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# Moments of quadratic twists of elliptic curve $L$ -functions over function fields

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We calculate the first and second moments of  $L$ -functions in the family of quadratic twists of a fixed elliptic curve  $E$  over  $\mathbb{F}_q[x]$ , asymptotically in the limit as the degree of the twists tends to infinity. We also compute moments involving derivatives of  $L$ -functions over quadratic twists, enabling us to deduce lower bounds on the correlations between the analytic ranks of the twists of two distinct curves.

## 1. Introduction and statement of results

The values of  $L$ -functions at the central point of the critical strip have been the subject of considerable interest in recent years. One way to study these central values is by considering moments in families of  $L$ -functions. There are now precise conjectured asymptotic formulas for such moments motivated by analogies with random matrix theory [Keating and Snaith 2000a; 2000b]. More precise asymptotic formulas containing lower order terms were conjectured in [Conrey et al. 2003; 2005; Diaconu et al. 2003]. In the case of the Riemann zeta-function, the analogue of these conjectures is now relatively well understood in terms of correlations of the divisor function [Conrey and Keating 2015a; 2015b; 2015c; 2016; 2019]. The moments of other degree-one  $L$ -functions have also been investigated intensively. It remains a challenge to extend these calculations to  $L$ -functions of degree two and higher.

The first moment of the family of derivatives of  $L$ -functions of quadratic twists of a fixed modular form was studied in [Bump et al. 1990b; Iwaniec 1990; Murty and Murty 1991]. Questions related to the nonvanishing of  $L$ -functions in this family were considered in [Bump et al. 1990a; 1990b; Murty and Murty 1991]. For example, it is shown independently in [Murty and Murty 1991] and [Bump et al. 1990a], using different techniques, that for a fixed elliptic curve with root number equal to 1 there are infinitely many fundamental discriminants  $d < 0$  such that its twist by  $d$  has analytic rank equal to 1.

The second moment of the family was considered by Soundararajan and Young [2010]. Unconditionally, they obtained a lower bound for the second moment which matches the asymptotic formula conjectured by Keating and Snaith [2000b] and assuming the generalized Riemann hypothesis (GRH) they established the conjectured formula. Using similar ideas, again under GRH, Petrow [2014] obtained several asymptotic formulas for moments of derivatives of these  $GL(2)$   $L$ -functions when the sign of the functional equation is  $-1$ .

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While no asymptotic formulas for moments larger than the second are known for this family, there are lower and upper bounds of the right order of magnitude. Rudnick and Soundararajan [2005; 2006] established unconditional lower bounds for all moments larger than the first, and, assuming GRH, the work of Soundararajan [2009] and its refinement by Harper [2013] produced upper bounds of the conjectured order of magnitude. Radziwiłł and Soundararajan [2015] proved upper bounds for moments below the first in this family of  $L$ -functions. Their techniques also allow them to obtain a one-sided central limit for the distribution of the logarithm of these central  $L$ -values. Their result supports a conjecture by Keating and Snaith [2000a] which can be viewed as the analogue of Selberg's central limit theorem for the distribution of  $\log|\zeta(\frac{1}{2} + it)|$ .

In this paper we study several moment problems of comparable difficulty to the moment computation of Soundararajan and Young in the function field family of quadratic twists of an elliptic curve. Since we are working over function fields, the results we obtain are unconditional due to the availability of GRH in this setting.

Recently there has been a good deal of work on computing moments of  $L$ -functions in the function field setting. Andrade and Keating [2012] obtained an asymptotic formula for the first moment in the symplectic family of quadratic  $L$ -functions when the degree of the  $L$ -functions (which is a polynomial in this case) goes to infinity and the size of the finite field is fixed (see also [Hoffstein and Rosen 1992] for a similar result). A lower order term of size approximately the cube root of the main term was computed in [Florea 2017c]. The second, third and fourth moments were computed in [Florea 2017b; 2017a] (see also [Diaconu 2019]). We note that the asymptotic formula for the fourth moment does not have a power savings error term, but recovers several of the expected leading order terms in the conjectured formula [Andrade and Keating 2014]. Obtaining an asymptotic formula with the leading order term for the fourth moment in the family of quadratic  $L$ -functions is comparable in difficulty to establishing an asymptotic formula for the second moment of  $L$ -functions of quadratic twists of an elliptic curve, and is one of the problems we consider in this paper.

We note that for all of our results, we fix the size  $q$  of the finite field we work in and let the degree of the  $L$ -functions go to infinity. If instead one fixes the degree and lets  $q \rightarrow \infty$ , then Katz and Sarnak [1999] showed that the  $L$ -functions become equidistributed in the orthogonal group, and hence computing the various moments reduces to computing several random matrix integrals (see for example [Keating and Snaith 2000a]). In the case of elliptic curve  $L$ -functions, the relevant equidistribution results were established in [Hall et al. 2017].

To state our results we first need some notation. Fix a prime power  $q$  with

$$(q, 6) = 1 \quad \text{and} \quad q \equiv 1 \pmod{4}.$$

Let  $K = \mathbb{F}_q(t)$  be the rational function field and  $\mathcal{O}_K = \mathbb{F}_q[t]$ . Let  $E/K$  be an elliptic curve defined by  $y^2 = x^3 + ax + b$ , with  $a, b \in \mathcal{O}_K$  and discriminant  $\Delta = 4a^3 + 27b^2$  such that  $\deg_t(\Delta)$  is minimal among models of  $E/K$  of this form.

The normalized  $L$ -function associated to the elliptic curve  $E/K$  has the following Euler product and Dirichlet series, which converge for  $\Re(s) > 1$ ,

$$\begin{aligned} L(E, s) &:= \mathcal{L}(E, u) \\ &= \sum_{f \in \mathcal{M}} \lambda(f) u^{\deg(f)} \\ &= \prod_{P \mid \Delta} (1 - \lambda(P) u^{\deg(P)})^{-1} \prod_{P \nmid \Delta} (1 - \lambda(P) u^{\deg(P)} + u^{2 \deg(P)})^{-1}, \end{aligned} \tag{1-1}$$

where we set  $u := q^{-s}$ , and  $\mathcal{M}$  denotes the set of monic polynomials over  $\mathbb{F}_q[t]$ . For a more detailed discussion of the coefficients  $\lambda(f)$  and how to define the  $L$ -function, see Section 2.2 in [Baig and Hall 2012]. The  $L$ -function is a polynomial in  $u$  with integer coefficients of degree

$$n := \deg(\mathcal{L}(E, u)) = \deg(M) + 2 \deg(A) - 4, \tag{1-2}$$

where for simplicity we denote by  $M$  the product of the finite primes where  $E$  has multiplicative reduction and by  $A$  the product of the finite primes where  $E$  has additive reduction. Moreover, the  $L$ -function satisfies a functional equation; namely, there exists  $\epsilon(E) \in \{\pm 1\}$  such that

$$\mathcal{L}(E, u) = \epsilon(E) (\sqrt{q}u)^n \mathcal{L}\left(E, \frac{1}{qu}\right).$$

For a more precise formula for the sign of the functional equation, see Lemma 2.3 in [Baig and Hall 2012]. Now for  $D \in \mathcal{O}_K$  with  $D$  square-free, monic of odd degree and  $(D, \Delta) = 1$ , we consider the twisted elliptic curve  $E \otimes \chi_D/K$  with the affine model  $y^2 = x^3 + D^2ax + D^3b$ . Then the  $L$ -function corresponding to the twisted elliptic curve has the following Dirichlet series and Euler product

$$\begin{aligned} \mathcal{L}(E \otimes \chi_D, u) &= \sum_{f \in \mathcal{M}} \lambda(f) \chi_D(f) u^{\deg(f)} \\ &= \prod_{P \mid \Delta} (1 - \lambda(P) \chi_D(P) u^{\deg(P)})^{-1} \prod_{P \nmid \Delta D} (1 - \lambda(P) \chi_D(P) u^{\deg(P)} + u^{2 \deg(P)})^{-1}. \end{aligned}$$

The new  $L$ -function is a polynomial of degree  $(n + 2 \deg(D))$  and satisfies the functional equation

$$\mathcal{L}(E \otimes \chi_D, u) = \epsilon(\sqrt{q}u)^{n+2 \deg(D)} \mathcal{L}\left(E \otimes \chi_D, \frac{1}{qu}\right), \tag{1-3}$$

where

$$\epsilon = \epsilon(E \otimes \chi_D) = \epsilon_{\deg(D)} \epsilon(E) \chi_D(M).$$

Here  $\epsilon_{\deg(D)} \in \{\pm 1\}$  is an integer which only depends on the degree of  $D$  (see Proposition 4.3 in [Baig and Hall 2012]).

Let  $\mathcal{H}_{2g+1}^*$  denote the set of monic, square free polynomials of degree  $(2g + 1)$  coprime to  $\Delta$ . Our first two theorems concern the first moments of  $L(E \otimes \chi_D, \frac{1}{2})$  and  $L'(E \otimes \chi_D, \frac{1}{2})$ .

**Theorem 1.1.** *Unless  $\epsilon_{2g+1}\epsilon(E) = -1$  and  $M = 1$ , we have*

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2}) = c_1(M) + O_\epsilon(q^{-g+\epsilon g}),$$

where the value  $c_1(M)$  is defined in (5-6) and (5-2). In particular, the constant  $c_1(M) \neq 0$  in this case and we obtain an asymptotic formula.

**Theorem 1.2.** *Unless  $\epsilon_{2g+1}\epsilon(E) = 1$  and  $M = 1$ , we have*

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})} = c_2(M)L(\text{Sym}^2 E, 1)g + c_3(M) + O_\epsilon(q^{-g+\epsilon g}),$$

where the values  $c_2(M)$  and  $c_3(M)$  are defined in (6-2), (6-3) and (5-2). In particular, the constant  $c_2(M) \neq 0$  in this case and we obtain an asymptotic formula.

Theorem 1.2 above should be compared to the number field analogous result in [Iwaniec 1990]. In the following theorems we obtain asymptotic formulas for the second moments of  $L(E \otimes \chi_D, \frac{1}{2})$  and  $L'(E \otimes \chi_D, \frac{1}{2})$ .

**Theorem 1.3.** *Unless  $\epsilon_{2g+1}\epsilon(E) = -1$  and  $M = 1$ , we have*

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2})^2 = c_4(M)L(\text{Sym}^2 E, 1)^3g + O_\epsilon(g^{1/2+\epsilon}),$$

where the value  $c_4(M)$  is defined in (7-2), (3-8) and (3-9). In particular, the constant  $c_4(M) \neq 0$  in this case and we obtain an asymptotic formula.

**Theorem 1.4.** *Unless  $\epsilon_{2g+1}\epsilon(E) = 1$  and  $M = 1$ , we have*

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})} = c_5(M)L(\text{Sym}^2 E, 1)^3g^3 + O_\epsilon(g^{2+\epsilon}),$$

where the value  $c_5(M)$  is defined in (8-2), (3-8) and (3-9). In particular, the constant  $c_5(M) \neq 0$  in this case and we obtain an asymptotic formula.

Note that Theorem 1.3 is the function field analogue of Theorem 1.2 in [Soundararajan and Young 2010]. Considering the smoothed second moment, Soundararajan and Young obtain an error term of size  $(\log X)^{3/4+\epsilon}$ , which would translate to  $g^{3/4+\epsilon}$  in the function field setting. Using slightly different techniques, Petrow [2014] states that the error term could be improved to  $(\log X)^{1/2+\epsilon}$  which is of the same quality we obtain in the result above.

Our Theorem 1.3 should also be compared to the asymptotic formula for the fourth moment of quadratic  $L$ -functions over function fields in [Florea 2017a]. We remark that for the symplectic family of quadratic  $L$ -functions, one can obtain lower order terms in the asymptotic formula by using an inductive argument and then obtaining upper bounds for moments of  $L$ -functions evaluated at points far from the critical point. The fact that one can compute a few lower order terms can be explained by the gap between

powers of  $g$  coming from evaluating moments at the critical point versus evaluating moments far from the critical point. When computing the fourth moment of quadratic  $L$ -functions close to the central point, one expects to obtain a power of  $g^{10}$ . As we move away from the central point, the family starts to behave like a family with unitary symmetry and one expects an upper bound of the magnitude  $g^4$ . The difference in powers of  $g$  gives one room to use a repetitive argument to rigorously compute lower order terms down to  $g^4$ . In the case of the orthogonal family we consider in this paper, note that the main term in [Theorem 1.3](#) is of size  $g$ , and the error term has size  $g^{1/2}$  coming from obtaining an upper bound for the second moment evaluated at a point far from the central point. The small difference between these powers of  $g$  does not give us enough room to compute a lower order term in this case.

We can also study the moment of the product of the quadratic twists of two elliptic curve  $L$ -functions. Let  $E_1$  and  $E_2$  be two elliptic curves over  $K$ . Let  $\Delta = \Delta_1 \Delta_2$ , where, for  $i = 1, 2$ ,  $\Delta_i$  denotes the discriminant of  $E_i$ . Let  $M_i$  denote the product of the finite primes where  $E_i$  has multiplicative reduction and

$$\epsilon_i = \epsilon_{\deg(D)} \epsilon(E_i) \chi_D(M_i).$$

Define

$$\epsilon_i^+ := \frac{1 + \epsilon_i}{2} \quad \text{and} \quad \epsilon_i^- := \frac{1 - \epsilon_i}{2}.$$

**Theorem 1.5.** *Unless  $\epsilon_{2g+1} \epsilon(E_1) = -1$  and  $M_1 = 1$ , or  $\epsilon_{2g+1} \epsilon(E_2) = 1$  and  $M_2 = 1$ , or  $\epsilon(E_1) = \epsilon(E_2)$  and  $M_1 = M_2$ , we have*

$$\begin{aligned} \frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ = c_6(M_1, M_2) L(\text{Sym}^2 E_1, 1) L(\text{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g + O_\epsilon(g^{1/2+\epsilon}), \end{aligned}$$

where the value  $c_6(M_1, M_2)$  is defined in (9-2), (3-8) and (3-10). In particular, the constant  $c_6(M_1, M_2) \neq 0$  in this case and we obtain an asymptotic formula.

We are not aware of the analogous number field result in the literature. We also prove the following.

**Theorem 1.6.** *Unless  $\epsilon_{2g+1} \epsilon(E_1) = 1$  and  $M_1 = 1$ , or  $\epsilon_{2g+1} \epsilon(E_2) = 1$  and  $M_2 = 1$ , or  $\epsilon(E_1) = -\epsilon(E_2)$  and  $M_1 = M_2$ , we have*

$$\begin{aligned} \frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_1^- \epsilon_2^- L'(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ = c_7(M_1, M_2) L(\text{Sym}^2 E_1, 1) L(\text{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g^2 + O_\epsilon(g^{1+\epsilon}), \end{aligned}$$

where the value  $c_7(M_1, M_2)$  is defined in (10-2), (3-8) and (3-10). In particular, the constant  $c_7(M_1, M_2) \neq 0$  in this case and we obtain an asymptotic formula.

Note that this result is the analogue of Theorem 2.2 in [\[Petrow 2014\]](#). An interesting problem would be to compute the average of  $L(E_1 \otimes \chi_D, \frac{1}{2}) L(E_2 \otimes \chi_D, \frac{1}{2})$  for distinct elliptic curves  $E_1$  and  $E_2$ . This would have applications to the question of simultaneous nonvanishing of  $L(E_1 \otimes \chi_D, \frac{1}{2})$  and  $L(E_2 \otimes \chi_D, \frac{1}{2})$ .

However our techniques do not allow us to obtain such an asymptotic formula, as the error term (coming from Proposition 4.7) would dominate the main term which would have constant size.

Define the analytic rank of a quadratic twist of an elliptic curve  $L$ -function  $L(E \otimes \chi_D, s)$  by

$$r_{E \otimes \chi_D} := \text{ord}_{s=1/2} L(E \otimes \chi_D, s).$$

Combining the upper bounds for moments of elliptic curve  $L$ -functions (see Section 4) with Theorems 1.5 and 1.6 leads to the following corollary.

**Corollary 1.7.** *Unless  $\epsilon_{2g+1}\epsilon(E_1) = -1$  and  $M_1 = 1$ , or  $\epsilon_{2g+1}\epsilon(E_2) = 1$  and  $M_2 = 1$ , or  $\epsilon(E_1) = \epsilon(E_2)$  and  $M_1 = M_2$ , we have*

$$\#\{D \in \mathcal{H}_{2g+1}^* : r_{E_1 \otimes \chi_D} = 0, r_{E_2 \otimes \chi_D} = 1\} \gg_\epsilon \frac{q^{2g}}{g^{6+\epsilon}}$$

as  $g \rightarrow \infty$ . Also, unless  $\epsilon_{2g+1}\epsilon(E_1) = 1$  and  $M_1 = 1$ , or  $\epsilon_{2g+1}\epsilon(E_2) = 1$  and  $M_2 = 1$ , or  $\epsilon(E_1) = -\epsilon(E_2)$  and  $M_1 = M_2$ , we have

$$\#\{D \in \mathcal{H}_{2g+1}^* : r_{E_1 \otimes \chi_D} = r_{E_2 \otimes \chi_D} = 1\} \gg_\epsilon \frac{q^{2g}}{g^{6+\epsilon}}$$

as  $g \rightarrow \infty$ .

As far as we are aware, Corollary 1.7 is the first result in literature where explicit lower bounds concerning the correlations between the ranks of two twisted elliptic curves are obtained. Following Harper’s argument [2013], for the upper bounds for moments of  $L$ -functions one may remove the exponents  $\epsilon$  in Corollary 1.7. We fail to obtain positive proportions in the above results because we are not able to use a mollifier. Note that the results of Heath-Brown [2004] adapted to the function field setting do not lead to positive proportions either.

## 2. Some useful lemmas

In this section we will gather a few useful lemmas we will need throughout the paper.

Recall that  $q$  is a prime power with  $q \equiv 1 \pmod{4}$  and  $(q, 6) = 1$ . Let  $\mathcal{M}$  denote the set of monic polynomials over  $\mathbb{F}_q[t]$  and  $\mathcal{H}$  be the set of monic, square-free polynomials. Let  $\mathcal{M}_n$  denote the set of monic polynomials of degree  $n$  over  $\mathbb{F}_q[t]$  and  $\mathcal{M}_{\leq n}$  be the set of monic polynomials of degree less than or equal to  $n$ . Let  $\mathcal{H}_n$  denote the monic, square-free polynomials of degree  $n$  and recall that  $\mathcal{H}_n^*$  denotes the set of monic, square-free polynomials of degree  $n$  coprime to  $\Delta$ . The norm of a polynomial  $f$  is defined by  $|f| = q^{\deg(f)}$ . Let  $\phi$  denote the Euler totient function and  $\tau$  the divisor function.

We define the zeta-function as

$$\zeta_q(s) = \sum_{f \in \mathcal{M}} \frac{1}{|f|^s}$$

for  $\Re(s) > 1$ . By counting monic polynomials of a given degree, one can easily show that

$$\zeta_q(s) = \frac{1}{1 - q^{1-s}},$$

and this provides a meromorphic continuation of  $\zeta_q$  with a simple pole at  $s = 1$ . As before, we will make the change of variables  $u = q^{-s}$  and so the zeta-function becomes

$$\mathcal{Z}(u) = \zeta_q(s) = \sum_{f \in \mathcal{M}} u^{\deg(f)} = \frac{1}{1 - qu},$$

with a simple pole at  $u = 1/q$ . Note that  $\mathcal{Z}(u)$  can also be written as an Euler product

$$\mathcal{Z}(u) = \prod_P (1 - u^{\deg(P)})^{-1},$$

where the product is over monic, irreducible polynomials in  $\mathbb{F}_q[t]$ .

The quadratic character over  $\mathbb{F}_q[t]$  is defined as follows. For  $P$  a monic, irreducible polynomial let

$$\left(\frac{f}{P}\right) = \begin{cases} 1 & \text{if } P \nmid f, f \text{ is a square modulo } P, \\ -1 & \text{if } P \nmid f, f \text{ is not a square modulo } P, \\ 0 & \text{if } P \mid f. \end{cases}$$

We extend the definition of the quadratic residue symbol above to any monic  $D \in \mathbb{F}_q[t]$  by multiplicativity, and define the quadratic character  $\chi_D$  by

$$\chi_D(f) = \left(\frac{D}{f}\right).$$

Since we assumed that  $q \equiv 1 \pmod{4}$ , note that the quadratic reciprocity law takes the following form: if  $A$  and  $B$  are two monic coprime polynomials, then

$$\left(\frac{A}{B}\right) = \left(\frac{B}{A}\right).$$

Throughout the paper, we will often make use of the Perron formula over function fields. If the series  $\sum_{f \in \mathcal{M}} a(f)u^{\deg(f)}$  is absolutely convergent for  $|u| \leq r < 1$  then

$$\sum_{f \in \mathcal{M}_n} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \left( \sum_{f \in \mathcal{M}} a(f)u^{\deg(f)} \right) \frac{du}{u^{n+1}},$$

and

$$\sum_{f \in \mathcal{M}_{\leq n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \left( \sum_{f \in \mathcal{M}} a(f)u^{\deg(f)} \right) \frac{du}{u^{n+1}(1-u)}.$$

Recall that the twisted elliptic curve  $L$ -function  $\mathcal{L}(E \otimes \chi_D, u)$  is a polynomial of degree  $(n + 2 \deg(D))$ , with  $n$  being defined in (1-2). Thus we can write

$$\mathcal{L}(E \otimes \chi_D, u) = \sum_{n=0}^{n+2 \deg(D)} c_n u^n,$$

where  $c_n = \sum_{f \in M_n} \lambda(f) \chi_D(f)$ .

**Lemma 2.1.** *The coefficients  $c_n$  of  $\mathcal{L}(E \otimes \chi_D, u)$  satisfy the following relation*

$$c_n = \epsilon q^{n-n/2-\deg(D)} c_{n+2 \deg(D)-n},$$

with  $\epsilon$  as in (1-3). In particular, if  $\epsilon = -1$  and  $n$  is even, then  $c_{n/2+\deg(D)} = 0$ .

*Proof.* From the functional equation (1-3) we have

$$\sum_{n=0}^{n+2 \deg(D)} c_n u^n = \epsilon \sum_{n=0}^{n+2 \deg(D)} c_n q^{n/2+\deg(D)-n} u^{n+2 \deg(D)-n}.$$

By setting  $k := n + 2 \deg(D) - n$  we get

$$\sum_{n=0}^{n+2 \deg(D)} c_n u^n = \epsilon \sum_{k=0}^{n+2 \deg(D)} c_{n+2 \deg(D)-k} q^{k-n/2-\deg(D)} u^k.$$

Comparing the coefficients we obtain the lemma. □

For  $D \in \mathcal{H}_{2g+1}^*$ , we can obtain the following exact formulas for  $L(E \otimes \chi_D, \frac{1}{2})$  and  $L'(E \otimes \chi_D, \frac{1}{2})$ . These are the analogues of the approximate functional equations in the number field setting.

**Lemma 2.2.** *Let  $D \in \mathcal{H}_{2g+1}^*$ . Then*

$$L(E \otimes \chi_D, \frac{1}{2}) = \sum_{f \in \mathcal{M}_{\leq [n/2]+\deg(D)}} \frac{\lambda(f) \chi_D(f)}{\sqrt{|f|}} + \epsilon \sum_{f \in \mathcal{M}_{\leq [(n-1)/2]+\deg(D)}} \frac{\lambda(f) \chi_D(f)}{\sqrt{|f|}},$$

with  $\epsilon$  as in (1-3).

*Proof.* We use Lemma 2.1 to get

$$\begin{aligned} \mathcal{L}(E \otimes \chi_D, u) &= \sum_{n=0}^{n+2 \deg(D)} c_n u^n \\ &= \sum_{n=0}^{[n/2]+\deg(D)} c_n u^n + \sum_{n=[n/2]+\deg(D)+1}^{n+2 \deg(D)} c_n u^n \\ &= \sum_{n=0}^{[n/2]+\deg(D)} c_n u^n + \epsilon \sum_{n=[n/2]+\deg(D)+1}^{n+2 \deg(D)} c_{n+2 \deg(D)-n} q^{n-n/2-\deg(D)} u^n. \end{aligned}$$

Changing the summation variable in the second sum leads to

$$\mathcal{L}(E \otimes \chi_D, u) = \sum_{n=0}^{[n/2]+\deg(D)} c_n u^n + \epsilon \sum_{n=0}^{[(n-1)/2]+\deg(D)} c_n (qu^2)^{n/2+\deg(D)-n} u^n.$$

Taking  $u = q^{-1/2}$  and recalling that  $c_n = \sum_{f \in \mathcal{M}_n} \lambda(f) \chi_D(f)$  concludes the proof. □

**Lemma 2.3.** *Let  $D \in \mathcal{H}_{2g+1}^*$ . If  $\epsilon = -1$ , then*

$$L'(E \otimes \chi_D, \frac{1}{2}) = 2(\log q) \sum_{f \in \mathcal{M}_{\leq [n/2]+\deg(D)}} \frac{([n/2] + \deg(D) - \deg(f)) \lambda(f) \chi_D(f)}{\sqrt{|f|}}.$$

*Proof.* The above formula follows simply by differentiating the last equation in the proof of [Lemma 2.2](#). Just note that as remarked in [Lemma 2.1](#), if  $\epsilon = -1$  and  $n$  is even, then  $c_{n/2+\deg(D)} = 0$ . □

We also have the following lemma which expresses a character sum over square-free polynomials in terms of sums over monics.

**Lemma 2.4.** *We have*

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1}^*} \chi_D(f) &= \sum_{C_1 | \Delta} \mu(C_1) \chi_{C_1}(f) \sum_{C_2 | (\Delta f)^\infty} \sum_{R \in \mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}} \chi_R(f) \\ &\quad - q \sum_{C_1 | \Delta} \mu(C_1) \chi_{C_1}(f) \sum_{C_2 | (\Delta f)^\infty} \sum_{R \in \mathcal{M}_{2g-1-\deg(C_1)-2\deg(C_2)}} \chi_R(f), \end{aligned}$$

where by  $C_2 | (\Delta f)^\infty$  we mean that the prime factors of  $C_2$  divide  $\Delta f$ .

*Proof.* Let

$$A(u) = \sum_{\substack{D \in \mathcal{H} \\ (D, \Delta)=1}} \chi_D(f) u^{\deg(D)}.$$

Then

$$\begin{aligned} A(u) &= \prod_{P \nmid \Delta f} (1 + \chi_P(f) u^{\deg(P)}) \\ &= \prod_{P \nmid \Delta f} (1 - u^{2\deg(P)}) (1 - \chi_P(f) u^{\deg(P)})^{-1} \\ &= (1 - qu^2) \mathcal{L}(u, \chi_f) \prod_{P | \Delta f} (1 - u^{2\deg(P)})^{-1} \prod_{\substack{P | \Delta \\ P \nmid f}} (1 - \chi_P(f) u^{\deg(P)}). \end{aligned}$$

Writing

$$\prod_{P | \Delta f} (1 - u^{2\deg(P)})^{-1} = \sum_{C_2 | (\Delta f)^\infty} u^{2\deg(C_2)}, \quad \prod_{\substack{P | \Delta \\ P \nmid f}} (1 - \chi_P(f) u^{\deg(P)}) = \sum_{C_1 | \Delta} \mu(C_1) \chi_{C_1}(f) u^{\deg(C_1)},$$

and comparing the coefficients of  $u^{2g+1}$ , the conclusion follows. □

As in [Hayes 1966] we define the exponential over function fields as follows. For  $a \in \mathbb{F}_q((1/t))$  let

$$e_q(a) = \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a_1)}{p}\right),$$

where  $a_1$  is the coefficient of  $1/t$  in the Laurent expansion of  $a$  and  $q$  is a power of the prime  $p$ . We define the generalized quadratic Gauss sum as

$$G(V, f) = \sum_{u \pmod{f}} \chi_f(u) e_q\left(\frac{uV}{f}\right),$$

where  $\chi_f$  is the quadratic character defined before. We gather here a few useful facts about  $G(V, f)$  whose proofs can be found in [Florea 2017c].

**Lemma 2.5.** (1) If  $(f, h) = 1$ , then  $G(V, fh) = G(V, f)G(V, h)$ .

(2) Write  $V = V_1 P^\alpha$  where  $P \nmid V_1$ . Then

$$G(V, P^j) = \begin{cases} 0 & \text{if } j \leq \alpha \text{ and } j \text{ odd,} \\ \phi(P^j) & \text{if } j \leq \alpha \text{ and } j \text{ even,} \\ -|P|^{j-1} & \text{if } j = \alpha + 1 \text{ and } j \text{ even,} \\ \chi_P(V_1)|P|^{j-1/2} & \text{if } j = \alpha + 1 \text{ and } j \text{ odd,} \\ 0 & \text{if } j \geq 2 + \alpha. \end{cases}$$

The following Poisson summation formula in function fields holds.

**Lemma 2.6.** Let  $f \in \mathcal{M}_n$ . If  $n$  is even, then

$$\sum_{R \in \mathcal{M}_m} \chi_R(f) = \frac{q^m}{|f|} \left( G(0, f) + (q-1) \sum_{V \in \mathcal{M}_{\leq n-m-2}} G(V, f) - \sum_{V \in \mathcal{M}_{n-m-1}} G(V, f) \right),$$

otherwise

$$\sum_{R \in \mathcal{M}_m} \chi_R(f) = \frac{q^m \overline{\tau(q)}}{|f|} \sum_{V \in \mathcal{M}_{n-m-1}} G(V, f),$$

where

$$\tau(q) = \sum_{c=1}^{q-1} \chi_f(c) \exp\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c)}{p}\right)$$

is the usual Gauss sum over  $\mathbb{F}_q$ .

**2A. Outline of the proof.** We will use the approximate functional equations for the  $L$ -functions involved in the moment computations and then truncate the Dirichlet series close to the endpoint. For the longer Dirichlet series, we will use Poisson summation and standard techniques to compute the main terms. For the tails, we will go back and write the Dirichlet series in terms of expressions involving moments and then use upper bounds for moments. The key in bounding the tails is the fact that the moments behave differently depending on the points where we evaluate them (the power of  $g$  gets smaller in different ranges).

### 3. Main proposition

For  $N \mid \Delta^\infty$ , let

$$S_{E_1, E_2}(N, X, Y; \alpha, \beta) := \sum_{D \in \mathcal{H}_{2g+1}^*} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y}}} \frac{\lambda_1(f)\lambda_2(h)\chi_D(Nfh)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}}, \tag{3-1}$$

where  $\lambda_i$  are the Dirichlet coefficients of  $L(E_i, s)$  as in (1-1).

**Proposition 3.1.** *Assume  $X \geq Y$ . We have*

$$\begin{aligned} S_E(N, X, Y) &:= S_{E, E}(N, X, Y; 0, 0) \\ &= |\mathcal{H}_{2g+1}|C_E(N; 1, 1, 1)L(\text{Sym}^2 E, 1)^3 Y + O(q^{2g}) + O(q^{2g-Y/5}g^2) + O(q^{g/2+3(X+Y)/8}g^{30}), \end{aligned}$$

and if  $E_1 \neq E_2$ , then

$$\begin{aligned} S_{E_1, E_2}(N, X, Y; \alpha, \beta) &= |\mathcal{H}_{2g+1}|C_{E_1, E_2}(N; 1, 1, 1, \alpha, \beta)L(\text{Sym}^2 E_1, 1 + 2\alpha)L(\text{Sym}^2 E_2, 1 + 2\beta)L(E_1 \otimes E_2, 1 + \alpha + \beta) \\ &\quad + O(q^{2g-Y/5}g^2) + O(q^{g/2+3(X+Y)/8}g^{30}) \end{aligned}$$

uniformly for  $|\alpha|, |\beta| \leq 1/g$ , where the values  $C_E(N; 1, 1, 1)$  and  $C_{E_1, E_2}(N; 1, 1, 1, \alpha, \beta)$  are defined in (3-8), (3-9) and (3-10).

We begin the proof of the proposition by applying Lemma 2.4 and rewriting  $S_{E_1, E_2}(N, X, Y; \alpha, \beta)$  as

$$\begin{aligned} &\sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1 \mid \Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{C_2 \mid (\Delta fh)^\infty} \sum_{R \in \mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh) \\ &- q \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1 \mid \Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{C_2 \mid (\Delta fh)^\infty} \sum_{R \in \mathcal{M}_{2g-1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh) \\ &= S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta) + T_{E_1, E_2}(N, X, Y, Z; \alpha, \beta), \tag{3-2} \end{aligned}$$

where  $S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$  denotes the contribution of the terms with  $\deg(C_2) \leq Z$  and where  $T_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$  denotes that with  $\deg(C_2) > Z$  for some  $Z \leq g$ . We first estimate  $T_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$ , which is easier.

#### 3A. The term $T_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$ .

**Lemma 3.2.** *We have*

$$T_{E_1, E_2}(N, X, Y, Z; \alpha, \beta) \ll_\varepsilon q^{2g-3Z/2}g^{6+\varepsilon}$$

uniformly for  $|\alpha|, |\beta| \leq 1/g$ .

*Proof.* It suffices to prove the bound for  $T'_{E_1, E_2}$ , which is

$$\sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1 | \Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{\substack{C_2 | (\Delta fh)^\infty \\ \deg(C_2) > Z}} \sum_{R \in \mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh).$$

We use the Perron formula for the sum over  $f$  and  $h$ . We write  $g(C_2)/(\text{rad}(C_2), \Delta) = C = AB$ , and replace  $f, h$  by  $Af$  and  $Bh$ , respectively, where  $g(f)$  is defined to be  $g(f) = \prod_{P|f} P$ . Then

$$\begin{aligned} T'_{E_1, E_2} &= \sum_{\substack{c_2 > Z \\ c_1 + 2c_2 \leq 2g+1}} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_2 \in \mathcal{M}_{c_2} \\ C_1 | \Delta, C=AB}} \frac{\mu(C_1)\chi_{C_1}(NAB)}{|A|^{1/2+\alpha}|B|^{1/2+\beta}} \sum_{R \in \mathcal{M}_{2g+1-c_1-2c_2}} \chi_R(NAB) \\ &\quad \times \frac{1}{(2\pi i)^2} \oint_{|u|=r} \oint_{|v|=r} \sum_{f, h \in \mathcal{M}} \frac{\chi_{C_1 R}(fh)\lambda_1(Af)\lambda_2(Bh)u^{\deg(f)}v^{\deg(h)}}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \\ &\quad \times \frac{dudv}{u^{X-\deg(A)+1}v^{Y-\deg(B)+1}(1-u)(1-v)} \end{aligned}$$

for any  $r < 1$ . The sum over  $f$  and  $h$  may be written as

$$\mathcal{D}_1(A, B, C_1 R; u, v, \alpha, \beta) \mathcal{L}\left(E_1 \otimes \chi_{C_1 R}, \frac{u}{q^{1/2+\alpha}}\right) \mathcal{L}\left(E_2 \otimes \chi_{C_1 R}, \frac{v}{q^{1/2+\beta}}\right),$$

where  $\mathcal{D}_1(A, B, C_1 R; u, v, \alpha, \beta)$  is some Euler product which is uniformly convergent provided that  $|u|, |v| \leq q^{-1/g}$ , and satisfies

$$\mathcal{D}_1(A, B, C_1 R; u, v, \alpha, \beta) \ll \tau(AB)$$

uniformly in this region. Moving the  $u$  and  $v$  contours to  $|u| = |v| = q^{-1/g}$  and using the bound

$$\begin{aligned} &\left| \mathcal{L}\left(E_1 \otimes \chi_{C_1 R}, \frac{u}{q^{1/2+\alpha}}\right) \mathcal{L}\left(E_2 \otimes \chi_{C_1 R}, \frac{v}{q^{1/2+\beta}}\right) \right| \\ &\ll \left| \mathcal{L}\left(E_1 \otimes \chi_{C_1 R}, \frac{u}{q^{1/2+\alpha}}\right) \right|^2 + \left| \mathcal{L}\left(E_2 \otimes \chi_{C_1 R}, \frac{v}{q^{1/2+\beta}}\right) \right|^2 \end{aligned}$$

we get

$$\begin{aligned} T'_{E_1, E_2} &\ll g^2 \sum_{\substack{c_2 > Z \\ c_1 + 2c_2 \leq 2g+1}} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_2 \in \mathcal{M}_{c_2} \\ C_1 | \Delta}} \frac{\tau(C)^2}{\sqrt{|C|}} \oint_{|u|=q^{-1/g}} \oint_{|v|=q^{-1/g}} \\ &\quad \sum_{R \in \mathcal{M}_{2g+1-c_1-2c_2}} \left( \left| \mathcal{L}\left(E_1 \otimes \chi_{C_1 R}, \frac{u}{q^{1/2+\alpha}}\right) \right|^2 + \left| \mathcal{L}\left(E_2 \otimes \chi_{C_1 R}, \frac{v}{q^{1/2+\beta}}\right) \right|^2 \right) dudv. \end{aligned}$$

Now let  $D = (C_1, R)$ . Write  $R = DR_1 = DEH^2$ , where  $E$  is square-free, and let  $C_1 = DC_3$ . Then

$$\left| \mathcal{L}\left(E_1 \otimes \chi_{C_1 R}, \frac{u}{q^{1/2+\alpha}}\right) \right| \ll_\varepsilon |DH|^\varepsilon \left| \mathcal{L}\left(E_1 \otimes \chi_{C_3 E}, \frac{u}{q^{1/2+\alpha}}\right) \right|.$$

Using upper bounds for moments (see [Remark 4.2](#) after [Theorem 4.1](#)), we get that

$$T'_{E_1, E_2} \ll_{\varepsilon} q^{2g} g^{3+\varepsilon} \sum_{\substack{C_2 \in \mathcal{M} \\ Z < \deg(C_2) \leq g}} \frac{\tau(C_2)^2}{\sqrt{|g(C_2)|C_2^2}} \ll_{\varepsilon} q^{2g-3Z/2} g^{6+\varepsilon},$$

and this finishes the proof of [Lemma 3.2](#). □

**3B. The term  $S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$ .** Define  $S'_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$  to be

$$\sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y}}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \sum_{C_1 | \Delta} \mu(C_1)\chi_{C_1}(Nfh) \sum_{\substack{C_2 | (\Delta fh)^{\infty} \\ \deg(C_2) \leq Z}} \sum_{R \in \mathcal{M}_{2g+1-\deg(C_1)-2\deg(C_2)}} \chi_R(Nfh),$$

and  $S''_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$  to be the same sum with  $g$  being replaced by  $(g - 1)$ . Then

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta) = S'_{E_1, E_2}(N, X, Y, Z; \alpha, \beta) - qS''_{E_1, E_2}(N, X, Y, Z; \alpha, \beta).$$

Using [Lemma 2.6](#) on the sum over  $R$ , it follows that  $S'_{E_1, E_2}(N, X, Y, Z; \alpha, \beta)$  equals

$$\begin{aligned} & q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ \deg(Nfh) \text{ even}}} \frac{\lambda_1(f)\lambda_2(h)}{|N||f|^{3/2+\alpha}|h|^{3/2+\beta}} \sum_{\substack{C_1 | \Delta \\ C_2 | (\Delta fh)^{\infty} \\ \deg(C_2) \leq Z \\ \deg(C_1)+2\deg(C_2) \leq 2g+1}} \frac{\mu(C_1)\chi_{C_1}(Nfh)}{|C_1||C_2|^2} \\ & \times \left( G(0, Nfh) + (q-1) \sum_{V \in \mathcal{M}_{\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-3}} G(V, Nfh) - \sum_{V \in \mathcal{M}_{\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-2}} G(V, Nfh) \right) \\ & + q^{2g+1} \overline{\tau(q)} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ \deg(Nfh) \text{ odd}}} \frac{\lambda_1(f)\lambda_2(h)}{|N||f|^{3/2+\alpha}|h|^{3/2+\beta}} \sum_{\substack{C_1 | \Delta \\ C_2 | (\Delta fh)^{\infty} \\ \deg(C_2) \leq Z \\ \deg(C_1)+2\deg(C_2) \leq 2g+1}} \frac{\mu(C_1)\chi_{C_1}(Nfh)}{|C_1||C_2|^2} \\ & \times \sum_{V \in \mathcal{M}_{\deg(Nfh)+\deg(C_1)+2\deg(C_2)-2g-2}} G(V, Nfh). \end{aligned}$$

Let  $S'_{E_1, E_2}(V = 0)$  denote the terms with  $V = 0$  above and  $S'_{E_1, E_2}(V \neq 0)$  be the terms with nonzero  $V$ . The terms  $S''_{E_1, E_2}(V = 0)$  and  $S''_{E_1, E_2}(V \neq 0)$  are similarly defined. Let

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V = 0) = S'_{E_1, E_2}(V = 0) - qS''_{E_1, E_2}(V = 0)$$

and

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V \neq 0) = S'_{E_1, E_2}(V \neq 0) - qS''_{E_1, E_2}(V \neq 0)$$

so that we have

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta) = S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V = 0) + S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V \neq 0). \quad (3-3)$$

We shall evaluate  $S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V = 0)$  in Section 3C and bound  $S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V \neq 0)$  in Section 3D.

**3C. The  $V = 0$  terms.**

**Lemma 3.3.** *We have*

$$S_E(N, X, Y, Z; V = 0) := S_{E, E}(N, X, Y, Z; 0, 0; V = 0) \\ = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; 1, 1, 1) L(\text{Sym}^2 E, 1)^3 Y + O(q^{2g}) + O(q^{2g-Y/5} g^2),$$

and if  $E_1 \neq E_2$ , then

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V = 0) \\ = |\mathcal{H}_{2g+1}| \mathcal{C}_{E_1, E_2}(N; 1, 1, 1, \alpha, \beta) L(\text{Sym}^2 E_1, 1 + 2\alpha) L(\text{Sym}^2 E_2, 1 + 2\beta) L(E_1 \otimes E_2, 1 + \alpha + \beta) \\ + O(q^{2g-Y/5} g^2) + O(q^{2g-3Z/2})$$

uniformly for  $|\alpha|, |\beta| \leq 1/g$ , where the values  $\mathcal{C}_E(N; 1, 1, 1)$  and  $\mathcal{C}_{E_1, E_2}(N; 1, 1, 1, \alpha, \beta)$  are defined in (3-8), (3-9) and (3-10).

*Proof.* Note that  $G(0, Nfh) \neq 0$  if and only if  $Nfh$  is a square polynomial, and in this case  $G(0, Nfh) = \phi(Nfh)$ . Hence

$$S'_{E_1, E_2}(V = 0) = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ Nfh = \square}} \frac{\lambda_1(f) \lambda_2(h) \phi(Nfh)}{|N| |f|^{3/2+\alpha} |h|^{3/2+\beta}} \sum_{\substack{C_2 | (\Delta fh)^\infty \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} \sum_{\substack{C_1 | \Delta \\ (C_1, Nfh) = 1 \\ \deg(C_1) \leq 2g+1-2 \deg(C_2)}} \frac{\mu(C_1)}{|C_1|}. \quad (3-4)$$

We have

$$\sum_{\substack{C_1 | \Delta \\ (C_1, Nfh) = 1 \\ \deg(C_1) \leq 2g+1-2 \deg(C_2)}} \frac{\mu(C_1)}{|C_1|} = \sum_{(C_1, Nfh) = 1} \frac{\mu(C_1)}{|C_1|} - \sum_{\substack{C_1 | \Delta \\ (C_1, Nfh) = 1 \\ \deg(C_1) > 2g+1-2 \deg(C_2)}} \frac{\mu(C_1)}{|C_1|} \\ = \prod_{\substack{P | \Delta \\ P \nmid Nfh}} \left( 1 - \frac{1}{|P|} \right) + O(q^{-2g} |C_2|^2) \\ = \frac{|Nfh|}{\phi(Nfh)} \prod_{P | \Delta fh} \left( 1 - \frac{1}{|P|} \right) + O(q^{-2g} |C_2|^2). \quad (3-5)$$

Note that

$$\sum_{\substack{C_2 \in \mathcal{M}_n \\ C_2 | (\Delta fh)^\infty}} 1 \ll_\varepsilon q^{\varepsilon n}. \quad (3-6)$$

Let  $N = N_1^2 N_2$  with  $N_2$  being square-free. The condition  $Nfh = \square$  is equivalent to  $fh = N_2 \ell^2$  for some polynomial  $\ell$ . Then we can write  $f = N_2' A$  and  $h = N_2'' B$ , with  $N_2' N_2'' = N_2$  and  $AB = \ell^2$ . It follows

that the contribution of the error term in (3-5) to (3-4) will be

$$\ll_{\varepsilon} q^{\varepsilon g} \sum_{\ell \in \mathcal{M}_{\leq (X+Y)/2}} \frac{1}{|\ell|} \sum_{AB=\ell^2} |\lambda_1(N'_2 A)\lambda_2(N''_2 B)| \ll_{\varepsilon} q^{\varepsilon g},$$

by using the bound  $|\lambda_i(f)| \leq \tau(f) \ll_{\varepsilon} |f|^{\varepsilon}$ . Thus we can rewrite (3-4) as

$$S'_{E_1, E_2}(V=0) = q^{2g+1} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ Nfh=\square}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \prod_{P|\Delta fh} \left(1 - \frac{1}{|P|}\right) \sum_{\substack{C_2 | (\Delta fh)^{\infty} \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} + O_{\varepsilon}(q^{\varepsilon g}).$$

We obtain a similar estimate for  $S''_{E_1, E_2}(V=0)$  with  $g$  being replaced by  $(g-1)$ , and hence

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V=0) = |\mathcal{H}_{2g+1}| \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ Nfh=\square}} \frac{\lambda_1(f)\lambda_2(h)}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \prod_{P|\Delta fh} \left(1 - \frac{1}{|P|}\right) \sum_{\substack{C_2 | (\Delta fh)^{\infty} \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} + O_{\varepsilon}(q^{\varepsilon g}). \quad (3-7)$$

From the Perron formula for the sum over  $C_2$ ,

$$\sum_{\substack{C_2 | (\Delta fh)^{\infty} \\ \deg(C_2) \leq Z}} \frac{1}{|C_2|^2} = \frac{1}{2\pi i} \oint_{|w|=r} \prod_{P|\Delta fh} \left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1} \frac{dw}{w^{Z+1}(1-w)}$$

for any  $r < 1$ , it follows that

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V=0) = \frac{|\mathcal{H}_{2g+1}|}{(2\pi i)^3} \oint_{|u|=r} \oint_{|v|=r} \oint_{|w|=r} \mathcal{A}_{E_1, E_2}(N; u, v, w, \alpha, \beta) \frac{dudvdw}{u^{X+1}v^{Y+1}w^{Z+1}(1-u)(1-v)(1-w)} + O_{\varepsilon}(q^{\varepsilon g}),$$

where

$$\mathcal{A}_{E_1, E_2}(N; u, v, w, \alpha, \beta) = \sum_{\substack{f, h \in \mathcal{M} \\ Nfh=\square}} \frac{\lambda_1(f)\lambda_2(h)u^{\deg(f)}v^{\deg(h)}}{|f|^{1/2+\alpha}|h|^{1/2+\beta}} \prod_{P|\Delta fh} \left(1 - \frac{1}{|P|}\right) \left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1}.$$

We can write down an Euler product for  $\mathcal{A}_{E_1, E_2}(N; u, v, w, \alpha, \beta)$  as follows.

$$\begin{aligned} &\mathcal{A}_{E_1, E_2}(N; u, v, w, \alpha, \beta) \\ &= \prod_{P \nmid \Delta} \left(1 + \left(1 - \frac{1}{|P|}\right) \left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1} \sum_{i+j \text{ even} \geq 2} \frac{\lambda_1(P^i)\lambda_2(P^j)u^i v^j \deg(P)^{i+j}}{|P|^{(1/2+\alpha)i+(1/2+\beta)j}}\right) \\ &\quad \times \prod_{P|\Delta} \left(\left(1 - \frac{1}{|P|}\right) \left(1 - \frac{w^{\deg(P)}}{|P|^2}\right)^{-1} \sum_{\substack{i, j \\ i+j+\text{ord}_P(N) \text{ even}}} \frac{\lambda_1(P^i)\lambda_2(P^j)u^i v^j \deg(P)^{i+j}}{|P|^{(1/2+\alpha)i+(1/2+\beta)j}}\right). \quad (3-8) \end{aligned}$$

Then

$$\begin{aligned} \mathcal{A}_E(N; u, v, w, 0, 0) &:= \mathcal{A}_{E,E}(N; u, v, w, 0, 0) \\ &= \mathcal{C}_E(N; u, v, w) \mathcal{L}\left(\text{Sym}^2 E, \frac{u^2}{q}\right) \mathcal{L}\left(\text{Sym}^2 E, \frac{v^2}{q}\right) \times \mathcal{L}\left(\text{Sym}^2 E, \frac{uv}{q}\right) \mathcal{Z}\left(\frac{uv}{q}\right), \end{aligned} \quad (3-9)$$

and

$$\mathcal{A}_{E_1, E_2}(N; u, v, w, \alpha, \beta) = \mathcal{C}_{E_1, E_2}(N; u, v, w, \alpha, \beta) \mathcal{L}\left(\text{Sym}^2 E_1, \frac{u^2}{q^{1+2\alpha}}\right) \mathcal{L}\left(\text{Sym}^2 E_2, \frac{v^2}{q^{1+2\beta}}\right) \mathcal{L}\left(E_1 \otimes E_2, \frac{uv}{q^{1+\alpha+\beta}}\right), \quad (3-10)$$

if  $E_1 \neq E_2$ , where  $\mathcal{C}_E(N; u, v, w)$  and  $\mathcal{C}_{E_1, E_2}(N; u, v, w, \alpha, \beta)$  are some Euler products which are uniformly bounded for example when  $|u|, |v| \leq q^{1/5}, |w| \leq q^{3/2}$ .

Consider the case  $E_1 = E_2 = E$ . We have

$$\begin{aligned} S_E(N, X, Y, Z; V = 0) &= \frac{|\mathcal{H}_{2g+1}|}{(2\pi i)^3} \oint_{|u|=r} \oint_{|v|=r} \oint_{|w|=r} \mathcal{C}_E(N; u, v, w) \mathcal{L}\left(\text{Sym}^2 E, \frac{u^2}{q}\right) \\ &\quad \times \mathcal{L}\left(\text{Sym}^2 E, \frac{v^2}{q}\right) \mathcal{L}\left(\text{Sym}^2 E, \frac{uv}{q}\right) \frac{dudvdw}{u^{X+1}v^{Y+1}w^{Z+1}(1-u)(1-v)(1-w)(1-uv)} + O_\varepsilon(q^{\varepsilon g}) \end{aligned} \quad (3-11)$$

for any  $r < 1$ . We choose  $r = q^{-1/g}$  and move the  $u$  contour to  $|u| = q^{1/5}$ , encountering two simple poles at  $u = 1$  and  $u = 1/v$ . The new integral is trivially bounded by  $O(q^{2g-X/5}g^2)$ .

Furthermore, the contribution from the residue at  $u = 1/v$  is

$$\begin{aligned} -|\mathcal{H}_{2g+1}| L(\text{Sym}^2 E, 1) \frac{1}{(2\pi i)^2} \oint_{|v|=q^{-1/g}} \oint_{|w|=q^{-1/g}} \mathcal{C}_E(N; 1/v, v, w) \\ \times \mathcal{L}\left(\text{Sym}^2 E, \frac{1}{v^2q}\right) \mathcal{L}\left(\text{Sym}^2 E, \frac{v^2}{q}\right) \frac{dvdw}{v^{Y-X}w^{Z+1}(1-v)^2(1-w)}, \end{aligned}$$

which is  $O(q^{2g})$ . This can be seen by first moving the  $v$  contour to  $|v| = q^{-1/5}$ , creating no poles, and then moving the  $w$  contour to  $|w| = q^{3/2}$ , crossing a simple pole at  $w = 1$ . Both the new integral and the residue at  $w = 1$  are  $O(q^{2g})$  as  $X \geq Y$ . So

$$\begin{aligned} S_E(N, X, Y, Z; V = 0) &= |\mathcal{H}_{2g+1}| L(\text{Sym}^2 E, 1) \frac{1}{(2\pi i)^2} \oint_{|v|=q^{-1/g}} \oint_{|w|=q^{-1/g}} \mathcal{C}_E(N; 1, v, w) \\ &\quad \times \mathcal{L}\left(\text{Sym}^2 E, \frac{v^2}{q}\right) \mathcal{L}\left(\text{Sym}^2 E, \frac{v}{q}\right) \frac{dvdw}{v^{Y+1}w^{Z+1}(1-v)^2(1-w)} \\ &\quad + O(q^{2g}) + O(q^{2g-X/5}g^2). \end{aligned}$$

We now move the  $v$  contour to  $|v| = q^{1/5}$ , encountering a double pole at  $v = 1$ . The new integral is bounded by  $O(q^{2g-Y/5}g)$ , and an argument similar to the above implies that the residue at  $v = 1$  is

$$|\mathcal{H}_{2g+1}| L(\text{Sym}^2 E, 1)^3 Y \frac{1}{2\pi i} \oint_{|w|=q^{-1/g}} \mathcal{C}_E(N; 1, 1, w) \frac{dw}{w^{Z+1}(1-w)} + O(q^{2g}).$$

Hence

$$S_E(N, X, Y, Z; V = 0) = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; 1, 1, 1) L(\text{Sym}^2 E, 1)^3 Y + O(q^{2g}) + O(q^{2g-Y/5} g^2).$$

For  $E_1 \neq E_2$ , we have that

$$\begin{aligned} S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V = 0) &= \frac{|\mathcal{H}_{2g+1}|}{(2\pi i)^3} \oint_{|u|=r} \oint_{|v|=r} \oint_{|w|=r} \mathcal{C}_{E_1, E_2}(N; u, v, w, \alpha, \beta) \mathcal{L}\left(\text{Sym}^2 E_1, \frac{u^2}{q^{1+2\alpha}}\right) \mathcal{L}\left(\text{Sym}^2 E_2, \frac{v^2}{q^{1+2\beta}}\right) \\ &\quad \times \mathcal{L}\left(E_1 \otimes E_2, \frac{uv}{q^{1+\alpha+\beta}}\right) \frac{dudvdw}{u^{X+1}v^{Y+1}w^{Z+1}(1-u)(1-v)(1-w)} + O_\varepsilon(q^{\varepsilon g}) \end{aligned}$$

for any  $r < 1$ . We choose  $r = q^{-1/g}$  and first shift the  $u$  contour to  $|u| = q^{1/5}$ , encountering a pole at  $u = 1$ . The new integral over  $|u| = q^{1/5}$ ,  $|v| = |w| = q^{-1/g}$  is bounded by  $q^{2g-X/5} g^2$ . To calculate the residue at  $u = 1$ , we move the  $v$  contour to  $|v| = q^{1/5}$ , crossing a pole at  $v = 1$ . The new integral is  $O(q^{2g-Y/5} g)$ . For the residue at  $u = v = 1$ , we move the  $w$  contour to  $|w| = q^{3/2}$ . In doing so we obtain

$$\begin{aligned} S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V = 0) &= |\mathcal{H}_{2g+1}| \mathcal{C}_{E_1, E_2}(N; 1, 1, 1, \alpha, \beta) L(\text{Sym}^2 E_1, 1 + 2\alpha) \\ &\quad \times L(\text{Sym}^2 E_2, 1 + 2\beta) L(E_1 \otimes E_2, 1 + \alpha + \beta) \\ &\quad + O(q^{2g-Y/5} g^2) + O(q^{2g-3Z/2}), \end{aligned}$$

and this concludes the proof of the lemma. □

### 3D. The $V \neq 0$ terms.

**Lemma 3.4.** *We have*

$$S_{E_1, E_2}(N, X, Y, Z; \alpha, \beta; V \neq 0) \ll q^{(X+Y+Z)/2} g^{30}$$

uniformly for  $|\alpha|, |\beta| \leq 1/g$ .

*Proof.* We will prove the bound for the term

$$\begin{aligned} S(V \neq 0) &= q^{2g+1} \overline{\tau(q)} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq Y} \\ \deg(Nfh) \text{ odd}}} \frac{\lambda_1(f) \lambda_2(h)}{|N| |f|^{3/2+\alpha} |h|^{3/2+\beta}} \sum_{\substack{C_1 | \Delta \\ C_2 | (\Delta fh)^\infty \\ \deg(C_2) \leq Z \\ \deg(C_1) + 2 \deg(C_2) \leq 2g+1}} \frac{\mu(C_1) \chi_{C_1}(Nfh)}{|C_1| |C_2|^2} \\ &\quad \times \sum_{V \in \mathcal{M}_{\deg(Nfh) + \deg(C_1) + 2 \deg(C_2) - 2g - 2}} G(V, Nfh), \quad (3-12) \end{aligned}$$

the treatment of the other terms being similar. We also assume for simplicity that  $\deg(N)$ ,  $X$  and  $Y$  are all odd. The other cases can be done similarly.

We use the Perron formula in the forms

$$\sum_{\substack{f \in \mathcal{M}_{\leq X} \\ \deg(f) \text{ odd}}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \left( \sum_{f \in \mathcal{M}} a(f) u^{\deg(f)} \right) \frac{du}{u^{X+1}(1-u^2)}$$

and

$$\sum_{\substack{f \in \mathcal{M}_{\leq X} \\ \deg(f) \text{ even}}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \left( \sum_{f \in \mathcal{M}} a(f) u^{\deg(f)} \right) \frac{du}{u^X(1-u^2)}$$

for the sums over  $f$  and  $h$ . We write  $V = V_1 V_2^2$  with  $V_1$  being a square-free polynomial and  $V_2 \in \mathcal{M}$ ,  $g(C_2)/(\text{rad}(C_2), \Delta) = C = AB$ , and replace  $f, h$  by  $Af$  and  $Bh$ , respectively. We then see that

$$\begin{aligned} S(V \neq 0) &= q^{2g+1} \overline{\tau(q)} \sum_{\substack{c_2 \leq Z \\ c_1+2c_2 \leq 2g+1}} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_2 \in \mathcal{M}_{c_2} \\ C_1 | \Delta, C=AB}} \frac{\mu(C_1) \chi_{C_1}(NAB)}{|N||A|^{3/2+\alpha} |B|^{3/2+\beta} |C_1||C_2|^2} \\ &\times \frac{1}{(2\pi i)^3} \oint_{|u|=r} \oint_{|v|=r} \oint_{|w|=r} \sum_{V_1 \in \mathcal{H}} \mathcal{B}(N, A, B, C_1, V_1; u/w, v/w, w, \alpha, \beta) \\ &\times \frac{(1+uv)dudvdw}{u^{X-\deg(A)+1} v^{Y-\deg(B)+1} w^{\deg(NAB)-\deg(V_1)+c_1+2c_2-2g-1} (1-u^2)(1-v^2)} \end{aligned} \tag{3-13}$$

for any  $r < 1$ , where  $\mathcal{B}(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$  equals

$$\sum_{f, h, V_2 \in \mathcal{M}} \frac{\chi_{C_1}(fh) \lambda_1(Af) \lambda_2(Bh) u^{\deg(f)} v^{\deg(h)} w^{2\deg(V_2)} G(V_1 V_2^2, NABfh)}{|f|^{3/2+\alpha} |h|^{3/2+\beta}}$$

To proceed we need to study the function  $\mathcal{B}(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$ .

**Lemma 3.5.** *The function  $\mathcal{B}(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$  defined above may be written as*

$$\mathcal{D}_2(N, A, B, C_1, V_1; u, v, w, \alpha, \beta) \mathcal{L}\left(E_1 \otimes \chi_{C_1 V_1}, \frac{u}{q^{1+\alpha}}\right) \mathcal{L}\left(E_2 \otimes \chi_{C_1 V_1}, \frac{v}{q^{1+\beta}}\right),$$

where  $\mathcal{D}_2(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$  is some Euler product which is uniformly convergent provided that  $|u|, |v| \leq q^{1/2-2/g}$ ,  $|w| \leq q^{-1/2-\varepsilon}$ , and satisfies

$$\mathcal{D}_2(N, A, B, C_1, V_1; u, v, w, \alpha, \beta) \ll g^{10} \tau_{10}(AB) \sqrt{AB}$$

uniformly in this region.

*Proof.* It is easy to see that  $\mathcal{B}(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$  converges absolutely if  $|u|, |v| \leq q^{-\varepsilon}$  and  $|w| \leq q^{-1/2-\varepsilon}$ . We claim that the sum over  $f, h$  and  $V_2$  is triply multiplicative. Indeed, one can easily see

that the double sum over  $f, h$  is multiplicative, so

$$\sum_{f, h \in \mathcal{M}} \frac{\chi_{C_1}(fh)\lambda_1(Af)\lambda_2(Bh)u^{\deg(f)}v^{\deg(h)}G(V_1V_2^2, NABfh)}{|f|^{3/2+\alpha}|h|^{3/2+\beta}}$$

$$= \prod_P \left( \sum_{i, j} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^{i+a_P})\lambda_2(P^{j+b_P})u^{i \deg(P)}v^{j \deg(P)}G(V_1V_2^2, P^{i+j+a_P+b_P+n_P})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}} \right),$$

where  $a_P, b_P$  and  $n_P$  denote the orders of  $A, B$  and  $N$  with respect to  $P$  respectively. Let  $A_P(V_2)$  denote the Euler product above. Note that when  $P \nmid V_2$ , we have  $A_P(V_2) = A_P(1)$ . Then we rewrite the double sum over  $f, h$  as

$$\prod_P A_P(1) \prod_{P \mid V_2} \frac{A_P(V_2)}{A_P(1)}.$$

We introduce the sum over  $V_2$  and use the observation that for  $(V_2, V_3) = 1$  and  $P \nmid V_3$  we have  $A_P(V_2V_3) = A_P(V_2)$ . Then

$$\prod_P A_P(1) \sum_{V_2 \in \mathcal{M}} w^{2 \deg(V_2)} \prod_{P \mid V_2} \frac{A_P(V_2)}{A_P(1)}$$

$$= \prod_P A_P(1) \prod_P \left( 1 + \frac{1}{A_P(1)} \sum_k A_P(P^k) w^{2k \deg(P)} \right)$$

$$= \prod_P \left( \sum_{i, j, k} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^{i+a_P})\lambda_2(P^{j+b_P})u^{i \deg(P)}v^{j \deg(P)}w^{2k \deg(P)}G(V_1P^{2k}, P^{i+j+a_P+b_P+n_P})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}} \right),$$

and hence the generating series for  $f, h, V_2$  is indeed triply multiplicative.

Now we rewrite  $\mathcal{B}(N, A, B, C_1, V_1; u, v, w, \alpha, \beta)$  as

$$\prod_{P \nmid C_1} \left( \sum_{i, j, k} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^{i+a_P})\lambda_2(P^{j+b_P})u^{i \deg(P)}v^{j \deg(P)}w^{2k \deg(P)}G(V_1P^{2k}, P^{i+j+a_P+b_P+n_P})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}} \right)$$

$$\times \prod_{P \mid C_1} \left( \sum_k \lambda_1(P^{a_P})\lambda_2(P^{b_P})w^{2k \deg(P)}G(V_1P^{2k}, P^{a_P+b_P+n_P}) \right).$$

We next compute the Euler factors at an irreducible  $P$  in the region  $|u|, |v| \leq q^{1/2-2/g}, |w| \leq q^{-1/2-\varepsilon}$ . Note that in this region,  $w^{2k \deg(P)} \ll_\varepsilon |P|^{-1-\varepsilon}$  if  $k \geq 1$ .

Consider first the case when  $P \nmid NABC_1V_1$ . The contribution of such an Euler factor is

$$\sum_{i, j, k} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^i)\lambda_2(P^j)u^{i \deg(P)}v^{j \deg(P)}w^{2k \deg(P)}G(V_1P^{2k}, P^{i+j})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}}.$$

In view of [Lemma 2.5](#), this is equal to

$$1 + \frac{\chi_{C_1V_1}(P)\lambda_1(P)u^{\deg(P)}}{|P|^{1+\alpha}} + \frac{\chi_{C_1V_1}(P)\lambda_2(P)v^{\deg(P)}}{|P|^{1+\beta}} + O\left(\frac{1}{|P|^{1+\varepsilon}}\right),$$

which justifies the two  $L$ -functions.

In the case  $P \mid V_1$  but  $P \nmid NABC_1$ , the Euler factor equals

$$\begin{aligned} & \sum_{i,j} \frac{\chi_{C_1}(P^{i+j})\lambda_1(P^i)\lambda_2(P^j)u^{i \deg(P)}v^{j \deg(P)}G(V_1, P^{i+j})}{|P|^{(3/2+\alpha)i+(3/2+\beta)j}} + O\left(\frac{1}{|P|^{1+\varepsilon}}\right) \\ &= 1 - \frac{\lambda_1(P^2)u^{2 \deg(P)}}{|P|^{2+2\alpha}} - \frac{\lambda_1(P)\lambda_2(P)(uv)^{\deg(P)}}{|P|^{2+\alpha+\beta}} - \frac{\lambda_2(P^2)v^{2 \deg(P)}}{|P|^{2+2\beta}} + O\left(\frac{1}{|P|^{1+\varepsilon}}\right) \\ &= 1 + O\left(\frac{1}{|P|^{1+2/g}}\right). \end{aligned}$$

Similarly, the corresponding Euler factor is

$$= \begin{cases} O(|P|^{1/2}) & \text{if } P \mid AB \text{ and } P \nmid NC_1V_1, \\ O(1) & \text{if } P \mid AB, P \mid V_1 \text{ and } P \nmid NC_1, \\ 1 + O(1/|P|^{1+\varepsilon}) & \text{if } P \mid C_1 \text{ and } P \nmid NAB, \\ O(|P|^{1/2}) & \text{if } P \mid C_1, P \mid AB \text{ and } P \nmid NV_1, \\ 0 & \text{if } P \mid C_1, P \mid AB, P \mid V_1 \text{ and } P \nmid N, \end{cases}$$

and is

$$\ll_{\varepsilon} \sum_{i+j \leq 2k} |P|^{1-(1+\varepsilon)k-(i+j)/2} \ll_{\varepsilon} |P|,$$

if  $P \mid N$ .

The lemma easily follows by combining these estimates. □

We now return to (3-13). In view of Lemma 3.5, we take  $r \leq q^{-1/2-\varepsilon}$  and move the  $u$  and  $v$  contours to  $|u| = |v| = rq^{1/2-2/g}$ . This creates no poles. Then, by the above result,

$$\begin{aligned} & \mathcal{B}(N, A, B, C_1, V_1; u/w, v/w, w, \alpha, \beta) \\ & \ll g^{10} \tau_{10}(AB)\sqrt{AB} \left| \mathcal{L}\left(E_1 \otimes \chi_{C_1V_1}, \frac{u/w}{q^{1+\alpha}}\right) \mathcal{L}\left(E_2 \otimes \chi_{C_1V_1}, \frac{v/w}{q^{1+\beta}}\right) \right| \\ & \ll g^{10} \tau_{10}(AB)\sqrt{AB} \left( \left| \mathcal{L}\left(E_1 \otimes \chi_{C_1V_1}, \frac{u/w}{q^{1+\alpha}}\right) \right|^2 + \left| \mathcal{L}\left(E_2 \otimes \chi_{C_1V_1}, \frac{v/w}{q^{1+\beta}}\right) \right|^2 \right). \end{aligned}$$

So

$$\begin{aligned} & S(V \neq 0) \\ & \ll q^{2g-(X+Y)/2} g^{10} \sum_{\substack{c_2 \leq Z \\ c_1+2c_2 \leq 2g+1}} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_2 \in \mathcal{M}_{c_2} \\ C_1 \mid \Delta}} \frac{\tau(C)\tau_{10}(C)\tau(C_1)}{\sqrt{|C||C_1||C_2|^2}} \\ & \times \oint_{|u|=q^{1/2-1/g}} \oint_{|v|=q^{1/2-1/g}} \oint_{|w|=r} \sum_{V_1 \in \mathcal{H}} \left| \mathcal{L}\left(E_1 \otimes \chi_{C_1V_1}, \frac{u}{q^{1+\alpha}}\right) \right|^2 \frac{dudvdw}{|w|^{X+Y+\deg(N)-\deg(V_1)+c_1+2c_2-2g+1}}. \end{aligned}$$

If  $\deg(V_1) \leq X + Y + c_1 + 2c_2 - 2g$ , then we move the  $w$  contour to  $|w| = q^{-3/4}$ , otherwise we move the  $w$  contour to  $|w| = q^{-5/4}$ . Using the upper bounds for moments as in Theorem 4.1 (see Remark 4.2) we

find that

$$\begin{aligned}
 S(V \neq 0) &\ll q^{(X+Y)/2} g^{11} \sum_{\substack{c_2 \leq Z \\ c_1 + 2c_2 \leq 2g+1}} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_2 \in \mathcal{M}_{c_2} \\ C_1 | \Delta}} \frac{\tau(C) \tau_{10}(C)}{\sqrt{|C|}} \\
 &\ll q^{(X+Y)/2} g^{11} \sum_{C_2 \in \mathcal{M}_{\leq Z}} \frac{\tau(C_2) \tau_{10}(C_2)}{\sqrt{|g(C_2)|}} \ll q^{(X+Y+Z)/2} g^{30},
 \end{aligned}$$

which finishes the proof of [Lemma 3.4](#). □

[Proposition 3.1](#) follows upon combining the estimates and choosing  $Z = g - (X + Y)/4$ .

### 4. Upper bounds for moments

The aim of this section is to bound the tails of the Dirichlet series in the approximate functional equations in [Lemmas 2.2](#) and [2.3](#). We start with the following upper bounds for moments.

**Theorem 4.1.** *Let  $k > 0$ ,  $u = e^{i\theta}$ ,  $v = e^{i\gamma}$  with  $\theta, \gamma \in [0, 2\pi]$  and let  $m = \deg(\mathcal{L}(E \otimes \chi_D, w))$ . Then*

$$\sum_{D \in \mathcal{H}_{2g+1}^*} \left| \mathcal{L}\left(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{q^{1/2+\beta}}\right) \right|^k \ll_{\varepsilon} q^{2g} g^{\varepsilon} \exp\left(k\mathcal{M}(u, v, m) + \frac{k^2}{2}\mathcal{V}(u, v, m)\right)$$

uniformly for  $|\alpha|, |\beta| \leq 1/g$ , where  $\mathcal{M}(u, v, m)$  and  $\mathcal{V}(u, v, m)$  are given by [\(4-9\)](#) and [\(4-11\)](#) respectively.

**Remark 4.2.** Note that the same upper bound as above holds if we replace  $\mathcal{L}(E \otimes \chi_D, w)$  with  $\mathcal{L}(E \otimes \chi_{D\ell}, w)$  for a fixed polynomial  $\ell$  with  $(\ell, \Delta) = 1$ . Since the proof of the upper bound for this twisted moment is the same as the proof of [Theorem 4.1](#), we only focus on  $\ell = 1$ .

We first need the following proposition, whose proof is similar to the proof of [Theorem 3.3](#) in [\[Altuğ and Tsimerman 2014\]](#).

**Proposition 4.3.** *Let  $D \in \mathcal{H}_{2g+1}^*$  and let  $m = \deg(\mathcal{L}(E \otimes \chi_D, w))$ . Then for  $h \leq m$  and  $z$  with  $\Re(z) \geq 0$  we have*

$$\log |L(E \otimes \chi_D, \frac{1}{2} + z)| \leq \frac{m}{h} + \frac{1}{h} \Re\left( \sum_{\substack{j \geq 1 \\ \deg(P^j) \leq h}} \frac{\chi_D(P^j)(\alpha(P)^j + \beta(P)^j) \log q^{h-j \deg(P)}}{|P|^j(1/2+z+1/(h \log q)) \log q^j} \right).$$

*Proof.* We write

$$L(E \otimes \chi_D, s) = \prod_{j=1}^m (1 - \alpha_j q^{1/2-s}),$$

where  $|\alpha_j| = 1$  (see [\[Hall et al. 2017\]](#)). Then

$$\frac{L'}{L}(E \otimes \chi_D, s) = \log q \left( -\frac{m}{2} + \sum_{j=1}^m \left( \frac{1}{1 - \alpha_j q^{1/2-s}} - \frac{1}{2} \right) \right). \tag{4-1}$$

We put  $s = \sigma + z$  and integrate (4-1) with respect to  $\sigma$  from  $\frac{1}{2}$  to  $\sigma_0$ , where  $\sigma_0 > \frac{1}{2}$ . Taking real parts gives

$$\begin{aligned} & \log|L(E \otimes \chi_D, \frac{1}{2} + z)| - \log|L(E \otimes \chi_D, \sigma_0 + z)| \\ &= \frac{m \log q}{2} \left( \sigma_0 - \frac{1}{2} \right) - \frac{1}{2 \log q} \sum_{j=1}^m \log \frac{q^{\sigma_0 - 1/2} - 2q^{-\Re(z)} \cos(\theta_j - \log q \Im(z)) + q^{1/2 - \sigma_0 - 2\Re(z)}}{1 + q^{-2\Re(z)} - 2q^{-\Re(z)} \cos(\theta_j - \log q \Im(z))}, \end{aligned}$$

where  $\alpha_j = e^{i\theta_j}$ . We use the inequality  $\log(1 + x) \geq x/(1 + x)$  for  $x > 0$  and get that

$$\begin{aligned} & \log|L(E \otimes \chi_D, \frac{1}{2} + z)| - \log|L(E \otimes \chi_D, \sigma_0 + z)| \\ & \leq \frac{m \log q}{2} \left( \sigma_0 - \frac{1}{2} \right) - \frac{1}{2 \log q} \sum_{j=1}^m \frac{(1 - q^{1/2 - \sigma_0 - 2\Re(z)})(1 - q^{1/2 - \sigma_0})}{1 - 2q^{1/2 - \sigma_0 - \Re(z)} \cos(\theta_j - \log q \Im(z)) + q^{1 - 2\sigma_0 - 2\Re(z)}} \\ & = \frac{m \log q}{2} \left( \sigma_0 - \frac{1}{2} \right) - \frac{G_z(\sigma_0)(1 - q^{1/2 - \sigma_0 - 2\Re(z)})(1 - q^{1/2 - \sigma_0})}{(1 - q^{1 - 2\sigma_0 - 2\Re(z)}) \log q}, \end{aligned} \tag{4-2}$$

where

$$G_z(\sigma) = \frac{1 - q^{1 - 2\sigma - 2\Re(z)}}{2} \sum_{j=1}^m \frac{1}{1 - 2q^{1/2 - \sigma - \Re(z)} \cos(\theta_j - \log q \Im(z)) + q^{1 - 2\sigma - 2\Re(z)}}. \tag{4-3}$$

Now similarly as in [Altuğ and Tsimmerman 2014] we compute the integral

$$\frac{1}{2\pi i} \int_2^{2+2\pi i/\log q} -\frac{L'}{L}(E \otimes \chi_D, \sigma + z + w) \frac{q^{hw} q^{-w}}{(1 - q^{-w})^2} dw$$

in two different ways. First we write

$$-\frac{L'}{L}(E \otimes \chi_D, s) = \sum_{n \geq 0} \frac{\lambda_D(n)}{q^{ns}},$$

and integrate term by term. Secondly we continue analytically to the left and pick up the residues. We integrate with respect to  $\sigma$  from  $\sigma_0$  to  $\infty$  and take real parts, which gives

$$\begin{aligned} & -\frac{1}{(\log q)^2} \Re \left( \sum_{n \leq h} \frac{\lambda_D(n) \log q^{h-n}}{q^{n(\sigma_0+z)} \log q^n} \right) \\ &= -\frac{h}{\log q} \log|L(E \otimes \chi_D, \sigma_0 + z)| - \frac{1}{(\log q)^2} \Re \left( \frac{L'}{L}(E \otimes \chi_D, \sigma_0 + z) \right) \\ & \quad + \sum_{j=1}^m \Re \left( \int_{\sigma_0}^{\infty} \frac{(\alpha_j q^{1/2 - \sigma_0 - z})^h \alpha_j^{-1} q^{\sigma+z-1/2}}{(1 - \alpha_j^{-1} q^{\sigma+z-1/2})^2} d\sigma \right). \end{aligned} \tag{4-4}$$

Now we have

$$\Re\left(\int_{\sigma_0}^{\infty} \frac{(\alpha_j q^{1/2-\sigma_0-z})^h \alpha_j^{-1} q^{\sigma+z-1/2}}{(1-\alpha_j^{-1} q^{\sigma+z-1/2})^2} d\sigma\right) \leq \int_{\sigma_0}^{\infty} \frac{q^{(1/2-\sigma-\Re(z))h} q^{\sigma+\Re(z)-1/2}}{|1-\alpha_j^{-1} q^{\sigma+z-1/2}|^2} d\sigma$$

$$= \int_{\sigma_0}^{\infty} \frac{q^{(1/2-\sigma-\Re(z))h} q^{1/2-\sigma-\Re(z)}}{1-2q^{1/2-\sigma-\Re(z)} \cos(\theta_j - \log q \Im(z)) + q^{1-2\sigma-2\Re(z)}} d\sigma.$$

By taking the derivative of

$$f(x) = \frac{q^{-x}}{1-2q^{1/2-x-\Re(z)} \cos(\theta_j - \log q \Im(z)) + q^{1-2x-2\Re(z)}}$$

we can see that  $f$  is decreasing on  $[\sigma_0, \infty)$ . Hence

$$\sum_{j=1}^m \Re\left(\int_{\sigma_0}^{\infty} \frac{(\alpha_j q^{1/2-\sigma_0-z})^h \alpha_j^{-1} q^{\sigma+z-1/2}}{(1-\alpha_j^{-1} q^{\sigma+z-1/2})^2} d\sigma\right)$$

$$\leq \sum_{j=1}^m \frac{q^{1/2-\sigma_0-\Re(z)}}{1-2q^{1/2-\sigma_0-\Re(z)} \cos(\theta_j - \log q \Im(z)) + q^{1-2\sigma_0-2\Re(z)}} \int_{\sigma_0}^{\infty} q^{(1/2-\sigma-\Re(z))h} d\sigma$$

$$= \frac{2G_z(\sigma_0) q^{(1/2-\sigma_0-\Re(z))h} q^{1/2-\sigma_0-\Re(z)}}{(1-q^{1-2\sigma_0-2\Re(z)}) \log q^h}$$

$$\leq \frac{2G_z(\sigma_0) q^{(1/2-\sigma_0-\Re(z))h}}{(\sigma_0 + \Re(z) - 1/2) \log q \log q^h},$$

with  $G_z(\sigma_0)$  as in (4-3). Now from (4-1) note that

$$\Re\left(\frac{L'}{L}(E \otimes \chi_D, \sigma_0 + z)\right) = \log q \left(-\frac{m}{2} + G_z(\sigma_0)\right).$$

Combining the equations above and (4-4) gives

$$\log|L(E \otimes \chi_D, \sigma_0 + z)|$$

$$\leq \frac{1}{\log q^h} \Re\left(\sum_{n \leq h} \frac{\lambda_D(n) \log q^{h-n}}{q^{n(\sigma_0+z)} \log q^n}\right) + \frac{1}{h} \left(\frac{m}{2} - G_z(\sigma_0)\right) + \frac{2G_z(\sigma_0) q^{(1/2-\sigma_0-\Re(z))h}}{h(\sigma_0 + \Re(z) - 1/2) \log q^h}.$$

This and (4-2) lead to

$$\log|L(E \otimes \chi_D, \frac{1}{2} + z)|$$

$$\leq \frac{m}{2h} + \frac{m \log q}{2} \left(\sigma_0 - \frac{1}{2}\right) + \frac{1}{\log q^h} \Re\left(\sum_{n \leq h} \frac{\lambda_D(n) \log q^{h-n}}{q^{n(\sigma_0+z)} \log q^n}\right)$$

$$+ G_z(\sigma_0) \left(-\frac{1}{h} + \frac{2q^{(1/2-\sigma_0-\Re(z))h}}{h(\sigma_0 + \Re(z) - 1/2) \log q^h} - \frac{(1-q^{1/2-\sigma_0-2\Re(z)}) (1-q^{1/2-\sigma_0})}{(1-q^{1-2\sigma_0-2\Re(z)}) \log q}\right).$$

Choosing  $\sigma_0 = \frac{1}{2} + 1/\log q^h$  ensures that the coefficient of  $G_z(\sigma_0)$  is negative. Since

$$\lambda_D(n) = \log q \sum_{j|n} \sum_{\deg(P^j)=n} \deg(P) \chi_D(P)^j (\alpha(P)^j + \beta(P)^j),$$

the conclusion follows. □

Before proving [Theorem 4.1](#), we also need the following lemma (see Lemma 8.4 in [\[Florea 2017a\]](#)).

**Lemma 4.4.** *Let  $h, l$  be integers such that  $2hl \leq 2g + 1$ . For any complex numbers  $a(P)$  we have*

$$\sum_{D \in \mathcal{H}_{2g+1}^*} \left| \sum_{\deg(P) \leq h} \frac{\chi_D(P) a(P)}{|P|^{1/2}} \right|^{2l} \ll q^{2g} \frac{(2l)!}{l!2^l} \left( \sum_{\deg(P) \leq h} \frac{|a(P)|^2}{|P|} \right)^l.$$

Let

$$N(V, u, v) = \left| \left\{ D \in \mathcal{H}_{2g+1}^* : \log \left| \mathcal{L} \left( E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}} \right) \mathcal{L} \left( E \otimes \chi_D, \frac{v}{q^{1/2+\beta}} \right) \right| \geq V + \mathcal{M}(u, v, m) \right\} \right|.$$

We will prove the following lemma.

**Lemma 4.5.** *If  $\sqrt{\log m} \leq V \leq \mathcal{V}(u, v, m)$ , then*

$$N(V, u, v) \ll q^{2g+1} \exp \left( -\frac{V^2}{2\mathcal{V}(u, v, m)} \left( 1 - \frac{8}{\log \log m} \right) \right);$$

*if  $\mathcal{V}(u, v, m) < V \leq \frac{1}{13} \log \log m \mathcal{V}(u, v, m)$ , then*

$$N(V, u, v) \ll q^{2g+1} \exp \left( -\frac{V^2}{2\mathcal{V}(u, v, m)} \left( 1 - \frac{8V}{\mathcal{V}(u, v, m) \log \log m} \right)^2 \right);$$

*if  $V > \frac{1}{13} \log \log m \mathcal{V}(u, v, m)$ , then*

$$N(V, u, v) \ll q^{2g+1} \exp \left( -\frac{V \log V}{4500} \right).$$

Using [Lemma 4.5](#) above we can prove [Theorem 4.1](#) as follows.

*Proof of [Theorem 4.1](#).* We have the following.

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1}^*} \left| \mathcal{L} \left( E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}} \right) \mathcal{L} \left( E \otimes \chi_D, \frac{v}{q^{1/2+\beta}} \right) \right|^k &= - \int_{-\infty}^{\infty} \exp(kV + k\mathcal{M}(u, v, m)) dN(V, u, v) \\ &= k \int_{-\infty}^{\infty} \exp(kV + k\mathcal{M}(u, v, m)) N(V, u, v) dV. \end{aligned}$$

In the equation above we use [Lemma 4.5](#) in the form

$$N(V, u, v) \ll \begin{cases} q^{2g+1} m^{o(1)} \exp(-V^2/2\mathcal{V}(u, v, m)) & \text{if } V \leq 8k\mathcal{V}(u, v, m), \\ q^{2g+1} m^{o(1)} \exp(-4kV) & \text{if } V > 8k\mathcal{V}(u, v, m), \end{cases}$$

which finishes the proof of [Theorem 4.1](#). □

*Proof of Lemma 4.5.* We assume without loss of generality that  $\alpha, \beta$  are positive and real. Indeed, notice that if  $\alpha \in \mathbb{C}$ , since  $|\alpha| \leq 1/g$ , we have  $|\Re(\alpha)| \leq 1/g$  and  $|\Im(\alpha)| \leq 1/g$ . The proof that follows goes through in exactly the same way, with  $\alpha$  replaced by  $\Re(\alpha)$  and  $\theta$  replaced by  $\theta - \Im(\alpha) \log q$ . Once we assume  $\alpha$  is real, we can also assume that  $\alpha$  is positive, since by the functional equation we have

$$\left| \mathcal{L}\left(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\right) \right| = q^{-\alpha(n+2\deg(D))} \left| \mathcal{L}\left(E \otimes \chi_D, \frac{1}{q^{1/2-\alpha}u}\right) \right|.$$

Let

$$\frac{m}{h} = \frac{V}{A} \quad \text{and} \quad h_0 = \frac{h}{\log m},$$

where

$$A = \begin{cases} \frac{1}{2} \log \log m & \text{if } V \leq \mathcal{V}(u, v, m), \\ \frac{1}{2V} (\log \log m) \mathcal{V}(u, v, m) & \text{if } \mathcal{V}(u, v, m) < V \leq \frac{1}{16} \log \log m \mathcal{V}(u, v, m), \\ 8 & \text{if } V > \frac{1}{16} \log \log m \mathcal{V}(u, v, m). \end{cases}$$

Using Proposition 4.3 gives

$$\begin{aligned} & \log \left| \mathcal{L}\left(E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{q^{1/2+\beta}}\right) \right| \\ & \leq \frac{2m}{h} + \frac{1}{h} \Re \left( \sum_{\substack{j \geq 1 \\ \deg(P^j) \leq h}} \frac{\chi_D(P^j) (\alpha(P)^j + \beta(P)^j) \log q^{h-j \deg(P)}}{|P|^{(1/2+1/h \log q)j} \log q^j} (|P|^{-(\alpha-i\theta/\log q)j} + |P|^{-(\beta-i\gamma/\log q)j}) \right). \end{aligned} \tag{4-5}$$

Note that the contribution of the terms with  $j \geq 3$  is bounded by  $O(1)$ .

The terms with  $j = 2$  in (4-5) will contribute, up to a term of size  $O(\log \log m)$  coming from those  $P$  with  $P \mid D$ ,

$$\frac{1}{2} \sum_{\deg(P) \leq h/2} \frac{(h - 2 \deg(P)) (\lambda(P^2) - 1)}{h |P|^{1+2/h \log q}} \left( \frac{\cos(2\theta \deg(P))}{|P|^{2\alpha}} + \frac{\cos(2\gamma \deg(P))}{|P|^{2\beta}} \right).$$

Let

$$F_\alpha(h, \theta) = \sum_{n=1}^h \frac{\cos(2n\theta)}{nq^{2n\alpha+n/h \log q}}.$$

Similarly as in [Florea 2017a, Lemma 9.1], we can show that

$$F_\alpha(h, \theta) = \log \min \left\{ h, \frac{1}{2\theta} \right\} + O(1), \tag{4-6}$$

where for  $\theta \in [0, 2\pi]$  we denote  $\bar{\theta} = \min\{\theta, 2\pi - \theta\}$ . Now using the fact that

$$\sum_{\deg(P) \leq h} \frac{\lambda(P^2) \cos(2\theta \deg(P))}{|P|^{1+1/h \log q}} = O(\log \log h), \tag{4-7}$$

it follows that the contribution from  $j = 2$  will be equal to

$$\begin{aligned}
 &-\frac{1}{2} \sum_{\deg(P) \leq \frac{h}{2}} \frac{h - 2 \deg(P)}{h|P|^{1+2/h \log q}} \left( \frac{\cos(2\theta \deg(P))}{|P|^{2\alpha}} + \frac{\cos(2\gamma \deg(P))}{|P|^{2\beta}} \right) + O(\log \log m) \\
 &= -\frac{1}{2} \left( F_\alpha \left( \frac{h}{2}, \theta \right) + F_\beta \left( \frac{h}{2}, \gamma \right) \right) + O(\log \log m) \\
 &\leq -\frac{1}{2} (F_\alpha(m, \theta) + F_\beta(m, \gamma)) + \frac{2m}{h} + O(\log \log m) = \mathcal{M}(u, v, m) + \frac{2m}{h} + O(\log \log m), \tag{4-8}
 \end{aligned}$$

where

$$\mathcal{M}(u, v, m) = -\frac{1}{2} \left( \log \min \left\{ m, \frac{1}{2\theta} \right\} + \log \min \left\{ m, \frac{1}{2\gamma} \right\} \right) \tag{4-9}$$

by formula (4-6). Note that in the second line of the equation above we used the fact that

$$F_\alpha(m, \theta) - F_\alpha \left( \frac{h}{2}, \theta \right) = \sum_{n=1}^m \frac{\cos(2n\theta)}{nq^{2n\alpha} e^{n/m}} - \sum_{n=1}^{h/2} \frac{\cos(2n\theta)}{nq^{2n\alpha} e^{2n/h}},$$

and since  $e^{-x} = 1 + O(x)$ , it follows that

$$F_\alpha(m, \theta) - F_\alpha \left( \frac{h}{2}, \theta \right) = \sum_{n=h/2+1}^m \frac{\cos(2n\theta)}{nq^{2n\alpha}} + O(1) \leq \frac{2m}{h} + O(1).$$

Hence, using (4-8) in (4-5) we get

$$\begin{aligned}
 &\log \left| \mathcal{L} \left( E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}} \right) \mathcal{L} \left( E \otimes \chi_D, \frac{v}{q^{1/2+\beta}} \right) \right| \\
 &\leq \mathcal{M}(u, v, m) + \frac{5m}{h} + \sum_{\deg(P) \leq h} \frac{(h - \deg(P)) \chi_D(P) \lambda(P)}{h|P|^{1/2+1/(h \log q)}} \left( \frac{\cos(\theta \deg(P))}{|P|^\alpha} + \frac{\cos(\gamma \deg(P))}{|P|^\beta} \right).
 \end{aligned}$$

Let  $S_1$  be the sum above truncated at  $\deg(P) \leq h_0$  and  $S_2$  be the sum over primes with  $h_0 < \deg(P) \leq h$ . If  $D$  is such that

$$\log \left| \mathcal{L} \left( E \otimes \chi_D, \frac{u}{q^{1/2+\alpha}} \right) \mathcal{L} \left( E \otimes \chi_D, \frac{v}{q^{1/2+\beta}} \right) \right| \geq \mathcal{M}(u, v, m) + V,$$

then

$$S_1 \geq V_1 := V \left( 1 - \frac{6}{A} \right) \quad \text{or} \quad S_2 \geq \frac{V}{A}.$$

Let

$$\mathcal{F}_1 = \{ D \in \mathcal{H}_{2g+1}^* : S_1 \geq V_1 \} \quad \text{and} \quad \mathcal{F}_2 = \left\{ D \in \mathcal{H}_{2g+1}^* : S_2 \geq \frac{V}{A} \right\}.$$

If  $D \in \mathcal{F}_2$ , then by Markov’s inequality and Lemma 4.4 it follows that

$$|\mathcal{F}_2| \ll q^{2g} \frac{(2l)!}{l!2^l} \left( \frac{A}{V} \right)^{2l} \left( \sum_{h_0 < \deg(P) \leq h} \frac{|a(P)|^2}{|P|} \right)^l,$$

for any  $l \leq g/h$  where

$$a(P) = \frac{(h - \deg(P))\lambda(P)}{h|P|^{1/(h \log q)}} \left( \frac{\cos(\theta \deg(P))}{|P|^\alpha} + \frac{\cos(\gamma \deg(P))}{|P|^\beta} \right).$$

Picking  $l = [g/h]$  and noting that  $a(P) \ll 1$  and  $m = 4g + O(1)$ , we get that

$$|\mathcal{F}_2| \ll q^{2g} \left(\frac{2l}{e}\right)^l \left(\frac{A}{V}\right)^{2l} (\log \log m)^l \ll q^{2g} \exp\left(-\frac{V \log V}{8A}\right). \tag{4-10}$$

If  $D \in \mathcal{F}_1$  then for any  $l \leq g/h_0$ , we have

$$|\mathcal{F}_1| \ll q^{2g} \frac{(2l)!}{l!2^l} \frac{1}{V_1^{2l}} \left( \sum_{\deg(P) \leq h_0} \frac{|a(P)|^2}{|P|} \right)^l.$$

Using the expression for  $a(P)$  and (4-7) we get that

$$|\mathcal{F}_1| \ll q^{2g} \frac{(2l)!}{l!2^l V_1^{2l}} (\mathcal{V}(u, v, m) + O(\log \log m))^l,$$

where

$$\mathcal{V}(u, v, m)$$

$$\begin{aligned} &= \log m + \frac{1}{2}(F_\alpha(m, \theta) + F_\beta(m, \gamma)) + F_{(\alpha+\beta)/2}\left(m, \frac{\overline{\theta+\gamma}}{2}\right) + F_{(\alpha+\beta)/2}\left(m, \frac{\overline{\theta-\gamma}}{2}\right) \\ &= \log m + \frac{1}{2}\left(\log \min\left\{m, \frac{1}{2\theta}\right\} + \log \min\left\{m, \frac{1}{2\gamma}\right\}\right) + \log \min\left\{m, \frac{1}{\theta+\gamma}\right\} + \log \min\left\{m, \frac{1}{\theta-\gamma}\right\}, \end{aligned} \tag{4-11}$$

and the last line of the equation above follows from (4-6). Then

$$|\mathcal{F}_1| \ll q^{2g} \left(\frac{2l}{eV_1^2} (\mathcal{V}(u, v, m) + O(\log \log m))\right)^l.$$

If  $V \leq \mathcal{V}(u, v, m)^2$ , then we pick  $l = [V_1^2/2\mathcal{V}(u, v, m)]$ , and if  $V > \mathcal{V}(u, v, m)^2$ , then we pick  $l = [10V]$ . In doing so we get

$$|\mathcal{F}_1| \ll q^{2g} \exp\left(-\frac{V_1^2}{2\mathcal{V}(u, v, m)}\right) + q^{2g} \exp(-V \log V). \tag{4-12}$$

Combining the bounds (4-12) and (4-10) finishes the proof of Lemma 4.5. □

The following is an immediate corollary of Theorem 4.1.

**Corollary 4.6.** *Let  $u = e^{i\theta}$  and  $v = e^{i\gamma}$  with  $\theta, \gamma \in [0, 2\pi]$ . Then*

$$\begin{aligned} &\sum_{D \in \mathcal{H}_{2g+1}^*} \left| \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{\sqrt{q}}\right) \right| \\ &\ll_\varepsilon q^{2g} g^{1/2+\varepsilon} \min\left\{g, \frac{1}{2\theta}\right\}^{-1/4} \min\left\{g, \frac{1}{2\gamma}\right\}^{-1/4} \min\left\{g, \frac{1}{(\theta-\gamma)}\right\}^{1/2} \min\left\{g, \frac{1}{(\theta+\gamma)}\right\}^{1/2}. \end{aligned}$$

For  $N \mid \Delta^\infty$  and fixed  $n \in \mathbb{N}$ , we define the truncated sums

$$\mathcal{E}_{1,E}(N, X, n) := \sum_{X < \deg(f) \leq n + \deg(D)} \frac{\lambda(f)\chi_D(Nf)}{\sqrt{|f|}} \tag{4-13}$$

and

$$\mathcal{E}_{2,E}(N, X, n) := \sum_{X < \deg(f) \leq n + \deg(D)} \frac{(n + \deg(D) - \deg(f))\lambda(f)\chi_D(Nf)}{\sqrt{|f|}}. \tag{4-14}$$

We are now ready to prove the following upper bounds for  $\mathcal{E}_{i,E}(N, X, n)$ .

**Proposition 4.7.** *For  $i = 1, 2$  and any fixed  $n \in \mathbb{N}$  we have*

$$\sum_{D \in \mathcal{H}_{2g+1}^*} |\mathcal{E}_{i,E}(N, X, n)|^2 \ll_\varepsilon q^{2g} g^{1/2+\varepsilon} (2g - X)^{2i}.$$

*Proof.* Using the Perron formula for the sum over  $f$  in (4-13), we get that

$$\mathcal{E}_{1,E}(N, X, n) = \frac{\chi_D(N)}{2\pi i} \oint_{|u|=1} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right) \left(\frac{1}{u^{n+\deg(D)}} - \frac{1}{u^X}\right) \frac{du}{u(1-u)}.$$

Note that there is no pole at  $u = 1$ . So we need to bound the following expression

$$\begin{aligned} & \frac{1}{(2\pi i)^2} \oint_{|u|=1} \oint_{|v|=1} \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{\sqrt{q}}\right) \\ & \times \left(\frac{1}{u^{n+\deg(D)}} - \frac{1}{u^X}\right) \left(\frac{1}{v^{n+\deg(D)}} - \frac{1}{v^X}\right) \frac{dudv}{uv(1-u)(1-v)}. \end{aligned} \tag{4-15}$$

We use Corollary 4.6 to bound the integral above and consider  $\theta$  and  $\gamma$  on different arcs on the unit circle. We bound the integral on these arcs and notice that we obtain the biggest upper bound when  $\theta, \gamma$  are not close to 0 (i.e.,  $u$  and  $v$  are not close to 1) and when  $\theta$  is not close to  $\gamma$  or to  $2\pi - \gamma$  (by close we mean on an arc of length on the scale of  $1/g$ ).

For example, if  $\theta$  and  $\gamma$  are both on an arc  $C_1$  of length on the scale of  $1/g$  around 0, then the double integral in (4-15) over the arcs  $C_1$  is  $O_\varepsilon(q^{2g} g^{-1+\varepsilon})$  (since from the corollary we get a power of  $g$  which gets multiplied by  $g^{-2}$ , the product of the sizes of the arcs.)

If  $\theta, \gamma$  are both on the complement of  $C_1$ , but close to each other (i.e.,  $\theta$  is within  $1/g$  of  $\gamma$ ), we get that the corresponding integral over the two arcs is  $O_\varepsilon(q^{2g} g^\varepsilon)$ . We get a similar bound if  $\theta$  is close to  $2\pi - \gamma$ , under the same conditions.

We are left with the case when  $\theta, \gamma$  are on the complement of  $C_1$  and  $\theta$  is far from  $\gamma$  and from  $2\pi - \gamma$ . In this case the corresponding integral will be  $O_\varepsilon(q^{2g} g^{1/2+\varepsilon})$ . This finishes the proof of the upper bound when  $i = 1$ .

When  $i = 2$ , using the Perron formula for the sum over  $f$  in (4-14), we have

$$\mathcal{E}_{2,E}(N, X, n) = \frac{\chi_D(N)}{2\pi i} \oint_{|u|=1} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right) \left(\frac{1}{u^{n+\deg(D)}} - \frac{(n + \deg(D) - X)(1-u) + u}{u^{X+1}}\right) \frac{du}{(1-u)^2}.$$

Hence

$$\begin{aligned} \mathcal{E}_{2,E}(N, X, n)^2 &= \frac{1}{(2\pi i)^2} \oint_{|u|=1} \oint_{|v|=1} \mathcal{L}\left(E \otimes \chi_D, \frac{u}{\sqrt{q}}\right) \mathcal{L}\left(E \otimes \chi_D, \frac{v}{\sqrt{q}}\right) \\ &\quad \times \left(\frac{1}{u^{n+\deg(D)}} - \frac{(n + \deg(D) - X)(1 - u) + u}{u^{X+1}}\right) \\ &\quad \times \left(\frac{1}{v^{n+\deg(D)}} - \frac{(n + \deg(D) - X)(1 - v) + v}{v^{X+1}}\right) \frac{dudv}{(1-u)^2(1-v)^2}. \end{aligned}$$

We proceed as before and keeping in mind that  $|u| = |v| = 1$ , it follows that

$$\sum_{D \in \mathcal{H}_{2g+1}^*} |\mathcal{E}_{2,E}(N, X, n)|^2 \ll_{\varepsilon} q^{2g} g^{1/2+\varepsilon} (2g - X)^4,$$

as required. □

### 5. Proof of Theorem 1.1

For  $N \mid \Delta^\infty$ , let

$$R_E(N, X) := \sum_{D \in \mathcal{H}_{2g+1}^*} \sum_{f \in \mathcal{M}_{\leq X}} \frac{\lambda(f) \chi_D(Nf)}{\sqrt{|f|}}.$$

We will prove the following lemma.

**Lemma 5.1.** *We have*

$$R_E(N, X) = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; 1) L(\text{Sym}^2 E, 1) + O_\varepsilon(q^{2g-X/2+\varepsilon g}) + O_\varepsilon(q^{X/2+\varepsilon g}),$$

where the value  $\mathcal{C}_E(N; 1)$  is defined in (5-2).

*Proof.* Note that

$$R_E(N, X) = S_{E,E}(N, X, 0; 0, 0),$$

where  $S_{E_1, E_2}(N, X, Y; \alpha, \beta)$  is defined as in (3-1). We proceed as in Section 3, see (3-2), (3-3) and (3-7), and write

$$R_E(N, X) = S_{E,E}(N, X, 0, g; 0, 0) = R_E(N, X; V = 0) + R_E(N, X; V \neq 0),$$

where  $R_E(N, X; V \neq 0) = S_{E,E}(N, X, 0, g; 0, 0; V \neq 0)$  and

$$\begin{aligned} R_E(N, X; V = 0) &= S_{E,E}(N, X, 0, g; 0, 0; V = 0) \\ &= |\mathcal{H}_{2g+1}| \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ Nf = \square}} \frac{\lambda(f)}{\sqrt{|f|}} \prod_{P \mid \Delta f} \left(1 - \frac{1}{|P|}\right) \sum_{\substack{C_2 \mid (\Delta f)^\infty \\ \deg(C_2) \leq g}} \frac{1}{|C_2|^2} + O_\varepsilon(q^{\varepsilon g}). \end{aligned}$$

We first evaluate  $R_E(N, X; V = 0)$ . From (3-6) we have

$$\sum_{\substack{C_2 | (\Delta f)^\infty \\ \deg(C_2) \leq g}} \frac{1}{|C_2|^2} = \prod_{P | \Delta f} \left(1 - \frac{1}{|P|^2}\right)^{-1} + O_\varepsilon(q^{-2g+\varepsilon g}).$$

The contribution of the error term to  $R_E(N, X; V = 0)$  is

$$\ll_\varepsilon q^{\varepsilon g} \sum_{l \in \mathcal{M}_{\leq X/2}} \frac{|\lambda(Nl^2)|}{|l|} \ll_\varepsilon q^{\varepsilon g}.$$

Hence

$$R_E(N, X; V = 0) = |\mathcal{H}_{2g+1}| \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ Nf = \square}} \frac{\lambda(f)}{\sqrt{|f|}} \prod_{P | \Delta f} \left(1 + \frac{1}{|P|}\right)^{-1} + O_\varepsilon(q^{\varepsilon g}).$$

Applying the Perron formula to the sum over  $f$  yields

$$R_E(N, X; V = 0) = \frac{|\mathcal{H}_{2g+1}|}{2\pi i} \oint_{|u|=r} \mathcal{B}_E(N; u) \frac{du}{u^{X+1}(1-u)} + O_\varepsilon(q^{\varepsilon g}) \tag{5-1}$$

for any  $r < 1$ , where

$$\mathcal{B}_E(N; u) = \sum_{\substack{f \in \mathcal{M} \\ Nf = \square}} \frac{\lambda(f)u^{\deg(f)}}{\sqrt{|f|}} \prod_{P | \Delta f} \left(1 + \frac{1}{|P|}\right)^{-1}.$$

We can write  $\mathcal{B}_E(N; u)$  in terms of its Euler product,

$$\begin{aligned} \mathcal{B}_E(N; u) &= \prod_{P \nmid \Delta} \left(1 + \left(1 + \frac{1}{|P|}\right)^{-1} \sum_{i \geq 1} \frac{\lambda(P^{2i})u^{2i \deg(P)}}{|P|^i}\right) \prod_{P | \Delta} \left(\left(1 + \frac{1}{|P|}\right)^{-1} \sum_{i + \text{ord}_P(N) \text{ even}} \frac{\lambda(P^i)u^{i \deg(P)}}{|P|^{i/2}}\right) \\ &= \mathcal{C}_E(N; u) \mathcal{L}\left(\text{Sym}^2 E, \frac{u^2}{q}\right), \end{aligned} \tag{5-2}$$

where  $\mathcal{C}_E(N; u)$  is some Euler product which is uniformly bounded for  $|u| \leq q^{1/2-\varepsilon}$ . We shift the contour in (5-1) to  $|u| = q^{1/2-\varepsilon}$ , encountering a simple pole at  $u = 1$ . Then

$$R_E(N, X; V = 0) = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; 1) L(\text{Sym}^2 E, 1) + O_\varepsilon(q^{2g-X/2+\varepsilon g}). \tag{5-3}$$

Now we will bound  $R_E(N, X; V \neq 0)$ . As in Section 3D, see (3-12), it suffices to bound the term

$$R(V \neq 0) = q^{2g+1} \overline{\tau(q)} \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ \deg(Nf) \text{ odd}}} \frac{\lambda(f)}{|N||f|^{3/2}} \sum_{\substack{C_1 | \Delta \\ C_2 | (\Delta f)^\infty \\ \deg(C_1)+2 \deg(C_2) \leq 2g+1}} \frac{\mu(C_1)\chi_{C_1}(Nf)}{|C_1||C_2|^2} \sum_{V \in \mathcal{M}_{\deg(Nf)+\deg(C_1)+2 \deg(C_2)-2g-2}} G(V, Nf).$$

Using the fact that

$$\sum_{\substack{C_2 \in \mathcal{M}_{c_2} \\ C_2 | (\Delta f)^\infty}} \frac{1}{|C_2|^2} = \frac{q^{-2c_2}}{2\pi i} \oint_{|w|=r} \prod_{P | \Delta f} (1 - w^{\deg(P)})^{-1} \frac{dw}{w^{c_2+1}}$$

for  $r < 1$ , and writing  $V = V_1 V_2^2$  with  $V_1$  a square-free polynomial, we have

$$\begin{aligned} R(V \neq 0) &= \frac{q^{2g+1} \overline{\tau(q)}}{|N|} \sum_{c_1+2c_2 \leq 2g+1} q^{-2c_2} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_1 | \Delta}} \frac{\mu(C_1) \chi_{C_1}(N)}{|C_1|} \\ &\quad \times \sum_{\substack{n \leq X \\ n+\deg(N) \text{ odd}}} \sum_{\substack{j \leq n+\deg(N)+c_1+2c_2-2g-2 \\ j+c_1 \text{ odd}}} \sum_{V_1 \in \mathcal{H}_j} \sum_{V_2 \in \mathcal{M}_{(n+\deg(N)+c_1-j)/2+c_2-g-1}} \\ &\quad \times \frac{1}{2\pi i} \oint_{|w|=r} \sum_{f \in \mathcal{M}_n} \frac{\chi_{C_1}(f) \lambda(f) G(V_1 V_2^2, Nf)}{|f|^{3/2}} \prod_{P | \Delta f} (1 - w^{\deg(P)})^{-1} \frac{dw}{w^{c_2+1}}. \end{aligned}$$

Now

$$\begin{aligned} \sum_{f \in \mathcal{M}} \frac{\chi_{C_1}(f) \lambda(f) G(V_1 V_2^2, Nf)}{|f|^{3/2}} \prod_{P | \Delta f} (1 - w^{\deg(P)})^{-1} u^{\deg(f)} \\ = \mathcal{H}(V_1; u, w) \mathcal{I}(V_1 V_2^2, N; u, w) \mathcal{J}(V_1 V_2^2; u, w), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}(V_1; u, w) &= \prod_{P \nmid V_1} \left( 1 + \frac{\chi_{C_1 V_1}(P) \lambda(P) u^{\deg(P)}}{|P|} (1 - w^{\deg(P)})^{-1} \right), \\ \mathcal{I}(V_1 V_2^2, N; u, w) &= \prod_{P | \Delta} \left( \sum_j \frac{\chi_{C_1}(P^j) \lambda(P^j) G(V_1 V_2^2, P^{j+n_P}) u^{j \deg(P)}}{|P|^{3j/2}} \right) (1 - w^{\deg(P)})^{-1} \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}(V_1 V_2^2; u, w) &= \prod_{\substack{P | V_1 V_2 \\ P \nmid \Delta}} \left( 1 + \sum_{j \geq 1} \frac{\chi_{C_1}(P^j) \lambda(P^j) G(V_1 V_2^2, P^j) u^{j \deg(P)}}{|P|^{3j/2}} (1 - w^{\deg(P)})^{-1} \right) \\ &\quad \times \prod_{\substack{P \nmid V_1 \\ P | \Delta V_2}} \left( 1 + \frac{\chi_{C_1 V_1}(P) \lambda(P) u^{\deg(P)}}{|P|} (1 - w^{\deg(P)})^{-1} \right)^{-1}. \end{aligned}$$

We use the Perron formula for the sum over  $f$  and obtain

$$\begin{aligned}
 R(V \neq 0) &= \frac{q^{2g+1} \overline{\tau(q)}}{|N|} \sum_{c_1+2c_2 \leq 2g+1} q^{-2c_2} \sum_{\substack{C_1 \in \mathcal{M}_{c_1} \\ C_1 | \Delta}} \frac{\mu(C_1) \chi_{C_1}(N)}{|C_1|} \\
 &\times \sum_{\substack{n \leq X \\ n+\deg(N) \text{ odd}}} \sum_{\substack{j \leq n+\deg(N)+c_1+2c_2-2g-2 \\ j+c_1 \text{ odd}}} \sum_{V_1 \in \mathcal{H}_j} \sum_{V_2 \in \mathcal{M}_{(n+\deg(N)+c_1-j)/2+c_2-g-1}} \\
 &\times \frac{1}{(2\pi i)^2} \oint_{|u|=r} \oint_{|w|=r} \mathcal{H}(V_1; u, w) \mathcal{I}(V_1 V_2^2, N; u, w) \mathcal{J}(V_1 V_2^2; u, w) \frac{du}{u^{n+1}} \frac{dw}{w^{c_2+1}}. \tag{5-4}
 \end{aligned}$$

Let  $r_1 = q^{1/2-\varepsilon}$ ,  $r_2 = q^{-\varepsilon}$ , and let  $k_0$  be minimal such that  $|r_1 r_2^{k_0}| < 1$ . Then we can write

$$\mathcal{H}(V_1; u, w) = \mathcal{L}\left(E \otimes \chi_{C_1 V_1}, \frac{u}{q}\right) \mathcal{L}\left(E \otimes \chi_{C_1 V_1}, \frac{uw}{q}\right) \cdots \mathcal{L}\left(E \otimes \chi_{C_1 V_1}, \frac{uw^{k_0-1}}{q}\right) \mathcal{K}(V_1; u, w), \tag{5-5}$$

where

$$\mathcal{K}(V_1; u, w) \ll_{\varepsilon} |C_1|^{\varepsilon}$$

uniformly for  $|u| \leq r_1$  and  $|w| \leq r_2$ . We also have

$$\mathcal{I}(V_1 V_2^2, N; u, w) \ll_{\varepsilon} |V_1 V_2|^{\varepsilon} \quad \text{and} \quad \mathcal{J}(V_1 V_2^2; u, w) \ll_{\varepsilon} |V_2|^{\varepsilon}$$

in this region. We now move the contours in (5-4) to  $|u| = r_1$  and  $|w| = r_2$ . We then use the Lindelöf bound for each  $L$ -function and trivially bound the rest of the expression to obtain that

$$R(V \neq 0) \ll_{\varepsilon} q^{X/2+\varepsilon g}.$$

Combining this with (5-3) finishes the proof of Lemma 5.1. □

To prove Theorem 1.1, note that from Lemma 2.2 we have

$$\sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2}) = R_E(1, [n/2] + 2g + 1) + \epsilon_{2g+1} \epsilon(E) R_E(M, [(n+1)/2] + 2g).$$

Using Lemma 5.1 and choosing  $X = g$  we have that

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2}) = c_1(M) L(\text{Sym}^2 E, 1) + O_{\varepsilon}(q^{-g+\varepsilon g}),$$

where

$$c_1(M) = (\mathcal{C}_E(1; 1) + \epsilon_{2g+1} \epsilon(E) \mathcal{C}_E(M; 1)) \prod_{P | \Delta} \frac{|P| + 1}{|P|}. \tag{5-6}$$

This finishes the proof of the theorem.

### 6. Proof of Theorem 1.2

For  $N \mid \Delta^\infty$ , let

$$R_E(N, X; \alpha) := \sum_{D \in \mathcal{H}_{2g+1}^*} \sum_{f \in \mathcal{M}_{\leq X}} \frac{\lambda(f) \chi_D(Nf)}{|f|^{1/2+\alpha}}.$$

Note that  $R_E(N, X; 0) = R_E(N, X)$  with  $R_E(N, X)$  as in Section 5. Similarly as in Section 5 and for  $|\alpha| \leq 1/g$  we get that

$$R_E(N, X; \alpha) = |\mathcal{H}_{2g+1}| \mathcal{C}_E(N; q^{-\alpha}) L(\text{Sym}^2 E, 1 + 2\alpha) + O_\varepsilon(q^{2g-X/2+\varepsilon g}) + O_\varepsilon(q^{X/2+\varepsilon g}). \tag{6-1}$$

Using Lemma 2.3 we obtain

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'}(E \otimes \chi_D, \tfrac{1}{2}) \\ &= (\log q)([n/2] + 2g + 1)(R_E(1, [n/2] + 2g + 1) - \epsilon_{2g+1} \epsilon(E) R_E(M, [n/2] + 2g + 1)) \\ & \quad + \frac{\partial}{\partial \alpha} (R_E(1, [n/2] + 2g + 1; \alpha) - \epsilon_{2g+1} \epsilon(E) R_E(M, [n/2] + 2g + 1; \alpha)) \Big|_{\alpha=0}. \end{aligned}$$

From Lemma 5.1 and (6-1) it follows that

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'}(E \otimes \chi_D, \tfrac{1}{2}) = c_2(M) L(\text{Sym}^2 E, 1) g + c_3(M) + O_\varepsilon(q^{-g+\varepsilon g}),$$

where

$$c_2(M) = 2(\log q)(\mathcal{C}_E(1; 1) - \epsilon_{2g+1} \epsilon(E) \mathcal{C}_E(M; 1)) \prod_{P \mid \Delta} \frac{|P| + 1}{|P|} \tag{6-2}$$

and

$$\begin{aligned} c_3(M) = & (((\log q)([n/2] + 1) L(\text{Sym}^2 E, 1) + 2L'(\text{Sym}^2 E, 1))(\mathcal{C}_E(1; 1) - \epsilon_{2g+1} \epsilon(E) \mathcal{C}_E(M; 1)) \\ & + L(\text{Sym}^2 E, 1)(\mathcal{C}'_E(1; 1) - \epsilon_{2g+1} \epsilon(E) \mathcal{C}'_E(M; 1))) \prod_{P \mid \Delta} \frac{|P| + 1}{|P|}. \end{aligned} \tag{6-3}$$

### 7. Proof of Theorem 1.3

Following Lemma 2.2, for  $X < 2g$ , we define

$$\mathcal{M}_{1,E}(X) := (1 + \epsilon) \sum_{f \in \mathcal{M}_{\leq X}} \frac{\lambda(f) \chi_D(f)}{\sqrt{|f|}},$$

so that

$$L(E \otimes \chi_D, \tfrac{1}{2}) = \mathcal{M}_{1,E}(X) + \mathcal{E}_{1,E}(1, X, [n/2]) + \epsilon_{2g+1} \epsilon(E) \mathcal{E}_{1,E}(M, X, [(n-1)/2]), \tag{7-1}$$

where recall expression (4-13) for  $\mathcal{E}_{1,E}(N, X, n)$ . Hence

$$\begin{aligned} & L(E \otimes \chi_D, \tfrac{1}{2})^2 \\ &= 2L(E \otimes \chi_D, \tfrac{1}{2}) \mathcal{M}_{1,E}(X) - \mathcal{M}_{1,E}(X)^2 + (\mathcal{E}_{1,E}(1, X, [n/2]) + \epsilon_{2g+1} \epsilon(E) \mathcal{E}_{1,E}(M, X, [(n-1)/2]))^2. \end{aligned}$$

By Cauchy’s inequality and [Proposition 4.7](#) we get

$$\sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2})^2 = 2 \sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2}) \mathcal{M}_{1,E}(X) - \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{M}_{1,E}(X)^2 + O_\epsilon(q^{2g} g^{1/2+\epsilon} (2g - X)^2).$$

Now using [Lemma 2.2](#) again and expanding  $\mathcal{M}_{1,E}(X)^2$ , the first line of the equation above is

$$2(S_E(1, [n/2] + 2g + 1, X) + S_E(1, [(n + 1)/2] + 2g, X) - S_E(1, X, X)) + 2\epsilon_{2g+1}\epsilon(E)(S_E(M, [n/2] + 2g + 1, X) + S_E(M, [(n + 1)/2] + 2g, X) - S_E(M, X, X)),$$

where recall the definition of  $S_E(M, X, Y)$  in [Section 3](#). Using [Proposition 3.1](#), this is equal to

$$|\mathcal{H}_{2g+1}^*| c_2(M) L(\text{Sym}^2 E, 1)^3 X + O(q^{2g}) + O(q^{2g-X/5} g^3) + O(q^{5g/4+3X/8} g^{30}),$$

where

$$c_2(M) = 2(C_E(1; 1, 1, 1) + \epsilon_{2g+1}\epsilon(E)C_E(M; 1, 1, 1)) \prod_{P|\Delta} \frac{|P| + 1}{|P|}, \tag{7-2}$$

and  $C_E(N; 1, 1, 1)$  is defined in [\(3-9\)](#). Thus

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2})^2 = c_4(M) L(\text{Sym}^2 E, 1)^3 X + O(q^{-X/5} g^3) + O(q^{-3g/4+3X/8} g^{30}) + O_\epsilon(g^{1/2+\epsilon} (2g - X)^2).$$

Choosing  $X = 2g - 100 \log g$  we obtain the theorem.

### 8. Proof of [Theorem 1.4](#)

Following [Lemma 2.3](#), for  $X < 2g$  and fixed  $n \in \mathbb{N}$ , we define

$$\mathcal{M}_{2,E}(X, n) := (1 - \epsilon) \sum_{f \in \mathcal{M}_{\leq X}} \frac{(n + \deg(D) - \deg(f))\lambda(f)\chi_D(f)}{\sqrt{|f|}},$$

so that

$$\epsilon^- L'(E \otimes \chi_D, \frac{1}{2}) = (\log q) \mathcal{M}_{2,E}(X, [n/2]) + (\log q)(\mathcal{E}_{2,E}(1, X, [n/2]) - \epsilon_{2g+1}\epsilon(E)\mathcal{E}_{2,E}(M, X, [n/2])), \tag{8-1}$$

where recall that  $\mathcal{E}_{2,E}(M, X, n)$  is given in [\(4-14\)](#). Then we get that

$$\epsilon^- L'(E \otimes \chi_D, \frac{1}{2})^2 = 2(\log q)\epsilon^- L'(E \otimes \chi_D, \frac{1}{2}) \mathcal{M}_{2,E}(X, [n/2]) - (\log q)^2 \mathcal{M}_{2,E}^2(X, [n/2]) + (\log q)^2 (\mathcal{E}_{2,E}(1, X, [n/2]) - \epsilon_{2g+1}\epsilon(E)\mathcal{E}_{2,E}(M, X, [n/2]))^2.$$

By Cauchy's inequality and [Proposition 4.7](#) we get that

$$\sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})^2} = 2(\log q) \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})} \mathcal{M}_{2,E}(X, [n/2]) - (\log q)^2 \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{M}_{2,E}(X, [n/2])^2 + O_\epsilon(q^{2g} g^{1/2+\epsilon} (2g - X)^4).$$

Now by [Lemma 2.3](#) we have

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})} \mathcal{M}_{2,E}(X, [n/2]) \\ &= 2(\log q)([n/2]+2g+1)^2 (S_E(1, [n/2]+2g+1, X) - \epsilon_{2g+1} \epsilon(E) S_E(M, [n/2]+2g+1, X)) \\ & \quad + 2([n/2]+2g+1) \frac{\partial}{\partial \beta} (S_E(1, [n/2]+2g+1, X; 0, \beta) - \epsilon_{2g+1} \epsilon(E) S_E(M, [n/2]+2g+1, X; 0, \beta)) \Big|_{\beta=0} \\ & \quad + 2([n/2]+2g+1) \frac{\partial}{\partial \alpha} (S_E(1, [n/2]+2g+1, X; \alpha, 0) - \epsilon_{2g+1} \epsilon(E) S_E(M, [n/2]+2g+1, X; \alpha, 0)) \Big|_{\alpha=0} \\ & \quad + \frac{2}{\log q} \frac{\partial^2}{\partial \alpha \partial \beta} (S_E(1, [n/2]+2g+1, X; \alpha, \beta) - \epsilon_{2g+1} \epsilon(E) S_E(M, [n/2]+2g+1, X; \alpha, \beta)) \Big|_{\alpha=\beta=0}, \end{aligned}$$

and similarly

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{M}_{2,E}(X, [n/2])^2 \\ &= 2([n/2] + 2g + 1)^2 (S_E(1, X, X) - \epsilon_{2g+1} \epsilon(E) S_E(M, X, X)) \\ & \quad + \frac{2}{\log q} ([n/2] + 2g + 1) \frac{\partial}{\partial \beta} (S_E(1, X, X; 0, \beta) - \epsilon_{2g+1} \epsilon(E) S_E(M, X, X; 0, \beta)) \Big|_{\beta=0} \\ & \quad + \frac{2}{\log q} ([n/2] + 2g + 1) \frac{\partial}{\partial \alpha} (S_E(1, X, X; \alpha, 0) - \epsilon_{2g+1} \epsilon(E) S_E(M, X, X; \alpha, 0)) \Big|_{\alpha=0} \\ & \quad + \frac{2}{(\log q)^2} \frac{\partial^2}{\partial \alpha \partial \beta} (S_E(1, X, X; \alpha, \beta) - \epsilon_{2g+1} \epsilon(E) S_E(M, X, X; \alpha, \beta)) \Big|_{\alpha=\beta=0}. \end{aligned}$$

Choosing  $X = 2g - 100 \log g$  and using [Proposition 3.1](#), we obtain that

$$\frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon^{-L'(E \otimes \chi_D, \frac{1}{2})^2} = c_5(M) L(\text{Sym}^2 E, 1)^3 g^3 + O_\epsilon(g^{2+\epsilon}),$$

where

$$c_5(M) = 16(\log q)^2 (\mathcal{C}_E(1; 1, 1, 1) - \epsilon_{2g+1} \epsilon(E) \mathcal{C}_E(M; 1, 1, 1)) \prod_{P|\Delta} \frac{|P|+1}{|P|}. \tag{8-2}$$

**9. Proof of Theorem 1.5**

By combining (7-1) and (8-1),

$$\begin{aligned} \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2}) &= \epsilon_2^- \mathcal{M}_{1,E_1}(X) L'(E_2 \otimes \chi_D, \frac{1}{2}) + (\log q) L(E_1 \otimes \chi_D, \frac{1}{2}) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) \\ &\quad - (\log q) \mathcal{M}_{1,E_1}(X) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) \\ &\quad + (\log q) (\mathcal{E}_{1,E_1}(1, X, [\mathfrak{n}_1/2]) + \epsilon_{2g+1} \epsilon(E_1) \mathcal{E}_{1,E_1}(M_1, X, [(\mathfrak{n}_1 - 1)/2])) \\ &\quad \times (\mathcal{E}_{2,E_2}(1, X, [\mathfrak{n}_2/2]) - \epsilon_{2g+1} \epsilon(E_2) \mathcal{E}_{2,E_2}(M_2, X, [\mathfrak{n}_2/2])). \end{aligned}$$

We bound the last term above using Cauchy’s inequality and Proposition 4.7. In doing so we get

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2}) &= \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- \mathcal{M}_{1,E_1}(X) L'(E_2 \otimes \chi_D, \frac{1}{2}) + (\log q) \sum_{D \in \mathcal{H}_{2g+1}^*} L(E_1 \otimes \chi_D, \frac{1}{2}) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) \\ &\quad - (\log q) \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{M}_{1,E_1}(X) \mathcal{M}_{2,E_2}(X, [\mathfrak{n}_2/2]) + O_\epsilon(q^{2g} g^{1/2+\epsilon} (2g - X)^3). \end{aligned} \tag{9-1}$$

We shall estimate the remaining three terms using Proposition 3.1. They all have similar forms. For the first term, by Lemma 2.3 again we have

$$\begin{aligned} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- \mathcal{M}_{1,E_1}(X) L'(E_2 \otimes \chi_D, \frac{1}{2}) &= (\log q) \sum_{D \in \mathcal{H}_{2g+1}^*} (1 + \epsilon_1)(1 - \epsilon_2) \sum_{\substack{f \in \mathcal{M}_{\leq X} \\ h \in \mathcal{M}_{\leq [\mathfrak{n}_2/2] + 2g + 1}}} \frac{([\mathfrak{n}_2/2] + 2g + 1 - \deg(h)) \lambda_1(f) \lambda_2(h) \chi_D(fh)}{\sqrt{|fh|}}. \end{aligned}$$

By expanding out, this equals

$$\begin{aligned} (\log q) ([\mathfrak{n}_2/2] + 2g + 1) (S_{E_1, E_2}(1, [\mathfrak{n}_2/2] + 2g + 1, X; 0, 0) &+ \epsilon_{2g+1} \epsilon(E_1) S_{E_1, E_2}(M_1, [\mathfrak{n}_2/2] + 2g + 1, X; 0, 0) \\ &- \epsilon_{2g+1} \epsilon(E_2) S_{E_1, E_2}(M_2, [\mathfrak{n}_2/2] + 2g + 1, X; 0, 0) \\ &- \epsilon(E_1) \epsilon(E_2) S_{E_1, E_2}(M_1 M_2, [\mathfrak{n}_2/2] + 2g + 1, X; 0, 0)) \\ &+ \frac{\partial}{\partial \beta} (S_{E_1, E_2}(1, [\mathfrak{n}_2/2] + 2g + 1, X; 0, \beta) \\ &+ \epsilon_{2g+1} \epsilon(E_1) S_{E_1, E_2}(M_1, [\mathfrak{n}_2/2] + 2g + 1, X; 0, \beta) \\ &- \epsilon_{2g+1} \epsilon(E_2) S_{E_1, E_2}(M_2, [\mathfrak{n}_2/2] + 2g + 1, X; 0, \beta) \\ &- \epsilon(E_1) \epsilon(E_2) S_{E_1, E_2}(M_1 M_2, [\mathfrak{n}_2/2] + 2g + 1, X; 0, \beta)) \Big|_{\beta=0}, \end{aligned}$$

which is, by [Proposition 3.1](#) and Cauchy’s residue theorem,

$$|\mathcal{H}_{2g+1}^*|c_6(M_1, M_2)L(\text{Sym}^2 E_1, 1)L(\text{Sym}^2 E_2, 1)L(E_1 \otimes E_2, 1)g + O(q^{2g}) + O(q^{5g/4+3X/8}g^{31}),$$

where

$$\begin{aligned} c_6(M_1, M_2) = & 2(\log q)\left(\mathcal{C}_{E_1, E_2}(1; 1, 1, 1, 0, 0) + \epsilon_{2g+1}\epsilon(E_1)\mathcal{C}_{E_1, E_2}(M_1; 1, 1, 1, 0, 0)\right. \\ & \left. - \epsilon_{2g+1}\epsilon(E_2)\mathcal{C}_{E_1, E_2}(M_2; 1, 1, 1, 0, 0)\right. \\ & \left. - \epsilon(E_1)\epsilon(E_2)\mathcal{C}_{E_1, E_2}(M_1M_2; 1, 1, 1, 0, 0)\right) \prod_{P|\Delta} \frac{|P|+1}{|P|}. \end{aligned} \tag{9-2}$$

The other two terms in [\(9-1\)](#) have the same asymptotics so we obtain

$$\begin{aligned} & \frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2})L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ & = c_6(M_1, M_2)L(\text{Sym}^2 E_1, 1)L(\text{Sym}^2 E_2, 1)L(E_1 \otimes E_2, 1)g + O(q^{-3g/4+3X/8}g^{31}) + O_\epsilon(g^{1/2+\epsilon}(2g-X)^3). \end{aligned}$$

Choosing  $X = 2g - 100 \log g$  we obtain the theorem.

### 10. Proof of [Theorem 1.6](#)

We argue as in the previous section. From [\(8-1\)](#) we have

$$\begin{aligned} & \epsilon_1^- \epsilon_2^- L'(E_1 \otimes \chi_D, \frac{1}{2})L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ & = (\log q)\left(\epsilon_2^- \mathcal{M}_{2, E_1}(X, [\mathbf{n}_1/2])L'(E_2 \otimes \chi_D, \frac{1}{2}) + \epsilon_1^- L'(E_1 \otimes \chi_D, \frac{1}{2})\mathcal{M}_{2, E_2}(X, [\mathbf{n}_2/2])\right) \\ & \quad - (\log q)^2 \mathcal{M}_{2, E_1}(X, [\mathbf{n}_1/2])\mathcal{M}_{2, E_2}(X, [\mathbf{n}_2/2]) \\ & \quad + (\log q)^2 (\mathcal{E}_{2, E_1}(1, X, [\mathbf{n}_1/2]) - \epsilon_{2g+1}\epsilon(E_1)\mathcal{E}_{2, E_1}(M_1, X, [\mathbf{n}_1/2])) \\ & \quad \quad \times (\mathcal{E}_{2, E_2}(1, X, [\mathbf{n}_2/2]) - \epsilon_{2g+1}\epsilon(E_2)\mathcal{E}_{2, E_2}(M_2, X, [\mathbf{n}_2/2])). \end{aligned}$$

Bounding the last term above using Cauchy’s inequality and [Proposition 4.7](#) leads to

$$\begin{aligned} & \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_1^- \epsilon_2^- L'(E_1 \otimes \chi_D, \frac{1}{2})L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ & = (\log q) \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_2^- \mathcal{M}_{2, E_1}(X, [\mathbf{n}_1/2])L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ & \quad + (\log q) \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_1^- L'(E_1 \otimes \chi_D, \frac{1}{2})\mathcal{M}_{2, E_2}(X, [\mathbf{n}_2/2]) \\ & \quad - (\log q)^2 \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{M}_{2, E_1}(X, [\mathbf{n}_1/2])\mathcal{M}_{2, E_2}(X, [\mathbf{n}_2/2]) + O_\epsilon(q^{2g}g^{1/2+\epsilon}(2g-X)^4). \end{aligned} \tag{10-1}$$

We shall illustrate the evaluation of the third term using [Proposition 3.1](#). The first two terms can be treated in the same way, and in fact they all have the same asymptotics. We have

$$(\log q)^2 \sum_{D \in \mathcal{H}_{2g+1}^*} \mathcal{M}_{2,E_1}(X, [n_1/2]) \mathcal{M}_{2,E_2}(X, [n_2/2]) = (\log q)^2 \sum_{D \in \mathcal{H}_{2g+1}^*} (1 - \epsilon_1)(1 - \epsilon_2) \sum_{f,h \in \mathcal{M}_{\leq X}} \frac{([n_1/2] + 2g + 1 - \deg(f))([n_2/2] + 2g + 1 - \deg(h)) \lambda_1(f) \lambda_2(h) \chi_D(fh)}{\sqrt{|fh|}}.$$

By expanding out, this equals

$$\begin{aligned} & (\log q)^2 ([n_1/2] + 2g + 1) ([n_2/2] + 2g + 1) (S_{E_1, E_2}(1, X, X; 0, 0) - \epsilon_{2g+1} \epsilon(E_1) S_{E_1, E_2}(M_1, X, X; 0, 0) \\ & \quad - \epsilon_{2g+1} \epsilon(E_2) S_{E_1, E_2}(M_2, X, X; 0, 0) + \epsilon(E_1) \epsilon(E_2) S_{E_1, E_2}(M_1 M_2, X, X; 0, 0)) \\ & + (\log q) ([n_1/2] + 2g + 1) \frac{\partial}{\partial \beta} (S_{E_1, E_2}(1, X, X; 0, \beta) - \epsilon_{2g+1} \epsilon(E_1) S_{E_1, E_2}(M_1, X, X; 0, \beta) \\ & \quad - \epsilon_{2g+1} \epsilon(E_2) S_{E_1, E_2}(M_2, X, X; 0, \beta) + \epsilon(E_1) \epsilon(E_2) S_{E_1, E_2}(M_1 M_2, X, X; 0, \beta)) \Big|_{\beta=0} \\ & + (\log q) ([n_2/2] + 2g + 1) \frac{\partial}{\partial \alpha} (S_{E_1, E_2}(1, X, X; \alpha, 0) - \epsilon_{2g+1} \epsilon(E_1) S_{E_1, E_2}(M_1, X, X; \alpha, 0) \\ & \quad - \epsilon_{2g+1} \epsilon(E_2) S_{E_1, E_2}(M_2, X, X; \alpha, 0) + \epsilon(E_1) \epsilon(E_2) S_{E_1, E_2}(M_1 M_2, X, X; \alpha, 0)) \Big|_{\alpha=0} \\ & + \frac{\partial^2}{\partial \alpha \partial \beta} (S_{E_1, E_2}(1, X, X; \alpha, \beta) - \epsilon_{2g+1} \epsilon(E_1) S_{E_1, E_2}(M_1, X, X; \alpha, \beta) \\ & \quad - \epsilon_{2g+1} \epsilon(E_2) S_{E_1, E_2}(M_2, X, X; \alpha, \beta) + \epsilon(E_1) \epsilon(E_2) S_{E_1, E_2}(M_1 M_2, X, X; \alpha, \beta)) \Big|_{\alpha=\beta=0}. \end{aligned}$$

In view of [Proposition 3.1](#) and Cauchy’s residue theorem, this is

$$|\mathcal{H}_{2g+1}^*| c_7(M_1, M_2) L(\text{Sym}^2 E_1, 1) L(\text{Sym}^2 E_2, 1) L(E_1 \otimes E_2, 1) g^2 + O(q^{2g} g) + O(q^{g/2+3X/4} g^{32}),$$

where

$$\begin{aligned} c_7(M_1, M_2) = & 4(\log q)^2 (\mathcal{C}_{E_1, E_2}(1; 1, 1, 1, 0, 0) - \epsilon_{2g+1} \epsilon(E_1) \mathcal{C}_{E_1, E_2}(M_1; 1, 1, 1, 0, 0) \\ & - \epsilon_{2g+1} \epsilon(E_2) \mathcal{C}_{E_1, E_2}(M_2; 1, 1, 1, 0, 0) \\ & + \epsilon(E_1) \epsilon(E_2) \mathcal{C}_{E_1, E_2}(M_1 M_2; 1, 1, 1, 0, 0)) \prod_{P|\Delta} \frac{|P| + 1}{|P|}. \end{aligned} \tag{10-2}$$

The other two terms in [\(10-1\)](#) have the same asymptotics so we obtain

$$\begin{aligned} & \frac{1}{|\mathcal{H}_{2g+1}^*|} \sum_{D \in \mathcal{H}_{2g+1}^*} \epsilon_1^- \epsilon_2^- L'(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2}) \\ & = c_7(M_1, M_2) L(\text{Sym}^2 E_1, 1) L(\text{Sym}^2 E_2, 1) L(E_1 E_2, 1) g^2 \\ & \quad + O(g) + O(q^{-3g/2+3X/4} g^{32}) + O_\epsilon(g^{1/2+\epsilon} (2g - X)^4). \end{aligned}$$

Choosing  $X = 2g - 100 \log g$  we obtain the theorem.

### 11. Proof of Corollary 1.7

The results in Section 4 imply that

$$\sum_{D \in \mathcal{H}_{2g+1}^*} L(E \otimes \chi_D, \frac{1}{2})^4 \ll_{\varepsilon} q^{2g} g^{6+\varepsilon}. \tag{11-1}$$

We next obtain some upper bounds for moments of the derivatives. We have

$$L^{(l)}(E \otimes \chi_D, \frac{1}{2})^k = \left(\frac{l!}{2\pi i}\right)^k \oint \cdots \oint \frac{L(E \otimes \chi_D, 1/2 + \alpha_1) \cdots L(E \otimes \chi_D, 1/2 + \alpha_k)}{\alpha_1^{l+1} \cdots \alpha_k^{l+1}} d\alpha_1 \cdots d\alpha_k,$$

where we are integrating along small circles of radii  $r$  around the origin. Then using Hölder’s inequality leads to

$$\sum_{D \in \mathcal{H}_{2g+1}^*} |L^{(l)}(E \otimes \chi_D, \frac{1}{2})|^k \ll \frac{(l!)^k}{r^{lk+1}} \oint_{|\alpha|=r} \sum_{D \in \mathcal{H}_{2g+1}^*} |L(E \otimes \chi_D, \frac{1}{2} + \alpha)|^k d\alpha.$$

Choosing  $r = 1/g$  and using upper bounds for moments of  $L$ -functions we get that

$$\sum_{D \in \mathcal{H}_{2g+1}^*} |L^{(l)}(E \otimes \chi_D, \frac{1}{2})|^k \ll_{\varepsilon} q^{2g} (l!)^k g^{lk+k(k-1)/2+\varepsilon}.$$

In particular, with  $l = 1$  and  $k = 4$ , we have

$$\sum_{D \in \mathcal{H}_{2g+1}^*} L'(E \otimes \chi_D, \frac{1}{2})^4 \ll_{\varepsilon} q^{2g} g^{10+\varepsilon}. \tag{11-2}$$

Now from Hölder’s inequality we have

$$\begin{aligned} & \left(\sum_{\substack{D \in \mathcal{H}_{2g+1} \\ (D, \Delta_1)=1}} L(E_1 \otimes \chi_D, \frac{1}{2})^4\right) \left(\sum_{\substack{D \in \mathcal{H}_{2g+1} \\ (D, \Delta_2)=1}} L'(E_2 \otimes \chi_D, \frac{1}{2})^4\right) \left(\sum_{\substack{D \in \mathcal{H}_{2g+1}, (D, \Delta_1 \Delta_2)=1 \\ \epsilon_2^- L(E_1 \otimes \chi_D, 1/2) L'(E_2 \otimes \chi_D, 1/2) \neq 0}} 1\right)^2 \\ & \geq \left(\sum_{\substack{D \in \mathcal{H}_{2g+1} \\ (D, \Delta_1 \Delta_2)=1}} \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2})\right)^4. \end{aligned}$$

Combining (11-1) and (11-2) with Theorem 1.5 we get

$$\#\{D \in \mathcal{H}_{2g+1}^* : \epsilon_2^- L(E_1 \otimes \chi_D, \frac{1}{2}) L'(E_2 \otimes \chi_D, \frac{1}{2}) \neq 0\} \gg_{\varepsilon} \frac{q^{2g}}{g^{6+\varepsilon}},$$

which implies the first statement. The second statement can be obtained similarly.

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