Global dimension of real-exponent polynomial rings

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The ring R of real-exponent polynomials in n variables over any field has global dimension $n + 1$ and flat dimension n. In particular, the residue field $\Bbbk = R/\mathfrak{m}$ of R modulo its maximal graded ideal $\mathfrak{m}$ has flat dimension n via a Koszul-like resolution. Projective and flat resolutions of all R-modules are constructed from this resolution of $\Bbbk$. The same results hold when R is replaced by the monoid algebra for the positive cone of any subgroup of $\mathbb{R}^n$ satisfying a mild density condition.

1. Introduction

Overview. The aim of this note is to prove that the commutative ring R of real-exponent polynomials in n variables over any field $\Bbbk$ has global dimension $n + 1$ and flat dimension n (Theorem 3.6 and Corollary 2.10). It might be unexpected that R has finite global dimension at all, but it should be more expected that the flat dimension is achieved by the residue field $\Bbbk = R/\mathfrak{m}$ of R modulo its maximal graded ideal $\mathfrak{m}$; a Koszul-like construction shows that it is (Proposition 2.4 along with Example 2.5). In one real-exponent variable the residue field $\Bbbk$ also achieves the global dimension bound of 2 (Lemma 3.2), and this calculation lifts to n variables by tensoring with an ordinary Koszul complex (Proposition 3.4), demonstrating global dimension at least $n + 1$. Projective and flat resolutions of all R-modules are constructed from resolutions of the residue field in the proofs of Theorems 3.6 and 2.9 to yield the respective upper bounds of $n + 1$ and n. The results extend to the monoid algebra for the positive cone of any subgroup of $\mathbb{R}^n$ satisfying a mild density condition (Definition 4.1 and Theorem 4.3).

Background. Global dimension measures how long projective resolutions of modules can get, or how high the homological degree of a nonvanishing Ext module can be [20, Theorem 4.1.2]. Finding rings of finite global dimension is of particular value, since they are considered to be smooth, generalizing the best-known case of local noetherian commutative rings [2; 19], which correspond to germs of functions on nonsingular algebraic varieties.

The related notion of flat dimension (also called Tor dimension or weak global dimension) measures how long flat resolutions of modules can get, or how high the homological degree of a nonvanishing Tor module can be. Flat dimension is bounded by global dimension because projective modules are flat. These two dimensions agree for noetherian commutative rings [20, Proposition 4.1.5]. Without the

noetherian condition equality can fail; commutative examples include von Neumann regular rings that are infinite products of fields (see [20, page 98]), but domains are harder to come by.

The cardinality of a real-exponent polynomial ring a priori indicates a difference between flat and projective dimension that could be as high as 1 plus the index on $\aleph$ in the cardinality of the real numbers [17, page 14]. In certain situations, such as in valuation rings, ideals generated by $\aleph_n$ and no fewer elements are known to cause global dimension at least $n + 2$ [16]; see also [17, page 14]. But despite R having an ideal minimally generated by all monomials with total degree 1, of which there are $2^{\aleph_0}$, the dimension of the positive cone of exponents is more pertinent than its cardinality. This remains the case when the exponent set is intersected with a suitably dense subgroup of $\mathbb{R}^n$: the rank of the subgroup is irrelevant (Section 4).

Methods. The increase from global dimension n to $n + 1$ in the presence of n variables is powered by the violation of condition 5 from [3, Theorem P]: a monomial ideal with an "open orthant" of exponents, such as the maximal ideal $\mathfrak{m}_1$ in one indeterminate, is a direct limit of principal monomial ideals (Lemma 3.1) but is not projective (Lemma 3.2). This phenomenon occurs already for Laurent polynomials L_1 in one integer-exponent variable. But although $\mathfrak{m}_1$ and L_1 both have projective dimension 1, the real-exponent maximal ideal $\mathfrak{m}_1$ is a submodule of a projective (actually, free) module; the inclusion has a cokernel, and its projective dimension is greater by 1.

The most nontrivial point is how to produce a projective resolution of length at most $n + 1$ for any module over the real-exponent polynomial ring R in n variables. Our approach takes two steps. The first is a length n Koszul-like complex (Definition 2.7) in $2n$ variables that resolves the residue field and can be massaged into a flat resolution of any module (Theorem 2.9). This "total Koszul" construction was applied to combinatorially resolve monomial ideals in ordinary (that is, integer-exponent) polynomial rings [7, Section 6]. The integer grading in the noetherian case makes this construction produce a Koszul double complex, which is key for the combinatorial purpose of minimalizing the resulting free resolution by splitting an associated spectral sequence. It is not obvious whether the double complex survives to the real-exponent setting, but the total complex does (Definition 2.7; see [20, Application 4.5.6]), and that suffices here because minimality is much more subtle — if it is even possible — in the presence of real exponents [13].

Motivations. Beyond basic algebra, there has been increased focus on nonnoetherian settings in, for example, noncommutative geometry and topological data analysis.

Quantum noncommutative toric geometry [9] is based on dense finitely generated additive subgroups of $\mathbb{R}^n$ instead of the discrete sublattices that the noetherian commutative setting requires. The situations treated by our main theorems, including especially Section 4, correspond to "smooth" affine quantum toric varieties and could have consequences for sheaf theory in that setting.

The question of finite global dimension over real-exponent polynomial rings has surfaced in topological data analysis (TDA), where modules graded by $\mathbb{R}^n$ are known as real multiparameter persistent homology; see [6; 12; 13], for example, or [18] for a perspective from quiver theory. The question of global dimension

arises because defining metrics for statistical analysis requires distances between persistence modules, many of which use derived categorical constructions [4; 8; 15]; see [6, Section 7.1] for an explicit mention of the finite global dimension problem.

Real-exponent modules that are graded by $\mathbb{R}^n$ and satisfy a suitable finiteness condition ("tameness") to replace the too-easily violated noetherian or finitely presented conditions admit finite multigraded resolutions by monomial ideals [14, Theorem 6.12], which are useful for TDA. But even in the tame setting no universal bound is known for the finite lengths of such resolutions [13, Remark 13.15]. The global dimension calculations here suggest but do not immediately imply a universal bound of $n + 1$.

Notation. The ordered additive group $\mathbb{R}$ of real numbers has its monoid $\mathbb{R}_+$ of nonnegative elements. The n-fold product $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$ has nonnegative cone $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$. The monoid algebra $R = R_n = \Bbbk[\mathbb{R}_+^n]$ over an arbitrary field $\Bbbk$ is the ring of real-exponent polynomials in n variables: finite sums $\sum_{\boldsymbol{a} \in \mathbb{R}_+^n} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$, where $\boldsymbol{x}^{\boldsymbol{a}} = x_1^{a_1} \cdots x_n^{a_n}$. Its unique monoid-graded maximal ideal $\mathfrak{m}$ is spanned over $\Bbbk$ by all nonunit monomials.

Unadorned tensor products are over $\Bbbk$. For example, $R \cong R_1 \otimes \cdots \otimes R_1$ is an n-fold tensor product over $\Bbbk$, where $R_1 = \Bbbk[\mathbb{R}_+]$ is the real-exponent polynomial ring in one variable with graded maximal ideal $\mathfrak{m}_1$.

2. Flat dimension n

Lemma 2.1. *The filtered colimit $\varinjlim_{\varepsilon > 0}(R_1 \hookleftarrow \langle x^\varepsilon \rangle)$ of the inclusions of the principal ideals generated by x^ε for positive $\varepsilon \in \mathbb{R}$ is a flat resolution $\mathring{\Bbbk}_\bullet^1 : R_1 \hookleftarrow \mathfrak{m}_1$ of $\Bbbk$ over R_1.*

Proof. Colimits commute with homology so the colimit is a resolution. Filtered colimits of free modules are flat by Lazard's criterion [11], so the resolution is flat. □

Definition 2.2. The *open Koszul complex* is the tensor product $\mathring{\Bbbk}_\bullet^{\boldsymbol{x}} = \bigotimes_{i=1}^n \mathring{\Bbbk}_\bullet^1$ over the field $\Bbbk$ of n copies of the flat resolution in Lemma 2.1. The 2^n summands of $\mathring{\Bbbk}_\bullet^{\boldsymbol{x}}$, each a tensor product of j copies of R_1 and $n - j$ copies of $\mathfrak{m}_1$, are *orthant ideals*.

Example 2.3. The open Koszul complex in two real-exponent variables is depicted in Figure 2. From a geometric perspective, take the ordinary Koszul complex from Figure 1, replace the free modules with their continuous versions, and push the generators as close to the origin as possible without meeting it. The four possible orthant ideals are rendered in Figure 2. From left to right, viewing them as tensor products, they correspond to the product of two closed rays $\Bbbk[\mathbb{R}_+]$, the product (in both orders) of a closed ray with an open ray $\mathfrak{m}$, and the product of two open rays. In n real-exponent variables the 2^n orthant ideals arise from all n-fold tensor products of closed and open rays.
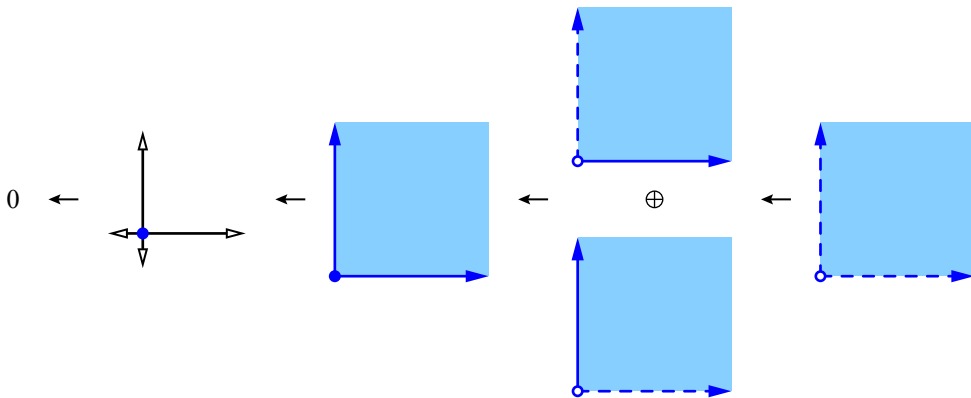
Proposition 2.4. *The open Koszul complex $\mathring{\Bbbk}_\bullet^{\boldsymbol{x}}$ is a flat resolution of $\Bbbk$ over R.*

Proof. Lemma 2.1 and the Künneth theorem [20, Theorem 3.6.3]. □

Limit-Koszul complexes similar to $\mathring{\Bbbk}_\bullet^{\boldsymbol{x}}$ have previously been used to compute flat dimensions of absolute integral closures [1] in the context of tight closure.
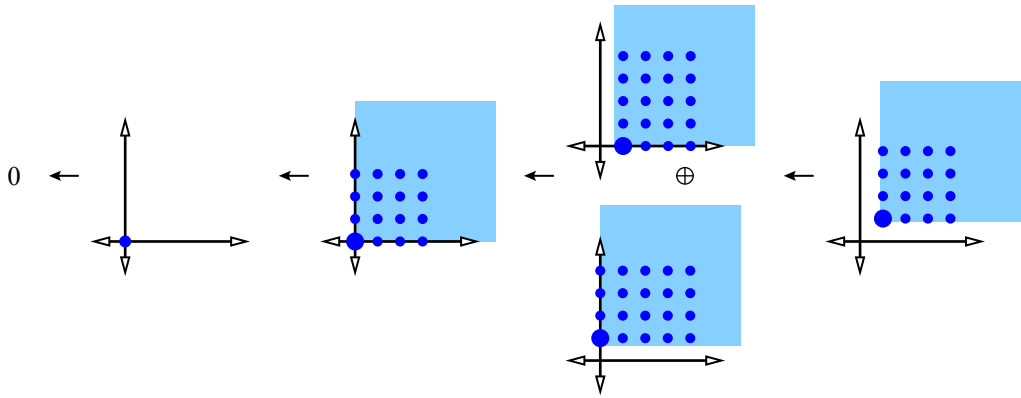
Figure 1. Ordinary Koszul complex in two variables.

Figure 2. Open Koszul complex in two real-exponent variables.

Example 2.5. The sequence $\boldsymbol{x}^{[\varepsilon]} = x_1^\varepsilon, \ldots, x_n^\varepsilon$ is regular in R [5, Chapter 1], so the usual Koszul complex $\mathbb{K}_\bullet(\boldsymbol{x}^{[\varepsilon]})$ is a length n free resolution of $B_n^\varepsilon = R/\langle \boldsymbol{x}^{[\varepsilon]} \rangle$ over R. Using this resolution, $\mathrm{Tor}_n^R(\Bbbk, B_n^\varepsilon) = \Bbbk$ because $\Bbbk \otimes_R \mathbb{K}_\bullet(\boldsymbol{x}^{[\varepsilon]})$ has vanishing differentials.

Lemma 2.6. *The real-exponent polynomial ring $R^{\otimes 2} = R \otimes R$ has $2n$ variables*

$$\boldsymbol{x} = x_1, \ldots, x_n = x_1 \otimes 1, \ldots, x_n \otimes 1 \quad and \quad \boldsymbol{y} = y_1, \ldots, y_n = 1 \otimes x_1, \ldots, 1 \otimes x_n.$$

Over $R^{\otimes 2}$ is a directed system of Koszul complexes $\mathbb{K}_\bullet(\boldsymbol{x}^{[\varepsilon]} - \boldsymbol{y}^{[\varepsilon]})$ on the sequences

$$\boldsymbol{x}^{[\varepsilon]} - \boldsymbol{y}^{[\varepsilon]} = x_1^\varepsilon - y_1^\varepsilon, \ldots, x_n^\varepsilon - y_n^\varepsilon$$

with $\varepsilon > 0$. The colimit $\mathring{\mathbb{K}}_\bullet^{\boldsymbol{x}-\boldsymbol{y}} = \varinjlim_{\varepsilon > 0} \mathbb{K}_\bullet(\boldsymbol{x}^{[\varepsilon]} - \boldsymbol{y}^{[\varepsilon]})$ is an $R^{\otimes 2}$-flat resolution of R.

Proof. The general case is the tensor product over $\Bbbk$ of n copies of the case $n = 1$, which in turn reduces to the calculation $R^{\otimes 2}/\langle x^\varepsilon - y^\varepsilon \mid \varepsilon > 0 \rangle \cong R$. $\qquad\square$

Definition 2.7. Denote by R^x and R^y the copies of R embedded in $R^{\otimes 2}$ as $R \otimes 1$ and $1 \otimes R$. Fix an R^x-module M:

(1) Write M^y for the corresponding R^y-module, with the x variables renamed to y.

(2) The *open total Koszul complex* of an R^x-module M is $\mathring{\mathbb{K}}_\bullet^{x-y}(M) = \mathring{\mathbb{K}}_\bullet^{x-y} \otimes_{R^y} M^y$.

Remark 2.8. By [Definition 2.2](), each of the 4^n summands of $\mathring{\mathbb{K}}_\bullet^{x-y}$ in [Lemma 2.6]() is the tensor product over $\Bbbk$ of an orthant R^x-ideal and an orthant R^y-ideal.

Theorem 2.9. *The open total Koszul complex $\mathring{\mathbb{K}}_\bullet^{x-y}(M)$ is a length n resolution of M over $R^{\otimes 2}$ for any R^x-module M. This resolution is flat over R^x; more precisely, as an R^x-module $\mathring{\mathbb{K}}_\bullet^{x-y}(M)$ is a direct sum of orthant R^x-ideals.*

Proof. The tensor product $\mathring{\mathbb{K}}_\bullet^{x-y} \otimes_{R^y} M^y$ is over R^y and hence converts the orthant R^x-ideal decomposition for $\mathring{\mathbb{K}}_\bullet^{x-y}$ afforded by [Remark 2.8]() into one for $\mathring{\mathbb{K}}_\bullet^{x-y}(M)$.

Since tensor products commute with colimits, $\mathring{\mathbb{K}}_\bullet^{x-y}(M) = \varinjlim_{\varepsilon>0} \mathbb{K}_\bullet^\varepsilon(M)$, where $\mathbb{K}_\bullet^\varepsilon(M) = \mathbb{K}_\bullet(x^{[\varepsilon]} - y^{[\varepsilon]}) \otimes_{R^y} M^y$. Each complex $\mathbb{K}_\bullet^\varepsilon(M)$ is the ordinary Koszul complex of the sequence $x^{[\varepsilon]} - y^{[\varepsilon]}$ on the $R^{\otimes 2}$-module $R^{\otimes 2} \otimes_{R^y} M^y$. But $x^{[\varepsilon]} - y^{[\varepsilon]}$ is a regular sequence on this module because the x variables are algebraically independent from the y variables. Thus $\mathbb{K}_\bullet^\varepsilon(M)$ is acyclic by exactness of colimits. Moreover, again by algebraic independence, the nonzero homology of $\mathbb{K}_\bullet^\varepsilon(M)$ is naturally the R^y-module M^y, with an action of $\Bbbk[x^{[\varepsilon]}]$ where x_i^ε acts the same way as y_i^ε due to the relation $x_i^\varepsilon - y_i^\varepsilon$. $\square$

Corollary 2.10. *The n-variable real-exponent polynomial ring has flat dimension n.*

Proof. [Example 2.5]() implies that fl.dim $R \geq n$, and fl.dim $R \leq n$ by [Theorem 2.9](). $\square$

3. Global dimension $n+1$

Lemma 3.1. *Fix an orthant ideal $\mathcal{O} \neq R$. Choose a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ such that $\varepsilon_k = (\varepsilon_{1k}, \dots, \varepsilon_{nk}) \in \mathbb{R}_+^n$ has*

- $\varepsilon_{ik} = 0$ *for all k if the i-th factor of $\mathcal{O}$ is R_1 and*

- $\{\varepsilon_{ik}\}_{k \in \mathbb{N}}$ *strictly decreases with limit 0 if the i-th factor of $\mathcal{O}$ is $\mathfrak{m}_1$.*

Let $F = \bigoplus_k \langle x^{\varepsilon_k} \rangle$ be the direct sum of the principal ideals in R generated by the monomials with degrees ε_k. Each summand $\langle x^{\varepsilon_k} \rangle$ is free with basis vector 1_k, and $\mathcal{O}$ has a free resolution $0 \leftarrow F \leftarrow F \leftarrow 0$ whose differential sends $1_k \in \langle x^{\varepsilon_k} \rangle$ to $1_k - x^{\varepsilon_k - \varepsilon_{k+1}} 1_{k+1}$.

Proof. The augmentation map $\mathcal{O} \xleftarrow{\alpha} F$ sends 1_k to x^{ε_k}. It is surjective by definition of $\mathcal{O}$. Since α is graded by the monoid $\mathbb{R}_+^n$, its kernel can be calculated degree by degree. In degree $a \in \mathbb{R}_+$ the kernel is spanned by all differences $x^{a - \varepsilon_k} 1_k - x^{a - \varepsilon_\ell} 1_\ell$ such that ε_k and ε_ℓ both weakly precede a; indeed, this subspace of the a-graded component F_a has codimension 1, and it is contained in the kernel because $x^{a - \varepsilon_k} x^{\varepsilon_k} = x^{a - \varepsilon_\ell} x^{\varepsilon_\ell}$. The differential is injective because each element $f \in F$ has nonzero coefficient on a basis vector 1_k with k maximal, and the image of f has nonzero coefficient on 1_{k+1}. $\square$

Lemma 3.2. $\Bbbk = R_1/\mathfrak{m}_1$ *has a free resolution of length* 2, *and* $\mathrm{Ext}^2_{R_1}(\Bbbk, F) \neq 0$.

Proof. The resolution of $\mathfrak{m}_1$ over R_1 in Lemma 3.1 (with $n = 1$) can be augmented and composed with the inclusion $R_1 \hookleftarrow \mathfrak{m}_1$ to yield a free resolution of $\Bbbk$ over R_1. The long exact sequence from $0 \leftarrow \Bbbk \leftarrow R_1 \leftarrow \mathfrak{m}_1 \leftarrow 0$ implies that $\mathrm{Ext}^{i+1}_{R_1}(\Bbbk, -) \cong \mathrm{Ext}^i_{R_1}(\mathfrak{m}_1, -)$ for $i \geq 1$. Now apply $\mathrm{Hom}(\mathfrak{m}_1, -)$ to the exact sequence $0 \to F \to F \to \mathfrak{m}_1 \to 0$. The first few terms are $0 \to \mathrm{Hom}(\mathfrak{m}_1, F) \to \mathrm{Hom}(\mathfrak{m}_1, F) \to R_1 \to \mathrm{Ext}^1(\mathfrak{m}_1, F)$. The image of $\mathrm{Hom}(\mathfrak{m}_1, F) \to R_1$ is $\mathfrak{m}_1$, so $\Bbbk \hookrightarrow \mathrm{Ext}^1(\mathfrak{m}_1, F) \cong \mathrm{Ext}^2(\Bbbk, F)$ is nonzero. $\square$

Remark 3.3. Any ideal that is a countable (but not finite) union of a chain of principal ideals has projective dimension 1 [17, page 14]. But it is convenient to have an explicit free resolution of $\mathfrak{m}_1$ over R_1, and it is no extra work to resolve all orthant ideals.

Proposition 3.4. *Set* $\mathfrak{m}_1 = \langle x_n^\varepsilon \mid \varepsilon > 0 \rangle$ *and* $J = \langle x_1, \dots, x_{n-1} \rangle \subseteq R$. *Using* $x = x_n$ *for* R_1, *consider the* R_1-*module* F *in Lemma 3.2 with* $n = 1$ *as an* R-*module via* $R \twoheadrightarrow R_1$, *where* $x_i^\varepsilon \mapsto 0$ *for all* $\varepsilon > 0$ *and* $i \leq n - 1$. *Then* $\mathrm{Ext}^{n+1}_R(R/I, F) \neq 0$ *when* $I = J + \mathfrak{m}_1$.

Proof. Let $\mathbb{F}_\bullet : 0 \leftarrow R_1 \leftarrow F \leftarrow F \leftarrow 0$ be the R_1-free resolution of $\Bbbk$ obtained by augmenting the resolution of $\mathfrak{m}_1$ in Lemma 3.1 with $n = 1$. Let $\mathbb{K}_\bullet = \mathbb{K}^{R_{n-1}}_\bullet(\boldsymbol{x}_{n-1})$ be the ordinary Koszul complex over R_{n-1} on the sequence $\boldsymbol{x}_{n-1} = x_1, \dots, x_{n-1}$, which is a free resolution of $R_{n-1}/\boldsymbol{x}_{n-1}R_{n-1}$ over R_{n-1}. Then $\mathrm{Tot}(\mathbb{F}_\bullet \otimes_\Bbbk \mathbb{K}_\bullet)$ is a free resolution of R/I over R. On the other hand,

$$\mathbb{F}_\bullet \otimes_\Bbbk \mathbb{K}_\bullet \cong \mathbb{F}_\bullet \otimes_{R_1} R_1 \otimes_\Bbbk R_{n-1} \otimes_{R_{n-1}} \mathbb{K}_\bullet$$
$$\cong \mathbb{F}_\bullet \otimes_{R_1} R \otimes_{R_{n-1}} \mathbb{K}_\bullet$$
$$\cong \mathbb{F}_\bullet \otimes_{R_1} R \otimes_R R \otimes_{R_{n-1}} \mathbb{K}_\bullet$$
$$= \mathbb{F}^R_\bullet \otimes_R \mathbb{K}^R_\bullet,$$

where $\mathbb{F}^R_\bullet = \mathbb{F}_\bullet \otimes_{R_1} R$ is an R-free resolution of $R/\mathfrak{m}_1 R$ and the ordinary Koszul complex $\mathbb{K}^R_\bullet = R \otimes_{R_{n-1}} \mathbb{K}_\bullet = \mathbb{K}^R_\bullet(\boldsymbol{x}_{n-1})$ of the sequence $\boldsymbol{x}_{n-1}$ in R is an R-free resolution of R/J.

Using $(-)^*$ to denote the free dual $\mathrm{Hom}_R(-, R)$, compute

$$\mathrm{Hom}_R(\mathbb{F}^R_\bullet \otimes_R \mathbb{K}^R_\bullet, F) \cong \mathrm{Hom}_R(\mathbb{F}^R_\bullet, \mathrm{Hom}_R(\mathbb{K}^R_\bullet, F))$$
$$\cong \mathrm{Hom}_R(\mathbb{F}^R_\bullet, (\mathbb{K}^R_\bullet)^* \otimes_R F)$$
$$\cong \mathrm{Hom}_R(\mathbb{F}^R_\bullet, (\mathbb{K}^R_\bullet)^* \otimes_R R_1 \otimes_{R_1} F), \qquad (3\text{-}1)$$

where the bottom isomorphism is because the R-action on F factors through R_1. The differentials of the complex $(\mathbb{K}^R_\bullet)^* \otimes_R R_1 \cong (\mathbb{K}^R_\bullet)^* \otimes_{R_{n-1}} \Bbbk$ all vanish, and this complex has cohomology $R_1^{\binom{n-1}{q}}$ in degree q.

Hence the total complex of Equation (3-1) has homology

$$\operatorname{Ext}^i_R(R/I, F) \cong \bigoplus_{p+q=i} H_p \operatorname{Hom}_R(\mathbb{F}^R_\bullet, F^{\binom{n-1}{q}})$$

$$\cong \bigoplus_{p+q=i} H_p \operatorname{Hom}_{R_1}(\mathbb{F}_\bullet, F^{\binom{n-1}{q}})$$

$$\cong \bigoplus_{p+q=i} \operatorname{Ext}^p_{R_1}(\Bbbk, F^{\binom{n-1}{q}}),$$

where the middle isomorphism is again because the R-action on F factors through R_1. Taking $p = 2$ and $q = n - 1$ yields the nonvanishing by Lemma 3.2. $\qquad\square$

Remark 3.5. The proof of Proposition 3.4 is essentially a Grothendieck spectral sequence for the derived functors of the composite $\operatorname{Hom}_{R_1}(\Bbbk, -) \circ \operatorname{Hom}_{R_{n-1}}(R_{n-1}/\boldsymbol{x}_{n-1}, -)$, but the elementary Koszul argument isn't more lengthy than verifying the hypotheses.

Theorem 3.6. *The n-variable real-exponent polynomial ring has global dimension $n + 1$.*

Proof. Proposition 3.4 yields the lower bound gl.dim $R \geq n + 1$. For the opposite bound, given any R-module M, each module in the length n flat resolution from Theorem 2.9 has a free resolution of length at most 1 by Lemma 3.1. By the comparison theorem for projective resolutions [20, Theorem 2.2.6], the differentials of the flat resolution lift to chain maps of these free resolutions. The total complex of the resulting double complex has length at most $n + 1$. $\qquad\square$

Remark 3.7. As an $\mathbb{R}^n$-graded module, the quotient R/I in Proposition 3.4 is nonzero only in degrees from $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$. Hence R/I is ephemeral [4], meaning, more or less, that its set of nonzero degrees has measure 0. The projective dimension exceeding n is not due solely to this ephemerality. Indeed, multiplication by x_n^1 induces an inclusion of R/I into R/I' for $I' = \langle x_1, \ldots, x_{n-1} \rangle + \langle x_n^\varepsilon \mid \varepsilon > 1 \rangle$, which is supported on a unit cube in $\mathbb{R}^n_+$ that is neither open nor closed. Theorem 3.6 implies that $\operatorname{Ext}^{n+1}_R(R/I', N) \twoheadrightarrow \operatorname{Ext}^{n+1}_R(x_n R/I, N)$ is surjective for all modules N, so R/I' has projective dimension $n + 1$. On the other hand, it could be the closed right endpoints [10] — that is, closed socle elements [13, Section 4.1] — that cause the problem. Thus it could be that sheaves in the conic topology ("γ-topology"; see [4; 8; 15]) have consistently lower projective dimensions.

4. Dense exponent sets

The results in Sections 2 and 3 extend to monoid algebras for positive cones of subgroups of $\mathbb{R}^n$ satisfying a mild density condition. Applications to noncommutative toric geometry should require restriction to subgroups of this sort.

Definition 4.1. Let $G \subseteq \mathbb{R}^n$ be a subgroup whose intersection with each coordinate ray ρ of $\mathbb{R}^n$ is dense. Write $G_+ = G \cap \mathbb{R}^n_+$ for the positive cone in G, set $\mathring{\rho} = \rho \cap \mathbb{R}^n_+ \smallsetminus \{\boldsymbol{0}\}$, and let $\mathring{G}_+ = \prod_\rho G \cap \mathring{\rho}_+$ be the set of points in G whose projections to all coordinate rays are strictly positive and still lie in G. Set

$R_G = \Bbbk[G_+]$, the monoid algebra of G_+ over $\Bbbk$. Let R_G^x and R_G^y be the copies of R_G embedded in $R_G^{\otimes 2}$ as $R_G \otimes 1$ and $1 \otimes R_G$. For $\boldsymbol{\varepsilon} \in \mathring{G}_+$ let $\boldsymbol{x}^{[\boldsymbol{\varepsilon}]} = x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n}$ be the corresponding sequence of elements in R_G.

(1) The *open Koszul complex* over R_G is the colimit $\mathring{\Bbbk}_\bullet^x = \varinjlim_{\boldsymbol{\varepsilon} \in \mathring{G}_+} \Bbbk_\bullet(\boldsymbol{x}^{[\boldsymbol{\varepsilon}]})$.

(2) Fix an R_G^x-module M. Write M^y for the corresponding R_G^y-module, with the $\boldsymbol{x}$ variables renamed to $\boldsymbol{y}$. With notation for variables as in Lemma 2.6, the *open total Koszul complex* of M is the colimit
$\mathring{\Bbbk}_\bullet^{x-y}(M) = \varinjlim_{\boldsymbol{\varepsilon} \in \mathring{G}_+} \Bbbk_\bullet(\boldsymbol{x}^{[\boldsymbol{\varepsilon}]} - \boldsymbol{y}^{[\boldsymbol{\varepsilon}]}) \otimes_{R^y} M^y$.

(3) Given a subset $\sigma \subseteq \{1, \ldots, n\}$, the *orthant ideal* $I_\sigma \subseteq R_G$ is generated by all monomials $\boldsymbol{x}^{\boldsymbol{\varepsilon}}$ for $\boldsymbol{\varepsilon} \in G_+$ such that $\varepsilon_i > 0$ for all $i \in \sigma$.

Example 4.2. Let G be generated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ e \end{bmatrix}$ as a subgroup of $\mathbb{R}^2$, so G consists of the integer linear combinations of these four vectors. The intersection $G \cap \rho^y$ with the y-axis ρ^y arises from integer coefficients α, β, γ, and δ such that

$$\begin{bmatrix} 0 \\ y \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} \pi \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ e \end{bmatrix}.$$

This occurs precisely when $2\alpha + \pi\beta + \gamma = 0$, and in that case $y = \gamma + \delta e$. Since π is irrational it is linearly independent from 1 over the integers, so $\beta = 0$ and hence $\gamma = -2\alpha$ is always an even integer. Since e is irrational, the only integer points in $G \cap \rho^y$ have even y-coordinate:

$$G \cap \rho^y = \left\langle \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ e \end{bmatrix} \right\rangle.$$

The point $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in G$ has strictly positive projection to ρ^y, but that projection lands outside of G. Hence $\mathring{G}_+ = G \cap \mathring{\rho}_+^x \times G \cap \mathring{\rho}_+^y$ is a proper subgroup of G, given the strictly positive point $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in G_+ \smallsetminus \mathring{G}_+$. Nonetheless, $\mathring{G}_+$ contains positive real multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ approaching the origin, which is all the colimit in the proof of Theorem 4.3 requires.

Theorem 4.3. *If a subgroup $G \subseteq \mathbb{R}^n$ is dense in every coordinate subspace of $\mathbb{R}^n$ as in Definition 4.1, then Theorem 2.9 holds verbatim with $R_G = \Bbbk[G \cap \mathbb{R}_+^n]$ in place of R. Consequently, the ring R_G has flat dimension n and global dimension $n + 1$.*

Proof. For $\sigma \subseteq \{1, \ldots, n\}$ and $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ let $\boldsymbol{\varepsilon}_\sigma \in \mathbb{R}^n$ be the restriction of $\boldsymbol{\varepsilon}$ to σ, so $\boldsymbol{\varepsilon}_\sigma$ has entry 0 in the coordinate indexed by every $j \notin \sigma$. The 2^n summands of $\mathring{\Bbbk}_\bullet^x$ are orthant ideals because $\Bbbk_i(\boldsymbol{x}^{[\boldsymbol{\varepsilon}]}) \cong \bigoplus_{|\sigma|=i} \langle \boldsymbol{x}^{\boldsymbol{\varepsilon}_\sigma} \rangle$ naturally with respect to the inclusions induced by the colimit defining $\mathring{\Bbbk}_\bullet^x$. Each orthant ideal is flat because this colimit is filtered: given two vectors $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathring{G}_+$, the coordinatewise minimum $\boldsymbol{\varepsilon}_1 \wedge \boldsymbol{\varepsilon}_2 \in \mathbb{R}_+^n$ lies in $\mathring{G}_+$ because its projection to each ray lies in G. Proposition 2.4 therefore generalizes to R_G by the exactness of colimits and the cokernel calculation $\Bbbk = R_G/\mathfrak{m}$ for the G-graded maximal ideal $\mathfrak{m} = \langle \boldsymbol{x}^{\boldsymbol{\varepsilon}} \mid \boldsymbol{0} \neq \boldsymbol{\varepsilon} \in G_+ \rangle$. Example 2.5 generalizes with no additional work. Lemma 2.6 generalizes by exactness of colimits and the cokernel calculation $R_G \cong R_G^{\otimes 2}/\langle \boldsymbol{x}^{[\boldsymbol{\varepsilon}]} - \boldsymbol{y}^{[\boldsymbol{\varepsilon}]} \mid \boldsymbol{0} \neq \boldsymbol{\varepsilon} \in G_+ \rangle$. The conclusion of Remark 2.8 generalizes, but the reason is direct calculation of $\mathring{\Bbbk}_\bullet(\boldsymbol{x}^{[\boldsymbol{\varepsilon}]} - \boldsymbol{y}^{[\boldsymbol{\varepsilon}]})$ as was done for $\mathring{\Bbbk}_i^x$.

The original proof of Theorem 2.9 uses that tensor products commute with colimits, but the generalized proof avoids that argument by simply defining $\mathbb{K}_\bullet^{\circ x-y}$ as the relevant colimit. The rest of the proof and the generalization of the flat dimension claim in Corollary 2.10 work mutatis mutandis, given the strengthened versions of the results they cite.

The orthant ideal resolution in Lemma 3.1 generalizes to R_G by the density hypothesis, including specifically the part about intersecting with coordinate subspaces. The Ext calculation in Lemma 3.2 works again by density of the exponent set of $\mathfrak{m}_1$ in $\mathbb{R}_+$. The statement and proof of Proposition 3.4 work mutatis mutandis for R_G in place of R as long as the power of x_i generating J lies in the intersection of G with the corresponding coordinate ray of $\mathbb{R}^n$. The proof of Theorem 3.6 then works verbatim, given the strengthened versions of the results it cites. $\qquad\square$

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