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Teppei Takamatsu

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On a cohomological generalization of the Shafarevich conjecture for K3 surfaces

Teppei Takamatsu

The Shafarevich conjecture for K3 surfaces asserts the finiteness of isomorphism classes of K3 surfaces over a fixed number field admitting good reduction away from a fixed finite set of finite places. André proved this conjecture for polarized K3 surfaces of fixed degree, and recently She proved it for polarized K3 surfaces of unspecified degree. We prove a certain generalization of their results, which is stated by the unramifiedness of ℓ-adic étale cohomology groups for K3 surfaces over finitely generated fields of characteristic 0. As a corollary, we get the original Shafarevich conjecture for K3 surfaces without assuming the extendability of polarization, which is stronger than the results of André and She. Moreover, as an application, we get the finiteness of twists of K3 surfaces via a finite extension of characteristic 0 fields.

1. Introduction

The Shafarevich conjecture for abelian varieties is a remarkable result which asserts the finiteness of isomorphism classes of abelian varieties of a fixed dimension over a fixed number field admitting good reduction away from a fixed finite set of finite places. This theorem was proved by Faltings [1983] in the polarized case, and Zarhin [1985] in the unpolarized case.

In this paper, we shall prove an analogue of this theorem for K3 surfaces. For any discrete valuation field K and a K3 surface X over K, we say X has good reduction if X admits a smooth proper model over the valuation ring of K, as an algebraic space (see [Liedtke and Matsumoto 2018, Section 1]).[1] Then one can formulate the analogue of the Shafarevich conjecture for K3 surfaces. Previously, this conjecture was studied in [André 1996; She 2017] for polarized K3 surfaces. The goal of this paper is to generalize these results in terms of the unramifiedness of ℓ-adic étale cohomology groups. Our main theorem is the following (for a more generalized form, see Theorem 6.1.1).

Theorem 1.0.1 (compare with Theorem 6.1.1). *Let F be a finitely generated field over $\mathbb{Q}$, and R be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field F. Then, the set*

$$\mathrm{Shaf}(F, R) := \left\{ X \;\middle|\; \begin{array}{l} X : K3 \text{ surface over } F, \\ \text{for any height } 1 \text{ prime ideal } \mathfrak{p} \in \operatorname{Spec} R, \\ \text{there exists a prime number } \ell \notin \mathfrak{p} \\ \text{such that } H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_\ell) \text{ is unramified at } \mathfrak{p} \end{array} \right\} / F\text{-isom}$$

is finite.

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As a corollary, we have the original Shafarevich conjecture for K3 surfaces over finitely generated fields of characteristic 0.

Corollary 1.0.2 (Corollary 6.1.4). *Let F be a finitely generated field over $\mathbb{Q}$, and R be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field F. Then, the set*

$$\left\{X \mid X : K3 \text{ surface over } F,\ X \text{ has good reduction at any height } 1 \text{ prime ideal } \mathfrak{p} \in \operatorname{Spec} R\right\}/F\text{-isom}$$

is finite.

Note that our results are stronger than results of André and She (see Remark 1.0.4 for details). Moreover, as an application of our cohomological generalization, we get the following corollary, which asserts the finiteness of twists of a K3 surface via a finite extension of characteristic 0 fields.

Corollary 1.0.3 (Corollary 6.2.1). *Let F be a field of characteristic 0, E/F be a finite extension, and X be a K3 surface over F. Then, the set*

$$\operatorname{Tw}_{E/F}(X) := \{Y : K3 \text{ surface over } F \mid Y_E \simeq_E X_E\}/F\text{-isom}$$

is finite. Here $Y_E \simeq_E X_E$ means the K3 surfaces $Y_E := Y \otimes_F E$ and $X_E := X \otimes_F E$ are isomorphic over E.

We note that our cohomological generalization is necessary for this application, i.e., the original statement of the Shafarevich conjecture (Corollary 1.0.2) is not enough to show Corollary 1.0.3.

Let us give some comments on the statement of Theorem 1.0.1. Theorem 1.0.1 is motivated by the good reduction criterion for K3 surfaces given by Liedtke and Matsumoto [2018]. For K3 surfaces over a Henselian discrete valuation field satisfying some assumptions, they showed the equivalence between the unramifiedness of ℓ-adic étale cohomology groups and admitting good reduction after a finite unramified extension [Liedtke and Matsumoto 2018, Theorem 1.3]. Note that the latter condition cannot be replaced by "admitting good reduction" (see [Liedtke and Matsumoto 2018, Theorem 1.6]), so our cohomological generalization is stronger than the original Shafarevich conjecture. Moreover, we deal with finitely generated fields of characteristic 0 rather than number fields, motivated by the application to Corollary 1.0.3. In fact, André also proved the Shafarevich conjecture for polarized K3 surfaces in this way (see [André 1996, Theorem 9.1.1], and also Remark 1.0.4).

Remark 1.0.4. Our results are stronger than previous results obtained by André and She. To explain this, we briefly recall their results. André [1996, Theorem 9.1.1] proved the Shafarevich conjecture for polarized K3 surfaces, i.e., the finiteness of isomorphism classes of polarized K3 surfaces of fixed degree over a fixed number field which admit good reduction away from a fixed finite set of finite places (actually, as stated above, André dealt with finitely generated fields of characteristic 0). Here, André said that a polarized K3 surface (X, L) admits good reduction if there exists a smooth proper model $\mathcal{X}$ of X as a scheme such that the ample line bundle L extends to an ample line bundle on $\mathcal{X}$. She [2017, Theorem 1.1.5] proved it for

[1] Note that it is natural to admit an integral model being an algebraic space rather than a scheme in the case of K3 surfaces (see [Matsumoto 2015, Section 5.2]).

polarized K3 surfaces of unspecified degree. More correctly, She proved the finiteness of K3 surfaces over a fixed number field which admit good reduction as polarized K3 surfaces (without fixing polarization degree) away from a fixed finite set of finite places. Here, we remark that She's result does not cover K3 surfaces admitting a smooth proper model only as an algebraic space. Moreover, there exists an example of a K3 surface admitting good reduction such that no smooth proper model has a polarization (therefore this K3 surface does not admit good reduction as polarized K3 surfaces) (see [Matsumoto 2015, Section 5.2]). Therefore, Corollary 1.0.2 is also stronger than previous results, even in the number field case.

The strategy of the proof of Theorem 1.0.1 is as follows. We basically take the approach of André and She. We first show the polarized version of Theorem 1.0.1 before dealing with the unpolarized case. To generalize the result obtained by André, we should formulate the Kuga–Satake construction as preserving the finiteness. We achieve this by using the moduli interpretation of the Kuga–Satake construction introduced by Rizov [2010]. In the perspective of the unpolarized case, we use the uniform Kuga–Satake construction introduced by She to study K3 surfaces of all degrees simultaneously. Our proof is slightly different from She's proof, and here we will sketch the differences. In [She 2017], it is crucial to show that K3 surfaces admitting good reduction are sent to abelian varieties admitting good reduction via the uniform Kuga–Satake map. She proves this using integral canonical models of certain Shimura varieties (the argument like "$\mathcal{O}$-valued points go to $\mathcal{O}$-valued points"). However, in our case, we do not assume that each K3 surface admits a smooth proper model, so instead of She's method, we use the Néron–Ogg–Shafarevich criterion for abelian varieties (this approach is already known by [André 1996]; see also [Imai and Mieda 2020]). For this purpose, we study She's uniform Kuga–Satake construction in detail in Section 3. Note that our proof does not require the theory of integral canonical models of Shimura varieties. To simplify the arguments, we need a section of the natural map from the GSpin Shimura variety to the SO Shimura variety (see Section 3.3). To get such a section, we should work with a level structure that has a sufficiently small $\mathbb{Z}_2$-component (see Remark 3.1.7). So we should suppose the unramifiedness of 2-adic representation to overcome that general K3 surfaces may not admit level structure (see Proposition 4.2.3). However, this assumption is not essential since ℓ-independence of the unramifiedness is true in a general situation (see Lemma 5.0.1). Note that Lemma 5.0.1 is essentially known by Madapusi Pera, Matsumoto [2016] and Imai and Mieda [2020] (see Remark 5.0.2).

The outline of this paper is as follows. In Section 2, we will recall the basic results on K3 surfaces, and define the moduli space of K3 surfaces introduced by Rizov and Madapusi Pera. In Section 3, we will define several algebraic groups to introduce the uniform Kuga–Satake abelian varieties, and study their basic properties. In Section 4, we will prove the main theorem in a little weaker form (i.e., only considering 2-adic cohomology) by using the results of Section 3 and the arguments given by André and She. In Section 5, we will see an ℓ-independence of the unramifiedness by using Matsumoto's result on weight filtrations [2016, Theorem 3.3]. We also prove a crystalline analogue of it by using Ochiai's ℓ-independence results [1999] and the Kuga–Satake abelian varieties (as above, this result is essentially proved in [Imai and Mieda 2020]). In Section 6, we will complete the proof of the main theorem combining the results in Sections 4 and 5, and prove the finiteness of twists via a finite extension of characteristic 0 fields.

2. K3 surfaces and their moduli

2.1. ***Basic definitions for K3 surfaces.*** In this subsection, we give definitions and basic notations about K3 surfaces.

Definition 2.1.1. (1) For any field k, a *K3 surface over k* is a smooth proper surface X over k with $\Omega^2_{X/k} \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

(2) For any scheme S, a *K3 family over S* is a smooth proper algebraic space $\mathcal{X}$ over S whose geometric fibers are K3 surfaces.

Remark 2.1.2. For any field k, a K3 family over k is automatically a K3 surface over k since smooth proper algebraic spaces of dimension 2 over a field are schemes.

Definition 2.1.3 [Rizov 2006, Definitions 3.2.2, 3.2.3]. (1) A *polarization* on a K3 family $\pi : \mathcal{X} \to S$ is an element $\lambda \in \mathrm{Pic}_{\mathcal{X}/S}(S)$ whose pullback by any geometric point of S is an ample line bundle. Here $\mathrm{Pic}_{\mathcal{X}/S}$ is the relative Picard functor.

(2) A polarization λ is *primitive* if its pullback by any geometric point of S is primitive, i.e., not divisible by an integer greater than 1.

(3) A polarization λ is *of degree* $2d$ if its pullback by any geometric point of S has degree $2d$, i.e., its self intersection number is $2d$.

Remark 2.1.4. Let F be a subfield of $\mathbb{C}$. For a K3 surface X over F, the relative Picard functor $\mathrm{Pic}_{X/F}$ is represented by a scheme, thus $\mathrm{Pic}_{X/F}(F) = \mathrm{Pic}(X_{\bar{F}})^{\mathrm{Gal}(\bar{F}/F)}$ is a primitive sublattice in $H^2(X(\mathbb{C}), \mathbb{Z}(1))$ via the Chern class map as in [She 2017, Lemma 2.2.3]. Hence there exists a primitive polarization for each X (by dividing a polarization by an integer greater than 1 if necessary). Note that the inclusion $\mathrm{Pic}(X) \subset \mathrm{Pic}_{X/F}(F)$ may be proper in general, though it always has a finite cokernel (see [Huybrechts 2016, Chapter 17, Section 2.2]).

Definition 2.1.5. (1) A *K3 lattice* $\mathcal{L}_{K3}$ is a unimodular lattice[2] of signature (19, 3) which is defined as

$$\mathcal{L}_{K3} := \mathbb{E}_8^{\oplus 2} \oplus \mathbb{H}^{\oplus 3},$$

where $\mathbb{E}_8$ is the (positive signature) E_8-lattice as in [Huybrechts 2016, Chapter 14, Example 0.3], and $\mathbb{H}$ is the hyperbolic plane.

(2) Consider the last component $\mathbb{H} \subset \mathcal{L}_{K3}$, and take $e, f \in \mathbb{H} \subset \mathcal{L}_{K3}$ satisfying

$$(e, f) = (f, e) = 1, \quad (e, e) = (f, f) = 0.$$

Let $v_d := e - df$. Then the *degree $2d$ primitive part of $\mathcal{L}_{K3}$* is defined as

$$\mathcal{L}_d := v_d^{\perp} \simeq \mathbb{E}_8^2 \oplus \mathbb{H}^2 \oplus \langle 2d \rangle.$$

The lattice $\mathcal{L}_d$ is a primitive sublattice of $\mathcal{L}_{K3}$, and $\mathrm{disc}(\mathcal{L}_d) = 2d$.

[2] In this paper, a ($\mathbb{Z}$-)lattice means a finite free $\mathbb{Z}$-module with a symmetric bilinear pairing valued in $\mathbb{Z}$, and $\langle c \rangle$ means the $\mathbb{Z}$-lattice of rank 1 given by $(a, b) = cab$.

Remark 2.1.6 [Rizov 2006, Remark 2.3.2]. For a K3 surface X over $\mathbb{C}$ and its primitive polarization L of degree $2d$, there exists an isomorphism

$$(H^2(X(\mathbb{C}), \mathbb{Z}(1)), -\cup) \simeq \mathcal{L}_{K3}$$

which sends $\mathrm{ch}_{\mathbb{Z}}(L)$ to v_d. Here $-\cup$ denotes the minus of the cup product. Therefore, for a primitively polarized K3 surface (X, L) of degree $2d$ over a field F which is contained in $\mathbb{C}$, we sometimes identify $H^2(X(\mathbb{C}), \mathbb{Z}(1))$ with $\mathcal{L}_{K3}$, $H^2_{\text{ét}}(X_{\bar{F}}, \widehat{\mathbb{Z}}(1))$ with $\mathcal{L}_{K3,\widehat{\mathbb{Z}}} := \mathcal{L}_{K3} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, $P^2((X(\mathbb{C}), L_{\mathbb{C}}), \mathbb{Z}(1))$ with $\mathcal{L}_d$, and $P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \widehat{\mathbb{Z}}(1))$ with $\mathcal{L}_{d,\widehat{\mathbb{Z}}} := \mathcal{L}_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. Here, we denote the primitive parts of the singular cohomology group and the étale cohomology group by

$$P^2((X(\mathbb{C}), L_{\mathbb{C}}), \mathbb{Z}(1)) := \mathrm{ch}(L_{\mathbb{C}})^{\perp} \subset H^2(X(\mathbb{C}), \mathbb{Z}),$$
$$P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \widehat{\mathbb{Z}}(1)) := \mathrm{ch}_{\widehat{\mathbb{Z}}}(L)^{\perp} \subset H^2_{\text{ét}}(X_{\bar{F}}, \widehat{\mathbb{Z}}(1)).$$

To simplify notation, we omit the pairing in the rest of the paper. We denote $(H^2(X(\mathbb{C}), \mathbb{Z}(1)), -\cup)$ by $H^2(X(\mathbb{C}), \mathbb{Z}(1))$, and same with others.

Definition 2.1.7. The *discriminant kernel of* $\mathcal{L}_d$ is

$$D_d := \{g \in \mathrm{SO}(\mathcal{L}_{d,\widehat{\mathbb{Z}}}) \mid g \text{ acts trivially on } \mathcal{L}^{\vee}_{d,\widehat{\mathbb{Z}}}/\mathcal{L}_{d,\widehat{\mathbb{Z}}}\}.$$

Note that D_d is a compact open subgroup of $\mathrm{SO}(\mathcal{L}_{d,\mathbb{A}_f})$. For any prime number ℓ, we denote its $\mathbb{Z}_\ell$-component by $(D_d)_\ell$.

Proposition 2.1.8 [Madapusi Pera 2016, Lemma 2.6]. *There is a natural identification*

$$D_d = \{\tilde{g} \in \mathrm{SO}(\mathcal{L}_{K3,\widehat{\mathbb{Z}}}) \mid \tilde{g}(v_d) = v_d\}.$$

Proof. This is proved in [Madapusi Pera 2016, Lemma 2.6]. We include its proof because we need to recall the identification explicitly. Let ℓ be any prime number, and we will verify this claim for each $\mathbb{Z}_\ell$-component. First, we will define a map from the left-hand side to the right-hand side. For

$$g_\ell \in (D_d)_\ell = \{g \in \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell}) \mid g \text{ acts trivially on } \mathcal{L}^{\vee}_{d,\mathbb{Z}_\ell}/\mathcal{L}_{d,\mathbb{Z}_\ell}\},$$

define $\tilde{g}_\ell$ as the image of g_ℓ via the composition of

$$\mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell}) \hookrightarrow \mathrm{SO}(\mathcal{L}_{d,\mathbb{Q}_\ell}) \hookrightarrow \mathrm{SO}(\mathcal{L}_{K3,\mathbb{Q}_\ell}).$$

Then we have $\tilde{g}_\ell v_d = v_d$. We will show that $\tilde{g}_\ell \mathcal{L}_{K3,\mathbb{Z}_\ell} = \mathcal{L}_{K3,\mathbb{Z}_\ell}$. Consider the morphisms

$$\mathcal{L}_{K3,\mathbb{Z}_\ell} \simeq \mathcal{L}^{\vee}_{K3,\mathbb{Z}_\ell} \hookrightarrow \mathcal{L}^{\vee}_{d,\mathbb{Z}_\ell} \oplus \langle v_d \rangle^{\vee} \twoheadrightarrow \mathcal{L}^{\vee}_{d,\mathbb{Z}_\ell}. \tag{1}$$

For any $v \in \mathcal{L}_{K3,\mathbb{Z}_\ell}$, denote its image in $\mathcal{L}^{\vee}_{d,\mathbb{Z}_\ell} \oplus \langle v_d \rangle^{\vee}$ by $u_1 + u_2$. Then we have

$$\tilde{g}_\ell(u_1 + u_2) = g_\ell(u_1) + u_2 = (g_\ell(u_1) - u_1) + (u_1 + u_2) \in \mathcal{L}_{K3,\mathbb{Z}_\ell},$$

because g_ℓ acts trivially on $\mathcal{L}^{\vee}_{d,\mathbb{Z}_\ell}/\mathcal{L}_{d,\mathbb{Z}_\ell}$. Hence $\tilde{g}_\ell^{\pm 1}\mathcal{L}_{K3,\mathbb{Z}_\ell} \subset \mathcal{L}_{K3,\mathbb{Z}_\ell}$, thus $\tilde{g}_\ell \mathcal{L}_{K3,\mathbb{Z}_\ell} = \mathcal{L}_{K3,\mathbb{Z}_\ell}$, and we can define the desired map.

Next, we will define a map from the right-hand side to the left-hand side. For $\tilde{h}_\ell \in \mathrm{SO}(\mathcal{L}_{K3,\mathbb{Z}_\ell})$ such that $\tilde{h}_\ell v_d = v_d$, we can associate $h_\ell \in \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell})$ as the restriction of $\tilde{h}_\ell$. We can show that h_ℓ acts trivially on $\mathcal{L}_{d,\mathbb{Z}_\ell}^\vee/\mathcal{L}_{d,\mathbb{Z}_\ell}$. Indeed, because the embedding $\mathcal{L}_d \hookrightarrow \mathcal{L}_{K3}$ is primitive, the composition of (1) is surjective, so for any $u_1 \in \mathcal{L}_{d,\mathbb{Z}_\ell}^\vee$, there exists $u_2 \in \langle v_d \rangle^\vee$ such that $u_1 + u_2 \in \mathcal{L}_{K3,\mathbb{Z}_\ell}$. Thus we have

$$h_\ell(u_1) - u_1 = \tilde{h}_\ell(u_1 + u_2) - (u_1 + u_2) \in \mathcal{L}_{K3,\mathbb{Z}_\ell} \cap \mathcal{L}_{d,\mathbb{Q}_\ell} = \mathcal{L}_{d,\mathbb{Z}_\ell}.$$

Clearly, the above maps are inverses of each other. This finishes the proof. □

2.2. ***Moduli spaces of K3 surfaces and the Torelli theorem.*** In this subsection, we recall the Torelli theorem in terms of moduli spaces. First, we recall the definition of the moduli space of K3 surfaces with oriented level structures. See [Rizov 2006, Section 6; Madapusi Pera 2015, Section 3; Ito et al. 2018, Section 5] for details.

We define a groupoid-valued moduli functor $M^\circ_{2d,\mathbb{Q}}$ by

$$M^\circ_{2d,\mathbb{Q}}(S) := \left\{(\pi : \mathcal{X} \to S, \lambda \in \mathrm{Pic}_{X/S}(S)) \,|\, \pi : \text{K3 family over } S,\ \lambda : \text{primitive polarization of degree } 2d\right\}$$

for any $\mathbb{Q}$-scheme S. Let $\widetilde{M}^\circ_{2d,\mathbb{Q}}$ be the twofold finite étale cover constructed by Madapusi Pera [2015, Section 5] which parametrizes orientations. Then, for any $S \to \widetilde{M}^\circ_{2d,\mathbb{Q}}$ we get (π, λ, ν), where (π, λ) is as above, and ν is an isometry of $\widehat{\mathbb{Z}}$-local systems

$$\nu : \underline{\det \mathcal{L}_{d,\widehat{\mathbb{Z}}}} \simeq \det P^2\pi_*\widehat{\mathbb{Z}}$$

such that for any $s \in S(\mathbb{C})$, the isometry ν restricts to an isometry

$$\nu_s : \det \mathcal{L}_d \simeq \det P^2(\mathcal{X}_s, \mathbb{Z}).$$

Here, we put

$$P^2\pi_*\widehat{\mathbb{Z}}(1) := \mathrm{ch}_{\widehat{\mathbb{Z}}}(\lambda)^\perp \subset R^2\pi_*\widehat{\mathbb{Z}}(1),$$

where $\mathrm{ch}_{\widehat{\mathbb{Z}}}(\lambda)$ is the Chern class of λ [Madapusi Pera 2015, Section 3.10]. Let $\mathbb{K} \subset D_d$ be a compact open subgroup. For any scheme S over $\widetilde{M}^\circ_{2d,\mathbb{Q}}$, one can define the étale sheaf I by

$$I(T) := \left\{ g : \underline{\mathcal{L}_{K3,\widehat{\mathbb{Z}}}} \to R^2\pi|_{T*}\widehat{\mathbb{Z}}(1) \;\middle|\; g : \text{isometry},\ g(v_d) = \mathrm{ch}_{\widehat{\mathbb{Z}}}(\lambda),\ \det g \text{ induces } \nu|_T \right\},$$

for any étale morphism $T \to S$. A $\mathbb{K}$-level structure on $S \to \widetilde{M}^\circ_{2d,\mathbb{Q}}$ is a section $\alpha \in H^0(S, I/\mathbb{K})$, where $\mathbb{K}$ acts on I through $\mathcal{L}_{K3,\widehat{\mathbb{Z}}}$. Then, one can define the moduli functor $M^\circ_{2d,\mathbb{K},\mathbb{Q}}$ over $\widetilde{M}^\circ_{2d,\mathbb{Q}}$ which parametrizes $\mathbb{K}$-level structures. For simplicity, we write an each element of $M^\circ_{2d,\mathbb{K},\mathbb{Q}}(S)$ as $(\mathcal{X}, \lambda, \nu, \alpha)$. Moreover, for any field F of characteristic 0, we denote the base change by $M^\circ_{2d,\mathbb{K},F}$.

Definition 2.2.1. (1) $\mathrm{SO}_{\mathcal{L}_d}$ is an algebraic group over $\mathbb{Q}$ whose R-valued points are given by

$$\mathrm{SO}_{\mathcal{L}_d}(R) := \{g \in \mathrm{SL}(\mathcal{L}_{d,R}) \mid (gv, gw) = (v, w),\ \text{for any } v, w \in \mathcal{L}_{d,R}\}.$$

(2) We put

$$\Omega^{\pm}_{\mathrm{SO}_{\mathcal{L}_d}} := \{\text{oriented negative definite planes in } \mathcal{L}_{d,\mathbb{R}}\}.$$

Then $\Omega^{\pm}_{\mathrm{SO}_{\mathcal{L}_d}}$ is naturally identified with $X_{\mathrm{SO}_{\mathcal{L}_d}}$ which gives the Shimura datum $(\mathrm{SO}_{\mathcal{L}_d}, X_{\mathrm{SO}_{\mathcal{L}_d}})$ with a reflex field $\mathbb{Q}$. Actually, $X_{\mathrm{SO}_{\mathcal{L}_d}}$ is isomorphic to $X_{\mathrm{GSpin}_{\mathcal{L}_d}}$ which is defined as in Definition 3.2.3 via the adjoint representation.

Here, we quickly state the moduli interpretation of the Torelli theorem over $\mathbb{Q}$.

Proposition 2.2.2 (the Torelli theorem [Madapusi Pera 2015, Corollary 5.4, Theorem 5.8]). *Let $\mathbb{K} \subset D_d$ be a compact open subgroup. Moreover, assume that $\mathbb{K}$ is contained in the principal level n congruence subgroup of $\mathrm{SO}(\mathcal{L}_{d,\widehat{\mathbb{Z}}})$ with $n \geq 3$. Then $M^{\circ}_{2d,\mathbb{K},\mathbb{Q}}$ is representable by a scheme, and moreover there is the period map which is an étale morphism between $\mathbb{Q}$-schemes*

$$j : M^{\circ}_{2d,\mathbb{K},\mathbb{Q}} \to \mathrm{Sh}_{\mathbb{K}}(\mathrm{SO}_{\mathcal{L}_d}, X_{\mathrm{SO}_{\mathcal{L}_d}}).$$

Here $\mathrm{Sh}_{\mathbb{K}}(\mathrm{SO}_{\mathcal{L}_d}, X_{\mathrm{SO}_{\mathcal{L}_d}})$ is the canonical model of the Shimura variety over $\mathbb{Q}$.

In Proposition 3.3.3, we will use the more detailed properties of the period map j.

3. The uniform Kuga–Satake construction

In this section, we recall the definition and properties of the Kuga–Satake construction. In this section, we use only the uniform Kuga–Satake construction introduced by She. In fact, the classical Kuga–Satake construction is enough for proving the polarized case (Theorem 4.1.3), but we need She's methods to prove the unpolarized case (Theorem 4.1.4). Hence we omit the classical Kuga–Satake construction to avoid some repetitions.

3.1. ***Preparation.*** In this subsection and the next, we will define several algebraic groups and their adelic subgroups which play an important role in the Kuga–Satake construction. In this subsection, we discuss objects related with the lattice $\mathcal{L}_d$.

For any algebra R and any quadratic space $\mathcal{N}$ over R, we denote the Clifford algebra (resp. even Clifford algebra) of $\mathcal{N}$ by $C(\mathcal{N})$ (resp. $C^+(\mathcal{N})$).

Definition 3.1.1. $\mathrm{GSpin}_{\mathcal{L}_d}$ is an algebraic group over $\mathbb{Q}$, whose R-valued points are given by

$$\mathrm{GSpin}_{\mathcal{L}_d}(R) := \{z \in C^+(\mathcal{L}_{d,R})^{\times} \mid z\mathcal{L}_{d,R}z^{-1} = \mathcal{L}_{d,R}\}.$$

Remark 3.1.2. (1) There exists the following natural homomorphism of algebraic groups over $\mathbb{Q}$

$$f_d : \mathrm{GSpin}_{\mathcal{L}_d} \to \mathrm{SO}_{\mathcal{L}_d};\ g \mapsto (l \mapsto glg^{-1}).$$

(2) For any $\mathbb{Z}$-algebra R, we put

$$\mathrm{GSpin}(\mathcal{L}_{d,R}) := \{z \in C^+(\mathcal{L}_{d,R})^{\times} \mid z\mathcal{L}_{d,R}z^{-1} = \mathcal{L}_{d,R}\}.$$

Then, for any prime number ℓ, we can define

$$f_d : \mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell}) \to \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell})$$

by the conjugation. Moreover, it is easy to confirm the identity

$$\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell}) = \mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Q}_\ell}) \cap C^+(\mathcal{L}_{d,\mathbb{Z}_\ell})^\times.$$

(3) For any $\mathbb{Z}$-algebra R, we will use the notation $\mathrm{GSpin}(\mathcal{L}_{K3,R})$ in a similar sense to (2). Moreover, for any prime number ℓ, we denote the conjugation map $\mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_\ell}) \to \mathrm{SO}(\mathcal{L}_{K3,\mathbb{Z}_\ell})$ by f_{K3}. As in (2), it follows that

$$\mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_\ell}) = \mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Q}_\ell}) \cap C^+(\mathcal{L}_{K3,\mathbb{Z}_\ell})^\times.$$

Lemma 3.1.3 [Madapusi Pera 2016, (2.6.1)]. *Let ℓ be any prime number. Through the natural inclusion $C^+(\mathcal{L}_{d,\mathbb{Z}_\ell}) \subset C^+(\mathcal{L}_{K3,\mathbb{Z}_\ell})$, we have*

$$C^+(\mathcal{L}_{d,\mathbb{Z}_\ell}) = \{z \in C^+(\mathcal{L}_{K3,\mathbb{Z}_\ell}) \mid v_d z = z v_d\}.$$

Moreover, the above inclusion induces an embedding

$$\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell}) \subset \mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_\ell}).$$

Proof. The first claim is essentially proved in [Madapusi Pera 2016, (2.6.1)]. For the sake of completeness, we recall the proof. For the first claim, both sides of the desired identity are primitive $\mathbb{Z}_\ell$-modules in $C^+(\mathcal{L}_{K3,\mathbb{Z}_\ell})$. Thus, it is enough to show that

$$C^+(\mathcal{L}_{d,\mathbb{Q}_\ell}) = \{z \in C^+(\mathcal{L}_{K3,\mathbb{Q}_\ell}) \mid v_d z = z v_d\}.$$

It can be easily verified by using a basis of $\mathcal{L}_{K3,\mathbb{Q}_\ell}$ which is given by a basis of $\mathcal{L}_{d,\mathbb{Q}_\ell}$ and v_d. For the second claim, by Remark 3.1.2 (2) and (3), we can reduce the problem to the obvious inclusion $\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Q}_\ell}) \subset \mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Q}_\ell})$. □

Definition 3.1.4 [André 1996, Section 4.4; Rizov 2006, Example 5.1.4]. For any positive integer n, we define a compact open subgroup $\mathbb{K}^{\mathrm{sp}}_{d,n} \subset \mathrm{GSpin}_{\mathcal{L}_d}(\mathbb{A}_f)$ by

$$\mathbb{K}^{\mathrm{sp}}_{d,n} := \{g \in \mathrm{GSpin}(\mathcal{L}_{d,\widehat{\mathbb{Z}}}) \mid g = 1 \text{ in } C^+(\mathcal{L}_{d,\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}})\}.$$

Proposition 3.1.5 (cf. [André 1996, Section 4.4; Madapusi Pera 2015, Section 4.4]).

$$D_d(n) := f_d(\mathbb{K}^{\mathrm{sp}}_{d,n}) \subset \mathrm{SO}(\mathcal{L}_{d,\widehat{\mathbb{Z}}})$$

is a compact open subgroup of D_d.

Proof. First, we shall show that $D_d(n)$ is contained in D_d. Lemma 3.1.3 shows that

$$f_{K3}(\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell})) \subset \{g \in \mathrm{SO}(\mathcal{L}_{K3,\mathbb{Z}_\ell}) \mid g v_d = v_d\},$$

thus the desired inclusion follows from Proposition 2.1.8.

For the openness, it is enough to show that for any ℓ not dividing $2dn$, the $\mathbb{Z}_\ell$-component of $D_d(n)$ is equal to $\mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell})$. This follows from [André 1996, Section 4.4]. □

The following proposition gives more information about $D_d(n)$.

Proposition 3.1.6. *For any odd prime number $\ell \neq 2$, we have*

$$f_d(\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell})) = (D_d)_\ell.$$

If $\ell = 2$, as a subset of $\mathrm{SO}(\mathcal{L}_{K3,\mathbb{Z}_2})$, we have

$$f_d(\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_2})) = (D_d)_2 \cap f_{K3}(\mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_2})).$$

Proof. If ℓ does not divide $2d$, these results are essentially shown in the proof of Proposition 3.1.5.

First, for any prime number ℓ, we have $f_d(\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell})) \subset (D_d)_\ell$ as in the proof of Proposition 3.1.5. We assume $\ell \neq 2$. For any $g \in (D_d)_\ell \subset \mathrm{SO}(\mathcal{L}_{K3,\mathbb{Z}_\ell})$, by the same argument as in [André 1996, Section 4.4] (here we use $\ell \neq 2$), there exists $z \in \mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_\ell})$ such that $f_{K3}(z) = g$. Proposition 2.1.8 implies $z v_d z^{-1} = v_d$, and so in fact, $z \in C^+(\mathcal{L}_{d,\mathbb{Z}_\ell})^\times$ by Lemma 3.1.3. By Proposition 2.1.8, z stabilizes $\mathcal{L}_{d,\mathbb{Z}_\ell}$ via conjugation; thus $z \in \mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell})$, finishing the proof of the first claim. If $\ell = 2$, the second claim follows by the same arguments. □

Remark 3.1.7. Unfortunately, if $\ell = 2$, we have $f_d(\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_2})) \neq (D_d)_2$. Indeed, there exists $g_2 \in (D_d)_2$ which is nontrivial in $\mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}/2\mathbb{Z}})$ (for example, permutation of two components $\mathbb{H}_{\mathbb{Z}_2} \subset \mathcal{L}_{d,\mathbb{Z}_2}$), though any element in the image of f_d is trivial there.

Corollary 3.1.8. *Let $(D_d(n))_\ell$ be the $\mathbb{Z}_\ell$-component of $D_d(n)$, and n_ℓ be the ℓ-part of n. Then, for any prime number $\ell \neq 2$, we have*

$$[(D_d)_\ell : (D_d(n))_\ell] \leq n_\ell^{(2^{20})}.$$

Moreover, there exists a positive integer N which is independent of d and n such that

$$[(D_d)_2 : (D_d(n))_2] \leq N \cdot n_2^{(2^{20})}.$$

Proof. Assume $\ell \neq 2$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell}) & \twoheadrightarrow & (D_d)_\ell \\ \uparrow & & \uparrow \\ (\mathbb{K}^{\mathrm{sp}}_{d,n})_\ell & \twoheadrightarrow & (D_d(n))_\ell \end{array}$$

Here we have

$$(\mathbb{K}^{\mathrm{sp}}_{d,n})_\ell = \{g \in \mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell}) \mid g = 1 \text{ in } C^+(\mathcal{L}_{d,\mathbb{Z}_\ell/n_\ell\mathbb{Z}_\ell})\}.$$

Since

$$\#(C^+(\mathcal{L}_{d,\mathbb{Z}_\ell/n_\ell\mathbb{Z}_\ell})^\times) \leq \#(C^+(\mathcal{L}_{d,\mathbb{Z}_\ell/n_\ell\mathbb{Z}_\ell})) = n_\ell^{(2^{20})},$$

the index of $(\mathbb{K}^{\mathrm{sp}}_{d,n})_\ell$ in $\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell})$ is bounded by $n_\ell^{(2^{20})}$. This finishes the proof of the first claim.

For the second claim, we put

$$N := [\mathrm{SO}(\mathcal{L}_{K3,\mathbb{Z}_2}) : f_{K3}(\mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_2}))].$$

Then, by the second claim of Proposition 3.1.6 and the above arguments, we have

$$[(D_d)_2 : D_d(n)_2] \leq [(D_d)_2 : (D_d)_2 \cap f_{K3}(\mathrm{GSpin}(\mathcal{L}_{K3,\mathbb{Z}_2}))] \cdot n_2^{(2^{20})} \leq N \cdot n_2^{(2^{20})}. \qquad \square$$

3.2. ***Preparation II.*** Here, we will introduce an even unimodular lattice $\mathcal{L}$ of signature $(26, 2)$ which contains all $\mathcal{L}_d$. Then we will define related objects as in the previous subsection.

Proposition 3.2.1 (see also [She 2017, Lemma 3.3.1]). *We put*

$$\mathcal{L} := \mathbb{E}_8^3 \oplus \mathbb{H}^2.$$

For any positive integer d, there exists a primitive embedding of lattices

$$i_d : \mathcal{L}_d \hookrightarrow \mathcal{L}.$$

Proof. See [Nikulin 1979, Corollary 1.12.3]. $\square$

Remark 3.2.2. (1) Since $\mathcal{L}$ is unimodular, the group $\mathrm{SO}(\mathcal{L}_{\widehat{\mathbb{Z}}})$ is the discriminant kernel of $\mathcal{L}$, which is defined as in Definition 2.1.7.

(2) To get a primitive embedding into a self-dual lattice, the lattice $\mathbb{E}_8^2 \oplus \mathbb{H}^2 \oplus \langle 1 \rangle^5$ is enough (see [She 2017, Lemma 3.3.1]). However, we require that $\mathcal{L}$ comes from a quadratic space for using the usual definition of Clifford algebras.

Next, we will define related algebraic groups and Shimura data for $\mathcal{L}$ as in Definitions 2.2.1 and 3.1.1.

Definition 3.2.3. (1) $\mathrm{GSpin}_{\mathcal{L}}$ is the algebraic group over $\mathbb{Q}$ whose R-valued points are given by

$$\mathrm{GSpin}_{\mathcal{L}}(R) := \{z \in C^+(\mathcal{L}_R)^\times \mid z\mathcal{L}_R z^{-1} = \mathcal{L}_R\}.$$

(2) Take a 2-dimensional negative definite subspace of $\mathcal{L}_{\mathbb{Q}}$, and let e_1, e_2 be its orthogonal basis. Let e_1', e_2' be an orthonormal basis over $\mathbb{R}$ which are given by constant multiples of e_1, e_2, and $J := e_1' e_2' \in C^+(\mathcal{L}_{\mathbb{R}})$. Let ψ be the map

$$\psi : \mathbb{S} \to \mathrm{GSpin}_{\mathcal{L},\mathbb{R}}; \alpha + \beta i \mapsto \alpha + \beta J,$$

and $X_{\mathrm{GSpin}_{\mathcal{L}}}$ be a $\mathrm{GSpin}_{\mathcal{L}}(\mathbb{R})$-conjugacy class containing ψ.

(3) $\mathrm{SO}_{\mathcal{L}}$ is the algebraic group over $\mathbb{Q}$ whose R-valued points are given by

$$\mathrm{SO}_{\mathcal{L}}(R) := \{g \in \mathrm{SL}(\mathcal{L}_R) \mid (gv, gw) = (v, w) \text{ for any } v, w \in \mathcal{L}_R\}.$$

(4) $X_{\mathrm{SO}_{\mathcal{L}}}$ is the (isomorphic) image of $X_{\mathrm{GSpin}_{\mathcal{L}}}$ via the adjoint representation $\mathrm{GSpin}_{\mathcal{L}} \to \mathrm{SO}_{\mathcal{L}}$.

(5) For $V := C(\mathcal{L})$ and a fixed $a \in V$ which is a constant multiple of e_1e_2, define $\phi_a : V \times V \to \mathbb{Z}$ as $\phi_a(x, y) := \mathrm{tr}_{V/\mathbb{Q}}(xay^*)$. Here $\mathrm{tr}_{V/\mathbb{Q}}(x)$ means the trace of a left multiplication map by x as in [Huybrechts 2016, Chapter 4, Section 2.2], and $*$ denotes the natural anti-automorphism on the Clifford algebra. Then ϕ_a is a nondegenerate alternative form. We denote its degree by r. Let $\mathrm{GSp}_{V,a}$ be the algebraic group over $\mathbb{Q}$ whose R-valued points are given by

$$\mathrm{GSp}_{V,a}(R) := \left\{ g \in \mathrm{GL}(V_R) \mid \text{there exists } c \in R^\times \text{ such that } \phi_a(gx, gy) = c\phi_a(x, y) \text{ for any } x, y \in V_R \right\}.$$

Let $(\mathrm{GSp}_{V,a}, X_{\mathrm{GSp}_{V,a}})$ be the Shimura datum associated with (V, ϕ_a).

Remark 3.2.4. (1) As in Remark 3.1.2(1), we can define a homomorphism

$$f : \mathrm{GSpin}_{\mathcal{L}} \to \mathrm{SO}_{\mathcal{L}};\ g \mapsto (l \mapsto glg^{-1}).$$

Moreover, it induces a morphism of Shimura data

$$(\mathrm{GSpin}_{\mathcal{L}}, X_{\mathrm{GSpin}_{\mathcal{L}}}) \to (\mathrm{SO}_{\mathcal{L}}, X_{\mathrm{SO}_{\mathcal{L}}}).$$

(2) We can define a homomorphism

$$h : \mathrm{GSpin}_{\mathcal{L}} \to \mathrm{GSp}_{V,a};\ g \mapsto (v \mapsto gv).$$

Moreover, it induces an embedding of Shimura data

$$(\mathrm{GSpin}_{\mathcal{L}}, X_{\mathrm{GSpin}_{\mathcal{L}}}) \to (\mathrm{GSpin}_{V,a}, X_{\mathrm{GSpin}_{V,a}})$$

by our definition of a (see [Huybrechts 2016, Chapter 4, Section 2.2]).

(3) We will use a similar notation as in Remark 3.1.2 (2), (3) for $\mathcal{L}$.

Definition 3.2.5. For any positive integer n, we define compact open subgroups $\mathbb{K}_n^{\mathrm{sp}} \subset \mathrm{GSpin}_{\mathcal{L}}(\mathbb{A}_f)$ and $\mathbb{K}_n \subset \mathrm{GSp}_{V,a}(\mathbb{A}_f)$ by

$$\mathbb{K}_n^{\mathrm{sp}} := \{g \in \mathrm{GSpin}(\mathcal{L}_{\widehat{\mathbb{Z}}}) \mid g = 1 \text{ in } C^+(\mathcal{L}_{\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}})\},$$
$$\mathbb{K}_n := \{g \in \mathrm{GSp}_{V,a}(\mathbb{A}_f) \mid gV_{\widehat{Z}} = V_{\widehat{\mathbb{Z}}},\ g \text{ acts trivial on } V_{\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}}\}.$$

Remark 3.2.6. (1) One can show that $h(\mathbb{K}_n^{\mathrm{sp}}) \subset \mathbb{K}_n$ and $h^{-1}(\mathbb{K}_n) = \mathbb{K}_n^{\mathrm{sp}}$. Moreover, our definition of $\mathbb{K}_n$ coincides with Λ_n in [Rizov 2010, Section 5.5]. Therefore, as in [Rizov 2010, Section 5.5], we have an embedding

$$\mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a}, X_{\mathrm{GSp}_{V,a}}) \hookrightarrow \mathcal{A}_{g,\sqrt{r},n,\mathbb{Q}}.$$

Here, we put $g := 2^{27}$, and $\mathcal{A}_{g,\sqrt{r},n,\mathbb{Q}}$ is the moduli space of g-dimensional degree r polarized abelian schemes with level n-structure.

(2) The lattice embedding $i_d : \mathcal{L}_d \hookrightarrow \mathcal{L}$ induces a morphism of algebraic groups $i_d : \mathrm{SO}_{\mathcal{L}_d} \to \mathrm{SO}_{\mathcal{L}}$. It induces an embedding of Shimura data

$$(\mathrm{SO}_{\mathcal{L}_d}, X_{\mathrm{SO}_{\mathcal{L}_d}}) \to (\mathrm{SO}_{\mathcal{L}}, X_{\mathrm{SO}_{\mathcal{L}}}).$$

(3) One can show that $D(n) := f(\mathbb{K}_n^{\mathrm{sp}})$ is a compact open subgroup of $\mathrm{SO}(\mathcal{L}_{\widehat{\mathbb{Z}}})$ as in Proposition 3.1.5. Moreover, it is clear that $i_d(D_d(n)) \subset D(n)$ because $\mathrm{GSpin}(\mathcal{L}_{d,\mathbb{Z}_\ell}) \subset \mathrm{GSpin}(\mathcal{L}_{\mathbb{Z}_\ell})$ as in Lemma 3.1.3.

3.3. ***The uniform Kuga–Satake construction.*** In this subsection, we assume that a positive integer n is sufficiently large (in our application, n would be a sufficiently large power of 2). The previous two subsections imply that there exists the following diagram of schemes over $\mathbb{Q}$:

$$\begin{array}{ccccccc} & & & & \mathrm{Sh}_{\mathbb{K}_n^{\mathrm{sp}}}(\mathrm{GSpin}_{\mathcal{L}}, X_{\mathrm{GSpin}_{\mathcal{L}}}) & & \\ & & & & \downarrow f & \searrow^{h} & \\ M^{\circ}_{2d,D_d(n),\mathbb{Q}} & \xrightarrow{j} & \mathrm{Sh}_{D_d(n)}(\mathrm{SO}_{\mathcal{L}_d}, X_{\mathrm{SO}_{\mathcal{L}_d}}) & \xrightarrow{i_d} & \mathrm{Sh}_{D(n)}(\mathrm{SO}_{\mathcal{L}}, X_{\mathrm{SO}_{\mathcal{L}}}) & & \mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a}, X_{\mathrm{GSp}_{V,a}}) \end{array}$$

Here $\mathrm{Sh}_{\mathbb{K}}(G, X)$ means the canonical model of a Shimura variety of level $\mathbb{K}$ associated with (G, X) over $\mathbb{Q}$, which is the reflex field of (G, X). Then, by the arguments in [Rizov 2010, Section 5.5], we can find δ which is a section of f over a certain number field E_n. Indeed, as in [Rizov 2010, Section 5.5], our definition of $D(n)$ guarantees that f in the above diagram induces isomorphisms between geometric connected components of the above Shimura varieties. Hence we can find a section of f over a number field on which all geometric connected components are defined.

In the following of this subsection, we fix a field F containing E_n. We consider the base change from $\mathbb{Q}$ to F of the above diagram.

$$\begin{array}{ccccccc} & & & & \mathrm{Sh}_{\mathbb{K}_n^{\mathrm{sp}}}(\mathrm{GSpin}_{\mathcal{L}}) & & \\ & & & & \delta \uparrow \downarrow f & \searrow^{h} & \\ M^{\circ}_{2d,D_d(n),F} & \xrightarrow{j} & \mathrm{Sh}_{D_d(n)}(\mathrm{SO}_{\mathcal{L}_d}) & \xrightarrow{i_d} & \mathrm{Sh}_{D(n)}(\mathrm{SO}_{\mathcal{L}}) & & \mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a}) \end{array} \qquad (*)$$

Here, and in the following of this paper, for simplicity, we denote $(\mathrm{Sh}_{\mathbb{K}}(G, X))_F$ by $\mathrm{Sh}_{\mathbb{K}}(G)$. Moreover, we denote the composition $h \circ \delta \circ i_d \circ j$ by Δ_d.

Remark 3.2.6 implies that there exists the universal abelian scheme $\mathcal{A}$ over $\mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a})$ possessing the degree r polarization and the level n-structure. Then, for $(X, L, \nu, \alpha) \in M^{\circ}_{2d,D_d(n),F}(F)$ which corresponds to a morphism $t : \mathrm{Spec}\, F \to M^{\circ}_{2d,D_d(n),F}$, we can associate an abelian variety $A^{(X,L,\alpha)}$ by pulling buck $\mathcal{A}$ via $\Delta_d \circ t$. We will quickly recall the properties of $A^{(X,L,\alpha)}$.

Definition 3.3.1. Let ℓ be any prime number.

(1) Let S be any (schematic) connected component of $\mathrm{Sh}_{D(n)}(\mathrm{SO}_{\mathcal{L}})$, and $\bar{s} \to S$ be a geometric point. Then, as in [Milne 1990, III, Remark 6.1], we can show that

$$\varprojlim_{\mathbb{K}} \mathrm{Sh}_{\mathbb{K}}(\mathrm{SO}_{\mathcal{L}}) \to \mathrm{Sh}_{D(n)}(\mathrm{SO}_{\mathcal{L}})$$

is a Galois covering with a Galois group $D(n)$, and so we can associate the representation

$$\pi_1(S, \bar{s}) \to (D(n))_\ell \to \mathrm{SO}(\mathcal{L}_{\mathbb{Z}_\ell}).$$

We define $\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell}$ as the corresponding $\mathbb{Z}_\ell$-sheaf on $\mathrm{Sh}_{D(n)}(\mathrm{SO}_{\mathcal{L}})$, which has a symmetric pairing structure.

(2) Similarly, we define $\mathcal{L}^{\mathrm{shf}}_{d,\mathbb{Z}_\ell}$ as the $\mathbb{Z}_\ell$-sheaf on $\mathrm{Sh}_{D_d(n)}(\mathrm{SO}_{\mathcal{L}_d})$ corresponding to the representation $(D_d(n))_\ell \to \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell})$. The sheaf $\mathcal{L}^{\mathrm{shf}}_{d,\mathbb{Z}_\ell}$ has a symmetric pairing structure.

(3) Similarly, we define $V^{\mathrm{shf}}_{\mathbb{Z}_\ell}$ as the $\mathbb{Z}_\ell$-sheaf on $\mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a})$ corresponding to the representation $(\mathbb{K}_n)_\ell \to \mathrm{GSp}(V_{\mathbb{Z}_\ell}, \phi_a)$. The sheaf $V^{\mathrm{shf}}_{\mathbb{Z}_\ell}$ has a symplectic pairing structure.

Lemma 3.3.2. (1) *There exists the natural injection of étale sheaves $\mathcal{L}^{\mathrm{shf}}_{d,\mathbb{Z}_\ell} \to i_d^*\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell}$ preserving the pairing. Moreover, $(\mathcal{L}^{\mathrm{shf}}_{d,\mathbb{Z}_\ell})^\perp$ is trivial as a $\mathbb{Z}_\ell$-sheaf.*

(2) *There exists the natural injection of étale sheaves $f^*\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell} \to \mathrm{End}(h^*(V^{\mathrm{shf}}_{\mathbb{Z}_\ell}))$, which induces a "left multiplication" on a stalk.*

Proof. For (1), it is enough to show that $i_d : \mathcal{L}_{d,\mathbb{Z}_\ell} \to \mathcal{L}_{\mathbb{Z}_\ell}$ is $\pi_1(S, \bar{s})$-equivariant, and $(\mathcal{L}_{d,\mathbb{Z}_\ell})^\perp$ is a trivial $\pi_1(S, \bar{s})$-module. Here S is any connected component of $\mathrm{Sh}_{D_d(n)}(\mathrm{SO}_{\mathcal{L}_d})$, $\bar{s}$ is a geometric point of S, and $\pi_1(S, \bar{s})$-module structures on $\mathcal{L}_{d,\mathbb{Z}_\ell}$, $\mathcal{L}_{\mathbb{Z}_\ell}$ correspond to $\mathcal{L}^{\mathrm{shf}}_{d,\mathbb{Z}_\ell}$, $i_d^*\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell}$. In regard to a $\mathbb{Z}_\ell$-sheaf given by a representation of adelic subgroup, a pullback of a $\mathbb{Z}_\ell$-sheaf corresponds to a pullback of a representation. Thus $\pi_1(S, \bar{s})$-module structure on $\mathcal{L}_{\mathbb{Z}_\ell}$ is given by

$$\pi_1(S, \bar{s}) \to (D_d(n))_\ell \hookrightarrow (D(n))_\ell \hookrightarrow \mathrm{SO}(\mathcal{L}_{\mathbb{Z}_\ell}).$$

Hence the desired claim is clear.

For (2), it is enough to show that the morphism

$$\mathcal{L}_{\mathbb{Z}_\ell} \to \mathrm{End}(V_{\mathbb{Z}_\ell}); v \mapsto (z \mapsto vz)$$

is $\pi_1(S, \bar{s})$-equivariant, where S is any connected component of $\mathrm{Sh}_{\mathbb{K}_n^{\mathrm{sp}}}(\mathrm{GSpin}_{\mathcal{L}})$, and $\pi_1(S, \bar{s})$-module structures on $\mathcal{L}_{\mathbb{Z}_\ell}$, $V_{\mathbb{Z}_\ell}$ correspond to $f^*(\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell})$, $h^*(V^{\mathrm{shf}}_{\mathbb{Z}_\ell})$. By the same reason as (1), these structures are given by

$$\pi_1(S, \bar{s}) \to (\mathbb{K}_n^{\mathrm{sp}})_\ell \xrightarrow{f} (D(n))_\ell \hookrightarrow \mathrm{SO}(\mathcal{L}_{\mathbb{Z}_\ell}),$$
$$\pi_1(S, \bar{s}) \to (\mathbb{K}_n^{\mathrm{sp}})_\ell \xrightarrow{h} (\mathbb{K}_n)_\ell \to \mathrm{GSp}(V_{\mathbb{Z}_\ell}, \phi_a).$$

Hence if we denote the first arrows of the both by σ, these actions are described as

$$\gamma(v) = \sigma(\gamma) v \sigma(\gamma)^{-1}, \quad \gamma(z) = \sigma(\gamma)(z),$$

for $\gamma \in \pi_1(S, \bar{s})$, $v \in \mathcal{L}_{\mathbb{Z}_\ell}$, and $z \in V_{\mathbb{Z}_\ell}$. Thus the desired equivariance is clear. □

Proposition 3.3.3. *Let ℓ be any prime number, $t : \mathrm{Spec}\, F \to M^\circ_{2d, D_d(n), F}$ be the point corresponding to $(X, L, \nu, \alpha) \in M^\circ_{2d, D_d(n), F}(F)$, $A^{(X,L,\alpha)}$ be the abelian variety given by $(\Delta_d \circ t)^*(\mathcal{A})$, and $\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}$ be the* $\mathrm{Gal}(\overline{F}/F)$*-lattice identified with $(i_d \circ j \circ t)^*(\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell})$. Then the following hold.*

(1) *There exists a Galois equivariant lattice embedding*

$$P^2_{\text{ét}}((X_{\overline{F}}, L_{\overline{F}}), \mathbb{Z}_\ell(1)) \subset \mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}$$

such that $\mathrm{Gal}(\overline{F}/F)$ *acts trivially on the orthogonal complement*

$$P^2_{\text{ét}}((X_{\overline{F}}, L_{\overline{F}}), \mathbb{Z}_\ell(1))^\perp \subset \mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}.$$

(2) *The abelian variety $A^{(X,L,\alpha)}$ has a level n-structure defined over F. Thus each n-torsion point of $A^{(X,L,\alpha)}$ is F-rational.*

(3) *The abelian variety $A^{(X,L,\alpha)}$ admits a left $C(\mathcal{L})$-action over F, and moreover there exists an isomorphism of $\mathbb{Z}_\ell$-modules*

$$H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\bar{F}}, \mathbb{Z}_\ell) \simeq C(\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)})$$

which identifies the algebra

$$C(\mathcal{L}_{\mathbb{Z}_\ell})^{\mathrm{op}} \subset \mathrm{End}(H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\bar{F}}, \mathbb{Z}_\ell))$$

with

$$C(\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)})^{\mathrm{op}} \subset \mathrm{End}(C(\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)})).$$

Here, the former inclusion of algebras is induced by the above $C(\mathcal{L})$-action, and the latter is induced by the right multiplication.

(4) *The left multiplication by $C(\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)})$ on the right-hand side of the isomorphism in* (3) *induces a Galois equivariant isomorphism*

$$C(\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}) \simeq \mathrm{End}_{C(\mathcal{L}_{\mathbb{Z}_\ell})^{\mathrm{op}}}(H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\bar{F}}, \mathbb{Z}_\ell)).$$

Here, the (left) $C(\mathcal{L}_{\mathbb{Z}_\ell})^{\mathrm{op}}$-module structure is induced by the left $C(\mathcal{L})$-action on $A^{(X,L,\alpha)}$ as in (3).

Proof. These results are essentially proved in [She 2017, Proposition 3.5.8].

Statement (1) follows from Lemma 3.3.2(1) and the fact

$$(j \circ t)^*(\mathcal{L}^{\mathrm{shf}}_{d,\mathbb{Z}_\ell}) \simeq P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_\ell(1))$$

(see [Madapusi Pera 2015, Proposition 5.6 (1)]).

Statement (2) is clear because the universal family $\mathcal{A}$ admits a level n-structure.

Before proving (3) and (4), we note that for the universal abelian scheme $u : \mathcal{A} \to \mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a})$, we have $R^1u_*\mathbb{Z}_\ell \simeq V^{\mathrm{shf}}_{\mathbb{Z}_\ell}$.

For (3), as in [Madapusi Pera 2016, Section 3.10], $h^*(\mathcal{A})_{\mathrm{Sh}_{\mathbb{K}_n^{\mathrm{sp}}}(\mathrm{GSpin}_{\mathcal{L}})_{\mathbb{C}}}$ admits a $C(\mathcal{L})$-action which corresponds to a right multiplication on the cohomology, since our definition of h guarantees that the right multiplication preserves the Hodge structure. This action descends to F by [Madapusi Pera 2016, Proposition 3.11] and induces a $C(\mathcal{L})$-action on $A^{(X,L,\alpha)}$ with desired properties.

For (4), the statement (3) of this proposition and Lemma 3.3.2(2) imply the well-definedness and the Galois equivariance of our morphism, and it is clearly bijective. □

4. Proof of the Main Theorems

4.1. *Statements.*

Lemma 4.1.1. *Let F be a finitely generated field over $\mathbb{Q}$, R be a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field F, and $\bar{s}$ be a geometric point corresponding to an algebraic*

closure $\overline{F}$ over F. For any $\pi_1(\mathrm{Spec}\, F, \bar{s})$-module M such that $\mathrm{Ker}(\pi_1(\mathrm{Spec}\, F, \bar{s}) \to \mathrm{Aut}(M))$ is closed, the following are equivalent.

(1) *The $\pi_1(\mathrm{Spec}\, F, \bar{s})$-action on M arises from a $\pi_1(\mathrm{Spec}\, R, \bar{s})$-action on M.*

(2) *For any height 1 prime ideal $\mathfrak{p} \in \mathrm{Spec}\, R$, the $\pi_1(\mathrm{Spec}\, F, \bar{s})$-action on M arises from a $\pi_1(\mathrm{Spec}\, R_{\mathfrak{p}}, \bar{s})$-action on M.*

(3) *For any height 1 prime ideal $\mathfrak{p} \in \mathrm{Spec}\, R$, M is unramified at $\mathfrak{p}$, i.e., if we take $\bar{v}$ which is an extension of valuation $\mathfrak{p}$ to $\overline{F}$, the inertia group $I_{\bar{v}}$ acts trivially on M.*

When M satisfies the above equivalent conditions, we say M is unramified over Spec R.

Proof. First, we recall that $\pi_1(\mathrm{Spec}\, F, \bar{s}) \to \pi_1(\mathrm{Spec}\, R, \bar{s})$ is surjective and its kernel is identified with $\mathrm{Gal}(\overline{F}/F_R^{\mathrm{ur}})$, where F_R^{ur} is the composite of finite extensions E/F which are unramified over Spec R [Fu 2015, Proposition 3.3.6]. Here, we say E/F is unramified over Spec R if the normalization of Spec R in E is unramified over Spec R. The same results hold for $R_{\mathfrak{p}}$.

(1)⇔(2) By the assumption on M, it suffices to show that

$$\mathrm{Ker}(\pi_1(\mathrm{Spec}\, F, \bar{s}) \to \pi_1(\mathrm{Spec}\, R, \bar{s}))$$

is generated by

$$(\mathrm{Ker}(\pi_1(\mathrm{Spec}\, F, \bar{s}) \to \pi_1(\mathrm{Spec}\, R_{\mathfrak{p}}, \bar{s})))_{\mathfrak{p}}$$

as a topological group. By the above remark, it is enough to show that $F_R^{\mathrm{ur}} = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} F_{R_{\mathfrak{p}}}^{\mathrm{ur}}$. The inclusion $F_R^{\mathrm{ur}} \subset \bigcap_{\mathrm{ht}(\mathfrak{p})=1} F_{R_{\mathfrak{p}}}^{\mathrm{ur}}$ is obvious, and the other direction follows from the Zariski–Nagata purity.

(2)⇔(3) By the assumption on M, it suffices to show that $\mathrm{Ker}(\pi_1(\mathrm{Spec}\, F, \bar{s}) \to \pi_1(\mathrm{Spec}\, R_{\mathfrak{p}}, \bar{s}))$ is generated by $(I_{\bar{v}})_{\bar{v}\ \mathrm{over}\ \mathfrak{p}}$ as a topological group, but it follows from the above remark. □

Remark 4.1.2. The condition "M is unramified at $\mathfrak{p}$" does not depend on a choice of $\bar{v}$. Indeed, for each $\mathfrak{p}$, the inertia group $I_{\bar{v}}$ is determined by $\mathfrak{p}$ up to conjugation in $\mathrm{Gal}(\overline{F}/F)$.

The following are the statements of results of this section (for more generalized statements, see Theorem 6.1.1).

Theorem 4.1.3. *Let F be a finitely generated field over $\mathbb{Q}$, R be a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field F, and d be a positive integer. Then, the set*

$$\mathrm{Shaf}(F, R, d) := \left\{ (X, L) \;\middle|\; \begin{array}{l} X : K3 \text{ surface over } F, \\ L \in \mathrm{Pic}_{X/F}(F) : \text{primitive ample}, \\ H^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_2) : \text{unramified over Spec } R, \\ \deg L = 2d \end{array} \right\} / F\text{-isom}.$$

is finite.

Theorem 4.1.4. *Let F be a finitely generated field over $\mathbb{Q}$, and R be a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field F. Then, the set*

$$\mathrm{Shaf}(F, R) := \left\{ X \mid X : K3 \text{ surface over } F,\ H^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_2) : \text{unramified over } \operatorname{Spec} R \right\} / F\text{-isom}$$

is finite.

Remark 4.1.5. For a nonempty open subscheme $\operatorname{Spec}(R') \subset \operatorname{Spec}(R)$, the finiteness of $\mathrm{Shaf}(F, R', d)$ (resp. $\mathrm{Shaf}(F, R')$) clearly implies the finiteness of $\mathrm{Shaf}(F, R, d)$ (resp. $\mathrm{Shaf}(F, R)$). Thus, to prove Theorems 4.1.3 and 4.1.4, we may assume $\frac{1}{2} \in R$. Note that it is equivalent to say that the residual characteristic at any point of $\operatorname{Spec} R$ is different from 2.

4.2. ***Proof of Theorem 4.1.3.*** In this subsection, we use the same notation as Theorem 4.1.3, unless otherwise noted. First, for using the Kuga–Satake construction, we will replace F by an appropriate finite extension of it to provide a level structure on $(X, L) \in \mathrm{Shaf}(F, R, d)$. The following lemma is essential for justifying this replacement.

Lemma 4.2.1. *Let E/F be a finite extension, X_0 be a $K3$ surface over F, and $L_0 \in \operatorname{Pic}_{X_0/F}(F)$ be a polarization. Then, the set*

$$\left\{ (X, L) \mid X \text{ is a } K3 \text{ surface over } F,\ L \in \operatorname{Pic}_{X/F}(F) : \text{ample},\ (X_E, L_E) \simeq_E (X_{0,E}, L_{0,E}) \right\} / F\text{-isom}$$

is finite.

Proof. Taking the Galois closure of E in $\bar{F}$, we may assume that E/F is a Galois extension. Then we can identify this set with the Galois cohomology group $H^1(\mathrm{Gal}(E/F), \mathrm{Aut}_E(X_0, L_0))$. The finiteness of this set follows from [Huybrechts 2016, Chapter 5, Proposition 3.3]. □

Lemma 4.2.2 (cf. [André 1996, Lemma 8.4.1]). *Let X be a $K3$ surface over F, $L \in \operatorname{Pic}_{X/F}(F)$ be a primitive polarization of degree $2d$ on X over F, and n be a positive integer. We put*

$$W_{\widehat{\mathbb{Z}}} := P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \widehat{\mathbb{Z}}(1)).$$

Let

$$\rho : \mathrm{Gal}(\bar{F}/F) \to \mathrm{O}(W_{\widehat{\mathbb{Z}}})$$

be the natural Galois representation. Fix an isometry

$$i_{(X,L)} : \mathcal{L}_{K3,\widehat{\mathbb{Z}}} \simeq H^2_{\text{ét}}(X_{\bar{F}}, \widehat{\mathbb{Z}}(1)),$$

which restricts to an isometry $\mathcal{L}_{K3} \simeq H^2(X(\mathbb{C}), \mathbb{Z}(1))$, and which sends v_d to $\mathrm{ch}_{\widehat{\mathbb{Z}}}(L)$ (see Remark 2.1.6). Using $i_{(X,L)}$, we identify $D_d(n)$ with a compact open subgroup of $\mathrm{SO}(W_{\widehat{\mathbb{Z}}})$. Then, for any finite extension E/F, we have

$$\rho(\mathrm{Gal}(\bar{F}/E)) \subset D_d(n) \Leftrightarrow \rho_\ell(\mathrm{Gal}(\bar{F}/E)) \subset (D_d(n))_\ell \quad \text{for every } \ell \mid 2n.$$

Proof. In the following, we identify $\mathrm{SO}(\mathcal{L}_{d,\widehat{\mathbb{Z}}})$ with $\mathrm{SO}(W_{\widehat{\mathbb{Z}}})$ vie $i_{(X,L)}$. This lemma is essentially shown in [André 1996, Lemma 8.4.1]. The following claim is shown in the proof of [André 1996, Lemma 8.4.1], using specialization arguments and the Weil conjecture.

Claim. *If there exists a prime number ℓ such that $\rho_\ell(\mathrm{Gal}(\overline{F}/E)) \subset \mathrm{SO}(W_{\mathbb{Z}_\ell})$, then $\rho(\mathrm{Gal}(\overline{F}/E)) \subset \mathrm{SO}(W_{\widehat{\mathbb{Z}}})$.*

André stated that the above claim implies the following result.

$$\rho(\mathrm{Gal}(\overline{F}/E)) \subset D_d(n) \Leftrightarrow \rho_\ell(\mathrm{Gal}(\overline{F}/E)) \subset (D_d(n))_\ell \quad \text{for every } \ell \mid 2dn.$$

Indeed, for $\ell \nmid 2dn$, we have $(D_d(n))_\ell = \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_\ell})$.

More generally, for $\ell \nmid 2n$, we have $(D_d(n))_\ell = (D_d)_\ell$ (see Corollary 3.1.8). Therefore, to generalize André's result to our lemma, it is enough to show that if $\rho_\ell(\mathrm{Gal}(\overline{F}/E)) \subset \mathrm{SO}(W_{\mathbb{Z}_\ell})$, then $\rho_\ell(\mathrm{Gal}(\overline{F}/E)) \subset (D_d)_\ell$. However, since $\mathrm{Gal}(\overline{F}/F)$ stabilizes $\mathrm{ch}_{\mathbb{Z}_\ell}(L)$, it follows from our description of the discriminant kernel

$$(D_d)_\ell = \{\tilde{g}_\ell \in \mathrm{SO}(H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}_\ell)) \mid \tilde{g}_\ell(\mathrm{ch}_{\mathbb{Z}_\ell}(L)) = \mathrm{ch}_{\mathbb{Z}_\ell}(L)\},$$

which follows from Proposition 2.1.8. □

In the rest of this section, fix a positive integer n which is a sufficiently large power of 2.

Proposition 4.2.3. *To prove Theorem 4.1.3, it is enough to show that*

$$\mathrm{Shaf}'(F, R, d) := \left\{ (X, L) \;\middle|\; \begin{array}{l} X : K3 \text{ surface over } F, \\ L \in \mathrm{Pic}_{X/F}(F) : \text{ primitive ample}, \\ H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_2) : \text{unramified over } \mathrm{Spec}\, R, \\ \deg L = 2d, \\ (X, L) \text{ admits a } D_d(n)\text{-level structure} \end{array} \right\} / F\text{-isom}$$

is a finite set for any F, R, d as in Theorem 4.1.3. Moreover, if we fix a number field F', it suffices to show only in the case where $F \supset F'$ and $\frac{1}{2} \in R$.

Here, "(X, L) admits a $D_d(n)$-level structure" means that there exists an element $(X, L, \nu_{(X,L)}, \alpha_{(X,L)})$ in $M^\circ_{2d, D_d(n), F}(F)$.

Proof. We should prove the finiteness of $\mathrm{Shaf}(F, R, d)$. By Remark 4.1.5, we may assume $\frac{1}{2} \in R$ (so the Tate twist $\otimes \mathbb{Z}_2(1)$ does not affect the unramifiedness over $\mathrm{Spec}\, R$; see Lemma 4.1.1).

First, we will show that there exists a finite extension E/F such that for any $(X, L) \in \mathrm{Shaf}(F, R, d)$, the pair (X_E, L_E) admits a $D_d(n)$-level structure. We fix $(X, L) \in \mathrm{Shaf}(F, R, d)$ and $i_{(X,L)}$, moreover we use the same identification as in Lemma 4.2.2. Let

$$\overline{\rho}_2 := \overline{\rho}_{(X,L),2} : \pi_1(\mathrm{Spec}\, R, \bar{s}) \to \mathrm{O}(P^2_{\text{ét}}((X_{\overline{F}}, L_{\overline{F}}), \mathbb{Z}_2(1)))$$

be the representation induced by

$$\rho := \rho_{(X,L)} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{O}(P^2_{\text{ét}}((X_{\overline{F}}, L_{\overline{F}}), \widehat{\mathbb{Z}}(1))).$$

The inverse image $\bar{\rho}_2^{-1}((D_d(n))_2)$ is a finite index subgroup, so we can associate a pointed finite étale cover $\operatorname{Spec}\widetilde{R} \to \operatorname{Spec} R$. Then we have $\bar{\rho}_2(\pi_1(\operatorname{Spec}\widetilde{R}, \bar{s})) \subset (D_d(n))_2$. The former equals $\rho_2(\mathrm{Gal}(\overline{F}/\mathrm{Frac}(\widetilde{R})))$, and by Lemma 4.2.2, we can get the $D_d(n)$-level structure on $(X_{\mathrm{Frac}(\widetilde{R})}, L_{\mathrm{Frac}(\widetilde{R})})$ by $i_{(X,L)}$.

Here, note that

$$[\pi_1(\operatorname{Spec} R, \bar{s}) : \bar{\rho}_2^{-1}((D_d(n))_2)] \leq C_d := [\mathrm{O}(\mathcal{L}_{d,\mathbb{Z}_2}) : (D_d(n))_2],$$

where C_d is independent of (X, L) and $i_{(X,L)}$. By the analogue of the Hermite–Minkowski theorem [Harada and Hiranouchi 2009, Proposition 2.3, Theorem 2.9], the family of subsets

$$\mathcal{C} := \{H \subset \pi_1(\operatorname{Spec} R, \bar{s}) : \text{open subgroup} \mid [\pi_1(\operatorname{Spec} R, \bar{s}) : H] \leq C_d\}$$

is finite, therefore

$$H_0 := \bigcap_{H \in \mathcal{C}} H$$

is an open subgroup. Let $\operatorname{Spec}\widetilde{R}_0 \to \operatorname{Spec} R$ be the corresponding pointed finite étale covering, then by the above argument, we can get a $D_d(n)$-level structure on $(X_{\mathrm{Frac}(\widetilde{R}_0)}, L_{\mathrm{Frac}(\widetilde{R}_0)})$. Hence we now get a desired finite extension $E := \mathrm{Frac}(\widetilde{R}_0)$.

Thus, by using the assumption for $\mathrm{Shaf}'(E, \widetilde{R}_0, d)$ and Lemma 4.2.1, we can show the finiteness of $\mathrm{Shaf}(F, R, d)$. Note that the latter statement is clear by Lemma 4.2.1. □

The following proposition is essentially known by [André 1996], and one can prove it as a corollary of the theory of potentially good loci of Shimura variety (see [Imai and Mieda 2020]).

Proposition 4.2.4. *Assume $F \supset E_n$, where E_n is as in Section 3.3. For $(X, L, \nu, \alpha) \in M^{\circ}_{2d, D_d(n), F}(F)$, let $A^{(X,L,\alpha)}$ be the Kuga–Satake abelian variety as in Proposition 3.3.3. Let R be a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field F, and assume $\frac{1}{2} \in R$. Assume that $H^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_2)$ is unramified over* $\operatorname{Spec} R$ *(its Tate twists are unramified too, because $\frac{1}{2} \in R$). Then, for any height* 1 *prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, the abelian variety $A^{(X,L,\alpha)}$ has good reduction at $\mathfrak{p}$.*[3]

Proof. We will follow the proof by André [1996, Lemma 9.3.1].[4] By the Néron–Ogg–Shafarevich criterion for abelian varieties (it is true whether a residue field is perfect or not), it is enough to show that $H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\overline{F}}, \mathbb{Z}_2)$ is unramified at $\mathfrak{p}$ (here we use $\frac{1}{2} \in R$). Let $\bar{v}$ be an extension on $\overline{F}$ of the valuation $\mathfrak{p}$, and $\varphi : I_{\bar{v}} \to \mathrm{Aut}(H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\overline{F}}, \mathbb{Z}_2))$ be a restriction of the Galois representation. Since $C(\mathcal{L})$-action on $A^{(X,L,\alpha)}$ is defined over F, for any $\gamma \in I_{\bar{v}}$, we have

$$\varphi(\gamma) \in \mathrm{End}_{C(\mathcal{L}_{\mathbb{Z}_2})^{\mathrm{op}}}(H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\overline{F}}, \mathbb{Z}_2)) \simeq C(\mathcal{L}_{\mathbb{Z}_2,(X,L,\alpha)})$$

(see Proposition 3.3.3 (3), (4)). Thus we also denote its image by $\varphi(\gamma) \in C(\mathcal{L}_{\mathbb{Z}_2,(X,L,\alpha)})$.

[3] If n is a sufficiently large power of ℓ, then the same argument work with ℓ in place of 2.

[4] The referee taught me another quick way of seeing this proposition. By the construction, the Galois representation on $H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\overline{F}}, \mathbb{Q}_2)$ has to factor through $\mathrm{GSpin}(PH^2_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_2))$ and hence the inertia representation has to factor through the center $\mathbb{Q}_2^{\times}$ and in fact through $\mathbb{Z}_2^{\times}$ by the compactness. Moreover, by the level assumption, it factors through $1 + 4\mathbb{Z}_2$, so it finishes proof since $1 + 4\mathbb{Z}_2$ has no nontrivial quasiunipotent elements.

On the other hand, $I_{\bar{v}}$ acts trivially on $P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_2(1))$ by our assumptions (see Lemma 4.1.1), and moreover acts trivially on $P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_2(1))^{\perp} \subset \mathcal{L}_{\mathbb{Z}_2,(X,L,\alpha)}$ by Proposition 3.3.3(1), thus $\gamma(c) = c$ for any $\gamma \in I_{\bar{v}}$ and $c \in C(\mathcal{L}_{\mathbb{Z}_2,(X,L,\alpha)})$. By Proposition 3.3.3(4), we have $\gamma(z \mapsto cz) = (z \mapsto cz)$ in $\mathrm{End}_{C(\mathcal{L}_{\mathbb{Z}_2})^{\mathrm{op}}}(H^1_{\text{ét}}(A^{(X,L,\alpha)}_{\bar{F}}, \mathbb{Z}_2))$, where the left-hand side is $(z \mapsto \varphi(\gamma)c\varphi(\gamma)^{-1}z)$. This implies $\varphi(\gamma)$ is contained in the center of $C(\mathcal{L}_{\mathbb{Q}_2,(X,L,\alpha)})$, which is a reduced algebra.

Proposition 3.3.3(2) and the Raynaud semiabelian reduction criterion [SGA 7_{I} 1972, Exposé IX, Proposition 4.7] imply that $A^{(X,L,\alpha)}$ has semiabelian reduction at $\mathfrak{p}$ (i.e., $A^{(X,L,\alpha)}$ extends to a semiabelian scheme over $\operatorname{Spec} R_{\mathfrak{p}}$). Here, we use that $n \geq 3$ is a power of 2, and the residual characteristic of $\mathfrak{p}$ is not 2. Thus for any $\gamma \in I_{\bar{v}}$, $\varphi(\gamma)$ is a unipotent element of a reduced algebra, it is identity. Hence it finishes the proof. □

We now complete the proof of Theorem 4.1.3. By Proposition 4.2.3, it is enough to show the finiteness of $\mathrm{Shaf}'(F, R, d)$ when $F \supset E_n$ and $1/2 \in R$. Here, we take E_n as in Section 3.3. In the following, we identify $(X, L) \in \mathrm{Shaf}'(F, R, d)$ with $(X, L, \nu, \alpha) \in M^{\circ}_{2d, D_d(n), F}(F)$ by choosing a level structure. Hence for $(X, L, \nu, \alpha) \in \mathrm{Shaf}'(F, R, d)$, we can associate $A^{(X,L,\alpha)}$, and since in the diagram $(*)$ of Section 3.3, each fiber of i_d is finite and h is injective (because they are induced by an embedding of Shimura data), it suffices to show the finiteness of $\Delta_d(\mathrm{Shaf}'(F, R, d))$. The image $\Delta_d(X, L, \nu, \alpha)$ corresponds to $A^{(X,L,\alpha)}$ with their degree r polarization and level n-structure. However, by Proposition 4.2.4, the abelian variety $A^{(X,L,\alpha)}$ has good reduction at any height 1 prime of $\operatorname{Spec} R$, so this set is finite by [Faltings 1983, Satz 6] (for finitely generated fields of characteristic 0, see [Faltings et al. 1984, VI, §1, Theorem 2]).

4.3. ***Proof of Theorem 4.1.4.*** In this subsection, we use the same notation as in Theorem 4.1.4, unless otherwise noted. The strategy is the same as [She 2017], i.e., we use Theorem 4.1.3 for reducing the problem to the finiteness of Picard lattices, and use the uniform Kuga–Satake maps for associating $\mathrm{Shaf}(F, R)$ with a finite set of abelian varieties.

Lemma 4.3.1 (cf. [She 2017, Corollary 4.1.3]). *For any $X_0 \in \mathrm{Shaf}(F, R)$, there exist only finitely many $X \in \mathrm{Shaf}(F, R)$ whose Picard lattice $\mathrm{Pic}_{X/F}(F)$ is isometric to the Picard lattice $\mathrm{Pic}_{X_0/F}(F)$.*

Proof. As in [She 2017, Proposition 4.1.2], a K3 surface X over F admits a primitive polarization whose degree bounded by a constant depending only on the isometry class of $\mathrm{Pic}_{X/F}(F)$. Hence this lemma follows from Theorem 4.1.3. □

Lemma 4.3.2 (cf. [She 2017, Lemma 4.1.4]). *Let E/F be a finite extension. For any $X_0 \in \mathrm{Shaf}(F, R)$, the set*

$$\{X \in \mathrm{Shaf}(F, R) \mid X_E \simeq_E X_{0,E}\}$$

is finite.

Proof. Taking a Galois closure, we may assume E/F is a Galois extension. By Lemma 4.3.1, it suffices to show the finiteness of isometry classes of Picard lattices $\mathrm{Pic}_{X/F}(F)$ associated with the considering set. Note that $\mathrm{Pic}_{X/F}(E)^{\mathrm{Gal}(E/F)} = \mathrm{Pic}_{X/F}(F)$ and $\mathrm{Pic}_{X/F}(E)$ is isometric to $\mathrm{Pic}_{X_0/F}(E)$. Since the set

of conjugacy classes of subgroups of $\mathrm{O}(\mathrm{Pic}_{X_0/F}(E))$ with the order $[E : F]$ is finite by [Borel 1963, Section 5, (a)], the desired finiteness follows. □

Proposition 4.3.3. *Recall that we fixed a positive integer n which is a power of 2. To show Theorem 4.1.4, it is enough to show that*

$$\mathrm{Shaf}'(F, R) := \left\{ X \;\middle|\; \begin{array}{l} X : K3 \text{ surface over } F, \\ H^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_2) : \text{unramified over } \mathrm{Spec}\, R, \\ \text{there exists } d_X, L_X, \nu_X, \alpha_X \text{ such that} \\ (X, L_X, \nu_X, \alpha_X) \in M^{\circ}_{2d_X, D_{d_X}(n), \mathbb{Q}}(F) \end{array} \right\} / F\text{-isom}$$

is a finite set for any (F, R) as in Theorem 4.1.4. Moreover, if we fix a number field F', it suffices to show only in the case where $F \supset F'$ and $\frac{1}{2} \in R$.

Proof. The proof is similar to Proposition 4.2.3, but we need more precise evaluation since we should discuss all degrees simultaneously.

As in the proof of Proposition 4.2.3, it is enough to show the finiteness of $\mathrm{Shaf}(F, R)$ with $\frac{1}{2} \in R$. First, note that every K3 surface over F admits some primitive polarization over F. Therefore, for any $X \in \mathrm{Shaf}(F, R)$, we can associate a primitive polarization L_X. Let $2d_X$ be the degree of L_X. We will show that there exists a finite extension E/F such that for any $X \in \mathrm{Shaf}(F, R)$, the pair $(X_E, L_{X,E})$ admits a $D_{d_X}(n)$-level structure. For each $X \in \mathrm{Shaf}(F, R)$, we fix $i_{(X,L_X)}$ as in Lemma 4.2.2, and we use the notation $\bar{\rho}_2 := \bar{\rho}_{(X,L_X),2}$ in the same sense as in Proposition 4.2.3. To get a desired extension, we should replace the bound C_{d_X} in the proof of Proposition 4.2.3 by a bound which is independent of X. For $\Gamma := \bar{\rho}_2(\pi_1(\mathrm{Spec}\, R, \bar{s}))$, we have

$$[\Gamma : \Gamma \cap (D_{d_X}(n))_2] = [\Gamma : \Gamma \cap \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_2})] \cdot [\Gamma \cap \mathrm{SO}(\mathcal{L}_{d_X,\mathbb{Z}_2}) : \Gamma \cap (D_{d_X}(n))_2] \leq 2N \cdot n^{(2^{20})}.$$

Here, we use $\Gamma \cap \mathrm{SO}(\mathcal{L}_{d,\mathbb{Z}_2}) \subset \Gamma \cap (D_d)_2$ (which follows from Proposition 2.1.8, see the proof of Lemma 4.2.2), and Corollary 3.1.8. We note that this bound is independent of X, L_X, and $i_{(X,L_X)}$. Hence replacing C_d by $2N \cdot n^{(2^{20})}$ in the arguments in the proof of Proposition 4.2.3, we get a pointed finite étale covering $\mathrm{Spec}\, \widetilde{R}_0 \to \mathrm{Spec}\, R$, whose fraction field E satisfies the desired property. Thus, by using the assumption for $\mathrm{Shaf}'(E, \widetilde{R}_0)$ and Lemma 4.3.2, we get the finiteness of $\mathrm{Shaf}(F, R)$. The latter statement is clear by Lemma 4.3.2. □

Definition 4.3.4 [She 2017, Definition 4.1.10]. Let F be a subfield of $\mathbb{C}$, X be a K3 surface over F, and ℓ be any prime number. We define (relative) *transcendental lattices* by

$$T(X) := \mathrm{Pic}_{X/F}(F)^{\perp} \subset H^2(X(\mathbb{C}), \mathbb{Z}(1)),$$
$$T(X)_{\mathbb{Z}_\ell} := \mathrm{Pic}_{X/F}(F)^{\perp} \subset H^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Z}_\ell(1)),$$
$$T(X)_{\widehat{\mathbb{Z}}} := \mathrm{Pic}_{X/F}(F)^{\perp} \subset H^2_{\text{ét}}(X_{\bar{F}}, \widehat{\mathbb{Z}}(1)).$$

Here we omit the Chern class map.

Remark 4.3.5 (cf. [She 2017, Corollary 4.1.13]). Recall that $M := H^2(X(\mathbb{C}), \mathbb{Z}(1)) \simeq \mathcal{L}_{K3}$ is unimodular, and $N := \mathrm{Pic}_{X/F}(F)$ is a primitive sublattice. In this situation, one can verify a canonical isomorphism

$$N^\vee/N \simeq M/(N + N^\perp) \simeq (N^\perp)^\vee/N^\perp.$$

Thus we get $\mathrm{disc}(\mathrm{Pic}_{X/F}(F)) = \mathrm{disc}(T(X))$.

Lemma 4.3.6 (cf. [She 2017, Proposition 4.1.11]). *For $(X, L, \nu, \alpha) \in M^\circ_{2d, D_d(n), F}(F)$ and any prime number ℓ, we have*

$$(\mathcal{L}^{\mathrm{Gal}(\bar{F}/F)}_{\mathbb{Z}_\ell, (X,L,\alpha)})^\perp = T(X)_{\mathbb{Z}_\ell}.$$

Here, the orthogonal complement of the left-hand side is taken in $\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}$, and the above equality is as a sublattice of $P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_\ell)$.

Proof. First, we can show that

$$T(X)_{\mathbb{Z}_\ell} = (P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_\ell(1))^{\mathrm{Gal}(\bar{F}/F)})^\perp$$

(the orthogonal complement of the right-hand side is taken in $P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_\ell(1))$). Indeed, since the both sides of this equality are primitive in $P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_\ell(1))$, it suffices to show this equality after inverting ℓ, which follows directly from the Tate conjecture over F [1994, Theorem 5.6(a)].

Hence we have to show that

$$(P^2_{\text{ét}}((X_{\bar{F}}, L_{\bar{F}}), \mathbb{Z}_\ell(1))^{\mathrm{Gal}(\bar{F}/F)})^\perp = (\mathcal{L}^{\mathrm{Gal}(\bar{F}/F)}_{\mathbb{Z}_\ell, (X,L,\alpha)})^\perp$$

(note that the $\perp$ in both sides have different meaning). However, since the both sides are primitive in $\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}$, we may invert ℓ for showing this equality, so it follows obviously from Proposition 3.3.3(1). □

Let us complete the proof of Theorem 4.1.4. As in the previous subsection, by Theorem 4.1.4, it suffices to show the finiteness of $\mathrm{Shaf}'(F, R)$ when $F \supset E_n$ and $1/2 \in R$. By Lemma 4.3.1 and the fact [Cassels 1982, Chapter 9, Theorem 1.1] which asserts the finiteness of isometry classes of lattices with bounded rank and discriminant, it is enough to show that $\mathrm{disc}(\mathrm{Pic}_{X/F}(F))$ $(X \in \mathrm{Shaf}'(F, R))$ is bounded. Using Remark 4.3.5, we can reduce the problem to the finiteness of $\{T(X)_{\widehat{\mathbb{Z}}} \mid X \in \mathrm{Shaf}'(F, R)\}$/isometry.

For $X \in \mathrm{Shaf}'(F, R)$, we choose an element

$$(X, L_X, \nu_X, \alpha_X) \in M^\circ_{2d_X, D_{d_X}(n), F}(F).$$

Then, by Proposition 4.2.4 and [Zarhin 1985, Theorem 1] (for finitely generated fields of characteristic 0, see [Faltings et al. 1984, VI, §1, Theorem 2]), the subset

$$\{\Delta_{d_X}(X, L_X, \nu_X, \alpha_X) \mid X \in \mathrm{Shaf}'(F, R)\} \subset \mathrm{Sh}_{\mathbb{K}_n}(\mathrm{GSp}_{V,a})(F)$$

is finite. We denote them by $t_1, \ldots t_m$, and we put

$$\mathrm{Shaf}'(F, R)_i := \{X \in \mathrm{Shaf}'(F, R) \mid \Delta_{d_X}(X, L_X, \nu_X, \alpha_X) = t_i\}.$$

Thus, the desired finiteness follows from the following lemma.

Lemma 4.3.7. *The $\widehat{\mathbb{Z}}$-lattices $T(X)_{\widehat{\mathbb{Z}}}$ ($X \in \mathrm{Shaf}'(F, R)_i$) are isometric to each other.*

Proof. By Lemma 4.3.6, it suffices to show that $\mathcal{L}_{\mathbb{Z}_\ell,(X,L_X,\alpha_X)}$ ($X \in \mathrm{Shaf}'(F, R)_i$) is unique up to a $\mathrm{Gal}(\overline{F}/F)$-equivariant isometry, for any ℓ. We denote the lift of t_i on $\mathrm{Sh}_{D(n)}(\mathrm{SO}_{\mathcal{L}})$ via $h \circ \delta$ (it exists by the definition of t_i, and it is unique because $h \circ \delta$ is injective) by $\tilde{t}_i$. Recall that we have the étale sheaf $\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell}$, which has a symmetric pairing structure, so we get the $\mathrm{Gal}(\overline{F}/F)$-lattice $\tilde{t}_i^*(\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell})$, which depends only on t_i. By our construction of $\mathcal{L}_{\mathbb{Z}_\ell,(X,L,\alpha)}$ in Proposition 3.3.3, for any $X \in \mathrm{Shaf}'(F, R)_i$, the $\mathrm{Gal}(\overline{F}/F)$-lattice $\mathcal{L}_{\mathbb{Z}_\ell,(X,L_X,\alpha_X)}$ is none other than $\tilde{t}_i^*(\mathcal{L}^{\mathrm{shf}}_{\mathbb{Z}_\ell})$, finishing the proof. □

5. ℓ-independence

In this section, we prove ℓ-independence of the unramifiedness for completing the proof of the main theorem.

Lemma 5.0.1. *Let K be a Henselian discrete valuation field, k be the residue field of K, p be the characteristic of k, and X be a smooth proper surface X over K. Then, the following are equivalent.*

(a) *The $\mathrm{Gal}(\overline{K}/K)$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for some $\ell \neq p$.*

(b) *The $\mathrm{Gal}(\overline{K}/K)$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified for all $\ell \neq p$.*

Moreover, if K is a complete discrete valuation field of mixed characteristic $(0, p)$ with the perfect residue field k and X is a $K3$ surface over K, then $(a)(\Leftrightarrow(b))$ is equivalent to the following.

(c) *The $\mathrm{Gal}(\overline{K}/K)$-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.*

Remark 5.0.2. (1) In fact, for K3 surfaces, Lemma 5.0.1 is already mentioned by Madapusi Pera in [Matsumoto 2015, Remark 4.3] (using the Kuga–Satake construction, we can reduce the problem to the case of abelian varieties). We note that such arguments also appeared in [Imai and Mieda 2020]. So we will prove only the ℓ versus ℓ' part for general smooth proper surfaces as a corollary of Matsumoto's ℓ-independence result [2016, Theorem 3.3(2)].

(2) If we assume that X admits a Kulikov model after a finite extension of K, then Lemma 5.0.1 is known as a corollary of a good reduction criterion for K3 surfaces (see [Chiarellotto et al. 2019, Theorem 1.1] for example).

(3) For any smooth proper surface X over K, one can easily prove the similar assertion for $H^i_{\text{ét}}$ ($i \neq 2$). Indeed, the case of $i = 0, 4$ is trivial. Moreover, for $i = 1, 3$, by using the Picard variety, we can reduce the problem to the case of abelian varieties. Therefore, it follows from the Néron–Ogg–Shafarevich criterion for abelian varieties (and its crystalline analogue [Coleman and Iovita 1999, Theorem 1]).

5.1. ***Proof of Lemma 5.0.1.*** In this subsection, we prove the Lemma 5.0.1. As in Remark 5.0.2, it is enough to show the equivalence (a)⇔(b) in Lemma 5.0.1. Let K be a Henselian discrete valuation field, k be the residue field of K, p be the characteristic of k, and X be a smooth proper surface over K.

First, we recall the definition of the monodromy operator.

Definition 5.1.1. Let ℓ be a prime number different from p. Consider the representation

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)).$$

By Grothendieck's monodromy theorem, there exists an open subgroup of the inertia subgroup $J \subset I_K$ and the nilpotent operator

$$N_\ell : H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(1) \to H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$$

such that for all $\sigma \in J$, we have $\rho_\ell(\sigma) = \exp(t_\ell(\sigma)N_\ell)$, where $t_\ell : I_K \to \mathbb{Z}_\ell(1)$ is a natural projection. By fixing an isomorphism $\mathbb{Q}_\ell(1) \simeq \mathbb{Q}_\ell$, we regard N_ℓ as a linear endomorphism of $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, which is called the *monodromy operator*.

Remark 5.1.2. By the definition, N_ℓ does not change if we replace K by a finite extension of it.

The following lemma is an elementary fact about ℓ-adic representations.

Lemma 5.1.3. *The following are equivalent.*

(1) *The ℓ-adic representation ρ_ℓ is unramified.*

(2) $N_\ell = 0$ *and* $\mathrm{tr}(\rho_\ell(\sigma)) = \dim(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ *for any* $\sigma \in I_K$.

Proof. (1) $\Rightarrow$ (2) is trivial. Therefore we prove the opposite direction. By the definition of the monodromy operator, we have $\rho_\ell(g) = 1$ for any $g \in J$, where J is an open subgroup of I_K. Hence for any $\sigma \in I_K$, we get $\rho_\ell(\sigma)$ is of finite order, and the trace condition implies that $\rho_\ell(\sigma) = 1$. □

Definition 5.1.4. (1) There exists a unique increasing filtration $\mathrm{M}_r(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ such that $\mathrm{M}_r = 0$ for $r \ll 0$, $\mathrm{M}_r = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for $r \gg 0$, $N(\mathrm{M}_r) \subset \mathrm{M}_{r-2}$ and N^r induces an isomorphism $\mathrm{gr}^{\mathrm{M}}_r \simeq \mathrm{gr}^{\mathrm{M}}_{-r}$ for any positive integer r. We call M_r the *monodromy filtration* on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$.

(2) If X admits a strictly semistable model over $\mathcal{O}_K$, we get the *weight filtration* $\mathrm{W}_r(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ by the weight spectral sequence (see [Saito 2003, Corollary 2.8]). For general X, one can also define the weight filtration W_r by using de Jong's alteration.

Lemma 5.1.5. (1) *For any integer r, we have*

$$\mathrm{M}_r(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)) = \mathrm{W}_{r+2}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)).$$

(2) *For any integer r, the dimension of* $\mathrm{gr}^{\mathrm{W}}_r(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ *is independent of ℓ.*

Proof. Part (1) is well-known as the weight monodromy conjecture for surfaces (see [Rapoport and Zink 1982, Satz 2.13; Saito 2003, Lemma 3.9)]. Part (2) follows from [Matsumoto 2016, Theorem 3.3(2)]. □

The proof of $(a) \Leftrightarrow (b)$ in Lemma 5.0.1. Taking a completion, we may assume K is complete. We shall prove $(a) \Rightarrow (b)$. Take prime numbers $\ell, \ell' \neq p$. By [Ochiai 1999, Corollary 2.5] (for imperfect residue fields, see [Vidal 2004, Proposition 4.2]), we have $\mathrm{tr}(\rho_\ell(\sigma)) = \mathrm{tr}(\rho_{\ell'}(\sigma))$ for any $\sigma \in I_K$. By the definition of the monodromy filtration, we have

$$N_\ell = 0 \Leftrightarrow \dim(\mathrm{gr}^M_0(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell))) = \dim(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)).$$

Therefore, by Lemmas 5.1.3 and 5.1.5, we get the desired implication.

6. Corollaries

6.1. ***Some remarks.*** First, combining Theorem 4.1.4 with Lemma 5.0.1, we obtain the main theorem in a more generalized form.

Theorem 6.1.1. *Let F be a finitely generated field over $\mathbb{Q}$, R be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field F, and d be a positive integer. Then, the set*

$$S(F, R) := \{X \mid X : K3 \text{ surface over } F \text{ satisfying the condition } (C)\}/F\text{-isom}$$

is finite. Here, the condition (C) is the following.

(C) *For any height 1 prime $\mathfrak{p} \in \operatorname{Spec} R$, take a discrete valuation field $E_{\mathfrak{p}}$ such that $E_{\mathfrak{p}}$ is an algebraic extension of discrete valuation fields over $F = \operatorname{Frac}(R_{\mathfrak{p}})$, the residue field of $E_{\mathfrak{p}}$ is the perfection of the residue field of $R_{\mathfrak{p}}$, and a uniformizer of $R_{\mathfrak{p}}$ is also a uniformizer of $E_{\mathfrak{p}}$. Then, there exists a prime number ℓ different from the residual characteristic of $\mathfrak{p}$ such that $H^2_{\text{ét}}(X_{\overline{E_{\mathfrak{p}}}}, \mathbb{Q}_\ell)$ is an unramified $\operatorname{Gal}(\overline{F}/E_{\mathfrak{p}})$-representation.*

Remark 6.1.2. (1) The field extension $E_{\mathfrak{p}}$ in the condition (C) always exists by [Matsumura 1989, Theorem 29.1].

(2) By Lemma 5.0.1, the unramifiedness assumption in the condition (C) is independent of ℓ. If the residual characteristic of $\mathfrak{p}$ is positive, replacing $E_{\mathfrak{p}}$ by the completion of it, we can replace this condition in terms of crystalline representations.

Proof. Shrinking Spec R if necessary, we may assume that R is smooth over $\mathbb{Z}$ since the generic fiber $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is generically smooth over $\mathbb{Q}$. Let M be the order of $\mathrm{GL}_{22}(\mathbb{F}_2)$. Shrinking Spec R again, we may assume that $1/M \in R$. Consider a height 1 prime $\mathfrak{p} \in \operatorname{Spec} R$, and we denote its residual characteristic by $p \geq 0$. Take an extension of valuation $\mathfrak{p}$ to $\overline{F}$, and we denote it by $\bar{v}$. We denote the inertia subgroups by $I_{\bar{v}} \subset \operatorname{Gal}(\overline{F}/F)$, $I'_{\bar{v}} \subset \operatorname{Gal}(\overline{F}/E_{\mathfrak{p}})$. We denote the $\operatorname{Gal}(\overline{F}/F)$-representation $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_2)$ by ρ. Then, by Remark 6.1.2(2), we get $\rho(I'_{\bar{v}}) = 1$. If $p = 0$, we have $\rho(I_{\bar{v}}) = \rho(I'_{\bar{v}}) = 1$. If $p > 0$, for any finite index open normal subgroup H of $\rho(I_{\bar{v}})$, we get $[\rho(I_{\bar{v}}) : H]$ is a p-group. (Here, we use that for any finite extension of discrete valuation fields of characteristic 0, the extension degree is equal to the product of the ramification index and the inertia degree). Therefore we get

$$\rho(I_{\bar{v}}) \cap (1 + 2 \cdot \mathrm{Mat}_{22}(\mathbb{Z}_2)) = 1$$

since the former is pro-p and the latter is pro-2. Moreover, the image of $\rho(I_{\bar{v}})$ in $\mathrm{GL}_{22}(\mathbb{F}_2)$ via the reduction map is trivial because p does not divide M. Therefore, we get $\rho(I_{\bar{v}}) = 1$ even if $p > 0$. Thus we have $S(F, R) \subset \mathrm{Shaf}(F, R)$, so $S(F, R)$ is a finite set. □

Next, as an immediate consequence of Theorem 6.1.1, we obtain the unpolarized Shafarevich conjecture for K3 surfaces over finitely generated fields of characteristic 0.

Definition 6.1.3. Let $R_{\mathfrak{p}}$ be a discrete valuation ring with maximal ideal $\mathfrak{p}$, and F be the fraction field of $R_{\mathfrak{p}}$. For a K3 surface X over F, we say X has good reduction at $\mathfrak{p}$ if there exists a smooth proper algebraic space over $R_{\mathfrak{p}}$ whose generic fiber is isomorphic to X. Note that such model would be automatically a K3 family over Spec $R_{\mathfrak{p}}$ (see Definition 2.1.1(2)).

Corollary 6.1.4. *Let F be a finitely generated field over $\mathbb{Q}$, and R be a finite type algebra over $\mathbb{Z}$ which is a normal domain with the fraction field F. Then, the set*

$$\left\{X \mid X : K3 \text{ surface over } F,\ X \text{ has good reduction at any height } 1 \text{ prime ideal } \mathfrak{p} \in \operatorname{Spec} R\right\}/F\text{-isom}$$

is finite.

6.2. ***The finiteness of twists.*** Here, we give the finiteness result of twists of K3 surfaces via a finite extension of characteristic 0 fields.

Corollary 6.2.1. *Let F be a field of characteristic 0, E/F be a finite extension, and X be a K3 surface over F. Then, the set*

$$\operatorname{Tw}_{E/F}(X) := \{Y : K3 \text{ surface over } F \mid Y_E \simeq_E X_E\}/F\text{-isom}$$

is finite.

Proof. Clearly, we may assume E/F is a finite Galois extension. First, we will reduce the problem to the case of finitely generated fields. Since $\operatorname{Aut}(X_{\bar{F}})$ is a finitely generated group [Sterk 1985, Proposition 2.2], extending E if necessary, we may assume $\operatorname{Aut}(X_E) = \operatorname{Aut}(X_{\bar{F}})$. We can take a finitely generated field $E' \subset E$ on which X and any elements of $\operatorname{Aut}(X_E)$ are defined. Moreover, by extending E' if necessary, we may assume E' is $\operatorname{Gal}(E/F)$-stable and $\operatorname{Gal}(E/F) \to \operatorname{Aut}(E')$ is injective. Let F' be the fixed subfield ${E'}^{\operatorname{Gal}(E/F)}$. Then, the description of twists

$$\operatorname{Tw}_{E/F}(X) \simeq H^1(\operatorname{Gal}(E/F), \operatorname{Aut}(X_E)) \simeq H^1(\operatorname{Gal}(E'/F'), \operatorname{Aut}(X_{E'})).$$

implies that the desired finiteness is reduced to the case of E'/F'.

Thus, in the following of this proof, we assume F is a finitely generated field and E/F is a finite Galois extension. One can take a smooth proper morphism of schemes $\mathcal{X} \to \operatorname{Spec} R$ whose generic fiber is X, where R is a smooth algebra over $\mathbb{Z}$ which is an integral domain with the fraction field F and $\frac{1}{2} \in R$. Then, via a monodromy action, we get $H^2_{\text{ét}}(X_{\bar{F}}, \mathbb{Z}_2)$ is unramified over Spec R. Let $\widetilde{R}$ be the normalization of R in E. Shrinking Spec R if necessary, we may assume Spec $\widetilde{R} \to$ Spec R is a finite étale covering. Since E is unramified over Spec R, by [Fu 2015, Proposition 3.3.6], we have

$$\operatorname{Ker}(\pi_1(\operatorname{Spec} E, \bar{s}) \to \pi_1(\operatorname{Spec} \widetilde{R}, \bar{s})) = \ker(\pi_1(\operatorname{Spec} F, \bar{s}) \to \pi_1(\operatorname{Spec} R, \bar{s})).$$

For any $Y \in \operatorname{Tw}_{E/F}(X)$, the isomorphism $Y_E \simeq_E X_E$ implies that the $\operatorname{Gal}(\bar{F}/E)$-action on $H^2_{\text{ét}}(Y_{\bar{F}}, \mathbb{Z}_2)$ arises from a $\pi_1(\operatorname{Spec} \widetilde{R}, \bar{s})$-action. Moreover, because of the above equality, the $\operatorname{Gal}(\bar{F}/F)$-action on $H^2_{\text{ét}}(Y_{\bar{F}}, \mathbb{Z}_2)$ also arises from a $\pi_1(\operatorname{Spec} R, \bar{s})$-action. Hence we get a natural inclusion $\operatorname{Tw}_{E/F}(X) \hookrightarrow \operatorname{Shaf}(F, R)$, and thus the desired finiteness follows from Theorem 4.1.4. □

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teppei@ms.u-tokyo.ac.jp *Graduate School of Mathematical Sciences, The University of Tokyo, Tokyo, Japan*

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