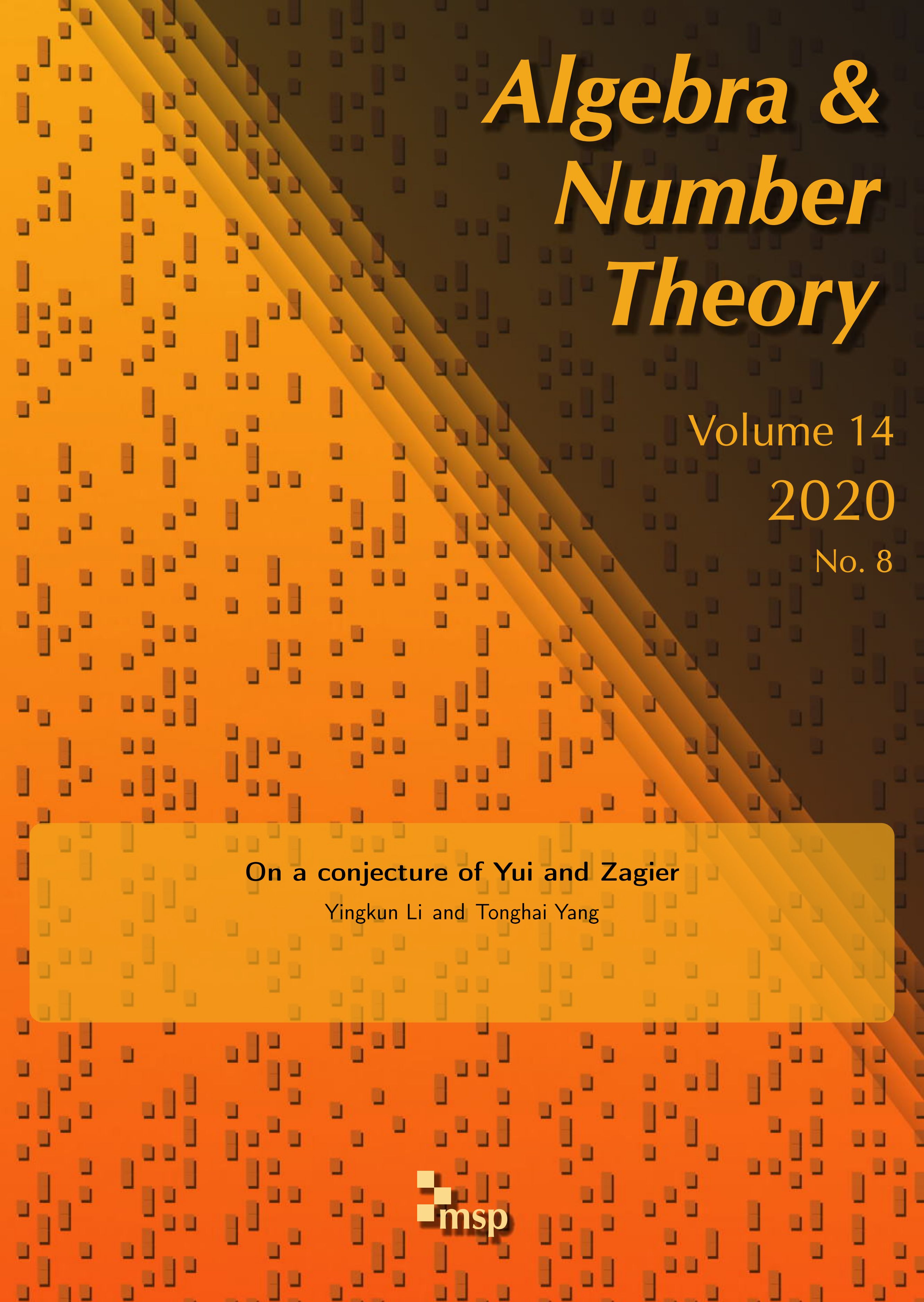

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On a conjecture of Yui and Zagier

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We prove the conjecture of Yui and Zagier concerning the factorization of the resultants of minimal polynomials of Weber class invariants. The novelty of our approach is to systematically express differences of certain Weber functions as products of Borcherds products.

1. Introduction

In his book, Weber [1908] proved the following well-known theorem in the theory of complex multiplication. For a fundamental discriminant $d < 0$, let $\mathcal{O}_d = \mathbb{Z}[\theta]$ be the ring of integers of an imaginary quadratic field $K_d = \mathbb{Q}(\sqrt{d})$. Then the CM value of the famous j-invariant $j(\tau)$ at $\tau = \theta$ is an algebraic integer generating the Hilbert class field of K_d. The number $j(\theta)$ is called *singular moduli* and plays an important role in the arithmetic of CM elliptic curves [Gross and Zagier 1985]. Weber also considered some special modular functions h of higher levels and observed that some of their CM values $h(\theta)$ still generate the Hilbert class field of K_d (for some choices of θ), not the larger class fields as expected for general h.

These amusing observations were later studied by various authors; see, for example, [Birch 1969; Yui and Zagier 1997; Gee 1999]. In particular, Gee gave a systematic proof of these facts using Shimura's reciprocity law. One of them concerns with the CM values of the three classical Weber functions of level 48, which are defined by the following quotients of η-functions:

$$\begin{aligned} \mathfrak{f}(\tau) &:= \zeta_{48}^{-1} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}), \\ \mathfrak{f}_1(\tau) &:= \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)} = q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}), \\ \mathfrak{f}_2(\tau) &:= \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)} = \sqrt{2} q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^{n}). \end{aligned} \tag{1-1}$$

Together, they form a 3-dimensional, vector-valued modular function for $\mathrm{SL}_2(\mathbb{Z})$; see (2-4). In fact, the same holds for integral powers of these modular functions; see [Milas 2007, p. 50]. Furthermore, $\mathfrak{f}_2$ is a modular function for $\Gamma_0(2)$ with character χ of order 24:

$$\mathfrak{f}_2(\gamma\tau) = \chi(\gamma)\mathfrak{f}_2(\tau), \quad \gamma \in \Gamma_0(2). \tag{1-2}$$

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The kernel of χ, denoted by $\Gamma_\chi \subset \Gamma_0(2)$, is a congruence subgroup containing $\Gamma(48)$; see (2-8). Yui and Zagier [1997] studied the CM values of these modular functions. The starting point of their work is the following result.

Proposition 1.1 [Yui and Zagier 1997, Proposition]. *Let $d < 0$ be a discriminant satisfying*

$$d \equiv 1 \bmod 8 \quad \text{and} \quad 3 \nmid d. \tag{1-3}$$

Denote $\varepsilon_d := (-1)^{(d-1)/8}$. For each proper ideal $\mathfrak{a} = \bigl[a, \frac{1}{2}(-b+\sqrt{d})\bigr]$ of the order $\mathcal{O}_d := \mathbb{Z}\bigl[\frac{1}{2}(1+\sqrt{d})\bigr]$ with $a > 0$, let $\tau_{\mathfrak{a}} = \frac{1}{2a}(-b+\sqrt{d})$ be the associated CM point and

$$f(\mathfrak{a}) = \begin{cases} \zeta_{48}^{b(a-c-ac^2)}\mathfrak{f}(\tau_{\mathfrak{a}}) & \text{if } 2 \mid (a,c), \\ \varepsilon_d \zeta_{48}^{b(a-c-ac^2)}\mathfrak{f}_1(\tau_{\mathfrak{a}}) & \text{if } 2 \mid a, 2 \nmid c, \\ \varepsilon_d \zeta_{48}^{b(a-c+a^2c)}\mathfrak{f}_2(\tau_{\mathfrak{a}}) & \text{if } 2 \nmid a, 2 \mid c. \end{cases} \tag{1-4}$$

Then $f(\mathfrak{a})$ is an algebraic integer depending only on the class of $\mathfrak{a}$ in the class group $\mathrm{Cl}(d)$ of $\mathcal{O}_d$, i.e., it is a class invariant. Moreover, $H_d := K_d(f(\mathfrak{a})) = K_d(j(\tau_{\mathfrak{a}}))$ is the ring class field of K_d corresponding $\mathcal{O}_d$.

Remark 1.2. The class invariant in [Yui and Zagier 1997] was defined using binary quadratic forms. It is a standard procedure to go between these and ideals in quadratic fields; see, e.g., [Cox 1989].

Remark 1.3. The sign ε_d in the definition of $f(\mathfrak{a})$ ensures that the class invariants behave nicely under the action of the Galois group. In particular when $d < 0$ is fundamental,

$$\sigma_{\mathfrak{a}_2}(f(\mathfrak{a}_1)) = f(\mathfrak{a}_1\mathfrak{a}_2^{-1}) \tag{1-5}$$

for any proper $\mathcal{O}_d$-ideals $\mathfrak{a}_1, \mathfrak{a}_2$, where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(H_d/K_d)$ is associated to the ideal class $[\mathfrak{a}] \in \mathrm{Cl}(d)$ by Artin's map. This was conjectured in [Yui and Zagier 1997] and proved in [Gee 1999, Proposition 22].

This class invariant is much better than the singular moduli in the sense that its minimal polynomial (class polynomial) has much smaller coefficients. This gives a generator of the Hilbert class field with small height, which is crucial in the speed of elliptic curve primality test [Atkin and Morain 1993]. For example, according to [Yui and Zagier 1997], the minimal polynomial of $j\bigl(\frac{1}{2}(1+\sqrt{-55})\bigr)$ is

$$x^4 + 3^3 5^3 29 \cdot 134219x^3 - 3^7 5^3 23 \cdot 101 \cdot 32987x^2 + 3^9 5^7 11^2 83 \cdot 101 \cdot 110641x - 3^{12} 5^6 11^3 29^3 41^3,$$

while the minimal polynomial of $f(\mathcal{O}_{-55})$ is simply

$$x^4 + x^3 - 2x - 1.$$

Yui and Zagier [1997] made conjectures about the prime factorizations of the discriminants and resultants of such polynomials. The goal of this paper is to prove the conjecture about the factorizations of the resultants, which also clears the path to prove the conjecture about the discriminant; see Remark 1.13.

For two co-prime, fundamental discriminants d_1 and d_2, Gross and Zagier [1985] proved a beautiful factorization formula for the resultant of the class polynomials of $j\big(\frac{1}{2}(d_1+\sqrt{d_1})\big)$ and $j\big(\frac{1}{2}(d_2+\sqrt{d_2})\big)$, which is the norm of the difference $j\big(\frac{1}{2}(d_1+\sqrt{d_1})\big)-j\big(\frac{1}{2}(d_2+\sqrt{d_2})\big)$. When $\big(\frac{d_1d_2}{p}\big)\neq -1$, set

$$\epsilon(p)=\begin{cases}\left(\dfrac{d_1}{p}\right) & \text{if } p\nmid d_1,\\[2ex] \left(\dfrac{d_2}{p}\right) & \text{if } p\nmid d_2.\end{cases}$$

Define in general $\epsilon(n)=\prod_{p\mid n}\epsilon(p)^{\operatorname{ord}_p(n)}$, where $\operatorname{ord}_p(n)$ is the power of p dividing n. For a positive integer m, if $\epsilon(m)=-1$, define

$$\mathfrak{F}(m)=\prod_{\substack{nn'=m\\ n,n'>0}} n^{\epsilon(n')}\in\mathbb{N}, \tag{1-6}$$

which is always a prime power. If $\epsilon(m)=1$ or is not defined, define $\mathfrak{F}(m)=1$. The result of Gross and Zagier can be stated as follows.

Theorem 1.4 [Gross and Zagier 1985, Theorem 1.3]. *Let $d_1, d_2<0$ be co-prime, fundamental discriminants, and $w_j=|\mathcal{O}_{d_j}^\times|$. In the notations above, we have*

$$J(d_1,d_2)^2:=\prod_{[\mathfrak{a}_j]\in\mathrm{Cl}(d_j),\, j=1,2}|j(\tau_{\mathfrak{a}_1})-j(\tau_{\mathfrak{a}_2})|^{8/(w_1w_2)}=\prod_{\substack{m\in\mathbb{N},\, a\in\mathbb{Z}\\ a^2+4m=d_1d_2}}\mathfrak{F}(m). \tag{1-7}$$

Inspired by this beautiful formula, Yui and Zagier [1997] gave a conjectural formula of the resultant of the minimal polynomials of the Weber class invariants defined above and provided numerical evidence. This conjecture was originally given using two tables with totally 48 entries (see [Yui and Zagier 1997, p. 1653]), but can be simplified and formulated in the following elegant way (see, e.g., $(14_?)$ in [Yui and Zagier 1997] for $d_1\equiv d_2\equiv 1 \bmod 24$).

Conjecture 1.5 [Yui and Zagier 1997, $(14_?)$]. *Let d_1, d_2 be co-prime, fundamental discriminants satisfying (1-3) and $s\mid 24$. Define the constant*

$$\kappa_3(s):=\begin{cases}\dfrac{1}{2} & \textit{if } \left(\dfrac{d_1}{3}\right)=\left(\dfrac{d_2}{3}\right)=-1 \textit{ and } 3\mid s,\\[1ex] 1 & \textit{otherwise,}\end{cases} \tag{1-8}$$

which only depends on d_1, d_2 and s. Then

$$f_s(d_1,d_2):=\prod_{[\mathfrak{a}_j]\in\mathrm{Cl}(d_j),\, j=1,2}|f(\mathfrak{a}_1)^{24/s}-f(\mathfrak{a}_2)^{24/s}|=\prod_{\substack{m,a\in\mathbb{N},\, r\mid s\\ a^2+16mr^2=d_1d_2\\ m\equiv 19(d_1+d_2-1) \bmod s/r}}\mathfrak{F}(m)^{\kappa_3(s)}. \tag{1-9}$$

Remark 1.6. Because of the relation $j(\tau)=(\mathfrak{f}_2^{24}(\tau)-16)^3/\mathfrak{f}_2^{24}(\tau)$, we know that $f_s(d_1,d_2)\mid J(d_1,d_2)$ for any co-prime, fundamental discriminants d_1, d_2 satisfying (1-3). Since the invariants are algebraic integers, it is also clear that $f_{s'}(d_1,d_2)\mid f_s(d_1,d_2)$ for any $s\mid s'\mid 24$. The conjecture above also reflects such divisibilities since $\mathfrak{F}(m/r^2) \mid \mathfrak{F}(m)$ for all $m, r \in \mathbb{N}$; see, e.g., the explicit formula of $\mathfrak{F}(m)$ on [Yui and Zagier 1997, p. 1651].

When $s = 1$, it was suggested in [Yui and Zagier 1997] that one can try to prove this conjecture by adapting the analytic approach in [Gross and Zagier 1985] with $\mathrm{SL}_2(\mathbb{Z})$ replaced by $\Gamma_0(2)$. This was later carried out in [Roskam 2003]. Yang and Yin [2019] gave another analytic proof of the conjecture for $s = 1$, where the new ingredients are Borcherds' regularized theta lift [1998] and the big CM formula in [Bruinier et al. 2012]. Although the spirits of the approaches are the same, the one in [Yang and Yin 2019] is conceptually easier to understand and opens the door to attack the conjecture for $s > 1$. In this paper, we complete the proof of the conjecture for all $s \mid 24$.

Theorem 1.7. *Conjecture 1.5 is true for every* $s \mid 24$.

For $s = 1$, the proof of Theorem 1.7 in [Yang and Yin 2019] consists of three steps:

(1) Relate $\mathfrak{f}_2(z_1)^{24} - \mathfrak{f}_2(z_2))^{24}$ to a Borcherds product on the Shimura variety associated to the rational quadratic space $(M_2(\mathbb{Q}), \det)$.[1]

(2) View a pair of CM points (τ_1, τ_2) as a big CM point on this Shimura variety in the sense of [Bruinier et al. 2012]. Apply the big CM value formula [Bruinier et al. 2012, Theorem 5.2] and express the CM value in terms of Fourier coefficients of incoherent Eisenstein series.

(3) Compute the Fourier coefficients in Step (2) and obtain the formula. This is a local calculation.

In the first step for $s = 1$, one can find a vector-valued modular function $\widetilde{F}_1$ and identify $\mathfrak{f}_2(z_1)^{24} - \mathfrak{f}_2(z_2)^{24}$ with the Borcherds product $\Psi(z_1, z_2, \widetilde{F}_1)$ associated to $\widetilde{F}_1$. Note $\mathfrak{f}_2(z)^{24} = 2^{12}(\Delta(2z)/\Delta(z))$ is a Hauptmodul of $\Gamma_0(2)$, and the Borcherds product $\Psi(z_1, z_2, \widetilde{F}_1)$ is well-known in the literature on VOA and moonshine (see, e.g., [Borcherds 1992; Scheithauer 2008]). In the second step, one suitably identifies the Galois orbit of CM points with the toric orbit of big CM points, and apply Theorem 5.2 in [Bruinier et al. 2012]. This reduces the proof to the third step, where the local calculations have been completed in many special cases (see [Yang 2005; Howard and Yang 2012, Section 4.6; Kudla and Yang 2010]) and the most general result can be found in Appendix A of [Yang et al. 2019].

To execute this strategy for $s > 1$, we first need to relate $\mathfrak{f}_2(z_1)^{24/s} - \mathfrak{f}_2(z_2)^{24/s}$ to Borcherds product. Since the function $\mathfrak{f}_2(z)^{24/s}$ is invariant with respect to $\Gamma_{\chi,s} := \langle \Gamma_\chi, T^s \rangle \supset \Gamma(2s)$, one would hope to find the analog of $\widetilde{F}_1$ in $M^!(\omega_s)$, with ω_s the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on the finite quadratic module associated to the lattice L_s (see (3-1)), which is the same as the lattice used in [Yang and Yin 2019] to produce $\Psi(z_1, z_2, \widetilde{F}_1)$, but with the quadratic form scaled by s. We have computationally decomposed the representation ω_s and analyzed the space of vector-valued modular functions. To our surprise, there is *no* modular function whose Borcherds product equals to $(\mathfrak{f}_2(z_1)^{24/s} - \mathfrak{f}_2(z_2)^{24/s})^s$! Our new idea then is to express $(\mathfrak{f}_2(z_1)^{24/s} - \mathfrak{f}_2(z_2)^{24/s})^s$ as *a product of Borcherds products*, which works out beautifully.

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[1] The Shimura variety is just the product of two modular curves in this case.

Theorem 1.8 (Theorems 4.4 and 4.5). *For every* $\mathrm{d} \mid 24$, *there is a vector-valued modular function* $\widetilde{F}_{\mathrm{d}} \in M^!(\omega_{\mathrm{d}})$ *with associated Borcherds product* $\Psi_{\mathrm{d}}(z_1, z_2) := \Psi(z_1, z_2, \widetilde{F}_{\mathrm{d}})$ *such that*

$$(\mathfrak{f}_2(z_1)^{24/s} - (\varepsilon \mathfrak{f}_2(z_2))^{24/s})^s = \prod_{\mathrm{d}\mid s} \Psi_{\mathrm{d}}(z_1, z_2)^{\varepsilon^{\frac{24}{\mathrm{d}}}}, \tag{1-10}$$

for every $s \mid 24$ *and any* $\varepsilon = \pm 1$.

Remark 1.9. The index $r \mid s$ in the product on the right-hand side of (1-9) is *not* directly related to the index $\mathrm{d} \mid s$ in the product above! Instead, it comes out of local calculation in Section 6.

Remark 1.10. Each Borcherds product $\Psi_{\mathrm{d}}(z_1, z_2)$ comes from a different quadratic space depending on d, and is a meromorphic function on the Shimura variety X^2_{d}, which admits a natural covering map from X^2_s when $\mathrm{d} \mid s$ (see Section 4). One can then pull back Ψ_{d} to a function on X^2_s. Notice that this decomposes the divisor of the left-hand side, which is a Heegner divisor on X^2_s, into a sum of pullbacks of Heegner divisors on X^2_{d} with $\mathrm{d} \mid s$. When $s > 1$, the product $\prod_{\mathrm{d}\mid s} \Psi_{\mathrm{d}}(z_1, z_2)$ is itself *not* a single Borcherds product on X^2_s.

Remark 1.11. Theorem 1.8 naturally leads one to speculate a generalization of the converse theorem in [Bruinier 2014], namely every principal Heegner divisor on an orthogonal Shimura variety associated to a lattice of signature $(n, 2)$ with Witt rank greater than or equal to 2 should be the divisor of a product of Borcherds products.

To arrive at this idea, we took $s = 2$ and started from the simple observation that

$$(\mathfrak{f}_2(z_1)^{12} - \mathfrak{f}_2(z_2)^{12})^2 = (\mathfrak{f}_2(z_1)^{24} - \mathfrak{f}_2(z_2)^{24}) \cdot \frac{\mathfrak{f}_2(z_1)^{12} - \mathfrak{f}_2(z_2)^{12}}{\mathfrak{f}_2(z_1)^{12} + \mathfrak{f}_2(z_2)^{12}}. \tag{1-11}$$

We already know that the first factor on the right-hand side is a Borcherds product. If we can realize the second factor as a Borcherds product, then the left-hand side would be a product of Borcherds products (with different quadratic forms). To do that, we can read off the divisor of the second factor, and deduce the principal part of the input to Borcherds' lift. In this case, it is of the form $q^{-1/2}\mathfrak{u}_2$ for a suitable vector $\mathfrak{u}_2$ in a 64 dimensional vector space $\mathbb{C}[\mathcal{A}_2]$, where $\mathrm{SL}_2(\mathbb{Z})$ acts via the Weil representation ω_2 (see Section 3A for details). Then we find the irreducible representation in ω_2 containing $\mathfrak{u}_2$, which is 3-dimensional, and hope to find the suitable vector-valued modular function $\widetilde{F}_2$ with this principal part. Miraculously, this function exists and its three components are the $(-24/2)$-th power of the three Weber functions $\frac{\mathfrak{f}}{\sqrt{2}}, \frac{\mathfrak{f}_1}{\sqrt{2}}, \frac{\mathfrak{f}_2}{\sqrt{2}}$.

The observation (1-11) generalizes to any $s \mid 24$ by substituting $X = (\varepsilon \mathfrak{f}_2(z_2)/\mathfrak{f}_2(z_1))^{24/s}$ into the following simple identity in $\mathbb{Q}(X)$

$$(1 - X)^s = \prod_{\mathrm{d}\mid s} \prod_{b\mid \mathrm{d}} (1 - X^{s/b})^{b\cdot\mu(\mathrm{d}/b)}, \tag{1-12}$$

where μ is the Möbius function, and multiplying by $\mathfrak{f}_2(z_1)^{24}$ on both sides. Note that the identity in (1-12) holds for any $s \in \mathbb{N}$ (see Lemma 4.3). Then the miracle continues to happen, and we find a family of vectors $\{\mathfrak{u}_{\mathrm{d}} : \mathrm{d} \mid 24\}$ (see (3-12)) and vector-valued modular functions $\widetilde{F}_{\mathrm{d}} = q^{-1/\mathrm{d}}\mathfrak{u}_{\mathrm{d}} + O(q^{1/(2\mathrm{d})})$ producing the Borcherds lifts Ψ_{d} (see (2-4) and Remark 3.7). For $\mathrm{d} > 1$, the vector $\mathfrak{u}_{\mathrm{d}}$ satisfies nice invariance properties (see Proposition 3.5) and is of independent interest, whereas the components of $\widetilde{F}_{\mathrm{d}}$ are simply the $(-24/\mathrm{d})$-th power of the three Weber functions $\frac{\mathfrak{f}}{\sqrt{2}}, \frac{\mathfrak{f}_1}{\sqrt{2}}, \frac{\mathfrak{f}_2}{\sqrt{2}}$!

Remark 1.12. The $\varepsilon = \pm 1$ in Theorem 1.8 is there for a good reason. To prove the Yui–Zagier conjecture, we need to choose $\varepsilon = \varepsilon_{d_1}\varepsilon_{d_2} = (-1)^{(d_1+d_2-2)/8}$ (see Proposition 5.5 and its proof). It is also amusing to see that the same ε appears when we calculate the Fourier coefficients of derivatives of certain Eisenstein series (see Theorem 6.2).

To complete the proof, we can now apply the second step to each Borcherds product, obtain a big CM value formula, and add them together. Note that the identification of the Galois orbit of $(\tau_{\mathfrak{a}_1}, \tau_{\mathfrak{a}_2})$ used in defining $f_s(d_1, d_2)$ with the big CM cycle in [Bruinier et al. 2012] depends on the input $\widetilde{F}_{\mathrm{d}}$ in Step (1). Therefore, it is not a priori clear that this will work out. We prove this in Proposition 5.5, which crucially depends on Lemma 5.2. This unexpected result was first observed with some computer calculations, and has been reduced to a computation with finite groups in $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z})$. Finally, we apply the local calculations in [Yang et al. 2019] to finish off Step (3).

Remark 1.13. With Theorem 1.8, one can now replace the big CM value formula in [Bruinier et al. 2012] with the small CM value formula in [Schofer 2009] to prove the conjectural factorization of the discriminant of the minimal polynomials of the Weber invariants in [Yui and Zagier 1997]. We plan to carry these out as a sequel to this work [Li and Yang $\geq$ 2021].

This paper is organized as follows. After setting up notation and defining basic terms in Section 2, we study in Section 3 the action of certain subgroup $H'_{\mathrm{d}} \subset \mathrm{SO}(L_{\mathrm{d}})/\Gamma_{L_{\mathrm{d}}}$ on the finite quadratic module $\mathcal{A}_{\mathrm{d}} := L_{\mathrm{d}}^{\vee}/L_{\mathrm{d}}$ and use it to decompose the Weil representation ω_{d} of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[\mathcal{A}_{\mathrm{d}}]^{H'_{\mathrm{d}}}$. The goal and main result is to construct certain element $\mathfrak{u}_{\mathrm{d}} \in \mathbb{C}[\mathcal{A}_{\mathrm{d}}]^{H'_{\mathrm{d}}}$ satisfying (3-13). This vector generates a 3-dimensional, H'_{d}-invariant subrepresentation of ω_{d}, and will be crucial in finding the input $\widetilde{F}_{\mathrm{d}}$ that produces the Borcherds product Ψ_{d}. In Section 4, we view product of two modular curves as a Shimura variety of orthogonal type $(2, 2)$ associated to L_{d}, construct the Borcherds product Ψ_{d}, and prove Theorem 1.8. In Section 5, we view the pair $(\tau_{\mathfrak{a}_1}, \tau_{\mathfrak{a}_2})$ as a big CM point on the product of two modular curves and study its Galois orbit. The upshot is Proposition 5.5, which relates the left-hand side of Conjecture 1.5 to the big CM value of Borcherds products. By the second step of strategy, Conjecture 1.5 is reduced to local calculation of certain Eisenstein series and its derivative, which we carry out in Section 6B using the results in the appendix of [Yang et al. 2019]. Finally in the Appendix, we explicitly write down the cosets in the finite quadratic module used in constructing the Borcherds products, and include a numerical example for $d_1 = -31$ and $d_2 = -127$.

2. Preliminaries

2A. ***Weil representation.*** Let (L, Q) be an even integral lattice of signature $(2, 2)$ and $V := L \otimes \mathbb{Q}$ the rational quadratic space. Denote L' the dual lattice and $\mathcal{A}_L := L'/L$ the finite quadratic module. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $U_L := \mathbb{C}[\mathcal{A}_L]$ via the Weil representation ω_L given by

$$\omega_L(T)\mathfrak{e}_h = \mathbf{e}(-Q(h))\mathfrak{e}_h, \quad \omega_L(S)\mathfrak{e}_h = \frac{1}{\sqrt{|\mathcal{A}_L|}} \sum_{\mu \in \mathcal{A}_L} \mathbf{e}((\mu, h))\mathfrak{e}_\mu, \tag{2-1}$$

where $\{\mathfrak{e}_\mu : \mu \in \mathcal{A}_L\}$ is the standard basis of U_L and

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2-2}$$

Note that this differs from the convention of Borcherds by complex conjugation.

Let $S(L) = \bigoplus_{\mu \in L'/L} \phi_\mu \subset S(V \otimes \mathbb{A}_f)$ with $\widehat{L} = L \otimes \widehat{\mathbb{Z}}$ and

$$\phi_\mu = \mathrm{Char}(\mu + \widehat{L}).$$

Under the isomorphism $U_L \to S(L)$ that maps $\mathfrak{e}_\mu$ to ϕ_μ, the representation ω_L becomes the restriction of the Weil representation $\omega = \omega_{V,\psi}$ (with the usual idelic character ψ of $\mathbb{Q}$) from $\mathrm{SL}_2(\mathbb{A})$ to (the diagonally embedded) $\mathrm{SL}_2(\mathbb{Z})$. We will sometimes switch the representation spaces between U_L and $S(L)$. Note that $S(L_1 \oplus L_2) = S(L_1) \otimes S(L_2)$ for any two sublattices $L_1, L_2 \subset L$ orthogonal to each other.

2B. ***Weber functions.*** For any finite-dimensional, $\mathbb{C}$-representation $\rho : \Gamma \to V$ of a finite index subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, denote $M^!(\rho, \Gamma)$ the space of weakly holomorphic, vector-valued modular function with respect to ρ. We drop ρ (resp. Γ) from the notation if ρ is trivial (resp. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$). For example, the three Weber functions defined by (1-1) form a vector-valued modular function

$$\begin{pmatrix} \mathfrak{f}_2 \\ \mathfrak{f}_1 \\ \mathfrak{f} \end{pmatrix} \in M^!(\overline{\varrho_{24}}).$$

Here, for a positive integer d and $j \in (\mathbb{Z}/2\mathrm{d}\mathbb{Z})^\times$, the representation $\varrho_{\mathrm{d},j} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{C})$ is defined by

$$\varrho_{\mathrm{d},j}(T) := \begin{pmatrix} \zeta_{\mathrm{d}}^{-j} & 0 & 0 \\ 0 & 0 & \zeta_{2\mathrm{d}}^{j} \\ 0 & \zeta_{2\mathrm{d}}^{j} & 0 \end{pmatrix}, \quad \varrho_{\mathrm{d},j}(S) := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2-3}$$

We simply write ϱ_{d} for $\varrho_{\mathrm{d},1}$. Finally, $\bar{\rho}(g) := \overline{\rho(g)}$. Later, the modular function

$$F_{\mathrm{d}}(\tau) := \sqrt{2}^{24/\mathrm{d}} \begin{pmatrix} \mathfrak{f}_2^{-24/\mathrm{d}}(\tau) \\ \mathfrak{f}_1^{-24/\mathrm{d}}(\tau) \\ \mathfrak{f}^{-24/\mathrm{d}}(\tau) \end{pmatrix} \in M^!(\varrho_{\mathrm{d}}), \ \mathrm{d} \mid 24 \tag{2-4}$$

will play an important role for us as the representation ϱ_{d} defined above is a subrepresentation of certain Weil representation that we will consider.

Remark 2.1. For convenience later, we will denote

$$\sqrt{2}^{24/\mathrm{d}} \mathfrak{f}_2^{-24/\mathrm{d}}(\tau) = \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{24/\mathrm{d}} = \sum_{l \geq -1, l \equiv -1 \bmod \mathrm{d}} c_{\mathrm{d}}(l) q^{l/\mathrm{d}} \in q^{-1/\mathrm{d}} \mathbb{Z}[\![q]\!]. \tag{2-5}$$

Clearly $c_{\mathrm{d}}(-1) = 1$ for every $\mathrm{d} \mid 24$. We will also denote $c_{-1}(l)$ the l-th Fourier coefficient of

$$2^{12}\mathfrak{f}_1^{-24}(2\tau) = \mathfrak{f}_2^{24}(\tau) = 2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)}.$$

Let $\chi : \Gamma_0(2) \to \mathbb{C}^\times$ be the character defined in (1-2). On the generators T, S^2 and TB of $\Gamma_0(2)$, where

$$B := ST^2S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \tag{2-6}$$

the character χ is explicitly given by

$$\chi(T) = \zeta_{24}, \quad \chi(S^2) = 1, \quad \text{and} \quad \chi(TB) = 1. \tag{2-7}$$

The kernel of χ is a normal subgroup of $\Gamma_0(2)$ defined by

$$\Gamma_\chi := \langle \Gamma_0(2)^{\mathrm{der}}, T^{24}, S^2, TB \rangle \subset \Gamma_0(2), \tag{2-8}$$

where $\Gamma_0(2)^{\mathrm{der}}$ is the derived subgroup of $\Gamma_0(2)$. We remark that Γ_χ is the group $\Phi_0^0(24)$ in [Yang and Yin 2016]. Furthermore, it contains the congruence subgroup $\Gamma_0(48) \cap \Gamma(24)$ and $\Gamma_0(2)/\Gamma_\chi \cong \mathbb{Z}/24$. More generally, for any divisor $\mathrm{d} \mid 24$, denote the kernel of $\chi^{24/\mathrm{d}}$ by

$$\Gamma_{\chi,\mathrm{d}} := \langle \Gamma_\chi, T^{\mathrm{d}} \rangle \subset \Gamma_0(2). \tag{2-9}$$

It has index d in $\Gamma_0(2)$ and contains $\Gamma_\chi = \Gamma_{\chi,24}$, as well as the congruence subgroup

$$\Gamma_{\mathrm{d}} := \Gamma_1(2\mathrm{d}) \cap \Gamma(\mathrm{d}). \tag{2-10}$$

In particular, $\Gamma_0(2) = \Gamma_{\chi,1}$. More generally for $\mathrm{d} \mid \mathrm{d}' \mid 24$, we have $\Gamma_{\mathrm{d}} \supset \Gamma_{\mathrm{d}'}$. For future convenience, we also write d_p for the p-primary part of d. Then clearly $\mathrm{d} = \mathrm{d}_2\mathrm{d}_3$.

3. Decomposition of Weil representations

3A. ***Lattice.*** For a divisor $\mathrm{d} \mid 24$, consider the quadratic lattice

$$L_{\mathrm{d}} = \left\{ \lambda = \begin{pmatrix} \lambda_{00} & \lambda_{01} \\ 2\lambda_{10} & \lambda_{11} \end{pmatrix} : \lambda_{ij} \in \mathbb{Z} \right\}, \quad Q_{\mathrm{d}}(\lambda) := \mathrm{d}\det(\lambda). \tag{3-1}$$

The dual lattice is given by

$$L'_{\mathrm{d}} = \left\{ \lambda = \frac{1}{\mathrm{d}} \begin{pmatrix} \lambda_{00} & \lambda_{01}/2 \\ \lambda_{10} & \lambda_{11} \end{pmatrix} : \lambda_{ij} \in \mathbb{Z} \right\}. \tag{3-2}$$

The finite quadratic module $L'_{\mathrm{d}}/L_{\mathrm{d}}$ is then isomorphic to

$$\mathcal{A}_{\mathrm{d}} := \{ h = [h_0, h_1, h_2, h_3] : h_0, h_3 \in \mathbb{Z}/\mathrm{d}\mathbb{Z}, h_1, h_2 \in \mathbb{Z}/(2\mathrm{d}\mathbb{Z}) \}, \tag{3-3}$$

where the isomorphism is fixed throughout and given by

$$L'_{\mathrm{d}}/L_{\mathrm{d}} \cong \mathcal{A}_{\mathrm{d}}, \qquad \frac{1}{\mathrm{d}} \begin{pmatrix} \lambda_{00} & \lambda_{01}/2 \\ \lambda_{10} & \lambda_{11} \end{pmatrix} + L_{\mathrm{d}} \mapsto [\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}]. \tag{3-4}$$

Via this isomorphism, the quadratic form Q_{d} on $\mathcal{A}_{\mathrm{d}}$ becomes

$$Q_{\mathrm{d}}(h) := \frac{2h_0h_3 - h_1h_2}{2\mathrm{d}} \in \frac{1}{2\mathrm{d}}\mathbb{Z}/\mathbb{Z} \tag{3-5}$$

for $h = [h_0, h_1, h_2, h_3] \in \mathcal{A}_{\mathrm{d}}$. We denote $U_{\mathrm{d}} := \mathbb{C}[\mathcal{A}_{\mathrm{d}}]$, which is acted on by $\mathrm{SL}_2(\mathbb{Z})$ via the Weil representation $\omega_{\mathrm{d}} := \omega_{L_{\mathrm{d}}}$.

Now, we can map L_{d} into $L'_{\mathrm{d}}/L_{\mathrm{d}} \cong \mathcal{A}_{\mathrm{d}}$ via

$$\kappa_{\mathrm{d}} : L_{\mathrm{d}} \to L'_{\mathrm{d}}/L_{\mathrm{d}}, \quad \lambda \mapsto \frac{1}{\mathrm{d}}\lambda + L_{\mathrm{d}}, \tag{3-6}$$

which is compatible with the left and right action of $\Gamma_0(2)$, i.e.,

$$g_1 \cdot \kappa_{\mathrm{d}}(\lambda \cdot g_2) = \kappa_{\mathrm{d}}(g_1 \cdot \lambda \cdot g_2) = \kappa_{\mathrm{d}}(g_1 \cdot \lambda) \cdot g_2 \tag{3-7}$$

for all $g_1, g_2 \in \Gamma_0(2)$ and $\lambda \in L_{\mathrm{d}}$. By viewing $\Gamma_{\chi,1} = \Gamma_0(2)$ as a subset of L_{d}, we can send it to a subset in $\mathcal{A}_{\mathrm{d}}$. If we denote

$$\mathcal{A}_{\mathrm{d}}^0 := \{[h_0, h_1, h_2, h_3] \in \mathcal{A}_{\mathrm{d}} : h_2 = 0 \in \mathbb{Z}/(2\mathrm{d}\mathbb{Z})\}, \tag{3-8}$$

it will be helpful to know the parts of $\Gamma_0(2)$ that land in $\mathcal{A}_{\mathrm{d}}^0$ under κ_{d} when we simplify the expression of Borcherds products. For this we need the following lemma, whose proof will follow from combining the corresponding local results in Lemmas 3.8 and 3.12.

Lemma 3.1. *For any $j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}$, we have (viewing $\Gamma_0(2) \subset L_{\mathrm{d}}$)*

$$\kappa_{\mathrm{d}}(T^j\Gamma_{\chi,\mathrm{d}}) \cap \mathcal{A}_{\mathrm{d}}^0 = \{\mathrm{d}_3^{-1}[r, r(2j + (r^2 - 1)), 0, r] : r \in (\mathbb{Z}/\mathrm{d}\mathbb{Z})^\times\}.$$

Remark 3.2. Note that $r^3 - r \bmod 2\mathrm{d}$ is well-defined for $r \in (\mathbb{Z}/\mathrm{d}\mathbb{Z})^\times$ when $\mathrm{d} \mid 24$. Furthermore

$$r^3 - r \equiv \begin{cases} \mathrm{d} \bmod 2\mathrm{d} & \text{if } 8 \mid \mathrm{d} \text{ and } r \equiv \pm 3 \bmod 8, \\ 0 \bmod 2\mathrm{d} & \text{otherwise.} \end{cases}$$

Let $\mathrm{GL}_2(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{Q})$ acts on $V_{\mathrm{d}} = L_{\mathrm{d}} \otimes \mathbb{Q} = M_2(\mathbb{Q})$ via

$$(g_1, g_2) \cdot X = g_1 X g_2^{-1}.$$

This action gives an identification of $\mathrm{GSpin}(V)$ with $H = \{(g_1, g_2) : \det g_1 = \det g_2\}$, and a commutative diagram of exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d] & \mathrm{SL}_2 \times \mathrm{SL}_2 \arrow[r] \arrow[d] & \mathrm{SO}(V) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & H \arrow[r] & \mathrm{SO}(V) \arrow[r] & 1.
\end{tikzcd}$$

For the particular lattice L_{d}, we have

$$\mathrm{SO}(L_{\mathrm{d}}) = \overline{\Gamma_0(2) \times \Gamma_0(2)} = \Gamma_0(2) \times \Gamma_0(2)/\{\pm(I_2, I_2)\}, \tag{3-9}$$

As $\{\pm(I_2, I_2)\}$ does not matter in this paper, we will simply identify $\mathrm{SO}(L_{\mathrm{d}})$ with $\Gamma_0(2)\times\Gamma_0(2)$ and drop the overline. Under this identification, we have $\Gamma_{L_{\mathrm{d}}} = \Gamma_{\mathrm{d}}\times\Gamma_{\mathrm{d}}$, where Γ_{d} is defined in (2-10). In particular, we are interested in the action of the subgroup of $\mathrm{SO}(L_{\mathrm{d}})$ generated by the images of (T, T) and $\Gamma_{\chi,\mathrm{d}}\times\Gamma_{\chi,\mathrm{d}}$. We let H'_{d} be its image in $H_{\mathrm{d}} := N_{\mathrm{d}}\times N_{\mathrm{d}}$, where $N_{\mathrm{d}} := \Gamma_0(2)/\Gamma_{\mathrm{d}}$. Let $N'_{\mathrm{d}} := \Gamma_{\chi,\mathrm{d}}/\Gamma_{\mathrm{d}}$. Then

$$N'_{\mathrm{d}}\times N'_{\mathrm{d}}\subset H'_{\mathrm{d}}\subset H_{\mathrm{d}} = N_{\mathrm{d}}\times N_{\mathrm{d}}. \tag{3-10}$$

Since $\Gamma_0(2)/\Gamma_{\chi,\mathrm{d}}\cong\mathbb{Z}/\mathrm{d}\mathbb{Z}$, the quotient group $H_{\mathrm{d}}/H'_{\mathrm{d}}$ is isomorphic to $\mathbb{Z}/\mathrm{d}\mathbb{Z}$. For prime p, let $N_{\mathrm{d},p}$ denote the quotient of the subgroups generated respectively by $\Gamma_0(2)$ and Γ_{d} in $\mathrm{SL}_2(\mathbb{Z}_p)$. Similarly, we can also define K_p for $K\in\{N_{\mathrm{d}}, N'_{\mathrm{d}}, H_{\mathrm{d}}, H'_{\mathrm{d}}\}$. Since d is only divisible by 2 and 3 in our case, the Chinese remainder theorem implies

$$H_{\mathrm{d}}\cong H_{\mathrm{d},2}\times H_{\mathrm{d},3},\quad H'_{\mathrm{d}}\cong H'_{\mathrm{d},2}\times H'_{\mathrm{d},3},\quad N'_{\mathrm{d},p}\times N'_{\mathrm{d},p}\subset H'_{\mathrm{d},p}\subset H_{\mathrm{d},p} = N_{\mathrm{d},p}\times N_{\mathrm{d},p}.$$

For the same reason, we have the decomposition

$$\mathcal{A}_{\mathrm{d}}\cong\mathcal{A}_{\mathrm{d},2}\times\mathcal{A}_{\mathrm{d},3},\quad \mathcal{A}_{\mathrm{d},p} := \mathcal{A}_{\mathrm{d}}\otimes_{\mathbb{Z}}\mathbb{Z}_p. \tag{3-11}$$

Using this isomorphism, we can write $\omega_{\mathrm{d}}\cong\omega_{\mathrm{d},2}\otimes\omega_{\mathrm{d},3}$ and $U_{\mathrm{d}}\cong U_{\mathrm{d},2}\otimes U_{\mathrm{d},3}$, where $\omega_{\mathrm{d},p}$ is the Weil representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ acting on $U_{\mathrm{d},p}$ associated to $\mathcal{A}_{\mathrm{d},p}$.

Now, we introduce the vector $\mathfrak{u}_{\mathrm{d}}\in U_{\mathrm{d}}$.

$$\begin{aligned}\mathfrak{u}_{\mathrm{d}} &:= \mathfrak{u}_{\mathrm{d},2}\otimes\mathfrak{u}_{\mathrm{d},3} = \sum_{j\in\mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j)\Bigg(\sum_{h\in\kappa_{\mathrm{d}}(T^j\Gamma_{\chi,\mathrm{d}})}\mathfrak{e}_h\Bigg),\\ a_{\mathrm{d}}(j) &:= \Bigg(\sum_{s\in(\mathbb{Z}/\mathrm{d}\mathbb{Z})^\times}\zeta_{\mathrm{d}}^{sj}\Bigg) = \mu\bigg(\frac{\mathrm{d}}{(\mathrm{d}, j)}\bigg)\frac{\varphi(\mathrm{d})}{\varphi(\mathrm{d}/(\mathrm{d}, j))}\in\mathbb{Z},\end{aligned} \tag{3-12}$$

where $\mathfrak{u}_{\mathrm{d},p} = \mathfrak{u}_{\mathrm{d},p}(1,\dots,1)\in U'_{\mathrm{d},p}\subset U_{\mathrm{d},p}$ is the vector defined in (3-23) and (3-33), μ and φ are the Möbius and Euler φ-function respectively. Note that $a_{\mathrm{d}}(j)$ is defined for any $\mathrm{d}\in\mathbb{N}$ and $j\in\mathbb{Z}/\mathrm{d}\mathbb{Z}$.

Remark 3.3. A natural question is where the element $\mathfrak{u}_{\mathrm{d},p}$ comes from and what it is good for? In the next two subsections, we will give some ideas where they come from. For now, we are satisfied to give its nice properties as below. See Proposition 3.6 below.

Lemma 3.4. *For any* $\mathrm{d}, r \mid 24$, *the vector* $\mathfrak{u}_{\mathrm{d}}\in\mathbb{Z}[\mathcal{A}_{\mathrm{d}}]$ *is invariant with respect to* $\Gamma_{\chi,r}\times\Gamma_{\chi,r}$ *if and only if* $\mathrm{d}\mid r$.

Proof. If $\mathrm{d}\mid r$, then $\Gamma_{\chi,r}\subset\Gamma_{\chi,\mathrm{d}}$ and we just need to prove the case when $r=\mathrm{d}$. Let $(g_1, g_2)\in\mathrm{SO}(L_{\mathrm{d}})$ with $g_j\in\Gamma_{\chi,\mathrm{d}}$. Then

$$(g_1, g_2)\cdot\mathfrak{u}_{\mathrm{d}} = \sum_{j\in\mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j)\Bigg(\sum_{h\in\kappa_{\mathrm{d}}(g_1T^j\Gamma_{\chi,\mathrm{d}}g_2^{-1})}\mathfrak{e}_h\Bigg) = \mathfrak{u}_{\mathrm{d}},$$

where we have used the fact that $\Gamma_{\chi,\mathrm{d}}$ is normal in $\Gamma_0(2)$ with coset representatives $\{T^j : j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}\}$. Similarly, $(T, T)\cdot\mathfrak{u}_{\mathrm{d}} = \mathfrak{u}_{\mathrm{d}}$. Thus $\mathfrak{u}_{\mathrm{d}}$ is $\Gamma_{\chi,\mathrm{d}} \times \Gamma_{\chi,\mathrm{d}}$-invariant. If $\mathrm{d} \nmid r$, then $(T^r, 1) \in (\Gamma_{\chi,r} \times \Gamma_{\chi,r}) \backslash (\Gamma_{\chi,\mathrm{d}} \times \Gamma_{\chi,\mathrm{d}})$. It is easy to see that

$$(T^r, 1)\cdot\mathfrak{u}_{\mathrm{d}} = \sum_{j\in\mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j-r)\Bigg(\sum_{h\in\kappa_{\mathrm{d}}(T^jN'_{\mathrm{d}})} \mathfrak{e}_h\Bigg) \neq \mathfrak{u}_d$$

since

$$\frac{a_{\mathrm{d}}(-r)}{\varphi(\mathrm{d})} = \frac{\mu(\mathrm{d}/(\mathrm{d}, r))}{\varphi(\mathrm{d}/(\mathrm{d}, r))} \neq 1 = \frac{a_{\mathrm{d}}(0)}{\varphi(\mathrm{d})}$$

when $\mathrm{d} \nmid r$. □

Proposition 3.5. *For any* $\mathrm{d} \mid 24$, *we have*

$$\omega_{\mathrm{d}}(g)\mathfrak{u}_{\mathrm{d}} = \chi(g)^{-24/\mathrm{d}}\mathfrak{u}_{\mathrm{d}} \tag{3-13}$$

for all $g \in \Gamma_0(2)$.

Proof. This follows directly from the local results 3.10 and 3.16 as

$$\omega_{\mathrm{d}}(g)\mathfrak{u}_{\mathrm{d}} = (\omega_{\mathrm{d},2}(g)\mathfrak{u}_{\mathrm{d},2}) \otimes (\omega_{\mathrm{d},3}(g)\mathfrak{u}_{\mathrm{d},3}) = \chi(g)^{-24(\mathrm{d}_2/\mathrm{d}_3+\mathrm{d}_3/\mathrm{d}_2)}\mathfrak{u}_{\mathrm{d},2} \otimes \mathfrak{u}_{\mathrm{d},3} = \chi(g)^{-24/\mathrm{d}}\mathfrak{u}_{\mathrm{d}}.$$

for all $g \in \Gamma_0(2)$. Here we have used $\frac{1}{\mathrm{d}} - \left(\frac{\mathrm{d}_2}{\mathrm{d}_3} + \frac{\mathrm{d}_3}{\mathrm{d}_2}\right) \in \mathbb{Z}$ when $\mathrm{d} \mid 24$. □

Now, define two further vectors

$$\mathfrak{v}_{\mathrm{d}} := \omega_{\mathrm{d}}(S)\mathfrak{u}_{\mathrm{d}}, \quad \mathfrak{w}_{\mathrm{d}} := \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}(T)\mathfrak{v}_{\mathrm{d}}. \tag{3-14}$$

Note that $\mathfrak{u}_{\mathrm{d}}$, $\mathfrak{v}_{\mathrm{d}}$ and $\mathfrak{w}_{\mathrm{d}}$ are linearly independent for all $\mathrm{d} \mid 24$. The key to the input of Borcherds lifting is then constructed using these vectors in the following result.

Proposition 3.6. *The representations* ϱ_{d} *defined in* (2-3) *is a subrepresentation of the Weil representation* ω_{d} *via the map*

$$\iota_{\mathrm{d}} : \mathbb{C}^3 \to U_{\mathrm{d}}^{H'_{\mathrm{d}}} \subset U_{\mathrm{d}}, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto a\mathfrak{u}_{\mathrm{d}} + b\mathfrak{v}_{\mathrm{d}} + c\mathfrak{w}_{\mathrm{d}}. \tag{3-15}$$

Let F_{d} *be the modular function defined in* (2-4). *The function* $\iota_{\mathrm{d}} \circ F_{\mathrm{d}}$ *is then in* $M^!(\omega_{\mathrm{d}})$ *and invariant with respect to the orthogonal group* $H'_{\mathrm{d}} \subset \mathrm{SO}(L_{\mathrm{d}})/\Gamma_{L_{\mathrm{d}}}$. *Furthermore, it has the principal part*

$$\iota_d \circ F_{\mathrm{d}}(\tau) = q^{-1/\mathrm{d}}\mathfrak{u}_{\mathrm{d}} + \begin{cases} O(q^{1/2}) & \textit{if } \mathrm{d} > 1, \\ O(1) & \textit{if } \mathrm{d} = 1, \end{cases} \tag{3-16}$$

Remark 3.7. When $\mathrm{d} = 1$, the function $\iota_{\mathrm{d}} \circ F_{\mathrm{d}}$ differs from the input in [Yang and Yin 2019] by a constant vector. To simplify the notation, we will write

$$\widetilde{F}_{\mathrm{d}} := \iota_{\mathrm{d}} \circ F_{\mathrm{d}} + \begin{cases} 24(\mathfrak{e}_{(0,0)} + \mathfrak{e}_{(1/2,0)}) & \text{if } \mathrm{d} = 1, \\ 0 & \text{if } \mathrm{d} > 1, \end{cases} \tag{3-17}$$

which is an element in $M^!(\omega_{\mathrm{d}})$ invariant with respect to H'_{d}.

Proof. It suffices to check on the generators T, S of $\mathrm{SL}_2(\mathbb{Z})$. From the definition and Proposition 3.5, it is clear that

$$\omega_{\mathrm{d}}(T)\mathfrak{u}_{\mathrm{d}} = \zeta_{\mathrm{d}}^{-1}\mathfrak{u}_{\mathrm{d}}, \qquad \omega_{\mathrm{d}}(T)\mathfrak{v}_{\mathrm{d}} = \zeta_{2\mathrm{d}}\mathfrak{w}_{\mathrm{d}},$$

$$\omega_{\mathrm{d}}(T)\mathfrak{w}_{\mathrm{d}} = \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}(T^2S)\mathfrak{u}_{\mathrm{d}} = \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}(SB)\mathfrak{u}_{\mathrm{d}} = \zeta_{2\mathrm{d}}\omega_{\mathrm{d}}(S)\mathfrak{u}_{\mathrm{d}} = \zeta_{2\mathrm{d}}\mathfrak{v}_{\mathrm{d}},$$

$$\omega_{\mathrm{d}}(S)\mathfrak{u}_{\mathrm{d}} = \mathfrak{v}_{\mathrm{d}}, \qquad \omega_{\mathrm{d}}(S)\mathfrak{v}_{\mathrm{d}} = \mathfrak{u}_{\mathrm{d}},$$

$$\omega_{\mathrm{d}}(S)\mathfrak{w}_{\mathrm{d}} = \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}(STS)\mathfrak{u}_{\mathrm{d}} = \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}((ST)^2BS^2)\mathfrak{u}_{\mathrm{d}} = \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}(S(TS)^3)\mathfrak{u}_{\mathrm{d}} = \zeta_{2\mathrm{d}}^{-1}\omega_{\mathrm{d}}(S)\mathfrak{u}_{\mathrm{d}} = \mathfrak{w}_{\mathrm{d}}. \qquad \square$$

In the following two subsections, we work at the 2-part and 3-part separately and construct $\mathfrak{u}_{\mathrm{d},p}$ for $p = 2, 3$. This will shed some light on where $\mathfrak{u}_{\mathrm{d}}$ comes from.

3B. *The case $p = 3$.* There are two possibilities for $\mathcal{A}_{\mathrm{d},3}$. If $3 \nmid \mathrm{d}$, then $\mathcal{A}_{\mathrm{d},3}$ is trivial. If $3 \mid \mathrm{d}$, we can identify the groups $\mathcal{A}_{\mathrm{d},3}$ and $\mathcal{A} := M_2(\mathbb{F}_3)$ via

$$\kappa_{\mathrm{d},3} : M_2(\mathbb{F}_3) \cong \mathcal{A}_{\mathrm{d},3}, \qquad \begin{pmatrix} h_0 & -h_1 \\ h_2 & h_3 \end{pmatrix} \bmod 3 \mapsto h = [h_0, h_1, h_2, h_3] \otimes \mathbb{Z}_3, \tag{3-18}$$

which is just the map κ_{d} in (3-6) tensored with $\mathbb{Z}_3$. This is an isomorphism of finite quadratic modules if we equip $M_2(\mathbb{F}_3)$ with the quadratic form $Q_{\mathrm{d},3} := (3\mathrm{d}_2)^{-1}\det$, which has value in $\frac{1}{3}\mathbb{Z}/\mathbb{Z}$. Then $H_3 \cong \mathrm{SL}_2(\mathbb{F}_3) \times \mathrm{SL}_2(\mathbb{F}_3)$, $H'_3 = \langle N'_3 \times N'_3, (T, T)\rangle$, where

$$\begin{aligned} N'_{\mathrm{d},3} &:= \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right) \right\} \\ &= \left\langle \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right) \right\rangle \subset \mathrm{SL}_2(\mathbb{F}_3) \subset M_2(\mathbb{F}_3) \end{aligned} \tag{3-19}$$

is isomorphic to the group of quaternions. Another way to characterize $N'_{\mathrm{d},3}$ is

$$N'_{\mathrm{d},3} = \left\{ \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\} \cup \left\{ g \in \mathrm{SL}_2(\mathbb{F}_3) : \mathrm{Tr}(g) = 0 \right\}. \tag{3-20}$$

From this, it is easy to check the following local analog of Lemma 3.1 at 3.

Lemma 3.8. *For any $j \in \mathbb{Z}/\mathrm{d}_3\mathbb{Z}$, we have*

$$\kappa_{\mathrm{d},3}(T^j N'_{\mathrm{d},3}) \cap \mathcal{A}^0_{\mathrm{d},3} = \{\pm[1, -j, 0, 1]\}.$$

Denote $\mathbf{0}_3 \in \mathcal{A}$ the zero matrix. Then $H_{\mathrm{d},3}$ acts on the set $\mathcal{A}\backslash\mathbf{0}_3$, and decomposes it into 3 orbits according to the norm of the elements. The subgroup $H'_{\mathrm{d},3} \subset H_{\mathrm{d},3}$ acts on $\mathcal{A}\backslash\mathbf{0}_3$ similarly and decomposes the three orbits into 5 orbits. We denote the sum of elements in each orbit by $\mathfrak{w}_i$ for $i = 0, 1, 2, 3, 4$. They are explicitly given as follows:

$$\mathfrak{w}_i := \begin{cases} \sum_{h \in \kappa_{\mathrm{d},3}(T^i N'_{\mathrm{d},3})} \mathfrak{e}_h & \text{if } i = 0, 1, 2. \\ \sum_{h \in \mathcal{A}\backslash\mathbf{0}_3,\, \det(h) \equiv -i \bmod 3} \mathfrak{e}_h & \text{if } i = 3, 4. \end{cases} \tag{3-21}$$

This gives $U_{\mathrm{d},3}^{H'_{\mathrm{d},3}} \cong \mathbb{C}\mathfrak{e}_{\mathbf{0}_3} + \sum_{j=0}^4 \mathbb{C}\mathfrak{w}_j \subset \mathbb{C}[\mathcal{A}]$. Moreover, $U_{\mathrm{d},3}^{H'_{\mathrm{d},3}}$ contains an $\mathrm{SL}_2(\mathbb{Z})$-invariant vector $4\mathfrak{e}_{\mathbf{0}_3} + \mathfrak{w}_3$, which is also in $U_{\mathrm{d},3}^{H_{\mathrm{d},3}}$. Its orthogonal complement in $U_{\mathrm{d},3}^{H'_{\mathrm{d},3}}$ is 5-dimensional and decomposes into $\chi_3^{-\mathrm{d}_2} \oplus \chi_3^{-\mathrm{d}_2} \oplus \varrho^{-\mathrm{d}_2}$, where χ_3 and ϱ are irreducible representations of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\chi_3(T) = \zeta_3, \quad \chi_3(S) = 1, \quad \varrho(T) = \begin{pmatrix} 1 & & \\ & \zeta_3 & \\ & & \zeta_3^2 \end{pmatrix}, \quad \varrho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \tag{3-22}$$

with respect to the basis $\{\mathfrak{w}_0 - \mathfrak{w}_1, \mathfrak{w}_0 - \mathfrak{w}_2, 8\mathfrak{e}_{\mathbf{0}_3} - \mathfrak{w}_3, \mathfrak{w}_0 + \mathfrak{w}_1 + \mathfrak{w}_2, 2\mathfrak{w}_4\}$. For any $m \in \mathbb{Z}$, we use $\varrho^{[m]}$ and $\chi_3^{[m]}$ to denote the representations of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\varrho^{[m]}(g) := \varrho(g)^m, \quad \chi_3^{[m]}(g) := \chi_3(g)^m.$$

Note that $\varrho^{[m]}$ and $\chi_3^{[m]}$ are well-defined and only depend on $m \bmod 3$. We remark $\chi_3|_{\Gamma_0(2)} = \chi^8$. In summary, we have:

Lemma 3.9. (1) *The subrepresentation* $\omega_{\mathrm{d},3}^{H_{\mathrm{d},3}} \subset \omega_{\mathrm{d},3}$ *fixed by* $H_{\mathrm{d},3}$ *decomposes as*

$$\omega_{\mathrm{d},3}^{H_{\mathrm{d},3}} \cong \mathbb{1} \oplus \varrho^{[-\mathrm{d}_2]}$$

with respect to the basis $\{4\mathfrak{e}_{\mathbf{0}_3} + \mathfrak{w}_3, 8\mathfrak{w}_0 - \mathfrak{w}_3, \mathfrak{w}_0 + \mathfrak{w}_1 + \mathfrak{w}_2, 2\mathfrak{w}_4\}$.

(2) *Denote* $U'_{\mathrm{d},3}$ *the orthogonal complement of* $U_{\mathrm{d},3}^{H_{\mathrm{d},3}}$ *in* $U_{\mathrm{d},3}^{H'_{\mathrm{d},3}}$ *and* $\omega'_{\mathrm{d},3}$ *the restriction of* $\omega_{\mathrm{d},3}$ *to* $U'_{\mathrm{d},3}$. *Then*

$$U'_{\mathrm{d},3} = \left\{ \sum_{j=0}^{2} a_j \mathfrak{w}_j : a_j \in \mathbb{C}, \sum_j a_j = 0 \right\}$$

and $\omega'_{\mathrm{d},3} \cong (\chi_3^{[-\mathrm{d}_2]})^{\oplus 2}$.

(3) *Under this identification,* $M^!(\omega_{\mathrm{d},3})^{H_{\mathrm{d},3}} \cong M^! \oplus M^!(\varrho^{[-\mathrm{d}_2]})$ *and*

$$M^!(\omega_{\mathrm{d},3})^{H'_{\mathrm{d},3}} \cong M^!(\omega_{\mathrm{d},3})^{H_{\mathrm{d},3}} \oplus M^!(\chi_3^{[-\mathrm{d}_2]})^{\oplus 2}.$$

The analog of $\mathfrak{u}_{\mathrm{d}}$ satisfying Lemma 3.4 and Proposition 3.5 is in the subspace

$$U'_{\mathrm{d},3} = \{\mathfrak{u}_{\mathrm{d},3}(\vec{c}) : \vec{c} = (c_s) \in \mathbb{C}^{\varphi(\mathrm{d}_3)}\},$$

where

$$\mathfrak{u}_{\mathrm{d},3}(\vec{c}) := \sum_{j \in \mathbb{Z}/\mathrm{d}_3\mathbb{Z}} \Bigg(\sum_{s \in (\mathbb{Z}/\mathrm{d}_3\mathbb{Z})^\times} c_s \zeta_{\mathrm{d}_3}^{sj} \Bigg) \Bigg(\sum_{h \in \kappa_{\mathrm{d},3}(T^j N'_{\mathrm{d},3})} \mathfrak{e}_h \Bigg). \tag{3-23}$$

As a consequence of Lemma 3.4, we have the following local analog of Proposition 3.5 at $p = 3$.

Proposition 3.10. *For any* $\mathrm{d} \mid 24$ *and* $\vec{c} \in \mathbb{C}^{\varphi(\mathrm{d}_3)}$, *we have*

$$\omega_{\mathrm{d},3}(g)\mathfrak{u}_{\mathrm{d},3}(\vec{c}) = \chi(g)^{-24\mathrm{d}_2/\mathrm{d}_3}\mathfrak{u}_{\mathrm{d},3}(\vec{c}) \tag{3-24}$$

for all $g \in \Gamma_0(2)$.

Proof. If $\mathrm{d}_3 = 1$, this is clear. Otherwise,

$$\omega_{\mathrm{d},3}(g)\mathfrak{u}_{\mathrm{d},3}(\vec{c}) = \chi_3(g)^{-\mathrm{d}_2}\mathfrak{u}_{\mathrm{d},3}(\vec{c}) = \chi(g)^{-8\mathrm{d}_2}\mathfrak{u}_{\mathrm{d},3}(\vec{c}).$$

□

If $\vec{c} = (1, \dots, 1) \in \mathbb{C}^{\varphi(\mathrm{d}_3)}$, then we simply denote $\mathfrak{u}_{\mathrm{d},3}(\vec{c})$ by $\mathfrak{u}_{\mathrm{d},3}$, which is explicitly given by

$$\mathfrak{u}_{\mathrm{d},3} = \begin{cases} 2\mathfrak{w}_0 - \mathfrak{w}_1 - \mathfrak{w}_2 & \text{if } \mathrm{d}_3 = 3, \\ \mathfrak{e}_{\mathfrak{o}_3} & \text{if } \mathrm{d}_3 = 1. \end{cases} \tag{3-25}$$

3C. ***The case $p = 2$.*** In this case, the finite quadratic module

$$\mathcal{A}_{\mathrm{d},2} = \mathbb{Z}/\mathrm{d}_2\mathbb{Z} \times \mathbb{Z}/(2\mathrm{d}_2)\mathbb{Z} \times \mathbb{Z}/(2\mathrm{d}_2)\mathbb{Z} \times \mathbb{Z}/\mathrm{d}_2\mathbb{Z}$$

has the quadratic form

$$Q_{\mathrm{d},2}([h_0, h_1, h_2, h_3]) := \frac{\mathrm{d}_3^{-1}}{2\mathrm{d}_2}(2h_0h_3 - h_1h_2) \in \frac{1}{2\mathrm{d}_2}\mathbb{Z}/\mathbb{Z}. \tag{3-26}$$

Even though the size of $\mathcal{A}_{\mathrm{d},2}$ can be large, the number of orbits under the suitable orthogonal group $H'_{\mathrm{d},2}$ is much smaller. More precisely, we have $H_{\mathrm{d},2} = N_{\mathrm{d},2} \times N_{\mathrm{d},2}$ and $H'_{\mathrm{d},2} \supset N'_{\mathrm{d},2} \times N'_{\mathrm{d},2}$, where

$$\begin{aligned} N_{\mathrm{d},2} &:= \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/(2\mathrm{d}_2\mathbb{Z})) \right\} / \langle T^{\mathrm{d}_2}, C^{\mathrm{d}_2/(2,\mathrm{d}_2)} \rangle, \\ N'_{\mathrm{d},2} &:= \langle A, C, D \rangle \cong (\mathbb{Z}/(\mathrm{d}_2(2,\mathrm{d}_2)/(4,\mathrm{d}_2))\mathbb{Z} \times \mathbb{Z}/(\mathrm{d}_2/(2,\mathrm{d}_2))\mathbb{Z}) \rtimes \mathbb{Z}/(4,\mathrm{d}_2)\mathbb{Z}. \end{aligned} \tag{3-27}$$

Here $A := \left(\begin{smallmatrix} 3 & 2 \\ 4 & 3 \end{smallmatrix}\right)$, $C := \left(\begin{smallmatrix} 5 & 4 \\ 16 & 13 \end{smallmatrix}\right)$, $D := \left(\begin{smallmatrix} -1 & 1 \\ -2 & 1 \end{smallmatrix}\right)$ are elements in $\mathrm{SL}_2(\mathbb{Z})$ projected into $N_{\mathrm{d},2}$. The commutation relation is given by $DAD^{-1} = A^3$. In particular $N'_{\mathrm{d},2}$ has size d_2^2 and is abelian for $\mathrm{d}_2 = 1, 2, 4$.

The group $N_{\mathrm{d},2}$ acts on the left on $\mathcal{A}_{\mathrm{d},2}$ via (simply coming from matrix multiplication)

$$\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \cdot [h_0, h_1, h_2, h_3] := [ah_0 + bh_2, ah_1 + 2(bh_3), 2(ch_0) + dh_2, ch_1 + dh_3] \tag{3-28}$$

for $\left(\begin{smallmatrix} a & b \\ 2c & d \end{smallmatrix}\right) \in N_{\mathrm{d},2}$ and $[h_0, h_1, h_2, h_3] \in \mathcal{A}_{\mathrm{d},2}$. The same holds for the right action. We can embed $N_{\mathrm{d},2}$ into $\mathcal{A}_{\mathrm{d},2}$ using the map $\kappa_{\mathrm{d},2} : N_{\mathrm{d},2} \to \mathcal{A}_{\mathrm{d},2}$ defined by

$$\kappa_{\mathrm{d},2}\left(\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}\right) := \mathrm{d}_3^{-1}[a \bmod \mathrm{d}_2, 2b, 2c, d \bmod \mathrm{d}_2]. \tag{3-29}$$

It is then easy to check that

$$\begin{gathered} \kappa_{\mathrm{d},2}(g_1g_2) = g_1 \cdot \kappa_{\mathrm{d},2}(g_2) = \kappa_{\mathrm{d},2}(g_1) \cdot g_2, \\ Q_{\mathrm{d},2}(\kappa_{\mathrm{d},2}(g)) = \frac{\mathrm{d}_3^{-1} \det(g) \bmod \mathrm{d}_2}{\mathrm{d}_2} \in \frac{2}{2\mathrm{d}_2}\mathbb{Z}/\mathbb{Z} \end{gathered} \tag{3-30}$$

for all $g, g_1, g_2 \in N_{\mathrm{d},2}$. From this, when $2 \mid \mathrm{d}$, it is easy to check that $\kappa_{\mathrm{d},2}$ is a two-to-one map since $(\mathrm{d}_2 + 1)\left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \in N'_{\mathrm{d},2}$ and

$$\kappa_{\mathrm{d},2}\left((\mathrm{d}_2 + 1)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) = \kappa_{\mathrm{d},2}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right).$$

To better describe $\kappa_{\mathrm{d},2}(N_{\mathrm{d},2})$, it is useful to know the smallest additive subgroup of $\mathcal{A}_{\mathrm{d},2}$ containing it. We describe it in the following lemma.

Lemma 3.11. *Let* $\mathcal{A}'_{\mathrm{d},2} \subset \mathcal{A}_{\mathrm{d},2}$ *be the smallest* (*additive*) *subgroup containing* $\kappa_{\mathrm{d},2}(N'_{\mathrm{d},2})$.

(1) *When* $8 \mid \mathrm{d}$, $\mathcal{A}'_{\mathrm{d},2} \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/8\mathbb{Z})^2$ *is the orthogonal complement of the subgroup generated by* $[6, 4, 0, 2], [0, 8, 0, 0], [0, 2, 2, 0] \in \mathcal{A}_{\mathrm{d},2}$, *and*

$$\kappa_{\mathrm{d},2}^{-1}(\mathcal{A}'_{\mathrm{d},2}) = N'_{\mathrm{d},2} \sqcup T^4 N'_{\mathrm{d},2}.$$

Furthermore, *we can distinguish the elements in* $N'_{\mathrm{d},2}$ *and* $T^4 N'_{\mathrm{d},2}$ *via*

$$\begin{aligned} N'_{\mathrm{d},2} &= \kappa_{\mathrm{d},2}^{-1}(\{[h_0, h_1, h_2, h_3] \in \mathcal{A}'_{\mathrm{d},2} : h_0^2 - \mathrm{d}_3^2 \equiv h_1 + h_2 \bmod 16\}), \\ T^4 N'_{\mathrm{d},2} &= \kappa_{\mathrm{d},2}^{-1}(\{[h_0, h_1, h_2, h_3] \in \mathcal{A}'_{\mathrm{d},2} : h_0^2 - \mathrm{d}_3^2 \equiv h_1 + h_2 + 8 \bmod 16\}). \end{aligned} \tag{3-31}$$

(2) *When* $8 \nmid \mathrm{d}$, $\mathcal{A}'_{\mathrm{d},2} \cong (\mathbb{Z}/\mathrm{d}_2\mathbb{Z})^2$ *is generated by* $\kappa_{\mathrm{d},2}\big(\big(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\big)\big)$, $\kappa_{\mathrm{d},2}\big(\big(\begin{smallmatrix}-1 & 1 \\ -2 & 1\end{smallmatrix}\big)\big)$ *and*

$$\kappa_{\mathrm{d},2}^{-1}(\mathcal{A}'_{\mathrm{d},2}) = N'_{\mathrm{d},2}.$$

Proof. This can be verified using the Appendix and some computer calculation. □

In addition, we record the following local analog of Lemma 3.1 at the prime 2.

Lemma 3.12. *For any* $j \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}$, *we have*

$$\kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2}) \cap \mathcal{A}^0_{\mathrm{d},2} = \{[r, r(2j + (\mathrm{d}_3 r)^2 - 1), 0, r] : r \in (\mathbb{Z}/\mathrm{d}_2\mathbb{Z})^\times\}.$$

Proof. For $j = 0$, this follows directly from Lemma 3.11. In general, it is easy to check that

$$T^j(\kappa_{\mathrm{d},2}(N'_{\mathrm{d},2}) \cap \mathcal{A}^0_{\mathrm{d},2}) = \kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2}) \cap \mathcal{A}^0_{\mathrm{d},2}$$

for any j since the action of T preserves $\mathcal{A}^0_{\mathrm{d}}$. □

Since $T^{\mathrm{d}_2} = 0 \in N'_{\mathrm{d},2}$ and $H'_{\mathrm{d},2}$ is generated by $N'_{\mathrm{d},2} \times N'_{\mathrm{d},2}$ and (T, T), the index of $H'_{\mathrm{d},2}$ in $H_{\mathrm{d},2}$ is d_2 and the sizes of $H_{\mathrm{d},2}$ and $H'_{\mathrm{d},2}$ are $\mathrm{d}_2^6/(2, \mathrm{d}_2)$ and $\mathrm{d}_2^5/(2, \mathrm{d}_2)$ respectively. The dimension of $U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$ is the number of orbits in $\mathcal{A}_{\mathrm{d},2}$ under the action of $H'_{\mathrm{d},2}$. Since the finite group $H'_{\mathrm{d},2}$ is explicitly given in (3-27), it is straightforward to calculate these orbits on a computer in practice. We did this in Sage [2019] and received the following results:

$$\dim U_{\mathrm{d},2}^{H'_{\mathrm{d},2}} = \begin{cases} 4 & \text{if } \mathrm{d}_2 = 1, \\ 16 & \text{if } \mathrm{d}_2 = 2, \\ 46 & \text{if } \mathrm{d}_2 = 4, \\ 118 & \text{if } \mathrm{d}_2 = 8. \end{cases} \tag{3-32}$$

With these calculations, one can already explicitly decompose the representation $\omega_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$ on $U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$. To find the desired vectors, we need to consider the following subspace of $U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$.

For $\mathrm{d}_2 = 1$, the vector $\mathfrak{e}_{(1/2,0)} - \mathfrak{e}_{(0,1/2)}$ generates a 1-dimensional $\mathrm{SL}_2(\mathbb{Z})$-invariant subspace. Denote $U'_{\mathrm{d},2} \subset U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$ its orthogonal complement. For $\mathrm{d}_2 \geq 2$, the subgroup $H'_{\mathrm{d},2}$ has index 2 in $H'_{\mathrm{d}/2,2} = \langle T^{\mathrm{d}_2/2}, H'_{\mathrm{d},2}\rangle$. Denote $U'_{\mathrm{d},2} \subset U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$ the orthogonal complement of $U_{\mathrm{d},2}^{H'_{\mathrm{d}/2,2}} \subset U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$. Then it is clear that

$$\dim U'_{\mathrm{d},2} = \tfrac{1}{2}\big(\text{numbers of } H'_{\mathrm{d},2}\text{-orbits of } \mathcal{A}_{\mathrm{d},2} - \text{numbers of } H'_{\mathrm{d}/2,2}\text{-orbits of } \mathcal{A}_{\mathrm{d},2}\big).$$

The following result comes out of the computer calculations.

Lemma 3.13. *For any* $\mathrm{d} \mid 24$, *the dimension of* $U'_{\mathrm{d},2}$ *is* $3\varphi(\mathrm{d}_2)$. *Furthermore, the support of any elements in* $U'_{\mathrm{d},2}$ *is contained in the union of* $\{h \in \mathcal{A}_{\mathrm{d},2} : 2Q_{\mathrm{d}}(h) = -\mathrm{d}_3^{-1}/\mathrm{d}_2\}$ *and* $\bigcup_{j \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2})$.

Now for $\mathrm{d} \mid 24$, define the following vectors

$$\begin{aligned} \mathfrak{u}_{\mathrm{d},2}(\vec{c}) &:= \sum_{j \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \Bigg(\sum_{s \in (\mathbb{Z}/\mathrm{d}_2\mathbb{Z})^\times} c_s \zeta_{\mathrm{d}_2}^{js} \Bigg) \Bigg(\sum_{h \in \kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2})} \mathfrak{e}_h \Bigg), \\ \mathfrak{v}_{\mathrm{d},2}(\vec{c}) &:= \omega_{\mathrm{d},2}(S)\mathfrak{u}_{\mathrm{d},2}(\vec{c}), \quad \mathfrak{w}_{\mathrm{d},2}(\vec{c}) := \zeta_{\mathrm{d}_2}^{-\mathrm{d}_3} \omega_{\mathrm{d},2}(T)\mathfrak{v}_{\mathrm{d},2}(\vec{c}) \end{aligned} \tag{3-33}$$

for all $\vec{c} = (c_s) \in \mathbb{C}^{\varphi(\mathrm{d}_2)}$. From Lemma 3.13, we can show that these vectors give a basis of $U'_{\mathrm{d},2}$.

Lemma 3.14. *For any* $\mathrm{d} \mid 24$ *with* $2 \mid \mathrm{d}$ *and* $\vec{c} \in \mathbb{C}^{\varphi(\mathrm{d}_2)}$, *the vectors* $\mathfrak{v}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{w}_{\mathrm{d},2}(\vec{c})$ *have the same support, which is disjoint from that of* $\mathfrak{u}_{\mathrm{d},2}(\vec{c}_1)$ *for any* $\vec{c}_1 \in \mathbb{C}^{\varphi(\mathrm{d}_2)}$.

Proof. Since the action of $\omega_{\mathrm{d},2}(T)$ does not change the support, we know that $\mathfrak{v}_{\mathrm{d},2}(\vec{c})$ and $\mathfrak{w}_{\mathrm{d},2}(\vec{c})$ have the same support. Now we have by definition

$$\mathfrak{v}_{\mathrm{d},2}(\vec{c}) = (2\mathrm{d}_2^2)^{-1} \sum_{s \in (\mathbb{Z}/\mathrm{d}_2\mathbb{Z})^\times} c_s \sum_{\mu \in \mathcal{A}_{\mathrm{d},2}} \sum_{j \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \zeta_{\mathrm{d}_2}^{js} \Bigg(\sum_{h \in \kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2})} \mathbf{e}((\mu, h))\mathfrak{e}_\mu \Bigg).$$

We want to show that the coefficient of $\mathfrak{e}_\mu$ is zero if $\mu = \kappa_{\mathrm{d},2}(T^{j'}g')$ with $j' \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}$ and $g' \in N'_{d,2}$. Now if $h = \kappa_{\mathrm{d},2}(g)$ with $g \in T^j N'_{d,2}$, then

$$(\mu, h) = \frac{\mathrm{d}_3^{-1}\,\mathrm{Tr}(g(T^{j'}g')^{-1})}{\mathrm{d}_2}$$

by (3-30). Since $N'_{\mathrm{d},2}$ is normal in $N_{\mathrm{d},2}$, which contains T, we have $T^j N'_{\mathrm{d},2} T^{-j'} = T^{j-j'} N'_{\mathrm{d},2}$. Therefore, it suffices to show that the sum below vanishes

$$\sum_{j \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \zeta_{\mathrm{d}_2}^{js} \sum_{h \in \kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2})} \mathbf{e}((\mu, h)) = \zeta_{\mathrm{d}_2}^{j's} \sum_{j'' \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \zeta_{\mathrm{d}_2}^{j''s} \sum_{h \in \kappa_{\mathrm{d},2}(T^{j''} N'_{\mathrm{d},2})} \zeta_{\mathrm{d}_2}^{\mathrm{d}_3^{-1}\mathrm{Tr}(h)}$$

with $j'' := j - j'$. Also for $h = [h_0, h_1, h_2, h_3] \in \kappa_{\mathrm{d},2}(N'_{\mathrm{d},2})$, we have $\mathrm{Tr}(T^j \cdot h) = \mathrm{Tr}(h) + j \cdot h_1$. Using this, we can rewrite

$$\sum_{j'' \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \zeta_{\mathrm{d}_2}^{j''s} \sum_{h \in \kappa_{\mathrm{d},2}(T^{j''} N'_{\mathrm{d},2})} \zeta_{\mathrm{d}_2}^{\mathrm{d}_3^{-1}\mathrm{Tr}(h)} = \sum_{h \in \kappa_{\mathrm{d},2}(N'_{\mathrm{d},2})} \zeta_{\mathrm{d}_2}^{\mathrm{d}_3^{-1}\mathrm{Tr}(h)} \sum_{j'' \in \mathbb{Z}/\mathrm{d}_2\mathbb{Z}} \zeta_{\mathrm{d}_2}^{j''(s+\mathrm{d}_3^{-1}h_1)}$$

By Lemma 3.11 (or inspecting the Appendix), we know that $h_1 \in 2\mathbb{Z}/2\mathrm{d}_2\mathbb{Z}$ for all $h \in \kappa_{\mathrm{d},2}(N'_{\mathrm{d},2})$. So $s + \mathrm{d}_3^{-1} h_1 \in (\mathbb{Z}/\mathrm{d}_2\mathbb{Z})^\times$ and the sum above vanishes. □

Lemma 3.15. *For any* $\mathrm{d} \mid 24$ *and any basis* $\mathcal{B}$ *of* $\mathbb{C}^{\varphi(\mathrm{d}_2)}$, *the set*

$$\bigcup_{\vec{c} \in \mathcal{B}} \{\mathfrak{u}_{\mathrm{d},2}(\vec{c}), \mathfrak{v}_{\mathrm{d},2}(\vec{c}), \mathfrak{w}_{\mathrm{d},2}(\vec{c})\} \tag{3-34}$$

is a basis of $U'_{\mathrm{d},2}$.

Proof. We know that dimension of $U'_{\mathrm{d},2}$ is $3\varphi(\mathrm{d}_2)$ from Lemma 3.13, and need to check linear independence of the vectors in the set above. Since the vectors $\mathfrak{u}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{v}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{w}_{\mathrm{d},2}(\vec{c})$ are defined linearly, it suffices to prove the lemma for $\mathcal{B} = \{\vec{e}(s_0) : s_0 \in (\mathbb{Z}/\mathrm{d}_2\mathbb{Z})^\times\}$ with $\vec{e}(s_0) \in \mathbb{C}^{\varphi(\mathrm{d}_2)}$ the standard basis vector with 0 everywhere except 1 at the s_0-th entry. It is easily checked from the definition that $\mathfrak{u}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{v}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{w}_{\mathrm{d},2}(\vec{c})$ are in $U'_{\mathrm{d},2}$ are eigenvectors of T with eigenvalue $\zeta_{\mathrm{d}_2}^{-s_0}$ when $\vec{c} = \vec{e}(s_0)$. Therefore, it suffices to check that the three vectors $\mathfrak{u}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{v}_{\mathrm{d},2}(\vec{c})$, $\mathfrak{w}_{\mathrm{d},2}(\vec{c})$ are linearly independent whenever $\vec{c} = \vec{e}(s_0)$.

When $\mathrm{d}_2 = 1$, this is easily checked by hand. When $\mathrm{d}_2 \geq 2$, it suffices to show that $\mathfrak{v}_{\mathrm{d},2}(\vec{e}(s_0))$ and $\mathfrak{w}_{\mathrm{d},2}(\vec{e}(s_0))$ are linearly independent by Lemma 3.14. Let us assume otherwise. Then the restriction of $\omega_{\mathrm{d},2}$ to $\mathbb{C}\mathfrak{u}_{\mathrm{d},2}(\vec{e}(s_0)) + \mathbb{C}\mathfrak{v}_{\mathrm{d},2}(\vec{e}(s_0))$ is a 2-dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$. In the basis $\{\mathfrak{u}_{\mathrm{d},2}(\vec{e}(s_0)), \mathfrak{v}_{\mathrm{d},2}(\vec{e}(s_0))\}$, it is given by the map

$$T \mapsto \begin{pmatrix} \zeta_{\mathrm{d}_2}^{-1} & \\ & \pm\zeta_{2\mathrm{d}_2} \end{pmatrix}, \quad S \mapsto \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

However, $(T \cdot S)^6$ is the identity, whereas

$$\left(\begin{pmatrix} \zeta_{\mathrm{d}_2}^{-1} & \\ & \pm\zeta_{2\mathrm{d}_2} \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right)^6 = \left(\pm \begin{pmatrix} \zeta_{2\mathrm{d}_2}^{-1} & \\ & \zeta_{2\mathrm{d}_2}^{-1} \end{pmatrix} \right)^3$$

is not the identity since $2 \mid \mathrm{d}_2$. This is a contradiction and finishes the proof. □

Proposition 3.16. *For* $\mathrm{d} \mid 24$, *let* $\omega'_{\mathrm{d},2}$ *denote the restriction of* $\omega_{\mathrm{d},2}$ *to* $U'_{\mathrm{d},2} \subset U_{\mathrm{d},2}^{H'_{\mathrm{d},2}}$. *Then* $\mathfrak{u}_{\mathrm{d},2}(\vec{c})$ *satisfies*

$$\omega'_{\mathrm{d},2}(g)\mathfrak{u}_{\mathrm{d},2}(\vec{c}) = \chi(g)^{-24\mathrm{d}_3/\mathrm{d}_2}\mathfrak{u}_{\mathrm{d},2}(\vec{c}) \tag{3-35}$$

for all $g \in \Gamma_0(2)$ *and* $\vec{c} \in \mathbb{C}^{\varphi(\mathrm{d}_2)}$. *Furthermore with respect to the basis in* (3-34), *we have*

$$\omega'_{\mathrm{d},2} \cong \varrho_{\mathrm{d}_2,\mathrm{d}_3}^{\oplus\varphi(\mathrm{d}_2)} \tag{3-36}$$

Here $\varrho_{\mathrm{d}_2,\mathrm{d}_3}$ *is the* 3*-dimensional representation defined in* (2-3).

Remark 3.17. If $\vec{c} = (1, \dots, 1) \in \mathbb{C}^{\varphi(\mathrm{d}_2)}$, we simply write $\mathfrak{u}_{\mathrm{d},2}$ for $\mathfrak{u}_{\mathrm{d},2}(\vec{c})$. They are explicitly given by

$$\mathfrak{u}_{\mathrm{d},2} = \begin{cases} \mathfrak{e}_{\mathbf{0}_2} & \text{if } \mathrm{d}_2 = 1, \\ 2^{\mathrm{d}_2/2}\left(\sum_{\kappa_{\mathrm{d},2}(N'_{\mathrm{d},2})} \mathfrak{e}_h - \sum_{\kappa_{\mathrm{d},2}(T^{\mathrm{d}_2/2}N'_{\mathrm{d},2})} \mathfrak{e}_h\right) & \text{if } \mathrm{d}_2 = 2, 4, 8. \end{cases} \tag{3-37}$$

Proof. For the first claim, it suffices to prove the cases $g = T, S^2, TB$, which are generators of $\Gamma_0(2)$. If $g = T$, then $\omega'_{\mathrm{d},2}(T)\mathfrak{e}_h = \mathbf{e}(-Q_{\mathrm{d},2}(h))\mathfrak{e}_h$. For $h \in \kappa_{\mathrm{d},2}(T^j N'_{\mathrm{d},2})$, we have $Q_{\mathrm{d},2}(h) = \mathrm{d}_3^{-1}/\mathrm{d}_2 =$

$\mathrm{d}_3/\mathrm{d}_2 \in \frac{1}{\mathrm{d}_2}\mathbb{Z}/\mathbb{Z}$. Therefore (3-35) holds for $g = T$. When $g = S^2$, since $\omega_{\mathrm{d}}(S^2)\mathfrak{e}_h = \mathfrak{e}_{-h}$ for all $h \in \mathcal{A}_{\mathrm{d}}$ and $-\left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right) \in N'_{\mathrm{d},2}$, we know that $-\kappa_{\mathrm{d}}(T^j N'_{\mathrm{d},2}) = \kappa_{\mathrm{d}}(T^j N'_{\mathrm{d},2})$ and (3-35) holds for $g = S^2$.

For $g = TB = TST^2S^{-1}$, it suffices to show the middle equation below

$$\omega'_{\mathrm{d},2}(S)\omega_{\mathrm{d},2}(T^{-1})\mathfrak{u}_{\mathrm{d},2}(\vec{c}) = \zeta_{\mathrm{d}_2}^{\mathrm{d}_3^{-1}}\mathfrak{v}_{\mathrm{d},2}(\vec{c}) = \omega_{\mathrm{d},2}(T^2)\mathfrak{v}_{\mathrm{d},2}(\vec{c}) = \omega_{\mathrm{d},2}(T^2)\omega'_{\mathrm{d},2}(S)\mathfrak{u}_{\mathrm{d},2}(\vec{c}).$$

This is easily checked by hand when $\mathrm{d}_2 = 1$. If $2 \mid \mathrm{d}_2$, we know by Lemmas 3.13 and 3.14 that the support of $\mathfrak{v}_{\mathrm{d},2}(\vec{c})$ is contained in $\{h \in \mathcal{A}_{\mathrm{d},2} : 2Q_{\mathrm{d}}(h) = -\mathrm{d}_3^{-1}/\mathrm{d}_2\}$. It is therefore an eigenvector of $\omega'_{\mathrm{d},2}(T^2)$ with eigenvalue $\zeta_{\mathrm{d}_2}^{\mathrm{d}_3^{-1}}$. This proves the first claim. As in the proof of Proposition 3.6, the vectors $\{\mathfrak{u}_{\mathrm{d},2}(\vec{c}), \mathfrak{v}_{\mathrm{d},2}(\vec{c}), \mathfrak{w}_{\mathrm{d},2}(\vec{c})\}$ generate a 3-dimensional subrepresentation of $\omega_{\mathrm{d},2}$ isomorphic to $\varrho_{\mathrm{d}_2,\mathrm{d}_3}$. The second claim then follows from Lemma 3.15. ☐

4. Borchers liftings

4A. ***Brief review of Borcherds liftings.*** We first set up notation and briefly review the Borcherds lifting, following [Yang and Yin 2019, Section 3]. Let $V = V_{\mathrm{d}}$ and H be as in Section 3.

Let

$$\mathcal{L} = \{w \in V_{\mathbb{C}} : (w, w) = 0,\ (w, \bar{w}) < 0\}. \tag{4-1}$$

and let $\mathbb{D}$ be the Hermitian symmetric domain of oriented negative 2-planes in $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$. Then one has an isomorphism

$$pr : \mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}, \qquad w = u + iv \mapsto \mathbb{R}u + \mathbb{R}(-v).$$

For the isotropic matrix $\ell = \left(\begin{smallmatrix}0 & -1\\ 0 & 0\end{smallmatrix}\right) \in L$ and $\ell' = \left(\begin{smallmatrix}0 & 0\\ 1/\mathrm{d} & 0\end{smallmatrix}\right) \in V$ with $(\ell, \ell') = 1$. We also have the associated tube domain

$$\mathcal{H}_{\ell,\ell'} = \left\{\begin{pmatrix} z_1 & 0\\ 0 & -z_2\end{pmatrix} : y_1y_2 > 0\right\}, \qquad y_i = \mathrm{Im}(z_i),$$

together with

$$w : \mathcal{H}_{\ell,\ell'} \to \mathcal{L}, \qquad w\left(\begin{pmatrix} z_1 & 0\\ 0 & -z_2\end{pmatrix}\right) = \begin{pmatrix} z_1 & -\mathrm{d}z_1z_2\\ 1/\mathrm{d} & -z_2\end{pmatrix}.$$

This gives an isomorphism $\mathcal{H}_{\ell,\ell'} \cong \mathcal{L}/\mathbb{C}^\times$. We also identity $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ with $\mathcal{H}_{\ell,\ell'}$ by

$$\psi_{\mathrm{d}} : z = (z_1, z_2) \mapsto \begin{pmatrix} z_1/\mathrm{d} & 0\\ 0 & -z_2/\mathrm{d}\end{pmatrix}.$$

Note that we use this identification in order to have the following compatibility property and it is also the identification used in the computation of Borcherds products. The following is a special case of [Yang and Yin 2019, Proposition 3.1].

Proposition 4.1. *Define*

$$w_{\mathrm{d}} : \mathbb{H}^2 \cup (\mathbb{H}^-)^2 \to \mathcal{L}, \qquad w_{\mathrm{d}}(z_1, z_2) = w \circ \psi_{\mathrm{d}}(z_1, z_2) = \begin{pmatrix} z_1/\mathrm{d} & -z_1z_2/\mathrm{d}\\ 1/\mathrm{d} & -z_2/\mathrm{d}\end{pmatrix}.$$

Then the composition $pr \circ w_{\mathrm{d}}$ *gives an isomorphism between* $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ *and* $\mathbb{D}$. *Moreover,* w_{d} *is* $H(\mathbb{R})$*-equivariant, where* $H(\mathbb{R})$ *acts on* $\mathbb{H}^2 \cup (\mathbb{H}^-)^2$ *via the usual linear fraction*:

$$(g_1, g_2)(z_1, z_2) = (g_1(z_1), g_2(z_2)),$$

and acts on $\mathcal{L}$ *and* $\mathbb{D}$ *naturally via its action on* V. *Moreover, one has*

$$(g_1, g_2)w_{\mathrm{d}}(z_1, z_2) = \frac{j(g_1, z_1)j(g_2, z_2)}{\nu(g_1, g_2)} w_{\mathrm{d}}(g_1(z_1), g_2(z_2)), \tag{4-2}$$

where $\nu(g_1, g_2) = \det g_1 = \det g_2$ *is the spin character of* $H \cong \mathrm{GSpin}(V)$, *and*

$$j(g, z) := cz + d, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the automorphy factor of weight 1.

For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, let X_Γ be the associated open modular curve over $\mathbb{Q}$ such that $X_\Gamma(\mathbb{C}) = \Gamma \backslash \mathbb{H}$. Assume $\Gamma \supset \Gamma(M)$ for some integer $M \geq 1$. Let

$$\nu : \mathbb{A}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{A}), \quad \nu(d) = \mathrm{diag}(1, d).$$

Let $K(\Gamma)$ be the product of $\nu(\widehat{\mathbb{Z}}^\times)$ and the preimage of $\Gamma/\Gamma(M)$ in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ (under the map $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$). Let $K = (K(\Gamma) \times K(\Gamma)) \cap H(\mathbb{A}_f)$. Then one has by the strong approximation theorem

$$X_K \cong X_\Gamma \times X_\Gamma.$$

In this way, we have identified the product of two copies of a modular curve X_Γ with a Shimura variety X_K.

Suppose that Γ acts on L'/L trivially, then for each $\mu \in L'/L$ and $m \in Q(\mu) + L$, the associated special divisor $Z_\Gamma(m, \mu)$ is given by

$$Z_\Gamma(m, \mu) = (\Gamma \times \Gamma) \backslash \{(z_1, z_2) : w_{\mathrm{d}}(z_1, z_2) \perp x \text{ for some } x \in \mu + L,\ Q(x) = m\}.$$

More generally, assume $\Gamma \supset \Gamma(M)$ preserves L, and $\mathfrak{u} = \sum a_\mu \mathfrak{e}_\mu \in \mathbb{C}[L'/L]$ is $\Gamma \times \Gamma$-invariant, the cycle

$$Z_{\Gamma(M)}(m, \mathfrak{u}) = \sum a_\mu Z_{\Gamma(M)}(m, \mu)$$

descends to a cycle $Z_\Gamma(m, \mathfrak{u})$ in $X_\Gamma \times X_\Gamma$. For our purpose, we will take

$$\mathrm{d} \mid 24, \quad \Gamma = \Gamma_{\chi,\mathrm{d}} \supset \Gamma_{\mathrm{d}} \supset \Gamma(2\mathrm{d}) \supset \Gamma(48)$$

from now on and write $X_{\mathrm{d}} := X_\Gamma = X_{\Gamma_{\chi,\mathrm{d}}}$. Notice that $X_1 = X_0(2)$ has two cusps, $i\infty$ and 0. Since $\{T^j : 1 \leq j \leq \mathrm{d}\}$ are coset representatives of $\Gamma_{\chi,\mathrm{d}}$ in $\Gamma_{\chi,1}$, the modular curve X_{d} has the same cusps as X_1.

Lemma 4.2 [Yang and Yin 2019, Corollary 3.3]. *For* $\mathrm{d} \mid \mathrm{d}' \mid 24$, *let* $\pi : X_{\Gamma(2\mathrm{d}')} \to X_{\mathrm{d}}$ *be the natural projection. Then*

$$(\pi \times \pi)^*(X_{\mathrm{d}}^\Delta) = \sum_{\gamma \in \Gamma/\Gamma(2\mathrm{d}')} Z_{\Gamma(2\mathrm{d}')}\Big(\frac{1}{\mathrm{d}'}, \frac{1}{\mathrm{d}'}\gamma + L\Big) \tag{4-3}$$

and the group $\Gamma(2\mathrm{d}')$ *can be replaced by* $\Gamma_{\mathrm{d}'}$.

Since Γ is normal in $\Gamma_0(2)=\Gamma_{\chi,1}$, the action of $T=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in\Gamma_0(2)$ on $\mathbb{H}$ factors through X_{d} and defines an isomorphism $X_{\mathrm{d}}\to X_{\mathrm{d}}$, which we also denote by T. Using this, we can define translates of the diagonal

$$X_{\mathrm{d}}^{\Delta}(j):=(T^j\times\mathbb{1})^*(X_{\mathrm{d}}^{\Delta})\subset X_{\mathrm{d}}\times X_{\mathrm{d}} \tag{4-4}$$

for $j\in\mathbb{Z}/\mathrm{d}\mathbb{Z}$. Equation (4-3) also generalizes to

$$(\pi\times\pi)^*(X_{\mathrm{d}}^{\Delta}(j))=\sum_{\gamma\in\Gamma/\Gamma(2\mathrm{d}')}Z_{\Gamma(2\mathrm{d}')}\Big(\frac{1}{\mathrm{d}'},\frac{1}{\mathrm{d}'}T^j\gamma+L\Big), \tag{4-5}$$

where one can replace $\Gamma(2\mathrm{d}')$ with $\Gamma_{\mathrm{d}'}$. From this, we see that the pull back of $X_{\mathrm{d}}^{\Delta}(j)$ along natural projection $X_{\mathrm{d}'}\times X_{\mathrm{d}'}\to X_{\mathrm{d}}\times X_{\mathrm{d}}$ is $\bigcup_{l\in\mathrm{d}\mathbb{Z}/\mathrm{d}'\mathbb{Z}}X_{\mathrm{d}'}^{\Delta}(j+l)$. Before proceeding further to state and prove the main result of this section, we record the following identity for convenience.

Lemma 4.3. *For any* $\mathrm{d}\in\mathbb{N}$, *we have the following identity in* $\mathbb{Q}(X)$

$$\mathrm{p}_{\mathrm{d}}(X):=\prod_{j\in\mathbb{Z}/\mathrm{d}\mathbb{Z}}(1-\zeta_{\mathrm{d}}^jX)^{a_{\mathrm{d}}(j)}=\prod_{b\mid\mathrm{d}}(1-X^{\mathrm{d}/b})^{b\cdot\mu(\mathrm{d}/b)}, \tag{4-6}$$

where $a_{\mathrm{d}}(j)$ *is the constant defined in* (3-12). *Furthermore for any* $s\in\mathbb{N}$, *we have*

$$\prod_{\mathrm{d}\mid s}\mathrm{p}_{\mathrm{d}}(X^{s/\mathrm{d}})=(1-X)^s. \tag{4-7}$$

Proof. To prove (4-6), it suffices to check that both sides have the same roots counting multiplicity, since they agree at $X=0$. The multiplicity of $X=\zeta_{\mathrm{d}}^j$ on the left-hand side is $a_{\mathrm{d}}(-j)=a_{\mathrm{d}}(j)$, whereas it is $\sum_{b\mid(\mathrm{d},j)}b\cdot\mu(\mathrm{d}/b)$ on the right-hand side. The equality is then a consequence of the identity

$$\sum_{b\mid n}b\cdot\mu\Big(\frac{\mathrm{d}}{b}\Big)=\mu(\mathrm{d}/n)\frac{\varphi(\mathrm{d})}{\varphi(\mathrm{d}/n)},\qquad n\mid\mathrm{d},$$

which is a standard exercise that we leave, along with (4-7), to the curious readers. ☐

Now, we can specialize Borcherds' far reaching lifting theorem [1998, Theorem 13.3] (see also [Yang and Yin 2019, Theorems 2.1 and 2.2]) to the modular function $\widetilde{F}_{\mathrm{d}}$ in (3-17) and the result below.

Theorem 4.4. *For every* $\mathrm{d}\mid 24$, *recall the modular function in* $M^!(\omega_{\mathrm{d}})^{H'_{\mathrm{d}}}$

$$\widetilde{F}_{\mathrm{d}}(\tau)=\sqrt{2}^{\mathrm{d}}(\mathfrak{f}_2^{-24/\mathrm{d}}(\tau)\mathfrak{u}_{\mathrm{d}}+\mathfrak{f}_1^{-24/\mathrm{d}}(\tau)\mathfrak{v}_{\mathrm{d}}+\mathfrak{f}^{-24/\mathrm{d}}(\tau)\mathfrak{w}_{\mathrm{d}})+\begin{cases}24(\mathfrak{e}_{(0,0)}+\mathfrak{e}_{(1/2,0)}) & \textit{if } \mathrm{d}=1,\\ 0 & \textit{if } \mathrm{d}>1,\end{cases}$$

defined in (3-17) *with* $\mathfrak{u}_{\mathrm{d}},\mathfrak{v}_{\mathrm{d}},\mathfrak{w}_{\mathrm{d}}\in U_{\mathrm{d}}$ *vectors defined in* (3-12) *and* (3-14). *Let* $\Psi_{\mathrm{d}}(z)$ *be the meromorphic modular function on* $X_{\mathrm{d}}\times X_{\mathrm{d}}$ (*with some characters*) *associated to* $\widetilde{F}_{\mathrm{d}}$ *via Borcherds multiplicative lifting, i.e.,* $-\log\|\Psi_{\mathrm{d}}(z)\|_{\mathrm{Pet}}^2$ *is the regularized theta lift of* $\widetilde{F}_{\mathrm{d}}$ *with* $\|\cdot\|_{\mathrm{Pet}}$ *a suitably normalized Petersson norm* (*see, e.g., Theorem 2.1 in* [*Yang and Yin 2019*]). *Then* $\Psi_{\mathrm{d}}(z)$ *has the following properties*:

(1) *On* $X_{\mathrm{d}} \times X_{\mathrm{d}}$,

$$\operatorname{Div}(\Psi_{\mathrm{d}}(z)) = \sum_{j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j) X_{\mathrm{d}}^{\Delta}(j).$$

(2) *When* $\mathrm{d} = 1$, $\Psi_{\mathrm{d}}(z)$ *has a product expansion of the form*

$$\Psi_1(z) = 2^{12}(q_1 - q_2) \prod_{m,n \geq 1} (1 - q_1^n q_2^m)^{c_1(mn)} (1 - q_1^{2n} q_2^{2m})^{c_{-1}(2mn)}$$

near the cusp $\mathbb{Q}\ell$ *of* X_K, *where* $q_j := e^{2\pi i z_j}$ *and* $c_{\mathrm{d}}(l)$ *are the Fourier coefficients defined in Remark 2.1.*

(3) *When* $\mathrm{d} > 1$, $\Psi_{\mathrm{d}}(z)$ *has a product expansion of the form*

$$\Psi_{\mathrm{d}}(z) = \prod_{b \mid \mathrm{d}} (q_1^{1/b} - q_2^{1/b})^{b \cdot \mu(\mathrm{d}/b)} \prod_{\substack{m,n \in \mathbb{N} \\ mn \equiv -1 \bmod \mathrm{d}}} \left(\prod_{b \mid \mathrm{d}} (1 - q_1^{n/b} q_2^{m/b})^{b \cdot \mu(\mathrm{d}/b)} \right)^{c_{\mathrm{d}}(mn)(-1)^{(n^2-1)/\mathrm{d}_2}}$$

near the cusp $\mathbb{Q}\ell$ *of* X_K, *where* μ *and* φ *are the Möbius and Euler* φ*-function respectively.*

Proof. This is a specialization of Borcherds' result to the input $\widetilde{F}_{\mathrm{d}} \in M^!(\omega_{\mathrm{d}})$. For this, we need to substitute the suitable parameters into Borcherds' result, which has been specialized to this case in Theorems 2.1 and 2.2 in [Yang and Yin 2019]. Using the specialization there, we see that the divisor of Ψ_{d} is

$$\sum_{j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j) \sum_{\mu \in \kappa(T^j N_{\mathrm{d}}')} Z_{\Gamma_{\mathrm{d}}}\Bigl(-\frac{1}{\mathrm{d}}, \mu\Bigr) = \sum_{j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j) \sum_{\gamma \in \Gamma/\Gamma(2\mathrm{d})} Z_{\Gamma(2\mathrm{d})}\Bigl(-\frac{1}{\mathrm{d}}, \frac{1}{\mathrm{d}} T^j \gamma + L\Bigr),$$

which gives us the first claim after applying Lemma 4.2.

For the second and third claim, we specialize Theorem 2.2 in [Yang and Yin 2019] and use the notations there. When $\mathrm{d} = 1$, this is rather classical and can be found in [Scheithauer 2008] (see also Proposition 5.3 in [Yang and Yin 2019]).[2] For $\mathrm{d} > 1$, the Weyl chambers for $\widetilde{F}_{\mathrm{d}}$ are the same as in the case $\mathrm{d} = 1$, and we choose the one $W = \mathbb{R}\bigl\{\left(\begin{smallmatrix} a & \\ & -1 \end{smallmatrix}\right) : a > 1\bigr\}$. Since $\mathfrak{u}_{\mathrm{d}}$, $\mathfrak{v}_{\mathrm{d}}$ and $\mathfrak{w}_{\mathrm{d}}$ do not have support on any isotropic vector, the associated form $\widetilde{F}_{\mathrm{d},P}$ is identically zero, and the Weyl vector $\rho(W, \widetilde{F}_{\mathrm{d}})$ is 0. Since $\widetilde{F}_{\mathrm{d}}$ does not have any constant term, the constant C in the product expansion is 1, For the infinite product, suppose $\lambda = \frac{1}{\mathrm{d}}\left(\begin{smallmatrix} -m & \\ & n \end{smallmatrix}\right)$ with $m, n \in \mathbb{Z}$. Then $(\lambda, W) > 0$ if and only if $m \geq -n$, $n \geq 0$ and $(m, n) \neq (0, 0)$.

The set of $\mu \in L_0'/L$ with $p(\mu) = \lambda$ consists then of $\frac{1}{\mathrm{d}}\left(\begin{smallmatrix} -m & -j \\ 0 & n \end{smallmatrix}\right)$ with $j \in \frac{1}{2}\mathbb{Z}/\mathrm{d}\mathbb{Z}$. For such λ, μ, we have

$$1 - e((\lambda, z) + (\mu, \ell')) = 1 - \zeta_{\mathrm{d}}^j q_1^{n/\mathrm{d}} q_2^{m/\mathrm{d}}.$$

By inspecting the q-expansion of F_{d}, we notice that

$$F_{\mathrm{d}}(\tau) = \Bigl(q^{-1/\mathrm{d}} + \sum_{l \in \mathbb{N}, l \equiv -1 \bmod \mathrm{d}} c_{\mathrm{d}}(l) q^{l/\mathrm{d}} \Bigr) \mathfrak{u}_{\mathrm{d}} + \sum_{\mu \in L'/L,\, Q_{\mathrm{d}}(\mu) \in \left\{\frac{1}{2\mathrm{d}}, \frac{1}{2\mathrm{d}} + \frac{1}{2}\right\}} F_{\mathrm{d},\mu}(\tau) \mathfrak{e}_{\mu}.$$

---

[2] Note that the Fourier expansion of f in [loc. cit.] is incorrect.

Therefore, the only pairs of (m, n) with $m < 0$ is $m = -n = -1$, and the only $\mu \in L_0'/L$ where $F_{\mathrm{d},\mu}$ could be nonzero are contained in the support of $\mathfrak{u}_{\mathrm{d}}$, hence

$$\mu = \frac{1}{\mathrm{d}}\begin{pmatrix} -m & j \\ 0 & n \end{pmatrix} + L \in \frac{1}{\mathrm{d}} T^{j'} \Gamma_{\chi,\mathrm{d}} + L$$

with $mn \equiv -1 \mod \mathrm{d}$ and $j' := nj - \frac{1}{2}(n^2-1) \in \mathbb{Z}/\mathrm{d}\mathbb{Z}$ by Lemma 3.1. The Fourier coefficient $c(-Q(\lambda), \mu)$ of the input is then $c_{\mathrm{d}}(mn) a_{\mathrm{d}}(j')$. It is easy to check that $a_{\mathrm{d}}(j') = a_{\mathrm{d}}(j)(-1)^{(n^2-1)/\mathrm{d}_2}$. By Theorem 2.2 in [Yang and Yin 2019], $\Psi_{\mathrm{d}}(z)$ has the product expansion

$$\Psi_{\mathrm{d}}(z) = \prod_{\substack{m \in \mathbb{Z}_{\geq -1}, n \in \mathbb{N} \\ mn \equiv -1 \bmod \mathrm{d}}} \prod_{j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}} (1 - \zeta_{\mathrm{d}}^{j} q_1^{n/\mathrm{d}} q_2^{m/\mathrm{d}})^{c_{\mathrm{d}}(mn) a_{\mathrm{d}}(j)(-1)^{(n^2-1)/8}}$$

Finally, applying Lemma 4.3 finishes the proof. ☐

4B. ***The Weber function differences as Borcherds liftings.*** Now, we are ready to state and prove the following main result of this section.

Theorem 4.5. *For* $\mathrm{d} \mid 24$, *let* $\Psi_{\mathrm{d}}(z_1, z_2)$ *be the Borcherds product of* $\widetilde{F}_{\mathrm{d}} \in M^!(\omega_{\mathrm{d}})$ *as in Theorem 4.4. Then for any* $s \mid 24$ *and* $\varepsilon \in \{\pm 1\}$, *we have*

$$(\mathfrak{f}_2(z_1)^{24/s} - (\varepsilon \mathfrak{f}_2(z_2))^{24/s})^s = \prod_{\mathrm{d}|s} \Psi_{\mathrm{d}}(z_1, z_2)^{\varepsilon^{24/\mathrm{d}}}. \tag{4-8}$$

Proof. We first look at their divisors in the open Shimura varieties $X_s \times X_s$. Suppose $\varepsilon = 1$. The left-hand side clearly has $s \cdot [X_s^\Delta]$ as its divisor, whereas the right-hand side has the divisor

$$\sum_{\mathrm{d}|s} \sum_{j \in \mathbb{Z}/\mathrm{d}\mathbb{Z}} a_{\mathrm{d}}(j) \sum_{l \in \mathrm{d}\mathbb{Z}/s\mathbb{Z}} [X_s^\Delta(j+l)] = \sum_{k \in \mathbb{Z}/s\mathbb{Z}} \Big(\sum_{\mathrm{d}|s} a_{\mathrm{d}}(k) \Big) [X_s^\Delta(k)] = s \cdot [X_s^\Delta],$$

as

$$\sum_{\mathrm{d}|s} a_{\mathrm{d}}(k) = \sum_{\mathrm{d}|s} \mu(\mathrm{d}/(\mathrm{d},k)) \varphi(\mathrm{d})/\varphi(\mathrm{d}/(\mathrm{d},k)) = \begin{cases} s & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

When $\varepsilon = -1$, the argument is the same unless $8 \mid s$. In that case, the divisor of the left-hand side is $s \cdot [X_s^\Delta(s/2)]$, whereas the divisor of the right-hand side is

$$\mathrm{Div} \prod_{\mathrm{d}|s/2} \Psi_{\mathrm{d}}(z_1, z_2)^2 - \mathrm{Div} \prod_{\mathrm{d}|s} \Psi_{\mathrm{d}}(z_1, z_2) = s \cdot ([X_s^\Delta] + [X_s^\Delta(s/2)]) - s \cdot [X_s^\Delta] = s \cdot [X_s^\Delta(s/2)].$$

Now let

$$g(z_1, z_2) = \frac{\prod_{\mathrm{d}|s} \varepsilon^{24/(\mathrm{d},s)} \Psi_{\mathrm{d}}(z_1, z_2)^2}{(\mathfrak{f}_2(z_1)^{24/s} - (\varepsilon \mathfrak{f}_2(z_2))^{24/s})^s}.$$

Then it is holomorphic and has no zeros on $X_s \times X_s$. So

$$\operatorname{Div}(g(z_1, z_2)) = a_{\infty,1}(\{\infty\} \times X_s) + a_{\infty,2}(X_s \times \{\infty\}) + a_{0,1}(\{0\} \times X_s) + a_{0,2}(X_s \times \{0\})$$

is supported on the boundary with $a_{i,j} \in \mathbb{Z}$. The product expansion of Ψ_d and the definition of $\mathfrak{f}_2$ imply that $a_{\infty,1} = a_{\infty,2} = 0$.

Next, fix $z_2 \in X_s$ the above argument shows that $g(z_1, z_2)$, as a function of z_1 on $X_s \cup \{0, \infty\}$ has only zeros or poles at the cusp $\{0\}$, which is impossible. So $g(z_1, z_2)$ has no zeros or poles in z_1, and is therefore independent of z_1, i.e, $g(z_1, z_2) = g(z_2)$ is purely a function of z_2 with no zeros or poles in $X_s \cup \{\infty\}$. This implies that $g(z_1, z_2) = g(z_2) = C$ is a constant.

Finally, looking at the q_1-leading term of the Fourier expansion, we see $C = 1$ and this proves the theorem. The last part of the proof follows from the argument in the proof of [Yang and Yin 2019, Theorem 3.4]. ☐

5. Big CM values

5A. ***Products of CM cycles as big CM cycles.*** Yang and Yin [2019, Section 3.2] have described how to view a pair of CM points as a big CM point, which we now briefly review for convenience and set up necessary notation. We modify a little for use in this paper. For $j = 1, 2$, let $d_j < 0$ be co-prime, fundamental discriminants satisfying (1-3). Denote $E_j = \mathbb{Q}(\sqrt{d_j})$ with ring of integers $\mathcal{O}_j = \mathbb{Z}\big[\frac{1}{2}(1 + \sqrt{d_j})\big]$, and class group $\mathrm{Cl}(d_j)$. Let $E = E_1 \otimes_{\mathbb{Q}} E_2 = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ with ring of integers $\mathcal{O}_E = \mathcal{O}_1 \otimes_{\mathbb{Z}} \mathcal{O}_2$. Then E is a biquadratic CM number field with real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$ and $D = d_1 d_2$.

For a positive integer d, we define $W = W_{\mathrm{d}} = E$ with the F-quadratic form $Q_F(x) = \mathrm{d}x\bar{x}/\sqrt{D}$. Let $W_{\mathbb{Q}} = W$ with the $\mathbb{Q}$-quadratic form $Q_{\mathbb{Q}}(x) = \operatorname{Tr}_{F/\mathbb{Q}} Q_F(x)$. Let σ_1 and σ_2 be two real embeddings of F with $\sigma_j(\sqrt{D}) = (-1)^{j-1}\sqrt{D}$. Then W has signature $(0, 2)$ at σ_2 and $(2, 0)$ at σ_1 respectively, and so $W_{\mathbb{Q}}$ has signature $(2, 2)$. Choose a $\mathbb{Z}$-basis of $\mathcal{O}_E$ as follows

$$e_1 = 1 \otimes 1, \quad e_2 = \frac{-1+\sqrt{d_1}}{2} = \frac{-1+\sqrt{d_1}}{2} \otimes 1, \quad e_3 = \frac{1+\sqrt{d_2}}{2} = 1 \otimes \frac{1+\sqrt{d_2}}{2}, \quad e_4 = e_2 e_3.$$

We will drop $\otimes$ when there is no confusion. Then it is easy to check that

$$(W_{\mathbb{Q}}, Q_{\mathbb{Q}}) \cong (V, Q) = (M_2(\mathbb{Q}), \mathrm{d}\det), \qquad \sum x_i e_i \mapsto \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix}. \tag{5-1}$$

We will identify $(W_{\mathbb{Q}}, Q_{\mathbb{Q}})$ with the quadratic space $(V, Q) = (M_2(\mathbb{Q}), \mathrm{d}\det)$. Under this identification, the lattice $M_2(\mathbb{Z})$ becomes $\mathcal{O}_E$, and the lattice L_{d} becomes $\mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}2e_4 \subset \mathcal{O}_E$, which we still denote by $L = L_{\mathrm{d}}$. Define T to be the maximal torus in H given by the following diagram:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & \operatorname{Res}_{F/\mathbb{Q}} \mathrm{SO}(W) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & H \arrow[r] & \mathrm{SO}(V) \arrow[r] & 1 \end{tikzcd} \tag{5-2}$$

Then T can be identified with ([Howard and Yang 2012; Bruinier et al. 2012, Section 6])

$$T(R)=\{(t_1,t_2)\in(E_1\otimes_{\mathbb{Q}}R)^\times\times(E_2\otimes_{\mathbb{Q}}R)^\times : t_1\bar{t}_1=t_2\bar{t}_2\},$$

for any $\mathbb{Q}$-algebra R, and the map from T to $\mathrm{SO}(W)$ is given by $(t_1,t_2)\mapsto t_1/\bar{t}_2$. The map from T to H is explicitly given as follows. Define the embeddings $\iota_j : E_j\to M_2(\mathbb{Q})$ by

$$(e_1,e_2)\iota_1(r)=(re_1,re_2),\quad \iota_2(r)(e_3,e_1)^t=(\bar{r}e_3,\bar{r}e_1)^t. \tag{5-3}$$

Then $\iota=(\iota_1,\iota_2)$ gives the embedding from T to H. If $r_j=\alpha_j e_1+(-1)^{j+1}\beta_j e_{j+1}\in E_j$, then

$$\iota_j(r_j)=\alpha_j\begin{pmatrix}1&0\\0&1\end{pmatrix}+\beta_j\begin{pmatrix}0&\frac{1}{4}(d_j-1)\\1&-1\end{pmatrix}. \tag{5-4}$$

Extend the two real embeddings of F into a CM type $\Sigma=\{\sigma_1,\sigma_2\}$ of E via

$$\sigma_1(\sqrt{d_i})=\sqrt{d_i}\in\mathbb{H},\quad \sigma_2(\sqrt{d_1})=\sqrt{d_1},\quad \sigma_2(\sqrt{d_2})=-\sqrt{d_2}.$$

Since $W_{\sigma_2}=W\otimes_{F,\sigma_2}\mathbb{R}\subset V_{\mathbb{R}}$ has signature $(0,2)$, it gives two points $z^\pm_{\sigma_2}$ in $\mathbb{D}$. In this case, the big CM cycles associated to T as defined in [Bruinier et al. 2012; Yang and Yin 2019] are given by

$$Z(W,z^\pm_{\sigma_2})=\{z^\pm_{\sigma_2}\}\times T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T\in Z^2(X_K), \tag{5-5}$$

and

$$Z(W)=Z(W,z^\pm_{\sigma_2})+\sigma_2(Z(W,z^\pm_{\sigma_2})). \tag{5-6}$$

For simplicity, we will denote z_{σ_2} for $z^+_{\sigma_2}$. The same calculation as in the proof of [Yang and Yin 2019, Lemma 3.4] gives the following result.

Lemma 5.1. *On $\mathbb{H}^2\cup(\mathbb{H}^-)^2$, one has $z_{\sigma_2}=(\tau_1,\tau_2)\in\mathbb{H}^2$ and $z^-_{\sigma_2}=(\bar{\tau}_1,\bar{\tau}_2)\in(\mathbb{H}^-)^2$, where*

$$\tau_j=\frac{1+\sqrt{d_j}}{2}.$$

For $\mathrm{d}\mid 24$, let $K_{\mathrm{d}}\subset H(\mathbb{A}_f)$ be the compact open subgroup generated by (T,T), $(\Gamma_{\chi,\mathrm{d}}\times\Gamma_{\chi,\mathrm{d}})\otimes\hat{\mathbb{Z}}\subset H(\mathbb{A}_f)$ and $(\nu(\hat{\mathbb{Z}}^\times)\times\nu(\hat{\mathbb{Z}}^\times))\cap H(\mathbb{A}_f)$. By the choice of $\Gamma_{\chi,\mathrm{d}}$, we actually have the following result.

Lemma 5.2. *Suppose $d_j<0$ are discriminants satisfying (1-3) for $j=1,2$. Then for any $\mathrm{d}\mid 24$, the preimage $\iota^{-1}(K_{\mathrm{d}})$ is independent of $\mathrm{d}\mid 24$.*

Remark 5.3. We will simply denote $\iota^{-1}(K_{\mathrm{d}})$ by K_T.

Remark 5.4. The lemma does not require d_j to be fundamental or co-prime.

Proof. Since $K_{24}\subset K_{\mathrm{d}}\subset K_1$ for any $\mathrm{d}\mid 24$, it suffices to check that $\iota^{-1}(K_1)=\iota^{-1}(K_{24})$. Furthermore, we know that $\Gamma(48)\subset\Gamma_{\chi,24}\subset\Gamma_{\chi,1}=\Gamma_0(2)$, so we only need to check the equality when tensoring with $\mathbb{Z}/3\mathbb{Z}$ and with $\mathbb{Z}/16\mathbb{Z}$. This then boils down to a short calculation with finite groups.

To check the case modulo 3, it suffices to show that $\iota(\iota^{-1}(K_1)\otimes\mathbb{Z}/3\mathbb{Z})\subset K_{24}\subset \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})\times\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$. Since $\Gamma_{\chi,1}\otimes\mathbb{Z}/3=\Gamma_0(2)\otimes\mathbb{Z}/3=\mathrm{SL}_2(\mathbb{Z}_3)$, we have $\iota^{-1}(K_1)\otimes\mathbb{Z}/3=\iota^{-1}(H(\mathbb{Z}/3\mathbb{Z}))$. Thus (5-4) implies

$$\iota(\iota^{-1}(K_1)\otimes\mathbb{Z}/3)=\left\{\left(\begin{pmatrix}\alpha_j & \beta_j(d_j-1)/4\\ \beta_j & \alpha_j-\beta_j\end{pmatrix}\right)_{j=1,2}\in H(\mathbb{Z}/3\mathbb{Z}):\alpha_j,\beta_j\in\mathbb{Z}/3\mathbb{Z},\right\}$$

and we need to show that this is contained in

$$\begin{aligned}K_{24}\otimes\mathbb{Z}/3\mathbb{Z}&=\langle\Gamma_{\chi,24}\times\Gamma_{\chi,24},(T,T),\nu(\hat{\mathbb{Z}}^\times)\times\nu(\hat{\mathbb{Z}}^\times)\rangle\otimes\mathbb{Z}/3\mathbb{Z}\\&=\langle\Gamma_{\chi,3}\times\Gamma_{\chi,3},(T,T),\nu(\hat{\mathbb{Z}}^\times)\times\nu(\hat{\mathbb{Z}}^\times)\rangle\otimes\mathbb{Z}/3\mathbb{Z}\\&=\left\langle N'_{\mathrm{d},3}\times N'_{\mathrm{d},3},(T,T),\left(\left(\begin{smallmatrix}1&\\&-1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&\\&-1\end{smallmatrix}\right)\right)\right\rangle\subset H(\mathbb{Z}/3\mathbb{Z}).\end{aligned}$$

Now given $r=(r_1,r_2)\in\iota^{-1}(K_1)\otimes\mathbb{Z}/3\mathbb{Z}$ with $\iota_j(r_j)=\left(\begin{smallmatrix}\alpha_j&\beta_j(d_j-1)/4\\\beta_j&\alpha_j-\beta_j\end{smallmatrix}\right)$, we know that

$$\delta:=\det(\iota_j(r_j))=\mathrm{Tr}(\iota_j(r_j))^2-\beta_j^2d_j\in(\mathbb{Z}/3\mathbb{Z})^\times \tag{5-7}$$

is independent of j. If $\beta_j=0$, then $\iota_j(r_j)=\pm\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right)\in N'_{\mathrm{d},3}$ and $\iota(r)\in K_{24}\otimes\mathbb{Z}/3\mathbb{Z}$. If $\beta_1=0$ and $\beta_2\neq 0$, then $\iota_1(r_1)\in N'_{\mathrm{d},3}$ and $\delta=1$, which implies $\mathrm{Tr}(\iota_2(r_2))=0$ by (5-7). That means $\iota_2(r_2)\in N'_{\mathrm{d},3}$ by (3-20). Finally suppose $\beta_j\neq 0$, then we can use $3\nmid d_j$ to show that $\epsilon:=\alpha_j\beta_j(\delta+1)$ is independent of j. It is then straightforward to check that $T^{1-\epsilon}\left(\begin{smallmatrix}1&\\&\delta\end{smallmatrix}\right)\iota_j(r_j)\in N'_{\mathrm{d},3}$. Therefore $\iota(r)\in K_{24}\otimes\mathbb{Z}/3\mathbb{Z}$.

To check the case modulo 16, suppose

$$r=(r_1,r_2)\in\iota^{-1}(K_1)\otimes\mathbb{Z}/16\mathbb{Z}$$

with $r_j=\alpha_je_1+(-1)^{j+1}\beta_je_{j+1}$, $\alpha_j,\beta_j\in\mathbb{Z}/16\mathbb{Z}$. Then simple calculation shows that $\alpha_j-1,\beta_j\in 2\mathbb{Z}/16\mathbb{Z}$. Furthermore, $\det(\iota_j(r_j))=\alpha_j(\alpha_j-\beta_j)-\beta_j^2(d_j-1)/4\in(\mathbb{Z}/16\mathbb{Z})^\times$ is independent of j since $\iota(r)\in H(\mathbb{A}_f)$, and $\beta_j^2(d_j-1)/4\equiv 0 \bmod 8$ since $d_j\equiv 1\bmod 8$ and $\beta_j\in 2\mathbb{Z}/16\mathbb{Z}$. Therefore,

$$\det(\iota_j(r_j))^{-1}=\alpha_j^{-1}(\alpha_j-\beta_j)^{-1}+\beta_j^2(d_j-1)/4.$$

Now $r\in\iota^{-1}(K_{24})$ if and only if $\iota(r)\in K_{24}\otimes\mathbb{Z}/16\mathbb{Z}$, which is generated by $\nu((\mathbb{Z}/16\mathbb{Z})^\times)\times\nu((\mathbb{Z}/16\mathbb{Z})^\times)$, $(T,T)\otimes\mathbb{Z}/16\mathbb{Z}$ and $(\Gamma_{\chi,24}\times\Gamma_{\chi,24})\otimes\mathbb{Z}/16\mathbb{Z}\cong(\Gamma_{\chi,8}/\Gamma(16)\times\Gamma_{\chi,8}/\Gamma(16))$. From the natural surjection $\Gamma_{\chi,8}/\Gamma(16)\to\Gamma_{\chi,8}/\Gamma_8=N'_8=N'_{8,2}$, we see that the following claim will finish the proof: the element

$$g_j:=\nu(\det(\iota_j(r_j)))^{-1}T^{\det(\iota_j(r_j)-1)/2}\iota_j(r_j)$$

is in $N'_8=N'_{8,2}$ for all $r_j=\alpha_je_j+(-1)^{j+1}\beta_je_{j+1}$ with $\alpha_j-1,\beta_j\in 2\mathbb{Z}/16\mathbb{Z}$. By dropping the subscript j in d_j,g_j,α_j and β_j, we can write

$$g=\alpha\begin{pmatrix}1&\frac{\alpha(\alpha-\beta)-1-\beta^2(d-1)/4}{2}\\0&\alpha^{-1}(\alpha-\beta)^{-1}+\frac{\beta^2(d-1)}{4}\end{pmatrix}+\beta\begin{pmatrix}\frac{\alpha(\alpha-\beta)-1-\beta^2(d-1)/4}{2}&\frac{d-1}{4}-\frac{\alpha(\alpha-\beta)-1-\beta^2(d-1)/4}{2}\\\alpha^{-1}(\alpha-\beta)^{-1}&-\alpha^{-1}(\alpha-\beta)^{-1}\end{pmatrix},$$

which is an element in $N_8 = N_{8,2}$. Denote $h = [h_0, h_1, h_2, h_3] := \kappa_{8,2}(g) \in \mathcal{A}_{8,2}$. To show that $g \in N_8' = N_{8,2}'$, it suffices check that $h \in \mathcal{A}_{8,2}'$, i.e.,

$$h \perp [6, 4, 0, 2], \quad h \perp [0, 2, 2, 0]$$

and $h_0^2 - 1 \equiv h_1 + h_2 \mod 16$ by Lemma 3.11. All of these can be checked by hand (assuming $d \equiv 1 \mod 8$ and $\alpha - 1 \equiv \beta \equiv 0 \mod 2$), and we leave the details to the reader. □

By [Yang and Yin 2019, Lemma 3.5], the map

$$p : T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_T \to \mathrm{Cl}(d_1) \times \mathrm{Cl}(d_2), \quad [t_1, t_2] \mapsto ([t_1], [t_2]) = ([\mathfrak{a}_1], [\mathfrak{a}_2]) \tag{5-8}$$

is injective. Here $\mathfrak{a}_j$ is the ideal of E_j associated to t_j. If d_1, d_2 are co-prime, then [Yang and Yin 2019, Lemma 3.8] tells us that it is an isomorphism. If $d_1 d_2$ is not a perfect square, this subgroup can be identified with $\mathrm{Gal}(H/E)$ with H the composite of the ring class fields H_{d_j} associated to the order of discriminant d_j (see Proposition 3.2 in [Li 2018]). This observation and the above lemma give the following corollary.

Proposition 5.5. *Let $d_j < 0$ be co-prime, fundamental discriminants satisfying (1-3). For $[\mathfrak{a}_j] \in \mathrm{Cl}(d_j)$, recall the class invariant $f(\mathfrak{a}_j)$ defined in (1-4). Then for any $s \mid 24$*

$$4s \sum_{[\mathfrak{a}_j] \in \mathrm{Cl}(d_j),\, j=1,2} \log |f(\mathfrak{a}_1)^{24/s} - f(\mathfrak{a}_2)^{24/s}| = \sum_{\mathrm{d} \mid s} \varepsilon^{24/\mathrm{d}} \log |\Psi_{\mathrm{d}}(Z(W))|, \tag{5-9}$$

where $\varepsilon := \varepsilon_{d_1}\varepsilon_{d_2} = (-1)^{(d_1+d_2-2)/8}$ and $Z(W)$ is the big CM cycle defined in (5-6).

Proof. We may assume $s = 24$ for simplicity, as the other cases are the same. By applying Shimura's reciprocity law, Proposition 22 in [Gee 1999] showed that class invariants $f(\mathfrak{a}_j)$ for $[\mathfrak{a}_j] \in \mathrm{Cl}(d_j)$ are conjugates of each other under the Galois group. In particular, [loc. cit., (18)] implies

$$f(\mathfrak{a}_j) = \varepsilon_{d_j} (\zeta_{48}^{-1} \mathfrak{f}_2(\tau_j))^{\sigma_{t_j}} = \varepsilon_{d_j} \zeta_{48}^{-\sigma_{t_j}} \mathfrak{f}_2^{\sigma_{t_j}}(\tau_j^{\sigma_{t_j}}) = \varepsilon_{d_j} \zeta_{48}^{-t_j \bar{t}_j} \delta(t_j) \mathfrak{f}_2(\tau_j^{\sigma_{t_j}})$$

where the class $t_j \in (E_j \otimes \mathbb{A}_f)^\times$ in $\mathrm{Cl}(d_j)$ is $[\mathfrak{a}_j]$. Here $t_j \bar{t}_j$ can be understood to be an integer modulo 48, and

$$\delta(t_j) = \frac{(\sqrt{2})^{\sigma_{t_j \bar{t}_j}}}{\sqrt{2}}$$

is an 8-th root of unit depending only on $t_j \bar{t}_j$ mod 8, coming from the Fourier coefficients of $\mathfrak{f}_2$. Note that $\sqrt{2} = \zeta_8 + \zeta_8^{-1}$. Thus for $t = (t_1, t_2) \in T(\mathbb{A}_f)$, we have $t_1 \bar{t}_1 = t_2 \bar{t}_2$ and

$$\log |f(\mathfrak{a}_1) - f(\mathfrak{a}_2)| = \log |(\zeta_{48}^{-1} \mathfrak{f}_2(\tau_1))^{\sigma_{t_1}} - \varepsilon (\zeta_{48}^{-1} \mathfrak{f}_2(\tau_2))^{\sigma_{t_2}}| = \log |\mathfrak{f}_2(\tau_1^{\sigma_{t_1}}) - \varepsilon \mathfrak{f}_2(\tau_2^{\sigma_{t_2}})|,$$

which depends only on the image $p(t) = ([\mathfrak{a}_1], [\mathfrak{a}_2]) \in \mathrm{Cl}(d_1) \times \mathrm{Cl}(d_2)$. So by the isomorphism (5-8),

$$\sum_{[\mathfrak{a}_j] \in \mathrm{Cl}(d_j),\, j=1,2} \log|f(\mathfrak{a}_1) - f(\mathfrak{a}_2)| = \sum_{t \in T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T} \log|\mathfrak{f}_2(\tau_1^{\sigma_{t_1}}) - \varepsilon \mathfrak{f}_2(\tau_2^{\sigma_{t_2}})|$$

$$= \sum_{(z,t) \in Z(W, \sigma_2^+)} \log|\mathfrak{f}_2(z_1) - \varepsilon \mathfrak{f}_2(z_2)|_{(z_1 z_2) = [(z,t)]}.$$

As the other three orbits are Galois conjugates of $Z(W, \sigma_2^+)$, the sums over the other orbits are the same as this one. Now the desired identity follows from Theorem 4.5. ☐

6. Incoherent Eisenstein Series and the proof of the Yui–Zagier conjecture

In this section, we will use the big CM value formula of Bruinier, Kudla and Yang [Bruinier et al. 2012] (see also [Yang and Yin 2019, Theorem 2.6]) to prove the factorization formula for $\Psi_{\mathrm{d}}(Z(W))$ and the Yui–Zagier conjecture. To do so, we need to review the associated incoherent Eisenstein series and compute their Fourier coefficients.

6A. ***Incoherent Eisenstein series.*** Let $F = \mathbb{Q}(\sqrt{D})$, $E = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, and $W = E$ with F-quadratic form $Q_F(x) = \mathrm{d}x\bar{x}/\sqrt{D}$ as in Section 5. Here $D = d_1 d_2$. Let $\chi_{E/F}$ be the quadratic Hecke character of F associated to E/F. Then there is a $\mathrm{SL}_2(\mathbb{A}_F)$-equivariant map

$$\lambda = \prod \lambda_v : S(W(\mathbb{A}_F)) \to I(0, \chi_{E/F}), \quad \lambda(\phi)(g) = \omega(g)\phi(0). \tag{6-1}$$

Here $I(s, \chi_{E/F}) = \mathrm{Ind}_{B_{\mathbb{A}_F}}^{\mathrm{SL}_2(\mathbb{A}_F)} \chi_{E/F}|\cdot|^s$ is the principal series, whose sections (elements) are smooth functions Φ on $\mathrm{SL}_2(\mathbb{A}_F)$ satisfying the condition

$$\Phi(n(b)m(a)g, s) = \chi(a)|a|^{s+1}\Phi(g, s), \quad b \in \mathbb{A}_F,\ a \in \mathbb{A}_F^\times.$$

Here $B = NM$ is the standard Borel subgroup of SL_2. Such a section is called factorizable if $\Phi = \bigotimes \Phi_v$ with $\Phi_v \in I(s, \chi_v)$. It is called standard if $\Phi|_{\mathrm{SL}_2(\widehat{\mathcal{O}}_F)\mathrm{SO}_2(\mathbb{R})^2}$ is independent of s. For a standard section $\Phi \in I(s, \chi)$, its associated Eisenstein series is defined as

$$E(g, s, \Phi) = \sum_{\gamma \in B_F \backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s)$$

for $\Re(s) \gg 0$.

For $\phi \in S(V_f) = S(W_f)$, let Φ_f be the standard section associated to $\lambda_f(\phi) \in I(0, \chi_f)$. For each real embedding $\sigma_i : F \hookrightarrow \mathbb{R}$, let $\Phi_{\sigma_i} \in I(s, \chi_{\mathbb{C}/\mathbb{R}}) = I(s, \chi_{E_{\sigma_i}/F_{\sigma_i}})$ be the unique "weight one" eigenvector of $\mathrm{SL}_2(\mathbb{R})$ given by

$$\Phi_{\sigma_i}(n(b)m(a)k_\theta) = \chi_{\mathbb{C}/\mathbb{R}}(a)|a|^{s+1}e^{i\theta},$$

for $b \in \mathbb{R}$, $a \in \mathbb{R}^\times$, and $k_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \in \mathrm{SO}_2(\mathbb{R})$. We define for $\vec{\tau} = (\tau_1, \tau_2) \in \mathbb{H}^2$

$$E(\vec{\tau}, s, \phi) = \mathrm{N}(\vec{v})^{-1/2} E\Big(g_{\vec{\tau}}, s, \Phi_f \otimes \Big(\bigotimes_{1 \le i \le 2} \Phi_{\sigma_i}\Big)\Big),$$

where $\vec{v} = \mathrm{Im}(\vec{\tau})$, $\mathrm{N}(\vec{v}) = \prod_i v_i$, and $g_{\vec{\tau}} = (n(u_i)m(\sqrt{v_i}))_{1 \le i \le 2}$. It is a (nonholomorphic) Hilbert modular form of parallel weight 1 for some congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_F)$. Following [Bruinier et al. 2012], we further normalize

$$E^*(\vec{\tau}, s, \phi) = \Lambda(s+1, \chi_{E/F}) E(\vec{\tau}, s, \phi),$$

where

$$\Lambda(s, \chi) = D^{s/2}\bigl(\pi^{-(s+1)/2}\Gamma\bigl(\tfrac{1}{2}(s+1)\bigr)\bigr)^2 L(s, \chi_{E/F}). \tag{6-2}$$

According to [Yang and Yin 2019], this Eisenstein series is incoherent in the sense of Kudla, and $E^*(\vec{\tau}, 0, \phi) = 0$ automatically. Write its central derivative via Fourier expansion

$$E^{*,\prime}(\vec{\tau}, 0, \phi) = \sum_{t \in F} a(\vec{v}, t, \phi) q^t, \qquad q^t = \mathbf{e}(\mathrm{Tr}(t\tau)), \tag{6-3}$$

with $\vec{v}$ the imaginary part of $\vec{\tau} \in \mathbb{H}^2$. Then it is known that $a(t, \phi) = a(\vec{v}, t, \phi)$ is independent of $\vec{v}$ when t is totally positive. Finally, when $\phi = \bigotimes_{\mathfrak{p}} \phi_{\mathfrak{p}} \in S(V_f)$ is factorizable, one has for $t \gg 0$ (the factor -4 comes from [Yang and Yin 2019, Proposition 2.7(1)(2)])

$$a(t, \phi) = -4 \frac{d}{ds}\biggl(\prod_{\mathfrak{p}} W_{t,\mathfrak{p}}(s, \phi)\biggr)\bigg|_{s=0} \tag{6-4}$$

where

$$W_{t,\mathfrak{p}}(s, \phi) := \int_{F_{\mathfrak{p}}} \omega(wn(b))(\phi_{\mathfrak{p}})(0) |a(wn(b))|_{\mathfrak{p}}^s \psi_{\mathfrak{p}}(-tb)\, db \tag{6-5}$$

are the local Whittaker functions. Specializing Theorem 5.2 in [Bruinier et al. 2012] gives us the following result.

Theorem 6.1 (Bruinier–Kudla–Yang). *Let $d_j < 0$ be fundamental discriminants satisfying $d_j \equiv 1 \bmod 8$ and $3 \nmid d_j$. For any $1 \ne \mathrm{d} \mid 24$, let $\phi_{\mathrm{d}} \in S(V_{\mathrm{d}}(\mathbb{A}_f))$ be associated to $\mathfrak{u}_{\mathrm{d}}$. Then we have*

$$-\log |\Psi_{\mathrm{d}}(Z(W))|^4 = C(W, K) \sum_{t \in F^\times, t \gg 0, \mathrm{Tr}(t) = 1/\mathrm{d}} a(t, \phi_{\mathrm{d}}), \tag{6-6}$$

where $Z(W)$ is the big CM 0-cycle associated to d_1, d_2 defined in (5-6), and

$$C(W, K) = \frac{\deg(Z(W, z_{\sigma_2}^{\pm}))}{\Lambda(0, \chi)} = 2.$$

The rest of this section is to compute $a(t, \phi_{\mathrm{d}})$ and prove the Yui–Zagier conjecture. Unfortunately, ϕ_{d} is not factorizable over F at the places dividing $(\mathrm{d}, 6)$. Instead, we have

$$\phi_{\mathrm{d}} = \phi_{\mathrm{d},2} \phi_{\mathrm{d},3} \otimes_{\mathfrak{p} \nmid 6} \phi_{\mathrm{d},\mathfrak{p}}.$$

Then for $\mathfrak{p} \nmid 6$, the contribution of $W_{t,\mathfrak{p}}(s, \phi_{\mathrm{d}})$ is the same as in the case of Gross-Zagier (see [Yang and Yin 2019]). Therefore, we are left with the local calculations at 2 and 3. Since 2 splits completely in $E/\mathbb{Q}$, we denote $\mathfrak{p}_1, \mathfrak{p}_2$ the two primes in F above 2. Also denote $\mathfrak{p}_3, \mathfrak{p}_3'$ the primes in F above 3. They are the same if and only if $\bigl(\frac{D}{3}\bigr) = -1$. The local calculations in Section 6B lead to the following result.

Theorem 6.2. *Let $d_j < 0$ and d be the same as in Theorem 6.1, and let $\varepsilon = \varepsilon_1\varepsilon_2 = (-1)^{(d_1+d_2-2)/8}$. Suppose $t = (a+\sqrt{D})/(2\mathrm{d}\sqrt{D}) \in F^\times$ is totally positive with $a \in \mathbb{Q}$. Then*

$$a(t,\phi_{\mathrm{d}}) = -\mathrm{d}_2\varepsilon^{24/\mathrm{d}}\delta_2(\mathrm{d}_2,t) \times \Bigg\{ \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p}\nmid 3}} (1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D}))\rho^{(6)}(t\sqrt{D}\mathfrak{p}^{-1})\delta_3(\mathrm{d}_3,t)\log(\mathrm{Nm}(\mathfrak{p})) + \log 3 \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p}\mid 3}} \rho^{(2)}(t\sqrt{D}\mathfrak{p}^{-1})\delta_3'(\mathrm{d}_3,t)\Bigg\} \tag{6-7}$$

if $a \in \mathbb{Z}$ and zero otherwise. The functions $\delta_p(\mathrm{d}_p,t)$ and $\delta_3'(\mathrm{d}_3,t)$ are defined by

$$\delta_2(1,t) := 2(v_2(\mathrm{Nm}(t))-1), \quad v_2(\mathrm{Nm}(t)) \geq 2,$$

$$\delta_2(2,t) := \begin{cases} 1 & \text{if } v_2(\mathrm{Nm}(t)) = 0, \\ v_2(\mathrm{Nm}(t))-3 & \text{if } v_2(\mathrm{Nm}(t)) \geq 1, \end{cases}$$

$$\delta_2(4,t) := \begin{cases} \mp 1 & \text{if } \mathrm{Nm}(2t) \equiv \pm 1 \bmod 4, \\ 1 & \text{if } v_2(\mathrm{Nm}(t)) = 0, \\ v_2(\mathrm{Nm}(t))-3 & \text{if } v_2(\mathrm{Nm}(t)) \geq 1, \end{cases}$$

$$\delta_2(8,t) := \begin{cases} 1 & \text{if } \mathrm{Nm}(4t) \equiv 3 \bmod 8, \\ -1 & \text{if } \mathrm{Nm}(4t) \equiv 7 \bmod 8, \\ \mp 1 & \text{if } \mathrm{Nm}(2t) \equiv \pm 1 \bmod 4, \\ 1 & \text{if } v_2(\mathrm{Nm}(t)) = 0, \\ v_2(\mathrm{Nm}(t))-3 & \text{if } v_2(\mathrm{Nm}(t)) \geq 1, \end{cases}$$

$$\delta_3(1,t) := \rho_3(t), \quad v_3(\mathrm{Nm}(t)) \geq 0,$$

$$\delta_3(3,t) := \begin{cases} 2-\frac{3}{4}\left(1-\left(\frac{d_1}{3}\right)\right)\left(1-\left(\frac{d_2}{3}\right)\right) & \text{if } \mathrm{Nm}(3t) \equiv 1 \bmod 3, \\ -1 & \text{if } \mathrm{Nm}(3t) \equiv 2 \bmod 3, \\ \left(1+\left(\frac{d_1}{3}\right)\right)v_3(\mathrm{Nm}(t))+1-\left(\frac{d_1}{3}\right)^{v_3(\mathrm{Nm}(t))-1} & \text{if } v_3(\mathrm{Nm}(3t)) \geq 1, \end{cases}$$

$$\delta_3'(\mathrm{d}_3,t) := \begin{cases} v_3(\mathrm{Nm}(t))+1 & \text{if } \mathrm{d}_3 = 1, \\ 2v_3(\mathrm{Nm}(t))+3 & \text{if } \mathrm{d}_3 = 3, \end{cases}$$

and zero otherwise. Here $\rho^{(M)}(\mathfrak{a}) := \rho(\mathfrak{a}^{(M)})$ is the number of integral ideals of E with relative norm (to F) $\mathfrak{a}^{(M)}$, $\rho_M(\mathfrak{a}) := \rho(\mathfrak{a}/\mathfrak{a}^{(M)})$, and $\mathfrak{a}^{(M)}$ is the prime to M part of an ideal $\mathfrak{a}$.

Proof. To evaluate $a(t,\phi)$, it is convenient to introduce the "Diff" set of Kudla. For a totally positive $t \in F^\times$, define

$$\mathrm{Diff}(W,t) := \{\mathfrak{p} : W_{\mathfrak{p}} \text{ does not represent } t\}.$$

Then $|\mathrm{Diff}(W,t)|$ is finite and odd. Furthermore if $\#\mathrm{Diff}(W,t) > 1$, then $a(t,\phi)$ vanishes. This is also the case with the expression on the right-hand side of (6-7), since $\delta_3(\mathrm{d}_3,t) = 0$ if $\mathfrak{p}_3, \mathfrak{p}_3' \in \mathrm{Diff}(W,t)$ and $\rho^{(6)}(t\sqrt{D}\mathfrak{p}) = 0$ for every inert $\mathfrak{p}$ if $\mathrm{Diff}(W,t)$ contains two primes coprime to 6. Therefore, we can suppose that $\mathrm{Diff}(W, t) = \{\mathfrak{p}_0\}$ for a single prime $\mathfrak{p}_0$ of F. In that case, every term with $\mathfrak{p} \neq \mathfrak{p}_0$ on the right-hand side of (6-7) vanishes. Given $t = (a + \sqrt{D})/(2\mathrm{d}\sqrt{D}) \in F$ totally positive, the Fourier coefficient $a(t, \phi)$ is given by

$$a(t, \phi_{\mathrm{d}}) = -4 \frac{d}{ds} \left(\frac{W^*_{t,2}(s, \phi_{\mathrm{d},2})}{\gamma(W_2)} \frac{W^*_{t,3}(s, \phi_{\mathrm{d},3})}{\gamma(W_3)} \prod_{\mathfrak{p} \nmid 6\infty} \frac{W^*_{t,\mathfrak{p}}(s, \phi_{\mathfrak{p}})}{\gamma(W_{\mathfrak{p}})} \right) \bigg|_{s=0},$$

where $\gamma(W_{\mathfrak{p}})$ is the Weil index of $W_{\mathfrak{p}}$ (see, e.g., Proposition 2.7 in [Yang and Yin 2019]).

Recall that $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3, \mathfrak{p}'_3$ are primes in F above 2 and 3 respectively. Since $\mathfrak{p}_1, \mathfrak{p}_2$ splits in E, they are not in $\mathrm{Diff}(W, t)$ for any t. However, $\mathfrak{p}_3$ and $\mathfrak{p}'_3$ could appear in some Diff set if they are inert in E/F. Now, if $\mathfrak{p}_0 \nmid 3$, then we can proceed as in the proof of Theorem 1.1 in [Yang and Yin 2019] to obtain

$$a(t, \phi_{\mathrm{d}}) = -2 \frac{W^*_{t,2}(0, \phi_{\mathrm{d},2})}{\gamma(W_2)} \frac{W^*_{t,3}(0, \phi_{\mathrm{d},3})}{\gamma(W_3)} \rho^{(6)}(\mathrm{d}\sqrt{D} t \mathfrak{p}_0^{-1})(1 + \mathrm{ord}_{\mathfrak{p}_0}(t\sqrt{D})) \log \mathrm{Nm}(\mathfrak{p}_0).$$

By Lemma 6.5 and (6-13), we can replace $2W^*_{t,2}(0, \phi_{\mathrm{d},2})/\gamma(W_2)$ with $\varepsilon^{24/\mathrm{d}} \mathrm{d}_2 \delta_2(\mathrm{d}_2, t)$. By Lemmas 6.7, 6.10 and 6.12, we can replace $W^*_{t,3}(0, \phi_{\mathrm{d},3})/\gamma(W_3)$ with $\delta_3(\mathrm{d}_3, t)$ and arrive at the right-hand side.

If $\mathrm{Diff}(W, t) = \{\mathfrak{p}_0\}$ with $\mathfrak{p}_0 \mid 3$, then $\left(\frac{d_j}{3}\right) = -1$ and we can write

$$a(t, \phi_{\mathrm{d}}) = -4 \frac{W^*_{t,2}(0, \phi_{\mathrm{d},2})}{\gamma(W_2)} \frac{W^{*,\prime}{}_{t,3}(0, \phi_{\mathrm{d},3})}{\gamma(W_3)} \rho^{(6)}(\mathrm{d}\sqrt{D} t).$$

We can again replace $2W^*_{t,2}(0, \phi_{\mathrm{d},2})/\gamma(W_2)$ with $\varepsilon^{24/\mathrm{d}} \mathrm{d}_2 \delta_2(\mathrm{d}_2, t)$ and apply Lemma 6.12 to replace $2W^{*,\prime}{}_{t,3}(0, \phi_{\mathrm{d},3})/\gamma(W_3)$ with $\delta'_3(3, t) \log 3$. This finishes the proof. □

Yui and Zagier [1997] derived the conjectural factorization of $\mathrm{Nm}_{H/\mathbb{Q}}(f(\tau_1)^{24/s} - f(\tau_2)^{24/s})$ from the conjectural factorization of $\mathrm{Nm}_{H/\mathbb{Q}}(\Phi_{24/s}(f(\tau_1), f(\tau_2)))$, where Φ_r the r-th cyclotomic polynomial. Since $\mathfrak{F}(m)$ is the power of a rational prime ℓ, we can define

$$\mathfrak{F}(m) = \ell^{\gamma(m)}, \tag{6-8}$$

where $\gamma(m) = \prod_{p \mid m} \gamma_p(m)$ with

$$\gamma_p(m) := \begin{cases} \mathrm{ord}_p(m) + 1 & \text{if } \varepsilon(p) = 1, \\ 1 & \text{if } \varepsilon(p) = -1 \text{ and } 2 \mid \mathrm{ord}_p(m), \\ \frac{1}{2}(\mathrm{ord}_p(m) + 1) & \text{if } \varepsilon(p) = -1 \text{ and } 2 \nmid \mathrm{ord}_p(m) \text{ (i.e., } p = \ell). \end{cases} \tag{6-9}$$

The conjecture is then expressed in terms of how $\gamma_2(m)$ and $\gamma_3(m)$ decomposes, which are summarized in two tables (see [Yui and Zagier 1997, p. 1653]). The theorem above is equivalent to this formulation of the conjecture. As in [Yui and Zagier 1997], one can give a conjecture with an equivalent, but simplified, expression. This is the content of Conjecture 1.5, which we prove now.

Proof of Theorem 1.7. By Proposition 5.5 and Theorems 6.1, 6.2, we can write

$$\begin{aligned}
4s &\sum_{[\mathfrak{a}_j]\in \mathrm{Cl}(d_j),\, j=1,2} \log|f(\mathfrak{a}_1)^{24/s} - f(\mathfrak{a}_2)^{24/s})|^4 \\
&= -2\sum_{\mathrm{d}|s} \varepsilon^{24/\mathrm{d}} \sum_{t\in F^\times, t\gg 0, \mathrm{Tr}(t)=1/\mathrm{d}} a(t,\phi_{\mathrm{d}}) \\
&= 2\sum_{\mathrm{d}|s} \sum_{t\in F^\times, t\gg 0, \mathrm{Tr}(t)=1/\mathrm{d}} \mathrm{d}_2\delta_2(\mathrm{d}_2,t) \\
&\qquad\qquad \times \Bigg\{\log 3 \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p}|3}} \rho^{(2)}(t\sqrt{D}\mathfrak{p}^{-1})\delta_3'(\mathrm{d}_3,t) \\
&\qquad\qquad\qquad + \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p}\nmid 3}} (1+\mathrm{ord}_{\mathfrak{p}}(t\sqrt{D}))\rho^{(6)}(t\sqrt{D}\mathfrak{p}^{-1})\delta_3(\mathrm{d}_3,t)\log(\mathrm{Nm}(\mathfrak{p}))\Bigg\} \\
&= 2 \sum_{4\sqrt{D}\tilde{t}\in\mathcal{O}_F, \tilde{t}\gg 0, \mathrm{Tr}(\tilde{t})=1/2} \sum_{\mathrm{d}|s} \mathrm{d}_2\delta_2\big(\mathrm{d}_2, \tfrac{2\tilde{t}}{\mathrm{d}}\big) \\
&\qquad\qquad \times \Bigg\{\log 3 \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p}|3}} \rho^{(2)}\big(\tfrac{2\tilde{t}}{\mathrm{d}}\sqrt{D}\mathfrak{p}^{-1}\big)\delta_3'\big(\mathrm{d}_3,\tfrac{2\tilde{t}}{\mathrm{d}}\big) \\
&\qquad\qquad\qquad + \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p}\nmid 3}} (1+\mathrm{ord}_{\mathfrak{p}}(\tilde{t}\sqrt{D}))\rho^{(6)}(\tilde{t}\sqrt{D}\mathfrak{p}^{-1})\delta_3\big(\mathrm{d}_3,\tfrac{2\tilde{t}}{\mathrm{d}}\big)\log(\mathrm{Nm}(\mathfrak{p}))\Bigg\}
\end{aligned}$$

By Theorem 6.2, we have

$$\sum_{\mathrm{d}_2|s_2} \mathrm{d}_2\delta_2\big(\mathrm{d}_2,\tfrac{2\tilde{t}}{\mathrm{d}}\big) = \sum_{\mathrm{d}_2|s_2} \mathrm{d}_2\delta_2\big(\mathrm{d}_2,\tfrac{2\tilde{t}}{\mathrm{d}_2}\big),$$

$$\sum_{\mathrm{d}_2|1} \mathrm{d}_2\delta_2\big(\mathrm{d}_2,\tfrac{2\tilde{t}}{\mathrm{d}_2}\big) = 2(v_2(\mathrm{Nm}(\tilde{t}))+1) = 2\gamma_2(\mathrm{Nm}(\tilde{t})),$$

$$\sum_{\mathrm{d}_2|2} \mathrm{d}_2\delta_2\big(\mathrm{d}_2,\tfrac{2\tilde{t}}{\mathrm{d}_2}\big) = 4\begin{cases} 1 & \text{if } v_2(\mathrm{Nm}(\tilde{t}))=0, \\ v_2(\mathrm{Nm}(\tilde{t}))-1 & \text{if } v_2(\mathrm{Nm}(\tilde{t}))\geq 1, \end{cases}$$

$$\sum_{\mathrm{d}_2|4} \mathrm{d}_2\delta_2\big(\mathrm{d}_2,\tfrac{2\tilde{t}}{\mathrm{d}_2}\big) = 8\begin{cases} 1 & \text{if } v_2(\mathrm{Nm}(\tilde{t}))\equiv -1 \bmod 4 \\ & \text{or } v_2(\mathrm{Nm}(\tilde{t}))=2, \\ v_2(\mathrm{Nm}(\tilde{t}))-3 & \text{if } v_2(\mathrm{Nm}(\tilde{t}))\geq 3, \end{cases}$$

$$\sum_{\mathrm{d}_2|8} \mathrm{d}_2\delta_2\big(\mathrm{d}_2,\tfrac{2\tilde{t}}{\mathrm{d}_2}\big) = 16\begin{cases} 1 & \text{if } v_2(\mathrm{Nm}(\tilde{t}))=4 \\ & \text{or } v_2(\mathrm{Nm}(\tilde{t}))\equiv 12 \bmod 16 \\ & \text{or } v_2(\mathrm{Nm}(\tilde{t}))\equiv 3 \bmod 8, \\ v_2(\mathrm{Nm}(\tilde{t}))-5 & \text{if } v_2(\mathrm{Nm}(\tilde{t}))\geq 5. \end{cases}$$

From this, it is easy to check that

$$\sum_{\mathrm{d}_2 \mid s_2} \mathrm{d}_2 \delta_2\bigl(\mathrm{d}_2, \tfrac{2\tilde{t}}{\mathrm{d}}\bigr) = 2s_2 \sum_{\substack{r_2 \mid s_2,\, m := D\mathrm{Nm}(\tilde{t}/r_2) \in \mathbb{Z}, \\ m \equiv 3 \bmod s_2/r_2}} \gamma_2(m), \tag{6-10}$$

where we write $s = s_2 s_3$ with s_p the p-part of s. Similarly, we also have

$$\begin{aligned} &\kappa_3(s) s_3 \sum_{\substack{r_3 \mid s_3,\, m := D\mathrm{Nm}(\tilde{t}/r_3) \in \mathbb{Z}, \\ m \equiv d_1 + d_2 - 1 \bmod s_3/r_3}} \gamma_3(m) \\ &\quad = \begin{cases} \frac{1}{2} \sum_{\mathrm{d}_3 \mid s_3} \sum_{\mathfrak{p} \mid 3} \rho_3(\mathfrak{p}^{-1}\tilde{t}/3) \delta_3'\bigl(\mathrm{d}_3, \frac{2\tilde{t}}{\mathrm{d}}\bigr) & \text{if } \bigl(\frac{d_1}{3}\bigr) = \bigl(\frac{d_2}{3}\bigr) = -1 \text{ and } 2 \nmid v_3(\mathrm{Nm}(\tilde{t})), \\ \sum_{\mathrm{d}_3 \mid s_3} \delta_3\bigl(\mathrm{d}_3, \frac{2\tilde{t}}{\mathrm{d}}\bigr) & \text{otherwise,} \end{cases} \end{aligned} \tag{6-11}$$

where $\kappa_3(s) \in \bigl\{1, \frac{1}{2}\bigr\}$ is the constant defined in (1-8). So suppose $\mathrm{Diff}(W, \tilde{t}) = \{\mathfrak{p}_0\}$ with $\ell = \mathrm{Nm}(\mathfrak{p}_0)$. Then substituting in these gives us

$$\begin{aligned} \sum_{\mathrm{d} \mid s} \mathrm{d}_2 \delta_2\bigl(\mathrm{d}_2, \tfrac{2\tilde{t}}{\mathrm{d}}\bigr) \Biggl\{ &\log 3 \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p} \mid 3}} \rho^{(2)}\bigl(\tfrac{2\tilde{t}}{\mathrm{d}} \sqrt{D} \mathfrak{p}^{-1}\bigr) \delta_3'\bigl(\mathrm{d}_3, \tfrac{2\tilde{t}}{\mathrm{d}}\bigr) \\ &+ \sum_{\substack{\mathfrak{p} \text{ inert in } E/F \\ \mathfrak{p} \nmid 3}} (1 + \mathrm{ord}_{\mathfrak{p}}(\tilde{t}\sqrt{D})) \rho^{(6)}(\tilde{t}\sqrt{D}\mathfrak{p}^{-1}) \delta_3\bigl(\mathrm{d}_3, \tfrac{2\tilde{t}}{\mathrm{d}}\bigr) \log(\mathrm{Nm}(\mathfrak{p})) \Biggr\} \\ &= 4s \sum_{\substack{r \mid s,\, m := D\mathrm{Nm}(\tilde{t}/r) \in \mathbb{Z} \\ m \equiv 19D \bmod s/r}} \log(\ell) \prod_{p \mid m} \gamma_p(m) = 4s \sum_{\substack{r \mid s,\, m := D\mathrm{Nm}(\tilde{t}/r) \in \mathbb{Z} \\ m \equiv 19D \bmod s/r}} \log \mathfrak{F}(m). \end{aligned}$$

After writing $\tilde{t} = (\sqrt{D} + a)/(4\sqrt{D})$ with $a \in \mathbb{Z}$ in the summation, we obtain (1-9). □

6B. *Local Calculations.* We first need to write $\phi_{\mathrm{d},p}$ as a linear combination of $\bigotimes_{\mathfrak{p} \mid p} \phi_{\mathfrak{p}}$ for some $\phi_{\mathfrak{p}} \in S(E_{\mathfrak{p}}) = S(W_{\mathfrak{p}})$.

6B1. $p = 2$. In this subsection, we deal with the case $p = 2$. Since $d_j \equiv 1 \bmod 8$, the prime 2 splits completely. We fix $\delta, \delta_j \in \mathbb{Z}_2^\times$ such that

$$\delta^2 = D, \quad \delta_j^2 = d_j, \quad \delta_1 \delta_2 = \delta. \tag{6-12}$$

We also denote

$$\overline{\delta_j} := -\delta_j, \quad \delta' := -\delta.$$

Note that

$$\varepsilon_{d_1} \varepsilon_{d_2} = \Bigl(\frac{2}{\delta}\Bigr). \tag{6-13}$$

For $i = 1, 2$, let $\mathfrak{p}_i$ be the two primes in F above 2, and $\mathfrak{P}_i, \overline{\mathfrak{P}_i}$ the two primes in E above $\mathfrak{p}_i$. Then the local fields $E_{\mathfrak{P}_i}$ and $E_{\overline{\mathfrak{P}_i}}$ are isomorphic to $\mathbb{Q}_2$ via the map

$$\begin{aligned} \sigma_i : F_{\mathfrak{p}_i} &\cong \mathbb{Q}_2, \quad \sqrt{D} \mapsto (-1)^i \delta, \\ \sigma_i : E_{\mathfrak{P}_i} &\cong \mathbb{Q}_2, \quad \sqrt{D} \mapsto (-1)^i \delta, \quad \sqrt{d_j} \mapsto (-1)^{(i-1)(j-1)} \delta_j, \\ \sigma_i : E_{\overline{\mathfrak{P}_i}} &\cong \mathbb{Q}_2, \quad \sqrt{D} \mapsto (-1)^i \delta, \quad \sqrt{d_j} \mapsto -(-1)^{(i-1)(j-1)} \delta_j. \end{aligned}$$

Under these identifications, $W_2 = W \otimes_{\mathbb{Q}} \mathbb{Q}_2 = W_{\mathfrak{p}_1} \times W_{\mathfrak{p}_2}$ with

$$W_{\mathfrak{p}_i} = E_{\mathfrak{p}_i} = E_{\mathfrak{P}_i} \times E_{\overline{\mathfrak{P}_i}} \cong \mathbb{Q}_2^2, \quad Q_{\mathfrak{p}_i}(y_1, y_2) = (-1)^i \frac{\mathrm{d}}{\delta} y_1 y_2.$$

Now we identify the $\mathbb{Q}_2$-quadratic space

$$\sigma : (V \otimes_{\mathbb{Q}} \mathbb{Q}_2, Q) \cong (E_{\mathfrak{p}_1}, Q_{\mathfrak{p}_1}) \times (E_{\mathfrak{p}_2}, Q_{\mathfrak{p}_2}), \quad \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix} \mapsto (\sigma_1(x), \sigma_1(\bar{x}), \sigma_2(x), \sigma_2(\bar{x})), \tag{6-14}$$

with

$$x = x_1 + x_2 \frac{-1+\sqrt{d_1}}{2} + x_3 \frac{1+\sqrt{d_2}}{2} + x_4 \frac{-1+\sqrt{d_1}}{2} \frac{1+\sqrt{d_2}}{2} \in W_2.$$

Under this isomorphism, we can identify $S(V \otimes \mathbb{Q}_2)$ with $S(E_{\mathfrak{p}_1} \times E_{\mathfrak{p}_2}) \cong S(E_{\mathfrak{p}_1}) \otimes S(E_{\mathfrak{p}_2})$, and map the lattice $L_{\mathrm{d},2} := L_{\mathrm{d}} \otimes \mathbb{Z}_2$ onto

$$\widetilde{L} := \Big\{ y = (y_1, y_2, y_3, y_4) \in \mathbb{Z}_2^4 : \sum y_i \in 2\mathbb{Z}_2 \Big\},$$

The $\mathbb{Q}_2$-quadratic form $\widetilde{Q}_{\mathrm{d}}$ on $\widetilde{L}$ is given by

$$\widetilde{Q}_{\mathrm{d}}(y) := -\frac{\mathrm{d}}{\delta}(y_1 y_2 - y_3 y_4) = Q_{\mathfrak{p}_1}(y_1, y_2) + Q_{\mathfrak{p}_2}(y_3, y_3).$$

Let

$$L_0 = (2\mathbb{Z}_2)^4 = 2\mathcal{O}_{E_{\mathfrak{p}_1}} \times 2\mathcal{O}_{E_{\mathfrak{p}_2}} = \widetilde{M}_1 \times \widetilde{M}_2$$

with $\widetilde{M}_i$ being the $\mathcal{O}_{F_{\mathfrak{p}_i}}$-lattice $2\mathcal{O}_{E_{\mathfrak{p}_i}}$. Then

$$L_0 \subset \widetilde{L} \subset \widetilde{L}' \subset L_0' = \frac{1}{4\mathrm{d}_2} L_0 \quad \text{and} \quad \widetilde{L}' = \left\{ y = \frac{1}{2\mathrm{d}_2}(y_1, y_2, y_3, y_4) \in \frac{1}{2\mathrm{d}_2} \mathbb{Z}_2^4 : y_i + y_j \equiv 0 \bmod 2 \right\}.$$

Notice that

$$\phi_{\widetilde{L}} \circ \sigma^{-1} = \sum_{\substack{y_i \in \mathbb{Z}/2\mathbb{Z} \\ \sum y_i = 0}} \phi_{(y_1, y_2) + \widetilde{M}_1} \otimes \phi_{(y_3, y_4) + \widetilde{M}_2},$$

where $\phi_A = \mathrm{Char}(A)$ for $A \subset W_2$. To apply the general formula in [Yang et al. 2019], we define $M_i = \mathbb{Z}_2^2$ with quadratic form $Q_i(y_1 y_2) = (-1)^i \frac{4\mathrm{d}}{\delta} y_1 y_2$. Then $(M_i, Q_i) \cong (\widetilde{M}_i, Q_{\mathfrak{p}_i})$ via scaling by 2. For any $\mu \in (\mathbb{Q}_2/\mathbb{Z}_2)^2$, we denote

$$\phi_\mu = \mathrm{char}(\mu + \mathbb{Z}_2^2)$$

and view it as an element in $S(M_i)$ for both $i = 1, 2$ if $\mu \in \big(\frac{1}{4\mathrm{d}_2} \mathbb{Z}_2/\mathbb{Z}_2\big)^2$.

Now, we can apply this scaling map to $\phi_{\mathrm{d},2} \circ \sigma^{-1}$, where $\phi_{\mathrm{d},2} \in S(L_{\mathrm{d},2})$ is the Schwartz function associated to $\mathfrak{u}_{\mathrm{d},2}$. We denote the result by $\widetilde{\phi}_{\mathrm{d},2} \in S(M_1 \times M_2) \cong S(M_1) \otimes S(M_2)$, which will depend on the choice of $\delta \bmod \mathrm{d}_2$. We have listed them as follows.

Lemma 6.3. *For $\delta_j \in \mathbb{Z}_2^\times$ and $\delta = \delta_1\delta_2 \in \mathbb{Z}_2^\times$, we have*

$$\widetilde{\phi}_{\mathrm{d},2} = \begin{cases} \phi_{\widetilde{L}}, & \mathrm{d}_2 = 1, \\ \phi_0 \otimes \phi_{1,\delta} + \phi_{1,-\delta} \otimes \phi_0, & \mathrm{d}_2 = 2, \\ 2\big((\phi_0 + \phi_{1,-\delta}) \otimes \phi_{2,\delta} + \phi_{2,-\delta} \otimes (\phi_0 + \phi_{1,\delta})\big), & \mathrm{d}_2 = 4, \\ \left(\frac{2}{\delta \mathrm{d}_3}\right) 4\big((\phi_0 + \phi_{1,\delta}) \otimes \phi_{3,\delta} + \phi_{3,-\delta} \otimes (\phi_0 + \phi_{1,-\delta}) + \phi_{2,\delta} \otimes \phi_{3,5\delta} + \phi_{3,-5\delta} \otimes \phi_{2,-\delta}\big), & \mathrm{d}_2 = 8, \end{cases} \tag{6-15}$$

where for $j = 1, 2, 3,\ r \in (\mathbb{Z}/2^{j+1}\mathbb{Z})^\times$

$$\phi_0 := \sum_{k \in \mathbb{Z}/2\mathbb{Z}} \phi_{\frac{1}{2}(k,k)} - \phi_{\frac{1}{2}(k,k+1)}, \quad \phi_{j,r} := \sum_{a \in (\mathbb{Z}/2^{j+1})^\times} \phi_{\mu(a;r,j)} - \phi_{\mu(a;r+2^j;j)},$$

are elements in $S(M_i)$ with

$$\mu(a; r, j) := \frac{1}{2^{j+1}}(a, ra^{-1}) \in (\mathbb{Q}_2/\mathbb{Z}_2)^2.$$

Remark 6.4. Note that the support of $\mathfrak{u}_{24,2}$ is the support of $\mathfrak{u}_{8,2}$ after scaling by d_3. This does not affect $\phi_{j,r}$ for $j = 1, 2$ but introduces the factor $\left(\frac{2}{\mathrm{d}_3}\right)$ when $j = 3$, since $\phi_{3,rc^2} = \left(\frac{2}{c}\right)\phi_{3,r}$ for any odd integer c. Therefore this factor appears above when $\mathrm{d}_2 = 8$.

Proof. One can use Lemma 3.11 to check that the cosets on the right indeed appear. Then we have all of them by counting. □

Now, we can apply the general Whittaker function formulas in [Yang et al. 2019] to obtain:

Lemma 6.5. *Let $\delta_2(\mathrm{d}_2, t)$ be defined as in Theorem 6.2. Then we have*

$$\frac{W_t^*(0, \widetilde{\phi}_{\mathrm{d},2})}{\gamma(W_2)} = \left(\frac{2}{\delta}\right)^{24/\mathrm{d}} \frac{\mathrm{d}_2}{2} \delta_2(\mathrm{d}_2, t)$$

for all totally positive $t \in F^\times$ with $\mathrm{Tr}(t) = \frac{1}{\mathrm{d}}$.

Proof. This can be checked case by case. For $\mathrm{d}_2 = 1$, this was already done in [Yang and Yin 2019]. Otherwise, we can apply Propositions 5.3 and 5.7 in [Yang et al. 2019] after scaling the lattice by 2 and the quadratic form by 4 (i.e., variant 2 in [Yang et al. 2019]). We write $t_i = \sigma_i(t) \in \mathbb{Q}_2$ and suppose $o(t_1) \geq o(t_2)$ with $o(t_i)$ the 2-adic valuation of $t_i \in \mathbb{Q}_2$. The case $o(t_1) \leq o(t_2)$ will be exactly the same. Tables 1–6 contain the nonzero values of $W_{t_i}^*(0, \phi_{\mu_i})/\gamma(W_{\mathfrak{p}_i})$ for $i = 1$.

$o(t_1)$	$\mu_1 = (0,0)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 0\right)$
1	0	1	0
≥ 2	$o(t_1) - 2$	0	1

Table 1. $\mathrm{d}_2 = 2$, $\beta = -8\mathrm{d}_3\delta^{-1}$.

$o(t_1)$	$\mu_1 = (0,0)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 0\right)$	$\left(\frac{a}{4}, -\frac{a^{-1}\delta}{4}\right)$	$\left(\frac{a}{4}, \frac{a^{-1}(-\delta+2)}{4}\right)$
$t_1 \in \mathrm{d}_3 + 4\mathbb{Z}_2$	0	0	0	$\frac{1}{2}$	0
$t_1 \in -\mathrm{d}_3 + 4\mathbb{Z}_2$	0	0	0	0	$\frac{1}{2}$
2	0	1	0	0	0
≥ 3	$o(t_1) - 3$	0	1	0	0

Table 2. $\mathrm{d}_2 = 4$, $\beta = -16\mathrm{d}_3\delta^{-1}$.

$o(t_1)$	$\mu_1 = (0,0)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 0\right)$	$\left(\frac{a}{4}, \frac{a^{-1}\delta}{4}\right)$	$\left(\frac{a}{4}, \frac{a^{-1}(\delta+2)}{4}\right)$	$\left(\frac{a}{8}, \frac{a^{-1}\delta^{-1}}{8}\right)$	$\left(\frac{a}{8}, \frac{a^{-1}(\delta^{-1}+4)}{8}\right)$
$t_1 \in \frac{1}{2}(-\mathrm{d}_3 + 8\mathbb{Z}_2)$	0	0	0	0	0	$\frac{1}{4}$	0
$t_1 \in \frac{1}{2}(3\mathrm{d}_3 + 8\mathbb{Z}_2)$	0	0	0	0	0	0	$\frac{1}{4}$
$t_1 \in 2(-\mathrm{d}_3 + 4\mathbb{Z}_2)$	0	0	0	$\frac{1}{2}$	0	0	0
$t_1 \in 2(\mathrm{d}_3 + 4\mathbb{Z}_2)$	0	0	0	0	$\frac{1}{2}$	0	0
3	0	1	0	0	0	0	0
≥ 4	$o(t_1) - 4$	0	1	0	0	0	0

Table 3. $\mathrm{d}_2 = 8$, $\beta = -32\mathrm{d}_3\delta^{-1}$.

$o(t_1)$	$\mu_2 = \left(\frac{a}{4}, \frac{a^{-1}\delta}{4}\right)$	$\left(\frac{a}{4}, \frac{a^{-1}(\delta+2)}{4}\right)$
≥ 1	$\frac{1}{2}$	0

Table 4. $\mathrm{d}_2 = 2$, $\beta = 8\mathrm{d}_3\delta^{-1}$.

$o(t_1)$	$\mu_2 = \left(\frac{a}{8}, \frac{a^{-1}\delta}{8}\right)$	$\left(\frac{a}{8}, \frac{a^{-1}(\delta+4)}{8}\right)$
0	0	$\frac{1}{4}$
≥ 1	$\frac{1}{4}$	0

Table 5. $\mathrm{d}_2 = 4$, $\beta = 16\mathrm{d}_3\delta^{-1}$.

$o(t_1)$	$\mu_2 = \left(\frac{a}{16}, \frac{a^{-1}\delta\mathrm{d}_3^2}{16}\right)$	$\left(\frac{a}{16}, \frac{a^{-1}(\delta\mathrm{d}_3^2+8)}{16}\right)$	$\left(\frac{a}{16}, \frac{5a^{-1}\delta\mathrm{d}_3^2}{16}\right)$	$\left(\frac{a}{16}, \frac{a^{-1}(5\delta\mathrm{d}_3^2+8)}{16}\right)$
-1	0	0	$\frac{1}{8}$	0
≥ 1	$\frac{1}{8}$	0	0	0

Table 6. $\mathrm{d}_2 = 8$, $\beta = 32\mathrm{d}_3\delta^{-1}$.

For $i=2$, we write $\alpha(\mu_2,t_2):=\beta\mu_2\overline{\mu_2}-t_2$ in the notation of [Yang et al. 2019]. When $\mathrm{d}_2=2$, we have $t_2\in\frac{1}{2}(\mathrm{d}_3^{-1}+4\mathbb{Z}_2)$ if $o(t_1)\geq 1$, since $t_1+t_2=1/(2\mathrm{d}_3)$. Then with $\beta=8\mathrm{d}_3\delta^{-1}$, we have $\alpha(\mu(a;\delta,1),t_2)=\beta\frac{a}{4}\frac{a^{-1}\delta}{4}-t_2\in 2\mathbb{Z}_2$. When $\mathrm{d}_2=4$, we have

$$t_2\in\begin{cases}\frac{1}{4}\mathrm{d}_3^{-1}+1+2\mathbb{Z}_2 & \text{if } o(t_1)=0,\\ \frac{1}{4}\mathrm{d}_3^{-1}+2\mathbb{Z}_2 & \text{if } o(t_1)\geq 1,\end{cases}$$

since $t_1+t_2=1/(4\mathrm{d}_3)$. Then with $\beta=16\mathrm{d}_3\delta^{-1}$, we have $\alpha(\mu(a;\delta,2),t_2)\in\mathbb{Z}_2^\times$, $\alpha(\mu(a;\delta+4,2),t_2)\in 2\mathbb{Z}_2$ if $o(t_1)=0$, and $\alpha(\mu(a;\delta,2),t_2)\in 2\mathbb{Z}_2$, $\alpha(\mu(a;\delta+4,2),t_2)\in\mathbb{Z}_2^\times$ if $o(t_1)\geq 1$. When $\mathrm{d}_2=8$, we have

$$t_2\in\begin{cases}\frac{1}{8}\mathrm{d}_3^{-1}+\frac{1}{2}\mathrm{d}_3+2\mathbb{Z}_2 & \text{if } o(t_1)=-1,\\ \frac{1}{8}\mathrm{d}_3^{-1}+2\mathbb{Z}_2 & \text{if } o(t_1)\geq 1,\end{cases}$$

since $t_1+t_2=1/(8\mathrm{d}_3)$. Then with $\beta=32\mathrm{d}_3\delta^{-1}$, we have

$$\alpha(\mu(a;\delta\mathrm{d}_3^2,3),t_2)\in\begin{cases}\frac{1}{2}\mathbb{Z}_2^\times & \text{if } o(t_1)=-1,\\ 2\mathbb{Z}_2 & \text{if } o(t_1)\geq 1,\end{cases}$$

$$\alpha(\mu(a;5\delta\mathrm{d}_3^2,3),t_2)\in\begin{cases}2\mathbb{Z}_2 & \text{if } o(t_1)=-1,\\ \frac{1}{2}\mathbb{Z}_2^\times & \text{if } o(t_1)\geq 1,\end{cases}$$

and $\alpha(\mu(a;\delta\mathrm{d}_3^2+8,3),t_2),\alpha(\mu(a;5\delta\mathrm{d}_3^2+8,3),t_2)\notin 2\mathbb{Z}_2$.

Putting these together, we see that when $\mathrm{d}_2=2$, we have

$$\frac{W_t^*(0,\tilde{\phi}_{\mathrm{d},2})}{\gamma(W_2)}=\begin{cases}1 & \text{if } o(t_1)=1,\\ o(t_1)-4 & \text{if } o(t_1)\geq 2.\end{cases}$$

Notice that $v_2(\mathrm{Nm}(t))=o(t_1t_2)=o(t_1)-1$. This proves the lemma for $\mathrm{d}_2=2$. When $\mathrm{d}_2=4$, we have

$$\frac{W_t^*(0,\tilde{\phi}_{\mathrm{d},2})}{\gamma(W_2)}=\begin{cases}\mp 1 & \text{if } t_1\in\pm\mathrm{d}_3+4\mathbb{Z}_2,\\ 1 & \text{if } o(t_1)=2,\\ o(t_1)-5 & \text{if } o(t_1)\geq 3.\end{cases}$$

Notice that $v_2(\mathrm{Nm}(t))=o(t_1t_2)=o(t_1)-2$. If $t_1\in\pm\mathrm{d}_3+4\mathbb{Z}_2$, then $4t_2\in\mathrm{d}_3^{-1}+4\mathbb{Z}_2$ and $\mathrm{Nm}(2t)=4t_1t_2\equiv\pm 1 \bmod 4$. This proves the lemma for $\mathrm{d}_2=4$. Finally when $\mathrm{d}_2=8$, we have

$$\left(\frac{2}{\delta}\right)\frac{W_t^*(0,\tilde{\phi}_{\mathrm{d},2})}{\gamma(W_2)}=\begin{cases}1 & \text{if } t_1\in\frac{1}{2}(-\mathrm{d}_3+8\mathbb{Z}_2),\\ -1 & \text{if } t_1\in\frac{1}{2}(3\mathrm{d}_3+8\mathbb{Z}_2),\\ \mp 1 & \text{if } t_1\in 2(\pm\mathrm{d}_3+4\mathbb{Z}_2),\\ 1 & \text{if } o(t_1)=3,\\ o(t_1)-6 & \text{if } o(t_1)\geq 4.\end{cases}$$

Notice that $v_2(\mathrm{Nm}(t))=o(t_1t_2)=o(t_1)-3$. If $t_1\in\frac{1}{2}(-\mathrm{d}_3+4\mathbb{Z}_2)$, then $8t_2\in\mathrm{d}_3^{-1}+4+8\mathbb{Z}_2$ and $\mathrm{Nm}(4t)=16t_1t_2\equiv 3 \bmod 8$. Similarly, if $t_1\in\frac{1}{2}(3\mathrm{d}_3+4\mathbb{Z}_2)$, then $8t_2\in\mathrm{d}_3^{-1}+4+8\mathbb{Z}_2$ and $\mathrm{Nm}(4t)=16t_1t_2\equiv 7 \bmod 8$. If $t_1\in 2(\pm\mathrm{d}_3+4\mathbb{Z}_2)$, then $8t_2\in\mathrm{d}_3^{-1}+8\mathbb{Z}_2$ and $\mathrm{Nm}(2t)=4t_1t_2\equiv\pm 1 \bmod 4$. This completes the proof. □

6B2. $p = 3$. If $\mathrm{d}_3 = 1$, then $\phi_{\mathrm{d},3} = \mathrm{Char}(\mathcal{O}_E \otimes \mathbb{Z}_3)$ and the calculations have been done before. So suppose $\mathrm{d}_3 = 3$. There are 3 cases to consider.

- $\left(\frac{d_i}{3}\right) = 1$.
- $\left(\frac{d_1}{3}\right) \neq \left(\frac{d_2}{3}\right)$.
- $\left(\frac{d_i}{3}\right) = -1$.

The first case is similar to the case $p = 2$ considered above. We again fix $\delta_i \in \mathbb{Z}_3^\times$ square roots of d_i and denote $\delta := \delta_1\delta_2$. Then the analog of the map in (6-14) for $p = 3$, which we also call σ, identifies $L_{\mathrm{d},3} = M_2(\mathbb{Z}_3)$ with $\widetilde{L}_3 := \mathbb{Z}_3^4$, which has the quadratic form $\widetilde{Q}_{\mathrm{d}}(y) = -\frac{3\mathrm{d}_2}{\delta}(y_1y_2 - y_3y_4)$. Denote $\widetilde{\phi}_{\mathrm{d},3} := \phi_{\mathrm{d},3} \circ \sigma^{-1} \in S(\widetilde{L}_3)$, where $\phi_{\mathrm{d},3}$ is the Schwartz function associated to $\mathfrak{u}_{\mathrm{d},3} \in \mathbb{C}[\mathcal{A}_{\mathrm{d},3}]$. Then the analog of Lemma 6.3 is as follows.

Lemma 6.6. *For* $\delta_i \in \mathbb{Z}_3^\times$ *and* $\delta = \delta_1\delta_2 \equiv \pm 1 \bmod 3$, *we have*

$$\widetilde{\phi}_{\mathrm{d},3} = \phi_0 \otimes \phi_\delta + \phi_{-\delta} \otimes \phi_0 + 2\phi_\delta \otimes \phi_{-\delta},$$

where

$$\phi_0 := 2\phi_{(0,0)} - (\phi_{\frac{1}{3}(0,1)} + \phi_{\frac{1}{3}(1,0)} + \phi_{\frac{1}{3}(0,2)} + \phi_{\frac{1}{3}(0,2)}), \quad \phi_{\pm 1} := \phi_{\frac{1}{3}(1,\pm 1)} + \phi_{\frac{1}{3}(2,\pm 2)}.$$

are in $S(\mathbb{Z}_3^2)$.

Proof. This follows from a straightforward calculations as in the case $p = 2$. ☐

Lemma 6.7. *Suppose* $\left(\frac{d_i}{3}\right) = 1$. *Then we have*

$$\frac{W_t^*(0, \widetilde{\phi}_{\mathrm{d},3})}{\gamma(W_3)} = \begin{cases} 2 & \text{if } v_3(\mathrm{Nm}(t)) = -2, \\ 2v_3(\mathrm{Nm}(t)) & \text{if } v_3(\mathrm{Nm}(t)) \geq -1, \end{cases}$$

for all totally positive $t \in F^\times$ *with* $\mathrm{Tr}(t) = \frac{1}{\mathrm{d}}$.

Proof. Apply Lemma 6.6 and Propositions 5.3, 5.7 in [Yang et al. 2019]. ☐

In the second case, the prime 3 is inert in F and splits into two primes $\mathfrak{P}, \overline{\mathfrak{P}}$ in E. We therefore fix $\delta \in \overline{\mathbb{Q}_3}$ such that $\delta^2 = D$, and denote $F_\delta := \mathbb{Q}_3(\delta)$ the quadratic extension of $\mathbb{Q}_3$ with $\mathcal{O}_\delta \subset F_\delta$ its ring of integers, where 3 is inert. For any choice of $\delta_j \in F_\delta$ such that $\delta_j^2 = d_j$ and $\delta_1\delta_2 = \delta$, we can identify $W \otimes \mathbb{Q}_3$ with $F_\delta \times F_\delta$ via

$$(a_1 + b_1\sqrt{d_1}) \otimes (a_2 + b_2\sqrt{d_2}) \mapsto ((a_1 + b_1\delta_1)(a_2 + b_2\delta_2), (a_1 - b_1\delta_1)(a_2 - b_2\delta_2)).$$

This identifies the $\mathbb{Q}_3$-vector spaces $V \otimes \mathbb{Q}_3$ and $F_\delta \times F_\delta$. The $\mathbb{Z}_3$-lattice $L_{3\mathrm{d}_2} \otimes \mathbb{Z}_3$ and its dual lattice $L'_{3\mathrm{d}_2} \otimes \mathbb{Z}_3$ in $V \otimes \mathbb{Q}_3$ are then mapped to

$$\widetilde{L}_3 := \mathcal{O}_\delta \times \mathcal{O}_\delta \quad \text{and} \quad \widetilde{L}'_3 := 3^{-1}\mathcal{O}_\delta \times 3^{-1}\mathcal{O}_\delta,$$

respectively. The finite $\mathbb{Z}_3$-modules $(L'_{3\mathrm{d}_2}/L_{3\mathrm{d}_2}) \otimes \mathbb{Z}_3$ and $\mathcal{O}_\delta/3\mathcal{O}_\delta \times \mathcal{O}_\delta/3\mathcal{O}_\delta$ are explicitly identified via

$$\begin{aligned} \frac{1}{3\mathrm{d}_2}\begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix} \otimes \mathbb{Z}_3 \mapsto \Big(&\mathrm{d}_2^{-1}(x_1 + x_2 - x_3 - x_4 + (x_4 - x_2)\delta_1 - (x_3 + x_4)\delta_2 + x_4\delta), \\ &\mathrm{d}_2^{-1}(x_1 + x_2 - x_3 - x_4 - (x_4 - x_2)\delta_1 + (x_3 + x_4)\delta_2 + x_4\delta)\Big). \end{aligned} \tag{6-16}$$

The latter can be viewed as the finite quadratic module of the $\mathcal{O}_\delta$-lattice $\mathcal{O}_E \otimes \mathbb{Z}_3 \cong \mathcal{O}_\delta \times \mathcal{O}_\delta$ with the F_δ-quadratic form $Q_{\mathrm{d},\delta}(y) := -\frac{3\mathrm{d}_2}{\delta} y_1 y_2$ for $y = (y_1, y_2) \in F_\delta \times F_\delta$. Note that $\mathcal{O}_\delta / 3\mathcal{O}_\delta = (\mathbb{Z}/3\mathbb{Z})[\delta]$ is a finite field of size 9.

Now let $\widetilde{\phi}_{\mathrm{d},3} \in S(\mathcal{O}_E \otimes \mathbb{Z}_3)$ be the Schwartz function associated to $\phi_{\mathrm{d},3} \in S(L_{\mathrm{d},3})$ under the map in (6-16). It is easy to check by hand the following lemma.

Lemma 6.8. *Let $\delta, \delta_1, \delta_2 \in \overline{\mathbb{Q}_3}$ be as above. Then*

$$\widetilde{\phi}_{\mathrm{d},3} = 2 \sum_{\mu \in S_0} \phi_\mu - \sum_{\mu \in S_1} \phi_\mu - \sum_{\mu \in S_{-1}} \phi_\mu, \tag{6-17}$$

where $S_j := \left\{ \mu \in \left(\tfrac{1}{3}\mathbb{Z}/\mathbb{Z}\right)[\delta] \times \left(\tfrac{1}{3}\mathbb{Z}/\mathbb{Z}\right)[\delta] : Q_{\mathrm{d},\delta}(\mu) = \tfrac{1}{3}(-\mathrm{d}_2 + j\delta) \in \left(\tfrac{1}{3}\mathbb{Z}/\mathbb{Z}\right)[\delta] \right\}$ for $j = 0, \pm 1$.

Remark 6.9. The size of S_j is 8 for every j.

We can now apply Proposition 5.3 in [Yang et al. 2019] to find the value of the Whittaker function.

Lemma 6.10. *Suppose $\left(\frac{d_1}{3}\right) \neq \left(\frac{d_1}{3}\right)$. Then we have*

$$\frac{W_t^*(0, \widetilde{\phi}_{\mathrm{d},3})}{\gamma(W_3)} = \begin{cases} 2 & \textit{if}\, \mathrm{Nm}(3t) \equiv 1 \bmod 3, \\ -1 & \textit{if}\, \mathrm{Nm}(3t) \equiv 2 \bmod 3, \end{cases}$$

for all totally positive $t \in F^\times$ with $\mathrm{Tr}(t) = \frac{1}{\mathrm{d}}$.

Proof. First, $\beta = -3\mathrm{d}_2/\delta$, the normalizing L-factor is $L(1, \chi) = \frac{9}{8}$ and the volume $\mathrm{vol}(\mathcal{O}_E, d_\beta x) = \frac{1}{9}$. Suppose $t = (\delta + a)/(6\mathrm{d}_2\delta) \in F_\delta$. For $\mu \in S_j$, the quantity $3\alpha(\mu, t)$ is

$$3\alpha(\mu, t) := 3(Q_{\mathrm{d},\delta}(\mu) - t) \equiv (-\mathrm{d}_2 + j\delta) - 2(\mathrm{d}_2\delta)^{-1}(\delta + a) \equiv (j - \mathrm{d}_2 a)\delta \bmod 3$$

since $\delta^2 = D \equiv 2 \bmod 3$. Now $3\alpha(\mu, t) \equiv 0 \bmod 3$ if and only if $3 \mid (j - \mathrm{d}_2 a)$. This happens when $3 \mid (j, a)$, in which case $\mu \in S_0$ and $\mathrm{Nm}(3t) \equiv 1 + a^2 \equiv 1 \bmod 3$. The value of $W_t^*(0, \widetilde{\phi}_{\mathrm{d},3})/\gamma(W_3)$ is 2. Otherwise if $3 \nmid a$ and $3 \mid (j - \mathrm{d}_2 a)$, then $\mu \in S_{\mathrm{d}_2 a}$ and $\mathrm{Nm}(3t) \equiv 2 \bmod 3$. The value of $W_t^*(0, \widetilde{\phi}_{\mathrm{d},3})/\gamma(W_3)$ is then -1. This finishes the proof. ☐

In the last case, we need to calculate both the value and derivative of the Whittaker function at $s = 0$ since 3 splits into the product of two inert primes $\mathfrak{p}_1, \mathfrak{p}_2$ in F. As in the setup of the previous two cases, we fix $\delta, \delta_i \in \overline{\mathbb{Q}_3}$ such that $\delta_i^2 = d_i$ and $\delta = \delta_1\delta_2 \in \mathbb{Z}_3$. Denote $\widetilde{E} := \mathbb{Q}_3(\delta_1) = \mathbb{Q}_3(\delta_2)$ the quadratic extension of $\mathbb{Q}_3$ with ring of integers $\widetilde{\mathcal{O}}$. This gives an identification

$$\sigma_i : F \otimes \mathbb{Q}_3 \cong \mathbb{Q}_3 : \sqrt{D} \mapsto (-1)^i \delta, \quad \sigma_i : E_{\mathfrak{p}_i} \cong \widetilde{E} : \sqrt{d_j} \mapsto (-1)^{(i-1)(j-1)} \delta_j.$$

Then the isomorphism in (5-1) induces $V \otimes \mathbb{Q}_3 \cong W \otimes \mathbb{Q}_3 = E_{\mathfrak{p}_1} \times E_{\mathfrak{p}_2} \cong \widetilde{E} \times \widetilde{E}$, with the quadratic form on $y \in E_{\mathfrak{p}_i}$ given by $Q_i(y) := (-1)^{i-1}(3\mathrm{d}_2)/(\sqrt{D})\mathrm{Nm}(y)$. The lattice $L_{\mathrm{d},3}$ is then isometric to

$$\widetilde{L}_{\mathrm{d},3} := \widetilde{\mathcal{O}} \times \widetilde{\mathcal{O}} \subset \widetilde{E} \times \widetilde{E},$$

whose dual lattice is $\widetilde{L}'_{\mathrm{d},3} := \frac{1}{3}\widetilde{\mathcal{O}} \times \frac{1}{3}\widetilde{\mathcal{O}} \subset \widetilde{E} \times \widetilde{E}$, with respect to the quadratic form $\widetilde{Q}_{\mathrm{d},\delta}(y) := -(3\mathrm{d}_2/\delta)(\mathrm{Nm}(y_1) - \mathrm{Nm}(y_2))$ for $y = (y_1, y_2) \in \widetilde{E} \times \widetilde{E}$. Under this identification, the Schwartz function $\widetilde{\phi}_{\mathrm{d},3} \in S(\widetilde{L}_{\mathrm{d},3})$ associated to $\phi_{\mathrm{d},3} \in S(L_{\mathrm{d},3})$ has the following decomposition.

Lemma 6.11. *Let $\delta, \delta_1, \delta_2 \in \overline{\mathbb{Q}_3}$ be as above. Then*

$$\widetilde{\phi}_{\mathrm{d},3} = 2\sum_{\mu\in S_1}\phi_\mu\otimes\phi_0 + 2\sum_{\mu\in S_{-1}}\phi_0\otimes\phi_\mu - \sum_{\mu_1\in S_{-1},\mu_2\in S_1}\phi_{\mu_1}\otimes\phi_{\mu_2} \tag{6-18}$$

where $S_j := \left\{\mu\in\frac{1}{3}\widetilde{\mathcal{O}}/\widetilde{\mathcal{O}} : -\frac{3}{\delta}\mathrm{Nm}(\mu)\equiv\frac{j}{3} \bmod \mathbb{Z}_3\right\}$ *for* $j=\pm 1$.

Now, we can again apply Proposition 5.3 in [Yang et al. 2019] to calculate the values and derivatives of the Whittaker function.

Lemma 6.12. *Suppose* $\left(\frac{d_1}{3}\right)=\left(\frac{d_1}{3}\right)=-1$. *Then we have*

$$\frac{W_t^*(0,\phi_{\mathrm{d},3})}{\gamma(W_3)} = \begin{cases} -1 & \textit{if } v_3(\mathrm{Nm}(t))=-2,\\ 2 & \textit{if } v_3(\mathrm{Nm}(t))\geq 0 \textit{ is even},\\ 0 & \textit{otherwise},\end{cases}$$

$$\frac{W^{*,\prime}{}_t(0,\phi_{\mathrm{d},3})}{\gamma(W_3)} = \left(v_3(\mathrm{Nm}(t))+\frac{3}{2}\right)\log 3 \quad \textit{if } v_3(\mathrm{Nm}(t))\geq -1 \textit{ is odd},$$

for all totally positive $t\in F^\times$ *with* $\mathrm{Tr}(t)=\frac{1}{\mathrm{d}}$.

Proof. Denote $t_i := \sigma_i(t)\in\mathbb{Q}_3$ and $o(t_i)$ its valuation. Since $\mathrm{Tr}(t)=1/(3\mathrm{d}_2)$, either $o(t_i)=-1$ for both $i=1,2$, or $o(t_i)\geq 0$ for exactly one of $i=1,2$. In the first case, it is easy to check that $W_{t_i}(s,\phi_\mu\otimes\phi_0)$ and $W_{t_i}(s,\phi_0\otimes\phi_\mu)$ are identically zero by Proposition 5.7 in [Yang et al. 2019]. If we write $t_1=(\delta-a)/(2\mathrm{d}_2 3\delta)$, $t_2=(\delta+a)/(2\mathrm{d}_2 3\delta)$ with $a\in\mathbb{Z}_3$, then we must have $a\in 3\mathbb{Z}_3$ since $\delta^2=D\in 1+3\mathbb{Z}_3$ and

$$-2=o(t_1)+o(t_2)=o(t_1t_2)=-2+o(\delta^2-a^2)=-2+o(1-a^2).$$

That means for $\mu_1\in S_{-1}$ and $\mu_2\in S_1$, we have

$$\alpha(\mu_1,t_1)=-\frac{3\mathrm{d}_2}{\delta}\mathrm{Nm}(\mu_1)-t_1\equiv-\frac{\mathrm{d}_2}{3}-\frac{\delta-a}{2\mathrm{d}_2 3\delta}\equiv 0 \bmod \mathbb{Z}_3,$$

$$\alpha(\mu_2,t_2)=\frac{3\mathrm{d}_2}{\delta}\mathrm{Nm}(\mu_2)-t_2\equiv-\frac{\mathrm{d}_2}{3}-\frac{\delta+a}{2\mathrm{d}_2 3\delta}\equiv 0 \bmod \mathbb{Z}_3.$$

By Proposition 5.3 in [Yang et al. 2019], $\gamma(W_3)^{-1}W_{t_i}(0,\phi_{\mu_1}\otimes\phi_{\mu_2})=\frac{1}{16}$ for any $(\mu_1,\mu_2)\in S_{-1}\times S_1$. Since S_j has size 4 for $j=\pm 1$, we obtain

$$\frac{W_t(0,\phi_{\mathrm{d},3})}{\gamma(W_3)}=-1$$

when $v_3(\mathrm{Nm}(t))=o(t_1)+o(t_2)=-2$.

In the second case, suppose $o(t_1)\geq 0$. Then Propositions 5.3 and 5.7 in [Yang et al. 2019] imply that $W_{t_i}(s,\phi_{\mu_1}\otimes\phi_{\mu_2})$ vanishes identically for $(\mu_1,\mu_2)\in S_{-1}\times S_1$ and

$$\frac{W_{t_1}^*(0,\phi_0)}{\gamma(W_{\mathfrak{p}_1})}=\frac{1+(-1)^{o(t_1)-1}}{2}, \qquad \frac{W_{t_2}^*(0,\phi_\mu)}{\gamma(W_{\mathfrak{p}_2})}=L(1,\chi_{\mathfrak{p}_2})3^{-1}=\frac{1}{4},$$

$$\frac{W^{*,\prime}_{t_1}(0,\phi_0)}{\gamma(W_{\mathfrak{p}_1})} = \frac{2o(t_1)+1}{2}\log(3) \qquad \text{when } 2 \mid o(t_1)$$

when $\mu \in S_1$ as $\alpha(\mu, t_2) = -(\mathrm{d}_2 \mathrm{Nm}(\mu))/(3\delta) - t_2 \in \frac{\mathrm{d}_2}{3} - t_2 + \mathbb{Z}_3 = \mathbb{Z}_3$. Since $v_3(\mathrm{Nm}(t)) = o(t_1 t_2) = o(t_1) - 1$, we obtain the lemma when $o(t_1) \geq 0$. The case $o(t_2) \geq 0$ holds similarly. $\square$

Appendix

We record here the set $\kappa_{\mathrm{d},2}(N'_{\mathrm{d},2}) \subset \mathcal{A}_{\mathrm{d},2} = \mathbb{Z}/\mathrm{d}_2\mathbb{Z} \times \mathbb{Z}/2\mathrm{d}_2\mathbb{Z} \times \mathbb{Z}/2\mathrm{d}_2\mathbb{Z} \times \mathbb{Z}/\mathrm{d}_2\mathbb{Z}$. Note that the group $N'_{\mathrm{d},2}$ and the map $\mathrm{d}_3 \cdot \kappa_{\mathrm{d},2}$ only depend on d_2. This helps with checking Lemma 3.11.

$$\kappa_{1,2}(N'_{1,2}) = \kappa_{3,2}(N'_{3,2}) = \{[0,0]\}, \quad \kappa_{2,2}(N'_{2,2}) = \kappa_{6,2}(N'_{6,2}) = \{[1,0,0,1],[1,2,2,1]\},$$

$$\begin{aligned}&\kappa_{4,2}(N'_{4,2}) = \kappa_{12,2}(N'_{12,2}) = \\ &\quad \big\{[1,2,6,3],[1,6,2,3],[1,0,0,1],[1,4,4,1],[3,6,2,1],[3,2,6,1],[3,0,0,3],[3,4,4,3]\big\}.\end{aligned}$$

$$\begin{aligned}&\kappa_{8,2}(N'_{8,2}) = \\ &\Big\{[1,2,14,7],[1,6,10,7],[1,10,6,7],[1,14,2,7],[1,0,0,1],[1,4,12,1],[1,8,8,1],[1,12,4,1],\\ &[3,14,10,5],[3,2,6,5],[3,6,2,5],[3,10,14,5],[3,8,0,3],[3,12,12,3],[3,0,8,3],[3,4,4,3],\\ &[5,2,6,3],[5,6,2,3],[5,10,14,3],[5,14,10,3],[5,8,0,5],[5,12,12,5],[5,0,8,5],[5,4,4,5],\\ &[7,14,2,1],[7,2,14,1],[7,6,10,1],[7,10,6,1],[7,0,0,7],[7,4,12,7],[7,8,8,7],[7,12,4,7]\Big\}.\end{aligned}$$

$$\begin{aligned}&\kappa_{24,2}(N'_{24,2}) = \kappa_{24,2}(N'_{8,2}) = 3^{-1}\cdot\kappa_{8,2}(N'_{8,2}) = \\ &\Big\{[3,6,10,5],[3,2,14,5],[3,14,2,5],[3,10,6,5],[3,0,0,3],[3,12,4,3],[3,8,8,3],[3,4,12,3],\\ &[1,10,14,7],[1,6,2,7],[1,2,6,7],[1,14,10,7],[1,8,0,1],[1,4,4,1],[1,0,8,1],[1,12,12,1],\\ &[7,6,2,1],[7,2,6,1],[7,14,10,1],[7,10,14,1],[7,8,0,7],[7,4,4,7],[7,0,8,7],[7,12,12,7],\\ &[5,10,6,3],[5,6,10,3],[5,2,14,3],[5,14,2,3],[5,0,0,5],[5,12,4,5],[5,8,8,5],[5,4,12,5]\Big\}.\end{aligned}$$

Here we also include an explicit example for Theorem 1.7. Let $d_1 = -31$, $d_2 = -127$, which have class numbers 3 and 5 respectively and satisfy $d_j \equiv 17 \bmod 24$. Then the minimal polynomials of the invariants $f\big(\big[1, \frac{1}{2}(1+\sqrt{d_j})\big]\big)$ are

$$g_1(x) = x^3 + x - 1, \quad g_2(x) = x^5 - x^4 - 2x^3 + x^2 + 3x - 1. \tag{6-19}$$

Table 7 lists the values of $\mathfrak{F}(m)$ for various m. By the Gross–Zagier theorem, one obtains $J(d_1, d_2)$ by simply takes the product of all the numbers in the fourth column. For $f_s(d_1, d_2)$, one takes product of the entries $\mathfrak{F}\big(\frac{m}{4r^2}\big)$ over all the m's in the table and $r \mid s$ satisfying $m \equiv 4 \cdot 19(d_1 + d_2 - 1) \bmod 4sr$. This congruence condition eliminates many entries, especially if s is large. For example, we have

$$f_{24}(d_1, d_2) = \Big(\mathfrak{F}\Big(\frac{2^2 3^5}{4}\Big)\mathfrak{F}\Big(\frac{2^4 3^3}{4\cdot 2^2}\Big)\mathfrak{F}\Big(\frac{2^2 3^5}{4\cdot 3^2}\Big)\mathfrak{F}\Big(\frac{2^4 3^3}{4\cdot 6^2}\Big)\Big)^{1/2} = 3^4$$

by Theorem 1.7. One can then immediately check that this is the absolute value of the resultant of the minimal polynomials g_1, g_2 in (6-19).

a	m	$m \bmod 96$	$\mathfrak{F}(m)$	$\mathfrak{F}\left(\frac{m}{2^2}\right)$	$\mathfrak{F}\left(\frac{m}{4^2}\right)$	$\mathfrak{F}\left(\frac{m}{8^2}\right)$	$\mathfrak{F}\left(\frac{m}{16^2}\right)$	$\mathfrak{F}\left(\frac{m}{6^2}\right)$	$\mathfrak{F}\left(\frac{m}{12^2}\right)$	$\mathfrak{F}\left(\frac{m}{24^2}\right)$	$\mathfrak{F}\left(\frac{m}{48^2}\right)$
1	$2^3 \cdot 3 \cdot 41$	24	3^8	3^4	1	1	1	1	1	1	1
3	$2 \cdot 491$	22	491^2	1	1	1	1	1	1	1	1
5	$2 \cdot 3 \cdot 163$	18	3^4	1	1	1	1	1	1	1	1
7	$2^2 \cdot 3^5$	12	3^9	3^3	1	1	1	3^2	1	1	1
9	$2^2 \cdot 241$	4	241^3	241	1	1	1	1	1	1	1
11	$2 \cdot 3^2 \cdot 53$	90	53^2	1	1	1	1	1	1	1	1
13	$2 \cdot 3 \cdot 157$	78	3^4	1	1	1	1	1	1	1	1
15	$2^5 \cdot 29$	64	29^6	29^4	29^2	1	1	1	1	1	1
17	$2^4 \cdot 3 \cdot 19$	48	3^{10}	3^6	3^2	1	1	1	1	1	1
19	$2 \cdot 3 \cdot 149$	30	3^4	1	1	1	1	1	1	1	1
21	$2 \cdot 19 \cdot 23$	10	23^4	1	1	1	1	1	1	1	1
23	$2^2 \cdot 3 \cdot 71$	84	3^6	3^2	1	1	1	1	1	1	1
25	$2^2 \cdot 3^2 \cdot 23$	60	23^3	23	1	1	1	23	1	1	1
27	$2 \cdot 401$	34	401^2	1	1	1	1	1	1	1	1
29	$2 \cdot 3^2 \cdot 43$	6	43^2	1	1	1	1	1	1	1	1
31	$2^3 \cdot 3 \cdot 31$	72	3^8	3^4	1	1	1	1	1	1	1
33	$2^3 \cdot 89$	40	89^4	89^2	1	1	1	1	1	1	1
35	$2 \cdot 3 \cdot 113$	6	3^4	1	1	1	1	1	1	1	1
37	$2 \cdot 3 \cdot 107$	66	3^4	1	1	1	1	1	1	1	1
39	$2^2 \cdot 151$	28	151^3	151	1	1	1	1	1	1	1
41	$2^2 \cdot 3 \cdot 47$	84	3^6	3^2	1	1	1	1	1	1	1
43	$2 \cdot 3^2 \cdot 29$	42	29^2	1	1	1	1	1	1	1	1
45	$2 \cdot 239$	94	239^2	1	1	1	1	1	1	1	1
47	$2^4 \cdot 3^3$	48	3^{10}	3^6	3^2	1	1	3^3	3	1	1
49	$2^7 \cdot 3$	0	3^8	3^6	3^4	3^2	1	1	1	1	1
51	$2 \cdot 167$	46	167^2	1	1	1	1	1	1	1	1
53	$2 \cdot 3 \cdot 47$	90	3^4	1	1	1	1	1	1	1	1
55	$2^2 \cdot 3 \cdot 19$	36	3^6	3^2	1	1	1	1	1	1	1
57	$2^2 \cdot 43$	76	43^3	43	1	1	1	1	1	1	1
59	$2 \cdot 3 \cdot 19$	18	3^4	1	1	1	1	1	1	1	1
61	$2 \cdot 3^3$	54	3^4	1	1	1	1	1	1	1	1

Table 7. Values of $\mathfrak{F}$ for $(d_1, d_2) = (-31, -127)$.

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