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p-adic cyclotomic multiple zeta values depend on the choice of a number of iterations of the crystalline Frobenius of the pro-unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$. In this paper we study how the iterated Frobenius depends on the number of iterations, in relation with the computation of p-adic cyclotomic multiple zeta values in terms of cyclotomic multiple harmonic sums. This provides new results on that computation and the definition of a new pro-unipotent harmonic action.

0. Introduction

0A. ***p-adic cyclotomic multiple zeta values, computation and iteration of the Frobenius.*** Cyclotomic multiple zeta values are the following iterated integrals: for any positive integers, d and n_i $(1 \le i \le d)$ and roots of unity ξ_i $(1 \le i \le d)$, such that $(n_d, \xi_d) \neq (1, 1)$,

$$\zeta((n_i)_d; (\xi_i)_d) = (-1)^d \int_{t_n=0}^{1} \frac{dt_n}{t_n - \epsilon_n} \int_{t_{n-1}=0}^{t_n} \cdots \int_{t_1=0}^{t_2} \frac{dt_1}{t_1 - \epsilon_1}, \tag{0-1}$$

where $n = \sum_{i=1}^{d} n_i$ and $(\epsilon_n, \dots, \epsilon_1) = (\overbrace{0, \dots, 0}^{n_d - 1}, 1, \dots, \overbrace{0, \dots, 0}^{n_1 - 1}, 1)$. We choose N such that the ϵ_i are N-th roots of unity. Let p be a prime number prime to N. p-adic cyclotomic multiple zeta values are defined as p-adic analogues of the above iterated integrals. They are elements of the extension K of

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$\mathbb{Q}_p$ generated by a primitive N-th root of unity. There are two types of p-adic cyclotomic multiple zeta values; both of the notions rely on the Frobenius of the crystalline pro-unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ at the base-points $\vec{1}_0$ and $\vec{1}_1$, as follows (see Section 1A3 for details):

(i) Numbers $\zeta_q^{\mathrm{KZ}}((n_i)_d; (\xi_i)_d) \in K$ defined by Coleman integration i.e., by using a Frobenius-invariant path [Furusho 2004; 2007; Yamashita 2010] (here, q is the cardinality of the residue field of K).

(ii) For each $\alpha \in \mathbb{Z} \setminus \{0\}$, numbers $\zeta_{p,\alpha}((n_i)_d; (\xi_i)_d) \in K$ defined by the image by Frobenius iterated α times of the canonical path in the de Rham fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ ([Jarossay 2019, Section 1] and, for particular values of α, [Yamashita 2010; Deligne and Goncharov 2005; Ünver 2004; 2016]).

p-adic cyclotomic multiple zeta values can be considered as canonical coefficients of the Frobenius, and conversely one can recover the Frobenius by knowing only p-adic cyclotomic multiple zeta values.

Cyclotomic multiple harmonic sums are the following numbers:

$$h_m((n_i)_d; (\xi_i)_{d+1}) = \sum_{0<m_1<\cdots<m_d<m} \frac{\left(\frac{\xi_2}{\xi_1}\right)^{n_1} \cdots \left(\frac{\xi_{d+1}}{\xi_d}\right)^{n_d} \left(\frac{1}{\xi_{d+1}}\right)^{m}}{m_1^{n_1} \cdots m_d^{n_d}}. \tag{0-2}$$

In the complex case we have $\zeta((n_i)_d; (\xi_i)_d) = \lim_{m\to\infty} h_m((n_i)_d; ((\xi_i)_d, 1))$. Similarly it is possible to compute p-adic cyclotomic multiple zeta values in terms of cyclotomic multiple harmonic sums [Jarossay 2015], thanks to a big combinatorial simplification proved by the main result of [Jarossay 2019]. (Two cases, $\alpha = -1,\ N = 1$ and $\alpha = -1,\ d \geq 2$, are handled in [Ünver 2015] and [Ünver 2016], respectively, through a different computation that does not use the simplification of [Jarossay 2019]; those results are more complicated and seem difficult to use.) In this paper we are going to study the following question: how does the iterated Frobenius depend on its number of iterations? More specifically, we are going to connect this question and the framework of [Jarossay 2015].

0B. ***Principles of the study.*** Most of the time, we are not going to consider directly the Frobenius but, instead, the *harmonic Frobenius*, defined in [Jarossay 2015, Definition 2.3.5], (we will reproduce it in Section 1D). It is a variant of the Frobenius which is much simpler and more natural from the point of view of multiple harmonic sums, and computing it suffices to compute the Frobenius.

Whereas the Frobenius is an isomorphism of bundles with connection, the harmonic Frobenius is a map on a space which contains the noncommutative generating series of weighted multiple harmonic sums $\mathrm{har}_m((n_i)_d; (\xi_i)_{d+1}) = m^{n_d+\cdots+n_1} h_m((n_i)_d; (\xi_i)_{d+1})$.

We will use the fact that the harmonic Frobenius can be expressed in two ways:

(a) One "in terms of integrals", i.e., in which the coefficients of the harmonic Frobenius are expressed in terms of p-adic cyclotomic multiple zeta values, which are integrals and which we want to compute.

(b) Another one "in terms of series", in which the coefficients of the harmonic Frobenius are certain sums of series expressed in terms of the numbers $\mathrm{har}_{p^\alpha}((n_i)_d; (\xi_i)_{d+1})$, which are explicit.

In [Jarossay 2015], by writing these two expressions and observing that they are equal, we get an expression for p-adic cyclotomic multiple zeta values in terms of the numbers $\text{har}_{p^\alpha}((n_i)_d;\,(\xi_i)_{d+1})$ and vice-versa. We are going to do something similar here, not for the harmonic Frobenius but for the study of the numbers $\text{har}_{q^{\tilde{\alpha}}}((n_i)_d;\,(\xi_i)_{d+1})$, as functions of $\tilde{\alpha} \in \mathbb{N}^*$. Since the harmonic Frobenius can be expressed in terms of these numbers, this will directly provide a study of the iterated harmonic Frobenius in terms of its number of iterations.

After some preliminaries (Section 1) we will do this study in terms of integrals (Section 2), in terms of series (Section 3), and we will use the fact that these two ways give the same result (Section 4). In Section 5 we will go back from the harmonic Frobenius to the Frobenius.

Moreover, we will keep track of the motivic structure underlying this framework. Indeed, p-adic cyclotomic multiple zeta values are reductions of p-adic periods [Yamashita 2010], and there is a motivic Galois action on the pro-unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ [Deligne and Goncharov 2005, Section 5].

The Frobenius is expressed by means of the Ihara action (1-1), which is the image of the motivic Galois action by a certain morphism (see Section 1A3). In [Jarossay 2015], the passage from the Frobenius to the harmonic Frobenius lifts to a passage from the Ihara action to an operation which we called the pro-unipotent harmonic action of integrals, and we also find a pro-unipotent harmonic action of series (see Section 1D). The interest of pro-unipotent harmonic actions is that, being byproducts of the motivic Galois action, they retain certain properties of motivic Galois actions; and having a computation which keeps track of the motivic Galois action is key for us. The pro-unipotent harmonic actions are the main objects in our papers [Jarossay 2014; 2016a; 2016b] in which we show the compatibility between our computation and the motivic Galois theory of p-adic cyclotomic multiple zeta values.

Establishing the definition of pro-unipotent harmonic actions requires enriching the pro-unipotent fundamental groupoid, which is a groupoid in affine schemes over $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ by turning it into a groupoid in ultrametric complete normed algebras [Jarossay 2015, Section 1].

0C. ***A few definitions.*** The study will require new definitions. First, we will define an ad hoc notion of contraction mapping (Definition 1B.2). We will show that the Frobenius at base-points $(\vec{1}_1, \vec{1}_0)$ is a contraction in our ad hoc sense. This will shed light on the dynamics of the Frobenius which has a unique fixed point. Thus, the ultrametric framework established in [Jarossay 2015] will be crucial here because in this framework we will introduce a notion of contraction mapping and we will see that the Frobenius is a contraction. Keeping track of the motivic structures will also require new definitions.

We will study $\text{har}_{q^{\tilde{\alpha}}}$ as a function of $\tilde{\alpha}$ "in terms of integrals" (Section 2) by viewing $\text{har}_{q^{\tilde{\alpha}}}$ via their expression in terms of p-adic cyclotomic multiple zeta values proved in the main theorem of [Jarossay 2015]. This will be done in two different ways, corresponding to the two types of p-adic cyclotomic multiple zeta values evoked in Section 0A:

(i) A way involving the fixed point of the Frobenius and the numbers ζ_q^{KZ}. It will lead us to introduce a pro-unipotent harmonic action of integrals at $(1, 0)$, $\circ_{\text{har}}^{\int_{1,0}}$ (Definition 2A.3), which is a variant of the notion introduced in [Jarossay 2015]. Another point of view on this object will be explained in the Appendix.

(ii) A way involving the numbers ζ_{p,α_0}. It will lead us to introduce a map $\mathrm{iter}^{\int}_{\mathrm{har}}(\mathbf{a}, \Lambda)$ of iteration of the harmonic Frobenius of integrals at $(1, 0)$, (Λ and $\mathbf{a}$ are formal variables which represent respectively $q^{\tilde{\alpha}}$ and $\tilde{\alpha}/\tilde{\alpha}_0$) (Definition 3B.1).

In the study of $\mathrm{har}_{q^{\tilde{\alpha}}}$ as a function of $\tilde{\alpha}$ in terms of series (Section 3), we do not have an analogue of the fixed point and the study will lead us to introduce a map of iteration of the harmonic Frobenius of series $\mathrm{iter}^{\Sigma}_{\mathrm{har}}(\mathbf{a}, \Lambda)$ (Definition 3B.1).

Finally, in Section 4, we will relate Sections 2 and 3 by defining a map of comparison between series and integrals, which will be injective thanks to the results of Sections 2 and 3.

As in [Jarossay 2015] these definitions enable us to express the computation not number by number, but as a new structure on the pro-unipotent fundamental group, which is more efficient. Indeed, this structure retains certain features of the motivic Galois theory of periods, which will be crucial in our subsequent papers [Jarossay 2014; 2016a; 2016b] in which we will relate the motivic Galois theory of p-adic cyclotomic multiple zeta values to our formulas.

0D. *Results.* The main result consists of three equations to express $\mathrm{har}_{q^{\tilde{\alpha}}}$ as a function of $\tilde{\alpha}$, and the comparison between them.

The first two equations (proved in Section 2), in which the harmonic Frobenius is thought of in terms of integrals, correspond to (i) and (ii) above. The first one (0-3) involves the fixed-point of the Frobenius and will be called the *fixed point equation of the harmonic Frobenius of integrals at* $(1, 0)$; the second one (0-4) will be the *iteration equation of the harmonic Frobenius of integrals at* $(1, 0)$.

Finally, the third equation (0-5) (proved in Section 3), in which the harmonic Frobenius is thought of in terms of series, will be the *iteration equation of the harmonic Frobenius of integrals at* $(1, 0)$.

In (0-4) and (0-5) the dependence of $\mathrm{har}_{q^{\tilde{\alpha}}}$ in $\tilde{\alpha}$ is via a power series in $K[[q^{\tilde{\alpha}}]][\tilde{\alpha}]$ and in (0-3) it is via a power series in $K[[q^{\tilde{\alpha}}]]$. We are going to see that these expansions are equal (0-6); in particular, the coefficients of $(q^{\tilde{\alpha}})^0\tilde{\alpha}^m$ for $m \geq 1$ will vanish.

In the statement below, for any $\tilde{\alpha} \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$, $\Phi_{q,\tilde{\alpha}}$ is the generating series of the numbers $\zeta_{q,\tilde{\alpha}}$ and $\Phi_{q,-\infty} = \Phi_q^{\mathrm{KZ}}$ is the generating series of the numbers ζ_q^{KZ} (Notation 1A.1), $\mathrm{har}_{q,\alpha}$ is a generating series of generalized prime weighted cyclotomic multiple harmonic sums (Definition 1C.2), and τ is defined in (1-2).

Theorem. *Let* $\tilde{\alpha}_0, \tilde{\alpha} \in \mathbb{N}^*$ *such that* $\tilde{\alpha}_0 \mid \tilde{\alpha}$:

(i) (*integrals*) *The pro-unipotent harmonic action of integrals at* $(1, 0)$, *denoted by* $\circ^{\int_{1,0}}_{\mathrm{har}}$, *is a continuous group action, and we have*

$$\mathrm{har}_{q,\tilde{\alpha}} = \tau(q^{\tilde{\alpha}})(\Phi^{-1}_{q,-\infty} e_1 \Phi_{q,-\infty}) \circ^{\int_{1,0}}_{\mathrm{har}} \mathrm{har}_{q,-\infty}\,. \tag{0-3}$$

The map of iteration of the Frobenius of integrals at $(1,0)$, *denoted by* $\mathrm{iter}_{\mathrm{har}}^{\int_{1,0}}$, *satisfies, at words* $w=((n_i)_d;(\xi_i)_{d+1})$ *such that* $\tilde{\alpha}/\tilde{\alpha}_0>d$,

$$\mathrm{har}_{q,\tilde{\alpha}}(w)=\mathrm{iter}_{\mathrm{har}}^{\int_{1,0}}\left(\frac{\tilde{\alpha}}{\tilde{\alpha}_0},q^{\tilde{\alpha}_0}\right)(\Phi_{q,\tilde{\alpha}_0}^{-1}e_1\Phi_{q,\tilde{\alpha}_0})(w). \tag{0-4}$$

(ii) (*series*) *The map of iteration of the Frobenius of series, denoted by* $\mathrm{iter}_{\mathrm{har}}^{\Sigma}$, *satisfies*

$$\mathrm{har}_{q,\tilde{\alpha}}=\mathrm{iter}_{\mathrm{har}}^{\Sigma}\left(\frac{\tilde{\alpha}}{\tilde{\alpha}_0},q^{\tilde{\alpha}_0}\right)(\mathrm{har}_{q,\tilde{\alpha}_0}). \tag{0-5}$$

(iii) (*comparison between integrals and series*) *We have the following equalities of formal power series with formal variables* a *and* Λ:

$$\tau(\Lambda)(\Phi_{q,-\infty}^{-1}e_1\Phi_{q,-\infty})\circ_{\mathrm{har}}^{\int_{1,0}}\mathrm{har}_{q,-\infty}=\mathrm{iter}_{\mathrm{har}}^{\int}(a,\Lambda)(\Phi_{q,\tilde{\alpha}_0}^{-1}e_1\Phi_{q,\tilde{\alpha}_0})=\mathrm{iter}_{\mathrm{har}}^{\Sigma}(a,\Lambda)(\mathrm{har}_{q,\tilde{\alpha}_0}). \tag{0-6}$$

The first terms of the equations of the theorem are written in Example 4A.2.

In Section 5, we deduce a similar result for the iteration of the Frobenius on the affinoid analytic subspace $\mathbb{P}^{1,\mathrm{an}}\setminus\bigcup_{\xi\in\mu_N(K)}B(\xi,1)$ of $\mathbb{P}^{1,\mathrm{an}}/K$, knowing that the fixed-point equation of the Frobenius is already given by Coleman integration. One of these equations uses the regularization of p-adic iterated integrals studied in [Jarossay 2019]. In the Appendix, we explain that the pro-unipotent harmonic action of integrals in $(1,0)$ corresponds to a certain Poisson bracket.

The main result provides a natural way to compute the fixed point $\Phi_{q,-\infty}$ i.e., p-adic cyclotomic multiple zeta values in the sense of Coleman integration. Indeed, we see that the fixed point of the Frobenius $\Phi_{q,-\infty}\in\Pi_{1,0}(K)$ appears naturally as a way to express the coefficients of the iteration equations, and that these iteration equations can be understood in terms of explicit sums of series. This gives a way to compute Coleman integration without directly doing Coleman integration.

The main result also allows us to replace the map of comparison from integrals to series defined in [Jarossay 2015] by a map which has the advantage of being injective.

From a dynamical point of view, the main result gives an asymptotic expansion at infinite order of the convergence of the iterated (harmonic) Frobenius towards its fixed point. More precise information would follow from nonvanishing results or results on the valuation of p-adic cyclotomic multiple zeta values, or of certain infinite sums of them. This gives a correspondence between certain arithmetical properties of p-adic cyclotomic multiple zeta values and dynamical properties of the Frobenius. A correspondence between dynamical properties of the Frobenius and analytic properties of cyclotomic multiple harmonic sums is also deduced in Section 5.

The pro-unipotent harmonic action $\circ_{\mathrm{har}}^{\int_{1,0}}$ which is defined in this paper will be central in our next papers [Jarossay 2014; 2016a; 2016b] on the explicit version of the algebraic theory of p-adic cyclotomic multiple zeta values. We will also see there that considering the iterates of the Frobenius, instead of only the Frobenius itself, is necessary to formulate an explicit version of the algebraic theory of p-adic cyclotomic multiple zeta values which is purely p-adic and not adelic. It will also find an application in [Jarossay 2017], where we will see that we can construct a structure on $\pi_1^{\mathrm{un,dR}}(\mathbb{P}^1\setminus\{0,\mu_{p^\alpha N},\infty\})$

which generalizes the crystalline Frobenius on $\pi_1^{\mathrm{un,dR}}(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\})$ iterated α times. Considering two parameters $\tilde{\alpha}_0$ and $\tilde{\alpha}$ with $\tilde{\alpha}_0 \mid \tilde{\alpha}$ and not just $\tilde{\alpha}$ will also be useful in [Jarossay 2017] to shed light on the computation of p-adic cyclotomic multiple zeta values.

1. Preliminaries

In this section we establish the framework of this paper. We review some definitions and properties about the pro-unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$, some results from [Jarossay 2015] and we add to them a few new definitions and notations. Throughout this paper, $\mathbb{N}$ and $\mathbb{N}^*$ will denote the set of nonnegative and positive integers, respectively.

1A. *The pro-unipotent fundamental groupoid of* $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$.

1A1. *The de Rham realization.* Let X be $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ over the p-adic field K, with the notations of Section 0A. The de Rham pro-unipotent fundamental groupoid $\pi_1^{\mathrm{un,dR}}(X)$, in the sense of [Deligne 1989], is a groupoid in pro-affine schemes over X. Its base points are the points of X and the nonzero tangent vectors at $\{0, \mu_N, \infty\} \subset \mathbb{P}^1$, called tangential base-points. The groupoid structure is defined by the morphisms $\pi_1^{\mathrm{un,dR}}(X_K, z, y) \times \pi_1^{\mathrm{un,dR}}(X_K, y, x) \to \pi_1^{\mathrm{un,dR}}(X_K, z, x)$ for any base-points x, y, z. By [loc. cit., Section 12.9], each $\pi_1^{\mathrm{un,dR}}(X, y, x)$ is canonically isomorphic to the spectrum of the shuffle Hopf algebra $\mathcal{O}^{ш, e_{0\cup\mu_N}}$ over the alphabet $e_{0\cup\mu_N} = \{e_x \mid x \in \{0\} \cup \mu_N(K)\}$. This isomorphism is compatible with the groupoid structure.

Following [Deligne and Goncharov 2005], for any N-th root of unity $\xi \in \mu_N(K)$, we denote by $\Pi_{\xi,0} = \pi_1^{\mathrm{un,dR}}(X, \vec{1}_\xi, \vec{1}_0)$. Let $f \mapsto f^{(\xi)}$ be the isomorphism $\Pi_{1,0} \to \Pi_{\xi,0}$ induced by the automorphism $x \mapsto \xi x$ of X by functoriality of $\pi_1^{\mathrm{un,dR}}$.

Let $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ be the noncommutative K-algebra of formal power series over the noncommuting variables e_x, $x \in \{0 \cup \mu_N(K)\}$. We will write an element $f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ as $f((e_x)_x) = f((e_x)_{x\in\{0\}\cup\mu_N(K)})$ or $f(e_0, (e_\xi)_\xi) = f(e_0, (e_\xi)_\xi)$. The coefficient in f of a word w on the alphabet $e_{0\cup\mu_N}$ is denoted by $f[w]$. This notation extends by linearity to linear combinations of words, and if for any $n \geq 0$ w_n is a linear combination of words of weight n, we denote by $f\big[\sum_{n=0}^{\infty} w_n\big] = \sum_{n=0}^{\infty} f[w_n]$ if this series converges. We have a canonical inclusion $\mathrm{Spec}(\mathcal{O}^{ш, e_{0\cup\mu_N}})(K) \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle$; namely, $\mathrm{Spec}(\mathcal{O}^{ш, e_{0\cup\mu_N}})(K)$ is the group of elements satisfying the shuffle equation: for all words w, w', $f[w]f[w'] = f[w ш w']$ where ш is the shuffle product of words on the alphabet $e_{0\cup\mu_N}$.

1A2. *Motivic Galois action and byproducts.* The motivic version of $\pi_1^{\mathrm{un}}(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\})$ is constructed in [Deligne and Goncharov 2005, Section 5].

Let G^ω be the motivic Galois group associated with the Tannakian category of mixed Tate motives over k_N unramified at p prime to N, and with the canonical fiber functor ω. There is a semidirect product decomposition $G_\omega = \mathbb{G}_m \ltimes U_\omega$ where U_ω is pro-unipotent.

One has an action of G^ω on $\Pi_{1,0}$. Let V^ω be the group of automorphisms defined in [Deligne and Goncharov 2005, Section 5.10]. There is a morphism $U^\omega \to V^\omega$ sending the action of U^ω on $\Pi_{1,0}$ to an action of V^{ω} on $\Pi_{1,0}$. This action makes $\Pi_{1,0}$ a torsor under V^{ω}. Thus one can consider the isomorphism of schemes $V^{\omega} \simeq \Pi_{1,0}$, $v \mapsto v({}_1 1_0)$ where ${}_1 1_0$ is the canonical de Rham path in the sense of [Deligne 1989, Section 12]. This isomorphism sends the action of V^{ω} on $\Pi_{1,0}$ to the Ihara action [Deligne and Goncharov 2005, Section 5.11], namely, the group law $\circ^{\int_{1,0}}$ on $\Pi_{1,0}$ defined by

$$g \circ^{\int_{1,0}} f = g(e_0, (e_\xi)_\xi) \times f(e_0, ({g^{(\xi)}}^{-1} e_\xi g^{(\xi)})_\xi). \tag{1-1}$$

The motivic Galois action of $\mathbb{G}_m$ on $\Pi_{1,0}$ is

$$\tau : (\lambda, f((e_x)_x)) \mapsto f((\lambda e_x)_x), \tag{1-2}$$

i.e., λ acts by multiplying the term of weight n in f by λ^n, for all $n \in \mathbb{N}$. Let the collection of maps $(\tau_n)_{n\in\mathbb{N}}$ be defined by the equality $\sum_{n\in\mathbb{N}} \tau_n(f)\lambda^n = \tau(\lambda)(f)$ for all λ. Namely, τ_n sends $f = \sum_{w \text{ word}} f[w]w$ to $\sum_{w \text{ word, weight}(w)=n} f[w]w$. These formulas also define an action on $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ for which we will use the same notations.

1A3. *The crystalline realization.* Let σ be the Frobenius automorphism of K. For $\alpha \in \mathbb{N}^*$, let $X^{(p^\alpha)}$ be the pull back of X by σ iterated α times.

Let ϕ be the Frobenius of the crystalline pro-unipotent fundamental groupoid of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ [Deligne 1989, Sections 11 and 13.6]. It is a σ-linear isomorphism of groupoids $\pi_1^{\text{un,dR}}(X^{(p)}) \xrightarrow{\sim} \pi_1^{\text{un,dR}}(X)$. For any $\alpha \in \mathbb{N}^*$, the Frobenius iterated α times is $\phi_\alpha = (\sigma^{\alpha-1})^*\phi \circ \cdots \circ \sigma^*(\phi) \circ \phi$. It is a σ^α-linear isomorphism of groupoids $\pi_1^{\text{un,dR}}(X^{(p^\alpha)}) \xrightarrow{\sim} \pi_1^{\text{un,dR}}(X)$. When α is divisible by $\mathfrak{o} = \log(p)/\log(q)$, then $\sigma^\alpha = \text{id}$, thus ϕ_α is K-linear in the usual sense, its source and target are the same, and it is equal to $\phi_{\mathfrak{o}}$ iterated $\alpha/\mathfrak{o}$ times: we will write $\alpha = \mathfrak{o}\tilde{\alpha}$, and $\phi_{\mathfrak{o}\tilde{\alpha}} = \phi_{\mathfrak{o}}^{\tilde{\alpha}}$. We denote by $\phi_{-\alpha} = \phi_\alpha^{-1}$.

Let us now consider the Frobenius at base-points $(1, 0) = (\vec{1}_1, \vec{1}_0)$: $\phi_\alpha : \Pi_{1,0}^{(p^\alpha)} \xrightarrow{\sim} \Pi_{1,0}$ where $\Pi_{1,0}^{(p^\alpha)} = \pi_1^{\text{un,dR}}(X^{(p^\alpha)}, \vec{1}_1, \vec{1}_0)$. The noncommutative generating series of p-adic cyclotomic multiple zeta values are $\Phi_{p,\alpha} = \tau(p^\alpha)\phi_\alpha(1) \in \Pi_{1,0}(K)$, and $\Phi_{p,-\alpha} = \phi_{-\alpha}(1) \in \Pi_{1,0}^{(p^\alpha)}(K)$. Let us denote again by σ the map $K\langle\langle e_{0\cup\mu_N}\rangle\rangle \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ defined by applying the Frobenius σ of K to each coefficient of a formal power series. The formal properties of the Frobenius imply the following formulas:

$$\tau(p^\alpha)\phi_\alpha : f \in \Pi_{1,0}^{(p^\alpha)}(K) \mapsto \Phi_{p,\alpha} \circ^{\int_{1,0}} \sigma^{-\alpha}(f) \in \Pi_{1,0}(K), \tag{1-3}$$

$$\phi_{-\alpha} : f \in \Pi_{1,0}(K) \ \mapsto \Phi_{p,-\alpha} \circ^{\int_{1,0}} \tau(p^\alpha)\sigma^\alpha(f) \in \Pi_{1,0}^{(p^\alpha)}(K). \tag{1-4}$$

One also has the other notion $\Phi_q^{\text{KZ}} \in \Pi_{1,0}(K)$ of a noncommutative generating series of p-adic cyclotomic multiple zeta values, defined by the following equality:

$$\phi_{\log(q)/\log(p)}(\Phi_q^{\text{KZ}}) = \Phi_q^{\text{KZ}}, \tag{1-5}$$

where the existence and uniqueness of a fixed point of $\phi_{\log(q)/\log(p)}$ in $\Pi_{1,0}(K)$ follows from the theory of Coleman integration [Coleman 1982; Besser 2002; Vologodsky 2003]. Item (ii) of the notation below will be justified by the results of Section 2.

Notation 1A.1. (i) For any $\tilde{\alpha} \in \mathbb{Z} \setminus \{0\}$, let $\Phi_{q,\tilde{\alpha}} = \Phi_{p,\log(q)/\log(p)\cdot\tilde{\alpha}}$.

(ii) Let $\Phi_{q,-\infty} = \Phi_q^{\mathrm{KZ}}$, and let $\Phi_{q,\infty}$ be the inverse of $\Phi_{q,-\infty}$ for the Ihara product $\circ^{\int_{1,0}}$.

(iii) For $\alpha \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$, the p-adic cyclotomic multiple zeta values are the numbers $\zeta_{q,\tilde{\alpha}}((n_i)_d; (\xi_i)_d) = (-1)^d \Phi_{q,\tilde{\alpha}}[e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}]$ (and similarly for $\zeta_{p,\alpha}$ and $\Phi_{p,\alpha}$.)

1A4. *Around the adjoint action* $\mathrm{Ad}(e_1)$. We use the convention that the adjoint action $\mathrm{Ad}_{(\cdot)}(x)$ on $\Pi_{1,0}$ is $f \mapsto f^{-1}xf$ (instead of the usual $f \mapsto fxf^{-1}$, due to our convention of reading the groupoid multiplication from the right to the left). The adjoint Ihara action, defined in [Jarossay 2015], is the group law on $\mathrm{Ad}_{\Pi_{1,0}}(e_1)$ defined by

$$h \circ_{\mathrm{Ad}}^{\int_{1,0}} f = f(e_0, (h^{(\xi)})_\xi). \tag{1-6}$$

Let $\tilde{\Pi}_{1,0}$ be the subgroup scheme of $\Pi_{1,0}$ defined by the equations $f[e_1] = f[e_0] = 0$ (see Section 1A1); $\mathrm{Ad}(e_1)$ induces an isomorphism of groups $(\tilde{\Pi}_{1,0}(K), \circ^{\int_{1,0}}) \xrightarrow{\sim} (\mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1), \circ_{\mathrm{Ad}}^{\int_{1,0}})$.

By (1-1) and (1-2), one has a semidirect product $\mathbb{G}_m \ltimes \tilde{\Pi}_{1,0}$, which acts on $\tilde{\Pi}_{1,0}$. Similarly, by (1-6) and (1-2), one has a semidirect product $\mathbb{G}_m \ltimes \mathrm{Ad}_{\tilde{\Pi}_{1,0}}(e_1)$, which acts on $\mathrm{Ad}_{\tilde{\Pi}_{1,0}}(e_1)$. The map $\mathrm{id} \times \mathrm{Ad}(e_1)$ induces an isomorphism between these two group actions. For all $f, g \in \Pi_{1,0}(K)$, $\lambda \in K^*$, $n \in \mathbb{N}$, we have

$$g \circ^{\int_{1,0}} (\tau(\lambda)(f)) = \sum_{n \in \mathbb{N}} \lambda^n g \circ^{\int_{1,0}} (\tau_n f). \tag{1-7}$$

We have

$$\tau_{n+1} \circ \mathrm{Ad}(e_1) = \mathrm{Ad}(e_1) \circ \tau_n, \tag{1-8}$$

$$\mathrm{Ad}_g(e_1) \circ_{\mathrm{Ad}}^{\int_{1,0}} \frac{\tau(\lambda)}{\lambda} \mathrm{Ad}_f(e_1) = \sum_{n \in \mathbb{N}} \lambda^n \mathrm{Ad}_g(e_1) \circ_{\mathrm{Ad}}^{\int_{1,0}} \tau_{n+1} \mathrm{Ad}_f(e_1). \tag{1-9}$$

1B. ***An ultrametric structure on the K-points of the de Rham pro-unipotent fundamental groupoid.*** As reviewed in Section 1A, each $\Pi_{y,x} = \pi_1^{\mathrm{un,dR}}(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}, y, x)$ is an affine scheme over K, and we have a canonical embedding $\Pi_{y,x}(K) \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle$. We consider now an enrichment of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ into a ultrametric complete normed K-algebra: we review facts from [Jarossay 2015], and we add a few complements. In particular, in Section 1B3 we add a notion of contraction and we apply it to the Frobenius at base-points $(1, 0)$.

1B1. *From affine schemes to ultrametric normed algebras over* K. For $n, d \in \mathbb{N}^*$, let $\mathrm{Wd}_{*,d}(e_{0\cup\mu_N})$, resp. $\mathrm{Wd}_{n,d}(e_{0\cup\mu_N})$ the set of words on $e_{0\cup\mu_N}$ that are of depth d, resp. of weight n and depth d. Let Λ and D be two formal variables.

Let $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{<\infty} \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle$, be the subset of the elements f such that, for each $d \in \mathbb{N}^*$, we have $\sup_{w \in \mathrm{Wd}_{*,d}(e_{0\cup\mu_N})} |f[w]|_p < \infty$. We say that the elements of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{<\infty}$ are bounded. Let $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)} \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{<\infty}$ be the subset of the elements f such that, for each $d \in \mathbb{N}^*$, we have $\sup w \in \mathrm{Wd}_{n,d}(e_{0\cup\mu_N}) |f[w]|_p \xrightarrow[n\to\infty]{} 0$. We say that the elements of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$ are summable.

Let

$$\mathcal{N}_{\Lambda,D} : f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle \mapsto \sum_{(n,d)\in\mathbb{N}^2} \Big(\max_{w\in \mathrm{Wd}_{n,d}(e_{0\cup\mu_N})} |f[w]|_p \Big) \Lambda^n D^d \in \mathbb{R}_+[\![\Lambda, D]\!],$$

and

$$\mathcal{N}_{D} : f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{<\infty} \mapsto \sum_{d\in\mathbb{N}} \Big(\sup_{w\in \mathrm{Wd}_{*,d}(e_{0\cup\mu_N})} |f[w]|_p \Big) D^d \in \mathbb{R}_+[\![D]\!].$$

One can check that these definitions give structures of complete normed ultrametric K-algebra on $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$, $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{<\infty}$ and $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$ [Jarossay 2015, Proposition 1.3.3].

1B2. *Compatibility between the ultrametric structure and the usual algebraic operations.* By [Jarossay 2015, Proposition 1.3.6], the Ihara product (1-1), the adjoint Ihara action (1-6), and the action τ (1-2) are continuous relative to the topologies defined by $\mathcal{N}_{\Lambda,D}$ and $\mathcal{N}_D$ on $\Pi_{1,0}(K)$ and the p-adic topology on K. And for all $f, g \in \Pi_{1,0}(K)$, $\lambda \in K^\times$, we have (by [Jarossay 2015, proof of Proposition 1.3.6])

$$\mathcal{N}_{\Lambda,D}(\mathrm{Ad}_f(e_1)) \leqslant \Lambda D \mathcal{N}_{\Lambda,D}(f), \tag{1-10}$$

$$\mathcal{N}_{\Lambda,D}(g \circ^{\int_{1,0}} f) \leqslant \mathcal{N}_{\Lambda,D}(g) \times \mathcal{N}_{\Lambda,D}(f), \tag{1-11}$$

$$\mathcal{N}_{\Lambda,D}(\tau(\lambda)(f))(\Lambda, D) = \mathcal{N}_{\Lambda,D}(f)(\lambda\Lambda, D). \tag{1-12}$$

These equations imply similar equations with $\mathcal{N}_D$ instead of $\mathcal{N}_{\Lambda,D}$ by passing to supremums.

Let us add another compatibility, which concerns the maps τ_n defined in Section 1A2.

Lemma 1B.1. (i) *For all $n \in \mathbb{N}^*$, for all $f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle$, we have $\mathcal{N}_{\Lambda,D}(\tau_n(f)) \leqslant \mathcal{N}_{\Lambda,D}(f)$, and in particular τ_n is a continuous linear map for the $\mathcal{N}_{\Lambda,D}$-topology.*

(ii) *$K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{<\infty}$ and $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$ are stable by τ_n.*

(iii) *For all $f, g \in \Pi_{1,0}(K)$ and $n \in \mathbb{N}$ we have*

$$\mathcal{N}_{\Lambda,D}(\mathrm{Ad}_{(g\circ^{\int_{1,0}}\tau_n(f))}(e_1)) \leqslant \Lambda D \mathcal{N}_{\Lambda,D}(g)\mathcal{N}_{\Lambda,D}(f). \tag{1-13}$$

Proof. (i) and (ii) are clear from the definitions. (iii) follows from (1-10), (1-11) and (i). □

1B3. *The weighted Ihara action and contraction mappings.* We define a notion of contraction mappings within the topological framework reviewed above. The exponent $-1_{\int_{1,0}}$ means the inverse for the Ihara product (1-1).

Definition 1B.2. Let $\kappa \in K^*$ with $|\kappa|_p < 1$. We say that a map $\psi : \Pi_{1,0}(K) \to \Pi_{1,0}(K)$ is a κ-contraction (with respect to $\mathcal{N}_{\Lambda,D}$ and $\circ^{\int_{1,0}}$) if, for all $f_1, f_2 \in \Pi_{1,0}(K)$, we have

$$\mathcal{N}_{\Lambda,D}(\psi(f_2)^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} \psi(f_1))(\Lambda, D) \leqslant \mathcal{N}_{\Lambda,D}(f_2^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} f_1)(\kappa\Lambda, D). \tag{1-14}$$

Indeed, let $\psi : \Pi_{1,0}(K) \to \Pi_{1,0}(K)$ be a contraction in the sense of Definition 1B.2. Then, by the submultiplicativity of the Ihara product with respect to $\mathcal{N}_{\Lambda,D}$ ((1-11)) and the fact that $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ is complete with respect to the distance defined by $\mathcal{N}_{\Lambda,D}$ (Section 1B), a standard proof tells us that ψ is continuous (with respect to $\mathcal{N}_{\Lambda,D}$) and has a unique fixed point, equal to $\mathrm{fix}_\psi = \lim_{a\to\infty} \psi^a(f)$ for all $f \in \Pi_{1,0}(K)$. Thus the contractions in the sense of Definition 1B.2 satisfy the usual properties of contractions regarding fixed points.

Definition 1B.3. Let $(\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{1,0}(K)$. We call weighted Ihara action by g with parameter λ and we denote by $(\lambda, g)\circ^{\int_{1,0}}$ the map

$$\Pi_{1,0}(K) \to \Pi_{1,0}(K), \quad f \mapsto (\lambda, g) \circ^{\int_{1,0}} f = g \circ^{\int_{1,0}} \tau(\lambda)(f).$$

We now relate the two previous definitions in the case where λ is small.

Proposition 1B.4. (i) *Let $(\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{1,0}(K)$ such that $|\lambda|_p < 1$. The map $(\lambda, g)\circ^{\int_{1,0}}$ is a λ-contraction. More precisely, the inequality* (1-14) *is an equality if $\kappa = \lambda$.*

(ii) *For all $(\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{1,0}(K)$, the Ihara action of g weighted by λ is an automorphism of the scheme $\Pi_{1,0} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K$, whose inverse is*

$$f \mapsto \tau(\lambda^{-1})(g^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} f).$$

Proof. (i) We have

$$\begin{aligned}((\lambda, g) \circ^{\int_{1,0}} (f_2))^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} ((\lambda, g) \circ^{\int_{1,0}} (f_1)) &= (g \circ^{\int_{1,0}} \tau(\lambda)(f_2))^{-1_{\int_{1,0}}} g \circ^{\int_{1,0}} \tau(\lambda)(f_1)\\ &= \tau(\lambda)(f_2)^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} g^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} g \circ^{\int_{1,0}} \tau(\lambda)(f_1)\\ &= \tau(\lambda)(f_2)^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} \tau(\lambda)(f_1)\\ &= \tau(\lambda)((f_2)^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} f_1).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{N}_{\Lambda,D}((\lambda, g) \circ^{\int_{1,0}} (f_2))^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} ((\lambda, g) \circ^{\int_{1,0}} (f_1))(\Lambda, D) &= \mathcal{N}_{\Lambda,D}\tau(\lambda)((f_2)^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} f_1)(\Lambda, D)\\ &= \mathcal{N}_{\Lambda,D}((f_2)^{-1_{\int_{1,0}}} \circ^{\int_{1,0}} f_1)(\lambda\Lambda, D)\end{aligned}$$

by (1-12).

(ii) Follows from the definitions. □

Knowing that we have a family of contractions, we consider their fixed points and their iterations.

Definition 1B.5. Let $(\lambda, g) \mapsto \mathrm{fix}_{\lambda,g}$ be the fixed point map which sends $(\lambda, g) \in \{z \in K \mid 0 < |z|_p < 1\} \times \Pi_{1,0}(K)$ to the unique fixed point of the weighted Ihara action $(\lambda, g)\circ^{\int_{1,0}}$.

It follows from the definitions that the map $\mathrm{fix}_\lambda : g \mapsto \mathrm{fix}_{\lambda,g}$ is an automorphism of the scheme $\Pi_{1,0} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K$, whose inverse is $\mathrm{fix}_\lambda^{-1} : f \mapsto f \circ^{\int_{1,0}} \tau(\lambda)(f)^{-1_{\int_{1,0}}}$. The fixed point map is characterized by the equation

$$g(e_0, (e_\xi)_\xi)\, \mathrm{fix}_{\lambda,g}(\lambda e_0, \lambda(g_\xi^{-1} e_\xi g_\xi)_\xi) = \mathrm{fix}_{\lambda,g}(e_0, (e_\xi)_\xi). \tag{1-15}$$

Note that the inversion for the Ihara product on $\Pi_{1,0}(K)$ is characterized by

$$g(e_0, (e_\xi)_\xi).g^{-1_{\int_{1,0}}}(e_0, (g_\xi^{-1} e_\xi g_\xi)_\xi) = 1.$$

Definition 1B.6. Let $a \in \mathbb{N}^*$. Let the map of iteration a times of the Ihara action weighted by λ, $\mathrm{iter}^{\int_{1,0}}_{a,\lambda} : \Pi_{1,0}(K) \to \Pi_{1,0}(K)$ be defined by $g \mapsto g^{a(\circ^{\int_{1,0}},\lambda)}$ where

$$g^{a(\circ^{\int_{1,0}},\lambda)} = \underbrace{(\lambda, g) \circ^{\int_{1,0}} \cdots \circ^{\int_{1,0}} (\lambda, g)}_{a} \circ^{\int_{1,0}} 1 = g \circ^{\int_{1,0}} \tau(\lambda)(g) \circ^{\int_{1,0}} \cdots \circ^{\int_{1,0}} \tau(\lambda^{a-1})(g). \tag{1-16}$$

Thus $g^{a(\circ^{\int_{1,0}},\lambda)}$ is the unique element of $\Pi_{1,0}(K)$ such that we have, for all $f \in \Pi_{1,0}(K)$,

$$\underbrace{(\lambda, g) \circ^{\int_{1,0}} \cdots \circ^{\int_{1,0}} (\lambda, g)}_{a} \circ^{\int_{1,0}} f = (\lambda^a, g^{a(\circ^{\int_{1,0}},\lambda)}) \circ^{\int_{1,0}} f.$$

The iteration map is expressed in terms of the usual de Rham multiplication on $\Pi_{1,0}(K)$ by

$$\mathrm{iter}^{\int_{1,0}}_{a,\lambda}(g) = g(e_0, (e_\xi)_{\xi \in \mu_N(K)}) g(\lambda e_0, (\lambda \operatorname{Ad}_{g_\xi}(e_\xi))_\xi) \cdots g(\lambda^{a-1} e_0, (\lambda^{a-1} \operatorname{Ad}_{g_\xi^{a-1}}(e_\xi))_\xi). \tag{1-17}$$

1B4. *Application to the Frobenius.* We apply the previous paragraphs to study the iteration of Frobenius at the base-points $(1, 0)$ which we view as a map $\phi : \Pi^{(p)}_{1,0}(K) \to \Pi_{1,0}(K)$.

Lemma 1B.7. *The map $\phi_{-\log(q)/\log(p)} : \Pi_{1,0}(K) \to \Pi_{1,0}(K)$ is a $(1/q)$-contraction. If $\Pi^{(p)}_{1,0}(K)$ is identified to $\Pi_{1,0}(K)$ by the isomorphism defined by $e_0 \mapsto e_0$ and $e_{\xi^{(p^\alpha)}} \mapsto e_\xi$ for all $\xi \in \mu_N(K)$, the map $\phi_{-1} : \Pi_{1,0}(K) \to \Pi^{(p)}_{1,0}(K)$ is a $(1/p)$-contraction.*

Proof. This follows from the formula (1-3), from Proposition 1B.4 and from the fact that σ is an isometry of K for the p-adic metric. □

In the rest of this paper, for simplicity, we will deal mostly with the iterations of $\phi_{-\log(q)/\log(p)}$. This is sufficient, knowing that, for any $\alpha \in \mathbb{N}^*$, writing the Euclidean division $\alpha = r + u \log(q)/\log(p)$, we have $\phi_\alpha = \phi^u_{\log(q)/\log(p)} \circ \phi_r$.

1C. ***Prime weighted cyclotomic multiple harmonic sums.*** The numbers $\mathrm{har}_{q^\alpha}((n_i)_d; (\xi_i)_{d+1})$ will play a central role; here, we formally explain how to study them.

1C1. *The three frameworks of computation.* Multiple polylogarithms are the solutions to the Knizhnik–Zamolodchikov differential equation, which is the universal connection associated with $\pi_1^{\mathrm{un,dR}}(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\})$ (in the sense of [Deligne 1989, Section 12]), and whose crystalline Frobenius structure is ϕ.

Their power series expansion at 0 is the following, which relates them to cyclotomic multiple harmonic sums:

$$\mathrm{Li}[(n_i)_d; (\xi_i)_d](z) = \sum_{0<m_1<\cdots<m_d} \frac{\left(\frac{\xi_2}{\xi_1}\right)^{m_1} \cdots \left(\frac{z}{\xi_d}\right)^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}}. \tag{1-18}$$

We proved in [Jarossay 2015, (0.3.7)] that prime weighted multiple harmonic sums are expressed in the following way (where $f \mapsto f^{(\xi)}$ is the natural map $\Pi_{1,0}(K) \to \Pi_{\xi,0}(K)$):

$$\mathrm{har}_{p^\alpha}((n_i)_d; (\xi_i)_{d+1}) = (-1)^d \sum_{l=0}^{\infty} \sum_{\xi} \xi^{-p^\alpha} \operatorname{Ad}_{\Phi^{(\xi)}_{p,\alpha}}(e_\xi)[e_0^l e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}]. \tag{1-19}$$

We have a total of three ways to deal with prime weighted multiple harmonic sums, which makes three frameworks for computations:

(i) Via their expression in terms of p-adic cyclotomic multiple zeta values (1-19).

(ii) Via their expression as coefficients of power series expansions of multiple polylogarithms (1-18).

(iii) Via their definition as elementary explicit iterated sums (0-2).

We will symbolize these three frameworks by, respectively, the notations $\int_{1,0}$, $\int$ and Σ.

In [Jarossay 2015] we have expressed the "harmonic Frobenius" in the frameworks $\int$ and Σ and we have compared the two expressions. Here we are going to express the "iteration of the harmonic Frobenius" in the frameworks $\int_{1,0}$ and Σ and compare the two expressions. Keeping in mind the distinction between these three frameworks $\int_{1,0}$, $\int$ and Σ will be essential in this paper and in subsequent ones. We note that the frameworks Σ and $\int$ make sense for all weighted cyclotomic multiple harmonic sums whereas the framework $\int_{1,0}$ makes sense only for the prime weighted cyclotomic multiple harmonic sums and follows from a theorem.

1C2. *The generalization to negative numbers of iterations of the Frobenius.* The indices of p-adic cyclotomic multiple zeta values, of the form $((n_i)_d;\,(\xi_i)_d)$, are distinct from the indices of cyclotomic weighted multiple harmonic sums (0-2), of the form $((n_i)_d;\,(\xi_i)_{d+1})$.

Definition 1C.1. A harmonic word over $e_{0\cup\mu_N}$, is a tuple $((n_i)_d;\,(\xi_i)_{d+1})$, with $d\in\mathbb{N}^*$ $(n_i)_d\in(\mathbb{N}^*)^d$, $(\xi_i)_{d+1}\in\mu_N(K)^{d+1}$. We sometimes identify it with $e_{\xi_{d+1}}e_0^{n_d-1}e_{\xi_d}\cdots e_0^{n_1-1}e_{\xi_1}$. Let us denote by $\mathrm{Wd}_{\mathrm{har}}(e_{0\cup\mu_N})$ the set of harmonic words over $e_{0\cup\mu_N}$.

We now define, using (1-19), an analogue of multiple harmonic sums associated with negative numbers of iterations of the Frobenius, and another analogue associated with the fixed point of the Frobenius. The notation "har" that we are going to use is justified by [Jarossay 2014; 2016a; 2016b] and the notion of "cyclotomic multiple harmonic values". Below, $f[1/(1-e_0)\cdot w]=\sum_{l=0}^{\infty}f[e_0^l w]$.

Definition 1C.2. For any $w=e_{\xi_{d+1}}e_0^{n_d-1}e_{\xi_d}\cdots e_0^{n_1-1}e_{\xi_1}=((n_i)_d;\,(\xi_i)_{d+1})$ we call generalized prime weighted multiple harmonic sums the following numbers:

(i) For any $\alpha\in\mathbb{N}^*$, let

$$\mathrm{har}_{p,\alpha}(w)=\mathrm{har}_{p^\alpha}(w)\quad\text{and}\quad\mathrm{har}_{p,-\alpha}(w)=(-1)^d\sum_{\xi}\xi^{-p^\alpha}\,\mathrm{Ad}_{\Phi^{(\xi)}_{p,-\alpha}}(e_\xi)\biggl[\frac{1}{1-e_0}\cdot w\biggr].$$

(ii) For $\epsilon\in\{\pm1\}$, let

$$\mathrm{har}_{q,\epsilon\infty}(w)=(-1)^d\sum_{\xi}\xi^{-1}\,\mathrm{Ad}_{\Phi^{(\xi)}_{q,\epsilon\infty}}(e_\xi)\biggl[\frac{1}{1-e_0}\cdot w\biggr].$$

(iii) If $p^\alpha=q^{\tilde{\alpha}}$, with $\tilde{\alpha}\in\mathbb{Z}$, we denote by $\mathrm{har}_{q,\tilde{\alpha}}=\mathrm{har}_{p,\alpha}$.

1C3. *The noncommutative generating series.* We now define generating series of prime weighted cyclotomic multiple harmonic sums, the numbers (0-2) with $m = p^{\alpha}$.

We have defined in [Jarossay 2015] two variants of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int}$ adapted to multiple harmonic sums: $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int}_{\text{har}} \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ the vector subspace of the elements f such that, for all words w on $e_{0\cup\mu_N}$, the sequence $(f[e_0^l w])_{l\in\mathbb{N}}$ is constant and $f[w'e_0] = 0$ for all words w'; and $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\Sigma}_{\text{har}} = K^{\text{Wd}_{\text{har}}(e_{0\cup\mu_N})}$. Here is the third variant:

Definition 1C.3. (i) Let

$$K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har}} = \{f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle \mid \forall l > 0, \forall r > 0, \forall w \text{ word on } e_{0\cup\mu_N},\ f[e_0^l w e_0^r] = 0\}.$$

(ii) Let

$$K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har},0} = \{f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har}} \mid \forall d \in \mathbb{N}^*, n_i \in \mathbb{N}^*(1 \leqslant i \leqslant d), \xi_i \in \mu_N(K)(1 \leqslant i \leqslant d+1),$$
$$f[e_{\xi\xi_{d+1}} e_0^{n_d-1} e_{\xi\xi_d} \cdots e_0^{n_1-1} e_{\xi\xi_1}] = \xi^{-1} f[e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}]\}.$$

The map $f \mapsto \sum_{w\in\text{Wd}_{\text{har}}(e_{0\cup\mu_N})} f[w]1/(1-e_0)\cdot w$ clearly defines an isomorphism

$$K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har}} \xrightarrow{\sim} K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int}_{\text{har}}$$

of topological K-vector spaces, with topology defined by $\mathcal{N}_D$. However, we denote $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har}}$ and $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int}_{\text{har}}$ differently in order to keep in mind the important distinction between the frameworks $\int_{1,0}$, $\int$, Σ.

Let us define the noncommutative generating series of the generalized prime weighted cyclotomic multiple harmonic sums.

Definition 1C.4. For any $\tilde{\alpha} \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$, let $\text{har}_{q,\tilde{\alpha}} = \sum_{w\in\text{Wd}_{\text{har}}(e_{0\cup\mu_N})} \text{har}_{q,\tilde{\alpha}}(w) w \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har}}$.

We note that, for $\tilde{\alpha} \in \mathbb{N}^*$, we have $\text{har}_{q,\tilde{\alpha}} \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har},0}$, because we have, for all $\xi \in \mu_N(K)$, $\xi^{-q^{\tilde{\alpha}}} = \xi^{-q} = \xi^{-1}$.

1D. ***The pro-unipotent harmonic actions and the harmonic Frobenius.*** We review definitions from [Jarossay 2015] of the pro-unipotent harmonic actions $\circ^{\int}_{\text{har}}$ and $\circ^{\Sigma}_{\text{har}}$, and the harmonic Frobeniuses $(\tau(p^{\alpha})\phi^{\alpha})^{\int}_{\text{har}}$ and $(\tau(p^{\alpha})\phi^{\alpha})^{\Sigma}_{\text{har}}$.

Let $\text{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1)_{o(1)} = \text{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1) \cap K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$; by [Jarossay 2015, Proposition 1.3.5], it is a subgroup of $\text{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1)$ for the usual group structure of $\text{Spec}(\mathcal{O}^{\text{Ш},e_{0\cup\mu_N}})$, and for the adjoint Ihara product $\circ^{\int_{1,0}}_{\text{Ad}}$; it is a complete topological group with the topology defined by $\mathcal{N}_D$, for both group structures.

1D1. *In the framework of integrals.* We review definitions from [Jarossay 2015] which will be useful in what follows.

Let $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\lim} \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ be the vector subspace consisting of the elements $f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ such that, for all words w on $e_{0\cup\mu_N}$, the sequence $(f[e_0^l w])_{l\in\mathbb{N}}$ has a limit in K, and $f[w'e_0] = 0$ for all words w'. Let the map $\lim : K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\lim} \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int}_{\text{har}}$ be defined by, for all words w over $e_{0\cup\mu_N}$,

$(\lim f)[w] = \lim_{l\to\infty} f[e_0^l w]$. The p-adic pro-unipotent harmonic action of integrals [Jarossay 2015, Definition 2.2.2] is the map $\circ_{\text{har}}^{\int} : \text{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \times (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\int})^{\mathbb{N}} \to (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\int})^{\mathbb{N}}$ defined by

$$(g, (h_m)_{m\in\mathbb{N}}) \mapsto g \circ_{\text{har}}^{\int} (h_m)_{m\in\mathbb{N}} = (\lim(h_m(e_0, (\tau(m)(g^{(\xi)})_\xi)))_{m\in\mathbb{N}})$$

The harmonic Frobenius of integrals [Jarossay 2015, Definition 2.3.5] is the map $(\tau(p^\alpha)\phi^\alpha)_{\text{har}}^{\int} : (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\int})^{\mathbb{N}} \to (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\int})^{\mathbb{N}}$ defined by

$$f \mapsto \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha} \circ_{\text{har}}^{\int} \sigma^\alpha(f).$$

1D2. *In the framework of series.* The p-adic pro-unipotent harmonic action of series [Jarossay 2015, Proposition–Definition 4.3.1] is a counterpart of $\circ_{\text{har}}^{\int}$ found in terms of series. It is a map

$$\circ_{\text{har}}^{\Sigma} : K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har},o(1)}^{\Sigma} \times (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\Sigma})^{\mathbb{N}} \to (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\Sigma})^{\mathbb{N}}$$

where $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har},o(1)}^{\Sigma}$ is defined in [Jarossay 2015, Definition 4.1.3].

The harmonic Frobenius of series is

$$f \mapsto \text{har}_{p^\alpha} \circ_{\text{har}}^{\Sigma} \sigma^\alpha(f).$$

We proved in [Jarossay 2015] that the harmonic Frobenius of integrals and the harmonic Frobenius of series are equal [loc. cit., (0.3.3) and (0.3.5)]. Thus, we see that the harmonic Frobenius is characterized by $\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha}$ or, equivalently, by har_{p^α}. This is why studying har_{p^α} is equivalent to studying the harmonic Frobenius and, in the next sections, we will see har_{p^α} as a function of α, which amounts to studying the harmonic Frobenius as a function of α.

Remark 1D.1. We can define natural analogues of all the structures which are at base-points $(1, 0)$ and which are mentioned in this Section 1, at base-points $(0, \xi)$ for any $\xi \in \mu_N(K)$, which are compatible with the morphism $(x \mapsto \xi x)_*$. All the results of this Section 1 have natural analogues at base-points $(0, \xi)$ for any $\xi \in \mu_N(K)$. This will be used implicitly in the proofs of the next section.

2. Iteration of the harmonic Frobenius of integrals at (1, 0)

We prove (0-3) in Section 2A, and we prove (0-4) in Section 2B.

2A. *The fixed point equation of the harmonic Frobenius of integrals at* (1, 0).

2A1. *The fixed point equation for the Frobenius of integrals.* We express the iterated Frobenius at base-points of $(1, 0)$ as a function of the number of iterations.

Proposition 2A.1. (a) *For all $\tilde{\alpha} \in \mathbb{N}^*$, we have $\Phi_{q,-\infty} = \text{fix}_{q^{\tilde{\alpha}_0},\Phi_{q,-\tilde{\alpha}}}$ i.e.,*

$$\Phi_{q,-\tilde{\alpha}} \circ^{\int_{1,0}} \tau(q^{\tilde{\alpha}})\Phi_{q,-\infty} = \Phi_{q,-\infty}, \quad \textit{i.e.,} \quad \tau(q^{\tilde{\alpha}})\Phi_{q,\infty} \circ^{\int_{1,0}} \Phi_{q,\tilde{\alpha}} = \Phi_{q,\infty}.$$

(b) *For the topology defined by $\mathcal{N}_D$ on $\Pi_{1,0}(K)$ we have*

$$\Phi_{q,-\tilde{\alpha}} \xrightarrow[\tilde{\alpha}\to\infty]{} \Phi_{q,-\infty}, \quad \textit{and} \quad \Phi_{q,\tilde{\alpha}} \xrightarrow[\tilde{\alpha}\to\infty]{} \Phi_{q,\infty}.$$

(c) *For any* $\tilde{\alpha} \in \mathbb{N}^*$, $f \in \Pi_{1,0}(K)$, $g \in \Pi_{1,0}^{(p^{\alpha})}(K)$,

$$\phi^{-\tilde{\alpha}}_{\frac{\log(q)}{\log(p)}}(f) = \sum_{n=0}^{\infty} \Phi_{q,\infty} \circ^{\int_{1,0}} (\tau_n(\Phi_{q,-\infty} \circ^{\int_{1,0}} f)).(q^{\tilde{\alpha}})^n,$$

$$\phi^{\tilde{\alpha}}_{\frac{\log(q)}{\log(p)}}(g) = \sum_{n=0}^{\infty} \Phi_{q,\infty} \circ^{\int_{1,0}} (\tau_n(\Phi_{q,-\infty} \circ^{\int_{1,0}} g)).(q^{\tilde{\alpha}})^n.$$

Proof. (a) (1-4) and (1-5) imply the first equation. (1-4) and (1-5) imply the second equation via Notation 1A.1(ii).

(b) The first equation follows from Lemma 1B.7 and the discussion after Proposition 1B.4, and the second equation is deduced from the first one by applying the inversion for $\circ^{\int_{1,0}}$, knowing the structure of topological group of $(\Pi_{1,0}(K), \circ^{\int_{1,0}})$ for $\mathcal{N}_D$. Alternatively, the two equations follow from (i), the fact that $\tau(q^{\tilde{\alpha}})\Phi_{q,\infty} \xrightarrow[\tilde{\alpha}\to\infty]{} 1$, $\tau(q^{\tilde{\alpha}})\Phi_{q,-\infty} \xrightarrow[\tilde{\alpha}\to\infty]{} 1$ and that structure of topological group.

(c) Follows from (1-3), (1-4), in which we replace $\Phi_{q,\tilde{\alpha}}$ and $\Phi_{q,-\tilde{\alpha}}$ by their expressions given by (i), and in which we express τ in terms of the maps τ_n defined in Section 1A2 just after (1-2). □

In particular, by Proposition 2B.1(i), the existence and uniqueness of a Frobenius-invariant path, which follows from the theory of Coleman integration, is reproved and made more precise in the very particular example of $\Pi_{1,0}(K)$.

2A2. *Pro-unipotent harmonic action and harmonic Frobenius of integrals at* $(1,0)$. We move from discussing the Frobenius at $(1,0)$ to discussing the harmonic Frobenius of integrals, in the framework $\int_{1,0}$ in the sense of Section 1C1. In view of this result, we introduce new objects.

Definition 2A.2. Let $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}} \subset K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ be the set of elements h such that all series of the type $\sum_{l=0}^{\infty} h[e_0^l e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}]$, where d and the n_i are positive integers and the ξ_i are N-th roots of unity, are convergent in K. Let the summation map be $S : K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}} \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har},0}^{\int_{1,0}}$ defined by

$$h \mapsto \sum_{\xi\in\mu_N(K)} \xi^{-1} \sum_{\substack{d\in\mathbb{N}^* \\ \xi_1,\dots,\xi_{d+1}\in\mu_N(K) \\ n_1,\dots,n_d\in\mathbb{N}^*}} h^{(\xi)}\left[\frac{1}{1-e_0} e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}\right] e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}.$$

We note that $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}}$ contains $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$ defined in Section 1B. We now define a variant of the pro-unipotent harmonic action of integrals $\circ^{\int}_{\text{har}}$. In the next statement, we denote by $\circ^{\int_{1,0}}_{\text{Ad}}$ the extension of the adjoint Ihara product $\circ^{\int_{1,0}}_{\text{Ad}}$ into a map $K\langle\langle e_{0\cup\mu_N}\rangle\rangle \times K\langle\langle e_{0\cup\mu_N}\rangle\rangle \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ defined again by the formula of (1-6).

Definition 2A.3. Let the map

$$\circ^{\int_{1,0}}_{\text{har},0} : \text{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \times K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har},0} \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\int_{1,0}}_{\text{har},0}$$

be characterized by the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \times K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}} & \xrightarrow{\circ_{\mathrm{Ad}}^{\int 1,0}} & K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}} \\ \big\downarrow \mathrm{id}\times S & & \big\downarrow S \\ \mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \times K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int 1,0} & \xrightarrow{\circ_{\mathrm{har}}^{\int 1,0}} & K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int 1,0} \end{array} \tag{2-1}$$

The pro-unipotent harmonic action of integrals at $(1, 0)$ is the map

$$\circ_{\mathrm{har}}^{\int 1,0} : \mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \times K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int 1,0} \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int 1,0}$$

defined by the same formula as the one of $\circ_{\mathrm{har}}^{\int 1,0}$.

The basic properties of $\circ_{\mathrm{har}}^{\int 1,0}$ are summarized in the next proposition.

Proposition 2A.4. (i) *$\circ_{\mathrm{har}}^{\int 1,0}$ is a well-defined group action of $(\mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}, \circ_{\mathrm{Ad}}^{\int 1,0})$ on $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int 1,0}$, continuous for the topology defined by $\mathcal{N}_D$.*

(ii) *The isomorphism $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int 1,0} \xrightarrow{\sim} K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int}$, $f \mapsto \sum_{w\in \mathrm{Wd}_{\mathrm{har}}(e_{0\cup\mu_N})} f[w]1/(1-e_0)\cdot w$ induces, for all $m \in \mathbb{N}^*$, a natural isomorphism of continuous group actions between the m-th term of $\circ_{\mathrm{har}}^{\int}$, namely $(g, h_m) \mapsto \lim(\tau(m)(g) \circ_{\mathrm{Ad}}^{\int 0,0} h_m)$, and the action $(g, h) \mapsto \tau(m)g \circ_{\mathrm{har}}^{\int 1,0} h$.*

Proof. (i) For any g in $\mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1)$, and any word w, the map $f \mapsto S(g \circ_{\mathrm{Ad}}^{\int 1,0} f)[w]$ factors in a natural way through the map $f \mapsto Sf$. This can be seen by writing the formula for the dual of $\circ_{\mathrm{Ad}}^{\int 1,0}$. The fact that $\circ_{\mathrm{Ad}}^{\int 1,0}$ sends $\mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \times K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}}$ to $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}}$ follows from the shuffle equation for elements of $\tilde{\Pi}_{1,0}(K)_{o(1)}$ and from the formula for the dual of $\circ_{\mathrm{Ad}}^{\int 1,0}$. Finally, S is surjective: any h in $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har},0}^{\int 1,0}$ is equal to

$$S\Bigg(\sum_{\substack{\xi_1,\dots,\xi_{d+1}\in\mu_N(K)\\ n_1,\dots,n_d\in\mathbb{N}^*}} h[e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}] e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}\Bigg).$$

This proves that $\circ_{\mathrm{har},0}^{\int 1,0}$ is well-defined, and that one can write a formula for it, which is linear with respect to the second argument; thus, $\circ_{\mathrm{har}}^{\int 1,0}$ is well-defined.

Let $g_1, g_2 \in \mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1)$ and $f \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$; we have

$$g_2 \circ_{\mathrm{Ad}}^{\int 1,0} (g_1 \circ_{\mathrm{Ad}}^{\int 1,0} f) = (g_2 \circ_{\mathrm{Ad}}^{\int 1,0} g_1) \circ_{\mathrm{Ad}}^{\int 1,0} f.$$

Applying the map S and the commutativity of (2-1) gives

$$g_2 \circ_{\mathrm{har}}^{\int 1,0} (S(g_1 \circ_{\mathrm{Ad}}^{\int 1,0} f)) = (g_2 \circ_{\mathrm{Ad}}^{\int 1,0} g_1) \circ_{\mathrm{har}}^{\int 1,0} S(f),$$

and applying again the commutativity of (2-1) we deduce

$$g_2 \circ_{\mathrm{har}}^{\int 1,0} (g_1 \circ_{\mathrm{har}}^{\int 1,0} S(f)) = (g_2 \circ_{\mathrm{Ad}}^{\int 1,0} g_1) \circ_{\mathrm{har}}^{\int 1,0} S(f).$$

This proves that $\circ_{\mathrm{har}}^{\int 1,0}$ is a group action.

The map $\circ_{\mathrm{Ad}}^{\int_{1,0}}$ is continuous and each sequence $(w_l)_{l\in\mathbb{N}}$ such that $w_l = e_0^l e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}$ for all l with $e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}$ independent of l satisfies $\limsup_{l\to\infty} \mathrm{depth}(w_l) < +\infty$ and $\mathrm{weight}(w_l) \xrightarrow[l\to\infty]{} +\infty$; thus the map S is continuous. By the commutativity of (2-1), this implies that $\circ_{\mathrm{har}}^{\int_{1,0}} \circ (\mathrm{id} \times S)$ is continuous. If a sequence $(h_u)_{u\in\mathbb{N}}$ in $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}}$ tends to $h \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}}$, we can find a sequence $(f_u)_{u\in\mathbb{N}}$ and f in $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}}$ such that $h_u = Sf_u$ for all u, $h = f$ and $f_u \to f$. We deduce that $\circ_{\mathrm{har}}^{\int_{1,0}}$ is continuous.

(ii) The relation between $\circ_{\mathrm{har}}^{\int_{1,0}}$ and $\circ_{\mathrm{har}}^{\int}$ follows from the definitions of these two objects and the following property of $\circ_{\mathrm{Ad}}^{\int_{1,0}}$, which is itself a consequence of the formula for the dual of $\circ_{\mathrm{Ad}}^{\int_{1,0}}$: for all $h \in \mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1)$ and $g \in \mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1)$, and for any word w over $e_{0\cup\mu_N}$, we have

$$(g \circ_{\mathrm{Ad}}^{\int_{1,0}} h)\left[\frac{1}{1-e_0} w\right] = \lim_{l\to\infty} (g \circ_{\mathrm{Ad}}^{\int_{1,0}} Sh)[e_0^l w]. \qquad \square$$

In the next example, the indices (n_1) and (n_1, n_2) are harmonic words in the sense of Definition 1C.1.

Example 2A.5. If $N = 1$, the terms of depth one and two of $g \circ_{\mathrm{har}}^{\int_{1,0}} h$ are given as follows:

$$(g \circ_{\mathrm{har}}^{\int_{1,0}} h)(n_1) = h(n_1) + g\left[\frac{1}{1-e_0} e_1 e_0^{n_1-1} e_1\right],$$

$$(g \circ_{\mathrm{har}}^{\int_{1,0}} h)(n_1, n_2) = h(n_1, n_2) + g\left[\frac{1}{1-e_0} e_1 e_0^{n_2-1} e_1 e_0^{n_1-1} e_1\right] + \sum_{r=0}^{n_1-1} g\left[\frac{1}{1-e_0} e_1 e_0^{n_2-1} e_1 e_0^{r}\right] h(n_1 - r) + \sum_{r=0}^{n_2-1} g[e_0^r e_1 e_0^{n_1-1} e_1] h(n_2 - r).$$

We now deduce from Definition 2A.3 the counterpart of the harmonic Frobenius of integrals in the framework $\int_{1,0}$.

Definition 2A.6. For any $\alpha \in \mathbb{N}^*$ divisible by $\log(q)/\log(p)$, let the harmonic Frobenius of integrals at $(1, 0)$, iterated α times, be the map $(\tau(p^\alpha)\phi_\alpha)_{\mathrm{har}}^{\int_{1,0}} : K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}} \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}}$ defined by $f \mapsto \mathrm{Ad}_{\Phi_{p,\alpha}}(e_1) \circ_{\mathrm{har}}^{\int_{1,0}} f$.

Corollary 2A.7. *The isomorphism*

$$K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}} \xrightarrow{\sim} K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int}, \quad f \mapsto \sum_{w\in \mathrm{Wd}_{\mathrm{har}}(e_{0\cup\mu_N})} f[w] \frac{1}{1-e_0} w$$

induces, for all $m \in \mathbb{N}$, an isomorphism of continuous maps between $(\tau(p^\alpha)\phi_\alpha)_{\mathrm{har}}^{\int_{1,0}}$ and the $m = 1$ term of $(\tau(p^\alpha)\phi_\alpha)_{\mathrm{har}}$.

Proof. Direct consequence of Proposition 2A.4 and the definitions of the two harmonic Frobeniuses. $\square$

2A3. *Fixed point equation for the harmonic Frobenius.* We can now deduce from Section 2A1, via Section 2A2, an expression of the iterated harmonic Frobenius of integrals at $(1,0)$ as a function of its number of iterations, whose coefficients are expressed in terms of the fixed point of the Frobenius at $(1,0)$; this is (0-3).

By Proposition 2B.1, we have $\tau(q^{\tilde{\alpha}})\Phi_{q,\infty} \circ^{\int 1,0} \Phi_{q,\tilde{\alpha}} = \Phi_{q,\infty}$. By definition, the inverse of $\Phi_{q,\infty}$ for $\circ^{\int 1,0}$ is $\Phi_{q,-\infty}$ thus the inverse of $\tau(q^{\tilde{\alpha}})(\Phi_{q,\infty})$ is $\tau(q^{\tilde{\alpha}})(\Phi_{q,-\infty})$. Whence $\Phi_{q,\tilde{\alpha}} = \tau(q^{\tilde{\alpha}})\Phi_{q,-\infty} \circ^{\int 1,0} \Phi_{q,\infty}$. By applying $\mathrm{Ad}(e_1)$, we deduce $\mathrm{Ad}_{\Phi_{q,\tilde{\alpha}}}(e_1) = \tau(q^{\tilde{\alpha}})\,\mathrm{Ad}_{\Phi_{q,-\infty}}(e_1) \circ^{\int 1,0}_{\mathrm{Ad}} \mathrm{Ad}_{\Phi_{q,\infty}}(e_1)$, whence $S\,\mathrm{Ad}_{\Phi_{q,\tilde{\alpha}}}(e_1) = S(\tau(q^{\tilde{\alpha}})\,\mathrm{Ad}_{\Phi_{q,-\infty}}(e_1) \circ^{\int 1,0}_{\mathrm{Ad}} \mathrm{Ad}_{\Phi_{q,\infty}}(e_1))$. By the commutative diagram in Definition 2A.3, this amounts to $S\,\mathrm{Ad}_{\Phi_{q,\tilde{\alpha}}}(e_1) = \tau(q^{\tilde{\alpha}})\,\mathrm{Ad}_{\Phi_{q,-\infty}}(e_1) \circ^{\int 1,0}_{\mathrm{har}} S\,\mathrm{Ad}_{\Phi_{q,\infty}}(e_1))$. By (1-19), we have $S\,\mathrm{Ad}_{\Phi_{q,\tilde{\alpha}}}(e_1) = \mathrm{har}_{q,\tilde{\alpha}}$ and $S\,\mathrm{Ad}_{\Phi_{q,\infty}}(e_1)) = \mathrm{har}_{q,\infty}$. Whence (0-3).

Remark 2A.8. The power series expansion of any $\mathrm{har}_{q^{\tilde{\alpha}}}((n_i)_d;(\xi_i)_{d+1})$ in terms of $q^{\tilde{\alpha}}$, given by (0-3) have coefficients of degrees in $\{1,\dots,\min_{1\leqslant i\leqslant d} n_i - 1\}$ equal to 0. This follows from $\Phi^{(\xi)}_{q,-\infty}[e_0] = 0$ which implies $\mathrm{Ad}_{\Phi^{(\xi)}_{q,-\infty}}(e_\xi) = e_\xi +$ terms of depth $\geqslant 2$, for all $\xi \in \mu_N(K)$.

Example 2A.9. In depth one and two and for $\mathbb{P}^1 \setminus \{0,1,\infty\}$, we have

$$\mathrm{har}_{q^{\tilde{\alpha}}}(n_1) = \mathrm{har}_{q^\infty}(n_1) + \sum_{n=n_1}^{\infty} (q^{\tilde{\alpha}})^n\, \mathrm{Ad}_{\Phi_{q,-\infty}}(e_1)[e_0^{n-n_1} e_1 e_0^{n_1-1} e_1],$$

$$\begin{aligned}\mathrm{har}_{q^{\tilde{\alpha}}}(n_1,n_2) = \mathrm{har}_{q^\infty}(n_1,n_2) &+ \sum_{n=n_1+n_2}^{\infty} (q^{\tilde{\alpha}})^n\, \mathrm{Ad}_{\Phi_{q,-\infty}}(e_1)[e_0^{n-n_1-n_2} e_1 e_0^{n_2-1} e_1 e_0^{n_1-1} e_1] \\ &+ \sum_{r_1=0}^{n_1-1} \sum_{n=n_2+r_1}^{\infty} (q^{\tilde{\alpha}})^n\, \mathrm{Ad}_{\Phi_{q,-\infty}}(e_1)[e_0^{n-n_2-r_1} e_1 e_0^{n_2-1} e_1 e_0^{r_1}]\, \mathrm{har}_{q^\infty}(n_1-r_1) \\ &+ \sum_{r_2=0}^{n_2-1} (q^{\tilde{\alpha}})^{n_1+r_2}\, \mathrm{Ad}_{\Phi_{q,-\infty}}(e_1)[e_0^{r} e_1 e_0^{n_1-1} e_1]\, \mathrm{har}_{q^\infty}(n_2-r_2).\end{aligned}$$

2B. *Iteration equation of the harmonic Frobenius of integrals at* (1, 0).

We are now going to reexpress the iterated harmonic Frobenius of integrals at $(1,0)$ as a function of the number of iterations, in a different way, without involving the fixed point.

2B1. *Iteration on the Frobenius at* $(1,0)$. As in Section 2A2, the first step is to describe the iterated Frobenius at base-points $(1,0)$ as a function of its number of iterations, this time without involving the fixed point but, instead, the map of iteration of the weighted Ihara action (Definition 1B.6).

Proposition 2B.1. *For all* $\tilde{\alpha}_0, \tilde{\alpha} \in \mathbb{N}^*$ *such that* $\tilde{\alpha}_0 \mid \tilde{\alpha}$, *we have*

$$\Phi_{q,\alpha} = \mathrm{iter}^{\int 1,0}_{\tilde{\alpha}/\tilde{\alpha}_0, q^{\tilde{\alpha}_0}}(\Phi_{q,\tilde{\alpha}_0}) \quad \text{and} \quad \Phi_{q,\tilde{\alpha}} = \Phi_{q,-\tilde{\alpha}}^{-1_{\int 1,0}}.$$

More generally, for all $\alpha \in \mathbb{N}^*$, *we have* $\Phi_{p,-\alpha} = \Phi_{p,-1} \circ^{\int 1,0} \tau(\lambda)\sigma(\Phi_{p,-1}) \cdots \circ^{\int 1,0} \tau(\lambda^{\alpha-1})\sigma^{\alpha-1}(\Phi_{p,-1})$ *and* $\Phi_{p,-\alpha} = \Phi_{p,\alpha}^{-1_{\int 1,0}}$.

Proof. This follows from (1-3), (1-4), and Definition 1B.3. □

This generalizes a statement appearing in [Furusho 2007, proof of Proposition 3.1].

Before continuing on the study of the iterated Frobenius, we remark that, by Proposition 2A.1 and Proposition 2B.1 considering the coefficients of these noncommutative formal power series series, one has equations relating the p-adic cyclotomic multiple zeta values $\zeta_{p,\alpha}(w)$ and $\zeta_{p,\alpha'}(w')$, for any $\alpha, \alpha' \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$, as follows:

Corollary 2B.2. *Let $\alpha \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$. For any $n \in \mathbb{N}^*$, let $\mathcal{Z}_{p,\alpha,n}$ be the $\mathbb{Q}$-vector space generated by the numbers $\zeta_{p,\alpha}(w)$ with w a word of weight n:*

(1) *For $\alpha \in \mathbb{N}^*$, $\mathcal{Z}_{p,\alpha,n} = \sigma^{-\alpha}(\mathcal{Z}_{p,1,n})$, and $\mathcal{Z}_{p,-\alpha,n} = \sigma^{\alpha}(\mathcal{Z}_{p,-1,n})$. If $\alpha \in \mathbb{Z} \setminus \{0\}$ is such that $p^{|\alpha|}$ is a power of q then $\mathcal{Z}_{p,\alpha,n} = \mathcal{Z}_{q,\infty,n} = \mathcal{Z}_{q,-\infty,n}$. In particular, the dimension of $\mathcal{Z}_{p,\alpha,n}$ is independent of $\alpha \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$.*

(2) *$\mathcal{N}_{\Lambda,D}(\Phi_{p,\alpha})$ is independent of $\alpha \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$.*

Proof. For any positive integers n, d, let $\mathcal{O}_{n,d}^{\text{Ш},e_{0\cup\mu_N}} \subset \mathcal{O}^{\text{Ш},e_{0\cup\mu_N}}$ be the subspace generated by words of weight n and depth d. Let

$$\mathcal{O}^2_{n,\leqslant d} = \sum_{\substack{r\geqslant 2\\ n_1+\cdots+n_r=n\\ d_1+\cdots+d_r\leqslant d}} \mathcal{O}_{n_1,d_1}^{\text{Ш},e_{0\cup\mu_N}} \,\text{Ш}\cdots\text{Ш}\, \mathcal{O}_{n_r,d_r}^{\text{Ш},e_{0\cup\mu_N}}.$$

Let $\lambda \in K$ such that $|\lambda|_p < 1$ and $a \in \mathbb{N}^*$. For each w word on $e_{0\cup\mu_N}$, of weight n and depth d, by (1-15), (1-16), (1-17) we have the following congruences, where the duals refer to the duality between $\mathcal{O}^{\text{Ш},e_{0\cup\mu_N}}$ and the points of the corresponding group scheme:

$$\begin{aligned}(1-\lambda^n)\,\mathrm{fix}^\vee_\lambda(w) &\equiv w \mod \mathcal{O}^2_{n,\leqslant d},\\ \mathrm{iter}^{f_{1,0}\,\vee}_{a,\lambda}(w) &\equiv \frac{1-\lambda^{an}}{1-\lambda^n} w \mod \mathcal{O}^2_{n,\leqslant d},\\ (-1_{f_{1,0}})^\vee(w) &\equiv -w \mod \mathcal{O}^2_{n,\leqslant d}.\end{aligned}$$

By induction on (n,d), this implies the following equalities, where ${}_{\mathbb{Z}(\lambda)}\mathcal{O}_{n,\leqslant d}^{\text{Ш},e_{0\cup\mu_N}}$ is the $\mathbb{Z}(\lambda)$-module generated by words of weight n and of depth $\leqslant d$:

$$\mathrm{fix}^\vee_\lambda({}_{\mathbb{Z}(\lambda)}\mathcal{O}_{n,\leqslant d}^{\text{Ш},e_{0\cup\mu_N}}) = \mathrm{iter}^\vee_{a_{\circ}f_{1,0},\lambda}({}_{\mathbb{Z}(\lambda)}\mathcal{O}_{n,\leqslant d}^{\text{Ш},e_{0\cup\mu_N}}) = (-1_{f_{1,0}})^\vee({}_{\mathbb{Z}(\lambda)}\mathcal{O}_{n,\leqslant d}^{\text{Ш},e_{0\cup\mu_N}}) = {}_{\mathbb{Z}(\lambda)}\mathcal{O}_{n,\leqslant d}^{\text{Ш},e_{0\cup\mu_N}},$$

and, that, for all $g \in \Pi_{\xi,0}(K)$, we have

$$\mathcal{N}_{\Lambda,D}(\mathrm{fix}_{\lambda,g}) = \mathcal{N}_{\Lambda,D}(g^{a_{\circ}f_{1,0,\lambda}}) = \mathcal{N}_{\Lambda,D}(g^{-1_{f_{1,0}}}) = \mathcal{N}_{\Lambda,D}(g).$$

This implies the result via Proposition 2B.1 and Proposition 2B.1. □

A particular case of Corollary 2B.2(i) can be found in [Yamashita 2010, Proposition 3.10] and an explicit example is given in [Furusho 2007, Example 2.10].

Remark 2B.3. The proof of Corollary 2B.2 indicates that the equations of Corollary 2B.2 are compatible with the depth filtration and the bounds on valuations of $p\mathrm{MZV}\mu_N$'s, which are two parameters in our computation. This can be viewed as a prerequisite for the next paragraphs in which we show a kind of compatibility between the iteration of the Frobenius and our computation of $p\mathrm{MZV}\mu_N$'s.

2B2. *The iteration of the harmonic Frobenius.* As in Section 2A2, we move from discussing the Frobenius at (1, 0), in Section 2A2, to discussing the harmonic Frobenius. We first describe how the map $\text{iter}^{\int_{1,0}}_{a,\lambda}$ depends on its parameters λ and a.

Definition 2B.4. For any $d \in \mathbb{N}^*$, let $\tau_{*,\leqslant d} : K\langle\langle e_{0\cup\mu_N}\rangle\rangle \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ be the map which sends $f = \sum_{w \text{ word}} f[w]w$ to $\sum_{w \text{ word}, \text{depth}(w)\leqslant d} f[w]w$.

Proposition 2B.5. *Let Λ, $\tilde{\Lambda}$, $\boldsymbol{a}$ be three formal variables. There exists a map*

$$\text{iter}^{\int_{1,0}}(\Lambda, \tilde{\Lambda}, \boldsymbol{a}) : \text{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1) \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle[\tilde{\Lambda}, \boldsymbol{a}](\Lambda), \tag{2-2}$$

such that, for any $f \in \text{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1)$, word w, $a \in \mathbb{N}^*$ such that $a > \text{depth}(w)$ and $\lambda \in K \setminus \{0\}$ which is not a root of unity, we have

$$\text{iter}^{\int_{1,0}}_{a,\lambda}(f)[w] = \text{iter}^{\int_{1,0}}(\lambda, \lambda^a, a)(f)[w].$$

Proof. With the assumptions of the statement, let $d = \text{depth}(w)$; knowing that $g[\varnothing] = 1$, dualizing the multiplication of the a factors in (1-17) gives

$$\text{iter}^{\int_{1,0}}_{a,\lambda}(g)[w] = \sum_{0\leqslant d'\leqslant d} \sum_{0\leqslant i_1<\cdots<i_{d'}\leqslant a-1} \sum_{\substack{w_{i_1}\neq\varnothing,\ldots,w_{i_{d'}}\neq\varnothing\\ w_{i_1}\cdots w_{i_{d'}}=w}} g(\lambda^{i_1-1}e_0, (\lambda^{i_1-1}\text{Ad}_{g^{(\xi)i_1-1}}(e_\xi))_\xi)[w_{i_1}] \times \cdots \times g(\lambda^{i_{d'}-1}e_0, (\lambda^{i_1-1}\text{Ad}_{g^{(\xi)i_{d'}-1}}(e_\xi))_\xi)[w_{i_{d'}}]. \tag{2-3}$$

We have assumed that $a > d$; let us thus separate the indices $i_j \leqslant d$ and $i_j > d$:

$$\sum_{0\leqslant d'\leqslant d} \sum_{0\leqslant i_1<\cdots<i_{d'}\leqslant a-1} = \sum_{0\leqslant d''\leqslant d'\leqslant d} \sum_{0\leqslant i_1<\cdots<i_{d''}\leqslant d} \sum_{d<i_{d''+1}<\cdots<i_{d'}\leqslant a-1}.$$

This yields an expression of (2-3), as a K-linear combination indexed by $\{(d'', d') \mid 0 \leqslant d'' \leqslant d' \leqslant d\} \times \{$deconcatenations of w in d' nonempty subwords$\}$ which is independent of a but depends polynomially of λ, and with coefficients as well independent of a and polynomial functions of λ, of the numbers

$$\sum_{d<i_{d''}<\cdots<i_{d'}\leqslant a-1} g(\lambda^{i_1-1}e_0, (\lambda^{i_{d''}-1}\text{Ad}_{g^{(\xi)i_{d''}-1}}(e_\xi))_\xi)[w_{i_{d''}}] \times \cdots \times g(\lambda^{i_{d'}-1}e_0, (\lambda^{i_{d''}-1}\text{Ad}_{g^{(\xi)i_{d''}-1}}(e_\xi))_\xi)[w_{i_{d'}}]. \tag{2-4}$$

Let $\epsilon^{(\xi)} = g^{(\xi)} - 1$ and $\tilde{\epsilon}^{(\xi)} = g^{(\xi)^{-1}} - 1$. For $0 \leqslant i_j \leqslant a-1$, we have (where $\tau_{*,\leqslant d}$ is defined in Definition 2B.4)

$$\tau_{*,\leqslant d}(\text{Ad}_{g^{(\xi)i_j-1}}(e_\xi)) = \sum_{\substack{m_j,m'_j\in\mathbb{N}\\ m_j\leqslant i_j, m'_j\leqslant i_j, m_j+m'_j+1\leqslant d}} \binom{i_j}{m_j}\binom{i_j}{m'_j} \tilde{\epsilon}_\xi^{m'_j} e_\xi \epsilon_\xi^{m_j}.$$

When $i_j > d$, the collection of conditions $\{m_j, m'_j \in \mathbb{N}, m_j \leqslant i_j, m'_j \leqslant i_j, m_j + m'_j + 1 \leqslant d\}$ is equivalent to $\{m_j, m'_j \in \mathbb{N}, m_j + m'_j + 1 \leqslant d\}$; thus, dualizing in (2-4) each factor $g(\lambda^{i_j-1}e_0, (\lambda^{i_j-1}\text{Ad}_{g^{(\xi)i_j-1}}(e_\xi))_\xi)$ tells us that (2-4) is a linear combination, independent of a and λ, of sums

$$\sum_{d<i_{d''}<\cdots<i_{d'}\leqslant a-1} \prod_{j=d''}^{d'} \binom{i_j}{m_j}\binom{i_j}{m'_j} \lambda^{i_j \text{ weight}_j},$$

where $\text{weight}_j \in \mathbb{N}^*$ arises as the weight of a certain quotient sequence of w_{i_j}, and $\binom{i_j}{m_j}\binom{i_j}{m'_j}$ are polynomials of i_j.

Finally, any function of a of the form $\sum_{L \leqslant I_1 < \cdots < I_\delta \leqslant a-1} \prod_{j=1}^{\delta} P_j(I_j) \lambda^{I_j C_j}$ with $L, \delta \in \mathbb{N}^*$, $C_1, \ldots, C_\delta \in \mathbb{N}^*$ and $P_1, \ldots, P_\delta \in K[T]$ polynomials, depends on a as a polynomial function of (a, λ^a): one can reduce this statement to $L = 0$ by splitting an iterated sum over $0 \leqslant I_1 < \cdots < I_\delta \leqslant a-1$ at L and by induction on δ, then use, again by induction on δ that, for all $\deg_j \in \mathbb{N}^*$, we have

$$\sum_{I_j=0}^{\deg_j} I_j^{\delta'} \lambda^{C_j I_j} = \left(\lambda^{C_j} \frac{d}{d(\Lambda^{C_j})}\right)^l \left(\frac{\lambda^{C_j \deg_j} - 1}{\Lambda^{C_j} - 1}\right). \qquad \square$$

Let us now define the map of iteration of the harmonic Frobenius of integrals at (1, 0), by using the above iteration map and the summation map S of Definition 2A.2.

Definition 2B.6. Let $\text{iter}_{\text{har}}^{\int_{1,0}}(a, \lambda) = S \circ \text{iter}^{\int_{1,0}}(\lambda, \lambda^a, a) : \text{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1) \to K\langle\langle e_{0 \cup \mu_N} \rangle\rangle_{\text{har}}^{\int_{1,0}}$.

We can now deduce from Section 2B1, and the previous proposition a second description of the iterated harmonic Frobenius as a function of its number of iterations; this is (0-4).

By Proposition 2B.1, we have $\Phi_{q,\tilde{\alpha}} = \text{iter}_{\tilde{\alpha}/\tilde{\alpha}_0, q; \tilde{\alpha}_0}^{\int_{1,0}}(\Phi_{q,\tilde{\alpha}_0})$, thus by Proposition 2B.5, we have $\Phi_{q,\tilde{\alpha}} = \text{iter}^{\int_{1,0}}(q^{\tilde{\alpha}_0}, q^{\tilde{\alpha}}, \tilde{\alpha}/\tilde{\alpha}_0)(\Phi_{q,\tilde{\alpha}_0})$, whence $S\Phi_{q,\tilde{\alpha}} = S\, \text{iter}^{\int_{1,0}}(q^{\tilde{\alpha}_0}, q^{\tilde{\alpha}}, \tilde{\alpha}/\tilde{\alpha}_0)(\Phi_{q,\tilde{\alpha}_0})$. By Definition 2B.6, this amounts to $S\Phi_{q,\tilde{\alpha}} = \text{iter}_{\text{har}}^{\int_{1,0}}(\tilde{\alpha}/\tilde{\alpha}_0, q^{\tilde{\alpha}})(\Phi_{q,\tilde{\alpha}_0})$. Finally, by (1-19), we have $S\Phi_{q,\tilde{\alpha}} = \text{har}_{q,\tilde{\alpha}}$. Whence (0-4).

3. Iteration of the harmonic Frobenius of series

In this section, we prove (0-5) and we discuss its meaning.

3A. *Prime weighted multiple harmonic sums as functions of the number of iterations of the Frobenius.*

In this section we study how $\text{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1})$ depends on $\tilde{\alpha}$.

The first step is to write a p-adic expression of $\text{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1})$, obtained by considering the $q^{\tilde{\alpha}_0}$-adic expansion of the indices $m_1, \ldots, m_d$ of the domain of summation of $\text{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1})$ as in (0-2).

Lemma 3A.1. *We have, for any $d \in \mathbb{N}^*$, positive integers n_i $(1 \leqslant i \leqslant d)$, and N-th roots of unity ξ_i $(1 \leqslant i \leqslant d+1)$,*

$$\text{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1}) = \sum_{\substack{(l_i)_d \in \mathbb{N}^d \\ (v_i)_d, (u_i)_d, (r_i)_d \in \mathbb{N}^d \times \mathbb{N}^d \times \{1, \ldots, q^{\tilde{\alpha}_0}-1\} \\ u_i \leqslant q^{\tilde{\alpha}_0 (v_{i+1} - v_i - 1)}(q^{\tilde{\alpha}_0} u_{i+1} + r_{i+1}) - 1 \text{ if } v_i < v_{i+1} \\ u_i \leqslant u_{i+1} \text{ if } v_i = v_{i+1} \text{ and } r_i < r_{i+1} \\ u_i \leqslant u_{i+1} - 1 \text{ if } v_i = v_{i+1} \text{ and } r_i \geqslant r_{i+1} \\ q^{\tilde{\alpha}_0 (v_i - v_{i+1} - 1)}(q^{\tilde{\alpha}_0} u_i + r_i) \leqslant u_{i+1} \text{ if } v_i > v_{i+1}}} (q^{\tilde{\alpha}})^{\sum_{i=1}^{d} n_i} \left(\frac{1}{\xi_{d+1}}\right)^{q^{\tilde{\alpha}}} \left(\prod_{j=1}^{d} \left(\frac{\xi_{j+1}}{\xi_j}\right)^{u_j + r_j} \binom{-n_j}{l_j} \frac{(q^{\tilde{\alpha}_0} u_j)^{l_j}}{r_j^{l_j + n_j}}\right). \quad (3\text{-}1)$$

Proof. Let $(m_1, \dots, m_d) \in \mathbb{N}^d$ such that $0 < m_1 < \cdots < m_d < q^{\tilde{\alpha}}$. There is a unique way to write $m_i = (q^{\tilde{\alpha}_0})^{v_i}(q^{\tilde{\alpha}_0}u_i + r_i)$ with $v_i \in \mathbb{N}$, $u_i \in \mathbb{N}$, $r_i \in \{1, \dots, q^{\tilde{\alpha}_0} - 1\}$: for each $(m_1, \dots, m_d)$ and each $i \in \{1, \dots, d\}$, v_i is the $q^{\tilde{\alpha}_0}$-adic valuation of m_i, and u_i and r_i are, respectively, the quotient and the remainder of the Euclidean division of $m_i q^{-\tilde{\alpha}_0 v_i}$ by $q^{\tilde{\alpha}_0}$.

We have, for any $\xi \in \mu_N(K)$, $\xi^{(q^{\tilde{\alpha}_0})^{v_i}(q^{\tilde{\alpha}_0}u_i+r_i)} = \xi^{(q^{\tilde{\alpha}_0}u_i+r_i)} = \xi^{u_i+r_i}$, and we write

$$(q^{\tilde{\alpha}_0}u_i + r_i)^{-n_i} = r_i^{-n_i}\left(\frac{q^{\tilde{\alpha}_0}u_i}{r_i} + 1\right)^{-n_i} = \sum_{l_i \in \mathbb{N}} \binom{-n_i}{l_i} (q^{\tilde{\alpha}_0}u_i)^{l_i} r_i^{-n_i-l_i}$$

for each i. This gives

$$\left(\frac{1}{\xi_{d+1}}\right)^{q^{\tilde{\alpha}}} \prod_{i=1}^{d} \left(\frac{\xi_{i+1}}{\xi_i}\right)^{m_i} m_i^{-n_i} = \left(\frac{1}{\xi_{d+1}}\right)^{q^{\tilde{\alpha}}} \prod_{j=1}^{d} \left(\sum_{l_j \in \mathbb{N}} \left(\frac{\xi_{j+1}}{\xi_j}\right)^{u_j+r_j} \binom{-n_j}{l_j} \frac{(q^{\tilde{\alpha}_0}u_j)^{l_j}}{r_j^{l_j+n_j}} \right).$$

Let $(v_i)_d \in \{0, \dots, \tilde{\alpha}/\tilde{\alpha}_0 - 1\}^d$, $(u_i)_d \in \{0, \dots, q^{\tilde{\alpha}-\tilde{\alpha}_0} - 1\}^d$, $(r_i)_d \in \{1, \dots, q^{\tilde{\alpha}_0} - 1\}^d$ such that, for all $i \in \{1, \dots, d\}$ we have $0 < q^{\tilde{\alpha}_0 v_i}(q^{\tilde{\alpha}_0}u_i + r_i) < q^{\tilde{\alpha}}$. Then, for all $i \in \{1, \dots, d-1\}$,

$$q^{\tilde{\alpha}_0 v_i}(q^{\tilde{\alpha}_0}u_i + r_i) < q^{\tilde{\alpha}_0 v_{i+1}}(q^{\tilde{\alpha}_0}u_{i+1} + r_{i+1}) \Leftrightarrow \begin{cases} u_i \leqslant q^{\tilde{\alpha}_0(v_{i+1}-v_i-1)}(q^{\tilde{\alpha}_0}u_{i+1} + r_{i+1}) - 1 & \text{if } v_i < v_{i+1}, \\ u_i \leqslant u_{i+1} & \text{if } v_i = v_{i+1}, r_i < r_{i+1}, \\ u_i \leqslant u_{i+1} - 1 & \text{if } v_i = v_{i+1}, r_i \geqslant r_{i+1}, \\ q^{\tilde{\alpha}_0(v_i-v_{i+1}-1)}(q^{\tilde{\alpha}_0}u_i + r_i) \leqslant u_{i+1} & \text{if } v_i > v_{i+1}. \end{cases} \tag{3-2}$$

This completes the proof. □

In the expression of $\mathrm{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1})$ of the Lemma 3A.1, we are going to sum over all the possible values of the parameters u_i and r_i, in order to have an expression which depends only on the v_i. In view of that, let the numbers $\mathcal{B}^l_m(\xi) \in K$, for $l, m \in \mathbb{N}$ such that $0 \leqslant m \leqslant l + 1$ and $\xi \in \mu_N(K)$, defined by the equation $\sum_{n_1=0}^{n-1} \xi^{n_1} n_1^l = \xi^n \sum_{m=0}^{l+1} \mathcal{B}^l_m(\xi) n^m$ for all $n \in \mathbb{N}$ (see [Jarossay 2019, Lemma 3.1.3]). We denote by $\mathcal{B}^l_m = \mathcal{B}^l_m(1)$. For $l, m \in \mathbb{N}^2$ such that $1 \leqslant m \leqslant l + 1$, we have $\mathcal{B}^l_m = 1/(l+1)\binom{l+1}{m} B_{l+1-m}$ where B denotes Bernoulli numbers, and the others $\mathcal{B}^l_m$ are 0. For $\xi \in \mu_N(K) \setminus \{1\}$, $n, l \in \mathbb{N}^*$, we have $\mathcal{B}^l_m(\xi) \in \mathbb{Z}[\xi, 1/\xi, 1/(\xi - 1)]$, and a formula for $\mathcal{B}^l_m(\xi)$ can be obtained by applying $\left(T\frac{d}{dT}\right)^l$ to the equation $(T^n - 1)/(T - 1) = \sum_{n_1=0}^{n-1} T^{n_1}$, where T is a formal variable.

Lemma 3A.2. *Let $w = ((n_i)_d; (\xi_i)_{d+1})$. We fix $(l_i)_d \in \mathbb{N}^d$ and $(v_i)_d \in \{0, \dots, \tilde{\alpha}/\tilde{\alpha}_0 - 1\}$.*

Let R be the ring generated by N-th roots of unity and numbers $1/(1 - \xi)$ where $\xi \neq 1$ is a root of unity. For any word w' over $e_{0 \cup \mu_N}$, there exists a polynomial

$$P_{w,w',(l_i)_d,(v_i)_d} \in R[(Q_{j,j+1})_{1 \leqslant i \leqslant d-1}, (B_{m,l,\xi})_{1 \leqslant l \leqslant l_1+\cdots+l_d+d,\ 0 \leqslant m \leqslant l+1,\ \xi \in \mu_N(K)}]$$

with degree at most $l_1 + \cdots + l_d + d$ in the variables $Q_{j,j+1}$, and with total degree at most d in the variables $B_{m,l,\xi}$, which is nonzero for finitely many w, and such that we have

$$\sum_{\substack{(u_i)_d \in \mathbb{N}^d \\ (r_i)_d \in \{1,\ldots,q^{\tilde{\alpha}_0}-1\} \\ 0 < q^{\tilde{\alpha}_0 v_1}(q^{\tilde{\alpha}_0}u_1 + r_1) < \cdots < q^{\tilde{\alpha}_0 v_d}(q^{\tilde{\alpha}_0}u_d + r_d) < q^{\tilde{\alpha}}}} \left(\prod_{j=1}^{d} \left(\frac{\xi_{j+1}}{\xi_j} \right)^{u_j + r_j} \frac{(q^{\tilde{\alpha}_0})^{l_i + n_i} u_i^{l_i}}{r_j^{l_j + n_j}} \right)$$

$$= \sum_{w' \text{ word on } e_{0 \cup \mu_N}} P_{w,w',(l_i)_d,(v_i)_d} \left((q^{\tilde{\alpha}_0(|v_{j+1} - v_j| - 1)})_{1 \leqslant j \leqslant d}, (\mathcal{B}^l_m(\xi))_{\substack{0 \leqslant m \leqslant \sum_{i=1}^d l_i + d + 1 \\ \xi \in \mu_N(K)}} \right) \mathrm{har}_{q^{\tilde{\alpha}_0}}(w'). \quad (3\text{-}3)$$

Proof. If $d = 1$ we can apply the definition of the numbers $\mathcal{B}^l_m(\xi)$, $\xi \in \mu_N(K)$ mentioned above.

If $d > 1$, let $i \in \{1, \ldots, d\}$ such that $v_i = \min(v_1, \ldots, v_d)$; we fix u_{i-1}, r_{i-1} and u_{i+1}, r_{i+1}. By (3-2), one has natural functions f_1, f_2, f_3, f_4 such that we can write

$$\sum_{q^{\tilde{\alpha}_0 v_{i-1}}(q^{\tilde{\alpha}_0}u_{i-1} + r_{i-1}) < q^{\tilde{\alpha}_0 v_i}(q^{\tilde{\alpha}_0}u_i + r_i) < q^{\tilde{\alpha}_0 v_{i+1}}(q^{\tilde{\alpha}_0}u_{i+1} + r_{i+1})}$$

$$= \sum_{\substack{f_1(u_{i-1}, r_{i-1}, u_{i+1}, r_{i+1}) \leqslant u_i \leqslant f_2(u_{i-1}, r_{i-1}, u_{i+1}, r_{i+1}) \\ r_i < r_{i+1}}} + \sum_{\substack{f_3(u_{i-1}, r_{i-1}, u_{i+1}, r_{i+1}) \leqslant u_i \leqslant f_4(u_{i-1}, r_{i-1}, u_{i+1}, r_{i+1}) \\ r_i \geqslant r_{i+1}}} .$$

Using that equality we can apply the result in depth 1 i.e., the definitions of the numbers $\mathcal{B}^l_m(\xi)$ to express the sum

$$\sum_{q^{\tilde{\alpha}_0 v_{i-1}}(q^{\tilde{\alpha}_0}u_{i-1} + r_{i-1}) < q^{\tilde{\alpha}_0 v_i}(q^{\tilde{\alpha}_0}u_i + r_i) < q^{\tilde{\alpha}_0 v_{i+1}}(q^{\tilde{\alpha}_0}u_{i+1} + r_{i+1})} \left(\frac{\xi_{i+1}}{\xi_i} \right)^{u_i + r_i} \frac{(q^{\tilde{\alpha}_0})^{l_i + n_i} u_i^{l_i}}{r_i^{l_i + n_i}}.$$

This gives an expression for the left-hand side of (3-3) as a sum over $d - 1$ variables u_i and $d - 1$ variables r_i, which is of a similar type. Continuing this procedure, we obtain an expression depending on the r_i via localized multiple harmonic sums, in the sense of [Jarossay 2015, Section 3.2]:

$$\sum_{0 < r_1 < \cdots < r_d < q^{\tilde{\alpha}_0}} \frac{\left(\frac{\rho_2}{\rho_1}\right)^{r_1} \cdots \left(\frac{\rho_{d+1}}{\rho_d}\right)^{r_d} \left(\frac{1}{\rho_{d+1}}\right)^{q^{\tilde{\alpha}_0}}}{r_1^{\tilde{n}_1} \cdots r_d^{\tilde{n}_d}}$$

with a positive integer d, for any N-th roots of unity ρ_i $(1 \leqslant i \leqslant d+1)$, $\tilde{n}_i \in \mathbb{Z}$ $(1 \leqslant i \leqslant d)$ which are not necessarily positive. This can be expressed in terms of $q^{\tilde{\alpha}_0}$ and usual prime weighted multiple harmonic sums $\mathrm{har}_{q^{\tilde{\alpha}_0}}$ by the main result of [Jarossay 2015, Section 3.2]. When we obtain products of numbers $\mathrm{har}_{q^{\tilde{\alpha}_0}}(w)$, we can linearize that expression by using the quasishuffle relation of [Hoffman 2000]. □

In the expression of $\mathrm{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1})$ obtained by combining Lemmas 3A.1 and 3A.2, we are going to sum over all the possible values of the parameters v_i in Lemma 3A.2. In view of that, we need another lemma.

Lemma 3A.3. *Let d, $M \in \mathbb{N}^*$; let $A_1, \dots, A_d \in R[T]$ be polynomials, with $R \subset \mathbb{Q}$, and $T_1, \dots, T_d$ formal variables. There exist coefficients $C_{\delta_1,\dots,\delta_d}(M) \in R[T]$ such that we have*

$$\sum_{0 \leqslant \tilde{v}_1 < \cdots < \tilde{v}_d \leqslant M-1} \prod_{i=1}^{d} T_i^{\tilde{v}_i} A_i(\tilde{v}_i) = \sum_{\substack{\forall i, l_1+\cdots+l_i \leqslant \deg(A_1)+\cdots+\deg A_i \\ (U_1,\dots,U_d) \in \mathbb{Q}(T_1,\dots,T_d)^d \text{ such that} \\ U_1 = T_1 \\ \forall i \in \{1,\dots,d-1\}, U_{i+1} \in \{U_i T_{i+1}, -T_{i+1}\} \\ \delta_1,\dots,\delta_d \geqslant 0}} C_{\delta_1,\dots,\delta_d}(M) \prod_{i=1}^{d} \frac{U_i^{\delta_i}}{(U_i - 1)^{\delta_i+1}}.$$

Proof. (a) For any $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, we have

$$\left(T \frac{\partial}{\partial T}\right) \frac{T^m}{(T-1)^n} = m \frac{T^m}{(T-1)^n} - n \frac{T^{m+1}}{(T-1)^{n+1}}.$$

Thus, by induction on α, for all $\alpha \in \mathbb{N}^*$,

$$\left(T \frac{\partial}{\partial T}\right)^{\alpha} \frac{T^m}{(T-1)^n} = \sum_{l=0}^{\alpha} (-n)(-n-1) \cdots (-n-l+1) \frac{T^{m+l}}{(T-1)^{n+l}}.$$

Moreover we have

$$\sum_{0 \leqslant \tilde{v} \leqslant W-1} T^{\tilde{v}} \tilde{v}^{\alpha} = \left(T \frac{\partial}{\partial T}\right)^{\alpha} \left(\sum_{0 \leqslant \tilde{v} \leqslant W-1} T^{\tilde{v}}\right) = \left(T \frac{\partial}{\partial T}\right)^{\alpha} \left(\frac{T^{\tilde{v}} - 1}{T-1}\right).$$

Whence

$$\sum_{0 \leqslant \tilde{v} \leqslant M-1} T^{v} v^{\alpha} = \sum_{l=0}^{\alpha} (-1)^l l! \frac{T^l}{(T-1)^{l+1}} (T^M - 1).$$

This gives the result for $d = 1$ by linearity with respect to A_1.

(b) Let us prove the result by induction on d. Assume that $A_1 = \sum_{\alpha_1=0}^{\deg A_1} u_{\alpha_1} \tilde{v}^{\alpha_1}$ with $u_{\alpha_1} \in \mathbb{Q}$. We have, for all $\alpha_1 \in \{0, \dots, \deg A_1\}$,

$$\sum_{0 \leqslant \tilde{v}_1 < \cdots < \tilde{v}_d \leqslant M-1} T_1^{\tilde{v}_1} \tilde{v}_1^{\alpha_1} \prod_{i=2}^{d} T_i^{\tilde{v}_i} A_i(\tilde{v}_i)$$
$$= \sum_{l=0}^{a_1} (-1)^l l! \frac{1}{(T_1 - 1)^{l+1}} \sum_{0 \leqslant \tilde{v}_2 < \cdots < \tilde{v}_d \leqslant M-1} \left((T_1 T_2)^{\tilde{v}_2} \prod_{i=3}^{d} T_i^{\tilde{v}_i} A_i(\tilde{v}_i) - \prod_{i=2}^{d} T_i^{\tilde{v}_i} A_i(\tilde{v}_i) \right).$$

Whence the result by induction and by linearity. □

Combining Lemmas 3A.1, 3A.2 and 3A.3 we can now sum over all the u_i, r_i and v_i and write an expression of $\mathrm{har}_{q^{\tilde{\alpha}}}((n_i)_d; (\xi_i)_{d+1})$ as a function of $\tilde{\alpha}$ as we wanted.

Proposition 3A.4. *Let a harmonic word $w = ((n_i)_d; (\xi_i)_{d+1})$. Let us fix $(l_i)_d \in \mathbb{N}^d$.*

Let R be the ring of Lemma 3A.2. For every word w' over $e_{0 \cup \mu_N}$, there exists a polynomial $P_{w,w',(l_i)_d} \in R[\tilde{Q}, A, (B_{m,l,\xi})_{1 \leqslant l \leqslant l_1+\cdots+l_d+d,\ 1 \leqslant m \leqslant l+1,\ \xi \in \mu_N(K)}]$ with degree at most $\sum_{i=1}^{d} l_i + d$ in $\tilde{Q}$, and with total *degree at most d in the variables $B_{m,l,\xi}$, such that we have*

$$\sum_{(v_i)_d\in\{1,\dots,q^{\tilde{\alpha}_0}-1\}} P_{w,w',(l_i)_d,(v_i)_d}((q^{\tilde{\alpha}_0(|v_{j+1}-v_j|-1)})_{1\leqslant j\leqslant d},\, (\mathcal{B}^l_m(\xi))_{0\leqslant m\leqslant \sum_{i=1}^d l_i+d+1,\ \xi\in\mu_N(K)}) = P_{w,w',(l_i)_d}\left(q^{\tilde{\alpha}},\, \frac{\tilde{\alpha}}{\tilde{\alpha}_0},\, (\mathcal{B}^l_m(\xi))_{0\leqslant m\leqslant \sum_{i=1}^d l_i+d+1,\ \xi\in\mu_N(K)}\right). \quad (3\text{-}4)$$

Proof. The set $\{0,\dots,\tilde{\alpha}/\tilde{\alpha}_0-1\}^d$ admits a partition, which depends only on d, indexed by the set of couples (E,σ), where E is a partition of $\{1,\dots,d\}$ and σ is a permutation of $\{1,\dots,\sharp E\}$, defined as follows: for each $(v_1,\dots,v_d)\in[0,k-1]^d$, and each such (E,σ), we say that $(v_1,\dots,v_d)\in(E,\sigma)$ if and only if, for all, i, i', a

$$v_i = v_{i'} \quad \text{for } i, i'\in P_{\sigma(a)},$$
$$v_i < v_{i'} \quad \text{for } i\in P_{\sigma(a)},\, i'\in P_{\sigma(a+1)}.$$

By the proof of Lemma 3A.2, the function $(v_i)_d\mapsto P_{w,w',(l_i)_d,(v_i)_d}$ is constant on each term of that partition (since $\xi^q=\xi$ for all $\xi\in\mu_N(K)$, we have $\xi^{q^{\tilde{\alpha}_0 v}}=\xi$ for all $v\in\mathbb{N}^*$). We split the left-hand side of (3-4) as $\sum_{(v_i)_d\in\{1,\dots,\tilde{\alpha}/\tilde{\alpha}_0-1\}}=\sum_{(E,\sigma)}\sum_{(v_i)_d\in(E,\sigma)}$ and we compute each subsum $\sum_{(v_i)_d\in(E,\sigma)}$. By multilinearity we can assume that $P_{w,w',(l_i)_d,(v_i)_d}$ is a monomial in the $q^{\tilde{\alpha}_0(|v_{j+1}-v_j-1|)}$. Thus the subsum is a function of the type

$$\sum_{0\leqslant\tilde{v}_1<\cdots<\tilde{v}_{d'}\leqslant M-1} T_{i_1}^{\tilde{v}_1}\cdots T_{i_r}^{\tilde{v}_r}, \quad (3\text{-}5)$$

applied to $\tilde{v}_i=v_{\sigma(i+1)}-v_{\sigma(i)}-1$ and $T_i=q^{\tilde{\alpha}_0}$, where $d', M\in\mathbb{N}^*$, $I=\{i_1,\dots,i_r\}\subset\{1,\dots,d'\}$ with $i_1<\cdots<i_r$, and $T_{i_1},\dots,T_{i_r}$ formal variables. Moreover, $\sum_{i\leqslant i+1<\cdots<j}1$ is a polynomial function of (i,j) with coefficients in $\mathbb{Z}$. Thus we can express (3-5) by Lemma 3A.3. This provides the result. ☐

3B. ***The relation of iteration of the harmonic Frobenius of series.*** Using the result of Section 3A we can now formalize the iteration of the harmonic Frobenius from the point of view of series. We refer to $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\Sigma}_{\text{har}}=\prod_{\text{w harmonic word}}K.w$ defined in [Jarossay 2015, Section 3].

Definition 3B.1. Let the map of iteration of the harmonic Frobenius of series be the map $\text{iter}^{\Sigma}_{\text{har}}(\Lambda,\Lambda^a,a)$: $K\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\Sigma}_{\text{har}}\to K[\Lambda,\Lambda^a,a]\langle\langle e_{0\cup\mu_N}\rangle\rangle^{\Sigma}_{\text{har}}$ defined by, for any word w,

$$\text{iter}^{\Sigma}_{\text{har}}(f)[w]=\sum_{w' \text{ word on } e_{0\cup\mu_N}} P_{w,w',(l_i)_d}(\Lambda^a,a,(\mathcal{B}^l_m(\xi))_{0\leqslant m\leqslant\sum_{i=1}^d l_i+d+1,\ \xi\in\mu_N(K)})f[w'].$$

Let us now finish the proof of (0-5). By Lemmas 3A.1, 3A.2 and Proposition 3A.4, the only thing to check is the convergence of the series, which are infinite sums over $(l_i)_d\in\mathbb{N}^d$. This follows from the following facts:

(a) For any $n\in\mathbb{N}^*$, it follows from $p^{v_p(n)}\leqslant n$ that $v_p(1/n)\geqslant-\log(n)/\log(p)$; moreover, for any $n\in\mathbb{N}^*$, we have $v_p(B_n)\geqslant-1$ (this is part of von Staudt–Clausen's theorem). Thus for all l, m, we have $v_p(\mathcal{B}^l_m)\geqslant-1-\log(l+1)/\log(p)$, and, given that $|\xi|_p=|1-\xi|_p=1$ for all $\xi\in\mu_N(K)\setminus\{1\}$, we have, for all l, m, and $\xi\in\mu_N(K)\setminus\{1\}$, $v_p(\mathcal{B}^l_m(\xi))\geqslant 0$.

(b) If T_1, T_2 are formal variables and $m \in \mathbb{N}^*$, we have $(T_1^m - 1)/(T_1 - 1) - (T_2^m - 1)/(T_2 - 1) = (T_1^m - T_1)/(T_1 - 1) - (T_2^m - T_2)/(T_2 - 1)$.

(c) For any $z \in K$ such that $v_p(z) \neq 0$, we have $v_p(1/(z-1)) > 0$ if $v_p(z) > 0$, and $v_p(1/(z^{-1}-1)) > v_p(z^{-1})$ if $v_p(z) < 0$.

Remark 3B.2. Equation (0-5) is related to the formula $\mathrm{har}_{p^\alpha \mathbb{N}} = \mathrm{har}_{p^\alpha} \circ^{\Sigma}_{\mathrm{har}} \mathrm{har}_{\mathbb{N}}^{(p^\alpha)}$ proved in [Jarossay 2015], where $\circ^{\Sigma}_{\mathrm{har}}$ is the pro-unipotent harmonic action of series introduced in [loc. cit., Section 4.3]: restricting that equation to har_m with m a power of p gives a functional equation satisfied by the map $\alpha \mapsto \mathrm{har}_{p^\alpha}$, which expresses $\mathrm{har}_{p^{\alpha+\beta}}$ in terms of har_{p^α} and $\mathrm{har}_{p^\beta}^{(p^\alpha)}$.

Remark 3B.3. As in [Jarossay 2015], the computation which leads to the above result remains true for the generalization of cyclotomic multiple harmonic sums obtained by replacing the factors $1/m_i^{n_i}$, $1 \leqslant i \leqslant d$ in (0-2) by, more generally, factors $\chi_i(m_i)$ where χ_i are locally analytic group endomorphisms of the multiplicative group K^*, which are analytic on disks of radius $p^{-\alpha}$.

Remark 3B.4. The main theorem gives formulas for p-adic cyclotomic multiple zeta values which depend on an additional parameter, a number of iterations of the Frobenius different from the one under consideration. Here is another way to obtain formulas with parameters. The computation of regularized iterated integrals in [Jarossay 2019, Section 3] can be done by replacing the Euclidean division by p^α in $\mathbb{N}$ by the Euclidean division by p^β with $\beta \geq \alpha$. This gives, for example, $\zeta_{p,\alpha}(n) = p^{\alpha n}/(n-1) \lim_{|m|_p \to 0} 1/(p^\beta m) \sum_{0 < m_1 < p^\beta m,\ p^\alpha \nmid m} 1/m_1^n$, and this gives formulas in which the prime weighted multiple harmonic sums are replaced by the following generalization, where $(l_1, \ldots, l_d) \in \mathbb{N}^d$, $I, I' \subset \{1, \ldots, d\}$ and β:

$$p^{\alpha \sum_{i=1}^d n_i + \beta \sum_{i=1}^d l_i} \sum_{\substack{0 = m_0 < m_1 < \cdots < m_d < p^\beta \\ \text{for } j \in I,\ m_{j-1} \equiv m_j [p^\alpha] \\ \text{for } j \in I',\ m_j \equiv 0 [p^\alpha]}} \frac{\left(\frac{\xi_2}{\xi_1}\right)^{m_1} \cdots \left(\frac{\xi_{d+1}}{\xi_d}\right)^{m_d} \left(\frac{1}{\xi_{d+1}}\right)^{p^\beta}}{m_1^{n_1+l_1} \cdots m_d^{n_d+l_d}}.$$

Example 3B.5. Let us consider the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ $(N = 1)$, for which we have $q = p$, $\tilde{\alpha} = \alpha$, $\tilde{\alpha}_0 = \alpha_0$, and depth one and two. Equation (0-5) is, for all $n \in \mathbb{N}^*$,

$$\mathrm{har}_{p^\alpha}(n) = -\sum_{l \geqslant 0} \binom{-n}{l} \mathrm{har}_{p^{\alpha_0}}(l+n) \sum_{u=1}^{l+1} \mathcal{B}_u^l \frac{1}{1 - p^{\alpha_0(u+n)}} - \sum_{u \geqslant n+1} p^{\alpha u} \frac{1}{1 - p^{\alpha_0 u}} \sum_{l \geqslant u-n-1} \binom{-n}{l} \mathrm{har}_{p^{\alpha_0}}(l+n) \mathcal{B}_{u-n}^l.$$

For all $n_1, n_2 \in \mathbb{N}^*$, $\mathrm{har}_{p^\alpha}(n_1, n_2)$ is the sum of the following terms, where the variables v_1, v_2 are those defined in Lemma 3A.1 and where, for a set E, 1_E means the characteristic function of E:

- The term "$v_1 = v_2$":

$$\sum_{\substack{u\geqslant 1\\ l_1,l_2\geqslant 0\\ l_1+l_2\geqslant u-1}} \frac{p^{\alpha(u+n_1+n_2)}-1}{p^{\alpha_0(u+n_1+n_2)}-1} \prod_{i=1}^{2}\binom{-n_i}{l_i}\Big(\mathcal{B}_u^{l_1,l_2}\prod_{i=1}^{2}\mathrm{har}_{p^{\alpha_0}}(n_i+l_i)+\mathcal{B}_u^{l_1+l_2}\,\mathrm{har}_{p^{\alpha_0}}(n_1+l_1,n_2+l_2)\Big).$$

- The term "$v_1 < v_2$":

$$\begin{aligned}\sum_{\substack{M_1,M_2\geqslant -1\\ u,t\geqslant 1}} \Bigg[& \frac{1_{t\neq u+n_2}}{p^{\alpha_0(n_2+u-t)}-1}\bigg(\frac{p^{\alpha(n_1+n_2+u)}-p^{\alpha_0(n_1+n_2+u)}}{p^{\alpha_0(n_2+n_1+u)}-1}-\frac{p^{\alpha(n_1+t)}-p^{\alpha_0(n_1+t)}}{p^{\alpha_0(n_1+t)}-1}\bigg)\\ & +1_{t=u+n_2}\bigg(\frac{\alpha p^{\alpha(n_1+n_2+u)}}{p^{\alpha_0(n_1+n_2+u)}-1}+\frac{1-p^{\alpha(n_1+n_2+u)}}{(p^{\alpha_0(n_1+n_2+u)}-1)^2}\bigg)\mathcal{B}_t^{M_1+t}\mathcal{B}_u^{M_2+u}\\ & \times\sum_{j=0}^{\min(t,M_2+u)}\binom{t}{j}\binom{-n_1}{M_1+t}\binom{-n_2}{M_2+u-j}\,\mathrm{har}_{p^{\alpha_0}}(n_1+M_1+t)\,\mathrm{har}_{p^{\alpha_0}}(n_2+M_2+u-t)\Bigg].\end{aligned}$$

- The term "$v_1 > v_2$": by the change of variable $(m_1,m_2)\mapsto(p^{\alpha}-m_1,\,p^{\alpha}-m_2)$, it is

$$\sum_{\substack{0<m_1<m_2<p^{\alpha}\\ v_p(n_1)>v_p(n_2)}}\frac{(p^{\alpha})^{n_1+n_2}}{n_1^{n_1}n_2^{n_2}}=\sum_{\substack{l_1,l_2\geqslant 0\\ 0<n_1<n_2<p^{\alpha}\\ v_p(n_1)<v_p(n_2)}}\binom{-n_1}{l_1}\binom{-n_2}{l_2}\frac{(p^{\alpha})^{l_1+l_2+n_1+n_2}}{n_1^{n_1}n_2^{n_2}}.$$

3C. ***Interpretation in terms of cyclotomic multiple harmonic sums viewed as functions of the upper bound of their domain of summation.*** The main result above gives a description of $\mathrm{har}_{q^{\tilde{\alpha}}}$ as a function of $q^{\tilde{\alpha}}$ (and $\tilde{\alpha}$) regarded as a p-adic integer. Let us extend the question and consider the study of har_m as a function of m, for any $m\in\mathbb{N}^*$ regarded as a p-adic integer. We are going to remove the factor m^{weight} in har_m, i.e., consider the (nonweighted) cyclotomic multiple harmonic sums $\mathfrak{h}_m((n_i)_d;(\xi_i)_{d+1})$ of (0-2).

Intuitively, $\mathfrak{h}_m$ is a highly discontinuous function of m, but we have proved by the main theorem that $\mathrm{har}_{q^{\tilde{\alpha}}}$ has a power series expansion in terms of $q^{\tilde{\alpha}}$. The goal of the next proposition is to write a decomposition of $\mathfrak{h}_m$ in a way which explains the relation between these two phenomena, by using the q-adic expansion of m, in order to clarify the dependence of har_m in m.

Below we use the following definition: an increasing connected partition of a subset of $\mathbb{N}$ is a partition of that set into sets J_i of consecutive integers, such that each element of J_i is less than each element of $J_{i'}$ when $i<i'$.

Proposition 3C.1. (i) *Let $m\in\mathbb{N}^*$, and let its q-adic expansion be*

$$m=a_{y_{d'}}q^{y_{d'}}+a_{y_{d'-1}}q^{y_{d'-1}}+\cdots+a_{y_1}q^{y_1},$$

with $y_{d'}>\cdots>y_1$, and $a_{y_{d'}},\ldots,a_{y_1}\in\{1,\ldots,q-1\}$. Let

$$\nu_{j'}=a_{y_{d'}}q^{y_{d'}}+\cdots+a_{y_{d'-j'+1}}q^{y_{d'-j'+1}}$$

for $j' \in \{1, \ldots, d'\}$. *We have*

$$\mathfrak{h}_m((n_i)_d; (\xi_i)_{d+1}) = \sum_{\substack{\mathfrak{n}=\{\nu_{j''_1},\ldots,\nu_{j''_{d''}}\}\subset\{\nu_1,\ldots,\nu_{d'}\} \\ \jmath:\mathfrak{n}\hookrightarrow\{1,\ldots,d\} \text{ injective} \\ J_0\amalg\cdots\amalg J_{d'}=\{1,\ldots,d\}-\jmath(\mathfrak{n}), \text{ satisfying } (*)}} \prod_{j''=1}^{d''} \frac{1}{\nu_{j''}^{n_{\jmath(\nu_{j''})}}} \prod_{j'=0}^{d'} \sum_{l_{j^{\max}_{j'}},\ldots,l_{j^{\min}_{j'}}\geqslant 0} \left(\prod_{u=j^{\min}_{j'}}^{j^{\max}_{j'}} \binom{-n_u}{l_u}\right)\left(\sum_{l=d'}^{d'-j'+1} a_{y_l} q^{y_l}\right)^{\sum_{u=j^{\min}_{j'}}^{j^{\max}_{j'}} l_u}$$

$$\times\, \mathfrak{h}_{a_{y_{d'-j'}} q^{y_{d'-j'}}}((n_j+l_j)_{j_{j',min}<j<j_{j',max}}; (\xi)_{j_{\min}<j<j_{\max}+1}), \quad (3\text{-}6)$$

where $(*)$ *is that* $J_0 \amalg \cdots \amalg J_{d'}$ *is an increasing connected partition of* $\{1, \ldots, d\} - \jmath(\mathfrak{n})$, *such that each* $J_{\jmath(\nu_{j''})} \amalg \cdots \amalg J_{\jmath(\nu_{j''+1})-1}$, $j'' = 1, \ldots, d''$, *is an increasing connected partition of* $(\{1, \ldots, d\} - \jmath(\mathfrak{n})) \cap\,]\jmath(\nu_{j''}), \jmath(\nu_{j''+1})[$.

(ii) *Let* $n \in \mathbb{N}^*$, *whose decomposition in base* q *is of the form* aq^y, *with* $a \in \{1, \ldots, q-1\}$ *and* $y \in \mathbb{N}^*$. *Let* $\nu_{j'} = j'q^y$ *for* $j' \in \{1, \ldots, a-1\}$. *We have*

$$\mathfrak{h}_{aq^y}((n_i)_d; (\xi_i)_{d+1}) = \sum_{\substack{\mathfrak{n}=\{\nu_{j''_1},\ldots,\nu_{j''_{d''}}\}\subset\{\nu_1,\ldots,\nu_{d'}\} \\ \jmath:\mathfrak{n}\hookrightarrow\{1,\ldots,d\} \text{ injective} \\ J_0\amalg\cdots\amalg J_{d'}=\{1,\ldots,d\}-\jmath(\mathfrak{n}), \text{ satisfying } (*)}} \prod_{j''=1}^{d''} \frac{1}{\nu_{j''}^{n_{\jmath(\nu_{j''})}}} \prod_{j'=0}^{d'} \sum_{l_{j^{\max}_{j'}},\ldots,l_{j^{\min}_{j'}}\geqslant 0} \left(\prod_{u=j^{\min}_{j'}}^{j^{\max}_{j'}} \binom{-n_u}{l_u}\right)$$

$$\times\, (j'q^y)^{\sum_{u=j^{\min}_{j'}}^{j^{\max}_{j'}} l_u}\, \mathfrak{h}_{q^{y_{d'-j'}}}((n_{j_{j'}}+l_{j_{j'}})_{j_{\min}<j'<j_{\max}}; (\xi_{j_{j'}})_{j_{\min}<j'<j_{\max}+1}). \quad (3\text{-}7)$$

Proof. (i) and (ii) We apply the "formula of splitting" of multiple harmonic sums of [Jarossay 2015, Section 3] at $\{\nu_1, \ldots, \nu_r\}$; this gives

$$\mathfrak{h}_m((n_i)_d; (\xi_i)_{d+1}) = \sum_{\substack{\mathfrak{n}=\{\nu_{j_1},\ldots,\nu_{j_{d''}}\}\subset\{\nu_1,\ldots,\nu_{d'}\} \\ \jmath:\mathfrak{n}\hookrightarrow\{1,\ldots,d\} \text{ injective} \\ J_0\amalg\cdots\amalg J_{d'}=\{1,\ldots,d\}\setminus\jmath(\mathfrak{n}), \text{ satisfying } (*)}} \prod_{j''=1}^{d''} \frac{1}{n_{\jmath(\nu_{j''})}^{n_{\jmath(\nu_{j''})}}} \prod_{j'=0}^{d'} \mathfrak{h}_{\nu_{j'},\nu_{j'+1}}(w|_{J_{j'}})$$

and we express each factor $\mathfrak{h}_{\nu_{j'},\nu_{j'+1}}(w|_{J_{j'}})$ in terms of $\mathfrak{h}_{\nu_{j'+1}-\nu_{j'}}$ by the p-adic formula of shifting of multiple harmonic sums of [Jarossay 2015, Section 4.1] writing $J_{j'} = [j^{\min}_{j'}, j^{\max}_{j'}]$. □

Example 3C.2 ($N = 1$ and $d = 1$). For all $n \in \mathbb{N}^*$, we have

$$\mathfrak{h}_m(n) = \frac{\mathfrak{h}_{a_{y_{d'}}+1}(n)}{(p^{y_{d'}})^n} + \sum_{i=1}^{d'-1} \sum_{l\geqslant 0} \frac{\mathfrak{h}_{a_{y_i}+1}(n+l)}{(p^{y_i})^{n+l}} \binom{-n}{l} (a_{y_{d'}} p^{y_{d'}} + \cdots + a_{y_{i+1}} p^{y_{i+1}})^l$$

$$+ \sum_{\substack{1\leqslant j\leqslant d' \\ 0\leqslant a'_{y_j}\leqslant a_{y_j}-1}} \sum_{l\geqslant 0} \binom{-n}{l} \mathfrak{h}_{y_j}(n+l) \left(\sum_{m=j+1}^{d'} a_{y_m} p^{y_m} + a'_j p^{y_j}\right)^l. \quad (3\text{-}8)$$

In the formulas of the proposition, there are terms which are analytic functions of a power of q by the main theorem, and certain factors which are "polar" in function of the q-adic expansion of m. This sheds light on the dependence of $\mathfrak{h}_m$ on m.

The reason why studying $\mathfrak{h}_m$ as a function of m is a natural comes from [Jarossay 2019], in which this question appeared implicitly. We have studied the map sending m to the coefficient of degree m in the power series expansion at 0 of the overconvergent p-adic multiple polylogarithm $\mathrm{Li}^{\dagger}_{p,\alpha}[w]$, for w any word on $e_{0\cup\mu_N}$. We have proved that it can be extended to a locally analytic map on $\mathbb{Z}_p^{(N)} = \varprojlim \mathbb{Z}/Np^u\mathbb{Z}$ [loc. cit., Section 3]. This map is a linear combination of multiple harmonic sums over the ring generated by p-adic cyclotomic multiple zeta values. Thus we can interpret them as a "regularization" of multiple harmonic sums. See also Appendix A of [loc. cit.].

4. Comparison between equations on integrals and equations on series

We prove (0-6) and we discuss more generally the comparison between integrals and series.

4A. *Uniqueness of the expansion of* $\mathrm{har}_{q^{\tilde{\alpha}}}$ *as a function of* $\tilde{\alpha}$ *and* $q^{\tilde{\alpha}}$.

Proposition 4A.1. *Let $\delta \in \mathbb{N}^*$, and a map $S : \mathbb{N}^* \cap [\delta, +\infty[\to K$ such that we have, for all $a \in \mathbb{N}^* \cap [\delta, +\infty[$, $S(a) = \sum_{n\in\mathbb{N}} \sum_{m=0}^{M} c_{n,m}(q^a)^n a^m$, where $M \in \mathbb{N}^*$, and $(c_{l,m})_{0\leqslant n,\ 0\leqslant m\leqslant M} \in K^{\mathbb{N}\times\{0,\dots,M\}}$ such that $\sum_{n\in\mathbb{N}} \sum_{m=0}^{M} |c_{n,m}q^n|_p < \infty$. If $S(a) = 0$ for all $a \in \mathbb{N}^* \cap [\delta, +\infty[$, then we have $c_{n,m} = 0$ for all $(n, m) \in \mathbb{N} \times \{0, \dots, M\}$.*

Proof. Let $a_0 \in \mathbb{N}^* \cap [\delta, +\infty[$ and $u \in \mathbb{N}$. By taking $a = a_0 + p^u$ in the equation $\sum_{n\in\mathbb{N}} \sum_{m=0}^{M} c_{n,m}(q^a)^n a^m = 0$ and by taking the limit $u \to \infty$, we get $\sum_{m=0}^{M} c_{0,m} a_0^m = 0$. Since this is true for infinitely many a_0, we get $c_{0,m} = 0$ for all m. This implies that, for all a,

$$\sum_{n\geqslant 1} \sum_{m=0}^{M} c_{n,m}(q^a)^n a^m = q^a \left(\sum_{n\geqslant 1} \sum_{m=0}^{M} c_{n,m}(q^a)^{n-1} a^m \right) = 0,$$

thus

$$\sum_{n\in\mathbb{N}} \sum_{m=0}^{M} c_{n+1,m}(q^a)^n a^m = 0.$$

Whence the result: by induction on n, we have a contradiction if there exists (n, m) such that $c_{n,m} \neq 0$. □

Let us now prove (0-6). By [Jarossay 2019], we have $\Phi_{q,\tilde{\alpha}} \in K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$ for any $\tilde{\alpha} \in \mathbb{N}^*$. By Corollary 2B.2, this implies that $|\Phi|_q = \sum_{w \text{ word on } e_{0\cup\mu_N}} \sup_{\tilde{\alpha}\in\mathbb{Z}\cup\{\pm\infty\}\setminus\{0\}} |\Phi_{q,\tilde{\alpha}}[w]|_p w$ is a well-defined element of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}$. We have a similar bound for the coefficients of the expansion of each $\mathrm{har}_{q,\tilde{\alpha}}[w]$ obtained in Section 3C. Thus we can apply Proposition 4A.1 to the power series expansion of each $\mathrm{har}_{q,\tilde{\alpha}}[w]$ in (0-3), (0-4), (0-5) to deduce that they are the same.

Example 4A.2. The term of depth one ($d = 1$), in the case of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (i.e., in the case $N = 1$) for which $p = q$, $\tilde{\alpha} = \alpha$ and $\tilde{\alpha}_0 = \alpha_0$, of the equations of the theorem is the following, respectively (with $\mathcal{B}_b^{L+b} = 1/(L+b+1)\binom{L+b+1}{L} B_{L+1}$ for $1 \leqslant b \leqslant L+1$):

- The fixed point equation of the harmonic Frobenius of integrals at $(1, 0)$ (equation (0-3)):

$$\text{har}_{p^{\alpha}}(n) = \sum_{b=1}^{\infty} (p^{\alpha})^{b+n} \operatorname{Ad}_{\Phi_{p,-\infty}}(e_1)[e_0^b e_1 e_0^{n-1} e_1] + \operatorname{Ad}_{\Phi_{p,\infty}}(e_1)\left[\frac{1}{1-e_0} e_1 e_0^{n-1} e_1\right]. \tag{4-1}$$

- The relation of iteration of the harmonic Frobenius of integrals at $(1, 0)$ (equation (0-4)):

$$\text{har}_{p^{\alpha}}(n) = \sum_{b=1}^{\infty} \frac{(p^{\alpha})^n (p^{\alpha_0})^b}{p^{\alpha_0}-1} \operatorname{Ad}_{\Phi_{p,\alpha_0}}(e_1)[e_0^b e_1 e_0^{n-1} e_1] - \frac{(p^{\alpha})^n}{p^{\alpha_0}-1} \operatorname{Ad}_{\Phi_{p,\alpha_0}}(e_1)\left[\frac{1}{1-e_0} e_1 e_0^{n-1} e_1\right]. \tag{4-2}$$

- The relation of iteration of the harmonic Frobenius of series (0-5):

$$\text{har}_{p^{\alpha}}(n) = \sum_{b=1}^{\infty} \frac{p^{\alpha(n+b)}-1}{p^{\alpha_0(n+b)}-1} \sum_{L=-1}^{\infty} \mathcal{B}_b^{L+b} \, \text{har}_{p^{\alpha_0}}(n+b+L). \tag{4-3}$$

- The comparison between these three expressions: (0-6)

$$\begin{aligned}\frac{(p^{\alpha_0})^b}{p^{\alpha_0}-1} \operatorname{Ad}_{\Phi_{p,\alpha_0}}(e_1)[e_0^b e_1 e_0^{n-1} e_1] &= \operatorname{Ad}_{\Phi_{p,\infty}}(e_1)[e_0^b e_1 e_0^{n-1} e_1] \\ &= \frac{1}{p^{\alpha_0(n+b)}-1} \sum_{L=-1}^{\infty} \mathcal{B}_b^{L+b} \, \text{har}_{p^{\alpha_0}}(n+b+L). \end{aligned} \tag{4-4}$$

Generalizing this example to higher depths gives a new way to compute p-adic cyclotomic multiple zeta values.

4B. ***The map of comparison for all number of iterations.*** In [Jarossay 2015, Definition 5.1.3] we have defined the map of comparison, from integrals to series, $\text{comp}^{\Sigma \int} : K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{o(1)}^{N} \to K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har},o(1)}$, by

$$\begin{aligned}(\text{comp}^{\Sigma\int}((g_\xi)_{\xi\in\mu_N(K)}))&[e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}] \\ &= (-1)^d \sum_{\xi\in\mu_N(K)} \xi^{-p^{\alpha}} g_\xi \left[\frac{1}{1-e_0} e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}\right].\end{aligned}$$

In the context of this paper, it is natural to define a variant of the map of comparison from integral to series, which takes into account the properties of the iterated Frobenius viewed as a function of its number of iterations, and which has the additional advantage of being injective.

Definition 4B.1. Let the map $\text{comp}_{\text{iter}}^{\Sigma\int} : \operatorname{Ad}_{\tilde{\Pi}_{1,0}(K)_{o(1)}}(e_1) \to (K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\text{har}}^{\int_{1,0}})^{q^{\mathbb{N}^*}}$ be defined by $f \mapsto (\tau(q^{\tilde{\alpha}})(f) \circ_{\text{har}}^{\int_{1,0}} \text{comp}^{\Sigma\int} f)_{\tilde{\alpha}\in\mathbb{N}^*}$.

Equation (0-3) can be restated as

$$\text{comp}_{\text{iter}}^{\Sigma\int}(\operatorname{Ad}_{\Phi_{q,-\infty}}(e_1)) = (\text{har}_{q,\tilde{\alpha}})_{\alpha\in\mathbb{N}^*}.$$

The key property of $\text{comp}_{\text{iter}}^{\Sigma\int}$, which a priori does not hold for $\text{comp}^{\Sigma\int}$, is the following:

Proposition 4B.2. $\text{comp}_{\text{iter}}^{\Sigma\int}$ *is injective.*

Proof. Let a word $w = e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}$. For $n \geqslant \text{weight}(w)$, let us consider the coefficient of $(q^a)^n$ in $\text{comp}^{\Sigma \int,\text{iter}}(f)[w]$. It is equal to $f[e_0^{n-(n_1+\cdots+n_d)} e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}]$ + terms of lower depth, where the depth is the one of coefficients of f. This gives the result by an induction on the depth. □

5. Iteration of the Frobenius on $\mathbb{P}^1 \setminus \bigcup_\xi B(\xi, 1)$

In the previous sections, we considered the Frobenius of $\pi_1^{\text{un,crys}}(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\})$ at base-points $(1, 0)$ and the harmonic Frobenius. We now consider the Frobenius of $\pi_1^{\text{un,crys}}(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\})$ on the affinoid subspace $U^{\text{an}} = \mathbb{P}^{1,\text{an}} \setminus \bigcup_{\xi^N=1} B(\xi, 1)$ of $\mathbb{P}^{1,\text{an}}$ over K, where $B(\xi, 1)$ is the open ball of center ξ and of radius 1. As in Section 2, we will have a fixed-point equation (Section 5A) and an iteration equation (Section 5B). Additionally, we will have a third equation coming from the study of regularized iterated integrals in [Jarossay 2019, Section 5.3].

5A. ***Fixed point equation.*** The fixed point equation of the Frobenius is known thanks to the theory of Coleman integration. It amounts to the definition of p-adic multiple polylogarithms Li_q^{KZ} as Coleman integrals, in [Furusho 2004; 2007] for $N = 1$ and in [Yamashita 2010] for any N; we have

$$\phi_{\log(q)/\log(p)}(\text{Li}_q^{\text{KZ}}) = \text{Li}_q^{\text{KZ}}. \tag{5-1}$$

Restricted to U^{an}, the fixed point equation amounts to the following equations [Jarossay 2019, Proposition 2.1.3], they involve the overconvergent p-adic multiple polylogarithms $\text{Li}_{p,\alpha}^\dagger[w]$ [Jarossay 2019, Definition 1.2.5], which are overconvergent analytic functions on U^{an}:

$$\text{Li}_{p,\alpha}^\dagger(z) = \text{Li}_q^{\text{KZ}}(z)(p^\alpha e_0, (p^\alpha e_\xi)_\xi)\text{Li}_q^{\text{KZ}(p^\alpha)}(z^{p^\alpha})(e_0, (\text{Ad}_{\Phi_{p,\alpha}^{(\xi)}}(e_\xi)_\xi))^{-1}, \tag{5-2}$$

$$\text{Li}_{p,-\alpha}^\dagger(z) = \text{Li}_q^{\text{KZ}(p^\alpha)}(z^{p^\alpha})(e_0, (e_{\xi^{(p^\alpha)}})_\xi)\, \text{Li}_q^{\text{KZ}}(z)(p^\alpha e_0, p^\alpha(\text{Ad}_{\Phi_{p,-\alpha}^{(\xi^{p^\alpha})}}(e_{\xi^{(p^\alpha)}})_\xi))^{-1}, \tag{5-3}$$

where $\text{Li}_q^{\text{KZ}(p^\alpha)}$ is the analogue of Li_q^{KZ} on $X^{(p^\alpha)}$ equal to the pull-back of $X = (\mathbb{P}^1 \setminus \{0, \mu_N, \infty\})/K$ by σ^α where σ is the Frobenius automorphism of K. When α is a multiple of $\log(q)/\log(p)$, $X^{(p^\alpha)} = X$ and $\text{Li}_q^{\text{KZ}(p^\alpha)} = \text{Li}_q^{\text{KZ}}$, and when $\alpha = \log(q)/\log(p)$, (5-2), (5-3) are directly equivalent to (5-1).

Notation 5A.1. For any $\tilde{\alpha} \in \mathbb{Z} \cup \{\pm\infty\} \setminus \{0\}$, let $\text{Li}_{q,\tilde{\alpha}}^\dagger = \text{Li}_{p,\alpha}^\dagger$ and $\text{Li}_{q,-\tilde{\alpha}}^\dagger = \text{Li}_{p,-\alpha}^\dagger$, with $p^\alpha = q^{\tilde{\alpha}}$.

The Frobenius on U^{an} is characterized by the couple $(\text{Li}_{p,\alpha}^\dagger, \Phi_{p,\alpha})$. We already know by Section 2 a description of $\Phi_{p,\alpha}$ as a function of α. If we combine it with (5-2), (5-3), we deduce a description of $\text{Li}_{p,\alpha}^\dagger$ as a function of α. This gives a description of the iterated Frobenius on U^{an} as a function of its number of iterations. We leave the details to the reader. One can also write an analogue for U^{an} of the notion of contraction mapping at base-points $(1, 0)$ of Definition 1B.2 and of the fact that the Frobenius at $(1, 0)$ is a contraction of Lemma 1B.7.

Let us just consider the convergence of the iterated Frobenius towards the fixed point when the number of iterations tends to ∞, in the unit ball $\text{B}(0, 1)$.

Proposition 5A.2. *For all $z \in K$ such that $|z|_p < 1$, we have, for the $\mathcal{N}_{\Lambda,D}$-topology,*

$$\tau(q^{-\tilde{\alpha}})\,\mathrm{Li}^{\dagger}_{q,\tilde{\alpha}}(z) \xrightarrow[\tilde{\alpha}\to\infty]{} \mathrm{Li}^{\mathrm{KZ}}_{q}(z)(e_0,(e_\xi)_\xi),$$
$$\tau(q^{-\tilde{\alpha}})\,\mathrm{Li}^{\dagger}_{q,-\tilde{\alpha}}(z) \xrightarrow[\tilde{\alpha}\to\infty]{} \mathrm{Li}^{\mathrm{KZ}}_{q}(z)(e_0,(\mathrm{Ad}_{\Phi^{(\xi)}_{q,-\infty}}(e_\xi))_\xi).$$

Moreover, these convergences are uniform on all the closed disks of center 0 *and radius* $\rho < 1$.

Proof. $\tau(q^{-\tilde{\alpha}})\,\mathrm{Li}^{\dagger}_{q,\tilde{\alpha}}(z)$ is the product of $\mathrm{Li}^{\mathrm{KZ}}_{q}(z)(e_0,(e_\xi)_\xi)$ by

$$\tau(q^{-\tilde{\alpha}})\,\mathrm{Li}^{\mathrm{KZ}}_{p,X_K^{(q^{\tilde{\alpha}})}}(z^{q^{\tilde{\alpha}}})(e_0,(\mathrm{Ad}_{\Phi^{(\xi)}_{q,\tilde{\alpha}}}(e_\xi))_\xi)^{-1}. \tag{5-4}$$

The coefficient of (5-4) at a word w is of the form,

$$q^{-\tilde{\alpha}\,\mathrm{weight}(w)} \sum_{(w_1,w_2)} \sum_{m=0}^{\infty} \mathfrak{h}_{q^{\alpha}m}(w_1)\frac{z^{q^{\tilde{\alpha}}m}}{(q^{\tilde{\alpha}}m)^L}\zeta_{q,\alpha}(w_2)$$

where $L \in \mathbb{N}^*$ and w_1, w_2 are in a finite set depending only on w, determined by the combinatorics of the composition of noncommutative formal power series. For all $m \in \mathbb{N}^*$ we have $-v_p(m) \geqslant -\log(m)/\log(p)$. Applying this to the m_i in (0-2) we deduce

$$v_p(\mathrm{har}_{q^{\tilde{\alpha}}m}(w)) \geqslant -\mathrm{weight}(w)\frac{\tilde{\alpha}\log(q)+\log(n)}{\log(p)}.$$

For all $C, C' \in \mathbb{R}^{+*}$, and $z \in K$ such that $|z|_p < 1$, we have $q^{\tilde{\alpha}} n v_p(z) - C\tilde{\alpha} - C'\log(n) \xrightarrow[\tilde{\alpha}\to\infty]{} +\infty$ and this convergence is uniform with respect to n. Indeed, let n_0 be an integer such that for all $n \geqslant n_0$, we have $C'\log(n) \leqslant \frac{1}{2} n v_p(z)$; then n_0 is independent of $\tilde{\alpha}$ and we have, for all $n \geqslant n_0$,

$$q^{\tilde{\alpha}} n v_p(z) - C\tilde{\alpha} - C'\log(n) \geqslant \frac{q^{\tilde{\alpha}} n}{2} v_p(z) - C\tilde{\alpha}.$$

Because of the bounds of valuations of cyclotomic p-adic multiple zeta values of [Jarossay 2019, Section 4], the sequence $(\mathcal{N}_{\Lambda,D}(\Phi_{q,-\alpha}))_{\tilde{\alpha}\in\mathbb{N}^*}$ is bounded. Thus, $\tau(q^{-\tilde{\alpha}})\,\mathrm{Li}^{\dagger}_{q,\alpha}(z)$ converges to $\mathrm{Li}^{\mathrm{KZ}}_{q}(z)$. Moreover, we can see that n_0 can be chosen independently from z in a closed disk of center 0 and radius $\rho < 1$.

The proofs of the statements concerning $\tau(q^{-\tilde{\alpha}})\,\mathrm{Li}^{\dagger}_{q,-\alpha}(z)$ are similar. □

Remark 5A.3. The convergence in Proposition 5A.2 does not a priori extend to a uniform convergence on U^{an}. Indeed, otherwise, in fact, the map $\mathrm{Li}^{\mathrm{KZ}}_{q}$ would be rigid analytic on U^{an}. By the main result of Appendix A of [Jarossay 2019], this would imply that, for any word w, the multiple harmonic sums functions $m \mapsto \mathrm{har}_m(w)$ restricted to classes of congruences modulo N should be continuous as a function $m \in \mathbb{N} \subset \varprojlim \mathbb{Z}/Np^l\mathbb{Z}$. This seems to contradict the results of Section 3C. More generally, we expect that the lack of regularity of the maps $m \mapsto \mathrm{har}_m(w)$ can be at least partially reflected in the mode of convergence of the sequences $\mathrm{Li}^{\dagger}_{q,\tilde{\alpha}}[w]$ when $\alpha \to \infty$.

Remark 5A.4. One can deduce a result similar to Proposition 5A.2 on the ball $B(\infty, 1)$ by applying the automorphism $z \mapsto 1/z$ of $\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ and the functoriality of $\pi_1^{\mathrm{un,crys}}$.

5B. ***Iteration equation.*** We now write an equation for the iteration of the Frobenius on the subspace $\mathbb{P}^{1,\mathrm{an}} \setminus \bigcup_{\xi\in\mu_N(K)} B(\xi,1)$. We restrict the statement to positive numbers of iterations for simplicity, but a similar result holds for negative numbers of iterations. If $f(z)=\sum_{m=0}^{\infty} c_m z^m$ is a power series with coefficients in K, let $f^{(p^\alpha)}(z)=\sum_{m=0}^{\infty}\sigma^\alpha(c_m)z^m$ where σ is the Frobenius automorphism of K.

Proposition 5B.1. *For any, $\alpha_0,\alpha\in\mathbb{N}^*$ with α_0 dividing α, we have*

$$\mathrm{Li}^{\dagger}_{p,\alpha}(e_0,(e_\xi)_\xi)=\mathrm{Li}^{\dagger}_{p,\alpha_0}(e_0,(e_\xi)_\xi)\mathrm{Li}^{\dagger}_{p,\alpha_0}{}^{(p^{\alpha_0})}(e_0,(\mathrm{Ad}_{\Phi^{(\xi)}_{p,\alpha_0}}(e_\xi)_\xi)\cdots\mathrm{Li}^{\dagger}_{p,\alpha_0}{}^{(p^{\alpha/\alpha_0-1})}(e_0,(\mathrm{Ad}_{\Phi^{\alpha-1}_{p,\alpha_0}(\xi)}(e_\xi)_\xi)).$$

Proof. The crystalline Frobenius of $\pi_1^{\mathrm{un}}(\mathbb{P}^1\setminus\{0,\mu_N,\infty\})$, restricted to the rigid analytic sections on $\mathbb{P}^{1,\mathrm{an}} \setminus \bigcup_{\xi\in\mu_N(K)} B(\xi,1)$, is given, with the conventions of [Jarossay 2019], by the formula

$$\tau(p^\alpha)\phi^\alpha : f(e_0,(e_\xi)_\xi)(z)\mapsto \mathrm{Li}^{\dagger}_{p,\alpha}(e_0,(e_\xi)_\xi)(z)\times f^{(p^\alpha)}(z^{p^\alpha})(e_0,\mathrm{Ad}_{\Phi^{(\xi)}_{p,\alpha}}(e_\xi)_\xi). \tag{5-5}$$

This implies the result. □

5C. ***Another iteration equation via regularized iterated integrals.*** The computation of overconvergent p-adic multiple polylogarithms in [Jarossay 2019], which was centered around a notion of regularization of iterated integrals, gives us another point of view on how they depend on α. Below, for a power series $S\in K[[z]]$, we denote by $S[z^m]$ the coefficient of z^m in S for all $m\in\mathbb{N}$. Again we restrict the statement to positive numbers of iteration of the Frobenius for simplicity, but a similar statement holds for negative number of iterations.

Proposition 5C.1. *For any word w on $e_{0\cup\mu_N}$ and any $m_0\in\mathbb{N}$, there exists a sequence*

$$(c^{(l,\xi,n)}[w](m_0))_{\substack{l\in\mathbb{N}\\ \xi\in\mu_N(K)\\ n\in\mathbb{N}}}$$

of elements of K such that, for any $\tilde{\alpha}\in\mathbb{N}$ such that $q^{\tilde{\alpha}}>m_0$ and $m\in\mathbb{N}^*$ satisfying $|m-m_0|_p\leqslant q^{-\tilde{\alpha}}$, we have

$$\mathrm{Li}^{\dagger}_{q,\tilde{\alpha}}[w][z^m]=\sum_{n=0}^{\infty}(q^{\tilde{\alpha}})^n\biggl(\sum_{l\in\mathbb{N}}\sum_{\xi}c^{(l,\xi,n)}[w](m_0)\xi^{-m}(m-m_0)^l\biggr).$$

Proof. In [Jarossay 2019, Section 3], we have defined a notion of regularized iterated integrals attached to any sequence of differential forms among $p^\alpha\frac{dz}{z}$, $\frac{p^\alpha dz}{z-\xi}$, $\xi\in\mu_N(K)$, $\frac{d(z^{p^\alpha})}{z^{p^\alpha}-\xi^{p^\alpha}}$, $\xi\in\mu_N(K)$. We have computed these regularized iterated integrals by induction on the depth, and this gives us information on how they depend on α. Namely, each regularized iterated integral is a rigid analytic function on $\mathbb{P}^{1,\mathrm{an}}\setminus\bigcup_{\xi\in\mu_N(K)}B(\xi,1)$ which has a power series expansion $\sum_{m=0}^{\infty}c_m z^m$ satisfying the following property: for any $m_0\in\{0,\dots,p^\alpha-1\}$, there exists a sequence $(c^{(l,\xi)}(m_0))_{\substack{l\in\mathbb{N}\\ \xi\in\mu_N(K)}}$ of elements of K such that, for all $m\in\mathbb{N}$ with $|m-m_0|_p\leqslant p^{-\alpha}$, we have $c_m=\sum_{l=0}^{\infty}\sum_\xi c^{(l,\xi)}(m_0)\xi^{-m}(m-m_0)^l$.

In [Jarossay 2019, Appendix B], we have showed that the numbers $c^{(l,\xi)}(m_0)$ have an expression as certain sums of series involving multiple harmonic sums, and particularly prime weighted multiple harmonic sums.

In [Jarossay 2019, Section 4], we have showed an expression of each $\mathrm{Li}^{\dagger}_{p,\alpha}[w]$ as a linear combination of regularized p-adic iterated integrals over the ring of p-adic cyclotomic multiple zeta values $\zeta_{p,\alpha}(w')$.

Combining these facts with the results of Sections 1 and 2 on how $\zeta_{q,\tilde{\alpha}}$ and $\mathrm{har}_{q^{\tilde{\alpha}}}$ depend on $\tilde{\alpha}$, we deduce the result. □

Appendix: A Poisson bracket corresponding to the pro-unipotent harmonic action of integrals at (1, 0)

We have seen that, by their definitions, the pro-unipotent harmonic actions (Section 1D and Definition 2A.3) are connected to the Ihara product (1-1). However, often in the literature, what is used is not the Ihara product but the corresponding Lie bracket, called the Ihara bracket, which is a Poisson bracket. In this section we explain that the pro-unipotent harmonic action of integrals at $(1, 0)$ (Definition 2A.3) corresponds naturally to a Poisson Lie bracket.

A.1. ***The Ihara bracket and the adjoint analogue.*** Let V^{ω} be the group of automorphisms defined in [Deligne and Goncharov 2005, Section 5.10]. The Ihara bracket is the Lie bracket of $\mathrm{Lie}(V^{\omega})$, regarded via the isomorphism $\mathrm{Lie}(V^{\omega}) \simeq \mathrm{Lie}(\Pi_{1,0})$, $v \mapsto v({}_1\Pi_0)$; namely, it is given by the following formula [loc. cit, Section 5.12–5.13]:

$$\{f, g\} = [f, g] + D_f(g) + D_g(f),$$

where D_f is the derivation which sends $e_0 \mapsto 0$ and $e_\xi \mapsto [f^{(\xi)}, e_\xi]$ for any $\xi \in \mu_N(K)$. The Ihara bracket is a Poisson bracket; namely it satisfies the equality $\{fg, h\} = \{f, h\}g + f\{g, h\}$.

Let $\tilde{V}^{\omega}$ be the preimage of $\tilde{\Pi}_{1,0}$ by the isomorphism $V^{\omega} \xrightarrow{\sim} \Pi_{1,0}$, $v \mapsto v({}_1\Pi_0)$.

We have defined in [Jarossay 2015, Definition 1.1.3] the adjoint Ihara product ((1-6)) and we have proved in [loc. cit., Proposition 1.1.4] that $\mathrm{Ad}(e_1)$ is an isomorphism of groups from $(\tilde{\Pi}_{1,0}(K), \circ^{\int_{1,0}}) \xrightarrow{\sim} (\mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1), \circ^{\int_{1,0}}_{\mathrm{Ad}})$.

We have viewed $\mathrm{Lie}(\tilde{V}^{\omega})$ as a subset of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$ and we can again view $\mathrm{Lie}(\mathrm{Ad}_{\tilde{V}^{\omega}}(e_1))$ as a subset of $K\langle\langle e_{0\cup\mu_N}\rangle\rangle$. The derivative of $f \mapsto f^{-1}e_1 f$, the map $f \mapsto [e_1, f]$ is the isomorphism of Lie algebras defined by $\mathrm{Lie}(\tilde{V}^{\omega}) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Ad}_{\tilde{V}^{\omega}}(e_1))$.

Proposition A.1.1. *The Lie bracket of* $\mathrm{Lie}(\mathrm{Ad}_{\tilde{V}^{\omega}}(e_1))$ *is* $\{f, g\}_{\mathrm{Ad}} = d_f(g) - d_g(f)$ *where* d_f *is the derivation sending* $e_0 \mapsto 0$, $e_\xi \mapsto f^{(\xi)}$ *for all* $\xi \in \mu_N(K)$.

Let the product $._{\mathrm{Ad}}^{\int_{1,0}}$ *on* $\mathrm{Ad}_{\tilde{\Pi}_{1,0}(K)}(e_1)$ *be defined by* $(g^{-1}e_1 g)._{\mathrm{Ad}}^{\int_{1,0}}(f^{-1}e_1 f) = (gf)^{-1}e_1(gf)$, *and its Lie version be* $[g, e_1]._{\mathrm{Ad}}^{\int_{1,0}}[f, e_1] = [gf, e_1]$.

Then, $\{\cdot\,,\cdot\}_{\mathrm{Ad}}$ *is a Poisson bracket, namely, it satisfies* $\{a._{\mathrm{Ad}}^{\int_{1,0}}b, c\} = \{a, c\}._{\mathrm{Ad}}^{\int_{1,0}}b + a._{\mathrm{Ad}}^{\int_{1,0}}\{b, c\}$.

Proof. Let us proceed as in the proof of Proposition 5.13 in [Deligne and Goncharov 2005]. We write $g = 1 + a\epsilon$. When $\epsilon \to 0$, we have $(1 + a\epsilon) \circ_{\mathrm{Ad}}^{\int_{1,0}} f = f + \epsilon d_g(f) + O(\epsilon^2)$. Thus, the action of $\mathrm{Lie}(V^{\omega})$ on $\mathrm{Ad}(\tilde{\Pi}_{1,0})(e_1)$ by $\circ_{\mathrm{Ad}}^{\int_{1,0}}$ is by $g \mapsto d_g$. This map is injective. By the injectivity of d, we only have to show that $[d_f, d_g] = d_{d_f(g) - d_g(f)}$. Since they are derivations, it is sufficient to prove that these two maps agree on e_1, and this follows directly from their definitions. The fact that it is a Poisson bracket follows from the isomorphism of Lie algebras $\mathrm{Lie}(\tilde{V}^{\omega}) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Ad}_{\tilde{V}^{\omega}}(e_1))$. □

Definition A.1.2. We call $\{\cdot,\cdot\}_{\mathrm{Ad}}$ the adjoint Ihara bracket.

A.2. ***Harmonic analogue of the Ihara bracket.*** We also have defined in Definition 2A.3 the pro-unipotent harmonic action of integrals at $(1,0)$, $\circ_{\mathrm{har}}^{\int_{1,0}}$, by pushing forward the adjoint Ihara action by the map $\mathrm{comp}^{\Sigma\int}$ (see Section 4B), which amounts to the map S of Definition 2A.2. We are now going to push-forward $\{\cdot,\cdot\}_{\mathrm{Ad}}$ by a linear and injective version of the map S of Definition 2A.2. Below, $K[[\Lambda]]\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}}$ is defined like $K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}}$ with coefficients in $K[[\Lambda]]$ instead of K.

Definition A.2.1. (i) Let $S_{\lambda} : K\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\widetilde{o(1)}} \to K[[\Lambda]]\langle\langle e_{0\cup\mu_N}\rangle\rangle_{\mathrm{har}}^{\int_{1,0}}$ defined by

$$h \mapsto \sum_{\substack{d\in\mathbb{N}^* \\ \xi_1,\dots,\xi_{d+1}\in\mu_N(K) \\ n_1,\dots,n_d\in\mathbb{N}^*}} h\left[\frac{1}{1-\Lambda e_0} e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}\right] e_{\xi_{d+1}} e_0^{n_d-1} e_{\xi_d} \cdots e_0^{n_1-1} e_{\xi_1}.$$

(ii) Let $L_{\mathrm{har}}(K) = S_{\lambda}\,\mathrm{Lie}\,\mathrm{Ad}_{\tilde{V}^{\omega}(K)}(e_1)$.

Proposition A.2.2. *$L_{\mathrm{har}}(K)$ has a canonical Lie bracket $\{\cdot,\cdot\}_{\mathrm{har}}$ defined by*

$$\{S_{\Lambda} f, S_{\Lambda} g\}_{\mathrm{har}} = S_{\Lambda}(\{f,g\}_{\mathrm{Ad}}).$$

It is a Poisson bracket, i.e., we have $\{a \cdot_{\mathrm{har}}^{\int_{1,0}} b, c\} = \{a,c\}\cdot_{\mathrm{har}}^{\int_{1,0}} b + a \cdot_{\mathrm{har}}^{\int_{1,0}} \{b,c\}$, where the product $\cdot_{\mathrm{har}}^{\int_{1,0}}$ is defined by $S_{\Lambda} g \cdot_{\mathrm{har}}^{\int_{1,0}} S_{\Lambda} f = S_{\Lambda}(g \cdot_{\mathrm{Ad}}^{\int_{1,0}} f)$.

Proof. Similar to the proof of Proposition A.1.1. □

Definition A.2.3. We call $\{\cdot,\cdot\}_{\mathrm{har}}$ the harmonic Ihara bracket.

Remark A.2.4. (i) The harmonic Ihara bracket $\{\cdot,\cdot\}_{\mathrm{har}}$ corresponds to the group law $\tilde{\circ}_{\mathrm{har}}^{\int_{1,0}}$ defined by $S_{\Lambda} g \tilde{\circ}_{\mathrm{har}}^{\int_{1,0}} S_{\Lambda} f = S_{\Lambda}(g \circ_{\mathrm{Ad}}^{\int_{1,0}} f)$. Because of the injectivity of $S_{\Lambda} g$, the group law $\tilde{\circ}_{\mathrm{har}}^{\int_{1,0}}$ can be thought of as another version of the pro-unipotent harmonic action of integrals $\circ_{\mathrm{har}}^{\int_{1,0}}$ of Definition 2A.3.

(ii) Another way to define a harmonic variant of the Ihara bracket would be to restrict to summable points $\mathrm{Ad}_{\tilde{\Pi}_{1,0}}(K)_{o(1)}(e_1) \subset \mathrm{Ad}_{\tilde{\Pi}_{1,0}}(K)_{o(1)}(e_1)$ and to use $\mathrm{comp}_{\mathrm{iter}}^{\Sigma\int}$, which is injective by Proposition 4B.2, instead of $\mathrm{comp}_{\Lambda}^{\Sigma\int}$.

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