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The basepoint-freeness threshold and syzygies of abelian varieties

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We show how a natural constant introduced by Jiang and Pareschi for a polarized abelian variety encodes information about the syzygies of the section ring of the polarization. As a particular case this gives a quick and characteristic-free proof of Lazarsfeld's conjecture on syzygies of abelian varieties, originally proved by Pareschi in characteristic zero.

1. Introduction

Throughout this paper we will work with abelian varieties over an algebraically closed field $\mathbb{K}$. Jiang and Pareschi [2017] introduced and studied the (generic) cohomological ranks $h^i(A, \mathcal{F}\langle x\underline{l}\rangle)$ of a (bounded complex of) $\mathbb{Q}$-twisted coherent sheaf on a polarized abelian variety $(A, \underline{l})$. This defines *cohomological rank functions* of $\mathcal{F}$ with respect to the polarization $\underline{l}$

$$h^i_{\mathcal{F},\underline{l}} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}, \text{ [1]}$$

as follows

$$h^i_{\mathcal{F},\underline{l}}(x) := h^i(A, \mathcal{F}\langle x\underline{l}\rangle).$$

In [loc. cit.] it is observed that these functions are already very interesting in the case $\mathcal{F} = \mathcal{I}_p$, where $\mathcal{I}_p$ is the ideal sheaf of a closed point $p \in A$. Indeed the *basepoint-freeness threshold*

$$\epsilon_1(\underline{l}) := \mathrm{Inf}\{x \in \mathbb{Q} \mid h^1_{\mathcal{I}_p,\underline{l}}(x) = 0\}, \text{ [2]}$$

has the following properties:

(a) $\epsilon_1(\underline{l}) \leq 1$ *and* $\epsilon_1(\underline{l}) < 1$ *if and only if the polarization* $\underline{l}$ *is basepoint-free*, i.e., any line bundle L representing $\underline{l}$ has no base points.

(b) $\epsilon_1(\underline{l}) < \frac{1}{2}$ *if and only if* $\underline{l}$ *is projectively normal,* meaning that L is projectively normal for all line bundles L representing the class $\underline{l}$ [Jiang and Pareschi 2017, Corollary E].

In this paper we go further on item (b), proving that $\epsilon_1(\underline{l})$ indeed encodes information about the syzygies of the section algebra of L. In recent years syzygies of abelian varieties has received considerable attention. On the one hand Pareschi [2000] (see also [Pareschi and Popa 2004]), building partially on previous works of Kempf [1989; 1991], proved, in characteristic zero, Lazarsfeld's conjecture on syzygies of

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[1] In [loc. cit.] such functions are extended to (continuous) real functions, but in this paper we don't need this.

[2] In [loc. cit.] this is denoted by $\beta(\underline{l})$.

abelian varieties endowed with a polarization which is a multiple of a given one. This was is in turn a generalization of classical results of Koizumi and Mumford. On the other hand, more recently Küronya and Lozovanu [2019], Ito [2018] and Lozovanu [2018], building on previous work of Hwang and To [2011] and Lazarsfeld, Pareschi and Popa [Lazarsfeld et al. 2011], used completely different methods — involving local positivity and Nadel vanishing theorem — to prove (over $\mathbb{C}$) effective statements for the syzygies of abelian varieties of dimension 2 and 3 endowed with *any* polarization, in particular with a primitive polarization.

In this paper we show a general result, Theorem 1.1 below, partially generalizing (b) to higher syzygies. This provides at the same time a surprisingly quick proof of Lazarsfeld's conjecture, extending it to abelian varieties defined over a ground field of arbitrary characteristic, and a proof of the criterion of [Lazarsfeld et al. 2011] relating local positivity and syzygies.

Turning to details, we first recall some terminology about syzygies of projective varieties. Let X be a projective variety and let L be an ample line bundle on X. For an integer $p \geq 0$, the line bundle L is said to *satisfy the property* (N_p) if the first p steps of the minimal graded free resolution of the section algebra $R_L = \bigoplus_m H^0(X, L^m)$ over the polynomial ring $S_L = \operatorname{Sym} H^0(X, L)$ are linear (we refer to Section 4 for the precise definition). Thus (N_0) means that R_L is generated in degree 0 as an S_L-module, i.e., that L is projectively normal (*normally generated* in Mumford's terminology [1970]); (N_1) means that in addition the homogeneous ideal $I_{X/\mathbb{P}}$ of X in $\mathbb{P} = \mathbb{P}(H^0(X, L)^\vee)$ is generated by quadrics (*normally presented* in [Mumford 1970]); (N_2) means that the relations among these quadrics are generated by linear ones (this is the first nonclassical condition) and so on. These notions were introduced by Green [1984] and the present terminology was introduced in [Green and Lazarsfeld 1986]. Our main result is the following:

Theorem 1.1. *Let $(A, \underline{l})$ be a polarized abelian variety defined over an algebraically closed field $\mathbb{K}$ and let p be a nonnegative integer. If*

$$\epsilon_1(\underline{l}) < \frac{1}{p+2},$$

then the property (N_p) holds for $\underline{l}$, i.e., it holds for any line bundle L representing $\underline{l}$.

Corollary 1.2. *Let $m \in \mathbb{N}$. If*

$$\epsilon_1(\underline{l}) < \frac{m}{p+2},$$

then the polarization $m\underline{l}$ satisfies the property (N_p).

Proof. By definition (see Section 2) we have $h^1_{\mathcal{I}_p, m\underline{l}}(x) = h^1_{\mathcal{I}_p, \underline{l}}(mx)$, therefore $\epsilon_1(m\underline{l}) = \epsilon_1(\underline{l})/m$. Now Theorem 1.1 applies to $m\underline{l}$, because $\epsilon_1(m\underline{l}) < 1/(p+2)$. □

A classical result of Koizumi [1976] states that if L is an ample line bundle on a complex abelian variety and $m \geq 3$, then L^m is projectively normal (see [Sekiguchi 1976; 1977; Sasaki 1977] for a proof of the analog result in positive characteristic, based on Mumford's ideas). Moreover, a well-known theorem of Mumford and Kempf says that, when $m \geq 4$, the homogeneous ideal of A in the embedding given by L^m is generated by quadrics [Mumford 1970; Kempf 1991, Theorem 6.13], i.e., L^m is normally presented. Based on these classical facts and motivated by a result of Green [1984] on higher syzygies for curves, Lazarsfeld conjectured that, for an ample line bundle L on an abelian variety, L^m satisfies the property (N_p) if $m \geq p+3$ [Lazarsfeld 1989, Conjecture 1.5.1]. This was proved by Pareschi [2000] in characteristic zero. Pareschi and Popa [2004] also proved a stronger version of it.

We have that Corollary 1.2 gives a very quick — and characteristic-free — proof of Lazarsfeld's conjecture. Indeed, by (a) above,

$$\epsilon_1(\underline{l}) \leq 1 < \frac{p+3}{p+2}.$$

Moreover it also implies that the polarization $m\underline{l}$ satisfies the property (N_p), as soon as $m \geq p+2$ and $\underline{l}$ is basepoint-free (see [Pareschi and Popa 2004] for a more precise result). Indeed, if $\underline{l}$ is basepoint-free, then

$$\epsilon_1(\underline{l}) < 1 = \frac{p+2}{p+2},$$

thanks again to (a) above.

More in general, defining

$$t(\underline{l}) := \max\{t \in \mathbb{N} \mid \epsilon_1(\underline{l}) \leq \tfrac{1}{t}\},$$

we have

Theorem 1.3. *Let p and t be nonnegative integers with $p+1 \geq t$. Let $\underline{l}$ be a basepoint-free polarization on A such that $t(\underline{l}) \geq t$. Then the property (N_p) holds for $m\underline{l}$, as soon as $m \geq p+3-t$.*

However, one of the main feature of Theorem 1.1 is the chance to be applied to *primitive* polarizations, i.e., those that cannot be written as a multiple of another one. This is one of the reasons why it would be quite interesting the compute, or at least bound from above, the invariant $\epsilon_1(\underline{l})$ of polarized abelian varieties $(A, \underline{l})$. In this perspective, as already mentioned, an interesting issue arises in connection with a criterion of Lazarsfeld, Pareschi and Popa [2011], where they prove

> *if there exists an effective $\mathbb{Q}$-divisor F such that its multiplier ideal $\mathcal{J}(A, F)$ is the ideal sheaf of the identity point of the abelian variety A and $\frac{1}{p+2}\underline{l} - F$ is ample, then $\underline{l}$ satisfies the property N_p* (see [Küronya and Lozovanu 2019; Ito 2018; Lozovanu 2018]).

Therefore one is lead to consider the threshold

$$r(\underline{l}) := \mathrm{Inf}\{r \in \mathbb{Q} \mid \exists \text{ an effective } \mathbb{Q}\text{-divisor } F \text{ on } A \text{ such that } r\underline{l} - F \text{ is ample and } \mathcal{J}(A, F) = \mathcal{I}_0\}.$$ [3]

The relation with the basepoint-freeness threshold is in the following proposition, based on Nadel's vanishing.

Proposition 1.4. *Assume $\mathbb{K} = \mathbb{C}$. Then $\epsilon_1(\underline{l}) \leq r(\underline{l})$.*

This, combined with Theorem 1.1, provides a different and simpler proof of the criterion of [Lazarsfeld et al. 2011].

Finally, we note that in the papers [Küronya and Lozovanu 2019; Ito 2018] for dimension 2 and [Lozovanu 2018] for dimension 3, the authors, in the spirit of Green's and Green and Lazarsfeld's conjectures on curves, show explicit geometric conditions ensuring the property (N_p) by means of upper bounds on the threshold $r(\underline{l})$ (or related invariants) and applying the criterion of [Lazarsfeld et al. 2011].

[3] Note that this set is nonempty, i.e., $r(\underline{l}) < +\infty$. Proof: Let k be a sufficiently large positive integer such that the Seshadri constant of $M = L^k$ is strictly bigger than $2 \dim A$. Such a k exists because of the homogeneity of the Seshadri constant. Then, by Lemma 1.2 of [Lazarsfeld et al. 2011], there exists an effective $\mathbb{Q}$-divisor F on A such that $\mathcal{J}(A, F) = \mathcal{I}_0$ and $F \equiv_{\mathrm{num}} \frac{1}{2}(1-c)M$, for some $0 < c \ll 1$. If we now take $r > \frac{1}{2}(1-c)k$, we have that $r\underline{l} - F$ is ample.

This suggests to look for similar estimates directly for the basepoint-freeness threshold $\epsilon_1(\underline{l})$. Namely one could ask if $\epsilon_1(\underline{l})$ is less or equal to

$$\mathrm{Inf}\{r \in \mathbb{Q}^+ \mid (D_r^{\dim Z} \cdot Z) > (\dim Z)^{\dim Z} \text{ for any abelian subvariety } \{0\} \neq Z \subseteq A\},$$

where $D_r := rL$ (see in particular [Ito 2018, Question 4.2]). This is true for complex abelian surfaces, thanks to the Proposition 1.4 and [Ito 2018].

The paper is organized as follows: In Section 2 we recall the definition and some basic properties of cohomological rank functions, and show that, despite the fact that in [Jiang and Pareschi 2017] the authors assume that the characteristic of the ground field is zero, the basic theory of cohomological rank functions works over an algebraically closed ground field of arbitrary characteristic as well. Finally, in this section we prove Proposition 1.4.

In Section 3 we prove the basic properties of the threshold $\epsilon_1(\underline{l})$ needed in the proof of the main results.

In Section 4 we show a criterion, due to Kempf [1989], reducing the property (N_p) of syzygies to the surjectivity of certain multiplication maps of global sections, inductively defined. This is easily proved and well-known in characteristic zero (see e.g., [Ein and Lazarsfeld 1993, proof of Corollary 2.2] or [Pareschi 2000, Lemma 4.1(a)]). Kempf's approach is more complicated, but has the advantage of working in arbitrary characteristic. Since Kempf's argument is somewhat obscure, we provide full details. We hope that this will be useful for extending to arbitrary characteristic some of known results concerning syzygies of projective varieties in characteristic zero.

In Section 5 we prove the Theorems 1.1 and 1.3.

Notation. Let A be an abelian variety over an algebraically closed field, and let $\dim A = g$. For $b \in \mathbb{Z}$,

$$\mu_b : A \to A, \quad x \mapsto bx$$

denotes the multiplication-by-b isogeny of degree b^{2g}. A polarization $\underline{l}$ on A is the class of an ample line bundle L in $\mathrm{NS}(A) = \mathrm{Pic}A/\mathrm{Pic}^0 A$. For a polarization $\underline{l}$ on A, the corresponding isogeny is denoted

$$\varphi_{\underline{l}} : A \to \hat{A},$$

where $\hat{A} = \mathrm{Pic}^0 A$ is the dual abelian variety. Recall that $\deg(\varphi_{\underline{l}}) = \chi(\underline{l})^2 = (h^0(\underline{l}))^2$. We denote by $\mathcal{P}$ the normalized Poincaré line bundle on $A \times \hat{A}$, and by $R\Phi_{\mathcal{P}} : \mathrm{D}^b(A) \to \mathrm{D}^b(\hat{A})$ the Fourier–Mukai–Poincaré equivalence [Mukai 1981]. Here $\mathrm{D}^b(A)$ denotes the bounded derived category of coherent sheaves on A. For $\alpha \in \hat{A}$, the corresponding line bundle on A is denoted by $P_\alpha = \mathcal{P}|_{A \times \{\alpha\}}$. Given a complex $\mathcal{F} \in \mathrm{D}^b(A)$, we denote by $\mathcal{F}^\vee = R\mathcal{H}om(\mathcal{F}, \mathcal{O}_A)$ its derived dual, and by $h^i_{\mathrm{gen}}(A, \mathcal{F})$ the dimension of the hypercohomology $H^i(A, \mathcal{F} \otimes P_\alpha)$, for α general in $\hat{A}$.

2. Cohomological rank functions on abelian varieties

Given $\mathcal{F} \in \mathrm{D}^b(A)$, $i \in \mathbb{Z}$ and a polarization $\underline{l}$ on A, Jiang and Pareschi [2017] considered cohomological rank functions

$$h^i_{\mathcal{F},\underline{l}} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$

defined as follows:

$$h^i_{\mathcal{F},\underline{l}}(x) = h^i_{\mathcal{F}}(x\underline{l}) := \frac{1}{b^{2g}} h^i_{\mathrm{gen}}(A, \mu_b^*(\mathcal{F}) \otimes L^{ab}),$$

where $x = a/b \in \mathbb{Q}$ and $b > 0$. Since $\mu_b^*(\underline{l}) = b^2\underline{l}$, the pullback via μ_b of the rational class $(a/b)\underline{l}$ is $ab\underline{l}$. Moreover $\deg(\mu_b) = b^{2g}$, therefore, as explained in Remark 2.2 of [loc. cit.], one may think of $h^i_{\mathcal{F},\underline{l}}(x)$ as the (generic) cohomological rank $h^i(A, \mathcal{F}\langle x\underline{l}\rangle)$ of the $\mathbb{Q}$-*twisted complex* $\mathcal{F}\langle x\underline{l}\rangle$, which is defined similarly to [Lazarsfeld 2004, Section 6.2A]. Namely, $\mathcal{F}\langle x\underline{l}\rangle$ is the equivalence class of the pair $(\mathcal{F}, x\underline{l})$, where the equivalence is by definition

$$(\mathcal{F}\otimes L^m, x\underline{l}) \sim (\mathcal{F}, (m+x)\underline{l}),$$

for *any* line bundle L representing $\underline{l}$ and $m \in \mathbb{Z}$. Note that an "untwisted" object $\mathcal{F}$ may be naturally seen as the $\mathbb{Q}$-twisted object $\mathcal{F}\langle 0\underline{l}\rangle$. Moreover we have that $\mathcal{F}\otimes P_\alpha\langle x\underline{l}\rangle = \mathcal{F}\langle x\underline{l}\rangle$, for any $\alpha \in \mathrm{Pic}^0(A)$.

In [Jiang and Pareschi 2017] the authors introduced such notion assuming that the characteristic of the ground field $\mathbb{K}$ is zero. However the above definition makes sense in any characteristic. The main point consists in showing that it does not depend on the representation $x = a/b$. To this purpose we need to verify that the quantity $h^i_{\mathrm{gen}}(A, \mathcal{F})$ is multiplicative with respect to any isogeny μ_m:

$$h^i_{\mathrm{gen}}(A, \mu_m^*\mathcal{F}) = m^{2g} h^i_{\mathrm{gen}}(A, \mathcal{F}). \tag{2-1}$$

This is checked in [loc. cit.] under the assumption that $\mathrm{char}(\mathbb{K}) = 0$. However the same thing can be checked removing such assumption as follows. By cohomology and base change, $h^i_{\mathrm{gen}}(A, \mu_m^*\mathcal{F})$ is the generic rank of the Fourier–Mukai–Poincaré transform $R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F})$. Moreover $R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F}) = \hat{\mu}_{m*}R^i\Phi_{\mathcal{P}}(\mathcal{F})$ [Mukai 1981, (3.4)], where $\hat{\mu}_m : \hat{A} \to \hat{A}$ is the dual isogeny of μ_m, i.e., it is the multiplication-by-m isogeny of $\hat{A}$. Since the morphism $\hat{\mu}_m$ is in any case flat, the generic rank of $\hat{\mu}_{m*}R^i\Phi_{\mathcal{P}}(\mathcal{F})$ is that of $R^i\Phi_{\mathcal{P}}(\mathcal{F})$ multiplied by the degree of $\hat{\mu}_m$. Therefore we get (2-1). Granting this, $h^i_{\mathcal{F}}(x\underline{l})$ is well-defined: if we take another representation of x, say $x = (am)/(bm)$, then

$$\begin{aligned} h^i_{\mathcal{F}}(x\underline{l}) &= \frac{1}{(bm)^{2g}} h^i_{\mathrm{gen}}(A, \mu_{bm}^*(\mathcal{F})\otimes L^{abm^2}) \\ &= \frac{1}{(bm)^{2g}} h^i_{\mathrm{gen}}(A, \mu_m^*(\mu_b^*(\mathcal{F})\otimes L^{ab})) \\ &= \frac{1}{b^{2g}} h^i_{\mathrm{gen}}(A, \mu_b^*(\mathcal{F})\otimes L^{ab}). \end{aligned}$$

Remark 2.1. Although we won't need this in this paper, we remark that from the above discussion it follows that the basic properties satisfied by the cohomological rank functions described in Section 2 of [Jiang and Pareschi 2017] — especially the fundamental transformation formula with respect to the Fourier–Mukai–Poincaré transform Proposition 2.3 of [loc. cit.] and its consequences — work in any characteristic.

Using the cohomological rank functions it is possible to introduce several invariants attached to a polarized abelian variety $(A, \underline{l})$. Let us recall that, given a line bundle L that represents the class $\underline{l}$, the *kernel bundle* M_L associated to L is by definition the kernel of the evaluation map $H^0(A, L)\otimes\mathcal{O}_A \to L$. If L is basepoint-free, then M_L sits in the exact sequence

$$0 \to M_L \to H^0(A, L)\otimes\mathcal{O}_A \to L \to 0.$$

Definition 2.2. Let $(A, \underline{l})$ be a polarized abelian variety. Then we consider

$$\epsilon_1(\underline{l}) := \operatorname{Inf}\{x \in \mathbb{Q} \mid h^1_{\mathcal{I}_p}(x\underline{l}) = 0\},$$

where $\mathcal{I}_p$ is the ideal sheaf of a closed point $p \in A$ and, if $\underline{l}$ is basepoint-free

$$\kappa_1(\underline{l}) := \operatorname{Inf}\{x \in \mathbb{Q} \mid h^1_{M_L}(x\underline{l}) = 0\},$$

where M_L is the kernel bundle associated to a line bundle L representing $\underline{l}$.

Remark 2.3. The above invariants are well-defined, i.e., $\epsilon_1(\underline{l})$ does not depend on the point p, and $\kappa_1(\underline{l})$ is independent from the representing line bundle L. We point out that — although there no examples so far — $\epsilon_1(\underline{l})$ and $\kappa_1(\underline{l})$ could be irrational. However, as will be clear later on, this does not create any trouble.

The relation between the above two constants was established by Jiang and Pareschi.

Theorem 2.4 [Jiang and Pareschi 2017, Theorem D]. *Let $\underline{l}$ be a basepoint-free polarization. Then*

$$\kappa_1(\underline{l}) = \frac{\epsilon_1(\underline{l})}{1-\epsilon_1(\underline{l})}.$$

Remark 2.5. From this result, in [loc. cit.] it is derived that $\kappa_1(\underline{l}) < 1$, i.e., $\underline{l}$ is projectively normal, if and only if $\epsilon_1(\underline{l}) < \frac{1}{2}$ (see in particular [loc. cit., Corollary 8.2(b)]). Our Theorem 1.1 is an extension of the "if" implication to higher syzygies.

Proof of Proposition 1.4. Only in this subsection we make the assumption that the ground field $\mathbb{K}$ is $\mathbb{C}$.

Let $r \in \mathbb{Q}$ such that there exists an effective $\mathbb{Q}$-divisor F on A with

$$rL - F \text{ ample}, \tag{2-2}$$

$$\mathcal{I}_0 = \mathcal{J}(A, F). \tag{2-3}$$

To prove the proposition we need to prove that

$$h^1_{\mathcal{I}_0}(r\underline{l}) = 0. \tag{2-4}$$

Writing $r = a/b$ with $b > 0$, this means that

$$h^1_{\text{gen}}(L^{ab} \otimes \mu_b^* \mathcal{I}_0) = 0.$$

But, by (2-3), the left-hand side is $h^1_{\text{gen}}(L^{ab} \otimes \mu_b^* \mathcal{J}(A, F)) = h^1_{\text{gen}}(L^{ab} \otimes \mathcal{J}(A, \mu_b^* F))$, where we used that forming multiplier ideals commutes with pulling back under étale morphism (see [Lazarsfeld 2004, Example 9.5.44]). Since $\mu_b^* F \equiv_{\text{num}} b^2 F$, it follows from (2-2) that $L^{ab} - \mu_b^* F$ is ample. Therefore (2-4) follows from Nadel's vanishing.

3. Generic vanishing of $\mathbb{Q}$-twisted sheaves on abelian varieties

Following Section 5 of [Jiang and Pareschi 2017], one can extend the usual notions of *generic vanishing* to the $\mathbb{Q}$-twisted setting.

Definition/Theorem 3.1 [Jiang and Pareschi 2017, Theorem 5.1]. (1) *A $\mathbb{Q}$-twisted sheaf $\mathcal{F}\langle x\underline{l}\rangle$, with $x = a/b$, is said to be GV if*

$$\operatorname{codim}_{\hat{A}}\operatorname{Supp}(R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})) \geq i, \quad \textit{for all } i > 0.$$

Equivalently the transform[4] $R\Phi_{\mathcal{P}^\vee}((\mu_b^*\mathcal{F}^\vee)\otimes L^{-ab})$ *is a sheaf concentrated in cohomological degree g, i.e.,*

$$R\Phi_{\mathcal{P}^\vee}((\mu_b^*\mathcal{F}^\vee)\otimes L^{-ab}) = R^g\Phi_{\mathcal{P}^\vee}((\mu_b^*\mathcal{F}^\vee)\otimes L^{-ab})[-g].$$

(2) *It is said to be $IT(0)$ if the transform*

$$R\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab}) = R^0\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})$$

is concentrated in cohomological degree 0.

Remark 3.2. (a) The above definitions do not depend on the representation $x = a/b$. For example for any i, $R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F}) = \hat{\mu}_{m*}R^i\Phi_{\mathcal{P}}(\mathcal{F})$ [Mukai 1981, (3.4)] where $\hat{\mu}_m$ is the dual isogeny of μ_m, therefore by cohomology and base change we see that $\operatorname{Supp}(R^i\Phi_{\mathcal{P}}(\mu_m^*\mathcal{F}))$ corresponds to the image of $\operatorname{Supp}(R^i\Phi_{\mathcal{P}}(\mathcal{F}))$ via the isogeny $\hat{\mu}_m$.

(b) They neither depend on the line bundle L representing the class $\underline{l}$. Indeed, thanks to the exchange of translations and tensor product by elements of $\operatorname{Pic}^0 A$ [Mukai 1981, (3.1)], if L_0 is another line bundle algebraically equivalent to L, then $R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L_0^{ab})$ is a translate of $R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})$.

By cohomology and base change one has that

$$\operatorname{Supp}(R^i\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})) \subseteq \{\alpha \in \hat{A} \mid H^i(A, (\mu_b^*\mathcal{F})\otimes L^{ab}\otimes P_\alpha) \neq 0\} =: V^i((\mu_b^*\mathcal{F})\otimes L^{ab}) \tag{3-1}$$

and, if $V^{i+1}((\mu_b^*\mathcal{F})\otimes L^{ab}) = \varnothing$, then equality holds. Moreover we have that the $\mathbb{Q}$-twisted sheaf $\mathcal{F}\langle x\underline{l}\rangle$ is GV if and only if

$$\operatorname{codim}_{\hat{A}} V^i((\mu_b^*\mathcal{F})\otimes L^{ab}) \geq i,$$

for all $i > 0$ and for any representation $x = a/b$ [Pareschi and Popa 2011a, Lemma 3.6]. By cohomology and base change again, $\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$ if and only if

$$V^i((\mu_b^*\mathcal{F})\otimes L^{ab}) = \varnothing$$

for all $i > 0$ and for any representation $x = a/b$. In particular we see that an $IT(0)$ $\mathbb{Q}$-twisted sheaf is GV.

These generic vanishing concepts are strongly related to the invariants introduced in Definition 2.2, as explained in Section 8 of [Jiang and Pareschi 2017]. Namely, we have:

Lemma 3.3 [Jiang and Pareschi 2017, page 25]. *Given two polarizations $\underline{l}$ and $\underline{n}$ — with $\underline{n}$ basepoint-free — and a rational number x, the fact that $\epsilon_1(\underline{l}) < x$ (resp. $\kappa_1(\underline{n}) < x$) is equivalent to the fact that the $\mathbb{Q}$-twisted sheaf $\mathcal{I}_p\langle x\underline{l}\rangle$ (resp. $M_N\langle x\underline{n}\rangle$) is $IT(0)$.*

For the reader's convenience we explicitly write down the case of $\epsilon_1(\underline{l})$: assume that $\epsilon_1(\underline{l}) < x \in \mathbb{Q}$ and fix a sufficiently small $\eta > 0$ such that $x_0 := \epsilon_1(\underline{l}) + \eta \in \mathbb{Q}$ and $x_0 < x$. By (3-1), $\mathcal{I}_p\langle x_0\underline{l}\rangle$ is GV, therefore Hacon's criterion (see [Jiang and Pareschi 2017, Theorem 5.2(a)]) implies that $\mathcal{I}_p\langle(x_0+(x-x_0))\underline{l}\rangle = \mathcal{I}_p\langle x\underline{l}\rangle$ is $IT(0)$. Conversely suppose that $\mathcal{I}_p\langle x\underline{l}\rangle$ is $IT(0)$, then $\mathcal{I}_p\langle(x-y)\underline{l}\rangle$ is still $IT(0)$, for a sufficiently small

[4] Recall that $R\Phi_{\mathcal{P}^\vee} = (-1_{\hat{A}})^* R\Phi_{\mathcal{P}}$.

$y \in \mathbb{Q}^+$ [Jiang and Pareschi 2017, Theorem 5.2(c)]. Then $\epsilon_1(\underline{l}) < x - y < x$. For $\kappa_1(\underline{n})$, the argument is similar.

The following is a $\mathbb{Q}$-twisted analog of a well-known property of "preservation of vanishing"[Pareschi and Popa 2011b, Proposition 3.1].

Proposition 3.4. *Assume that $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves, and that one of them is locally free. If $\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$ and $\mathcal{G}\langle y\underline{l}\rangle$ is GV, then $\mathcal{F}\langle x\underline{l}\rangle \otimes \mathcal{G}\langle y\underline{l}\rangle := (\mathcal{F}\otimes\mathcal{G})\langle (x+y)\underline{l}\rangle$ is $IT(0)$.*

Proof. Let $x = a/b$ and $y = c/d$, with $b, d > 0$. So $x + y = (ad + bc)/(bd)$. We want to prove that $\mu_{bd}^*(\mathcal{F}\otimes\mathcal{G})\otimes L^{(ad+bc)bd}$ is an $IT(0)$ sheaf. By hypothesis $\mathcal{F}\langle x\underline{l}\rangle$ is $IT(0)$, hence

$$\mu_d^*((\mu_b^*\mathcal{F})\otimes L^{ab}) = (\mu_{bd}^*\mathcal{F})\otimes L^{abd^2}$$

is an $IT(0)$ sheaf, because $R\Phi_{\mathcal{P}}(\mu_d^*((\mu_b^*\mathcal{F})\otimes L^{ab})) = \hat{\mu}_{d*}R\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab}) = \hat{\mu}_{d*}R^0\Phi_{\mathcal{P}}((\mu_b^*\mathcal{F})\otimes L^{ab})$ [Mukai 1981, (3.4)] is concentrated in degree 0, where $\hat{\mu}_d : \hat{A} \to \hat{A}$ is the dual isogeny of μ_d. Likewise, if $\mathcal{G}\langle y\underline{l}\rangle$ is GV, by using the equivalence in Definition/Theorem 3.1(1), we have that

$$\mu_b^*((\mu_d^*\mathcal{G})\otimes L^{cd}) = (\mu_{bd}^*\mathcal{G})\otimes L^{b^2cd}$$

is a GV sheaf. Since

$$\mu_{bd}^*(\mathcal{F}\otimes\mathcal{G})\otimes L^{(ad+bc)bd} = ((\mu_{bd}^*\mathcal{F})\otimes L^{abd^2})\otimes((\mu_{bd}^*\mathcal{G})\otimes L^{b^2cd}),$$

we conclude by applying the "preservation of vanishing" for (untwisted) coherent sheaves [Pareschi and Popa 2011b, Proposition 3.1]. □

For our purposes, the central result of this section is the following:

Proposition 3.5. *Let p be a nonnegative integer. If*

$$\epsilon_1(\underline{l}) < \frac{1}{p+2},$$

then $M_L^{\otimes(p+1)}\otimes L^h$ is $IT(0)$ for all $h \geq 1$.

Proof. Let L be a line bundle representing $\underline{l}$, and let M_L be the kernel of the evaluation morphism $H^0(A, L)\otimes\mathcal{O}_A \to L$. The assumption on $\epsilon_1(\underline{l})$ implies, in particular, that $\underline{l}$ is basepoint-free and, using Theorem 2.4, we get

$$\kappa_1(\underline{l}) = \frac{\epsilon_1(\underline{l})}{1-\epsilon_1(\underline{l})} = -1 + \frac{1}{1-\epsilon_1(\underline{l})} < -1 + \frac{p+2}{p+1} = \frac{1}{p+1}.$$

By Lemma 3.3, this is equivalent to say that $M_L\langle 1/(p+1)\underline{l}\rangle$ is an $IT(0)$ $\mathbb{Q}$-twisted sheaf. Fix now an integer $h \geq 1$ and write $M_L^{\otimes(p+1)}\otimes L^h = M_L^{\otimes(p+1)}\otimes L\otimes L^{h-1}$ as the $\mathbb{Q}$-twisted sheaf

$$M_L^{\otimes(p+1)}\left\langle\left(\frac{p+1}{p+1}+h-1\right)\underline{l}\right\rangle = \left(M_L\left\langle\frac{1}{p+1}\underline{l}\right\rangle\right)^{\otimes(p+1)}\otimes\mathcal{O}_A\langle(h-1)\underline{l}\rangle.$$

Since L^{h-1} is ample — hence $IT(0)$ — if $h > 1$, or it is trivial if $h = 1$, and $M_L\langle 1/(p+1)\underline{l}\rangle$ is $IT(0)$, we have that $M_L^{\otimes(p+1)}\otimes L^h$ is $IT(0)$ thanks to the "preservation of vanishing" (Proposition 3.4). □

4. Syzygies and the property (N_p)

We recall the definition and geometric meaning of the property (N_p) in more detail. Let X be a projective variety, defined over an algebraically closed field $\mathbb{K}$. If L gives an embedding

$$\phi_{|L|} : X \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(X, L)^\vee),$$

then L is said to *satisfy the property* (N_p) if the first p steps of the minimal graded free resolution $E_\bullet(L)$ of the algebra $R_L := \bigoplus_m H^0(X, L^m)$ over the polynomial ring $S_L := \mathrm{Sym} H^0(X, L)$ are linear, i.e., of the form

$$\begin{array}{ccccccccccc} S_L(-p-1)^{\oplus i_p} & \longrightarrow & S_L(-p)^{\oplus i_{p-1}} & \longrightarrow & \cdots & \longrightarrow & S_L(-2)^{\oplus i_1} & \longrightarrow & S_L & \longrightarrow & R_L \longrightarrow 0 \\ \| & & \| & & & & \| & & \| & & \\ E_p(L) & & E_{p-1}(L) & & & & E_1(L) & & E_0(L) & & \end{array}$$

Thus (N_0) means that L is projectively normal (and in this case a resolution of the homogeneous ideal $I_{X/\mathbb{P}}$ of X in $\mathbb{P}$ is given by $\cdots \to E_1(L) \to I_{X/\mathbb{P}} \to 0$); (N_1) means that $I_{X/\mathbb{P}}$ is generated by quadrics; (N_2) means that the relations among these quadrics are generated by linear ones and so on.

Writing $\mathbb{K} = S_L/S_{L+}$ as the quotient of the polynomial ring S_L by the irrelevant maximal ideal $S_{L+} := \bigoplus_{m\geq 1} \mathrm{Sym}^m H^0(X, L)$, it is well-known that $\dim_{\mathbb{K}}(\mathrm{Tor}_i^{S_L}(R_L, \mathbb{K})_j)$ computes the cardinality of any minimal set of homogeneous generators of $E_i(L)$ of degree j, therefore

$$E_i(L) = \bigoplus_j \mathrm{Tor}_i^{S_L}(R_L, \mathbb{K})_j \otimes_{\mathbb{K}} S_L(-j)$$

and L satisfies the property (N_p) if and only if

$$\mathrm{Tor}_p^{S_L}(R_L, \mathbb{K})_j = 0 \quad \text{for all } j \geq p+2.\text{[5]} \tag{4-1}$$

A well established condition ensuring the property (N_p) for L in *characteristic zero* is the vanishing

$$H^1(X, M_L^{\otimes(p+1)} \otimes L^h) = 0 \quad \text{for all } h \geq 1. \tag{4-2}$$

Indeed, tensoring the Koszul resolution of $\mathbb{K}$ by R_L and taking graded pieces, we see that the property (N_p) for L is equivalent to the exactness in the middle of the Koszul complex

$$\Lambda^{p+1} H^0(X, L) \otimes H^0(X, L^h) \to \Lambda^p H^0(X, L) \otimes H^0(X, L^{h+1}) \to \Lambda^{p-1} H^0(X, L) \otimes H^0(X, L^{h+2})$$

for all $h \geq 1$ (see [Lazarsfeld 1989, pages 510–511] for details). This can be expressed in terms of the kernel bundle of L. Namely, taking wedge products of the exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0,$$

we get

$$0 \to \Lambda^{p+1} M_L \to \Lambda^{p+1} H^0(X, L) \otimes \mathcal{O}_X \to \Lambda^p M_L \otimes L \to 0.$$

[5] $\mathrm{Tor}_0^{S_L}(R_L, \mathbb{K})_1$ is always trivial, because we are dealing with the complete linear series $|L|$ and the corresponding embedding is linearly normal. Moreover, the vanishing $\mathrm{Tor}_p^{S_L}(R_L, \mathbb{K})_j = 0$ for all $j \geq p+2$, forces $\mathrm{Tor}_i^{S_L}(R_L, \mathbb{K})_j = 0$ for all $0 \leq i \leq p$ and $j \geq i+2$ (see the proof of Proposition 1.3.3 in [Lazarsfeld 1989] for details).

Tensoring it by L^h and taking global section, we see that the exactness of the Koszul complex above is equivalent to the surjectivity of the map

$$\Lambda^{p+1} H^0(X, L)\otimes H^0(X, L^h) \to H^0(X, \Lambda^p M_L \otimes L^{h+1}),$$

that in turn follows from the vanishing

$$H^1(X, \Lambda^{p+1} M_L \otimes L^h) = 0 \quad \text{for all } h \geq 1. \tag{4-3}$$

Now, if char$(\mathbb{K}) = 0$, $\Lambda^{p+1} M_L$ is a *direct summand* of $M_L^{\otimes(p+1)}$ and in particular (4-2) implies (4-3); otherwise said L satisfies the property (N_p). If char$(\mathbb{K}) > 0$, the exterior power $\Lambda^{p+1} M_L$ may no longer be a direct summand of the tensor power $M_L^{\otimes(p+1)}$, hence the above discussion does not apply. Nevertheless in this section, following an approach essentially due to G. Kempf, we prove that (4-2) implies the property (N_p) for L, even in *positive characteristic*.

Proposition 4.1. *Let X be a projective variety defined over an algebraically closed field $\mathbb{K}$. Let L be an ample and globally generated line bundle on X, and let p be a nonnegative integer. If*

$$H^1(X, M_L^{\otimes(p+1)} \otimes L^h) = 0 \quad \textit{for all } h \geq 1,$$

then the property (N_p) holds for L.

Let us start by recalling two definitions and an algebraic lemma of Kempf [1989] (see also [Rubei 2000, Section 2]).

Definition 4.2. For any L_i (not necessarily ample) line bundles on X, let $K(L_1) = H^0(X, L_1)$ and, for $n > 1$, define $K(L_1, \dots, L_n)$ inductively by the exact sequence:

$$0 \to K(L_1, \dots, L_n) \to K(L_1, L_3, \dots, L_n)\otimes K(L_2) \to K(L_1 \otimes L_2, L_3, \dots, L_n).$$

In particular, $K(L_1, L_2)$ is the kernel of the multiplication map of global sections

$$H^0(X, L_1)\otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2).$$

Definition 4.3. Let S be a polynomial ring over $\mathbb{K}$ and let R be a finitely generated graded S-module:

(1) Define $T^0(R) := R$, $T^1(R) := \mathrm{Ker}[R(-1)\otimes_{\mathbb{K}} S_1 \to R]$ and inductively

$$T^i(R) := T^{i-1}(T^1(R)).$$

(2) Define

$$d^i(R) := \min\{d \in \mathbb{Z} \mid T^i(R) \text{ is generated over } S \text{ by elements of degree} \leq d\}.$$

Lemma 4.4 [Kempf 1989, Lemma 16]. *Let $S = \mathbb{K}[x_0, \dots, x_r]$ be a polynomial ring, graded in the standard way, over $\mathbb{K} = S/(x_0, \dots, x_r)$. Let R be a finitely generated graded S-module. If $j > p - i + d^i(R)$ for all $0 \leq i \leq p$, then*

$$\mathrm{Tor}_p^S(R, \mathbb{K})_j = 0.$$

Due to some obscurities in Kempf's argument and for the sake of self-containedness, we prefer to give a proof of the above lemma, which closely follows that of Kempf.

Proof of Lemma 4.4. Consider the exact sequence

$$0 \to T^1(R) \to R(-1) \otimes_{\mathbb{K}} S_1 \xrightarrow{\alpha} R.$$

The image R' of α is a graded submodule of R. The quotient module $Q = R/R'$ is of finite length, hence its Castelnuovo–Mumford regularity $\mathrm{reg}(Q) = \max\{d \mid Q_d \neq 0\}$ (see [Eisenbud 2005, Corollary 4.4]). Moreover Q is zero in degrees $> d^0(R)$, therefore

$$\mathrm{Tor}_p^S(Q, \mathbb{K}) \text{ is zero in degrees } > p + d^0(R). \tag{4-4}$$

Indeed, if $\mathrm{Tor}_p^S(Q, \mathbb{K})_j \neq 0$ for a $j > p + d^0(R)$, then $\mathrm{reg}(Q) \leq d^0(R) < j - p$. But, by definition, $\mathrm{reg}(Q) = \mathrm{Sup}\{k - i \mid \dim_{\mathbb{K}}(\mathrm{Tor}_i^S(Q, \mathbb{K})_k) \neq 0\}$ and so we get a contradiction. Now (4-4) implies that the map

$$\mathrm{Tor}_p^S(R', \mathbb{K}) \to \mathrm{Tor}_p^S(R, \mathbb{K})$$

is surjective in degrees $> p + d^0(R)$. Therefore, in order to prove the statement, it is enough to prove that $\mathrm{Tor}_p^S(R', \mathbb{K})_j = 0$, if $j > p + d^0(R)$. From the long exact sequence associated to

$$0 \to T^1(R) \to R(-1) \otimes_{\mathbb{K}} S_1 \xrightarrow{\alpha} R' \to 0,$$

we get

$$\mathrm{Tor}_p^S(R(-1) \otimes_{\mathbb{K}} S_1, \mathbb{K}) \xrightarrow{\alpha_*} \mathrm{Tor}_p^S(R', \mathbb{K}) \xrightarrow{\delta} \mathrm{Tor}_{p-1}^S(T^1(R), \mathbb{K}).$$

Note that α_* is the multiplication by S_1 in the first variable. Since α_* is also the multiplication by S_1 in the second variable, it is the zero map. Therefore δ gives an inclusion

$$\mathrm{Tor}_p^S(R', \mathbb{K}) \subseteq \mathrm{Tor}_{p-1}^S(T^1(R), \mathbb{K})$$

and we may repeat this procedure p times, obtaining

$$\mathrm{Tor}_{-1}^S(T^{p+1}(R), \mathbb{K}) = 0. \qquad \square$$

If now L is an ample line bundle on X, $S = S_L$ and $R = R_L$, the link between the previous definitions is given by

$$T^i(R_L) = \bigoplus_{m \geq i} K(L^{m-i}, \underbrace{L, \dots, L}_{i}). \tag{4-5}$$

Proof. If $i = 0$, then $T^0(R_L) = R_L$ and $K(L^m) = H^0(X, L^m)$. So (4-5) is true. By definition

$$T^i(R_L) = T^{i-1}(T^1(R_L)) = T^{i-1}(\mathrm{Ker}[R_L(-1) \otimes_{\mathbb{K}} H^0(X, L) \to R_L]),$$

and

$$0 \to \bigoplus_{m \geq i} K(L^{m-i}, \underbrace{L, \dots, L}_{i}) \to \bigoplus_{m \geq i} K(L^{m-i}, \underbrace{L, \dots, L}_{i-1}) \otimes H^0(X, L) \to \bigoplus_{m \geq i} K(L^{m-i+1}, \underbrace{L, \dots, L}_{i-1}).$$

Therefore (4-5) holds, by induction on i. $\square$

The next lemma allows to reduce the property (N_p) for L to the vanishing (4-2), in a way that avoids the exterior power of M_L.

Lemma 4.5. (1) *For all $n \geq 0$ and $h \geq 1$, one has* $H^0(X, M_L^{\otimes n} \otimes L^h) = K(L^h, \underbrace{L, \dots, L}_{n})$, *if L is basepoint-free.*

(2) *Let $i \geq 0$ and $h \geq 1$. If L is basepoint-free and $H^1(X, M_L^{\otimes(i+1)} \otimes L^h) = 0$, then the multiplication map*

$$K(L^h, \underbrace{L, \dots, L}_{i}) \otimes H^0(X, L) \to K(L^{h+1}, \underbrace{L, \dots, L}_{i})$$

is surjective.

(3) [Rubei 2000, page 2578] *If the multiplication maps*

$$K(L^h, \underbrace{L, \dots, L}_{i}) \otimes H^0(X, L) \to K(L^{h+1}, \underbrace{L, \dots, L}_{i})$$

are surjective for all $h \geq 1$, then $d^i(R_L) = i + 1$.

Proof. (1) If $n = 0$, then by definition $H^0(X, L^h) = K(L^h)$ for all $h \geq 1$. Suppose $n \geq 1$. The kernel bundle M_L sits in the short exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0. \tag{4-6}$$

Tensoring it by $M_L^{\otimes(n-1)} \otimes L^h$, one obtains

$$0 \to M_L^{\otimes n} \otimes L^h \to H^0(X, L) \otimes M_L^{\otimes(n-1)} \otimes L^h \to M_L^{\otimes(n-1)} \otimes L^{h+1} \to 0. \tag{4-7}$$

Taking global sections of (4-7) and using the inductive hypothesis, we obtain

$$0 \to H^0(X, M_L^{\otimes n} \otimes L^h) \to H^0(X, L) \otimes K(L^h, \underbrace{L, \dots, L}_{n-1}) \to K(L^{h+1}, \underbrace{L, \dots, L}_{n-1}).$$

Therefore, by definition, $H^0(X, M_L^{\otimes n} \otimes L^h) = K(L^h, \underbrace{L, \dots, L}_{n})$.

(2) Tensoring (4-6) by $M_L^{\otimes i} \otimes L^h$, we have

$$0 \to M_L^{\otimes(i+1)} \otimes L^h \to H^0(X, L) \otimes M_L^{\otimes i} \otimes L^h \to M_L^{\otimes i} \otimes L^{h+1} \to 0. \tag{4-8}$$

From the long exact sequence in cohomology associated to (4-8), and thanks to the point (1), one has

$$H^0(X, L) \otimes K(L^h, \underbrace{L, \dots, L}_{i}) \xrightarrow{\alpha} K(L^{h+1}, \underbrace{L, \dots, L}_{i}) \to H^1(X, M_L^{\otimes(i+1)} \otimes L^h) = 0.$$

Therefore the multiplication map α is surjective.

(3) By (4-5) and the hypothesis we have that $T^i(R_L)$ is generated over S_L by

$$K(\underbrace{L, \dots, L}_{i+1}).$$

This means that it is generated by the piece of degree m with $m - i = 1$, i.e., $m = i + 1$. Therefore $d^i(R_L) = i + 1$. □

5. Proof of the Theorems 1.1 and 1.3

Proof of Theorem 1.1. Let L be a representative of the class $\underline{l}$. For all $0 \le i \le p$, we have

$$\epsilon_1(\underline{l}) < \frac{1}{p+2} \le \frac{1}{i+2}.$$

Therefore L is basepoint-free and, thanks to the Proposition 3.5, we know that $M_L^{\otimes(i+1)} \otimes L^h$ is $IT(0)$, for all $h \ge 1$. This implies, in particular, that $H^1(A, M_L^{\otimes(i+1)} \otimes L^h) = 0$ for all $h \ge 1$. Hence, by Lemma 4.5(2) and (3), we obtain

$$d^i(R_L) = i+1.$$

Now, if $j > p - i + d^i(R_L) = p+1$, Kempf's Lemma 4.4 implies that

$$\mathrm{Tor}_p^{S_L}(R_L, \mathbb{K})_j = 0.$$

As explained in (4-1), this is equivalent to the property (N_p) for L. □

Proof of Theorem 1.3. Note that we have already proved the $t = 0$ case — even without the basepoint-freeness assumption — and the $t = 1$ case (Corollary 1.2). Hence we may assume $t > 1$. By Theorem 1.1, it suffices to show that $\epsilon_1(m\underline{l}) < 1/(p+2)$. We have

$$\epsilon_1(m\underline{l}) = \frac{\epsilon_1(\underline{l})}{m} \le \frac{\epsilon_1(\underline{l})}{p+3-t} \le \frac{1}{t(p+3-t)},$$

where the last inequality follows by definition. Let us impose now the inequality

$$\frac{1}{t(p+3-t)} < \frac{1}{p+2},$$

or equivalently

$$t^2 - (p+3)t + p + 2 < 0.$$

This is satisfied if and only if $1 < t < p+2$ and, by hypothesis, we have $1 < t \le p+1$. □

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