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Hybrid subconvexity bounds for twists of GL(3) × GL(2) L-functions.

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We prove hybrid subconvexity bounds for GL(3) × GL(2) L-functions twisted by a primitive Dirichlet character modulo M (prime) in the M- and t-aspects. We also improve hybrid subconvexity bounds for twists of GL(3) L-functions in the M- and t-aspects.

1. Introduction

The subconvexity problem of automorphic L-functions on the critical line is one of the central problems in number theory. In general, let $\mathcal{C}$ denote the analytic conductor of the relevant L-function; see, e.g., Iwaniec and Kowalski [2004, Section 5.1]), then one hopes to obtain a subconvexity bound $\mathcal{C}^{1/4-\delta}$ for some $\delta > 0$ on the critical line. Subconvexity bounds have many very important applications such as the equidistribution problems; see, e.g., Michel and Venkatesh [2010].

For the GL(1) case, i.e., the Riemann zeta function and Dirichlet L-functions, subconvexity bounds have been known for a long time thanks to Weyl [1921] and Burgess [1963]. In the last decades, many cases of GL(2) L-functions have been treated; see Michel and Venkatesh [2010]. In the last ten years, people have made progress on GL(3) L-functions; see [Blomer 2012; Li 2011; Munshi 2015a; 2015b; 2022; Sharma 2022]. In this paper, we extend the techniques in [Lin and Sun 2021; Munshi 2022; Sharma 2022] to prove, for the first time, hybrid subconvexity bounds for GL(3) × GL(2) L-functions twisted by a primitive Dirichlet character modulo M (prime), which reach the best known bounds in the M- and t-aspects simultaneously. Our method also improves hybrid subconvexity bounds for twists of GL(3) L-functions due to Huang [2021a] and Lin [2021].

Let π be a Hecke–Maass cusp form of type (ν_1, ν_2) for $SL(3, \mathbb{Z})$ with the normalized Fourier coefficients $A(m, n)$. The L-function of π is defined as

$$L(s, \pi) = \sum_{n\geq 1} \frac{A(1, n)}{n^s}, \quad \mathrm{Re}(s) > 1.$$

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Let f be a Hecke–Maass cusp form with the spectral parameter t_f for $\mathrm{SL}(2,\mathbb{Z})$, with the normalized Fourier coefficients $\lambda_f(n)$. The L-function of f is defined by

$$L(s,f)=\sum_{n\geq 1}\frac{\lambda_f(n)}{n^s},\qquad \mathrm{Re}(s)>1.$$

Let χ be a primitive Dirichlet character modulo M. The $\mathrm{GL}(3)\times\mathrm{GL}(2)\times\mathrm{GL}(1)$ Rankin–Selberg L-function is defined as

$$L(s,\pi\times f\times\chi)=\sum_{m\geq 1}\sum_{n\geq 1}\frac{A(m,n)\lambda_f(n)\chi(m^2n)}{(m^2n)^s},\qquad \mathrm{Re}(s)>1.$$

These L-functions have analytic continuation to the whole complex plane. In this paper, we consider the L-values at the point $\frac{1}{2}+it$ with $t\in\mathbb{R}$. The Phragmén–Lindelöf principle and the functional equation imply the convexity bounds

$$L\big(\tfrac{1}{2}+it,\pi\times f\times\chi\big)\ll_{\pi,f,\varepsilon}(M(1+|t|))^{3/2+\varepsilon},$$

for any $\varepsilon>0$. It is known that the Riemann hypothesis for $L(s,\pi\times f\times\chi)$ implies the Lindelöf hypothesis, i.e., $L\big(\frac{1}{2}+it,\pi\times f\times\chi\big)\ll_{\pi,f,\varepsilon}(M(1+|t|))^{\varepsilon}$. For $M=1$, the first subconvex exponent in t-aspect was obtained by Munshi [2022]. Recently, Lin and Sun [2021] proved that

$$L\big(\tfrac{1}{2}+it,\pi\times f\big)\ll_{\pi,f,\varepsilon}(1+|t|)^{3/2-3/20+\varepsilon}.$$

For $t=0$ and prime M, Sharma [2022] proved that

$$L\big(\tfrac{1}{2},\pi\times f\times\chi\big)\ll_{\pi,f,\varepsilon}M^{3/2-1/16+\varepsilon}.$$

In the context of L-functions, obtaining hybrid bounds that perfectly combine the two aspects is a difficult problem; see [Blomer and Harcos 2008; Fan and Sun 2022; Heath-Brown 1978; Lin 2021; Huang 2021c; Petrow and Young 2020; 2023]. Our main result in this paper is the following hybrid subconvexity bounds.

Theorem 1.1. *With the notation as above. Let $t\in\mathbb{R}$ and M be prime. Then we have*

$$L\big(\tfrac{1}{2}+it,\pi\times f\times\chi\big)\ll_{\pi,f,\varepsilon}M^{3/2-1/16+\varepsilon}(1+|t|)^{3/2-3/20+\varepsilon}.$$

Remark 1.2. Below we will carry out the proof under the assumption $t\geq M^{\varepsilon}$ for some small $\varepsilon>0$. For the case $t\ll M^{\varepsilon}$, one can extend the method of Sharma [2022] to prove $L\big(\frac{1}{2}+it,\pi\times f\times\chi\big)\ll_{t,\pi,f,\varepsilon}M^{3/2-1/16+\varepsilon}$ with polynomial dependence on t. For the case $t\leq -M^{\varepsilon}$, the same result follows from the case $t\geq M^{\varepsilon}$ by the functional equation.

Remark 1.3. Let π, χ and t be the same as above and f be a weight k Hecke modular form for $\mathrm{SL}(2,\mathbb{Z})$. The same hybrid subconvexity bounds for $L\big(\frac{1}{2}+it,\pi\times f\times\chi\big)$ can be proved by our method. The only thing need to be changed is the $\mathrm{GL}(2)$ Voronoi summation formula.

Note that by the Hecke relation of the Fourier coefficients (see [Goldfeld 2006, Theorem 6.4.11]), we have

$$A(1,m)A(1,n) = \sum_{d\,|\,(m,n)} A\left(d, \frac{mn}{d^2}\right).$$

Hence we have

$$L(s,\pi\times\chi)^2 = \sum_{m\geq 1}\sum_{n\geq 1} \frac{A(m,n)\tau(n)\chi(m^2n)}{(m^2n)^s}, \quad \mathrm{Re}(s) > 1,$$

where $\tau(n) = \sum_{d\,|\,n} 1$ is the divisor function which is the coefficient of the Eisenstein series for $\mathrm{SL}(2,\mathbb{Z})$. The subconvexity bounds for $L\left(\frac{1}{2}+it,\pi\times\chi\right)$ follow from bounds for $L\left(\frac{1}{2}+it,\pi\times f\times\chi\right)$ with f being a GL(2) Eisenstein series.

Theorem 1.4. *With the notation as above. Let $t \in \mathbb{R}$ and M be prime. Then we have*

$$L\big(1/2+it,\pi\times\chi\big) \ll_{\pi,\varepsilon} M^{3/4-1/32+\varepsilon}(1+|t|)^{3/4-3/40+\varepsilon}.$$

Remark 1.5. The only difference in the proofs of Theorem 1.4 and Theorem 1.1 is that we need to use the Voronoi summation formula for $\tau(n)$ instead of those for Fourier coefficients of a GL(2) cusp form. This will give us another zero frequency contribution in the dual sum. This contribution will not have any effect on the final result. Indeed, in the generic case, the weight function for the sum of $\tau(n)$ is oscillating. By integration by parts, we can show its contribution is negligibly small.

Theorem 1.4 improves the hybrid subconvexity bounds for twists of GL(3) L-functions due to Huang [2021a] and Lin [2021], and also reaches the best known bounds in the M- and t-aspects simultaneously; see Sharma [2022] and Aggarwal [2021]. Recall that under the same assumptions Lin [2021] proved that

$$L\left(\tfrac{1}{2}+it,\pi\times\chi\right) \ll_{\pi,\varepsilon} (M(1+|t|))^{3/4-1/36+\varepsilon}.$$

One may give a quick comparison with Lin's work [2021]. Actually, we have a different structure from Lin's paper. Theorem 1.4 can be viewed as a subconvexity result for GL(3) × GL(2) × GL(1) L-functions, where the GL(2)-item is the Eisenstein series. Lin's work is to consider the $L\left(\frac{1}{2}+it,\pi\times\chi\right)$ directly.[1]

Heath-Brown [1978] proved the first hybrid subconvexity bounds for Dirichlet L-functions by extending the Burgess method and van der Corput method to give good estimates for hybrid sums $\sum\chi(n)n^{it}$. Recently, Petrow and Young [2020; 2023] proved the Weyl bound in both aspects by estimating moments of L-functions. For the GL(2) case, Blomer and Harcos [2008] proved the first hybrid subconvexity bounds in the M- and t-aspects by using moments of L-functions. Recently, Fan and Sun [2022] improved the bounds by using a delta method. Our method can also provide hybrid subconvexity bounds in the GL(1) and GL(2) settings, but are weaker than the best known results.

The basic observation is that the subconvexity bounds for GL(3) × GL(2) × GL(1) L-functions in individual M-aspect or t-aspect were proved by applying the Duke–Friedlander–Iwaniec delta method

[1] Kıral, Kuan and Lesesvre [Kıral et al. 2022] further improved subconvexity bounds for twisted GL(3) L-functions under the restriction $M < t^{8/7}$.

to separate oscillatory factors. This suggests to us that in order to prove a hybrid subconvexity bound one may use the same method as the starting point. This philosophy may allow us to make progress in other hybrid settings; see [Huang 2021c]. However, technically speaking, to estimate the complicated sums (e.g., (3-1) below) is much more difficult. We have to take care of both aspects carefully. It is worth mentioning that, as in [Aggarwal 2021; Huang 2021b; Lin and Sun 2021], we drop the conductor-lowering trick which was used in Munshi [2015a] for the t-aspect, but we still use the conductor-lowering trick for the M-aspect as in Munshi [2015b] and Sharma [2022].

1A. ***Sketch of the proof.*** We give a brief sketch of the proof. By the approximate functional equation and some standard analysis, we need to get

$$\sum_{n\sim N} A(r,n)\lambda_f(n)\chi(n)n^{-it} \ll N^{1/2+\varepsilon}M^{3/2-1/16}t^{3/2-3/20},$$

where $N \ll (Mt)^{3+\varepsilon}/r^2$, $r \ll M^{1/8}t^{3/10}$ and $(r,M)=1$ (see Proposition 3.1). We will apply the Duke–Friedlander–Iwaniec delta method with moduli $q \le Q$ (see Lemma 2.6). For simplicity let us focus on the generic case, i.e., $N = M^3t^3$, $r = 1$ and $q \sim Q = (LN/MK)^{1/2}$ for some parameters L and $K \ll t^{1-\varepsilon}$ which will be chosen later. After applying the Duke–Friedlander–Iwaniec delta method and the conductor-lowering trick for the M-aspect by Munshi (see Sharma [2022]), the main object of study is given by

$$\frac{1}{L}\sum_{\ell\sim L}\overline{A(1,\ell)}\int_{x\sim 1}\frac{1}{M}\sum_{b \bmod M}^{\star}\frac{1}{Q}\sum_{\substack{q\sim Q\\(q,\ell M)=1}}\frac{1}{q}\sum_{a \bmod q}^{\star}\sum_{n\sim LN}A(1,n)e\Big(\frac{n(aM+bq)}{qM}\Big)e\Big(\frac{nx}{MqQ}\Big)$$
$$\cdot\sum_{m\sim N}\lambda_f(m)\chi(m)e\Big(\frac{-m\ell(aM+bq)}{qM}\Big)m^{-it}e\Big(\frac{-m\ell x}{MqQ}\Big)\,\mathrm{d}x.$$

By using the Ramanujan conjecture on average, trivially estimating at this stage gives $O(LN^2)$. So we want to save LN plus a “little more” in the above sum. Note that here we don’t need the conductor-lowing trick for the t-aspect as observed in [Aggarwal 2021; Huang 2021b; Lin and Sun 2021]. In fact, the x-integral above plays the same role as the v-integral in Munshi [2015a].

We apply the Voronoi summation formulas to both n and m sums. For the n sum, by the GL(3) Voronoi, we get essentially

$$qM\sum_{n_2=1}^{\infty}\frac{A(1,n_2)}{n_2}S(\overline{(aM+bq)},n_2;qM)\Psi_x\Big(\frac{n_2}{q^3M^3}\Big),$$

for certain weight function Ψ_x depending on x. The conductor in the above n_2-sum is $K^3M^3Q^3$, and the length is about LN. Hence the dual length becomes $n_2 \asymp K^3M^3Q^3/(LN) = L^2N^2/Q^3$. By Lemma 4.1, the current bound for this dual sum is $QM\cdot(QM)^{1/2}\cdot(LN/(MQ^2))^{3/2}$. So we save $(LN)^{1/4}/(MK)^{3/4}$.

In the GL(2) Voronoi, the dual sum becomes essentially

$$\frac{N}{Mq\tau(\overline{\chi})}\sum_{\substack{u \bmod M\\ u\not\equiv b \bmod M}}\overline{\chi}(u\ell)\sum_{m\geq 1}\lambda_f(m)e\bigg(\pm\frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\bigg)H^{\pm}\bigg(\frac{mN}{M^2q^2}\bigg)$$

for certain weight function $H^{\pm}$. The conductor in the m-sum is $t^2Q^2M^2$, so the dual length becomes $m \asymp t^2Q^2M^2/N = LMt^2/K$. By Lemma 2.4 and the square root cancellation in the u sum, the trivial bound for this dual sum is $(N/QM)\cdot(M^{1/2}Q^{1/2}/N^{1/4})\cdot(t^2Q^2M^2/N)^{3/4}\cdot(1/t^{1/2})$. Hence we save $N^{1/2}K^{1/2}/(L^{1/2}M^{1/2}t)$. By the stationary phase method, we save $K^{1/2}$ from the x-integral. We also save $Q^{1/2}$ in the a sum and $M^{1/2}$ in the b sum. Hence in total we have saved

$$\frac{(LN)^{1/4}}{(MK)^{3/4}}\cdot\frac{N^{1/2}K^{1/2}}{L^{1/2}M^{1/2}t}\cdot K^{1/2}Q^{1/2}M^{1/2}=\frac{N}{Mt}.$$

Hence we still need to save LMt plus a "little more". Generally we arrive at

$$\frac{N^{13/12}}{M^2LQ}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)\chi(\ell)}\ell^{1/3}\sum_{q\sim Q}\frac{1}{q^{3/2}}\sum_{n_2\asymp L^2N^2/Q^3}\frac{A(1,n_2)}{n_2^{2/3}}\sum_{m\asymp M^2Q^2t^2/N}\frac{\lambda_f(m)}{m^{1/4}}\mathcal{C}\mathcal{J},$$

for certain character sum $\mathcal{C}$ and integral transform $\mathcal{J}$ (see (4-11)).

Next applying the Cauchy inequality we arrive at

$$\Bigg(\sum_{n_2\asymp L^2N^2/Q^3}\bigg|\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)\chi(\ell)}\ell^{1/3}\sum_{q\sim Q}\frac{1}{q^{3/2}}\sum_{m\asymp M^2Q^2t^2/N}\frac{\lambda_f(m)}{m^{1/4}}\mathcal{C}\mathcal{J}\bigg|^2\Bigg)^{1/2},$$

where we seek to save LMt plus extra. Opening the absolute value square we apply the Poisson summation formula on the sum over n_2. For the zero frequency we save $(LQM^2Q^2t^2/N)^{1/2}$. This gives a bound of size $N^{3/4}M^{3/4}K^{3/4}/L^{1/4}$. We save enough in the zero frequency if $K<t$ and $L>1$.

For the non-zero frequencies, the conductor is of size Q^2MK, hence the length of the dual sum is $O((Q^2MK/(L^2N^2/Q^3))^{1/2})=O(L^{1/4}N^{1/4}/(M^{3/4}K^{3/4}))$. In the integral transform we save $K^{1/4}$ and the character sums save $(Q^2M^{1/2})^{1/2}=QM^{1/4}$. Hence in total in the non-zero frequencies we save $(M^{3/4}K^{3/4}/L^{1/4}N^{1/4})K^{1/4}QM^{1/4}$. This gives a bound of size $N^{1/4}QL^{1/4}Mt=N^{3/4}L^{3/4}M^{1/2}t/K^{1/2}$. We save enough in the non-zero frequencies if $L<M^{1/3}$ and $K>t^{1/2}$. We also have different bounds from other cases. In fact, the best choice is $L=M^{1/4}$ and $K=t^{4/5}$ which gives $O(N^{1/2+\varepsilon}M^{3/2-1/16}t^{3/2-3/20})$ as claimed.

1B. *Plan for this paper.* The rest of this paper is organized as follows. In Section 2, we introduce some notation and present some lemmas that we will need later. The approximate functional equation allows us to reduce the subconvexity problem to estimating certain convolution sums. In Section 3, we apply the delta method to the convolution sums. In Section 4, we apply the Voronoi summation formulas and estimate the integral transforms by the stationary phase method. In Section 5, we apply the Cauchy–Schwarz inequality and Poisson summation formula, and then analyze the integrals. Then we deal with character sums and the zero frequency contribution in Section 6. In Section 7, we give the contribution from non zero frequencies. Finally, in Section 8, we balance parameters optimally and prove Proposition 3.1 which leads to Theorem 1.1.

Notation. Throughout the paper, ε is an arbitrarily small positive number; all of them may be different at each occurrence. By a smooth dyadic subdivision of a sum $\sum_{n\geq 1} A(n)$, we will mean

$$\sum_{(V,N)} \sum_{n\geq 1} A(n) V\left(\frac{n}{N}\right),$$

where

$$\sum_{(V,N)} V\left(\frac{n}{N}\right) = 1$$

with V being a smooth function supported on $[1, 2]$ and satisfying $V^{(j)}(x) \ll_j 1$. The weight functions U, V, W may also change at each occurrence. As usual, $e(x) = e^{2\pi i x}$ and $n \sim N$ means $N \leq n < 2N$.

2. Preliminaries

2A. ***Automorphic forms.*** Let f be a Hecke–Maass cusp form with the spectral parameter t_f for $\mathrm{SL}(2, \mathbb{Z})$, with the normalized Fourier coefficients $\lambda_f(n)$. Let θ_2 be the bound toward to the Ramanujan conjecture and we have $\theta_2 \leq \frac{7}{64}$ due to Kim and Sarnak [2003]. It is well known that, by the Rankin–Selberg theory, one has

$$\sum_{n\leq N} |\lambda_f(n)|^2 \ll_f N. \tag{2-1}$$

Let π be a Hecke–Maass cusp form of type (ν_1, ν_2) for $\mathrm{SL}(3, \mathbb{Z})$ with the normalized Fourier coefficients $A(r, n)$. Similarly, Rankin–Selberg theory gives

$$\sum_{r^2 n\leq N} |A(r, n)|^2 \ll_\pi N. \tag{2-2}$$

We record the Hecke relation

$$A(r, n) = \sum_{d\,|\,(r,n)} \mu(d) A\left(\frac{r}{d}, 1\right) A\left(1, \frac{n}{d}\right)$$

which follows from the Möbius inversion and [Goldfeld 2006, Theorem 6.4.11]. Hence we have the individual bounds

$$A(r, n) \ll (rn)^{\theta_3+\varepsilon}, \tag{2-3}$$

where $\theta_3 \leq \frac{5}{14}$ is the bound toward to the Ramanujan conjecture on GL(3); see [Kim 2003]. So we have

$$\sum_{n\sim N} |A(r, n)| \ll \sum_{n_1\,|\,r^\infty} \sum_{\substack{n\sim N/n_1 \\ (n,r)=1}} |A(r, nn_1)| \leq \sum_{n_1\,|\,r^\infty} |A(r, n_1)| \sum_{\substack{n\sim N/n_1 \\ (n,r)=1}} |A(1, n)| \ll r^{\theta_3+\varepsilon} N \tag{2-4}$$

and

$$\sum_{n\sim N}|A(r,n)|^2 \ll \sum_{n_1\mid r^\infty}\sum_{\substack{n\sim N/n_1\\(n,r)=1}}|A(r,nn_1)|^2 \le \sum_{n_1\mid r^\infty}|A(r,n_1)|^2\sum_{\substack{n\sim N/n_1\\(n,r)=1}}|A(1,n)|^2 \ll r^{2\theta_3+\varepsilon}N. \tag{2-5}$$

Here we have used (2-2) and the fact $\sum_{d\mid r^\infty} d^{-\sigma} \ll r^\varepsilon$, for $\sigma > 0$.

2B. ***L-functions.*** The Rankin–Selberg L-function $L(s,\pi\times f\times\chi)$ has the following functional equation

$$\Lambda(s,\pi\times f\times\chi)=\epsilon_{\pi\times f\times\chi}\Lambda(1-s,\tilde\pi\times f\times\overline{\chi}),$$

where

$$\Lambda(s,\pi\times f\times\chi)=M^{3s}\pi^{-3s}\prod_{j=1}^{3}\prod_{\pm}\Gamma\Big(\frac{s-\alpha_j\pm it_f}{2}\Big)L(s,\pi\times f\times\chi)$$

is the completed L-function and $\epsilon_{\pi\times f\times\chi}$ is the root number. Here α_j are the Langlands parameters of π, and $\tilde\pi$ is the contragredient representation of π. By [Iwaniec and Kowalski 2004, Section 5.2], we can obtain the approximate functional equation which leads us to the following result.

Lemma 2.1. *We have*

$$L\big(\tfrac12+it,\pi\times f\times\chi\big) \ll (M(|t|+1))^\varepsilon \sup_{N\ll(M(|t|+1))^{3+\varepsilon}}\frac{|S(N)|}{\sqrt N}+(M(|t|+1))^{-A},$$

where

$$S(N)=\sum_{r\ge1}\sum_{n\ge1}A(r,n)\lambda_f(n)\chi(r^2n)(r^2n)^{-it}V\Big(\frac{r^2n}{N}\Big),$$

with some compactly supported smooth function V such that $\operatorname{supp} V\subset[1,2]$ *and* $V^{(j)}\ll_j 1$.

We first estimate the contribution from large values of r. By (2-1) and (2-5) we have

$$\begin{aligned}\sum_{r\ge M^{1/8}(|t|+1)^{3/10}}&\Big|\sum_{n\ge1}A(r,n)\lambda_f(n)\chi(n)(r^2n)^{-it}V\Big(\frac{r^2n}{N}\Big)\Big|\\ &\ll \sum_{M^{1/8}(|t|+1)^{3/10}\le r\ll\sqrt N}\Big(\sum_{n\asymp N/r^2}|A(r,n)|^2\Big)^{1/2}\Big(\sum_{n\asymp N/r^2}|\lambda_f(n)|^2\Big)^{1/2}\\ &\ll \sum_{M^{1/8}(|t|+1)^{3/10}\le r\ll\sqrt N} r^{\theta_3+\varepsilon}\frac{N}{r^2}\\ &\ll N\sum_{M^{1/8}(|t|+1)^{3/10}\le r\ll\sqrt N} r^{-3/2-\varepsilon}\\ &\ll N^{1/2}M^{3/2-1/16}(|t|+1)^{3/2-3/20+\varepsilon},\end{aligned}\tag{2-6}$$

for $N\ll(M(|t|+1))^{3+\varepsilon}$. The contribution from those terms to $L\big(\frac12+it,\pi\times f\times\chi\big)$ is bounded by $M^{3/2-1/16+\varepsilon}(|t|+1)^{3/2-3/20+\varepsilon}$.

Therefore, combining this together with Lemma 2.1, we prove the following lemma.

Lemma 2.2. *We have*

$$L\big(\tfrac{1}{2}+it,\pi\times f\times\chi\big)\ll t^{\varepsilon}\sum_{\substack{r\le M^{1/8}t^{3/10}\\(r,M)=1}}\frac{1}{r}\sup_{N\ll(Mt)^{3+\varepsilon}/r^2}\frac{|S(r,N)|}{\sqrt{N}}+M^{3/2-1/16}t^{3/2-3/20+\varepsilon},$$

where

$$S(r,N):=\sum_{n\ge1}A(r,n)\lambda_f(n)\chi(n)n^{-it}V\Big(\frac{n}{N}\Big).$$

2C. *Summation formulas.* We first recall the Poisson summation formula over an arithmetic progression.

Lemma 2.3. *Let $\beta\in\mathbb{Z}$ and $c\in\mathbb{Z}_{\ge1}$. For a Schwartz function $f:\mathbb{R}\to\mathbb{C}$, we have*

$$\sum_{\substack{n\in\mathbb{Z}\\ n\equiv\beta \bmod c}}f(n)=\frac{1}{c}\sum_{n\in\mathbb{Z}}\hat f\Big(\frac{n}{c}\Big)e\Big(\frac{n\beta}{c}\Big),$$

where $\hat f(y)=\int_{\mathbb{R}}f(x)e(-xy)\,\mathrm{d}x$ is the Fourier transform of f.

Proof. See, e.g., [Iwaniec and Kowalski 2004, (4.24)]. □

We recall the Voronoi summation formula for $\mathrm{SL}(2,\mathbb{Z})$. Let g be a smooth compactly supported function on $(0,\infty)$.

Lemma 2.4. *With the notation as above. Then we have*

$$\sum_{n\ge1}\lambda_f(n)e\Big(\frac{an}{q}\Big)g\Big(\frac{n}{N}\Big)=\frac{N}{q}\sum_{\pm}\sum_{n\ge1}\lambda_f(n)e\Big(\mp\frac{\bar a n}{q}\Big)H^{\pm}\Big(\frac{nN}{q^2}\Big)\tag{2-7}$$

where

$$H^{+}(y)=\frac{-\pi}{\sin(\pi i t_f)}\int_0^\infty g(\xi)(J_{2it_f}(4\pi\sqrt{y\xi})-J_{-2it_f}(4\pi\sqrt{y\xi}))\,\mathrm{d}\xi,\tag{2-8}$$

and

$$H^{-}(y)=4\epsilon_f\cosh(\pi t_f)\int_0^\infty g(\xi)K_{2it_f}(4\pi\sqrt{y\xi})\,\mathrm{d}\xi.\tag{2-9}$$

For $y\gg T^{\varepsilon}$, we have

$$H^{+}(y)=y^{-1/4}\int_0^\infty g(\xi)\xi^{-1/4}\sum_{j=0}^{J}\frac{c_je(2\sqrt{y\xi})+d_je(-2\sqrt{y\xi})}{(y\xi)^{j/2}}\,\mathrm{d}\xi+O(T^{-A})\tag{2-10}$$

for some constant $J=J(A)$ and

$$H^{-}(y)\ll_{t_f,A}y^{-A}.\tag{2-11}$$

Proof. See, e.g., [Lin and Sun 2021, Section 3.1]. □

Notice that (2-10) and (2-11) are only valid for $y \gg T^{\varepsilon}$. So we also need the facts which state that, for $y > 0$, $k \geq 0$ and $\operatorname{Re} \nu = 0$, one has (see [Kowalski et al. 2002, Lemma C.2])

$$\begin{aligned} y^k J_\nu^{(k)}(y) &\ll_{k,\nu} \frac{1}{(1+y)^{1/2}}, \\ y^k K_\nu^{(k)}(y) &\ll_{k,\nu} \frac{e^{-y}(1+|\log y|)}{(1+y)^{1/2}}. \end{aligned} \tag{2-12}$$

We now recall the Voronoi summation formula for $\mathrm{SL}(3,\mathbb{Z})$. Let ψ be a smooth compactly supported function on $(0,\infty)$, and let $\tilde{\psi}(s) := \int_0^\infty \psi(x)x^s \,\mathrm{d}x/x$ be its Mellin transform. For $\sigma > \frac{5}{14}$, we define

$$\Psi^{\pm}(z) := z\frac{1}{2\pi i}\int_{(\sigma)} (\pi^3 z)^{-s}\gamma_3^{\pm}(s)\tilde{\psi}(1-s)\,\mathrm{d}s, \tag{2-13}$$

with

$$\gamma_3^{\pm}(s) := \prod_{j=1}^{3}\frac{\Gamma((s+\alpha_j)/2)}{\Gamma((1-s-\alpha_j)/2)} \pm \frac{1}{i}\prod_{j=1}^{3}\frac{\Gamma((1+s+\alpha_j)/2)}{\Gamma((2-s-\alpha_j)/2)}, \tag{2-14}$$

where α_j are the Langlands parameters of π as above. Note that changing $\psi(y)$ to $\psi(y/N)$ for a positive real number N has the effect of changing $\Psi^{\pm}(z)$ to $\Psi^{\pm}(zN)$. The Voronoi formula on GL(3) was first proved by Miller and Schmid [2006]. The present version is due to Goldfeld and Li [2006] with slightly renormalized variables; see Blomer [2012, Lemma 3].

Lemma 2.5. *Let $c, d, \bar{d} \in \mathbb{Z}$ with $c \neq 0$, $(c,d) = 1$, and $d\bar{d} \equiv 1 \pmod{c}$. Then we have*

$$\sum_{n=1}^{\infty} A(m,n)e\left(\frac{n\bar{d}}{c}\right)\psi(n) = \frac{c\pi^{3/2}}{2}\sum_{\pm}\sum_{n_1\,|\,cm}\sum_{n_2=1}^{\infty}\frac{A(n_2,n_1)}{n_1 n_2}S\left(md, \pm n_2; \frac{mc}{n_1}\right)\Psi^{\pm}\left(\frac{n_1^2 n_2}{c^3 m}\right),$$

where $S(a,b;c) := \sum^*_{d(c)} e((ad+b\bar{d})/c)$ is the classical Kloosterman sum.

2D. *The delta method.* There are several oscillatory factors contributing to the convolution sums. Our method is based on separating these oscillations using the circle method. In the present situation we will use a version of the delta method of Duke, Friedlander and Iwaniec. More specifically we will use the expansion (20.157) given in [Iwaniec and Kowalski 2004, Section 20.5]. Let $\delta : \mathbb{Z} \to \{0,1\}$ be defined by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

We seek a Fourier expansion which matches with $\delta(n)$.

Lemma 2.6. *Let Q be a large positive number. Then we have*

$$\delta(n) = \frac{1}{Q}\sum_{1\leq q\leq Q}\frac{1}{q}\sum_{a \bmod q}^{\star} e\left(\frac{na}{q}\right)\int_{\mathbb{R}} g(q,x)e\left(\frac{nx}{qQ}\right)\mathrm{d}x, \tag{2-15}$$

where $g(q,x)$ is a weight function satisfying that

$$g(q,x) = 1 + O\left(\frac{Q}{q}\left(\frac{q}{Q}+|x|\right)^{A}\right), \qquad g(q,x) \ll |x|^{-A}, \text{ for any } A > 1, \tag{2-16}$$

and

$$\frac{\partial^j}{\partial x^j} g(q,x) \ll |x|^{-j} \min(|x|^{-1}, Q/q) \log Q, \qquad j \geq 1. \tag{2-17}$$

Here the ⋆ on the sum indicates that the sum over a is restricted by the condition $(a,q)=1$.

Proof. See [Huang 2021b, Lemma 15] and [Iwaniec and Kowalski 2004, Section 20.5]. □

In applications of (2-15), we can first restrict to $|x| \ll Q^{\varepsilon}$. If $q \gg Q^{1-\varepsilon}$, then by (2-17) we get $\frac{\partial^j}{\partial x^j} g(q,x) \ll Q^{\varepsilon}|x|^{-j}$, for any $j \geq 1$. If $q \ll Q^{1-\varepsilon}$ and $Q^{-\varepsilon} \ll |x| \ll Q^{\varepsilon}$, then by (2-17) we also have $\frac{\partial^j}{\partial x^j} g(q,x) \ll Q^{\varepsilon}|x|^{-j}$, for any $j \geq 1$. Finally, if $q \ll Q^{1-\varepsilon}$ and $|x| \ll Q^{-\varepsilon}$, then by (2-16), we can replace $g(q,x)$ by 1 with a negligible error term. So in all cases, we can view $g(q,x)$ as a nice weight function.

We remark that there is no restrictions on Q, so we can choose Q to be any large positive number. Recall that in Sharma [2022] and Lin and Sun [2021], the authors took Q to be $(NL/M)^{1/2}$ and $(N/t^{4/5})^{1/2}$, respectively. This motivates us to choose $Q=(NL/MK)^{1/2}$. As we will see, after balancing finally, we can take $L=M^{1/4}$ and $K=t^{4/5}$ optimally, which coincides with Sharma [2022] and Lin and Sun [2021].

2E. *Oscillatory integrals.* Let $\mathcal{F}$ be an index set and $X=X_T:\mathcal{F}\to\mathbb{R}_{\geq 1}$ be a function of $T\in\mathcal{F}$. A family of $\{w_T\}_{T\in\mathcal{F}}$ of smooth functions supported on a product of dyadic intervals in $\mathbb{R}^d_{>0}$ is called X-inert if for each $j=(j_1,\dots,j_d)\in\mathbb{Z}^d_{\geq 0}$ we have

$$\sup_{T\in\mathcal{F}} \sup_{(x_1,\dots,x_d)\in\mathbb{R}^d_{>0}} X_T^{-j_1-\cdots-j_d} |x_1^{j_1}\cdots x_d^{j_d} w_T^{(j_1,\dots,j_d)}(x_1,\dots,x_d)| \ll_{j_1,\dots,j_d} 1.$$

We will use the following stationary phase lemma several times.

Lemma 2.7. *Suppose $w=w_T(t)$ is a family of X-inert functions, with compact support on $[Z,2Z]$, so that $w^{(j)}(t)\ll (Z/X)^{-j}$. Also suppose that ϕ is smooth and satisfies $\phi^{(j)}\ll Y/Z^j$ for some $Y/X^2\geq R\geq 1$ and all t in the support of w. Let*

$$I=\int_{-\infty}^{\infty} w(t)e^{i\phi(t)}\,dt.$$

(i) *If $|\phi'(t)|\gg Y/Z$ for all t in the support of w, then $I\ll_A ZR^{-A}$ for A arbitrarily large.*

(ii) *If $|\phi''(t)|\gg Y/Z^2$ for all t in the support of w, and there exists $t_0\in\mathbb{R}$ such that $\phi'(t_0)=0$ (note that t_0 is necessarily unique), then*

$$I=\frac{e^{i\phi(t_0)}}{\sqrt{\phi''(t_0)}}F_T(t_0)+O_A(ZR^{-A}),$$

where F_T is a family of X-inert functions (depending on A) supported on $t_0\asymp Z$.

Proof. See [Blomer et al. 2013, Section 8] and [Kıral et al. 2019, Lemma 3.1]. □

3. Reduction

Now we start to prove Theorem 1.1. We assume $t \geq M^{\varepsilon}$. Recall that, Lemma 2.2, we are considering $S(r, N)$ with $N \ll (Mt)^{3+\varepsilon}/r^2$, $r \ll M^{1/8}t^{3/10}$, and $(r, M) = 1$. We will prove the following proposition.

Proposition 3.1. *We have*

$$S(r, N) \ll N^{1/2+\varepsilon} M^{3/2-1/16} t^{3/2-3/20},$$

for $N \ll (Mt)^{3+\varepsilon}/r^2$, $r \ll M^{1/8}t^{3/10}$ *and* $(r, M) = 1$.

Let $\mathcal{L}$ be the set of primes in $[L, 2L]$. Assume $M \notin [L, 2L]$. For $\ell \in \mathcal{L}$ and $n \geq 1$, by the Hecke relation, we have

$$A(1, \ell)A(r, n) = A(r, \ell n) + \delta_{\ell \mid r} A(r/\ell, n) + \delta_{\ell \mid n} A(r\ell, n/\ell).$$

By the prime number theorem for $L(s, \pi \times \tilde{\pi})$ we have

$$L^* := \sum_{\ell \in \mathcal{L}} |A(1, \ell)|^2 \gg L^{1-\varepsilon}.$$

We have

$$\begin{aligned} S(r, N) &= \frac{1}{L^*} \sum_{\ell \in \mathcal{L}} \overline{A(1, \ell)} \sum_{n \geq 1} A(r, n) A(1, \ell) \lambda_f(n) \chi(n) n^{-it} V\left(\frac{n}{N}\right) \\ &= S_1(N) + S_2(N) + S_3(N), \end{aligned}$$

where

$$S_1(N) = \frac{1}{L^*} \sum_{\ell \in \mathcal{L}} \overline{A(1, \ell)} \sum_{n \geq 1} A(r, n\ell) \lambda_f(n) \chi(n) n^{-it} V\left(\frac{n}{N}\right),$$

$$S_2(N) = \frac{1}{L^*} \sum_{\ell \in \mathcal{L}} \overline{A(1, \ell)} \sum_{n \geq 1} \delta_{\ell \mid r} A(r/\ell, n) \lambda_f(n) \chi(n) n^{-it} V\left(\frac{n}{N}\right),$$

and

$$S_3(N) = \frac{1}{L^*} \sum_{\ell \in \mathcal{L}} \overline{A(1, \ell)} \sum_{n \geq 1} \delta_{\ell \mid n} A(r\ell, n/\ell) \lambda_f(n) \chi(n) n^{-it} V\left(\frac{n}{N}\right).$$

We only consider $S_1(N)$, since the same method works for the other two sums and will give better bounds as the lengths of those sums are smaller. Actually, in S_2, since $\ell \mid r$, only $\tau(r)$ ℓ's contribute; in S_3, since $\ell \mid n$, the length of the n-sum is of size N/L. As the structures of sums in S_2 and S_3 are the same as in S_1, we can get better bounds than S_1.

Now we apply $(1/M) \sum_{b \bmod M} e((n - m\ell)b/M)$ to detect the condition $M \mid (n - m\ell)$, and then use the delta method, obtaining

$$\begin{aligned} S_1(N) = \frac{1}{L^*} \sum_{\ell \in \mathcal{L}} \overline{A(1, \ell)} \frac{1}{M} \sum_{b \bmod M} \sum_{n \geq 1} A(r, n) W\left(\frac{n}{\ell N}\right) \cdot \sum_{m \geq 1} \lambda_f(m) \chi(m) m^{-it} V\left(\frac{m}{N}\right) e\left(\frac{(n-m\ell)b}{M}\right) \\ \cdot \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{(n-m\ell)a}{Mq}\right) \int_{\mathbb{R}} g(q, x) e\left(\frac{(n-m\ell)x}{MqQ}\right) \mathrm{d}x. \end{aligned}$$

Rearranging the order of the sums and integrals we get

$$S_1(N)=\frac{1}{L^*}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)}\frac{1}{M}\sum_{b \bmod M}\frac{1}{Q}\sum_{1\le q\le Q}\int_{\mathbb{R}}g(q,x)\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q}\cdot\sum_{n\ge1}A(r,n)e\Big(\frac{n(bq+a)}{Mq}\Big)W\Big(\frac{n}{\ell N}\Big)e\Big(\frac{nx}{pqQ}\Big)\cdot\sum_{m\ge1}\lambda_f(m)\chi(m)m^{-it}e\Big(\frac{-m\ell(bq+a)}{Mq}\Big)V\Big(\frac{m}{N}\Big)e\Big(\frac{-m\ell x}{MqQ}\Big)\,\mathrm{d}x.$$

Inserting a smooth partition of unity for the x-integral and a dyadic partition for the q-sum, we get

$$S_1(N)\ll N^{\varepsilon}\sup_{t^{-B}\ll X\ll t^{\varepsilon}}\sup_{1\ll R\ll Q}\sum_{1\le j\le3}|S_{1j}^{\pm}(N,X,R)|+O(t^{-A}),$$

for any large constant $A>0$ and some large constant $B>0$ depending on A, where $S_{11}^{\pm}(N,X,R)$, $S_{12}^{\pm}(N,X,R)$ and $S_{13}^{\pm}(N,X,R)$ denote the terms with $(b,M)=1$, $M\mid b$ and $(q,\ell M)>1$, respectively. More precisely, we have

$$S_{11}^{\pm}(N,X,R)j=\frac{1}{L^*}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)}\int_{\mathbb{R}}\frac{1}{M}\sideset{}{^\star}\sum_{b \bmod M}\frac{1}{Q}\sum_{\substack{q\sim R\\(q,\ell M)=1}}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q}g(q,x)U\Big(\frac{\pm x}{X}\Big)\cdot\sum_{n\ge1}A(r,n)e\Big(\frac{n(aM+bq)}{qM}\Big)W\Big(\frac{n}{\ell N}\Big)e\Big(\frac{nx}{MqQ}\Big)\cdot\sum_{m\ge1}\lambda_f(m)\chi(m)e\Big(\frac{-m\ell(aM+bq)}{qM}\Big)m^{-it}V\Big(\frac{m}{N}\Big)e\Big(\frac{-m\ell x}{MqQ}\Big)\,\mathrm{d}x,\tag{3-1}$$

$$S_{12}^{\pm}(N,X,R)=\frac{1}{L^*}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)}\int_{\mathbb{R}}\frac{1}{M}\frac{1}{Q}\sum_{\substack{q\sim R\\(q,\ell M)=1}}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q}g(q,x)U\Big(\frac{\pm x}{X}\Big)\cdot\sum_{n\ge1}A(r,n)e\Big(\frac{na}{q}\Big)W\Big(\frac{n}{\ell N}\Big)e\Big(\frac{nx}{MqQ}\Big)\cdot\sum_{m\ge1}\lambda_f(m)\chi(m)e\Big(\frac{-m\ell a}{q}\Big)m^{-it}V\Big(\frac{m}{N}\Big)e\Big(\frac{-m\ell x}{MqQ}\Big)\,\mathrm{d}x,$$

and

$$S_{13}^{\pm}(N,X,R)=\frac{1}{L^*}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)}\int_{\mathbb{R}}\frac{1}{M}\sum_{b \bmod M}\frac{1}{Q}\sum_{\substack{q\sim R\\(q,\ell M)>1}}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q}g(q,x)U\Big(\frac{\pm x}{X}\Big)\cdot\sum_{n\ge1}A(r,n)e\Big(\frac{n(a+bq)}{qM}\Big)W\Big(\frac{n}{\ell N}\Big)e\Big(\frac{nx}{MqQ}\Big)\cdot\sum_{m\ge1}\lambda_f(m)\chi(m)e\Big(\frac{-m\ell(a+bq)}{qM}\Big)m^{-it}V\Big(\frac{m}{N}\Big)e\Big(\frac{-m\ell x}{MqQ}\Big)\,\mathrm{d}x.$$

Note that in $S_{11}^{\pm}(N,X,R)$ and $S_{12}^{\pm}(N,X,R)$, we have made a change of variable $a \to aM$. Here U is a fixed compactly supported 1-inert function with $\operatorname{supp} U \subset (0,\infty)$. We will only give details for the treatment of $S_{11}^{\pm}(N,X,R)$, since the same method works for $S_{12}^{\pm}(N,X,R)$ and $S_{13}^{\pm}(N,X,R)$ and will give a better upper bound. More precisely, in $S_{12}^{\pm}(N,X,R)$, we do not have the b-sum. In $S_{13}^{\pm}(N,X,R)$, we have the condition $(q,\ell M) > 1$. In fact, we should have the following cases:

(i) $b \equiv 0 \bmod M$ and $q = \ell^j q'$ with $j \geq 1$ and $(q',\ell M) = 1$.

(ii) $b \equiv 0 \bmod M$ and $q = M^k q'$ with $k \geq 1$ and $(q',\ell M) = 1$.

(iii) $b \equiv 0 \bmod M$ and $q = \ell^j M^k q'$ with $j,k \geq 1$ and $(q',\ell M) = 1$.

(iv) $(b,M) = 1$ and $q = \ell^j q'$ with $j \geq 1$ and $(q',\ell M) = 1$.

(v) $(b,M) = 1$ and $q = M^k q'$ with $k \geq 1$ and $(q',\ell M) = 1$.

(vi) $(b,M) = 1$ and $q = \ell^j M^k q'$ with $j,k \geq 1$ and $(q',\ell M) = 1$.

4. Applying Voronoi

We first apply the Voronoi summation formula (see Lemma 2.5) to the sum over n in $S_{11}^{\pm}(N,X,R)$, getting

$$\sum_{n\geq 1} A(r,n) e\left(\frac{n(bq+aM)}{qM}\right) W\left(\frac{n}{\ell N}\right) e\left(\frac{nx}{MqQ}\right)$$
$$= qM \sum_{\eta_1=\pm 1} \sum_{n_1 \mid qMr} \sum_{n_2=1}^{\infty} \frac{A(n_1,n_2)}{n_1 n_2} S(r\overline{(aM+bq)}, \eta_1 n_2; qMr/n_1) \Psi_x^{\operatorname{sgn}(\eta_1)}\left(\frac{n_1^2 n_2}{q^3 M^3 r}\right), \tag{4-1}$$

where $\Psi_x^{\operatorname{sgn}(\eta_1)}(z)$ is defined as in Lemma 2.5 with $\psi(y)$ replaced by $W(y/\ell N)e(xy/MqQ)$.

Lemma 4.1. (i) *If $zNL \gg t^{\varepsilon}$, then $\Psi_x^{\eta_1}(z)$ is negligibly small unless $\operatorname{sgn}(x) = -\operatorname{sgn}(\eta_1)$ and $N\ell(-\eta_1 x)/(MqQ) \asymp (zN\ell)^{1/3}$, in which case we have*

$$\Psi_x^{\operatorname{sgn}(\eta_1)}(z) = (zN\ell)^{1/2} e\left(\eta_1 \frac{2(zMqQ)^{1/2}}{(-\eta_1 x)^{1/2}}\right) \mathcal{W}\left(\frac{z^{1/2}(MqQ)^{3/2}}{N\ell(-\eta_1 x)^{3/2}}\right) + O(t^{-A}),$$

where $\mathcal{W}$ is a certain compactly supported 1-inert function depending on A.

(ii) *If $zNL \ll t^{\varepsilon}$ and $(NLX)/(MRQ) \gg t^{\varepsilon}$, then $\Psi_x^{\operatorname{sgn}(\eta_1)}(z) \ll t^{-A}$.*

(iii) *If $zNL \ll t^{\varepsilon}$ and $(NLX)/(MRQ) \ll t^{\varepsilon}$, then $\Psi_x^{\operatorname{sgn}(\eta_1)}(z) \ll t^{\varepsilon}$.*

Proof. See [Huang 2021b, 5.3]. □

In the last case, by taking $\sigma = \frac{1}{2}$ and making a change of variable, we get

$$\Psi_x^{\pm}(z) = (z\ell N)^{1/2} \frac{1}{2\pi^{5/2}} \int_{\mathbb{R}} (\pi^3 z\ell N)^{-i\tau} \gamma_3^{\pm}(1/2+i\tau) \int_0^{\infty} W(\xi) e\left(\frac{x\ell N\xi}{MqQ}\right) \xi^{-1/2-i\tau} \, d\xi \, d\tau.$$

We can truncate τ at $\tau \ll t^{\varepsilon}$ with a negligibly small error by repeated integration by parts for the ξ-integral above. That is, we have

$$\Psi_x^{\pm}(z) = (z\ell N)^{1/2} W_{x,\ell}^{\pm}(z) + O(t^{-A}), \tag{4-2}$$

where

$$W_{x,\ell}^{\pm}(z) = \frac{1}{2\pi^{5/2}} \int_{|\tau|\le t^{\varepsilon}} (\pi^3 z\ell N)^{-i\tau} \gamma_3^{\pm}(1/2+i\tau) \int_0^{\infty} W(\xi) e\left(\frac{x\ell N\xi}{MqQ}\right) \xi^{-1/2-i\tau}\, \mathrm{d}\xi\, \mathrm{d}\tau.$$

The contribution from the error to $S_{11}^{\pm}(N, X, R)$ is also negligibly small. Note that the function $W_{x,\ell}^{\pm}(z)$ satisfies that

$$\frac{\partial^j}{\partial z^j} W_{x,\ell}^{\pm}(z) \ll_j t^{\varepsilon} z^{-j}. \tag{4-3}$$

Now we consider the m-sum. By

$$\chi(m) = \bar{\chi}(\ell)\chi(m\ell) = \frac{\bar{\chi}(\ell)}{\tau(\bar{\chi})} \sum_{u \bmod M} \bar{\chi}(u) e\left(\frac{um\ell}{M}\right),$$

one has

$$\begin{aligned}
\sum_{m\ge 1} \lambda_f(m)\chi(m)m^{-it} e\left(\frac{-m\ell(bq+aM)}{Mq}\right) V\left(\frac{m}{N}\right) e\left(\frac{-m\ell x}{MqQ}\right)& \\
= \frac{1}{\tau(\bar{\chi})} \sum_{m\ge 1} \lambda_f(m) m^{-it} V\left(\frac{m}{N}\right) e\left(-\frac{m\ell x}{MqQ}\right)& \\
\cdot \left(\sum_{\substack{u \bmod M\\ u\not\equiv b \bmod M}} \bar{\chi}(u\ell)\left(e\left(-\frac{m\ell(aM+(b-u)q)}{Mq}\right) + e\left(-\frac{m\ell a}{q}\right)\right)\right)& \\
=: \Sigma_1 + \Sigma_2,&
\end{aligned}$$

say. From now on, we only deal with the terms involving Σ_1, since the treatment of Σ_2 is similar and in fact simpler. With the help of Lemma 2.4, we obtain

$$\Sigma_1 = \frac{N^{1-it}}{Mq\tau(\bar{\chi})} \sum_{\substack{u \bmod M\\ u\not\equiv b \bmod M}} \bar{\chi}(u\ell) \sum_{\pm} \sum_{m\ge 1} \lambda_f(m) e\left(\pm \frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\right) H^{\pm}\left(\frac{mN}{M^2q^2}\right), \tag{4-4}$$

where $H^{\pm}$ is defined as in Lemma 2.4 with $g(\xi)$ replaced by $V(\xi)\xi^{-it}e(-N\ell x\xi/MqQ)$.

Lemma 4.2. *If $z \ll t^{\varepsilon}$, then $H^{\pm}(z)$ is negligible unless $t \asymp (N\ell X)/(MqQ)$ and $x < 0$.*

Proof. If $z \ll t^{\varepsilon}$, then, in view of (2-8) and (2-9), we may regard $H^{\pm}(z)$ as

$$\mathcal{I}(z) := \int_0^{\infty} V(\xi) e\left(-\frac{t\log\xi}{2\pi} - \frac{N\ell x\xi}{MqQ}\right) J_f(z\xi)\, \mathrm{d}\xi, \tag{4-5}$$

where $J_f(z) = (-\pi/\sin(\pi i t_f))(J_{2it_f}(4\pi\sqrt{z}) - J_{-2it_f}(4\pi\sqrt{z}))$ or $J_f(z) = 4\epsilon_f \cosh(\pi t_f) K_{2it_f}(4\pi\sqrt{z})$. Then, by partial integration together with (2-12), $\mathcal{I}_1(z)$ is negligible unless $x < 0$ and $(NLX)/(MRQ) \asymp t$. □

If $mN/M^2q^2 \gg t^\varepsilon$, then, in view of (2-11), $H^-(mN/M^2q^2)$ is negligible. For the term in (4-4) involving H^+, with the help of (2-10), we may replace it by

$$\frac{N^{3/4-it}}{M^{1/2}q^{1/2}\tau(\bar\chi)} \sum_{\substack{u \bmod M\\ u\neq b}} \bar\chi(u\ell) \sum_{\eta_2=\pm1}\sum_{m\geq 1} \frac{\lambda_f(m)}{m^{1/4}} e\left(\frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\right) \cdot \int_{\mathbb{R}} \xi^{-1/4}V(\xi)e\left(-\frac{t\log\xi}{2\pi}+\eta_2\frac{2\sqrt{mN\xi}}{Mq}-\frac{N\ell x\xi}{MqQ}\right)\mathrm{d}\xi. \tag{4-6}$$

Note that we have $\ell \asymp L$, $|x| \asymp X$ and $q \asymp R$. By Lemma 4.1 and Lemma 4.2 and according to the size of $(N\ell x)/(MqQ)$, $(n_1^2n_2N\ell)/(q^3M^3r)$ and $(mN)/(M^2q^2)$, we can reduce $S_1^{\pm}(N,X,R)$ to the following three cases:

Case a. $$\frac{NLX}{MRQ} \asymp \left(\frac{n_1^2n_2NL}{R^3M^3r}\right)^{1/3} \gg t^\varepsilon, \qquad \frac{mN}{M^2R^2} \gg t^\varepsilon.$$

In this case, we insert (4-1) and (4-4) into (3-1) and use Lemma 4.1(i) and (4-6), so that it is sufficient to estimate

$$\begin{aligned}\frac{N^{5/4-it}}{\tau(\bar\chi)M^2LQr^{1/2}} &\sum_{\ell\in\mathcal{L}} \overline{A(1,\ell)\chi(\ell)}\ell^{1/2} \sideset{}{^\star}\sum_{b \bmod M} \sum_{\substack{q\sim R\\(q,\ell M)=1}} \frac{1}{q^2} \sideset{}{^\star}\sum_{a \bmod q} \sum_{\substack{u \bmod M\\ u\neq b}} \bar\chi(u)\\
&\cdot \sum_{m\geq1}\frac{\lambda_f(m)}{m^{1/4}} e\left(\frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\right) \sum_{\eta_1,\eta_2=\pm1}\sum_{n_1\mid qMr}\sum_{n_2\asymp \frac{N_0}{n_1^2}} \frac{A(n_1,n_2)}{n_2^{1/2}}\\
&\cdot S(r\overline{(aM+bq)},\eta_1n_2;qMr/n_1)\int_{\mathbb{R}}\xi^{-1/4}V(\xi)e\left(-\frac{it\log\xi}{2\pi}+\eta_2\frac{2\sqrt{mN\xi}}{Mq}\right)\\
&\cdot\int_{\mathbb{R}} g(q,x)e\left(-\frac{N\ell x\xi}{MqQ}+\eta_1\frac{2(n_1^2n_2Q)^{1/2}}{Mq((-\eta_1rx))^{1/2}}\right)\mathcal{W}\left(\frac{Q^{3/2}(n_1^2n_2)^{1/2}}{r^{1/2}(-\eta_1x)^{3/2}N\ell}\right)U\left(\frac{-\eta_1x}{X}\right)\mathrm{d}x\,\mathrm{d}\xi,\end{aligned} \tag{4-7}$$

where $N_0 = N^2L^2X^3r/Q^3$. Let $x = -\eta_1Xv$. Then the resulting x-integral becomes

$$-\eta_1X\int_{\mathbb{R}} e\left(\eta_1\frac{N\ell X\xi v}{MqQ}+\eta_1\frac{2(n_1^2n_2Q)^{1/2}}{Mq(rXv)^{1/2}}\right)g(q,-\eta_1Xv)U(v)W\left(\frac{Q^{3/2}(n_1^2n_2)^{1/2}}{r^{1/2}(Xv)^{3/2}N\ell}\right)\mathrm{d}v. \tag{4-8}$$

Let

$$h(v) = \eta_1\frac{N\ell X\xi v}{MqQ}+\eta_1\frac{2(n_1^2n_2Q)^{1/2}}{Mq(rXv)^{1/2}}.$$

Then

$$h'(v)=\eta_1\frac{N\ell X\xi}{MqQ}-\eta_1\frac{(n_1^2n_2Q)^{1/2}}{Mq(rX)^{1/2}}v^{-3/2}, \qquad h''(v)=\eta_1\frac{3(n_1^2n_2Q)^{1/2}}{2Mq(rX)^{1/2}}v^{-5/2}. \tag{4-9}$$

Note that the solution of $h'(v_0)=0$ is $v_0 = (n_1^2n_2)^{1/3}Q/(r^{1/3}(N\ell\xi)^{2/3}X) \asymp 1$, and

$$h(v_0)=\eta_1\frac{3(n_1^2n_2N\ell\xi)^{1/3}}{r^{1/3}Mq}, \qquad h''(v_0)=\frac{3\eta_1}{2v_0^2}\cdot\frac{(n_1^2n_2Q)^{1/2}}{Mq(rXv_0)^{1/2}}=\frac{3\eta_1}{2v_0^2}\cdot\frac{(n_1^2n_2N\ell\xi)^{1/3}}{r^{1/3}Mq}.$$

By the argument below Lemma 2.6, we can think $g(q,x)$ as a nice function which satisfies

$$\frac{\partial^j}{\partial x^j} g(q,x) \ll Q^{\varepsilon_1}|x|^{-j}, \tag{4-10}$$

up to a negligible error. Here ε_1 is a small positive number such that $t^{\varepsilon}/Q^{2\varepsilon_1} \gg t^{\varepsilon/2}$. Then, by applying Lemma 2.7, we have (4-8) is equal to

$$\frac{r^{1/6}(qM)^{1/2}X}{(n_1^2 n_2 N\ell\xi)^{1/6}} e\Big(\eta_1 \frac{3(n_1^2 n_2 N\ell\xi)^{1/3}}{r^{1/3}Mq}\Big) g(q,-\eta_1 X v_0)\mathcal{U}(v_0) W\Big(\frac{Q^{3/2}(n_1^2 n_2)^{1/2}}{r^{1/2}(Xv_0)^{3/2}N\ell}\Big) + O(t^{-A}),$$

where $\mathcal{U}$ is a certain compactly supported 1-inert function depending on A. We may assume $(n_1, M) = 1$, since otherwise we have $M \mid n_1$ which leads to a simpler case. Hence, by letting

$$\mathcal{V}(\xi) = \xi^{-5/12} V(\xi) g(q,-\eta_1 X v_0)\mathcal{U}(v_0) W\Big(\frac{Q^{3/2}(n_1^2 n_2)^{1/2}}{r^{1/2}(Xv_0)^{3/2}N\ell}\Big),$$

at the cost of a negligible error, we can rewrite (4-7) as

$$\frac{N^{13/12-it}X}{\tau(\bar\chi)M^{3/2}LQr^{1/3}} \sum_{\ell\in\mathcal{L}} \overline{A(1,\ell)\chi(\ell)}\ell^{1/3} \sum_{\substack{q\sim R\\ (q,\ell M)=1}} \frac{1}{q^{3/2}}$$
$$\cdot \sum_{\eta_1,\eta_2=\pm1} \sum_{n_1\mid qr} \frac{1}{n_1^{1/3}} \sum_{n_2 \asymp \frac{N_0}{n_1^2}} \frac{A(n_1,n_2)}{n_2^{2/3}} \sum_{m\geq 1} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(m,n_1,n_2,\ell,q)\mathcal{J}_{\boldsymbol{a}}(m,n_1,n_2,\ell,q), \tag{4-11}$$

where

$$\mathcal{J}_{\boldsymbol{a}}(m,n_1,n_2,\ell,q) = \int_{\mathbb{R}} \mathcal{V}(\xi) e\Big(-\frac{t}{2\pi}\log\xi + \eta_1 \frac{3(n_1^2 n_2 N\ell\xi)^{1/3}}{r^{1/3}Mq} + \eta_2 \frac{2\sqrt{mN\xi}}{Mq}\Big)\, d\xi,$$

and

$$\mathcal{C}(m,n_1,n_2,\ell,q) = \sideset{}{^\star}\sum_{b \bmod M} \sideset{}{^\star}\sum_{a \bmod q} S(r\overline{(aM+bq)}, \eta_1 n_2, qMr/n_1)$$
$$\cdot \sum_{\substack{u \bmod M\\ u\neq b}} \bar\chi(u) e\Big(\frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\Big). \tag{4-12}$$

By partial integration, one can truncate the m-sum at

$$m \ll \max\{t^2R^2M^2/N,\, NL^2X^2/Q^2\}.$$

We have

$$\mathcal{C}(m,n_1,n_2,\ell,q) = \sideset{}{^\star}\sum_{\alpha \bmod qMr/n_1} f(\alpha, m\bar\ell, q)\tilde S(\alpha, m\bar\ell, q) e\Big(\eta_1 \frac{\bar\alpha n_1 n_2}{qMr}\Big), \tag{4-13}$$

where

$$\tilde S(\alpha,m,q) = \sideset{}{^\star}\sum_{b \bmod M} \sum_{\substack{u \bmod M\\ u\neq b}} \bar\chi(u) e\Big(\frac{\bar q^2(n_1\alpha\bar b + m\overline{(b-u)})}{M}\Big),$$

and

$$f(\alpha,m,q)=\sum_{\substack{d\mid q\\ n_1\alpha\equiv -m \pmod d}} d\mu(q/d).$$

Case b.
$$\frac{NLX}{MRQ}\asymp\Big(\frac{n_1^2n_2NL}{R^3M^3r}\Big)^{1/3}\asymp t,\qquad \frac{mN}{M^2R^2}\ll t^{\varepsilon}.$$

In this case, we replace $H^{\pm}(z)$ by $\mathcal{I}(z)$ as defined in (4-5). Hence, we are led to estimate

$$\frac{N^{3/2-it}}{\tau(\overline{\chi})M^{5/2}LQr^{1/2}}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)\chi(\ell)}\ell^{1/2}\sideset{}{^\star}\sum_{b \bmod M}\sum_{\substack{q\sim R\\(q,\ell M)=1}}\frac{1}{q^{5/2}}\sideset{}{^\star}\sum_{a \bmod q}\sum_{\substack{u \bmod M\\ u\neq b}}\overline{\chi}(u)$$
$$\cdot\sum_{m\geq 1}\frac{\lambda_f(m)}{m^{1/4}}e\Big(\frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\Big)\sum_{\eta_1=\pm1}\sum_{n_1\mid qMr}\sum_{n_2\asymp N_0/n_1^2}\frac{A(n_1,n_2)}{n_2^{1/2}}$$
$$\cdot S(r\overline{(aM+bq)},\eta_1n_2;qMr/n_1)\int_{\mathbb{R}}\xi^{-1/4}V(\xi)J_f\Big(\frac{mN\xi}{M^2q^2}\Big)e\Big(-\frac{it\log\xi}{2\pi}Mq\Big)$$
$$\cdot\int_{\mathbb{R}}g(q,x)e\Big(-\frac{N\ell x\xi}{MqQ}+\eta_1\frac{2(n_1^2n_2Q)^{1/2}}{Mq((-\eta_1rx))^{1/2}}\Big)\mathcal{W}\Big(\frac{Q^{3/2}(n_1^2n_2)^{1/2}}{r^{1/2}(-\eta_1x)^{3/2}N\ell}\Big)U\Big(\frac{-\eta_1x}{X}\Big)\,\mathrm{d}x\,\mathrm{d}\xi.$$

By doing a similar treatment as in Case a, one can equate the above with (up to a negligible error and another term with $M\mid n_1$)

$$\frac{N^{4/3-it}X}{\tau(\overline{\chi})M^2LQr^{1/3}}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)\chi(\ell)}\ell^{1/3}\sum_{\substack{q\sim R\\(q,\ell M)=1}}\frac{1}{q^2}$$
$$\cdot\sum_{\eta_1,\eta_2=\pm1}\sum_{n_1\mid qr}\frac{1}{n_1^{1/3}}\sum_{n_2\asymp N_0/n_1^2}\frac{A(n_1,n_2)}{n_2^{2/3}}\sum_{m\geq1}\frac{\lambda_f(m)}{m^{1/4}}\mathcal{C}(m,n_1,n_2,\ell,q)\mathcal{J}_{\boldsymbol{b}}(m,n_1,n_2,\ell,q),\tag{4-14}$$

where $\mathcal{C}$ is defined as in (4-12) and

$$\mathcal{J}_{\boldsymbol{b}}(m,n_1,n_2,\ell,q)=\int_{\mathbb{R}}\xi^{-1/4}V(\xi)J_f\Big(\frac{mN\xi}{M^2q^2}\Big)e\Big(-\frac{t}{2\pi}\log\xi+\eta_1\frac{3(n_1^2n_2N\ell\xi)^{1/3}}{r^{1/3}Mq}\Big)\,\mathrm{d}\xi.\tag{4-15}$$

Case c.
$$\frac{n_1^2n_2}{R^3M^3r}LN\ll t^{\varepsilon},\qquad \frac{NLX}{MRQ}\ll t^{\varepsilon},\qquad \frac{mN}{M^2R^2}\gg t^{\varepsilon}.$$

Since $(NLX)/(MRQ)\ll t^{\varepsilon}$, we first deal with the ξ-integral in (4-6). Making a change of variable $\xi\rightsquigarrow\xi^2$, we have

$$\mathcal{J}_{\boldsymbol{c}}(m,\ell,q)=2\int_{\mathbb{R}}\xi^{-1/2}V(\xi^2)e\Big(-\frac{N\ell x\xi^2}{MqQ}\Big)e\Big(-\frac{t\log\xi}{\pi}+\eta_2\frac{2\sqrt{mN}}{Mq}\xi\Big)\,\mathrm{d}\xi.$$

Let

$$h(\xi)=-\frac{t\log\xi}{\pi}+\eta_2\frac{2\sqrt{mN}}{Mq}\xi.$$

Then we have

$$h'(\xi) = -\frac{t}{\pi\xi} + \eta_2 \frac{2\sqrt{mN}}{Mq}, \quad h''(\xi) = \frac{t}{\pi\xi^2}, \quad h^{(j)}(\xi) \asymp_j t, \quad j \geq 2.$$

Note that

$$\frac{t}{1+(NLX/MRQ)^2} \gg t^{1-\varepsilon}.$$

Hence, by Lemma 2.7, the integral is negligibly small unless $mN/(M^2R^2) \asymp t$ and $\eta_2 = 1$, in which case we have the stationary phase point $\xi_0 = tMq/(2\pi\sqrt{mN})$ and

$$\mathcal{J}_{\mathbf{c}}(m,\ell,q) = \frac{1}{t^{1/2}} e\left(-\frac{t}{\pi}\log\frac{tMq}{2\pi e\sqrt{mN}}\right) V_{x,\ell}\left(\frac{tMq}{\sqrt{mN}}\right) + O(t^{-A}),$$

where $V_{x,\ell}$ is a t^ε-inert function.

Together with (4-1) and (4-6), we have $S_{11}^{\pm}(N,X,R)$ is equal to (up to a negligibly small error term and another term with $u=b$)

$$\begin{aligned}\frac{1}{L^*}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)}\int_{\mathbb{R}}\frac{1}{M}\sideset{}{^\star}\sum_{b \bmod M}\frac{1}{Q}\sum_{\substack{q\sim R\\(q,\ell M)=1}}\frac{1}{q}\sideset{}{^\star}\sum_{a \bmod q} g(q,x)U\left(\frac{\pm x}{X}\right)qM\sum_{\eta_1=\pm1}\sum_{n_1\mid qMr}\\ \cdot\sum_{n_2=1}^{\infty}\frac{A(n_1,n_2)}{n_1n_2}S(r\overline{(aM+bq)},\eta_1n_2;qMr/n_1)\left(\frac{n_1^2n_2\ell N}{q^3M^3r}\right)^{1/2}W_{x,\ell}^{\operatorname{sgn}(\eta_1)}\left(\frac{n_1^2n_2}{q^3M^3r}\right)\\ \cdot\frac{N^{3/4-it}}{M^{1/2}q^{1/2}\tau(\overline{\chi})}\sum_{\substack{u \bmod M\\u\neq b}}\overline{\chi}(u\ell)\sum_{m\geq1}\frac{\lambda_f(m)}{m^{1/4}}e\left(\frac{m\overline{\ell(aM+(b-u)q)}}{Mq}\right)\\ \cdot\frac{1}{t^{1/2}}e\left(-\frac{t}{\pi}\log\frac{tMq}{2\pi e\sqrt{mN}}\right)V_{x,\ell}\left(\frac{tMq}{\sqrt{mN}}\right)\mathrm{d}x.\end{aligned}$$

We assume $(n_1,M)=1$, since otherwise we have $M\mid n_1$ which leads to a simpler case. Rearranging the sums, inserting a dyadic partition for the n_2-sum, and estimating the x-integral trivially, the above is bounded by

$$N^\varepsilon \sup_{1\ll N_0\ll(R^3M^3r/LN)t^\varepsilon}\sup_{x\asymp X}|S_{11}^{\pm}(N,X,R,N_0)|,$$

where

$$\begin{aligned}S_{11}^{\pm}(N,X,R,N_0) = \frac{N^{5/4}X}{M^{5/2}LQr^{1/2}}\frac{1}{t^{1/2}}\sum_{\eta_1=\pm1}\sum_{n_1\leq Rr}\sum_{n_2\asymp N_0/n_1^2}\frac{A(n_1,n_2)}{n_2^{1/2}}\sum_{\ell\in\mathcal{L}}\overline{A(1,\ell)\chi(\ell)}\ell^{1/2}\\ \cdot\sum_{\substack{q\sim R\\n_1\mid qr\\(q,\ell M)=1}}\frac{1}{q^{2+2it}}\sum_{m\asymp R^2M^2t^2/N}\frac{\lambda_f(m)}{m^{1/4-it}}\mathcal{C}(m,n_1,n_2,\ell,q)W_{x,\ell}^{\operatorname{sgn}(\eta_1)}\left(\frac{n_1^2n_2}{q^3M^3r}\right)V_{x,\ell}\left(\frac{tMq}{\sqrt{mN}}\right),\end{aligned}$$

and $\mathcal{C}$ is defined as in (4-12).

5. Applying Cauchy and Poisson

5A. *Case a.* In this subsection, we assume Case a. Write

$$q = q_1 q_2 \quad \text{with } q_1 \mid (rn_1)^\infty \text{ and } (q_2, rn_1) = 1,$$

then we have

$$\text{(4-11)} \ll \frac{N^{13/12+\varepsilon} X}{M^2 L Q r^{1/3}} \sum_{\eta_1,\eta_2=\pm 1} \sum_{n_1 \ll Rr} \frac{1}{n_1^{1/3}} \sum_{(n_1/(n_1,r))|q_1|(rn_1)^\infty} \frac{1}{q_1^{3/2}} \sum_{n_2 \asymp N_0/n_1^2} \frac{|A(n_1,n_2)|}{n_2^{2/3}} \cdot \Bigg| \sum_{\substack{\ell\in\mathcal{L}\\(\ell,q_1)=1}} \overline{A(1,\ell)\chi(\ell)}\ell^{1/3} \sum_{\substack{q_2\sim R/q_1\\(q_2,rn_1\ell M)=1}} \frac{1}{q_2^{3/2}} \sum_{m \ll \max\{t^2R^2M^2/N, NL^2X^2/Q^2\}} \frac{\lambda_f(m)}{m^{1/4}} \cdot \mathcal{C}(m,n_1,n_2,\ell,q_1q_2)\mathcal{J}_{\boldsymbol{a}}(m,n_1,n_2,\ell,q_1q_2)\Bigg|.$$

Now we use the Cauchy–Schwarz inequality and (2-5) to get

$$\ll \frac{N^{3/4+\varepsilon}X^{1/2}}{M^2 L^{4/3} Q^{1/2} r^{1/2}} \sum_{\eta_1,\eta_2=\pm1} \sup_{M_1 \ll \max\{t^2R^2M^2/N, NL^2X^2/Q^2\}} \sum_{n_1 \ll Rr} n_1^{\theta_3} \sum_{(n_1/(n_1,r))|q_1|n_1^\infty} \frac{1}{q_1^{3/2}} \Omega_{\boldsymbol{a}}^{1/2}, \tag{5-1}$$

where

$$\Omega_{\boldsymbol{a}} = \sum_{n_2 \asymp N_0/n_1^2} \Bigg| \sum_{\substack{\ell\in\mathcal{L}\\(\ell,q_1)=1}} \overline{A(1,\ell)\chi(\ell)}\ell^{1/3} \sum_{\substack{q_2\sim R/q_1\\(q_2,rn_1\ell M)=1}} \frac{1}{q_2^{3/2}} \sum_{m\sim M_1} \frac{\lambda_f(m)}{m^{1/4}} \mathcal{C}(m,n_1,n_2,\ell,q_1q_2)\mathcal{J}_{\boldsymbol{a}}(m,n_1,n_2,\ell,q_1q_2)\Bigg|^2.$$

Opening the absolute square, we get

$$\Omega_{\boldsymbol{a}} \ll \sum_{n_2\geq 1} W\left(\frac{n_1^2 n_2}{N_0}\right) \sum_{\substack{\ell\in\mathcal{L}\\(\ell\ell',q_1)=1}} \sum_{\ell'\in\mathcal{L}} \overline{A(1,\ell)\chi(\ell)} A(1,\ell')\chi(\ell')(\ell\ell')^{1/3} \cdot \sum_{m\sim M_1} \frac{\lambda_f(m)}{m^{1/4}} \sum_{m'\sim M_1} \frac{\lambda_f(m')}{m'^{1/4}} \sum_{\substack{q_2\sim R/q_1\\(q_2,\ell)=1\\(q_2q_2',rn_1M)=1}} \sum_{\substack{q_2'\sim R/q_1\\(q_2',\ell')=1}} \frac{1}{(q_2q_2')^{3/2}} \cdot \mathcal{C}(m,n_1,n_2,\ell,q_1q_2)\mathcal{J}_{\boldsymbol{a}}(m,n_1,n_2,\ell,q_1q_2)\overline{\mathcal{C}(m',n_1,n_2,\ell',q_1q_2')}\,\overline{\mathcal{J}_{\boldsymbol{a}}(m',n_1,n_2,\ell',q_1q_2')},$$

where W is supported on $[1,2]$ and satisfies $W^{(j)}(x) \ll 1$. We apply the Poisson summation formula on n_2, getting

$$\Omega_{\boldsymbol{a}} \ll \frac{N_0 q_1^3 L^{2/3}}{n_1^2 M_1^{1/2} R^3} \sum_{\substack{\ell \in \mathcal{L} \\ (\ell\ell', q_1)=1}} \sum_{\ell' \in \mathcal{L}} |A(1,\ell)A(1,\ell')| \sum_{m \sim M_1} \sum_{m' \sim M_1} |\lambda_f(m)||\lambda_f(m')| \cdot \sum_{\substack{q_2 \sim R/q_1 \\ (q_2,\ell)=1}} \sum_{\substack{q_2' \sim R/q_1 \\ (q_2',\ell')=1 \\ (q_2 q_2', r n_1 M)=1}} \sum_{n_2 \geq 1} |\mathfrak{C}(n_2)||\mathfrak{J}_{\boldsymbol{a}}(n_2)|,$$

where

$$\mathfrak{C}(n_2) = \sum_{b \bmod M}^{\star} \sum_{b' \bmod M}^{\star} \Bigg(\sum_{\substack{u \bmod M \\ u \neq b}} \bar{\chi}(u) e\bigg(\frac{m\overline{q_1^2 q_2^2 \ell (b-u)}}{M} \bigg) \Bigg) \cdot \Bigg(\sum_{\substack{u' \bmod M \\ u' \neq b'}} \chi(u') e\bigg(\frac{-m'\overline{q_1^2 q_2'^2 \ell' (b'-u')}}{M} \bigg) \Bigg) \Bigg(\sum_{d \mid q_1 q_2} \sum_{d' \mid q_1 q_2'} d d' \mu(q_1 q_2/d) \mu(q_1 q_2'/d') \cdot \sum_{\alpha \ (\bmod M r q_1 q_2/n_1)}^{\star} \sum_{\substack{\alpha' \ (\bmod M r q_1 q_2'/n_1) \\ q_2' \bar{\alpha} - q_2 \overline{\alpha'} \equiv -\eta_1 n_2 (M r q_1 q_2 q_2'/n_1) \\ n_1 \alpha \equiv -m \bar{\ell} (d) \\ n_1 \alpha' \equiv -m' \overline{\ell'} (d')}}^{\star} e\bigg(\frac{n_1 \alpha \overline{b q_1^2 q_2^2} - n_1 \alpha' \overline{b' q_1^2 q_2'^2}}{M} \bigg) \Bigg), \tag{5-2}$$

and

$$\mathfrak{J}_{\boldsymbol{a}}(n_2) = \int_{\mathbb{R}} W(w) \mathcal{I}_{\boldsymbol{a}}(N_0 w, m, q_2) \overline{\mathcal{I}_{\boldsymbol{a}}(N_0 w, m', q_2')} e\bigg(-\frac{N_0 n_2 w}{q_1 q_2 q_2' M n_1 r} \bigg) \mathrm{d}w$$

with

$$\mathcal{I}_{\boldsymbol{a}}(w, n, q_2) = \int_{\mathbb{R}} \mathcal{V}(\xi) e\bigg(-\frac{t}{2\pi} \log \xi + \eta_1 \frac{3(wN\ell\xi)^{1/3}}{r^{1/3} M q_1 q_2} + \eta_2 \frac{2\sqrt{mN\xi}}{M q_1 q_2} \bigg) \mathrm{d}\xi.$$

5A1. $(NLX)/(MRQ) \ll t^{1-\varepsilon}$. We first consider $\mathcal{I}(N_0 w, m, q_2)$. Let

$$g(\xi) = -\frac{t}{2\pi} \log \xi + \eta_1 \frac{3(N_0 w N \ell \xi)^{1/3}}{r^{1/3} M q_1 q_2} + \eta_2 \frac{2\sqrt{mN\xi}}{M q_1 q_2}. \tag{5-3}$$

There exists a stationary phase point ξ_* if and only if $m \asymp t^2M^2R^2/N$ and $\eta_2 = 1$, in which case ξ_* can be written as $\xi_0 + \xi_1 + \xi_2 + \cdots$ with

$$\xi_0 = \frac{t^2M^2q_1^2q_2^2}{4\pi^2 mN} = \left(\frac{t}{\pi C}\right)^2 \asymp 1,$$

$$\xi_1 = -\eta_1 \frac{4\pi Bw^{1/3}}{3t}\xi_0^{4/3} \asymp \frac{B}{t},$$

$$\xi_2 = \frac{28\pi^2B^2w^{2/3}}{27t^2}\xi_0^{5/3} \asymp \frac{B^2}{t^2},$$

$$\xi_i = f_i(t, C)\left(\eta_1\frac{Bw^{1/3}}{t}\right)^i \ll \left(\frac{B}{t}\right)^i, \quad i \geq 3,$$

where $B = 3(N_0N\ell)^{1/3}/(r^{1/3}Mq_1q_2) \asymp NLX/(MRQ)$, $C = 2\sqrt{mN}/(Mq_1q_2)$ and $f_i(t, C) \asymp 1$ is a function. Recall that $\mathcal{V}(\xi) = \xi^{-5/12}V(\xi)g(q, -\eta_1 Xv_0)\mathcal{U}(v_0)W(Q^{3/2}(n_1^2n_2)^{1/2}/(r^{1/2}(Xv_0)^{3/2}N\ell))$, $v_0 = (n_1^2n_2)^{1/3}Q/(r^{1/3}(N\ell\xi)^{2/3}X) \asymp 1$ and (4-10). So it is easy to check the conditions in Lemma 2.7. By using this lemma together with the Taylor expansion, $\mathcal{I}_{\boldsymbol{a}}(N_0w, m, q_2)$ is essentially reduced to

$$\frac{1}{t^{1/2}}\xi_0^{-it}e\left(Bw^{1/3}g_1(C) + B^2w^{2/3}g_2(C) + O\left(\frac{|B|^3}{t^2}\right)\right), \tag{5-4}$$

where $g_1(C) = \eta_1\xi_0^{1/3} = \eta_1 t^{2/3}/(\pi C)^{2/3} \asymp 1$ and $g_2(C) = -4\pi/(9t)\xi_0^{2/3} \ll 1/t$. To estimate $\mathfrak{J}_a(n_2)$, we use the strategy in [Lin and Sun 2021, Lemma 4.3] and [Munshi 2022, Lemma 5] to get the following result.

Lemma 5.1. *Let* $N_2 = Q^2Rn_1/(NLX^2q_1)t^\varepsilon$ *and* $N_2' = t^\varepsilon(NLn_1/(M^2Rt^2q_1) + R^2Q^3Mn_1/(N^2L^2X^3q_1))$. *Assume* $(NLX)/(MRQ) \ll t^{1-\varepsilon}$:

(i) *We have* $\mathfrak{J}_{\boldsymbol{a}}(n_2) \ll t^{-A}$ *unless* $n_2 \ll N_2$, *in which case one has*

$$\mathfrak{J}_a(n_2) \ll \frac{1}{t^{1-\varepsilon}}. \tag{5-5}$$

(ii) *If* $N_2' \ll n_2 \ll N_2$, *we have*

$$\mathfrak{J}_{\boldsymbol{a}}(n_2) \ll \frac{RQ^{3/2}M^{1/2}n_1^{1/2}}{t^{1-\varepsilon}NLX^{3/2}q_1^{1/2}n_2^{1/2}}. \tag{5-6}$$

(iii) *If* $q_2 = q_2'$, *we have* $\mathfrak{J}_{\boldsymbol{a}}(0) \ll t^{-A}$ *unless* $\ell m' - \ell' m \ll t^\varepsilon(M_1N^2L^3X^2/(M^2R^2Q^2t^2) + M_1MRQ/(NX))$.

Proof. Let $w = u^3$. Then we may equate the w-integral in $\mathfrak{J}_{\boldsymbol{a}}$ with

$$\int_{\mathbb{R}} W(u^3)u^2e\left(-\frac{N_0n_2u^3}{q_1q_2q_2'Mn_1r} + (Bg_1(C) - B'g_1(C'))u + (B^2g_2(C) - B'^2g_2(C'))u^2 + O\left(\frac{B^3}{t^2}\right)\right) du,$$

where $B' = 3(N_0N\ell')^{1/3}/(r^{1/3}Mq_1q_2')$, $C' = 2\sqrt{m'N}/(Mq_1q_2')$. Applying integration by parts, the above integral is $\ll t^{-A}$ if $n_2 \gg N_2$, which gives the first result in (i). The second result in (i) is obvious, since we may save $t^{1/2}$ in both $\mathcal{I}_{\boldsymbol{a}}(N_0w, m, q_2)$ and $\overline{\mathcal{I}_{\boldsymbol{a}}(N_0w, m', q_2')}$ according to (5-4).

It is easy to see that

$$B^2 g_2(C) - B'^2 g_2(C') \ll \frac{B\xi_0^{1/3} + B'\xi_0'^{1/3}}{t} |B\xi_0^{1/3} - B'\xi_0'^{1/3}| \ll |Bg_1(C) - B'g_1(C')| t^{-\varepsilon}, \tag{5-7}$$

where we have used $\xi_0' = (t/(\pi C'))^2 \asymp 1$ and $B,\ B' \asymp (NLX)/(MRQ) \ll t^{1-\varepsilon}$. Therefore, if $N_2' \ll n_2 \ll N_2$, the u-integral is $O(t^{-A})$ unless $|Bg_1(C) - B'g_1(C')| \asymp N_0 n_2/(q_1 q_2 q_2' M n_1 r)$. By the second derivative test and (5-4), we get (5-6).

For $n_2 = 0$ and $q_2 = q_2'$, we may rewrite the above u-integral as

$$\int_{\mathbb{R}} W(u^3) u^2 e\biggl((Bg_1(C) - B'g_1(C'))u + (B^2 g_2(C) - B'^2 g_2(C'))u^2 + O\biggl(\frac{B^3}{t^2}\biggr)\biggr)\, du.$$

Notice that

$$\frac{Bg_1(C)}{(m'\ell)^{1/3}} = \frac{B'g_1(C')}{(m\ell')^{1/3}} \quad \text{and} \quad Bg_1(C) - B'g_1(C') = \frac{Bg_1(C)}{(m'\ell)^{1/3}}((m'\ell)^{1/3} - (m\ell')^{1/3}).$$

So by partial integration and (5-7), the u-integral is $O(t^{-A})$ unless

$$(m'\ell)^{1/3} - (m\ell')^{1/3} \ll \biggl(\frac{B^3}{t^2} + 1\biggr) \frac{(M_1 L)^{1/3} t^{\varepsilon}}{B}.$$

This actually proves the result in (iii). □

5A2. $(NLX)/(MRQ) \gg t^{1-\varepsilon}$. It is easy to see that $R \ll N^{1+\varepsilon} LX/(MtQ)$. We have the following Lemma 5.2.

Lemma 5.2. *Let N_2 be defined as in Lemma 5.1. Then, if $(NLX)/(MRQ) \gg t^{1-\varepsilon}$, one has the following estimates*:

(1) *If $n_2 \gg N_2$, we have $\mathfrak{J}_{\boldsymbol{a}}(n_2) \ll N^{-A}$.*

(2) *If $n_2 \ll N_2$, we have*

$$\mathfrak{J}_{\boldsymbol{a}}(n_2) \ll \frac{MRQ}{N^{1-\varepsilon} LX}.$$

Proof. The first result can be done by applying integration by parts with respect to the w-integral. For $n_2 \ll N_2$, we can use the arguments as in [Munshi 2022, Lemma 1] to see

$$\int_{\mathbb{R}} W(w) |\mathcal{I}_{\boldsymbol{a}}(N_0 w, n, \ell, q_2)|^2\, dw \ll \frac{MRQ}{N^{1-\varepsilon} LX},$$

which implies (ii). □

Remark 5.3. In the case of $(NLX)/(MRQ) \gg t^{1+\varepsilon}$, we remark that one may replace it by a more explicit version like Lemma 5.1. However, the present result is enough for our purpose.

5B. *Case b.* After a similar treatment, and noting that $m \ll M^2R^2t^\varepsilon/N$, we have

$$\text{(4-14)} \ll \frac{N^{1+\varepsilon}X^{1/2}}{M^{5/2}L^{4/3}Q^{1/2}r^{1/2}} \sum_{\eta_1=\pm 1} \sup_{M_1 \ll (M^2R^2t^\varepsilon/N)} \sum_{n_1 \ll Rr} n_1^{\theta_3} \sum_{(n_1/(n_1,r))|q_1|n_1^\infty} \frac{1}{q_1^2}\Omega_{\boldsymbol{b}}^{1/2}, \tag{5-8}$$

where

$$\Omega_{\boldsymbol{b}} \ll \frac{N_0q_1^4L^{2/3}}{n_1^2M_1^{1/2}R^4} \sum_{\substack{\ell\in\mathcal{L} \\ (\ell\ell',q_1)=1}}\sum_{\ell'\in\mathcal{L}} |A(1,\ell)A(1,\ell')| \sum_{m\sim M_1}\sum_{m'\sim M_1} |\lambda_f(m)||\lambda_f(m')| \sum_{\substack{q_2\sim R/q_1 \\ (q_2,\ell)=1}} \sum_{\substack{q_2'\sim R/q_1 \\ (q_2',\ell')=1 \\ (q_2q_2',n_1M)=1}} \sum_{n_2\geq 1} |\mathfrak{C}(n_2)||\mathfrak{J}_{\boldsymbol{b}}(n_2)|,$$

with $\mathfrak{C}(n_2)$ defined as in (5-2) and

$$\mathfrak{J}_{\boldsymbol{b}}(n_2) = \int_{\mathbb{R}} W(w)\mathcal{I}_{\boldsymbol{b}}(N_0w,m,q_2)\overline{\mathcal{I}_{\boldsymbol{b}}(N_0w,m',q_2')}e\left(-\frac{N_0n_2w}{q_1q_2q_2'Mn_1r}\right)\mathrm{d}w,$$

$$\mathcal{I}_{\boldsymbol{b}}(w,n,q_2) = \int_{\mathbb{R}} \xi^{-1/4}V(\xi)J_f\left(\frac{mN\xi}{M^2q^2}\right)e\left(-\frac{t}{2\pi}\log\xi + \eta_1\frac{3(wN\ell\xi)^{1/3}}{r^{1/3}Mq_1q_2}\right)\mathrm{d}\xi.$$

By the exactly treatment, we have the following lemma.

Lemma 5.4. *The results in Lemma 5.2 hold when replacing $\mathfrak{J}_{\boldsymbol{a}}$ by $\mathfrak{J}_{\boldsymbol{b}}$.*

5C. *Case c.* After a similar treatment, we have

$$S_{11}^{\pm}(N,X,R,N_0) \leq \frac{N^{5/4}X}{M^{5/2}LQr^{1/2}}\frac{1}{t^{1/2}} \sum_{\eta_1=\pm 1}\sum_{n_1\leq Rr} \sum_{(n_1/(n_1,r))\,|\,q_1\,|\,(rn_1)^\infty} \frac{1}{q_1^2} \sum_{n_2\asymp N_0/n_1^2} \frac{|A(n_1,n_2)|}{n_2^{1/2}}$$
$$\cdot\left|\sum_{\ell\in\mathcal{L}} \overline{A(1,\ell)\chi(\ell)}\ell^{1/2} \sum_{\substack{q_2\sim R/q_1 \\ (q_2,\ell Mrn_1)=1}} \frac{1}{q_2^{2+2it}}\right.$$
$$\left.\cdot\sum_{m\asymp R^2M^2t^2/N} \frac{\lambda_f(m)}{m^{1/4-it}} \mathcal{C}(m,n_1,n_2,\ell,q)W_{x,\ell}^{\mathrm{sgn}(\eta_1)}\left(\frac{n_1^2n_2}{q^3M^3r}\right)V_{x,\ell}\left(\frac{tMq}{\sqrt{mN}}\right)\right|.$$

By the Cauchy–Schwarz inequality and (2-5) we have

$$S_{11}^{\pm}(N,X,R,N_0) \ll \frac{N^{5/4+\varepsilon}X}{M^{5/2}LQr^{1/2}}\frac{1}{t^{1/2}} \sum_{\eta_1=\pm 1}\sum_{n_1\leq Rr} n_1^{\theta_3} \sum_{(n_1/(n_1,r))|q_1|(rn_1)^\infty} \frac{1}{q_1^2}\Omega_{\boldsymbol{c}}^{1/2}, \tag{5-9}$$

where

$$\Omega_{\boldsymbol{c}} = \sum_{n_2\asymp N_0/n_1^2} \left|\sum_{\ell\in\mathcal{L}} \overline{A(1,\ell)\chi(\ell)}\ell^{1/2} \sum_{\substack{q_2\sim R/q_1 \\ (q_2,\ell Mrn_1)=1}} \frac{1}{q_2^{2+2it}} \sum_{m\asymp R^2M^2t^2/N} \frac{\lambda_f(m)}{m^{1/4-it}}\right.$$
$$\left.\cdot\,\mathcal{C}(m,n_1,n_2,\ell,q)W_{x,\ell}^{\mathrm{sgn}(\eta_1)}\left(\frac{n_1^2n_2}{q^3M^3r}\right)V_{x,\ell}\left(\frac{tMq}{\sqrt{mN}}\right)\right|^2.$$

Opening the square we get

$$\Omega_{\boldsymbol{c}} \ll L \sum_{\ell\in\mathcal{L}} |A(1,\ell)| \sum_{\substack{q_2\sim R/q_1\\(q_2,\ell Mrn_1)=1}} \frac{1}{q_2^2} \sum_{m\asymp R^2M^2t^2/N} \frac{|\lambda_f(m)|}{m^{1/4}} \cdot \sum_{\ell'\in\mathcal{L}} |A(1,\ell')| \sum_{\substack{q_2'\sim R/q_1\\(q_2',\ell' Mrn_1)=1}} \frac{1}{q_2'^2} \sum_{m'\asymp R^2M^2t^2/N} \frac{|\lambda_f(m')|}{m'^{1/4}} \cdot \left| \sum_{n_2\geq 1} W\left(\frac{n_1^2n_2}{N_0}\right) \mathcal{C}(m,n_1,n_2,\ell,q)\overline{\mathcal{C}(m',n_1,n_2,\ell',q')} \right|,$$

where $W(n_1^2n_2/N_0)$ is a smooth compactly supported function which contains the weight function $W_{x,\ell}^{\mathrm{sgn}(\eta_1)}(n_1^2n_2/(q^3M^3r))\overline{W_{x,\ell}^{\mathrm{sgn}(\eta_1)}(n_1^2n_2/(q^3M^3r))}$. Note that by (4-3) we have

$$\frac{\partial^j}{\partial n_2^j} W\left(\frac{n_1^2n_2}{N_0}\right) \ll_j t^\varepsilon n_2^{-j}, \quad j\geq 0.$$

By the Poisson summation formula modulo $Mrq_1q_2q_2'/n_1$ we get

$$\Omega_{\boldsymbol{c}} \ll \frac{N_0}{n_1^2}\frac{Lq_1^4}{R^4}\frac{N^{1/2}}{RMt} \sum_{\ell\in\mathcal{L}} |A(1,\ell)| \sum_{\substack{q_2\sim R/q_1\\(q_2,\ell Mrn_1)=1}} \sum_{m\asymp R^2M^2t^2/N} \cdot \sum_{\ell'\in\mathcal{L}} |A(1,\ell')| \sum_{\substack{q_2'\sim R/q_1\\(q_2',\ell' Mrn_1)=1}} \sum_{m'\asymp R^2M^2t^2/N} |\lambda_f(m')|^2 \sum_{n_2\in\mathbb{Z}} |\mathfrak{C}(n_2)||\mathfrak{I}_{\boldsymbol{c}}(n_2)|,$$

where $\mathfrak{C}(n_2)$ is defined as in (5-2) and

$$\mathfrak{I}_{\boldsymbol{c}}(n_2) = \frac{n_1^2}{N_0}\int_{\mathbb{R}} W\left(\frac{n_1^2u}{N_0}\right) e\left(-\frac{un_2}{Mrq_1q_2q_2'/n_1}\right) \mathrm{d}u = \int_{\mathbb{R}} W(\xi) e\left(-\frac{N_0n_2\xi}{Mrq_1q_2q_2'n_1}\right) \mathrm{d}\xi.$$

By repeated integration by parts we have

$$\mathfrak{I}_{\boldsymbol{c}}(n_2) \ll \begin{cases} t^{-A} & \text{if } n_2 \gg (MrR^2n_1)/(q_1N_0)t^\varepsilon, \\ t^\varepsilon & \text{if } n_2 \ll (MrR^2n_1)/(q_1N_0)t^\varepsilon. \end{cases} \tag{5-10}$$

6. The zero frequency

In this section we estimate the contribution from the terms with $n_2 = 0$. Denote the contribution of this part to Ω_* by Ω_0, where $* \in \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$. Note that $q_2'\bar{\alpha} - q_2\overline{\alpha'} \equiv 0 \pmod{Mq_2q_2'}$. So we have

$q_2' = (q_2'\bar{\alpha}, Mq_2q_2') = (q_2\overline{\alpha'}, Mq_2q_2') = q_2$, and hence $\alpha = \alpha'$. We have

$$\mathfrak{C}(0) = \delta_{q=q'} \sideset{}{^\star}\sum_{b \pmod M} \sideset{}{^\star}\sum_{b' \pmod M} \Bigg(\sum_{\substack{u \pmod M \\ u \neq b}} \bar{\chi}(u) e\bigg(\frac{m\overline{q^2\ell(b-u)}}{M}\bigg)\Bigg)\Bigg(\sum_{\substack{u' \pmod M \\ u' \neq b'}} \chi(u') e\bigg(\frac{-m'\overline{q^2\ell'(b'-u')}}{M}\bigg)\Bigg)$$

$$\cdot \sum_{d\,|\,q} \sum_{d'\,|\,q} dd'\mu(q/d)\mu(q/d') \sideset{}{^\star}\sum_{\substack{\alpha \pmod{Mrq/n_1} \\ n_1\alpha \equiv -m\bar{\ell}(d) \\ n_1\alpha \equiv -m'\bar{\ell'}(d')}} e\bigg(\frac{n_1\alpha\overline{bq^2} - n_1\alpha\overline{b'q^2}}{M}\bigg).$$

6A. *Case a:* $t^\varepsilon \ll (LNX)/(MRQ) \ll t^{1-\varepsilon}$.

6A1. $M \mid (m\bar{\ell} - m'\bar{\ell'})$. Denote the contribution of this part to Ω_0 by Ω_{01}. Moreover, the α-sum depends on either $b \equiv b' \pmod M$ or $b \not\equiv b' \pmod M$. The character sum becomes

$$\begin{aligned}\mathfrak{C}(0) &\ll M|\mathfrak{C}_1'| \sum_{d\,|\,q} \sum_{d'\,|\,q} dd' \sum_{\substack{\alpha \pmod{rq/n_1} \\ n_1\alpha \equiv -m\bar{\ell} \pmod d \\ n_1\alpha \equiv -m'\bar{\ell'} \pmod d}} 1 + |\mathfrak{C}_1''| \sum_{d\,|\,q} \sum_{d'\,|\,q} dd' \sum_{\substack{\alpha \pmod{rq/n_1} \\ n_1\alpha \equiv -m\bar{\ell} \pmod d \\ n_1\alpha \equiv -m'\bar{\ell'} \pmod d}} 1 \\ &\ll (M|\mathfrak{C}_1'| + |\mathfrak{C}_1''|) \sum_{d\,|\,q} \sum_{d'\,|\,q} (d, d') rq\, \delta_{(d,d')\,|\,(m\ell' - m'\ell)},\end{aligned} \tag{6-1}$$

where

$$\mathfrak{C}_1' = \sideset{}{^\star}\sum_{b \pmod M} \sum_{\substack{u \pmod M \\ u \neq b}} \sum_{\substack{u' \pmod M \\ u' \neq b}} \bar{\chi}(u)\chi(u') e\bigg(\frac{m\overline{q^2\ell(b-u)}}{M}\bigg) e\bigg(-\frac{m'\overline{q^2\ell'(b-u')}}{M}\bigg),$$

and

$$\mathfrak{C}_1'' = \sideset{}{^\star}\sum_{b \pmod M} \sideset{}{^\star}\sum_{\substack{b' \pmod M \\ b' \not\equiv b \pmod M}} \sum_{\substack{u \pmod M \\ u \neq b}} \sum_{\substack{u' \pmod M \\ u' \neq b'}} \bar{\chi}(u)\chi(u') e\bigg(\frac{m\overline{q^2\ell(b-u)}}{M}\bigg) e\bigg(-\frac{m'\overline{q^2\ell'(b'-u')}}{M}\bigg).$$

Since $M \mid (m\bar{\ell} - m'\bar{\ell'})$, similar to [Sharma 2022, (6.3)], we have square root cancellation in the sum over u and u', and hence we obtain

$$\mathfrak{C}_1' \ll M^2 \quad \text{and} \quad \mathfrak{C}_1'' \ll M^3.$$

Hence

$$\mathfrak{C}(0) \ll M^3 \sum_{d\,|\,q} \sum_{d'\,|\,q} (d, d') rq\, \delta_{(d,d')\,|\,(m\ell' - m'\ell)}.$$

Note that $(M, (d, d')) = 1$ and

$$|A(1, \ell)A(1, \ell')\lambda_f(m)\lambda_f(m')| \ll |A(1, \ell)\lambda_f(m')|^2 + |A(1, \ell')\lambda_f(m)|^2.$$

By Lemma 5.1, we have

$$\sum_{\ell}\sum_{\ell'}\sum_{m}\sum_{m'}|\mathfrak{I}(0)| \ll \sum_{\ell}|A(1,\ell)|^2\sum_{m'}|\lambda_f(m')|^2 \sum_{\substack{\ell' \quad m\\ M(d,d')\,|\,(m\ell'-m'\ell)}}\sum |\mathfrak{I}(0)| + \sum_{\ell'}|A(1,\ell')|^2\sum_{m}|\lambda_f(m)|^2 \sum_{\substack{\ell \quad m'\\ M(d,d')\,|\,(m\ell'-m'\ell)}}\sum |\mathfrak{I}(0)|$$

$$\ll N^{\varepsilon}LM_1\left(\frac{LM_1((LNX)/(MRQt))^2+LM_1(MRQ)/(LNX)}{M(d,d')}+1\right)\frac{1}{t}. \tag{6-2}$$

Hence we have

$$\Omega_{01} \ll N^{\varepsilon}\frac{N_0M^3}{n_1^2M_1^{1/2}}\frac{L^{2/3}q_1^3}{R^3}\left(\sum_{\substack{q_2\sim R/q_1\\(q_2,rn_1)=1}}\sum_{\substack{d\,|\,q\\d'\,|\,q}} rqLM_1\left(\frac{LM_1(LNX/MRQt)^2+LM_1(MRQ/LNX)}{M}+q\right)\frac{1}{t}\right)$$

$$\ll N^{\varepsilon}\frac{N_0M^3}{n_1^2M_1^{1/2}}\frac{L^{2/3}q_1^2}{R^2}rRLM_1\left(\frac{LM_1(LNX/MRQt)^2+LM_1(MRQ/LNX)}{Mt}+\frac{R}{t}\right).$$

By using $N_0=N^2L^2X^3r/Q^3$ and $M_1 \ll t^2R^2M^2/N$, we get

$$\Omega_{01} \ll N^{\varepsilon}\frac{r^2N^{3/2}L^{11/3}Rq_1^2M^4X^3}{n_1^2Q^3}\left(\frac{L^3NX^2}{MRQ^2}+\frac{M^2R^2Qt^2}{N^2X}+1\right).$$

Hence, the contribution from Ω_{01} to (5-1) is

$$\ll \frac{N^{3/4+\varepsilon}X^{1/2}}{M^2L^{4/3}Q^{1/2}r^{1/2}}\sum_{n_1\ll RMr} n_1^{\theta_3}\sum_{(n_1/(n_1,r))|q_1|(rn_1)^\infty}\frac{1}{n_1q_1^{1/2}}\left(\frac{r^2N^{3/2}L^{11/3}RM^4X^3}{Q^3}\right)^{1/2} \cdot\left(\frac{L^3NX^2}{MRQ^2}+\frac{M^2R^2Qt^2}{N^2X}+1\right)^{1/2}$$

$$\ll N^{\varepsilon}r^{1/2}\frac{N^{3/2}L^{1/2}R^{1/2}X^2}{Q^2}\left(\frac{L^{3/2}N^{1/2}X}{M^{1/2}R^{1/2}Q}+\frac{MRQ^{1/2}t}{NX^{1/2}}+1\right).$$

Recall $Q=(NL/MK)^{1/2}$. Thus, by $X\ll t^{\varepsilon}$ and $R\le Q$, we arrive at

$$\begin{aligned}&\ll N^{\varepsilon}r^{1/2}\frac{N^2L^2}{M^{1/2}Q^3}+N^{\varepsilon}r^{1/2}N^{1/2}L^{1/2}Mt+N^{\varepsilon}r^{1/2}\frac{N^{3/2}L^{1/2}}{Q^{3/2}}\\ &\ll N^{\varepsilon}r^{1/2}N^{1/2}L^{1/2}MK^{3/2}+N^{\varepsilon}r^{1/2}N^{1/2}L^{1/2}Mt+N^{\varepsilon}r^{1/2}\frac{N^{3/4}M^{3/4}K^{3/4}}{L^{1/4}}.\end{aligned} \tag{6-3}$$

6A2. $M\nmid(m\bar{\ell}-m'\bar{\ell}')$. Denote the contribution of this part to Ω_0 by Ω_{02}. In this case, we also have $q_2=q_2'$ and $\alpha=\alpha'$. So we can estimate the character as in (6-1). Since $M\nmid(m\bar{\ell}-m'\bar{\ell}')$, the nondegeneracy holds for the variables b, u, u' in $\mathfrak{C}_1'$ and $\mathfrak{C}_1''$ and hence we have

$$\mathfrak{C}_1' \ll M^{3/2} \quad\text{and}\quad \mathfrak{C}_1'' \ll M^{5/2}.$$

Thus we get

$$\mathfrak{C}(0) \ll M^{5/2} \sum_{d \mid q_1q_2} \sum_{d' \mid q_1q_2} dd' \frac{rq_1q_2}{[d,d']} \delta_{(d,d') \mid (m\ell' - m'\ell)}. \tag{6-4}$$

As in (6-2), by Lemma 5.1, we have

$$\sum_{\ell} \sum_{\ell'} \sum_{m} \sum_{\substack{m' \\ (d,d') \mid (m\ell'-m'\ell)}} |\mathfrak{I}(0)| \ll N^{\varepsilon} LM_1 \left(\frac{LM_1(LNX/MRQt)^2 + LM_1MRQ/LNX}{(d,d')} + 1 \right) \frac{1}{t}.$$

Hence, similar to the estimate for Ω_{01}, we have

$$\Omega_{02} \ll N^{\varepsilon} \frac{r^2 N^{3/2} L^{11/3} R q_1^2 M^{7/2} X^3}{n_1^2 Q^3} \left(\frac{L^3 N X^2}{RQ^2} + \frac{M^3 R^2 Q t^2}{N^2 X} + 1 \right).$$

Hence, similar to the estimate for (6-3), the contribution from Ω_{02} to (5-1) is

$$\ll N^{\varepsilon} r^{1/2} N^{1/2} L^{1/2} M^{5/4} K^{3/2} + N^{\varepsilon} r^{1/2} N^{1/2} L^{1/2} M^{5/4} t + N^{\varepsilon} r^{1/2} \frac{N^{3/4} M^{1/2} K^{3/4}}{L^{1/4}}. \tag{6-5}$$

6B. ***Case a:*** $\boldsymbol{(LNX)/(MRQ) \gg t^{1-\varepsilon}}$**.** By the same argument as in the Section 6A and Lemma 5.2 we have

$$\begin{aligned} \Omega_0 \ll{}& N^{\varepsilon} \frac{N_0 M^3}{n_1^2 M_1^{1/2}} \frac{L^{2/3} q_1^3}{R^3} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1) = 1}} \sum_{\substack{d \mid q \\ d' \mid q}} rqLM_1 \left(\frac{LM_1}{M} + q \right) \frac{MRQ}{NLX} \\ &+ N^{\varepsilon} \frac{N_0 M^{5/2}}{n_1^2 M_1^{1/2}} \frac{L^{2/3} q_1^3}{R^3} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1) = 1}} \sum_{\substack{d \mid q \\ d' \mid q}} rqLM_1(LM_1 + q) \frac{MRQ}{NLX} \\ \ll{}& N^{\varepsilon} \frac{N_0}{n_1^2 M_1^{1/2}} \frac{L^{2/3} q_1^3}{R^3} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1) = 1}} rqLM_1(LM_1 M^{5/2} + qM^3) \frac{MRQ}{NLX} \\ \ll{}& N^{\varepsilon} \frac{r^2 N^{3/2} M q_1^2 L^{11/3} X^3}{n_1^2 Q^3} \left(\frac{NL^3 M^{5/2} X^2}{Q^2} + RM^3 \right). \end{aligned}$$

Here we have used $N_0 = N^2 L^2 X^3 r / Q^3$ and $M_1 \ll (NL^2 X^2 / Q^2) N^{\varepsilon}$. Therefore, the contribution from Ω_0 to (5-1) is

$$\begin{aligned} &\ll \frac{N^{3/4+\varepsilon} X^{1/2}}{M^2 L^{4/3} Q^{1/2} r^{1/2}} \sum_{n_1 \ll RMr} n_1^{\theta_3} \sum_{(n_1/(n_1,r)) \mid q_1 \mid (rn_1)^{\infty}} \frac{1}{n_1 q_1^{1/2}} \left(\frac{r^2 N^{3/2} M L^{11/3} X^3}{Q^3} \left(\frac{NL^3 M^{5/2} X^2}{Q^2} + RM^3 \right) \right)^{1/2} \\ &\ll N^{\varepsilon} r^{1/2} \frac{N^{3/2} L^{1/2} X^2}{M^{3/2} Q^2} \left(\frac{N^{1/2} L^{3/2} M^{5/4} X}{Q} + R^{1/2} M^{3/2} \right). \end{aligned}$$

Note that we have $R \ll NLX/(MQt^{1-\varepsilon})$ now. By this and inserting $Q=(NL/MK)^{1/2}$, one can bound the above by

$$\ll N^{\varepsilon} r^{1/2} N^{1/2} L^{1/2} M^{5/4} K^{3/2} + N^{\varepsilon} r^{1/2} \frac{N^{3/4} M^{3/4} K^{5/4}}{L^{1/4} t^{1/2}}. \tag{6-6}$$

6C. ***Case b:*** $(LNX)/(MRQ) \asymp t$. By the same argument as in the Section 6A and Lemma 5.4 we have

$$\begin{aligned}\Omega_0 \ll N^{\varepsilon} \frac{N_0 M^3}{n_1^2 M_1^{1/2}} \frac{L^{2/3} q_1^4}{R^4} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1)=1}} \sum_{\substack{d \mid q \\ d' \mid q}} rqLM_1 \left(\frac{LM_1}{M} + q\right) \frac{MRQ}{NLX} \\ + N^{\varepsilon} \frac{N_0 M^{5/2}}{n_1^2 M_1^{1/2}} \frac{L^{2/3} q_1^4}{R^4} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1)=1}} \sum_{\substack{d \mid q \\ d' \mid q}} rqLM_1 (LM_1 + q) \frac{MRQ}{NLX} \\ \ll N^{\varepsilon} \frac{N_0}{n_1^2} \frac{L^{2/3} q_1^3}{R^3} rRLM_1^{1/2} (LM_1 M^{5/2} + RM^3) \frac{MRQ}{NLX}.\end{aligned}$$

By $N_0 = N^2 L^2 X^3 r / Q^3$ and $M_1 \ll (M^2 R^2 / N) t^{\varepsilon}$ we obtain

$$\begin{aligned}\Omega_0 &\ll N^{\varepsilon} \frac{1}{n_1^2} \frac{N^2 L^2 X^3 r}{Q^3} \frac{MRQ}{NLX} \frac{L^{2/3} q_1^3}{R^3} rRL \frac{MR}{N^{1/2}} \left(LM^{5/2} \frac{M^2 R^2}{N} + RM^3 \right) \\ &\ll N^{\varepsilon} \frac{r^2 N^{1/2} M^2 q_1^3 L^{8/3} X^2}{n_1^2 Q^2} \left(\frac{LM^{9/2} R^2}{N} + RM^3 \right).\end{aligned}$$

Thus, the contribution from Ω_0 to (5-8) is

$$\begin{aligned}&\ll \frac{N^{1+\varepsilon} X^{1/2}}{M^{5/2} L^{4/3} Q^{1/2} r^{1/2}} \sum_{n_1 \ll RMr} n_1^{\theta_3} \sum_{(n_1/(n_1,r)) \mid q_1 \mid (rn_1)^{\infty}} \frac{1}{n_1 q_1^{1/2}} \left(\frac{r^2 N^{1/2} M^2 L^{8/3} X^2}{Q^2} \left(\frac{LM^{9/2} R^2}{N} + RM^3 \right) \right)^{1/2} \\ &\ll N^{\varepsilon} r^{1/2} \frac{N^{3/4} L^{1/2} M^{3/4} R X^{3/2}}{Q^{3/2}} + N^{\varepsilon} r^{1/2} \frac{N^{5/4} R^{1/2} X^{3/2}}{Q^{3/2}}.\end{aligned}$$

By $Q=(NL/MK)^{1/2}$ again and noting that $R \asymp (NLX)/(MQt)$, we deduce that the above is dominated by

$$\ll N^{\varepsilon} r^{1/2} N^{1/2} L^{1/4} M \frac{K^{5/4}}{t} + N^{\varepsilon} r^{1/2} \frac{N^{3/4} M^{1/2} K}{L^{1/2} t^{1/2}}. \tag{6-7}$$

6D. ***Case c:*** $(LNX)/(MRQ) \ll t^{\varepsilon}$. By the same argument as in the Section 6A and (5-10) we have (taking $M_1 \asymp R^2 M^2 t^2 / N$)

$$\begin{aligned}\Omega_0 &\ll N^{\varepsilon} \frac{N_0 M^3}{n_1^2 M_1^{1/2}} \frac{L q_1^4}{R^4} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1)=1}} \sum_{\substack{d \mid q \\ d' \mid q}} rqLM_1 \left(\frac{LM_1}{M} + q \right) + N^{\varepsilon} \frac{N_0 M^{5/2}}{n_1^2 M_1^{1/2}} \frac{L q_1^4}{R^4} \sum_{\substack{q_2 \sim R/q_1 \\ (q_2, rn_1)=1}} \sum_{\substack{d \mid q \\ d' \mid q}} rqLM_1 (LM_1 + q) \\ &\ll N^{\varepsilon} \frac{N_0}{n_1^2} \frac{L q_1^3}{R^3} rRLM_1^{1/2} (LM_1 M^{5/2} + RM^3).\end{aligned}$$

By $N_0 \ll (R^3M^3r/LN)t^\varepsilon$ and $M_1 \asymp R^2M^2t^2/N$, one has

$$\Omega_0 \ll N^\varepsilon \frac{1}{n_1^2}\frac{R^3M^3r}{LN}\frac{Lq_1^3}{R^3}rRL\frac{RMt}{N^{1/2}}\left(LM^{5/2}\frac{R^2M^2t^2}{N}+RM^3\right)$$
$$\ll N^\varepsilon \frac{r^2q_1^3R^2LM^4t}{n_1^2N^{3/2}}\left(\frac{LM^{9/2}R^2t^2}{N}+RM^3\right).$$

So the contribution from Ω_0 to (5-9) is

$$\ll \frac{N^{5/4+\varepsilon}X}{M^{5/2}LQr^{1/2}t^{1/2}}\left(\frac{r^2R^2LM^4t}{N^{3/2}}\left(\frac{LM^{9/2}R^2t^2}{N}+RM^3\right)\right)^{1/2}$$
$$\ll N^\varepsilon r^{1/2}\frac{M^{7/4}R^2tX}{Q}+N^\varepsilon r^{1/2}\frac{N^{1/2}MR^{3/2}X}{L^{1/2}Q}.$$

Now we have the condition $X \ll (MRQ/LN)t^\varepsilon$, so one computes the above as

$$\ll N^\varepsilon r^{1/2}N^{1/2}L^{1/2}M^{5/4}\frac{t}{K^{3/2}}+N^\varepsilon r^{1/2}\frac{N^{3/4}M^{3/4}}{L^{1/4}K^{5/4}}. \tag{6-8}$$

Combining (6-3), (6-5), (6-6), (6-7) and (6-8), we see that the contribution of the zero frequency is dominated by

$$\ll N^\varepsilon r^{1/2}N^{1/2}L^{1/2}M^{5/4}K^{3/2}+N^\varepsilon r^{1/2}N^{1/2}L^{1/2}M^{5/4}t$$
$$+N^\varepsilon r^{1/2}\frac{N^{3/4}M^{3/4}K^{3/4}}{L^{1/4}}+N^\varepsilon r^{1/2}\frac{N^{3/4}M^{3/4}K^{5/4}}{L^{1/4}t^{1/2}}. \tag{6-9}$$

7. The nonzero frequencies

7A. $n_2 \not\equiv 0$ (mod M). Denote the contribution from $n_2 \not\equiv 0 \pmod M$ in Ω_* by $\Omega_{*,1}$, where $* \in \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$. We have

$$\mathfrak{C}(n_2) \ll |\mathfrak{C}_1(n_2)\mathfrak{C}_2(n_2)\mathfrak{C}_3(n_2)|,$$

where

$$\mathfrak{C}_1(n_2)=\sideset{}{^\star}\sum_{b \bmod M}\ \sideset{}{^\star}\sum_{b' \bmod M}\left(\sum_{\substack{u \bmod M\\ u\neq b}}\bar\chi(u)e\left(\frac{m\overline{q_2^2\ell(b-u)}}{M}\right)\right)$$
$$\cdot\left(\sum_{\substack{u' \bmod M\\ u'\neq b'}}\chi(u')e\left(-\frac{m'\overline{q_2'^2\ell(b'-u')}}{M}\right)\right)\left(\sideset{}{^\star}\sum_{\substack{\alpha,\alpha' \bmod M\\ q_2'\bar\alpha-q_2\overline{\alpha'}\equiv-\eta_1n_2(M)}}e\left(\frac{\alpha b\overline{q_2^2}-\alpha'b'\overline{q_2'^2}}{M}\right)\right),$$

$$\mathfrak{C}_2(n_2)=\sum_{d_1\mid q_1}\sum_{d_1'\mid q_1}d_1d_1'\sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod rq_1/n_1\\ n_1\alpha_1\equiv-m\bar\ell \bmod d_1}}\ \sideset{}{^\star}\sum_{\substack{\alpha_1' \bmod rq_1/n_1\\ n_1\alpha_1'\equiv-m'\bar\ell' \bmod d_1'\\ q_2'\alpha_1-q_2\overline{\alpha_1'}\equiv-\eta_1n_2\ (\mathrm{mod}\ rq_1/n_1)}}1,$$

and

$$\mathfrak{C}_3(n_2) = \sum_{d_2 \mid q_2} \sum_{d_2' \mid q_2'} d_2 d_2' \sideset{}{^\star}\sum_{\substack{\alpha_2(q_2), \alpha_2'(q_2') \\ q_2'\overline{\alpha_2} - q_2\overline{\alpha_2'} \equiv -\eta_1 n_2 \bmod q_2 q_2' \\ n_1\alpha_2 \equiv -m\bar{\ell} \bmod d_2 \\ n_1\alpha_2' \equiv -m'\bar{\ell'} \bmod d_2'}} 1.$$

For $\mathfrak{C}_2(n_2)$, the congruence condition determines at most one solution of α_1' in terms of α_1, and hence we have

$$\mathfrak{C}_2(n_2) \leq \sum_{d_1 \mid q_1} d_1 \sum_{d_1' \mid q_1} d_1' \sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod rq_1/n_1 \\ -m\bar{\ell} \equiv n_1\alpha_1 \bmod d_1}} 1.$$

Note that α_1 is uniquely determined modulo $d_1/(d_1, n_1)$. Since $(d_1/(d_1, n_1), n_1/(d_1, n_1)) = 1$, $(d_1/(d_1, n_1)) \mid (q_1/(d_1, n_1))$ and $n_1/(d_1, n_1) \mid rq_1/(d_1, n_1)$, we have $d_1/(d_1, n_1) \mid rq_1/n_1$. Hence we have

$$\mathfrak{C}_2(n_2) \ll \frac{rq_1}{n_1} \sum_{d_1 \mid q_1} \sum_{d_1' \mid q_1} d_1'(d_1, n_1) \delta_{(d_1, n_1) \mid m}.$$

Similarly by considering α_1-sum first we have

$$\mathfrak{C}_2(n_2) \ll \frac{rq_1}{n_1} \sum_{d_1 \mid q_1} \sum_{d_1' \mid q_1} d_1(d_1', n_1) \delta_{(d_1', n_1) \mid m'}.$$

For $\mathfrak{C}_3(n_2)$, from the congruence $q_2'\bar{\alpha} - q_2\overline{\alpha'} \equiv -\eta_1 n_2 \bmod q_2 q_2'$ we have $(q_2, q_2') \mid n$. Since $(n_1, q_2) = 1$, we have $\alpha \equiv -m\bar{\ell}\bar{n}_1 \bmod d_2$ and hence $q_2'\bar{\alpha} \equiv -\eta_1 n_2 \bmod d_2$. Therefore we get $n_1 q_2' \equiv \eta_1 m n_2 \bar{\ell} \bmod d_2$. Similarly we have $-n_1 q_2 \equiv \eta_1 m' n_2 \bar{\ell'} \bmod d_2'$. Note that the congruence determines $\alpha_2 \bmod [q_2/(q_2, q_2'), d_2]$ and for each given α_2 we have at most one solution of $\alpha_2' \bmod q_2'$. Hence we have

$$\mathfrak{C}_3(n) \ll \sum_{\substack{d_2 \mid (q_2, -q_2' n_1 \ell + \eta_1 m n_2) \\ d_2' \mid (q_2', q_2 n_1 \ell' + \eta_1 m' n_2)}} \sum d_2 d_2' \frac{q_2}{[q_2/(q_2, q_2'), d_2]} \delta_{(q_2, q_2') \mid n}.$$

Similarly we have

$$\mathfrak{C}_3(n_2) \ll \sum_{\substack{d_2 \mid (q_2, -q_2' n_1 \ell + \eta_1 m n_2) \\ d_2' \mid (q_2', q_2 n_1 \ell' + \eta_1 m' n_2)}} \sum d_2 d_2' \frac{q_2'}{[q_2'/(q_2, q_2'), d_2']} \delta_{(q_2, q_2') \mid n_2}.$$

Together with [Sharma 2022, (5.6)] and [Lin et al. 2023, Proposition 4.4], we have

$$\mathfrak{C}_1(n_2) \ll M^{5/2},$$
$$\mathfrak{C}_2(n_2) \ll \frac{q_1 r}{n_1} \sum_{d_1 \mid q_1} \sum_{d_1' \mid q_1} \min\{d_1'(d_1, n_1)\delta_{(d_1, n_1) \mid m}, d_1(d_1', n_1)\delta_{(d_1', n_1) \mid m'}\},$$
$$\mathfrak{C}_3(n_2) \ll \sum_{\substack{d_2 \mid (q_2, -q_2' n_1 \ell + \eta_1 m n_2) \\ d_2' \mid (q_2', q_2 n_1 \ell' + \eta_1 m' n_2)}} \sum d_2 d_2' \min\left\{\frac{q_2}{[q_2/(q_2, q_2'), d_2]}, \frac{q_2'}{[q_2'/(q_2, q_2'), d_2']}\right\} \delta_{(q_2, q_2') \mid n_2}.$$

Now, we need some careful counting to estimate $\Omega_{*,1}$; see [Munshi 2022, Section 6; Sharma 2022, Section 5; Lin et al. 2023, Section 6; Lin and Sun 2021, Section 4.5].

7A1. *Case a.* It is obvious that, for fixed tuple (n_1, α, n_2), the congruence

$$-q_2' n_1 \ell + \eta_1 m n_2 \equiv 0 \bmod d_2$$

has a solution if and only if $(d_2, n_2) \mid q_2' \ell$, in which case m is uniquely determined modulo $d_2/(d_2, n_2)$. Combining this together with the condition $\delta_{(d_1,n_1)\mid m}$ in $\mathfrak{C}_2(n_2)$, the number of m $(\sim M_1)$ is dominated by $\delta_{(d_2,n_2)\mid q_2'} O(1 + M_1(d_2, n_2)/((d_1, n_1)d_2))$. Then, we get

$$\begin{aligned}\Omega_{\boldsymbol{a},1} \ll \frac{q_1^4 r N_0 L^{2/3} M^{5/2}}{n_1^3 M_1^{1/2} R^3} &\sum_{d_1 \mid q_1} \sum_{d_1' \mid q_1} d_1'(d_1, n_1) \sum_{\substack{\ell \in \mathcal{L} \\ (\ell\ell', q_1)=1}} \sum_{\ell' \in \mathcal{L}} |A_\pi(1, \ell) A_\pi(1, \ell')| \\ &\cdot \sum_{\substack{q_2 \sim R/q_1 \\ (q_2,\ell)=1 \\ (q_2 q_2', n_1 M)=1}} \sum_{\substack{q_2' \sim R/q_1 \\ (q_2',\ell')=1}} \sum_{d_2 \mid q_2} \sum_{d_2' \mid q_2'} \sum_{\substack{1 \le n_2 \le N_2 \\ (d_2,n_2) \mid q_2'\ell \\ (q_2,q_2') \mid n_2}} d_2 d_2' \left(1 + \frac{M_1(d_2, n_2)}{(d_1, n_1) d_2}\right) \\ &\cdot \min\left\{\frac{q_2}{[q_2/(q_2, q_2'), d_2]}, \frac{q_2'}{[q_2'/(q_2, q_2'), d_2']}\right\} \sum_{\substack{m' \sim M_1 \\ q_2 n_1 \ell' + \eta_1 m' n_2 \equiv 0 \ (\mathrm{mod}\ d_2')}} |\lambda_f(m')|^2 |\mathfrak{J}_{\boldsymbol{a}}(n_2)|.\end{aligned}$$

Let us make the following notation:

$$\begin{gathered}(q_2, q_2') = q_3, \quad q_2 = q_3 q_4, \quad q_2' = q_3 q_4' \\ d_2 = d_0 d_3 d_4, \quad d_0 \mid (q_3, q_4), \quad d_3 \mid q_3, \quad (d_3, q_4) = 1, \quad (d_4, q_3) = 1, \quad d_4 \mid q_4, \\ d_2' = d_3' d_4', \quad d_3' \mid q_3, \quad d_4' \mid q_4.\end{gathered}$$

It is easy to see that

$$\begin{aligned}(d_2, n_2) &\le (d_0 d_3, n_2)(d_4, n_2) \\ &\le d_0 d_3 (d_4, n_2) \\ &= d_0 d_3 (d_4, n_2/q_3), \, q_2/[q_2/(q_2, q_2'), d_2] \\ &= q_3 q_4/[q_4, d_2] \\ &\le q_3/d_3,\end{aligned}$$

and

$$q_2'/[q_2'/(q_2, q_2'), d_2'] = q_3 q_4'/[q_4', d_2'] \le q_3 q_4'/d_2'.$$

Then, breaking the n_2-sum into dyadic segments $n_2 \sim \tilde{N}_2$ with $\tilde{N}_2 \ll N_2$ and using Lemmas 5.1 and 5.2, one has

$$\Omega_{\boldsymbol{a},1} \ll \sup_{\substack{1 \ll \tilde{N}_2 \ll N_2 \\ \text{dyadic}}} \frac{N^{\varepsilon} q_1^4 r N_0 L^{2/3} M^{5/2}}{n_1^3 M_1^{1/2} R^3} \sum_{d_1 \mid q_1} \sum_{d_1' \mid q_1} (d_1, n_1) d_1' \sum_{\substack{\ell \in \mathcal{L} \\ (\ell\ell', q_1)=1}} \sum_{\ell' \in \mathcal{L}} |A_\pi(1,\ell) A_\pi(1,\ell')|$$
$$\cdot \sum_{\substack{q_3 \le R/q_1 \\ (q_3, n_1 \ell\ell' M)=1}} \sum_{\substack{q_4 \sim R/q_3 q_1 \\ (q_4,\ell)=1}} \sum_{\substack{q_4' \sim R/q_3 q_1 \\ (q_4',\ell')=1 \\ (q_4 q_4', n_1 M)=1}} \sum_{d_0 \mid (q_3,q_4)} \sum_{\substack{d_3 \mid q_3 \\ (d_3,q_4)=1}} \sum_{\substack{d_4 \mid q_4 \\ (d_4,q_3)=1}} d_0 d_3 d_4$$
$$\cdot \sum_{d_3' \mid q_3} \sum_{d_4' \mid q_4'} d_3' d_4' \sum_{\substack{n_2 \sim \tilde{N}_2 \\ (d_2,n_2) \mid q_3 q_4' \ell \\ q_3 \mid n_2}} \left(1 + \frac{M_1 (d_4, n_2/q_3)}{(d_1, n_1) d_4}\right) C(\tilde{N}_2)$$
$$\cdot \min\left\{\frac{q_3}{d_3}, \frac{q_3 q_4'}{d_3' d_4'}\right\} \sum_{\substack{m' \sim M_1 \\ q_3 q_4 n_1 \ell' + \eta_2 m' n_2 \equiv 0 \ (\mathrm{mod}\ d_3' d_4')}} |\lambda_f(m')|^2,$$

where

$$C(\tilde{N}_2) = \begin{cases} (R Q^{3/2} M^{1/2} n_1^{1/2})/(t N L X^{3/2} q_1^{1/2} \tilde{N}_2^{1/2}) & N_2' \ll \tilde{N}_2 \ll N_2 \text{ and } NLX/MRQ \ll t^{1-\varepsilon}, \\ 1/t & \tilde{N}_2 \ll N_2' \text{ and } NLX/MRQ \ll t^{1-\varepsilon}, \\ MRQ/NLX & \tilde{N}_2 \ll N_2 \text{ and } NLX/MRQ \gg t^{1-\varepsilon}. \end{cases} \tag{7-1}$$

Case (i): $q_3 q_4 n_1 \ell' + \eta_2 m' n_2 \neq 0$. Denote the contribution from this part in $\Omega_{\boldsymbol{a},1}$ by $\Omega_{\boldsymbol{a},11}$. Write

$$q_3 = d_3' q_5, \quad q_4 = d_0 q_6 \quad \text{and} \quad q_4' = d_4' q_6',$$

then we have

$$\Omega_{\boldsymbol{a},11} \ll \sup_{\substack{1 \ll \tilde{N}_2 \ll N_2 \\ \text{dyadic}}} \frac{N^{\varepsilon} q_1^5 r N_0 L^{2/3} M^{5/2}}{n_1^3 M_1^{1/2} R^3} \sum_{d_1 \mid q_1} (d_1, n_1) \sum_{\ell \in \mathcal{L}} \sum_{\ell' \in \mathcal{L}} |A_\pi(1,\ell) A_\pi(1,\ell')|$$
$$\cdot \sum_{d_3' \le R/q_1} d_3' \sum_{q_5 \le R/q_1 d_3'} \sum_{\substack{d_0 \le R/q_1 d_3' q_5 \\ d_0 \mid d_3' q_5}} d_0 \sum_{\substack{d_3 \mid d_3' q_5 \\ (d_3,q_4)=1}} \sum_{q_6 \sim R/q_1 d_3' d_0 q_5} \sum_{\substack{d_4 \mid d_0 q_6 \\ (d_4, d_3' q_5)=1}}$$
$$\cdot \sum_{\substack{n_2 \sim \tilde{N}_2 \\ d_3' q_5 \mid n_2}} d_3' q_5 \left(d_4 + \frac{M_1 (d_4, n_2/d_3' q_5)}{(d_1, n_1)}\right) C(\tilde{N}_2) \sum_{m' \sim M_1} |\lambda_f(m')|^2$$
$$\cdot \sum_{\substack{d_4' \le R/q_1 d_3' q_5 \\ 0 \neq d_3' d_4 q_5 q_6 n_1 \ell' + \eta_1 m' n_2 \equiv 0 \ (\mathrm{mod}\ d_3' d_4')}} d_4' \sum_{q_6' \sim R/q_1 d_3' q_5 d_4'} 1,$$

By the well known bound of the divisor function, the number of the tuple (d_0, d_3, d_4, d_4') is bounded by $O(N^{\varepsilon})$. Combining this together with (2-1) and (2-2), we get

$$\Omega_{\boldsymbol{a},11} \ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} \frac{N^{\varepsilon} q_1^3 r N_0 M^{5/2} M_1^{1/2} L^{8/3} \tilde{N}_2 C(\tilde{N}_2)}{n_1^3 R}(R+M_1) \tag{7-2}$$

Case (ii): $q_3q_4n_1\ell' + \eta_2 m'n_2 = 0$. Denote the contribution from this part in $\Omega_{\boldsymbol{a},1}$ by $\Omega_{\boldsymbol{a},12}$. In this subsection, we use $(d_2, n_2) \le (q_2'\ell, q_2) = q_3$. Therefore we have

$$\begin{aligned}
\Omega_{\boldsymbol{a},12} \ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} & \frac{N^{\varepsilon} q_1^4 r N_0 L^{2/3} M^{5/2}}{n_1^3 M_1^{1/2} R^3} \sum_{d_1\mid q_1}\sum_{d_1'\mid q_1} d_1'(d_1,n_1)\\
&\cdot \sum_{\ell\in\mathcal{L}}\sum_{\ell'\in\mathcal{L}} |A_\pi(1,\ell)A_\pi(1,\ell')| \sum_{q_3\le R/q_1}\sum_{q_4,q_4'\sim R/q_1q_3}\sum_{d_0\mid(q_3,q_4)}\sum_{\substack{d_3\mid q_3\\(d_3,q_4)=1}}\sum_{\substack{d_4\mid q_4\\(d_4,q_3)=1}} d_0d_3d_4\\
&\cdot \sum_{d_3'\mid q_3}\sum_{d_4'\mid q_4'} d_3'd_4' \sum_{\substack{n_2\sim\tilde{N}_2\\ q_3\mid n_2}} \left(1+\frac{M_1q_3}{(d_1,n_1)d_0d_3d_4}\right) C(\tilde{N}_2)\\
&\cdot \min\left\{\frac{q_3}{d_3}, \frac{q_3q_4'}{d_3'd_4'}\right\} \sum_{\substack{m'\sim M_1\\ q_3q_4n_1\ell'+\eta_1m'n_2=0}} |\lambda_f(m')|^2\\
\ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} & \frac{N^{\varepsilon} q_1^5 r N_0 L^{2/3} M^{5/2}}{n_1^3 M_1^{1/2} R^3} \sum_{d_1\mid q_1}(d_1,n_1)\sum_{m'\sim M_1}|\lambda_f(m')|^2 \sum_{q_3\le R/q_1} q_3 \sum_{\substack{n_2\sim\tilde{N}_2\\ q_3\mid n_2}} C(\tilde{N}_2)\\
&\cdot \sum_{q_4\sim R/q_1q_3}\sum_{\ell\in\mathcal{L}}|A_\pi(1,\ell)|\left(\frac{R}{q_1}+\frac{M_1q_3}{(d_1,n_1)}\right)\\
&\cdot \sum_{d_0\mid(q_3,q_4)}\sum_{\substack{d_3\mid q_3\\(d_3,q_4)=1}}\sum_{\substack{d_4\mid q_4\\(d_4,q_3)=1}}\sum_{d_3'\mid q_3}\sum_{\ell'\in\mathcal{L}}|A_\pi(1,\ell')|\delta_{q_3q_4\ell'\mid m'n_2}\sum_{q_4'\sim R/q_1q_3}\sum_{d_4'\mid q_4'} q_4'.
\end{aligned}$$

Now, we estimate the last two sums trivially, and then use the condition $\delta_{q_3q_4\ell'\mid m'n_2}$ together with (2-2) and (2-3), obtaining

$$\begin{aligned}
&\sum_{q_4\sim R/q_1q_3}\sum_{\ell\in\mathcal{L}}|A_\pi(1,\ell)|\left(\frac{R}{q_1}+\frac{M_1q_3}{(d_1,n_1)}\right)\\
&\quad\cdot \sum_{d_0\mid(q_3,q_4)}\sum_{\substack{d_3\mid q_3\\(d_3,q_4)=1}}\sum_{\substack{d_4\mid q_4\\(d_4,q_3)=1}}\sum_{d_3'\mid q_3}\sum_{\ell'\in\mathcal{L}}|A_\pi(1,\ell')|\delta_{q_3q_4\ell'\mid m'n_2}\sum_{q_4'\sim R/q_1q_3}\sum_{d_4'\mid q_4'} q_4'\\
&\qquad\ll \frac{R^2L^{1+\theta_3}}{q_1^2q_3^2}\left(\frac{R}{q_1}+\frac{M_1q_3}{(d_1,n_1)}\right),
\end{aligned}$$

where $\theta_3 \le \frac{5}{14}$. Therefore, it follows that

$$\Omega_{\boldsymbol{a},12} \ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} \frac{N^{\varepsilon} q_1^4 r N_0 L^{2/3} M^{5/2}}{n_1^3 M_1^{1/2} R^3} \sum_{d_1 \mid q_1} (d_1, n_1) \sum_{m\sim M_1} |\lambda_f(m')|^2 \sum_{q_3 \le R/q_1} q_3 \sum_{\substack{n_2\sim\tilde{N}_2\\ q_3 \mid n_2}} C(\tilde{N}_2) \cdot \frac{R^2 L^{1+\theta_3}}{q_1^2 q_3^2}\left(\frac{R}{q_1} + \frac{M_1 q_3}{(d_1, n_1)}\right)$$
$$\ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} \frac{q_1^3 r N^{\varepsilon} N_0 M^{5/2} M_1^{1/2} L^{5/3+\theta_3} \tilde{N}_2 C_1(\tilde{N}_2)}{n_1^3 R}(R + M_1). \tag{7-3}$$

Recall

$$Q = \left(\frac{NL}{MK}\right)^{1/2}, \quad N_0 = \frac{N^2 L^2 X^3 r}{Q^3}, \quad N_2 = \frac{Q^2 R n_1}{NLX^2 q_1} t^{\varepsilon},$$
$$N_2' = t^{\varepsilon}\left(\frac{NLn_1}{M^2 R t^2 q_1} + \frac{R^2 Q^3 M n_1}{N^2 L^2 X^3 q_1}\right), \quad N \ll \frac{(Mt)^{3+\varepsilon}}{r^2}. \tag{7-4}$$

For $NLX/(MRQ) \ll t^{1-\varepsilon}$, we have $M_1 \ll t^2 R^2 M^2/N$. By taking $L = M^{1/4}$ and $K = t^{4/5}$, one has $R + M_1 \ll t^2 M^2 RQ/N$. Hence, by applying these bounds into (7-2) and (7-3), we derive that

$$\Omega_{\boldsymbol{a},1} \ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} \frac{N^{\varepsilon} q_1^3 r N_0 M^{11/2} RQt^3 L^{8/3} \tilde{N}_2 C(\tilde{N}_2)}{n_1^3 N^{3/2}}.$$

Combining this together with (7-1) and (7-4), we get

$$\Omega_{\boldsymbol{a},1} \ll \frac{N^{\varepsilon} q_1^2 r^2}{n_1^2}\left(\frac{Q^3 L^{19/6} M^6 t^2}{N} + \frac{N^{3/2} L^{17/3} M^{7/2}}{Q^2} + \frac{Q^4 L^{8/3} M^{13/2} t^2}{N^{3/2}}\right).$$

For $NLX/(MRQ) \gg t^{1-\varepsilon}$, we have $M_1 \ll NL^2X^2/Q^2$ and $R \ll NLX/(MQt^{1-\varepsilon})$. Thus, in this case, we arrive at

$$\Omega_{\boldsymbol{a},1} \ll \frac{N^{\varepsilon} q_1^2 r^2}{n_1^2}\left(\frac{N^{5/2} L^{17/3} M^{3/2}}{Q^3 t^2} + \frac{N^{5/2} L^{20/3} M^{5/2}}{Q^4 t}\right).$$

Therefore, the contribution from $\Omega_{\boldsymbol{a},1}$ to (5-1) is

$$\ll \frac{N^{3/4+\varepsilon} X^{1/2}}{M^2 L^{4/3} Q^{1/2} r^{1/2}} \sum_{n_1 \ll RMr} n_1^{\theta_3} \sum_{(n_1/(n_1,r)) | q_1 | (rn_1)^\infty} \frac{r}{n_1 q_1^{1/2}} \cdot\left(\frac{Q^3 L^{19/6} M^6 t^2}{N} + \frac{N^{3/2} L^{17/3} M^{7/2}}{Q^2} + \frac{Q^4 L^{8/3} M^{13/2} t^2}{N^{3/2}} + \frac{N^{5/2} L^{17/3} M^{3/2}}{Q^3 t^2} + \frac{N^{5/2} L^{20/3} M^{5/2}}{Q^4 t}\right)^{1/2}$$
$$\ll r^{1/2} N^{3/4+\varepsilon} M^{1/2} L^{3/4}\left(\frac{t}{K^{1/2}} + K^{3/4} + \frac{K^{5/4}}{t^{1/2}}\right) + r^{1/2} N^{1+\varepsilon} \frac{L^{1/2} K}{M^{1/4} t}. \tag{7-5}$$

7A2. *Case b.* By the same arguments, we obtain

$$\Omega_{\boldsymbol{b},1} \ll \sup_{\substack{1\ll\tilde{N}_2\ll N_2\\ \text{dyadic}}} \frac{N^{\varepsilon} q_1^4 r N_0 M^{5/2} M_1^{1/2} L^{8/3} \tilde{N}_2 C(\tilde{N}_2)}{n_1^3 R^2}(R+M_1),$$

where $M_1 \ll M^2R^2t^{\varepsilon}/N$ and $R \asymp (NLX)/(MQt)$. So we see that

$$\Omega_{\boldsymbol{b},1} \ll \frac{N^{\varepsilon} q_1^3 r^2}{n_1^2}\left(\frac{N^{3/2}L^{14/3}M^{5/2}}{Q^2t^2} + \frac{N^{3/2}L^{17/3}M^{7/2}}{Q^3t^3}\right),$$

which contributes (5-8) at most

$$\ll \frac{N^{1+\varepsilon}X^{1/2}r^{1/2}}{M^{5/2}L^{4/3}Q^{1/2}} \sum_{n_1\ll RMr} \frac{1}{n_1^{1-\theta_3}} \sum_{(n_1/(n_1,r))|q_1|n_1^{\infty}} \frac{1}{q_1^{1/2}} \left(\frac{N^{3/2}L^{14/3}M^{5/2}}{Q^2t^2} + \frac{N^{3/2}L^{17/3}M^{7/2}}{Q^3t^3}\right)^{1/2}$$
$$\ll \frac{N^{1+\varepsilon}r^{1/2}L^{1/4}K^{3/4}}{M^{1/2}t} + \frac{N^{3/4+\varepsilon}r^{1/2}L^{1/2}M^{1/4}K}{t^{3/2}}. \tag{7-6}$$

7A3. *Case c.* Similarly, by the same treatment and the results in Section 5C, we have

$$\Omega_{\boldsymbol{c},1} \ll \frac{N^{\varepsilon} q_1^4 r N_0 M^{5/2} M_1^{1/2} L^3 \tilde{N}_2}{n_1^3 R^2}(R+M_1),$$

where $N_0 \ll (R^3M^3r/LN)t^{\varepsilon}$, $\tilde{N}_2 = (MrR^2n_1/q_1N_0)$ and $M_1 = R^2M^2t^2/N$. It is easy to see that

$$\Omega_{\boldsymbol{c},1} \ll \frac{q_1^3r^2}{n_1^2}\left(\frac{R^3M^{13/2}t^3L^3}{N^{3/2}} + \frac{R^2M^{9/2}tL^3}{N^{1/2}}\right).$$

Notice that $X \ll (MRQ/NL)t^{\varepsilon}$ now. Hence, the contribution from $\Omega_{\boldsymbol{c},1}$ to (5-9) is

$$\ll \frac{N^{5/4+\varepsilon}Xr^{1/2}}{M^{5/2}LQ}\frac{1}{t^{1/2}} \sum_{\eta_1=\pm1}\sum_{n_1\le Rr} \frac{1}{n_1^{1-\theta_3}} \sum_{(n_1/(n_1,r))\,|\,q_1\,|\,(rn_1)^{\infty}} \frac{1}{q_1^{1/2}} \left(\frac{R^3M^{13/2}t^3L^3}{N^{3/2}} + \frac{R^2M^{9/2}tL^3}{N^{1/2}}\right)^{1/2}$$
$$\ll \frac{r^{1/2}N^{3/4}M^{1/2}L^{3/4}t}{K^{5/4}} + \frac{r^{1/2}NL^{1/2}}{M^{1/4}K}. \tag{7-7}$$

7B. $n_2 \equiv 0 \pmod{M}$, $n_2 \neq 0$. Denote the contribution of this part to Ω by Ω_2. By the congruence condition $q_2'\bar{\alpha} - q_2\overline{\alpha'} \equiv -n_2 \pmod{M}$, we have $\alpha' \equiv \overline{q_2'}q_2\alpha \pmod{M}$. Hence,

$$\mathfrak{C}(n_2) \ll |\mathfrak{C}_1(n_2)||\mathfrak{C}_2(n_2)||\mathfrak{C}_3(n_2)|,$$

where $\mathfrak{C}_2(n_2)$ and $\mathfrak{C}_3(n_2)$ are defined as in Section 7A, and

$$\mathfrak{C}_1(n_2) = \sideset{}{^\star}\sum_{b \bmod M}\ \sideset{}{^\star}\sum_{b' \bmod M}\left(\sum_{\substack{u \bmod M\\ u\neq b}} \bar{\chi}(u)e\left(\frac{m\overline{q_2^2\ell(b-u)}}{M}\right)\right)$$
$$\cdot\left(\sum_{\substack{u' \bmod M\\ u'\neq b'}} \chi(u')e\left(-\frac{m'\overline{q_2'^2\ell'(b'-u')}}{M}\right)\right)\left(\sideset{}{^\star}\sum_{\alpha \bmod M} e\left(\frac{\alpha\overline{bq_2^2} - \alpha q_2\overline{b'q_2'^3}}{M}\right)\right)$$

Note that the innermost α-sum is a Ramanujan sum. We get

$$\mathfrak{C}_1(n_2) \ll M\bigg|\sideset{}{^\star}\sum_{b \bmod M}\bigg(\sum_{\substack{u \bmod M\\ u\neq b}}\bar{\chi}(u)e\bigg(\frac{\overline{nq_2^2\ell(b-u)}}{M}\bigg)\bigg)\bigg(\sum_{\substack{u' \bmod M\\ u'\neq bq_2^3\overline{q_2'}^3}}\chi(u')e\bigg(\frac{\overline{n'q_2'^2\ell'(bq_2^3\overline{q_2'}^3-u')}}{M}\bigg)\bigg)\bigg|$$
$$+\bigg|\sideset{}{^\star}\sum_{b \bmod M}\ \sideset{}{^\star}\sum_{\substack{b' \bmod M\\ b'\not\equiv bq_2^3\overline{q_2'}^3 \bmod M}}\bigg(\sum_{\substack{u \bmod M\\ u\neq b}}\bar{\chi}(u)e\bigg(\frac{\overline{nq_2^2\ell(b-u)}}{M}\bigg)\bigg)$$
$$\cdot\bigg(\sum_{\substack{u' \bmod M\\ u'\neq b'}}\chi(u')e\bigg(\frac{\overline{n'q_2'^2\ell'(b'-u')}}{M}\bigg)\bigg)\bigg|.$$

As in [Sharma 2022, Section 6.2], there is a square root cancellation in the sum over u and u', so we arrive at

$$\mathfrak{C}_1(n_2) \ll M^3.$$

Therefore, by the same treatment as in Section 7A together with the condition $n_2 \equiv 0 \pmod M$, we can get a better result than that in Section 7A.

Combining the above argument together with (7-5), (7-6) and (7-7), the contribution of the non-zero frequencies can be dominated by

$$\ll r^{1/2}N^{3/4+\varepsilon}M^{1/2}L^{3/4}\bigg(\frac{t}{K^{1/2}}+K^{3/4}+\frac{K^{5/4}}{t^{1/2}}\bigg)+\frac{r^{1/2}N^{1+\varepsilon}L^{1/2}}{M^{1/4}}\bigg(\frac{K}{t}+\frac{1}{K}\bigg). \tag{7-8}$$

8. Proof of Proposition 3.1

Now we are ready to give an upper bound for $S_{11}^{\pm}(N, X, R)$ when $(r, M) = 1$. By (6-9) and (7-8), we get

$$S_{11}^{\pm}(N, X, R) \ll r^{1/2}N^{1/2+\varepsilon}L^{1/2}M^{5/4}(K^{3/2}+t)+\frac{r^{1/2}N^{3/4+\varepsilon}M^{3/4}}{L^{1/4}}\bigg(K^{3/4}+\frac{K^{5/4}}{t^{1/2}}\bigg)$$
$$+r^{1/2}N^{3/4+\varepsilon}M^{1/2}L^{3/4}\bigg(\frac{t}{K^{1/2}}+K^{3/4}+\frac{K^{5/4}}{t^{1/2}}\bigg)+\frac{r^{1/2}N^{1+\varepsilon}L^{1/2}}{M^{1/4}}\bigg(\frac{K}{t}+\frac{1}{K}\bigg).$$

Noting that $N \ll (Mt)^{3+\varepsilon}/r^2$ and $r \ll M^{1/8}t^{3/10}$, and assuming $K < t$, we obtain

$$S_{11}^{\pm}(N, X, R) \ll N^{1/2+\varepsilon}\bigg(M^{1/16}t^{3/20}L^{1/2}M^{5/4}(K^{3/2}+t)+\frac{L^{1/2}}{M^{7/4}}\bigg(Kt^{1/2}+\frac{t^{3/2}}{K}\bigg)$$
$$+\frac{M^{3/2}}{L^{1/4}}t^{3/4}K^{3/4}+M^{5/4}L^{3/4}t^{3/4}\bigg(\frac{t}{K^{1/2}}+K^{3/4}\bigg)\bigg).$$

To balance the terms in the second line, the best choice of K is to satisfy $t/K^{1/2} = K^{3/4}$ and the best choice of L is to satisfy $M^{3/2}/L^{1/4} = M^{5/4}L^{3/4}$. Hence we should take

$$L = M^{1/4}, \quad K = t^{4/5},$$

from which we deduce that

$$S_{11}^{\pm}(N, X, R) \ll N^{1/2+\varepsilon} M^{3/2-1/16} t^{3/2-3/20}.$$

As we point out in Section 3, all the other cases (such as $S_{12}^{\pm}(N, X, R)$, $S_{13}^{\pm}(N, X, R)$, $S_2(N)$, $S_3(N)$) are similar and in fact easier. Hence, we finally prove Proposition 3.1.

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