0)^{\mathrm{sm}}$, so we get again (d).

We are left with the case where both D_1 and D_2 are of type $(3, 2)$, and we can assume that $D_1 \cdot C_{D_2} = 0$ by Lemma 4.6. If $\delta_X \geq 3$, then Theorem 3.2 implies that X is a product of surfaces; in this case it is easy to check directly that $D_2 \cdot C_{D_1} = 0$. If $\delta_X \leq 2$, then we get $D_2 \cdot C_{D_1} = 0$ by [Casagrande 2013b, Lemma 2.2(b)]. So we have (c). ☐

4C. ***Special rational contractions of Fano 4-folds with $\rho_X \geq 7$.*** Given a Fano 4-fold X with $\rho_X \geq 7$, and a special rational contraction of fiber type $f \colon X \dashrightarrow Y$, in this subsection we show that, for every prime divisor B of Y, f^*B has at most two irreducible components. Moreover we give conditions on the type of the fixed prime divisors in f^*B, when f^*B is reducible.

Lemma 4.10. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, or $\rho_X = 6$ and $\delta_X \leq 2$, and $f \colon X \dashrightarrow Y$ a special rational contraction; we use Notation 2.14. Let $i \in \{1, \dots, m\}$:*

- *If $\dim Y = 3$, then every fixed divisor in f^*B_i is of type $(3, 2)$.*
- *If $\dim Y = 2$, then every fixed divisor in f^*B_i is of type $(3, 2)$ or $(3, 1)^{\mathrm{sm}}$.*

Proof. Let E_0 be an irreducible component of f^*B_i. By Lemma 2.19 there are a SQM $X \dashrightarrow \tilde{X}$ and an elementary divisorial contraction $\sigma \colon \tilde{X} \to Z$ such that $\mathrm{Exc}(\sigma)$ is the transform of E_0, and $\dim \sigma(\mathrm{Exc}(\sigma)) \geq \dim Y - 1$. Theorem–Definition 4.4 yields the statement. ☐

Lemma 4.11. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, and $f \colon X \dashrightarrow Y$ a special rational contraction; we use Notation 2.14. Then $r_i = 2$ for every $i = 1, \dots, m$.*

Proof. We consider for simplicity $i = 1$.

Claim. *For every irreducible component D of f^*B_1, there exists another component E of f^*B_1 such that $E \cdot C_D > 0$.*

Let us first show that the Claim implies the statement. Assume by contradiction that $r_1 > 2$, and let us consider a component D_1 of f^*B_1. By the Claim, there exists a second component D_2 with $D_2 \cdot C_{D_1} > 0$, and since $r_1 \geq 3$, we have $\langle [D_1], [D_2] \rangle \cap \mathrm{Mov}(X) = \{0\}$ by Corollary 2.18. Applying Lemma 4.9, we conclude that D_1 is not of type $(3, 2)$, and D_2 is of type $(3, 2)$.

Now we restart with D_2, and we deduce that D_2 is not of type $(3, 2)$, a contradiction. Hence $r_1 = 2$.

We prove the Claim. By Lemma 2.19, there exists a diagram:

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "f"'] & \tilde{X} \arrow[d, "\sigma"] \\
Y & Z \arrow[l, "g"]
\end{tikzcd}$$

where φ is a SQM and σ is an elementary divisorial contraction with $\operatorname{Exc}(\sigma) = \tilde{D}$, the transform of D in $\tilde{X}$.

Since $g \circ \sigma$ is special, we have $g(\sigma(\tilde{D})) = B_1$ and hence $\sigma(\tilde{D}) \subset g^{-1}(B_1)$; let $E_Z \subset Z$ be an irreducible component of $g^{-1}(B_1)$ containing $\sigma(\tilde{D})$. Let $\tilde{E} \subset \tilde{X}$ and $E \subset X$ be the transforms of E_Z, so that E is an irreducible component of f^*B_1. Note that $\tilde{E} \cdot \mathrm{NE}(\sigma) > 0$ by construction.

Now let $\Gamma \subset \tilde{D}$ be a general minimal irreducible curve contracted by σ; by Theorem–Definition 4.4(d) and (e), the transform of Γ in X is the curve C_D, and Γ is contained in the open subset where $\varphi^{-1} \colon \tilde{X} \dashrightarrow X$ is an isomorphism. Therefore $E \cdot C_D = \tilde{E} \cdot \Gamma > 0$. ☐

5. Fano 4-folds to surfaces

In this section we study rational contractions from a Fano 4-fold to a surface, and show the following.

Theorem 5.1. *Let X be a smooth Fano 4-fold having a rational contraction $f \colon X \dashrightarrow S$ with $\dim S = 2$. Then one of the following holds*:

(i) *X is a product of surfaces.*

(ii) $\rho_X \leq 12$.

(iii) $13 \leq \rho_X \leq 17$, *S is a smooth del Pezzo surface, the general fiber F of f is a smooth del Pezzo surface with* $4 \leq \dim \mathcal{N}_1(F, X) \leq \rho_F \leq 8$, *and* $\rho_X \leq 9 + \dim \mathcal{N}_1(F, X)$.

(iv) *$S \cong \mathbb{P}^2$ and f is special.*

Lemma 5.2. *Let X be a smooth Fano 4-fold with $\rho_X \geq 7$, and $f \colon X \dashrightarrow S$ a special rational contraction with* $\dim S = 2$; *we use Notation 2.14. Then for every $i = 1, \ldots, m$ the divisor f^*B_i has two irreducible components, one a fixed divisor of type* $(3, 2)$, *and the other one of type* $(3, 2)$ *or* $(3, 1)^{\mathrm{sm}}$.

Proof. We consider for simplicity $i = 1$. By Lemma 4.11 f^*B_1 has two irreducible components, and by Lemma 4.10 they are of type $(3, 2)$ or $(3, 1)^{\mathrm{sm}}$. We have to show that they cannot be both of type $(3, 1)^{\mathrm{sm}}$.

Let us choose a SQM $\varphi \colon X \dashrightarrow \tilde{X}$ such that $\tilde{f} := f \circ \varphi^{-1} \colon \tilde{X} \to S$ is regular, K-negative, and special (see Lemma 4.2). Let $E, \hat{E} \subset \tilde{X}$ be the irreducible components of $\tilde{f}^*(B_1)$, and $F \subset \tilde{X}$ a general fiber of $\tilde{f}$ over the curve B_1.

Suppose that E is of type $(3, 1)^{\mathrm{sm}}$. By Theorems 2.19 and 4.4, we have a diagram:

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi"] \arrow[rrd, dashed, "f"'] & \tilde{X} \arrow[r, dashed, "\psi"] \arrow[rd, "\tilde{f}"] & \hat{X} \arrow[r, "k"] \arrow[d, "\hat{f}"] & \tilde{X}_1 \arrow[ld, "f_1"] \\
 & & S &
\end{tikzcd}$$

where ψ is SQM and k is the blow-up of a smooth irreducible curve $C \subset \tilde{X}_1$, with exceptional divisor the transform of $E \subset \tilde{X}$, and $f_1(C) = B_1$.

Recall from the proof of Lemma 2.19 that ψ arises from a MMP for E, relative to $\tilde{f}$. Since $\tilde{f}$ is K-negative, one can use a MMP with scaling of $-K_{\tilde{X}}$ (see [Birkar et al. 2010, Section 3.10], and for this specific case [Casagrande 2012, Proposition 2.4] which can be adapted to the relative setting), so that ψ factors as a sequence of K-negative flips, relative to $\tilde{f}$. Then by Lemma 4.1(b) and (c), the indeterminacy locus of ψ is a disjoint union of exceptional planes, and is disjoint from the indeterminacy locus of φ^{-1}.

In particular, the indeterminacy locus of ψ is contracted to points by $\tilde{f}$. Since F is a general fiber of $\tilde{f}$ over B_1, it must be contained in the open subset where ψ is an isomorphism, and $\hat{F} := \psi(F) \subset \hat{X}$ is a general fiber of $\hat{f}$ over B_1. We also note that F is contained in the open subset where φ^{-1} is an isomorphism; otherwise there should be an exceptional line contained in E, and this would give an exceptional line contained in $\mathrm{Exc}(k)$, contradicting [Casagrande 2017, Remark 5.6].

Every irreducible component of $\mathrm{Exc}(k) \cap \hat{F}$ is a fiber of k over C. We deduce that the transform in X of any curve in $E \cap F$ has class in $\mathbb{R}_{\geq 0}[C_E]$.

We have $\dim F \cap E \cap \hat{E} \geq 1$, let Γ be an irreducible curve in $F \cap E \cap \hat{E}$. If $\hat{E}$ were of type $(3, 1)^{\mathrm{sm}}$ too, the transform of Γ in X should have class in both $\mathbb{R}_{\geq 0}[C_E]$ and $\mathbb{R}_{\geq 0}[C_{\hat{E}}]$. This would imply that the classes of C_E and $C_{\hat{E}}$ are proportional, and this is impossible by Theorem–Definition 4.4(e). Therefore E and $\hat{E}$ cannot be both of type $(3, 1)^{\mathrm{sm}}$. ☐

Proof of Theorem 5.1. We can assume that $\rho_X \geq 13$, otherwise we have (ii).

By Proposition 2.13 f factors as a special rational contraction $g \colon X \dashrightarrow T$ followed by a birational map $T \to S$. There exists a SQM $\varphi \colon X \dashrightarrow \tilde{X}$ such that $\tilde{X}$ is smooth and the composition $\tilde{g} := g \circ \varphi^{-1} \colon \tilde{X} \to T$ is regular, K-negative and special (see Lemma 4.2); in particular T is a smooth surface by Lemma 4.3.

$$\begin{tikzcd}
X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "f"'] \arrow[rd, dashed, "g"] & \tilde{X} \arrow[d, "\tilde{g}"] \\
S & T \arrow[l]
\end{tikzcd}$$

Finally g has $r_i = 2$ for every $i = 1, \dots, m$ (we use Notation 2.14) by Lemma 4.11.

Suppose that $m = 0$, equivalently that $\tilde{g}$ is quasielementary. If g is regular, then [Casagrande 2008, Theorem 1.1(i)] together with $\rho_X \geq 13$ yield that X is a product of surfaces, so we have (i).

Assume instead that g is not regular, and let $F \subset X$ be a general fiber of f, which is also a general fiber of g. Since the indeterminacy locus of φ^{-1} has dimension 1 (see Lemma 4.1(a)), it does not meet a general fiber of $\tilde{g}$. This means that F is contained in the open subset where φ is an isomorphism, and $\varphi(F)$ is a general fiber of $\tilde{g}$. By Lemma 2.17 and [Casagrande 2013a, Corollary 3.9 and its proof] we have that F is a smooth del Pezzo surface with $\rho_F \leq 8$ and

$$\rho_X = \dim \mathcal{N}_1(F, X) + \rho_T \leq \rho_F + \rho_T \leq 8 + \rho_T.$$

In particular $\rho_T \geq 13 - 8 = 5$. Then [loc. cit., Proposition 4.1 and its proof] imply that g is not elementary and that T is a del Pezzo surface. Therefore $\rho_X \leq 17$, $\dim \mathcal{N}_1(F, X) = \rho_X - \rho_T \geq 13 - 9 = 4$, and S is a smooth del Pezzo surface too. So we have (iii).

Suppose now that $m \geq 1$. By Lemma 5.2, $(\tilde{g})^* B_1$ has an irreducible component E which is a fixed divisor of type $(3, 2)$. We have $(\tilde{g})_* \mathcal{N}_1(E, \tilde{X}) = \mathbb{R}[B_1]$, so that $\operatorname{codim} \mathcal{N}_1(E, \tilde{X}) \geq \rho_T - 1$. If $\rho_T > 1$, then we get (i) by Lemma 4.5.

Let us assume that $\rho_T = 1$. Then $T \cong \mathbb{P}^2$, because T is a smooth rational surface. Moreover the birational map $T \to S$ must be an isomorphism, hence $S \cong \mathbb{P}^2$ and f is special, and we get (iv). ☐

6. Fano 4-folds to 3-folds

In this section we study rational contractions from a Fano 4-fold to a 3-dimensional target, and show the following.

Theorem 6.1. *Let X be a smooth Fano 4-fold. If there exists a rational contraction $X \dashrightarrow Y$ with $\dim Y = 3$, then either X is a product of surfaces, or $\rho_X \leq 12$.*

Proof. If $\delta_X \geq 3$ the statement follows from Theorem 3.2, so we can assume that $\delta_X \leq 2$; we also assume that $\rho_X \geq 7$. By Proposition 2.13, we can suppose that the map $X \dashrightarrow Y$ is special. Moreover by Lemma 4.2 we can factor it as

$$X \overset{\varphi}{\dashrightarrow} \tilde{X} \xrightarrow{f} Y,$$

where φ is a SQM, $\tilde{X}$ is smooth, and f is regular, K-negative and special.

By Lemmas 3.4 and 3.5 we have $\rho_X = \rho_Y + m + 1$, $r_1 = \cdots = r_m = 2$, and the divisors $B_1, \ldots, B_m$ are pairwise disjoint in Y (we use Notation 2.14). For $i = 1, \ldots, m$ the irreducible components of $f^* B_i$ are fixed divisors of type $(3, 2)$ by Lemma 4.10.

If $\rho_X - \rho_Y \geq 3$, then $m \geq 2$. Let E_1, E_2 be the irreducible components of $f^* B_1$, and W an irreducible component of $f^* B_2$. Since $B_1 \cap B_2 = \varnothing$, we have $E_1 \cap W = \varnothing$, so that $\mathcal{N}_1(E_1, \tilde{X}) \subsetneq \mathcal{N}_1(\tilde{X})$ by Remark 1.3, and this implies the statement by Lemma 4.5.

If instead $\rho_X - \rho_Y = 1$, then f is elementary, and $\rho_X \leq 11$ by [Casagrande 2013a, Theorem 1.1].

We are left with the case where $\rho_X - \rho_Y = 2$ and $m = 1$, which we assume from now on. We will adapt the proof of [loc. cit., Theorem 1.1] of the elementary case to the case $\rho_X - \rho_Y = 2$, and divide the proof in several steps. Since $m = 1$, we set for simplicity $B := B_1$.

6.2. If $\mathcal{N}_1(E_1, \tilde{X}) \subsetneq \mathcal{N}_1(\tilde{X})$ we conclude as before, so we can assume that $\mathcal{N}_1(E_1, \tilde{X}) = \mathcal{N}_1(\tilde{X})$; this implies that $\mathcal{N}_1(B, Y) = \mathcal{N}_1(Y)$.

By Lemma 3.4, $E_1 \cup E_2$ is covered by curves of anticanonical degree 1. Since an exceptional line cannot meet such curves (see Lemma 4.1(b)), we deduce that $\ell \cap (E_1 \cup E_2) = \varnothing$ for every exceptional line $\ell \subset \tilde{X}$.

Notice that even if f is not elementary, by specialty it does not have fibers of dimension 3, and has at most isolated fibers of dimension 2. Moreover Y is locally factorial and has (at most) isolated canonical singularities, by Lemma 4.3. More precisely, $\operatorname{Sing}(Y)$ is contained in the images of the 2-dimensional fibers of f (this is due to Ando, see [Andreatta and Wiśniewski 1997, Theorem 4.1 and references therein]).

Since $\tilde{X}$ is smooth and Y is locally factorial, it is easy to see that $f^*B = E_1 + E_2$.

Finally, since X is Fano, by [Prokhorov and Shokurov 2009, Lemma 2.8] there exists a $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is a klt log Fano, so that $-K_Y$ is big.

6.3. Let $g\colon Y \to Y_0$ be a small elementary contraction. Then $\operatorname{Exc}(g)$ is the disjoint union of smooth rational curves lying in the smooth locus of Y, with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$; in particular $K_Y \cdot \operatorname{NE}(g) = 0$.

Proof. Exactly the same proof as the one of [Casagrande 2013a, Lemma 4.5] applies, with the only difference that, in the notation of [loc. cit., Lemma 4.5], $\dim \mathcal{N}_1(\tilde{U}/U)$ could be bigger than 2. We take τ to be any extremal ray of $\operatorname{NE}(\tilde{U}/U)$ not contained $\operatorname{NE}(g_{|\tilde{U}})$. ☐

6.4. Let $g\colon Y \to Y_0$ be an elementary divisorial contraction. Then g is the blow-up of a smooth point of Y_0; in particular $-K_Y \cdot \operatorname{NE}(g) > 0$.

Proof. Set $G := \operatorname{Exc}(g) \subset Y$. Since g is elementary and $\dim g(G) \leq 1$, we have $\dim \mathcal{N}_1(G, Y) \leq 2$; on the other hand $\dim \mathcal{N}_1(B, Y) = \rho_Y = \rho_X - 2 \geq 5$ (see 6.2), so $G \neq B$, and $D := f^*G$ is a prime divisor in $\tilde{X}$, different from E_1 and E_2, with $\dim \mathcal{N}_1(D, \tilde{X}) \leq \dim \ker f_* + \dim \mathcal{N}_1(G, Y) \leq 2 + 2 = 4$.

Since G is fixed, also D is a fixed divisor in $\tilde{X}$; let $D_X \subset X$ be the transform of D.

6.4.1. We show that D is not of type $(3, 2)$. Otherwise, as in the proof of Lemma 4.5 we see that $\dim \mathcal{N}_1(D_X, X) = \dim \mathcal{N}_1(D, \tilde{X}) \leq 4$. On the other hand we have $\delta_X \leq 2$ and $\rho_X \geq 7$, a contradiction.

6.4.2. We show that g is of type $(2, 0)$. By contradiction, suppose that g is of type $(2, 1)$. As in [Casagrande 2013a, proof of Lemma 4.6], we show that there is an open subset $\tilde{U} \subseteq \tilde{X}$ such that $D \cap \tilde{U}$ is covered by curves of anticanonical degree 1. By [Casagrande 2017, Lemma 2.8(3)], D_X still has a nonempty open subset covered by curves of anticanonical degree 1; this implies that D_X and D are of type $(3, 2)$ by [loc. cit., Lemma 2.18], a contradiction to 6.4.1.

6.4.3. Thus g is of type $(2, 0)$; set $p := g(G) \in Y_0$.

Since $\mathcal{N}_1(B, Y) = \mathcal{N}_1(Y)$ by 6.2, we must have $G \cap B \neq \varnothing$ by Remark 1.3. Therefore $p \in g(B)$, hence $g^*(g(B)) = B + aG$ with $a > 0$, and $(g \circ f)^*(g(B)) = E_1 + E_2 + aD$ (see again 6.2).

As in [Casagrande 2013a, proof of Lemma 4.6], we get a diagram:

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "\psi"] \arrow[d, "f"] & \hat{X} \arrow[r, "k"] & \tilde{X}_1 \arrow[dl, "f_1"] \\
Y \arrow[r, "g"] & Y_0 &
\end{tikzcd}$$

where ψ is a sequence of D-negative flips relative to $g \circ f$, k is an elementary divisorial contraction with exceptional divisor the transform $\hat{D} \subset \hat{X}$ of D, and f_1 is a contraction of fiber type with $\dim \ker(f_1)_* = 2$. By 6.4.1 and Theorem–Definition 4.4, k is of type $(3, 0)^{\mathrm{sm}}$, $(3, 0)^{Q}$, or $(3, 1)^{\mathrm{sm}}$; in particular $\tilde{X}_1$ has at most one isolated locally factorial and terminal singularity. Moreover f_1 is special, so that Y_0 has locally factorial, canonical singularities by Lemma 2.21.

6.4.4. Let us consider the factorization of ψ as a sequence of D-negative flips relative to $g \circ f$:

$$\begin{tikzcd}
\tilde{X} = Z_0 \arrow[r, dashed, "\sigma_1"] \arrow[drr, "g\circ f"'] & \cdots \arrow[r, dashed] & Z_{i-1} \arrow[r, dashed, "\sigma_i"] \arrow[d, "\zeta_{i-1}"'] & Z_i \arrow[r, dashed] \arrow[dl, "\zeta_i"] & \cdots \arrow[r, dashed, "\sigma_n"] & Z_n = \hat{X} \arrow[dlll, "f_1\circ k"] \\
 & & Y_0 & & &
\end{tikzcd}$$

With a slight abuse of notation, we still denote by D, E_1, E_2 the transforms of these divisors in Z_i, for $i = 0, \ldots, n$.

We show by induction on $i = 0, \ldots, n$ that σ_i is K-negative and that $(E_1 + E_2) \cdot \ell \leq 0$ for every exceptional line $\ell \subset Z_i$. For $i = 0$, this holds by 6.2.

Suppose that the statement is true for $i - 1$. Let R and R' be the small extremal rays of $\mathrm{NE}(Z_{i-1})$ and $\mathrm{NE}(Z_i)$ respectively corresponding to the flip σ_i. By the commutativity of the diagram above and by 6.4.3, we have $E_1 + E_2 + aD = \zeta_{i-1}^*(g(B))$, hence $(E_1 + E_2 + aD) \cdot R = 0$, where $a > 0$. On the other hand $D \cdot R < 0$, thus $(E_1 + E_2) \cdot R > 0$ and $(E_1 + E_2) \cdot R' < 0$.

If $-K_{Z_{i-1}} \cdot R \leq 0$, then by [Casagrande 2013a, Remark 3.6(2)] there exists an exceptional line $\ell_0 \subset Z_{i-1}$ such that $[\ell_0] \in R$, therefore $(E_1 + E_2) \cdot \ell_0 > 0$, contradicting the induction assumption. Hence $-K_{Z_{i-1}} \cdot R > 0$ and σ_i is K-negative.

Finally if $\ell \subset Z_i$ is an exceptional line, by [loc. cit., Remark 4.2] we have either $\ell \subset \mathrm{dom}\, \sigma_i^{-1}$, or $\ell \cap \mathrm{dom}\, \sigma_i^{-1} = \varnothing$. In the first case $\sigma_i^{-1}(\ell)$ is an exceptional line in Z_{i-1}, and we deduce that $(E_1 + E_2) \cdot \ell \leq 0$. In the second case, we must have $[\ell] \in R'$ and hence $(E_1 + E_2) \cdot \ell < 0$.

6.4.5. By 6.4.4, ψ factors as a sequence of K-negative flips, and Lemma 4.1(c) yields that the indeterminacy locus of ψ^{-1} is a disjoint union of exceptional lines $\ell_1, \ldots, \ell_s$.

6.4.6. Set $F_p := f_1^{-1}(p)$. We show that $\dim F_p = 1$.

Note that $\tilde{X}$ and $\hat{X}$ are isomorphic outside the fibers of $g \circ f$ and $f_1 \circ k$ over p, respectively. In $\tilde{X}$ we have $(g \circ f)^{-1}(p) = D$, and the indeterminacy locus of ψ must be contained in D. In $\hat{X}$ we have $(f_1 \circ k)^{-1}(p) = k^{-1}(F_p) = \hat{D} \cup \overline{F}_p$, where $\overline{F}_p$ is the transform of the components of F_p not contained in $k(\hat{D})$. On the other hand, by 6.4.5 we also have $k^{-1}(F_p) = \hat{D} \cup \ell_1 \cup \cdots \cup \ell_s$. This shows that $\overline{F}_p \subseteq \ell_1 \cup \cdots \cup \ell_s$, in particular $\dim \overline{F}_p \leq 1$, and since $\dim k(\hat{D}) \leq 1$ (see 6.4.3), we conclude that $\dim F_p = 1$.

We have also shown that the transform in $\hat{X}$ of any irreducible component of F_p not contained in $k(\hat{D})$ must be one of the ℓ_i.

6.4.7. We show that f_1 is K-negative. Since f is K-negative and $f_{|\tilde{X}\smallsetminus D} \cong (f_1)_{|\tilde{X}_1\smallsetminus F_p}$, we only have to check the fiber F_p. Let Γ be an irreducible component of F_p.

If $\Gamma \not\subseteq k(\hat{D})$, then by 6.4.6 we can assume that the transform of Γ in $\hat{X}$ is ℓ_1. Since $k^{-1}(F_p)$ is connected and $\ell_1, \dots, \ell_s$ are pairwise disjoint, we have $\hat{D}\cdot\ell_1 > 0$; notice also that $K_{\hat{X}}\cdot\ell_1 = 1$. Thus $-K_{\tilde{X}_1}\cdot\Gamma > 0$ because $k^*(-K_{\tilde{X}_1}) = -K_{\hat{X}} + b\hat{D}$ with $b \in \{2, 3\}$ (see 6.4.3).

If instead $\Gamma \subseteq k(\hat{D})$, then by 6.4.3 k must be of type $(3, 1)^{\mathrm{sm}}$ and $\Gamma = k(\hat{D})$. By [Casagrande 2017, Lemma 5.25] there is a SQM $\varphi_1 \colon \tilde{X}_1 \dashrightarrow X_1$ where X_1 is a Fano 4-fold, and Γ is contained in the open subset where φ_1 is an isomorphism, so that $-K_{\tilde{X}_1}\cdot\Gamma = -K_{X_1}\cdot\varphi_1(\Gamma) > 0$.

6.4.8. By 6.4.3, 6.4.6, and 6.4.7, $\tilde{X}_1$ has isolated locally factorial and terminal singularities, Y_0 has locally factorial canonical singularities, f_1 is K-negative, and $\dim F_p = 1$. Then [Ou 2018, Lemma 5.5] yields that p is a smooth point of Y_0 (note that in [loc. cit.] the contraction is supposed to be elementary, but this is used only to conclude that Y_0 is locally factorial, which here we already know).

In particular p is a terminal singularity, hence g is K-negative. The possibilities for $(G, -K_{\tilde{X}_1|G})$ are given in [Andreatta and Wiśniewski 1997, Theorem 1.19]; moreover we know that G is Gorenstein, and by adjunction that $-K_G\cdot C \geq 2$ for every curve $C \subset G$. Going through the list, it is easy to see that the possibilities for G are $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$, and the quadric cone. In the first two cases, $G \subset Y_{\mathrm{reg}}$, and it follows from [Mori 1982, Corollary 3.4] that $G \cong \mathbb{P}^2$ and g is the blow-up of p.

Suppose instead that G is isomorphic to a quadric cone Q. Then the normal bundle of G has to be $\mathcal{O}_Q(-1)$, and as in [Mori 1982, page 164] and [Cutkosky 1988, proof of Theorem 5] one sees that $\mathcal{I}_p\mathcal{O}_Y = \mathcal{O}_Y(-G)$ where $\mathcal{I}_p$ is the ideal sheaf of p in Y_0, so that $g^{-1}(p) = G$ scheme-theoretically. Then g factors through the blow-up of p, and being g elementary, it must be the blow-up of p, which yields $G \cong \mathbb{P}^2$ and hence a contradiction. ☐

6.5. If Y has an elementary rational contraction of fiber type $Y \dashrightarrow Z$, then $\rho_Z = \rho_X - 3 \geq 4$, in particular Z is a surface. The composition $X \dashrightarrow Z$ is a rational contraction with $\rho_X - \rho_Z = 3$, and we can apply Theorem 5.1. If (i) or (ii) hold, we have the statement. If (iii) holds, then $\rho_X \geq 13$ and Z is a del Pezzo surface, so that $\rho_Z \leq 9$, which is impossible. Finally (iv) cannot hold because $\rho_Z > 1$.

Therefore we can assume that Y does not have elementary rational contractions of fiber type.

6.6. Let R be an extremal ray of $\mathrm{NE}(Y)$. By 6.5 the associated contraction cannot be of fiber type, thus it is birational, either small of divisorial. By 6.3 and 6.4, $-K_Y\cdot R \geq 0$. Since Y is log Fano, $\mathrm{NE}(Y)$ is closed and polyhedral, and we conclude that $-K_Y$ is nef and Y is a weak Fano variety (see 6.2).

6.7. Let $Y \dashrightarrow \tilde{Y}$ be a SQM. Then the composition $X \dashrightarrow \tilde{Y}$ is again a special rational contraction with $\rho_X - \rho_{\tilde{Y}} = 2$, so all the previous steps apply to $\tilde{Y}$ as well. As in [Casagrande 2013a, page 622], using 6.3 and 6.4 one shows that if $E \subset Y$ is a fixed prime divisor, then E can contain at most finitely many curves of anticanonical degree zero.

6.8. Let us consider all the contracting birational maps $Y \dashrightarrow Y_1$ with $\mathbb{Q}$-factorial target, and choose one with ρ_{Y_1} minimal.

Suppose that $\rho_{Y_1} \geq 3$. By minimality, Y_1 has an elementary rational contraction of fiber type $Y_1 \dashrightarrow Z$, and Z must be a surface with $\rho_Z = \rho_{Y_1} - 1 \geq 2$. The composition $X \dashrightarrow Z$ is a rational contraction, let $F \subset X$ be a general fiber. The general fiber of $Y \dashrightarrow Z$ is a smooth rational curve $\Gamma \subset Y$, and $\dim \mathcal{N}_1(F, X) \leq \dim \mathcal{N}_1(\Gamma, Y) + (\rho_X - \rho_Y) = 3$. Thus we get the statement by Theorem 5.1.

Therefore we can assume that $\rho_{Y_1} \leq 2$.

6.9. By [Casagrande 2017, Lemma 4.18], we can factor the map $Y \dashrightarrow Y_1$ as $Y \dashrightarrow Y' \to Y_1$, where $Y \dashrightarrow Y'$ is a SQM, and $Y' \to Y_1$ is a sequence of elementary divisorial contractions. Now notice that the composition $X \dashrightarrow Y'$ is again a special rational contraction with $\rho_X - \rho_{Y'} = 2$, so up to replacing Y with Y', we can assume that the map $a\colon Y \dashrightarrow Y_1$ is regular and is a sequence of $r := \rho_Y - \rho_{Y_1}$ elementary divisorial contractions:

$$Y = W_0 \xrightarrow{a_1} W_1 \xrightarrow{a_2} W_2 \to \cdots \to W_r = Y_1.$$

Let us show that the exceptional loci of these maps are all disjoint, so that a is just the blow-up of r distinct smooth points of Y_1.

We know by 6.4 that a_1 is the blow-up of a smooth point $w_1 \in W_1$, and since $-K_Y$ is nef, it is easy to see that if $C \subset W_1$ is an irreducible curve containing w_1, then $-K_{W_1} \cdot C \geq 2$.

Suppose that $\operatorname{Exc}(a_2)$ contains w_1. Then a_2 is K-negative, and $\operatorname{Exc}(a_2)$ cannot be covered by curves of anticanonical degree one. By [Andreatta and Wiśniewski 1997, Theorem 1.19] this implies that $\operatorname{Exc}(a_2) \cong \mathbb{P}^2$ and $(-K_{W_1})_{|\operatorname{Exc}(a_2)} \cong \mathcal{O}_{\mathbb{P}^2}(2)$. Then the transform of $\operatorname{Exc}(a_2)$ would be a fixed prime divisor covered by curves of anticanonical degree zero, which is impossible by 6.7. Proceeding in the same way, we conclude that the exceptional loci of the maps a_i are all disjoint.

Now Y_1 is weak Fano with isolated locally factorial, canonical singularities, and we have $(-K_{Y_1})^3 \leq 72$ by [Prokhorov 2005]. Therefore

$$0 < (-K_Y)^3 = (-K_{Y_1})^3 - 8r,$$

which yields $r \leq 8$ and $\rho_X = \rho_{Y_1} + r + 2 \leq 12$. □

Theorem 1.1 is a straightforward consequence of Theorems 5.1 and 6.1.

Proof of Theorem 1.2. Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be open subsets such that $f_0 := f_{|X_0}\colon X_0 \to Y_0$ is a projective morphism. Up to taking the Stein factorization, we can assume that f_0 is a contraction. Let $A \in \operatorname{Pic}(Y)$ be ample and consider $H := f^*A \in \operatorname{Pic}(X)$. Then H is a movable divisor, hence it yields a rational contraction $f'\colon X \dashrightarrow Y'$. It is easy to see that $f'_{|X_0} = f_0$, in particular $\dim Y' = 3$. Then the statement follows from Theorem 6.1. □

7. Fano 4-folds to $\mathbb{P}^1$

Let X be a Fano 4-fold and $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational contraction; notice that f is always special. In the following proposition we collect the information that we can give on f.

Proposition 7.1. *Let X be a smooth Fano 4-fold and $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational contraction. Let $F_1, \dots, F_m$ be the reducible fibers of f. Then one of the following hold*:

(i) $\rho_X \leq 12$.

(ii) *X is a product of surfaces.*

(iii) $\rho_X \leq m + 10$, *f is not regular*, *and every F_i has two irreducible components*, *which are fixed divisors of type* $(3, 1)^{\mathrm{sm}}$ *or* $(3, 0)^{Q}$.

Proof. We can assume that $\rho_X \geq 7$, so that $r_i = 2$ for $i = 1, \dots, m$ by Lemma 4.11. By Lemma 4.2 we can factor f as $X \overset{\varphi}{\dashrightarrow} X' \xrightarrow{f'} \mathbb{P}^1$ where φ is a SQM, X' is smooth, and f' is regular and K-negative.

If some F_i has a component of type $(3, 0)^{\mathrm{sm}}$, then we get (i) by [Casagrande 2017, Theorem 5.40].

If some F_i has a component of type $(3, 2)$, let $E \subset X'$ be its transform. Then $\mathcal{N}_1(E, X') \subseteq \ker(f')_* \subsetneq \mathcal{N}_1(X')$, so we get (i) or (ii) by Lemma 4.5.

We are left with the case where every component of every F_i is of type $(3, 1)^{\mathrm{sm}}$ or $(3, 0)^{Q}$. The general fiber F of f' is a smooth Fano 3-fold, so that $\rho_F \leq 10$ by Mori and Mukai's classification (see [Iskovskikh and Prokhorov 1999, Corollary 7.1.2]). If f is regular, then φ is an isomorphism, and $\rho_X \leq \rho_F + \delta_X$, so we get (i) or (ii) by Theorem 3.2.

If instead f is not regular, then as in [Casagrande 2013a, proof of Corollary 3.9] one shows that in fact $\rho_F \leq 9$. Therefore Corollary 2.16 yields $\rho_X \leq m + 10$, and we have (iii). ☐

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