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The Prasad conjectures for GSp_4 and PGSp_4

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We use the theta correspondence between $\mathrm{GSp}_4(E)$ and $\mathrm{GO}(V)$ to study the GSp_4-distinction problems over a quadratic extension E/F of nonarchimedean local fields of characteristic 0. With a similar strategy, we investigate the distinction problem for the pair $(\mathrm{GSp}_4(E), \mathrm{GSp}_{1,1}(F))$, where $\mathrm{GSp}_{1,1}$ is the unique inner form of GSp_4 defined over F. Then we verify the Prasad conjecture for a discrete series representation $\bar{\tau}$ of $\mathrm{PGSp}_4(E)$.

1. Introduction

Let F be a finite field extension over $\mathbb{Q}_p$ and E be a quadratic extension over F with associated Galois group $\mathrm{Gal}(E/F) = \{1, \sigma\}$ and associated quadratic character $\omega_{E/F}$ of $F^\times$. Let W_F be the Weil group of F and WD_F be the Weil–Deligne group. Then $\omega_{E/F}$ is a quadratic character of W_F with kernel W_E. Let $\boldsymbol{G}$ be a connected reductive group defined over F and $\boldsymbol{G}(F)$ (resp. $\boldsymbol{G}(E)$) be the F-rational (resp. E-rational) points. Let $\mathrm{Irr}(\boldsymbol{G}(E))$ denote the set of irreducible smooth representations of $\boldsymbol{G}(E)$. Given a representation $\tau \in \mathrm{Irr}(\boldsymbol{G}(E))$ and a character χ of $\boldsymbol{G}(F)$, we say that τ is $(\boldsymbol{G}(F), \chi)$-distinguished or has a nonzero $(\boldsymbol{G}(F), \chi)$-period if

$$\mathrm{Hom}_{\boldsymbol{G}(F)}(\tau, \chi) \neq 0.$$

If χ is the trivial character, then τ is called $\boldsymbol{G}(F)$-distinguished. There exists a rich literature, such as [Beuzart-Plessis 2018; Flicker 1991; Gan and Raghuram 2013; Lu 2017b; Matringe 2011; Prasad 2015], trying to classify all $\boldsymbol{G}(F)$-distinguished representations of $\boldsymbol{G}(E)$. The method often used to study the distinction problems is the relative trace formula, such as in [Beuzart-Plessis 2018; Flicker and Hakim 1994], which is powerful especially for the global period problems. This paper focuses on the local period

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problems for $\boldsymbol{G} = \mathrm{GSp}_4$, PGSp_4 and their inner forms. The main tool in this paper is the local theta correspondence appearing in [Gan and Takeda 2011b; Kudla and Rallis 1992; Yamana 2011].

Let V be the unique nonsplit quaternion algebra D_E with quadratic form N_{D_E} over E, or the split 6-dimensional quadratic space $\mathbb{H}_E^3$ over E. Then

$$\mathrm{GSO}(V) \cong \begin{cases} \mathrm{GSO}_{4,0}(E) = D_E^\times(E) \times D_E^\times(E)/\{(t, t^{-1}) : t \in E^\times\} & \text{if } V = D_E, \\ \mathrm{GSO}_{3,3}(E) = \mathrm{GL}_4(E) \times E^\times/\{(t^{-1}, t^2) : t \in E^\times\} & \text{if } V = \mathbb{H}_E^3, \end{cases}$$

and any irreducible representation of $\mathrm{GSO}(V)$ must be of the form

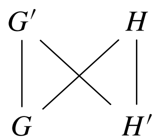
- $\pi_1 \boxtimes \pi_2$ with $\omega_{\pi_1} = \omega_{\pi_2}$ if $V = D_E$;
- $\Pi \boxtimes \mu$ with $\omega_\Pi = \mu^2$ if $V = \mathbb{H}_E^3$.

Here for each i, π_i is an irreducible representation of $D_E^\times(E)$.

Gan and Takeda [2011b] have studied the explicit theta correspondence between $\mathrm{GSO}(V)$ and $\mathrm{GSp}_4(E)$ and proved that any irreducible representation τ of $\mathrm{GSp}_4(E)$ falls into one of the following two disjoint families of representations:

- $\tau = \theta(\pi_1 \boxtimes \pi_2)$ with $\omega_{\pi_1} = \omega_{\pi_2}$;
- $\tau = \theta(\Pi \boxtimes \mu)$ with $\mu = \omega_\tau$ and $\omega_\Pi = \mu^2$.

The see-saw identity (sometimes called the local Siegel–Weil identity) plays a vital role in the proof of our main theorems. More precisely, suppose that $G \times H$ is a reductive dual pair, with a Weil representation ω_ψ over F. Let $H' \times G'$ be another dual pair contained in the same ambient group, with $G \subset G'$ and $H' \subset H$. Via a so-called see-saw diagram



we have

$$\dim \mathrm{Hom}_G(\Theta_\psi(\chi), \pi) = \dim \mathrm{Hom}_{G \times H'}(\omega_\psi, \pi \boxtimes \chi) = \dim \mathrm{Hom}_{H'}(\Theta_\psi(\pi), \chi)$$

for a representation $\pi \in \mathrm{Irr}(G)$ and a character χ of H'. Typically, $\Theta_\psi(\chi)$ is a simpler representation, such as a degenerate principal series representation of G', and the multiplicity $\dim \mathrm{Hom}_G(\Theta_\psi(\chi), \pi)$ has a better chance of being understood; see [Gan 2019]. In order to use the see-saw identity, we need to study the big theta lift $\Theta(\tau)$ to $\mathrm{GO}(V)$ of a generic representation τ of $\mathrm{GSp}_4(E)$. In fact, we have studied the general (almost equal rank) case for the irreducibility of big theta lifts to $\mathrm{GO}_{n+1,n+1}(F)$ of a generic representation of $\mathrm{GSp}_{2n}(F)$ in Section 3C. After computing the big theta lifts following [Gan and Ichino 2014; Gan and Takeda 2011b], we use the local theta correspondences between $\mathrm{GSp}_4(E)$ and $\mathrm{GSO}(V)$ and the see-saw identities to discuss GSp_4-period problems, by transferring the period problem for GSp_4 to various analogous period problems for GL_2, GL_4 and their various forms (not necessarily inner). Then we obtain the following results:

Theorem 1.1 (Theorem 4.4.9). *Suppose that $\tau \in \mathrm{Irr}(\mathrm{GSp}_4(E))$ with a central character ω_τ and $\omega_\tau|_{F^\times} = \mathbf{1}$.*

(i) *If $\tau = \theta(\Sigma)$ is an irreducible representation of $\mathrm{GSp}_4(E)$, where Σ is an irreducible representation of $\mathrm{GO}_{4,0}(E)$, then the representation τ is not $\mathrm{GSp}_4(F)$-distinguished.*

(ii) *If $\tau = \theta(\pi_1 \boxtimes \pi_2)$, where $\pi_1 \boxtimes \pi_2$ is a generic representation of $\mathrm{GSO}_{2,2}(E)$, then*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = \begin{cases} 2 & \text{if } \pi_i \ncong \pi_0 \text{ are both } \mathrm{GL}_2(F)\text{-distinguished}, \\ 1 & \text{if } \pi_1 \ncong \pi_2 \text{ but } \pi_1^\sigma \cong \pi_2^\vee, \\ 1 & \text{if } \pi_1 \cong \pi_2 \text{ is } \mathrm{GL}_2(F)\text{-distinguished but not } (\mathrm{GL}_2(F), \omega_{E/F})\text{-distinguished}, \\ 1 & \text{if } \pi_2 \text{ is } \mathrm{GL}_2(F)\text{-distinguished and } \pi_1 \cong \pi_0, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\pi_0 = \pi(\chi_1, \chi_2)$ with $\chi_1 \neq \chi_2$, $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$ is a principal series representation of $\mathrm{GL}_2(F)$. Note that these conditions are mutually exclusive.

(iii) *Assume that τ is not in case* (i) *or* (ii) *and that $\tau = \theta(\Pi \boxtimes \chi)$ is generic, where $\Pi \boxtimes \chi$ is a representation of $\mathrm{GSO}_{3,3}(E)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = \begin{cases} 1 & \text{if } \Pi \text{ is } \mathrm{GL}_4(F)\text{-distinguished}, \\ 0 & \text{otherwise}. \end{cases}$$

The full local Langlands conjecture for GSp_4 (see Theorem 4.4.7) has been proved by Gan and Takeda [2011a]. Then we can verify the Prasad conjecture for GSp_4 in Section 6C. More precisely, let G_0 be a quasisplit group defined over F (denoted by G^{op} in [Prasad 2015]) such that

$$^L G_0 = \mathrm{GSp}_4(\mathbb{C}) \rtimes \mathrm{Gal}(E/F),$$

where the nontrivial element $\sigma \in \mathrm{Gal}(E/F)$ acts on $\mathrm{GSp}_4(\mathbb{C})$ by

$$\sigma(g) = \mathrm{sim}(g)^{-1} \cdot g.$$

Here sim: $\mathrm{GSp}_4(\mathbb{C}) \to \mathbb{C}^\times$ is the similitude character. Let ϕ_τ be the Langlands parameter of τ. Define

$$F(\phi_\tau) = \{\tilde{\phi} : WD_F \to {}^L G_0 \mid \tilde{\phi}|_{WD_E} = \phi_\tau\}. \tag{1-1}$$

Theorem 1.2 (the Prasad conjecture for GSp_4). *Let τ be an irreducible smooth representation of $\mathrm{GSp}_4(E)$ with enhanced Langlands parameter $(\phi_\tau, \lambda_\tau)$ (called the Langlands-Vogan parameter). Assume that the L-packet Π_{ϕ_τ} is generic. Then*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \omega_{E/F}) = \begin{cases} |F(\phi_\tau)| & \text{if } \tau \text{ is generic, i.e., } \lambda_\tau \text{ is trivial}, \\ 0 & \text{otherwise}, \end{cases}$$

where $F(\phi_\tau)$ is defined in (1-1) and $|F(\phi_\tau)|$ denotes the cardinality of the set $F(\phi_\tau)$.

We will prove analogous results for the inner form in Section 5. Let D be the 4-dimensional quaternion division algebra of F. In a similar way, we study the period problem for the inner form $\mathrm{GU}_2(D)=\mathrm{GSp}_{1,1}$, i.e., try to figure out the multiplicity

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau,\mathbb{C})$$

for a representation $\tau\in\mathrm{Irr}(\mathrm{GSp}_4(E))$. We will not state the results of the inner form case in the introduction; the precise results can be found in Theorem 5.3.1.

Combining Theorem 1.1 and its analog for inner forms, we can verify the conjecture of Dipendra Prasad [2015, Conjecture 2] for PGSp_4. Given a quasisplit reductive group $\boldsymbol{G}$ defined over F and a quadratic extension E/F, assuming the Langlands–Vogan conjectures for $\boldsymbol{G}$, Prasad [2015] used the recipes from the Galois side to give a formula for the individual multiplicity

$$\dim \mathrm{Hom}_{G_\alpha(F)}(\tau,\chi_{\boldsymbol{G}}),$$

where

- τ is an irreducible discrete series representation of $\boldsymbol{G}(E)$;
- $\chi_{\boldsymbol{G}}$ is a quadratic character of $\boldsymbol{G}(F)$ depending on $\boldsymbol{G}$ and E;
- G_α is any pure inner form of $\boldsymbol{G}$ defined over F satisfying $G_\alpha(E)=\boldsymbol{G}(E)$.

In Section 7, we will focus on the case $\boldsymbol{G}=\mathrm{PGSp}_4$. Then $H^1(F,\boldsymbol{G})=\{\mathrm{PGSp}_4,\mathrm{PGU}_2(D)\}$ and $\chi_{\boldsymbol{G}}=\omega_{E/F}$. The local Langlands correspondences for the quasisplit groups SO_n and Sp_{2n} over a nonarchimedean local field have been verified by Arthur [2013] under certain assumptions which have been removed by Mœglin and Waldspurger [2016a; 2016b; 2018]. We can use the results from the local Langlands correspondence for $\mathrm{SO}_5=\mathrm{PGSp}_4$ freely. More precisely, if $\tau\in\mathrm{Irr}(\mathrm{GSp}_4(E))$ with a trivial central character, then τ corresponds to a representation of $\mathrm{PGSp}_4(E)$, denoted by $\bar\tau$. Given a discrete series representation $\bar\tau$ of $\mathrm{PGSp}_4(E)$ with the enhanced L-parameter $(\phi_{\bar\tau},\lambda_{\bar\tau})$, where $\lambda_{\bar\tau}$ is a character of the component group $\pi_0(Z(\phi_{\bar\tau}))$, set

$$F(\phi_{\bar\tau})=\{\tilde\phi: WD_F\to\mathrm{Sp}_4(\mathbb{C})\mid \tilde\phi|_{WD_E}=\phi_{\bar\tau}\}.$$

Up to the twisting by the quadratic character $\omega_{E/F}$, there are several orbits in $F(\phi_{\bar\tau})$, denoted by $\sqcup_{i=1}^r\mathcal{O}(\tilde\phi_i)$. Each orbit $\mathcal{O}(\tilde\phi_i)$ corresponds to a unique subset $\mathcal{C}_i$ of $H^1(W_F,\boldsymbol{G})$. (See Section 6A for more details.)

Theorem 1.3. *Let notation be as above. Given a discrete series representation $\bar\tau$ of $\mathrm{PGSp}_4(E)$, we have*

$$\dim \mathrm{Hom}_{G_\alpha(F)}(\bar\tau,\omega_{E/F})=\sum_{i=1}^r m(\lambda_{\bar\tau},\tilde\phi_i)\mathbf{1}_{\mathcal{C}_i}(G_\alpha)/d_0(\tilde\phi_i), \tag{1-2}$$

where

- $\mathbf{1}_{\mathcal{C}_i}$ *is the characteristic function of the set* $\mathcal{C}_i$;
- $m(\lambda_{\bar\tau},\tilde\phi)$ *is the multiplicity for the trivial representation contained in the restricted representation* $\lambda_{\bar\tau}|_{\pi_0(Z(\tilde\phi))}$;
- $d_0(\tilde\phi)=|\mathrm{Coker}\{\pi_0(Z(\tilde\phi))\to\pi_0(Z(\phi_{\bar\tau}))^{\mathrm{Gal}(E/F)}\}|$, *where* $|-|$ *denotes its cardinality.*

Remark 1.4. We would like to highlight the fact that the square-integrable representation $\bar{\tau}$ may be nongeneric and so $\bar{\tau}$ is not $\mathrm{PGSp}_4(F)$-distinguished (see Theorem 5.3.1) but $\bar{\tau}$ contains a nonzero period for the pure inner form $\mathrm{PGSp}_{1,1}(F)$. It is different from the case $\boldsymbol{G} = \mathrm{PGL}_2$ that if a representation $\bar{\pi}$ of $\mathrm{PGL}_2(E)$ is $\mathrm{PD}^\times(F)$-distinguished, then $\bar{\pi}$ must be $\mathrm{PGL}_2(F)$-distinguished (see Lemma 4.4.5).

In fact, we have shown that the equality (1-2) holds for almost all generic representations in Section 7, except that the Langlands parameter $\phi_{\bar{\tau}} = 2\chi_F|_{W_E} \oplus \phi_2$ with ϕ_2 conjugate-symplectic (in the sense of [Gan et al. 2012, §3]) and $\chi_F^2 = \omega_{E/F}$. However, there is a weaker version of the Prasad conjecture which determines the sum of $\dim \mathrm{Hom}_{G_\alpha(F)}(\bar{\tau}, \chi_{\boldsymbol{G}})$ as G_α runs over all pure inner forms of $\boldsymbol{G}$ satisfying $G_\alpha(E) = \boldsymbol{G}(E)$. It involves the degree of the base change map

$$\Phi : \mathrm{Hom}(WD_F, \mathrm{Sp}_4(\mathbb{C})) \to \mathrm{Hom}(WD_E, \mathrm{Sp}_4(\mathbb{C}))$$

for the exception case, i.e., the identity

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}) + \dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = \sum_{\tilde{\phi} \in F(\phi_{\bar{\tau}})} m(\lambda_{\bar{\tau}}, \tilde{\phi}) \frac{\deg \Phi(\tilde{\phi})}{d_0(\tilde{\phi})} \tag{1-3}$$

when the L-packet $\Pi_{\phi_{\bar{\tau}}}$ is generic, which is the original identity formulated by Prasad.

There is a brief introduction to the proof of Theorem 1.3. After introducing the local theta correspondence between quaternionic unitary groups following [Yamana 2011], we use the isomorphism $\mathrm{GU}_2(R) = \mathrm{GSp}_{1,1}(E) \cong \mathrm{GSp}_4(E)$, where $R \cong \mathrm{Mat}_{2,2}(E)$ is the split quaternion algebra over E, to embed the group $\mathrm{GSp}_{1,1}(F)$ into $\mathrm{GSp}_4(E)$. Then one can use the see-saw identity to transfer the inner form $\mathrm{GSp}_{1,1}$-period problem to $\mathrm{GO}^*_{3,0}$ or $\mathrm{GO}^*_{1,1}$ side, which are closely related to GL_n-period problems. But we need to be very careful when we use the see-saw identity for a pair of quaternionic unitary groups. (See Remark 5.2.4.) Once the see-saw identity for the quaternionic unitary groups has been set up, the rest of the proof for the inner form case is similar to the case for GSp_4-period. Then we obtain the results for the distinction problems for the automorphic side. For the Galois side, i.e., the right-hand side of (1-3), it will be checked case by case in Section 7.

Remark 1.5. Raphaël Beuzart-Plessis [2018, Theorem 1] used the local trace formula to deal with the distinction problems for the Galois pair $(G'(E), G'(F))$ for the stable square-integrable representations, where G' is an inner form of $\boldsymbol{G}$ defined over F, which generalizes [Prasad 1992, Theorem C].

The paper is organized as follows. In Section 2, we set up the notation about the local theta correspondence. In Section 3, we will study the irreducibility for the big theta lift of a generic representation in the almost equal rank case, which generalizes the results of [Gan and Ichino 2014, Proposition C.4] for the tempered representations. The detailed computation for the explicit big theta lift $\Theta(\tau)$ to $\mathrm{GO}(V)$ will be given in Section 3E. In Section 4, we will study the distinction problems for GSp_4 over a quadratic extension E/F. The proof of Theorem 1.1 will be given in Section 4D. The analogous results for the inner form $\mathrm{GSp}_{1,1}$ will be given in Section 5. In Section 6A, we will introduce the Prasad conjecture for a reductive quasisplit group $\boldsymbol{G}$ defined over F. Then we will verify the Prasad conjecture for GSp_4 in Section 6C. Finally, the proof of Theorem 1.3 will be given in Section 7.

2. The local theta correspondences for similitudes

In this section, we will briefly recall some results about the local theta correspondence, following [Gan and Takeda 2011b; Kudla 1996; Roberts 2001].

Let F be a nonarchimedean local field of characteristic zero. Consider the dual pair $\mathrm{O}(V)\times\mathrm{Sp}(W)$. For simplicity, we may assume that $\dim V$ is even. Fix a nontrivial additive character ψ of F. Let ω_ψ be the Weil representation for $\mathrm{O}(V)\times\mathrm{Sp}(W)$. If π is an irreducible smooth representation of $\mathrm{O}(V)$ (resp. $\mathrm{Sp}(W)$), the maximal π-isotypic quotient of ω_ψ has the form

$$\pi\boxtimes\Theta_\psi(\pi)$$

for some smooth representation $\Theta_\psi(\pi)$ of $\mathrm{Sp}(W)$ (resp. some smooth representation $\Theta_\psi(\pi)$ of $\mathrm{O}(V)$). We call $\Theta_\psi(\pi)$ or $\Theta_{V,W,\psi}(\pi)$ the big theta lift of π. It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ or $\theta_{V,W,\psi}(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$, which is called the small theta lift of π.

Theorem 2.1 (Howe duality conjecture [Gan and Takeda 2016a; 2016b]).

- *$\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is nonzero.*
- *The map $\pi\mapsto\theta_\psi(\pi)$ is injective on its domain.*

This has been proved by Waldspurger [1990] when $p\neq 2$.

We extend the Weil representation to the case of similitude groups. Let λ_V and λ_W be the similitude factors of $\mathrm{GO}(V)$ and $\mathrm{GSp}(W)$ respectively. We shall consider the group

$$R=\mathrm{GO}(V)\times\mathrm{GSp}^+(W),$$

where $\mathrm{GSp}^+(W)$ is the subgroup of $\mathrm{GSp}(W)$ consisting of elements g such that $\lambda_W(g)$ lies in the image of λ_V. Define

$$R_0=\{(h,g)\in R\mid \lambda_V(h)\lambda_W(g)=1\}$$

to be the subgroup of R. The Weil representation ω_ψ extends naturally to the group R_0 via

$$\omega_\psi(g,h)\phi=|\lambda_V(h)|_F^{-\frac{1}{8}\dim V\cdot\dim W}\omega(g_1,1)(\phi\circ h^{-1}),$$

where $|-|_F$ is the absolute value on F and

$$g_1=g\begin{pmatrix}\lambda_W(g)^{-1} & 0\\ 0 & 1\end{pmatrix}\in\mathrm{Sp}(W).$$

Here the central elements $(t,t^{-1})\in R_0$ acts by the quadratic character $\chi_V(t)^{(\dim W)/2}$, which is slightly different from the normalization used in [Roberts 2001].

Now we consider the compactly induced representation

$$\Omega=\mathrm{ind}_{R_0}^{R}\,\omega_\psi.$$

As a representation of R, Ω depends only on the orbit of ψ under the evident action of $\operatorname{Im}\lambda_V \subset F^\times$. For example, if λ_V is surjective, then Ω is independent of ψ. For any irreducible representation π of $\mathrm{GO}(V)$ (resp. $\mathrm{GSp}^+(W)$), the maximal π-isotropic quotient of Ω has the form

$$\pi \otimes \Theta_\psi(\pi),$$

where $\Theta_\psi(\pi)$ is some smooth representation of $\mathrm{GSp}^+(W)$ (resp. $\mathrm{GO}(V)$). Similarly, we let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$. Note that though $\Theta_\psi(\pi)$ may be reducible, it has a central character $\omega_{\Theta_\psi(\pi)}$ given by

$$\omega_{\Theta_\psi(\pi)} = \chi_V^{(\dim W)/2}\omega_\pi.$$

There is an extended Howe conjecture for similitude groups, which says that $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is nonzero and the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain. It was shown by Roberts [1996] that this follows from Theorem 2.1.

If λ_V is surjective, we have $\mathrm{GSp}^+(W) = \mathrm{GSp}(W)$.

Proposition 2.2 [Gan and Takeda 2011a, Proposition 2.3]. *Suppose that π is a supercuspidal representation of* $\mathrm{GO}(V)$ (*resp.* $\mathrm{GSp}(W)$). *Then $\Theta_\psi(\pi)$ is either zero or is an irreducible representation of* $\mathrm{GSp}^+(W)$ (*resp.* $\mathrm{GO}(V)$).

2A. ***First occurrence indices for pairs of orthogonal Witt towers.*** Let W_n ($n \geq 1$) be the $2n$-dimensional symplectic vector space with associated symplectic group $\mathrm{Sp}(W_n)$ and consider the two towers of orthogonal groups attached to the quadratic spaces with trivial discriminant. More precisely, let $\mathbb{H}$ be the split 2-dimensional quadratic space over F and D be the quaternion division algebra over F. Let

$$V_{2r}^+ = \mathbb{H}^r \quad \text{and} \quad V_{2r}^- = D(F) \oplus \mathbb{H}^r,$$

and denote the orthogonal groups by $\mathrm{O}(V_{2r}^+) = \mathrm{O}_{r,r}$ and $\mathrm{O}(V_{2r}^-) = \mathrm{O}_{r+4,r}$, respectively. For an irreducible representation π of $\mathrm{Sp}(W_n)$, one may consider the theta lifts $\theta_{2r}^+(\pi)$ and $\theta_{2r}^-(\pi)$ to $\mathrm{O}(V_{2r}^+)$ and $\mathrm{O}(V_{2r}^-)$ respectively, with respect to a fixed nontrivial additive character ψ. Set

$$\begin{cases} r^+(\pi) = \inf\{r : \theta_{2r}^+(\pi) \neq 0\}, \\ r^-(\pi) = \inf\{r : \theta_{2r}^-(\pi) \neq 0\}. \end{cases}$$

Then Kudla and Rallis [2005] and Sun and Zhu [2015] showed:

Theorem 2.3 (conservation relation). *For any irreducible representation π of* $\mathrm{Sp}(W_n)$, *we have*

$$r^+(\pi) + r^-(\pi) = 2n = \dim W_n.$$

On the other hand, one may consider the mirror situation, where one fixes an irreducible representation π of $\mathrm{O}(V_{2r})$ and consider its theta lift $\theta_n(\pi)$ to the tower of symplectic groups $\mathrm{Sp}(W_n)$. Then, with $n(\pi)$ defined in the analogous fashion

$$n(\pi) = \inf\{n : \theta_n(\pi) \neq 0\},$$

one has

$$n(\pi) + n(\pi \otimes \det) = 2r = \dim V_{2r}.$$

For similitude groups, this implies that

$$n(\pi)+n(\pi\otimes\nu)=2r,$$

where ν is the nontrivial character of $\mathrm{GO}(V_{2r})/\mathrm{GSO}(V_{2r})$.

3. The irreducibility of the big theta lift

Let τ be an irreducible representation of $\mathrm{Sp}_{2n}(F)$. Gan and Ichino [2014, Proposition C.4] showed that the big theta lift $\Theta^{+}_{2n+2}(\tau)$ to $\mathrm{O}_{n+1,n+1}(F)$ (called the almost equal rank case) is irreducible if τ is tempered. This includes the case $p=2$ since the Howe duality conjecture has been proved in [Gan and Takeda 2016b]. We will use the generalized standard module [Heiermann 2016, Theorem 3.2] to study the case when Π_{ϕ_τ} is generic (see Theorem 3.2).

In Section 3C, we mainly study the big theta lift to the split group $\mathrm{O}_{n+1,n+1}(F)$ from a representation τ of $\mathrm{Sp}_{2n}(F)$ when the associated L-packet Π_{ϕ_τ} is generic. Then we will focus on the computation for $n=2$.

3A. *Notation.* Let us introduce the notation used in this section.

- $|-|_F$ (resp. $|-|_E$) is an absolute value defined on F (resp. E).
- $P_{\vec{n}}$ (resp. $Q_{\vec{n}}$) is a parabolic subgroup of Sp_{2n} (resp. $\mathrm{O}_{n+1,n+1}$) defined over F.
- ϕ_τ is the Langlands parameter or L-parameter of τ and $\phi_\tau^\vee$ is the dual parameter of ϕ_τ.
- $\tau^\vee$ is the contragredient representation of τ.
- Π_{ϕ_τ} is the L-packet containing τ.
- $\mathcal{W}_r$ is the symplectic vector space over E of dimension $2r$.
- Z is a line in $\mathcal{W}_2$ and Y is a maximal isotropic subspace in $\mathcal{W}_2$.
- $Q(Z)$ (resp. $P(Y)$) is the Klingen (resp. Siegel) parabolic subgroup of $\mathrm{GSp}_4(E)=\mathrm{GSp}(\mathcal{W}_2)$.
- B (resp. B_0) is the Borel subgroup of $\mathrm{GSp}_4(E)$ (resp. $\mathrm{GL}_4(E)$).
- P is the parabolic subgroup of $\mathrm{GL}_4(E)$ with Levi component $\mathrm{GL}_2(E)\times\mathrm{GL}_2(E)$.
- $\Theta^{+}_{2r}(\tau)$ (resp. $\Theta_6(\tau)$) is the big theta lift to $\mathrm{GO}_{r,r}(E)$ (resp. $\mathrm{GSO}_{3,3}(E)$) of τ of $\mathrm{GSp}_4(E)$.
- $\theta^{+}_6(\tau)$ (resp. $\theta_6(\tau)$) is the small theta lift to $\mathrm{GO}_{3,3}(E)$ (resp. $\mathrm{GSO}_{3,3}(E)$) of τ of $\mathrm{GSp}_4(E)$.

3B. *The standard module conjecture.* Let $\boldsymbol{G}$ be a quasisplit reductive group defined over F. Fix a Borel subgroup $\boldsymbol{B}=\boldsymbol{T}\boldsymbol{U}$ of $\boldsymbol{G}$. Let π be an irreducible smooth representation of $\boldsymbol{G}(F)$. If there exists a nondegenerate character ψ_U of $\boldsymbol{U}(F)$ such that $\mathrm{Hom}_{\boldsymbol{U}(F)}(\pi,\psi_U)\neq 0$, then we say π is ψ_U-generic or generic. If the L-packet Π_{ϕ_π} contains a generic representation, then we call Π_{ϕ_π} a generic L-packet. Let $\boldsymbol{P}=\boldsymbol{M}\boldsymbol{N}$ be a standard parabolic subgroup of $\boldsymbol{G}$. Suppose that there exists a generic tempered representation ρ of $\boldsymbol{M}(F)$ such that π is isomorphic to the Langlands quotient $J(\rho,\chi)$, where χ is a character of $\boldsymbol{M}(F)$ and lies in the positive Weyl chamber with respect to $\boldsymbol{P}(F)$. (See [Heiermann and Opdam 2013, p. 777] for more details.)

Theorem 3.1 (the standard module conjecture). *If $\pi = J(\rho, \chi)$ is a generic representation of $\boldsymbol{G}(F)$, then $\mathrm{Ind}_{\boldsymbol{P}(F)}^{\boldsymbol{G}(F)}(\rho \otimes \chi)$ (normalized induction) is irreducible. Moreover, for any irreducible representation ρ' of $\boldsymbol{M}(F)$ lying inside the L-packet Π_{ϕ_ρ}, $\mathrm{Ind}_{\boldsymbol{P}(F)}^{\boldsymbol{G}(F)}(\rho' \otimes \chi)$ is irreducible.*

Heiermann and Opdam [2013] proved the standard module conjecture. Later Heiermann [2016, Theorem 3.2] proved its generalized version i.e., the "moreover" part of Theorem 3.1. The following subsection will focus on the cases $\boldsymbol{G} = \mathrm{Sp}_{2n}$ and $\boldsymbol{G} = \mathrm{O}_{n+1,n+1}$.

3C. *Theta lift from $\mathrm{Sp}_{2n}(F)$ to $\mathrm{O}_{n+1,n+1}(F)$.* Suppose that τ is a generic irreducible admissible representation of $\mathrm{Sp}_{2n}(F)$. Assume that there exists a parabolic subgroup $P_{\vec{n}} = M_{\vec{n}} N_{\vec{n}}$ of Sp_{2n} and an irreducible representation $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_r \otimes \tau_0$ of $M_{\vec{n}}(F) \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \mathrm{Sp}_{2n_0}(F)$ (for $n_1 + n_2 + \cdots + n_r + n_0 = n$) such that τ is the unique irreducible quotient of the standard module

$$\mathrm{Ind}_{P_{\vec{n}}(F)}^{\mathrm{Sp}_{2n}(F)}(\pi_1|-|_F^{s_1} \otimes \cdots \otimes \pi_r|-|_F^{s_r} \otimes \tau_0)(\text{normalized induction}), \tag{3-1}$$

where $s_1 > s_2 > \cdots > s_r > 0$, $n \geq n_0$, each π_i is a tempered representation of $\mathrm{GL}_{n_i}(F)$ and τ_0 is a tempered representation of $\mathrm{Sp}_{2n_0}(F)$. Moreover, the Langlands parameter $\phi_\tau : WD_F \to \mathrm{SO}_{2n+1}(\mathbb{C})$ is given by

$$\phi_\tau = \phi_{\pi_1}|-|_F^{s_1} \oplus \cdots \oplus \phi_{\pi_r}|-|_F^{s_r} \oplus \phi_{\tau_0} \oplus \phi_{\pi_r}^\vee|-|_F^{-s_r} \oplus \cdots \oplus \phi_{\pi_1}^\vee|-|_F^{-s_1},$$

where each ϕ_{π_i} is the Langlands parameter of π_i and ϕ_{τ_0} is the Langlands parameter of τ_0. Here we identify the characters of $F^\times$ and the characters of the Weil group W_F by the local class field theory. Due to Theorem 3.1, the generic representation τ is isomorphic to the standard module, i.e., the standard module is irreducible. Thanks to [Gan and Ichino 2014, Proposition C.4], the small theta lift $\theta_{2n+2}^+(\tau)$ is the unique irreducible quotient of the standard module

$$\mathrm{Ind}_{Q_{\vec{n}}(F)}^{\mathrm{O}_{n+1,n+1}(F)}(\pi_1|-|_F^{s_1} \otimes \cdots \otimes \pi_r|-|_F^{s_r} \otimes \Theta_{2n_0+2}^+(\tau_0)), \tag{3-2}$$

where $Q_{\vec{n}}(F)$ is the parabolic subgroup of $\mathrm{O}_{n+1,n+1}(F)$ with Levi component $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \times \mathrm{O}_{n_0+1,n_0+1}(F)$. We will show that (3-2) equals $\theta_{2n+2}^+(\tau)$ under certain conditions.

Theorem 3.2. *Let $P_{\vec{n}}$ (resp. $Q_{\vec{n}}$) be a parabolic subgroup of Sp_{2n} (resp. $\mathrm{O}_{n+1,n+1}$) defined as above. If the irreducible representation τ is generic and so τ is isomorphic to the standard module (3-1), and the standard L-function of τ is regular at $s = 1$, then $\Theta_{2n+2}^+(\tau)$ is irreducible.*

There is another key input in the proof of Theorem 3.2.

Theorem 3.3. *Let $\boldsymbol{G}$ be Sp_{2n} or $\mathrm{SO}_{n+1,n+1}$. Let π be an irreducible representation of $\boldsymbol{G}(F)$. The L-packet Π_{ϕ_π} is generic if and only if the adjoint L-function $L(s, \phi_\pi, \mathrm{Ad})$ is regular at $s = 1$.*

Proof. See [Liu 2011, Theorem 1.2; Jantzen and Liu 2014, Theorem 1.5]. □

Proof of Theorem 3.2. We will show that $\Theta_{2n+2}^+(\tau)|_{\mathrm{SO}_{n+1,n+1}(F)}$ is irreducible. If $n = n_0$, then it follows from [Gan and Ichino 2014, Proposition C.4]. Assume that $s_1 > 0$. Then there exists a surjection

$$\mathrm{Ind}_{Q_{\vec{n}}(F)}^{\mathrm{O}_{n+1,n+1}(F)}(\pi_1|-|_F^{s_1} \otimes \cdots \otimes \pi_r|-|_F^{s_r} \otimes \Theta_{2n_0+2}^+(\tau_0)) \twoheadrightarrow \Theta_{2n+2}^+(\tau).$$

Due to [Gan and Ichino 2014, Proposition C.4], if τ_0 is tempered, then $\Theta^+_{2n_0+2}(\tau_0)$ is irreducible and generic. Moreover, if

$$\phi_{\tau_0} : WD_F \to \mathrm{SO}_{2n_0+1}(\mathbb{C})$$

is the Langlands parameter of τ_0, then $\phi_{\theta^+_{2n_0+2}(\tau_0)} = \phi_{\tau_0} \oplus \mathbb{C}$ due to [Gan and Ichino 2014, Theorem C.5]. Assume that $\phi_\tau = \phi_0 \oplus \phi_{\tau_0} \oplus \phi_0^\vee$ with ϕ_{τ_0} tempered and $\phi_0 = \bigoplus_i \phi_{\pi_i}|-|^{s_i}$. Then due to [Gan and Ichino 2014, Proposition C.4], we have $\phi_{\theta^+_{2n+2}(\tau)} = \phi_0 \oplus (\phi_{\tau_0} \oplus \mathbb{C}) \oplus \phi_0^\vee$. Observe that

$$L(s, \mathrm{Ad}_{\mathrm{SO}_{2n+2}} \circ \phi_{\theta^+_{2n+2}(\tau)}) = L(s, \mathrm{Ad}_{\mathrm{SO}_{2n+1}} \circ \phi_\tau) \cdot L(s, \phi_\tau, \mathrm{Std}),$$

where $L(s, \phi_\tau, \mathrm{Std})$ is the standard L-function of τ. By [Liu 2011, Theorem 1.2] and the assumption that τ is generic, we obtain that $L(s, \mathrm{Ad}_{\mathrm{SO}_{2n+1}} \circ \phi_\tau)$ is regular at $s = 1$. So $L(s, \mathrm{Ad}_{\mathrm{SO}_{2n+2}} \circ \phi_{\theta^+_{2n+2}(\tau)})$ is regular at $s = 1$. Thanks to [Jantzen and Liu 2014, Theorem 1.5], the L-packet $\Pi_{\phi_{\theta^+_{2n+2}(\tau)}}$ is generic. By the generalization of the standard module conjecture [Heiermann 2016, Theorem 3.2] that the standard module with a generic quotient is irreducible,

$$\theta^+_{2n+2}(\tau) = \Theta^+_{2n+2}(\tau) = \mathrm{Ind}^{\mathrm{O}_{n+1,n+1}(F)}_{Q_{\bar n}(F)} (\pi_1|-|^{s_1}_F \otimes \cdots \otimes \pi_r|-|^{s_r}_F \otimes \Theta^+_{2n_0+2}(\tau_0)),$$

i.e., $\Theta^+_{2n+2}(\tau)$ is irreducible. □

Remark 3.4. Similarly, if Σ is a generic representation of $\mathrm{O}_{n,n}(F)$ and $L(s, \Sigma, \mathrm{Std})$ is regular at $s = 1$, then the big theta lift $\Theta_n(\Sigma)$ to $\mathrm{Sp}_{2n}(F)$ is irreducible. However, if τ is a generic representation of $\mathrm{Sp}_{2n}(F)$ and $L(s, \tau, \mathrm{Std})$ is regular at $s = 1$, the big theta lift to nonsplit group $\mathrm{O}(V_F)$ may be reducible when V_F is a $(2n + 2)$-dimensional quadratic space over F with nontrivial discriminant. (See [Lu 2017b, Proposition 3.8(iii)].)

Remark 3.5. There exists an isomorphism between the characters $\lambda_{\theta^+_{2n+2}(\tau)} \cong \lambda_{\theta^+_{2n_0+2}(\tau_0)}$, the latter of which is given in [Atobe and Gan 2017, Theorem 4.3] in terms of the character λ_{τ_0}, conjectured in [Prasad 1993].

Corollary 3.6. *Let Π_{ϕ_τ} be the L-packet of $\mathrm{Sp}_{2n}(F)$ containing τ. Suppose that Π_{ϕ_τ} is generic. If the standard L-function $L(s, \phi_\tau, \mathrm{Std})$ is a factor of the adjoint L-function $L(s, \mathrm{Ad} \circ \phi_\tau)$, then the big theta lift $\Theta^+_{2n+2}(\tau)$ to $\mathrm{O}_{n+1,n+1}(F)$ is irreducible for any $\tau \in \Pi_{\phi_\tau}$.*

For the rest of this section, we will compute the big theta lifts between $\mathrm{GSp}_4(E)$ and $\mathrm{GO}(V)$ explicitly when $\dim_E V = 4$ or 6.

3D. *Representations of* GO(V). Let π_i be an irreducible representations of $\mathrm{GL}_2(E)$ with central character ω_{π_i} and $\omega_{\pi_1} = \omega_{\pi_2}$. Then $\pi_1 \boxtimes \pi_2$ is an irreducible representation of the similitude group

$$\mathrm{GSO}_{2,2}(E) \cong \mathrm{GL}_2(E) \times \mathrm{GL}_2(E)/\{(t, t^{-1}) : t \in E^\times\}.$$

If $\pi_1 \neq \pi_2$, then $\Sigma = \mathrm{Ind}^{\mathrm{GO}_{2,2}(E)}_{\mathrm{GSO}_{2,2}(E)}(\pi_1 \boxtimes \pi_2)$ is an irreducible smooth representation of $\mathrm{GO}_{2,2}(E)$ and $\Sigma \cong \Sigma \otimes \nu$, where $\nu|_{\mathrm{O}_{2,2}(E)} = \det$. If $\pi_1 = \pi_2$, then there are two extensions $(\pi_1 \boxtimes \pi_1)^\pm$ and only one of them participates in the theta lift between $\mathrm{GSp}_4(E)$ and $\mathrm{GO}_{2,2}(E)$, denoted by $(\pi_1 \boxtimes \pi_1)^+ = \Sigma$. Moreover, we have $(\pi_1 \boxtimes \pi_1)^+ \otimes \nu \cong (\pi_1 \boxtimes \pi_1)^-$. (See [Gan and Takeda 2011b, §6].)

Any irreducible representation of

$$\mathrm{GSO}_{3,3}(E)=\mathrm{GL}_4(E)\times \mathrm{GL}_1(E)/\{(t,t^{-2}):t\in E^\times\}$$

is of the form

$$\Pi\boxtimes\chi,$$

where Π is a representation of $\mathrm{GL}_4(E)$ with central character ω_Π, χ is a character of $E^\times$ and $\chi^2=\omega_\Pi$.

3E. *Representations of* $\mathbf{GSp_4}(E)$. Assume that $\tau=\theta(\pi_1\boxtimes\pi_2)$ is a representation of $\mathrm{GSp}_4(E)$ and $\pi_1\boxtimes\pi_2\in\mathrm{Irr}(\mathrm{GSO}_{2,2}(E))$. Then τ is generic if and only if $\pi_1\boxtimes\pi_2$ is generic due to [Gan and Takeda 2011b, Corollary 4.2(ii)]. We follow the notation in [Gan and Takeda 2011b] to describe the nondiscrete series representations of $\mathrm{GSp}_4(E)$. Thanks to [Gan and Takeda 2011b, Proposition 5.3], the nondiscrete series representations of $\mathrm{GSp}_4(E)$ fall into the following three families:

- $\tau\hookrightarrow I_{Q(Z)}(\chi|-|_E^{-s},\pi)$ with χ a unitary character, $s\geq 0$ and π a discrete series representation of $\mathrm{GL}_2(E)$ up to twist;
- $\tau\hookrightarrow I_{P(Y)}(\pi|-|_E^{-s},\chi)$ with χ an arbitrary character, $s\geq 0$ and π a unitary discrete series representation of $\mathrm{GL}(Y)$;
- $\tau\hookrightarrow I_B(\chi_1|-|_E^{-s_1},\chi_2|-|_E^{-s_2};\chi)$, where χ_1,χ_2 are unitary and $s_1\geq s_2\geq 0$.

Note that if τ itself is generic and nontempered, then those embeddings are in fact isomorphisms due to the standard module conjecture for GSp_4, except

$$\tau\hookrightarrow I_{Q(Z)}(\mathbf{1},\pi).$$

For instance, $\tau=J_{P(Y)}(\pi|-|_E^{s},\chi)$ with $s\geq 0$. If τ is generic, then $I_{P(Y)}(\pi|-|_E^{s},\chi)$ is irreducible and so

$$\tau=I_{P(Y)}(\pi|-|_E^{s},\chi)\cong I_{P(Y)}(\pi^\vee|-|_E^{-s},\chi\omega_\pi|-|_E^{2s})$$

with $s\geq 0$. (See [Gan and Takeda 2011b, Lemma 5.2].)

If the big theta lift $\Theta_6^+(\tau)$ to $\mathrm{GO}_{3,3}(E)$ of τ is irreducible, the restricted representation $\Theta_6^+(\tau)|_{\mathrm{GSO}_{3,3}(E)}$ is irreducible due to [Prasad 1993, §5, p. 282]. We use $\Theta_6(\tau)$ to denote the big theta lift to $\mathrm{GSO}_{3,3}(E)$ of τ.

Proposition 3.7. *Let τ be a generic irreducible representation of $\mathrm{GSp}_4(E)$. Then the big theta lift $\Theta_6(\tau)$ to $\mathrm{GSO}_{3,3}(E)$ of τ is an irreducible representation unless $\tau=I_{Q(Z)}(|-|_E,\pi)$ with π essentially square-integrable. If $\tau=I_{Q(Z)}(|-|_E,\pi)$, then $\Theta_6(\tau)=I_P(\pi|-|_E,\pi)\boxtimes\omega_\pi|-|_E$ is reducible.*

Proof. If τ is a tempered representation, then $\Theta_6^+(\tau)$ is irreducible due to [Gan and Ichino 2014, Proposition C.4] (which holds even for $p=2$ since the Howe duality conjecture holds) and so $\Theta_6(\tau)$ is irreducible. Assume that the generic representation τ is not essentially tempered. There are 4 cases:

- If $\tau=I_B(\chi_1,\chi_2;\chi)$ is irreducible, then none of the characters $\chi_1,\chi_2,\chi_1/\chi_2,\chi_1\chi_2$ is $|-|_E^{\pm 1}$ and so $I_{B_0}(\mathbf{1},\chi_2,\chi_1,\chi_1\chi_2)$ has a generic quotient where B_0 is a Borel subgroup of $\mathrm{GL}_4(E)$. Thus $\Theta_6(\tau)=I_{B_0}(\mathbf{1},\chi_2,\chi_1,\chi_1\chi_2)\cdot\chi\boxtimes\chi^2\chi_1\chi_2$ is irreducible due to the standard module conjecture for GL_4.

- If $\tau = I_{P(Y)}(\pi, \chi)$, then $\Theta_6(\tau)$ is a quotient of
$$I_Q(\mathbf{1}, \pi, \omega_\pi) \cdot \chi \boxtimes \chi^2\omega_\pi,$$
where Q is a parabolic subgroup of $\mathrm{GL}_4(E)$ with Levi subgroup $\mathrm{GL}_1(E)\times\mathrm{GL}_2(E)\times\mathrm{GL}_1(E)$. Due to [Gan and Takeda 2011b, Proposition 13.2], the adjoint L-function $L(s, As\circ\phi_\tau)$ is regular at $s=1$. Since the standard L-function $L(s, \tau, \mathrm{Std})$ is a factor of $L(s, \mathrm{Ad}\circ\phi_\tau)$, we have $L(s, \tau, \mathrm{Std})$ is regular at $s=1$. Then $I_Q(\mathbf{1}, \pi, \omega_\pi)$ is irreducible and so $\Theta_6(\tau) = I_Q(\mathbf{1}, \pi, \omega_\pi)\cdot\chi\boxtimes\chi^2\omega_\pi$ is irreducible.
- If $\tau = I_{Q(Z)}(\chi, \pi)$ with $\chi \neq \mathbf{1}$, then there is an epimorphism
$$I_P(\pi\cdot\chi, \pi)\boxtimes\omega_\pi\chi \twoheadrightarrow \Theta_6(\tau)$$
of $\mathrm{GSO}_{3,3}(E)$-representations, where P is a parabolic subgroup of $\mathrm{GL}_4(E)$ with Levi subgroup $\mathrm{GL}_2(E)\times\mathrm{GL}_2(E)$. Gan and Takeda [2011b, Proposition 13.2] have proved that $I_P(\pi\cdot\chi, \pi)$ is irreducible if $I_{Q(Z)}(\chi, \pi)$ is irreducible and $\chi \neq |-|_E$. If $\chi = |-|_E$ and π is essentially square-integrable, applying [Gan and Takeda 2011b, Corollary 4.4] that τ is generic implies that $\Theta_6(\tau)$ is generic, then $\Theta_6(\tau) = I_P(\pi\cdot\chi, \pi)\boxtimes\omega_\pi\chi$ and $\theta_6(\tau) = J_P(\pi\cdot\chi, \pi)\boxtimes\omega_\pi\chi$ is the Langlands quotient.
- If $\tau \hookrightarrow I_{Q(Z)}(\mathbf{1}, \pi)$, then $\Theta_6(\tau)$ is either zero or $I_P(\pi, \pi)\boxtimes\omega_\pi$, where P is a parabolic subgroup of $\mathrm{GL}_4(E)$ with Levi subgroup $\mathrm{GL}_2(E)\times\mathrm{GL}_2(E)$. In fact, $\Theta_6(\tau) = 0$ only when τ is a nongeneric constituent representation of $I_{Q(Z)}(\mathbf{1}, \pi)$.

This finishes the proof of Proposition 3.7. □

Remark 3.8. Similarly one can prove that if Σ is a generic representation of $\mathrm{GSO}_{2,2}(E)$ and $L(s, \Sigma, \mathrm{Std})$ is regular at $s=1$, then the big theta lift $\Theta_2(\Sigma)$ to $\mathrm{GSp}_4(E)$ is an irreducible representation.

Let us turn the table around. The rest of this subsection focuses on the computation of local theta lifts to $\mathrm{GO}_{2,2}(E)$ from $\mathrm{GSp}_4(E)$.

Proposition 3.9. *Let τ be a generic irreducible representation of $\mathrm{GSp}_4(E)$. Assume that $\theta_4^+(\tau)\neq 0$.*

(i) *If $\tau = I_{Q(Z)}(\mathbf{1}, \pi(\mu_1, \mu_2))$, then the big theta lift $\Theta_4^+(\tau)$ to $\mathrm{GO}_{2,2}(E)$ of τ is $\mathrm{Ext}^1_{\mathrm{GO}_{2,2}(E)}(\Sigma^+, \Sigma^-)$, where $\Sigma^\pm$ are two distinct extensions of $\pi(\mu_1, \mu_2)\boxtimes\pi(\mu_1, \mu_2)$ from $\mathrm{GSO}_{2,2}(E)$ to $\mathrm{GO}_{2,2}(E)$.*

(ii) *If $\tau \neq I_{Q(Z)}(\mathbf{1}, \pi(\mu_1, \mu_2))$, then $\Theta_4^+(\tau)$ is an irreducible representation of $\mathrm{GO}_{2,2}(E)$.*

Proof. (i) If $\tau = I_{Q(Z)}(\mathbf{1}, \pi(\mu_1, \mu_2))$, then the small theta lift $\theta_4^+(\tau)$ equals Σ^+ by the Howe duality, where Σ^+ is the extension to $\mathrm{GO}_{2,2}(E)$ of $\pi(\mu_1, \mu_2)\boxtimes\pi(\mu_1, \mu_2)$. Let ψ_U be a nondegenerate character of the standard unipotent subgroup U of $\mathrm{GO}_{2,2}(E)$. Then
$$\dim \mathrm{Hom}_U(\Theta_4^+(\tau), \psi_U) = \dim \mathrm{Hom}_{H(\mathcal{W}_1)\rtimes\mathrm{Sp}(\mathcal{W}_1)}(\tau, \omega_\psi) = 2, \tag{3-3}$$
where $\mathcal{W}_2 = Z\oplus\mathcal{W}_1\oplus Z^*$, $H(\mathcal{W}_1)$ is the Heisenberg group of $\mathcal{W}_1$ equipped with the Weil representation ω_ψ and τ is the representation of $\mathrm{GSp}(\mathcal{W}_2)$. Thus the big theta lift $\Theta_4^+(\tau)$ to $\mathrm{GO}_{2,2}(E)$ is reducible. There is

a short exact sequence of $GO_{2,2}(E)$-representations

$$\Sigma^- \oplus \Sigma^+ \longrightarrow \Theta_4^+(\tau) \longrightarrow \Sigma^+ \longrightarrow 0. \tag{3-4}$$

However, we can not determine $\Theta_4^+(\tau)$ at this moment. Note that

$$\dim \mathrm{Ext}^1_{GSO_{2,2}(E)}(\pi(\mu_1,\mu_2) \boxtimes \pi(\mu_1,\mu_2), \pi(\mu_1,\mu_2) \boxtimes \pi(\mu_1,\mu_2)) = 1$$

due to [Adler and Prasad 2012, Theorem 1]. Here Ext^1 is the extension functor defined on the category of all smooth representations with a fixed central character. Then $\dim \mathrm{Ext}^1_{GO_{2,2}(E)}(\Sigma^+, \Sigma^- \oplus \Sigma^+) = 1$ by Frobenius reciprocity, which implies that either $\mathrm{Ext}^1_{GO_{2,2}(E)}(\Sigma^+, \Sigma^-)$ or $\mathrm{Ext}^1_{GO_{2,2}(E)}(\Sigma^+, \Sigma^+)$ is zero. Assume that B is the Borel subgroup of $GSO_{2,2}(E)$. Set $\widetilde{B} = B \rtimes \mu_2$ to be a subgroup of $GO_{2,2}(E)$ and $\widetilde{B} \cap GSO_{2,2}(E) = B$. Since

$$\pi(\mu_1,\mu_2) \boxtimes \pi(\mu_1,\mu_2) = \mathrm{Ind}_B^{GSO_{2,2}(E)} \chi \,(\text{normalized induction}),$$

there are two extensions $\chi^\pm$ to $\widetilde{B}$ of χ of B. We may assume without loss of generality that $\Sigma^+ = \mathrm{Ind}_{\widetilde{B}}^{GO_{2,2}(E)} \chi^+$ and $\Sigma^- = \mathrm{Ind}_{\widetilde{B}}^{GO_{2,2}(E)} \chi^-$. Note that $\mathrm{Ext}^1_{\widetilde{B}}(\chi^+, \chi^-) \neq 0$. Then there is a short exact sequence of $GO_{2,2}(E)$-representations

$$0 \longrightarrow \Sigma^- \longrightarrow \mathrm{Ind}_{\widetilde{B}}^{GO_{2,2}(E)}(\mathrm{Ext}^1_{\widetilde{B}}(\chi^+, \chi^-)) \longrightarrow \Sigma^+ \longrightarrow 0,$$

which is not split. Hence $\mathrm{Ext}^1_{GO_{2,2}(E)}(\Sigma^+, \Sigma^-) \neq 0$. Together with (3-3) and (3-4), one can obtain the desired equality $\Theta_4^+(\tau) = \mathrm{Ext}^1_{GO(2,2)(E)}(\Sigma^+, \Sigma^-)$.

(ii) If τ is a (essentially) discrete series representation, then it follows from [Atobe and Gan 2017, Proposition 5.4].

- If $\tau = I_{Q(Z)}(\mu_0, \pi(\mu_1,\mu_2))$ with $\mu_0 \neq \mathbf{1}$, then there exists only one orbit in the double coset $Q(Z)\backslash GSp_4(E)/H(\mathcal{W}_1) \rtimes Sp(\mathcal{W}_1)$ that contributes to the multiplicity

$$\dim \mathrm{Hom}_{H(W_1) \rtimes Sp(\mathcal{W}_1)}(\tau, \omega_\psi),$$

  and so $\Theta_4^+(\tau)$ is irreducible.

- If $\tau \subset I_{Q(Z)}(\mathbf{1}, \pi)$ with π square-integrable, then τ is tempered. Due to [Atobe and Gan 2017, Proposition 5.5], $\Theta_4^+(\tau)$ is tempered. Note that $\theta_4^+(\tau)$ is a discrete series representation which is projective in the category of the tempered representations. Thus $\Theta_4^+(\tau) = \theta_4^+(\tau)$ is irreducible. Otherwise, it will contradict the Howe duality conjecture (see Theorem 2.1).

- If $\tau = I_{P(Y)}(\pi, \chi)$, then $\dim \mathrm{Hom}_U(\Theta_4^+(\tau), \psi_U) = 1$ and so $\Theta_4^+(\tau)$ is irreducible.

This finishes the proof of Proposition 3.9. □

4. The $\mathrm{GSp}_4(F)$-distinguished representations

This section focuses on the proof of Theorem 1.1. First, we will introduce the see-saw identity in the similitude group in Section 4B. Then we will study the filtrations of various degenerate principal series representations restricted to reductive subgroups in Section 4C, which involves the complicated computation for the double coset decompositions. The proof of Theorem 1.1 will be given in the last subsection.

4A. ***Notation.***

- $\mathbb{C}$ or $\mathbf{1}$ is the trivial representation.
- $\mathbb{H}$ (resp. $\mathbb{H}_E$) is the split 2-dimensional quadratic space over F (resp. E).
- $(-,-)_E$ is the Hilbert symbol on $E^\times \times E^\times$.
- $\mathrm{Res}_{E/F} V$ is a quadratic space over F while V is a quadratic space over E.
- $\mathrm{GSp}(W_n) = \mathrm{GSp}_{2n}(F)$ is the symplectic similitude group.
- $\mathrm{GU}_2(D) = \mathrm{GSp}_{1,1}$ is the unique inner form of GSp_4.
- λ_W (resp. λ_V) is the similitude character of $\mathrm{GSp}_4(E)$ (resp. $\mathrm{GO}(V)$).
- $\mathrm{GSp}_4(E)^\natural = \{g \in \mathrm{GSp}_4(E) \mid \lambda_W(g) \in F^\times\}$ is the subgroup of $\mathrm{GSp}_4(E)$ and similarly for $\mathrm{GO}_{2,2}(E)^\natural$.
- P' (resp. $P^\natural$) is a parabolic (resp. Siegel parabolic) subgroup of $\mathrm{GSp}_4(E)^\natural$ and $Q^\natural$ is the Siegel parabolic subgroup of $\mathrm{GO}_{2,2}(E)^\natural$. And $R_{\bar{P}'}$ (resp. $R_{\bar{P}^\natural}$) is the Jacquet functor with respect to the parabolic subgroup opposite to P' (resp. $P^\natural$).
- ind denotes the compact induction.
- $R_r(\mathbf{1})$ is the big theta lift to $\mathrm{GO}_{4,4}(F)$ of the trivial representation of $\mathrm{GSp}(W_r)$.
- $R^{m,n}(\mathbf{1})$ is the big theta lift to $\mathrm{GSp}_8(F)$ of the trivial representation of $\mathrm{GO}_{m,n}(F)$.
- Σ is a generic representation of $\mathrm{GO}(V)$.
- Q_r is the Siegel parabolic subgroup of $H_r = \mathrm{GO}_{r,r}(F)$.
- $I_{Q_r}^{H_r}(s)$ is the degenerate Siegel principal series of H_r.
- $X_4 = Q_4\backslash H_4$ is the projective variety.
- $\mathcal{I}(s)$ is the degenerate Siegel principal series of $\mathrm{GSp}_8(F)$.
- $\mathrm{Mat}_{m,n}(F)$ is the matrix space over F consisting of all $m \times n$ matrices.

4B. ***See-saw identity for orthogonal-symplectic dual pairs.*** Following the notation in [Prasad 1996], for a quadratic space (V, q) of even dimension over E, let $\mathrm{Res}_{E/F} V$ be the same space V but now thought of as a vector space over F with a quadratic form

$$q_F(v) = \tfrac{1}{2}\,\mathrm{tr}_{E/F}\, q(v).$$

If W_0 is a symplectic vector space over F, then $W_0 \otimes_F E$ is a symplectic vector space over E. Then we have the following isomorphism of symplectic spaces over F:

$$\mathrm{Res}_{E/F}[(W_0 \otimes_F E) \otimes_E V] \cong W_0 \otimes_F \mathrm{Res}_{E/F} V =: \boldsymbol{W}.$$

There is a pair

$$\big(\mathrm{GSp}(W_0), \mathrm{GO}(\mathrm{Res}_{E/F} V)\big) \quad \text{and} \quad \big(\mathrm{GSp}(W_0 \otimes_F E), \mathrm{GO}(V)\big)$$

of similitude dual reductive pairs in the symplectic similitude group $\mathrm{GSp}(\boldsymbol{W})$. A pair (G, H) and (G', H') of dual reductive pairs in a symplectic similitude group is called a see-saw pair if $H \subset G'$ and $H' \subset G$. The following lemma is quite useful in this section. See [Prasad 1996, Lemma, p. 6].

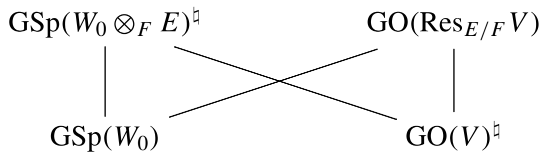
Lemma 4.2.1. *For a see-saw pair of dual reductive pairs (G, H) and (G', H'), let π be an irreducible representation of H and π' of H'. Then we have the following isomorphism*:

$$\mathrm{Hom}_H(\Theta_\psi(\pi'), \pi) \cong \mathrm{Hom}_{H'}(\Theta_\psi(\pi), \pi').$$

Let $\mathrm{GSp}(W_0 \otimes_F E)^\natural$ be the subgroup of $\mathrm{GSp}(W_0 \otimes_F E)$ where the similitude factor takes values in $F^\times$. Similarly we define

$$\mathrm{GO}(V)^\natural = \{h \in \mathrm{GO}(V) \mid \lambda_V(h) \in F^\times\}.$$

Then we have a see-saw diagram



Replace W_0 by a 4-dimensional symplectic space W_2 over F with a symplectic similitude group $\mathrm{GSp}_4(F)$. Then there is a see-saw pair

$$\big(\mathrm{GSp}_4(E)^\natural, \mathrm{GO}(V)^\natural\big) \quad \text{and} \quad \big(\mathrm{GSp}_4(F), \mathrm{GO}(\mathrm{Res}_{E/F} V)\big)$$

in the similitude symplectic group $\mathrm{GSp}(\boldsymbol{W})$, where $\boldsymbol{W} = \mathrm{Res}_{E/F}((W_2 \otimes_F E) \otimes_E V)$ and

$$\mathrm{GSp}_4(E)^\natural = \{g \in \mathrm{GSp}_4(E) | \lambda_W(g) \in F^\times\}.$$

Remark 4.2.2. Let V_F be a quadratic space over F. If the image of the similitude character λ_{V_F} is not surjective, then we need to consider the dual pair $R = \mathrm{GSp}_{4n}(F)^+ \times \mathrm{GO}(V_F)$. Moreover, $\mathrm{GSp}_{4n}(F) \times \mathrm{GO}(V_F)$ is not a dual pair in the usual sense. However, for our purpose (see Lemma 4.4.1), we will consider the induction in stages (see [Gan 2011, §9.7])

$$\mathrm{Ind}_R^{\mathrm{GSp}_{4n}(F)\times\mathrm{GO}(V_F)} \Omega_\psi = \mathrm{Ind}_R^{\mathrm{GSp}_{4n}(F)\times\mathrm{GO}(V_F)} \mathrm{ind}_{R_0}^R \omega_\psi,$$

where Ω_ψ (resp. ω_ψ) is the Weil representation of R (resp. R_0) defined in Section 2. Suppose that $V_F \otimes_E E$ is a split quadratic space over E. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GO}(V_F)}(\Theta_\psi(\tau), \chi) &= \mathrm{Hom}_{\mathrm{GSp}_{2n}(E)^\natural \times \mathrm{GO}(V_F)}(\mathrm{Ind}_R^{\mathrm{GSp}_{4n}(F)\times\mathrm{GO}(V_F)} \Omega_\psi, \tau \boxtimes \chi) \\ &= \mathrm{Hom}_{\mathrm{GSp}_{2n}(E)^\natural}(\mathrm{Ind}_{\mathrm{GSp}_{4n}(F)^+}^{\mathrm{GSp}_{4n}(F)} \Theta_\psi(\chi), \tau)\end{aligned}$$

for a representation $\tau \in \mathrm{Irr}(\mathrm{GSp}_{2n}(E)^\natural)$ and a character χ of $\mathrm{GO}(V_F)$.

In order to use Lemma 4.2.1, we need to figure out the discriminant and Hasse invariant of the quadratic space $\mathrm{Res}_{E/F} V$ over F.

Assume that $E = F(\sqrt{d})$ is a quadratic field extension of F, where $d \in F^\times \setminus F^{\times 2}$. Let D_E be the nonsplit quaternion algebra with involution $*$ defined over E with a norm map N_{D_E}, which is a 4-dimensional quadratic space V over E. More precisely, D_E is a noncommutative E-algebra generated by $1, i$ and j, denoted by $\left(\frac{a,b}{E}\right)$, where $i^2 = a$, $j^2 = b$, $ij = -ji$, $a, b \in E^\times$ and $(a,b)_E = -1$. Here $(-,-)_E$ is the Hilbert symbol defined on $E^\times \times E^\times$. Then there is an isomorphism for the vector space $\mathrm{Res}_{E/F} V$,

$$\mathrm{Res}_{E/F} D_E \cong \mathrm{Span}_F\{1, \sqrt{d}, i, \sqrt{d}i, j, \sqrt{d}j, ij, \sqrt{d}ij\}$$

as F-vector spaces. Given a vector $v \in V$, set

$$q_F(v) = \tfrac{1}{2}\,\mathrm{tr}_{E/F} \circ N_{D_E}(v) \quad \text{and} \quad (v_i, v_j) = q_F(v_i + v_j) - q_F(v_i) - q_F(v_j).$$

Lemma 4.2.3. *The quadratic space* $\mathrm{Res}_{E/F} D_E$ *with quadratic form* $\frac{1}{2}\mathrm{tr}_{E/F} \circ N_{D_E}$ *over* F *has dimension* 8, *discriminant* 1 *and Hasse-invariant* -1.

Proof. The nonsplit quaternion algebra over a nonarchimedean local field is unique. We may assume that

$$i^2 = a \in F^\times$$

and $j^2 = b = b_1 + b_2\sqrt{d}$, $N_{E/F}(b) = b_1^2 - b_2^2 d$, $b_i \in F$.

For an element $v = x_1 + x_2 i + x_3 j + x_4 ij$ in D_E with $x_i \in E$, we have

$$\tfrac{1}{2}(v,v) = N_{D_E}(v) = vv^* = x_1^2 - a x_2^2 - b x_3^2 + ab x_4^2$$

and the corresponding matrix for the quadratic space $(\mathrm{Res}_{E/F} D_E, q_F)$ is

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2ad & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2b_1 & -2b_2 d & 0 & 0 \\ 0 & 0 & 0 & 0 & -2b_2 d & -2b_1 d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2ab_1 & 2dab_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2dab_2 & 2dab_1 \end{pmatrix}.$$

The discriminant algebra of $\mathrm{Res}_{E/F} D_E$ is trivial in $F^\times / F^{\times 2}$. If $b_1 = 0$, then the Hasse-invariant is

$$(-d, a) = -1$$

since $(b_2\sqrt{d}, a)_E = -1$, where $(-,-)$ is the Hilbert symbol defined on $F^\times \times F^\times$. If $b_1 \neq 0$, then the Hasse-invariant is

$$(d,d)(-a,-ad)\left(-b_1, \frac{N_{E/F}(b)d}{-b_1}\right)(N_{E/F}(b)d, -1)\left(ab_1, \frac{N_{E/F}(b)d}{ab_1}\right) = (a, N_{E/F}(b)) = (a,b)_E = -1,$$

because $(a,b)_E = (a, N_{E/F}(b))$ for all $a \in F^\times$ and $b \in E^\times$. □

Now let V be the split $2n$-dimensional quadratic space $\mathbb{H}_E^n$ over E. There is a basis $\{e_i, e_j'\}_{1\le i,j\le n}$ for the quadratic space V satisfying $\langle e_i, e_j'\rangle = \delta_{ij}$ and the other inner products are zero. Then we fix the basis

$$\{e_i, \sqrt{d}e_i, e_j', e_j'/\sqrt{d}\}_{1\le i,j\le n}$$

for $\mathrm{Res}_{E/F}V$. It is straightforward to check that the vector space $\mathrm{Res}_{E/F}V$ is isomorphic to the split $4n$-dimensional quadratic space $\mathbb{H}^{2n}$ over F.

4C. ***The structure of degenerate principal series.*** In this subsection, we follow the notation in [Gan and Ichino 2011; Kudla 1996]. Let $H_n = \mathrm{GO}(\mathbb{H}^n)$ be the orthogonal similitude group. Define the quadratic character ν to be

$$\nu(h) = \det(h)\cdot\lambda_V^{-n}(h) \text{ for } h\in \mathrm{GO}(\mathbb{H}^n)$$

so that $\nu|_{\mathrm{O}(\mathbb{H}^n)} = \det$. Define

$$\mathrm{GSO}(\mathbb{H}^n) = \ker\nu = \{h\in \mathrm{GO}(\mathbb{H}^n) | \lambda(h)^n = \det(h)\}.$$

Assume that Q_n is the standard Siegel parabolic subgroup of H_n, i.e.,

$$Q_n = \left\{ \begin{pmatrix} A^{-1} & \\ & \lambda A^t \end{pmatrix} \begin{pmatrix} I & X \\ & I \end{pmatrix} \,\middle|\, A\in \mathrm{GL}_n(F),\ X\in \mathrm{Mat}_{n,n}(F) \text{ and } X + X^t = 0 \right\}$$

with modular character $|\det A|_F^{1-n}|\lambda|_F^{-n(n-1)/2}$. Then $Q_n\backslash H_n$ is a projective variety and a homogenous space equipped with H_n-action. Each point on $Q_n\backslash H_n$ corresponds to an isotropic subspace in $\mathbb{H}^n$ of dimension n. Set the degenerate principal series representation $I_{Q_n}^{H_n}(s)$ as

$$I_{Q_n}^{H_n}(s) = \left\{ f : H_n \to \mathbb{C} \mid f(xg) = \delta_{Q_n}(x)^{1/2+s/(n-1)} f(g) \text{ for } x\in Q_n,\ g\in H_n \right\}.$$

Let W_r be the symplectic space with a symplectic similitude group $\mathrm{GSp}(W_r)$. Set $\mathbf{1}_W$ to be the trivial representation of $\mathrm{GSp}(W_r)$. Then the big theta lift $\Theta_r(\mathbf{1}_W)$ to H_n of the trivial representation $\mathbf{1}_W$ is isomorphic to a subrepresentation of $I_{Q_n}^{H_n}(s_0)$, where

$$s_0 = r - \tfrac{1}{2}(n-1).$$

The image of $\Theta_r(\mathbf{1}_W)$ in $I_{Q_n}^{H_n}(s_0)$ is denoted by $R_r(\mathbf{1})$, i.e.,

$$\Theta_r(\mathbf{1}_W) = R_r(\mathbf{1}) \subset I_{Q_n}^{H_n}(s_0).$$

Let us come back to the GSp_4-cases. Assume that $r = 2$ and $n = 4$.

Proposition 4.3.1. *There is an exact sequence of H_4-modules*

$$0 \longrightarrow R_2(\mathbf{1}) \longrightarrow I_{Q_4}^{H_4}(\tfrac{1}{2}) \longrightarrow R_1(\mathbf{1})\otimes\nu \longrightarrow 0.$$

Proof. Note that $R_2(\mathbf{1})|_{\mathrm{O}_{4,4}(F)}$ is isomorphic to the big theta lift of the trivial representation $\mathbf{1}_W$ from $\mathrm{Sp}_4(F)$ to $\mathrm{O}_{4,4}(F)$, and similarly for the big theta lift $R_1(\mathbf{1})$. There is only one orbit for the double coset

$$Q_4\backslash H_4/\mathrm{O}_{4,4}(F) = (Q_4\cap \mathrm{O}_{4,4}(F))\backslash \mathrm{O}_{4,4}(F)/\mathrm{O}_{4,4}(F).$$

Applying Mackey theory, we have $I_{Q_4}^{H_4}(\frac{1}{2})|_{O_{4,4}(F)} \cong I_{Q_4\cap O_{4,4}(F)}^{O_{4,4}(F)}(\frac{1}{2})$. Then the sequence is still the same when restricted to the orthogonal group $O_{4,4}(F)$. The sequence is exact when restricted to the orthogonal group $O_{4,4}(F)$ due to the structure of degenerate principal series (see [Gan and Ichino 2014, Proposition 7.2]). By the construction of the extended Weil representation, the sequence is exact as H-modules. □

Similarly, let $P_4 = M_4N_4$ be the Siegel parabolic subgroup of $\mathrm{GSp}(W_4) = \mathrm{GSp}_8(F)$ where $M_4 \cong \mathrm{GL}_1(F) \times \mathrm{GL}_4(F)$ is the Levi part of the parabolic subgroup. Let $\mathcal{I}(s)$ be the degenerate normalized induced representation of $\mathrm{GSp}_8(F)$ associated to P_4, i.e.,

$$\mathcal{I}(s) = \left\{ f : \mathrm{GSp}_8(F) \to \mathbb{C} \mid f(pg) = \delta_{P_4}(p)^{(1/2)+(s/5)} f(g) \text{ for } p \in P_4,\, g \in \mathrm{GSp}_8(F) \right\}.$$

Then we have:

Proposition 4.3.2. *There is an exact sequence of* $\mathrm{GSp}_8(F)$*-modules*

$$0 \longrightarrow R^{3,3}(\mathbf{1}) \longrightarrow \mathcal{I}(\tfrac{1}{2}) \longrightarrow R^{4,0}(\mathbf{1}) \longrightarrow 0,$$

where $\mathcal{I}(s)$ *is the degenerate normalized induced representation of* $\mathrm{GSp}_8(F)$ *and* $R^{3,3}(\mathbf{1})$ (*resp.* $R^{4,0}(\mathbf{1})$) *is the big theta lift to* $\mathrm{GSp}_8(F)$ *of the trivial representation of* $\mathrm{GO}_{3,3}(F)$ (*resp.* $\mathrm{GO}_{4,0}(F)$).

Now we use Mackey theory to study $I_{Q_4}^{H_4}(\frac{1}{2})|_{\mathrm{GO}_{2,2}(E)^\natural}$ which involves the computation for the double coset $Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural$. Denote $X_4 = Q_4\backslash H_4$ as the projective variety.

4C1. *Double cosets.* Now let us consider the double coset

$$Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural.$$

Assume that $V = \mathbb{H}_E^2$ with basis $\{e_i, e_j'\}_{1\le i,j\le 2}$ and $\langle e_i, e_j'\rangle = \delta_{ij}$. Fix the basis

$$\{e_1, \sqrt{d}e_1, e_2, \sqrt{d}e_2, e_1', e_1'/\sqrt{d}, e_2', e_2'/\sqrt{d}\}$$

for $V_F = \mathrm{Res}_{E/F} V$. The inner product $\langle\langle -, -\rangle\rangle$ on V_F is given by

$$\langle\langle x, y\rangle\rangle := \tfrac{1}{2}\,\mathrm{tr}_{E/F}(\langle x, y\rangle)$$

for $x, y \in V$. Let us fix an embedding $i : \mathrm{GO}_{2,2}(E)^\natural \to \mathrm{GSO}_{4,4}(F)$.

The double coset decomposition for the case at hand can be obtained from more general case. Assume that $\boldsymbol{V}$ is a symplectic space or a split quadratic space over E of dimension $2n$, with a nondegenerate bilinear form $B : \boldsymbol{V} \times \boldsymbol{V} \to E$. Let $U(\boldsymbol{V})$ be the isometry group, i.e.,

$$U(\boldsymbol{V}) = \left\{ g \in \mathrm{GL}(\boldsymbol{V}) \mid B(gx, gy) = B(x, y) \text{ for all } x, y \in \boldsymbol{V} \right\}$$

which is a symplectic group or an orthogonal group. Then $\mathrm{Res}_{E/F}\boldsymbol{V}$ is a vector space over F of dimension $4n$ with a nondegenerate bilinear form $\frac{1}{2}\,\mathrm{tr}_{E/F} \circ B$.

Lemma 4.3.3. *Let P be a Siegel parabolic subgroup of $U(\mathrm{Res}_{E/F}\boldsymbol{V})$. Then each point in the homogeneous space $X = P\backslash U(\mathrm{Res}_{E/F}\boldsymbol{V})$ corresponds to a $2n$-dimensional maximal isotropic subspace in $\mathrm{Res}_{E/F}\boldsymbol{V}$ and the finite double cosets $X/U(\boldsymbol{V})$ can be parametrized by a pair*

$$(\dim_E E\cdot L,\, B_L),$$

where $L\subset \mathrm{Res}_{E/F}\boldsymbol{V}$ is a maximal isotropic subspace with respect to the inner product $\langle\langle -,-\rangle\rangle$ over F,

$$E\cdot L := \{e\cdot x \mid e\in E, x\in L\}$$

is a linear E-subspace in $\boldsymbol{V}$ and

$$B_L : L/L_0\times L/L_0 \to \sqrt{d}\cdot F \tag{4-1}$$

is a nondegenerate bilinear form inherited from V, where

$$L_0=\{x\in L : B(x,y)=0 \text{ for all } y\in L\}.$$

Moreover, if $L=L_0$, then L lies in the closed orbit. If $L_0=0$, then L lies in the open orbit.

Proof. Under a suitable basis for L, the bilinear form for $B|_L$ corresponds to a matrix $\sqrt{d}\cdot T$, where $T\in M_{2n}(F)$. Moreover, we can choose T such that it is a diagonal (resp. an anti-diagonal) matrix if $B(x,y)=B(y,x)$ (resp. $B(y,x)=-B(x,y)$). Then

$$\dim_E E\cdot L = n+\tfrac{1}{2}\cdot \mathrm{rank}(T),$$

which is invariant under $U(\boldsymbol{V})$-action. The bilinear form B_L corresponds to a matrix $\sqrt{d}\cdot T'$, i.e.,

$$T=\begin{pmatrix}0&0&0\\0&T'&0\\0&0&0\end{pmatrix}$$

where T' is invertible and $\mathrm{rank}(T)=\mathrm{rank}(T')$.

Assume that there are two isotropic subspaces L_1 and L_2 satisfying

$$\dim_E E\cdot L_1=\dim_E E\cdot L_2 = l \quad\text{and}\quad B_{L_1}\cong B_{L_2}.$$

This means that there exists $g\in \mathrm{GL}_l(E)$ such that $g : E\cdot L_1\to E\cdot L_2$ satisfying

$$B_{L_1}(x,y)=B_{L_2}(gx,gy).$$

It is easy to lift g to $g_E\in U(\boldsymbol{V})$ such that $g_E L_1 = L_2$.

In fact, $g=\left(\begin{smallmatrix}g_1&0\\0&g_2\end{smallmatrix}\right)$ lies in a subgroup of $\mathrm{GL}_l(E)$, which can be regarded as a Levi subgroup of $U(\boldsymbol{V})$, and

$$B_L(gx,gy)=B_L(g_2x',g_2y')$$

when $x-x', y-y'\in L_0$. Then $g_E=\begin{pmatrix}g_1&&\\&g_2&\\&&g_1^*\end{pmatrix}\in U(\boldsymbol{V})$, where g_1^* depends on g_1 and $\boldsymbol{V}$. □

Remark 4.3.4. There is only one closed orbit in the double coset $P\backslash U(\mathrm{Res}_{E/F}\boldsymbol{V})/U(\boldsymbol{V})$. When $T=0$, the subspace $E\cdot L$ is the maximal isotropic subspace of $\boldsymbol{V}$ and so $U(\boldsymbol{V})$ acts on the subvariety $\{L : L=L_0\}\subset X$ transitively.

Consider the double coset decomposition of

$$Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural.$$

There are several F-rational orbits in $Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural$. By Lemma 4.3.3, there are two invariants for the orbit $\mathrm{GO}_{2,2}(E)^\natural \cdot L$:

- the dimension $\dim_E(E\cdot L)$, and
- the bilinear form B_L (defined in (4-1)) up to scaling in $F^\times$.

By the classification of 4-dimensional quadratic spaces over F, there are 4 elements lying in the kernel

$$\ker\{H^1(F,\mathrm{O}_4)\to H^1(E,\mathrm{O}_4)\},$$

which are

- the split quaternion algebra $\mathrm{Mat}_{2,2}(F)$ with $q(v)=\det(v)$ for $v\in\mathrm{Mat}_{2,2}(F)$,
- the quaternion division algebra $D(F)$ with the norm map $N_{D/F}$,
- the nonsplit 4-dimensional quadratic space $V_3=E\oplus\mathbb{H}$ with $q(e,x,y)=N_{E/F}(e)-xy$, and
- $V_4=\epsilon V_3$ with $\epsilon\in F^\times\setminus N_{E/F}(E^\times)$.

However, we consider the double coset

$$Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural$$

for the similitude groups and observe that V_3 and V_4 are in the same orbit in $Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural$. More precisely, $\mathrm{Mat}_{2,2}(F)$, $D(F)$ and $E\oplus\mathbb{H}$ are three representatives in the union of the open orbits $\mathrm{GO}_{2,2}(E)^\natural\cdot L$ in $X_4/\mathrm{GO}_{2,2}(E)^\natural$.

Proposition 4.3.5. *Pick a point $L\in X_4/\mathrm{GO}_{2,2}(E)^\natural$ lying in an open orbit. Then the stabilizer of L in $\mathrm{GO}_{2,2}(E)^\natural$ is isomorphic to the similitude group $\mathrm{GO}(L)$.*

Proof. For $g\in\mathrm{GO}_{2,2}(E)^\natural$ with $g(L)=L$, we have

$$\langle gl_1, gl_2\rangle=\lambda(g)\cdot\langle l_1,l_2\rangle$$

and so $\langle\langle gl_1,gl_2\rangle\rangle=\lambda(g)\cdot\langle\langle l_1,l_2\rangle\rangle$. This means $g\in\mathrm{GO}(L)$. Conversely, if $h\in\mathrm{GO}(L,(1/\sqrt{d})q_E|_L)$, set

$$h_E: x\otimes e\mapsto h(x)\otimes e$$

for $x\otimes e\in L\otimes E\cong L\cdot E=V$. Then $h_E(L)=L$ and

$$\langle h_E(x_1\otimes e_1),h_E(x_2\otimes e_2)\rangle=e_1e_2\lambda(h)\langle\langle x_1,x_2\rangle\rangle=\lambda(h)\langle x_1\otimes e_1,x_2\otimes e_2\rangle,$$

i.e., $h_E\in\mathrm{GO}_{2,2}(E)^\natural$. Then we get a bijection between the similitude orthogonal group $\mathrm{GO}(L)$ and the stabilizer of L in $\mathrm{GO}_{2,2}(E)^\natural$. Observe that the map $h\mapsto h_E$ is a group homomorphism. Then $\mathrm{GO}(L)$ is isomorphic to the stabilizer of L via the map $h\mapsto h_E$. □

There are three F-rational open orbits $\mathrm{GO}_{2,2}(E)^\natural\cdot L$ where L represents one of $\mathrm{Mat}_{2,2}(F)$, $D(F)$ or $E\oplus\mathbb{H}$, whose stabilizers are $\mathrm{GO}_{2,2}(F)$, $\mathrm{GO}_{4,0}(F)$ and $\mathrm{GO}_{3,1}(F)$ respectively. There is one closed orbit $\mathrm{GO}_{2,2}(E)^\natural\cdot L$ which has stabilizer

$$\mathrm{GO}_{2,2}(E)^\natural\cap Q_4=:Q^\natural\cong\left\{\begin{pmatrix}A^{-1}&*\\0&\lambda A^t\end{pmatrix}\;\middle|\;A\in\mathrm{GL}_2(E),\lambda\in F^\times\right\}.$$

There are two intermediate orbits with representatives L_1, L_2 and $\dim_E(E\cdot L_i)=3$. The stabilizers are isomorphic to

$$(\mathrm{GL}_1(E)\times\mathrm{GO}_{1,1}(F))\cdot\mathrm{Mat}_{2,2}(F)\quad\text{and}\quad(\mathrm{GL}_1(E)\times\mathrm{GO}(\mathcal{V}_E))\cdot\mathrm{Mat}_{2,2}(F),$$

where $\mathcal{V}_E$ is the 2-dimensional quadratic space over F whose discriminant algebra is E.

Remark 4.3.6. For $(g,t)\in\mathrm{GL}_2(E)\times F^\times$, we set

$$\beta((g,t))=(g,\sigma(g)\cdot t)\in\mathrm{GL}_2(E)\times\mathrm{GL}_2(E).$$

Then $\beta:\mathrm{GSO}_{3,1}(F)\to\mathrm{GSO}_{2,2}(E)^\natural$ is an embedding due to the exact sequences

$$\begin{array}{ccccccccc}1&\longrightarrow&E^\times&\xrightarrow{i_1}&\mathrm{GL}_2(E)\times F^\times&\longrightarrow&\mathrm{GSO}_{3,1}(F)&\longrightarrow&1\\&&\|&&\downarrow\beta&&\downarrow&&\\1&\longrightarrow&E^\times&\xrightarrow{i_2}&\mathrm{GL}_2(E)\times\mathrm{GL}_2(E)&\longrightarrow&\mathrm{GSO}_{2,2}(E)&\longrightarrow&1\end{array}$$

where $i_1(e)=(e,N_{E/F}(e)^{-1})$ and $i_2(e)=(e,e^{-1})$ for $e\in E^\times$.

There are several orbits for $X_4/\mathrm{GO}_{2,2}(E)^\natural$. By Mackey theory, there is a decreasing filtration of $\mathrm{GO}_{2,2}(E)^\natural$-modules for $I_{Q_4}^{H_4}(s)|_{\mathrm{GO}_{2,2}(E)^\natural}$.

4C2. *Filtration.* Consider the filtration

$$I_{Q_4}^{H_4}(s)=I_2(s)\supset I_1(s)\supset I_0(s)\supset 0$$

of $I_{Q_4}^{H_4}(s)|_{\mathrm{GO}_{2,2}(E)^\natural}$ with a sequence of subquotients

$$\begin{aligned}I_0(s)&=\mathrm{ind}_{\mathrm{GO}_{2,2}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus\mathrm{ind}_{\mathrm{GO}_{4,0}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus\mathrm{ind}_{\mathrm{GO}_{3,1}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C},\\I_2(s)/I_1(s)&\cong\mathrm{ind}_{Q^\natural}^{\mathrm{GO}_{2,2}(E)^\natural}\delta_{Q^\natural}^{s+1},\end{aligned}$$

where $Q^\natural$ is the Siegel parabolic subgroup of $\mathrm{GO}_{2,2}(E)^\natural$ with modular character $\delta_{Q^\natural}$ and

$$I_1(s)/I_0(s)\cong\mathrm{ind}_{(\mathrm{GL}_1(E)\times\mathrm{GO}_{1,1}(F))\cdot N}^{\mathrm{GO}_{2,2}(E)^\natural}\delta_Q^{\frac{1}{2}+\frac{s}{3}}\delta_1^{-\frac{1}{2}}\oplus\mathrm{ind}_{Q'}^{\mathrm{GO}_{2,2}(E)^\natural}\delta_Q^{\frac{1}{2}+\frac{s}{3}}\delta_2^{-\frac{1}{2}}$$

where $Q'=(\mathrm{GL}_1(E)\times\mathrm{GO}(\mathcal{V}_E))\cdot N$, $N\cong\mathrm{Mat}_{2,2}(F)$ and

$$\delta_i(t,h)=|N_{E/F}(t^2)\cdot\lambda_V(h)^{-2}|_F$$

for $t\in\mathrm{GL}_1(E)$ and $h\in\mathrm{GO}_{1,1}(F)$ or $\mathrm{GO}(\mathcal{V}_E)$, where $\mathcal{V}_E$ is the nonsplit 2-dimensional quadratic space.

Remark 4.3.7. We would like to highlight the fact that on the open orbits related to $I_0(s)$, the group embedding $\mathrm{GO}_{2,2}(F) \hookrightarrow \mathrm{GO}_{2,2}(E)^\natural$ (and similarly for the other two group embeddings) is not induced from the geometric embedding $i : \mathrm{GO}(L) \hookrightarrow \mathrm{GO}(L \otimes_F E)$, but the composite map $\mathrm{Ad}_{h^\delta} \circ i$ of the adjoint map Ad_{h^δ} and the geometric embedding i where

$$h^\delta = \begin{pmatrix} \sqrt{d} & \\ & 1 \end{pmatrix} \in \mathrm{GO}(2,2)(E).$$

However, it does not affect the results when we consider the distinction problems for the similitude groups. In Section 4D, we will show that the results on the open orbits determine the distinction problems $\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}\big(\tfrac{1}{2}\big), \Sigma\big)$ when Σ is a generic representation.

Recall that

$$\mathrm{GSp}_4(E)^\natural = \{g \in \mathrm{GSp}_4(E) \mid \lambda_W(g) \in F^\times\}.$$

When we deal with the case

$$\mathrm{Ind}_{P_4}^{\mathrm{GSp}_8(F)} \delta_{P_4}^{s/5}|_{\mathrm{GSp}_4(E)^\natural},$$

where P_4 is the Siegel parabolic subgroup of $\mathrm{GSp}_8(F)$ with modular character δ_{P_4}, the above results still hold. More precisely, set

$$\mathcal{I}(s) = \big\{ f : \mathrm{GSp}_8(F) \to \mathbb{C} \mid f(xg) = \delta_{P_4}(x)^{(1/2)+(s/5)} f(g) \text{ for } x \in P_4,\, g \in \mathrm{GSp}_8(F) \big\}.$$

There is a filtration

$$\mathcal{I}_0(s) \subset \mathcal{I}_1(s) \subset \mathcal{I}_2(s) = \mathcal{I}(s)|_{\mathrm{GSp}_4(E)^\natural}$$

of $\mathcal{I}(s)|_{\mathrm{GSp}_4(E)^\natural}$ such that

- $\mathcal{I}_0(s) \cong \mathrm{ind}_{\mathrm{GSp}_4(F)}^{\mathrm{GSp}_4(E)^\natural} \mathbb{C}$,
- $\mathcal{I}_1(s)/\mathcal{I}_0(s) \cong \mathrm{ind}_{M'N'}^{\mathrm{GSp}_4(E)^\natural} \delta_{P_4}^{(1/2)+(s/5)} \delta_{M'N'}^{-1/2}$ and
- $\mathcal{I}_2(s)/\mathcal{I}_1(s) \cong \mathrm{ind}_{P^\natural}^{\mathrm{GSp}_4(E)^\natural} \delta_{P^\natural}^{(s+1)/3}$,

where $P^\natural$ is the Siegel parabolic subgroup of $\mathrm{GSp}_4(E)^\natural$,

$$M' \cong \mathrm{GL}_1(E) \times \mathrm{GL}_2(F), \qquad N' \cong \mathrm{Mat}_{1,1}(E) \oplus \mathrm{Mat}_{2,2}(F)$$

and

$$\delta_{M'N'}(t, g) = |N_{E/F}(t)^4 \cdot \lambda_W(g)^{-4}|_F$$

for $(t, g) \in \mathrm{GL}_1(E) \times \mathrm{GL}_2(F)$. Here the group embedding $\mathrm{GSp}_4(F) \hookrightarrow \mathrm{GSp}_4(E)^\natural$ in $\mathcal{I}_0(s)$ is the composition map $\mathrm{Ad}_{g^\delta} \circ i'$ where $i' : \mathrm{GSp}(W_2) \hookrightarrow \mathrm{GSp}(W_2 \otimes_F E)$ is the geometric embedding and

$$g^\delta = \begin{pmatrix} \sqrt{d} & \\ & 1 \end{pmatrix} \in \mathrm{GSp}_4(E).$$

4D. ***The distinction problem for*** $\mathbf{GSp_4}$**.** Let us recall what we have obtained. Let $\tau \in \mathrm{Irr}(\mathrm{GSp}_4(E))$. Since $\tau|_{\mathrm{Sp}_4(E)}$ is multiplicity-free due to [Adler and Prasad 2006, Theorem 1.4], $\tau|_{\mathrm{GSp}_4(E)^\natural}$ is multiplicity-free. Assume that $\tau = \theta(\pi_1 \boxtimes \pi_2)$ participates in the theta correspondence with $\mathrm{GSO}_{2,2}(E)$. Then the see-saw identity implies that

$$\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) \subset \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\Theta_2(\Sigma), \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_2(\mathbf{1}), \Sigma),$$

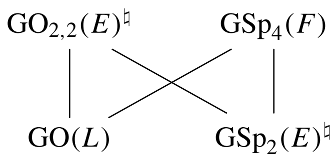
where $R_2(\mathbf{1})$ is the image of the big theta lift to H_4 of the trivial representation of $\mathrm{GSp}_4(F)$ in $I_{Q_4}^{H_4}\left(\frac{1}{2}\right)$ and Σ is the irreducible representation of $\mathrm{GO}_{2,2}(E)$ such that $\tau = \theta(\Sigma)$. In fact, if $\pi_1 \not\cong \pi_2$, then $\Sigma = \mathrm{Ind}_{\mathrm{GSO}_{2,2}(E)}^{\mathrm{GO}_{2,2}(E)}(\pi_1 \boxtimes \pi_2)$. If $\pi_1 \cong \pi_2$, then there are two extensions to $\mathrm{GO}_{2,2}(E)$ of $\pi_1 \boxtimes \pi_2$. The representation Σ is the unique extension of $\pi_1 \boxtimes \pi_1$ which participates into the theta correspondence with $\mathrm{GSp}_4(E)$, denoted by $(\pi_1 \boxtimes \pi_1)^+$.

Lemma 4.4.1. *Assume that* $\pi_1 \boxtimes \pi_2 \in \mathrm{Irr}(\mathrm{GSO}_{2,2}(E))$. *Let* $\Sigma \in \mathrm{Irr}(\mathrm{GO}_{2,2}(E))$ *such that* $\Sigma|_{\mathrm{GSO}_{2,2}(E)} \supset \pi_1 \boxtimes \pi_2$ *and* Σ *has a nonzero theta lift to* $\mathrm{GSp}_4(E)$. *Then*

$$\dim \mathrm{Hom}_{\mathrm{GO}(L)}(\Sigma, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSO}(L)}(\pi_1 \boxtimes \pi_2, \mathbb{C}),$$

where $\mathrm{GO}(L) \hookrightarrow \mathrm{GO}(L \otimes_F E) = \mathrm{GO}_{2,2}(E)$ *and the* 4*-dimensional quadratic space* L *is one of* $\mathrm{Mat}_{2,2}(F)$, $D(F)$ *or* $E \oplus \mathbb{H}_F$.

Proof. If $\pi_1 \neq \pi_2$, then it follows from Frobenius reciprocity. If $\pi_1 = \pi_2$ and L is either $\mathrm{Mat}_{2,2}(F)$ or $D(F)$, then we consider the see-saw diagram



where $\mathrm{GSp}_2(E)^\natural = \{g \in \mathrm{GSp}_2(E) | \lambda_W(g) \in F^\times\}$. We have

$$\mathrm{Hom}_{\mathrm{GO}(L)}(\Sigma \otimes \nu, \mathbb{C}) = \mathrm{Hom}_{\mathrm{GO}(L)}(\Sigma, \nu) = \mathrm{Hom}_{\mathrm{GSp}_2(E)^\natural}(\Theta_2(\nu), \pi_1) = 0,$$

because the big theta lift $\Theta_2(\nu)$ to $\mathrm{GSp}_4(F)$ is zero by the conservation relation. If $\pi_1 = \pi_2$ and L is $E \oplus \mathbb{H}_F$, then

$$\mathrm{Hom}_{\mathrm{GO}(L)}(\Sigma, \nu) = \mathrm{Hom}_{\mathrm{GSp}_2(E)^\natural}(\mathrm{Ind}_{\mathrm{GSp}_4(F)^+}^{\mathrm{GSp}_4(F)} \Theta_2(\nu), \mathbb{C}) = 0.$$

(See Remark 4.2.2.) Hence

$$\mathrm{Hom}_{\mathrm{GSO}(L)}(\pi_1 \boxtimes \pi_2, \mathbb{C}) = \mathrm{Hom}_{\mathrm{GO}(L)}(\Sigma \oplus (\Sigma \otimes \nu), \mathbb{C}) = \mathrm{Hom}_{\mathrm{GO}(L)}(\Sigma, \mathbb{C}).$$

This finishes the proof. □

Lemma 4.4.2. *Given a representation* $\tau \in \mathrm{Irr}(\mathrm{GSp}_4(E))$ *with* $\omega_\tau|_{F^\times} = \mathbf{1}$, *we have*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau^g, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau^\vee, \mathbb{C}),$$

where $\tau^g(x) = \tau(gxg^{-1})$ *for* $g \in \mathrm{GSp}_4(E)$.

Proof. Note that $\tau^g \cong \tau$ and so $\dim \operatorname{Hom}_{\mathrm{GSp}_4(F)}(\tau^g, \mathbb{C}) = \dim \operatorname{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C})$. Since $\omega_\tau|_{F^\times}$ is trivial and $\tau^\vee \cong \tau \otimes \omega_\tau^{-1}$, we have

$$\operatorname{Hom}_{\mathrm{GSp}_4(F)}(\tau^\vee, \mathbb{C}) = \operatorname{Hom}_{\mathrm{GSp}_4(F)}(\tau \otimes \omega_\tau^{-1}, \mathbb{C}) = \operatorname{Hom}_{\mathrm{GSp}_4(F)}(\tau, \omega_\tau|_{F^\times}) = \operatorname{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}). \quad \square$$

Remark 4.4.3. We have a similar statement for the group $\mathrm{GO}(V)$ when V is a 4-dimensional split quadratic space over E.

There is another key input for the GL_4-distinction problems in our proof of Theorem 1.1.

Theorem 4.4.4 [Matringe 2011, Theorem 5.2]. *Given a generic representation π of $\mathrm{GL}_n(E)$ with a Langlands parameter $\phi_\pi = \triangle_1 \oplus \triangle_2 \oplus \cdots \oplus \triangle_t$ with $\triangle_i : WD_E \to \mathrm{GL}_{n_i}(\mathbb{C})$ irreducible and $\sum_{i=1}^t n_i = n$, then π is $\mathrm{GL}_n(F)$-distinguished if and only if there is a reordering of $\triangle_i'$s and an integer r between 1 and $\frac{1}{2}t$ such that $\triangle_{i+1}^\sigma = \triangle_i^\vee$ for $i = 1, 3, \ldots, 2r-1$ and $\triangle_i$ is conjugate-orthogonal for $i > 2r$.*

Lemma 4.4.5. *Let π be a square-integrable representation of $\mathrm{GL}_2(E)$. Then π is $\mathrm{GL}_2(F)$-distinguished if and only if π is $D^\times(F)$-distinguished. If $\pi = \pi(\chi^{-1}, \chi^\sigma)$, then π is both $\mathrm{GL}_2(F)$-distinguished and $D^\times(F)$-distinguished. Let $\pi_0 = \pi(\chi_1, \chi_2)$ with $\chi_1 \neq \chi_2$, $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$ be an irreducible smooth representation of $\mathrm{GL}_2(E)$. Then π_0 is $\mathrm{GL}_2(F)$-distinguished but not $D^\times(F)$-distinguished. These exhaust all generic $\mathrm{GL}_2(F)$-distinguished representations of $\mathrm{GL}_2(E)$.*

Proof. If π is square-integrable, then it follows from [Prasad 1992, Theorem C]. Let $\pi_0 = \pi(\chi_1, \chi_2)$. By Mackey theory, we know that

$$\dim \operatorname{Hom}_{D^\times(F)}(\pi_0, \mathbb{C}) = \dim \operatorname{Hom}_{E^\times}(\chi_1\chi_2^\sigma, \mathbb{C}) = \begin{cases} 1 & \text{if } \chi_1\chi_2^\sigma = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\chi_1 \neq \chi_2$ and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$, then $\chi_1\chi_2^\sigma \neq \mathbf{1}$. Thus π_0 is not $D^\times(F)$-distinguished. Since the Langlands parameter $\phi_\pi = \chi^{-1} \oplus \chi^\sigma$ (resp. ϕ_{π_0}) is conjugate-orthogonal in the sense of [Gan et al. 2012, §3], π (resp. π_0) is $\mathrm{GL}_2(F)$-distinguished due to [Gan and Raghuram 2013, Theorem 6.2] or Theorem 4.4.4. The last claim follows from Theorem 4.4.4. $\square$

Lemma 4.4.6. *Let π be an essentially discrete series representation of $\mathrm{GL}_2(E)$. Let $\Pi = J_P(\pi|-|_E, \pi)$ be the nongeneric representation of $\mathrm{GL}_4(E)$. Then the following statements are equivalent*:

(i) *Π is either $\mathrm{GL}_4(F)$-distinguished or $(\mathrm{GL}_4(F), \omega_{E/F})$-distinguished.*

(ii) $\Pi^\vee \cong \Pi^\sigma$.

(iii) *$I_P(\pi|-|_E, \pi)$ is both $\mathrm{GL}_4(F)$-distinguished and $(\mathrm{GL}_4(F), \omega)$-distinguished.*

Proof. See [Gurevich et al. 2018, Theorem 6.5]. $\square$

4D1. *The Langlands correspondence for* GSp_4. In this part, we will recall the Langlands correspondence for GSp_4 which has been set up in [Gan and Takeda 2011a].

Let $\Pi(\mathrm{GSp}_4)$ be the set of (equivalence classes of) irreducible smooth representation of $\mathrm{GSp}_4(F)$. Let $\operatorname{Hom}(WD_F, \mathrm{GSp}_4(\mathbb{C}))$ be the set of (equivalence classes of) admissible homomorphisms

$$WD_F \to \mathrm{GSp}_4(\mathbb{C}).$$

Theorem 4.4.7 (Gan–Takeda). *There is a surjective finite to one map*

$$L:\Pi(\mathrm{GSp}_4)\to \mathrm{Hom}(WD_F,\mathrm{GSp}_4(\mathbb{C}))$$

with the following properties:

(i) *τ is a (essentially) discrete series representation of $\mathrm{GSp}_4(F)$ if and only if its L-parameter $\phi_\tau=L(\tau)$ does not factor through any proper Levi subgroup of $\mathrm{GSp}_4(\mathbb{C})$.*

(ii) *For an L-parameter $\phi\in\mathrm{Hom}(WD_F,\mathrm{GSp}_4(\mathbb{C}))$, its fiber Π_ϕ can be naturally parametrized by the set of irreducible characters of the component group*

$$\pi_0(Z(\mathrm{Im}(\phi))/Z_{\mathrm{GSp}_4(\mathbb{C})}).$$

This component group is either trivial or equal to $\mathbb{Z}/2\mathbb{Z}$. When it is $\mathbb{Z}/2\mathbb{Z}$, exactly one of the two representations in Π_ϕ is generic and it is the one indexed by the trivial character of $\pi_0(Z(\mathrm{Im}(\phi))/Z_{\mathrm{GSp}_4(\mathbb{C})})$.

(iii) *The similitude character $\mathrm{sim}(\phi_\tau)$ of ϕ_τ equals the central character ω_τ of τ. Here $\mathrm{sim}:\mathrm{GSp}_4(\mathbb{C})\to\mathbb{C}^\times$ is the similitude character of $\mathrm{GSp}_4(\mathbb{C})$.*

(iv) *The L-parameter of $\tau\otimes(\chi\circ\lambda_W)$ is equal to $\phi_\tau\otimes\chi$. Here $\lambda_W:\mathrm{GSp}_4(F)\to F^\times$ is the similitude character of $\mathrm{GSp}_4(F)$, and we have regarded χ as both a character of $F^\times$ and a character W_F by local class field theory.*

Definition 4.4.8. An irreducible representation τ of $\mathrm{GSp}_4(E)^\natural$ occurs on the boundary of $\mathcal{I}(s)$ if

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(\mathcal{I}_{i+1}(s)/\mathcal{I}_i(s),\tau)\neq 0\quad \text{for } i=0 \text{ or } 1.$$

In [Lu 2017a], we have verified the Prasad conjecture for GSp_4 when τ is a tempered representation by showing that τ does not occur on the boundary of $\mathcal{I}(\frac{1}{2})$. After discussing with Dmitry Gourevitch, we realized that [Gourevitch et al. 2019, Proposition 4.9] can imply the Prasad conjecture for GSp_4 when the L-packet Π_{ϕ_τ} is generic. Thus we will give a slightly different proof of Theorem 1.1 from the one in [Lu 2017a].

We repeat the statements of Theorem 1.1 as below.

Theorem 4.4.9. *Assume that $\tau\in\mathrm{Irr}(\mathrm{GSp}_4(E))$ with a central character ω_τ satisfying $\omega_\tau|_{F^\times}=\mathbf{1}$.*

(i) *If $\tau=\theta(\Sigma)$ is an irreducible representation of $\mathrm{GSp}_4(E)$, where Σ is an irreducible representation of $\mathrm{GO}_{4,0}(E)$, then τ is not $\mathrm{GSp}_4(F)$-distinguished.*

(ii) *Suppose $\Sigma=(\pi_1\boxtimes\pi_1)^+$ is an irreducible representation of $\mathrm{GO}_{2,2}(E)$ and $\Sigma=\mathrm{Ind}_{\mathrm{GSO}_{2,2}(E)}^{\mathrm{GO}_{2,2}(E)}(\pi_1\boxtimes\pi_2)$ if $\pi_1\neq\pi_2$. If $\tau=\theta(\Sigma)$ is generic, then*

$$\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau,\mathbb{C})=\begin{cases}2 & \text{if } \pi_i\not\cong\pi_0 \text{ are both } \mathrm{GL}_2(F)\text{-distinguished},\\ 1 & \text{if } \pi_1\not\cong\pi_2 \text{ but } \pi_1^\sigma\cong\pi_2^\vee,\\ 1 & \text{if } \pi_1\cong\pi_2 \text{ is } \mathrm{GL}_2(F)\text{-distinguished but not } (\mathrm{GL}_2(F),\omega_{E/F})\text{-distinguished},\\ 1 & \text{if } \pi_2 \text{ is } \mathrm{GL}_2(F)\text{-distinguished and } \pi_1\cong\pi_0,\\ 0 & \text{otherwise}.\end{cases}$$

Here $\pi_0 = \pi(\chi_1, \chi_2)$ *with* $\chi_1 \neq \chi_2$, $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$.

(iii) *Assume that* τ *is not in case* (i) *or* (ii), *so that* $\tau = \theta(\Pi \boxtimes \chi)$, *where* $\Pi \boxtimes \chi$ *is a representation of* $\mathrm{GSO}_{3,3}(E)$. *If* τ *is generic, then*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = \begin{cases} 1 & \text{if } \phi_\Pi \text{ is conjugate-orthogonal}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) If Σ is a representation of $\mathrm{GO}_{4,0}(E)$, then $\tau = \theta(\Sigma) = \Theta(\Sigma)$ and

$$\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\Theta(\Sigma), \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}(\Theta_{W,D',\psi}(\mathbf{1}), \Sigma^+),$$

where $D' = \mathrm{Res}_{E/F} D_E = D(F) \oplus \mathbb{H}^2$ is the 8-dimensional quadratic vector space over F with determinant 1 and Hasse invariant -1 due to Lemma 4.2.3 and $\Theta_{W,D',\psi}(\mathbf{1})$ is the big theta lift to $\mathrm{GO}(V')$ of the trivial representation $\mathbf{1}$. Note that the first occurrence of the trivial representation is $\dim_F W = 4$ in the Witt tower $D \oplus \mathbb{H}^r$, which is bigger than 2. Thus $\Theta_{W,D',\psi}(\mathbf{1}) = 0$. Hence

$$\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\Theta(\Sigma), \mathbb{C}) = 0$$

and so $\tau = \theta(\Sigma)$ is not $\mathrm{GSp}_4(F)$-distinguished.

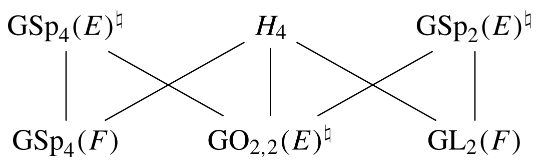
(ii) By Proposition 4.3.1, there is an exact sequence

$$0 \longrightarrow R_2(\mathbf{1}) \longrightarrow I_{Q_4}^{H_4}\left(\tfrac{1}{2}\right) \longrightarrow \nu \otimes R_1(\mathbf{1}) \longrightarrow 0 \tag{4-2}$$

of H_4-representations, where $R_i(\mathbf{1})$ is the big theta lift to H_4 of the trivial representation $\mathbf{1}$ of $\mathrm{GSp}_{2i}(F)$. We take the right exact contravariant functor $\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(-, \Sigma)$ with respect to (4-2) and get a short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_1(\mathbf{1})) \otimes \nu, \Sigma) \to \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}\left(\tfrac{1}{2}\right), \Sigma\big) \to \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_2(\mathbf{1}), \Sigma). \tag{4-3}$$

Consider the following double see-saw diagrams:



Note that $\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_2(\mathbf{1}), \Sigma) \cong \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C})$. There is a key observation due to Wee Teck Gan that $\mathrm{GO}_{2,2}(E)^\natural$ is a subgroup of $\mathrm{GSO}_{4,4}(F)$. One has

$$\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_1(\mathbf{1}) \otimes \nu, \Sigma) = \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_1(\mathbf{1}), \Sigma) \cong \mathrm{Hom}_{\mathrm{GSp}_2(F)}(\Theta_1(\Sigma), \mathbb{C}).$$

Here $\Theta_1(\Sigma)$ is the big theta lift to $\mathrm{GSp}_2(E)$ of Σ, which is zero unless $\pi_1 = \pi_2$. Then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) + \dim \mathrm{Hom}_{\mathrm{GSp}_2(F)}(\Theta_2(\Sigma), \mathbb{C}) \geq \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}\left(\tfrac{1}{2}\right), \Sigma\big). \tag{4-4}$$

Observe that $\mathrm{GO}_{2,2}(E)^\natural$ is the fixed point of a involution on H_4, which is given by the scalar matrix

$$h=\sqrt{d}\in \mathrm{GO}_{2,2}(E)^\natural\subset H_4$$

acting on H_4 by conjugation. Due to [Ólafsson 1987, Theorem 2.5], there exists a polynomial f on H_4 such that the complements of the open orbits in the double coset $Q_4\backslash H_4/\mathrm{GO}_{2,2}(E)^\natural$ is the zero set of f. Thanks to [Gourevitch et al. 2019, Proposition 4.9], the multiplicity $\dim\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}(\frac12),\Sigma\big)$ is at least $\dim\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_0(\frac12),\Sigma\big)$ where the submodule I_0 corresponds to the open orbits. More precisely,

$$I_0\big(\tfrac12\big)\cong \mathrm{ind}_{\mathrm{GO}_{4,0}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus \mathrm{ind}_{\mathrm{GO}_{2,2}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus \mathrm{ind}_{\mathrm{GO}_{3,1}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}$$

and

$$\dim\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}(\tfrac12),\Sigma\big)\ge \dim\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(\mathrm{ind}_{\mathrm{GO}_{4,0}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus \mathrm{ind}_{\mathrm{GO}_{2,2}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus \mathrm{ind}_{\mathrm{GO}_{3,1}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C},\Sigma\big). \quad (4\text{-}5)$$

Together with (4-4), we have

$$\begin{aligned}&\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GSp}_2(F)}(\Theta_2(\Sigma),\mathbb{C})\\ &\quad\ge \dim\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}(\tfrac12),\Sigma\big)\\ &\quad\ge \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(\mathrm{ind}_{\mathrm{GO}_{4,0}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus\mathrm{ind}_{\mathrm{GO}_{2,2}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C}\oplus\mathrm{ind}_{\mathrm{GO}_{3,1}(F)}^{\mathrm{GO}_{2,2}(E)^\natural}\mathbb{C},\Sigma\big)\\ &\quad=\dim\mathrm{Hom}_{\mathrm{GO}_{4,0}(F)}(\Sigma,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GO}_{2,2}(F)}(\Sigma,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GO}_{3,1}(F)}(\Sigma,\mathbb{C})\\ &\quad=\dim\mathrm{Hom}_{\mathrm{GSO}_{4,0}(F)}(\pi_1\boxtimes\pi_2,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GSO}_{2,2}(F)}(\pi_1\boxtimes\pi_2,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GSO}_{3,1}(F)}(\pi_1\boxtimes\pi_2,\mathbb{C}).\end{aligned}\quad (4\text{-}6)$$

The last equality of (4-6) holds due to Lemma 4.4.1, which also equals

$$\dim\mathrm{Hom}_{D^\times(F)}(\pi_1,\mathbb{C})\dim\mathrm{Hom}_{D^\times(F)}(\pi_2,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1,\mathbb{C})\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_2,\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi_1^\sigma,\pi_2^\vee).$$

In order to get the upper bound for the multiplicity $\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau,\mathbb{C})$, let us turn the table around. There is an exact sequence

$$0\longrightarrow R^{3,3}(\mathbf{1})\longrightarrow \mathcal{I}\big(\tfrac12\big)\longrightarrow R^{4,0}(\mathbf{1})\longrightarrow 0$$

of $\mathrm{GSp}_8(F)$-representations, where $\mathcal{I}(s)$ is the degenerate principal series of $\mathrm{GSp}_8(F)$ and $R^{m,n}(\mathbf{1})$ is the big theta lift to $\mathrm{GSp}_8(F)$ of the trivial representation $\mathbf{1}$ of $\mathrm{GO}_{m,n}(F)$. There is only one open orbit in the double coset decomposition $P_4\backslash\mathrm{GSp}_8(F)/\mathrm{GSp}_4(E)^\natural$. In a similar way, by Lemma 4.4.2, [Ólafsson 1987, Theorem 2.5] and [Gourevitch et al. 2019, Proposition 4.9],

$$\begin{aligned}\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau,\mathbb{C})&=\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_0(\tfrac12),\tau\big)\le\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}(\tfrac12),\tau\big)\\ &\le\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{3,3}(\mathbf{1}),\tau)+\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{4,0}(\mathbf{1}),\tau)\\ &=\dim\mathrm{Hom}_{\mathrm{GO}_{3,3}(F)}(\Theta_6^+(\tau),\mathbb{C})+\dim\mathrm{Hom}_{\mathrm{GO}_{4,0}(F)}(\Theta_4^+(\tau),\mathbb{C}).\end{aligned}\quad (4\text{-}7)$$

Now we separate them into two cases: $\pi_1 \not\cong \pi_2$ and $\pi_1 \cong \pi_2$.

(A) If $\pi_1 \not\cong \pi_2$, then the theta lift $\Theta_1(\Sigma)$ to $\mathrm{GSp}_2(E)$ of Σ is zero,

$$\mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(R_1(\mathbf{1}) \otimes \nu, \Sigma) = \mathrm{Hom}_{\mathrm{GSp}_2(F)}(\Theta_1(\Sigma), \mathbb{C}) = 0$$

and $\Sigma = \mathrm{Ind}_{\mathrm{GSO}(2,2)(E)}^{\mathrm{GO}(2,2)(E)}(\pi_1 \boxtimes \pi_2)$. There are several subcases:

(A1) If $\pi_i (i = 1, 2)$ are both $D^\times(F)$-distinguished, which implies that ϕ_{π_i} are conjugate-orthogonal and so that π_i are both $\mathrm{GL}_2(F)$-distinguished due to Lemma 4.4.5, then $\pi_1^\vee \not\cong \pi_2^\sigma$. Otherwise, $\pi_1^\sigma \cong \pi_1^\vee \cong \pi_2^\sigma$, which contradicts the assumption $\pi_1 \not\cong \pi_2$. Then the inequality (4-6) can be rewritten as

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) \geq \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}(\tfrac{1}{2}), \Sigma\big) \geq 2. \tag{4-8}$$

Flicker [1991] proved that $(\mathrm{GL}_n(E), \mathrm{GL}_n(F))$ is a Gelfand pair, which implies that

$$1 \geq \mathrm{Hom}_{\mathrm{GSO}_{3,3}(F)}(\Theta_6^+(\tau), \mathbb{C}) = \mathrm{Hom}_{\mathrm{GO}_{3,3}(F)}(\Theta_6^+(\tau), \mathbb{C}).$$

Thus

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) \leq 1 + 1 \tag{4-9}$$

due to the upper bound (4-7). Then (4-8) and (4-9) imply

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = 2.$$

(A2) If $\pi_1 = \pi(\chi_1, \chi_2)$, $\chi_1 \neq \chi_2$, $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$ and π_2 is $\mathrm{GL}_2(F)$-distinguished, then Lemma 4.4.5 implies that both ϕ_{π_1} and ϕ_{π_2} are conjugate-orthogonal, $\pi_1^\vee \not\cong \pi_2^\sigma$ and

$$\mathrm{Hom}_{\mathrm{GO}_{4,0}(F)}(\Sigma, \mathbb{C}) = 0 = \mathrm{Hom}_{\mathrm{GO}_{3,1}(F)}(\Sigma, \mathbb{C}).$$

Moreover, $\mathrm{Hom}_{\mathrm{GSO}_{3,3}(F)}(\Theta_6^+(\tau), \mathbb{C}) \neq 0$. Since

$$\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}(\tfrac{1}{2}), \Sigma\big) \geq \dim \mathrm{Hom}_{\mathrm{GO}(2,2)(F)}(\Sigma, \mathbb{C}) + 0 = 1,$$

the desired equality $\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = 1$ follows from (4-6) and (4-7).

(A3) If $\pi_1^\sigma \cong \pi_2^\vee$, then Lemma 4.4.1 implies

$$\dim \mathrm{Hom}_{\mathrm{GO}_{3,1}(F)}(\Sigma, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSO}_{3,1}(F)}(\pi_1 \boxtimes \pi_2, \mathbb{C}) = 1.$$

By the previous arguments, we know that $\mathrm{Hom}_{\mathrm{GO}_{2,2}(F)}(\Sigma, \mathbb{C}) = 0$ in this case. Therefore

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = 1.$$

In other cases, if $\pi_1^\sigma \not\cong \pi_2^\vee$ and either ϕ_{π_1} or ϕ_{π_2} is not conjugate-orthogonal, then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = 0.$$

If not, then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSO}_{3,3}(F)}(\Theta_6^+(\tau), \mathbb{C}) = 1.$$

Set $\Pi\boxtimes\chi=\Theta_6^+(\tau)|_{GSO_{3,3}(E)}$ as a representation of $GSO_{3,3}(E)$, which is irreducible due to Proposition 3.7. Then Π is $GL_4(F)$-distinguished and so ϕ_Π is conjugate-orthogonal.

We consider the following cases:

- If ϕ_{π_1} is conjugate-orthogonal, then ϕ_{π_2} is conjugate-orthogonal by Theorem 4.4.4.
- If ϕ_{π_1} is irreducible, by the assumption $\pi_1^\sigma\ncong\pi_2^\vee$ and Theorem 4.4.4, then ϕ_{π_1} is conjugate-orthogonal, which will imply that ϕ_{π_2} is conjugate-orthogonal as well.
- Now suppose that both ϕ_{π_1} and ϕ_{π_2} are reducible and that neither ϕ_{π_1} nor ϕ_{π_2} is conjugate-orthogonal. Assume that $\phi_{\pi_i}=\chi_{i1}+\chi_{i2}$ $(i=1,2)$. Then

$$\phi_\Pi=\chi_{11}+\chi_{12}+\chi_{21}+\chi_{22},\qquad \chi_{11}\chi_{12}=\chi_{21}\chi_{22}:E^\times/F^\times\to\mathbb{C}^\times.$$

Thanks to Theorem 4.4.4, $\chi_{11}\chi_{21}^\sigma=\mathbf{1}$ and $\chi_{12}\neq\chi_{22}$ but $\chi_{12}|_{F^\times}=\mathbf{1}=\chi_{22}|_{F^\times}$. Furthermore, $\chi_{21}\chi_{22}\cdot(\chi_{21}\chi_{22})^\sigma=\mathbf{1}$ implies

$$\chi_{21}^\sigma\chi_{21}=\mathbf{1}.$$

Similarly $\chi_{11}^\sigma\chi_{11}=\mathbf{1}$. Thus, $\chi_{21}^\sigma=\chi_{21}^{-1}$ and $\chi_{11}=\chi_{21}$. This implies that $\chi_{12}=\chi_{22}$ which contradicts the condition $\chi_{12}\neq\chi_{22}$.

Hence the Langlands parameter ϕ_Π can not be conjugate-orthogonal. Thus $\mathrm{Hom}_{GSp_4(F)}(\tau,\mathbb{C})=0$ if $\pi_1^\sigma\ncong\pi_2^\vee$ and either ϕ_{π_1} or ϕ_{π_2} is not conjugate-orthogonal.

(B) If $\pi_1=\pi_2$ is a discrete series representation, then $\Theta_1(\Sigma)=\pi_1$ due to [Atobe and Gan 2017, Proposition 5.4]. If $\pi_1=\pi_2$ is an irreducible principal series representation, applying the functor $\mathrm{Hom}_{GO_4(E)}(-,\Sigma)$ on the Kudla filtration (see [Gan and Takeda 2011b, Theorem A1]), we have

$$\Theta_1(\Sigma)=\pi_1$$

except for $\pi_1=\pi(\chi,\chi)$. If $\pi_1=\pi(\chi,\chi)$, then there is an exact sequence

$$\pi_1\longrightarrow\Theta_1(\pi_1\boxtimes\pi_1)\longrightarrow\pi_1\longrightarrow 0$$

of $GL_2(E)$-representations, where we can not deduce $\Theta_1(\pi_1\boxtimes\pi_1)$ directly. There are two choices that $\Theta_1(\pi_1\boxtimes\pi_1)$ is either π_1 or $\mathrm{Ext}_{GL_2(E)}(\pi_1,\pi_1)$. We will show that $\Theta_1(\pi_1\boxtimes\pi_1)$ has a unique Whittaker model which can imply that $\Theta_1(\pi_1\boxtimes\pi_1)=\pi_1$. Let $N=\left\{\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\mid n\in E\right\}$ be the subgroup of $GSp_2(E)$. Let ψ_N be a nontrivial character of N. Consider the Whittaker model of $\Theta_1(\pi_1\boxtimes\pi_1)$,

$$\dim\mathrm{Hom}_N(\Theta_1(\pi_1\boxtimes\pi_1),\psi_N)=\dim\mathrm{Hom}_{PGL_2(E)}(\pi_1\boxtimes\pi_1,\mathbb{C})\le 1$$

due to [Lu 2017b, Proposition 3.4], which implies that $\Theta_1(\Sigma)=\pi_1$. Therefore the exact sequence (4-3) implies the inequality

$$\dim\mathrm{Hom}_{GSp_4(F)}(\tau,\mathbb{C})\ge\dim\mathrm{Hom}_{GO_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}\big(\tfrac12\big),\Sigma\big)-\dim\mathrm{Hom}_{GSp_2(F)}(\pi_1,\mathbb{C}).\tag{4-10}$$

We separate them into the following cases:

(B1) If π_1 is $D^\times(F)$-distinguished, then $\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_0\big(\tfrac{1}{2}\big), \Sigma\big) = 3$. Again, we consider the upper bound (4-7) and the lower bound (4-10) to obtain the equality

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(\mathbb{C})}(\tau, \mathbb{C}) = 2.$$

(B2) If $\pi_1 \cong \pi_0 = \pi(\chi_1, \chi_2)$ with $\chi_1 \neq \chi_2$ and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$, then

$$\dim \mathrm{Hom}_{\mathrm{GO}_{4,0}(F)}(\Sigma, \mathbb{C}) = 0.$$

In a similar way, we can get $\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = 1$.

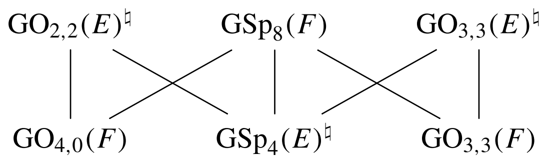
(B3) If π_1 is not $\mathrm{GL}_2(F)$-distinguished but $(\mathrm{GL}_2(F), \omega_{E/F})$-distinguished, then

$$\mathrm{Hom}_{\mathrm{GSp}_2(F)}(\pi_1, \mathbb{C}) = 0 \text{ and } \mathrm{Hom}_{\mathrm{GO}_{3,1}(F)}(\Sigma, \mathbb{C}) \neq 0,$$

which implies that $\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I_{Q_4}^{H_4}\big(\tfrac{1}{2}\big), \Sigma\big) \geq 1 = \dim \mathrm{Hom}_{\mathrm{GSO}_{3,3}(F)}(\Theta_6^+(\tau), \mathbb{C})$. Thus we can deduce that $\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) = 1$.

(iii) If τ is not in case (i) or (ii), then the first occurrence index of τ of $\mathrm{GSp}_4(E)$ in the Witt tower $\mathbb{H}_E^r$ is 3. Observe that $\Theta_6^+(\tau)|_{\mathrm{GSO}_{3,3}(E)}$ is irreducible unless $\tau = \mathrm{Ind}_{Q(Z)}^{\mathrm{GSp}_4(E)}(\chi, \pi)$ with $\chi = |-|_E$.

Suppose that $\tau \neq \mathrm{Ind}_{Q(Z)}^{\mathrm{GSp}_4(E)}(|-|_E, \pi)$. Consider the double see-saw diagrams



By [Kudla and Rallis 1992, p. 211] and Proposition 4.3.1, there are two exact sequences

$$0 \longrightarrow R^{3,3}(\mathbf{1}) \longrightarrow \mathcal{I}\big(\tfrac{1}{2}\big) \longrightarrow R^{4,0}(\mathbf{1}) \longrightarrow 0$$

and

$$0 \longrightarrow R^{4,0}(\mathbf{1}) \oplus R^{2,2}(\mathbf{1}) \longrightarrow \mathcal{I}\big(-\tfrac{1}{2}\big) \longrightarrow R^{5,1}(\mathbf{1}) \cap R^{3,3}(\mathbf{1}) \longrightarrow 0$$

of $\mathrm{GSp}_8(F)$-modules, where $\mathcal{I}(s)$ is the degenerate principal series of $\mathrm{GSp}_8(F)$ and $R^{m,n}(\mathbf{1})$ is the big theta lift to $\mathrm{GSp}_8(F)$ of the trivial representation $\mathbf{1}$ of $\mathrm{GO}_{m,n}(F)$. Assume that τ is generic and its theta lift to $\mathrm{GO}_{2,2}(E)$ is zero. Then

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{4,0}(\mathbf{1}), \tau) = \mathrm{Hom}_{\mathrm{GO}_{4,0}(F)}(\Theta_4^+(\tau), \mathbb{C}) = 0,$$

so that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}\big(-\tfrac{1}{2}\big), \tau\big) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{5,1}(\mathbf{1}) \cap R^{3,3}(\mathbf{1}), \tau).$$

Thus applying Lemma 4.4.2,

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) &= \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_0\big(\tfrac{1}{2}\big), \tau\big)\\ &\le \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}\big(\tfrac{1}{2}\big), \tau\big)\\ &\le \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{3,3}(\mathbf{1}), \tau)\\ &= \dim \mathrm{Hom}_{\mathrm{GO}_{3,3}(F)}(\Theta_6^+(\tau), \mathbb{C})\\ &= \dim \mathrm{Hom}_{\mathrm{GO}_{3,3}(F)}((\Pi \boxtimes \chi)^+, \mathbb{C}) \end{aligned} \tag{4-11}$$

where $(\Pi \boxtimes \chi)^{\pm}$ are two extensions to $\mathrm{GO}_{3,3}(E)$ of $\Pi \boxtimes \chi$. On the other hand, one has

$$\mathrm{Hom}_{\mathrm{GO}_{3,3}(F)}((\Pi \boxtimes \chi)^-, \mathbb{C}) = \mathrm{Hom}_{\mathrm{GO}_{3,3}(F)}(\Theta_6^+(\tau) \otimes \nu, \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(\Theta(\nu), \tau) = 0.$$

Then we have an inequality

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}) \le \dim \mathrm{Hom}_{\mathrm{GSO}_{3,3}(F)}(\Pi \boxtimes \chi, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GL}_4(F)}(\Pi, \mathbb{C}). \tag{4-12}$$

Now we want to obtain the reverse inequality. Note that

$$1 \longrightarrow R^{5,1}(\mathbf{1}) \cap R^{3,3}(\mathbf{1}) \longrightarrow R^{3,3}(\mathbf{1}) \longrightarrow R^{2,2}(\mathbf{1}) \longrightarrow 1$$

is exact (see [Gan and Ichino 2014, Proposition 7.2]). There is an injection

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{3,3}(\mathbf{1}), \tau) \hookrightarrow \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(R^{5,1}(\mathbf{1}) \cap R^{3,3}(\mathbf{1}), \tau) = \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}\big(-\tfrac{1}{2}\big), \tau\big) \tag{4-13}$$

since the theta lifts to $\mathrm{GO}_{2,2}(E)$ and $\mathrm{GO}_{4,0}(E)$ of τ are both zero by the assumption.

We will show that τ does not occur on the boundary of $\mathcal{I}\big(-\frac{1}{2}\big)$ under the assumptions. If τ is nondiscrete, then $\tau = J_{Q(Z)}(\chi, \pi)$, $\chi \ne \mathbf{1}$, due to [Gan and Takeda 2011b, Table 1]. Note that

$$\mathcal{I}_1(s)/\mathcal{I}_0(s) = \mathrm{ind}_{(E^\times \times \mathrm{GSp}_2(F))N'}^{\mathrm{GSp}_4(E)^\natural} \chi',$$

where $N' \cong E \oplus \mathrm{Mat}_{2,2}(F)$ and $\chi'(t, g) = |N_{E/F}(t)^{s+\frac{1}{2}} \cdot \lambda(g)^{-2s-3}|_F$. Set

$$P' = (\mathrm{GL}_1(E) \times \mathrm{GSp}_2(E)^\natural) \cdot N'.$$

Thanks to the second adjoint theorem due to Bernstein, we have

$$\mathrm{Hom}\big(\mathcal{I}_1\big(-\tfrac{1}{2}\big)/\mathcal{I}_0\big(-\tfrac{1}{2}\big), \tau\big) = \mathrm{Hom}_{E^\times \times \mathrm{Sp}_2(E) \times F^\times}\big(\mathbf{1} \otimes \mathrm{ind}_{\mathrm{Sp}_2(F)}^{\mathrm{Sp}_2(E)} \mathbb{C} \otimes |-|_F^{-2}, R_{\bar{P}'}(J_{Q(Z)}(\chi, \pi))\big) = 0,$$

because $R_{\bar{P}'}(J(\chi, \pi)) = \chi \otimes \pi + \chi^{-1} \otimes \pi\chi$ and $\chi \ne \mathbf{1}$. Moreover, the cuspidal supports of $J_{Q(Z)}(\chi, \pi)$ and $\mathcal{I}_2\big(-\frac{1}{2}\big)/\mathcal{I}_1\big(-\frac{1}{2}\big)$ are disjoint. Therefore $\tau = J_{Q(Z)}(\chi, \pi)$ does not occur on the boundary of $\mathcal{I}\big(-\frac{1}{2}\big)$ and so

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}\big(-\tfrac{1}{2}\big), \tau\big) \le \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_0\big(-\tfrac{1}{2}\big), \tau\big) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \mathbb{C}).$$

Note that if τ is a discrete series representation, then we have

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_{i+1}\big(-\tfrac{1}{2}\big)/\mathcal{I}_i\big(-\tfrac{1}{2}\big),\tau\big)=0$$

for $i=0,1$. If not, then we will get a contradiction. Suppose that

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_1\big(-\tfrac{1}{2}\big)/\mathcal{I}_0\big(-\tfrac{1}{2}\big),\tau\big)\neq 0.$$

Then $\mathrm{Hom}_{\mathrm{GL}_1(E)}(\mathbf{1},R_{\bar{P}'}(\tau))\neq 0$, which contradicts Casselman's criterion [Casselman and Miličić 1982] for the discrete series representation that

$$\mathrm{Hom}_{\mathrm{GL}_1(E)}(|-|_E^s,R_{\bar{P}'}(\tau))\neq 0$$

implies $s<0$. Similarly,

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_2\big(-\tfrac{1}{2}\big)/\mathcal{I}_1\big(-\tfrac{1}{2}\big),\tau\big)=\mathrm{Hom}_{\mathrm{GL}_2(E)\times F^\times}(\delta_{P^\natural}^{1/6},R_{\bar{P}^\natural}(\tau))=0$$

and so

$$\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}\big(-\tfrac{1}{2}\big),\tau\big)\leq\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_0\big(-\tfrac{1}{2}\big),\tau\big).\tag{4-14}$$

Therefore one can combine (4-12)–(4-14) to obtain that

$$\begin{aligned}\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau,\mathbb{C})&=\dim\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathcal{I}_0\big(-\tfrac{1}{2}\big),\tau\big)\\&=\dim\mathrm{Hom}_{\mathrm{GSO}_{3,3}(F)}(\Theta_6^+(\tau),\mathbb{C})\\&=\dim\mathrm{Hom}_{\mathrm{GL}_4(F)}(\Pi,\mathbb{C}).\end{aligned}\tag{4-15}$$

Thus the left-hand side is 1 if and only if Π is $\mathrm{GL}_4(F)$-distinguished.

If $\tau=\mathrm{Ind}_{Q(Z)}^{\mathrm{GSp}_4(E)}(|-|_E,\pi)$ is irreducible, then $\theta_6(\tau)=J_P(\pi|-|_E,\pi)\boxtimes\omega_\pi|-|_E$. It suffices to show that $I_P(\pi|-|_E,\pi)$ is $\mathrm{GL}_4(F)$-distinguished if and only if ϕ_Π is conjugate-self-dual. This follows from Lemma 4.4.6.

Hence we have finished the proof. □

Remark 4.4.10. We can also show that if $\tau=\theta(\pi_1\boxtimes\pi_2)$ with $\pi_1^\vee\cong\pi_2^\sigma$ is generic, then $\phi_\Pi=\phi_{\pi_1}\oplus\phi_{\pi_2}$ is not only conjugate-orthogonal but also conjugate-symplectic. Keeping this fact in mind will be helpful when we verify the Prasad conjecture for GSp_4 in Section 6C.

Corollary 4.4.11. *The pair* $(\mathrm{GSp}_4(E)^\natural,\mathrm{GSp}_4(F))$ *is not a Gelfand pair.*

For a generic representation τ of $\mathrm{GSp}_4(E)$ with $\omega_\tau|_{F^\times}=\chi_F^2$, we may consider the multiplicity

$$\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\pi,\chi_F)$$

which is equal to $\dim\mathrm{Hom}_{\mathrm{GSp}_4(F)}(\pi\otimes\chi_E^{-1},\mathbb{C})$, where χ_E is a character of $E^\times$ and $\chi_F=\chi_E|_{F^\times}$. We will focus on the case $\chi_F=\omega_{E/F}$ when we verify the Prasad conjecture for GSp_4 in Section 6C.

5. The $\mathrm{GSp}_{1,1}(F)$-distinguished representations

5A. *Notation.*

- D (resp. D_E) is a quaternion division algebra over F (resp. E) with a standard involution $*$.
- π^{D_E} is the Jacquet–Langlands lift to $D_E^\times(E)$ of π and $\pi^{D_E}\boxtimes\pi^{D_E}$ is a representation of $\mathrm{GSO}_{4,0}(E)$.
- $\mathfrak{W}$ (resp. $\mathfrak{V}$) is a right skew-Hermitian (resp. left Hermitian) D-vector space with isometry group $U(\mathfrak{W})$ (resp. $U(\mathfrak{V})$).
- $\mathfrak{U}^*$ is the dual D-vector space of $\mathfrak{U}$ in $\mathrm{Res}_{R/D}V_R$.
- $\mathfrak{W}\otimes_D\mathfrak{V}$ is a symplectic F-vector space.
- $\mathrm{GO}^*_{3,0}=\mathrm{GL}_1(D_4)\times\mathbb{G}_m/\{(t^{-1},t^2)\}$ (resp. $\mathrm{GO}^*_{r,r}$) is the inner form of $\mathrm{GO}_{3,3}$ (resp. $\mathrm{GO}_{2r,2r}$) defined over F and D_4 is the division F-algebra of degree 4.
- $\mathfrak{I}(s)$ (resp. $I(s)$) is the degenerate principal series of $\mathrm{GSp}_{2,2}(F)$ (resp. $\mathrm{GO}^*_{2,2}(F)$).
- $\mathrm{GSO}^*_{2,0}$ is the inner form of $\mathrm{GSO}_{3,1}$ defined over F.
- $\mathrm{GO}_{5,1}=\mathrm{GL}_2(D_E)\times\mathbb{G}_m/\{(t^{-1},t^2)\}$ is the pure inner form of $\mathrm{GO}_{3,3}$ defined over E and $\Pi^D\boxtimes\chi$ is a representation of $\mathrm{GSO}_{5,1}(E)$.
- B_1 is the minimal parabolic subgroup of $\mathrm{GL}_2(D_E)(E)$.
- $\mathrm{GSp}_{1,0}=D^\times$ (resp. $\mathrm{Sp}_{1,0}$) is the inner form of GL_2 (resp. SL_2).
- $P(Y_D)$ (resp. $\mathfrak{Q}$) is the Siegel parabolic subgroup of $\mathrm{GU}(\mathfrak{V})$ (resp. $\mathrm{GO}^*_{2,2}(F)$).
- $\mathfrak{R}^3(\mathbf{1})$ (resp. $\mathfrak{R}^2(\mathbf{1})$) is the big theta lift to $\mathrm{GSp}_{2,2}(F)$ of the trivial representation of $\mathrm{GO}^*_{3,0}(F)$ (resp. $\mathrm{GO}^*_{1,1}(F)$) and $\mathfrak{R}^{1,j}(\mathbf{1})$ is the big theta lift to $\mathrm{GO}^*_{2,2}(F)$ from $\mathrm{GSp}_{1,j}(F)$.
- $\theta_2^-(\tau)$ (resp. $\Theta_2^-(\tau)$) is the small (resp. big) theta lift to $\mathrm{GO}_{5,1}(E)$ of τ of $\mathrm{GSp}_4(E)$.
- $\Theta_{\mathfrak{W},\mathfrak{V},\psi}(\pi)$ is the big theta lift to $\mathrm{GU}(\mathfrak{V})$ of π of $\mathrm{GU}(\mathfrak{W})$.
- γ_F is the Weil index and $\gamma_F(\psi\circ q)\in\mu_8$ for the character of second degree $x\mapsto\psi(q(x,x))$, where q is a nondegenerate symmetric F-bilinear form.

5B. ***Theta lifts for quaternionic unitary groups.*** In order to study the $\mathrm{GSp}_{1,1}$-distinction problems, we need to introduce the local theta lift for quaternionic unitary groups, following [Gan and Tantono 2014; Gurevich and Szpruch 2015; Yamana 2011].

5B1. *Morita equivalence.* Let $R=\mathrm{Mat}_{2,2}(E)$ be the split quaternion algebra over E. Any left Hermitian (resp. right skew-Hermitian) free R-module (W_R,h_R) corresponds to a symplectic (resp. orthogonal) space (W_E,h_E) over E and

$$\dim_E W_E=2\cdot\dim_R W_R,\ \mathrm{Aut}(W_R,h_R)=\mathrm{Aut}(W_E,h_E).$$

See [Gurevich and Szpruch 2015, §2.1] for more details.

5B2. *Dual pairs.* Let D be the unique nonsplit quaternion algebra over F, with a standard involution $*$. Then $D \otimes_F E \cong R$. There is a D-linear map

$$\mathrm{tr}_{R/D} : R \to D$$

such that $\mathrm{tr}_{R/D}(d) = 2d$ for $d \in D$. Given a 4-dimensional symplectic space $(\mathcal{W}_2, h_E)$ over E, corresponding to a 2-dimensional left Hermitian space (W_R, h_R), we set

$$h_D(x, y) = \tfrac{1}{2}\,\mathrm{tr}_{R/D}(h_R(x, y)) \in D$$

for all $x, y \in W_R$. Then h_D is a nondegenerate Hermitian form on $\mathfrak{V} = \mathrm{Res}_{R/D} W_R$ and $\dim_D \mathfrak{V} = 4$.

Given a left Hermitian space $(\mathfrak{V}, h_D)$ and a right skew-Hermitian space $(\mathfrak{W}, s_D)$, the tensor product space $\mathfrak{W} \otimes_D \mathfrak{V}$ admits a symplectic form defined by

$$\langle w \otimes v, w' \otimes v' \rangle := \tfrac{1}{2}\,\mathrm{tr}_{D/F}((w, w') \cdot (v, v')^*).$$

This gives an embedding of F-groups

$$U(\mathfrak{W}) \times U(\mathfrak{V}) \to \mathrm{Sp}(\mathfrak{W} \otimes_D \mathfrak{V}).$$

Then we can define the Weil representation ω_ψ on $U(\mathfrak{W}) \times U(\mathfrak{V})$, using the complete polarization $\mathfrak{V} = Y_D + Y_D^*$ of $\mathfrak{V}$.

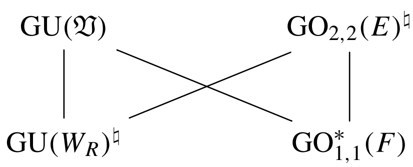
Theorem 5.2.1 [Gan and Sun 2017, Theorem 1.2]. *The Howe duality conjecture holds for the dual pair* $U(\mathfrak{W}) \times U(\mathfrak{V})$.

We can extend it to the similitude group $\mathrm{GU}(\mathfrak{W}) \times \mathrm{GU}(\mathfrak{V})$ following Roberts. (See [Gan and Tantono 2014, §3].)

5B3. *The see-saw diagram.* Let us fix the polarization $W_R = Y_R + Y_R^*$. Then

$$\mathfrak{V} = \mathrm{Res}_{R/D} W_R = Y_D + Y_D^*.$$

Consider the following see-saw diagram:



Here $\mathrm{GU}(W_R)^\natural = \mathrm{GSp}_4(E)^\natural$.

Proposition 5.2.2 [Gurevich and Szpruch 2015, Theorem 8.2]. *Let τ be an irreducible representation of* $\mathrm{GSp}(\mathcal{W}_2) \cong \mathrm{GU}(W_R)$. *Assume that π is an irreducible representation of* $\mathrm{GO}^*_{1,1}(F)$. *Then*

$$\mathrm{Hom}_{\mathrm{GU}(W_R)^\natural}(\Theta_{\mathfrak{W},\mathfrak{V},\psi}(\pi), \tau) = \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Theta_4^+(\tau), \pi).$$

Assume that V_R is a skew-Hermitian free module over R of rank 2, associated to the anisotropic 4-dimensional quadratic space over E given by (D_E, N_{D_E}) such that

$$\mathrm{GU}(V_R) \cong \mathrm{GO}_{4,0}(E).$$

Then $\mathrm{Res}_{R/D} V_R$ is a 4-dimensional skew-Hermitian D-vector space with trivial discriminant. There is a natural embedding

$$\mathrm{SU}(V_R) \cong \mathrm{SO}_{4,0}(E) \hookrightarrow \mathrm{SO}^*_{2,2}(F) = \mathrm{SU}(\mathrm{Res}_{R/D} V_R).$$

Given a 1-dimensional Hermitian vector space $\mathfrak{V}_1$ over D, we consider the theta lift from $\mathrm{GU}(\mathfrak{V}_1) = \mathrm{GSp}_{1,0}(F)$ to $\mathrm{GO}^*_{2,2}(F)$ and the theta lift from $\mathrm{GSO}_{4,0}(E)$ to $\mathrm{GU}(R \otimes_D \mathfrak{V}_1) = \mathrm{GL}_2(E)$. Consider the see-saw diagram

$$\begin{array}{ccc} \mathrm{GU}(\mathrm{Res}_{R/D} V) & & \mathrm{GL}_2(E)^\natural \\ | & \times & | \\ \mathrm{GSO}_{4,0}(E)^\natural & & \mathrm{GSp}_{1,0}(F) \end{array}$$

which is different from the situation in [Gurevich and Szpruch 2015, Theorem 8.2], since there does not exist a natural polarization in the symplectic F-vector space $\mathbb{V} = (\mathrm{Res}_{R/D} V_R) \otimes_D \mathfrak{V}_1$.

Assume that $\mathbb{V} = \mathbb{X} \oplus \mathbb{Y}$ is a polarization. Set the group

$$\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}} = \mathrm{Sp}(\mathbb{V}) \times \mathbb{C}^\times$$

with group law

$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, z_1 z_2 \cdot z_{\mathbb{Y}}(g_1, g_2)),$$

where $z_{\mathbb{Y}}(g_1, g_2) = \gamma_F\big(\tfrac{1}{2}\psi \circ q(\mathbb{Y}, g_2^{-1}\mathbb{Y}, g_1\mathbb{Y})\big)$ is a 2-cocycle (called Rao cocyle) associated to $\mathbb{Y}$ and $q(\mathbb{Y}, g_2^{-1}\mathbb{Y}, g_1\mathbb{Y})$ is the Leray invariant. (See [Kudla 1996, §I.3].)

Suppose that $\mathbb{V} = \mathbb{X}' \oplus \mathbb{Y}'$ is another polarization of $\mathbb{V}$. There is an isomorphism

$$\mathcal{S}(\mathbb{X}) \cong \mathcal{S}(\mathbb{X}').$$

Given $\varphi \in \mathcal{S}(\mathbb{X})$ and $\varphi' \in \mathcal{S}(\mathbb{X}')$, due to [Ichino and Prasanna 2016, Lemma 3.3], we have

$$\varphi(x) = \int_{\mathbb{Y} \cap \mathbb{Y}' \backslash \mathbb{Y}} \psi\big(\tfrac{1}{2}\langle x', y'\rangle - \tfrac{1}{2}\langle x, y\rangle\big)\varphi'(x')\,dy$$

where $x' \in \mathbb{X}'$ and $y' \in \mathbb{Y}'$ are given by $x' + y' = x + y \in \mathbb{V}$.

Lemma 5.2.3 (local Siegel–Weil identity). *Assume that π is an irreducible discrete series representation of $\mathrm{GL}_2(E)$ so that the big theta lift $\Theta(\pi)$ to $\mathrm{GSO}_{4,0}(E)$ is isomorphic to $\pi^{D_E} \boxtimes \pi^{D_E}$, where π^{D_E} is the Jacquet–Langlands lift to $D_E^\times(E)$ of π. Let ϱ be an irreducible representation of $\mathrm{GSp}_{1,0}(F)$. Then*

$$\dim \mathrm{Hom}_{\mathrm{GSO}_{4,0}(E)^\natural}(\Theta(\varrho), \pi^{D_E} \boxtimes \pi^{D_E}) = \dim \mathrm{Hom}_{\mathrm{GSp}_{1,0}(F)}(\pi, \varrho),$$

where $\Theta(\varrho)$ is the big theta lift to $\mathrm{GO}^*_{2,2}(F)$ of ϱ.

Proof. It suffices to show that two splittings of $\mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_{1,0}(F)$ in $\mathrm{Mp}(\mathbb{V})$ are compatible. Let us fix two polarizations $\mathrm{Res}_{R/D}V_R=\mathfrak{U}\oplus\mathfrak{U}^*$ and $R\otimes_D\mathfrak{V}_1=X\oplus Y$. Then

$$\mathbb{V}=\mathbb{X}\oplus\mathbb{Y}=(\mathfrak{U}\otimes_D\mathfrak{V}_1)\oplus(\mathfrak{U}^*\otimes_D\mathfrak{V}_1)\quad\text{and}\quad\mathbb{V}=\mathbb{X}'\oplus\mathbb{Y}'=(D_E\otimes_E X)\oplus(D_E\otimes_E Y).$$

Choose a fixed element $h_0\in\mathrm{Sp}(\mathbb{V})$ such that

$$\mathbb{X}'=h_0\mathbb{X}\quad\text{and}\quad\mathbb{Y}'=h_0\mathbb{Y}.$$

By [Ichino and Prasanna 2016, Appendix B.4], there is an isomorphism $\alpha_0:\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}'}\to\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}}$ via

$$(h,z)\mapsto(\alpha_0(h),z),$$

where $\alpha_0(h)=h^{-1}\cdot g\cdot h$ for all $h\in\mathrm{Sp}(\mathbb{V})$. Moreover,

$$z_{\mathbb{Y}'}(h_1,h_2)=z_{\mathbb{Y}}(\alpha_0(h_1),\alpha_0(h_2)).$$

Now we fix the splitting $i_{\mathbb{Y}}:\mathrm{O}^*_{2,2}(F)\times\mathrm{Sp}_{1,0}(F)\hookrightarrow\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}}$ and

$$i_{\mathbb{Y}'}:\mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_2(E)\hookrightarrow\mathrm{Mp}(\mathbb{V})_{\mathbb{Y}'},$$

where the splitting $i_{\mathbb{Y}}(y,z)=((y,z),\beta_{\mathbb{Y}}(z))$ is defined in [Kudla 1994, Theorem 3.1].

We will show that $i_{\mathbb{Y}}(h)=\alpha_0\circ i_{\mathbb{Y}'}(h)$ for all $h=(y,z)\in\mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_{1,0}(F)$. Consider

$$\begin{array}{ccccc}
\mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_{1,0}(F) & \hookrightarrow & \mathrm{O}^*_{2,2}(F)\times\mathrm{Sp}_{1,0}(F) & \xrightarrow{i_{\mathbb{Y}}} & \mathrm{Mp}(\mathbb{V})_{\mathbb{Y}}\\
\Big\| & & & & \uparrow{\scriptstyle\alpha_0}\\
\mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_{1,0}(F) & \hookrightarrow & \mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_2(E) & \xrightarrow{i_{\mathbb{Y}'}} & \mathrm{Mp}(\mathbb{V})_{\mathbb{Y}'}
\end{array}$$

Set $i_{\mathbb{Y}}(h)=(h,\beta_{\mathbb{Y}}(h))$. Then $\beta_{\mathbb{Y}}(z)=1$ for all $z\in\mathrm{Sp}_{1,0}(F)$. Similarly, we have

$$\beta_{\mathbb{Y}'}(y)=1$$

for all $y\in\mathrm{SO}_{4,0}(E)$. In order to show that

$$\beta_{\mathbb{Y}}(h)=\beta_{\mathbb{Y}'}(h)$$

for all $h=(y,z)\in\mathrm{SO}_{4,0}(E)\times\mathrm{Sp}_{1,0}(F)$, we will show that $\beta_{\mathbb{Y}}(y)=1=\beta_{\mathbb{Y}'}(z)$.

- If $y\in\mathrm{SO}_{4,0}(E)\subset\mathrm{O}^*_{2,2}(F)=\bigsqcup_{i=0}^{2}\mathfrak{P}\omega_i\mathfrak{P}$, say $y\in\mathfrak{P}\omega_i\mathfrak{P}$, where $\mathfrak{P}$ is the Siegel parabolic subgroup of $\mathrm{O}^*_{2,2}(F)$, $\omega_0=\mathbf{1}_4$ (the identity matrix in $\mathrm{O}^*_{2,2}(F)$),

$$\omega_1=\begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1\end{pmatrix}\quad\text{and}\quad\omega_2=\begin{pmatrix} & & 1 & \\ & & & 1\\ 1 & & & \\ & 1 & & \end{pmatrix},$$

  then $\beta_{\mathbb{Y}}(y)=(-1)^i$. Since ω_1 switches a pair of vectors e_1 and e_1' in a basis $\{e_1,e_2,e_1',e_2'\}$, which corresponds to an element $h\in\mathrm{O}_{4,0}(E)$ with determinant -1, where $\mathfrak{P}$ stabilizes the maximal isotropic subspace $\{e_1,e_2\}$, it follows that

$$\mathrm{SO}_{4,0}(E)\cap\mathfrak{P}\omega_1\mathfrak{P}=\varnothing,$$

  i.e., $\beta_{\mathbb{Y}}(y)=1$.

- If $z \in \mathrm{Sp}_{1,0}(F)$ and so $z = g \in \mathrm{SL}_2(E)$, then $\beta_{\mathbb{Y}'}(z) = \gamma_F\big(x(g), \tfrac{1}{2}\psi\big)^4 \cdot \gamma_F\big(\tfrac{1}{2}\psi \circ N_{D_E}\big)^4 = 1$, where

$$x(g) = \begin{cases} N_{E/F}(a_{21}) \pmod{F^{\times 2}} & \text{if } g = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right) \text{ with } a_{21} \neq 0, \\ N_{E/F}(a_{22}) \pmod{F^{\times 2}} & \text{otherwise.} \end{cases}$$

Therefore we have finished the proof. □

Remark 5.2.4. From the proof above, we can see that the see-saw identity does not hold if one replaces $\mathrm{SO}_{4,0}(E)$ by $\mathrm{O}_{4,0}(E)$ in this case.

Let V be a free R-module of rank 2 corresponding to the quadratic space $\mathbb{H}_E^2$ by the Morita equivalence. Then $\mathrm{Res}_{R/D} V$ is a skew-Hermitian D-vector space of dimension 4.

Lemma 5.2.5. *Let Σ be an irreducible representation of $\mathrm{GO}_{2,2}(E)$. Let ϱ be an irreducible representation of $\mathrm{GSp}_{1,j}(F)$ for $j = 0$ or 1. Then*

$$\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(\Theta(\varrho), \Sigma) = \dim \mathrm{Hom}_{\mathrm{GSp}_{1,j}(F)}(\Theta_{1+j}(\Sigma \cdot \nu^{1+j}), \varrho),$$

where ν is the nontrivial character of $\mathrm{GO}_{2,2}(E)/\mathrm{GSO}_{2,2}(E)$ and $\nu|_{\mathrm{O}_{2,2}(E)} = \det$.

Proof. Consider the see-saw diagram

$$\begin{array}{ccc} \mathrm{GO}^*_{2,2}(F) & & \mathrm{GSp}_{2+2j}(E)^\natural \\ | & \times & | \\ \mathrm{GO}_{2,2}(E)^\natural & & \mathrm{GSp}_{1,j}(F) \end{array}$$

Assume that $\mathfrak{W} = \mathrm{Res}_{R/D} V$. Let us fix the polarization $\mathfrak{W} = \mathfrak{U} + \mathfrak{U}^*$ and $\mathbb{H}_E^2 = Y + Y^*$, where Y^* is the dual space of Y. Let $\mathfrak{V}$ be a Hermitian D-vector space with isometric group $\mathrm{GSp}_{1,j}(F)$. Then there exists a natural polarization

$$\mathfrak{W} \otimes_D \mathfrak{V} = \mathfrak{U} \otimes_D \mathfrak{V} + \mathfrak{U}^* \otimes_D \mathfrak{V}.$$

Similarly, $\mathbb{H}_E^2 \otimes_E \mathcal{W}_{1+j} = Y \otimes_E \mathcal{W}_{1+j} + Y^* \otimes_E \mathcal{W}_{1+j}$, where $\mathcal{W}_r$ is the symplectic vector space over E of dimension $2r$. Set $\mathbb{Y} = \mathfrak{U}^* \otimes_D \mathfrak{V}$ and $\mathbb{Y}' = Y^* \otimes_E \mathcal{W}_{1+j}$. Then we have the splitting $i_{\mathbb{Y}}$ and $i_{\mathbb{Y}'}$ defined in [Kudla 1994, Theorem 3.1]. For instance, $i_{\mathbb{Y}'}(y, z) = ((y, z), \beta_{\mathbb{Y}'}(y))$ for $(y, z) \in \mathrm{O}_{2,2}(E) \times \mathrm{Sp}_{2+2j}(E)$ and

$$i_{\mathbb{Y}}(y, z) = ((y, z), \beta_{\mathbb{Y}}(y)) \in \mathrm{Mp}(\mathfrak{W} \otimes_D \mathfrak{V})_{\mathbb{Y}}$$

for $y \in \mathrm{O}^*_{2,2}(F)$ and $z \in \mathrm{Sp}_{1,j}(F)$. Note that $\beta_{\mathbb{Y}'}(y) = 1$ for $y \in \mathrm{O}_{2,2}(E)$ and

$$\beta_{\mathbb{Y}}(y) = (-1)^{(1+j)i}$$

if $y \in \mathfrak{P}\omega_i\mathfrak{P}$, where $\mathrm{O}^*_{2,2}(F) = \bigcup_i \mathfrak{P}\omega_i\mathfrak{P}$ and $\mathfrak{P}$ is the Siegel parabolic subgroup of $\mathrm{O}^*_{2,2}(F)$. Thus

$$\beta_{\mathbb{Y}}(h) = \beta_{\mathbb{Y}'}(h) \cdot (\nu(h))^{1+j}$$

for $h \in \mathrm{O}_{2,2}(E)$. Hence

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(\Theta(\varrho), \Sigma) &= \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural \times \mathrm{GSp}_{1,j}(F)}(\omega_{\psi,\mathbb{Y}}, \Sigma \otimes \varrho)\\ &= \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural \times \mathrm{GSp}_{1,j}(F)}(\omega_{\psi,\mathbb{Y}'}, \Sigma \cdot \nu^{1+j} \otimes \varrho)\\ &= \dim \mathrm{Hom}_{\mathrm{GSp}_{1,j}(F)}(\Theta_{1+j}(\Sigma \cdot \nu^{1+j}), \varrho),\end{aligned}$$

where $\omega_{\psi,\mathbb{Y}}$ (resp. $\omega_{\psi,\mathbb{Y}'}$) is the Weil representation on $\mathrm{Mp}(\mathfrak{W} \otimes_D \mathfrak{V})$ emphasizing the splitting $\mathbb{Y} + \mathbb{Y}^*$ (resp. $\mathbb{Y}' + \mathbb{Y}'^*$). This finishes the proof. □

5B4. *Degenerate principal series.* Let us fix the complete polarization

$$\mathfrak{V} = Y_D + Y_D^*.$$

Suppose $\dim_D \mathfrak{V} = 4$. Assume that $\mathfrak{I}(s)$ is the degenerate principal series of $\mathrm{GU}(\mathfrak{V}) = \mathrm{GSp}_{2,2}(F)$ associated to a Siegel parabolic subgroup $P(Y_D)$, i.e.,

$$\mathfrak{I}(s) = \left\{ f : \mathrm{GU}(\mathfrak{V}) \to \mathbb{C} \mid f(pg) = \delta_{P(Y_D)}(p)^{(1/2)+(s/5)} f(g) \text{ for all } p \in P(Y_D), g \in \mathrm{GU}(\mathfrak{V}) \right\},$$

where $\delta_{P(Y_D)}$ is the modular character. Similar to Proposition 4.3.1, we have

Lemma 5.2.6. *Assume that* $\mathfrak{R}^3(\mathbf{1})$ *is the big theta lift to* $\mathrm{GU}(\mathfrak{V})$ *of the trivial representation of* $\mathrm{GO}^*_{3,0}(F)$. *Then there is an exact sequence*

$$0 \longrightarrow \mathfrak{R}^3(\mathbf{1}) \longrightarrow \mathfrak{I}\left(\tfrac{1}{2}\right) \longrightarrow \mathfrak{R}^2(\mathbf{1}) \longrightarrow 0,$$

where $\mathfrak{R}^2(\mathbf{1})$ *is the big theta lift to* $\mathrm{GU}(\mathfrak{V})$ *of the trivial representation of* $\mathrm{GO}^*_{1,1}(F)$.

Proof. By [Yamana 2011, Theorem 1.4], we may give a similar proof as in Proposition 4.3.1. So we omit it here. □

5B5. *Double cosets.* Assume that $P(Y_D)$ is the Siegel parabolic subgroup of $\mathrm{GU}(\mathfrak{V}) = \mathrm{GSp}_{2,2}(F)$. Then the homogeneous space $X_D = P(Y_D)\backslash\mathrm{GSp}_{2,2}(F)$ corresponds to the set of maximal isotropic subspaces in $\mathfrak{V}$. We consider the double coset $X_D/\mathrm{GU}(W_R)^\natural = X_D/\mathrm{GSp}_4(E)^\natural$, similar to Lemma 4.3.3.

Proposition 5.2.7. *In the double cosets* $X_D/\mathrm{GSp}_4(E)^\natural$, *there are*

- *one closed orbit with stabilizer* $P(Y_D) \cap \mathrm{GSp}_4(E)^\natural$,
- *one open orbit with stabilizer* $\mathrm{GU}_2(D)(F) = \mathrm{GSp}_{1,1}(F) \subset \mathrm{GSp}_4(E)^\natural$ *and*
- *one intermediate orbit with a representative*

$$L = Dr(\sqrt{d}e + f) + D\left(e - \frac{1}{\sqrt{d}} f\right) \in X_D,$$

which is a nonfree R*-module with stabilizer* $(\mathrm{GL}_1(E) \times \mathrm{GSp}_{1,0}(F)) \cdot N$, $N \cong E \oplus D$, *where* $r = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) = r^2 \in R$ *and* $W_R = Re + Rf$ *with* $h_R(e, f) = 1$.

Lemma 5.2.8. *Let τ be an irreducible representation of $\mathrm{GU}(W_R)^\natural = \mathrm{GSp}_4(E)^\natural$ and $\mathrm{GSp}_4(E)^\natural \hookrightarrow \mathrm{GSp}_{2,2}(F)$ be a natural embedding. Then*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(\tfrac{1}{2}\big), \tau\big) \geq \dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau^\vee, \mathbb{C}).$$

Proof. Note that there are three orbits for $P(Y_D)\backslash \mathrm{GSp}_{2,2}(F)/\mathrm{GSp}_4(E)^\natural$. There is a filtration for $\mathfrak{I}\big(\tfrac{1}{2}\big)|_{\mathrm{GSp}_4(E)^\natural}$ as follows:

$$\mathrm{ind}_{\mathrm{GSp}_{1,1}(F)}^{\mathrm{GSp}_4(E)^\natural} \mathbb{C} = \mathfrak{I}_0\big(\tfrac{1}{2}\big) \subset \mathfrak{I}_1\big(\tfrac{1}{2}\big) \subset \mathfrak{I}_2\big(\tfrac{1}{2}\big) = \mathfrak{I}\big(\tfrac{1}{2}\big)|_{\mathrm{GSp}_4(E)^\natural},$$

where $\mathfrak{I}_2\big(\tfrac{1}{2}\big)/\mathfrak{I}_1\big(\tfrac{1}{2}\big) \cong \mathrm{ind}_{P^\natural}^{\mathrm{GSp}_4(E)^\natural} \delta_{P^\natural}^{1/2}$ and $\mathfrak{I}_1\big(\tfrac{1}{2}\big)/\mathfrak{I}_0\big(\tfrac{1}{2}\big) \cong \mathrm{ind}_{MN}^{\mathrm{GSp}_4(E)^\natural} \delta_{P(Y_D)}^{3/5}\delta_3^{-1/2}$, where

$$M \cong \mathrm{GL}_1(E) \times \mathrm{GSp}_{1,0}(F), \quad N \cong D \oplus E \quad \text{and} \quad \delta_3(t, x) = |N_{E/F}(t)^4 \cdot \lambda(x)^{-4}|_F$$

for $(t, d) \in M$. There exists an involution on $\mathrm{GSp}_{2,2}(F)$ such that the fixed points coincides with $\mathrm{GSp}_4(E)^\natural$. Applying [Ólafsson 1987, Theorem 2.5; Gourevitch et al. 2019, Proposition 4.9], we obtain the inequality

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(\tfrac{1}{2}\big), \tau\big) \geq \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}_0\big(\tfrac{1}{2}\big), \tau\big) = \dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau^\vee, \mathbb{C}).$$

This finishes the proof. □

5C. *The distinction problem for* $\mathbf{GSp_{1,1}}$. Let $\mathrm{GU}_2(D) = \mathrm{GSp}_{1,1}$ be the inner form of GSp_4 defined over F, whose E-points coincide with $\mathrm{GSp}_4(E)$. Assume that $\tau \in \mathrm{Irr}(\mathrm{GSp}_4(E))$ with $\omega_\tau|_{F^\times} = \mathbf{1}$. In this subsection, we will study the multiplicity

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}).$$

Theorem 5.3.1. *Let τ be an irreducible representation of $\mathrm{GSp}_4(E)$ such that Π_{ϕ_τ} is generic.*

(i) *If $\tau = \theta(\pi_1 \boxtimes \pi_2)$ is a nongeneric tempered representation of $\mathrm{GSp}_4(E)$, where $\pi_1 \boxtimes \pi_2$ is an irreducible smooth representation of $\mathrm{GSO}_{4,0}(E)$, then $\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = 1$ if and only if one of the following holds*:

- *$\pi_1 \ncong \pi_2$ but $\pi_1^\vee \cong \pi_2^\sigma$;*
- *$\pi_1 \cong \pi_2$ are both $(D^\times(F), \omega_{E/F})$-distinguished.*

(ii) *If $\tau = \theta(\pi_1 \boxtimes \pi_2) = \theta(\pi_2 \boxtimes \pi_1)$ is generic, then*

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \begin{cases} 2 & \text{if } \pi_1 = \pi_2 = \pi(\chi^{-1}, \chi^\sigma), \\ 1 & \text{if } \pi_1 = \pi_2 \text{ are square-integrable and } D^\times(F)\text{-distinguished}, \\ 1 & \text{if } \pi_1 \text{ is } D^\times(F)\text{-distinguished and } \pi_2 = \pi_0, \\ 2 & \text{if } \pi_1 \neq \pi_2 \text{ are both } D^\times(F)\text{-distinguished}, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\pi_0 = \pi(\chi_1, \chi_2)$ with $\chi_1 \neq \chi_2$, $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \mathbf{1}$. Note that these conditions are mutually exclusive.

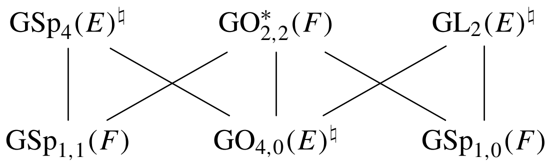
(iii) *Assume that τ is not as in case* (i) *or* (ii), *so that $\tau = \theta(\Pi^D \boxtimes \chi)$ is generic, where $\Pi^D \boxtimes \chi$ is an irreducible representation of* $\mathrm{GSO}_{5,1}(E)$. *Then* $\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = 1$ *if and only if one of the following holds*:

- ϕ_Π *is irreducible and conjugate-orthogonal or*
- $\phi_\Pi = \phi_\rho + \phi_\rho \mu$ *with* $\rho^\sigma \cong \rho^\vee \mu^{-1}$,

where $\Pi = JL(\Pi^D)$ *is the Jacquet–Langlands lift to* $\mathrm{GL}_4(E)$ *of* Π^D.

Proof. The proof is very similar to the proof of Theorem 4.4.9.

(i) Assume that V_R is a skew-Hermitian free module over R of rank 2, corresponding to D_E by the Morita equivalence. Then $\mathrm{Res}_{R/D} V_R$ is a 4-dimensional skew-Hermitian vector space over D with trivial discriminant. Fix a polarization $\mathrm{Res}_{R/D} V = \mathfrak{U} \oplus \mathfrak{U}^*$. Consider the diagram



There is an exact sequence of $\mathrm{GO}^*_{2,2}(F)$-representations

$$0 \longrightarrow \mathfrak{R}^{1,1}(\mathbf{1}) \longrightarrow I\big(\tfrac{1}{2}\big) \longrightarrow \mathfrak{R}^{1,0}(\mathbf{1}) \longrightarrow 0,$$

where $I(s)$ is the degenerate principal series of $\mathrm{GO}^*_{2,2}(F)$ and $\mathfrak{R}^{1,j}(\mathbf{1})$ is the theta lift to $\mathrm{GO}^*_{2,2}(F)$ the trivial representation of $\mathrm{GSp}_{1,j}(F)$. Set $\tau = \Theta_2(\Sigma)$, where

$$\Sigma = \begin{cases} \mathrm{Ind}^{\mathrm{GO}_{4,0}(E)}_{\mathrm{GSO}_{4,0}(E)}(\pi_1 \boxtimes \pi_2) & \text{if } \pi_1 \ncong \pi_2, \\ (\pi_1 \boxtimes \pi_1)^+ & \text{if } \pi_1 \cong \pi_2. \end{cases}$$

Note that $\mathrm{GO}_{4,0}(E)$ is an anisotropic group. Using the contravariant exact functor

$$\mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}(-, \Sigma),$$

we obtain a short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}(\mathfrak{R}^{1,0}(\mathbf{1}), \Sigma) \to \mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}\big(I\big(\tfrac{1}{2}\big), \Sigma\big) \to \mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}(\mathfrak{R}^{1,1}(\mathbf{1}), \Sigma) \to 0.$$

Applying Lemma 5.2.5, we have

$$0 \to \mathrm{Hom}_{\mathrm{GSp}_{1,0}(F)}(\Theta_1(\Sigma \otimes \nu), \mathbb{C}) \to \mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}\big(I\big(\tfrac{1}{2}\big), \Sigma\big) \to \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) \to 0, \tag{5-1}$$

where $\Theta_1(\Sigma \otimes \nu)$ is the big theta lift to $\mathrm{GL}_2(E)$ of $\Sigma \otimes \nu$. There are no F-rational points on the nonidentity connected component of $\mathrm{GO}^*_{2,2}$ (see [Mœglin et al. 1987, pp. 21–22]), so that

$$\mathrm{GO}^*_{2,2}(F) = \mathrm{GSO}^*_{2,2}(F) = \mathfrak{Q} \cdot \mathrm{GO}_{4,0}(E)^\natural,$$

where $\mathfrak{Q}$ is the Siegel parabolic subgroup of $\mathrm{GO}^*_{2,2}(F)$. Then

$$\mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}\big(I\big(\tfrac{1}{2}\big), \Sigma\big) = \mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}(\mathrm{ind}^{\mathrm{GO}_{4,0}(E)^\natural}_{\mathrm{GO}^*_{2,0}(F)} \mathbb{C}, \Sigma) = \mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma, \mathbb{C}). \tag{5-2}$$

Here $\mathrm{GSO}^*_{2,0}(F)$ sits in the exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E^\times & \xrightarrow{\ i\ } & D_E^\times(E)\times F^\times & \longrightarrow & \mathrm{GSO}^*_{2,0}(F) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & D_E^\times(E)\times D_E^\times(E) & \longrightarrow & \mathrm{GSO}_{4,0}(E) & \longrightarrow & 1 \end{array}$$

where $i(e)=(e, N_{E/F}(e)^{-1})$ and the embedding $\mathrm{GSO}^*_{2,0}(F)\hookrightarrow \mathrm{GSO}_{4,0}(E)$ is given by

$$(x,t)\mapsto (x, t\cdot x^\sigma)$$

for $x\in D_E^\times(E)$ and $t\in F^\times$. The σ-action on $D_E^\times(E)$ is induced from the isomorphism $D_E(E)\cong D_E(E)\otimes_E (E,\sigma)$. There are two subcases:

- If $\pi_1\ncong\pi_2$, then $\pi_1\boxtimes\pi_2$ does not participate in theta correspondence with $\mathrm{GL}_2(E)$. The short exact sequence (5-1) implies that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau,\mathbb{C}) = \dim\mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}\big(I\big(\tfrac12\big),\Sigma\big) = \dim\mathrm{Hom}_{\mathrm{GSO}^*_{2,0}(F)}(\pi_1\boxtimes\pi_2,\mathbb{C}). \tag{5-3}$$

  Hence one can get

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau,\mathbb{C}) = \dim\mathrm{Hom}_{D_E^\times(E)}(\pi_2^\vee,\pi_1^\sigma),$$

  where $\pi_1^\sigma = JL^{-1}(JL(\pi_1)^\sigma)$.

- If $\pi_1=\pi_2$, then the short exact sequence (5-1) implies that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau,\mathbb{C}) = \dim\mathrm{Hom}_{\mathrm{GO}_{4,0}(E)^\natural}\big(I\big(\tfrac12\big),\Sigma\big) = \dim\mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma,\mathbb{C})$$

  because $\Theta_1(\Sigma\otimes\nu)=0$. Note that

$$\dim\mathrm{Hom}_{\mathrm{GSO}^*_{2,0}(F)}(\pi_1\boxtimes\pi_1,\mathbb{C}) = \dim\mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma,\mathbb{C}) + \dim\mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma\otimes\nu,\mathbb{C}).$$

  In a similar way, $\dim\mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma\otimes\nu,\mathbb{C}) = \dim\mathrm{Hom}_{\mathrm{GSp}_{1,0}(F)}(JL(\pi_1),\mathbb{C})$. Therefore, if $JL(\pi_1)$ is $D^\times(F)$-distinguished, then $\pi_1^\sigma\cong\pi_1^\vee$ and so

$$\dim\mathrm{Hom}_{\mathrm{GSO}^*_{2,0}(F)}(\pi_1\boxtimes\pi_1,\mathbb{C}) = 1 = \dim\mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma\otimes\nu,\mathbb{C}).$$

  Then $\dim\mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau,\mathbb{C}) = \dim\mathrm{Hom}_{\mathrm{GO}^*_{2,0}(F)}(\Sigma,\mathbb{C})=0$ if $JL(\pi_1)$ is $D^\times(F)$-distinguished. Also, τ is $\mathrm{GSp}_{1,1}(F)$-distinguished if and only if $JL(\pi_1)^\vee\cong JL(\pi_1)^\sigma$ which is not $D^\times(F)$-distinguished. Thus τ is $\mathrm{GSp}_{1,1}(F)$-distinguished if and only if $JL(\pi_1)$ is $(D^\times(F),\omega_{E/F})$-distinguished, in which case ϕ_{π_1} is conjugate-symplectic.

(Similarly, one can show that

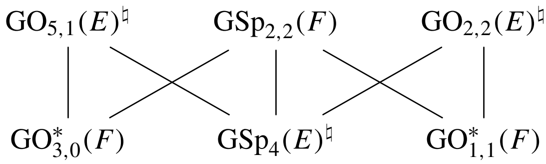
$$\dim\mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau,\omega_{E/F}) = \dim\mathrm{Hom}_{D_E^\times(E)}(\pi_2^\vee,\pi_1^\sigma) - \dim\mathrm{Hom}_{D^\times(F)}(\Theta_1(\Sigma\otimes\nu),\omega_{E/F}).$$

Here we use the fact

$$\omega_{E/F}\circ\lambda_V|_{\mathrm{GO}^*_{2,0}(F)} = \mathbf{1}.$$

Hence $\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \omega_{E/F}) = 1$ if and only if either $JL(\pi_1) = JL(\pi_2)$ are both $D^\times(F)$-distinguished or $\pi_1 \not\cong \pi_2$ but $\pi_1^\vee = \pi_2^\sigma$. It will be useful when we verify the Prasad conjecture for PGSp_4 in Section 7.)

(ii) We will use a similar argument. Assume that V_R corresponds to $\mathbb{H}_E^2$ by the Morita equivalence. By the conversation relation, we have $\theta_2^-(\tau) = 0$. Via the see-saw diagrams



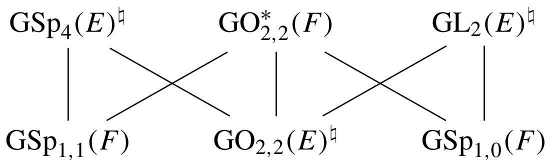
applying Lemma 5.2.6 and Proposition 5.2.2, we have

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(\tfrac{1}{2}\big), \tau\big) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(\mathfrak{R}^2(\mathbf{1}), \tau) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Theta_4^+(\tau), \mathbb{C}),$$

where $\mathfrak{I}(s)$ is the degenerate principal series of $\mathrm{GSp}_{2,2}(F)$. Due to Lemma 5.2.8,

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) \le \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(\tfrac{1}{2}\big), \tau\big) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Theta_4^+(\tau), \mathbb{C}).$$

We want to get the reverse inequality. Consider the diagrams



There is an exact sequence of $\mathrm{GO}^*_{2,2}(F)$-representations

$$0 \longrightarrow \mathfrak{R}^{1,0}(\mathbf{1}) \longrightarrow I\big(-\tfrac{1}{2}\big) \longrightarrow \mathfrak{R}^{1,1}(\mathbf{1}) \longrightarrow 0.$$

Note that $\dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(\mathfrak{R}^{1,0}(\mathbf{1}), \Sigma) = \dim \mathrm{Hom}_{\mathrm{GSp}_{1,0}(F)}(\Theta_1(\Sigma \otimes \nu), \mathbb{C}) = 0$. Thanks to [Ólafsson 1987, Theorem 2.5; Gourevitch et al. 2019, Proposition 4.9], we have

$$\begin{aligned}
\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) &= \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(\mathfrak{R}^{1,1}(\mathbf{1}), \Sigma) \\
&= \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}\big(I\big(-\tfrac{1}{2}\big), \Sigma\big) \\
&\ge \dim \mathrm{Hom}_{\mathrm{GO}_{2,2}(E)^\natural}(\mathrm{ind}_{\mathrm{GO}^*_{1,1}(F)}^{\mathrm{GO}_{2,2}(E)^\natural} \mathbb{C}, \Sigma) \\
&= \dim \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Sigma, \mathbb{C}).
\end{aligned}$$

Therefore $\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Theta_4^+(\tau), \mathbb{C})$ unless $\Theta_4^+(\tau)$ is reducible. There is no F-rational points on the nonidentity connected component of $\mathrm{GO}^*_{1,1}$, so that

$$\mathrm{GO}^*_{1,1}(F) = \mathrm{GSO}^*_{1,1}(F).$$

There are two cases: $\pi_1 \not\cong \pi_2$ and $\pi_1 = \pi_2$.

Assume that $\pi_1 \not\cong \pi_2$. Since

$$\mathrm{GO}^*_{1,1}(F) = \mathrm{GSO}^*_{1,1}(F) \cong \mathrm{GL}_2(F) \times D^\times(F)/\{(t, t^{-1}) : t \in F^\times\},$$

for $\pi_1 \neq \pi_2$ one can obtain that $\Theta_4^+(\tau) = \mathrm{Ind}^{\mathrm{GO}(2,2)(E)}_{\mathrm{GSO}(2,2)(E)}(\pi_1 \boxtimes \pi_2)$ and

$$\mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Sigma, \mathbb{C}) = \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\pi_1 \boxtimes \pi_2, \mathbb{C}) \oplus \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\pi_2 \boxtimes \pi_1, \mathbb{C}). \tag{5-4}$$

There are two subcases:

- If π_i $(i = 1, 2)$ are both $D^\times(F)$-distinguished, then (5-4) implies that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Sigma, \mathbb{C}) = 2.$$

- If π_1 is $D^\times(F)$-distinguished and $\pi_2 = \pi(\chi_1, \chi_2)$ with $\chi_1 \neq \chi_2$, $\chi_1|_{F^\times} = \chi_2|_{F^\times} = 1$, then π_2 is $\mathrm{GL}_2(F)$-distinguished but not $D^\times(F)$-distinguished (see Lemma 4.4.5). So (5-4) implies that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = 1.$$

If $\pi_1 = \pi_2$ are both square-integrable representations, then

$$\mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Sigma, \mathbb{C}) = \mathrm{Hom}_{\mathrm{GSO}^*_{1,1}(F)}(\pi_1 \boxtimes \pi_1, \mathbb{C}) = \begin{cases} 1 & \text{if } \pi_1 \text{ is } D^\times(F)\text{-distinguished}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\pi_1 = \pi_2 = \pi(\chi^{-1}, \chi^\sigma)$, then $\Theta_4^+(\tau)$ is reducible. We will show that $\tau = I_{Q(Z)}(\mathbf{1}, \pi_1)$ does not occur on the boundary of $\mathfrak{I}\left(\frac{1}{2}\right)$ and hence that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{1,1}(F)}(\Theta_4^+(\tau), \mathbb{C}).$$

There is a filtration

$$\mathrm{ind}^{\mathrm{GSp}_4(E)^\natural}_{\mathrm{GSp}_{1,1}(F)} \mathbb{C} = \mathfrak{I}_0(s) \subset \mathfrak{I}_1(s) \subset \mathfrak{I}_2(s) = \mathfrak{I}(s)|_{\mathrm{GSp}_4(E)^\natural}$$

of $\mathfrak{I}(s)|_{\mathrm{GSp}_4(E)^\natural}$ such that $\mathfrak{I}_2(s)/\mathfrak{I}_1(s) = \mathrm{ind}^{\mathrm{GSp}_4(E)^\natural}_{P^\natural} \delta^{(s+1)/3}_{P^\natural}$ and

$$\mathfrak{I}_1(s)/\mathfrak{I}_0(s) = \mathrm{ind}^{\mathrm{GSp}_4(E)^\natural}_{MN} \delta^{(1/2)+(s/5)}_{P(Y_D)} \delta_3^{-1/2},$$

where $\delta_3(t, x) = |N_{E/F}(t)^4 \lambda(d)^{-4}|_F$ for $(t, x) \in M = \mathrm{GL}_1(E) \times \mathrm{GSp}_{1,0}(F)$. If

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\left(\mathfrak{I}_1\left(\tfrac{1}{2}\right)/\mathfrak{I}_0\left(\tfrac{1}{2}\right), \tau\right) \neq 0,$$

then

$$\mathrm{Hom}_{\mathrm{GL}_1(E)}(|-|_E, R_{\bar{P}''}(I_{Q(Z)}(\mathbf{1}, \pi_1))) \neq 0,$$

which is impossible, where $P'' = (\mathrm{GL}_1(E) \times \mathrm{GL}_2(E)^\natural) \ltimes N$ is a parabolic subgroup of $\mathrm{GSp}_4(E)^\natural$ and $R_{\bar{P}''}$ denotes the Jacquet functor associate to the parabolic opposite to P''. So

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\left(\mathfrak{I}_1\left(\tfrac{1}{2}\right)/\mathfrak{I}_0\left(\tfrac{1}{2}\right), \tau\right) = 0.$$

It is quite straightforward to see that

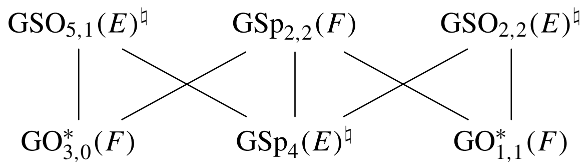
$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(\mathrm{ind}^{\mathrm{GSp}_4(E)^\natural}_{P^\natural} \delta^{1/2}_{P^\natural}, I_{Q(Z)}(\mathbf{1}, \pi_1)) = 0$$

by applying the Jacquet functor. Hence $\tau = I_{Q(Z)}(\mathbf{1}, \pi_1)$ does not occur on the boundary of $\mathfrak{I}\left(\frac{1}{2}\right)$.

The big theta lift to $\mathrm{GSO}_{2,2}(E)$ of $\tau = I_{Q(Z)}(\mathbf{1}, \pi_1)$ of $\mathrm{GSp}_4(E)$ is

$$\mathrm{Ext}^1_{\mathrm{GSO}(2,2)(E)}(\pi_1 \boxtimes \pi_1, \pi_1 \boxtimes \pi_1).$$

From the see-saw pairs diagram



one can use the fact $\theta_2^-(\tau) = 0$ to obtain that

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSO}^*_{1,1}(F)}(\mathrm{Ext}^1_{\mathrm{GSO}_{2,2}(E)}(\pi_1 \boxtimes \pi_1, \pi_1 \boxtimes \pi_1), \mathbb{C}) = 2.$$

(iii) Assume that $\theta_4^+(\tau) = 0$. Note that $0 \to \mathfrak{R}^2(\mathbf{1}) \to \mathfrak{I}\big(-\frac{1}{2}\big) \to \mathfrak{R}^3(\mathbf{1}) \to 0$ is exact. Then we can use the same method appearing in (ii) to show that

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{GO}^*_{3,0}(F)}(\Theta_2^-(\tau), \mathbb{C}) &= \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(\mathfrak{R}^3(\mathbf{1}), \tau) \\ &= \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(-\tfrac{1}{2}\big), \tau\big) \geq \dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}).\end{aligned}$$

We will show that τ does not occur on the boundary of $\mathfrak{I}\big(-\frac{1}{2}\big)$ in this case. Then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(-\tfrac{1}{2}\big), \tau\big) \leq \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}_0\big(-\tfrac{1}{2}\big), \tau\big) = \dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C})$$

and so

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}\big(-\tfrac{1}{2}\big), \tau\big).$$

In order to show that τ does not occur on the boundary of $\mathfrak{I}\big(-\frac{1}{2}\big)$, we separate them into two cases.

- If $\tau = I_{Q(Z)}(\chi, \pi)$ with $\chi \neq \mathbf{1}$, then

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}_2\big(-\tfrac{1}{2}\big)/\mathfrak{I}_1\big(-\tfrac{1}{2}\big), \tau\big) = \mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}(\mathrm{ind}_{P^\natural}^{\mathrm{GSp}_4(E)^\natural} \delta_{P^\natural}^{1/6}, \tau) = 0.$$

  If $\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}_1\big(-\frac{1}{2}\big)/\mathfrak{I}_0\big(-\frac{1}{2}\big), \tau\big) \neq 0$, then $\mathrm{Hom}_{\mathrm{GL}_1(E)}(\mathbf{1}, R_{\overline{P}''}(\tau)) \neq 0$ which is impossible since $R_{\overline{P}''}(\tau) = \chi \otimes \pi \oplus \chi^{-1} \otimes \pi\chi$ and $\chi \neq \mathbf{1}$, where $P'' = (\mathrm{GL}_1(E) \times \mathrm{GL}_2(E)^\natural) \rtimes N$.

- If τ is square-integrable, then $\mathrm{Hom}_{\mathrm{GL}_1(E)}(\mathbf{1}, R_{\overline{P}''}(\tau)) = 0$ due to the Casselman criterion in [Casselman and Miličić 1982] for a discrete series representation that $\mathrm{Hom}_{\mathrm{GL}_1(E)}(|-|_E^s, R_{\overline{P}''}(\tau)) \neq 0$ implies that $s < 0$. Hence $\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}_1\big(-\frac{1}{2}\big)/\mathfrak{I}_0\big(-\frac{1}{2}\big), \tau\big) = 0$. In a similar way,

$$\mathrm{Hom}_{\mathrm{GSp}_4(E)^\natural}\big(\mathfrak{I}_2\big(-\tfrac{1}{2}\big)/\mathfrak{I}_1\big(-\tfrac{1}{2}\big), \tau\big) = \mathrm{Hom}_{\mathrm{GL}_2(E) \times F^\times}(\delta_{P^\natural}^{1/6}, R_{\overline{P}^\natural}(\tau)) = 0.$$

Hence τ does not occur on the boundary of $\mathfrak{I}\big(-\frac{1}{2}\big)$. Moreover, if $\tau \neq I_{Q(Z)}(|-|_E, \rho)$, then $\Theta_2^-(\tau) = \Pi^D \boxtimes \chi$ is irreducible. Then there exists an identity

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{3,0}(F)}(\Pi^D \boxtimes \chi, \mathbb{C}) = \dim \mathrm{Hom}_{D_4^\times(F)}(\Pi^D, \mathbb{C}),$$

where D_4 is the division algebra over F of degree 4.

- If $\Pi = JL(\Pi^D)$ is a square-integrable representation of $\mathrm{GL}_4(E)$, then [Beuzart-Plessis 2018, Theorem 1] and Theorem 4.4.4 imply that

$$\dim \mathrm{Hom}_{\mathrm{GL}_4(F)}(\Pi, \omega_{E/F}) = \dim \mathrm{Hom}_{D_4^\times(F)}(\Pi^D, \omega_{E/F}) = \begin{cases} 1 & \text{if } \phi_\Pi \text{ is conjugate-symplectic,} \\ 0 & \text{otherwise.} \end{cases}$$

  Thus $\dim \mathrm{Hom}_{D_4^\times(F)}(\pi^D, \mathbb{C}) = 1$ if and only if ϕ_Π is conjugate-orthogonal.

- If Π^D is an induced representation $\pi(\rho_D, (\rho_D)^\vee \otimes \mu)$ with $\mu \neq \omega_{\rho_D}$, then we use the orbit decomposition $B_1\backslash \mathrm{GL}_2(D_E)(E)/\mathrm{GL}_1(D_4)(F)$ and Mackey theory to get that

$$\begin{aligned}\dim \mathrm{Hom}_{D_4^\times(F)}(\Pi^D, \mathbb{C}) &= \dim \mathrm{Hom}_{D_E^\times(E)}(\rho_D^\sigma \otimes \rho_D^\vee \cdot \mu, \mathbb{C}) = \dim \mathrm{Hom}_{D_E^\times(E)}(\rho_D^\sigma, \rho_D \cdot \mu^{-1}) \\ &= \begin{cases} 1 & \text{if } \rho_D^\sigma \cong \rho_D \mu^{-1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5-5}$$

  In this case, $\rho^\sigma = \rho\mu^{-1}$ where $\rho = JL(\rho_D)$ is the Jacquet–Langlands lift to $\mathrm{GL}_2(E)$ and $\phi_\Pi = \phi_\rho \oplus \phi_\rho^\vee \cdot \mu$, which is conjugate-orthogonal due to Theorem 4.4.4.

- If $\Pi^D = \mathrm{Sp}(\rho_D|-|_E^{1/2})$ is a generalized Speh representation and $\tau = I_{Q(Z)}(|-|_E, \rho)$, then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{GO}^*_{3,0}(F)}(\Theta_2^-(\tau), \mathbb{C}) = \begin{cases} 1 & \text{if } \rho^\sigma \cong \rho^\vee|-|_E^{-1}, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

6. The Prasad conjecture for GSp_4

6A. ***The Prasad conjecture.*** In this subsection, we give a brief introduction to the Prasad conjecture [2015, Conjecture 2]. One may refer to [Prasad 2015, §13] for more details.

Let $\boldsymbol{G}$ be a quasisplit reductive group defined over a local field F with characteristic zero. Let W_F be the Weil group of F and WD_F be the Weil–Deligne group of F. Let E be a quadratic extension over F. A quadratic character $\chi_{\boldsymbol{G}}$ is introduced in [Prasad 2015, §8] and another quasisplit reductive group G^{op} defined over F is introduced in [Prasad 2015, §7]. Then there is a relation between the fibers of the base change map

$$\Phi : \mathrm{Hom}(WD_F, {}^LG^{op}) \to \mathrm{Hom}(WD_E, {}^LG^{op})$$

from the Galois side and the $\chi_{\boldsymbol{G}}$-distinction problems for $\boldsymbol{G}(E)/\boldsymbol{G}(F)$ from the automorphic side.

More precisely, assume the Langlands–Vogan conjecture in [Vogan 1993]. Given an irreducible representation π of $\boldsymbol{G}(E)$ with an enhanced L-parameter (ϕ_π, λ), where λ is an irreducible representation of the component group $\pi_0(Z(\phi_\pi))$ and the L-packet Π_{ϕ_π} is generic, we have

$$\sum_\alpha \dim \mathrm{Hom}_{G_\alpha(F)}(\pi, \chi_{\boldsymbol{G}}) = \sum_i m(\lambda, \tilde{\phi}_i) \deg \Phi(\tilde{\phi}_i)/d_0(\tilde{\phi}_i),$$

where

- $\alpha \in H^1(W_F, \boldsymbol{G})$ runs over all pure inner forms of $\boldsymbol{G}$ satisfying $G_\alpha(E) = \boldsymbol{G}(E)$;
- $\tilde{\phi}_i \in \mathrm{Hom}(WD_F, {}^LG^{op})$ runs over all parameters of ${}^LG^{op}$ satisfying $\tilde{\phi}_i|_{WD_E} = \phi_\pi$;
- $m(\lambda, \tilde{\phi}) = \dim \mathrm{Hom}_{\pi_0(Z(\tilde{\phi}))}(\mathbf{1}, \lambda)$ is the multiplicity of the trivial representation contained in the restricted representation $\lambda|_{\pi_0(Z(\tilde{\phi}))}$;
- $d_0(\tilde{\phi}) = |\mathrm{Coker}\{\pi_0(Z(\tilde{\phi})) \to \pi_0(Z(\phi_\pi))^{\mathrm{Gal}(E/F)}\}|$.

Remark 6.1.1. If $H^1(F, \boldsymbol{G})$ is trivial such as $\boldsymbol{G} = \mathrm{GSp}_{2n}$, then the automorphic side contains only one term. The Prasad conjecture gives a precise formula for the multiplicity

$$\dim \mathrm{Hom}_{\boldsymbol{G}(F)}(\pi, \chi_{\boldsymbol{G}}).$$

Remark 6.1.2. There exists a counterexample even for GL_2 when Π_{ϕ_π} is not generic. Let $\boldsymbol{G} = \mathrm{GL}_2$, $\chi_{\boldsymbol{G}} = \omega_{E/F}$ and $\pi = \mathbf{1}$ be the trivial representation. Then the automorphic side is zero however the Galois side is nonzero.

Remark 6.1.3. If $\tilde{\phi}$ comes from a square-integrable representation, then $\deg \Phi(\tilde{\phi}) = 1$. The reason, due to Prasad, is that $\tilde{\phi}$ represents a singleton in $\mathrm{Hom}(WD_F, {}^L G^{op})$.

If π is square-integrable, then we have a refined version, i.e., the formula for each dimension

$$\dim \mathrm{Hom}_{G_\alpha(F)}(\pi, \chi_{\boldsymbol{G}}).$$

Let $Z(\widehat{G}^{op})$ be the center of the dual group $\widehat{G}^{op}$. There is a perfect pairing

$$H^1(\mathrm{Gal}(E/F), Z(\widehat{G}^{op})^{W_E}) \times H^1(\mathrm{Gal}(E/F), \boldsymbol{G}(E)) \to \mathbb{Q}/\mathbb{Z}$$

for Prasad's studies [2015, §11] of the character twists. Set $\Omega_{\boldsymbol{G}}(E) = H^1(\mathrm{Gal}(E/F), Z(\widehat{G}^{op})^{W_E})$. Given a parameter $\tilde{\phi} \in H^1(W_F, \widehat{G}^{op})$, we consider the stabilizer $\Omega_{\boldsymbol{G}}(\tilde{\phi}, E) \subset \Omega_{\boldsymbol{G}}(E)$ under the pairing

$$H^1(W_F, Z(\widehat{G}^{op})) \times H^1(W_F, \widehat{G}^{op}) \to H^1(W_F, \widehat{G}^{op}).$$

Set

$$A_{\boldsymbol{G}}(\tilde{\phi}) \subset H^1(\mathrm{Gal}(E/F), \boldsymbol{G}(E)) \cong \Omega_{\boldsymbol{G}}(E)^\vee$$

to be the annihilator of the stabilizer $\Omega_{\boldsymbol{G}}(\tilde{\phi}, E)$. Then there is another perfect pairing

$$\Omega_{\boldsymbol{G}}(E)/\Omega_{\boldsymbol{G}}(\tilde{\phi}, E) \times A_{\boldsymbol{G}}(\tilde{\phi}) \to \mathbb{Q}/\mathbb{Z},$$

meaning that in the orbit $\Omega_{\boldsymbol{G}}(E)/\Omega_{\boldsymbol{G}}(\tilde{\phi}, E)$ of character twists of $\tilde{\phi}$ (which go to a particular parameter under the base change to E) there are exactly as many parameters as there are certain pure inner forms of $\boldsymbol{G}$ over F which trivialize after base change to E.

Consider

$$F(\phi_\pi) = \{\tilde{\phi} : WD_F \to {}^L G^{op} \mid \tilde{\phi}|_{WD_E} = \phi_\pi\} = \sqcup_{i=1}^r \mathcal{O}(\tilde{\phi}_i).$$

Each orbit $\mathcal{O}(\tilde{\phi}_i)$ of $\Omega_{\boldsymbol{G}}(E)$-action on $F(\phi_\pi)$ is associated to a coset $\mathcal{C}_i$ of $A_{\boldsymbol{G}}(\tilde{\phi}_i)$ in $H^1(\mathrm{Gal}(E/F), \boldsymbol{G}(E))$ defining a set of certain pure inner forms G_α of $\boldsymbol{G}$ over F such that $G_\alpha(E) = \boldsymbol{G}(E)$. Then

$$\dim \mathrm{Hom}_{G_{\alpha(F)}}(\pi, \omega_{\boldsymbol{G}}) = \sum_{i=1}^r m(\lambda, \tilde{\phi}_i) \cdot 1_{\mathcal{C}_i}(G_\alpha)/d_0(\tilde{\phi}_i),$$

where

- $1_{\mathcal{C}_i}$ is the characteristic function of the coset $\mathcal{C}_i$;
- $m(\lambda, \tilde{\phi})$ is the multiplicity for the trivial representation contained in the restricted representation $\lambda|_{\pi_0(Z(\tilde{\phi}))}$, which may be zero;
- $d_0(\tilde{\phi}) = |\,\mathrm{Coker}\{\pi_0(Z(\tilde{\phi})) \to \pi_0(Z(\phi_\pi))^{\mathrm{Gal}(E/F)}\}|$.

6B. ***The Prasad conjecture for*** **GL_2.** Before we give the proof of Theorem 1.2, let us recall the Prasad conjecture for $\boldsymbol{G}=\mathrm{GL}_2=\mathrm{GSp}_2$. Set $\boldsymbol{G}=\mathrm{GL}_2$. Then $\chi_{\boldsymbol{G}}=\omega_{E/F}$ and $G^{op}=\mathrm{U}(2, E/F)$ is the quasisplit unitary group, where E is a quadratic field extension over a p-adic field F. Denote

$$^{L}G^{op}=\mathrm{GL}_2(\mathbb{C})\rtimes\langle\sigma\rangle,$$

where σ-action on $\mathrm{GL}_2(\mathbb{C})$ is given by

$$\sigma(g)=\omega_0(g^t)^{-1}\omega_0^{-1}=g\cdot\det(g)^{-1},$$

$\omega_0=\begin{pmatrix} & 1\\ -1 & \end{pmatrix}$ and $g\in\mathrm{GL}_2(\mathbb{C})$, g^t denotes its transpose matrix. Given an irreducible representation π of $\mathrm{GL}_2(E)$ with $\phi=\phi_\pi$ irreducible (for simplicity), there is no other pure inner form for GL_2. Then

$$\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi,\omega_{E/F})=|F(\phi)|,$$

where $F(\phi)=\{\tilde{\phi}:WD_F\to{}^{L}G^{op}\mid\tilde{\phi}|_{WD_E}=\phi\}$ and $|F(\phi)|$ denotes its cardinality.

Proposition 6.2.1. *The following statements are equivalent*:

(i) $\dim\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi,\omega_{E/F})=1$.

(ii) *The Langlands parameter ϕ is conjugate-symplectic.*

(iii) *There is only one extension $\tilde{\phi}\in F(\phi)$.*

Proof. We only prove the direction (ii)⇒(iii) and the rest follows from Flicker's results [1991]. If ϕ is conjugate-symplectic, then

$$\phi^s=\phi^\vee=\phi(\det\phi)^{-1},$$

where $s\in W_F\setminus W_E$ is fixed. There exists $A\in\mathrm{GL}_2(\mathbb{C})$ such that

$$\phi(sts^{-1})=\phi^s(t)=A\cdot\phi(t)\det(\phi(t))^{-1}\cdot A^{-1}$$

for all $t\in WD_E$. Set

$$\tilde{\phi}(s)=A\cdot\sigma$$

and $\tilde{\phi}(t)=\phi(t)$ for $t\in WD_E$. Then

$$\tilde{\phi}(sts^{-1})=\tilde{\phi}(s)\cdot\phi(t)\cdot\tilde{\phi}(s)^{-1}$$

and $\tilde{\phi}(s^2)=\phi(s^2)=(\tilde{\phi}(s))^2$ due to the sign of ϕ. More precisely, assuming that $\langle -,-\rangle$ is the WD_E-equivariant bilinear form associated to $\phi:WD_E\to\mathrm{GSp}(V,\langle -,-\rangle)$, we define

$$B:V\times V\to\mathbb{C}$$

by $B(v_1,v_2)=\langle v_1, A^{-1}v_2\rangle$ for $v_1,v_2\in V$. Then

$$B(\phi(t)v_1,\phi^s(t)v_2)=\langle\phi(t)v_1,\phi^\vee(t)A^{-1}v_2\rangle=B(v_1,v_2)$$

and so B gives a conjugate-self-dual bilinear form on V. By Schur's lemma, B has sign -1, i.e.,

$$B(v_1,\phi(s^2)v_2)=-B(v_2,v_1)$$

for all $v_1, v_2 \in V$. Thus $B(Av_1, \phi(s^2)v_2) = -B(v_2, Av_1)$, i.e.,

$$\langle Av_1, A^{-1}\phi(s^2)v_2\rangle = -\langle v_2, A^{-1}Av_1\rangle = \langle v_1, v_2\rangle$$

for all $v_i \in V$. Then $\det(A)\cdot A^{-2}\phi(s^2) = 1$, i.e., $\phi(s^2) = A\cdot\det(A)^{-1}A = (\tilde{\phi}(s))^2$.

Therefore $\tilde{\phi} \in F(\sigma)$. If there are two extensions $\tilde{\phi}_i$ with $A_i \in \mathrm{GL}_2(\mathbb{C})$ such that $\tilde{\phi}_i|_{WD_E} = \phi$, then $A_1A_2^{-1} \in Z(\phi) \cong \mathbb{C}^\times$ by Schur's lemma, so that $\phi_1 = \phi_2$. □

Remark 6.2.2. This method will appear again when we study the Prasad conjecture for $\boldsymbol{G} = \mathrm{GSp}_4$ in Section 6C1 The key idea is to choose a proper element A such that the lift

$$\tilde{\phi} : WD_F \to {}^L G_0$$

satisfies $\tilde{\phi}(s) = A\cdot\sigma$ and $\tilde{\phi}|_{WD_E} = \phi$.

6C. *The Prasad conjecture for* GSp₄. The aim of this subsection is to verify the Prasad conjecture for GSp_4. Now we consider the generic representation $\tau = \theta(\Pi \boxtimes \chi)$ of $\mathrm{GSp}_4(E)$, with ϕ_Π conjugate-symplectic and $\chi|_{F^\times} = 1$. Note that the Langlands parameter ϕ_Π is equal to $i\circ\phi_\tau$, where

$$i : \mathrm{GSp}_4(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$$

is the embedding between L-groups. Furthermore, χ is the similitude character of ϕ_τ. If ϕ_Π is conjugate-symplectic (resp. conjugate-orthogonal), we say that ϕ_τ is conjugate-symplectic (resp. conjugate-orthogonal). There are two cases: ϕ_Π is irreducible and ϕ_Π is reducible.

Lemma 6.3.1. *Assume that $\tau = \theta(\Pi\boxtimes\chi)$ is a generic representation of* $\mathrm{GSp}_4(E)$ *and $\omega_\tau|_{F^\times} = \mathbf{1}$. Then τ is* $(\mathrm{GSp}_4(F), \omega_{E/F})$*-distinguished if and only if ϕ_Π is conjugate-symplectic.*

Proof. Due to Theorem 4.4.9, the following are equivalent:

- τ is $\mathrm{GSp}_4(F)$-distinguished.
- Π is $\mathrm{GL}_4(F)$-distinguished.
- ϕ_Π is conjugate-orthogonal.

Fix a character χ_E of $E^\times$ such that $\chi_E|_{F^\times} = \omega_{E/F}$. Then τ is $(\mathrm{GSp}_4(F), \omega_{E/F})$-distinguished if and only if $\tau\otimes\chi_E\circ\lambda_W$ is $\mathrm{GSp}_4(F)$-distinguished, which is equivalent to that $\phi_\Pi\otimes\chi_E$ is conjugate-orthogonal. Note that χ_E^{-1} is conjugate-symplectic. Hence τ is $(\mathrm{GSp}_4(F), \omega_{E/F})$-distinguished if and only if ϕ_Π is conjugate-symplectic. □

Recall that if $\boldsymbol{G} = \mathrm{GSp}_{2n}$, then $\chi_G = \omega_{E/F}$ and

$$G^{op}(F) = \{g \in \mathrm{GSp}_{2n}(E) | \sigma(g) = \theta(g)\},$$

where $\theta(g) = \lambda_W(g)^{-1}g$ is the involution. Note that the σ-actions on $\mathrm{GSp}_4(E)$ and $\mathrm{GSp}_4(\mathbb{C})$ are totally different. (We hope that this will not confuse the reader.) Observe that $H^1(\mathrm{Gal}(E/F), Z(\widehat{G}^{op})^{W_E}) = 1$, which corresponds to the fact that the pure inner form of GSp_{2n} is trivial.

According to Theorem 4.4.9, we will divide the proof of Theorem 1.2 into four parts:

- $i \circ \phi_\tau$ is irreducible;
- $i \circ \phi_\tau = \rho \oplus \rho\nu$ with $\nu \neq \mathbf{1}$;
- the endoscopic case $i \circ \phi_\tau = \phi_{\pi_1} \oplus \phi_{\pi_2}$ and τ is generic;
- $i \circ \phi_\tau = \phi_{\pi_1} \oplus \phi_{\pi_2}$ and τ is nongeneric.

See Section 6C1–Section 6C4.

6C1. *The irreducible L-parameter ϕ_τ.* Given a conjugate-symplectic L-parameter $\phi = \phi_\tau$, which is irreducible, we want to extend ϕ to

$$\tilde{\phi} : WD_F \to {}^L G_0 = \mathrm{GSp}_4(\mathbb{C}) \rtimes \langle \sigma \rangle,$$

where σ acts on $\mathrm{GSp}_4(\mathbb{C})$ by

$$\sigma(g) = g \cdot \mathrm{sim}(g)^{-1}.$$

Let $s \in W_F \setminus W_E$. The parameter ϕ is conjugate-symplectic, so that $\phi^\vee = \phi^s$ and $\phi^\vee = \phi\chi^{-1}$. Hence there exists an element $A \in \mathrm{GSp}_4(\mathbb{C})$ such that

$$\phi(sts^{-1}) = \phi^s(t) = A \cdot \phi(t)\chi^{-1}(t) \cdot A^{-1} \tag{6-1}$$

for all $t \in WD_E$. Set

$$\tilde{\phi}(s) = A \cdot \sigma \quad \text{and} \quad \tilde{\phi}(t) = \phi(t)$$

for $t \in WD_E$. Then $\phi(sts^{-1}) = A\phi(t)\chi^{-1}(t)A^{-1} = \tilde{\phi}(s) \cdot \phi(t) \cdot \tilde{\phi}(s)^{-1}$. Moreover, we will show that

$$\tilde{\phi}(s^2) = \phi(s^2) = (\tilde{\phi}(s))^2.$$

Then $\tilde{\phi} \in \mathrm{Hom}(WD_F, {}^L G_0)$ and $\tilde{\phi}|_{WD_E} = \phi$.

Assume that $\langle -, - \rangle$ is the WD_E-equivariant bilinear form associated to

$$\phi_\tau : WD_E \to \mathrm{GSp}_4(\mathbb{C}) = \mathrm{GSp}(V, \langle -, - \rangle).$$

Set

$$B(v, w) = \langle v, A^{-1} w \rangle$$

for $v, w \in V$. Then (6-1) implies that

$$B(\phi(t)v, \phi(sts^{-1})w) = \langle \phi(t)v, \phi(t)\chi^{-1}(t)A^{-1}w \rangle = \chi(t) \cdot \langle v, \chi^{-1}(t)A^{-1}w \rangle = B(v, w).$$

Thus B is a conjugate-self-dual bilinear form on ϕ and hence it has sign -1 by Schur's lemma, i.e.,

$$-B(w, v) = B(v, \phi(s^2)w).$$

Therefore we have

$$\begin{aligned} \langle v, w \rangle = -\langle w, v \rangle = -B(w, Av) = B(Av, \phi(s^2)w) \\ = \langle Av, A^{-1}\phi(s^2)w \rangle = \langle v, \mathrm{sim}(A)A^{-2}\phi(s^2)w \rangle \end{aligned}$$

and so $\phi(s^2) = A \cdot \mathrm{sim}(A)^{-1} A = (\tilde{\phi}(s))^2$.

Proposition 6.3.2. *Assume that $\tau=\theta(\Pi\boxtimes\chi)$ with ϕ_Π irreducible. Then there exists at most one extension $\tilde{\phi}: WD_F\to {}^LG_0$ such that $\tilde{\phi}|_{WD_E}=\phi_\tau$.*

Proof. If there are two extensions $\tilde{\phi}_i (i=1,2)$ such that $\tilde{\phi}_i(s)=A_i\cdot\sigma$ with $A_i\in \mathrm{GSp}_4(\mathbb{C})$ and

$$\tilde{\phi}_i(sts^{-1})=\tilde{\phi}_i(s)\cdot\phi_\tau(t)\cdot\tilde{\phi}_i(s)^{-1}$$

for all $t\in WD_E$, then $A_1A_2^{-1}$ commutes with ϕ_τ. So $A_1A_2^{-1}$ is a scalar by Schur's lemma. Thus $\tilde{\phi}_1=\tilde{\phi}_2$. □

Hence, if $\tau=\theta(\Pi\boxtimes\chi)$ with ϕ_Π irreducible and conjugate-symplectic, then there is one extension $\tilde{\phi}\in F(\phi_\tau)$ and

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau,\omega_{E/F})=1.$$

If $\phi=\phi_\tau$ is conjugate-symplectic and reducible, then there are several cases.

6C2. *$\phi_\tau=\rho+\rho\nu$ with $\nu\neq\mathbf{1}$ and ρ irreducible.* If $\phi_\Pi=\rho+\rho\nu$ with ρ irreducible and $\chi=\nu\cdot\det\rho$ conjugate-orthogonal, then $\chi\chi^s=\mathbf{1}$. Thanks to Theorem 4.4.4, there are two subcases:

- ρ and $\rho\nu$ are both conjugate-symplectic or
- $\rho^s=\rho^\vee\nu^{-1}$.

(i) If ρ and $\rho\nu$ are both conjugate-symplectic, then ν is conjugate-orthogonal and there exist

$$\tilde{\rho}_i: WD_F\to \mathrm{GL}_2(\mathbb{C})\rtimes\langle\sigma\rangle$$

such that $\tilde{\rho}_1|_{WD_E}=\rho,\ \tilde{\rho}_2|_{WD_E}=\rho\nu$ and $\tilde{\rho}_i(s)=A_i\cdot\sigma$ for $A_i\in\mathrm{GL}_2(\mathbb{C})$ due to Proposition 6.2.1. Note that ρ is irreducible. Then given $t\in WD_E$,

$$\tilde{\rho}_1^s(t)\nu^s(t)=\tilde{\rho}_2^s(t)=A_2\sigma(\rho(t)\nu(t))(A_2\sigma)^{-1}=A_2\rho^\vee(t)A_2^{-1}\cdot\nu^{-1}(t)$$

and so $A_1\cdot\sigma\cdot\rho(t)\sigma^{-1}A_1^{-1}=A_2\rho^\vee(t)A_2^{-1}$ (since $\nu\nu^s=\mathbf{1}$) which implies $A_1A_2^{-1}\in\mathbb{C}^\times$. Set

$$\tilde{\phi}(s)=\begin{pmatrix} & A_1\\ A_1 & \end{pmatrix}\cdot\sigma\in\mathrm{GSp}_4(\mathbb{C})\rtimes\langle\sigma\rangle \quad\text{and}\quad \tilde{\phi}(t)=\begin{pmatrix}\rho(t) & \\ & \rho(t)\nu(t)\end{pmatrix}$$

for $t\in WD_E$. Then $\tilde{\phi}\in F(\phi)$ is the unique extension of ϕ_τ.

(ii) If $\rho^s\cong\rho^\vee\nu^{-1}$, there exists an $A\in\mathrm{GL}_2(\mathbb{C})$ such that

$$\rho^s(t)\nu(t)=(\det\rho(t))^{-1}\cdot A\rho(t)A^{-1}$$

for $t\in WD_E$. Then

$$\det\rho^s\cdot\det\rho\cdot\nu^2=\mathbf{1},$$

which implies that $\nu=\nu^s$. Observe that

$$\begin{aligned}\rho^s(sts^{-1})\nu(sts^{-1})&=(\det\rho(sts^{-1}))^{-1}A\rho(sts^{-1})A^{-1}\\&=\det\rho^s(t)^{-1}A\cdot\nu(t)^{-1}\det\rho(t)^{-1}A\rho(t)A^{-1}\cdot A^{-1}\\&=\nu(t)^{-1}\det\rho^s(t)^{-1}\det\rho(t)^{-1}A^2\rho(t)A^{-2}.\end{aligned}$$

Then $\rho(s^2)\rho(t)\rho(s^2)^{-1} = A^2\rho(t)A^{-2}$ since the character $\nu \det\rho$ is conjugate-orthogonal. Note that ρ is irreducible. Then $A^{-2}\rho(s^2)$ is a scalar. Choose a proper A such that $A^{-2}\rho(s^2) = 1$. Set

$$\tilde\phi(s) = \begin{pmatrix} A & \\ & A\cdot\det(A^{-1}) \end{pmatrix}\cdot\sigma \quad\text{and}\quad \tilde\phi(t) = \begin{pmatrix} \rho(t) & \\ & \rho(t)\nu(t) \end{pmatrix}$$

for $t \in WD_E$. Then

$$\begin{aligned}\tilde\phi(s)\cdot\phi(t)\cdot\tilde\phi(s)^{-1} &= \begin{pmatrix} A & \\ & A\cdot\det(A)^{-1} \end{pmatrix}\cdot\sigma\cdot\begin{pmatrix} \rho(t) & \\ & \rho(v)\nu(t) \end{pmatrix}\cdot\left(\sigma^{-1}\cdot\begin{pmatrix} A^{-1} & \\ & A^{-1}\det(A) \end{pmatrix}\right)\\ &= \begin{pmatrix} A & \\ & A\cdot\det(A)^{-1} \end{pmatrix}\begin{pmatrix} \rho^\vee(t)\nu(t)^{-1} & \\ & \rho^\vee(t) \end{pmatrix}\cdot\sigma\cdot\sigma^{-1}\cdot\begin{pmatrix} A^{-1} & \\ & A^{-1}\det(A) \end{pmatrix}\\ &= \begin{pmatrix} A\rho^\vee(t)\nu(t)^{-1}A^{-1} & \\ & A\rho^\vee(t)A^{-1} \end{pmatrix} = \begin{pmatrix} \rho^s(t) & \\ & \rho^s(t)\nu(t) \end{pmatrix} = \tilde\phi^s(t) \end{aligned} \tag{6-2}$$

and $(\tilde\phi(s))^2 = \phi(s^2)$. Thus $\tilde\phi$ is a homomorphism from WD_F to ${}^LG_0$ and $\tilde\phi|_{WD_E} = \phi$.

Remark 6.3.3. The key point here is to find a proper element $\tilde\phi(s)$ such that $\tilde\phi \in \mathrm{Hom}(WD_F, {}^LG_0)$. Hence we always need to check the following two conditions: $\tilde\phi^s(t) = \tilde\phi(s)\cdot\phi(t)\cdot\tilde\phi(s)^{-1}$ and $(\tilde\phi(s))^2 = \phi(t^2)$. Following the definition, the computation like (6-2) is quite straightforward and we may skip it sometimes.

6C3. *Endoscopic case.* If $\phi_\tau = \rho_1 + \rho_2$ is the endoscopic case, then $\det\rho_1 = \det\rho_2$ are both conjugate-orthogonal. There are several subcases. Assume that $\tau = \theta(\pi_1 \boxtimes \pi_2)$ is generic, $\rho_i = \phi_{\pi_i} (i = 1, 2)$ and $\rho_0 = \chi_1 + \chi_2$, with $\chi_1 \neq \chi_2$ and $\chi_1|_{F^\times} = \chi_2|_{F^\times} = \omega_{E/F}$. There are also 2 cases: $\rho_1 \neq \rho_2$ and $\rho_1 = \rho_2$.

Assume that $\rho_1 \neq \rho_2$. Then

(i) If ρ_1 and ρ_2 are both conjugate-symplectic and $\rho_i \neq \rho_0$ $(i = 1, 2)$, so that both π_1 and π_2 are $(D^\times(F), \omega_{E/F})$-distinguished due to Lemma 4.4.5, then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \omega_{E/F}) = 2.$$

Thanks to Proposition 6.2.1, there exist $\tilde\rho_1$ and $\tilde\rho_2$ of U(2, E/F) such that $\tilde\rho_i|_{WD_E} = \rho_i$. (Here we need to choose A_i properly such that $\det A_1 = \det A_2$ if $\tilde\rho_i(s) = A_i\cdot\sigma$.)

If ρ_1 and ρ_2 are both irreducible, then every lift of ϕ should be of the form

$$s \mapsto \begin{pmatrix} \lambda_1\tilde\rho_1(s) & \\ & \lambda_2\tilde\rho_2(s) \end{pmatrix} \in \mathrm{GSp}_4(\mathbb{C}) \rtimes \langle\sigma\rangle$$

with $\lambda_1^2 = \lambda_2^2$. It is known that $\tilde\phi = \omega_{E/F}\cdot\tilde\phi$ as parameters of ${}^LG_0$ since

$$\omega_{E/F}\cdot\tilde\phi = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\tilde\phi\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}^{-1}.$$

Thus there are two lifts $\tilde\phi_1 = \tilde\rho_1 + \tilde\rho_2$ and $\tilde\phi_2 = \tilde\rho_1\omega_{E/F} + \tilde\rho_2$ such that $\tilde\phi_i|_{WD_E} = \phi$.

If $\rho_1 = \chi^{-1} + \chi^s$, then the centralizer $Z_{\mathrm{GL}_2(\mathbb{C})}(\rho_1)$ is $\mathbb{C}^\times \times \mathbb{C}^\times$ or $\mathrm{GL}_2(\mathbb{C})$. Moreover,

$$\tilde{\rho}_1(s) = \begin{pmatrix} 1 & \\ & \chi(s^2) \end{pmatrix} \cdot \sigma.$$

In this case, $\tilde{\rho}_1 + \tilde{\rho}_2 \neq \tilde{\rho}_1 \omega_{E/F} + \tilde{\rho}_2$, which will be a different story if $\rho_1 = \rho_0$.

(ii) If $\rho_1 = \rho_0$ and ρ_2 is conjugate-symplectic, then $\tilde{\rho}_1(s) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \cdot \sigma$. Because

$$\begin{pmatrix} \omega_{E/F}\tilde{\rho}_1 & \\ & \tilde{\rho}_2 \end{pmatrix} = \begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\rho}_1 & \\ & \tilde{\rho}_2 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix},$$

where $a = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, we have $\tilde{\phi}_1 = \tilde{\phi}_2$.

(iii) If $\rho_1^\vee = \rho_2^s$, then there exists an $A \in \mathrm{SL}_2(\mathbb{C})$ such that

$$A^{-1} \rho_1^\vee(t) A = \rho_2^s(t)$$

for $t \in WD_E$. Set

$$\tilde{\phi}(s) = \begin{pmatrix} & A\rho_2(s^2) \\ A^{-1} & \end{pmatrix} \cdot \sigma \in \mathrm{Sp}_4(\mathbb{C}) \rtimes \sigma.$$

Then $\tilde{\phi}(sts^{-1}) = \tilde{\phi}(s) \cdot \tilde{\phi}(t) \cdot \tilde{\phi}(s^{-1})$ and

$$[\tilde{\phi}(s)]^2 = \begin{pmatrix} & A\rho_2(s^2) \\ A^{-1} & \end{pmatrix}^2 = \begin{pmatrix} A\rho_2(s^2)A^{-1} & \\ & \rho_2(s^2) \end{pmatrix} = \begin{pmatrix} \rho_1^\vee(s^2) & \\ & \rho_2(s^2) \end{pmatrix} = \phi(s^2).$$

The last equality holds because $\det \rho_1$ is conjugate-orthogonal and so $\det \rho_1(s^2) = 1$.

Now we assume $\rho_1 = \rho_2$. According to ρ_1, we still separate it into 3 cases in a similar way.

(i) If ρ_1 is conjugate-symplectic but $\rho_1 \neq \rho_0$, then $\tilde{\phi}_1 = \tilde{\rho}_1 + \tilde{\rho}_1$ and $\tilde{\phi}_2 = \tilde{\rho}_1 + \tilde{\rho}_1 \omega_{E/F}$, where $\tilde{\rho}_1 : WD_F \to \mathrm{GL}_2(\mathbb{C}) \rtimes \langle \sigma \rangle$ satisfies $\tilde{\rho}_1|_{WD_E} = \rho_1$.

(ii) If $\rho_1 = \rho_0$, there is only one lift $\tilde{\phi} = \tilde{\rho}_1 + \tilde{\rho}_1$.

(iii) If ρ_1 is not conjugate-symplectic but conjugate-orthogonal, set

$$\tilde{\phi}(s) = \begin{pmatrix} & -A \\ A & \end{pmatrix} \cdot \sigma \in \mathrm{GSp}_4(\mathbb{C}) \rtimes \langle \sigma \rangle$$

where $A \in \mathrm{GL}_2(\mathbb{C})$ satisfies $A\rho_1^\vee(t)A^{-1} = \rho_1^s(t)$. Let us verify

$$\phi(s^2) = \tilde{\phi}(s^2) = \tilde{\phi}(s)^2,$$

i.e., $-A^2 \det(A)^{-1} = \rho_1(s^2)$.

- Suppose that ρ_1 is irreducible. Let $\langle -, - \rangle$ be the WD_E-equivariant bilinear form associated to $\rho_1 : WD_E \to \mathrm{GSp}(V, \langle -, - \rangle)$. Set

$$B(m, n) = \langle m, A^{-1} n \rangle$$

for $m, n \in V$. We have

$$B(\rho_1(t)m, \rho_1^s(t)n) = \langle \rho_1(t)m, \rho_1^\vee(t) A^{-1} n \rangle = B(m, n).$$

Note that ρ_1 is conjugate-orthogonal. By Schur's lemma, the conjugate-self-dual bilinear form B has sign 1, i.e.,

$$B(m, \rho_1(s^2)n) = B(n, m)$$

for all $m, n \in V$. Replacing m by Am, we have

$$\langle Am, A^{-1}\rho_1(s^2)n \rangle = \langle n, A^{-1}Am \rangle = \langle n, m \rangle = \langle m, -n \rangle.$$

Therefore $\det(A) \cdot A^{-2}\rho_1(s^2) = -1$. In this case,

$$\begin{aligned}\tilde{\phi}(s)\tilde{\phi}(t)\tilde{\phi}(s)^{-1} &= \begin{pmatrix} & -A \\ A & \end{pmatrix} \begin{pmatrix} \rho_1^\vee(t) & \\ & \rho_1^\vee(t) \end{pmatrix} \begin{pmatrix} & -A \\ A & \end{pmatrix}^{-1} \\ &= \begin{pmatrix} A\rho_1^\vee(t)A^{-1} & \\ & A\rho_1^\vee(t)A^{-1} \end{pmatrix} = \tilde{\phi}^s(t)\end{aligned}$$

for all $t \in WD_E$.

- If $\rho_1 = \mu_1 + \mu_2$ with $\mu_1\mu_2^s = \mathbf{1}$, then ρ_1 is conjugate-symplectic, which contradicts the assumption.
- If $\rho_1 = \mu_1 + \mu_2$ with $\mu_1 \neq \mu_2$ and $\mu_1|_{F^\times} = \mu_2|_{F^\times} = \mathbf{1}$, then $A = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $A^2 = 1 = \rho_1(s^2)$.

6C4. *Nongeneric tempered.* Let τ be an irreducible nongeneric tempered representation of $\mathrm{GSp}_4(E)$ and $\tau = \theta(\pi_1 \boxtimes \pi_2)$, where each π_i is an irreducible representations of $D_E^\times(E)$. If the enhanced L-parameter of τ is (ϕ_τ, λ), where $\phi_\tau = \rho_1 + \rho_2$, $\rho_i = \phi_{\pi_i}$ and λ is the nontrivial character of the component group $\pi_0(Z_{\phi_\tau}/Z_{\mathrm{GSp}_4(\mathbb{C})})$, then

$$\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \omega_{E/F}) = 0.$$

On the Galois side, if $\phi_\pi = \rho_1 + \rho_2$, then for arbitrary parameter $\tilde{\phi}$ satisfying $\tilde{\phi}|_{WD_E} = \phi_\tau$, the restricted representation $\lambda|_{\pi_0(Z(\tilde{\phi}))}$ does not contain the trivial character $\mathbf{1}$, i.e.,

$$m(\lambda, \tilde{\phi}) = 0.$$

Finally we can prove Theorem 1.2.

Proof of Theorem 1.2. It is obvious if τ is a nongeneric tempered representation of $\mathrm{GSp}_4(E)$. (See Section 6C4.) Since the Levi subgroup of a parabolic subgroup in GSp_4 are GL-type, [Prasad 2015, Lemma 10] implies that $\deg \Phi(\tilde{\phi}) = 1$ in our case. By the above discussions, we know that if τ is generic, then the multiplicity $\dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \omega_{E/F})$ equals to the number of inequivalent lifts $|F(\phi_\tau)|$. □

7. Proof of Theorem 1.3

This section focuses on the Prasad conjecture for PGSp_4. Let $\bar{\tau}$ be a representation of $\mathrm{PGSp}_4(E)$, i.e., a representation τ of $\mathrm{GSp}_4(E)$ with trivial central character. If the multiplicity

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}) = \dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\tau, \omega_{E/F})$$

is nonzero, then we say $\bar{\tau}$ is $(\mathrm{PGSp}_4(F), \omega_{E/F})$-distinguished. Let $\mathrm{PGSp}_{1,1} = \mathrm{PGU}_2(D)$ be the pure inner form of PGSp_4 defined over F. Similarly,

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = \dim \mathrm{Hom}_{\mathrm{GSp}_{1,1}(F)}(\tau, \omega_{E/F})$$

for a representation τ of $\mathrm{GSp}_4(E)$ with trivial central character.

7A. *Notation.*

- $\bar{\tau}, \pi^{++}, \pi^{--}, \pi^{+}$ and π^{-} are representations of $\mathrm{PGSp}_4(E)$.
- $s \in W_F \setminus W_E$ and $\phi_\tau^s(t) = \phi_\tau(sts^{-1})$ for $t \in WD_E$.
- $S_\phi = \pi_0(Z(\phi))$ is the component group associated to ϕ.
- $\tilde{\phi} : WD_F \to \mathrm{Sp}_4(\mathbb{C})$ and $\tilde{\phi}_i$ are Langlands parameters of $\mathrm{PGSp}_4(F)$.
- $\mathcal{C}_i$ is a coset of $A_G(\tilde{\phi}_i)$ in $H^1(F, \mathrm{PGSp}_4)$ and $1_{\mathcal{C}_i}$ denotes its characteristic function.
- $\mathrm{PGSp}_{1,1}$ (resp. $PD^\times$) is the pure inner form of PGSp_4 (resp. PGL_2) defined over F.

7B. *The Prasad conjecture for* PGL_2.

If $\boldsymbol{G} = \mathrm{PGL}_2$, then $\chi_{\boldsymbol{G}} = \omega_{E/F}$ and $G^{op} = \mathrm{PGL}_2$.

Theorem 7.2.1. *Let $\bar{\pi}$ be a generic irreducible representation of* $\mathrm{PGL}_2(E)$. *Then the following are equivalent*:

(i) $\dim \mathrm{Hom}_{\mathrm{PGL}_2(F)}(\bar{\pi}, \omega_{E/F}) = 1$.

(ii) *The Langlands parameter $\phi_{\bar{\pi}}$ is conjugate-symplectic.*

(iii) *There exists a parameter $\tilde{\phi} : WD_F \to \mathrm{SL}_2(\mathbb{C})$ such that $\tilde{\phi}|_{WD_E} = \phi_{\bar{\pi}}$.*

(iv) *$\bar{\pi}$ is $(PD^\times(F), \omega_{E/F})$-distinguished or $\bar{\pi} = \pi(\chi_E, \chi_E^{-1})$ with $\chi_E|_{F^\times} = \omega_{E/F}$ and $\chi_E^2 \neq \mathbf{1}$.*

Proof. See [Gan and Raghuram 2013, Theorem 6.2; Lu 2017b, Main Theorem (local)]. □

7C. *The Prasad conjecture for* PGSp_4.

Recall that if $\boldsymbol{G} = \mathrm{PGSp}_4$, then $\widehat{\boldsymbol{G}} = \mathrm{Spin}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C})$, $G^{op} = \mathrm{PGSp}_4$ and $\chi_{\boldsymbol{G}} = \omega_{E/F}$. Let $\bar{\tau}$ be a representation of $\mathrm{PGSp}_4(E)$ with enhanced L-parameter $(\phi_{\bar{\tau}}, \lambda_{\bar{\tau}})$. Assume that the L-packet $\Pi_{\phi_{\bar{\tau}}}$ is generic. The Prasad conjecture for PGSp_4 implies the following:

P(i) If $\bar{\tau}$ is $(\mathrm{PGSp}_4(F), \omega_{E/F})$-distinguished, then

- $\Pi_{\phi_{\bar{\tau}}^s} = \Pi_{\phi_{\bar{\tau}}^\vee}$, an equality of L-packets and
- $\phi_{\bar{\tau}} = \tilde{\phi}|_{WD_E}$ for some parameter $\tilde{\phi} : WD_F \to \mathrm{Sp}_4(\mathbb{C})$.

P(ii) If $\bar{\tau}$ is generic and there exists $\tilde{\phi}: WD_F \to \mathrm{Sp}_4(\mathbb{C})$ such that $\tilde{\phi}|_{WD_E} = \phi_{\bar{\tau}}$, then we have that $\bar{\tau}$ is $(\mathrm{PGSp}_4(F), \omega_{E/F})$-distinguished.

P(iii) Assume that $\phi_{\bar{\tau}} = \tilde{\phi}|_{WD_E}$ for some parameter $\tilde{\phi}: WD_F \to \mathrm{Sp}_4(\mathbb{C})$. If $\bar{\tau}$ is a discrete series representation, then we set

$$F(\phi_{\bar{\tau}}) = \{\tilde{\phi} : \tilde{\phi}|_{WD_E} = \phi_{\bar{\tau}}\} = \bigsqcup_i \mathcal{O}(\tilde{\phi}_i),$$

where $\mathcal{O}(\tilde{\phi}_i) = \{\tilde{\phi}_i, \omega_{E/F} \cdot \tilde{\phi}_i\}$ which may be a singleton. Given a parameter $\tilde{\phi}_i : W_F \to \mathrm{Sp}_4(\mathbb{C})$ with $\phi_{\bar{\tau}}$ its restriction to WD_E and $\tilde{\phi}_i \cdot \omega_{E/F} = \tilde{\phi}_i$, there exists an element $g_i \in Z(\phi_{\bar{\tau}})$ such that

$$(\tilde{\phi}_i \cdot \omega_{E/F})(x) = g_i \tilde{\phi}_i(x) g_i^{-1}$$

for all $x \in WD_F$ and so g_i normalizes $Z(\tilde{\phi}_i)$. Then $\mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}) \neq 0$ if $\lambda_{\bar{\tau}}(g_i) = 1$ and $\mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) \neq 0$ if $\lambda_{\bar{\tau}}(g_i) = -1$. In this case, $A_G(\tilde{\phi}_i) \subset H^1(F, \mathrm{PGSp}_4)$ is trivial and

$$\mathcal{C}_i = \begin{cases} \{\mathrm{PGSp}_4\} & \text{if } \lambda_{\bar{\tau}}(g_i) = 1, \\ \{\mathrm{PGSp}_{1,1}\} & \text{if } \lambda_{\bar{\tau}}(g_i) = -1. \end{cases}$$

If $\tilde{\phi}_i \neq \tilde{\phi}_i \cdot \omega_{E/F}$, then $A_G(\tilde{\phi}_i) = H^1(F, \mathrm{PGSp}_4)$ and $\mathcal{C}_i = \{\mathrm{PGSp}_4, \mathrm{PGSp}_{1,1}\}$. Set G_α to be PGSp_4 or $\mathrm{PGSp}_{1,1}$. Then

$$\dim \mathrm{Hom}_{G_\alpha(F)}(\bar{\tau}, \omega_{E/F}) = \sum_i m(\lambda_{\bar{\tau}}, \tilde{\phi}_i) 1_{\mathcal{C}_i}(G_\alpha)/d_0(\tilde{\phi}_i),$$

where $m(\lambda_{\bar{\tau}}, \tilde{\phi}_i)$ is the multiplicity of the trivial representation contained in the restricted representation $\lambda_{\bar{\tau}}|_{\pi_0(Z(\tilde{\phi}_i))}$.

P(iv) If $\Pi_{\phi_{\bar{\tau}}}$ is generic, then we have (1-3), i.e.,

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}) + \dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = \sum_{\varphi \in F(\phi_{\bar{\tau}})} m(\lambda_{\bar{\tau}}, \varphi) \cdot \frac{\deg \Phi(\varphi)}{d_0(\varphi)}.$$

Let us start to verify the Langlands functoriality lift in the Prasad conjecture for PGSp_4, i.e., part **P**(i) and **P**(ii) listed above. Part **P**(iii) is the same with Theorem 1.3. Part **P**(iv) will be studied in detail in the next subsection.

Theorem 7.3.1. *Let $\bar{\tau}$ be a generic representation of* $\mathrm{PGSp}_4(E)$. *It is* $(\mathrm{PGSp}_4(F), \omega_{E/F})$*-distinguished if and only if there exists a parameter $\tilde{\phi}: WD_F \to \mathrm{Sp}_4(\mathbb{C})$ such that $\tilde{\phi}|_{WD_E} = \phi_{\bar{\tau}}$.*

Proof. Assume that $\tau = \theta(\Pi \boxtimes \chi)$ with $\chi = \mathbf{1}$, i.e., $\omega_\tau = \mathbf{1}$. Fix $s \in W_F \setminus W_E$.

(i) If $\bar{\tau}$ is $(\mathrm{PGSp}_4(F), \omega_{E/F})$-distinguished, then ϕ_Π is conjugate-symplectic and so $\Pi_{\phi^s_{\bar{\tau}}} = \Pi_{\phi^\vee_{\bar{\tau}}} = \Pi_{\phi_{\bar{\tau}}}$. If ϕ_Π is irreducible, then we can repeat the process in Section 6C1 to obtain that there exists a parameter $\tilde{\phi}: WD_F \to \mathrm{Sp}_4(\mathbb{C})$ such that $\tilde{\phi}|_{WD_E} = \phi_{\bar{\tau}}$. If $\phi_\Pi = \rho_1 \oplus \rho_2$ is reducible and ρ_1 is irreducible, then

$$\rho_1 \oplus \rho_2 = \rho_1^\vee \oplus \rho_2^\vee = \rho_1^s \oplus \rho_2^s$$

and either $\rho_1^s = \rho_2^\vee$ or both ρ_1 and ρ_2 are conjugate-symplectic.

- If $\rho_1^s=\rho_2^\vee$, then there are two subcases. If $\rho_2^\vee=\rho_2$, then $\rho_1^s=\rho_2$. Set $\tilde{\phi}=\mathrm{Ind}_{WD_E}^{WD_F}\rho_1$ if $\rho_1\neq\rho_2$. If $\rho_1=\rho_2=\rho_2^\vee$, then $\rho_1^s=\rho_1$ and so there exists a parameter $\tilde{\rho}_1:WD_F\to \mathrm{GL}_2(\mathbb{C})$ such that $\tilde{\rho}_1|_{WD_E}=\rho_1$. Set $\tilde{\phi}=\tilde{\rho}_1\oplus\tilde{\rho}_1^\vee$. If $\rho_2^\vee\neq\rho_2$, then $\rho_2^\vee=\rho_1$. Thus $\rho_1^s=\rho_1$ and $\tilde{\phi}=\tilde{\rho}_1\oplus\tilde{\rho}_1^\vee$.
- If both ρ_1 and ρ_2 are conjugate-symplectic, then

$$\tilde{\phi}=\begin{cases}\mathrm{Ind}_{WD_E}^{WD_F}\rho_1 & \text{if } \rho_1^s=\rho_2\neq\rho_1,\\ \tilde{\rho}_1\oplus\tilde{\rho}_1^\vee & \text{if } \rho_1^s=\rho_1.\end{cases}$$

If neither ρ_1 nor ρ_2 is irreducible, then $\phi_{\bar{\tau}}$ belongs to the endoscopic case. Thanks to Theorem 4.4.9(ii), either $\rho_1^s=\rho_2^\vee$ or both ρ_1 and ρ_2 are conjugate-symplectic. The argument is similar and we omit it here. Therefore, there exists $\tilde{\phi}:WD_F\to \mathrm{Sp}_4(\mathbb{C})$ such that $\tilde{\phi}|_{WD_E}=\phi_{\bar{\tau}}$.

(ii) Conversely, if there exists $\tilde{\phi}:WD_F\to\mathrm{Sp}_4(\mathbb{C})$ such that $\tilde{\phi}|_{WD_E}=\phi_{\bar{\tau}}$, then it suffices to show that ϕ_Π is conjugate-symplectic. (See Lemma 6.3.1.) The nongeneric member in the L-packet $\Pi_{\phi_{\bar{\tau}}}$ is not $(\mathrm{GSp}_4(F),\omega_{E/F})$-distinguished due to Theorem 4.4.9(i) if $|\Pi_{\phi_{\bar{\tau}}}|=2$. Assume that

$$\phi_{\bar{\tau}}:WD_E\to \mathrm{Sp}(V,\langle -,-\rangle)=\mathrm{Sp}_4(\mathbb{C})\quad\text{and}\quad \phi_\Pi=i\circ\phi_{\bar{\tau}}:WD_E\to\mathrm{GL}(V),$$

where $i:\mathrm{Sp}_4(\mathbb{C})\to\mathrm{GL}(V)$ is the embedding between the L-groups. Then we set

$$B(m,n)=\langle m,\tilde{\phi}(s)^{-1}n\rangle$$

for $m,n\in V$. It is easy to check that $B(\phi_\Pi(t)m,\phi_\Pi^s(t)n)=B(m,n)$ and

$$B(m,\phi_\Pi(s^2)n)=\langle m,\tilde{\phi}(s)n\rangle=-\langle\tilde{\phi}(s)n,m\rangle=-\langle n,\tilde{\phi}(s)^{-1}m\rangle=-B(n,m).$$

Therefore, the bilinear form B on V implies that ϕ_Π is conjugate-symplectic.

We have finished the proof. □

However, in order to verify (1-3), we will need many more results from Theorems 4.4.9 and 5.3.1. We will give the full detail in the next subsection.

7D. ***Proof of Theorem 1.3.*** This subsection focuses on the proof of Theorem 1.3. Before we give the proof of Theorem 1.3, we will use the results in Theorems 4.4.9 and 5.3.1 to study the equality (1-3) in detail. Then Theorem 1.3 will follow automatically. According to the Langlands parameter $\phi_{\bar{\tau}}$, we divide them into three cases:

- the endoscopic case,
- the discrete series but nonendoscopic case and
- $\phi_{\bar{\tau}}=\rho+\rho\nu$ with $\nu\neq\mathbf{1}$ and $\nu\det\rho=\mathbf{1}$.

Set $S_\phi=\pi_0(Z(\phi))$ to be the component group. We identify the characters of W_F and the characters of $F^\times$ via the local class field theory.

7D1. *Endoscopic case.* Given $\phi_{\bar{\tau}} = \phi_1 \oplus \phi_2$, there are two cases: $\phi_1 = \phi_2$ and $\phi_1 \neq \phi_2$.

(A) If $\phi_1 = \phi_2 = \rho$ are irreducible, then the L-packet $\Pi_{\phi_{\bar{\tau}}}$ equals $\{\pi^+, \pi^-\}$ and $S_{\phi_{\bar{\tau}}}$ equals $\mathbb{Z}/2\mathbb{Z}$, where π^- (resp. π^+) is a nongeneric (resp. generic) representation of $\mathrm{PGSp}_4(E)$. There are two subcases:

(A1) If ρ is conjugate-orthogonal, then

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^+, \omega_{E/F}) = 0 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^-, \omega_{E/F})$$

and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^-, \omega_{E/F}) = 1 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^+, \omega_{E/F}).$$

On the Galois side, there is only one extension $\tilde{\phi} = \bar{\rho} \oplus \bar{\rho} \cdot \omega_{E/F}$ with

$$\deg \Phi(\tilde{\phi}) = 2 \quad \text{and} \quad S_{\tilde{\phi}} = \{\mathbf{1}\} \to S_{\phi_{\bar{\tau}}},$$

where $\bar{\rho} : WD_F \to \mathrm{GL}_2(\mathbb{C}) \times W_F$ with $\det \bar{\rho} = \omega_{E/F}$. Note that $\tilde{\phi} = \tilde{\phi} \cdot \omega_{E/F}$. Then π^+ supports a period on the trivial pure inner form and π^- supports a period on a nontrivial pure inner form.

(A2) If ρ is conjugate-symplectic, then

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^-, \omega_{E/F}) = 0 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^-, \omega_{E/F})$$

and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^+, \omega_{E/F}) = 1, \quad \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^+, \omega_{E/F}) = 2.$$

In this case, ρ has two extensions $\bar{\rho}$ and $\bar{\rho} \cdot \omega_{E/F}$, where $\bar{\rho} : WD_F \to \mathrm{SL}_2(\mathbb{C})$. There are three choices for the extension $\tilde{\phi} : WD_F \to \mathrm{Sp}_4(\mathbb{C})$ with $\deg \Phi(\tilde{\phi}) = 1$:

- $\tilde{\phi}^{++} = \bar{\rho} \oplus \bar{\rho}$ with $S_{\tilde{\phi}^{++}} = \mathbb{Z}/2\mathbb{Z} \cong S_{\phi_{\bar{\tau}}}$;
- $\tilde{\phi}^{+-} = \bar{\rho} \oplus \bar{\rho} \cdot \omega_{E/F}$ with $S_{\tilde{\phi}^{+-}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to S_{\phi_{\bar{\tau}}}$(sum map);
- $\tilde{\phi}^{--} = \bar{\rho} \cdot \omega_{E/F} \oplus \bar{\rho} \cdot \omega_{E/F}$ with $S_{\tilde{\phi}^{--}} = \mathbb{Z}/2\mathbb{Z} \cong S_{\phi_{\bar{\tau}}}$.

The parameters $\tilde{\phi}^{++}$ and ϕ^{--} are in the same orbit under the twisting by $\omega_{E/F}$, which corresponds to both pure inner forms. The parameter $\tilde{\phi}^{+-}$ is fixed under twisting by $\omega_{E/F}$, which supports a period on the trivial pure inner form.

(A3) If ρ is not conjugate-self-dual, then both the Galois side and the automorphic side are 0.

(B) If $\phi_1 \neq \phi_2$ are both irreducible, then the L-packet of PGSp_4 is $\Pi_{\phi_{\bar{\tau}}} = \{\pi^{++}, \pi^{--}\}$ and

$$S_{\phi_{\bar{\tau}}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

(B1) If ϕ_1 and ϕ_2 both extend to L-parameters of $\mathrm{PGL}_2(F)$, i.e., both are conjugate-symplectic, then one has $\phi_1^s \neq \phi_2$,

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^{++}, \omega_{E/F}) = 2 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^{++}, \omega_{E/F})$$

and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^{--}, \omega_{E/F}) = 0 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^{--}, \omega_{E/F}).$$

On the Galois side, there are also four ways of extending $\phi_{\bar{\tau}}$. For each such extension $\tilde{\phi}$, one has $\deg \Phi(\tilde{\phi}) = 1$ and the equality of component group

$$S_{\tilde{\phi}} = S_{\phi_{\bar{\tau}}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Therefore only the representation π^{++} in the L-packet can support a period. And there are 2 orbits in $F(\phi_{\bar{\tau}})$ under twisting by $\omega_{E/F}$, each of size 2.

(B2) If ϕ_1 and ϕ_2 do not extend to L-parameters of $\mathrm{PGL}_2(F)$, but $\phi_1^s = \phi_2 = \phi_2^\vee$, then

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^{++}, \omega_{E/F}) = 0 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^{--}, \omega_{E/F})$$

and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\pi^{--}, \omega_{E/F}) = 1 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\pi^{++}, \omega_{E/F})$$

There is a unique way of extending $\phi_{\bar{\tau}} = \phi_1 \oplus \phi_2$ to $\tilde{\phi} : WD_F \to \mathrm{Sp}_4(\mathbb{C})$. Namely, $\tilde{\phi} = \mathrm{Ind}_{WD_E}^{WD_F} \phi_1$ is an irreducible 4-dimensional symplectic representation, with a component group

$$S_{\tilde{\phi}} = \mathbb{Z}/2\mathbb{Z} \hookrightarrow S_{\phi_{\bar{\tau}}} (\text{diagonal embedding}).$$

And $S_{\phi_{\bar{\tau}}}^{\mathrm{Gal}(E/F)} = S_{\tilde{\phi}}$. Thus π^{++} supports a period on the trivial pure inner form and π^{--} supports a period on the nontrivial pure inner form.

(C) If $\phi_1 = \chi_1 \oplus \chi_1^{-1}$ is reducible, then there is only one element in the L-packet, i.e., $|\Pi_{\phi_{\bar{\tau}}}| = 1$. There are two cases: $\phi_1 = \phi_2$ and $\phi_1 \neq \phi_2$.

(C1) If $\phi_1 = \phi_2$, there are three subcases.

(C1.i) If $\chi_1 = \chi_1^s = \chi_F|_{W_E}$, then $S_{\phi_{\bar{\tau}}} = 1$ and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = 2 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}).$$

- If $\chi_F^2 \neq \omega_{E/F}$, then there are two ways to extend L-parameters of $\mathrm{PGL}_2(F)$, denoted by $\bar{\rho}$ and $\bar{\rho} \cdot \omega_{E/F}$. Thus there are 3 ways of extending $\phi_{\bar{\tau}}$, which are $\tilde{\phi}^{++}, \tilde{\phi}^{--}$ and $\tilde{\phi}^{+-}$. Moreover, $\deg \Phi(\tilde{\phi}^{++}) = 1 = \deg \Phi(\tilde{\phi}^{--})$ and $\deg \Phi(\tilde{\phi}^{+-}) = 2$.
- If $\chi_F^2 = \omega_{E/F}$, then there is only one way to extend $\phi_{\bar{\tau}}$. Denote it by $\tilde{\phi}$. Then

$$\deg \Phi(\tilde{\phi}) = 4.$$

(C1.ii) If $\chi_1 \neq \chi_1^{-1}$ but $\chi_1|_{F^\times} = \omega_{E/F}$, then $S_{\phi_{\bar{\tau}}} = 1$ and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = 0 \text{ and } \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}) = 1.$$

There is only one way to extend ϕ_1, denoted by

$$\bar{\rho} = \mathrm{Ind}_{WD_E}^{WD_F} \chi_1 : WD_F \to \mathrm{SL}_2(\mathbb{C}).$$

Then $\tilde{\phi} = \bar{\rho} \oplus \bar{\rho}$ with $S_{\tilde{\phi}} = \mathbb{Z}/2\mathbb{Z}$ and $\deg \Phi(\tilde{\phi}) = 1$. Note that $\tilde{\phi} \cdot \omega_{E/F} = \tilde{\phi}$. Then $\tilde{\phi}$ supports a period on the trivial pure inner form.

(C1.iii) If $\chi_1 \neq \chi_1^{-1}$ but $\chi_1|_{F^\times} = \mathbf{1}$, then $S_{\phi_{\bar{\tau}}} = 1$ and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = 0 \text{ and } \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}) = 1.$$

On the Galois side, there is only one choice $\tilde{\phi} = \bar{\rho} \oplus \bar{\rho}$ and $S_{\tilde{\phi}} = \mathbf{1}$, where

$$\bar{\rho} = \mathrm{Ind}_{WD_E}^{WD_F} \chi_1 : WD_F \to \mathrm{GL}_2(\mathbb{C})$$

with $\det \rho = \omega_{E/F}$. Since $\tilde{\phi} = \tilde{\phi} \cdot \omega_{E/F}$, it picks up only the trivial pure inner form.

(C2) If $\phi_1 \neq \phi_2$, there are several subcases:

(C2.i) If $\chi_1 = \chi_1^s = \chi_F|_{W_E}$ and ϕ_2 is irreducible and conjugate-symplectic, then $S_{\phi_{\bar{\tau}}} = \mathbb{Z}/2\mathbb{Z}$ and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = 2 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}).$$

- If $\chi_F^2 \neq \omega_{E/F}$, then there are four ways of extending $\phi_{\bar{\tau}}$ and for each such extension $\tilde{\phi}$, one has $S_{\tilde{\phi}} = \mathbb{Z}/2\mathbb{Z} \cong S_{\phi_{\bar{\tau}}}$. There are two orbits under the twisting by $\omega_{E/F}$, each of size 2.
- If $\chi_F^2 = \omega_{E/F}$, then there are two ways of extending $\phi_{\bar{\tau}}$. For each such extension $\tilde{\phi}$, one has $\deg \Phi(\tilde{\phi}) = 2$. There is one orbit under the twisting by $\omega_{E/F}$.

In this case, the identity

$$\dim \mathrm{Hom}_{G_\alpha(F)}(\bar{\tau}, \chi_G) = \sum_i m(\lambda, \tilde{\phi}_i) \mathbf{1}_{C_i}(G_\alpha) \cdot \frac{\deg \Phi(\tilde{\phi}_i)}{d_0(\tilde{\phi}_i)} \tag{7-1}$$

holds for $G_\alpha = \mathrm{PGSp}_4$ and $\mathrm{PGSp}_{1,1}$.

(C2.ii) If $\chi_1 = \chi_1^s = \chi_F|_{W_E}$ and $\chi_2 = \chi_2^s = \chi'_F|_{W_E}$, where $\phi_2 = \chi_2 \oplus \chi_2^{-1}$, then $S_{\phi_{\bar{\tau}}} = 1$ and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = 2 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}).$$

- If neither χ_F^2 nor ${\chi'_F}^2$ equals $\omega_{E/F}$, then there are four ways of extending $\phi_{\bar{\tau}}$. There are two orbits under the twisting by $\omega_{E/F}$, each of size 2.
- If $\chi_F^2 = \omega_{E/F}$ and $\chi_F'^2 \neq \omega_{E/F}$, then there are two ways to extend $\phi_{\bar{\tau}}$ and for each such extension $\tilde{\phi}$, one has $S_{\tilde{\phi}} = 1 = S_{\phi_{\bar{\tau}}}$ and $\deg \Phi(\tilde{\phi}) = 2$. There is one orbit under the twisting by $\omega_{E/F}$, which corresponds to both pure inner forms.
- If $\chi_F^2 = \chi_F'^2 = \omega_{E/F}$, then there is only one way to extend $\phi_{\bar{\tau}}$. For this extension $\tilde{\phi}$, one has $\deg \Phi(\tilde{\phi}) = 4$.

(C2.iii) If $\chi_1 \neq \chi_1^{-1}$ but χ_1 is conjugate-symplectic, and ϕ_2 is irreducible and conjugate-symplectic, then $S_{\phi_{\bar{\tau}}} = \mathbb{Z}/2\mathbb{Z}$ and

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau}, \omega_{E/F}) = 1 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau}, \omega_{E/F}).$$

There are two extensions $\tilde{\phi}=\bar{\rho}_1\oplus\bar{\rho}_2$ or $\bar{\rho}_1\oplus\bar{\rho}_2\omega_{E/F}$ with $S_{\tilde{\phi}}=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, where $\bar{\rho}_i : WD_F\to \mathrm{SL}_2(\mathbb{C})$ satisfies $\bar{\rho}_i|_{WD_E}=\phi_i$. Here the map $S_{\tilde{\phi}}\to S_{\phi_{\bar{\tau}}}$ is given by

$$(x,y)\mapsto x+y.$$

There is one orbit under the twisting by $\omega_{E/F}$, which corresponds to both pure inner forms.

(C2.iv) If $\chi_1\neq\chi_1^{-1}$ but χ_1 is conjugate-symplectic, and $\chi_2=\chi_2^s=\chi'_F|_{W_E}$ where $\phi_2=\chi_2\oplus\chi_2^{-1}$, then $S_{\phi_{\bar{\tau}}}=1$ and

$$\dim\mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau},\omega_{E/F})=1=\dim\mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau},\omega_{E/F}).$$

- If $\chi_F'^2\neq\omega_{E/F}$, then there are two ways to extend $\phi_{\bar{\tau}}$. Set $\tilde{\phi}=\bar{\rho}_1\oplus\bar{\rho}_2$ or $\bar{\rho}_1\oplus\bar{\rho}_2\omega_{E/F}$ with $S_{\tilde{\phi}}=\mathbb{Z}/2\mathbb{Z}$. There is one orbit under the twisting by $\omega_{E/F}$, which corresponds to both pure inner forms.
- If $\chi_F'^2=\omega_{E/F}$, there is one way to extend $\phi_{\bar{\tau}}$. Set $\tilde{\phi}=\bar{\rho}_1\oplus\chi'_F\oplus\chi'_F\omega_{E/F}$, and

  $$\deg\Phi(\tilde{\phi})=2.$$

  Note that the identity (7-1) fails in this case while the identity (1-3) still holds.

(C2.v) If ϕ_1 and ϕ_2 are reducible and four different characters χ_1, χ_1^{-1}, χ_2 and χ_2^{-1} satisfy

$$\chi_1|_{F^\times}=\omega_{E/F}=\chi_2|_{F^\times},$$

then $S_{\phi_{\bar{\tau}}}$ is trivial,

$$\dim\mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau},\omega_{E/F})=0,$$

and $\dim\mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau},\omega_{E/F})=1$. There is only one extension $\tilde{\phi}=\bar{\rho}_1\oplus\bar{\rho}_2$ with $S_{\tilde{\phi}}=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. Since $\tilde{\phi}=\tilde{\phi}\cdot\omega_{E/F}$, it picks up the trivial pure inner form.

(C2.vi) If $\phi_1^s=\phi_2^\vee=\phi_2$ and ϕ_1 is not conjugate-symplectic, then $S_{\phi_{\bar{\tau}}}=1$ and

$$\dim\mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau},\omega_{E/F})=0,\quad \dim\mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau},\omega_{E/F})=1.$$

There is only one extension

$$\tilde{\phi}=\mathrm{Ind}_{WD_E}^{WD_F}\phi_1 : WD_F\to\mathrm{Sp}_4(\mathbb{C})$$

with the component group $S_{\tilde{\phi}}=\mathbb{Z}/2\mathbb{Z}$. Since $\tilde{\phi}=\tilde{\phi}\cdot\omega_{E/F}$, it picks up the trivial pure inner form.

It is easy to check that the identity (1-3) holds when $\Pi_{\phi_{\bar{\tau}}}$ is generic, i.e.,

$$\dim\mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar{\tau},\omega_{E/F})+\dim\mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar{\tau},\omega_{E/F})=\sum_{\tilde{\phi}\in F(\phi_{\bar{\tau}})}m(\lambda,\tilde{\phi})\cdot\frac{\deg\Phi(\tilde{\phi})}{d_0(\tilde{\phi})}.$$

7D2. *Discrete and nonendoscopic case.* Assume that $\phi_{\bar\tau}$ is irreducible and so $\Pi_{\phi_{\bar\tau}}$ is a singleton. Given a parameter $\phi_{\bar\tau}$, which is nonendoscopic, the theta lift $\Theta_4^+(\tau)$ from $\mathrm{PGSp}_4(E)$ to $\mathrm{PGSO}_{2,2}(\mathrm{E})$ is zero.

If $\phi_{\bar\tau}$ is conjugate-symplectic, then

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar\tau, \omega_{E/F}) = 1 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar\tau, \omega_{E/F}).$$

There are two extensions $\tilde\phi$ and $\tilde\phi\cdot\omega_{E/F}$ with a component group $S_{\tilde\phi} = S_{\phi_{\bar\tau}} = \mathbb{Z}/2\mathbb{Z}$. There is one orbit under the twisting by $\omega_{E/F}$, which corresponds to both pure inner forms.

7D3. *Generic but neither discrete nor endoscopic case.* If $\phi_{\bar\tau} = \rho \oplus \rho\nu$, $\det \rho = \nu^{-1} \neq \mathbf{1}$, then $S_{\phi_{\bar\tau}} = 1$. There are two cases:

- If $\phi_{\bar\tau}$ is conjugate-symplectic and $\rho^s = \rho$, then

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar\tau, \omega_{E/F}) = 1 = \dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar\tau, \omega_{E/F}).$$

  There are two extensions $\tilde\phi = \tilde\rho + \tilde\rho^\vee$ and $\tilde\phi\cdot\omega_{E/F}$ where $\tilde\rho : WD_F \to \mathrm{GL}_2(\mathbb{C})$ satisfies $\tilde\rho|_{WD_E} = \rho$.

- If $\phi_{\bar\tau}$ is conjugate-symplectic and $\rho^s \neq \rho$, then

$$\dim \mathrm{Hom}_{\mathrm{PGSp}_4(F)}(\bar\tau, \omega_{E/F}) = 1 \text{ and } \dim \mathrm{Hom}_{\mathrm{PGSp}_{1,1}(F)}(\bar\tau, \omega_{E/F}) = 0.$$

  There is only one extension $\tilde\phi = \mathrm{Ind}_{WD_E}^{WD_F}\rho$ such that $\tilde\phi|_{WD_E} = \phi_{\bar\tau}$.

Proof of Theorem 1.3. It follows from the discussions in the endoscopic cases (B)enumz in Section 7D1 and the discrete and nonendoscopic case in Section 7D2. □

7E. *Further discussion.* Let E be a quadratic extension over a nonarchimedean local field F. Let $\boldsymbol{G}$ be a quasisplit reductive group defined over F. Let τ be an irreducible representation of $\boldsymbol{G}(E)$ with an enhanced L-parameter (ϕ_τ, λ). Assume that $F(\phi_\tau) = \sqcup_i \mathcal{O}(\tilde\phi_i)$ where $\tilde\phi_i|_{WD_E} = \phi_\tau$.

If for each orbit $\mathcal{O}(\tilde\phi_i)$, the coset $\mathcal{C}_i \subset H^1(W_F, \boldsymbol{G})$ contains all pure inner forms satisfying $G_\alpha(E) = \boldsymbol{G}(E)$, then ϕ_τ is called a "full" L-parameter of $\boldsymbol{G}(E)$, in which case $1_{\mathcal{C}_i}(G_\alpha) \equiv 1$ in (7-1).

Assume that τ belongs to a generic L-packet with Langlands parameter $\phi_\tau : WD_E \to {}^L\boldsymbol{G}$ and that ϕ_τ is "full". Then there is a conjectural identity

$$\dim \mathrm{Hom}_{G_\alpha}(\tau, \chi_{\boldsymbol{G}}) = \sum_i m(\lambda, \tilde\phi_i)\cdot \frac{\deg \Phi(\tilde\phi_i)}{d_0(\tilde\phi_i)} \tag{7-2}$$

for any pure inner form $G_\alpha \in H^1(W_F, \boldsymbol{G})$ satisfying $G_\alpha(E) = \boldsymbol{G}(E)$.

If $H^1(W_F, \boldsymbol{G})$ is trivial, then any L-parameter ϕ_τ is "full". So the conjectural identity (7-2) holds for $\boldsymbol{G} = \mathrm{GL}_2$. In fact, it holds for $\boldsymbol{G} = \mathrm{PGL}_2$ as well.

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