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and the cup product**

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Deformations of smooth complete toric varieties: obstructions and the cup product

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Let X be a complete $\mathbb{Q}$ -factorial toric variety. We explicitly describe the space $H^2(X, \mathcal{T}_X)$ and the cup product map $H^1(X, \mathcal{T}_X) \times H^1(X, \mathcal{T}_X) \rightarrow H^2(X, \mathcal{T}_X)$ in combinatorial terms. Using this, we give an example of a smooth projective toric threefold for which the cup product map does not vanish, showing that in general, smooth complete toric varieties may have obstructed deformations.

1. Introduction

Background and motivation. Let X be any variety over an algebraically closed field $\mathbb{K}$ of characteristic not equal to two or three. The deformation theory of X provides useful information on how X might fit into a moduli space. The abstract theory guarantees that in good situations (e.g., X complete or X an isolated singularity) X will possess a versal deformation, from which all deformations of X can be induced. However in practice, the versal deformation of a given variety may be very difficult to describe in its entirety. It is thus interesting to study classes of varieties for which one may more explicitly understand the deformation theory.

One special class of varieties whose deformation theory has been studied are *toric varieties*. Deformations of such varieties have applications ranging from mirror symmetry [Mavlyutov 2004; Coates et al. 2013] to Kähler–Einstein and extremal metrics [Rollin and Tipler 2014; Ilten and Süß 2017]. The deformation theory of *affine* toric varieties has been described extensively by Altmann. Combinatorial formulas exist for the tangent and obstruction spaces T_X^1 and T_X^2 as well as a combinatorial description of the cup product map [Altmann 1994; 1997a]; see also [Filip 2018]. A combinatorial recipe may be used to construct deformations of X over affine space [Altmann 1995], and in some cases (e.g., isolated Gorenstein singularities) there is an explicit combinatorial description of the entire versal deformation [Altmann 1997b].

In this paper, we will instead continue the program initiated by the first author in [Ilten 2011] of describing the deformation theory of *smooth, complete* toric varieties. Let X be a smooth complete

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toric variety corresponding to a fan Σ . Ilten [2011] gave a combinatorial description of the space $T_X^1 = H^1(X, \mathcal{T}_X)$ of first order deformations:

$$H^1(X, \mathcal{T}_X) = \bigoplus_{\rho \in \Sigma^{(1)}} \bigoplus_{\substack{u \in M \\ \rho(u) = -1}} \tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K}), \quad (1)$$

where ρ ranges over all rays of the fan Σ , M is the character lattice of the torus of X , $\rho(u) \in \mathbb{Z}$ denotes the pairing between the primitive generator of ρ and u , and $\Gamma_{\rho,u}$ is a certain graph; see Section 3. Here, $\tilde{H}$ denotes reduced cohomology.

Generalizing Altmann's construction in the affine case, Ilten and Vollmert [2012] gave a recipe for producing deformations of any toric variety X over affine spaces from combinatorial data; see also [Mavlyutov 2009; Petracci 2018]. In particular, when X is smooth and complete, each connected component of a graph $\Gamma_{\rho,u}$ appearing in (1) gives rise to a one-parameter deformation (over $\mathbb{A}^1$) lifting the corresponding first order deformation in T_X^1 [Ilten and Vollmert 2012, Theorem 6.5]. In fact, for any character $u \in M$, one may use this construction to produce a deformation over $\mathbb{A}^m$ whose image in T_X^1 spans the entire degree u piece. This is evidence that, despite in general having nonvanishing obstruction spaces, smooth complete toric varieties might have unobstructed deformations, similar to the situation of, e.g., Calabi–Yau varieties [Tian 1987; Todorov 1989]. However, we will see below that this is not the case.

Results. Throughout, X will be a complete $\mathbb{Q}$ -factorial toric variety corresponding to a fan Σ with character lattice M . The description (1) of $H^1(X, \mathcal{T}_X)$ in the case X smooth also holds when X is only $\mathbb{Q}$ -factorial; see Section 3. There is also a straightforward generalization of (1) for $H^2(X, \mathcal{T}_X)$:

Proposition 1.1 (Proposition 3.1). *The cohomology group $H^2(X, \mathcal{T}_X)$ may be decomposed as*

$$H^2(X, \mathcal{T}_X) = \bigoplus_{\rho \in \Sigma^{(1)}} \bigoplus_{\substack{u \in M \\ \rho(u) = -1}} H^1(K_{\rho,u}, \mathbb{K}), \quad (2)$$

where each $K_{\rho,u}$ is a simplicial complex determined from Σ ; see Section 3.

Our main result is then to give a combinatorial description of the cup product map

$$H^1(X, \mathcal{T}_X) \times H^1(X, \mathcal{T}_X) \rightarrow H^2(X, \mathcal{T}_X)$$

using (1) and (2). When X is smooth, $H^2(X, \mathcal{T}_X)$ is the obstruction space T_X^2 , and the cup product may be used to obtain the quadratic terms in the obstruction equations for the versal deformation of X . To describe the cup product, we will use Čech cohomology (with respect to a closed covering) to describe elements of the cohomology groups $\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K})$ and $H^1(K_{\rho,u}, \mathbb{K})$. The closed covering we consider will be indexed by maximal cones $\sigma \in \Sigma$; the corresponding closed sets will be the intersections of σ with either $\Gamma_{\rho,u}$ or $K_{\rho,u}$.

Theorem 1.2 (Theorem 4.3). *Fix $\rho, \rho' \in \Sigma^{(1)}$ and $u, u' \in M$ with $\rho(u) = \rho'(u') = -1$.*

(1) *The image of*

$$\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K}) \times \tilde{H}^0(\Gamma_{\rho',u'}, \mathbb{K})$$

in $H^2(X, \mathcal{T}_X)$ under the cup product via (1) is 0 unless $\rho(u') = 0$ or $\rho'(u) = 0$.

(2) *Assume that $\rho(u') = 0$, and let $f = (f_\sigma)$ and $f' = (f'_\sigma)$ be Čech zero-cycles of $\Gamma_{u,\rho}$ and $\Gamma_{u',\rho'}$. Then the cup product of $(\bar{f}, \bar{f}')$ is contained in $H^1(K_{\rho,u+u'})$ via (2) and may be represented by the Čech one-cocycle $g = (g_{\sigma\tau})$, where*

$$g_{\sigma\tau} = \begin{cases} \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f_\tau f'_\sigma) & \text{if } K_{\rho,u+u'} \cap \sigma \cap \tau \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

A similar formula holds when $\rho'(u) = 0$.

While this theorem gives an explicit description of the cup product on the combinatorial level, it is perhaps not always immediately obvious when the one-cocycle $(g_{\sigma\tau})$ is nontrivial. To remedy this, we proceed as follows. Assume as in the second part of the theorem that $\rho(u') = 0$. Consider any simple cycle α in $K_{\rho,u+u'}$, and connected components Z and Z' of $\Gamma_{\rho,u}$ and $\Gamma_{\rho',u'}$. Then $H^1(\alpha, \mathbb{K}) = \mathbb{K}$ with canonical generator α_{fun} , and Z and Z' induce elements of $\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K})$ and $\tilde{H}^0(\Gamma_{\rho',u'}, \mathbb{K})$. The pullback of the cup product of these elements to $H^1(\alpha, \mathbb{K})$ is $(Z *_\alpha Z') \cdot \alpha_{\text{fun}}$, where $Z *_\alpha Z'$ is determined from the intersection behavior of Z and Z' along α ; see Theorem 5.3 and the subsection before it for a precise statement.

This leads to a straightforward method to determine when the cup product vanishes. In particular, we may easily use this to construct examples of smooth toric threefolds where the cup product does not vanish:

Corollary 1.3 (Corollary 6.1). *There exists a smooth complete toric threefold with obstructed deformations.*

Murphy's law and future directions. Is this obstructedness result (Corollary 1.3) surprising? We would argue that although perhaps not surprising, it is far from obvious. On the one hand, Vakil [2006] has shown *Murphy's Law* for several classes of deformation problems, that is, that arbitrarily bad singularities of finite type over $\mathbb{Z}$ can occur in the versal deformations. For example, this is true for smooth projective n -folds ($n \geq 2$) with ample canonical class. Vakil writes that his results suggest that “unless there is some natural reason for the [deformation] space to be well-behaved, it will be arbitrarily badly behaved.”

On the other hand, toric varieties are so special that there may well be a natural reason for the deformation space to be well-behaved. In fact, Murphy's Law is false for smooth toric varieties! This follows, e.g., from [Ilten and Vollmert 2012, Theorem 6.5], which implies in particular that the versal deformation space of a smooth complete toric variety cannot be a fat point.

This means that the deformation theory of smooth complete toric varieties may belong to the small class of deformation problems which are obstructed, yet one can still hope to completely describe in some explicit manner. The next natural question to address is:

Question 1.4. *Is the versal deformation of a smooth complete toric variety cut out by quadrics?*

In fact, if we knew that the versal deformation was cut out by quadrics, then our results here would completely determine those equations. At the moment, we have far too little evidence to posit an answer one way or the other.

The remainder of this paper is organized as follows. In Section 2, we recall basic facts of Čech cohomology and toric geometry. In Section 3, we prove Proposition 1.1, describing T_X^2 combinatorially. The main work of this paper is contained in Section 4, where we prove our combinatorial description of the cup product (Theorem 1.2). In Section 5 we show how the cup product pulls back to simple cycles α . Finally, in Section 6, we present an example of an obstructed smooth toric threefold, proving Corollary 1.3.

2. Preliminaries

Čech cohomology. We begin by recalling basics of Čech cohomology and fixing notation; see, e.g., [Bosch 2013, §7.6] for more details. Let X be a topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ either an open or closed cover of X . For any sheaf $\mathcal{F}$ of abelian groups on X , the group $\check{C}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$ of singular p -th Čech cochains is

$$\check{C}_{\text{sing}}^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}).$$

The differential $d^p : \check{C}_{\text{sing}}^p(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}_{\text{sing}}^{p+1}(\mathcal{U}, \mathcal{F})$ is defined by $d^p(f) = g$, where

$$g_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k (f_{i_0 \dots \widehat{i_k} \dots i_{p+1}})|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}.$$

The p -th singular Čech cohomology group of $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is the p -th cohomology $\check{H}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$ of the complex $(\check{C}_{\text{sing}}^\bullet, d^\bullet)$. Elements of the kernel of d^p are called singular Čech cocycles.

It is more common to work with either *alternating* or *ordered* Čech cohomology, since these have bounded length and involve fewer terms. We will opt to consistently work with alternating Čech cohomology: if we do not explicitly specify that we are talking about *singular* Čech cohomology, then we are referring to alternating Čech cohomology. This is defined as follows.

The group of (alternating) p -th Čech cochains $\check{C}^p(\mathcal{U}, \mathcal{F})$ is the subgroup of $\check{C}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$ consisting of elements f satisfying

$$f_{i_{\pi(0)} \dots i_{\pi(p)}} = \text{sign}(\pi) f_{i_0 \dots i_p}$$

for any permutation $\pi \in S_{p+1}$ of $0, \dots, p$, and

$$f_{i_0 \dots i_p} = 0$$

if any index i_j is repeated. After eliminating terms with doubled indices, the differential d^p on the singular Čech complex also gives a differential on the subcomplex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$. The p -th (alternating) Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ is the p -th cohomology $\check{H}^p(\mathcal{U}, \mathcal{F})$ of this subcomplex. Elements of $\check{C}^p(\mathcal{U}, \mathcal{F})$ in the kernel of d^p are called (alternating) Čech cocycles.

The inclusion of complexes $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \hookrightarrow \check{C}_{\text{sing}}^\bullet(\mathcal{U}, \mathcal{F})$ induces homomorphisms of cohomology groups $\check{H}^p(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$. In fact, on the level of cohomology, these maps are isomorphisms; see [Bosch

2013, §7.6, Lemma 1]. For our purposes, we need a map $\check{C}_{\text{sing}}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^\bullet(\mathcal{U}, \mathcal{F})$ which on cohomology induces the inverse of this isomorphism:

Lemma 2.1. *Assume that $\mathcal{F}$ is a sheaf of $\mathbb{Q}$ -modules. The maps*

$$\phi^p : \check{C}_{\text{sing}}^p(\mathcal{U}, \mathcal{F}) \rightarrow \check{C}^p(\mathcal{U}, \mathcal{F})$$

defined by

$$\phi^p(f)_{i_0 \dots i_p} = \frac{1}{(p+1)!} \sum_{\pi \in S_{p+1}} \text{sign}(\pi) f_{i_{\pi(0)} \dots i_{\pi(p)}}$$

give a homomorphism of complexes. The induced map on cohomology is an isomorphism inverse to the map induced by the inclusion of $\check{C}^\bullet$ in $\check{C}_{\text{sing}}^\bullet$.

Proof. To show that $\phi^\bullet$ is a homomorphism of complexes, by linearity it suffices to consider images of elements $f \in \check{H}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$ which are contained in a single summand. The equality $d^p(\phi(f)) = \phi^{p+1}(d^p(f))$ then follows from a direct computation.

It is straightforward to check that ϕ^p is a section to the inclusion of $\check{H}^p(\mathcal{U}, \mathcal{F})$ in $\check{H}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$. Since this inclusion induces an isomorphism on cohomology, it follows that ϕ^p does as well. $\square$

Remark 2.2. If $\mathcal{F}$ is a sheaf of modules over a field $\mathbb{K}$ of characteristic $q \neq 0$, one may still define the map ϕ as in Lemma 2.1 for those p such that $p < q - 1$. It follows that it will still induce an isomorphism of cohomology for $p < q - 2$. In particular, since we are always assuming that our base field $\mathbb{K}$ doesn't have characteristic two or three, we will always obtain isomorphisms in cohomology for $p = 0, 1, 2$.

In the following, we will be using Čech cohomology in two situations. The first is when X is an algebraic variety, $\mathcal{F}$ is a coherent sheaf, and $\mathcal{U}$ is a particular open affine cover. In this case $\check{H}^p(\mathcal{U}, \mathcal{F})$ is canonically isomorphic to the sheaf cohomology $H^p(X, \mathcal{F})$ [Hartshorne 1977, Theorem 4.5], so we will usually just write $H^p(X, \mathcal{F})$. The second situation is when X is a finite simplicial complex, $\mathcal{F}$ is the constant sheaf with coefficients in $\mathbb{K}$, and $\mathcal{U}$ is a particular cover by closed simplices, all of whose intersections are contractible. In this case, $\check{H}^p(\mathcal{U}, \mathcal{F})$ is canonically isomorphic to the simplicial cohomology groups $H^p(X, \mathbb{K})$ [Godement 1958, §II.5.2], and we will again usually just write $H^p(X, \mathbb{K})$.

Cup products. Assume now that $\mathcal{F}$ is a sheaf of algebras on a topological space X with covering $\mathcal{U}$. The multiplication in $\mathcal{F}$ induces a *cup product* in cohomology

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \times \check{H}^q(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^{p+q}(\mathcal{U}, \mathcal{F}).$$

This is described for the *singular* Čech cohomology groups as follows; see, e.g., [Bosch 2013, §7.6 Exercise 6]. Given singular p - and q -cocycles $f = (f_{i_0 \dots i_p})$ and $f' = (f'_{i_0 \dots i_q})$, the cup product of the cohomology classes represented by f and f' is represented by the $p+q$ cocycle $g = (g_{i_0 \dots i_{p+q}})$ with

$$g_{i_0 \dots i_{p+q}} = f_{i_0 \dots i_p} * f'_{i_p \dots i_{p+q}}, \quad (3)$$

where $*$ denotes the product on $\mathcal{F}$. This product gives $\bigoplus_p \check{H}_{\text{sing}}^p(\mathcal{U}, \mathcal{F})$ the structure of a graded associative algebra.

For our purposes, we desire a description similar to (3) for the cup product between *alternating* Čech cohomology groups. This may be obtained by appropriately composing the maps between $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{C}_{\text{sing}}^\bullet(\mathcal{U}, \mathcal{F})$ with the cup product on singular Čech cohomology.

We will do this explicitly for the case of interest to us, namely, when $\mathcal{F} = \mathcal{T}_X$ is the tangent sheaf on an algebraic variety X with product induced by the Lie bracket $[\cdot, \cdot]$, and $p = q = 1$:

Lemma 2.3. *Let $f = (f_{ij})$ and $f' = (f'_{ij})$ be Čech one-cycles in $\check{C}^1(\mathcal{U}, \mathcal{T}_X)$. Then the image of their cohomology classes under the cup product map*

$$\check{H}^1(\mathcal{U}, \mathcal{T}_X) \times \check{H}^1(\mathcal{U}, \mathcal{T}_X) \rightarrow \check{H}^2(\mathcal{U}, \mathcal{T}_X)$$

is represented by the two-cycle $g = (g_{ijk})$ with

$$g_{ijk} = \frac{1}{6}([f_{ij}, f'_{jk}] + [f_{ij}, f'_{ik}] + [f_{ik}, f'_{jk}] - [f_{ik}, f'_{ij}] - [f_{jk}, f'_{ik}] - [f_{jk}, f'_{ij}]).$$

Proof. To compute the cup product, we first include f and f' in the group of singular Čech cochains $\check{C}_{\text{sing}}^1(\mathcal{U}, \mathcal{T}_X)$, and then apply (3) to find a representative $\tilde{g}$ of the cup product as a singular two-cycle. We get

$$\tilde{g}_{ijk} = [f_{ij}, f'_{jk}].$$

The claim now follows from Lemma 2.1 and a straightforward computation by setting $g = \phi^2(\tilde{g})$. $\square$

Remark 2.4. Choosing a section to the inclusion of $\check{C}^\bullet(\mathcal{U}, \mathcal{T}_X) \hookrightarrow \check{C}_{\text{sing}}^\bullet(\mathcal{U}, \mathcal{T}_X)$ that is different from our preferred section ϕ would lead to a representation of the cup product on the cocycle level that is different from that of Lemma 2.3. Our choice of section ϕ was motivated by the symmetry of the expression for g_{ijk} in this lemma.

Toric varieties. We now fix notation and review some basic facts from toric geometry. See [Fulton 1993] or [Cox et al. 2011] for a more thorough introduction. Throughout the paper we will fix a lattice M which is the character lattice of the algebraic torus $T = \text{Spec } \mathbb{K}[M]$. The lattice $N = \text{Hom}(M, \mathbb{Z})$ is the lattice of one-parameter subgroups of T . We denote the $\mathbb{Q}$ -vector spaces associated to M, N by $M_{\mathbb{Q}}$ and $N_{\mathbb{Q}}$.

Given a fan Σ in $N_{\mathbb{Q}}$, we associate a toric variety X_{Σ} ; see [Cox et al. 2011, §3.1]. The variety X_{Σ} is covered by open affine varieties U_{σ} as σ ranges over maximal cones in the fan Σ , where

$$U_{\sigma} = \text{Spec } \mathbb{K}[M \cap \sigma^{\vee}]; \quad \sigma^{\vee} = \{u \in M_{\mathbb{Q}} \mid v(u) \geq 0 \text{ for all } v \in \sigma\}.$$

We denote the regular function on U_{σ} associated to $u \in M \cap \sigma^{\vee}$ by χ^u .

Important geometric properties of X_{Σ} can be translated into properties on Σ . For example, the variety X_{Σ} is complete if and only if the fan Σ is complete, that is, the union of all cones in Σ is all of $N_{\mathbb{Q}}$ [loc. cit., Theorem 3.4.6]. Likewise, the variety X_{Σ} is smooth if and only if Σ is smooth, that is, each maximal $\sigma \in \Sigma$ has rays whose primitive generators are a subset of a lattice basis of N [loc. cit., Theorem 3.1.19]. Slightly more generally, the variety X is $\mathbb{Q}$ -factorial if and only if Σ is a simplicial fan, that is, each maximal $\sigma \in \Sigma$ has rays whose primitive generators are linearly independent [loc. cit.,

Proposition 4.2.7]. We will henceforth always assume that Σ is complete and simplicial. In other words, we will assume that X is $\mathbb{Q}$ -factorial and complete.

We denote the rays of Σ by $\Sigma^{(1)}$; to any ray $\rho \in \Sigma^{(1)}$ and $u \in M$ we denote by $\rho(u)$ the evaluation of the primitive lattice generator of ρ at u . Prime torus invariant divisors of X_Σ are in bijection with rays in $\Sigma^{(1)}$ [loc. cit., §4.1]. We denote the divisor corresponding to ρ by D_ρ . Any torus invariant divisor D may be written uniquely as a sum

$$D = \sum_{\rho \in \Sigma^{(1)}} a_\rho D_\rho.$$

The sheaf $\mathcal{O}(D)$ has the following local description: the function χ^u is in $H^0(U_\sigma, \mathcal{O}(D))$ if and only if

$$\rho(u) + a_\rho \geq 0$$

for all $\rho \in \sigma \cap \Sigma^{(1)}$. In particular, fixing a ray ρ , $\chi^u \in H^0(U_\sigma, \mathcal{O}(D_\rho))$ if and only if for all $\epsilon \in \sigma \cap \Sigma^{(1)}$,

$$\epsilon(u) \geq \begin{cases} 0 & \text{if } \epsilon \neq \rho, \\ -1 & \text{if } \epsilon = \rho. \end{cases} \quad (4)$$

The Euler sequence. The fundamental tool for understanding the tangent bundle on a smooth toric variety is the *Euler sequence*. For $X = X_\Sigma$ complete and $\mathbb{Q}$ -factorial, there is an exact sequence of sheaves

$$0 \rightarrow N^1 \otimes \mathcal{O}_X \rightarrow \bigoplus_{\rho \in \Sigma^{(1)}} \mathcal{O}_X(D_\rho) \xrightarrow{\eta} \mathcal{T}_X \rightarrow 0,$$

where N^1 is a finite-dimensional vector space; see [Cox et al. 2011, Theorem 8.1.6] (and dualize). This generalizes the standard Euler sequence on projective space. We will need an explicit description of the map η . Following through the construction in [loc. cit.] and dualizing, one obtains that

$$\eta(\chi^u) = \partial(\rho, u) \quad (5)$$

for χ^u a local section of $\mathcal{O}(D_\rho)$, where the derivation $\partial(\rho, u)$ is defined via

$$\partial(\rho, u)(\chi^v) = \rho(v)\chi^{u+v}$$

for any $v \in M$.

We will be interested in the cohomology groups of $\mathcal{T}_X$. The following was first observed by Jaczewski in the smooth case:

Lemma 2.5 [Jaczewski 1994]. *For $p \geq 1$, the map η induces isomorphisms*

$$\bigoplus_{\rho \in \Sigma^{(1)}} H^p(X, \mathcal{O}_X(D_\rho)) \rightarrow H^p(X, \mathcal{T}_X).$$

Proof. This follows directly from the Euler sequence, the long exact sequence of cohomology, and the vanishing of $H^p(X, \mathcal{O}_X)$ for $p \geq 1$; see [Cox et al. 2011, Theorem 9.2.3]. $\square$

Cohomology of divisorial sheaves on toric varieties. In order to understand the cohomology of $\mathcal{T}_X$, Lemma 2.5 implies that it will be useful to have a combinatorial description of the cohomology groups of the sheaves $\mathcal{O}(D_\rho)$. Since T acts on $X = X_\Sigma$, it will also act on the spaces of sections of $\mathcal{T}_X$ and $\mathcal{O}(D)$ for any torus invariant divisor D . This induces an M -grading on the respective cohomology groups. We will follow [Cox et al. 2011, §9.1] to describe the graded pieces of these cohomology groups. We go into what might seem more detail than necessary since we will later need explicit descriptions of the maps between various isomorphic cohomology groups.

Let $D = \sum a_\rho D_\rho$ be any torus invariant divisor. Fixing some $u \in M$, we define the simplicial complex

$$V_{D,u} = \bigcup_{\sigma \in \Sigma} \text{conv}\{n_\rho \mid \rho \in \sigma \cap \Sigma^{(1)} \text{ and } \rho(u) + a_\rho < 0\} \subset N_{\mathbb{Q}},$$

where n_ρ is the primitive generator of any ray ρ . For each $\sigma \in \Sigma$, there is a natural exact sequence

$$0 \rightarrow H^0(U_\sigma, \mathcal{O}_X(D))_u \rightarrow \mathbb{K} \rightarrow H^0(V_{D,u} \cap \sigma, \mathbb{K}) \rightarrow 0;$$

see [Cox et al. 2011, equation 9.1.10]. Here, the degree u piece of $H^0(U_\sigma, \mathcal{O}_X(D))$ is denoted by $H^0(U_\sigma, \mathcal{O}_X(D))_u$. Let I be the set of maximal cones in Σ ; we consider the open cover $\mathcal{U} = \{U_\sigma\}_{\sigma \in I}$ of X_Σ . Likewise, we have a closed cover $\mathcal{V} = \{V_\sigma\}_{\sigma \in I}$ of $V_{D,u}$, where $V_\sigma = V_{D,u} \cap \sigma$. The above exact sequence thus leads to an exact sequence

$$0 \rightarrow \check{C}^p(\mathcal{U}, \mathcal{O}(D))_u \rightarrow \check{C}^p(\mathcal{V}, \mathbb{K}) \rightarrow \check{C}^p(\mathcal{V}, \mathbb{K}) \rightarrow 0, \quad (6)$$

where $\mathcal{W} = \{W_\sigma\}_{\sigma \in I}$ is the trivial closed cover of a single point x with each $W_\sigma = x$. This sequence is compatible with the Čech differentials, so we obtain an exact sequence of Čech complexes. Since $H^0(x, \mathbb{K}) = \mathbb{K}$ and $H^p(x, \mathbb{K}) = 0$ for $p > 0$, the long exact sequence of cohomology implies that the connecting homomorphisms

$$\check{H}^{p-1}(\mathcal{V}, \mathbb{K}) \rightarrow \check{H}^p(\mathcal{U}, \mathcal{O}(D))_u \quad (7)$$

are isomorphisms if $p \geq 2$, and for $p = 1$ we have the exact sequence

$$0 \rightarrow H^0(X, \mathcal{O}(D))_u \rightarrow \mathbb{K} \rightarrow H^0(V_{D,u}, \mathbb{K}) \rightarrow H^1(X, \mathcal{O}(D))_u \rightarrow 0.$$

This final exact sequence induces an isomorphism between the reduced cohomology $\tilde{H}^0(V_{D,u}, \mathbb{K})$ and $H^1(X, \mathcal{O}(D))_u$; see [Cox et al. 2011, Theorem 9.1.3].

3. Tangent and obstruction spaces

As before, we consider a complete $\mathbb{Q}$ -factorial toric variety $X = X_\Sigma$. For $\rho \in \Sigma^{(1)}$ and $u \in M$, we define

$$V_{\rho,u} := V_{D_\rho,u}$$

and note that the vertices of $V_{\rho,u}$ have a concrete description: for $\epsilon \in \Sigma^{(1)}$, $n_\epsilon \in V_{\rho,u}$ if and only if

- (1) $\epsilon = \rho$ and $\epsilon(u) < -1$; or
- (2) $\epsilon \neq \rho$ and $\epsilon(u) < 0$.

We define $\Gamma_{\rho,u}$ and $K_{\rho,u}$ to be the one- and two-skeleta of $V_{\rho,u}$, respectively. More generally, let $V_{\rho,u}^{(p)}$ denote the p -skeleton of $V_{\rho,u}$. Below we will come to see that we only need to consider the special case when $\rho(u) = -1$, in which case the description of the vertices of $V_{\rho,u}$ simplifies and n_ρ itself is never a vertex of $V_{\rho,u}$.

We briefly comment on the decomposition

$$H^1(X, \mathcal{T}_X) = \bigoplus_{\rho \in \Sigma^{(1)}} \bigoplus_{\substack{u \in M \\ \rho(u) = -1}} \tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K}).$$

This was shown in [Ilten 2011] in the smooth case; it was noted in [Mavlyutov 2009] that this also holds in the $\mathbb{Q}$ -factorial case. The decomposition arises by combining Lemma 2.5 with the isomorphism between $\tilde{H}^0(V_{D_\rho,u}, \mathbb{K})$ and $H^1(X, \mathcal{O}(D_\rho))$ described afterwards. One then notes that $\tilde{H}^0(V_{D_\rho,u}, \mathbb{K}) = \tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K})$, and this is nonzero only if $\rho(u) = -1$.

A similar argument to the one above yields a description of $H^p(X, \mathcal{T}_X)$ for all $p \geq 1$:

Proposition 3.1. *For $p \geq 1$, the space $H^p(X, \mathcal{T}_X)$ may be decomposed as*

$$H^p(X, \mathcal{T}_X) = \bigoplus_{\substack{(\rho,u) \in \Sigma^{(1)} \times M \\ \rho(u) = -1}} \tilde{H}^{p-1}(V_{\rho,u}, \mathbb{K}) = \bigoplus_{\substack{(\rho,u) \in \Sigma^{(1)} \times M \\ \rho(u) = -1}} \tilde{H}^{p-1}(V_{\rho,u}^{(p)}, \mathbb{K}).$$

In particular, the space $H^2(X, \mathcal{T}_X)$ may be decomposed as

$$H^2(X, \mathcal{T}_X) = \bigoplus_{\rho \in \Sigma^{(1)}} \bigoplus_{\substack{u \in M \\ \rho(u) = -1}} H^1(K_{\rho,u}, \mathbb{K}).$$

Proof. By Lemma 2.5, we have an M -graded isomorphism

$$\bigoplus_{\rho \in \Sigma^{(1)}} H^p(X, \mathcal{O}_X(D_\rho)) \rightarrow H^p(X, \mathcal{T}_X).$$

Coupled with (7), we obtain

$$H^p(X, \mathcal{T}_X)_u \cong \bigoplus_{\rho \in \Sigma^{(1)}} \tilde{H}^{p-1}(V_{\rho,u}, \mathbb{K}).$$

We now show that $\tilde{H}^{p-1}(V_{\rho,u}, \mathbb{K}) = 0$ unless $\rho(u) = -1$. From the explicit description of $V_{\rho,u}$ above, we observe that if $\rho(u) \neq -1$, then $V_{\rho,\lambda \cdot u}$ is the same for any $\lambda \in \mathbb{Z}_{>0}$. In particular, if $\tilde{H}^{p-1}(V_{\rho,u}, \mathbb{K}) \neq 0$, $H^p(X, \mathcal{O}(D_\rho))$ would be an infinite-dimensional $\mathbb{K}$ -vector space, which is impossible since X is complete. We conclude that we must only consider those pairs (ρ, u) such that $\rho(u) = -1$.

Finally, the $(p-1)$ -st reduced cohomology of $V_{\rho,u}$ is the same as that of its p -skeleton $V_{\rho,u}^{(p)}$. $\square$

We will be interested in special zero-cocycles $f = (f_\sigma)$ representing elements of $\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K})$ coming from a connected component Z of $\Gamma_{\rho,u}$. For such a connected component Z , we define $f(Z) = (f(Z)_\sigma)$ by

$$f(Z)_\sigma = \begin{cases} 1 & \text{if } \sigma \cap Z \neq \emptyset, \\ 0 & \text{if } \sigma \cap Z = \emptyset. \end{cases} \quad (8)$$

These will be useful cocycles for us, since the classes of $\{f(Z)\}$ form a basis for $H^0(\Gamma_{\rho,u}, \mathbb{K})$ as Z ranges over all connected components of $\Gamma_{\rho,u}$. In particular, they provide a spanning set for $\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K})$. If we are instead considering a connected component Z' of $\Gamma_{\rho',u'}$, we will use the notation $f'(Z')$.

4. Combinatorial description of cup product

Mapping to $H^2(X, \mathcal{T}_X)$. Fix $\rho, \rho' \in \Sigma^{(1)}$ and $u, u' \in M$ satisfying $\rho(u) = \rho'(u') = -1$. We now describe the map

$$\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K}) \times \tilde{H}^0(\Gamma_{\rho',u'}, \mathbb{K}) \rightarrow H^2(X, \mathcal{T}_X)_{u+u'}$$

induced by the cup product in terms of Čech cocycles:

Lemma 4.1. *Let $f = (f_\sigma)$, $f' = (f'_\sigma)$ be Čech zero-cocycles of $\Gamma_{\rho,u}$ and $\Gamma_{\rho',u'}$. The image in $H^2(X, \mathcal{T}_X)$ of the corresponding reduced cohomology classes under the cup product is represented by the Čech two-cycle $\theta = (\theta_{\sigma\tau\gamma})$, where*

$$\theta_{\sigma\tau\gamma} = \frac{1}{2}(f_\sigma f'_\tau - f_\tau f'_\sigma + f_\gamma f'_\sigma - f_\sigma f'_\gamma + f_\tau f'_\gamma - f_\gamma f'_\tau) \cdot (\rho(u')\partial(\rho', u+u') - \rho'(u)\partial(\rho, u+u')).$$

Proof. We just need to trace through the inclusions of $\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K})$ and $\tilde{H}^0(\Gamma_{\rho',u'}, \mathbb{K})$ in $H^1(X, \mathcal{T}_X)$ and compose with the description of the cup product found in Lemma 2.3. First, mapping f to a cohomology class in $H^1(X, \mathcal{O}(D_\rho))_u$, we must use the first connecting homomorphism of (6). We do this by sending f to $a = (a_\sigma) \in \check{C}^0(\mathcal{W}, \mathbb{K})$ with $a_\sigma = f_\sigma$ and applying the differential d^0 to obtain $b = d^0(a) \in \check{C}^1(\mathcal{W}, \mathbb{K})$ with

$$b_{\sigma\tau} = a_\tau - a_\sigma = f_\tau - f_\sigma.$$

By construction, this is the image of the element $c \in \check{C}^1(\mathcal{U}, \mathcal{O}(D_\rho))_u$ where

$$c_{\sigma\tau} = (f_\tau - f_\sigma) \cdot \chi^u.$$

Mapping further to $H^1(X, \mathcal{T}_X)_u$ using Lemma 2.5, we obtain the cocycle $g = (g_{\sigma\tau})$, where

$$g_{\sigma\tau} = (f_\tau - f_\sigma)\partial(\rho, u).$$

A similar computation holds for f' ; we denote the corresponding one-cocycle in $\check{C}^1(\mathcal{U}, \mathcal{T}_X)_{u'}$ by g' .

Before applying Lemma 2.3, we note the straightforward calculation

$$\begin{aligned} [\partial(\rho, u), \partial(\rho', u')] &= \partial(\rho, u) \circ \partial(\rho', u') - \partial(\rho', u') \circ \partial(\rho, u) \\ &= \rho(u')\partial(\rho', u+u') - \rho'(u)\partial(\rho, u+u'). \end{aligned}$$

Taking this into account while applying the lemma to g and g' , we obtain the two-cocycle θ with

$$\begin{aligned} \theta_{\sigma\tau\gamma} &= \frac{1}{6} \left((f_\tau - f_\sigma)(f'_\gamma - f'_\tau) + (f_\tau - f_\sigma)(f'_\gamma - f'_\sigma) + (f_\gamma - f_\sigma)(f'_\gamma - f'_\tau) - (f_\gamma - f_\sigma)(f'_\tau - f'_\sigma) \right. \\ &\quad \left. - (f_\gamma - f_\tau)(f'_\gamma - f'_\sigma) - (f_\gamma - f_\tau)(f'_\tau - f'_\sigma) \right) \cdot (\rho(u')\partial(\rho', u+u') - \rho'(u)\partial(\rho, u+u')). \end{aligned}$$

This simplifies to the expression in the claim. $\square$

Lifting to $H^2(X, \mathcal{O}(D_\rho))$. We now show how to lift the cocycle θ of Lemma 4.1 to a cocycle representing an element of

$$H^2(X, \mathcal{O}(D_\rho))_{u+u'} \oplus H^2(X, \mathcal{O}(D_{\rho'}))_{u+u'}.$$

Lemma 4.2. *Assume that $\rho \neq \rho'$. With f, f', θ as in Lemma 4.1, define*

$$\begin{aligned} \kappa &= (\kappa_{\sigma\tau\gamma}); & \kappa' &= (\kappa'_{\sigma\tau\gamma}); \\ \kappa_{\sigma\tau\gamma} &= \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f_\tau f'_\sigma + f_\gamma f'_\sigma - f_\sigma f'_\gamma + f_\tau f'_\gamma - f_\gamma f'_\tau) \cdot \chi^{u+u'}; \\ \kappa'_{\sigma\tau\gamma} &= \frac{\rho(u')}{2} (f_\sigma f'_\tau - f_\tau f'_\sigma + f_\gamma f'_\sigma - f_\sigma f'_\gamma + f_\tau f'_\gamma - f_\gamma f'_\tau) \cdot \chi^{u+u'}. \end{aligned}$$

Then κ and κ' are two-cocycles in $\check{C}^2(\mathcal{U}, \mathcal{O}(D_\rho))$ and $\check{C}^2(\mathcal{U}, \mathcal{O}(D_{\rho'}))$, and the image of $\kappa' - \kappa$ under the map to $\check{C}^2(\mathcal{U}, \mathcal{T}_X)$ induced by η is exactly θ .

Proof. It follows from the explicit description of η in (5) that the image of $\kappa' - \kappa$ is indeed θ . So we just need to show that κ and κ' are two-cocycles. We will show below that for all $\sigma\tau\gamma$, $\kappa_{\sigma\tau\gamma}$ is an element of $H^0(U_{\sigma\cap\tau\cap\gamma}, \mathcal{O}(D_\rho))$. A similar statement will also hold for κ' . It then remains to show that κ and κ' are in the kernel of the differential d^2 . But since for $\rho \neq \rho'$, $\partial(\rho, u + u')$ and $\partial(\rho', u + u')$ are linearly independent over $\mathbb{K}$, the images of κ and κ' under the map induced by η must lie in the kernel of the differential. It follows that κ and κ' must as well.

So we are left to show the claim that $\kappa_{\sigma\tau\gamma}$ is an element of $H^0(U_{\sigma\cap\tau\cap\gamma}, \mathcal{O}(D_\rho))$ for arbitrary choice of σ, τ, γ . By bilinearity of the cup product, it suffices to do this for the special cases when $f = f(Z)$ and $f' = f'(Z')$ as in (8) for connected components Z, Z' of $\Gamma_{\rho, u}$ and $\Gamma_{\rho', u'}$. In the following, we shall fix such components Z, Z' .

In order to show that $\kappa_{\sigma\tau\gamma}$ is a regular section as desired, we need to show that either $\rho'(u) = 0$, $\chi^{u+u'} \in H^0(U_{\sigma\cap\tau\cap\gamma}, \mathcal{O}(D_\rho))$, or

$$f_\sigma f'_\tau - f_\tau f'_\sigma + f_\gamma f'_\sigma - f_\sigma f'_\gamma + f_\tau f'_\gamma - f_\gamma f'_\tau = f_\sigma (f'_\tau - f'_\gamma) + f_\tau (f'_\gamma - f'_\sigma) + f_\gamma (f'_\sigma - f'_\tau) = 0. \quad (9)$$

Let us thus assume that neither $\rho'(u)$ nor the expression in (9) is zero. Thus, all of $f'_\sigma, f'_\tau, f'_\gamma$ can not be equal, and by symmetry the same is true for $f_\sigma, f_\tau, f_\gamma$. Using that $f = f(Z)$ and $f' = f'(Z')$ and the symmetry of the expression, we may assume without loss of generality that we are in one of two cases:

- (1) $f_\sigma = 1, f_\tau = f_\gamma = 0$; or
- (2) $f_\sigma = f_\tau = 1, f_\gamma = 0$.

In the first case, we must thus have $f'_\tau \neq f'_\gamma$; without loss of generality $f'_\tau = 1$ and $f'_\gamma = 0$. In other words, our connected component Z intersects σ but not τ and γ , whereas Z' intersects τ but not γ .

Consider any ray ϵ of $\sigma \cap \tau \cap \gamma$. By (4) we must show that $\epsilon(u + u') \geq 0$ if $\epsilon \neq \rho$, and $\epsilon(u + u') \geq -1$ if $\epsilon = \rho$. Suppose $\epsilon = \rho$; then $\epsilon(u + u') = \epsilon(u') - 1$. If $\epsilon(u') < 0$, then ϵ is a vertex of $\Gamma_{\rho', u'}$ (note we are assuming $\rho \neq \rho'$). Now, ϵ is in τ , and Z' intersects τ , so by convexity, $\epsilon \in Z'$. But ϵ is also in γ , contradicting $Z' \cap \gamma = \emptyset$. Hence, $\epsilon(u') \geq 0$, implying $\epsilon(u + u') \geq -1$ as required.

Suppose instead that $\epsilon = \rho'$; then $\epsilon(u + u') = \rho'(u + u') = \epsilon(u) - 1$. If $\epsilon(u) < 0$, then ϵ is a vertex of $\Gamma_{\rho,u}$. Since $\sigma \cap Z \neq \emptyset$, convexity again implies that $\epsilon \in Z$, but this contradicts $Z \cap \tau = \emptyset$. We have also assumed that $\rho'(u) \neq 0$, so we conclude that $\epsilon(u) \geq 1$, and $\epsilon(u + u') \geq 0$.

Finally, supposed that $\epsilon \neq \rho, \rho'$. Arguing similarly to above, we cannot have $\epsilon(u) < 0$, since then we would obtain $\epsilon \in Z$, contradicting $Z \cap \tau = \emptyset$. Likewise, we cannot have $\epsilon(u') < 0$. We thus conclude $\epsilon(u + u') \geq 0$. This concludes the argument for the first case.

In the second case, we notice that we cannot have $f'_\tau = f'_\sigma$, and may argue as in the first case after appropriately permuting the roles of σ, τ , and γ . $\square$

Lifting to simplicial cohomology. We are now able to come to our main result:

Theorem 4.3. Fix $\rho, \rho' \in \Sigma^{(1)}$ and $u, u' \in M$ with $\rho(u) = \rho'(u') = -1$.

(1) The image of

$$\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K}) \times \tilde{H}^0(\Gamma_{\rho',u'}, \mathbb{K})$$

in $H^2(X, \mathcal{T}_X)$ under the cup product via (1) is 0 unless $\rho(u') = 0$ or $\rho'(u) = 0$.

(2) Assume that $\rho(u') = 0$, and let $f = (f_\sigma)$ and $f' = (f'_\sigma)$ be Čech zero-cycles of $\Gamma_{\rho,u}$ and $\Gamma_{\rho',u'}$. Then the cup product of $(\bar{f}, \bar{f}')$ is contained in $H^1(K_{\rho,u+u'})$ via (2) and may be represented by the Čech one-cocycle $g = (g_{\sigma\tau})$, where

$$g_{\sigma\tau} = \begin{cases} \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f_\tau f'_\sigma) & \text{if } K_{\rho,u+u'} \cap \sigma \cap \tau \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

A similar formula holds when $\rho'(u) = 0$.

Proof. Let f, f' be Čech zero-cocycles of $\Gamma_{\rho,u}$ and $\Gamma_{\rho',u'}$. We first show item (1). If $\rho = \rho'$, then Lemma 4.1 implies that the image of the cup product of the classes of f and f' in $H^2(X, \mathcal{T}_X)$ is zero. So we may henceforth assume that $\rho \neq \rho'$. By Lemma 4.2, the cup product of the classes of f and f' may be represented by a cocycle in

$$\bigoplus_{\rho \in \Sigma^{(1)}} H^2(X, \mathcal{O}(D_\rho))_{u+u'}$$

living entirely in the ρ and ρ' summands. But it follows from Proposition 3.1 that these vanish respectively unless $\rho(u + u') = -1$ or $\rho'(u + u') = -1$. Given $\rho(u) = \rho'(u') = -1$, this is equivalent to $\rho(u') = 0$ or $\rho'(u) = 0$. This completes the proof of the first item.

For item (2), assume now that $\rho(u') = 0$. By Lemma 4.2, it follows that the cup product of the classes of f and f' may be thought of as a class in $H^2(X, \mathcal{O}(D_\rho))_{u+u'}$, represented by the cocycle $-\kappa$ with

$$\kappa_{\sigma\tau\gamma} = \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f_\tau f'_\sigma + f_\gamma f'_\sigma - f_\sigma f'_\gamma + f_\tau f'_\gamma - f_\gamma f'_\tau) \cdot \chi^{u+u'}.$$

We consider the element $\tilde{g} \in \check{C}^1(\mathcal{W}, \mathbb{K})$ defined by

$$\tilde{g}_{\sigma\tau} = \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f_\tau f'_\sigma).$$

On the one hand, the image of $\tilde{g}$ in $\check{C}^1(\mathcal{V}, \mathbb{K})$ is exactly the one-cochain g from the statement of the theorem. On the other hand, we may compute that

$$d^1(\tilde{g})_{\sigma\tau\gamma} = \frac{\rho'(u)}{2}(f_\tau f'_\sigma - f_\sigma f'_\tau + f_\sigma f'_\gamma - f_\gamma f'_\sigma + f_\gamma f'_\tau - f_\tau f'_\gamma)$$

in $\check{C}^2(\mathcal{W}, \mathbb{K})$. This is exactly the image of $-\kappa$ under the inclusion

$$\check{C}^2(\mathcal{U}, \mathcal{O}(D_\rho))_{u+u'} \hookrightarrow \check{C}^2(\mathcal{W}, \mathbb{K}).$$

The exact sequence (6) and its compatibility with the Čech differentials implies that g is a cocycle in $\check{C}^1(\mathcal{V}, \mathbb{K})$, and that the image of its cohomology class under the connecting homomorphism is represented by $-\kappa$. This completes the proof of the second claim. $\square$

Remark 4.4. With notation as in Theorem 4.3, one might wonder what happens to the cup product when both $\rho(u') = 0$ and $\rho'(u) = 0$. It follows directly from the second part of the theorem that the image of this part of the cup product vanishes.

5. Pulling back to cycles

Setup. Let α be a simple cycle in $K_{\rho, u+u'}$, that is, an oriented connected subgraph of the edges of $K_{\rho, u+u'}$ in which no edges are repeated and every vertex has degree 2. Such a cycle α gives rise to a one-cycle $[\alpha]$ in the simplicial homology $H_1(K_{\rho, u+u'}, \mathbb{K})$ by considering the sum

$$[\alpha] = \sum_{E \in \alpha} \pm E$$

with signs depending on the orientation of α and the chosen orientation of $K_{\rho, u+u'}$. This similarly determines a distinguished generator of $H_1(\alpha, \mathbb{K})$ which we will also denote by $[\alpha]$. We denote the element in $H^1(\alpha, \mathbb{K})$ dual to the class in $H_1(\alpha, \mathbb{K})$ of $[\alpha]$ by α_{fun} .

Definition 5.1. A Σ -reduced cycle is a simple cycle α in $K_{\rho, u+u'}$, with $[\alpha]$ not homologous to zero, such that no edges of α are contained in a common cone of Σ .

By the following lemma, it will suffice to consider only Σ -reduced cycles:

Lemma 5.2. Any class in $H_1(K_{\rho, u+u'}, \mathbb{K})$ may be written as a sum of classes of Σ -reduced cycles.

Proof. It suffices to show the lemma for classes represented by $[\beta]$, for some simple cycle β . Suppose that β is not Σ -reduced, with edges E_1 and E_2 contained in a cone σ . If $E_1 = [v_1, v_2]$ and $E_2 = [v_2, v_3]$, then we may replace these edges in β by $E = [v_1, v_3]$ and obtain an equivalent homology class. The resulting simple cycle β' has one fewer edge.

Similarly, if $E_1 \cap E_2 = \emptyset$, we may split β into simple cycles β', β'' such that $[\beta]$ is homologous to $[\beta'] + [\beta'']$ and both these cycles have fewer edges.

The claim now follows by infinite descent. $\square$

Given a Σ -reduced cycle α , the closed cover $\mathcal{V}$ of $K_{\rho, u+u'}$ induces a closed cover $\mathcal{V}^\alpha = \{V_\sigma^\alpha\}$ of α :

$$V_\sigma^\alpha = \alpha \cap \sigma.$$

This closed cover is contractible, so $H^p(\mathcal{V}^\alpha, \mathbb{K}) = H^p(\alpha, \mathbb{K})$.

There is a natural map of complexes

$$\check{C}^\bullet(\mathcal{V}, \mathbb{K}) \rightarrow \check{C}^\bullet(\mathcal{V}^\alpha, \mathbb{K})$$

induced by restriction. The corresponding map of cohomology

$$\iota_\alpha^* : H^p(K_{\rho, u+u'}, \mathbb{K}) \rightarrow H^p(\alpha, \mathbb{K})$$

is the pullback morphism corresponding to the inclusion $\iota_\alpha : \alpha \hookrightarrow K_{\rho, u+u'}$. In particular, for any $\omega \in H^1(K_{\rho, u+u'}, \mathbb{K})$,

$$\iota_\alpha^*(\omega) = \langle \alpha, \omega \rangle \cdot \alpha_{\text{fun}}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between H_1 and H^1 .

Computing the cup product. Fix a Σ -reduced cycle α , and let Z and Z' be connected components of $\Gamma_{\rho, u}$ and $\Gamma_{\rho', u'}$. As in Theorem 4.3, assume that $\rho(u') = 0$. We now show how to compute $\iota_\alpha^*(\omega)$ directly, where ω is the class of $H^1(K_{\rho, u+u'}, \mathbb{K})$ corresponding to the cup product of the classes of $f(Z)$ and $f'(Z')$.

Let $E(\alpha)$ denote the set of edges α . We will write

$$E(\alpha) = \{E_1, \dots, E_k\}$$

with the edges ordered cyclically modulo k (E_{i+1} is the edge following E_i). For each edge E_i of α we fix a maximal cone σ_i with $\sigma_i \cap \alpha = E_i$. These cones exist since α is Σ -reduced.

To Z we associate a subset $\alpha(Z)$ of $E(\alpha)$:

$$\alpha(Z) = \{E_i \in E(\alpha) \mid Z \cap \sigma_i \neq \emptyset\}.$$

The set $\alpha(Z')$ is defined analogously. An index $1 \leq i \leq k$ is *relevant* if

$$\alpha(Z) \cap \{E_i, E_{i+1}\}, \quad \alpha(Z') \cap \{E_i, E_{i+1}\}$$

are not equal but both nonempty, and their union contains both E_i and E_{i+1} . Here, indices are taken modulo k .

We set

$$Z *_\alpha Z' := \frac{\rho'(u)}{2} \cdot \sum_{i \text{ relevant}} b_i, \quad b_i = \begin{cases} 1 & \text{if } E_i \in \alpha(Z), E_{i+1} \in \alpha(Z'), \\ -1 & \text{if } E_i \in \alpha(Z'), E_{i+1} \in \alpha(Z). \end{cases}$$

See Figure 1 for an illustration of various values of b_i : for each relevant index i , the value of b_i is recorded in parenthesis next to the edge E_i . For example in the leftmost case, there are two relevant indices, each with $b_i = 1$.

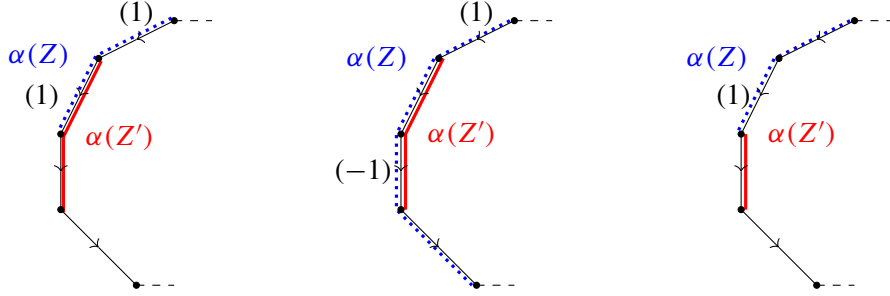


Figure 1. Values of b_i .

Theorem 5.3. *With the above notation,*

$$\iota_\alpha(\omega)^* = (Z *_\alpha Z') \cdot \alpha_{\text{fun}}.$$

*In particular, the image of $\tilde{H}^0(\Gamma_{\rho,u}, \mathbb{K}) \times \tilde{H}^0(\Gamma_{\rho',u'}, \mathbb{K})$ under the cup product is zero if and only if for all Σ -reduced cycles α and all connected components Z, Z' of $\Gamma_{\rho,u}$ and $\Gamma_{\rho',u'}$, $Z *_\alpha Z' = 0$.*

Proof. We know by Theorem 4.3 that ω is represented in $H^1(K_{\rho,u+u'}, \mathbb{K})$ by the cocycle $g = (g_{\sigma\tau})$, where

$$g_{\sigma\tau} = \begin{cases} \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f'_\sigma f_\tau) & \text{if } K_{\rho,u+u'} \cap \sigma \cap \tau \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The image of this cocycle under the map

$$\check{C}^1(\mathcal{V}, \mathbb{K}) \rightarrow \check{C}^1(\mathcal{V}^\alpha, \mathbb{K})$$

is thus $(h_{\sigma\tau})$, where

$$h_{\sigma\tau} = \begin{cases} \frac{\rho'(u)}{2} (f_\sigma f'_\tau - f'_\sigma f_\tau) & \text{if } \alpha \cap \sigma \cap \tau \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

There is an easier closed cover $\mathcal{W}^\alpha = (E_i)$ of α that we would like to use; it is indexed by $1 \leq i \leq k$. This is again a closed cover with all intersections contractible, so it also computes $H^1(\alpha, \mathbb{K})$. The assignment $i \mapsto \sigma_i$ lets us view $\mathcal{V}^\alpha$ as a refinement of $\mathcal{W}^\alpha$, and induces a map of Čech complexes

$$\check{C}^p(\mathcal{W}^\alpha, \mathbb{K}) \rightarrow \check{C}^p(\mathcal{V}^\alpha, \mathbb{K});$$

see, e.g., [Hartshorne 1977, Exercise III.4.4]. This map has a natural section given by forgetting entries with indices not among the σ_i ; both maps induce isomorphisms on cohomology.

Hence, we may represent $\iota_\alpha^*(\omega)$ as a Čech one-cocycle with respect to $\mathcal{W}^\alpha$ by (a_{ij}) with

$$a_{ij} = \begin{cases} \frac{\rho'(u)}{2} (f_{\sigma_i} f'_{\sigma_j} - f_{\sigma_j} f'_{\sigma_i}) & \text{if } i - j \equiv \pm 1 \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

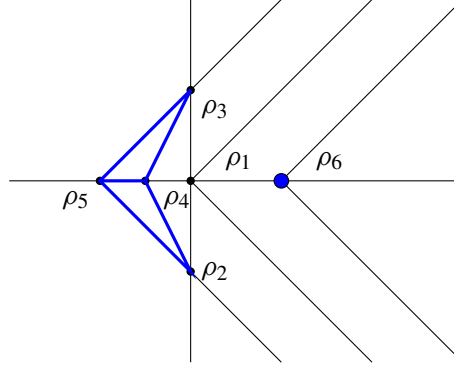


Figure 2. $\Sigma \cap [u = -1]$ for $u = (-1, 0, 0)$ and $\Gamma_{\rho_1, u}$.

On the other hand, a straightforward computation shows that for any Čech one-cocycle $c = (c_{ij})$ with respect to $\mathcal{W}^\alpha$, the class represented by c in $H^1(\alpha, \mathbb{K})$ is

$$\left(\sum_{i=1}^k c_{i(i+1)} \right) \cdot \alpha_{\text{fun}},$$

where indices are taken modulo k .

We thus compute

$$\begin{aligned} \sum_{i=1}^k a_{i(i+1)} &= \sum_{i=1}^k \frac{\rho'(u)}{2} (f_{\sigma_i} f'_{\sigma_{i+1}} - f_{\sigma_{i+1}} f'_{\sigma_i}) \\ &= \sum_{i \text{ relevant}} \frac{\rho'(u)}{2} (f_{\sigma_i} f'_{\sigma_{i+1}} - f_{\sigma_{i+1}} f'_{\sigma_i}) = \sum_{i \text{ relevant}} \frac{\rho'(u)}{2} b_i = Z *_{\alpha} Z'. \end{aligned} \quad \square$$

6. An obstructed example

We now consider the following concrete example. Let $N = \mathbb{Z}^3$, and define rays

$$\begin{aligned} \rho_1 &= \mathbb{Q}_{\geq 0} \cdot (1, 0, 0), & \rho_2 &= \mathbb{Q}_{\geq 0} \cdot (1, 0, -1), & \rho_3 &= \mathbb{Q}_{\geq 0} \cdot (1, 0, 1), \\ \rho_4 &= \mathbb{Q}_{\geq 0} \cdot (2, -1, 0), & \rho_5 &= \mathbb{Q}_{\geq 0} \cdot (1, -1, 0), & \rho_6 &= \mathbb{Q}_{\geq 0} \cdot (1, 1, 0), \\ \rho_7 &= \mathbb{Q}_{\geq 0} \cdot (0, 1, -1), & \rho_8 &= \mathbb{Q}_{\geq 0} \cdot (0, 1, 1), & \rho_9 &= \mathbb{Q}_{\geq 0} \cdot (-1, 0, 0). \end{aligned}$$

These ρ_i form the rays of a smooth complete fan Σ whose maximal cones are spanned by

$$\begin{array}{lll} \rho_1, \rho_2, \rho_4, & \rho_1, \rho_2, \rho_7, & \rho_1, \rho_3, \rho_4, \\ \rho_1, \rho_3, \rho_8, & \rho_1, \rho_6, \rho_7, & \rho_1, \rho_6, \rho_8, \\ \rho_2, \rho_4, \rho_5, & \rho_2, \rho_5, \rho_9, & \rho_2, \rho_7, \rho_9, \\ \rho_3, \rho_4, \rho_5, & \rho_3, \rho_5, \rho_9, & \rho_3, \rho_8, \rho_9, \\ & \rho_6, \rho_7, \rho_9, & \rho_6, \rho_8, \rho_9. \end{array}$$

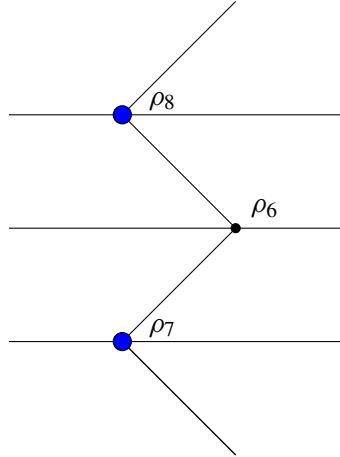


Figure 3. $\Sigma \cap [u' = -1]$ for $u' = (0, -1, 0)$ and $\Gamma_{\rho_6, u'}$.

We will see using Theorem 5.3 that X_Σ has nonvanishing cup-product, and hence obstructed deformations. This will show:

Corollary 6.1. *There exists a smooth complete toric threefold with obstructed deformations.*

The degrees where we will look for first-order deformations are $u = (-1, 0, 0)$ and $u' = (0, -1, 0)$. We picture the intersection of Σ with the hyperplane

$$\{v \in N_{\mathbb{Q}} \mid v(u) = -1\}$$

in Figure 2. The graph $\Gamma_{\rho_1, u}$ is also pictured in the figure in blue bold lines. It has two connected components: one component contains the generators of the rays $\rho_2, \rho_3, \rho_4, \rho_5$, and the other contains the generator of ρ_6 . We will denote the first component by Z . Note that for any other choice of ray ρ , $\Gamma_{\rho, u}$ is connected. Hence, $H^1(X, \mathcal{T}_X)_u$ is one-dimensional.

In Figure 3 we picture the intersection of Σ with the hyperplane

$$\{v \in N_{\mathbb{Q}} \mid v(u') = -1\}$$

along with the graph $\Gamma_{\rho_6, u'}$. This graph has two connected components, consisting of the primitive generators of ρ_7 and ρ_8 . We denote the first of these components by Z' . Again for any other choice of ray ρ , $\Gamma_{\rho, u'}$ is connected, so $H^1(X, \mathcal{T}_X)_{u'}$ is also one-dimensional.

We will now compute the cup product ω of the first-order deformations corresponding to Z and Z' . By Theorem 4.3, it is possible that this is nonzero, since $\rho_1(u') = 0$. This class will live in degree $u + u'$, so we picture the intersection of Σ with the hyperplane

$$\{v \in N_{\mathbb{Q}} \mid v(u + u') = -1\}$$

in Figure 4.

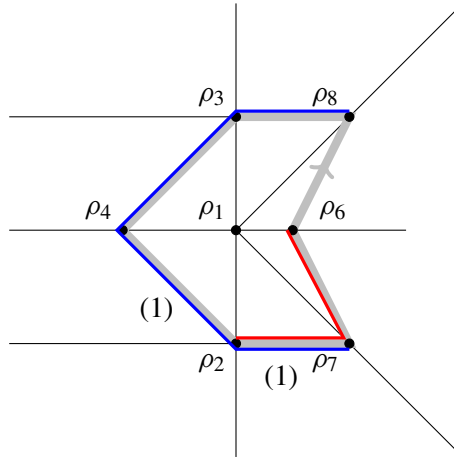


Figure 4. $\Sigma \cap [u + u' = -1]$, α , and $\alpha(Z), \alpha(Z')$.

Let α be the Σ -reduced cycle pictured in gray oriented in counter-clockwise direction. That is, α has vertices (in order) at the primitive generators of

$$\rho_8, \rho_3, \rho_4, \rho_2, \rho_7, \rho_6, \rho_8.$$

This is indeed Σ -reduced. To apply Theorem 5.3, we must choose a maximal cone $\sigma_i \in \Sigma$ for each edge of α . In this instance there is a canonical choice: for the edge corresponding to rays ρ_i and ρ_j , we take the cone generated by ρ_i, ρ_j, ρ_1 .

This gives rise to the sets $\alpha(Z)$ and $\alpha(Z')$, pictured in the figure in blue and red, respectively. The set $\alpha(Z)$ is a subgraph with vertices equal to the primitive generators of $\rho_8, \rho_3, \rho_4, \rho_2, \rho_7$, and the set $\alpha(Z')$ is a subgraph with vertices equal to the primitive generators of ρ_2, ρ_7, ρ_6 .

The only *relevant* indices are for the edges corresponding to ρ_4, ρ_2 and ρ_2, ρ_7 ; each contributes a value of $b_i = 1$, so we obtain by Theorem 5.3 that

$$\iota_\alpha^*(\omega) = 1 \cdot \alpha_{\text{fun}}.$$

In particular, the cup product ω of the first-order deformations corresponding to Z and Z' is nonzero.

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