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On self-correspondences on curves

Joël Bellaïche

À la toute fin du processus de révision, le comité éditorial d'Algebra & Number Theory a eu l'immense tristesse d'apprendre le décès de Joël Bellaïche. Joël faisait des maths comme il respirait, avec une immense joie. Les mathématiques n'étaient pas pour lui séparées de la vie, elles étaient la vie même — la vie vraiment vivante.

At the very end of the reviewing process, the editorial board of Algebra & Number Theory learned with great sadness that Joël Bellaïche had passed away. Joël did math as he breathed, with immense joy. For him mathematics was not disconnected from life; it was life itself, and his way of living it to the fullest.

We study the algebraic dynamics of self-correspondences on a curve. A self-correspondence on a (proper and smooth) curve C over an algebraically closed field is the data of another curve D and two nonconstant separable morphisms π_1 and π_2 from D to C. A subset S of C is *complete* if $\pi_1^{-1}(S)=\pi_2^{-1}(S)$. We show that self-correspondences are divided into two classes: those that have only finitely many finite complete sets, and those for which C is a union of finite complete sets. The latter ones are called *finitary*, and happen only when $\deg \pi_1 = \deg \pi_2$ and have a trivial dynamics. For a nonfinitary self-correspondence in characteristic zero, we give a sharp bound for the number of étale finite complete sets.

Introduction

Let k be a field, and let C a be smooth, proper and geometrically irreducible curve over k. By a *self-correspondence* on C (defined over k),[1] we mean the data of a smooth and proper scheme D over k, such that every connected component of D is a geometrically irreducible curve, and two k-morphisms

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[1]We adopt the definition of [Bullett and Penrose 2001] and [Krishnamoorthy 2018]. In part of the literature, a self-correspondence is defined instead as a divisor in the surface $C \times C$. The two notions are equivalent. To get a divisor of $C \times C$ using our definition, take $(\pi_1 \times \pi_2)(D)$, with multiplicities if several components of D have the same image in $C \times C$. To get from a divisor Δ of $C \times C$ a self-correspondence according to our definition, take the union of the normalization of each component of Δ repeated according to multiplicity. Our definition makes clearer the concepts of *étale* or *equiramified* complete sets, which are central in our study.

π_1 and π_2 from D to C, nonconstant and separable on every connected component of D. We denote by (D, π_1, π_2), or often simply by D, that self-correspondence.

Fixing an algebraic closure $\bar{k}$ of k, the intuitive way to think of a self-correspondence is as a mulitvalued map from $C(\bar{k})$ to itself defined by polynomial equations with coefficients in k, namely the map $x \mapsto \pi_2(\pi_1^{-1}(x))$. We call this mulitvalued map the *forward map* of the self-correspondence D.

Self-correspondences generalize endomorphisms: given an endomorphism f of a curve C, one can think of it as the self-correspondence $D_f := (C, \mathrm{Id}_C, f)$. Like endomorphisms, two self-correspondences D and D' on C can be composed into a self-correspondence DD' (see Section 1.10) whose forward map is the composition of the (mulitvalued) forward maps of D and D'. Thus, like endomorphisms, a self-correspondence can be iterated. Better than endomorphisms, a self-correspondence $D = (D, \pi_1, \pi_2)$ always has a *transpose* (which plays in part the role of an inverse), denoted by ${}^tD$ and defined by ${}^tD = (D, \pi_2, \pi_1)$. The forward-map of ${}^tD$, namely $x \mapsto \pi_1(\pi_2^{-1}(x))$ is called the *backward map* of D.

Let us introduce our fundamental terminology. A *forward-complete* (resp. *backward-complete*) set is a subset S of $C(\bar{k})$ that is stable by the forward (resp. backward) map (i.e., $\pi_1^{-1}(S) \subset \pi_2^{-1}(S)$, resp. $\pi_2^{-1}(S) \subset \pi_1^{-1}(S)$), and a *complete* set is a set which is both backward and forward-complete. An *irreducible complete set* is a minimal nonempty complete set.

The aim of this article is to answer elementary questions about finite complete sets for self-correspondences, such as when can there be infinitely many finite complete sets? This is a basic and fundamental question on the dynamics of self-correspondence. There is a relatively extensive literature on the subject of dynamics of self-correspondences on curves and even, less often, algebraic varieties. Most of this literature is concerned about correspondences over the complex numbers, see for instance [Fatou 1922; Bullett 1988; 1991; 1992; Bullett and Penrose 1994; 2001; Bullett and Lomonaco 2020; Bharali and Sridharan 2016; Pakovich 1996; 1995; 2008; Dinh 2002] (on $\mathbb{P}^1$), [Dinh 2005] (on $\mathbb{P}^k$), [Dinh and Sibony 2006] (on general varieties), [Dinh et al. 2020] (on general curves), but also over number fields; see [Autissier 2004; Ingram 2017; 2019], finite fields [Hallouin and Perret 2014] and general fields [Truong 2020]. To the best of our knowledge, the question we have in mind has not been solved, and not even asked, except in some very particular cases by Pakovich (see [Pakovich 1995] and Remark 3.3.7 below). A partial exception is the recent article of Krishnamoorthy [2018], essentially the second chapter of his PhD thesis at Columbia University. Though the focus of the paper is different, as Krishnamoorthy is concerned with general correspondences rather than self-correspondences, we borrow several ideas and concepts from him, and we gladly acknowledge our debt to his work.

Our first main result concerning finite complete sets is the following (see Theorem 2.2.1).

Theorem 1. *A self-correspondence (D, π_1, π_2) on a curve C over k has infinitely many finite complete sets if and only if there exists a nonconstant k-morphism*

$$f : C \to \mathbb{P}^1_k$$

such that

$$f \circ \pi_1 = f \circ \pi_2.$$

The method of proof of Theorem 1 is number-theoretic. Specifically, we use the famous theorem of Mordell, Weil and Néron asserting that the group of rational points of an abelian variety over a finitely generated field is a finitely generated abelian group. Theorem 1 seems difficult to prove by purely algebro-geometric methods or by complex-analytic ones in the case $k = \mathbb{C}$.

As a trivial consequence of Theorem 1, one sees that as soon as a self-correspondence has infinitely many finite complete sets, then in fact all its irreducible complete sets are finite, and moreover they have cardinality bounded by some integer M. A self-correspondence satisfying this property will be said *finitary*; its dynamical study is essentially trivial, in the sense that it reduces to dynamical questions over finite sets.

It is clear that only self-correspondences for which $\deg \pi_1 = \deg \pi_2$ (such a self-correspondence is said to be *balanced*) can be finitary. This explains why the notion of being finitary does not appear in the classical theory of complex dynamics, where one consider endomorphisms of $\mathbb{P}^1$, which as correspondences have $(\deg \pi_1, \deg \pi_2) = (1, d)$, the case $d = 1$ being excluded as trivial. Even among balanced self-correspondences, the notion of finitary self-correspondence is very restrictive. They are in practice exceptions to all interesting general statements about the dynamics of self-correspondences. Nonfinitary self-correspondences are the natural domain of study of the dynamics of self-correspondences. For instance, an interesting result on the existence of a canonical invariant measure for a balanced self-correspondence on a curve over $\mathbb{C}$ has recently been proved by Dinh, Kaufmann and Wu [Dinh et al. 2020] but only under a quite restrictive condition on the self-correspondence (see Remark 2.2.6). We believe that their main result (that the iterated pull-back of every smooth measure converges to the canonical measure) holds for all nonfinitary correspondences, and we plan to come back to this question in a subsequent work.

For a nonbalanced correspondence, there are no étale finite complete set, and finitely many nonétale ones. Moreover one can give an upper bound (in terms of the genera and degrees of the curves and maps involved) on the size of their union (see Proposition 2.1.1).

Balanced nonfinitary correspondences are much more subtle: they may have (finitely many) étale and nonétale finite complete sets; it is easy to give a bound on the number of finite nonétale complete sets, but we do not know how to bound their size. Our main objective is to bound the number of finite étale complete sets. With methods similar to those of the proof of Theorem 1, we are able to offer a bound (which happens to be optimal) only in some specific cases: when k is algebraic over a finite field (see Proposition 3.2.1); when k is arbitrary but $C = \mathbb{P}^1_k$ (see Proposition 3.2.3); and two other results when D is symmetric, that is $D \simeq {}^tD$ (see Propositions 3.2.5 and 3.3.3). But for more general results we need different, operator-theoretic, methods.

A self-correspondence D over C defines in a natural way a k-linear endomorphism T_D of the field of rational functions $k(C)$ of C, see Section 4.1. Whenever S is a forward-complete set, T_D stabilizes the subring B_S of $k(C)$ of functions whose all poles are in S. If moreover π_1 is étale on S, T_D stabilizes as well the natural filtration $(B_{S,n})_{n\geq 0}$, "by the order of the poles" of B_S.

The dynamical study of that action of T_D on the filtered ring B_S, when S is in particular étale complete, was the original motivation of this work. In fact, Hecke operators appearing in the theory of modular forms are of this type. Surprising results concerning the dynamics of the operators T_D for self-correspondences over finite fields have been obtained in a recent work by Medvedovsky [2018], and, applied to Hecke operators, those results provide a new and elementary proof of certain deep modularity result of Gouvêa and Mazur [1998]. We plan to come back to these questions on a subsequent work. But in this paper, we content ourselves to use the operators T_D to obtain new informations on the dynamics of D.

We say that D is *linearly finitary* if there is a monic polynomial Q in $k[X]$ such that $Q(T_D) = 0$ as endomorphisms of $k(C)$. That D is linearly finitary means that the dynamics of T_D is trivial, in the sense that it is similar to the one of an operator on a finite-dimensional vector space. We prove the following:

Proposition 1. *A finitary self-correspondence is linearly finitary. The converse is true in characteristic zero.*

The direct sense is very easy. We prove the converse using graph-theoretic methods and Theorem 1 (see Proposition 4.4.7).

Our second main result is the following (see Theorem 4.5.3 which is slightly more precise).

Theorem 2. *If a self-correspondence D has three irreducible étale finite complete sets, then it is linearly finitary. In particular, in characteristic zero, a nonfinitary correspondence has at most two irreducible finite étale complete sets.*

We give a brief description of the idea of the proof, which uses only elementary methods: the theorem of Riemann and Roch and linear algebra. The first étale finite complete set S is used to define, as above, the natural filtration $(B_{S,n})_{n\geq 0}$, "by the order of the poles" of B_S, which is stabilized by T_D. Is this filtration split as a $k[T_D]$-filtration? In general, this has no reason to be true. But with a second irreducible étale complete set S', one can show that this filtration is "almost split", namely that there exists for every n a T_D-stable subspace $V_{S,S',n}$ in $B_{S,n+1}$ such that $B_{S,n} + V_{S,S',n} = B_{S,n+1}$ and $\dim V_{S,S',n}$ bounded independently of n.[2] We prove this by defining $V_{S,S',n}$ as the space of functions in $B_{S,n+1}$ that vanishes on S' at a suitable order, and using Riemann–Roch. Now a third finite étale complete set S'' give a second quasisplitting $V_{S,S'',n}$ of the filtration $B_{S,n}$. Using Riemann–Roch, we prove that the two filtrations are orthogonal, in the sense that $V_{S,S',n} \cap V_{S,S'',n} = 0$ for n large enough. Then, a linear algebra argument shows that all eigenvalues of T_D appearing in $B_{S,n+1}$ (for n large enough) already appear in $B_{S,n}$ and Theorem 2 follows.

1. Self-correspondences

1.1. ***Curves.*** Let k be a field. By a *curve* over k we shall mean a nonempty proper and smooth scheme over $\operatorname{Spec} k$ which is equidimensional of dimension 1 and geometrically connected.

If C is a curve over k, we denote by $k(C)$ the function field of C. If C and C' are two curves over k, a nonconstant morphism of k-schemes $\pi : C \to C'$ is finite and flat, hence surjective and thus defines a

[2] The filtration $(B_{S,n})$ of B_S would be T_D-split if in addition of $B_{S,n} + V_{S,S',n} = B_{S,n+1}$ we required $B_{S,n} \cap V_{S,S',n} = 0$, or equivalently $\dim V_{S,S',n} = \dim B_{S,n+1}/B_{S,n}$. One has $\dim B_{S,n+1}/B_{S,n} \leq |S|$ for every n (with equality for n large enough), so requiring that $\dim V_{S,S',n}$ is bounded independently of n is a qualitative version of that property. Hence the phrase "almost split".

morphism of k-extensions $\pi^* : k(C') \to k(C)$, which makes $k(C)$ a finite extension of $k(C')$. Explicitly, if $f \in k(C)$ is seen as a morphism from C to $\mathbb{P}^1$, $\pi^* f = f \circ \pi$. We say that π is *separable* if $k(C)$ is a separable extension of $k(C')$.

1.2. *Self-correspondences.* Given a curve C over k, a *self-correspondence* (D, π_1, π_2) on C is the data of a noetherian reduced k-scheme D whose connected components are curves D_i over k, with two morphisms π_1 and π_2 from D to C whose restrictions to each D_i are nonconstant and separable. Often, the morphisms π_1 and π_2 will be implicit and we shall simply denote by D the self-correspondence (D, π_1, π_2)

Self-correspondences on a fixed curve C over k naturally form a category: a morphism from (D, π_1, π_2) to (D', π_1', π_2') is a surjective k-morphism $h : D \to D'$ such that $\pi_i' \circ h = \pi_i$ for $i = 1, 2$. In particular we have a notion of *isomorphism* of self-correspondences.

Example 1.2.1. Let k be a field of characteristic p (a prime or zero), $P(x, y) \in k[x, y]$ be a polynomial in two variables x and y, which has at least one nonzero monomial of the form $x^a y^b$ with $p \nmid a$ and one of the form $x^{a'} y^{b'}$ with $p \nmid b'$. Let D be the normalization of the projective curve defined by the affine curve D_0 of equation $P(x, y) = 0$. Then the maps $(x, y) \mapsto x$, and $(x, y) \mapsto y$, are algebraic functions on D_0, and they define rational functions on D, or equivalently maps $\pi_1, \pi_2 : D \to \mathbb{P}^1$. These maps are nonconstant and separable in view of our condition on P. Thus (D, π_1, π_2) is a self-correspondence over $\mathbb{P}^1$.

Example 1.2.2 (the arithmetic-geometric mean; see [Bullett 1991]). If a and b are two positive real numbers with $a < b$, we define two recurrence sequences $a_0 = a$, $b_0 = b$, and for $n \geq 1$, $a_n = \sqrt{a_{n-1} b_{n-1}}$, $b_n = (a_{n-1} + b_{n-1})/2$. Then it is well-known (Gauss) and elementary that (a_n) is strictly increasing, (b_n) is strictly decreasing, and both converge to a same positive real number $M(a, b)$ called *the arithmetic geometric mean of a and b*. Since the formulas giving a_n and b_n are homogenous of degree 1, nothing is lost by fixing all the a_n equal to 1. That is, if we define $c_n = b_n/a_n$ for any n, we have that $c_0 = b/a$ and for $n \geq 1$, $c_n = (1 + c_{n-1})/2\sqrt{c_{n-1}}$, and from c_n we can retrieve a_n by $a_n = a\sqrt{c_0} \cdots \sqrt{c_{n-1}}$ and $b_n = c_n a_n$. The sequence c_n is strictly decreasing and converges to 1.

Inspired by these classical facts, one define the arithmetic-geometric correspondence (D_{agm}, π_1, π_2) on $C = \mathbb{P}^1_{\mathbb{C}}$, with $D_{agm} = \mathbb{P}^1_{\mathbb{C}}$, $\pi_1(z) = z^2$ and $\pi_2(z) = (1 + z^2)/2z$. Thus $\pi_2(\pi_1^{-1}(z)$ is the multiset $\{(1 + z)/2\sqrt{z}\}$ where $\sqrt{z}$ is allowed to be either of the two determinations of the square root of z. Its dynamics, studied in [Bullett 1991], is related to works of Gauss ant his successors on the arithmetic-geometric mean.

Note that (D_{agm}, π_1, π_2) is a special case of the correspondences of Example 1.2.1, namely it is equivalent to the self-correspondence over $\mathbb{P}^1$ attached to the equation $4xy^2 = (x + 1)^2$.

Example 1.2.3. Let C be the complete Igusa curves of level N (with $(N, p) = 1$) over $\mathbb{F}_p$. Recall ([Gross 1990, pages 460–462], where the curve is denoted by $I_1(N)$) that C is the smooth completion of the affine Igusa curve, which is defined as the moduli space for triples (E, α, β), where E is an elliptic curve over a scheme of characteristic p, α an embedding $\mu_N \hookrightarrow E$ and β an embedding $\mu_p \hookrightarrow E$.

For l a prime number not dividing Np, we define the *Hecke correspondence* D_l, moduli space for quadruples (E, α, β, H) where (E, α, β) is as above and H is a subgroup scheme of E locally of order l. Define $\pi_1 : D_l \to C$ as just forgetting H, and π_2 as sending (E, α, β, H) to $(E/H, \alpha', \beta')$ where α' and β' are defined in the obvious manner. Then (D_l, π_1, π_2) is a self-correspondence, called the Hecke correspondence at l, on the Igusa curve C.

1.3. ***Bidegree.*** If D is a finite disjoint union of curves $D = \coprod_i D_i$, C a curve, and $\pi : D \to C$ a map nonconstant on every component D_i of D, then we define $\deg \pi$ as $\sum_i \deg \pi_{|D_i}$.

The *bidegree* of a self-correspondence (D, π_1, π_2) on a curve C is the ordered pair of integers $(\deg \pi_1, \deg \pi_2)$. It is often denoted (d_1, d_2). A self-correspondence is *balanced* when $d_1 = d_2$. For example, the bidegree of the arithmetic-geometric self-correspondence of Example 1.2.1 is (2, 2), and the bidegree of the Hecke correspondence D_l of Example 1.2.3 is $(l+1, l+1)$.

1.4. ***Transpose.*** If (D, π_1, π_2) is a self-correspondence on C, so is (D, π_2, π_1), called the *transpose* of (D, π_1, π_2). We shall denote this correspondence by ${}^tD$. Its bidegree is (d_2, d_1), if the degree of D is (d_1, d_2).

1.5. ***Self-correspondence of morphism type.*** Let f be a nonconstant separable morphism from C to C. We denote by D_f the self-correspondence (C, Id_C, f). A self-correspondence D over C is *of morphism type* if it is isomorphic to some D_f. Equivalently, D is of morphism type if and only if its bidegree has the form $(1, d)$. Thus we see that the transpose of a self-correspondence of the form D_f is of morphism type only when f is an isomorphism, and in this case ${}^tD_f \simeq D_{f^{-1}}$.

1.6. ***Minimal self-correspondences.*** A self-correspondence is *minimal* if the map $\pi_1 \times \pi_2 : D \to C^2$ is generically injective, that is there are only finitely many points of C^2 such that the fiber of $\pi_1 \times \pi_2$ has more than one element. Equivalently, D is minimal if $k(D)$ is generated, as a k-algebra, by its two subfields $\pi_1^*(k(C))$ and $\pi_2^*(k(C))$.

For any self-correspondence D on C, there is a unique pair $(D_{\min}, h)$ where $D_{\min}$ is a minimal self-correspondence over C and $h : D_{\min} \to D$ a morphism of self-correspondences on C: take $D_{\min}$ the normalization of the image of D by $\pi_1 \times \pi_2$.

A minimal self-correspondence is rigid, i.e., has a trivial group of automorphism. Indeed, an automorphism σ of a self-correspondence D stabilizes every fibers of $\pi_1 \times \pi_2$. If D is minimal, all those fibers but finitely many are singletons, so σ fixes all points of D but finitely many, which implies that $\sigma = \mathrm{Id}_D$.

1.7. ***Symmetric self-correspondences.*** A self-correspondence D is *symmetric* if ${}^tD \simeq D$. Obviously a symmetric self-correspondence is balanced. A self-correspondence is symmetric if and only if there exists an automorphism η of the k-scheme D such that $\pi_1 \circ \eta = \pi_2$. When D is minimal, this automorphism η is necessarily an involution, since η^2 is an automorphism of the self-correspondence D, which is rigid.

The Hecke correspondence D_l on the Igusa curve (see Example 1.2.3) is symmetric with $\eta(E, \alpha, \beta, H) = (E/H, \alpha', \beta', E[l]/H)$.

1.8. ***Terminology on directed graphs.*** By a *directed graph* we shall mean the data $\Gamma = (V, Z, s, t)$ of two sets V and Z and two maps $s, t : Z \to V$. Elements of V are called *vertices*, elements of Z are called *edges*, and for $z \in Z$, $s(z)$ is the *source* and $t(z)$ the *target* of z. In particular, *self-loops* (i.e., edges z such that $s(z) = t(z)$) and *repeated edges* (i.e., edges $z_1 \neq z_2$ such that $s(z_1) = s(z_2)$ and $t(z_1) = t(z_2)$) are allowed.

We use the usual terminology for a directed graph: a *forward-neighbor* (resp. *backward-neighbor*) of a vertex x is a vertex y such that there is an edge z with source x and target y (resp. with source y and target x). More generally, we say that y is a *k-forward-neighbor* of x if $y = x$ when $k = 0$, or if y is a forward-neighbor of a $(k-1)$-forward-neighbor when $k \geq 1$.

A subset S of V is said to be *forward-complete* (resp. *backward-complete*) if it contains every forward-neighbors (resp. backward-neighbors) of its vertices.[3] A subset S of V is *complete* if it is both backward-complete and forward-complete. A complete subset S is *irreducible* if it is nonempty and has no complete proper nonempty subset.[4]

It is clear that the irreducible complete subsets of a directed graph are the connected components of the nondirected graph it defines, and that the complete sets are the union of connected components. A union and intersection of complete sets is complete, as is the complement of any complete set. Every complete set is a disjoint union of irreducible complete sets.

If x, y are in V, a *directed path of length* n from x to y is a sequences of n edges $p = (z_1, \dots, z_n)$ such that $s(z_1) = x$, $t(z_n) = y$ and $t(z_i) = s(z_{i+1})$ for $i = 1, \dots, n-1$. A directed path from x to x is called a *directed cycle*.

We shall denote by $\mathrm{np}_{x,y,n}$ the number of directed paths from x to y of length n.

If k is a ring, and $\Gamma = (V, Z, s, t)$ is a directed graph, we define the *adjacency operator* of Γ, $A_\Gamma : \mathcal{C}(V, k) \to \mathcal{C}(V, k)$ on the k-module $\mathcal{C}(V, k)$ of maps from V to k, by the formula

$$(A_\Gamma f)(y) = \sum_{z \in Z, t(z) = y} f(s(z)).$$

The matrix of A_Γ in the canonical basis of $\mathcal{C}(V, k)$ is the *adjacency matrix* of Γ. By induction, we check that if $x, y \in V$ and $n \geq 1$ an integer, then

$$(A_\Gamma^n(\delta_x))(y) = \mathrm{np}_{x,y,n}\,. \tag{1}$$

We shall sometimes consider functions $f : V \to k \cup \{\infty\}$ where ∞ is a symbol not in k. For such a function, we define $A_\Gamma f$ by the same formula as above, with the convention that the sum of the right hand-side is ∞ if exactly one of its term is ∞, and is undefined if two or more of its terms are ∞.

[3] Another widely used terminology for the same notion is *forward-invariant*.

[4] An irreducible complete set S is sometimes called a *grand orbit* of the self-correspondence. Indeed, such a set S can be obtained by starting with any of its point and applying the forward (mulitvalued) map $\pi_2 \circ \pi_1^{-1}$ or the backward map $\pi_1 \circ \pi_2^{-1}$ repeatedly, which may explain this terminology.

1.9. ***The directed graph attached to a self-correspondence.*** If (D, π_1, π_2) is a self-correspondence of C over k, and $\bar{k}$ is a fixed algebraic closure of k, we define the oriented graph Γ_D attached to D as $(C(\bar{k}), D(\bar{k}), \pi_1, \pi_2)$. It is clear that for a subset S of $C(\bar{k})$, the notions of being forward-complete, backward-complete, or complete, as defined in the introduction (first page), and the same notions as defined for a subset of a directed graph defined in Section 1.8, coincide.

If $z \in D(\bar{k})$ is an edge, we write $e_{i,z}$ for the index of ramification of π_i at z. An edge $z \in D(\bar{k})$ is said *ramification-increasing* (resp. *equiramified*, resp. *étale*) if $e_{1,z} \le e_{2,z}$ (resp. $e_{1,z} = e_{2,z}$, resp. $e_{1.z} = e_{2,z} = 1$). We observe that there are only finitely many edges that are not étale (and a fortiori, not equiramified or ramification-increasing). This is because π_1 and π_2 are assumed separable.

A subset S of $C(\bar{k})$ is said *ramification-increasing*, *equiramified*, *étale* if all the edges whose both source and target are in S are *ramification-increasing*, etc.

For any vertex $x \in C(\bar{k})$, we have the formula

$$\sum_{z \in D(\bar{k}), \pi_1(z) = x} e_{1,z} = d_1, \tag{2}$$

$$\sum_{z \in D(\bar{k}), \pi_2(z) = x} e_{2,z} = d_2. \tag{3}$$

In particular, there are at most d_1 edges with source x and d_2 edges with target x, and the directed graph Γ_D is locally finite. Given a finite set of vertices $S \subset C(\bar{k})$, one has by summing the above formula

$$\sum_{z \in \pi_1^{-1}(S)} e_{1,z} = d_1 |S|, \tag{4}$$

$$\sum_{z \in \pi_2^{-1}(S)} e_{2,z} = d_2 |S|. \tag{5}$$

Remark 1.9.1. The directed graph of ${}^tD$ is the directed graph of D with source and target maps exchanged.

Lemma 1.9.2. *Let k be a finitely generated extension of a prime field. Then k has extensions of arbitrary large prime degrees.*

Proof. Let $\mathbb{F}$ be the prime subfield of k, and let $T_1, \dots, T_n$ be a transcendence basis of k over $\mathbb{F}$. Thus if $k_0 = \mathbb{F}(T_1, \dots, T_n)$, k has finite degree d over k_0. The field k_0 admits extension of any prime degree p: if $n = 0$, then $k_0 = \mathbb{F}_p$ or $\mathbb{Q}$ and the result is well-known, and if $n \ge 1$, for p prime, the polynomial $X^p - T_1$ has no root in k_0 hence is irreducible over k_0; see [Lang 2002, Theorem 9.1]. If $p \nmid d$, the composition of an extension of degree p of k_0 with k is an extension of k of degree p. □

Proposition 1.9.3. *The directed graph Γ_D has infinitely many irreducible complete sets. All but finitely many of them are étale.*

Proof. Let us consider C and D as embedded in a projective space over k (say $\mathbb{P}^3_k$), and let k_0 be the subfield of k generated over the prime subfield of by the coefficients of the projective equations of C and

D and the coefficients of the polynomials defining π_1 and π_2. Replacing k by k_0 we may assume that k is of finite type over its prime subfield.

If k is a finite type extension of its prime subfield, and $x \in C(k)$ is a vertex, then if $z \in D(\bar{k})$ is an edge with source (resp. target) x, one has $z \in D(k')$ for k' some finite extension of k of degree $\leq d_1$ (resp. $\leq d_2$). Indeed, z belongs to the schematic fiber of π_1 at x, which is a finite k_1-scheme of degree d_1. It follows that any forward-neighbor (resp. backward-neighbor) of $x \in C(k)$ is defined on an extension of degree $\leq d_1$ (resp. $\leq d_2$) of k'. By induction, any vertex in the same irreducible complete set as x is defined over an extension of k of degree whose all prime factors are $\leq \max(d_1, d_2)$.

Given a finite family $S_1, \dots, S_l$ of irreducible complete sets in $C(\bar{k})$, pick points $x_1 \in S_1, \dots, x_l \in S_l$. By replacing k by a finite extension, we may assume that $x_1, \dots, x_l$ all belong to $C(k)$, and thus every points x in $S_1 \cup \dots \cup S_l$ belong to $C(k')$ for k' a finite extension of k (depending on x) of some degree whose all prime factors are less than $\max(d_1, d_2)$. By the above lemma, k has extensions of arbitrary large prime degrees, hence has an extension k'' of prime degree $p > \max(d_1, d_2)$ and C has a point whose field of definition contains k''. Such a point cannot belong to $S_1 \cup \dots \cup S_l$, which shows that there are other irreducible complete sets in $C(\bar{k})$. Therefore the number of irreducible complete sets is infinite.

The second assertion is clear since there are only finitely many nonétale edges. □

Example 1.9.4. The directed graph of the Hecke correspondence D_l on the Igusa curve C (see Example 1.2.3) is well understood. Since D_l is symmetric, we loose no information by forgetting the orientation of the edges and looking at Γ_{D_l} as an undirected graph.

There are two obvious finite completes sets, the set of supersingular points and the sets of cusps. The complete set of supersingular points is étale (easy since $l \neq p$) and irreducible (this can be proved by direct analysis, or, as Krishnamoorthy notes in [Krishnamoorthy 2018], simply as a consequence of Proposition 3.2.5 below). The complete set of cusps may be reducible, and none of its irreducible components are étale (again a consequence of Proposition 3.2.5). The other complete sets are all infinite and have been called *isogeny volcanoes*: they consist in a cycle of some order n, with an infinite tree of valence $l+1$ attached to each vertices of that cycle; see [Sutherland 2013; Kohel 1996].

1.10. ***Sum and composition of self-correspondences.*** Let (D, π_1, π_2) and (D', π'_1, π'_2) be two self-correspondences on a curve C, of bidegrees (d_1, d_2) and (d'_1, d'_2).

The *sum* of D and D', denoted $D + D'$, is by definition the self-correspondence

$$\left(D \coprod D', \pi_1 \coprod \pi'_1, \pi_2 \coprod \pi'_2\right)$$

on C. It is obvious that the oriented graph $\Gamma_{D+D'}$ has the same vertices as Γ_D and $\Gamma_{D'}$ and for set of edges the disjoint union of their set of edges. The bidegree of $D + D'$ is $(d_1 + d'_1, d_2 + d'_2)$.

We define $D' \circ D$ as the scheme $D \times_{\pi_2, C, \pi'_1} D'$. The two projections pr_1 and pr_2 of this fibered product over D and D' are finite and flat, and its total ring of fractions is étale over $k(C)$, since it is the tensor product of the two separable extensions $k(D)$ and $k(D')$ over $k(C)$ (they are seen as extensions of $k(C)$ through π_2^* and π'^*_1 respectively). In particular, $D' \circ D$ is proper over k, and all of its irreducible component have dimension 1. We denote by $D'D$ the normalization of the reduced scheme attached to $D' \circ D$, and by $n : D'D \to D \circ D'$ the natural map:

$$\begin{tikzcd}[column sep=small]
 & & & D'D \arrow[d, "n"] & & & \\
 & & & D' \circ D \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & & & \\
 & & D \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & D' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & & \\
 & C & & C & & C &
\end{tikzcd}$$

Thus $D'D$ is a proper and smooth scheme of dimension 1, that is a disjoint union of curves. Since n is surjective, the restriction of $\mathrm{pr}_1 \circ n$ and $\mathrm{pr}_2 \circ n$ to every connected component of $D'D$ are surjective onto D and D' respectively and they are separable since n induces an isomorphism on the total rings of fractions. Thus, $(D'D, \pi_1 \circ \mathrm{pr}_1 \circ n, \pi_2' \circ \mathrm{pr}_2 \circ n)$ is a self-correspondence on C, which we shall call *the composition of D' with D*. Its bidegree is $(d_1 d_1', d_2 d_2')$.

Example 1.10.1. If f, g are morphisms from C to C, then $D_f D_g = D_{f \circ g}$.

The directed graph $\Gamma_{D' \circ D}$ is the graph whose vertices are those of Γ_D or $\Gamma_{D'}$, and edges (z, z') where z is an edge of Γ_D, z' is an edge of $\Gamma_{D'}$ such that the source of z' is the target of z. The source (resp. target) of the edge (z, z') in $\Gamma_{D' \circ D}$ is the source of z in Γ_D (resp. the target of z' in $\Gamma_{D'}$). Hence

$$A_{\Gamma_{D' \circ D}} = A_{\Gamma_{D'}} \circ A_{\Gamma_D}. \tag{6}$$

Since n is surjective, generically an isomorphism, the directed graph $\Gamma_{D'D}$ is that of $\Gamma_{D' \circ D}$ described above, with finitely many new edges added, all those new edges having the same source and target that an already existing edge in $\Gamma_{D' \circ D}$.

Lemma 1.10.2. *If S is a forward-complete* (*resp. backward-complete*, *resp. complete*) *set for D and D' then it is forward-complete* (*resp. etc.*) *for $D'D$.*

If S is a complete étale set for D and D', then the restrictions of the directed graphs of $D' \circ D$ and of $D'D$ to S coincide, S is also étale for $D'D$ and $A_{\Gamma_{D'D}}$ coincides with $A_{\Gamma_{D'}} \circ A_{\Gamma_D}$ on $\mathcal{C}(S, k)$.

Proof. The first assertion follows from what was said above. For the second, if $x \in S$, the fibers of D and D' at x are both étale, and so is their tensor product $D_x \times_k D'_x$, which is $(D' \circ D)_x$, so $D' \circ D$ is étale at points above x. But then it is smooth, and $n : D'D \to D' \circ D$ is thus an isomorphism on some neighborhood of the points above x, so $D'D$ is also étale over x, and the last assertion follows from (6). $\square$

It is clear that the composition of self-correspondences is associative up to obvious canonical identifications. We denote by D^n the composition of n copies of the self-correspondence D.

2. Finite complete sets

2.1. *The unbalanced case.*

Proposition 2.1.1. *Let (D, π_1, π_2) be a self-correspondence over C of bidegree (d_1, d_2) with $d_1 < d_2$. Denote by g_D, g_C the genera of D and C.*[5] *Then any finite backward-complete set S satisfies*

$$|S| \leq 2\frac{g_D - d_2 g_C + d_2 - 1}{d_2 - d_1}.$$

Moreover if such a set S is ramification-decreasing, then it is empty.

Proof. For the first assertion,

$$\begin{aligned}
|S|d_2 &= \sum_{z\in\pi_2^{-1}(S)} e_{2,z} && \text{(by formula (5))}\\
&= |\pi_2^{-1}(S)| + \sum_{z\in\pi_2^{-1}(S)} (e_{2,z} - 1)\\
&\leq |\pi_2^{-1}(S)| + \sum_{z\in D(\bar{k})} (e_{2,z} - 1)\\
&\leq |\pi_2^{-1}(S)| + 2g_D - 2d_2 g_C + 2d_2 - 2 && \text{(by Hurwitz's formula)}\\
&\leq |\pi_1^{-1}(S)| + 2g_D - 2d_2 g_C + 2d_2 - 2 && \text{(since } S \text{ is backward-complete)}\\
&\leq |S|d_1 + 2g_D - 2d_2 g_C + 2d_2 - 2,
\end{aligned}$$

from which the bound of the statement immediately follows.

Now if S is a ramification-decreasing backward-complete finite set,

$$\begin{aligned}
|S|d_2 &= \sum_{z\in\pi_2^{-1}(S)} e_{2,z}\\
&\leq \sum_{z\in\pi_2^{-1}(S)} e_{1,z} && \text{(since } S \text{ is ramification-decreasing)}\\
&\leq \sum_{z\in\pi_1^{-1}(S)} e_{1,z} && \text{(since } S \text{ is backward-complete)}\\
&= |S|d_1 && \text{(by formula (4))}
\end{aligned}$$

which under our assumption $d_2 > d_1$ implies $S = \varnothing$. □

We record for later use a consequence of the proof:

Scholium 2.1.2. Let (D, π_1, π_2) be a self-correspondence with $d_1 \leq d_2$. If S is a ramification-decreasing backward-complete finite set for D, then S is complete and equiramified. Moreover, if a self-correspondence admits a complete equiramified nonempty finite set, then it is balanced.

[5]The genus g_D of a finite disjoint union of curves $D = \coprod_i D_i$ is defined here as the sum of the genera of D_i. With this definition, and that of degree given in Section 1.3, Hurwitz formula is still valid for a map $\pi : D \to C$.

Proof. The second chain of inequalities in the proof of the proposition is an equality by assumption $d_1 \le d_2$, hence all intermediate inequalities must be equalities. The rest is clear. □

Applying the proposition (resp. the scholium) to ${}^tD$, we get a dual statement concerning forward-complete sets in the cases $d_1 > d_2$ (resp. $d_1 = d_2$) that we let the reader make explicit. Combining the proposition and its dual statement, we get:

Corollary 2.1.3. *Let (D, π_1, π_2) be a self-correspondence over C of bidegree (d_1, d_2) with $d_1 \neq d_2$ (that is, D is unbalanced). Let $d = max(d_1, d_2)$. Then any finite complete set S is not equiramified, in particular is not étale, and satisfies $|S| \le 2(g_D - dg_C + d - 1)/|d_2 - d_1|$.*

2.2. ***Finitary self-correspondences.***

Theorem 2.2.1. *For a self-correspondence D on a curve C over a field k, the following are equivalent:*

(i) *There exists a nonconstant k-morphism $h : C \to \mathbb{P}^1_k$ such that $h \circ \pi_1 = h \circ \pi_2$.*

(ii) *There exists a nonconstant $\bar{k}$-morphism $h : C_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ such that $h \circ \pi_1 = h \circ \pi_2$.*

(iii) *There is an integer M such that every irreducible complete set has cardinality less or equal than M.*

(iv) *All irreducible complete sets of D are finite.*

(v) *All irreducible complete sets of D but possibly finitely many are finite.*

(vi) *There are infinitely many finite complete sets.*

Proof. When D is unbalanced, (i) and (ii) are false by additivity of the degree, and (iii), (iv), (v) and (vi) are false by Proposition 2.1.1. The assertions (i) to (vi) are thus equivalent. We may therefore assume that D is balanced of degree d.

The implication (i) ⇒ (ii) is trivial. For (ii) implies (iii), we just note that the fibers $h^{-1}(t)$, $t \in \mathbb{P}^1(\bar{k})$ are complete since $\pi_1^{-1}(h^{-1}(t)) = (h \circ \pi_1)^{-1}(t) = (h \circ \pi_2)^{-1}(t) = \pi_2^{-1}(h^{-1}(t))$, and they all have size $\le \deg h$. Therefore every irreducible complete set has cardinality less or equal than $\deg h$, which gives (iii).

That (iii) implies (iv) and (iv) implies (v) is trivial; and (v) implies (vi) by Proposition 1.9.3.

It is also not hard to prove that (ii) implies (i), as in [Krishnamoorthy 2018, Proposition 3.8]. In fact, assuming (ii), we know that there exists a finite extension k' of k and a map h' from $C_{k'}$ to $\mathbb{P}^1_{k'}$ defined over k' such that $h' \circ \pi_1 = h' \circ \pi_2$. Let V be the Weil's restriction of scalars of $\mathbb{P}^1_{k'}$ to k, and $\tilde{h} : C \to V$ the k-morphism corresponding to h according to the universal property of Weil's restriction. The image C' of C by $\tilde{h} \circ \pi^1 = \tilde{h} \circ \pi_2$ in V, with its reduced scheme structure, is a closed subscheme of V defined over k and which has positive Krull dimension. There is therefore a rational function $u \in k(V)$ that induces a nonconstant map on C'. Thus, if $h := u \circ \tilde{h}$, h is a k-morphism $C \to \mathbb{P}^1_k$ such that $h \circ \pi_1 = h \circ \pi_2$ and (i) holds.

It only remains to prove that (vi) implies (ii).

To prove this, we first reduce to the case where k is of finite type over its prime field.

In any case, there is a subfield k_0 of k, of finite type over the prime subfield of k such that C, D, π_1 and π_2 are defined over k_0. Assuming (vi), that is that $C(\bar{k})$ contains infinitely many finite irreducible complete sets, we must show that (vi) holds for k_0, that is $C(\bar{k}_0)$ also contains infinitely many finite irreducible complete sets. If all the irreducible finite complete sets in $C(\bar{k})$ are in $C(\bar{k}_0)$, we are obviously done. Otherwise, there is one irreducible finite complete set S in $C(\bar{k})$ which is not in $C(\bar{k}_0)$.

Let k' be any subfield of $\bar{k}$ containing k_0 and of finite type over k_0 such that all points of S are defined over k'. There obviously exist such k', and since S is not included into $C(\bar{k}_0)$, k' is transcendental over k_0. Let $V = \operatorname{Spec} A$ be an affine algebraic variety over k_0 whose field of fraction is k' and such that S spreads out to V, that is that all points of S are defined over A. Then V has positive Krull's dimension, so it has infinitely many $\bar{k}_0$-points. Any $\bar{k}_0$-point t in V gives by specialization of S a finite complete set S_t in $C(\bar{k}_0)$. It follows that $C(\bar{k}_0)$ contains at least one finite irreducible complete S_0 in $C(\bar{k}_0)$. Replacing k_0 by a finite extension, we may assume that $S_0 \subset C(k_0)$. We may now replace C by $C - S(k_0)$, D by $D - \pi_1^{-1}(S_0) = D - \pi_2^{-1}(S_0)$, and π_1 and π_2 by their suitable restriction to have a new self-correspondence (C, D, π_1, π_2) defined over k_0, but with now C and D *affine*. Let us write $C = \operatorname{Spec} R$, where R is a k-algebra

We now redo the same reasoning from the beginning, namely we prove that we may assume that there is a irreducible finite complete set S' in $C(\bar{k})$ which is not contained in $C(\bar{k}_0)$. On can write $S' = \{x_1, \ldots, x_n\}$ where each x_i is a k_0-morphism from R to $\bar{k}$. We now define A as the k_0-subalgebra of $\bar{k}$ generated by the images of the x_i's in k', $V = \operatorname{Spec} A$, and k' the field of fraction of A. We can thus see the x_i as k_0-morphisms from R to A. With these notation, for $t \in V(\bar{k}_0) = \operatorname{Hom}_{k_0}(A, \bar{k}_0) = \operatorname{Hom}_{k_0}(k', \bar{k}_0)$, the specialization S_t of S is by definition the subset $\{t \circ x_1, \ldots, t \circ x_n\}$. Now we claim that for $t \neq t' \in V(\bar{k}_0)$, one has $S_t \neq S_{t'}$ as subsets of $C(\bar{k}_0)$; for if $S_t = S_{t'}$, then the subalgebra of $\bar{k}$ generated by the $t \circ x_i$ $(1 = 1, \ldots, n)$ on the one hand, by the $t' \circ x_i$ $(1 = 1, \ldots, n)$ on the other hand, are equal, and thus $t = t'$ on the subalgebra generated by the image of x_i $(i = 1, \ldots, n)$, which is A, and this means $t = t'$. We thus obtain infinitely many distinct finite complete subset in $C(\bar{k}_0)$, namely the S_t where t moves in the infinite set $V(\bar{k}_0)$.

Replacing k by k_0, we may henceforth assume that k is of finite type over its prime field.

We now continue the proof of (vi) $\Rightarrow$ (ii) assuming k is finitely generated over the prime field. Let k^{sep} and k^{perf} be the separable and perfect closures of k in $\bar{k}$. Then $\bar{k} = k^{\text{sep}} \otimes_k k^{\text{perf}}$ and $\operatorname{Gal}(k^{\text{sep}}/k) = \operatorname{Gal}(\bar{k}/k^{\text{perf}})$. We denote this Galois group by G.

Let J be the Jacobian of C over k. Since k is of finite type over its prime field, $J(k)$ is a finitely generated abelian group by the theorem of Néron [1952]; see also [Lang and Néron 1959].

Moreover, one has

$$J(k^{\text{perf}}) \otimes_{\mathbb{Z}} \mathbb{Q} = J(k) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{7}$$

Indeed, there is nothing to prove in characteristic 0, and in characteristic $p > 0$, $J(k^{\text{perf}}) = \cup_n J(k^{1/p^n})$, so by induction it suffices to prove that $pJ(k^{1/p}) \subset J(k)$. But

$$pJ(k^{1/p}) = VFJ(k^{1/p}) \subset VJ^{(p)}(k) \subset J(k) \tag{8}$$

where $J^{(p)} = J \otimes_{k, x \mapsto x^p} k$, $F : J \to J^{(p)}$ and $V : J^{(p)} \to J$ are the relative Frobenius and Verschiebung k-morphisms, which completes the proof of (7). (We refer the reader to [Edixhoven et al. 2012, Section 5.2], for instance, for the definitions of Frobenius and Verschiebung, and to [Edixhoven et al. 2012, Proposition 5.19] for the fundamental relation $p = VF$ used above. Since the map F raises the coordinates of a point to the power p, it sends $J(k^{1/p})$ into $J^{(p)}(k)$, hence the middle inclusion in (8).)

Let r be the finite dimension of $J(k^{\mathrm{perf}}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

By assumption, there are infinitely many finite complete sets in $C(\bar{k})$, and therefore infinitely many étale finite complete sets. Each of them is a finite subset of $C(\bar{k})$, hence has a finite G-orbit. By grouping the irreducible complete sets by G-orbits, we see that there are still infinitely many disjoint étale finite complete sets invariant by G. Let us chose $r+2$ of them, say $S_0, \ldots, S_{r+1}$. To every S_i we attach the Weil divisor

$$\Delta_i = \sum_{x \in S_i} [x] \in \operatorname{Div} C_{\bar{k}}$$

and let $\delta_i = |S_i|$ be its degree. The $r+1$ divisors $\delta_0 \Delta_i - \delta_i \Delta_0$ have degree zero, hence they define points in $\operatorname{Pic}^0(C_{\bar{k}}) = J(\bar{k})$, where J is the Jacobian of C, and those points are G-invariant, hence in $J(k^{\mathrm{perf}})$. Therefore those points are $\mathbb{Q}$-linearly dependent, hence $\mathbb{Z}$-linearly dependent, which means that there are integers n_i, not all zero, $i = 0, \ldots, r+1$, and a nonconstant k^{perf}-map $h : C_{k^{\mathrm{perf}}} \to \mathbb{P}^1_{k^{\mathrm{perf}}}$ such that

$$\sum_{i=0}^{r+1} n_i \Delta_i = \operatorname{div} h.$$

Since the S_i are étale, for $j = 1, 2$ one has

$$\pi_j^* \Delta_i = \sum_{z \in \pi_j^{-1}(S_s)} [z],$$

and since the S_i are complete, it follows that

$$\pi_1^* \Delta_i = \pi_2^* \Delta_i.$$

It follows that

$$\operatorname{div}(h \circ \pi_1) = \pi_1^* \operatorname{div} h = \pi_2^* \operatorname{div} h = \operatorname{div}(h \circ \pi_2),$$

hence that there exists $\lambda \in (k^{\mathrm{perf}})^*$ such that

$$\lambda h \circ \pi_1 = h \circ \pi_2.$$

This implies that (reasoning as in the proof of (ii) implies (iii)) if S is any complete set, $h(S)$ is stable by multiplication by λ and λ^{-1}. By assumption, there exists a finite complete S such that $h(S)$ is not contained in $\{0, \infty\}$. This implies that λ is a root of unity, and if $\lambda^n = 1$, replacing h by h^n gives (ii). This completes the proof of the implication (vi) implies (ii), hence of the theorem. □

Remember from the introduction that when the equivalent assertions of the preceding theorem are satisfied, we say that D is *finitary*.

Example 2.2.2. The arithmetic-geometric mean self-correspondence introduced in Example 1.2.2 is not finitary. To see it, it suffices to notice that the irreducible complete set S containing a real number $c > 1$ also contains all elements $c_0 = c$, $c_n = (1 + c_{n-1})/2\sqrt{c_{n-1}}$ for any n, and that the sequence (c_n), being strictly decreasing, take infinitely many value. Thus S is infinite, and D_{agm} is not finitary.

Remark 2.2.3. A self-correspondence D is finitary if and only if its transpose ${}^tD$ is finitary. This is seen trivially on any of the assertion (i) to (vi).

Also, if k' is any field containing k, a self-correspondence D is finitary if and only if its base change $D_{k'}$ is finitary. The implication D finitary $\Rightarrow D_{k'}$ finitary is clear on (i), and its converse is clear on (iii) or on (iv).

Remark 2.2.4. A correspondence of morphism type, say D_f, is finitary if and only if f is an automorphism of finite order. Indeed, if D_f is finitary, then $\deg f = 1$ by (i) and f is an automorphism, whose action on the generic fiber of h is a bijection of a finite set, so some power of f acts trivially on the generic fiber of h, hence on C, and f has finite order. The converse is trivial.

Remark 2.2.5. The method of using Jacobians in the proof of the theorem is inspired by Krishnamoorthy, [2018, Chapter 9], and our condition (a) is inspired by the condition he calls "having a core" which gives the title to his article.[6] For Krishnamoorthy, a (general, not self-) correspondence $(D, \pi_1 : D \to C, \pi_2 : D \to C')$ *has a core* if there exist nonconstant maps $f : C \to \mathbb{P}^1$, $g : C' \to \mathbb{P}^1$ such that $f \circ \pi_1 = g \circ \pi_2$. For a self-correspondence, his notion is much weaker than our notion of being finitary, where we require $f = g$. Indeed, finitary implies balanced, while there are plenty of unbalanced self-correspondences that have a core, in particular all those of morphism type. Even for balanced self-correspondences, having a core does not imply being finitary, as the example of D_f when f is an automorphism of infinite order shows (or for less trivial examples, a suitable étale self-correspondence on a curve C of genus 1, for such a correspondence always have a core in the sense of Krishnamoorthy, but in general is not finitary). In the case of a symmetric self-correspondence D, however, one can show that D is finitary if and only if it has a core in the sense of Krishnamoorthy: see Lemma 3.2.4.

Remark 2.2.6. One finds in the literature yet another property akin to "having a core" or "being finitary", namely what Dinh, Kaufmann et Wu calls "weakly modular" in [Dinh et al. 2020]. They work in the case $k = \mathbb{C}$ and they say that a balanced self-correspondence D over a curve C is *weakly modular* if there are two probability measures m_1 and m_2 over $C(\mathbb{C})$ such that $\pi_1^* m_1 = \pi_2^* m_2$. To "complete the square", let us say that D is *weakly finitary* if there is one probability measure m on $C(\mathbb{C})$ such that $\pi_1^* m = \pi_2^* m$.

[6] Krishnamoorthy himself is inspired by Mochizuki, which introduces the same notion in the hyperbolic case, under the name "having an hyperbolic core"; see [Mochizuki 1998]. In the case $k = \mathbb{C}$, a closely related notion has also been considered by Bullett, Penrose [2001, Section 2.5], and their coauthors, under the name of *separable* (self-)correspondence. This notion is equivalent to "having a core" in the minimal case. Apparently, the two sets of authors (Bullett et al., Mochizuki and Krishnamoorthy) were unaware of each other's works.

Thus one has a following square of implications for D a balanced self-correspondence over $\mathbb{C}$, none of which being an equivalence:

$$\begin{array}{ccc} D \text{ is finitary} & \Longrightarrow & D \text{ has a core} \\ \Downarrow & & \Downarrow \\ D \text{ is weakly finitary} & \Longrightarrow & D \text{ is weakly modular} \end{array}$$

Note that any self-correspondence which has a nonempty finite complete set S is weakly finitary, hence weakly modular, as taking m the normalized counting measure on S shows. Thus to be weakly modular is a weak condition.

Theorem 1.1 of [Dinh et al. 2020] states that for any balanced nonweakly modular self-correspondence D on a curve C over $\mathbb{C}$, there exists a measure μ_D on $C(\mathbb{C})$ which does not charge polar sets (in particular finite sets), and such that for every smooth measure μ on $C(\mathbb{C})$, $\left(\frac{1}{d}(\pi_1)_*(\pi_2)^*\right)^n\mu \to \mu_D$ as $n \to \infty$.[7] In the case of a nonbalanced self-correspondence with $d_1 < d_2$, the theorem was known earlier [Bharali and Sridharan 2016]. One may ask whether this theorem holds more generally for any D that is not finitary. We conjecture the answer to be yes. (See also Remark 3.4.2.)

3. The exceptional set of a nonfinitary self-correspondence

3.1. ***Bounding the exceptional set.*** Let (D, π_1, π_2) be a self-correspondence over a curve C. The *exceptional set* E of C of D is the union of all finite complete sets of $C(\bar{k})$. Obviously, by Theorem 2.2.1, D is nonfinitary if and only if E is finite. But we may ask if in this case one can give an effective bound on the size of E. Unfortunately, the proof of Theorem 2.2.1 does not provide such a bound, even involving not only g_C, g_D, and g, but the Mordell–Weil rank r of the Jacobian variety of C over k_0, because the step where we group together finite complete sets to get G-invariant complete sets is not effective.

We can ask two different questions on the exceptional set: we can ask for an upper bound on the number of elements of E, and for an upper bound on the number of irreducible components of E, that is the number of complete irreducible finite sets for D. The second question reduces to bounding the number of irreducible components of E that are étale, because the number of nonétale ones is less than the number of critical values of π_1 and π_2, which in turns is bounded, using Hurwitz's formula, by $2g_D - 2 - d(2g_C - 2)$.

In the case of an unbalanced self-correspondence, the two questions are solved by Proposition 2.1.1. We therefore limit ourselves to the balanced case.

3.2. ***The number of irreducible complete finite sets, I.*** Given a balanced nonfinitary self-correspondence D over C of bidegree (d, d), can we give a bound to the number of étale irreducible complete finite sets

[7] We recall the standard measure-theoretic notation used here: let $\pi : D(\mathbb{C}) \to C(\mathbb{C})$ be an holomorphic map, nonconstant on every component of D; if μ is a Borel measure on $D(\mathbb{C})$, then $\pi_*\mu$ is the measure on $C(\mathbb{C})$ defined by $\pi_*(\mu)(B) = \mu(\pi^{-1}(B))$; if μ is a Borel measure on $C(\mathbb{C})$, then $\pi^*\mu$ is the Borel measure on $D(\mathbb{C})$ defined by $\int f\,d\pi^*\mu = \int \pi_* f\,d\mu$ for every continuous function f on $D(\mathbb{C})$, where $\pi_* f(x) = \sum_{z\in\pi^{-1}(x)} f(x)$.

in terms of the genera g_C and g_D of the curve involved and the degree d? We shall give several such bounds in particular but important situations (namely the case where k is a finite field, or when $C = \mathbb{P}^1$, or when the correspondence D is symmetric) using effective variants of the proof of Theorem 2.2.1. In the next section, using completely different methods we give a general result in characteristic zero (and also under a weaker form in characteristic p), namely that a nonfinitary correspondence has at most 2 étale (or even equiramified) finite irreducible sets (see Section 4.5).

Proposition 3.2.1. *Assume that k is algebraic over a finite field. If a correspondence D on a curve C over k is not finitary, then it has at most one nonempty finite equiramified complete set, and in particular at most one étale nonempty complete set.*

Note that the proposition implies that in case there is one nonempty equiramified complete set, it is automatically irreducible. Of course, D can very well have no nonempty finite complete set, as in the case of D_f, where f is a the translation by a nontorsion element on an elliptic curve.

Remark 3.2.2. The hypothesis made on k is necessary. Indeed, assume that k is not algebraic over a finite field. Then k has an element $t \neq 0$ which is of infinite multiplicative order (take an element which is transcendental over $\mathbb{F}_p$ if k has characteristic p and $t = 2$ if k has characteristic 0). Consider the map $f(x) = tx$ from $\mathbb{P}^1$ to $\mathbb{P}^1$, and the correspondence D_f it defines. This correspondence is of bidegree (1, 1), and has two complete finite sets, obviously étale: $\{0\}$ and $\{\infty\}$. Yet D_f is not finitary, because f is not an automorphism of finite order.

Proof. Suppose that there are two distinct finite equiramified nonempty complete sets S and S'. If $S \subset S'$, replace S' by $S' - S$. If $S \not\subset S'$, replace S by $S - (S' \cap S)$. This way we may assume that S and S' are not only distinct, but disjoint.

In view of our hypothesis on k, we may assume that the self-correspondence is defined over a finite field k_0, that S and S' are subsets of $C(k_0)$, and that $\pi_1^{-1}(S)$ and $\pi_2^{-1}(S')$ are subsets of $D(k_0)$.

Let $\Delta_S = \sum_{s \in S}[s]$ and $\Delta_{S'} = \sum_{s \in S'}[s]$ be the effective Weil's divisors attached to S and S'. The divisor $|S'|\Delta_S - |S|\Delta_{S'}$ has degree zero, hence is torsion in $\mathrm{Pic}_{k_0}(C)$ (since it is an element of the finite group $\mathrm{Pic}^0_{k_0}(C)$, the group of k_0-rational points of the Jacobian of C). Therefore, there exist n and m such that $n\Delta_S - m\Delta_{S'}$ is a principal divisor; in other words, there exists a rational function h in $k(C)$ such that $\operatorname{div} h = n\Delta_S - m\Delta_{S'}$.

We claim that there exists $\lambda \in k^*$ such that $h \circ \pi_1 = \lambda h \circ \pi_2$ as functions in $k(D)$, up to multiplication by a nonzero scalar. Indeed, for $i = 1, 2$, $\operatorname{div} h \circ \pi_i = \pi_i^*(n\Delta_S - m\Delta_{S'}) = \sum_{t \in \pi_i^{-1}(S)} n e_{i,t}[t] - \sum_{t \in \pi_i^{-1}(S')} m e_{i,t'}[t']$ where $e_{i,t}$ is the ramification index of π_i at t. Using that S and S' are equiramified complete sets, we see that $\operatorname{div} h \circ \pi_1 = \operatorname{div} h \circ \pi_2$, hence the claim.

Now since λ belongs to a finite field, there is an n such that $\lambda^n = 1$. Replacing h by h^n, we see that D satisfies assertion (i) of Theorem 2.2.1. □

Proposition 3.2.3. *Let k be any field. If D a self-correspondence over $\mathbb{P}^1_k$ which is not finitary, there are at most two irreducible finite equiramified complete sets, and when there are two of such, there is no other irreducible finite complete set at all.*

Proof. Arguing as in the case of a finite base field k, but using the fact that $\mathrm{Pic}^0_k \mathbb{P}^1$ is trivial, we see that if S and S' are two disjoint irreducible equiramified sets, there is a function $f : \mathbb{P}^1 \to \mathbb{P}^1$ with divisor $|S'|\Delta_S - |S|\Delta_{S'}$. Such a function satisfies $f \circ \pi_1 = \lambda f \circ \pi_2$ for some $\lambda \in k^*$, and also by definition $f^{-1}(\infty) = S$, $f^{-1}(0) = S'$. Since D is not finitary, λ is not a root of unity, and as in the proof of Theorem 2.2.1, we see that for T complete, $f(T)$ is stable by multiplication by λ, and if T is also finite, this implies $f(T) \subset \{0, \infty\}$. Since $f^{-1}(0) = S$ and $f^{-1}(\infty) = S'$ are irreducible, they are the two only irreducible complete sets. □

Finally, we give two more results in the same vein but for symmetric self-correspondences. They are based on earlier results in the literature and the following lemma.

Lemma 3.2.4. *If a self-correspondence has a core in the sense of Krishnamoorthy* (*see Remark 2.2.5*, *or* [Krishnamoorthy 2018, Definition 3.5]) *then ${}^tDD$ is finitary. If moreover D is symmetric then it is itself finitary.*

Proof. If (D, π_1, π_2) has a core, that is if there exists $f, g : C \to \mathbb{P}^1$ such that $f \circ \pi_1 = g \circ \pi_2$, then the forward map of D sends fibers of f to fibers of g, and the backward map of D, i.e., the forward map of ${}^tD$, sends fibers of g to fibers of f. Thus ${}^tDD$ preserves the fibers of f, and is therefore finitary, which proves the first assertion.

If moreover D is symmetric, then D^2 is finitary and has infinitely many irreducible complete finite sets. But every irreducible complete set S of D breaks down in at most two irreducible complete sets of D^2, namely the set of points of S which are connected to a given point x_0 of S by a path of even length, and its complement if nonempty. Therefore D must have infinitely many finite complete sets, and is therefore finitary. □

Proposition 3.2.5 (Krishnamoorthy). *If D is a nonfinitary symmetric self-correspondence on a curve C over a field k, then D has at most one irreducible finite equiramified complete set. If k has characteristic zero and π_1, π_2 are étale, then D has no irreducible finite complete set.*

Proof. Krishnamoorthy proves that if D is a self-correspondence without a core, it has at most one finite irreducible finite étale complete set; see [Krishnamoorthy 2018, Theorem 9.6]. In fact his proofs work with "étale complete" replaced with "equiramified complete", as in the proofs of Proposition 3.2.1 and Proposition 3.2.3. By the lemma above, this implies the proposition. □

Corollary 3.2.6. *Let D be a nonfinitary symmetric self-correspondence on a curve C over a field k. Let S be an irreducible finite complete étale set for D. Then the undirected graph of S* (*obtained by forgetting the orientation of the edges*) *is not bipartite.*

Proof. If the undirected graph Γ_S attached to S was bipartite, then the graph $\Gamma_{S,2}$ with the same set of vertices S but whose edges are paths of degree 2 in S would be disconnected. But D^2 is also not finitary (by Lemma 3.2.4), and its graph on the set of vertices S is $\Gamma_{S,2}$ (by Lemma 1.10.2, using that S is étale), in contradiction with Proposition 3.2.5. □

Remark 3.2.7. Let $S \in C(\bar{k})$ be a finite complete set for a self-correspondence D on C. We shall say that the set S is *consistently ramified* if for every undirected cycle with edges $z_1, \dots, z_n$, the rational number $\prod_{i=1}^{n}(e_{2,z_i}/e_{1,z_i})^{\epsilon_i}$ is 1, where the signs ϵ_i (for $i = 1, \dots, n$) are defined to be $+1$ is the edge z_i has a compatible orientation with z_{i+1} (i.e., the target of z_i is the source of z_{i+1} or the source of z_i is the target of z_{i+1}) or -1 otherwise (we use the convention that $z_{n+1} = z_1$). If S is equiramified, it is consistently ramified, since all factors in the above product are 1; but clearly the converse is false.

We claim that Propositions 3.2.1, 3.2.3, and 3.2.5 are still true with the phrase "finite equiramified complete sets" replaced with "finite consistently ramified complete sets". Indeed, if S is consistently ramified, it is easy to see that one can attach to every $s \in S$ a positive integer n_s such that for every $z \in \pi_1^{-1}(S) = \pi_2^{-1}(S)$, one has $n_{\pi_1(z)}e_{1,z} = n_{\pi_2(z)}e_{2,z}$. In the proof of Proposition 3.2.1 (for example), it suffices to change the definition of Δ_S to be the divisor $\sum_{s\in S} n_s[s]$, to obtain a divisor with support S and such that $\pi_1^*\Delta_S = \pi_2^*\Delta_S$, and the rest of the proof may remain unchanged.

3.3. *Bounding the exceptional set.*

Proposition 3.3.1 (Krishnamoorthy). *If* (D, π_1, π_2) *is a nonfinitary symmetric self-correspondence on a curve C over a field k of characteristic* 0, *and if* π_1 *and* π_2 *are étale*, *then* $E = \varnothing$.

Proof. Krishnamoorthy proves that if π_1 and π_2 ate étale and D have no core, and k has characteristic 0, then $E = \varnothing$; see [Krishnamoorthy 2018, Corollary 9.2]. The result follows using Lemma 3.2.4. □

We shall now prove a generalization of the above result. A self-correspondence is *critically finite* (see [Bullett 1992]) if every ramification point in $C(\bar{k})$ of π_1 or π_2 belongs to a finite complete set. In other words, the union E_{crit} of the irreducible nonétale complete sets is finite.

Example 3.3.2. For the arithmetic geometric mean correspondence (see Example 1.2.2), it is easy to see that $E_{\text{crit}} = \{0, -1, 1, \infty\}$ and this correspondence is critically finite.

Proposition 3.3.3. *Let k be a field of characteristic zero. Let D be a critically finite self-correspondence on C over k having no core* (*in particular*, *symmetric and nonfinitary*)*. Assume that the curve D is irreducible. Then D has no nonempty étale finite complete set. In other words*, $E = E_{\text{crit}}$.

Proof. (Inspired by Section 3 of [Mochizuki 1998].)

We may assume that $k = \bar{k} = \mathbb{C}$. If E_{crit} is empty, then π_1, π_2 are étale, and the result follows from the Proposition 3.2.5. Let $C_0 := C - E_{\text{crit}}$, and $D_0 = D - \pi_1^{-1}(E_{\text{crit}}) = D - \pi_2^{-1}(E_{\text{crit}})$, and we still denote by π_1 and π_2 the restriction of π_1 and π_2 to D_0. They are étale maps, and (D_0, π_1, π_2) is, in an obvious sense, a self-correspondence over C_0 in the category of open curves.

If $C = \mathbb{P}^1$, and E_{crit} has 1 or 2 elements, then C_0 is the affine line or punctured affine line, and it has at most one étale finite cover of any degree d. Thus $\pi_1 = \pi_2$ contradicting the assumption that D is not finitary.

In the remaining cases, the open curve $C_0 := C - E_{\text{crit}}$ is hyperbolic. Let $D_0 = D - \pi_1^{-1}(E_{\text{crit}}) = D - \pi_2^{-1}(E_{\text{crit}})$. Let us identify the universal cover of D_0 (which is also a universal cover of C_0) with the upper half-plane $\mathcal{H}$.

Fix some $x \in C_0$, z_1 in $\pi_1^{-1}(x) \in D_0$ and $h_1 \in \mathcal{H}$ a point that maps to z_1 in D_0. The fundamental groups $\pi_1(C_0, x)$ and $\pi_1(D_0, z_1)$ are canonically identified, after those choices, with discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ that we shall denote respectively by Γ_C and Γ_D; we have $\Gamma_D \subset \Gamma_C$, the inclusion being of finite index. Choose also a $z_2 \in \pi_2^{-1}(z) \in D_0$, $h_2 \in \mathcal{H}$ that maps to z_2, and a $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $gz_1 = z_2$. Thus $\pi_1(D_0, z_2)$ is canonically identified with $g\Gamma_D g^{-1}$, which also is a subgroup of finite index of Γ_C, and we have a commutative diagram

$$\begin{tikzcd}
 & \mathcal{H}/\Gamma_D \arrow[rrrr, "\sim"] \arrow[d] & & & & D_0 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & \mathcal{H}/(\Gamma_C \cap g^{-1}\Gamma_C g) \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & & & & & \\
\mathcal{H}/\Gamma_C \arrow[rrrr, bend right, "\sim"'] & & \mathcal{H}/\Gamma_C \arrow[rrrr, bend right, "\sim"'] & & C_0 & & C_0
\end{tikzcd}$$

where the unnamed horizontal maps (curved or straight) are the identifications fixed above, and the diagonal maps s_1 and s_2 are given by respectively the inclusion and the conjugation by g of $\Gamma_C \cap g^{-1}\Gamma_C g$ into Γ_C, the vertical map being given by the inclusion

$$\Gamma_D \subset \Gamma_C \cap g^{-1}\Gamma_C g.$$

This inclusion shows that $\Gamma_C \cap g^{-1}\Gamma_C g$ has finite index in Γ_C and thus that g belongs to the commensurator of Γ_C in $\mathrm{PSL}_2(\mathbb{R})$.

Let Γ be the closure of the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by Γ_C and $g^{-1}\Gamma_C g$. We claim that Γ has infinite index in Γ_C. Otherwise, Γ would also be a lattice, and we would have two finite étale maps (surjective, of analytic stacks of dimension 1) $\mathcal{H}/\Gamma_C \to \mathcal{H}/\Gamma$ given by the inclusion and the conjugation by g^{-1} of Γ_C into Γ. We could then give an algebraic structure on $\mathcal{H}/\Gamma$, making it an algebraic stacks, and choose a nonconstant map of $\mathcal{H}/\Gamma$ to $\mathbb{P}^1$, which composed with the two finite étale maps $\mathcal{H}/\Gamma_C \to \mathcal{H}/\Gamma$ gives two morphisms of Riemann surfaces $f, g : \mathcal{H}/\Gamma_C = C_0 \to \mathbb{P}^1$ such that $f \circ \pi_1 = g \circ \pi_2$. Extending f, g to the complete smooth curves C, we thus see that the symmetric correspondence (D, π_1, π_2) has a core (f, g), hence is finitary by Lemma 3.2.4, contradicting our hypothesis.

We just need to show that the self-correspondence of Riemann surfaces (D_0, π_0, π_1) has no finite complete sets, and by the commutativity of the diagram above it suffices clearly to show hat the self-correspondence $\mathcal{H}/(\Gamma_C \cap g\Gamma_C g^{-1}, s_1, s_2)$ on $\mathcal{H}/\Gamma_C$ has no finite complete set. If S was such a complete set, its preimage $\tilde{S}$ in $\mathcal{H}$ would be invariant by Γ_D (obviously) and by $g\Gamma_D g^{-1}$, hence by Γ. The set $\tilde{S}$ would thus be an infinite union of Γ_C-orbits, contradicting the finiteness of S. □

We conclude this subsection by giving one result for self-correspondence on $\mathbb{P}^1$ in characteristic zero. It is a rephrasing of a result due to Pakovich.

Proposition 3.3.4. *Let k be a field of characteristic zero, (D, π_1, π_2) a self-correspondence over $\mathbb{P}^1_k$ of bidegree (d, d). Let g_D be the genus of D. We assume that*:

(i) *The singleton $\{\infty\}$ is a complete equiramified set.*

(ii) *There exists a $\lambda \in k^*$ such that for every $z \in \pi_1^{-1}(\infty) = \pi_2^{-1}(\infty)$,*

$$\operatorname{ord}_z(\pi_1 - \lambda\pi_2) > \operatorname{ord}_z \pi_1 = \operatorname{ord}_z \pi_2.$$

(iii) *D is not finitary.*

Then $|E| \leq 3 + (2g_D - 1)/d$, with equality possible only in the case $g_D = 0$, $d = 1$.

Proof. This is essentially Théorème 1 of [Pakovich 1996]. More precisely, to prove our proposition we reduce to the case $k = \mathbb{C}$. Assuming that the proposition is false, there is a finite complete set in $\mathbb{P}^1_{\mathbb{C}}$ of cardinality $> 3 + (2g_D - 1)/d$, hence, removing ∞, there is a finite complete set $K \subset \mathbb{C}$ of cardinality $> 2 + (2g_D - 1)/d$. Conditions (i) and (ii) allow us to apply Theorem 1 of [Pakovich 1996] which tells us that there is a rotation σ of the plane $\mathbb{C}$ such that $\sigma(K) = K$ and $\pi_1 = \sigma \circ \pi_2$. Since K is finite, $\sigma^{(\#K)!}$ is the identity of K, hence of the real affine closure of K, and since $\#K \geq 2$, the rotation $\sigma^{(\#K)!}$ must fix a real line in $\mathbb{C}$, hence is the identity of $\mathbb{C}$. Thus $h(z) = z^{(\#K)!}$ satisfies $h \circ \pi_1 = h \circ \pi_2$, in contradiction with (iii). □

Remark 3.3.5. The condition (ii) is the most restrictive in practice. Assuming (i), it is clear that for every $z \in \pi_1^{-1}(\infty) = \pi_2^{-1}(\infty)$, there exists a $\lambda \in k^*$ such that $\operatorname{ord}_z(\pi_1 - \lambda\pi_2) > \operatorname{ord}_z \pi_1 = \operatorname{ord}_z \pi_2$, but the existence of such a λ *independent of* $z \in \pi_1^{-1}(\infty)$ is problematic — except of course when $\pi_1^{-1}(\infty)$ is a singleton.

Moreover, the theorem is false without condition (ii), as the following example given by Pakovich shows: $D = \mathbb{P}^1$, $\pi_1(z) = (z^2 - z - 1)/(z^2 + z + 1)$, $\pi_2(z) = -(z^2 + 3z + 1)/(z^2 + z + 1)$. Then $\pi_1^{-1}(\{\infty\}) = \pi_2^{-1}(\{\infty\}) = \{j, j^2\}$ and $\{\infty\}$ is complete étale so (i) is satisfied, and one can check that (iii) is also satisfied. But $\{-1, 1\}$ is also a finite complete set (since $\pi_1^{-1}(\{-1, 1\}) = \{-1, 0, \infty\} = \pi_2^{-1}(\{-1, 1\})$). This provides an example where $C = D = \mathbb{P}^1$, and $|E| \geq 3$.

The following special case was proved by Pakovich [1995] in an earlier paper.

Corollary 3.3.6 (Pakovich). *Let k be a field of characteristic zero and suppose that $(\mathbb{P}^1_k, \pi_1, \pi_2)$ is a nonfinitary correspondence over $\mathbb{P}^1_k$, where π_1 and π_2 are polynomials. Then $|E| \leq 2$.*

Proof. We can apply the proposition above: (i) is satisfied because $\pi^{-1}(\infty) = \pi^{-2}(\infty) = \{\infty\}$ and (ii) is satisfied because $\pi^{-1}(\infty)$ is a singleton. It tells us that $|E| \leq 3 + (2g_D - 1)/d = 3 - 1/d < 3$. □

Remark 3.3.7. This result can be considered as an effective version of Theorems 2.2.1 and 4.5.3 in a very special case; it states that for a nonfinitary correspondence $(\mathbb{P}^1, \pi_1, \pi_2)$ over $\mathbb{P}^1$, with π_i polynomials, in characteristic zero, (thus having $S = \{\infty\}$ as equiramified complete set), then there is at most one other complete irreducible set S', which moreover is also a singleton (if it exists). By contrast, our Theorem 4.5.3 below, under the same hypothesis will also tell us that there is at most one other complete irreducible set, but without affirming that it has to be a singleton. However our conclusion will also hold without assuming that $C = D = \mathbb{P}^1$, nor any condition on π_1, π_2 and S.

3.4. *Complements and questions.*

Backward exceptional kernels. If $\Gamma = (V, E, s, t)$ is a directed graph, we define the *backward exceptional kernel* K_{backward} as the union in V of all finite backward-complete sets. A symmetric definition could of course be given for *forward exceptional sets* and we will let the interested reader reformulate the results below in this case.

If D is a self-correspondence over C of bidegree (d_1, d_2), its *backward exceptional kernel* is the one of its associated directed graph Γ_D. A natural question for a self-correspondence is then: when is K_{backward} finite? We cannot offer a complete answer to this question. Here is what the author knows on K_{backward}.

First note that it is no restriction to assume that D is minimal, for there is always a minimal self-correspondence associated to D which has the same dynamics as D, in particular the same K_{backward}. So we assume below that D is minimal

When $d_1 < d_2$, Proposition 2.1.1 shows that K_{backward} is finite and gives a bound to its size.

What about K_{backward} when $d_1 \geq d_2$?

Consider first the case $d_2 = 1$, that is of a transpose of self-correspondence of morphism type: $D \simeq {}^tD_f$. If $d_1 = 1$, then f is an automorphism of C, of infinite order, and K_{backward} is finite. If $d_1 > 1$ however, then K_{backward} is always countable infinite. In the case where $d_1 > d_2 > 1$, K_{backward} may be finite or infinite: for instance, consider the case $D = C = \mathbb{P}^1$, $\pi_1(x) = x^3$, $\pi_2(x) = x^2$ of bidegree $(3, 2)$, where it is easy to see that $K_{\text{backward}} = \{0, \infty\}$, or for a balanced case, any symmetric nonfinitary correspondence (e.g., an Hecke correspondence D_l on the Igusa curve), where by symmetry, $K_{\text{backward}} = E$ which is finite. For an example with K_{backward} infinite consider the sum $D = {}^tD_f + {}^tD_g$, where f and g are the endomorphisms of $\mathbb{P}^1$, $z \mapsto z^2$ and $z \mapsto z^3$. In this example K_{backward} is the set of all roots of unity while this self-correspondence is of bidegree $(5, 2)$, hence minimal since 5 and 2 are relatively prime.

Remains the balanced case $d_1 = d_2 > 1$. The finitary self-correspondences trivially have $K_{\text{backward}} = C(\bar{k})$. So let us consider a nonfinitary self-correspondence. Such a self-correspondence, if symmetric, will have $K_{\text{backward}} = E$ finite. The only question that remains is:

Question 3.4.1. Does there exist a nonfinitary balanced self-correspondence with K_{backward} infinite?

To analyze this question, note that for S a complete irreducible subset of $C(\bar{k})$, $S \cap K_{\text{backward}} = K_{\text{backward}}(S)$, and $K_{\text{backward}} = \coprod_S K_{\text{backward}}(S)$ when S run among irreducible complete subsets. If S is equiramified, then $K_{\text{backward}}(S)$ is a union of equiramified finite backward-complete subsets of S, and those subsets are complete by Scholium 2.1.2; thus, if $K_{\text{backward}}(S)$ is not empty it is finite and equal to S. This shows that the answer to the question is no when K_{backward} is equiramified, and because the number of nonequiramified irreducible complete sets S is finite, in general the question reduces to "is $K_{\text{backward}}(S)$ finite when S is a nonequiramified complete set?"

Backward exceptional and forward exceptional sets. If D is a self-correspondence over C of bidegree (d_1, d_2), we define the *backward exceptional set* E_{backward} as the smallest forward-complete set containing K_{backward}. Be careful that the *backward exceptional set* is forward-complete by definition, but not in general backward-complete.

If D is not finitary, E_{backward} is always "small": it contains the finite exceptional set E and is contained in the union of E and finitely many nonequiramified irreducible complete sets. In particular, E_{backward} is at most countable, and its complement contains an infinite union of irreducible complete sets. This follows easily from our study of K_{backward}.

If K_{backward} is étale, then it is complete and $E_{\text{backward}} = K_{\text{backward}}$ is finite. However, E_{backward} may be infinite in general. An example in bidegree (2, 3) due to Dinh and Favre showing this is given in [Dinh 2005, Exemples 3.11]. A similar balanced example is given by $\pi_2(z) = z^2 - z$, $\pi_2(z) = z^2$, for which we have $K_{\text{backward}} = \{0, \infty\}$, but E_{backward} contains a strictly increasing sequence $0, 1, 2\sqrt{5}, \dots$ (each term x_{n+1} being the larger number in the pair $\pi_2(\pi_1^{-1}(x_n))$), hence is infinite.

Remark 3.4.2. The significance of the set E_{backward} appears most clearly in the ergodic theory of self-correspondences on curves over $\mathbb{C}$. More precisely, for a balanced nonweakly modular self-correspondence, it is shown in [Dinh et al. 2020, Theorem 1.2] that for every $x \in C(\mathbb{C})$, $\bigl(\frac{1}{d_2}(\pi_1)_*(\pi_2)^*\bigr)^n(\delta_x) \to \mu_D$, where the definition of the measure μ_D on C is recalled in Remark 2.2.6. Note that in the nonweakly modular case, it is easy to see that $K_{\text{backward}} = E_{\text{backward}} = \varnothing$. When $d_1 < d_2$, it is proved in [Dinh 2005; Dinh and Sibony 2006] that $\bigl(\frac{1}{d_2}(\pi_1)_*(\pi_2)^*\bigr)^n(\delta_x) \to \mu_D$ for every $x \notin E_{\text{backward}}$. It is therefore natural to conjecture that in every case $d_1 \le d_2$, if D is not finitary, $\bigl(\frac{1}{d_2}(\pi_1)_*(\pi_2)^*\bigr)^n(\delta_x) \to \mu_D$ if and only if $x \notin E_{\text{backward}}$. At any rate, it is not hard to see that if $x \in E_{\text{backward}}$, if the limit $\lim_{n\to\infty}\bigl(\frac{1}{d_2}(\pi_1)_*(\pi_2)^*\bigr)^n(\delta_x)$ exists as a measure, then it charges at least one point in K_{backward}, and thus it cannot be μ_D.

Polarized self-correspondences.

Definition 3.4.3. Let (D, π_1, π_2) be a self-correspondence on a curve C over a field k. A *polarization* of D is an ample line bundle $\mathcal{L}$ on C such that $\pi_1^*\mathcal{L}^n = \pi_2^*\mathcal{L}^m$ for some positive integers n and m. If D admits a polarization we say that D is *polarized*.

Recall that on a curve, a line bundle is ample if and only if its degree is positive. If $\mathcal{L}$ is a polarization, one must have $d_1 n = d_2 m$.

A self-correspondence D has a polarization in each of the following cases:

(i) $D = \mathbb{P}^1$ (in which case necessarily $C = \mathbb{P}^1$)

(ii) k is algebraic over a finite field.

(iii) D has a finite nonempty equiramified complete set $S \subset C(k)$.

Indeed, let us consider the group homomorphism $h : \operatorname{Pic} C \to \operatorname{Pic}^0 D$, $\mathcal{L} \to (\pi_1^*\mathcal{L})^{d_2} \otimes (\pi_2^*\mathcal{L})^{-d_1}$. Clearly D is polarizable if and only if $\ker h \not\subset \operatorname{Pic}^0 C$. In particular, D is polarizable in case (i) since in this case $\operatorname{Pic}^0 D = 0$ and $\ker h = \operatorname{Pic} C = \mathbb{Z} \not\subset \operatorname{Pic}^0 C = 0$). Also D is polarizable in case (ii) for in this case $\operatorname{Pic}^0 D$

is torsion while $\operatorname{Pic} C/\operatorname{Pic}^0 C = \mathbb{Z}$ is torsion free. In case (iii), if the self-correspondence D has a finite equiramified complete set $\varnothing \neq S \subset C(k)$, take $\mathcal{L}$ the line bundle attached to the divisor $\Delta_S = \sum_{s\in S}[s]$.

We now recall the basics of the general height theory of Lang and Néron, following the exposition of [Chambert-Loir 2011, Section 3] and [Serre 1989]. Assume that k is either a number field or a nontrivial finite type extension of an algebraically closed field. It is known that we can choose a family $M(k)$ of pairwise inequivalent absolute values on k and numbers $\lambda_v > 0$ such that for $a \in k^*$, $|a|_v = 1$ for almost all $v \in M(k)$ and the product formula holds: $\prod_{v\in M(k)} |a|_v^{\lambda_v} = 1$. Moreover, if k' is a finite extension of k, one can choose $M(k')$ in such a way that there is a surjective map $M(k') \to M(k)$ with finite fibers, such that for $a \in k$, $v \in M(k)$, $|a|_v^{\lambda_v} = \prod_{v'\in M(k'),\pi(v')=v} |a|_{v'}^{\lambda_{v'}}$, and the product formula hods. The choice of $M(k)$ allows one to define the *height function* on $\mathbb{P}^n_{\bar{k}}$ by

$$h([x_0,\dots,x_n]) = \log\Biggl(\prod_{v\in M(k')} \max(|x_0|_v,\dots,|x_n|_v)^{\lambda_v}\Biggr),$$

if $x_0,\dots,x_n \in k'$ (the result is independent of the choice of the finite extension k' containing $x_0,\dots,x_n$).

For a k-projective variety V, let us denote by $\mathcal{F}(V)$ the $\mathbb{R}$-vector space of maps from $V(\bar{k})$ to $\mathbb{R}$ and by $\mathcal{F}b(V)$ the subspace consisting of these maps that are bounded. There is a unique morphism $\operatorname{Pic} V \to \mathcal{F}(V)/\mathcal{F}b(V)$, $\mathcal{L} \mapsto h_{\mathcal{L}}$, such that if $\mathcal{L}$ is very ample and ϕ is one of the embedding $V \to \mathbb{P}^n$ defined by $\mathcal{L}$, then $h_{\mathcal{L}} = h \circ \phi$. It follows that if $f : V \to W$ is a morphism of k-projective variety, $h_{f^*\mathcal{L}} = h_{\mathcal{L}} \circ f$.

Lemma 3.4.4. *If $\mathcal{L}$ is ample, then $h_{\mathcal{L}} \neq 0$ in $\mathcal{F}(V)/\mathcal{F}b(V)$.*

Proof. We may assume that $\mathcal{L}$ is very ample, and this reduces us to prove that for a projective subvariety in $\mathbb{P}^n_k$, the function h is unbounded on $C(\bar{k})$. Up to a linear change of variables, the map $\pi([x_0,\dots,x_n]) = [x_0, x_1]$ is surjective from V to $\mathbb{P}^1$, and it is clear that $h(x) \geq h(\pi(x))$. It suffices therefore to show that h is unbounded on $\mathbb{P}^1$, which is clear. □

Proposition 3.4.5. *Let (D, π_1, π_2) be an unbalanced polarized self-correspondence on a curve C over a field k that is not algebraic over a finite field. Then there are infinitely many vertices in Γ_D that do not belong to any directed cycle.*

Proof. Let $\mathcal{L}$ be a polarization on D. The self-correspondence D together with $\mathcal{L}$ are defined over a subfield k_0 of k which is of finite type over the prime subfield of k, and if k_1 is any field such that $k_0 \subset k_1 \subset \bar{k}$ it suffices obviously to prove the result for D considered as a self-correspondence over k_1. Since $\bar{k}$ is not the algebraic closure of a finite field, one can assume that k_1 is either a number field, or a nontrivial extension of finite type of an algebraic closed field. Replacing k_1 by k, we can now use the theory of height reminded above.

Let (d_1, d_2) be the bidegree of D. By symmetry of the statement to prove, we may and do assume that $d_1 < d_2$.

Since $\mathcal{L}$ is a polarization of D, there exists an integer $n > 0$ such that $\pi_1^*(\mathcal{L}^{d_2 n}) = \pi_2^*(\mathcal{L}^{d_1 n})$. Thus $d_2 h_{\mathcal{L}} \circ \pi_1 = d_1 h_{\mathcal{L}} \circ \pi_2$ in $\mathcal{F}(D)/\mathcal{F}b(D)$. In other word, if h is any lift of $h_{\mathcal{L}}$ in $\mathcal{F}(C)$, there exists a positive real constant M such that for all $z \in D(\bar{k})$, $|d_2 h(\pi_1(z)) - d_1 h(\pi_2(z))| < M$.

If $x_0, \dots, x_n$ is a directed cycle, then one has

$$|h(x_0) - (d_1/d_2)h(x_1)| < M/d_2$$

$$\dots < \dots$$

$$|h(x_{n-1}) - (d_1/d_2)h(x_0)| < M/d_2$$

so

$$|h(x_0) - (d_1/d_2)^n h(x_0)| < \frac{M}{d_2(1 + d_1/d_2 + \dots + (d_1/d_2)^{n-1})} < \frac{M}{d_2 - d_1}.$$

It follows that if x_0 belongs to a directed cycle then $h(x_0)$ is bounded by a constant (independent of the length of the cycle).

By the lemma, h is unbounded on $C(\bar{k})$. There is therefore infinitely many points in $C(\bar{k})$ that are not part of any directed cycle. $\square$

Remark 3.4.6. The proposition is obviously false for a finitary self-correspondence but we do not know whether the proposition holds for a polarized balanced nonfinitary self-correspondence, nor whether the polarized hypothesis may be dropped (even in the unbalanced case).

4. The operator attached to a self-correspondence

4.1. ***Definition of the operator T_D.*** If C is a curve, and (D, π_1, π_2) a self-correspondence of C, we denote by T_{D,π_1,π_2} or simply T_D, the map $k(C) \to k(C)$ which sends f to

$$T_D f = \mathrm{tr}_{k(D)/_{\pi_2^*} k(C)} \, \pi_1^*(f).$$

The notations $\mathrm{tr}_{k(D)/_{\pi_2^*} k(C)}$ means the trace map from $k(D)$ to $k(C)$, where $k(D)$ is seen as an algebra over $k(C)$ of dimension d_2 through the map π_2^*. The map T_D is thus a k-linear endomorphism of $k(C)$.

4.2. ***Local description of the operator T_D.*** First recall some basic terminology. If C/k is a curve, or a disjoint union of curves, and if $f \in k(C)^*$, $x \in C(\bar{k})$, we denote by $\mathrm{ord}_x(f)$ the order of vanishing of f at x. Thus $\mathrm{ord}_x(f) > 0$ if $f(x) = 0$, $\mathrm{ord}_x(f) = 0$ if $f(x) \in k^*$, and $\mathrm{ord}_x(f) < 0$ if $f(x) = \infty$. By convention, we set $\mathrm{ord}_x 0 = +\infty$. For S a subset of $C(\bar{k})$, we set

$$\mathrm{ord}_S(f) := \inf_{x \in S} \mathrm{ord}_x(f).$$

If $S = C(\bar{k})$, we simply write $\mathrm{ord}\, f$ for $\mathrm{ord}_S f$. One has $\mathrm{ord}\, f \le 0$ for any $f \in k(C)^*$, and $\mathrm{ord}\, f = 0$ if and only if f is a nonzero constant.

Now suppose given a self-correspondence (D, π_1, π_2) of C over k.

Given $y \in C(\bar{k})$, and $z \in \pi_2^{-1}(y)$, we note K_y and K_z the fraction fields of the completions $\hat{\mathcal{O}_{C,y}}$ and $\hat{\mathcal{O}_{D,z}}$ of the local rings $\mathcal{O}_{C,y}$ and $\mathcal{O}_{D,z}$. The valuation ord_y on $k(C)$ (resp. ord_z on $k(D)$) extends uniquely to K_y (resp. K_z), and $\hat{\mathcal{O}_{C,y}}$ (resp. $\hat{\mathcal{O}_{D,z}}$) is the ring of integers attached to this valuation; it is a complete discrete valuation ring. The map π_2^* induces an injective morphism $K_y \to K_z$, which makes K_z a finite separable extension of K_y, totally ramified of degree $e_{2,z}$ (which means that $\mathrm{ord}_y(f) = e_{2,z}\,\mathrm{ord}_z(f)$ for any $f \in K_y$ seen as an element of K_z).

Lemma 4.2.1. *Given $y \in C(\bar{k})$, $z \in \pi_2^{-1}(y)$ and $g \in K_z$, let $P_g(X) = X^{d_2} + \sum_{i=1}^{d_2} a_{i,g} X^{d_2-i} \in K_y[X]$ be the characteristic polynomial of the multiplication by g on K_z seen as a K_y-vector space of dimension d_2 through π_2^*. Then one has, for $i = 1, \ldots, d_2 - 1$, $\mathrm{ord}_z(a_{i,g}) \geq i\,\mathrm{ord}_z(g)$ and $\mathrm{ord}_z(a_{d_2,g}) = d_2\,\mathrm{ord}_z(g)$. In particular,*

$$\mathrm{ord}_y\,\mathrm{tr}_{K_z/K_y}(g) \geq \lceil \mathrm{ord}_z(g)/e_{2,z} \rceil.$$

Proof. Since K_z/K_y is separable, $P_{g.z}(X) = \prod_{i=1}^{d_2}(X - \sigma_i(g))$ where the σ_i runs amongst the embedding on K_z into some normal closure L of K_z over K_y. If w is the valuation on L extending ord_z on K_z, then $w(\sigma_i(g)) = \mathrm{ord}_z(g)$ and it follows that $\mathrm{ord}_z(a_{i,g}) = w(a_{i,g}) \geq i\,w(g) = i\,\mathrm{ord}_z(g)$, with equality if $i = d_2$. The last assertion follows since $a_{1,g} = \pm\,\mathrm{tr}_{K_z/K_y}(g)$ and $\mathrm{ord}_z(a_{1,g}) = e_{2,z}\,\mathrm{ord}_y(a_{1,g})$. □

Recall (Section 1.9) that $e_{1,z}$ and $e_{2,z}$ are the degrees of ramification of π_1 and π_2 at a point $z \in D(k)$.

Proposition 4.2.2. *Let $f \in k(C)$ and $y \in C(\bar{k})$. Set*

$$n = \min_{z \in D(k), \pi_2(z) = y} \left\lceil \frac{e_{1,z}\,\mathrm{ord}_{\pi_1(z)} f}{e_{2,z}} \right\rceil.$$

Then

$$\mathrm{ord}_y\, T_D f \geq n.$$

Moreover, if for any z such that $\pi_2(z) = y$, f has no pole at $\pi_1(z)$, then

$$f(y) = \sum_{z \in D(k), \pi_2(z) = y} e_{2,z} f(\pi_1(z)).$$

Proof. To compute the image of $T_D f$ in K_y we may extend the scalars from $k(C)$ to K_y since the formation of the trace commutes with base change. This means that $T_D f = \mathrm{tr}_{K_y \otimes_{k(C)} k(D)/K_y}\, \pi_1^*(f)$ in K_y. But $K_y \otimes_{k(C)} k(D) = \prod_{z \in \pi_2^{-1}(y)} K_z$ (see, e.g., [Serre 1979, Chapter II, Section 3, Theorem 1]) hence

$$T_D f = \sum_{z \in \pi_2^{-1}(y)} \mathrm{tr}_{K_z/K_y}\, \pi_1^*(f). \tag{9}$$

For $z \in \pi_2^{-1}(y)$, setting $x = \pi_1(z)$, we have $\mathrm{ord}_z\, \pi_1^* f = e_{1,z} \mathrm{ord}_x\, f$ hence by Lemma 4.2.1

$$\mathrm{ord}_y\,\mathrm{tr}_{K_z/K_y}\, \pi_1^*(f) \geq \left\lceil \frac{e_{1,z}\,\mathrm{ord}_{\pi_1(z)} f}{e_{2,z}} \right\rceil \geq n.$$

By (9), $\mathrm{ord}_y\, T_D f \geq n$.

To prove the second assertion, note that under its assumption, for any z such that $\pi_2(z) = y$, the image of $\pi_1^* f$ in K_z belongs to the complete d.v.r. $\hat{\mathcal{O}_{D,z}}$ and its image in the residue field $\bar{k}$ is $f(\pi_1(z))$. Thus

$\mathrm{tr}_{K_z/K_y}\,\pi_1^*(f) \in \hat{\mathcal{O}_{D,z}}$ and the image of that element in the residue field $\bar{k}$ is $e_{2,z} f(\pi_1(z))$. The formula $f(y) = \sum_{z \in D(k), \pi_2(z)=y} e_{2,z} f(\pi_1(z))$ then follows from (9). □

Recall (Section 1.8) that A_{Γ_D} is the adjacency operator of the graph D, and that for f a map from $C(\bar{k})$ to $\bar{k} \cup \{\infty\}$, $T_D(f)$ is a map from $C(\bar{k})$ minus a finite number of points to $\bar{k} \cup \{\infty\}$, the value of $T_D(f)$ being left undefined at any point where its computation requires to add the values of f at more than one pole.

Corollary 4.2.3. *Let $f \in k(C)$ seen as a map $f : C(\bar{k}) \to \bar{k} \cup \{\infty\}$. Then the functions $T_D f$ and $A_{\Gamma_D} f$ agree on all points of $C(\bar{k})$ but finitely many. They agree in particular on all étale complete sets on which f has no pole.*

Proof. Indeed, the last formula of the above proposition shows that the two functions agree at all points $y \in C(\bar{k})$ which are étale and not neighbors of a point where f has a pole. □

Corollary 4.2.4. *if D' and D are two self-correspondence on C, $T_{D'D} = T_{D'} \circ T_D$.*

Proof. For $f \in k(C)$, $T_{D'D} f$ agrees almost everywhere with $A_{\Gamma_{D'D}} f$, which agrees almost everywhere with $A_{\Gamma_{D'}} A_{\Gamma_D} f$, which agrees almost everywhere with $T'_D T_D f$, and those two functions in $k(C)$ must then be equal. □

4.3. *The filtered ring B_S attached to a set of vertices S.* Now fix a self-correspondence (D, π_1, π_2) on C over k and a set S of $C(\bar{k})$. We denote by $B_S \subset k(C)$ the rings of rational functions on C whose poles are all in S. Thus,

$$B_S = \{f \in k(C), \mathrm{ord}_x(f) \geq 0 \text{ for all } x \in C(\bar{k}) - S\}.$$

If $S = \varnothing$, $B_S = k$. If S is not empty, the ring of fractions of B_S is $k(C)$. For $n \geq 0$, we set

$$B_{S,n} = \{f \in B_S, \mathrm{ord}_S(f) \geq -n\}.$$

Lemma 4.3.1. *The k-subspaces $B_{S,n}$ for $n = 0, 1, \ldots$ form an increasing exhaustive filtration of B_S. One has $B_{S,0} = k$. The quotients $B_{S,n}/B_{S,n-1}$ for $n \geq 1$ are spaces of dimensions $\leq |S|$, and of dimension exactly $|S|$ when S is finite and n is large enough relatively to $|S|$.*

Proof. The first two sentences are trivial. The last one follows from Riemann–Roch. □

Proposition 4.3.2. *If S is forward-complete, the subring B_S of $k(C)$ is stable by T_D. If S is forward-complete and ramification-increasing, the filtration $(B_{S,n})$ is stable by T_D. The converses of both these statement hold if* $\mathrm{char}\, k = 0$ *or* $\mathrm{char}\, k > d_2$.

Proof. The first two statements follow from Proposition 4.2.2. The converse statements are left to the reader. □

Remark 4.3.3. Remember (Proposition 2.1.1) that a forward-complete and ramification-increasing set S, nonempty and finite, may only exist if $d_1 \leq d_2$. In the balanced case $d_1 = d_2$, such a set S has to be complete and equiramified (Scholium 2.1.2).

Example 4.3.4. If C is the Igusa curve, and S the supersingular complete set, which is étale, B_S is the space of modular forms of level N, all weights, over $\bar{\mathbb{F}}_p$ and $(B_{S,n})$ is the *weight filtration* on that space; see [Gross 1990].

4.4. ***Linearly finitary self-correspondences.*** Let D be a self-correspondence on a curve C over k.

Definition 4.4.1. We say that D is *linearly finitary* if there is a nonzero polynomial $Q \in k[X]$ such that $Q(T_D) = 0$ on $k(C)$.

Proposition 4.4.2. *If D is finitary, then there is a monic polynomial $Q \in \mathbb{Z}[X]$ such that $Q(T_D) = 0$ on $k(C)$. In particular D is linearly finitary.*

Proof. If D is finitary, there exists an $M \geq 0$ such that every irreducible finite complete set of D is a directed graph with $\leq M$ vertices, with $\leq d_1$ (resp. $\leq d_2$) arrows starting (ending) at each point. There are finitely many such graphs up to isomorphism, so infinitely many irreducible complete sets S must be isomorphic to some finite directed graph Γ. If $Q(X)$ is the characteristic polynomial of the adjacency matrix of Γ, we see that for every $f \in k(C)$, $Q(T_D)f$ is zero on infinitely many irreducible complete sets, so $Q(T_D)f = 0$, and therefore $Q(T_D) = 0$. □

Lemma 4.4.3. *Given two distinct points p, q in $C(\bar{k})$, a finite set $Z \subset C(\bar{k})$ not containing p or q, and an integer $n \geq 0$, there exists a rational function $f \in \bar{k}(C)$ such that $f(q) = 1$, f vanishes at every point of Z at order at least n, and f has no pole outside p.*

This follows from Riemann–Roch.

Lemma 4.4.4. *Let $Q(X) = \sum_{i=0}^{n} a_i X^i \in k[X]$ be a polynomial. The following are equivalent:*

(i) *One has $Q(T_D) = 0$ in $k(C)$.*

(ii) *There exists a nonempty forward-complete S such that $Q(T_D) = 0$ on B_S.*

(iii) *There exists infinitely many irreducible étale complete sets S' such that $Q(A_{\Gamma_D}) = 0$ on $\mathcal{C}(S', k)$.*

(iii') *There exists infinitely many irreducible étale complete set S' such that for every $x, x' \in S'$, one has $\sum_{i=0}^{n} a_i \operatorname{np}_{x,x',i} = 0$ in k, where $\operatorname{np}_{x,x',i}$ is the number of paths of length i from x to x' in Γ_D, as defined in Section 1.8.*

(iv) *There exists an infinite étale complete set S' such that $Q(A_{\Gamma_D}) = 0$ on $\mathcal{C}(S', k)$.*

(iv') *There exists an infinite étale complete set S' such that for every $x, x' \in S'$, one has $\sum_{i=0}^{n} a_i \operatorname{np}_{x,x',i} = 0$ in k.*

(v) *There exists an infinite backward-complete set S' such that $Q(A_{\Gamma_D})f$ has finite support for every $f \in \mathcal{C}(S', k)$.*

NB, the sets S in (ii) and S' in (iii) and (iii') are not assumed to be finite.

Proof. To prove that (i) implies (ii) is easy: take $S = C(\bar{k})$.

To prove that (ii) implies (iii), let p be a point in S. By Proposition 1.9.3, there exists infinitely many étale complete sets S' that do not contain p. Let S' be one of then, and let x be a point in S'. Let $\delta_x \in \mathcal{C}(S', k)$ be the function whose value at x is 1 and is 0 elsewhere. We provide Γ_D with the distance induced by its undirected graph structure. By Lemma 4.4.3, and since Γ_D is locally finite, there exists a function $f \in \bar{k}(C)$ such that $f = \delta_x$ on all points at distance $\leq 2n$ of x, and whose only possible pole is at p. In particular, $f \in B_S$, so $Q(T_D)f = 0$. By Corollary 4.2.3, $Q(A_{\Gamma_D})f = 0$ on S'. Clearly $Q(A_{\Gamma_D})f$ and $Q(A_{\Gamma_D})\delta_x$ agree on all points x' at distance $\leq n$ of x. Since $Q(A_{\Gamma_D})\delta_x$ has support in the sets of points at distance $\leq n$ of x, this implies that $Q(A_{\Gamma_D})\delta_x = 0$. Since this is true for an arbitrary point x of S', $Q(A_{\Gamma_D}) = 0$ on $\mathcal{C}(S', \bar{k})$, hence (iii).

It is clear that (iii) implies (iv), by taking S' in (iv) to be the union of all the S' in (iii). Also, the equivalences between (iii) and (iii') and (iv) and (iv') is just Formula (1).

Finally, (iv) implies (v) is clear, so it just remains to prove that (v) implies (i). Let $f \in k(C)$. Since S' is backward-complete, $Q(A_{S'})f$ and $T_D f$ agree on every points of S' at distance $> n$ of a pole of f. Since the number of such points is finite, $Q(A_{S'})f$ and $T_D f$ agree on every point of S' except a finite number of them, and this implies that $T_D f$ has finite support on S', and since S' is infinite, that $T_D f$ has infinitely many zeros. Hence $T_D f = 0$. □

Proposition 4.4.5. *If $Q(X) \in k[X]$ is a polynomial, then $Q(T_D) = 0$ if and only if $Q(T_{{}^tD}) = 0$. In particular, D is linearly finitary if and only if ${}^tD$ is. Moreover, if k' is any field extension of k, then D is linearly finitary if and only if $D_{k'}$ is linearly finitary.*

Proof. All is clear using condition (iii') or (iv') of Lemma 4.4.4. □

Lemma 4.4.6. *If S is an infinite irreducible étale complete set in Γ_D, then for every integer m and for every $x \in S$ there exists $x' \in S$ with a directed path from x to x' of length $m+1$ and no directed path from x to x' of length $\leq m$.*

Proof. By symmetry of the statement to be proved we may assume that $d_1 \geq d_2$ to begin with. For $x \in S$, denote by F_x the smallest forward-complete set containing x, that is the set of all end points of directed paths starting at x. If F_x is finite, then it is complete by Scholium 2.1.2, contradicting the irreducibility of S which is infinite. Thus F_x is infinite.

Let $F_{x,m}$ be the subset of F_x consisting of all end points of directed paths of length $\leq m$ starting at x. Then $F_{x,m} \subset F_{x,m+1}$ and $F_x = \cup_m F_{x,m}$. If for some m, $F_{x,m} = F_{x,m+1}$, then $F_{x,m}$ is forward-complete, hence $F_x = F_{x,m}$ and F_x is finite, a contradiction. This for every m there exists $x' \in F_{x,m+1} - F_{x,m}$, which proves the lemma. □

Proposition 4.4.7. *If k has characteristic zero, and if D is linearly finitary, then D is finitary.*

Proof. Assume $Q(T_D) = 0$ for $Q \in k[X]$ a nonzero polynomial of degree n and dominant term $a_n \neq 0$. By Lemma 4.4.4, there exists infinitely many étale irreducible complete sets S' such that $Q(A_{S'}) = 0$. For S' any of them, we prove by contradiction that S' is finite. Indeed, choose $x \in S'$, and let x' be a point in S' with a directed path of length n from x to x' but no shorter path, which exists by the lemma above if

S' is infinite; then $0 = (Q(A_{S'})\delta_x) = a_n \,\mathrm{np}_{n,x,x'}$, a contradiction since $\mathrm{np}_{n,x,x'} \geq 1$ is nonzero in k. Thus D has infinitely many finite complete sets, and is therefore finitary. $\square$

Remark 4.4.8. This result is false in positive characteristic p. Take any self-correspondence D on a curve C which is nonfinitary. Then the self-correspondence $D' := D + D + \cdots + D$ (p times) on C is still nonfinitary (it has the same complete sets as D), but the linear operator attached to this correspondence is $T_{D'} = pT_D = 0$, so D' is linearly finitary.

Corollary 4.4.9. *If D is linearly finitary, there is a monic polynomial $Q \in \mathbb{Z}[X]$ such that $Q(T_D) = 0$.*

Proof. There is a nonzero monic polynomial $Q(X) = \sum a_i X^i \in k[X]$ such that $Q(T_D) = 0$. Let k_0 be the prime subfield of k. Let l be a k_0-linear form on k such that $l(1) = 1$ for some i. Then $Q_l(X) = \sum l(a_i) X^i \in k_0[X]$ is monic, and by the equivalence between (i) and (iii') in Lemma 4.4.4, one has $Q_l(T_D) = 0$. If k_0 is $\mathbb{F}_p$ for some p, it suffices to lift Q_l into a monic polynomial in $\mathbb{Z}[X]$. If not, then k has characteristic zero, so D is finitary by Proposition 4.4.7, and there exists a monic polynomial in $\mathbb{Z}[X]$ that kills T_D by Proposition 4.4.2. $\square$

4.5. *The number of irreducible complete finite sets, II.*

Riemann–Roch calculations. Let k be an algebraic closed field, C a curve of genus $g \geq 1$ (just to avoid modifying the formulas in the case $g = 0$) and $S \subset C(k)$ a finite set. We denote as above by Δ_S the effective Weil divisor $\sum_{s \in S}[s]$ in $\operatorname{Pic} C$.

Given a second finite nonempty set S', disjoint from S, and any integer $n \geq 0$, there is clearly a unique relative integer $n' = n'(S, S', n)$ such that

$$2g - 2 + |S'| \geq (n-1)|S| - n'|S'| > 2g - 2. \tag{10}$$

We denote by $V_{S,S',n}$ the subspace of $B_{S,n}$ of functions such that $\operatorname{ord}_{S'} f \geq n'$ where n' is that integer, that is

$$V_{S,S',n} = \{f \in k(C), \operatorname{div} f \geq -(n\Delta_S - n'\Delta_{S'})\}. \tag{11}$$

In plain English, $V_{S,S',n}$ is the space of all algebraic functions on C with poles of order at most n at every point of S, and at most n' at every point of S'. Remember that n' (defined by (10)) may be negative, so that the later requirement may mean that f has *zeros* of order at least $-n'$ on S'.

Lemma 4.5.1. *One has $B_{S,n-1} + V_{S,S',n} = B_{S,n}$ and* $\dim V_{S,S',n} \leq g - 1 + |S| + |S'|$.

Proof. The divisor $D = (n-1)\Delta_S - n'\Delta_{S'}$ has degree $\deg D = (n-1)|S| - n'|S'| > 2g-2$ by assumption. Let $s_0 \in S$. By Riemann–Roch, $h(D + [s_0]) > h(D)$, so there exists a function $f_{s_0} \in k(C)$ such that $\operatorname{div} f_{s_0} \geq -D - [s_0]$ and $\operatorname{ord}_{s_0} f = -n$. Obviously $f_{s_0} \in V_{S,S',n}$ and the functions f_{s_0}, when s_0 runs in S, generate $B_{S,n}/B_{S,n-1}$. Hence the first assertion.

The second assertion also follows from Riemann–Roch, which says that

$$\begin{aligned}\dim V_{S,S',n} &= \deg(n\Delta_S - n'\Delta_{S'}) - g + 1 \qquad (\text{since } \deg(n\Delta_S - n'\Delta_{S'}) > 2g-2 \text{ and } g > 0)\\ &= |S| + (n-1)|S| - n'|S'| - g + 1\\ &\leq |S| + 2g - 2 + |S'| - g + 1 \qquad (\text{by } (10))\\ & g - 1 + |S| + |S'|.\end{aligned}$$

□

Now let S'' a third finite nonempty complete set, disjoint from S and from S'. Let n'' be defined by (10) with S' replaced by S'', and let $V_{S,S'',n'}$ be defined similarly with (11). One has:

Lemma 4.5.2. *There exists an integer n_0 such that for $n > n_0$, $V_{S,S',n} \cap V_{S,S'',n} = 0$.*

Proof. The space $V_{S,S',n} \cap V_{S,S'',n}$ is the space of functions f such that $\operatorname{div} f \geq -(n\Delta S - n'\Delta_{S'} - n''\Delta_{S''})$. Using (10) for n' and n'', one computes

$$\begin{aligned}\deg(n\Delta_S - n'\Delta_{S'} - n''\Delta_{S''}) &= n|S| - n'|S'| - n''|S''|\\ &= (n|S| - n'|S'|) + (n|S| - n''|S''|) - n|S|\\ &\leq 2g - 2 + |S'| + |S''| - n|S|.\end{aligned}$$

This number is negative for $n > (2g - 2 + |S'| + |S''|)/|S|$, and therefore $V_{S,S',n} \cap V_{S,S'',n} = 0$. □

The bound.

Theorem 4.5.3. *Let D be a self-correspondence on a curve C over an arbitrary field k. Assume that T_D is not linearly finitary. Then D has at most two irreducible complete equiramified finite sets.*

Proof. We may and do assume that k is algebraically closed. Also assume (just for simplicity in the formula) that the genus g of C is ≥ 1, the case $C = \mathbb{P}^1$ being taken care of by Proposition 3.2.3. Assume that D admits three irreducible complete equiramified finite sets, S, S' and S''. By Proposition 4.2.2, $V_{S,S',n}$ and $V_{S,S'',n}$ are stable by T_D. Let n_0 be as in Lemma 4.5.2. We claim that for $n > n_0$, every eigenvalue λ of T_D in $B_{S,n}$ is also an eigenvalue of T_D in $B_{S,n-1}$. We may assume that λ is an eigenvalue of T_D in the quotient $B_{S,n}/B_{S,n-1}$, otherwise there is nothing to prove. Let us call $m_\lambda \geq 1$ its multiplicity in $B_{S,n}/B_{S,n-1}$. By Lemma 4.5.1, λ is also an eigenvalue of T_D in V_{S,S'_n} (resp. in $V_{S,S'',n}$) with multiplicity $\geq m_\lambda$. Thus λ appears as an eigenvalue of T_D in $V_{S,S',n} + V_{S,S'',n}$ with multiplicity $\geq 2m_\lambda$, since the sum is direct by Lemma 4.5.2. Thus λ appears as an eigenvalue of T_D with multiplicity $\geq 2m_\lambda > m_\lambda$ in $B_{S,n}$, and it must appear in $B_{S,n-1}$.

By induction, all the eigenvalues of T_D on $B_{S,n}$ (hence on $V_{S,S',n}$) for any n already appear in B_{S,n_0}. Thus there are finitely many eigenvalues of T_D, say $\lambda_1, \ldots, \lambda_l$, appearing in $V_{S,S',n}$ for any n. Define $Q(X) = (X - \lambda_1)^{g-1+|S|+|S'|} \cdots (X - \lambda_l)^{g-1+|S|+|S'|}$. It is clear using Lemma 4.5.1 that $Q(T_D)$ kills $V_{S,S',n}$ for any n, hence, by Lemma 4.5.1, $B_{S,n}$ for any n, and thus $Q(T_D)$ kills B_S, and by Lemma 4.4.4, $Q(T_D)$ kills $k(C)$ contradicting the assumption that D is not linearly finitary. □

Corollary 4.5.4. *Let D be a self-correspondence on a curve C over a field k of characteristic zero. Assume that T_D is not finitary. Then D has at most two irreducible complete equiramified finite sets.*

This follows from the theorem and Proposition 4.4.7.

Remark 4.5.5. The upper bound, two, given by Theorem 4.5.3 for the number of irreducible complete equiramified finite sets for a nonlinearly finite self-correspondence is tight, in any characteristic: this is shown by the example considered in Remark 3.2.2, namely the self-correspondence attached to an automorphism of infinite order of $\mathbb{P}^1_k$, when there exists one (which is the case if and only if k is not algebraic over a finite field).

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Fitting ideals of class groups for CM abelian extensions

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Let K be a finite abelian CM-extension of a totally real field k and T a suitable finite set of finite primes of k. We determine the Fitting ideal of the minus component of the T-ray class group of K, except for the 2-component, assuming the validity of the equivariant Tamagawa number conjecture. As an application, we give a necessary and sufficient condition for the Stickelberger element to lie in that Fitting ideal.

1. Introduction

In number theory, the relationship between class groups and special values of L-functions is of great importance. We discuss such a phenomenon for a finite abelian CM-extension K/k, that is, a finite abelian extension such that k is a totally real field and K is a CM-field. We focus on the minus components of the (ray) class groups of K, except for the 2-components, and study the Fitting ideals of them.

Let Cl_K denote the ideal class group of K. For a $\mathbb{Z}[\mathrm{Gal}(K/k)]$-module M, let M^- denote the minus component after inverting the multiplication by 2. When $k = \mathbb{Q}$, Kurihara and Miura [2011] succeeded in proving a conjecture of Kurihara [2003a] on a description of the Fitting ideal of Cl_K^- using the Stickelberger elements. However, for a general totally real field k, the problem to determine the Fitting ideal of Cl_K^- is still open.

There seems to be an agreement that the *Pontryagin duals* (denoted by $(-)^\vee$) of the class groups are easier to deal with; see Greither and Kurihara [2008]. Greither [2007] determined the Fitting ideal of $\mathrm{Cl}_K^{\vee,-}$, assuming that the minus component of the equivariant Tamagawa number conjecture for $\mathbb{G}_m$ (eTNC for short) holds and that the group of roots of unity in K is cohomologically trivial. Subsequently, Kurihara [2021] generalized the results of Greither on $\mathrm{Cl}_K^{\vee,-}$ to results on $\mathrm{Cl}_K^{T,\vee,-}$, where Cl_K^T denotes the T-ray class group, for a finite set T of finite primes of k. This enables us, by taking suitably large T, to remove the assumption that the group of roots of unity is cohomologically trivial, though we still need to assume the validity of the eTNC. In recent work Dasgupta and Kakde [2023] succeeded in proving *unconditionally* the same formula as Kurihara on the Fitting ideal of $\mathrm{Cl}_K^{T,\vee,-}$ (see (1-2) below for the formula).

In this paper, for a general totally real field k, we determine the Fitting ideal of $\mathrm{Cl}_K^{T,-}$ *without the Pontryagin dual*, assuming the eTNC, except for the 2-component. This problem has been considered to be harder than that on $\mathrm{Cl}_K^{T,\vee,-}$ and actually our result is more complicated. Our main tool is the technique of shifts of Fitting ideals, which was established by Kataoka [2020].

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As an application of the description, we will obtain a necessary and sufficient condition for the Stickelberger element to be in the Fitting ideal of $\mathrm{Cl}_K^{T,-}$ (still assuming the eTNC). Note that the question for the dualized version $\mathrm{Cl}_K^{T,\vee,-}$ is called the strong Brumer–Stark conjecture and is answered affirmatively by Dasgupta and Kakde [2023] unconditionally.

Though we mainly assume the validity of the eTNC in this paper, we also obtain interesting unconditional results. For instance, in Theorem 1.6 we will show that the Fitting ideal of $\mathrm{Cl}_K^{T,-}$ is always contained in that of $\mathrm{Cl}_K^{T,\vee,-}$, and that the inclusion is often proper.

In the rest of this section, we give precise statements of the main results.

1A. ***Description of the Fitting ideal.*** Let K/k be a finite abelian CM-extension and put $G = \mathrm{Gal}(K/k)$. Let $S_\infty(k)$ be the set of archimedean places of k. Let $S_{\mathrm{ram}}(K/k)$ be the set of places of k which are ramified in K/k, including $S_\infty(k)$. For each finite prime $v \in S_{\mathrm{ram}}(K/k)$, let $I_v \subset G$ denote the inertia group of v in G and $\varphi_v \in G/I_v$ the arithmetic Frobenius of v. We then define elements g_v and h_v by

$$g_v = 1 - \varphi_v^{-1} + \#I_v \in \mathbb{Z}[G/I_v], \quad h_v = 1 - \frac{\nu_{I_v}}{\#I_v}\varphi_v^{-1} + \nu_{I_v} \in \mathbb{Q}[G],$$

where we put $\nu_{I_v} = \sum_{\tau \in I_v} \tau$. These elements are introduced in [Greither 2007, Lemmas 6.1 and 8.3] and [Kurihara 2021, Section 2.2, Equations (2.7) and (2.10)] (though in [Kurihara 2021] the same symbols g_v and h_v denote the involutions of ours). Note that $g_v = h_v$ if v is unramified in K/k. Moreover, we define a $\mathbb{Z}[G]$-module A_v by

$$A_v = \mathbb{Z}[G/I_v]/(g_v).$$

We write $\mathbb{Z}[G]^- = \mathbb{Z}[1/2][G]/(1+j)$, where j is the complex conjugation in G. For any $\mathbb{Z}[G]$-module M, we also define the minus component by $M^- = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]^-$. Note that we are implicitly inverting the action of 2. For any $x \in M$, we write x^- for the image of x under the natural map $M \to M^-$.

In general, for a set S of places of k, we write S_K for the set of places of K which lie above places in S. We take and fix a finite set T of finite primes of k satisfying the following:

- $T \cap S_{\mathrm{ram}}(K/k) = \varnothing$.
- $K_T^\times = \{x \in K^\times \mid \mathrm{ord}_w(x-1) > 0 \text{ for all primes } w \in T_K\}$ is torsion free. Here, ord_w denotes the normalized additive valuation.

Note that, if we fix an odd prime number p and are concerned with the p-components, the last condition can be weakened to that $K_T^\times$ is p-torsion-free. We consider the T-ray ideal class group of K defined by

$$\mathrm{Cl}_K^T = \mathrm{Cok}\left(K_T^\times \xrightarrow{\oplus \mathrm{ord}_w} \bigoplus_{w \notin T_K} \mathbb{Z}\right),$$

where w runs over the finite primes of K which are not in T_K.

For a character ψ of G, we write $L(s, \psi)$ for the primitive L-function for ψ. For any finite prime v of k, we put $N(v) = \#\mathbb{F}_v$, where $\mathbb{F}_v$ is the residue field of v. We then define the T-modified L-function by

$$L_T(s, \psi) = \left(\prod_{v \in T}(1 - \psi(\varphi_v)N(v)^{1-s})\right)L(s, \psi).$$

We define

$$\omega_T = \sum_{\psi} L_T(0, \psi)e_{\psi^{-1}} \in \mathbb{Q}[G], \tag{1-1}$$

where ψ runs over the characters of G and $e_\psi = \frac{1}{\#G}\sum_{\sigma \in G}\psi(\sigma)\sigma^{-1}$ is the idempotent of the ψ-component. We actually have $\omega_T \in \mathbb{Q}[G]$ instead of $\omega_T \in \mathbb{C}[G]$, thanks to the Siegel–Klingen theorem.

Now the first main theorem of this paper is the following, whose proof will be given in Section 3.

Theorem 1.1. *Assume that the eTNC for K/k holds. Then we have*

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-}) = \left(\prod_{v \in S_{\mathrm{ram}}(K/k)\setminus S_\infty(k)} h_v^- \,\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]^-}(A_v^-)\right)\omega_T^-,$$

where $\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]^-}$ *is the first shift of the Fitting ideal* (*see Definition 2.3*).

In the second main result below, we will obtain a concrete description of $h_v^- \,\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]^-}(A_v^-)$, which completes the description of the Fitting ideal of $\mathrm{Cl}_K^{T,-}$. We do not review the precise statement of the eTNC; see e.g., [Burns et al. 2016, Conjecture 3.6].

In order to compare with Theorem 1.1, we recall the result for the dualized version:

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,\vee,-}) = \left(\prod_{v \in S_{\mathrm{ram}}(K/k)\setminus S_\infty(k)}\left(\nu_{I_v}, 1 - \frac{\nu_{I_v}}{\#I_v}\varphi_v^{-1}\right)^-\right)\omega_T^-. \tag{1-2}$$

As already mentioned, Kurihara [2021, Corollary 3.7] showed this formula under the validity of the eTNC, and recently Dasgupta and Kakde [2023, Theorem 1.4] removed the assumption. Here, for a general G-module M, we equip the Pontryagin dual $M^\vee$ with the G-action by $(\sigma f)(x) = f(\sigma x)$ for $\sigma \in G$, $f \in M^\vee$, and $x \in M$. This convention is the opposite of [Kurihara 2021] and [Dasgupta and Kakde 2023], so the right-hand side of the formula (1-2) differs from those by the involution.

We now briefly outline the proof of Theorem 1.1. An important ingredient is an exact sequence of $\mathbb{Z}[G]^-$-modules of the form

$$0 \to \mathfrak{A}^- \to W_{S_\infty}^- \to \mathrm{Cl}_K^{T,-} \to 0$$

as in Proposition 3.2, where $\mathfrak{A}^-$ is a projective module of finite rank $\#S'$. Here, S' is an auxiliary finite set of places of k. This sequence was constructed by Kurihara [2021], based on preceding work such as Ritter and Weiss [1996] and Greither [2007], and played a key role in proving (1-2) under the eTNC. Our novel idea is to construct an explicit injective homomorphism from $W_{S_\infty}^-$ to $(\mathbb{Z}[G]^-)^{\oplus \#S'}$ whose cokernel is isomorphic to the direct sum of A_v^- for $v \in S' \setminus S_\infty(k)$. Moreover, assuming the eTNC, we will compute the determinant of the composite map $\mathfrak{A}^- \hookrightarrow W_{S_\infty}^- \hookrightarrow (\mathbb{Z}[G]^-)^{\oplus \#S'}$. By using these observations, we obtain an exact sequence to which the theory of shifts of Fitting ideals can be applied, and then Theorem 1.1 follows.

1B. ***Computation of the shift of Fitting ideal.*** In order to make the formula of Theorem 1.1 more explicit, in Section 4, we will compute $\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A_v)$. This will be accomplished by using a method similar to Greither and Kurihara [2015, Section 1.2], which was actually a motivation for introducing the shifts of Fitting ideals in [Kataoka 2020].

As the problem is purely algebraic, we deal with a general situation as follows (it should be clear from the notation how to apply the results below to the arithmetic situation; simply add subscripts v appropriately). Let G be a finite abelian group. Let I and D be subgroups of G such that $I \subset D \subset G$ and that the quotient D/I is a cyclic group. We choose a generator φ of D/I and put

$$g = 1 - \varphi^{-1} + \#I \in \mathbb{Z}[G/I], \quad h = 1 - \frac{\nu_I}{\#I}\varphi^{-1} + \nu_I \in \mathbb{Q}[G],$$

which are not a zero divisor. We define a finite $\mathbb{Z}[G]$-module A by

$$A = \mathbb{Z}[G/I]/(g).$$

In order to state the result, we introduce some notations. We choose a decomposition

$$I = I_1 \times \cdots \times I_s \tag{1-3}$$

as an abelian group such that I_l is a cyclic group for each $1 \le l \le s$. Here, we do not assume any minimality on this decomposition, so we allow even the extreme case where I_l is trivial for some l.

For each $1 \le l \le s$, we put $\nu_l = \nu_{I_l} = \sum_{\sigma \in I_l} \sigma \in \mathbb{Z}[G]$. We also put $\mathcal{I}_D = \mathrm{Ker}(\mathbb{Z}[G] \to \mathbb{Z}[G/D])$.

Definition 1.2. For $0 \le i \le s$, we define Z_i as the ideal of $\mathbb{Z}[G]$ generated by $\nu_{l_1} \cdots \nu_{l_{s-i}}$ where $(l_1, \dots, l_{s-i})$ runs over all tuples of integers satisfying $1 \le l_1 < \cdots < l_{s-i} \le s$, that is,

$$Z_i = (\nu_{l_1} \cdots \nu_{l_{s-i}} \mid 1 \le l_1 < \cdots < l_{s-i} \le s).$$

We clearly have $Z_0 = (\nu_I) \subset Z_1 \subset \cdots \subset Z_s = (1)$. We then define an ideal $\mathcal{J}$ of $\mathbb{Z}[G]$ by

$$\mathcal{J} = \sum_{i=1}^{s} Z_i \mathcal{I}_D^{i-1}.$$

Note that the definition of Z_i does depend on the choice of the decomposition (1-3). On the other hand, it can be shown directly that the ideal $\mathcal{J}$ is independent from the choice. We omit the direct proof because, at any rate, the independency can be deduced from the discussion in Section 4.

Example 1.3. When $s = 1$, we have

$$\mathcal{J} = (1).$$

When $s = 2$, we have

$$\mathcal{J} = (\nu_1, \nu_2) + \mathcal{I}_D.$$

When $s = 3$, we have

$$\mathcal{J} = (\nu_1\nu_2, \nu_2\nu_3, \nu_3\nu_1) + (\nu_1, \nu_2, \nu_3)\mathcal{I}_D + \mathcal{I}_D^2.$$

In this setting, we can describe $\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A)$ as follows. It is convenient to state the result after multiplying by h.

Theorem 1.4. *We have*

$$h\,\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A) = \left(\nu_I, \left(1 - \frac{\nu_I}{\#I}\varphi^{-1}\right)\mathcal{J}\right)$$

as fractional ideals of $\mathbb{Z}[G]$.

1C. ***Stickelberger element and Fitting ideal.*** As an application of Theorems 1.1 and 1.4, we shall discuss the problem whether or not the Stickelberger element lies in the Fitting ideal of $\mathrm{Cl}_K^{T,-}$.

We return to the setup in Section 1A. Let p be a fixed odd prime number and we shall work over $\mathbb{Z}_p$. Let G' denote the maximal subgroup of G of order prime to p. We put $k_p = K^{G'}$, which is the maximal p-extension of k contained in K. For each character χ of G', we regard $\mathcal{O}_\chi = \mathbb{Z}_p[\mathrm{Im}(\chi)]$ as a $\mathbb{Z}_p[G']$-module via χ, and put $\mathbb{Z}_p[G]^\chi = \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G']} \mathcal{O}_\chi$. For a $\mathbb{Z}_p[G]$-module M, we put $M^\chi = M \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[G]^\chi$, which is a $\mathbb{Z}_p[G]^\chi$-module. For an element $x \in M$, we write x^χ for the image of x by the natural map $M \to M^\chi$. We note that $\mathbb{Z}_p[G]$ is isomorphic to the direct product of $\mathbb{Z}_p[G]^\chi$ if χ runs over the equivalence classes of characters of G'.

From now on, we fix an odd character χ of G'. We define $K_\chi = K^{\mathrm{Ker}(\chi)}$. Then K_χ is a CM-field, $K_\chi \supset k_p$, and K_χ/k_p is a cyclic extension of order prime to p.

We put $S_\chi = S_{\mathrm{ram}}(K_\chi/k)$ and consider the χ-component of the Stickelberger element defined by

$$\theta^\chi_{K/k,T} = \sum_{\psi|_{G'}=\chi} L_{S_\chi,T}(0, \psi)e_{\psi^{-1}} \in \mathbb{Z}_p[G]^\chi, \tag{1-4}$$

where ψ runs over characters of G whose restriction to G' coincides with χ and we write

$$L_{S_\chi,T}(s, \psi) = \left(\prod_{v \in S_\chi \setminus S_\infty(k)} (1 - \psi(\varphi_v))\right)\left(\prod_{v \in T}(1 - \psi(\varphi_v)N(v)^{1-s})\right)L(s, \psi).$$

Note that, comparing (1-1) and (1-4), we have

$$\theta^\chi_{K/k,T} = \left(\prod_{v \in S_\chi \setminus S_\infty(k)} \left(1 - \frac{\nu_{I_v}}{\#I_v}\varphi_v^{-1}\right)^\chi\right)\omega_T^\chi. \tag{1-5}$$

Concerning the dualized version, by the work of Dasgupta and Kakde [2023, Theorem 1.3], we always have

$$\theta^\chi_{K/k,T} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^\chi}((\mathrm{Cl}_K^T \otimes \mathbb{Z}_p)^{\vee,\chi}).$$

This is called the strong Brumer–Stark conjecture. More precisely, the displayed claim is a bit stronger than [Dasgupta and Kakde 2023, Theorem 1.3] as we took S_χ instead of $S_{\mathrm{ram}}(K/k)$ in the definition of the Stickelberger element, but in any case it is an immediate consequence of the formula (1-2).

On the other hand, the corresponding claim without dual is known to be false in general; see [Greither and Kurihara 2008]. However, we had only partial results and an exact condition was unknown. The following theorem is strong as it gives a *necessary and sufficient* condition.

Theorem 1.5. *Assume that the eTNC for K/k holds* (*indeed*, *the p-part of the eTNC suffices*). *Then*, *for each odd character χ of G'*, *the following are equivalent*:

(i) *We have $\theta_{K/k,T}^{\chi} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^{\chi}}((\mathrm{Cl}_K^T \otimes \mathbb{Z}_p)^{\chi})$.*

(ii) *We have either $\theta_{K/k,T}^{\chi} = 0$ or, for any $v \in S_{\chi} \setminus S_{\infty}(k)$, one of the following holds.*

  (a) *v does not split completely in K_{χ}/k_p.*

  (b) *The inertia group I_v is cyclic.*

This theorem will be proved in Section 5 as an application of Theorems 1.1 and 1.4. Note that there is an elementary equivalent condition for $\theta_{K/k,T}^{\chi} = 0$ as in Lemma 5.3.

Theorem 1.5 indicates that the failure of the inertia groups to be cyclic is an obstruction for studying the Fitting ideal of the class group *without dual*. The same phenomenon will appear again in Theorem 1.6 below. We should say that this kind of phenomenon had been observed in preceding work, such as [Greither and Kurihara 2008]. Nickel [2011, Section 4] studied much the same subject when all the p-adic primes are tamely ramified. In that case, the inertia groups are indeed cyclic, so a main result [Nickel 2011, Section 4.2, Theorem 5] is now a part of Theorem 1.5.

It is also remarkable that the obstruction does not occur in the absolutely abelian case (i.e., when $k = \mathbb{Q}$), since in that case the inertia groups are automatically cyclic, apart from the 2-parts. This seems to fit the fact that Kurihara [2003a] and Kurihara and Miura [2011] succeeded in studying the class groups without dual in the absolutely abelian case.

Let us outline the proof of Theorem 1.5. We assume that χ is a faithful character of G' (i.e., $K_{\chi} = K$); actually we can deduce the general case from this case. Since ω_T^{χ} is a not a zero divisor of $\mathbb{Z}_p[G]^{\chi}$, by Theorem 1.1 and (1-5), we have $\theta_{K/k,T}^{\chi} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^{\chi}}((\mathrm{Cl}_K^T \otimes \mathbb{Z}_p)^{\chi})$ if and only if

$$\prod_{v} \left(1 - \frac{\nu_{I_v}}{\# I_v} \varphi_v^{-1}\right)^{\chi} \subset \prod_{v} (h_v \, \mathrm{Fitt}_{\mathbb{Z}_p[G]}^{[1]}(A_v \otimes \mathbb{Z}_p))^{\chi} \tag{1-6}$$

holds as fractional ideals of $\mathbb{Z}_p[G]^{\chi}$, where on both sides v runs over the elements of $S_{\mathrm{ram}}(K/k) \setminus S_{\infty}(k)$.

Obviously we may assume that $\theta_{K/k,T}^{\chi} \neq 0$. The proof of (ii) $\Rightarrow$ (i) is the easier part. We will show that, under the assumption (ii), the inclusion of (1-6) holds even for every v before taking the product. On the other hand, the opposite direction (i) $\Rightarrow$ (ii) is the harder part. That is because, roughly speaking, we have to work over the ring $\mathbb{Z}_p[G]^{\chi}$, whose ring theoretic properties are not very nice. A key idea to overcome this issue is to reduce to a computation in a discrete valuation ring. More concretely, we make use of a character ψ of G which satisfies $\psi|_{G'} = \chi$ and a certain additional condition, whose existence is verified by Lemma 5.3, and then we consider the $\mathbb{Z}_p[G]^{\chi}$-algebra $\mathcal{O}_{\psi} = \mathbb{Z}_p[\mathrm{Im}(\psi)]$. By investigating the ideals in (1-6) after base change from $\mathbb{Z}_p[G]^{\chi}$ to $\mathcal{O}_{\psi}$, we will show (i) $\Rightarrow$ (ii).

1D. ***Unconditional consequences.*** Even if we do not assume the validity of the eTNC, our argument shows the following.

Theorem 1.6. *We have an inclusion*

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-}) \subset \mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,\vee,-}).$$

Moreover, the inclusion is an equality if I_v is cyclic for every $v \in S_{\mathrm{ram}}(K/k) \setminus S_\infty(k)$.

This theorem follows immediately from Corollaries 3.7 and 4.2. Furthermore, by similar arguments as the proof of Theorem 1.5, we can observe that the inclusion is often proper.

As already remarked, Dasgupta and Kakde [2023] proved the formula (1-2) *unconditionally*. Therefore, if I_v is cyclic for every $v \in S_{\mathrm{ram}}(K/k) \setminus S_\infty(k)$, we can also deduce from Theorem 1.6 that $\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-})$ also coincides with that ideal, and this removes the assumption on the eTNC in Theorem 1.1. However, in Theorem 1.1 we still need to assume the eTNC when I_v is not cyclic for some v.

2. Definition of Fitting ideals and their shifts

In this section, we fix our notations concerning Fitting ideals.

2A. ***Fitting ideals.*** Let R be a noetherian ring.

Definition 2.1 [Northcott 1976]. We define the Fitting ideals as follows:

(i) Let A be a matrix over R with m rows and n columns. For each integer $0 \le i \le n$, we define $\mathrm{Fitt}_{i,R}(A)$ as the ideal of R generated by the $(n-i) \times (n-i)$ minors of A. For each integer $i > n$, we also define $\mathrm{Fitt}_{i,R}(A) = (1)$.

(ii) Let X be a finitely generated R-module. We choose a finite presentation A of X with m rows and n columns, that is, an exact sequence

$$R^m \xrightarrow{\times A} R^n \to X \to 0.$$

Here and henceforth, as a convention, we deal with row vectors, so we multiply matrices from the right. Then, for each $i \ge 0$, we define the i-th Fitting ideal of X by

$$\mathrm{Fitt}_{i,R}(X) = \mathrm{Fitt}_{i,R}(A).$$

It is known that this ideal does not depend on the choice of A. When $i = 0$, we also write $\mathrm{Fitt}_R(X) = \mathrm{Fitt}_{0,R}(X)$ and call it the initial Fitting ideal.

We will later make use of the following elementary lemma. We omit the proof; see [Kurihara 2003b, Lemma 3.3].

Lemma 2.2. *Let X be a finitely generated R-module and I be an ideal of R. If X is generated by n elements over R, then we have*

$$\mathrm{Fitt}_{0,R}(X/IX) = \sum_{i=0}^{n} I^i \,\mathrm{Fitt}_{i,R}(X).$$

2B. *Shifts of Fitting ideals.* In this subsection, we review the definition of shifts of Fitting ideals introduced by Kataoka [2020].

Although we can deal with a more general situation, for simplicity we consider the following. Let Λ be a Dedekind domain (e.g., $\Lambda = \mathbb{Z}$, $\mathbb{Z}\big[\frac{1}{2}\big]$, or $\mathbb{Z}_p$). Let Δ be a finite abelian group and consider the ring $R = \Lambda[\Delta]$.

We define $\mathcal{C}$ as the category of R-modules of finite length. We also define a subcategory $\mathcal{P}$ of $\mathcal{C}$ by

$$\mathcal{P} = \{P \in \mathcal{C} \mid \mathrm{pd}_R(P) \le 1\},$$

where pd_R denotes the projective dimension over R. Note that any module M in $\mathcal{C}$ satisfies $\mathrm{pd}_\Lambda(M) \le 1$.

Definition 2.3. Let X be an R-module in $\mathcal{C}$ and $d \ge 0$ an integer. We take an exact sequence

$$0 \to Y \to P_1 \to \cdots \to P_d \to X \to 0$$

in $\mathcal{C}$ with $P_1, \dots, P_d \in \mathcal{P}$. Then we define

$$\mathrm{Fitt}_R^{[d]}(X) = \left(\prod_{i=1}^{d} \mathrm{Fitt}_R(P_i)^{(-1)^i} \right) \mathrm{Fitt}_R(Y).$$

The well-definedness (i.e., the independence from the choice of the d-step resolution) is proved in [Kataoka 2020, Theorem 2.6 and Proposition 2.7].

We also introduce a variant for the case where d is negative.

Definition 2.4. Let X be an R-module in $\mathcal{C}$ and $d \le 0$ an integer. We take an exact sequence

$$0 \to X \to P_{-d} \to \cdots \to P_1 \to Y \to 0$$

in $\mathcal{C}$ with $P_1, \dots, P_{-d} \in \mathcal{P}$. Then we define

$$\mathrm{Fitt}_R^{\langle d \rangle}(X) = \left(\prod_{i=1}^{-d} \mathrm{Fitt}_R(P_i)^{(-1)^i} \right) \mathrm{Fitt}_R(Y).$$

The well-definedness is proved in [Kataoka 2020, Theorem 3.19 and Propositions 2.7 and 3.17].

3. Fitting ideals of ideal class groups

In this section, we prove Theorem 1.1, which describes the Fitting ideal of $\mathrm{Cl}_K^{T,-}$ using shifts of Fitting ideals. We keep the notation in Section 1A.

3A. ***Brief review of work of Kurihara.*** We first review necessary ingredients from [Kurihara 2021], which in turn relies on preceding work, in particular [Ritter and Weiss 1996] and [Greither 2007].

For each place w of K, let D_w and I_w denote the decomposition subgroup and the inertia subgroup of w in G, respectively. These subgroups depend only on the place of k which lies below w.

Let us introduce local modules W_v. For any finite group H, we define ΔH as the augmentation ideal in $\mathbb{Z}[H]$.

Definition 3.1. For each finite prime w of K, we define a $\mathbb{Z}[D_w]$-module W_{K_w} by

$$W_{K_w} = \{(x, y) \in \Delta D_w \oplus \mathbb{Z}[D_w/I_w] \mid \bar{x} = (1 - \varphi_v^{-1})y\},$$

where $\bar{x}$ denotes the image of x in $\mathbb{Z}[D_w/I_w]$. For each finite prime v of k, we define the $\mathbb{Z}[G]$-module W_v by taking the direct sum as

$$W_v = \bigoplus_{w \mid v} W_{K_w},$$

where w runs over the finite primes of K which lie above v. Alternatively, W_v can be regarded as the induced module of W_{K_w} from D_w to G, as long as we choose a place w of K above v.

We take an auxiliary finite set S' of places of k satisfying the following conditions:

- $S' \supset S_{\mathrm{ram}}(K/k)$.
- $S' \cap T = \varnothing$.
- $\mathrm{Cl}_{K,S'}^T = 0$, where $\mathrm{Cl}_{K,S'}^T = \mathrm{Cok}(K_T^\times \xrightarrow{\oplus \mathrm{ord}_w} \bigoplus_{w \notin S'_K \cup T_K} \mathbb{Z})$.
- G is generated by the decomposition groups D_v of v for all $v \in S'$.

We define a $\mathbb{Z}[G]$-module W_{S_∞} by

$$W_{S_\infty} = \bigoplus_{w \in S_\infty(K)} \Delta D_w \oplus \bigoplus_{v \in S' \setminus S_\infty(k)} W_v.$$

By using local and global class field theory, Kurihara constructed an exact sequence of the following form.

Proposition 3.2 [Kurihara 2021, Section 2.2, sequence (2.5)]. *We have an exact sequence*

$$0 \to \mathfrak{A}^- \to W_{S_\infty}^- \to \mathrm{Cl}_K^{T,-} \to 0,$$

where $\mathfrak{A}^-$ is a projective $\mathbb{Z}[G]^-$-module of rank $\#S'$.

Remark 3.3. Kurihara [2021] took the linear dual of this sequence, and the resulting sequence played an important role to study $\mathrm{Cl}_K^{T,\vee,-}$. In this paper, we do not take the linear dual but instead study the sequence itself for the proof of Theorem 1.1.

3B. ***Definition of f_v.*** Our key ingredient for the proof of Theorem 1.1 is the following homomorphism f_v.

Definition 3.4. For a finite prime w of K, we define a $\mathbb{Z}[D_w]$-homomorphism

$$f_w : W_{K_w} \to \mathbb{Z}[D_w]$$

by $f_w(x, y) = x + \nu_{I_w}(y)$ (recall the definition of W_{K_w} in Definition 3.1). For a finite prime v of k, we then define a $\mathbb{Z}[G]$-homomorphism $f_v : W_v \to \mathbb{Z}[G]$ by

$$f_v : W_v = \bigoplus_{w|v} W_{K_w} \xrightarrow{\oplus f_w} \bigoplus_{w|v} \mathbb{Z}[D_w] \simeq \mathbb{Z}[G],$$

where the last isomorphism depends on a choice of w.

In Section 1A we introduced a finite $\mathbb{Z}[G]$-module $A_v = \mathbb{Z}[G/I_v]/(g_v)$ with $g_v = 1 - \varphi_v^{-1} + \#I_v$. It is actually motivated by the following.

Lemma 3.5. *For any finite prime v of k, the map f_v is injective and*

$$\operatorname{Cok} f_v \simeq A_v.$$

Proof. It is enough to show that f_w is injective and $\operatorname{Cok} f_w \simeq \mathbb{Z}[D_w/I_w]/(g_v)$ for any finite prime w of K. Put $J_w = \operatorname{Ker}(\mathbb{Z}[D_w] \to \mathbb{Z}[D_w/I_w])$. We define a homomorphism $\alpha_w : J_w \to W_{K_w}$ by $\alpha_w(x) = (x, 0)$. Let us consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & J_w \arrow[r, "\alpha_w"] \arrow[d, equal] & W_{K_w} \arrow[r] \arrow[d, "f_w"] & \operatorname{Cok}\alpha_w \arrow[r] \arrow[d, "f'_w"] & 0 \\
0 \arrow[r] & J_w \arrow[r] & \mathbb{Z}[D_w] \arrow[r] & \mathbb{Z}[D_w/I_w] \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is the trivial one, the commutativity of the left square is easy, and the right vertical arrow is the induced one. By the definition of W_{K_w}, we have

$$\operatorname{Cok}\alpha_w = \{(\bar{x}, y) \in \Delta(D_w/I_w) \times \mathbb{Z}[D_w/I_w] \mid \bar{x} = (1 - \varphi_v^{-1})y\}.$$

Since D_w/I_w is a cyclic group generated by φ_v^{-1}, the $\mathbb{Z}[D_w/I_w]$-module $\operatorname{Cok}\alpha_w$ is free of rank 1 with a basis $(1 - \varphi_v^{-1}, 1)$. Moreover, f'_w sends this basis to $g_v = 1 - \varphi_v^{-1} + \#I_v$. Therefore, f'_w is injective with cokernel isomorphic to $\mathbb{Z}[D_w/I_w]/(g_v)$. Then by the diagram f_w also satisfies the desired properties. □

For any $v \in S' \setminus S_\infty(k)$, we consider the homomorphism $f_v^- : W_v^- \to \mathbb{Z}[G]^-$ which is the minus component of f_v. For any $v \in S_\infty(k)$, we have $(\oplus_{w|v} \Delta D_w)^- \simeq \mathbb{Z}[G]^-$ by choosing w, so we fix this isomorphism and write f_v^- for it. Using these f_v^-, we consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{A}^- \arrow[r] \arrow[d, equal] & W_{S_\infty}^- \arrow[r] \arrow[d, "\oplus_{v\in S'} f_v^-"] & \mathrm{Cl}_K^{T,-} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{A}^- \arrow[r, "\psi"] & \bigoplus_{v\in S'} \mathbb{Z}[G]^- \arrow[r] & \operatorname{Cok}\psi \arrow[r] & 0,
\end{tikzcd}$$

where the upper sequence is that in Proposition 3.2 and the map ψ is defined by the commutativity. By Lemma 3.5 and the snake lemma, we get the following proposition.

Proposition 3.6. *We have an exact sequence*

$$0 \to \mathrm{Cl}_K^{T,-} \to \mathrm{Cok}\,\psi \to \bigoplus_{v \in S' \setminus S_\infty(k)} A_v^- \to 0.$$

Moreover, the $\mathbb{Z}[G]^-$*-module* $\mathrm{Cok}\,\psi$ *is finite with* $\mathrm{pd}_{\mathbb{Z}[G]^-}(\mathrm{Cok}\,\psi) \le 1$.

Then we can describe the Fitting ideals of $\mathrm{Cl}_K^{T,-}$ and of $\mathrm{Cl}_K^{T,\vee,-}$ as follows.

Corollary 3.7. *We have*

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-}) = \mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cok}\,\psi) \prod_{v \in S' \setminus S_\infty(k)} \mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]^-}(A_v^-)$$

and

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,\vee,-}) = \mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cok}\,\psi) \prod_{v \in S' \setminus S_\infty(k)} \mathrm{Fitt}^{\langle -1 \rangle}_{\mathbb{Z}[G]^-}(A_v^-).$$

Proof. The first formula follows directly from Proposition 3.6 and Definition 2.3. For the second formula, by [Kataoka 2020, Proposition 4.7], we have

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,\vee,-}) = \mathrm{Fitt}^{\langle -2 \rangle}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-}).$$

By Proposition 3.6 and Definition 2.4, we also have

$$\mathrm{Fitt}^{\langle -2 \rangle}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-}) = \mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cok}\,\psi) \prod_{v \in S' \setminus S_\infty(k)} \mathrm{Fitt}^{\langle -1 \rangle}_{\mathbb{Z}[G]^-}(A_v^-).$$

This completes the proof. □

3C. *Fitting ideal of* Cok ψ.

Recall the definitions of ω_T and of h_v in Section 1A.

Theorem 3.8. *Assume that the eTNC for K/k holds. Then we have*

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cok}\,\psi) = \Bigg(\bigg(\prod_{v \in S' \setminus S_\infty(k)} h_v^-\bigg)\omega_T^-\Bigg).$$

Proof. For each $v \in S' \setminus S_\infty(k)$, we define a basis e_v of $\mathrm{Hom}_{\mathbb{Q}[G]}(W_v \otimes \mathbb{Q}, \mathbb{Q}[G])$ as in [Kurihara 2021, Section 2.2, Equation (2.9)] (we do not recall the precise definition here). Then we can see that its dual basis e'_v of $W_v \otimes \mathbb{Q}$ is given by

$$e'_v = \frac{1}{1 - \widetilde{\varphi_v}^{-1} + N_{I_v}}(1 - \widetilde{\varphi_v}^{-1}, 1),$$

where $\widetilde{\varphi_v}$ is a lift of φ_v. Then, by the definition of f_v, this element satisfies $f_v(e'_v) = 1$, where by abuse of notation f_v denotes the homomorphism $W_v \otimes \mathbb{Q} \to \mathbb{Q}[G]$ induced by $f_v : W_v \to \mathbb{Z}[G]$. For $v \in S_\infty(k)$, as a basis over $\mathbb{Z}[G]^-$, we take the element $e_v^{\prime,-}$ of $\big(\bigoplus_{w \mid v} \Delta D_w\big)^-$ which is characterized by $f_v^-(e_v^{\prime,-}) = 1$.

Let us consider the isomorphism $\Psi : \mathfrak{A}^- \otimes \mathbb{Q} \to W_{S_\infty}^- \otimes \mathbb{Q}$ induced by the sequence in Proposition 3.2. Then, under the eTNC, Kurihara [2021, Theorem 3.6] proved that $\mathfrak{A}^-$ is a free $\mathbb{Z}[G]^-$-module (a priori we only know $\mathfrak{A}^-$ is projective) and

$$\det(\Psi) = \biggl(\prod_{v \in S' \setminus S_\infty(k)} h_v^-\biggr)\omega_T^-$$

with respect to a certain basis of $\mathfrak{A}^-$ as a $\mathbb{Z}[G]^-$-module and the basis $(e_v'^{,-})_{v \in S'}$ of $W_{S_\infty}^-$. Actually this is an easy reformulation of the result of Kurihara, which concerns the determinant of the linear dual of Ψ.

Therefore, the determinant of the composite map ψ of Ψ and $\bigoplus_{v \in S'} f_v^-$, with respect to the basis of $\mathfrak{A}^-$ and the standard basis of $(\mathbb{Z}[G]^-)^{\oplus \# S'}$, also coincides with $\bigl(\prod_{v \in S' \setminus S_\infty(k)} h_v^-\bigr)\omega_T^-$. This shows the theorem. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Corollary 3.7 and Theorem 3.8, we have

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,-}) = \biggl(\prod_{v \in S' \setminus S_\infty(k)} h_v^- \,\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]^-}(A_v^-)\biggr)\omega_T^-.$$

For $v \in S' \setminus S_{\mathrm{ram}}(K/k)$, we have $A_v = \mathbb{Z}[G]/(h_v)$, so

$$\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]^-}(A_v^-) = (h_v^-)^{-1}.$$

Then Theorem 1.1 follows. $\square$

Remark 3.9. Similarly, under the validity of the eTNC, Corollary 3.7 and Theorem 3.8 also imply a formula

$$\mathrm{Fitt}_{\mathbb{Z}[G]^-}(\mathrm{Cl}_K^{T,\vee,-}) = \biggl(\prod_{v \in S_{\mathrm{ram}}(K/k) \setminus S_\infty(k)} h_v^- \,\mathrm{Fitt}^{\langle -1 \rangle}_{\mathbb{Z}[G]^-}(A_v^-)\biggr)\omega_T^-.$$

Combining this with Proposition 4.1 below, we can recover the formula (1-2). This argument may be regarded as a reinterpretation of the work [Kurihara 2021] by using the shifts of Fitting ideals.

4. Computation of shifts of Fitting ideals

In this section, we prove Theorem 1.4 on the description of $\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A)$. We keep the notations as in Section 1B.

4A. ***Computation of*** $\mathbf{Fitt}^{\langle -1 \rangle}_{\mathbb{Z}[G]}(A)$. Before $\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A)$, we determine $\mathrm{Fitt}^{\langle -1 \rangle}_{\mathbb{Z}[G]}(A)$, which is actually much easier.

We choose a lift $\widetilde{\varphi} \in D$ of φ and put

$$\tilde{g} = 1 - \widetilde{\varphi}^{-1} + \# I \in \mathbb{Z}[G],$$

which is again a not a zero divisor. Obviously, g is then the natural image of $\tilde{g}$ to $\mathbb{Z}[G/I]$.

Proposition 4.1. *We have*

$$\mathrm{Fitt}_{\mathbb{Z}[G]}^{\langle -1\rangle}(A) = (1, \nu_I g^{-1}).$$

Therefore, *we also have*

$$h\,\mathrm{Fitt}_{\mathbb{Z}[G]}^{\langle -1\rangle}(A) = \left(\nu_I, 1 - \frac{\nu_I}{\#I}\varphi^{-1}\right).$$

Proof. We have an exact sequence

$$0 \to \mathbb{Z}[G/I] \xrightarrow{\nu_I} \mathbb{Z}[G] \to \mathbb{Z}[G]/(\nu_I) \to 0.$$

Since multiplication by $\tilde{g}$ is injective on each of these modules, applying the snake lemma, we obtain an exact sequence

$$0 \to A \to \mathbb{Z}[G]/(\tilde{g}) \to \mathbb{Z}[G]/(\tilde{g}, \nu_I) \to 0.$$

By Definition 2.4, we then have

$$\mathrm{Fitt}_{\mathbb{Z}[G]}^{\langle -1\rangle}(A) = (\tilde{g})^{-1}(\tilde{g}, \nu_I) = (1, \nu_I g^{-1}).$$

This proves the former formula of the proposition.

Since we have $\nu_I g = \nu_I h$, the former formula implies $h\,\mathrm{Fitt}_{\mathbb{Z}[G]}^{\langle -1\rangle}(A) = (\nu_I, h)$. Then the latter formula follows from $h \equiv 1 - \frac{\nu_I}{\#I}\varphi^{-1} \pmod{(\nu_I)}$. ☐

Before proving Theorem 1.4, we show a corollary.

Corollary 4.2. *We have an inclusion*

$$\mathrm{Fitt}_{\mathbb{Z}[G]}^{[1]}(A) \subset \mathrm{Fitt}_{\mathbb{Z}[G]}^{\langle -1\rangle}(A).$$

Moreover, *if I is a cyclic group*, *the inclusion is an equality.*

Proof. By Definition 1.2, the ideal $\mathcal{J}$ is contained in $\mathbb{Z}[G]$ and we have $\mathcal{J} = \mathbb{Z}[G]$ if I is cyclic. Hence this corollary immediately follows from Theorem 1.4 and Proposition 4.1. ☐

4B. ***Computation of*** $\mathbf{Fitt}_{\mathbb{Z}[G]}^{[1]}(A)$**.** This subsection is devoted to the proof of Theorem 1.4.

We fix the decomposition (1-3) of I. For each $1 \le l \le s$, we choose a generator σ_l of I_l and put $\tau_l = \sigma_l - 1 \in \mathbb{Z}[G]$. Note that we then have $\nu_l = 1 + \sigma_l + \sigma_l^2 + \cdots + \sigma_l^{\#I_l - 1}$ and $\tau_l \nu_l = 0$. As in Section 4A, we put $\tilde{g} = 1 - \widetilde{\varphi}^{-1} + \#I$ after choosing $\widetilde{\varphi}$.

We recall $\mathcal{I}_D = \mathrm{Ker}(\mathbb{Z}[G] \to \mathbb{Z}[G/D])$ and also put $\mathcal{I}_I = \mathrm{Ker}(\mathbb{Z}[G] \to \mathbb{Z}[G/I])$. Then we have $\mathcal{I}_I = (\tau_1, \dots, \tau_s)$ and $\mathcal{I}_D = (\mathcal{I}_I, 1 - \widetilde{\varphi}^{-1})$.

We begin with a proposition.

Proposition 4.3. *We have*

$$\mathrm{Fitt}_{\mathbb{Z}[G]}^{[1]}(A) = \sum_{i=0}^{s} \tilde{g}^{i-1}\,\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I).$$

Proof. We have the tautological exact sequence

$$0 \to \mathcal{I}_I \to \mathbb{Z}[G] \to \mathbb{Z}[G/I] \to 0.$$

Since multiplication by $\tilde{g}$ is injective on each of these modules, by applying snake lemma, we obtain an exact sequence

$$0 \to \mathcal{I}_I/\tilde{g}\mathcal{I}_I \to \mathbb{Z}[G]/(\tilde{g}) \to A \to 0.$$

Then Definition 2.3 implies

$$\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A) = \tilde{g}^{-1}\,\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathcal{I}_I/\tilde{g}\mathcal{I}_I).$$

Since $\mathcal{I}_I$ is generated by the s elements $\tau_1, \dots, \tau_s$, we have

$$\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathcal{I}_I/\tilde{g}\mathcal{I}_I) = \sum_{i=0}^{s} \tilde{g}^i\,\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I)$$

by Lemma 2.2. Thus we obtain the proposition. □

Our next task is to determine $\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I)$ for $0 \le i \le s$. The result will be Proposition 4.9 below. For that purpose, we construct a concrete free resolution of $\mathbb{Z}$ over $\mathbb{Z}[I]$, using an idea of Greither and Kurihara [2015, Section 1.2]; one may also refer to [Kataoka 2020, Section 4.3]. We will have to deal with a large matrix denoted by $M_s(\nu_1, \dots, \nu_s, \tau_1, \dots, \tau_s)$, but it is not surprising; in a relevant study Greither, Kurihara and Tokio [Greither et al. 2020] dealt with an even more complicated matrix.

For each $1 \le l \le s$, we have a homological complex

$$C^l : \cdots \xrightarrow{\tau_l} \mathbb{Z}[I_l] \xrightarrow{\nu_l} \mathbb{Z}[I_l] \xrightarrow{\tau_l} \mathbb{Z}[I_l] \to 0$$

over $\mathbb{Z}[I_l]$, concentrated at degrees ≥ 0. Let C^l_n be the degree n component of C^l, so $C^l_n = \mathbb{Z}[I_l]$ if $n \ge 0$ and $C^l_n = 0$ otherwise. Then the homology groups are $H_n(C^l) = 0$ for $n \ne 0$ and $H_0(C^l) \simeq \mathbb{Z}$.

We define a complex C over $\mathbb{Z}[I]$ by

$$C = \bigotimes_{l=1}^{s} C^l,$$

which is the tensor product of complexes over $\mathbb{Z}$ (we do not specify the sign convention as it does not matter to us; we define it appropriately so that the descriptions of d_1 and d_2 below are valid). Explicitly, the degree n component C_n of C is defined as

$$C_n = \bigoplus_{n_1+\cdots+n_s=n} C^1_{n_1} \otimes \cdots \otimes C^s_{n_s}.$$

Clearly the tensor product is zero unless $n_1, \dots, n_s \ge 0$, and in that case

$$C^1_{n_1} \otimes \cdots \otimes C^s_{n_s} = \mathbb{Z}[I_1] \otimes \cdots \otimes \mathbb{Z}[I_s] \simeq \mathbb{Z}[I].$$

It is convenient to write $x_1^{n_1} \cdots x_s^{n_s}$ for the basis of $C^1_{n_1} \otimes \cdots \otimes C^s_{n_s}$ for each $n_1, \dots, n_s \geq 0$, following [Greither and Kurihara 2015]. Then, for each $n \geq 0$, the module C_n is a free module on the set of monomials of $x_1, \dots, x_s$ of degree n.

A basic property of tensor products of complexes implies that $H_n(C) = 0$ for $n \neq 0$ and $H_0(C) \simeq \mathbb{Z}$. Therefore, C is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[I]$.

It will be necessary to investigate some components of C of low degrees. Note that C_0 is free of rank one with a basis $1(= x_1^0 \cdots x_s^0)$, C_1 is a free module on the set

$$S_1 = \{x_1, \dots, x_s\},$$

and C_2 is a free module on the set $S_2 \cup S_2'$ where

$$S_2 = \{x_1^2, \dots, x_s^2\}, \quad S_2' = \{x_l x_{l'} \mid 1 \leq l < l' \leq s\}.$$

Moreover, the differential $d_n : C_n \to C_{n-1}$ for $n = 1, 2$ are described as follows. We have

$$d_1(x_l) = \tau_l \cdot 1$$

for each $1 \leq l \leq s$,

$$d_2(x_l^2) = \nu_l x_l$$

for each $1 \leq l \leq s$, and

$$d_2(x_l x_{l'}) = \tau_l x_{l'} - \tau_{l'} x_l$$

for each $1 \leq l < l' \leq s$.

Let M denote the presentation matrix of d_2. For clarity, we define M formally as follows.

Definition 4.4. We define a matrix

$$M = M_s(\nu_1, \dots, \nu_s, \tau_1, \dots, \tau_s)$$

with the columns (resp. the rows) indexed by S_1 (resp. $S_2 \cup S_2'$), by

$$\begin{cases} \text{the } (x_l^2, x_l) \text{ component is } \nu_l & \text{for } 1 \leq l \leq s, \\ \text{the } (x_l x_{l'}, x_l) \text{ component is } -\tau_{l'} & \text{for } 1 \leq l < l' \leq s, \\ \text{the } (x_l x_{l'}, x_{l'}) \text{ component is } \tau_l & \text{for } 1 \leq l < l' \leq s, \\ \text{and the other components are zero.} \end{cases}$$

Here, we do not specify the orders of the sets S_1 and $S_2 \cup S_2'$. The ambiguity does not matter for our purpose.

For later use, we also define a matrix

$$N_s(\tau_1, \dots, \tau_s)$$

as the submatrix of M with the rows in S_2 removed. More precisely, we define the matrix $N_s(\tau_1, \dots, \tau_s)$ with the columns (resp. rows) indexed by S_1 (resp. S_2'), by

$$\begin{cases} \text{the } (x_l x_{l'}, x_l) \text{ component is } -\tau_{l'} & \text{for } 1 \le l < l' \le s, \\ \text{the } (x_l x_{l'}, x_{l'}) \text{ component is } \tau_l & \text{for } 1 \le l < l' \le s, \\ \text{and the other components are zero.} \end{cases}$$

Therefore, by choosing appropriate orders of rows and columns, we have:

$$M_s(\nu_1, \dots, \nu_s, \tau_1, \dots, \tau_s) = \begin{pmatrix} \nu_1 & & \\ & \ddots & \\ & & \nu_s \\ & N_s(\tau_1, \dots, \tau_s) & \end{pmatrix}$$

Example 4.5. When $s = 3$, we have:

$$M = \begin{pmatrix} \nu_1 & & \\ & \nu_2 & \\ & & \nu_3 \\ & -\tau_3 & \tau_2 \\ -\tau_3 & & \tau_1 \\ -\tau_2 & \tau_1 & \end{pmatrix}$$

Here, we use the order x_2x_3, x_1x_3, x_1x_2 for the set S_2'.

Proposition 4.6. *The matrix $M_s(\nu_1, \dots, \nu_s, \tau_1, \dots, \tau_s)$, over $\mathbb{Z}[G]$, is a presentation matrix of the module $\mathcal{I}_I$.*

Proof. By the construction, M is a presentation matrix of $\mathrm{Ker}(\mathbb{Z}[I] \to \mathbb{Z})$ over $\mathbb{Z}[I]$. Since $\mathbb{Z}[G]$ is flat over $\mathbb{Z}[I]$, we obtain the proposition by base change. ☐

Proposition 4.7. *For each $0 \le i \le s$, we have*

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(M) = \sum_{j=0}^{s-i} \sum_{\substack{\boldsymbol{a} \subset \{1,2,\dots,s\} \\ \#\boldsymbol{a} = j}} \nu_{a_1} \cdots \nu_{a_j} \, \mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_{s-j}(\tau_{a_{j+1}}, \dots, \tau_{a_s})).$$

Here, for each j, in the second summation $\boldsymbol{a}$ runs over subsets of $\{1, 2, \dots, s\}$ of j elements, and for each $\boldsymbol{a}$ we define $a_1, \dots, a_s$ by requiring

$$\boldsymbol{a} = \{a_1, \dots, a_j\}, \quad \{a_1, \dots, a_s\} = \{1, 2, \dots, s\}, \quad a_1 < \cdots < a_j, \quad a_{j+1} < \cdots < a_s.$$

The matrix $N_{s-j}(\tau_{a_{j+1}}, \dots, \tau_{a_s})$ is defined as in Definition 4.4 for $s - j$ and $\tau_{a_{j+1}}, \dots, \tau_{a_s}$ instead of s and $\tau_1, \dots, \tau_s$.

Proof. By the definition of higher Fitting ideals, $\mathrm{Fitt}_{i,\mathbb{Z}[G]}(M)$ is generated by $\det(H)$ for square submatrices H of M of size $s - i$. Such a matrix H is in one-to-one correspondence with choices of a subset

$A_H^{\text{column}} \subset S_1 = \{x_1, \dots, x_s\}$ with $\# A_H^{\text{column}} = s-i$ and a subset $A_H^{\text{row}} \subset S_2 \cup S_2' = \{x_1^2, \dots, x_s^2, x_1x_2, \dots, x_{s-1}x_s\}$ with $\# A_H^{\text{row}} = s - i$. We only have to deal with H satisfying $\det(H) \neq 0$.

For each H, we define j and $\boldsymbol{a}$ by

$$j = \#(A_H^{\text{row}} \cap S_2)$$

(so clearly $0 \le j \le s - i$) and

$$A_H^{\text{row}} \cap S_2 = \{x_{a_1}^2, \dots, x_{a_j}^2\}.$$

Recall that the x_l^2 row in the matrix M contains a unique nonzero component ν_l in the x_l column. Therefore, the assumption $\det(H) \neq 0$ forces $x_{a_1}, \dots, x_{a_j} \in A_H^{\text{column}}$ and

$$\det(H) = \pm \nu_{a_1} \cdots \nu_{a_j} \det(H'),$$

where H' is the square submatrix of H of size $(s-i) - j$, with rows in $A_{H'}^{\text{row}} = A_H^{\text{row}} \setminus \{x_{a_1}^2, \dots, x_{a_j}^2\} = A_H^{\text{row}} \cap S_2'$ and columns in $A_{H'}^{\text{column}} = A_H^{\text{column}} \setminus \{x_{a_1}, \dots, x_{a_j}\}$.

Let $N_{\boldsymbol{a}}$ denote the submatrix of $N_s(\tau_1, \dots, \tau_s)$ obtained by removing the $x_{a_1}, \dots, x_{a_j}$ columns. Then it is clear that the $\det(H')$ (for fixed j and $\boldsymbol{a}$) as above generate $\mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_{\boldsymbol{a}})$. The argument so far implies

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(M) = \sum_{j=0}^{s-i} \sum_{\substack{\boldsymbol{a} \subset \{1,2,\dots,s\} \\ \#\boldsymbol{a} = j}} \nu_{a_1} \cdots \nu_{a_j} \mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_{\boldsymbol{a}}).$$

By the formula $\tau_l \nu_l = 0$, we may remove the components $\pm\tau_{a_1}, \dots, \pm\tau_{a_j}$ from the matrix $N_{\boldsymbol{a}}$ in the right hand side. It is easy to check that the resulting matrix is nothing but $N_{s-j}(\tau_{a_{j+1}}, \dots, \tau_{a_s})$ (with several zero rows added). This completes the proof. □

Proposition 4.8. *For $s \ge 0$ and $i \ge 0$, we have*

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_s(\tau_1, \dots, \tau_s)) = \begin{cases} (1) & (i \ge s), \\ 0 & (s \ge 1, i = 0), \\ (\tau_1, \dots, \tau_s)^{s-i} & (1 \le i < s). \end{cases}$$

Proof. Since $N_s(\tau_1, \dots, \tau_s)$ has s columns, the case for $i \ge s$ is clear.

We show the vanishing when $s \ge 1$ and $i = 0$. Let $R = \mathbb{Z}[T_1, \dots, T_s]$ be the polynomial ring over $\mathbb{Z}$. Then we have a ring homomorphism $f : R \to \mathbb{Z}[G]$ defined by sending T_l to τ_l. We define a matrix $N_s(T_1, \dots, T_s)$ over R in the same way as in Definition 4.4, with $\tau_\bullet$ replaced by $T_\bullet$. Then, by the base change via f, we have

$$\mathrm{Fitt}_{\mathbb{Z}[G]}(N_s(\tau_1, \dots, \tau_s)) = f(\mathrm{Fitt}_R(N_s(T_1, \dots, T_s)))\mathbb{Z}[G].$$

Hence the left hand side would vanish if we show that $\mathrm{Fitt}_R(N_s(T_1, \dots, T_s)) = 0$.

For each $1 \le l \le s$, we consider the complex

$$\widetilde{C}^l : 0 \to \mathbb{Z}[T_l] \xrightarrow{T_l} \mathbb{Z}[T_l] \to 0,$$

over $\mathbb{Z}[T_l]$, which satisfies $H_n(\widetilde{C}^l) = 0$ for $n \neq 0$ and $H_0(\widetilde{C}^l) \simeq \mathbb{Z}$. Similarly as previous, by taking the tensor product of the complexes $\widetilde{C}^l$ over $\mathbb{Z}$, we obtain an exact sequence

$$\cdots \to \widetilde{C}_2 \xrightarrow{N_s(T_1,\dots,T_s)} \to \widetilde{C}_1 \xrightarrow{\begin{pmatrix} T_1 \\ \vdots \\ T_s \end{pmatrix}} \to \widetilde{C}_0 \to \mathbb{Z} \to 0$$

over R. (Alternatively, this exact sequence is obtained from the Koszul complex for the regular sequence $T_1, \dots, T_s$.) This implies that $\mathrm{Fitt}_R(N_s(T_1, \dots, T_s))$ is the Fitting ideal of the augmentation ideal of R. Since $s \geq 1$, the augmentation ideal of R is generically of rank one, so we obtain $\mathrm{Fitt}_R(N_s(T_1, \dots, T_s)) = 0$, as desired.

Finally we show the case where $1 \leq i < s$. Since the components of the matrix $N_s(\tau_1, \dots, \tau_s)$ are either 0 or one of $\tau_1, \dots, \tau_s$, the inclusion $\subset$ is clear. In order to show the other inclusion, we use the induction on s.

For a while we fix an arbitrary $1 \leq l \leq s$. Then, by permuting the rows and columns, the matrix $N_s(\tau_1, \dots, \tau_s)$ can be transformed into:

$$\begin{pmatrix} N_{s-1}(\tau_1, \dots, \check{\tau}_l, \dots, \tau_s) & & & & & \\ -\tau_l & & & & & \tau_1 \\ & -\tau_l & & & & \vdots \\ & & \ddots & & & \check{\tau}_l \\ & & & -\tau_l & & \vdots \\ & & & & -\tau_l & \tau_s \end{pmatrix}$$

(The symbol $(\check{-})$ means omitting that term.) Here, the x_l column is placed in the right-most, and the $x_1x_l, \dots, x_{l-1}x_l, x_lx_{l+1}, \dots, x_lx_s$ rows are placed in the lower. We also reversed the signs of some rows for readability as that does not matter at all.

This expression implies

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_s(\tau_1, \dots, \tau_s)) \supset (\tau_1, \dots, \check{\tau}_l, \dots, \tau_s)\, \mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_{s-1}(\tau_1, \dots, \check{\tau}_l, \dots, \tau_s)).$$

By the induction hypothesis (note that $1 \leq i \leq s-1$), we have

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_s(\tau_1, \dots, \tau_s)) \supset (\tau_1, \dots, \check{\tau}_l, \dots, \tau_s)(\tau_1, \dots, \check{\tau}_l, \dots, \tau_s)^{s-1-i} = (\tau_1, \dots, \check{\tau}_l, \dots, \tau_s)^{s-i}.$$

Now we vary l and then obtain

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_s(\tau_1, \dots, \tau_s)) \supset \sum_{l=1}^{s} (\tau_1, \dots, \check{\tau}_l, \dots, \tau_s)^{s-i} = (\tau_1, \dots, \tau_s)^{s-i},$$

where the last equality follows from $s - i < s$. This completes the proof of the proposition. $\square$

Now we incorporate Propositions 4.6, 4.7 and 4.8 to prove the following.

Proposition 4.9. *For $0 \le i \le s$, we define an ideal J_i of $\mathbb{Z}[G]$ by*

$$J_i = \begin{cases} (\nu_1 \cdots \nu_s) = (\nu_I) & (i = 0), \\ \sum_{j=0}^{s-i} Z_{i+j} \mathcal{I}_I^j = Z_i + Z_{i+1}\mathcal{I}_I + \cdots + Z_s \mathcal{I}_I^{s-i} & (1 \le i \le s). \end{cases}$$

Then we have

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I) = J_i.$$

Proof. By Propositions 4.6 and 4.7, we have

$$\begin{aligned} \mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I) &= \mathrm{Fitt}_{i,\mathbb{Z}[G]}(M) \\ &= \sum_{j=0}^{s-i} \sum_{\substack{\boldsymbol{a} \subset \{1,2,\dots,s\} \\ \#\boldsymbol{a} = j}} \nu_{a_1} \cdots \nu_{a_j} \, \mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_{s-j}(\tau_{a_{j+1}}, \dots, \tau_{a_s})). \end{aligned}$$

When $i = 0$, Proposition 4.8 implies

$$\mathrm{Fitt}_{0,\mathbb{Z}[G]}(N_{s-j}(\tau_{a_{j+1}}, \dots, \tau_{a_s})) = \begin{cases} (1) & (j = s), \\ 0 & (0 \le j < s). \end{cases}$$

Clearly, $j = s$ forces $\boldsymbol{a} = \{1, 2, \dots, s\}$, so we obtain

$$\mathrm{Fitt}_{0,\mathbb{Z}[G]}(\mathcal{I}_I) = (\nu_1 \cdots \nu_s) = J_0.$$

When $1 \le i \le s$, since $1 \le i \le s - j$ by the choice of j, Proposition 4.8 implies

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(N_{s-j}(\tau_{a_{j+1}}, \dots, \tau_{a_s})) = (\tau_{a_{j+1}}, \dots, \tau_{a_s})^{s-i-j}.$$

Then we obtain

$$\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I) = \sum_{j=0}^{s-i} \sum_{\substack{\boldsymbol{a} \subset \{1,2,\dots,s\} \\ \#\boldsymbol{a} = j}} \nu_{a_1} \cdots \nu_{a_j} (\tau_{a_{j+1}}, \dots, \tau_{a_s})^{s-i-j}.$$

Using the relation $\nu_l \tau_l = 0$, for each $0 \le j \le s - i$, we have

$$\sum_{\substack{\boldsymbol{a} \subset \{1,2,\dots,s\} \\ \#\boldsymbol{a} = j}} \nu_{a_1} \cdots \nu_{a_j} (\tau_{a_{j+1}}, \dots, \tau_{a_s})^{s-i-j} = \sum_{\substack{\boldsymbol{a} \subset \{1,2,\dots,s\} \\ \#\boldsymbol{a} = j}} \nu_{a_1} \cdots \nu_{a_j} \mathcal{I}_I^{s-i-j} = Z_{s-j} \mathcal{I}_I^{s-i-j}.$$

These formulas imply $\mathrm{Fitt}_{i,\mathbb{Z}[G]}(\mathcal{I}_I) = J_i$. □

We are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Propositions 4.3 and 4.9, we have

$$\mathrm{Fitt}^{[1]}_{\mathbb{Z}[G]}(A) = \sum_{i=0}^{s} \tilde{g}^{i-1} J_i.$$

Then, noting $J_0 = (\nu_I)$, we can deduce

$$h \operatorname{Fitt}^{[1]}_{\mathbb{Z}[G]}(A) = \left(\nu_I, \left(1 - \frac{\nu_I}{\# I}\varphi^{-1}\right) \sum_{i=1}^{s} \tilde{g}^{i-1} J_i\right)$$

in the same way as in the proof of Proposition 4.1. Then it is enough to show

$$\mathcal{J} = \sum_{i=1}^{s} \tilde{g}^{i-1} J_i. \tag{4-1}$$

We claim that

$$(\mathcal{I}_I, \# I) J_{i+1} \subset J_i \tag{4-2}$$

holds for $1 \le i \le s-1$. We first see

$$\mathcal{I}_I J_{i+1} = \mathcal{I}_I \sum_{j=0}^{s-i-1} Z_{i+1+j} \mathcal{I}_I^j = \sum_{j=1}^{s-i} Z_{i+j} \mathcal{I}_I^j \subset J_i.$$

We also have $\nu_I J_{i+1} \subset (\nu_I) \subset J_0 \subset J_i$. Since $(\mathcal{I}_I, \# I) = (\mathcal{I}_I, \nu_I)$ as an ideal, these show the claim (4-2).

Using (4-2), we next show

$$\sum_{i=1}^{s} \tilde{g}^{i-1} J_i = \sum_{i=1}^{s} (1 - \widetilde{\varphi}^{-1})^{i-1} J_i. \tag{4-3}$$

More generally we actually show

$$\sum_{i=1}^{s'} \tilde{g}^{i-1} J_i = \sum_{i=1}^{s'} (1 - \widetilde{\varphi}^{-1})^{i-1} J_i$$

by induction on s', for each $0 \le s' \le s$. The case $s' = 0$ is trivial. For $1 \le s' \le s$, we have

$$\begin{aligned}
\sum_{i=1}^{s'} \tilde{g}^{i-1} J_i &= \tilde{g}^{s'-1} J_{s'} + \sum_{i=1}^{s'-1} \tilde{g}^{i-1} J_i \\
&= \left(\sum_{i=1}^{s'} (1 - \widetilde{\varphi}^{-1})^{i-1} (\# I)^{s'-i}\right) J_{s'} + \sum_{i=1}^{s'-1} (1 - \widetilde{\varphi}^{-1})^{i-1} J_i.
\end{aligned}$$

Here, the second equality follows from the induction hypothesis and expanding the power $\tilde{g}^{s'-1}$. By (4-2), for $1 \le i \le s'-1$, we have $(\# I)^{s'-i} J_{s'} \subset J_i$. Therefore, we obtain

$$\begin{aligned}
\sum_{i=1}^{s'} \tilde{g}^{i-1} J_i &= (1 - \widetilde{\varphi}^{-1})^{s'-1} J_{s'} + \sum_{i=1}^{s'-1} (1 - \widetilde{\varphi}^{-1})^{i-1} J_i \\
&= \sum_{i=1}^{s'} (1 - \widetilde{\varphi}^{-1})^{i-1} J_i.
\end{aligned}$$

This completes the proof of (4-3).

The right hand side of (4-3) can be computed as

$$\begin{aligned}\sum_{i=1}^{s}(1-\widetilde{\varphi}^{-1})^{i-1}J_i &= \sum_{i=1}^{s}\sum_{j=0}^{s-i} Z_{i+j}\mathcal{I}_I^j(1-\widetilde{\varphi}^{-1})^{i-1}\\ &= \sum_{k=1}^{s}\sum_{j=0}^{k} Z_k\mathcal{I}_I^j(1-\widetilde{\varphi}^{-1})^{k-j-1}\\ &= \sum_{k=1}^{s} Z_k\mathcal{I}_D^{k-1} = \mathcal{J}.\end{aligned}$$

Here, the first equality follows from the definition of J_i, the second by putting $i+j=k$, the third by $\mathcal{I}_D=(\mathcal{I}_I, 1-\widetilde{\varphi}^{-1})$, and the final by the definition of $\mathcal{J}$. Then, combining this with (4-3), we obtain the formula (4-1). This completes the proof of Theorem 1.4. □

5. Stickelberger element and Fitting ideal

In this section, we prove Theorem 1.5. As explained after the statement, we need to compare the ideals in the both sides of (1-6) for each v before taking the product. That task will be done in Section 5A, and after that we complete the proof of Theorem 1.5 in Section 5B.

In this section we fix an odd prime number p and always work over $\mathbb{Z}_p$.

5A. ***Comparison of ideals.*** In this subsection, we again consider the general algebraic situation as in Section 1B. Our task in this subsection is to compare the two fractional ideals

$$\mathcal{A} = h\,\mathrm{Fitt}^{[1]}_{\mathbb{Z}_p[G]}(A\otimes\mathbb{Z}_p), \quad \mathcal{B} = \left(1-\frac{\nu_I}{\#I}\varphi^{-1}\right)$$

of $\mathbb{Z}_p[G]$. In Lemma 5.1 (resp. Lemma 5.2) below, we deal with the case where D *is not* (resp. *is*) a p-group. We will make use of the concrete description of $\mathcal{A}$ in Theorem 1.4. As we always work over $\mathbb{Z}_p$ instead of $\mathbb{Z}$, by abuse of notation, in this subsection we simply write $\mathcal{I}_I$, $\mathcal{I}_D$, Z_i, and $\mathcal{J}$ for the extensions of those ideals from $\mathbb{Z}[G]$ to $\mathbb{Z}_p[G]$. We have no afraid of confusion due to this.

Let G' denote the maximal subgroup of G whose order is prime to p.

Lemma 5.1. *Let χ be a faithful character of G' (note that this requires that G' be cyclic). Suppose that D is not a p-group. Then we have*

$$\mathcal{A}^\chi = \mathcal{B}^\chi$$

as fractional ideals of $\mathbb{Z}_p[G]^\chi$.

Proof. We write $D' = D\cap G'$ and $I' = I\cap G'$. We first note that $\mathcal{I}_D^\chi = (1)$. This is because χ is nontrivial on D' by the assumptions. Then we have $\mathcal{J}^\chi = (1)$ by Definition 1.2, so Theorem 1.4 implies

$$\mathcal{A}^\chi = \left(\nu_I, \left(1-\frac{\nu_I}{\#I}\varphi^{-1}\right)\right)^\chi.$$

We have to show

$$\nu_I^\chi \in \left(1 - \frac{\nu_I}{\# I}\varphi^{-1}\right)^\chi.$$

When I' is nontrivial, then $\nu_I^\chi = 0$ as χ is nontrivial on I', so this is obvious. Let us suppose that I' is trivial. Since $\nu_I^2 = (\# I)\nu_I$, we have

$$\nu_I\left(1 - \frac{\nu_I}{\# I}\varphi^{-1}\right) = \nu_I(1 - \varphi^{-1}).$$

The element $(1 - \widetilde{\varphi}^{-1})^\chi$ of $\mathbb{Z}_p[G]^\chi$ is a unit since $\mathcal{I}_D = (\mathcal{I}_I, 1 - \widetilde{\varphi}^{-1})$, $\mathcal{I}_D^\chi = (1)$, and $\mathcal{I}_I^\chi \subsetneqq (1)$. This completes the proof. □

Lemma 5.2. *Suppose that I is nontrivial and that D is a p-group. Let $s = \operatorname{rank}_p(I)$ be the p-rank of I, that is, the number of minimal generators of I (note that $s \geq 1$):*

(1) *We have*

$$\mathcal{A} \supset \mathcal{I}_D^{s-1}\mathcal{B}$$

as fractional ideals of $\mathbb{Z}_p[G]$.

(2) *Let ψ be a character of G such that $\psi|_{G'}$ is faithful on G' and that ψ is nontrivial on D. Then we have*

$$\psi(\mathcal{A}) = \psi(\mathcal{I}_D)^{s-1}\psi(\mathcal{B})$$

as ideals of $\mathcal{O}_\psi$.

Proof. We may take a decomposition (1-3) of I so that s coincides with the p-rank of I as the lemma, and then I_l is nontrivial for each $1 \leq l \leq s$:

(1) By Definition 1.2, we have $\mathcal{I}_D^{s-1} \subset \mathcal{J}$ (by the $i = s$ term as $Z_s = (1)$). Then Theorem 1.4 shows the claim (1).

(2) We first show $\psi(\mathcal{J}) = \psi(\mathcal{I}_D)^{s-1}$. By the claim (1) above, the inclusion $\psi(\mathcal{J}) \supset \psi(\mathcal{I}_D)^{s-1}$ holds. For each $1 \leq l \leq s$, we observe $(\psi(\nu_l)) \subset (p)$ since $\psi(\nu_l)$ is either 0 or $\# I_l$. Moreover, we have $(p) \subset \psi(\mathcal{I}_D)$ since ψ is nontrivial on D and we have $(p) \subset (1-\zeta)$ if ζ is any nontrivial root of unity. These observations imply $\psi(Z_i) \subset \psi(\mathcal{I}_D)^{s-i}$ for $1 \leq i \leq s$. By the definition of $\mathcal{J}$, we then have $\psi(\mathcal{J}) \subset \psi(\mathcal{I}_D)^{s-1}$ as claimed.

By Theorem 1.4 and the above claim, we have

$$\psi(\mathcal{A}) = (\psi(\nu_I), \psi(\mathcal{B})\psi(\mathcal{I}_D)^{s-1}).$$

We have to show $\psi(\nu_I) \in \psi(\mathcal{B})\psi(\mathcal{I}_D)^{s-1}$. This is obvious if ψ is nontrivial on I. If ψ is trivial on I, we have

$$\psi(\nu_I) = \# I \in (p^s) \subset \psi(\mathcal{B})\psi(\mathcal{I}_D)^{s-1},$$

where the last inclusion follows from $\psi(\mathcal{B}) = \psi(\mathcal{I}_D) = (1 - \psi(\varphi)^{-1}) \supset (p)$. This completes the proof of (2). □

5B. ***Proof of Theorem 1.5.*** Now we consider the setup in Section 1C. In particular, we fix an odd prime number p and an odd character χ of G'. Recall the χ-component of the Stickelberger element $\theta^{\chi}_{K/k,T}$ defined as (1-4)

Lemma 5.3. *We have $\theta^{\chi}_{K/k,T} \neq 0$ if and only if there exists a character ψ of G such that $\psi|_{G'} = \chi$ and that ψ is nontrivial on D_v for any $v \in S_\chi \setminus S_\infty(k)$.*

Proof. By (1-5) and the fact that ω^{χ}_T is a not a zero divisor, we have $\theta^{\chi}_{K/k,T} \neq 0$ if and only if there exists a character ψ of G such that $\psi|_{G'} = \chi$ and, for every $v \in S_\chi \setminus S_\infty(k)$, we have $\psi(1 - (\nu_{I_v}/\#I_v)\varphi_v^{-1}) \neq 0$. The last condition is equivalent to that ψ is nontrivial on D_v. This proves the lemma. □

We begin the proof of Theorem 1.5.

Proof of Theorem 1.5. As already remarked in the outline of the proof after Theorem 1.5, we may and do assume that χ is a faithful character of G'. This is because we have $(\mathrm{Cl}^T_K \otimes \mathbb{Z}_p) \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p[\mathrm{Gal}(K_\chi/k)] \simeq \mathrm{Cl}^T_{K_\chi} \otimes \mathbb{Z}_p$ as the degree of K/K_χ is prime to p. Moreover, to simplify the notation, we write $S = S_\chi = S_{\mathrm{ram}}(K/k)$ and $S_{\mathrm{fin}} = S \setminus S_\infty(k)$. Recall that, by Theorem 1.1, the condition (i) is equivalent to (1-6). As in Section 5A, for each $v \in S_{\mathrm{fin}}$, we consider the fractional ideals of $\mathbb{Z}_p[G]$

$$\mathcal{A}_v = h_v \, \mathrm{Fitt}^{[1]}_{\mathbb{Z}_p[G]}(A_v \otimes \mathbb{Z}_p), \quad \mathcal{B}_v = \left(1 - \frac{\nu_{I_v}}{\#I_v}\varphi_v^{-1}\right).$$

We first suppose (ii) and aim at showing (i). The case where $\theta^{\chi}_{K/k,T} = 0$ is trivial, so we assume that, for any $v \in S_{\mathrm{fin}}$, either (a) or (b) holds. Then we obtain $\mathcal{B}^{\chi}_v \subset \mathcal{A}^{\chi}_v$ for any $v \in S_{\mathrm{fin}}$, by applying Lemma 5.1 (resp. Lemma 5.2(1)) if (a) (resp. (b)) holds. Thus (1-6) holds.

We now prove that (i) implies (ii). Suppose that both (i) and the negation of (ii) hold. Since $\theta^{\chi}_{K/k,T} \neq 0$, we may take a character ψ as in Lemma 5.3. By applying ψ to (1-6), we obtain

$$\prod_{v \in S_{\mathrm{fin}}} \psi(\mathcal{B}_v) \subset \prod_{v \in S_{\mathrm{fin}}} \psi(\mathcal{A}_v).$$

On the other hand, by Lemmas 5.1 and 5.2(2), for each $v \in S_{\mathrm{fin}}$, we have $\psi(\mathcal{A}_v) \subset \psi(\mathcal{B}_v)$. Moreover, the inclusion is proper if and only if both the conditions (a) and (b) in (ii) are false. Therefore, by the hypothesis that (ii) fails, we obtain

$$\prod_{v \in S_{\mathrm{fin}}} \psi(\mathcal{A}_v) \subsetneqq \prod_{v \in S_{\mathrm{fin}}} \psi(\mathcal{B}_v).$$

Thus we get a contradiction. This completes the proof of Theorem 1.5. □

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The behavior of essential dimension under specialization, II

Zinovy Reichstein and Federico Scavia

Let G be a linear algebraic group over a field. We show that, under mild assumptions, in a family of primitive generically free G-varieties over a base variety B, the essential dimension of the geometric fibers may drop on a countable union of Zariski closed subsets of B and stays constant away from this countable union. We give several applications of this result.

1. Introduction

Let X be a complex algebraic variety (that is, a separated reduced $\mathbb{C}$-scheme of finite type) equipped with a faithful action of a finite group G. We will refer to X as a G-variety. Assume that the G-action on X is primitive, that is, G transitively permutes the irreducible components of X. In this paper, we will be interested in the essential dimension $\mathrm{ed}_{\mathbb{C}}(X; G)$ and how it behaves in families. Essential dimension is an integer-valued birational invariant of the G-variety X. Its definition can be found in Section 2; for a more detailed discussion, see [Reichstein 2010] or [Merkurjev 2013]. When the group G is clear from the context, we will simply write $\mathrm{ed}_{\mathbb{C}}(X)$ for $\mathrm{ed}_{\mathbb{C}}(X; G)$. Clearly, $\mathrm{ed}_{\mathbb{C}}(X) \leqslant \dim(X)$; if equality holds, we will say that the G-variety X is *incompressible*.

To date, the study of essential dimension has been primarily concerned with understanding versal G-varieties (once again, see Section 2 for the definition). A complete versal G-variety X has the following special property: X has an A-fixed rational point for every abelian subgroup $A \subset G$; see [Merkurjev 2013, Corollary 3.21].

At the other extreme are complete G-varieties X, where the action of G is free, i.e., no nontrivial element has a fixed point. Existing methods for proving lower bounds on $\mathrm{ed}_{\mathbb{C}}(X)$ usually fail here. Until recently, there was only one family of interesting examples of complete incompressible G-varieties with a free G-action. These examples concern the action of $G = (\mathbb{Z}/p\mathbb{Z})^n$ on the product of elliptic curves $X = E_1 \times \cdots \times E_n$ over $\mathbb{C}$. Here, p is a prime; the generator of the i-th copy of $\mathbb{Z}/p\mathbb{Z}$ acts on E_i via translation by a point $x_i \in E_i(\mathbb{C})$ of order p, and trivially on E_j for $j \neq i$. Colliot-Thélène and Gabber [Colliot-Thélène 2002, Appendice] showed that for a very general choice of the elliptic curves E_i and torsion points $x_i \in E_i[p]$, a certain degree n cohomological invariant of G does not vanish on $\mathbb{C}(X)^G$.

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(Here, "very general" means "away from a countable union of proper subvarieties" in the moduli space of elliptic curves with a marked torsion point.) This implies that $\mathrm{ed}_{\mathbb{C}}(X) = n = \mathrm{ed}_{\mathbb{C}}(G)$. Additional examples of complete incompressible G-varieties X with a free G-action can be found in the recent work of Farb, Kisin and Wolfson [2021a; 2021b] and Fakhruddin and Saini [2022].

In this paper, we will give a systematic procedure for constructing such examples for an arbitrary finite group. In fact, we will work in a more general setting, where G is a linear algebraic group over an algebraically closed field k. Essential dimension and versality make sense in this more general setting, provided that we require our G-actions to be generically free and not just faithful; see Section 2.

If $\dim(G) > 0$, by Borel's fixed point theorem, G cannot act freely on a complete variety. Nevertheless, the notion of a free action of a finite group on a projective variety can be generalized to the case of an arbitrary linear algebraic group G as follows: a generically free primitive G-variety X is said to be *strongly unramified* if X is G-equivariantly birationally isomorphic to the total space X' of a G-torsor $X' \to P$ over some smooth projective irreducible k-variety P. We are now ready to state our first main result.

Theorem 1.1. *Let G be a linear algebraic group over an algebraically closed field k of good characteristic (see Definition 2.1) and of infinite transcendence degree over its prime field, and let X be a generically free primitive G-variety. Then there exists a strongly unramified G-variety Y such that* $\dim(Y) = \dim(X)$ *and* $\mathrm{ed}_k(Y) = \mathrm{ed}_k(X)$.

Applying Theorem 1.1 to a versal G-variety X, we obtain a strongly unramified G-variety Y of maximal essential dimension, i.e., such that $\mathrm{ed}_k(Y) = \mathrm{ed}_k(G)$. When G is finite, Y is itself smooth and projective. Thus, by starting with an incompressible G-variety X, we obtain examples analogous to those of Colliot-Thélène and Gabber; Farb, Kisin and Wolfson; and Fakhruddin and Saini for an arbitrary finite group G. Note, however, that Farb, Kisin and Wolfson, as well as Fakhruddin and Saini, produce examples over $k = \overline{\mathbb{Q}}$, whereas Theorem 1.1 requires k to be of infinite transcendence degree over the prime field.

Our proof of Theorem 1.1 will rely on Theorems 1.2 and 1.4 below, which are of independent interest.

Theorem 1.2. *Let G be a linear algebraic group over a field k of good characteristic (see Definition 2.1). Let B be a noetherian k-scheme, and let $f : \mathcal{X} \to B$ be a flat G-equivariant morphism of finite type such that G acts trivially on B and the geometric fibers of f are generically free and primitive G-varieties (in particular, reduced). Then for any fixed integer $n \geqslant 0$ the subset of $b \in B$ such that* $\mathrm{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})}) \leqslant n$ *for every (equivalently, some) geometric point $\bar{b}$ above b is a countable union of closed subsets of B.*

Furthermore, assume that k is algebraically closed and of infinite transcendence degree over its prime field. (In particular, the latter condition is satisfied if k is uncountable.) Let $m \geqslant 0$ be the maximum of $\mathrm{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})})$, *where $\bar{b}$ ranges over all geometric points of B. Then the set of those $b \in B(k)$ such that* $\mathrm{ed}_k(\mathcal{X}_b; G) = m$ *is Zariski dense in B.*

Informally, Theorem 1.2 can be restated as follows: in a family of G-varieties $\mathcal{X} \to B$, the essential dimension of the geometric fibers drops on a countable union of Zariski closed subsets of B, and stays constant away from this countable union. Several remarks concerning Theorem 1.2 are in order.

Remarks 1.3. (1) The assumption that the G-action on every geometric fiber $\mathcal{X}_b$ of f is generically free and primitive ensures that $\mathrm{ed}_{k(b)}(\mathcal{X}_b)$ is well defined.

(2) The countable union in the statement of Theorem 1.2 (a) cannot be replaced by a finite union, in general; see Propositions 5.2 and 5.3.

(3) The assumption that f is flat is necessary; see Remark 4.5. On the other hand, this assumption is rather mild. For example, when $\mathcal{X}$ and B are smooth k-varieties, by "miracle flatness", f is flat if and only if all of its fibers have the same dimension; see [Matsumura 1989, Theorem 23.1]. In the applications, one is usually interested in showing that the maximal value of $\mathrm{ed}_k(\mathcal{X}_b)$ is attained at a very general point $b \in B(k)$. This can be done under a weaker flatness assumption on f; see Theorem 9.1. (Here, once again, "very general" means "away from a countable union of proper subvarieties".)

(4) If k is not algebraically closed, then the k-points $b \in B(k)$ such that $\mathrm{ed}_k(\mathcal{X}_b) \leqslant n$ do not necessarily lie on a countable union of closed subvarieties of B; see Section 6. In other words, Theorem 1.2 fails if we consider fibers of arbitrary closed points instead of just geometric fibers.

Our proof of Theorem 1.2 proceeds as follows: First we choose a subfield $k_0 \subset k$ finitely generated over the prime field, such that $G = G_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$, $f = f_0 \times_{\mathrm{Spec}(k_0)} \mathrm{Spec}(k)$, and the assumptions of Theorem 1.2 hold for k_0, G_0 and $f_0 \colon \mathcal{X}_0 \to B_0$. Then using arguments inspired by Gabber's appendix [Colliot-Thélène 2002], we reduce Theorem 1.2 to the specialization property Proposition 3.1 and the rigidity property Lemma 4.2.

Note that the rigidity property may fail if k is not algebraically closed. This is the reason why in Theorem 1.2 we only consider the geometric fibers; see Remarks 1.3 (4).

Theorem 1.4. *Let k be an infinite field, G be a finite group, and let X_0 be an equidimensional generically free G-variety of dimension $e \geqslant 1$ (not necessarily primitive). Then there exist a smooth irreducible quasiprojective k-variety B, a smooth irreducible quasiprojective G-variety $\mathcal{X}$ and a smooth G-equivariant morphism $f \colon \mathcal{X} \to B$ of constant relative dimension e defined over k such that*

- (i) *G acts trivially on B and freely on $\mathcal{X}$,*
- (ii) *there exists a dense open subscheme $U \subset B$ such that for every $b \in U$, the fiber $\mathcal{X}_b$ is smooth, projective and geometrically irreducible,*
- (iii) *there exists $b_0 \in B(k)$ such that the fiber $\mathcal{X}_{b_0}$ of f over b_0 is G-equivariantly birationally isomorphic to X_0.*

In particular, for any geometric point b of U, the G-action on the fiber $\mathcal{X}_b$ is strongly unramified.

Our proof of Theorem 1.4 was motivated by Serre's construction of a smooth projective n-dimensional complete intersection with a free G-action, for an arbitrary finite group G and an arbitrary positive integer n; see [Serre 1958, Proposition 15].

The remainder of this paper is structured as follows. In Section 2, we set up notational conventions and recall some basic definitions concerning essential dimension and versality. Sections 3 and 4 are devoted to the proof of Theorem 1.2. As we mentioned above, the proof is in two steps. The specialization property is proved in Section 3. In Section 4, we prove the rigidity property and complete the proof of Theorem 1.2. Sections 5 and 6 are devoted to examples, showing that various assumptions in the statement of Theorem 1.2 cannot be dropped. Section 7 collects elementary results concerning transversal intersection in projective space; these results are then used in the proof of Theorem 1.4 in Section 8. Theorem 1.1 is deduced from Theorems 1.2 and 1.4 in Section 9. Along the way we prove Theorem 9.1, a variant of Theorem 1.2, which is often easier to use in applications. In Section 10, we show that Proposition 3.1 and Theorems 1.1, 1.2 and 9.1 remain valid (even under slightly weaker assumptions) if essential dimension is replaced by essential dimension at a prime q. We also give an application of one of these results motivated by an open question from [Duncan and Reichstein 2014]; see Theorem 10.8. Another application can be found in Section 11.

This paper is a sequel to [Reichstein and Scavia 2022]. The main result of [Reichstein and Scavia 2022] is used in the proof of Proposition 3.1 (the specialization property of essential dimension). Other than that, this paper can be read independently of [Reichstein and Scavia 2022].

2. Notation and preliminaries

Group actions and essential dimension. Let k be a field, $\bar{k}$ be an algebraic closure of k, G be a linear algebraic group over k and X be a G-variety, i.e., a separated geometrically reduced k-scheme of finite type endowed with a G-action over k. We will say that a G-variety X is primitive if $X \neq \varnothing$ and $G(\bar{k})$ transitively permutes the irreducible components of $X_{\bar{k}} := X \times_k \bar{k}$. We will say that a G-variety X is generically free if there exists a dense open subscheme $U \subset X$ such that for every $u \in U$ the scheme-theoretic stabilizer G_u of u is trivial.

By a G-compression of X, we will mean a dominant G-equivariant rational map $X \dashrightarrow Y$, where the G-action on Y is again generically free and primitive. The essential dimension of X, denoted by $\mathrm{ed}_k(X; G)$, or $\mathrm{ed}_k(X)$ if G is clear from the context, is defined as the minimal value of $\dim(Y)$, where the minimum is taken over all G-compressions $X \dashrightarrow Y$. The essential dimension $\mathrm{ed}_k(G)$ of the group G is defined as the supremum of $\mathrm{ed}_k(X)$, where X ranges over all faithful primitive G-varieties.

Good characteristic.

Definition 2.1. Let G be a linear algebraic group defined over a field k. We will say that G is in *good characteristic* if one of the following conditions holds:

- $\mathrm{char}\, k = 0$, or
- $\mathrm{char}\, k = p > 0$, the connected component G^0 is smooth reductive and there exists a finite subgroup $S \subset G(\bar{k})$ of order prime to p such that the induced map $H^1(K, S) \to H^1(K, G)$ is surjective for every field extension $K/\bar{k}$, or
- G is a finite discrete group, and if $\mathrm{char}\, k = p > 0$, then the only normal p-subgroup of G is the trivial subgroup (that is, G is weakly tame in the sense of [Brosnan et al. 2018]).

Here are two large families of examples in prime characteristic:

Example 2.2. Suppose G is a smooth group over a field k of characteristic $p > 0$. Assume that the connected component G^0 of G is reductive. Let T be a maximal torus in G^0, $r = \dim(T) \geqslant 0$, and let $W = N_G(T)/T$ be the Weyl group. If

(a) G^0 is a split reductive group and p does not divide $2^r|W|$, or

(b) G is connected and p does not divide $|W|$,

then G is in good characteristic. For a proof of (a), see [Reichstein and Scavia 2022, Proposition 5.1]. For a proof of (b), see [Chernousov et al. 2006, Theorem 1.1 (c)] and [Chernousov et al. 2008, Remark 4.1].

The following example shows that conditions (a) and (b) above can sometimes be relaxed.

Example 2.3. The split orthogonal group O_n, special orthogonal group SO_n and the spin group Spin_n over a field k are in good characteristic as long as $\operatorname{char} k \neq 2$. Indeed, let S be the group of diagonal $n \times n$ matrices of the form $\operatorname{diag}(\epsilon_1, \dots, \epsilon_n)$, where each $\epsilon_i = \pm 1$, let $S_0 = S \cap \mathrm{SL}_n$ and let $\tilde{S}$ be the preimage of S_0 under the natural map $\mathrm{Spin}_n \to \mathrm{SO}_n$. Then $|S| = |\tilde{S}| = 2^n$ and $|S_0| = 2^{n-1}$. Moreover, if $\operatorname{char} k \neq 2$, then the natural maps

$$H^1(K, S) \to H^1(K, \mathrm{O}_n), \quad H^1(K, S_0) \to H^1(K, \mathrm{SO}_n) \quad \text{and} \quad H^1(K, \tilde{S}) \to H^1(K, \mathrm{Spin}_n)$$

are all surjective. The surjectivity of the first two maps follows from the fact that every quadratic form over a field of characteristic $\neq 2$ can be diagonalized. The surjectivity of the third map is proved in [Brosnan et al. 2007, Lemma 13.2].

Versality and essential dimension at q. A G-variety X is called *weakly versal* if every generically free primitive G-variety T admits a G-equivariant rational map $T \dashrightarrow X$. We will say that X is versal if every dense open G-invariant subvariety $U \subset X$ is weakly versal. In particular, if V is a vector space with a generically free action of G, then V is versal; see [Merkurjev 2013, Proposition 3.10]. If X is a generically free primitive versal G-variety, then X has maximal possible essential dimension,

$$\mathrm{ed}_k(X; G) = \mathrm{ed}_k(G); \tag{2.4}$$

see [Merkurjev 2013, Proposition 3.11].

Recall that a correspondence $X \rightsquigarrow Z$ between G-varieties X and Z of degree d is a diagram of G-equivariant rational maps of G-varieties having the form

$$\begin{tikzcd} X' \arrow[d, dashed, "\text{degree } d"'] \arrow[dr, dashed] & \\ X & Z \end{tikzcd} \tag{2.5}$$

where the vertical map is dominant of degree d. We say that $X \rightsquigarrow Z$ is *dominant* if the rational map $X' \dashrightarrow Z$ in the above diagram is dominant. Dominant correspondences may be composed in the obvious way. A correspondence of degree 1 is the same thing as a rational map.

The notions of essential dimension and versality have local versions at a prime q. These are obtained by replacing G-compressions by dominant correspondences of degree prime to q. Specifically, let G be a linear algebraic group defined over a field k and X be a generically free primitive G-variety. The essential dimension $\mathrm{ed}_{k,q}(X;G)$ of X at q is defined as the minimal value of $\dim(Y)$, where the minimum is taken over all G-equivariant dominant correspondences $X \rightsquigarrow Y$ of degree prime to q. When the reference to G is clear from the context, we sometimes abbreviate $\mathrm{ed}_{k,q}(X;G)$ to $\mathrm{ed}_{k,q}(X)$. The maximal value of $\mathrm{ed}_{k,q}(X;G)$ is called the essential dimension of G at q and is denoted by $\mathrm{ed}_{k,q}(G)$. Here, the maximum is taken over all generically free primitive G-varieties X.

A G-variety X is called *weakly q-versal* if every generically free primitive G-variety T admits a G-equivariant rational map $T \rightsquigarrow X$ of degree prime to q. We will say that X is *q-versal* if every dense open G-invariant subvariety $U \subset X$ is weakly q-versal. If a generically free primitive G-variety X is q-versal, then

$$\mathrm{ed}_{k,q}(X;G) = \mathrm{ed}_{k,q}(G). \tag{2.6}$$

The proof of (2.6) is the same as the proof of (2.4), with rational maps replaced by correspondences of degree prime to q.

3. A specialization property

The purpose of this section is to prove the following specialization property of essential dimension:

Proposition 3.1. *Let G be a linear algebraic group over a field k of good characteristic. Let R be a discrete valuation ring containing k having residue field k, and let l be the fraction field of R. Fix algebraic closures $\bar{k}$ and $\bar{l}$ of k and l, respectively. Let X be a flat R-scheme of finite type endowed with a G-action over R, whose fibers are generically free and primitive G-varieties. Then* $\mathrm{ed}_{\bar{l}}(X_{\bar{l}}) \geqslant \mathrm{ed}_{\bar{k}}(X_{\bar{k}})$.

For the proof, we will first reduce to the case where X is the total space of a G-torsor $X \to Y$, and then replace Y by $\mathrm{Spec}(A)$, where A is a suitable discrete valuation ring containing R such that the inclusion $R \subset A$ is local. (Note that after the second reduction X is no longer of finite type over R.) In the latter case, the inequality $\mathrm{ed}_{\bar{l}}(X_{\bar{l}}) \geqslant \mathrm{ed}_{\bar{k}}(X_{\bar{k}})$ of Proposition 3.1 is established in [Reichstein and Scavia 2022, Theorem 6.4].

Proof of Proposition 3.1. By assumption, X_k (respectively, X_l) is a primitive generically free G_k-variety (respectively, G_l-variety). Our proof will be in several steps.

Claim 3.2. *There exists an integer $d \geqslant 0$ such that the irreducible components of $X_{\bar{k}}$ and of $X_{\bar{l}}$ are all of dimension d.*

Proof of Claim 3.2. For any finite field extensions $k' \supset k$ and $l' \supset l$, there exists a discrete valuation ring $R' \supset R$, finite and free over R, such that the residue field of R' contains k' and the fraction field of R' contains l'; see [Serre 1979, I.4, Proposition 9 and Remark] and [Serre 1979, I.6, Proposition 15]. Thus, extending R if necessary, we may assume that the irreducible components of X_k (respectively, X_l) are geometrically irreducible and transitively permuted by $G(k)$ (respectively, $G(l)$).

After this reduction, the problem becomes to show that there exists an integer $d \geqslant 0$ such that the irreducible components of X_k and of X_l are all of dimension d. Since G acts transitively on the irreducible components of the fibers, it suffices to exhibit one irreducible component of X_k and one irreducible component of X_l of the same dimension.

Since X is flat over R, by [Liu 2002, Lemma 4.3.7] every irreducible component of X dominates $\mathrm{Spec}(R)$. In other words, the open subscheme $X_l \subset X$ is dense. Therefore, each irreducible component of X is the closure of an irreducible component of X_l. Thus, since $X_k \neq \varnothing$, there exists an irreducible component $X' \subset X$ such that X'_k contains some irreducible component Z of X_k and such that X'_l is an irreducible component of X_l.

The composition $X' \hookrightarrow X \to \mathrm{Spec}(R)$ is surjective, hence [Stacks 2005–, Tag 0B2J] implies that every irreducible component of X'_k has dimension $\dim(X'_l)$. In particular, $\dim(Z) = \dim(X'_l)$, as desired □

Claim 3.3. *There exists a G-invariant R-fiberwise dense open subscheme $U \subset X$ such that G acts freely (i.e., with trivial stabilizers) on U.*

Proof of Claim 3.3. Since G_l acts generically freely on X_l, there exists a closed nowhere dense G_l-invariant subscheme $Z \subset X_l$ such that G_l acts freely on $X_l \setminus Z$. Let $W \subset Z$ be an irreducible component, and let $\overline{W}$ be the closure of W in X. By [Stacks 2005–, Tag 0B2J], either $(\overline{W})_k$ is empty, or

$$\dim((\overline{W})_k) = \dim(W) \leqslant \dim(X_k) - 1.$$

It now follows from Claim 3.2 that $\overline{W}$ does not contain any irreducible component of X_k. Therefore, the closure $\bar{Z}$ of Z does not contain any irreducible component of X_k.

Since G_k acts generically freely on X_k, there exists a closed nowhere dense G_k-invariant subscheme $Z' \subset X_k$ such that G_k acts freely on $X_k \setminus Z'$. It follows that

$$U := X \setminus (\bar{Z} \cup Z')$$

is a fiberwise dense G-invariant open subscheme of X, such that G acts freely on U_l and U_k. To prove Claim 3.3, it remains to show that G acts freely on U, i.e., that the stabilizer U-group scheme

$$\mathcal{G} := U \times_{(U \times_R U)} (G \times_R U)$$

is trivial. Here, the fibered product is taken over the diagonal morphism $U \to U \times_R U$ and the action morphism $G \times_R U \to U \times_R U$. Since G acts freely on U_l and U_k, the U_l-group scheme $\mathcal{G}_l$ and the U_k-group scheme $\mathcal{G}_k$ are both trivial. In particular, for every $u \in U$, the fiber of the structure morphism $\pi : \mathcal{G} \to U$ at u is $k(u)$-isomorphic to $\mathrm{Spec}(k(u))$. Thus, [EGA IV$_4$ 1967, Proposition 17.2.6] implies that π is a monomorphism. Let $e\colon U \to \mathcal{G}$ be the identity section. We have $\pi \circ e = \mathrm{id}_U$, hence $\pi \circ e \circ \pi = \pi$, which implies $e \circ \pi = \mathrm{id}_{\mathcal{G}}$ because π is a monomorphism. Therefore, π is an isomorphism (with inverse e), that is, $\mathcal{G}$ is trivial. □

Claim 3.4. *For the purpose of proving Proposition 3.1, we may assume that X is the total space of a G-torsor $X \to Y$, where Y is a flat R-scheme of finite type, and that $\mathrm{ed}_l(X_l) = \mathrm{ed}_{\bar{l}}(X_{\bar{l}})$.*

Proof of Claim 3.4. After replacing X by the open R-fiberwise dense subscheme U constructed in Claim 3.3, we may assume that G acts freely on X. We write $a\colon G \times_R X \to X \times_R X$ for the action morphism, given by $(g, x) \mapsto (gx, x)$, and $\mathcal{R} := (G \times_R X \rightrightarrows X)$ for the action groupoid induced by a; see [Stacks 2005–, Tag 03LK]. By Claim 3.3, the map a is an isomorphism, and hence in particular it is quasi-finite. We apply [Anantharaman 1973, Appendice I, Corollaire 3] to the equivalence relation determined by $\mathcal{R}$ and to a closed point x in X_k. We deduce that there exist a G-invariant dense open subscheme $V \subset X$ containing x, and a locally closed subscheme $Z \subset V$, such that the restriction $\mathcal{R}_Z$ of $\mathcal{R}$ to Z is flat, quasi-finite, finitely presented (equivalently, finite and locally free, see [Stacks 2005–, Tag 02KB]), and such that the natural morphism of fppf-sheaves $Z/\mathcal{R}_Z \to V/G$ is an isomorphism. Since V contains x and the G-variety X_k is primitive, V is R-fiberwise dense in X. We will prove that $Z/\mathcal{R}_Z$ is a scheme by applying [Stacks 2005–, Tag 07S6]. Therefore, we need to verify that assumptions (1)-(3) of [Stacks 2005–, Tag 07S6] are satisfied by $\mathcal{R}_Z$. We have already checked (1) and (2), and so it remains to check (3): for a dense set of points of $z \in Z$, the $\mathcal{R}_Z$-equivalence class of z is contained in an affine open subscheme of Z. By [Berhuy and Favi 2003, Theorem 4.7], there exist an open subscheme $U' \subset V_l$ and a quotient map $U' \to U'/G$ in the category of schemes which is a G-torsor. Let $V' \subset U'$ be the inverse image of a dense affine open subscheme of U'/G. Then V' is an everywhere dense G-invariant affine open subscheme of V. Since the map $Z/\mathcal{R}_Z \to V/G$ is an isomorphism, it is in particular surjective, and hence every G-orbit intersects Z. Thus $V' \cap Z$ is a dense affine open subscheme of Z which is a union of $\mathcal{R}_Z$-equivalence classes. In other words, (3) is satisfied by all the points in the dense open subscheme $V' \cap Z \subset Z$. We may now apply [Stacks 2005–, Tag 07S6] to deduce that $Z/\mathcal{R}_Z$ is represented by a scheme. Thus $V/G \simeq Z/\mathcal{R}_Z$ is also a scheme. Therefore, replacing X by V', we may assume that X is the total space of an fppf G-torsor of schemes $X \to Y$. Since G is smooth, $X \to Y$ is in fact an étale G-torsor. The fact that Y is flat over R now follows from [Stacks 2005–, Tag 02JZ], since X is flat over R (this is one of the assumptions of Proposition 3.1) and over Y (because $X \to Y$ is a G-torsor). Since every $G_{\bar{l}}$-equivariant compression of $X_{\bar{l}}$ over $\bar{l}$ is defined over some finite extension of l, there is a finite subextension $l \subset l' \subset \bar{l}$ such that $\mathrm{ed}_{l'}(X_{l'}) = \mathrm{ed}_{\bar{l}}(X_{\bar{l}})$. Let $R' \supset R$ be a discrete valuation ring with fraction field l', and let $k' \supset k$ be the residue field of R'. The G_R-torsor $X \to Y$ over R lifts to a $G_{R'}$-torsor on $X_{R'} \to Y_{R'}$, which is R'-fiberwise generically free and primitive. Since $\mathrm{ed}_{k'}(X_{k'}) \geqslant \mathrm{ed}_{\bar{k}}(X_{\bar{k}})$, we are allowed to replace R by R' and thus assume that $\mathrm{ed}_l(X_l) = \mathrm{ed}_{\bar{l}}(X_{\bar{l}})$. □

We are now ready to complete the proof of Proposition 3.1. By Claim 3.4, we may assume that X is the total space of a G-torsor $X \to Y$, where Y is a flat R-scheme of finite type, and that $\mathrm{ed}_l(X_l) = \mathrm{ed}_{\bar{l}}(X_{\bar{l}})$.

Let $\eta \in Y$ be the generic point of Y_k, and let $A := \mathcal{O}_{Y,\eta}$. Then A is a discrete valuation ring with residue field $k(Y_k)$ and fraction field $l(Y_l)$. We have a Cartesian diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & Y \arrow[d] \\ & \mathrm{Spec}(R) \end{tikzcd}$$

where $P := X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(A)$. By construction, the morphism $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ sends the closed point of $\operatorname{Spec}(A)$ to the closed point of $\operatorname{Spec}(R)$, and so it is local. Since G is in good characteristic, P satisfies the assumptions of [Reichstein and Scavia 2022, Theorem 6.4], hence $\operatorname{ed}_l(P_l) \geqslant \operatorname{ed}_k(P_k)$. Therefore,

$$\operatorname{ed}_l(P_l) \geqslant \operatorname{ed}_k(P_k) \geqslant \operatorname{ed}_{\bar{k}}(P_{\bar{k}}).$$

The morphism $\operatorname{Spec}(A) \to Y$ induces isomorphisms at the level of residue fields of the special and generic point. Thus $\operatorname{ed}_l(X_l) = \operatorname{ed}_l(P_l)$ and $\operatorname{ed}_{\bar{k}}(X_{\bar{k}}) = \operatorname{ed}_{\bar{k}}(P_{\bar{k}})$. Therefore,

$$\operatorname{ed}_{\bar{l}}(X_{\bar{l}}) = \operatorname{ed}_l(X_l) = \operatorname{ed}_l(P_l) \geqslant \operatorname{ed}_{\bar{k}}(P_{\bar{k}}) = \operatorname{ed}_{\bar{k}}(X_{\bar{k}}). \qquad \square$$

4. Proof of Theorem 1.2

Lemma 4.1. *Let k/k_0 be a field extension of infinite transcendence degree such that k is algebraically closed. Let B_0 be an irreducible k_0-variety and $B := B_0 \times_{k_0} k$. Then the set of k-rational points of B mapping to the generic point of B_0 is dense in B.*

Proof. Let U be a nonempty open k-subscheme of B. It suffices to prove that U has a k-point which maps to the generic point of B_0.

To prove this claim, note that the open embedding $U \hookrightarrow B$ is defined over some intermediate subfield $k_0 \subset k_1 \subset k$ such that the extension k_1/k_0 is finitely generated. In other words, $U \hookrightarrow B$ is obtained by base change from an affine open embedding $U_1 \hookrightarrow B_0 \times_{k_0} k_1$ defined over k_1. In particular, the morphism $U_1 \to B_0$ is dominant. Let $\eta_1 \colon \operatorname{Spec}(K_1) \to U_1$ be the generic point of U_1.

Now consider a subfield $k_1 \subset L_1 \subset K_1$ such that L_1/k_1 is purely transcendental of finite transcendence degree and K_1/L_1 is finite. Since k/k_1 has infinite transcendence degree, there exists a field embedding $\iota \colon L_1 \hookrightarrow k$ compatible with the k_1-algebra structures of L_1 and k. Since k is algebraically closed and K_1/L_1 is finite, we may extend ι to a field embedding $K_1 \hookrightarrow k$, again compatible with the k_1-algebra structures of K_1 and k. This gives rise to a scheme morphism

$$u_1 \colon \operatorname{Spec}(k) \to \operatorname{Spec}(K_1) \xrightarrow{\eta_1} U_1.$$

Since $U = U_1 \times_{k_1} k$, the point u_1 uniquely lifts to a k-point u of U mapping to the generic point of U_1. Since the morphism $U_1 \to B_0$ is dominant, the k-point u maps to the generic point of B_0. This completes the proof of claim, and also completes the proof of Lemma 4.1. $\square$

We will make use of the following "rigidity property" of essential dimension. For a proof, see [Reichstein and Scavia 2022, Lemma 2.2].

Lemma 4.2. *Let k be an algebraically closed field, G be a k-group and X be a generically free primitive G-variety defined over k. Then $\operatorname{ed}_k(X) = \operatorname{ed}_l(X_l)$ for any field extension l/k.*

Now let $f : \mathcal{X} \to B$ be as in Theorem 1.2. For every integer n, we set

$$\Phi_f(n) := \left\{ b \in B \mid \operatorname{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}) \leqslant n \text{ for some geometric point } \bar{b} \text{ with image } b \right\}.$$

Lemma 4.3. (a) *A point $b \in B$ belongs to $\Phi_f(n)$ if and only if* $\mathrm{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}) \leqslant n$ *for every geometric point $\bar{b}$ with image b.*

(b) *Let $\pi\colon B' \to B$ be a morphism of schemes, and let $f'\colon X \times_B B' \to B'$ be the base of change of f along π. Then $\Phi_{f'}(n) = \pi^{-1}(\Phi_f(n))$.*

Proof. (a) Let $\bar{b}_1$ and $\bar{b}_2$ be two geometric points of B with image b. The ring $A := k(\bar{b}_1) \otimes_{k(b)} k(\bar{b}_2)$ is not zero. If $\mathfrak{m}$ is a maximal ideal of A, the quotient $A/\mathfrak{m}$ is a field containing $k(\bar{b}_1)$ and $k(\bar{b}_2)$. By considering an algebraic closure of $A/\mathfrak{m}$, we are thus reduced to the case when there is a field homomorphism $k(\bar{b}_1) \hookrightarrow k(\bar{b}_2)$. We may, thus, assume that $k(\bar{b}_1) \subset k(\bar{b}_2)$. In this case, (a) follows from Lemma 4.2.

(b) Let $b' \in B'$ and $b \in B$ be such that $\pi(b') = b$. Let $\bar{b}'$ be a geometric point of B with image b', so that $\bar{b} := \pi \circ \bar{b}'$ is geometric point of B with image b. Then there is a natural isomorphism $\mathcal{X}_{\bar{b}'} \simeq \mathcal{X}_{\bar{b}} \times_{k(\bar{b})} k(\bar{b}')$ of $G_{\bar{b}'}$-varieties, and

$$\mathrm{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}) = \mathrm{ed}_{k(\bar{b}')}(\mathcal{X}_{\bar{b}} \times_{k(\bar{b})} k(\bar{b}')) = \mathrm{ed}_{k(\bar{b}')}(\mathcal{X}_{\bar{b}'})$$

by Lemma 4.2. In particular, $b \in \Phi_f(n)$ if and only if $b' \in \Phi_{f'}(n)$, as desired. $\square$

Proof of Theorem 1.2. We must show that $\Phi_f(n) \subset B$ is a union of countably many closed subsets of B. By noetherian approximation (see [EGA IV$_3$ 1966, IV, §8.10] or [Thomason and Trobaugh 1990, Appendix C]), the G-action on $\mathcal{X}$ descends to a subfield k_0 of k which is finitely generated over its prime field. In other words, there exist

- a field $k_0 \subset k$ finitely generated over its prime field,
- a smooth group scheme G_0 of finite type over k_0,
- k_0-schemes of finite type B_0 and $\mathcal{X}_0$,
- a G_0-action on $\mathcal{X}_0$ over k_0,
- a flat G_0-invariant morphism $f_0\colon \mathcal{X}_0 \to B_0$ and
- a Cartesian diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "f"'] & \mathcal{X}_0 \arrow[d, "f_0"] \\ B \arrow[r, "\pi"] & B_0, \end{tikzcd} \tag{4.4}$$

such that $G = G_0 \times_{k_0} k$, and the base change of the G_0-action on $\mathcal{X}_0/B_0$ along π is isomorphic to the G-action on $\mathcal{X}/B$.

By Lemma 4.3 (b), we have $\Phi_f(n) = \pi^{-1}(\Phi_{f_0}(n))$. Thus, since π is continuous, it suffices to prove that $\Phi_{f_0}(n)$ is a countable union of closed subsets of B_0. In other words, we may assume that k is finitely generated over its prime field and that B is of finite type over k. In this case, the underlying topological space of B is countable, hence $\Phi_f(n)$ is countable. It remains to show that $\Phi_f(n)$ is a union of closed subsets of B. By elementary topology, it suffices to show that $\Phi_f(n)$ is closed under specialization;

see [Stacks 2005–, Tag 0EES]. In other words, if $b' \in B$ is a specialization of $b \in \Phi_f(n)$, i.e., $b' \in \overline{\{b\}}$, then we want to show that $b' \in \Phi_f(n)$.

By [EGA II 1961, Proposition 7.1.4], there exist a discrete valuation ring R with closed point s and generic point η, and a morphism $\mathrm{Spec}(R) \to B$ sending s to b' and η to b. Precomposing with the completion map $\mathrm{Spec}(\hat{R}) \to \mathrm{Spec}(R)$, we may assume that R is complete. Since B is a k-scheme, the residue fields of b, b', s, η all have the same characteristic as k. Thus, R is complete and equicharacteristic and hence, by Cohen's structure theorem we have an isomorphism $R \simeq k(s)[[t]]$. (In characteristic 0, this is an isomorphism of k-algebras, whereas in characteristic p, it is only an isomorphism of rings. For the argument below, we only need an isomorphism of rings.) In particular, the residue field $k(s)$ is contained in R. By Proposition 3.1, letting $\bar{\eta}$ and $\bar{s}$ be geometric points of $\mathrm{Spec}(R)$ lying above η and s, respectively, we deduce that

$$\mathrm{ed}_{k(\bar{\eta})}(X_{k(\bar{\eta})}) \geqslant \mathrm{ed}_{k(\bar{s})}(X_{k(\bar{s})}).$$

Now, Lemma 4.3 (a) tells us that

$$n \geqslant \mathrm{ed}_{k(\bar{b})}(X_{k(\bar{b})}) \geqslant \mathrm{ed}_{k(\bar{b}')}(X_{k(\bar{b}')}),$$

where $\bar{b}$ and $\bar{b}'$ are geometric points of B lying above b and b', respectively. This shows that $\Phi_f(n)$ is closed under specialization.

Assume now that k is algebraically closed and of infinite transcendence degree over its prime field, and let m be the maximum of $\mathrm{ed}_{k(\bar{b})}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})})$, where $\bar{b}$ ranges over all geometric points of B. Consider the diagram (4.4). Since $\Phi_{f_0}(m-1)$ is a union of closed subsets of B_0 and it does not equal B_0, it does not contain the generic point of B_0. By Lemma 4.3 (b), we have $\Phi_f(m-1) = \pi^{-1}(\Phi_{f_0}(m-1))$, hence for every k-point b of B mapping to the generic point of B_0, we have $\mathrm{ed}_k(\mathcal{X}_b) = m$. By Lemma 4.1, the set of such k-points is Zariski dense in B. □

Remark 4.5. The following example shows that the flatness assumption in Theorem 1.2 is necessary.

Let n be a positive integer, and let k be an algebraically closed field of characteristic not dividing n. Consider the affine plane $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$, with coordinates x, y. Let $\mathcal{X} \subset \mathbb{A}^2$ be defined by the equation $x(y^n - 1) = 0$, let $B = \mathbb{A}^1_k = \mathrm{Spec}(k[x])$ and let $f : \mathcal{X} \to B$ be the projection given by $(x, y) \mapsto x$. The group $\mu_n = \mathbb{Z}/n\mathbb{Z}$ acts on $\mathcal{X}$ by $\zeta \cdot (x, y) \mapsto (x, \zeta y)$ and trivially on B. Clearly, f is μ_n-equivariant, and the μ_n-action on the fibers of f is generically free and primitive. We have $\mathrm{ed}_k(X_a) = 0$ for every $a \in k^\times$, but $\mathrm{ed}_k(X_0) = \mathrm{ed}_k(\mu_n) = 1$.

Remark 4.6. To put Theorem 1.2 in perspective, we will conclude this section by recalling an analogous result for cohomological invariants from [Colliot-Thélène 2002, Appendix]. For an overview of the theory of cohomological invariants, see [Garibaldi et al. 2003].

Let k be a field, G be a linear algebraic k-group and X be a generically free primitive G-variety. After passing to a G-invariant open subvariety of X, we may assume that X is the total space of a G-torsor $\tau : X \to Y$, where Y is irreducible with function field $K = k(Y) = k(X)^G$; see [Berhuy and Favi 2003, Theorem 4.7]. Note that $k(X)^G$ is a field, since X is primitive. Pulling back to the generic point $\eta : \mathrm{Spec}(K) \to Y$ of Y, we obtain a G-torsor $\tau_\eta : \mathcal{T}_\eta \to \mathrm{Spec}(K)$. We denote the class of τ_η in $H^1(K, G)$ by $[X]$.

Now, suppose $f: \mathcal{X} \to B$ is a family of generically free primitive G-varieties, as in Theorem 1.2. Let i be a nonnegative integer, C be a finite $\mathrm{Gal}(k_s/k)$-module of order prime to the characteristic of k and $F \in \mathrm{Inv}^i(G, C)$ be a cohomological invariant over k with values in the Galois cohomology ring $H^i(-, C)$. We will be interested in how $F([\mathcal{X}_b])$ behaves as b varies over B. First consider the generic point b_{gen} of B. Passing to a dense open subscheme of B if necessary, we may assume that $F([X_{b_{\mathrm{gen}}}])$ comes from a cohomology class $\alpha \in H^i_{\text{ét}}(B, C)$. In this case, by the compatibility of the specialization map in étale and Galois cohomology [Garibaldi et al. 2003, p. 15, Footnote], $\alpha_{\bar{b}} = F(k(X_{\bar{b}}))$ (up to sign) for every geometric point $\bar{b}$ of B. From [Colliot-Thélène 2002, Proposition A7], we deduce that

$$B_0 := \{b \in B : F(k(X_{\bar{b}})) = 0 \text{ for some geometric point } \bar{b} \text{ above } b\}$$

is a countable union of closed subsets of B. Note that by the rigidity property for étale cohomology [Milne 1980, Corollary VI.2.6], one may replace "some" by "every" in the definition of B_0, as in Lemma 4.3 (a).

5. Example: Elliptic curves with marked torsion points

In this section we will consider an example, showing that in Theorem 1.2 one may not replace "countable union" by "finite union".

Let A be a commutative algebraic group over $\mathbb{C}$. Any choice of $v_1, \dots, v_r \in A[\ell]$ gives rise to a $(\mathbb{Z}/\ell\mathbb{Z})^r$-action on A via

$$(n_1, \dots, n_r) : a \mapsto a + n_1 v_1 + \cdots + n_r v_r.$$

This action is free if and only if $v_1, v_2, \dots, v_r$ are linearly independent over $\mathbb{Z}/\ell\mathbb{Z}$. When we view A as a $(\mathbb{Z}/\ell\mathbb{Z})^r$-variety via this action, we will denote it by $(A; v_1, \dots, v_r)$. We will focus on the case, where $r = 2$ and $A = E \times E$ is the direct product of two copies of a complex elliptic curve E. More specifically, we will investigate how $\mathrm{ed}(E \times E; v_1, v_2)$ depends on the choice of E, v_1 and v_2.

Recall that for every prime integer ℓ, there exists a complex curve B and a family of elliptic curves $\mathcal{E} \to B$, together with a nowhere zero ℓ-torsion section, such that every pair $(E; q)$ where E is a complex elliptic curve and $q \in E(\mathbb{C})[\ell] \setminus \{0\}$ arises as a fiber of $\mathcal{E} \to B$; see [Colliot-Thélène 2002, Proposition A4]. The group $\mathbb{Z}/\ell\mathbb{Z}$ acts freely on $\mathcal{E}$ over B by translations by the ℓ-torsion section, and so $(\mathbb{Z}/\ell\mathbb{Z})^2$ acts freely on the fiber product

$$\Phi : \mathcal{X} = \mathcal{E} \times_B \mathcal{E} \to B \tag{5.1}$$

by translation. The fibers of Φ are $(\mathbb{Z}/\ell\mathbb{Z})^2$-varieties having the form $(E \times E; (q, 0), (0, q))$, where $q \in E(\mathbb{C})[\ell] \setminus \{0\}$.

Proposition 5.2. *Let ℓ be an odd prime integer, E be an elliptic curve over $\mathbb{C}$, and $q \in E(\mathbb{C})[\ell] \setminus \{0\}$.*

(i) *Suppose there exists an endomorphism $\phi : E \to E$ such that ϕ^2 is multiplication by an integer d, where d is not a square modulo ℓ. Then* $\mathrm{ed}(E \times E; (q, 0), (0, q)) = 1$.

(ii) *Assume* $\mathrm{End}(E) = \mathbb{Z}$. *Then* $\mathrm{ed}(E \times E; (q, 0), (0, q)) = 2$.

(iii) *Consider the map* $\Phi\colon \mathcal{X} \to B$ *from* (5.1). *Then there are countably infinite subsets* $B'_{\mathrm{cm}} \subset B_{\mathrm{cm}} \subsetneq B(\mathbb{C})$ *such that*

$$\operatorname{ed}(\mathcal{X}_b) = \begin{cases} 1, & \textit{if } b \in B'_{\mathrm{cm}}, \\ 2, & \textit{if } b \notin B_{\mathrm{cm}}. \end{cases}$$

Proof. (i) It is obvious from the definition that $\operatorname{ed}(E \times E : (q, 0), (0, q)) \geqslant 1$, so we only need to show that $\operatorname{ed}(E \times E; (q, 0), (0, q)) \leqslant 1$. Since ϕ is an endomorphism, it restricts to a group homomorphism $E(\mathbb{C})[\ell] \to E(\mathbb{C})[\ell]$. Fixing a $(\mathbb{Z}/\ell\mathbb{Z})$-basis of $E(\mathbb{C})[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$, ϕ gives rise to a matrix $A \in \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$. The matrix A does not have any eigenvalues in $\mathbb{Z}/\ell\mathbb{Z}$. Indeed, if $Av = \lambda v$ for some nonzero $v \in (\mathbb{Z}/\ell\mathbb{Z})^2$ and $\lambda \in \mathbb{Z}/\ell\mathbb{Z}$, then $dv = A^2 v = \lambda^2 v$, hence $d = \lambda^2$ in $\mathbb{Z}/\ell\mathbb{Z}$, which is impossible as d is not a square modulo ℓ. It follows that q and $\phi(q)$ are linearly independent, and so form a basis of $E(\mathbb{C})[\ell]$. Now, $\phi\colon (E; q) \to (E; \phi(q))$ is a $\mathbb{Z}/\ell\mathbb{Z}$-equivariant morphism and the composition

$$\big(E \times E; (q, 0), (0, q)\big) \xrightarrow{(\mathrm{id},\phi)} \big(E \times E; (q, 0), (0, \phi(q))\big) \xrightarrow{+} \big(E; (q, \phi(q))\big)$$

is a $(\mathbb{Z}/\ell\mathbb{Z})^2$-compression. Thus, $\operatorname{ed}(E \times E; (q, 0), (0, q)) \leqslant 1$, as desired.

(ii) Assume the contrary. Then there exists a dominant $(\mathbb{Z}/\ell\mathbb{Z})^2$-equivariant rational map

$$f\colon E \times E \dashrightarrow C,$$

where $E \times E$ stands for the $(\mathbb{Z}/\ell\mathbb{Z})^2$-variety $(E \times E; (q, 0), (0, q))$ and C is some curve with a faithful action of $(\mathbb{Z}/\ell\mathbb{Z})^2$. We may assume that C is smooth and projective. Since ℓ is odd, $(\mathbb{Z}/\ell\mathbb{Z})^2$ cannot act faithfully on $\mathbb{P}^1$. Thus, C is not isomorphic to $\mathbb{P}^1$. For all but finitely many v, the map f restricts to a well-defined surjective morphism $E \simeq E \times \{v\} \to C$. We deduce from Hurwitz's formula that C has genus 1. After suitably choosing an origin for C, the map f becomes an everywhere defined homomorphism of abelian varieties. The restrictions of f to $E \times \{0\}$ and $\{0\} \times E$ give isogenies $f_1, f_2\colon E \to C$ such that the element $(1, 0)$ of $(\mathbb{Z}/\ell\mathbb{Z})^2$ acts on C via translation by $f_1(q)$, and the element $(0, 1)$ acts on C via translations by $f_2(q)$. Since the $(\mathbb{Z}/\ell\mathbb{Z})^2$-action on C is faithful, we conclude that $f_1(q)$ and $f_2(q)$ form a basis of $C[\ell]$. On the other hand, recall from [Silverman 2009, Lemma III.4.2 (b)] that $\operatorname{Hom}(E, C)$ is torsion-free $\mathbb{Z}$-module. Since

$$\operatorname{Hom}(E, C) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \operatorname{Hom}(E, E) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q},$$

we conclude that $\operatorname{Hom}(E, C) = \mathbb{Z}$. This implies that there exists homomorphism $h\colon E \to C$ such that f_1 and f_2 are multiples of h. In particular, $f_1(q)$ and $f_2(q)$ are linearly dependent, a contradiction. We conclude that C does not exist, and thus $\operatorname{ed}(E \times E; (q, 0), (0, q)) = 2$, as claimed.

(iii) Recall that the endomorphism ring of an elliptic curve E over $\mathbb{C}$ is either $\mathbb{Z}$ (in which case we say that E has no complex multiplication) or the ring of integers in the field $\mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ and $d < 0$; see [Silverman 2009, Theorem III.9.3, Remark III.9.4.1].

Let B_{cm} be the set $b \in B(\mathbb{C})$ such that $\mathcal{E}_b = (E, q)$, where E is an elliptic curve with complex multiplication and q is an ℓ-torsion point. Let B'_{cm} be the set of $b \in B(\mathbb{C})$ such that $\mathcal{E}_b = (E, q)$, where $\operatorname{End}(E)$ is the ring of integers in $\mathbb{Q}[\sqrt{d}]$, where $d < 0$ is a negative integer which is not a square modulo ℓ and $d \equiv 2$ or 3 modulo 4. For d of this form, the ring of integers in $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\sqrt{d}]$.

By the Chinese remainder theorem, there exist infinitely many integers d of this form. Moreover, every ring of the form $\mathbb{Z}[\sqrt{d}]$, with d as above, arises as the endomorphism ring of a complex elliptic curve; see [Silverman 2009, Proposition VI.4.1]. Since there are only countably many complex elliptic curves with complex multiplication (see [Silverman 2009, Corollary 11.1.1 on p. 426]), we conclude that B_{cm} and B'_{cm} are both countably infinite subsets of $B(\mathbb{C})$. By part (i), we have $\mathrm{ed}(\mathcal{X}_b) = 1$ for every $b \in B'_{\mathrm{cm}}$ and by part (ii), we have $\mathrm{ed}(\mathcal{X}_b) = 2$ for every $b \notin B_{\mathrm{cm}}$. □

The case where $\ell = 2$ is a bit more complicated but the end result is similar.

Proposition 5.3. *Consider the map* $\Phi\colon X \to B$ *from* (5.1). *Then*:

(i) *There are countably infinitely many* $b \in B(\mathbb{C})$ *such that* $\mathrm{ed}(\mathcal{X}_b) = 1$.

(ii) *For a very general* $b \in B(\mathbb{C})$, *we have* $\mathrm{ed}(\mathcal{X}_b) = 2$.

Proof. (i) Let E be an complex elliptic curve, corresponding to a lattice $\mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$, where $\mathbb{Q}(\tau)$ is an imaginary quadratic extension of $\mathbb{Q}$. Multiplication by τ respects $\mathbb{Z} \oplus \mathbb{Z}\tau$; hence, it induces an endomorphism $\phi\colon E \to E$. Since multiplication by τ sends $1/2$ to $\tau/2$, there exists a point $q \in E(\mathbb{C})[2] \setminus \{0\}$ such that $\tau(q) \neq q$. We may now conclude (as we did in the proof of Proposition 5.2 (i)) that the morphism $\phi\colon (E; q) \to (E; \phi(q))$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant and the composition

$$(E \times E; (q, 0), (0, q)) \xrightarrow{(\mathrm{id}, \phi)} (E \times E; (q, 0), (0, \phi(q))) \xrightarrow{+} (E; (q, \phi(q)))$$

is a $(\mathbb{Z}/2\mathbb{Z})^2$-compression. Thus, $\mathrm{ed}(E \times E; (q, 0), (0, q)) = 1$. Since there are countably many such elliptic curves E, the proof of (i) is complete.

(ii) Suppose that there exists a dominant $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant rational map $f\colon E \times E \dashrightarrow C$, where C is a curve on which $(\mathbb{Z}/2\mathbb{Z})^2$ acts faithfully. In particular, there exists a surjective morphism $E \to C$. By the Hurwitz formula, C has genus 0 or 1. Since E is very general, we may suppose that $\mathrm{End}(E) = \mathbb{Z}$. In the proof of Proposition 5.2 (ii), we showed that C cannot have genus 0, because $(\mathbb{Z}/p\mathbb{Z})^2$ cannot act faithfully on $\mathbb{P}^1$ for any odd prime p. We then used the condition that $\mathrm{End}(E) = \mathbb{Z}$ to show that C cannot have genus 1. The latter argument goes through for $p = 2$ unchanged. Hence, we may assume that C has genus 0, i.e., $C = \mathbb{P}^1$. However, $(\mathbb{Z}/2\mathbb{Z})^2$ has a faithful action on $\mathbb{P}^1$, so the case where $C = \mathbb{P}^1$ requires additional consideration. We cannot dismiss it quite as easily as we did in the proof of Proposition 5.2 (ii). To analyze the case where $C = \mathbb{P}^1$, we begin by recalling a construction due to Garcia-Armas [Garcia-Armas 2016]. Let G be a finite group, V be a finite-dimensional complex vector space, and $\rho\colon G \to \mathrm{PGL}(V)$ be a faithful complex projective representation. Garcia-Armas [2016, §4] constructed a cohomological invariant

$$\Delta_\rho\colon H^1(K, G) \to H^2(K, \mathbb{G}_{\mathrm{m}}) = \mathrm{Br}(K),$$

where $K/\mathbb{C}$ ranges over all field extensions. If X is a generically free primitive G-variety, we will denote by $[X] \in H^1(\mathbb{C}(X)^G, G)$ the class of the G-torsor corresponding to X, as in Remark 4.6. Suppose now that V is 2-dimensional, so that $\mathbb{P}(V) = \mathbb{P}^1$. Then by [Garcia-Armas 2016, Corollary 4.4], for any generically free primitive G-variety X, we have $\Delta_\rho([X]) = 0$ if and only if there exists a G-equivariant compression $X \dashrightarrow \mathbb{P}(V)$. If the finite group G is abelian, then Δ_ρ is a group homomorphism and, in particular, it factors through $H^2(K, \mu_e) = \mathrm{Br}(K)[e]$, where $e \geqslant 1$ is the exponent of G. Writing $C = \mathbb{P}^1$ as $C = \mathbb{P}(V)$, where V is a 2-dimensional complex vector space, we observe that there is only one possible faithful action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^1$ (up to an automorphism of $\mathbb{P}^1$), given by the unique faithful complex projective representation $\rho\colon (\mathbb{Z}/2\mathbb{Z})^2 \hookrightarrow \mathrm{PGL}(V) = \mathrm{PGL}_2(\mathbb{C})$. "

The family $\mathcal{E} \to B$ considered above contains a special fiber $(\mathbb{Z}/2\mathbb{Z})$-equivariantly isomorphic to $(\mathbb{G}_\mathrm{m}; -1)$. It follows that the family $\Phi\colon \mathcal{X} = \mathcal{E} \times_B \mathcal{E} \to B$ contains a special fiber $\mathcal{X}_0$ that is $(\mathbb{Z}/2\mathbb{Z})^2$-equivariantly isomorphic to $(\mathbb{G}_\mathrm{m}^2; (-1, 1), (1, -1))$. Replacing $\mathbb{G}_\mathrm{m}^2$ by $\mathbb{A}^2$, we see that $\mathcal{X}_0$ is $(\mathbb{Z}/2\mathbb{Z})^2$-equivariantly birationally isomorphic to a linear action of $(\mathbb{Z}/2\mathbb{Z})^2$ on a 2-dimensional vector space. Hence, $\mathcal{X}_0$ is a versal $(\mathbb{Z}/2\mathbb{Z})^2$-variety. Consequently, $\mathrm{ed}_\mathbb{C}(\mathcal{X}_0) = \mathrm{ed}_\mathbb{C}((\mathbb{Z}/2\mathbb{Z})^2) = 2$; see (2.4). In particular, there does not exist a $(\mathbb{Z}/2\mathbb{Z})^2$-compression $\mathbb{G}_\mathrm{m}^2 \dashrightarrow \mathbb{P}^1$. Now, [Garcia-Armas 2016, Corollary 4.4] implies that $\Delta_\rho([\mathcal{X}_0]) \neq 0$ in $H^2(K, \mu_2) = \mathrm{Br}(K)[2]$, where $K = \mathbb{C}(\mathbb{G}_\mathrm{m}^2)^{(\mathbb{Z}/2\mathbb{Z})^2}$. By Remark 4.6, it follows that $\Delta_\rho([\mathcal{X}_b]) \neq 0$ for all but countably many $b \in B(\mathbb{C})$. Applying [Garcia-Armas 2016, Corollary 4.4] one more time, we deduce that for any such b, a $(\mathbb{Z}/2\mathbb{Z})^2$-equivariant rational map $f\colon \mathcal{X}_b \dashrightarrow \mathbb{P}^1$ does not exist. Consequently, $\mathrm{ed}(\mathcal{X}_b) = 2$ for all but countably many $b \in B(\mathbb{C})$. □

6. Example: Variety of quadratic forms

The following example shows that Theorem 1.2 fails if $\mathrm{ed}_{k(\bar{b})}$ is replaced by $\mathrm{ed}_{k(b)}$. In particular, if k is not algebraically closed, then the k-points $s \in B(k)$, where $\mathrm{ed}_k(\mathcal{X}_s) \leqslant n$ do not necessarily lie on a countable union of closed subvarieties of B.

Let $k = \mathbb{R}$ be the field of real numbers and G be the orthogonal group O_2 defined over $\mathbb{R}$. Consider the action of $G = \mathrm{O}_2$ on $\mathcal{X} = \mathrm{GL}_2$ via multiplication on the right. Note that $\mathcal{X}$ is the total space of a G-torsor $\tau : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y} = \mathrm{GL}_2 / O_2$ is naturally identified with the space of symmetric 2×2 matrices via $\tau : A \mapsto AA^T$.

Now consider the morphism

$$f : \mathcal{X} = \mathrm{GL}_2 \longrightarrow B = \mathbb{A}^1 \setminus \{0\},$$

sending a matrix A to $\det(A)^2$. This morphism factors through τ as

$$f : \mathcal{X} \xrightarrow{\tau} \mathcal{Y} \xrightarrow{\det} B = \mathbb{A}^1 \setminus \{0\}.$$

Denote that fibers of $\mathcal{X}$ and $\mathcal{Y}$ over $s \in B$ by $\mathcal{X}_s$ and $\mathcal{Y}_s$, respectively. Then $\mathcal{X}_s$ is a G-torsor over $\mathcal{Y}_s$.

Proposition 6.1. *View a nonzero real number s as an $\mathbb{R}$-point of B. Then*

$$\mathrm{ed}(\mathcal{X}_s) = \begin{cases} 0, & \text{if } s < 0, \\ 1, & \text{if } s > 0. \end{cases}$$

Proof. Note that $\mathcal{Y}_s$ is the variety of symmetric matrices $B = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ such that $\det(B) = s$. Thus $\mathcal{Y}_s$ is a rational surface over $\mathbb{R}$ whose function field can be identified with $\mathbb{R}(a, b)$. Passing to the generic point of $\mathcal{Y}_s$, we see that $\mathrm{ed}_{\mathbb{R}}(\mathcal{X}_s) = \mathrm{ed}_{\mathbb{R}}(\tau_s)$, where $\tau_s \in H^1(\mathbb{R}(a,b), \mathrm{O}_2)$ is the O_2-torsor over $\mathbb{R}(a,b)$ obtained by pulling back τ to the generic point of $\mathcal{Y}_s$. Examining the long exact cohomology sequence associated to the exact sequence $1 \to \mathrm{O}_2 \to \mathrm{GL}_2$ of algebraic groups and remembering that $H^1(\mathbb{R}(a,b), \mathrm{GL}_2) = 1$ by Hilbert's theorem 90, we see that $H^1(\mathbb{R}(a,b), \mathrm{O}_2)$ is in a natural bijective correspondence with the set of 2-dimensional nonsingular quadratic forms over $\mathbb{R}(a,b)$, up to equivalence, and the quadratic form q_s corresponding to τ_s is the form whose Gram matrix is $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, where $c = (s+b^2)/a$. Note that, by definition, $\mathrm{ed}_{\mathbb{R}}(\tau_s) = \mathrm{ed}_{\mathbb{R}}(q_s)$ and the discriminant of q_s is s.

Since q assumes the value a and has discriminant s, the quadratic form q_s is isomorphic to $\langle a, as\rangle$. Here, $\langle a, as\rangle$ denote the 2-dimensional quadratic form $q_s(z,w) = az^2 + asw^2$ over $\mathbb{R}(a,b)$. If $s < 0$, then q is isotropic over $\mathbb{R}(a,b)$. Hence, q is hyperbolic over $\mathbb{R}(a,b)$, i.e., q_s is isomorphic to $\langle 1, -1\rangle$; see [Lam 2005, Theorem I.3.2]. In particular, q_s descends to $\mathbb{R}$ and hence, $\mathrm{ed}_{\mathbb{R}}(q_s) = 0$.

On the other hand, suppose that $s > 0$. Then s is a complete square in $\mathbb{R}(a,b)$, so $q \simeq \langle a, a\rangle$. Clearly, q_s descends to $\mathbb{R}(a) \subset K$, so $\mathrm{ed}_{\mathbb{R}}(q_s) \leqslant 1$. In order to complete the proof of Proposition 6.1, it remains to show that $\mathrm{ed}_{\mathbb{R}}(q_s) \neq 0$. We argue by contradiction. Assume $\mathrm{ed}_{\mathbb{R}}(q_s) = 0$, i.e., q_s descends to some intermediate extension $\mathbb{R} \subset K \subset \mathbb{R}(a,b)$, where $\mathrm{trdeg}_{\mathbb{R}}(K) = 0$. In other words, K is algebraic over $\mathbb{R}$. Since $\mathbb{R}$ is algebraically closed in $\mathbb{R}(a,b)$, this is only possible if $K = \mathbb{R}$, i.e., q descends to a 2-dimensional form q_0 defined over $\mathbb{R}$. Since $s > 0$, q is anisotropic over $\mathbb{R}(a,b)$, and hence, so is q_0. Let $\nu_a \colon K^\times \to \mathbb{Z}$ be the valuation associated to the variable a. It is now easy to see that for any $(0,0) \neq (f,g) \in (K^\times)^2$, $\nu_a(q_0(f,g))$ is even, whereas $\nu_a(q(f,g))$ is odd. This tells us that q and q_0 have no values in common, contradicting our assumption that q descends to q_0. $\square$

7. Transversal intersections in projective space

This section contains several preliminary results which will be used in the proof of Theorem 1.4. The common theme is transversal intersections of projective varieties with linear subspaces in projective space. Note that there are no algebraic groups or group actions here; they will come into play in the next section.

Recall that a commutative ring with identity is said to be *regular* if it is noetherian and all its localizations at prime ideals are regular local rings.

Lemma 7.1. *Let A be a regular semilocal noetherian ring and $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$ be the maximal ideals of A. For each $1 \leqslant i \leqslant r$, let $P_i \subset \mathfrak{m}_i$ be a prime ideal such that $P_i \not\subset \mathfrak{m}_j$ for any $j \neq i$ and such that each local ring A/P_i is regular. Assume that the prime ideals $P_1, \ldots, P_r$ have the same height,* $\mathrm{ht}(P_1) = \cdots = \mathrm{ht}(P_r) = c$. *Then there exist $h_1, h_2, \ldots, h_c \in A$ such that $P_i A_{\mathfrak{m}_i} = (h_1, \ldots, h_c)A_{\mathfrak{m}_i}$ and $h_1, \ldots, h_c$ forms a regular sequence in A_{m_i} for each i.*

Proof. We claim that, for all $1 \leqslant i \leqslant r$, there exist $f_{i,1}, \ldots, f_{i,c} \in P_i$ whose images in the $A/\mathfrak{m}_i$-vector space $\mathfrak{m}_i/\mathfrak{m}_i^2$ are linearly independent. Indeed, letting $\overline{\mathfrak{m}}_i = \mathfrak{m}_i/P_i$ be the maximal ideal of A/P_i, we have a short exact sequence

$$0 \to P_i/(P_i \cap \mathfrak{m}_i^2) \to \mathfrak{m}_i/\mathfrak{m}_i^2 \to \overline{\mathfrak{m}}_i/\overline{\mathfrak{m}}_i^2 \to 0.$$

In view of the isomorphism $P_i/(P_i\cap\mathfrak{m}_i^2)\simeq(P_i+\mathfrak{m}_i^2)/\mathfrak{m}_i^2$ and the regularity of $A_{\mathfrak{m}_i}$ and A/P_i, this gives

$$\dim_k((P_i+\mathfrak{m}_i^2)/\mathfrak{m}_i^2)=\dim_k(\mathfrak{m}_i/\mathfrak{m}_i^2)-\dim_k(\overline{\mathfrak{m}}_i/\overline{\mathfrak{m}}_i^2)=\mathrm{Kdim}(A_{\mathfrak{m}_i})-\mathrm{Kdim}(A/P_i)=c,$$

where Kdim denotes the Krull dimension. Therefore, we may choose $f_{i,1},\dots,f_{i,c}\in P_iA_{\mathfrak{m}_i}$ whose images in $\mathfrak{m}_i/\mathfrak{m}_i^2$ are linearly independent. Multiplying each $f_{i,j}$ by a suitable unit in $A_{\mathfrak{m}_i}$, we may suppose that $f_{i,j}\in P_i$ for all j. This proves the claim.

For any $a\neq b$, the ideal P_a+P_b is not contained in any $\mathfrak{m}_i$, hence $P_a+P_b=A$. By the Chinese remainder theorem the natural ring homomorphism

$$A/(P_1^2\cdots P_r^2)\to(A/P_1^2)\times\cdots\times(A/P_r^2)$$

is an isomorphism; see [Stacks 2005–, Tag 00DT] or [Eisenbud 1995, Exercise 2.6]. In particular, the quotient map $A\to(A/P_1^2)\times\cdots\times(A/P_r^2)$ is surjective. Therefore, for all $1\leqslant i\leqslant r$ and $1\leqslant j\leqslant c$, we may find $h_{i,j}\in P_i$ whose reduction modulo P_i^2 coincides with the reduction of $f_{i,j}$, and whose reduction modulo P_s^2 is equal to 1 for all $s\neq i$. Then $h_{i,1},\dots,h_{i,c}\in P_i$ satisfy the following conditions:

(1) their images $\bar{h}_{i,1},\dots,\bar{h}_{i,c}$ in the $A/\mathfrak{m}_i$-vector space $\mathfrak{m}_i/\mathfrak{m}_i^2$ form a basis of the subspace $(P_i+\mathfrak{m}_i^2)/\mathfrak{m}_i^2$,

(2) $h_{i,j}-1\in P_s^2$ for all $s\neq i$.

For each $1\leqslant j\leqslant c$, set $h_j:=\prod_{i=1}^r h_{i,j}$. Then $h_j\in\cap_{i=1}^r P_i$ for all j. Moreover, for all $1\leqslant i\leqslant r$, the images of $h_1,\dots,h_c$ in $\mathfrak{m}_i/\mathfrak{m}_i^2$ are equal to $\bar{h}_{i,1},\dots,\bar{h}_{i,c}$, and so form a basis of $(P_i+\mathfrak{m}_i^2)/\mathfrak{m}_i^2$. Thus, for each $i=1,\dots,r$, the images of $h_1,\dots,h_c$ in $\mathfrak{m}_i/\mathfrak{m}_i^2$ may be completed to a basis of $\mathfrak{m}_i/\mathfrak{m}_i^2$, hence $h_1,\dots,h_c$ may be completed to a regular system of parameters for $\mathfrak{m}_iA_{\mathfrak{m}_i}$. Thus the elements $h_1,\dots,h_c$ form a regular sequence in $\mathfrak{m}_iA_{\mathfrak{m}_i}$.

Finally, we claim that $P_iA_{\mathfrak{m}_i}=(h_1,\dots,h_c)A_{\mathfrak{m}_i}$ for all $1\leqslant i\leqslant r$. Since $h_1,\dots,h_c$ are a regular sequence in $A_{\mathfrak{m}_i}$, by [Stacks 2005–, Tag 00NQ] the local ring $B_i:=A_{\mathfrak{m}_i}/(f_{i,1},\dots,f_{i,c})$ is a regular local ring of Krull dimension

$$\mathrm{Kdim}(B_i)=\mathrm{Kdim}(A_{\mathfrak{m}_i})-c=\mathrm{Kdim}(A/P_i).$$

Consider the surjection $\phi_i\colon B_i\to A/P_i$. We are going to show that ϕ_i is injective. Indeed, suppose that $0\neq x\in B_i$ is in the kernel of ϕ_i. By [Stacks 2005–, Tag 00NP], the regular local ring B_i is a domain, hence by [Stacks 2005–, Tag 00KW], we have $\mathrm{Kdim}(B_i/(x))=\mathrm{Kdim}(B_i)-1<\mathrm{Kdim}(A/P_i)$. On the other hand, ϕ_i factors through a surjection $B_i/(x)\to A/P_i$, hence $\mathrm{Kdim}(B_i/(x))\geqslant\mathrm{Kdim}(A/P_i)$, a contradiction. Therefore, ϕ_i is injective. This means that $P_iA_{\mathfrak{m}_i}$ is generated by $h_1,\dots,h_c$ as an ideal of $A_{\mathfrak{m}_i}$, as claimed. □

Let k be a field. For every $0\leqslant d\leqslant n$, we denote by $\mathrm{Gr}(n,n-d)$ the Grassmannian of codimension d hyperplanes of $\mathbb{P}_k^n$. If $W\subset\mathbb{P}_k^n$ is a k-subspace of codimension d, we will denote by $[W]\in\mathrm{Gr}(n,n-d)(k)$ the k-point representing W in the Grassmannian. If $Z\subset\mathbb{P}_k^n$ is a closed subscheme, we will say that W intersects Z transversely at a smooth k-point $z\in Z$ if $z\in W$ and the tangent space $T_z(Z)$ intersects W transversely. Equivalently, W intersects Z transversely at z if W can be cut out by linear forms $h_1,\dots,h_d\in\Gamma(\mathbb{P}_k^n,\mathcal{O}(1))$ such that $h_1(z)=\cdots=h_d(z)=0$ and $h_1/h,\dots,h_d/h$ form a regular sequence in the local ring $\mathcal{O}_{Z,z}$ for some (and, thus, any) $h\in\Gamma(\mathbb{P}_k^n,\mathcal{O}(1))$ with $h(z)\neq 0$; see [Eisenbud 1995, Section 10.3].

If $z \in Z$ is a smooth closed point, not necessarily k-rational, we say that W intersects Z transversely at z if $W_{\bar{k}}$ intersects $Z_{\bar{k}}$ transversely at every point $\bar{k}$-point of $Z \cap W$ lying above z.

Lemma 7.2. *Let Z be a closed subscheme of $\mathbb{P}^n_k$, $0 \leqslant c \leqslant n$ be an integer,*

$$I_{Z,c} \subset \mathrm{Gr}(n, n-c) \times Z$$

be the incidence correspondence parametrizing pairs $([W], v)$ such that $v \in W \cap Z$, and

$$\phi\colon I_{Z,c} \to \mathrm{Gr}(n, n-c), \quad ([V], v) \mapsto V$$

be the projection to the first component. Let z be a smooth closed point of Z, and let $W_0 \subset \mathbb{P}^n_k$ be a codimension c linear subspace such that W_0 and Z intersect transversely at z. Then ϕ is smooth at $([W_0], z)$.

Proof. Since smoothness may be verified after an fpqc base change [Stacks 2005–, Tag 02VL] and the morphism $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$ is fpqc, we may replace k by $\bar{k}$ and assume that k is algebraically closed.

We claim that ϕ is flat at $([W_0], z)$. Note that the fiber $W \cap Z$ of ϕ over $[W]$ is smooth at z, so the lemma follows from this claim by [Stacks 2005–, Tag 01V8].

To prove the claim, we argue by induction on c. In the base case, $c = 0$, $\mathrm{Gr}(n, n-c)$ is a point, ϕ is the identity map, and the claim is obvious.

For the induction step, assume that $c \geqslant 1$ and the claim holds when c is replaced by $c-1$, for every $n \geqslant c$ and every closed subscheme Z of $\mathbb{P}^n$. Choose linear forms $h_1, \dots, h_c \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ such that $h_1, \dots, h_c$ cut out W_0, and $h_1/h, \dots, h_c/h$ form a regular sequence in the local ring $\mathcal{O}_{Z,z}$ for some $h \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ such that $h(z) \neq 0$.

Denote the zero locus of h_1 by $\mathbb{P}^{n-1}$, the intersection $Z \cap \mathbb{P}^{n-1}$ by Z', the preimage of Z' under ϕ by $I'_{Z,c}$, and the restriction of ϕ to $I_{Z,c}$ by ϕ'. By [Eisenbud 1995, Corollary 6.9], it suffices to show that $\phi'\colon I'_{Z,c} \to \mathrm{Gr}(n, n-c)'$ is flat at $([W_0], z)$. Here, $\mathrm{Gr}(n, n-c)'$ denotes the hypersurface in $\mathrm{Gr}(n, n-c)$ consisting of $(n-c)$-dimensional linear subspaces of $\mathbb{P}^n$ which are contained in $\mathbb{P}^{n-1}$. We will view Z' as a closed subscheme of $\mathbb{P}^{n-1}$. Since h_1 cuts Z transversely at z, we see that z is a smooth point of Z'. Now observe that $\mathrm{Gr}(n, n-c)'$ is naturally isomorphic to $\mathrm{Gr}(n-1, n-c)$ and $I'_{Z,c}$ is naturally isomorphic to $I_{Z',c-1}$ over Z' so that the following diagram commutes:

$$\begin{tikzcd}
I_{Z',c-1} \arrow[r, "\simeq"] \arrow[d] & I'_{Z,c} \arrow[r, hook] \arrow[d, "\phi'"] & I_{Z,c} \arrow[d, "\phi"] \\
\mathrm{Gr}(n-1, n-c) \arrow[r, "\simeq"] & \mathrm{Gr}(n, n-c)' \arrow[r, hook] & \mathrm{Gr}(n, n-c)
\end{tikzcd}$$

Since $\mathrm{Gr}(n-1, n-c) = \mathrm{Gr}(n-1,\ (n-1)-(c-1))$, we can use the induction assumption to conclude that ϕ' is flat at $([W_0], z)$, as desired. □

Lemma 7.3. *Let Z be an irreducible quasiprojective k-variety, and let $Y \subset Z$ be a closed equidimensional subvariety, with irreducible components $Y_1, \dots, Y_r$. For every $1 \leqslant i \leqslant r$, let $z_i \in Y_i$ be a closed point such that Y and Z are smooth at z_i, and c be the codimension of Y_i in Z. Then there exist an integer $n \geqslant 0$, a closed embedding $Z \hookrightarrow \mathbb{P}^n_k$ and a codimension c subspace $W_0 \subset \mathbb{P}^n_k$ such that W_0 intersects Z transversely at z_i, and locally around z_i we have $Y = Y_i = Z \cap W_0$ (scheme-theoretically) for each $i = 1, \dots r$.*

We emphasize that in the statement of Lemma 7.3 the closed points $z_1, \dots, z_r$ are not assumed to be k-rational but the closed embedding of $Z \hookrightarrow \mathbb{P}^n_k$ and the codimension c subspace $W_0 \subset \mathbb{P}^n_k$ are defined over k.

Proof. Since Z is quasiprojective, there exists an affine open subset $\operatorname{Spec}(B) \subset Z$ containing $z_1, \dots, z_r$. Let $\operatorname{Spec}(A) \subset Z$ be the semilocalization of Z at $\{z_1, \dots, z_r\}$. By definition, the ring A is the semilocal ring obtained from B by localizing at the multiplicative subset consisting of elements which do not belong to the maximal ideal $\mathfrak{m}_{z_i}$ for any $i = 1, \dots, r$. By Lemma 7.1, there exist $f_1, \dots, f_c \in A$, such that $f_1, \dots, f_c$ form a regular sequence in $\mathcal{O}_{Z,z_i}$ and generate the ideal of Y_i in $\mathcal{O}_{Z,z_i}$, for each $1 \leqslant i \leqslant r$.

Since Z is a quasiprojective, there exists a locally closed embedding $\iota\colon Z \hookrightarrow \mathbb{P}^n_k$. For every $0 \leqslant i \leqslant c$, f_i is the restriction of a rational function P_i/Q_i on $\mathbb{P}^n_k$, where P_i and Q_i are homogeneous polynomials of the same degree and Q_i does not vanish at $z_1, \dots, z_r$. We deduce that locally near each of the points $z_1, \dots, z_r$ the variety Y is the scheme-theoretic intersection of Z and the closed subscheme defined by $P_1, \dots, P_c$. By a refinement of the graded prime avoidance lemma [Gabber et al. 2013, Lemma 4.11], there exist an integer $d \geqslant 1$ and homogeneous polynomials F_1, F_2 of degrees d and $d+1$, respectively, such that F_1 and F_2 do not vanish at $z_1, \dots, z_r$. Since d and $d+1$ are coprime, any sufficiently large positive integer is of the form $n_1 d + n_2(d+1)$ for some integers $n_1, n_2 \geqslant 0$. Therefore, multiplying the P_i and Q_i by suitable powers of F_1 and F_2, we may assume that $P_1, \dots, P_c$ have the same degree $D \geqslant 1$. After composing ι with the D-fold Veronese embedding of $\mathbb{P}^n_k$, we may further assume that each P_i has degree 1, that is, that each P_i cuts out a hyperplane in $\mathbb{P}^n_k$. Now the embedding $\iota\colon Z \hookrightarrow \mathbb{P}^n_k$ and the linear subspace W_0 of $\mathbb{P}^n_k$ given by $P_1 = \cdots = P_c = 0$ have the properties claimed in the lemma. $\square$

8. Proof of Theorem 1.4

Let $X_0 = X_0^{(1)} \cup \cdots \cup X_0^{(r)}$ be the irreducible decomposition of X_0. Then choose rational functions $\alpha_1, \dots, \alpha_d\colon X_0 \dashrightarrow \mathbb{A}^1_k$ such that the restriction of $\alpha_1, \dots, \alpha_d$ to $X_0^{(i)}$ generate the function field $k(X_0^{(i)})$ for every i. After adjoining all G-translates of $\alpha_1, \dots, \alpha_d$ to this set, we may assume that G permutes $\alpha_1, \dots, \alpha_d$. Consider the G-equivariant rational map $\alpha\colon X_0 \dashrightarrow \mathbb{P}(V)$, with $x \mapsto (1 : \alpha_1(x) : \cdots : \alpha_d(x))$, for a suitable linear (permutation) representation of G on $V = k^{d+1}$. Note that since the G-action on X_0 is assumed to be faithful, the G-action on $\mathbb{P}(V)$ is faithful as well. Moreover, by our construction, α induces a G-equivariant birational isomorphism between X_0 and $\alpha(X_0)$. After replacing X_0 with the closure of $\alpha(X_0)$ in $\mathbb{P}(V)$, we may assume that X_0 is a closed subvariety of $\mathbb{P}(V)$.

Let $\mathbb{P}(V)_{\text{nonfree}}$ be the nonfree locus for the G-action on V, i.e., the union of the fixed point loci $\mathbb{P}(V)^g$ as g ranges over the nontrivial elements of G. Since G acts faithfully on $\mathbb{P}(V)$, we have

$$\dim \mathbb{P}(V) > \dim \mathbb{P}(V)_{\text{nonfree}}. \tag{8.1}$$

Let V^m be the direct sum of m copies of V (as a G-representation). We claim that

$$\text{the codimension of } \mathbb{P}(V^m)_{\text{nonfree}} \text{ in } \mathbb{P}(V^m) \text{ is} \geqslant m. \tag{8.2}$$

Indeed, for each $1 \neq g \in G$, let $\operatorname{mult}(g, V)_\lambda$ be the dimension of the λ-eigenspace of g and $\operatorname{mult}(g, V)$ be the maximal value of $\operatorname{mult}(g, V)_\lambda$, where λ ranges over $\bar{k}$. Then $\dim \mathbb{P}(V)_{\text{nonfree}}$ is the maximal value of $\operatorname{mult}(g, V) - 1$, as g ranges over the nontrivial elements of G. Clearly, $\operatorname{mult}(g, V^m) = \operatorname{mult}(g, V)\, m$ for every g. Thus,

$$\dim \mathbb{P}(V^m)_{\text{nonfree}} = (\dim \mathbb{P}(V)_{\text{nonfree}} + 1)\, m - 1,$$

and the codimension of $\mathbb{P}(V^m)_{\text{nonfree}}$ in $\mathbb{P}(V^m)$ is

$$\begin{aligned}\dim \mathbb{P}(V^m) - \dim \mathbb{P}(V^m)_{\text{nonfree}} &= (\dim \mathbb{P}(V) + 1)\, m - 1 - \big((\dim \mathbb{P}(V)_{\text{nonfree}} + 1)\, m - 1\big)\\ &= \big(\dim \mathbb{P}(V) - \dim \mathbb{P}(V)_{\text{nonfree}}\big)\, m \geqslant m,\end{aligned}$$

where the last inequality follows from (8.1). This completes the proof of (8.2).

The G-equivariant linear embedding $V \hookrightarrow V^m$, given by $v \mapsto (v, 0, \dots, 0)$, induces a G-equivariant closed embedding

$$X_0 \subset \mathbb{P}(V) \hookrightarrow \mathbb{P}(V^m).$$

This allows us to view X_0 as a G-invariant subvariety of $\mathbb{P}(V^m)$. By [Popov and Vinberg 1994, Theorem 4.14] or [SGA 1 1971, Exposé V, Propositions 1.8 and 3.1], there exists a geometric quotient map $\pi : \mathbb{P}(V^m) \to \mathbb{P}(V^m)/G$. Explicitly, write

$$\mathbb{P}(V^m) = \operatorname{Proj}(k[V^m]),$$

where $k[V^m]$ is a polynomial ring in $\dim(V)m = (d+1)m$ variables. Then $\mathbb{P}(V^m)/G \simeq \operatorname{Proj}(A)$, where $A = k[V^m]^G$ is the ring of invariants and π is induced by the inclusion $A = k[V^m]^G \hookrightarrow k[V^m]$ of graded rings. Restricting π to $X_0 \subset \mathbb{P}(V^m)$, we obtain the geometric quotient map $X_0 \to X_0/G$. Note that

$$\dim \mathbb{P}(V^m)/G = \dim \mathbb{P}(V^m) = (d+1)m - 1$$

and $\dim X_0/G = \dim X_0 = e$. Therefore, every irreducible component of X_0/G is of codimension $c = (d+1)m - 1 - e$ in $\mathbb{P}(V^m)/G$.

By assumption (see Section 2), X_0 is geometrically reduced. By generic smoothness [Stacks 2005–, Tag 056V], there exists a dense open subscheme of X_0 which is smooth over k. Since the G-action on X_0 is assumed to be generically free, we may choose smooth closed points $x_1, \dots, x_r$, one on each irreducible component of X_0, such that the (scheme-theoretic) stabilizer G_{x_i} is trivial for every i. We now apply Lemma 7.3 to $Z = \mathbb{P}(V^m)/G$, $Y = X_0/G$, $z_1, \dots, z_r$, where $z_i = \pi(x_i)$ for each i. Note that by our choice of $x_1, \dots, x_r$, both Y and Z are smooth at each z_i, for $i = 1, \dots, r$. We deduce that there exist a closed embedding $\mathbb{P}(V^m)/G \hookrightarrow \mathbb{P}^n$ defined over k and a subspace $W_0 \subset \mathbb{P}^n$ of codimension c such that $X_0/G = W_0 \cap (\mathbb{P}(V^m)/G)$ locally around z_i for each i. Consider the diagram

$$\begin{tikzcd}
T \arrow[rr, hook] \arrow[dr] \arrow[ddrr, bend right, "\bar{f}"'] & & \mathrm{Gr}(n, n-c) \times \mathbb{P}(V^m) \arrow[d, "\mathrm{id}\times\pi"] \\
& I_{Z,c} \arrow[r, hook] \arrow[dr, "\phi"'] & \mathrm{Gr}(n, n-c) \times Z \arrow[d, "\mathrm{pr}_2"] \\
& & \mathrm{Gr}(n, n-c)
\end{tikzcd}$$

Here, $I_{Z,c}$ is the incidence correspondence parametrizing pairs $([W], q)$, where W is a linear subspace of $\mathbb{P}^n_k$ having codimension c and $q \in W \cap Z$, as in Lemma 7.2, and T is the preimage of $I_{Z,e}$ in $\mathrm{Gr}(n, n-c) \times \mathbb{P}(V^m)$.

By Lemma 7.2, ϕ is smooth at $([W_0], z_i)$ for each $i = 1, \dots, r$. On the other hand, by our choice of $x_1, \dots, x_r$, the map π is smooth at each of these points; hence, $\bar{f} = \phi \circ (\pi \times \mathrm{id})\colon T \to \mathrm{Gr}(n, n-c)$ is smooth at $([W], x_i)$ for each i.

From now on, we will assume that $m > e$. Note that e is given in the statement of Theorem 1.4, whereas m is a feature of our construction, which we are free to choose. We will construct the family $f : \mathcal{X} \to B$ by restricting $\bar{f}$ to a dense open subset $\mathcal{X} = T \setminus C$, where $C = T_{\mathrm{sing}} \cup T_{\mathrm{nonfree}}$. Here, T_{sing} is the singular locus of f and

$$T_{\mathrm{nonfree}} \subset \mathrm{Gr}_{n,n-c} \times \mathbb{P}(V^m)_{\mathrm{nonfree}}$$

is the nonfree locus for the G-action in T. Recall that $\mathbb{P}(V^m)_{\mathrm{nonfree}}$ was defined at the beginning of this section. The base B of our family is obtained by removing from $\mathrm{Gr}(n, n-c)$ the locus of points $b \in \mathrm{Gr}(n, n-c)$ such that the entire fiber T_b lies in C. In particular, $\bar{f}(\mathcal{X}) \subset B$. Since C is closed in T, B is open in $\mathrm{Gr}(n, n-c)$. Note also that since $\bar{f}$ is a proper morphism, $\bar{f}(C)$ is closed in B.

Let $b_0 := [W_0] \in \mathrm{Gr}(n, n-c)(k)$. By our choice of $x_1, \dots, x_r$, none of the points $([W_0], x_i)$ lie in C. Hence, $b_0 \in B(k)$ and the union of the irreducible components of $\mathcal{X}_{b_0}$ passing through $x_1, \dots, x_r$ remains birationally isomorphic to X_0. This is close to condition (iii) but a little weaker; we will return to this point at the end of the proof.

Note that by our construction $\mathcal{X}$ and B are irreducible, the G-action on $\mathcal{X}$ is free, f is smooth of constant relative dimension $e = \dim(X_0) = \dim(\mathcal{X}_{b_0})$. In particular, condition (i) of Theorem 1.4 holds for f. To prove that condition (ii) also holds for f, it suffices to check that when $m > e$, we have

(a) $\bar{f}(T_{\mathrm{nonfree}}) \neq \mathrm{Gr}(n, n-c)$,

(b) $\bar{f}(T_{\mathrm{sing}}) \neq \mathrm{Gr}(n, n-c)$,

(c) there exists a dense open subset $U \subset \mathrm{Gr}(n, n-c)$ such that the fibers of f over U are projective and irreducible.

To prove (a), recall that by (8.2), the codimension of $P(V^m)_{\mathrm{nonfree}}$ in $P(V^m)$ is $\geqslant m$. Remembering that $m > e$, $\dim(\mathbb{P}(V^m)) = (d+1)m - 1$, and $c = (d+1)m - 1 - e$, we obtain

$$\dim \pi(\mathbb{P}(V^m)_{\mathrm{nonfree}}) = \dim \mathbb{P}(V^m)_{\mathrm{nonfree}} \leqslant \dim(\mathbb{P}(V^m)) - m < \dim(\mathbb{P}(V^m)) - e = c.$$

Consequently, an $(n-c)$-dimensional linear subspace W in $\mathbb{P}^n$ in general position will intersect $\pi(\mathbb{P}(V^m)_{\mathrm{nonfree}}) = \mathbb{P}(V^m)_{\mathrm{nonfree}}/G$ trivially. We conclude that $\bar{f}^{-1}(W) \cap T_{\mathrm{nonfree}} = \varnothing$. This proves (a).

To prove (b), recall that the fiber of the morphism $\phi\colon I_{Z,c} \to \mathrm{Gr}(n, n-c)$ over $[W]$ is $W \cap Z$. By Bertini's theorem, there exists a dense open subset $U \subset \mathrm{Gr}(n, n-c)$ consisting of $(n-c)$-dimensional linear subspaces W of $\mathbb{P}^n$ such that $W \cap Z$ is smooth. By generic flatness [Stacks 2005–, Tag 0529], after replacing U by a smaller open subset, we may assume that $\phi\colon \phi^{-1}(U) \to U$ is flat. Appealing to [Stacks 2005–, Tag 01V8], as we did in Lemma 7.2, we see that since $\phi\colon \phi^{-1}(U) \to U$ is a flat map with smooth fibers, it is smooth. Finally, after intersecting U with the complement of $\bar{f}(T_{\mathrm{nonfree}})$ (which we know is a dense open subset of $\mathrm{Gr}(n, n-c)$ by (a)), we may assume that $\pi \times \mathrm{id}$ is smooth over $\phi^{-1}(U)$. Hence, the map $\bar{f}\colon \bar{f}^{-1}(U) \to U$, being a composition of two smooth maps, $\pi \times \mathrm{id}\colon \bar{f}^{-1}(U) \to \phi^{-1}(U)$ and $\phi\colon \phi^{-1}(U) \to U$, is smooth. This proves (b).

To prove (c), recall that by definition, for $b = [W] \in B(k)$, the fiber $\phi^{-1}(b) = W \cap Z = W \cap \mathbb{P}(V^m)/G$ is a complete intersection in $\mathbb{P}(V^m)/G$. The fiber $\mathcal{X}_{[W']} = f^{-1}(W')$ is cut out by the same homogeneous polynomials, now viewed as elements of $k[V^m]$ instead of $k[V^m]^G$. Thus, $\mathcal{X}_b$ is a smooth complete intersection in $\mathbb{P}(V^m)$. Since $\dim \mathcal{X}_b = e \geqslant 1$, by [Serre 1955, n. 78], $\mathcal{X}_b$ is connected, hence irreducible. This completes the proof of (c), and thus of condition (ii) of Theorem 1.4.

As we mentioned above, condition (iii) may not hold for the family $f\colon \mathcal{X} \to B$ we have constructed. Our construction only ensures that X_0 is G-equivariantly birationally isomorphic to a union of irreducible components of the fiber $\mathcal{X}_{b_0}$, and condition (iii) requires X_0 to be birationally isomorphic to the entire fiber $\mathcal{X}_{b_0}$. To bridge the gap between the two, we will slightly modify $\mathcal{X}$ as follows. (Note that B will remain unchanged.) Let Y_0 denote the union of all other components of $\mathcal{X}_{b_0}$, the ones that do not pass through any of the points $x_1, \ldots, x_r$. Then the open embedding $\mathcal{X} \setminus Y_0 \hookrightarrow \mathcal{X}$ is flat. After replacing $\mathcal{X}$ by $\mathcal{X} \setminus Y_0$ and $f\colon \mathcal{X} \to B$ by its restriction to $\mathcal{X} \setminus Y_0$, we obtain a family satisfying (i), (ii) and (iii). This completes the proof of Theorem 1.4. □

Remark 8.3. The family $f\colon \mathcal{X} \to B$ constructed in the proof of Theorem 1.4 has the additional property that the fibers of f over the dense open subset $U \subset B$ (the open subset in part (ii) of the statement of the theorem) are complete intersections in the projective space $\mathbb{P}^n$. With a bit of extra effort one can ensure that every fiber of f over U is of general type. Since we will not need this assertion in this paper, we leave the proof as an exercise for the interested reader.

We conclude this section with the following consequence of Theorem 1.4, where the group G is not necessarily finite.

Corollary 8.4. *Let k be an infinite field, G be a linear algebraic group, $e \geqslant 1$ be an integer and X_0 be an equidimensional generically free G-variety of dimension $e + \dim(G)$ (not necessarily primitive). Suppose that there exist a finite subgroup $S \subset G(k)$, a generically free S-variety Y_0 and a G-equivariant birational isomorphism $Y_0 \times^S G \sim X_0$.*

Then there exist a smooth irreducible k-variety B, a smooth G-variety $\mathcal{X}$ and a smooth G-equivariant morphism $f\colon \mathcal{X} \to B$ of constant relative dimension $e + \dim(G)$ defined over k such that

(a) *G acts trivially on B and freely on $\mathcal{X}$,*

(b) *there exists a dense open subscheme $U \subset B$ such that for every $b \in U$ the fiber $\mathcal{X}_b$ is a primitive G-variety and the total space of a G-torsor $\mathcal{X}_b \to \mathcal{X}_b/G$, where $\mathcal{X}_b/G$ is smooth projective,*

(c) *there exists a $b_0 \in B(k)$ such that the fiber $\mathcal{X}_{b_0}$ of f over b_0 is G-equivariantly birationally isomorphic to X_0.*

In particular, for any geometric point b of U, the G-action on the fiber $\mathcal{X}_b$ is strongly unramified.

Proof. Note that Y_0 is equidimensional of dimension

$$\dim(Y_0) = \dim(X_0) - \dim(G) = e.$$

Let $h\colon \mathcal{Y} \to B$ be a family obtained by applying Theorem 1.4 to the group S and the S-variety Y_0. This is possible because S is a finite group, Y_0 is equidimensional and generically free, and $e \geqslant 1$.

Set $\mathcal{X} := \mathcal{Y} \times^S G$. A priori, $\mathcal{X}$ is an algebraic space. However, since the k-variety $\mathcal{Y}$ is quasiprojective and S is finite, by [SGA 1 1971, Exposé V, Proposition 1.8 and Proposition 3.1], the quotient $\mathcal{X}/G \simeq \mathcal{Y}/S$ exists as a scheme. Since G is affine, the morphism of algebraic spaces $\mathcal{X} \to \mathcal{X}/G$ is affine, hence representable (see [Stacks 2005–, Tag 03WG]), and so $\mathcal{X}$ is a scheme. Let $f\colon \mathcal{X} \to B$ be the natural projection induced by h. Since h satisfies properties (i), (ii) and (iii) of Theorem 1.4, f satisfies properties (i), (ii) and (iii) of Corollary 8.4. □

9. Proof of Theorem 1.1

Let $d := \dim(X)$ and $e := \mathrm{ed}_k(X)$. We must show that there exists an irreducible smooth projective variety Z and a G-torsor $Y \to Z$ such that $\dim(Y) = \dim(X)$ and $\mathrm{ed}_k(Y) = \mathrm{ed}_k(X)$. If $e = 0$, Theorem 1.1 is obvious: we can take $Y = G \times \mathbb{P}^d$, where G acts by translations on the first factor and trivially on the second. Thus, we may assume without loss of generality that $e \geqslant 1$ and in particular, $d \geqslant 1$. In this case, we use the following strategy: construct a family $\mathcal{X} \to B$ as in Theorem 1.4 with $\mathcal{X}_{b_0} = X$, then take $Y = \mathcal{X}_b$, where b is a k-point of B in very general position. Theorem 1.4 tells us that $\dim(Y) = d$ and the G-action on Y is strongly unramified. We would like to appeal to Theorem 1.2 to conclude that $\mathrm{ed}_k(Y) = e$. One difficulty in implementing this strategy is that

(i) Theorem 1.4 requires G to be a finite group, whereas in Theorem 1.1, G is an arbitrary linear algebraic group over a field of good characteristic.

Even if we assume that G is a finite group, there is another problem:

(ii) Theorem 1.2 requires all fibers of $f\colon \mathcal{X} \to B$ to be primitive G-varieties, whereas if f is as in Theorem 1.4, we only have control over fibers $\mathcal{X}_b$ when $b \in U$.

We will overcome (i) by using Corollary 8.4 in place of Theorem 1.4 and (ii) by using Theorem 9.1 below in place of Theorem 1.2.

Theorem 9.1. *Let G be a linear algebraic group over an algebraically closed field k of good characteristic (see Definition 2.1), $f\colon \mathcal{X} \to B$ be a G-equivariant morphism of k-varieties such that B is irreducible, G act trivially on B and the generic fiber of f is a primitive and generically free $G_{k(B)}$-variety. Let $b_0 \in B(k)$, $x_0 \in \mathcal{X}_{b_0}(k)$ and X_0 be a G-invariant reduced open subscheme of $\mathcal{X}_{b_0}$ containing x_0 such that*

(1) *X_0 is a generically free primitive G-variety and*

(2) *f is flat at x_0.*

Then for a very general $b \in B(k)$, $\mathcal{X}_b$ is generically free and primitive, and $\mathrm{ed}_k(\mathcal{X}_b) \geqslant \mathrm{ed}_k(\mathcal{X}_{b_0})$. *Furthermore, if k is of infinite transcendence degree over its prime field (in particular, if k is uncountable), then the set of those $b \in B(k)$ such that* $\mathrm{ed}_k(\mathcal{X}_b) \geqslant \mathrm{ed}_k(X_0)$ *is Zariski dense in B.*

Here, as always, "very general" means "away from a countable union of proper subvarieties".

Proof. The proof is in several steps.

Claim 9.2. *There exists a dense open subscheme $V \subset B$ such that for all points $v \in V$ the fiber $\mathcal{X}_v$ is a generically free $G_{k(v)}$-variety.*

Proof. By [EGA IV$_3$ 1966, Proposition 9.6.1 (iii), Théorème 9.7.7 (iii)], the locus of points $b \in B$ such that the $k(b)$-scheme of finite type $\mathcal{X}_b$ is separated and geometrically irreducible, that is, a $G_{k(b)}$-variety, is a locally constructible subset of B. (By [EGA III$_1$ 1961, Chapitre 0, Proposition 9.1.12], a subset of B is locally constructible if and only if it is constructible.) Since $\mathcal{X}_{k(B)}$ is a $G_{k(B)}$-variety by assumption, we deduce the existence of a dense open subscheme $V_1 \subset B$ such that for all $v \in V_1$ the fiber $\mathcal{X}_v$ is a $G_{k(v)}$-variety.

Consider the stabilizer $\mathcal{X}$-group scheme

$$\mathcal{G} := \mathcal{X} \times_{(\mathcal{X} \times_B \mathcal{X})} (G \times_B \mathcal{X}),$$

where the fibered product is taken over the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_B \mathcal{X}$ and the action morphism $G \times_B \mathcal{X} \to \mathcal{X} \times_B \mathcal{X}$. Since the $G_{k(B)}$-variety $\mathcal{X}_{k(B)}$ is generically free, the geometric fibers of $\mathcal{G} \to \mathcal{X}$ at the generic points of the irreducible components of $\mathcal{X}_{k(B)}$ are trivial. By [EGA IV$_3$ 1966, Proposition 9.6.1 (xi)], the locus of points $x \in \mathcal{X}$ such that $\mathcal{G}_x \to \mathrm{Spec}(k(x))$ is an isomorphism is locally constructible. (Again by [EGA III$_1$ 1961, Chapitre 0, Proposition 9.1.12], a subset of $\mathcal{X}$ is locally constructible if and only if it is constructible.) Therefore, there exists an open subscheme $\mathcal{Y} \subset \mathcal{X}$ such that $\mathcal{Y}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense and such that for all $y \in \mathcal{Y}$ the scheme-theoretic stabilizer G_y is trivial. Since $\mathcal{Y}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense, by [Stacks 2005–, Tag 054X], there exists a dense open subscheme $V_2 \subset B$ such that $\mathcal{Y}_v \subset \mathcal{X}_v$ is dense for all $v \in V_2$. Letting $V := V_1 \cap V_2$, we conclude that $\mathcal{X}_v$ is a generically free $G_{k(v)}$-variety for all $v \in V$. □

By [Brion 2015, Theorem 1.1], there exists a finite subgroup $S \subset G(k)$ such that the projection $S \to G/G^0$ is surjective. By assumption, $G_{k(B)}$ acts primitively on $\mathcal{X}_{k(B)}$, hence so does S.

Claim 9.3. *There exists an S-invariant affine open subscheme $\mathcal{U} \subset \mathcal{X}$ such that $\mathcal{U}_{k(B)}$ is dense in $\mathcal{X}_{k(B)}$ and the quotient map $\mathcal{U} \to \mathcal{U}/S$ is an étale S-torsor.*

Proof. By [Stacks 2005–, Tag 03J1], there exists a separated dense open subscheme $\mathcal{X}_1 \subset \mathcal{X}$. The intersection $\mathcal{X}_2$ of all the S-translates of $\mathcal{X}_1$ is an S-invariant dense open subscheme of $\mathcal{X}$, and it is separated by [Stacks 2005–, Tag 01L8]. Let $\mathcal{X}_3 \subset \mathcal{X}_2$ be the S-free locus, which is a separated open subscheme of $\mathcal{X}$. By [EGA I 1971, Chapitre 0, (2.1.8)], the generic points of the irreducible components of $\mathcal{X}_{k(B)}$ are also generic points of irreducible components of $\mathcal{X}$. Recall that S acts generically freely on $\mathcal{X}_{k(B)}$. Therefore, while $\mathcal{X}_3$ is not necessarily dense in $\mathcal{X}$, $(\mathcal{X}_3)_{k(B)}$ is dense in $\mathcal{X}_{k(B)}$.

By [Stacks 2005–, Tag 01ZV], we can find an affine dense open subscheme $\mathcal{X}_4 \subset \mathcal{X}_3$. Since $\mathcal{X}_3$ is separated, the intersection of any two affine open subschemes of $\mathcal{X}_3$ is affine. Therefore, the intersection $\mathcal{U}$

of all the S-translates of $\mathcal{X}_4$ is an S-invariant affine open subscheme of $\mathcal{X}$ on which S acts freely, and the open embedding $\mathcal{U}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense. In particular, the quotient scheme $\mathcal{U}/S$ exists. Since S acts freely on $\mathcal{U}$, the quotient map $\mathcal{U} \to \mathcal{U}/S$ is étale, hence $\mathcal{U}$ is the total space of an étale S-torsor. □

Claim 9.4. *There exists a dense open subscheme $V \subset B$ such that for all points $v \in V$ the fiber $\mathcal{X}_v$ is a generically free and primitive $G_{k(v)}$-variety.*

Proof. Let $\mathcal{U} \subset \mathcal{X}$ be as in Claim 9.3. Since $\mathcal{U} \to \mathcal{U}/S$ is an S-torsor, the formation of the quotient of $\mathcal{U}$ by the S-action commutes with arbitrary base change $B' \to B$, and in particular, for every $b \in B$ the canonical morphism $\mathcal{U}_b/S \to (\mathcal{U}/S)_b$ is an isomorphism. Therefore, for every $b \in B$ the S-variety $\mathcal{U}_b$ is either empty or an S-torsor.

By our assumptions $\mathcal{U}_{k(B)}$ is dense open in $\mathcal{X}_{k(B)}$ and S acts transitively on the geometric irreducible components of $\mathcal{X}_{k(B)}$. Thus S acts transitively on the geometric irreducible components of $\mathcal{U}_{k(B)}$, that is, the $k(B)$-variety $\mathcal{U}_{k(B)}/S$ is geometrically irreducible. By [Stacks 2005–, Tag 0559], this implies the existence of a dense open subscheme $V \subset B$ such that the restriction $(\mathcal{U}/S)_V \to V$ has geometrically irreducible fibers, that is, such that for every $v \in V$ the S-variety $\mathcal{U}_v$ over $k(v)$ is nonempty and primitive. Since $\mathcal{U}_{k(B)} \subset \mathcal{X}_{k(B)}$ is dense, we deduce from [Stacks 2005–, Tag 054X] that, possibly after shrinking V, the open embedding $\mathcal{U}_v \subset \mathcal{X}_v$ is dense for all $v \in V$, and so $\mathcal{X}_v$ is primitive for all $v \in V$. By Claim 9.2, after shrinking V one more time, we may suppose that $\mathcal{X}_v$ is also a generically free $G_{k(v)}$-variety for all $v \in V$. □

We are now ready to complete the proof of Theorem 9.1. Let X_0' be the union of irreducible components of X_{b_0} which are not contained in X_0. By openness of the flat locus [Stacks 2005–, Tag 0398], possibly after replacing x_0 by another k-point of X_0, we may assume that $x_0 \notin X_0'$ and f remains flat at x_0.

Furthermore, after replacing $\mathcal{X}$ by $\mathcal{Y} = \mathcal{X} \setminus X_0'$, we may assume without loss of generality that $\mathcal{X}_{b_0} = X_0$. Indeed, the new projection $f' \colon \mathcal{Y} \to B$ is the composition of f (which is flat at x_0) with the open immersion $\mathcal{Y} \hookrightarrow \mathcal{X}$ (which is flat everywhere). Hence, f' is flat at x_0.

Let $V \subset B$ be as in Claim 9.4. By generic flatness [Stacks 2005–, Tag 0529], there exists a dense open subscheme $U \subset V$ such that f is flat over U. If $b_0 \in U(k)$, the conclusion follows from Theorem 1.2. We may, thus, assume that b_0 does not lie in U, and let $u \in U$ be the generic point of U. Since U is dense in B, b_0 is a specialization of u. By [EGA II 1961, Proposition 7.1.9], there exist a discrete valuation ring R and a separated morphism $\operatorname{Spec}(R) \to B$ mapping the closed point s of $\operatorname{Spec}(R)$ to b_0 and the generic point η of $\operatorname{Spec}(R)$ to u.

Since f is flat at x_0 and G acts primitively on $\mathcal{X}_{b_0}$, by the openness of the flat locus [Stacks 2005–, Tag 0399], the flat locus of the base change $f_R \colon \mathcal{X}_R \to \operatorname{Spec}(R)$ is dense in the component of the special fiber of f_R containing (the preimage of) x_0. Since we are assuming that the generic fiber is primitive, we conclude that the flat locus of f_R is dense in the generic fiber. Therefore, after removing the complement of the flat locus from $\mathcal{X}_R$, we may assume that f_R is flat. Let $\bar{\eta}$ and $\bar{s}$ be geometric points lying above η and s, respectively. By Proposition 3.1, we have

$$\mathrm{ed}_{k(\bar{\eta})}(\mathcal{X}_{\bar{\eta}}) \geqslant \mathrm{ed}_{k(\bar{s})}((X_0)_{\bar{s}}).$$

On the other hand, by Lemma 4.2,

$$\mathrm{ed}_{k(\bar{s})}((X_0)_{\bar{s}}) = \mathrm{ed}_k(X_0).$$

Therefore, $\mathrm{ed}_{k(\bar{\eta})}(X_{\bar{\eta}}) \geqslant \mathrm{ed}_k(X_0)$. Since $U \subset V$, the restriction of f to $f^{-1}(U)$ satisfies the assumptions of Theorem 1.2. The conclusion of Theorem 9.1 now follows from an application of Theorem 1.2 to the restriction of f to $f^{-1}(U)$. □

Conclusion to the proof of Theorem 1.1. As we mentioned at the beginning of this section, in the course of proving Theorem 1.1, we may assume that $e \geqslant 1$.

We will now reduce the theorem to the case, where X is incompressible, i.e., $d = e + \dim(G)$. Indeed, by the definition of essential dimension there exists a G-equivariant dominant rational map $X \dashrightarrow X'$ such that $\dim(X') = d' := e + \dim(G)$. Suppose we know that Theorem 1.1 holds for X'. In other words, there exists a G-variety Y' such that $\dim(Y') = d'$, $\mathrm{ed}(Y') = e$, and Y' is the total space of a G-torsor $t \colon Y' \to P'$ over a smooth projective variety P. Clearly $d \geqslant d'$. Let $Y = \mathbb{P}^{d-d'} \times Y'$, where G acts trivially on $\mathbb{P}^{d-d'}$. Then $\dim(Y) = d$ and Y is a G-torsor $\mathrm{id} \times t \colon Y = \mathbb{P}^{d-d'} \times Y' \to \mathbb{P}^{d-d'} \times P$ over the smooth projective variety $\mathbb{P}^{d-d'} \times P$. Moreover, by [Reichstein and Scavia 2022, Corollary 8.5], $\mathrm{ed}(Y) = \mathrm{ed}(Y') = e$, as desired (see also [Reichstein 2000, Lemma 3.3 (d)]).

From now on we will assume that $d = e + \dim(G)$ and $e \geqslant 1$. Denote by $[X]$ the class in $H^1(k(X)^G, G)$ associated to the generically free primitive G-variety X, as in Remark 4.6.

Now observe that there exists a finite subgroup S of G, such that for every field extension K/k, the natural map $H^1(K, S) \to H^1(K, G)$ is surjective. This follows from the definition of good characteristic if $\mathrm{char}(k) > 0$ and from [Chernousov et al. 2006, Theorem 1.1 (a)] if $\mathrm{char}(k) = 0$. (Recall that we are assuming that k is algebraically closed.) Suppose the S-torsor $T \to \mathrm{Spec}(K)$ represents a class in $H^1(K, S)$ in the preimage of $[X]$. Spreading out the S-torsor $T \to \mathrm{Spec}(K)$, we obtain a generically free primitive S-variety X' such that $X' \times^S G$ is G-equivariantly birationally equivalent to X. Therefore, X satisfies the assumptions of Corollary 8.4.

We let $f \colon \mathcal{X} \to B$ be a family obtained by applying Corollary 8.4 with $X_0 = X$. Then f is a smooth morphism of constant relative dimension $d = e + \dim(G)$, G acts freely on $\mathcal{X}$, $\mathcal{X}_b$ is a primitive G-variety for every $b \in U(k)$, the G-action on $\mathcal{X}_b$ is strongly unramified and X is G-equivariantly birationally isomorphic to $\mathcal{X}_{b_0}$. Clearly,

$$\mathrm{ed}(\mathcal{X}_b) \leqslant \dim(\mathcal{X}_b) - \dim(G) = \dim(X) - \dim(G) = d - \dim(G) = e.$$

On the other hand, since k is algebraically closed and of infinite transcendence degree over its prime field, by Theorem 9.1 there exists a k-rational point b of B such that $\mathrm{ed}_k(\mathcal{X}_b) \geqslant \mathrm{ed}_k(X) = e$. Setting $Y := \mathcal{X}_b$, we obtain a generically free primitive G-variety Y such that $\dim(Y) = d$, $\mathrm{ed}_k(Y) = e$ and the G-action on Y is strongly unramified, as desired. □

Corollary 9.5. *Let G be a linear algebraic group over an algebraically closed field k of good characteristic and of infinite transcendence degree over its prime field. Then there exists a strongly unramified generically free G-variety Y such that* $\dim(Y) = \mathrm{ed}_k(G) + \dim G$ *and* $\mathrm{ed}_k(Y) = \mathrm{ed}_k(G)$.

Proof. Let V be a generically free G-variety over k of essential dimension $e = \mathrm{ed}_k(G)$. Then there is a G-compression $V \dashrightarrow X$, where X is a generically free G-variety of dimension $e + \dim(G)$. Clearly, $\mathrm{ed}_k(X) \leqslant e$; on the other hand, $\mathrm{ed}_k(X, G) \leqslant \mathrm{ed}_k(V, G) = e = \dim(X)$. We conclude that $\dim(X) = e + \dim(G)$ and $\mathrm{ed}_k(X) = e$. By Theorem 1.1 there exists a strongly unramified generically free variety Y of dimension $e + \dim(G)$ and essential dimension e. □

10. Essential dimension at a prime

In this section, we extend some of the main results of this paper to essential dimension at a prime q and give an application in this setting.

We will now show that Proposition 3.1 and Theorems 1.1, 1.2 and 9.1 continue to hold if we replace essential dimension by essential dimension at a prime q. The proofs are largely unchanged; see below.

Note that while Proposition 3.1 and Theorems 1.1, 1.2 and 9.1 all require that the base field k should be of good characteristic, their q-analogues, Proposition 10.1 and Theorems 10.2, 10.3 and 10.5, do not need this assumption. These results hold whenever $\mathrm{char}(k) \neq q$.

Proposition 10.1. *Let k be an algebraically closed field, R be a discrete valuation ring containing k and with residue field k, and l be the fraction field of R. Let G be a linear algebraic group over k and q be a prime number invertible in k. Let X be a flat R-scheme of finite type endowed with a G-action over R, whose fibers are generically free and primitive G-varieties. Then* $\mathrm{ed}_{\bar{l},q}(X_{\bar{l}}) \geqslant \mathrm{ed}_{k,q}(X_k)$.

Proof. The proof is the same as that of Proposition 3.1, except that instead of the [Reichstein and Scavia 2022, Theorem 1.2], one should use [Reichstein and Scavia 2022, Theorem 11.1] which gives an analogous assertion for essential dimension at q. □

Theorem 10.2. *Let G be a linear algebraic group over an algebraically closed field k, and let q be a prime number invertible in k. Let B be a noetherian k-scheme, $f : \mathcal{X} \to B$ be a flat G-equivariant morphism of finite type such that G acts trivially on B and the geometric fibers of f are generically free and primitive G-varieties (in particular, reduced). Then for any fixed integer $n \geqslant 0$, the subset of $b \in B$ such that* $\mathrm{ed}_{k(\bar{b}),q}(\mathcal{X}_{\bar{b}}; G_{k(\bar{b})}) \leqslant n$ *for every (equivalently, some) geometric point $\bar{b}$ above b is a countable union of closed subsets of B.*

Proof. Analogous to that of Theorem 1.2, replacing Proposition 3.1 by Proposition 10.1. □

Theorem 10.3. *Let G be a linear algebraic group over an algebraically closed field k of infinite transcendence degree over its prime field, q be a prime number invertible in k, and X be a generically free primitive G-variety. Then there exists an irreducible smooth projective variety Z and a G-torsor $Y \to Z$ such that* $\dim(Y) = \dim(X)$ *and* $\mathrm{ed}_{k,q}(Y) = \mathrm{ed}_{k,q}(X)$.

Proof. Analogous to that of Theorem 1.1, replacing Theorem 1.2 by Theorem 10.2. □

The next lemma is the analogue of Lemma 4.2 for essential dimension at a prime.

Lemma 10.4. *Let k be an algebraically closed field, q be a prime number, G be a k-group and X be a generically free primitive G-variety defined over k. Then* $\mathrm{ed}_{k,q}(X) = \mathrm{ed}_{l,q}(X_l)$ *for any field extension l/k.*

Proof. The proof is analogous to that of [Reichstein and Scavia 2022, Lemma 2.2], replacing G-compressions by G-equivariant (rational) correspondences of degree prime to q. We sketch it for the sake of completeness.

By a base change, every G-equivariant correspondence $X \rightsquigarrow Y$ gives rise to a G_l-equivariant correspondence $X_l \rightsquigarrow Y_l$ of the same degree. Moreover, if the G-variety Y is generically free and primitive, so is the G_l-variety Y_l. It follows that $\mathrm{ed}_{k,q}(X) \geqslant \mathrm{ed}_{l,q}(X_l)$.

Conversely, let $f\colon X_l \rightsquigarrow Y$ be a G_l-equivariant correspondence

$$\begin{tikzcd} Z \arrow[d, dashed] \arrow[dr, dashed] & \\ X_l & Y \end{tikzcd}$$

of degree prime to q defined over l, where the G_l-variety Y is generically free and primitive of smallest dimension $\dim(G) + \mathrm{ed}_{l,q}(X_l)$. Since Z, Y and f are defined over a finitely generated subextension of l/k, we may suppose that l is finitely generated over k. Then there exist a k-variety U with function field $k(U) \simeq l$, G-invariant morphisms $\mathcal{Y} \to U$ and $\mathcal{Z} \to U$ whose generic fibers are Y and Z, respectively, and G-equivariant dominant rational maps

$$\begin{tikzcd} \mathcal{Z} \arrow[d, dashed] \arrow[dr, dashed] & \\ X \times_k U & \mathcal{Y} \end{tikzcd}$$

over U, whose base change along the generic point $\mathrm{Spec}(l) \to U$ is f. Since k is algebraically closed, $U(k)$ is dense in U. If $u \in U(k)$ is a k-point in general position in U, the G-variety $\mathcal{Y}_u$ is generically free and primitive, $\dim(\mathcal{Y}_u) = \dim(Y)$ and base change along $\mathrm{Spec}(k(u)) \to U$ yields a G-equivariant correspondence $X \rightsquigarrow \mathcal{Y}_u$ of the same degree as f. We conclude that

$$\mathrm{ed}_{k,q}(X) \leqslant \dim(\mathcal{Y}_u) - \dim(G) = \dim(Y) - \dim G = \mathrm{ed}_{l,q}(X_l). \qquad \square$$

Theorem 10.5. *Let G be a linear algebraic group over an algebraically closed field and q be a prime number invertible in k. Let $f\colon \mathcal{X} \to B$ be a G-equivariant morphism of k-varieties such that B is irreducible, G acts trivially on B and the generic fiber of f is a primitive and generically free $G_{k(B)}$-variety. Let $b_0 \in B(k)$, $x_0 \in \mathcal{X}_{b_0}(k)$, and X_0 be a G-invariant reduced open subscheme of $\mathcal{X}_{b_0}$ containing x_0 such that*

(1) *X_0 is a generically free primitive G-variety and*

(2) *f is flat at x_0.*

Then, for a very general $b \in B(k)$, $\mathcal{X}_b$ is generically free and primitive, and $\mathrm{ed}_{k,q}(\mathcal{X}_b) \geqslant \mathrm{ed}_{k,q}(X_0)$. Furthermore, if k is of infinite transcendence degree over its prime field (in particular, if k is uncountable), then the set of those $b \in B(k)$ such that $\mathrm{ed}_{k,q}(\mathcal{X}_b) \geqslant \mathrm{ed}_{k,q}(X_0)$ is Zariski dense in B.

Proof. Analogous to that of Theorem 9.1, replacing Proposition 3.1, Lemma 4.2 and Theorem 1.2 by Proposition 10.1, Lemma 10.4 and Theorem 10.2, respectively. $\square$

We will now give an application of Theorem 10.5. Recall from Section 2 that

$$\mathrm{ed}_k(G) = \max\ \mathrm{ed}_k(X; G) = \mathrm{ed}_{k,q}(V; G)$$

and

$$\mathrm{ed}_{k,q}(G) = \max_q\ \mathrm{ed}_{k,q}(X; G) = \mathrm{ed}_{k,q}(W; G),$$

where X ranges over all generically free primitive G-varieties and V (respectively, W) is a versal (respectively, q-versal) generically free primitive G-variety. By definition,

$$\mathrm{ed}_k(G) \geqslant \max_q\ \mathrm{ed}_{k,q}(G), \tag{10.6}$$

where the maximum is taken over all primes q. One can think of $\mathrm{ed}_{k,q}(G)$ as a local version of $\mathrm{ed}_k(G)$. There is some tension between the two. On the one hand, the problem of computing $\mathrm{ed}_k(G)$ is more natural and, in some cases (e.g., for $G = \mathrm{S}_n$ or PGL_n) is directly motivated by classical problems. On the other hand, $\mathrm{ed}_{k,q}(G)$ is usually more accessible: virtually all known techniques for bounding $\mathrm{ed}_k(G)$ from below actually yield a lower bound on $\mathrm{ed}_{k,q}(G)$ for some prime q. This means that when the obvious inequality (10.6) is not sharp, the exact value of $\mathrm{ed}_k(G)$ is difficult to establish.

In an attempt to probe the "gray area" between $\mathrm{ed}_k(G)$ and $\max_q\ \mathrm{ed}_{k,q}(G)$, Duncan and Reichstein [2014] defined poor man's essential dimension, $\mathrm{pmed}_k(G)$, as the maximal value of $\mathrm{ed}_k(V; G)$, where V is a generically free primitive G-variety, which is q-versal for every prime q (but not necessarily versal). Clearly,

$$\mathrm{ed}_{k,q}(G) \leqslant \mathrm{pmed}_k(G) \leqslant \mathrm{ed}_k(G)$$

for every prime q. It is shown in [Duncan and Reichstein 2014, Theorem 1.4, Proposition 11.1] that

$$\mathrm{pmed}_k(G) = \max_q\ \mathrm{ed}_{k,q}(G) \tag{10.7}$$

for many finite groups G. Here the maximum is taken over all primes q, and $|G|$ is assumed to be invertible in k. Conjecturally, (10.7) holds for every finite group G; see [Duncan and Reichstein 2014, Conjecture 11.5] (again, under the assumption that $|G|$ is invertible in k). We will now use Theorem 10.5 to prove the following result in a similar spirit:

Theorem 10.8. *Let G be a linear algebraic group defined over an uncountable algebraically closed field k. Suppose that at least one of the following conditions is satisfied*:

(i) $\mathrm{char}\, k = 0$,

(ii) $\mathrm{char}\, k = p > 0$, G^0 *is reductive*, *and there exists a finite subgroup $S \subset G(k)$ of order prime to p such that for every $q \neq p$ and every q-closed field L containing k the natural map $H^1(L, S) \to H^1(L, G)$ is surjective.*

Then there exists a strongly unramified generically free primitive G-variety X of dimension

$$\max_{q \neq \mathrm{char}(k)}\ \mathrm{ed}_{k,q}(G) + \dim(G)$$

such that $\mathrm{ed}_{k,q}(X; G) = \mathrm{ed}_{k,q}(G)$ *for every prime q that is invertible in k.*

To explain the relationship between Theorem 10.8 and the conjectural identity (10.7), assume for a moment that the G-variety X of Theorem 10.8 has been constructed. Then, on the one hand,

$$\mathrm{ed}_k(X; G) \leqslant \dim(X) - \dim(G) = \max_{q \neq \mathrm{char}(k)} \mathrm{ed}_{k,q}(G)$$

and on the other hand,

$$\mathrm{ed}_k(X; G) \geqslant \mathrm{ed}_{k,q}(X; G) = \mathrm{ed}_{k,q}(G)$$

for every prime q invertible in k. Thus,

$$\mathrm{ed}_k(X; G) = \dim(X) - \dim(G) = \max_{q \neq \mathrm{char}(k)} \mathrm{ed}_{k,q}(G). \tag{10.9}$$

If we knew that X is q-versal for every prime q, then this equality would imply (10.7). Unfortunately, since X is strongly unramified, it will usually not be q-versal for every prime q different from $\mathrm{char}(k)$, and the conjectural equality (10.7) remains open. On the other hand, one can think of the equality $\mathrm{ed}_{k,q}(X; G) = \mathrm{ed}_{k,q}(G)$ as a weaker substitute for q-versality. In this sense, one may view (10.9) (and thus, Theorem 10.8) as a weaker substitute for (10.7), valid for a wider class of algebraic groups G (not necessarily finite).

Our proof of Theorem 10.8 relies on the following well-known lemma. We include a proof for the sake of completeness.

Lemma 10.10. *Let G be a linear algebraic group defined over a field k. Then $\mathrm{ed}_{k,q}(G) = 0$ for all but at most finitely many primes q.*

Proof. By [Merkurjev 2009, Proposition 4.4], $\mathrm{ed}_{k,q}(G) > 0$ if and only if q is a torsion prime for G. It remains to show that every linear algebraic group G has only finitely many torsion primes.

Recall that q is a torsion prime for G if and only if q divides n_T for some G-torsor $T \to \mathrm{Spec}(K)$, where K is a field containing k. Here, n_T denotes the index of $T \to \mathrm{Spec}(K)$, i.e., the greatest common divisor of $[L : K]$, as L ranges over finite field extensions of K such that T splits over $\mathrm{Spec}(L)$. For any G-torsor $T \to \mathrm{Spec}(K)$ as above, the index n_T divides the so-called torsion index $t(G)$ of G; see [Grothendieck 1958, Theorem 2]. Recall that the torsion index $t(G)$ is $\mathrm{ind}(T_{\mathrm{vers}})$, where $T_{\mathrm{vers}} \to \mathrm{Spec}(K_{\mathrm{vers}})$ is a versal G-torsor. For an alternative definition and a discussion of the properties of the torsion index, see [Totaro 2005].

In summary, $\mathrm{ed}_{k,q}(G) > 0$ only if q is a prime factor of $t(G)$, and the lemma follows. □

Proof of Theorem 10.8. By Lemma 10.10, there are only finitely many primes q such that $\mathrm{ed}_{k,q}(G) > 0$. Denote them by $q_1, \dots, q_n$. If $n = 0$, i.e., $\mathrm{ed}_{k,q}(G) = 0$ for every prime $q \neq \mathrm{char}(k)$, we can take $X = G$, viewed as a G-variety with respect to the translation action of G on itself. Then

$$\max_{q \neq \mathrm{char}(k)} \mathrm{ed}_{k,q}(G) = 0 = \dim(X) - \dim(G) \quad \text{and} \quad \mathrm{ed}_{k,q}(X; G) = 0 = \mathrm{ed}_{k,q}(G)$$

for every prime $q \neq \mathrm{char}(k)$, as required.

We may thus assume without loss of generality that $n \geqslant 1$. Let $d_i = \mathrm{ed}_{k,q_i}(G) + \dim(G)$, and let $d = \max(d_1, \dots, d_n)$. For each $i = 1, \dots, n$, let X_i be a q_i-versal G-variety of minimal possible dimension d_i. Let X_0 be the disjoint union of d-dimensional G-varieties $X_i \times \mathbb{P}^{d-d_i}$, where G acts trivially on $\mathbb{P}^{d-d_i}$. The variety X_0 is equidimensional and the G-action on it is generically free. Note, however, that this action is not primitive unless $n = 1$.

There exists a subgroup $S \subset G(k)$ of order invertible in k such that for every $q \neq p$ and every q-closed field L containing k the natural map $H^1(L, S) \to H^1(L, G)$ is surjective. Indeed, if $\operatorname{char} k = p > 0$, this follows from assumption (ii) of Theorem 10.8. If $\operatorname{char}(k) = 0$, then S exist by [Chernousov et al. 2006, Theorem 1.1 (a)]. In other words, for every $i = 1, \dots, n$ there exists a G-equivariant dominant rational correspondence $X_i' \dashrightarrow X_i$ of degree prime to q and a generically free primitive S-variety Y_i such that $Y_i \times^S G$ is G-equivariantly birationally equivalent to X_i'. Now, let the S-variety Y_0 be the disjoint union of the $Y_i \times \mathbb{P}^{d-d_i}$, where S acts trivially on $\mathbb{P}^{d-d_i}$ for each i. By our construction, $Y_0 \times^S G$ and X_0 are G-equivariantly birationally equivalent. Since $X_i' \dashrightarrow X_i$ has degree prime to q, we have $\operatorname{ed}_{k,q}(X_i'; G) = \operatorname{ed}_{k,q}(X_i; G)$ for all $i = 1, \dots, n$. Therefore, replacing X_i by X_i', we may assume that X_0 admits reduction of structure to S, so that the assumptions of Corollary 8.4 apply to X_0.

Let $f \colon \mathcal{X} \to B$ be a morphism constructed in Corollary 8.4, with respect to the G-variety X_0 we have just defined. We now want to take $X = \mathcal{X}_b$ to be the fiber of X over a very general point $b \in B(k)$. By Corollary 8.4, X is strongly unramified (again, for a very general $b \in B(k)$). By our construction,

$$\dim(\mathcal{X}_b) = d = \max\bigl(\operatorname{ed}_{k,q_1}(G), \dots, \operatorname{ed}_{k,q_i}(G)\bigr) + \dim(G) = \max_{q \neq \operatorname{char}(k)} \operatorname{ed}_{k,q}(G) + \dim(G),$$

as desired. Moreover, by Theorem 10.5,

$$\operatorname{ed}_{k,q_i}(\mathcal{X}_b) \geqslant \operatorname{ed}_{k,q_i}(X_i \times \mathbb{P}^{d-d_i}) = \operatorname{ed}_{k,q_i}(X_i) = \operatorname{ed}_{k,q_i}(G).$$

Here, the equality $\operatorname{ed}_{k,q_i}(X_i \times \mathbb{P}^{d-d_i}) = \operatorname{ed}_{k,q_i}(X_i)$ follows from [Reichstein and Scavia 2022, Corollary 10.3]. (It can also be deduced directly from the definition of essential dimension at q_i.) The opposite inequality, $\operatorname{ed}_{k,q_i}(\mathcal{X}_b) \leqslant \operatorname{ed}_{k,q_i}(G)$, follows from the definition of $\operatorname{ed}_{k,q_i}(G)$. Thus,

$$\operatorname{ed}_{k,q_i}(\mathcal{X}_b) = \operatorname{ed}_{k,q_i}(G), \qquad \text{for each } i = 1, \dots, n.$$

In summary, a very general fiber $X = \mathcal{X}_b$ of $f \colon \mathcal{X} \to B$ has the properties claimed in Theorem 10.8. □

11. Example: Finite group actions on hypersurfaces

In this section, we give yet another application of Theorems 1.2 and 10.2.

Proposition 11.1. *Let k be an algebraically closed field of infinite transcendence degree over its prime field, W be a finite-dimensional k-vector space, G be a finite group and $G \hookrightarrow \mathrm{GL}(W)$ be a faithful linear representation over k. Set $V := W \oplus k$, where G acts trivially on k. Let $f(x_1, \dots, x_n) \in k[W]$ be a G-invariant homogeneous polynomial of degree d and $f_0(x_{n+1})$ be an (inhomogeneous) polynomial of degree d with d distinct roots in k. Assume that the G-action on the affine hypersurface $Z(f) \subset \mathbb{A}(V) \simeq \mathbb{A}^{n+1}$ is primitive. Then*

(a) *the G-action on the hypersurface $Z(f + f_0)$ is generically free and primitive,*

(b) $\operatorname{ed}_k(Z(f + f_0)) = \operatorname{ed}_k(G)$,

(c) $\operatorname{ed}_{k,q}(Z(f + f_0)) = \operatorname{ed}_{k,q}(G)$ *for every prime integer q.*

Proof. For $(0, 0) \neq (s, t) \in \mathbb{A}^2$, set

$$f_{(s,t)} = sf(x_1, \ldots, x_n) + tf_0(x_{n+1})$$

and $Z_{[s:t]} = V(f_{(s,t)})$. Note that $Z_{[s:t]}$ depends only on the projective point $[s : t] \in \mathbb{P}^1$. The 2-dimensional torus $\mathbb{G}_{\mathrm{m}}^2$ acts on $V = W \oplus k$ by scalar multiplication in W and k. This action commutes with the action of G on V. The hypersurfaces $Z_{[s:t]}$ with $s, t \neq 0$ all lie in the same orbit of this $\mathbb{G}_{\mathrm{m}}^2$-action; hence, they are isomorphic as G-varieties.

We now proceed with the proof of part (a). We claim that $Z_{[s:t]}$ is generically free and primitive whenever $s, t \neq 0$. As we saw above, the G-variety $Z_{[s:t]}$ is isomorphic to $Z_{[1,1]}$ whenever $s, t \neq 0$. Hence, the claim is equivalent to part (a). It will, however, be convenient for us to let $[s : t]$ vary over $\mathbb{P}^1$, rather than focusing solely on $Z_{[1:1]}$.

To show that $Z_{[s,t]}$ is primitive, assume the contrary. Then $f_{(s,t)}$ can be written as a product of two polynomials, α and $\beta \in k[x_0, \ldots, x_{n+1}]$ of degree $\leqslant d - 1$ such that G preserves α and β up to a scalar. Setting $x_{n+1} = 0$, we see that $f(x_1, \ldots, x_n) = \alpha(x_1, \ldots, x_n) \cdot \beta(x_1, \ldots, x_n)$, which contradicts our assumption that $Z(f) = Z_{[1:0]}$ is primitive.

To prove that the G-action on $Z_{[s:t]}$ is generically free, assume the contrary. Then $Z_{[s:t]}$ is contained in $V_{\mathrm{nonfree}} = W_{\mathrm{nonfree}} \times k$, where V_{nonfree} the union of the fixed point loci V^g as g ranges over the nontrivial elements of G, as in Section 8, and similarly for W_{nonfree}. Consider the family $\mathcal{X} \to B = \mathbb{P}^1$ of G-varieties, where $\mathcal{X} \subset \mathbb{A}(V) \times \mathbb{P}^1$ consists of pairs $(v, [s : t])$ such that $v \in Z_{[s:t]}$. In other words, the fiber over $[s : t]$ is $Z_{[s:t]}$. The set of $[s : t]$ such that the entire fiber $\mathcal{X}_{[s:t]} = Z_{[s:t]}$ lies in V_{nonfree} is closed in $\mathbb{P}^1$. On the other hand, note that $Z_{[0:1]} = Z(f_0)$ is a disjoint union of d copies of W. Since the G-action on W is faithful, hence generically free, we see that $Z_{[0:1]} = Z(f_0)$ is not contained in V_{nonfree}. Thus, the G-action on $Z_{[s:t]}$ can be nonfree for only finitely many $[s : t] \in \mathbb{P}^1$. Since $Z_{[s:t]} \simeq Z_{[1,1]}$ whenever $s, t \neq 0$, we conclude that the G-action on $Z_{[s:t]}$ is generically free whenever $s, t \neq 0$. This completes the proof of the claim and, hence, of part (a).

To prove part (b), let us examine the family $\mathcal{X} \to B = \mathbb{P}^1$ more closely. This family is obtained by a pullback from the universal family of affine hypersurfaces of degree $\leqslant d$ in V. In particular, it is flat. As we pointed out above, the fiber $\mathcal{X}_{[0:1]} = Z_{[0:1]}$ is a disjoint union of d copies of W (as a G-variety). Every copy of W has essential dimension equal to $\mathrm{ed}_k(G)$; see (2.4). Let $\mathcal{X}' \subset \mathcal{X}$ be the Zariski open subvariety obtained by removing all irreducible components of $\mathcal{X}_{[0:1]}$ but one. Since the inclusion map $\mathcal{X}' \hookrightarrow \mathcal{X}$ is an open embedding, and open embeddings are flat, we conclude that the composition $\mathcal{X}' \hookrightarrow \mathcal{X} \to B$ is also flat. Theorem 1.2 now tells us that for a very general $[s : t]$ in $\mathbb{P}^1$, we have

$$\mathrm{ed}_k(Z_{[s:t]}) = \mathrm{ed}_k(\mathcal{X}_{[s:t]}) = \mathrm{ed}_k(\mathcal{X}'_{[s:t]}) \geqslant \mathrm{ed}_k(\mathcal{X}'_{[0:1]}) = \mathrm{ed}_k(G).$$

In particular, $\mathrm{ed}_k(Z_{[s:t]}) \geqslant \mathrm{ed}_k(G)$ for some $s, t \neq 0$. The opposite inequality, $\mathrm{ed}_k(Z_{[s:t]}) \leqslant \mathrm{ed}_k(G)$, is immediate from the definition of $\mathrm{ed}_k(G)$. Thus, $\mathrm{ed}_k(Z_{[s:t]}) = \mathrm{ed}_k(G)$ for some $s, t \neq 0$. As we saw above, $Z_{[s,t]} \simeq Z_{[1:1]}$ for any $s, t \neq 0$. We conclude that $\mathrm{ed}_k(Z_{[1:1]}) = \mathrm{ed}_k(G)$. This proves part (b).

The proof of part (c) is the same, except that we use Theorem 10.2 in place of Theorem 1.2. □

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Differential operators, retracts, and toric face rings

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We give explicit descriptions of rings of differential operators of toric face rings in characteristic 0. For quotients of normal affine semigroup rings by radical monomial ideals, we also identify which of their differential operators are induced by differential operators on the ambient ring. Lastly, we provide a criterion for the Gorenstein property of a normal affine semigroup ring in terms of its differential operators.

Our main technique is to realize the $\Bbbk$-algebras we study in terms of a suitable family of their algebra retracts in a way that is compatible with the characterization of differential operators. This strategy allows us to describe differential operators of any $\Bbbk$-algebra realized by retracts in terms of the differential operators on these retracts, without restriction on char($\Bbbk$).

Introduction

Differential operators play a notable role in branches of mathematics as seemingly disparate as partial differential equations and local cohomology, dynamical systems and invariant theory. Recently, there have been many exciting new developments concerning differential operators and their applications in commutative algebra, including to topics such as Bernstein–Sato polynomials, connections between singularities and local cohomology, equivariant D-modules, and Hodge ideals; see, for example, [1; 2; 9; 16; 20; 21; 23; 24; 25; 31; 36; 37; 38; 39; 40; 41]. One obstruction to the even greater use of rings of differential operators is the notorious difficulty of computing them explicitly. In fact, there are very few classes of rings whose differential operators are systematically computed, namely polynomial rings, Stanley–Reisner rings, affine semigroup rings, and coordinate rings of curves; see, for example, [11; 12;

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22; 26; 27; 28; 29; 30; 32; 42; 43; 44; 47; 48]. The goal of this article is to give a new class of explicitly computed rings of differential operators.

A *retract* of a $\mathbb{k}$-algebra R is a subring of R which is isomorphic to a quotient of R. We call R a $\mathbb{k}$-*algebra realized by retracts* if it can be embedded into a finite direct sum of domains, each of which is a retract of R. Algebra retracts are of considerable interest; it is common in the literature to prove desirable properties of retracts using information from the ambient ring; see, for example, [3; 6; 7; 10; 14; 15; 46]. In this work, we do the opposite; we use knowledge about the differential operators on algebra retracts to compute differential operators of an ambient ring (see Theorems 2.6 and 2.8). Naturally, this approach is most productive when the differential operators on the retracts are known. This is the case for a combinatorially interesting class of $\mathbb{k}$-algebras realized by retracts known as *toric face rings*.

Toric face rings were first introduced by Stanley [45], and further developed in [3; 17; 18; 33; 34; 35; 49], among others. They include both Stanley–Reisner rings and affine semigroup rings as special cases, thus bringing under a single umbrella two of the mainstays of combinatorial commutative algebra. In this setting, the retracts we are interested in are affine semigroup rings, whose rings of differential operators are known in characteristic zero; here we use the presentation given by Saito and Traves in [42; 43]. Thus we can directly apply Theorem 2.8 to compute the ring of differential operators of a toric face ring in Proposition 3.3. As a consequence, we recover results on differential operators on Stanley–Reisner rings given by Tripp, Eriksson, and Traves in [12; 47; 48] over arbitrary fields.

Theorem 2.6 finds the differential operators of a $\mathbb{k}$-algebra realized by retracts as a subring of the direct sum of the rings of differential operators of the retracts. This direct sum of the retracts is in general much larger than the original ring. On the other hand, the differential operators on Stanley–Reisner rings and affine semigroup rings are induced from differential operators over their natural ambient rings. In general, however, the richness of the direct sum is really needed. To illustrate this, we provide a description of which differential operators on a quotient of a normal affine semigroup ring by a radical monomial ideal are induced from the differential operators on the ambient semigroup ring (see Theorem 4.3) and show that this does not necessarily give the whole ring of differential operators.

Outline. In Section 2, we describe the ring of differential operators of a $\mathbb{k}$-algebra realized by retracts in terms of differential operators on these retracts. In Section 3, we apply the results of Section 2 in characteristic zero to compute the rings of differential operators of Stanley–Reisner rings and of toric face rings more generally. In Section 4, we consider the quotient of a normal affine semigroup ring R by a radical monomial ideal J in characteristic zero and provide an explicit formula for the differential operators on R/J that are induced by operators on R. Finally, in Section 5, we use differential operators to provide a new condition for the Gorenstein property of an affine normal semigroup ring.

Throughout this paper, let $\mathbb{k}$ denote an algebraically closed field; in Sections 3, 4, and 5, assume that the characteristic of $\mathbb{k}$ is zero.

1. $\Bbbk$-algebras realized by retracts

In this section we introduce $\Bbbk$-*algebras realized by retracts* and provide examples. Let S and R be $\Bbbk$-algebras. An injective $\Bbbk$-algebra homomorphism $\iota\colon S \to R$ is called an *algebra retract* of R if there exists a surjective homomorphism of $\Bbbk$-algebras $\pi : R \to S$ such that $\pi \circ \iota = \mathrm{id}_S$. When S and R are graded, we also assume that the homomorphisms ι and π are also graded. See [15] for a local version of this definition.

Definition 1.1. Let R be a $\Bbbk$-algebra, and let $\{S_\ell\}_{\ell\in\Lambda}$ be a finite collection of domains that are algebra retracts of R with defining $\Bbbk$-algebra homomorphisms $\iota_\ell\colon S_\ell \hookrightarrow R$ and $\pi_\ell\colon R \twoheadrightarrow S_\ell$ satisfying $\pi_\ell\circ\iota_\ell = \mathrm{id}_{S_\ell}$. For each ℓ, let $P_\ell := \ker \pi_\ell$. We call R a $\Bbbk$-*algebra realized by retracts* and say that R is *realized by the retracts* $\{S_\ell \mid \ell \in \Lambda\}$ when the following two conditions hold:

(i) The following map is injective:

$$\phi\colon R \to \bigoplus_{\ell\in\Lambda} S_\ell \quad \text{given by } f \mapsto (\pi_\ell(f) \mid \ell \in \Lambda); \tag{1.2}$$

in other words, $\bigcap_{\ell\in\Lambda} P_\ell = 0$.

(ii) The S_ℓ are irredundant for (1.2); more precisely, for each $i \in \Lambda$, we have $\bigcap_{\substack{\ell\neq i\\ \ell\in\Lambda}} P_\ell \neq 0$.

Since ι_ℓ is injective, if $f \in S_\ell$, we write $f = \iota_\ell(f)$ in R. For $f \in R$, we use $\phi(f)_i$ to denote the i-th coordinate of $\phi(f)$ in $\bigoplus_{\ell\in\Lambda} S_\ell$.

For example, suppose that R is a quotient of a $\Bbbk$-algebra T by a radical ideal I with associated primes $P_1, \dots, P_r$. Now if T/P_i are algebra retracts of R for all $1 \le i \le r$, then R is realized by the retracts $\{S_\ell \mid \ell \in \Lambda\} = \{T/P_1, T/P_2, \dots, T/P_r\}$. In the remainder of this section, we show that Stanley–Reisner rings and toric face rings are $\Bbbk$-algebras realized by retracts.

1A. *Stanley–Reisner rings as $\Bbbk$-algebras realized by retracts.* Let Δ be a simplicial complex on a finite vertex set $V = \{1, 2, \dots, d\}$. The *Stanley–Reisner ring of* Δ is the $\Bbbk$-algebra given by

$$\Bbbk[\Delta] := \frac{\Bbbk[t_1, t_2, \dots, t_d]}{\langle t^{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathbb{N}^d, \operatorname{supp}(\boldsymbol{a}) \notin \Delta\rangle},$$

where $t^{\boldsymbol{a}} = t_1^{a_1}t_2^{a_2}\cdots t_d^{a_d}$ and $\operatorname{supp}(\boldsymbol{a}) := \{i \in V \mid a_i \neq 0\}$. Retracts have been previously studied in this context. For example, in [10], Epstein and Nguyen show that every graded algebra retract of a Stanley–Reisner ring $\Bbbk[\Delta]$ is a Stanley–Reisner ring. Further, all such retracts are isomorphic to $\Bbbk[\Delta|_W]$, where $\Delta|_W$ is the restriction of Δ to a subset W of V.

One way to view $\Bbbk[\Delta]$ as a $\Bbbk$-algebra realized by retracts is via the facets of Δ. If $\{F_\ell\}_\ell \in \Lambda$ denotes the collection of facets of Δ, then $\Bbbk[F_\ell] \cong \Bbbk[t_i \mid i \in F_\ell]$, and

$$\iota_\ell\colon \Bbbk[F_\ell] \hookrightarrow \Bbbk[\Delta] \quad\text{and}\quad \pi_\ell\colon \Bbbk[\Delta] \twoheadrightarrow \Bbbk[F_\ell]$$

$$t_i \mapsto \begin{cases} t_i & \text{if } i \in F_\ell,\\ 0 & \text{otherwise,}\end{cases}$$

are the maps needed to see that

$$\phi\colon \Bbbk[\Delta] \to \bigoplus_{\ell \in \Lambda} \Bbbk[F_\ell] \quad \text{given by } t_i \mapsto (\pi_\ell(t_i) \mid \ell \in \Lambda)$$

is an injective map. Since the retracts are domains, this implies that $\Bbbk[\Delta]$ is a $\Bbbk$-algebra realized by retracts.

1B. ***Toric face rings as $\Bbbk$-algebras realized by retracts.*** The building blocks of toric face rings are affine semigroup rings, so we begin with those.

Notation 1.3. Let M be a finitely generated submonoid of $\mathbb{Z}^d$. The *affine semigroup ring defined by M* is

$$\Bbbk[M] := \bigoplus_{\boldsymbol{a} \in M} \Bbbk \cdot t^{\boldsymbol{a}},$$

where $t^{\boldsymbol{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\boldsymbol{a} = (a_1, a_2, \ldots, a_d)$.

Finitely generated submonoids of $\mathbb{Z}^d$ are usually called affine semigroups. It is convenient to view the generators of an affine semigroup M as the columns of a $d \times n$ integer matrix A; in this case, we use the notation $M = \mathbb{N}A$. Throughout this article, we assume that the group generated by the columns of A is the full ambient lattice, so $\mathbb{Z}A = \mathbb{Z}^d$ and also that the real positive cone over A, $\mathbb{R}_{\geq 0}A$, is pointed, meaning that it contains no lines.

A semigroup $\mathbb{N}A$ is *normal* if $\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$. In this case, the semigroup ring $\Bbbk[\mathbb{N}A]$ is normal in the sense of commutative algebra.

A hyperplane H in $\mathbb{R}^d$ is a supporting hyperplane of $\mathbb{R}_{\geq 0}A$ if this cone lies entirely in one of the closed half spaces defined by H. A *face* σ of $\mathbb{R}_{\geq 0}A$ (or A or $\mathbb{N}A$) is the intersection of $\mathbb{N}A$ with a supporting hyperplane of $\mathbb{R}_{\geq 0}A$. Such a face is called a *facet* if its linear span has dimension $d-1$. This is somewhat nonstandard terminology, as our faces and facets are submonoids of $\mathbb{N}A$ instead of cones.

Recall that the $\mathbb{Z}^d$-graded prime ideals in $\Bbbk[\mathbb{N}A]$ are in one-to-one correspondence with the faces of A (or $\mathbb{R}_{\geq 0}A$) [19, Proposition 1.5], as a face τ of A corresponds to the multigraded prime $\Bbbk[\mathbb{N}A]$-ideal $P_\tau = \langle t^{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathbb{N}A \setminus \tau \rangle$.

Next, let $\Sigma \subset \mathbb{R}^d$ be a rational polyhedral fan consisting of strongly convex (or pointed) cones. A *monoidal complex* $\mathscr{M}$ supported on Σ is a collection of monoids $\{M_\sigma \mid \sigma \in \Sigma\}$ such that:

(i) $M_\tau \subseteq \tau \cap \mathbb{Z}^d$ and $\mathbb{R}_{\geq 0} M_\tau = \tau$.

(ii) if $\sigma, \tau \in \Sigma$ and $\sigma \subseteq \tau$, then $M_\sigma = \sigma \cap M_\tau$.

Denote $|\mathscr{M}| = \bigcup_{\tau \in \Sigma} M_\tau$.

Definition 1.4. The *toric face ring* of $\mathscr{M}$ over $\Bbbk$ is given as a graded vector space by

$$\Bbbk[\mathscr{M}] = \bigoplus_{\boldsymbol{a} \in |\mathscr{M}|} \Bbbk \cdot t^{\boldsymbol{a}}$$

with multiplication defined by

$$t^{\boldsymbol{a}} \cdot t^{\boldsymbol{b}} = \begin{cases} t^{\boldsymbol{a}+\boldsymbol{b}} & \text{if there is } \tau \in \Sigma \text{ such that } \boldsymbol{a}, \boldsymbol{b} \in \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Given $\tau \in \Sigma$, the semigroup ring $\Bbbk[M_\tau]$ is a subring of $\Bbbk[\mathscr{M}]$; we denote by

$$\iota_\tau \colon \Bbbk[M_\tau] \hookrightarrow \Bbbk[\mathscr{M}]$$

the natural inclusion. For $\tau \in \Sigma$, let

$$P_\tau := \langle t^{\boldsymbol{a}} \mid \boldsymbol{a} \in |\mathscr{M}| \setminus M_\tau \rangle.$$

Then $\Bbbk[\mathscr{M}]/P_\tau$ is isomorphic to the semigroup ring $\Bbbk[M_\tau]$, so that P_τ is a monomial prime ideal of $\Bbbk[\mathscr{M}]$. We identify $\Bbbk[\mathscr{M}]/P_\tau$ with $\Bbbk[M_\tau]$, and denote by

$$\pi_\tau \colon \Bbbk[\mathscr{M}] \twoheadrightarrow \Bbbk[M_\tau]$$

the natural projection onto the quotient. Clearly, $\Bbbk[M_\tau]$ is a retract of $\Bbbk[\mathscr{M}]$.

A *facet* of Σ is a cone in Σ which is maximal with respect to inclusion among all elements of Σ. Let $\mathscr{F}(\Sigma)$ denote the collection of all facets of Σ. Since $\bigcap_{\tau \in \mathscr{F}(\Sigma)} P_\tau = 0$, the ring homomorphism

$$\phi \colon \Bbbk[\mathscr{M}] \to \bigoplus_{\tau \in \mathscr{F}(\Sigma)} \Bbbk[M_\tau] \quad \text{given by } f \mapsto (\pi_\tau(f) \mid \tau \in \mathscr{F}(\Sigma)) \tag{1.5}$$

is injective. It follows that $\Bbbk[\mathscr{M}]$ is a $\Bbbk$-algebra realized by retracts.

In general, the algebra retracts of toric face rings of $\Bbbk[\mathscr{M}]$ are given by restricting $\mathscr{M}$ to a subfan Γ of Σ [10, Proposition 4.5]. The retracts that we consider here are those given by the restriction of Σ to one of its maximal cones Γ, yielding an affine semigroup ring.

Example 1.6. Stanley–Reisner rings of simplicial complexes are toric face rings. To see this, let Δ be a simplicial complex on the vertex set $V = \{1, 2, \ldots, d-1\}$. For each subset F of V, associate the pointed cone C_F generated by the set of elements of the form $\boldsymbol{e}_i + \boldsymbol{e}_d$ for $i \in F$, where $\boldsymbol{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^d$. If Σ denotes the fan in $\mathbb{R}^d$ consisting of the cones C_F for $F \in \Delta$, then the toric face ring $\Bbbk[\mathscr{M}]$ is isomorphic to the Stanley–Reisner ring $\Bbbk[\Delta]$. □

Example 1.7. On the other hand, when Σ has a unique maximal cone, then $\Bbbk[\mathscr{M}]$ is simply an affine semigroup ring. An affine semigroup ring $\Bbbk[\mathbb{N}A]$ modulo a radical monomial ideal J is also a toric face ring. In this case, if the ideal $J = \bigcap_{i=1}^{r} P_{\tau_i}$, then the fan Σ consists of faces of the cone $\mathbb{R}_{\geq 0}A$ that are contained in τ_i for some $1 \leq i \leq r$. We will examine this case more closely in Sections 4 and 5. □

2. Rings of differential operators on $\Bbbk$-algebras realized by retracts

Fix a $\Bbbk$-algebra R and R-module M, and note that in this section we have no requirements on the characteristic of $\Bbbk$ unless explicitly mentioned. The $\Bbbk$-linear differential operators $D(R, M)$ are defined inductively by the degree i of the operator. The degree 0 differential operators are

$$D^0(R, M) := \operatorname{Hom}_R(R, M),$$

and, for $i > 0$, the degree i differential operators are

$$D^i(R, M) := \{\delta \in \operatorname{Hom}_{\Bbbk}(R, M) \mid [\delta, r] \in D^{i-1}(R, M) \text{ for all } r \in R\},$$

where $[\delta, r] := \delta \circ r - r \circ \delta$ is the commutator. The module of differential operators on M, denoted $D(R, M)$, is $\bigcup_{i \geq 0} D^i(R, M)$. We write $D(R)$ for $D(R, R)$. Note that $D(R, -)$ is a left-exact functor, and that, in particular, for any R-ideal J, $D(R, J) = \{\delta \in D(R) \mid \delta(R) \subset J\}$.

Example 2.1. If $\operatorname{char}(\Bbbk) = 0$, the ring of differential operators of the polynomial ring in d variables with coefficients in $\Bbbk$ is the Weyl algebra

$$W = \Bbbk[t_1, \dots, t_d]\langle \partial_1, \dots, \partial_d \rangle,$$

where the relations defined on the generators are $t_i t_j - t_j t_i = 0 = \partial_i \partial_j - \partial_j \partial_i$, and $\partial_i t_j - t_j \partial_i = \delta_{ij}$, the Kronecker-$\delta$ function. The ring of differential operators of the ring of Laurent polynomials in d variables is the extended Weyl algebra $\Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle$ with the relations as in the ordinary Weyl algebra together with the additional relation $t_i^{-1} \partial_j - \partial_j t_i^{-1} = t_i^{-2} \delta_{ij}$. □

Let R be a $\Bbbk$-algebra realized by the retracts $\{S_\ell \mid \ell \in \Lambda\}$, as in Definition 1.1. Given $\ell \in \Lambda$, and $\delta \in D(R)$, set $\delta_\ell = \pi_\ell \circ \delta \circ \iota_\ell$. In this section, we show that the map:

$$D(R) \to \bigoplus_{\ell \in \Lambda} D(S_\ell) \quad \text{given by } \delta \mapsto (\delta_\ell := \pi_\ell \circ \delta \circ \iota_\ell \mid \ell \in \Lambda), \tag{2.2}$$

is injective and compute the ring of differential operators $D(R)$ in terms of the rings of differential operators $D(S_\ell)$ for $\ell \in \Lambda$. We do this in two ways (see Theorems 2.6 and 2.8). First, we include two lemmas.

Lemma 2.3. *Assume that the $\Bbbk$-algebra R is realized by the retracts $\{S_\ell \mid \ell \in \Lambda\}$. Let $\delta \in D^i(R)$. If $\ell \in \Lambda$, then $\delta_\ell \in D^i(S_\ell)$.*

Proof. We use induction on i. If $\delta \in D^0(R) = \operatorname{Hom}_R(R, R)$, then δ is given by multiplication by a fixed element of R, say r. But then δ_ℓ is given by multiplication by $\pi_\ell(r)$, and so $\delta_\ell \in D^0(S_\ell)$.

Now assume the result is true for operators of order $i - 1$, and let $\delta \in D^i(R)$. The $\Bbbk$-linearity of δ_ℓ follows from $\Bbbk$-linearity of δ, since ι_ℓ and π_ℓ are $\Bbbk$-linear. To show that $\delta_\ell \in D^i(S_\ell)$, it is enough to verify that $[\delta_\ell, f] \in D^{i-1}(S_\ell)$ for all $f \in S_\ell$. This follows by induction since, for $f \in S_\ell$, $[\delta_\ell, f] = [\delta, \iota_\ell(f)]_\ell$.

To see this, note that $\iota_\ell \circ f$ is the map $\iota_\ell(f)\iota_\ell$ and for any $g \in R$, $\pi_\ell \circ g$ is the map $\pi_\ell(g)\pi_\ell$. Then we have

$$\begin{aligned}[\delta_\ell, f] &= \pi_\ell \circ \delta \circ \iota_\ell \circ f - f \circ \pi_\ell \circ \delta \circ \iota_\ell \\ &= \pi_\ell \circ \delta \circ \iota_\ell(f)\iota_\ell - \pi_\ell(\iota_\ell(f))\pi_\ell \circ \delta \circ \iota_\ell \\ &= \pi_\ell \circ \delta \circ \iota_\ell(f)\iota_\ell - \pi_\ell \circ \iota_\ell(f)\delta \circ \iota_\ell \\ &= \pi_\ell \circ (\delta \circ \iota_\ell(f) - \iota_\ell(f)\delta) \circ \iota_\ell \\ &= [\delta, \iota_\ell(f)]_\ell. \end{aligned}$$

□

Lemma 2.4. *Assume that the $\Bbbk$-algebra R is realized by the retracts $\{S_\ell \mid \ell \in \Lambda\}$. Let $\delta \in D^i(R)$, $\ell \in \Lambda$, and $f \in R$. If $\pi_\ell(f) = 0$, then $\pi_\ell(\delta(f)) = 0$.*

Example 2.5. Lemma 2.4 is a key ingredient for the main results of this section and provides an easily checked necessary condition for a linear operator on R to be a differential operator on R. To see this more concretely, consider $R = \Bbbk[x, y]/\langle xy\rangle$. This Stanley–Reisner ring is realized by the retracts $\Bbbk[x] \cong R/\langle y\rangle$ and $\Bbbk[y] \cong R/\langle x\rangle$, where the canonical projections are denoted π_x and π_y, respectively. The ring of differential operators on R is generated as a $\Bbbk$-algebra by the operators $x^i\partial_x^j$ and $y^k\partial_y^\ell$ such that $i \ge j$ and $k \ge \ell$. To see why ∂_x is not a differential operator on R, note that $\pi_y(x) = 0$, but $\pi_y(\partial_x(x)) = \pi_y(1) \ne 0$, contradicting the conclusion of Lemma 2.4. □

Proof of Lemma 2.4. By induction on i as before, if $i = 0$, then $\delta(f) = r \cdot f$ for a fixed $r \in R$. If $\pi_\ell(f) = 0$, it follows that $\pi_\ell(r \cdot f) = 0$, since π_ℓ is a ring homomorphism.

Now assume the result is true for operators of order $i - 1$, and let $\delta \in D^i(R)$. Suppose $\pi_\ell(f) = 0$ and let $P = \bigcap_{f \notin P_j} P_j$ where $P_j = \ker \pi_j$. Let g be a nonzero element in P such that $\pi_\ell(g) \ne 0$. Such an element exists since R is realized by the retracts $\{S_\ell \mid \ell \in \Lambda\}$. Then since $f \cdot g \in \big(\bigcap_{f \in P_i} P_i\big)\big(\bigcap_{f \notin P_j} P_j\big) \subseteq \bigcap_{\ell \in \Lambda} P_\ell = 0$, we have $f \cdot g = 0$ in R. By the inductive hypothesis, $\pi_\ell([\delta, g](f)) = 0$. Then

$$[\delta, g](f) = \delta(g \cdot f) - g\delta(f) = -g\delta(f),$$

where the last equality holds because $\delta(g \cdot f) = \delta(0) = 0$. Hence

$$0 = \pi_\ell([\delta, g](f)) = \pi_\ell(-g\delta(f)) = -\pi_\ell(g)\pi_\ell(\delta(f)).$$

As g was chosen so that $\pi_\ell(g) \ne 0$, it follows that $\pi_\ell(\delta(f)) = 0$ since S_ℓ is a domain. □

We are now ready to give a first description of $D(R)$.

Theorem 2.6. *Assume that the $\Bbbk$-algebra R is realized by the retracts $\{S_\ell \mid \ell \in \Lambda\}$ with the injective map $\phi\colon R \to \bigoplus_{\ell \in \Lambda} S_\ell$. Let $\delta\colon R \to R$ be $\Bbbk$-linear. Then $\delta \in D^i(R)$ if and only if the following two conditions hold*:

(1) *$\delta_\ell \in D^i(S_\ell)$ for all $\ell \in \Lambda$.*

(2) *If $\ell \in \Lambda$ and $\pi_\ell(f) = 0$, then $\pi_\ell(\delta(f)) = 0$.*

Proof. If $\delta \in D^i(R)$, then δ satisfies Conditions (1) and (2) by Lemmas 2.3 and 2.4.

Now assume that δ is $\Bbbk$-linear and satisfies Conditions (1) and (2). Fix $f \in R$. For each $\ell \in \Lambda$, set $f_\ell = \iota_\ell(\pi_\ell(f))$. Then $\pi_\ell(f) = \pi_\ell(f_\ell)$ for all $\ell \in \Lambda$, and so $\pi_\ell(f - f_\ell) = 0$ for all $\ell \in \Lambda$. By Condition (2), we have $\pi_\ell(\delta(f - f_\ell)) = 0$. Hence,

$$\pi_\ell(\delta(f)) = \pi_\ell(\delta(f_\ell)) \quad \text{for each } \ell \in \Lambda. \tag{2.7}$$

We now show that $\delta \in D^i(R)$ by induction on i. Set $r = \delta(1)$, and assume that, for $\ell \in \Lambda$, $\delta_\ell \in D^0(S_\ell)$ by Condition (1). Note that $\delta_\ell = \pi_\ell \circ \delta \circ \iota_\ell$. As an S_ℓ-module homomorphism on $D^0(S_\ell)$, δ_ℓ is determined by the image of the identity element 1_{S_ℓ} in S_ℓ under δ_ℓ. That is, for any $g \in S_\ell$,

$$\delta_\ell(g) = \delta_\ell(1_{S_\ell} \cdot g) = \delta_\ell(1_{S_\ell})g.$$

Applying (2.7) to $f = 1$ in R, we have $\pi_\ell(\delta(1_\ell)) = \pi_\ell(\delta(1)) = \pi_\ell(r)$. On the other hand,

$$\pi_\ell(\delta(1_\ell)) = (\pi_\ell \circ \delta)(\iota_\ell(\pi_\ell(1))) = (\pi_\ell \circ \delta \circ \iota_\ell)(\pi_\ell(1)) = \delta_\ell(\pi_\ell(1)) = \delta_\ell(1_{S_\ell}).$$

It follows that δ_ℓ is given by multiplication by $\pi_\ell(r)$ for each $\ell \in \Lambda$. Next note that

$$\pi_\ell(\delta(f_\ell)) = \pi_\ell \circ \delta \circ \iota_\ell(\pi_\ell(f)) = \delta_\ell(\pi_\ell(f)) = \pi_\ell(r)\pi_\ell(f) = \pi_\ell(rf).$$

Combining these two observations, we have, by definition of ϕ,

$$\phi(\delta(f)) = (\pi_\ell(\delta(f)))_{\ell\in\Lambda} = (\pi_\ell(rf))_{\ell\in\Lambda} = \phi(rf).$$

Since ϕ is injective, $\delta(f) = rf$, which implies that $\delta \in D^0(R)$.

Assume now that the result is true for operators of order $i - 1$, and let δ satisfy the required conditions for order i: (1) $\delta_\ell \in D^i(S_\ell)$ for all $\ell \in \Lambda$. (2) If $\ell \in \Lambda$ and $\pi_\ell(g) = 0$ then $\pi_\ell(\delta(g)) = 0$.

We need to show that $[\delta, f] \in D^{i-1}(R)$ for $f \in R$. By induction, it suffices to show that Conditions (1) and (2) hold for $[\delta, f]$. To address Condition (1), fix $\ell \in \Lambda$. Since $[\delta, f]_\ell = \pi_\ell \circ [\delta, f] \circ \iota_\ell$, we will show that $\pi_\ell \circ [\delta, f] \circ \iota_\ell \in D^{i-1}(S_\ell)$ for all $\ell \in \Lambda$. Let us now expand:

$$\begin{aligned}
\pi_\ell \circ [\delta, f] \circ \iota_\ell &= \pi_\ell \circ (\delta \circ f - f\delta) \circ \iota_\ell \\
&= \pi_\ell \circ \delta \circ f \circ \iota_\ell - \pi_\ell(f\delta) \circ \iota_\ell \\
&= \pi_\ell \circ \delta \circ f_\ell \circ \iota_\ell - \pi_\ell(f)\pi_\ell \circ \delta \circ \iota_\ell && \text{by (2.7)} \\
&= \pi_\ell \circ \delta \circ \iota_\ell(\pi_\ell(f)) \circ \iota_\ell - \pi_\ell(f)\pi_\ell \circ \delta \circ \iota_\ell && \text{by the definition of } f_\ell \\
&= \pi_\ell \circ \delta \circ \iota_\ell \circ \pi_\ell(f) - \pi_\ell(f)\pi_\ell \circ \delta \circ \iota_\ell && \text{as } \iota_\ell \text{ is a homomorphism} \\
&= \delta_\ell \circ \pi_\ell(f) - \pi_\ell(f)\delta_\ell && \text{by the definition of } \delta_\ell \\
&= [\delta_\ell, \pi_\ell(f)].
\end{aligned}$$

Since $\delta_\ell \in D^i(S_\ell)$ by assumption, $[\delta_\ell, \pi_\ell(f)] \in D^{i-1}(S_\ell)$ by the definition of an order i differential operator.

To address Condition (2), assume $\ell \in \Lambda$ and $\pi_\ell(g) = 0$ for some $g \in R$. Because π_ℓ is a homomorphism and δ satisfies Condition (2), $\pi_\ell(fg) = \pi_\ell(f)\pi_\ell(g) = 0$, and so $\pi_\ell(\delta(fg)) = 0$. Then

$$\pi_\ell \circ [\delta, f](g) = \pi_\ell(\delta(fg)) - \pi_\ell(f\delta(g)) = 0 - \pi_\ell(f)\pi_\ell(\delta(g)) = 0.$$

Hence $[\delta, f]$ satisfied Condition (2) as well as Condition (1), and so, by induction, $[\delta, f] \in D^{i-1}(R)$, and then $\delta \in D^i(R)$. □

To state our second characterization of $D(R)$, if $\lambda \subset \Lambda$, we need some additional notation. Set $S_\lambda = \bigcap_{\ell \in \lambda} S_\ell$. Since S_ℓ is an algebra retract of R for all $\ell \in \lambda$, S_λ is also an algebra retract of R in a natural way (and indeed S_λ is also an algebra retract of S_ℓ for all $\ell \in \lambda$). We define $\iota_{\lambda,\ell}$ to be the natural inclusion that identifies S_λ as a subring of S_ℓ, and $\pi_{\ell,\lambda}$ is the natural projection from S_ℓ to S_λ, where the latter is considered as a quotient of the former.

Theorem 2.8. *Assume that the $\Bbbk$-algebra R is realized by the retracts $\{S_\ell \mid \ell \in \Lambda\}$. The map*

$$D(R) \to \bigoplus_{\ell \in \Lambda} D(S_\ell) \quad \text{given by } \delta \mapsto (\delta_\ell = \pi_\ell \circ \delta \circ \iota_\ell \mid \ell \in \Lambda)$$

from (2.2) *is an injective ring homomorphism. A tuple $(\rho_\ell \mid \ell \in \Lambda) \in \bigoplus_{\ell \in \Lambda} D(S_\ell)$ is in the image of* (2.2) *if and only if it satisfies the following two conditions*:

(a) *If $\lambda \subset \Lambda$ and $j, k \in \lambda$, then $\pi_{j,\lambda} \circ \rho_j \circ \iota_{\lambda,j} = \pi_{k,\lambda} \circ \rho_k \circ \iota_{\lambda,k}$.*

(b) *If $\lambda = \{j, k\}$, then $\pi_{j,\lambda}(\rho_j(f)) = 0$ if $\pi_k(f) = 0$.*

Proof. Since (2.2) is given by composition and direct sums of ring homomorphisms, it is itself a ring homomorphism.

First, for $\delta \in D(R)$, its image satisfies Conditions (a) (by construction) and (b) by Condition (2) of Theorem 2.6. To prove injectivity, as well as verify the description of the image of (2.2), we construct an inverse.

For $\lambda \subset \Lambda$ and $\rho \in D(S_\lambda)$, set $\bar{\rho} = \iota_\lambda \circ \rho \circ \pi_\lambda$. When ρ satisfies Condition (b), then, by an argument similar to that at the end of the proof of Theorem 2.6, $\bar{\rho} \in D(R)$.

If $(\rho_\ell \mid \tau \in \Lambda) \in \bigoplus_{\ell \in \Lambda} D(S_\ell)$ satisfies Condition (a) and $\varnothing \neq \lambda \subset \Lambda$, choose $\ell \in \lambda$, and set $\rho_\lambda = \pi_{\ell,\lambda} \circ \rho_\ell \circ \iota_{\lambda,\ell}$. By Condition (a), ρ_λ is independent of the choice of $\lambda \in \Lambda$. Now let

$$\rho = \sum_{\varnothing \neq \lambda \subset \Lambda} (-1)^{|\lambda|-1} \overline{\rho_\lambda}.$$

We claim that if $(\rho_\ell \mid \ell \in \Lambda) \in \bigoplus_{\ell \in \Lambda} D(S_\ell)$ satisfies Conditions (a) and (b), then $\rho \in D(R)$, and the image of ρ under the map (2.2) is $(\rho_\ell \mid \ell \in \Lambda)$. This gives the desired inverse and finishes the proof. □

3. Applications

In this section, we apply the results of Section 2 to compute rings of differential operators for examples of $\Bbbk$-algebras realized by retracts discussed in Section 1. Throughout this section, we assume that the characteristic of $\Bbbk$ is zero.

3A. ***Differential operators of Stanley–Reisner rings.*** Traves [47] gave a nice classification of the ring of differential operators for Stanley–Reisner ring. In particular, he proved that $D(\Bbbk[\Delta])$ is generated as a $\Bbbk$-algebra by

$$\{t^{\boldsymbol{a}}\partial^{\boldsymbol{b}} \mid t^{\boldsymbol{a}} \in P \text{ or } t^{\boldsymbol{b}} \notin P \text{ for each minimal prime } P \text{ of } R\}.$$

where $t^{\boldsymbol{a}}\partial^{\boldsymbol{b}} = t_1^{a_1} \dots t_d^{a_d}\partial_1^{b_1} \dots \partial_d^{b_d}$ and $\partial_i = \frac{\partial}{\partial t_i}$.

One can see directly that this description matches our description of the ring of differential operators in terms of algebra retracts. In particular, for a Stanley–Reisner ring $\Bbbk[\Delta]$, recall that $\Bbbk[\Delta]$ is realized by the retracts $\{\Bbbk[F_\ell] \mid \ell \in \Lambda\}$, where $\{F_\ell \mid \ell \in \Lambda\}$ are the facets of Δ. Further, we know that the minimal primes of $\Bbbk[\Delta]$ are exactly those corresponding to the facets, namely $P_{F_\ell} = \langle t^{\boldsymbol{a}} \mid \boldsymbol{a} \in \mathbb{N}^d, \operatorname{supp}(\boldsymbol{a}) \notin F_\ell\rangle$; see for example [13].

For any $\Bbbk$-linear map $\delta : \Bbbk[\Delta] \to \Bbbk[\Delta]$, Theorem 2.6 tells us that $\delta \in D^i(\Bbbk[\Delta])$ if and only if

(1) $\delta_\ell \in D^i(\Bbbk[F_\ell])$ for all $\ell \in \Lambda$ and

(2) $\pi_\ell(\delta(f)) = 0$ for all $\ell \in \Lambda$ and $f \notin \Bbbk[F_\ell]$.

Now for any facet F_ℓ of Δ, notice that $\Bbbk[F_\ell] \cong \Bbbk[t_i \mid i \in F_\ell]$, so that $D(\Bbbk[F_\ell])$ is the standard Weyl algebra on the variables $\{t_i \mid i \in F_\ell\}$. By Condition (1), $D(\Bbbk[\Delta])$ must be generated by elements of the form $x^{\boldsymbol{a}}\partial^{\boldsymbol{b}}$.

Further, we have that $\pi_\ell(t^{\boldsymbol{a}}\partial^{\boldsymbol{b}}(f)) = 0$ for all $f \notin \Bbbk[F_\ell]$ if and only if $t^{\boldsymbol{a}} \notin \Bbbk[F_\ell]$ or $\partial^{\boldsymbol{b}}(f) = 0$ for all $f \notin \Bbbk[F_\ell]$. This happens if and only if $\operatorname{supp}(\boldsymbol{a}) \notin F_\ell$ or $\operatorname{supp}(\boldsymbol{b}) \in F_\ell$. In other words, we have Condition (2) if and only if $t^{\boldsymbol{a}} \in P_{F_\ell}$ or $t^{\boldsymbol{b}} \notin P_{F_\ell}$ for every $\ell \in \Lambda$.

3B. ***Differential operators of toric face rings.*** The algebra retracts of toric face rings that we will consider are affine semigroup rings. Saito and Traves [42] described the ring of differential operators for an affine semigroup ring $\Bbbk[\mathbb{N}A]$ over the complex numbers, when viewed as a subring of the ring of differential operators of the Laurent polynomials, i.e.,

$$D(\Bbbk[\mathbb{Z}^d]) = \Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d\rangle,$$

where ∂_i denotes the differential operator $\frac{\partial}{\partial t_i}$. Setting $\theta_j = t_j\partial_j$ for $1 \le j \le d$ and noting that $\theta_i\theta_j = \theta_j\theta_i$ for all i and j, the ring $\Bbbk[\theta] = \Bbbk[\theta_1, \theta_2, \dots, \theta_d]$ is a polynomial ring. Set

$$\Omega(\boldsymbol{m}) := \{\boldsymbol{a} \in \mathbb{N}A \mid \boldsymbol{a} + \boldsymbol{m} \notin \mathbb{N}A\} = \mathbb{N}A \setminus (-\boldsymbol{m} + \mathbb{N}A).$$

The *idealizer of* $\Omega(\boldsymbol{m})$ is defined to be the $\Bbbk[\theta]$-ideal

$$\mathbb{I}(\Omega(\boldsymbol{m})) := \langle f(\theta) \in \Bbbk[\theta] \mid f(\boldsymbol{a}) = 0 \text{ for all } \boldsymbol{a} \in \Omega(\boldsymbol{m})\rangle,$$

with the θ_i of degree $\mathbf{0}$. In fact, $\mathbb{I}(\Omega(\boldsymbol{m}))$ consists of $f(\theta)$ such that $t^{\boldsymbol{m}} f(\theta) \in D(\Bbbk[\mathbb{N}A])$. This is a consequence of the work of Saito and Traves [42, Theorem 2.1], where they show that

$$D(\Bbbk[\mathbb{N}A]) = \bigoplus_{\boldsymbol{m}\in\mathbb{Z}^d} t^{\boldsymbol{m}} \cdot \mathbb{I}(\Omega(\boldsymbol{m})).$$

To compute $\mathbb{I}(\Omega(\boldsymbol{m}))$ for a normal semigroup ring, consider a facet σ of A, recalling that by this we mean a submonoid of $\mathbb{N}A$ whose linear span has dimension $d-1$. The *primitive integral support function* (or simply *support function*) F_σ is the unique linear form on $\mathbb{R}^d$ such that

$$(1)\ F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0, \quad (2)\ F_\sigma(\sigma) = 0, \quad \text{and} \quad (3)\ F_\sigma(\mathbb{Z}^d) = \mathbb{Z}.$$

Saito and Traves used the $F_\sigma(\boldsymbol{m})$ to determine the precise form of $\mathbb{I}(\Omega(\boldsymbol{m}))$ for normal $\Bbbk[\mathbb{N}A]$. The ideas of the computation in [42] can be traced back to [22] and [28]. Set

$$G_{\boldsymbol{m}}(\theta) := \prod_{F_\sigma(\boldsymbol{m})<0} \prod_{i=0}^{-F_\sigma(\boldsymbol{m})-1} (F_\sigma(\theta) - i). \tag{3.1}$$

Theorem 3.2 [42, Theorem 3.2.2]. *Let R be a normal affine semigroup ring of dimension d. Let $F_{\sigma_1}, \ldots, F_{\sigma_r}$ be the support functions of the facets σ_i of the semigroup defining R. Let $\boldsymbol{m}$ be a multidegree in $\mathbb{Z}^d$. Then*

$$D(R)_{\boldsymbol{m}} = t^{\boldsymbol{m}} \cdot \langle G_{\boldsymbol{m}}(\theta)\rangle.$$

We now provide a graded description of $D^i(R)$ when $R = \Bbbk[\mathscr{M}]$ is a toric face ring. Let $\sigma \in \Sigma$, and let $\rho \in D^i(\Bbbk[M_\sigma])$. Then ρ is a sum of operators of order i each of which is homogeneous with respect to the natural $\mathbb{Z}\sigma$-grading (where $\mathbb{Z}\sigma$ is the abelian group generated by M_σ). We claim that if ρ satisfies Condition (b) of Theorem 2.8, then each homogeneous component does as well. This holds since applying an operator of multidegree $\boldsymbol{b} \in \mathbb{Z}\sigma$ to a monomial $t^{\boldsymbol{a}}$ yields a (possibly zero) scalar multiple of $t^{\boldsymbol{a}+\boldsymbol{b}}$. As ρ is a finite sum of operators of multidegrees $\boldsymbol{b}_1, \ldots \boldsymbol{b}_n$, $\rho(t^{\boldsymbol{a}}) = 0$ if and only if the constant multiple of $t^{\boldsymbol{a}+\boldsymbol{b}_i}$ is 0 for all $1 \leq i \leq n$.

The following result then provides the final key to describe differential operators on toric face rings, as we illustrate in examples later.

Proposition 3.3. *Let $\sigma \in \mathscr{F}(\Sigma)$, and let $\rho \in D^i(\Bbbk[M_\sigma])$ be homogeneous of degree $\boldsymbol{b} \in \mathbb{Z}\sigma$ satisfying Condition (b) from Theorem 2.8. Then $\rho = t^{\boldsymbol{b}} q(\theta)$ for some $q \in \Bbbk[\theta]$ such that*

(i) *$q(\boldsymbol{a}) = 0$ if $\boldsymbol{a}+\boldsymbol{b} \notin M_\sigma$ when $\boldsymbol{a} \in M_\sigma$, and*

(ii) *for $\tau \in \mathscr{F}(\Sigma) \setminus \{\sigma\}$ and $\boldsymbol{a} \in M_\sigma \setminus M_{\sigma\cap\tau}$ such that $\boldsymbol{a}+\boldsymbol{b} \in M_{\sigma\cap\tau}$, we have $q(\boldsymbol{a}) = 0$.*

Proof. Condition (i) follows from the description of $D(\Bbbk[M_\tau])$. Condition (ii) follows from the assumption that Condition (b) of Theorem 2.8 is satisfied for ρ. □

Remark 3.4. The two conditions in the previous result arise from the multigraded nature of our descriptions of rings of differential operators. For a given multidegree $\boldsymbol{b}$, operators of this degree are spanned by operators of the form $\delta = t^{\boldsymbol{b}} q(\theta)$, where q is a polynomial. Then $\delta(t^{\boldsymbol{a}}) = q(\boldsymbol{a}) t^{\boldsymbol{a}+\boldsymbol{b}}$. If $\boldsymbol{a}$ belongs to our semigroup but $\boldsymbol{a}+\boldsymbol{b}$ does not, then we must have $q(\boldsymbol{a}) = 0$ if δ is to be a differential operator on the semigroup ring. This is the first condition in Proposition 3.3.

Similarly, if we are working with a semigroup coming from a monoidal complex, it may happen that $\boldsymbol{a}$ belongs to a semigroup, and $\boldsymbol{a}+\boldsymbol{b}$ belongs to both the semigroup and some other semigroup in the complex. In this case the projection of $\delta(t^{\boldsymbol{a}}) = q(\boldsymbol{a}) t^{\boldsymbol{a}+\boldsymbol{b}}$ onto this second semigroup will not vanish unless $q(\boldsymbol{a}) = 0$. This explains the second condition in Proposition 3.3.

For each $\boldsymbol{b}$, the set of $\boldsymbol{a}$ at which the polynomials q are forced to vanish lie on a finite collection of translates of the linear spans of the faces of the maximal cones in the fan Σ. This explains why their vanishing ideal is generated by products of shifts of linear forms corresponding to supporting hyperplanes of those faces. □

We now include concrete examples of differential operators in the toric face ring setting.

Example 3.5. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Two of the facets of the integral cone $\mathbb{N}A$ are $\sigma = \mathbb{N}B$ and $\tau = \mathbb{N}C$. Define

$$R := \frac{\Bbbk[\mathbb{N}A]}{P_\sigma \cap P_\tau} = \frac{\Bbbk[x, xy, xy^2, xz, xz^2]}{\langle x^2yz, x^2y^2z, x^2yz^2, x^2y^2z^2 \rangle}.$$

Note that there is a natural map

$$\phi\colon R \to \frac{R}{P_\sigma} \oplus \frac{R}{P_\tau} \cong \Bbbk[\mathbb{N}B] \oplus \Bbbk[\mathbb{N}C] \cong \Bbbk[x, xy, xy^2] \oplus \Bbbk[x, xz, xz^2].$$

The differential operators defined on the face $\sigma \cap \tau$ in each multidegree $(u, 0, 0)$ can be realized by operators in $\Bbbk[\mathbb{N}B] \oplus \Bbbk[\mathbb{N}C]$ in multidegree $((u, 0), (u, 0))$ generated by

$$(\rho_u, \delta_u) := \left(x^u \cdot \prod_{i=0}^{-2u-1} (2\theta_x - \theta_y - i), x^u \cdot \prod_{i=0}^{-2u-1} (2\theta_x - \theta_z - i) \right) \tag{3.6}$$

This is the case by Theorem 2.8 since

$$\pi_{\sigma,\sigma\cap\tau} \circ \rho_u \circ \iota_{\sigma\cap\tau,\sigma} = x^u \cdot \prod_{i=0}^{-2u-1} (2\theta_x - i) = \pi_{\tau,\sigma\cap\tau} \circ \delta_u \circ \iota_{\sigma\cap\tau,\tau}$$

and

$$\pi_{\sigma,\sigma\cap\tau}(\rho_u(x^u y^v)) = 0 \quad \text{for } (u, v) \in \sigma \setminus \tau,$$

as well as

$$\pi_{\tau,\sigma\cap\tau}(\rho_u(x^u z^v)) = 0 \quad \text{for } (u, v) \in \tau \setminus \sigma.$$

The operators on $\sigma \setminus \sigma \cap \tau$ in each multidegree $(u, v, 0)$ (for $v \neq 0$) can be realized by operators in $\Bbbk[\mathbb{N}B] \oplus \Bbbk[\mathbb{N}C]$ in multidegree $((u, v), (0, 0))$ generated by

$$(\rho_{u,v}, 0) := \left(x^u y^v \cdot \left(\prod_{i=0}^{-2u+v-1} (2\theta_x - \theta_y - i) \right) \left(\prod_{i=0}^{-v} (\theta_y - i) \right), 0 \right). \tag{3.7}$$

Since

$$\pi_{\sigma,\sigma\cap\tau}(\rho_{u,v}(x^u y^v)) = 0 \quad \text{for } (u, v) \in \sigma,$$

both Conditions (a) and (b) of Theorem 2.8 are clearly satisfied.

A similar argument shows that the operators on $\tau \setminus \sigma \cap \tau$ in each multidegree $(u, 0, v)$ (for $v \neq 0$) can be realized by operators in $\Bbbk[\mathbb{N}B] \oplus \Bbbk[\mathbb{N}C]$ in multidegree $((0, 0), (u, v))$ generated by

$$(0, \delta_{u,v}) := \left(0, x^u z^v \cdot \left(\prod_{i=0}^{-2u+v-1} (2\theta_x - \theta_z - i) \right) \left(\prod_{i=0}^{-v} (\theta_z - i) \right) \right). \tag{3.8}$$

Combinations of operators of the forms represented by (3.6), (3.7) and (3.8) also produce operators in our ring of differential operators for all $v \neq 0$. □

Example 3.9. Set

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Recall $\mathbb{N}A$ gives the integral cone of the ring $S = \Bbbk[\mathbb{N}A] = \Bbbk[x, xy, xz, xyz]$. The facets of $\mathbb{N}A$ are

$$\sigma_1 = \mathbb{N}\langle \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_2 \rangle, \quad \sigma_3 = \mathbb{N}\langle \boldsymbol{e}_1 + \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 \rangle,$$
$$\sigma_2 = \mathbb{N}\langle \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_3 \rangle, \quad \sigma_4 = \mathbb{N}\langle \boldsymbol{e}_1 + \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 \rangle.$$

The support functions of the facets of $\mathbb{N}A$ are $\theta_z, \theta_y, \theta_x - \theta_y, \theta_x - \theta_y$, respectively. By Theorem 3.2, the differential operators on R in multidegree $\boldsymbol{m}$ are given by

$$x^{m_1} y^{m_2} z^{m_3} \cdot \left\langle \left(\prod_{i=1}^{-m_3-1} (\theta_z - i) \right) \left(\prod_{i=1}^{-m_2-1} (\theta_y - i) \right) \left(\prod_{i=1}^{-m_1+m_3-1} (\theta_x - \theta_z - i) \right) \left(\prod_{i=1}^{-m_1+m_2-1} (\theta_x - \theta_y - i) \right) \right\rangle.$$

Set

$$R = \frac{S}{P_{\sigma_1} \cap P_{\sigma_2} \cap P_{\sigma_3} \cap P_{\sigma_4}} = \frac{\Bbbk[x, xy, xz, xyz]}{\langle x^2 yz \rangle}.$$

Note that $T := \Bbbk[a, b, c, d]/\langle ad, bc \rangle \cong R$. We know the differential operators of T by [47] since T is a Stanley–Reisner ring. The generators of $D(T)$ in multidegree $-\boldsymbol{e}_1$ are

$$a^{-1}\theta_a(\theta_a - 1), \quad a^{-1}\theta_a\theta_b, \quad a^{-1}\theta_a\theta_c.$$

There are not enough operators in the extended Weyl algebra in three variable to express these three operators. Given the mappings

$$\begin{tikzcd} D(T) \arrow[r] \arrow[d, "\cong"] & D(\Bbbk[a,b]) \oplus D(\Bbbk[a,c]) \oplus D(\Bbbk[b,d]) \oplus D(\Bbbk[c,d]) \arrow[d, "\cong"] \\ D(R) \arrow[r] & D(\Bbbk[x,xy]) \oplus D(\Bbbk[x,xz]) \oplus D(\Bbbk[xy,xyz]) \oplus D(\Bbbk[xz,xyz]) \end{tikzcd}$$

with horizontal maps as in Theorem 2.8, we see that

$$\begin{aligned} a^{-1}\theta_a(\theta_a-1) &\mapsto (a^{-1}\theta_a(\theta_a-1), a^{-1}\theta_a(\theta_a-1), 0, 0) \cong \left(x^{-1}\prod_{i=0}^{1}(\theta_x-\theta_y-i), x^{-1}\prod_{i=0}^{1}(\theta_x-\theta_z-i), 0, 0\right) \\ a^{-1}\theta_a\theta_b &\mapsto (a^{-1}\theta_a\theta_b, 0, 0, 0) \cong (x^{-1}(\theta_x-\theta_y)\theta_y, 0, 0, 0) \\ a^{-1}\theta_a\theta_c &\mapsto (0, a^{-1}\theta_a\theta_c, 0, 0) \cong (0, x^{-1}(\theta_x-\theta_z)\theta_z, 0, 0). \end{aligned}$$

To get an operator acting as $a^{-1}\theta_a(\theta_a-1)$, $a^{-1}\theta_a\theta_b$, or $a^{-1}\theta_a\theta_c$ in terms of the linear support functions on S, we would need fractional expressions of the forms

$$x^{-1}\cdot\frac{\prod_{i=0}^{1}(\theta_x-\theta_y-i)\prod_{i=0}^{1}(\theta_x-\theta_z-i)}{\prod_{i=0}^{1}(\theta_x-i)}, \quad x^{-1}\cdot\frac{(\theta_x-\theta_y)\theta_y(\theta_x-\theta_z)}{\theta_x}, \quad x^{-1}\cdot\frac{(\theta_x-\theta_y)(\theta_x-\theta_z)\theta_z}{\theta_x},$$

which do not come from the extended Weyl algebra. □

4. Rings of differential operators of quotient rings

In this section, we consider a special class of toric face rings, namely quotients of normal affine semigroup rings by radical monomial ideals. Our main goal is to characterize which differential operators on the quotient arise from operators on the ambient affine semigroup ring. We build off the techniques of [44, Proposition 1.6] and provide a careful inductive argument on the number of facets of the Newton polytope in order to generalize beyond the case where the ambient ring S is regular. In Section 5, we show that, if S is regular (or even Gorenstein) and J is the interior ideal of S, then $JD(S) = D(S, J)$, which gives a direct argument that our result agrees with [44, Proposition 1.6] in that case.

For any $\delta \in D(R)$, we will use δJ to denote the set consisting of products $\delta \circ f$ of differential operators for any $f \in J$ and $\delta * f$ to denote the action $\delta(f)$ to avoid any possible confusion.

Proposition 4.1. *Let R be commutative $\Bbbk$-algebra and J be an ideal of R. Let*

$$\mathbb{I}(J) := \{\delta \in D(R) \mid \delta * J \subset J\}.$$

The differential operators on R that induce maps on R/J are precisely those in $\mathbb{I}(J)$. Further, there is an embedding of rings

$$\frac{\mathbb{I}(J)}{D(R,J)} \hookrightarrow D(R/J).$$

Proof. The first statement follows from the universal property of quotients. In fact, for any $\delta \in D(R)$, if δ induces an operator in $D(R/J)$, i.e., a map from R/J to R/J, then we must have $\delta * J \subset J$. So, δ belongs to $\mathbb{I}(J)$ by definition.

Every operator in $\mathbb{I}(J)$ induces a differential operator from R/J to itself. Now consider the map

$$\rho\colon \mathbb{I}(J) \to D(R/J) \quad \text{given by } \rho(\delta) = \delta'.$$

The kernel is $\ker(\rho) = \{\delta \mid \delta * R \subseteq J\} = D(R, J)$ by [28, 1.2]. Hence ρ induces an injective map $\bar{\rho}\colon \mathbb{I}(J)/D(R, J) \hookrightarrow D(R/J)$ given by $\bar{\rho}(\bar{\delta}) = \delta'$, as desired. □

Before stating the main theorem, we will need some notation. Let R be the normal affine semigroup ring defined by a matrix A. When referring to an arbitrary face or facet of A, we will use τ or σ, respectively. Every face of A can be expressed as an intersection of facets. Recall the correspondence between the $\mathbb{Z}^d$-graded primes of R and the faces of A. A face τ of A corresponds to the prime ideal $P_\tau := \langle t^{\boldsymbol{m}} \mid \boldsymbol{m} \in \mathbb{N}A \setminus \mathbb{N}\tau\rangle$.

Consider the $\mathbb{Z}^d$-graded radical monomial ideal $J := \bigcap_{i=1}^r P_{\tau_i}$, with the face $\tau_i := \bigcap_{j=1}^{k_i} \sigma_{i,j}$ for facets $\sigma_{i,j}$ of A. For a fixed facet σ of A, set

$$H_{\sigma,\boldsymbol{m}}(\theta) := F_\sigma(\theta) + F_\sigma(\boldsymbol{m}). \tag{4.2}$$

When referring to a facet $\sigma_{i,j}$ from the definition of J, we will replace σ in F_σ and $H_{\sigma,\boldsymbol{m}}(\theta)$ with i, j, writing instead $F_{i,j}$ and $H_{i,j,\boldsymbol{m}}$. Lastly, for an integer k, let $[k]$ denote the set $\{1, 2, \dots, k\}$, and let $\boldsymbol{j} = (j_1, j_2, \dots, j_r) \in K_J := [k_1] \times [k_2] \times \cdots \times [k_r]$ denote an r-tuple of integers in the allowable range with respect to the ideal J.

Finally whenever we have a product L of linear factors, let $\operatorname{rad}(L)$ denote the (monic) generator of the radical of the ideal generated by L; in other words, $\operatorname{rad}(L)$ is a product of distinct linear polynomials in θ.

Theorem 4.3. *Let R be a normal affine semigroup ring defined by the $d \times n$ matrix A, and let $J = \bigcap_{i=1}^r P_{\tau_i}$ be the radical monomial ideal corresponding to the faces $\tau_i = \bigcap_{j=1}^{k_i} \sigma_{ij}$. Let $G_{\boldsymbol{m}}(\theta)$ and $H_{i,j,\boldsymbol{m}}(\theta)$ be as in* (3.1) *and* (4.2), *respectively. Then for $\boldsymbol{m} \in \mathbb{Z}^d$,*

$$\left[\frac{\mathbb{I}(J)}{D(R, J)}\right]_{\boldsymbol{m}} = t^{\boldsymbol{m}} \cdot \frac{\bigl\langle G_{\boldsymbol{m}}(\theta)\cdot \operatorname{rad}\bigl(\prod_{i=1}^r \prod_{F_{i,j}(\boldsymbol{m})<0} H_{i,j,\boldsymbol{m}}(\theta)\bigr) \mid \boldsymbol{j} \in K_J\bigr\rangle}{\bigl\langle G_{\boldsymbol{m}}(\theta)\cdot \operatorname{rad}\bigl(\prod_{i=1}^r \prod_{F_{i,j}(\boldsymbol{m})\le 0} H_{i,j,\boldsymbol{m}}(\theta)\bigr) \mid \boldsymbol{j} \in K_J\bigr\rangle},$$

which is precisely the contribution from $D(R)$ within the $\boldsymbol{m}$-th graded piece of $D(R/J)$ induced by the operators in $D(R)$.

To clarify notation, we will consider an example before proceeding with the proof of Theorem 4.3.

Example 4.4. Let $R = \Bbbk[x_1, x_2, x_3]$ and $J = \langle x_1x_2, x_1x_3\rangle$, in which case $r = 2$, $k_1 = 1$, $k_2 = 2$, and $K_J = \{1\} \times \{1, 2\}$. The prime decomposition of J is $J = \langle x_1\rangle \cap \langle x_2, x_3\rangle = P_{\sigma_{1,1}} \cap P_{\sigma_{2,1}\cap\sigma_{2,2}}$, and an element of K_J corresponds to a choice of one prime from the set $\{\langle x_1\rangle\}$ and one prime from the set $\{\langle x_2\rangle, \langle x_3\rangle\}$.

In the notation of Theorem 4.3,

$$\mathbb{I}(J)_{\boldsymbol{m}} = x^{\boldsymbol{m}} \cdot \left\langle G_{\boldsymbol{m}}(\theta) \prod_{\substack{F_{1,1}(\boldsymbol{m})<0,\\ F_{2,1}(\boldsymbol{m})<0}} H_{1,1,\boldsymbol{m}}(\theta) H_{2,1,\boldsymbol{m}}(\theta),\, G_{\boldsymbol{m}}(\theta) \prod_{\substack{F_{1,1}(\boldsymbol{m})<0,\\ F_{2,2}(\boldsymbol{m})<0}} H_{1,1,\boldsymbol{m}}(\theta) H_{2,2,\boldsymbol{m}}(\theta) \right\rangle.$$

It is the presence of $H_{1,1,\boldsymbol{m}}(\theta)$ as a factor of each generator f of $\mathbb{I}(J)_{\boldsymbol{m}}$ that guarantees that whenever $x^{\boldsymbol{m}'} \in J$, we have $f * x^{\boldsymbol{m}'} \in \langle x_1 \rangle$. Similarly, it is the presence of either $H_{2,1,\boldsymbol{m}}(\theta)$ or $H_{2,2,\boldsymbol{m}}(\theta)$ that guarantees that $f * x^{\boldsymbol{m}'} \in \langle x_2, x_3 \rangle$. Similarly, we have

$$D(R, J)_{\boldsymbol{m}} = x^{\boldsymbol{m}} \cdot \left\langle G_{\boldsymbol{m}}(\theta) \prod_{\substack{F_{1,1}(\boldsymbol{m})\le 0,\\ F_{2,1}(\boldsymbol{m})\le 0}} H_{1,1,\boldsymbol{m}}(\theta) H_{2,1,\boldsymbol{m}}(\theta),\, G_{\boldsymbol{m}}(\theta) \prod_{\substack{F_{1,1}(\boldsymbol{m})\le 0,\\ F_{2,2}(\boldsymbol{m})\le 0}} H_{1,1,\boldsymbol{m}}(\theta) H_{2,2,\boldsymbol{m}}(\theta) \right\rangle.$$

As in the general formula, the difference between the computations of $\mathbb{I}(J)_{\boldsymbol{m}}$ and $D(R, J)_{\boldsymbol{m}}$ is seen in the difference between the strict inequalities $F_{i,j}(\boldsymbol{m}) < 0$ of $\mathbb{I}(J)_{\boldsymbol{m}}$ and the weak inequalities $F_{i,j}(\boldsymbol{m}) \le 0$ of $D(R, J)_{\boldsymbol{m}}$. Finally, the differential operators in $D(R)_{\boldsymbol{m}}$ which induce maps on R/J are precisely those in $\mathbb{I}(J)_{\boldsymbol{m}}$ making $\mathbb{I}(J)_{\boldsymbol{m}}/D(R, J)_{\boldsymbol{m}}$ a submodule of $D(R/J)_{\boldsymbol{m}}$, which respects the grading on numerator and denominator. □

Proof of Theorem 4.3. Observe that

$$t^{\boldsymbol{m}} \cdot \mathbb{I}(\Omega(\boldsymbol{m})) = t^{\boldsymbol{m}} \cdot \left\langle \prod_{F_\sigma(\boldsymbol{m})<0} \prod_{i=0}^{-F_\sigma(\boldsymbol{m})-1} (F_\sigma(\theta) - i) \right\rangle.$$

Now for each $\boldsymbol{m}' \in \mathbb{Z}^d$, $t^{\boldsymbol{m}'} \in J$ if and only if $F_\sigma(\boldsymbol{m}') > 0$ for all facets σ of A. Because $\theta_i * t^{\boldsymbol{m}'} = m_i' t^{\boldsymbol{m}'}$ for each i,

$$\left[t^{\boldsymbol{m}} \cdot \prod_{F_\sigma(\boldsymbol{m})<0} \prod_{i=0}^{-F_\sigma(\boldsymbol{m})-1} (F_\sigma(\theta) - i) \right] * t^{\boldsymbol{m}'} = t^{\boldsymbol{m}+\boldsymbol{m}'} \cdot \prod_{F_\sigma(\boldsymbol{m})<0} \prod_{i=0}^{-F_\sigma(\boldsymbol{m})-1} (F_\sigma(\boldsymbol{m}') - i).$$

Hence,

$$[t^{\boldsymbol{m}} \cdot G_{\boldsymbol{m}}(\theta)] * t^{\boldsymbol{m}'} = t^{\boldsymbol{m}+\boldsymbol{m}'} \cdot G_{\boldsymbol{m}}(\boldsymbol{m}') \in J$$

exactly when at least one of the following two conditions is satisfied:

(1) $t^{\boldsymbol{m}+\boldsymbol{m}'} \in J$.

(2) $G_{\boldsymbol{m}}(\boldsymbol{m}') = 0$.

First, for each $\boldsymbol{m}'$ such that $t^{\boldsymbol{m}'} \in J$, we must have that $F_\sigma(\boldsymbol{m}') + F_\sigma(\boldsymbol{m}) = F_\sigma(\boldsymbol{m}+\boldsymbol{m}') \ge 0$ for all facets σ and that, for each $i \in [r]$, there exists some $j \in [k_i]$ so that $F_{i,j}(\boldsymbol{m}+\boldsymbol{m}') > 0$. We consider two conditions:

(i) $\boldsymbol{m}$ satisfies $F_{\sigma'}(\boldsymbol{m}) < 0$ for some facet σ'.

(ii) $F_{\sigma'}(\boldsymbol{m}) \ge 0$ for all facets σ' and there exists some $i \in [r]$ for which $F_{i,j}(\boldsymbol{m}+\boldsymbol{m}') = 0$ for all $j \in [k_i]$.

We claim that if either Condition (i) or Condition (ii) holds, then there exists some $\boldsymbol{m}'$ with $t^{\boldsymbol{m}'} \in J$ and $G_{\boldsymbol{m}}(\boldsymbol{m}') \neq 0$. If Condition (i) holds, then $G_{\boldsymbol{m}}(\boldsymbol{m}') \neq 0$ for exactly the $\boldsymbol{m}'$ that satisfy $F_{\sigma'}(\boldsymbol{m}) = -F_{\sigma'}(\boldsymbol{m}')$ and $F_\sigma(\boldsymbol{m}') \gg F_\sigma(\boldsymbol{m})$ for all facets $\sigma \neq \sigma'$. However, for all such $\boldsymbol{m}$, $H_{\sigma',\boldsymbol{m}}(\boldsymbol{m}')G_{\boldsymbol{m}}(\boldsymbol{m}') = 0$ for all $\boldsymbol{m}'$ that satisfy $F_{\sigma'}(\boldsymbol{m}) = -F_{\sigma'}(\boldsymbol{m}')$ and $F_\sigma(\boldsymbol{m}') \gg -F_\sigma(\boldsymbol{m})$ for all facets $\sigma \neq \sigma'$.

If Condition (ii) holds, then $G_{\boldsymbol{m}}(\boldsymbol{m}')$ will fail to vanish exactly for the $\boldsymbol{m}'$ for which there exists an i with $F_{i,j}(\boldsymbol{m}) = -F_{i,j}(\boldsymbol{m}')$ for all $j \in [k_i]$ and $F_{i',j}(\boldsymbol{m}') \gg F_{i',j}(\boldsymbol{m})$ for all facets $\sigma_{i',j}$ with $i \neq i'$. However, for all such $\boldsymbol{m}$, $H_{i,j,\boldsymbol{m}}(\boldsymbol{m}')G_{\boldsymbol{m}}(\boldsymbol{m}') = 0$ for all $i \in [r]$, $j \in [k_i]$, and $\boldsymbol{m}'$ satisfying the hypotheses that $F_{i,j}(\boldsymbol{m}) = -F_{i,j}(\boldsymbol{m}')$ for all $j \in [k_i]$ and $F_{i',j}(\boldsymbol{m}') \gg F_{i',j}(\boldsymbol{m})$ for all facets $\sigma_{i',j}$ for $i \neq i'$.

Combining these calculations,

$$\mathbb{I}(J)_{\boldsymbol{m}} = t^{\boldsymbol{m}} \cdot \left\langle G_{\boldsymbol{m}}(\theta) \cdot \operatorname{rad}\left(\prod_{i=1}^{r} \prod_{F_{i,j}(\boldsymbol{m})<0} H_{i,j,\boldsymbol{m}}(\theta) \right) \,\middle|\, \boldsymbol{j} \in K_J \right\rangle.$$

We compute $D(R, J)$ similarly. Now we begin with an arbitrary $\boldsymbol{m} \in \mathbb{Z}^d$ and $t^{\boldsymbol{m}'} \in R$. The only distinction between this calculation and the previous calculations for $\mathbb{I}(J)$ is that $\boldsymbol{m}'$ is now taken from a larger set. Namely, $F_\sigma(\boldsymbol{m}')$ can now be 0 as well. Then, Condition (1) holds when $F_\sigma(\boldsymbol{m}+\boldsymbol{m}') > 0$ whenever $t^{\boldsymbol{m}'} \in R$, a condition automatically satisfied when $F_\sigma(\boldsymbol{m}) > 0$. Again, in order to address Condition (2), we consider the two Conditions (i) and (ii), above. We note that, by the same argument used to compute $\mathbb{I}(J)$, if either Condition (i) or Condition (ii) holds, then there exists some vector $\boldsymbol{j} \in K_J$ for which

$$G_{\boldsymbol{m}}(\boldsymbol{m}') \cdot \operatorname{rad}\left(\prod_{i=1}^{r} \prod_{F_{i,j(\boldsymbol{m})}<0} H_{i,j,\boldsymbol{m}}(\boldsymbol{m}') \right)$$

is nonzero for some $\boldsymbol{m}'$ with $t^{\boldsymbol{m}'} \in R$. Hence

$$D(R, J)_{\boldsymbol{m}} = t^{\boldsymbol{m}} \cdot \left\langle G_{\boldsymbol{m}}(\theta) \cdot \operatorname{rad}\left(\prod_{i=1}^{r} \prod_{F_{i,j}(\boldsymbol{m})\leq 0} H_{i,j,\boldsymbol{m}}(\theta) \right) \,\middle|\, \boldsymbol{j} \in K_J \right\rangle.$$

It now follows from Proposition 4.1 that

$$\frac{\mathbb{I}(J)_{\boldsymbol{m}}}{D(R, J)_{\boldsymbol{m}}} = t^{\boldsymbol{m}} \cdot \frac{\left\langle G_{\boldsymbol{m}}(\theta) \cdot \operatorname{rad}\left(\prod_{i=1}^{r} \prod_{F_{i,j}(\boldsymbol{m})<0} H_{i,j,\boldsymbol{m}}(\theta)\right) \mid \boldsymbol{j} \in K_J \right\rangle}{\left\langle G_{\boldsymbol{m}}(\theta) \cdot \operatorname{rad}\left(\prod_{i=1}^{r} \prod_{F_{i,j}(\boldsymbol{m})\leq 0} H_{i,j,\boldsymbol{m}}(\theta)\right) \mid \boldsymbol{j} \in K_J \right\rangle},$$

as desired. □

5. Characterizing Gorenstein rings via differential operators

In this section, we compare $D(R, J)$ and $JD(R)$ when R is a normal affine semigroup ring. We find that the equality $D(R, J) = JD(R)$ holds if J is a principal monomial ideal (see Proposition 5.5). We then restrict to the special case of $J = \omega_R$, the intersection of all graded height one prime ideals of R. That is, ω_R is the defining ideal of the union of all facets. We choose the notation ω_R for this ideal, sometimes called the interior ideal, because it is a canonical module for R; see, for example, [8, Proposition 8.2.9].

Recall that R is Gorenstein if and only if ω_R is principal; see, for example, [5, Theorem 3.3.7]. The main result of the section is as follows:

Theorem 5.1. *Let R be a normal affine semigroup ring, and let ω_R be the intersection of all graded height one prime ideals of R. Then R is Gorenstein if and only if $\omega_R D(R) = D(R, \omega_R)$.*

Before giving the proof of Theorem 5.1, we include two examples.

Example 5.2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad R = \Bbbk[\mathbb{N}A] = \Bbbk[s, st, st^2] \quad \text{and} \quad J = \omega_R = \langle st \rangle.$$

Note that R is Gorenstein since ω_R is principal. We will see in this case that $\omega_R D(R) = D(R, \omega_R)$.

We denote the faces of A by $\sigma_{1,1}$ and $\sigma_{2,1}$, and so that the primitive integral support functions are

$$F_{1,1}(\theta) = \theta_2 \quad \text{and} \quad F_{2,1}(\theta) = 2\theta_1 - \theta_2.$$

Then recall that

$$H_{1,1,\boldsymbol{m}}(\theta) := F_{1,1}(\theta) + F_{1,1}(\boldsymbol{m}) \quad \text{and} \quad H_{2,1,\boldsymbol{m}}(\theta) := F_{2,1}(\theta) + F_{2,1}(\boldsymbol{m})$$

and

$$G_{\boldsymbol{m}}(\theta) := \prod_{i=0}^{-F_{1,1}(\boldsymbol{m})-1} (F_{1,1}(\theta) - i) \prod_{j=0}^{-F_{2,1}(\boldsymbol{m})-1} (F_{2,1}(\theta) - j).$$

By (the proof of) Theorem 4.3,

$$\begin{aligned} D(R, \omega_R)_{\boldsymbol{m}} &= s^{m_1} t^{m_2} \left\langle G_{\boldsymbol{m}}(\theta) \cdot \left(\prod_{i=1}^{2} \prod_{F_{i,1}(\boldsymbol{m}) \leq 0} H_{i,1,\boldsymbol{m}}(\theta) \right) \right\rangle \\ &= s^{m_1} t^{m_2} \left\langle \prod_{i=0}^{-F_{1,1}(\boldsymbol{m})} (F_{1,1}(\theta) - i) \prod_{j=0}^{-F_{2,1}(\boldsymbol{m})} (F_{2,1}(\theta) - j) \right\rangle \\ &= s^{m_1} t^{m_2} \left\langle \prod_{i=0}^{-m_2} (\theta_2 - i) \prod_{j=0}^{-2m_1 + m_2} (2\theta_1 - \theta_2 - j) \right\rangle, \end{aligned}$$

where, by convention, an empty product is 1 and $\langle - \rangle$ denotes an ideal in $\Bbbk[\theta_1, \theta_2]$. Multiplying the expression for $D(R)$ given in Theorem 3.2 by ω_R, we obtain

$$\omega_R D(R) = \langle st \rangle \cdot \bigoplus_{m \in \mathbb{Z}^2} s^{m_1} t^{m_2} \cdot \langle G_{\boldsymbol{m}}(\theta) \rangle.$$

We define $\mathbf{1} := (1, 1)$. Then we have

$$\begin{aligned}(\omega_R D(R))_{\boldsymbol{m}} &= s^{m_1}t^{m_2} \cdot \langle G_{\boldsymbol{m}-\mathbf{1}}(\theta)\rangle \\ &= s^{m_1}t^{m_2} \cdot \left\langle \prod_{i=0}^{-F_{1,1}(\boldsymbol{m}-\mathbf{1})-1} (F_{1,1}(\theta)-i) \prod_{i=0}^{-F_{2,1}(\boldsymbol{m}-\mathbf{1})-1} (F_{2,1}(\theta)-i)\right\rangle \\ &= s^{m_1}t^{m_2} \cdot \left\langle \prod_{i=0}^{-F_{1,1}(\boldsymbol{m})+F_{1,1}(\mathbf{1})-1} (F_{1,1}(\theta)-i) \prod_{i=0}^{-F_{2,1}(\boldsymbol{m})+F_{2,1}(\mathbf{1})-1} (F_{2,1}(\theta)-i)\right\rangle \\ &= s^{m_1}t^{m_2} \cdot \left\langle \prod_{i=0}^{-F_{1,1}(\boldsymbol{m})} (F_{1,1}(\theta)-i) \prod_{i=0}^{-F_{2,1}(\boldsymbol{m})} (F_{2,1}(\theta)-i)\right\rangle \\ &= s^{m_1}t^{m_2}\left\langle \prod_{i=0}^{-m_2}(\theta_2-i) \prod_{j=0}^{-2m_1+m_2} (2\theta_1-\theta_2-j)\right\rangle.\end{aligned}$$

It follows that $\omega_R D(R) = D(R, \omega_R)$. □

Example 5.3. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}, \quad R = \Bbbk[\mathbb{N}A] = \Bbbk[s, st, st^2, st^3] \quad \text{and} \quad J = \omega_R = \langle st, st^2\rangle.$$

Note that R is not Gorenstein. We denote the faces of A by $\sigma_{1,1}$ and $\sigma_{2,1}$, and so that the primitive integral support functions are

$$F_{1,1}(\theta) = \theta_2 \quad \text{and} \quad F_{2,1}(\theta) = 3\theta_1 - \theta_2.$$

Then for $\boldsymbol{m} = (m_1, m_2)$, we have

$$H_{1,1,\boldsymbol{m}}(\theta) = F_{1,1}(\theta) + F_{1,1}(\boldsymbol{m}) \quad \text{and} \quad H_{2,1,\boldsymbol{m}}(\theta) = F_{2,1}(\theta) + F_{2,1}(\boldsymbol{m})$$

and

$$G_{\boldsymbol{m}}(\theta) = \prod_{i=0}^{-F_{1,1}(\boldsymbol{m})-1} (F_{1,1}(\theta)-i) \prod_{i=0}^{-F_{2,1}(\boldsymbol{m})-1} (F_{2,1}(\theta)-i).$$

Multiplying the expression for $D(R)$ given in Theorem 3.2 by ω_R, we obtain

$$\omega_R D(R) = \langle st, st^2\rangle \cdot \bigoplus_{m\in\mathbb{Z}^2} s^{m_1}t^{m_2} \cdot \langle G_{\boldsymbol{m}}(\theta)\rangle,$$

so that

$$(\omega_R D(R))_{\boldsymbol{m}} = s^{m_1}t^{m_2} \cdot \langle G_{\boldsymbol{m}-(1,1)}(\theta), G_{\boldsymbol{m}-(1,2)}(\theta)\rangle,$$

where

$$\begin{aligned} G_{\boldsymbol{m}-(1,1)}(\theta) &= \prod_{j=0}^{-F_{1,1}(\boldsymbol{m}-(1,1))-1} (F_{1,1}(\theta)-j) \prod_{j=0}^{-F_{2,1}(\boldsymbol{m}-(1,1))-1} (F_{2,1}(\theta)-j) \\ &= \prod_{j=0}^{-F_{1,1}(\boldsymbol{m})} (F_{1,1}(\theta)-j) \prod_{j=0}^{-F_{2,1}(\boldsymbol{m})+1} (F_{2,1}(\theta)-j) \\ &= \prod_{j=0}^{-m_2} (\theta_2-j) \prod_{j=0}^{-3m_1+m_2+1} (3\theta_2-\theta_1-j) \end{aligned}$$

when $F_{1,1}(\boldsymbol{m}) = m_2 \leq 0$ and $F_{2,1}(\boldsymbol{m}) = 3m_1 - m_2 \leq 1$ and

$$\begin{aligned} G_{\boldsymbol{m}-(1,2)}(\theta) &= \prod_{j=0}^{-F_{1,1}(\boldsymbol{m}-(1,2))-1} (F_{1,1}(\theta)-j) \prod_{j=0}^{-F_{2,1}(\boldsymbol{m}-(1,2))-1} (F_{2,1}(\theta)-j) \\ &= \prod_{j=0}^{-F_{1,1}(\boldsymbol{m})+1} (F_{1,1}(\theta)-j) \prod_{j=0}^{-F_{2,1}(\boldsymbol{m})} (F_{2,1}(\theta)-j) \\ &= \prod_{j=0}^{-m_2+1} (\theta_2-j) \prod_{j=0}^{-3m_1+m_2} (3\theta_1-\theta_2-j) \end{aligned}$$

when $F_{1,1}(\boldsymbol{m}) = m_2 \leq 1$ and $F_{2,1}(\boldsymbol{m}) = 3m_1 - m_2 \leq 0$.

On the other hand, by Theorem 4.3 and a computation similar to the one in Example 5.2,

$$\begin{aligned} D(R,\omega_R)_{\boldsymbol{m}} &= s^{m_1}t^{m_2} \cdot \left\langle \prod_{i=0}^{-F_{1,1}(\boldsymbol{m})} (F_{1,1}(\theta)-i) \prod_{i=0}^{-F_{2,1}(\boldsymbol{m})} (F_{2,1}(\theta)-i) \right\rangle \\ &= s^{m_1}t^{m_2} \cdot \left\langle \prod_{i=0}^{-m_2} (\theta_2-i) \prod_{i=0}^{-3m_1+m_2} (3\theta_1-\theta_2-i) \right\rangle \end{aligned}$$

when $F_{1,1}(\boldsymbol{m}) = m_2 \leq 0$ and $F_{2,1}(\boldsymbol{m}) = 3m_1 - m_2 \leq 0$.

To see $\omega_R D(R)$ and $D(R,\omega_R)$ are different, consider any degree, say $\boldsymbol{m} = (-1,-1)$. Note that the degree $\boldsymbol{m}$ piece of $D(R,\omega_R)$ is generated by one element, namely

$$s^{-1}t^{-1} \prod_{i=0}^{1} (F_{1,1}(\theta)-i) \prod_{i=0}^{2} (F_{2,1}(\theta)-i)$$

On the other hand, the degree $\boldsymbol{m}$ piece of $\omega_R D(R)$ is generated by two elements; it is generated in $\Bbbk[\theta_1,\theta_2]$ by

$$s^{-1}t^{-1} \prod_{i=0}^{1} (F_{1,1}(\theta)-i) \prod_{i=0}^{3} (F_{2,1}(\theta)-i) \quad \text{and} \quad s^{-1}t^{-1} \prod_{i=0}^{2} (F_{1,1}(\theta)-i) \prod_{i=0}^{2} (F_{2,1}(\theta)-i)$$

In particular,

$$s^{-1}t^{-1}\prod_{i=0}^{1}(F_{1,1}(\theta)-i)\prod_{i=0}^{2}(F_{2,1}(\theta)-i)\in D(R,\omega_R)$$

but

$$s^{-1}t^{-1}\prod_{i=0}^{1}(F_{1,1}(\theta)-i)\prod_{i=0}^{2}(F_{2,1}(\theta)-i)\notin \omega_R D(R). \qquad \square$$

In order to prove Theorem 5.1, we will need some preliminary results.

Lemma 5.4 [4, Theorem 6.33]. *Let R be a normal affine semigroup ring of dimension d, and let $F_{\sigma_1},\dots,F_{\sigma_r}$ be the support functions of the facets σ_i of the semigroup defining R. Let ω_R be the intersection of all graded height one prime ideals of R. Then ω_R is a principal ideal if and only if there exists $t^{\boldsymbol{c}}\in\omega$ for some $\boldsymbol{c}\in\mathbb{Z}^d$ such that $F_{\sigma_i}(\boldsymbol{c})=1$ for all $i=1,\dots,k$.*

We first show that principal monomial ideals J behave well (i.e., $JD(R)=D(R,J)$) for affine normal semigroup rings.

Proposition 5.5. *Let R be a normal affine semigroup ring and let $J=\langle t^{\boldsymbol{c}}\rangle$, then*

$$JD(R)=D(R,J).$$

Proof. Since $JD(R)\subseteq D(R,J)$, it suffices to prove that $JD(R)_{\boldsymbol{m}}=D(R,J)_{\boldsymbol{m}}$ for all multidegrees $\boldsymbol{m}$. We do so by showing that they are both cyclic and are generated by the same differential operator.

Since $J=\langle t^{\boldsymbol{c}}\rangle$, by the expression obtained in Theorem 3.2, we have

$$JD(R)_{\boldsymbol{m}}=t^{\boldsymbol{m}}D(R)_{\boldsymbol{m}-\boldsymbol{c}}=t^{\boldsymbol{m}}\langle G_{\boldsymbol{m}-\boldsymbol{c}}(\theta)\rangle.$$

Since F_i is linear, we have

$$F_i(-(\boldsymbol{m}-\boldsymbol{c}))-F_i(\boldsymbol{c})=-F_i(\boldsymbol{m})+F_i(\boldsymbol{c})-F_i(\boldsymbol{c})=-F_i(\boldsymbol{m}).$$

Hence the single generator for the graded piece $JD(R)_{\boldsymbol{m}}$ can be viewed as

$$t^{\boldsymbol{m}}\cdot\left\langle G_{\boldsymbol{m}}(\theta)\prod_{i=1}^{r}H_{i,\boldsymbol{m}}(\theta)\right\rangle,$$

which is exactly the generator of $D(R,J)_{\boldsymbol{m}}$, completing the proof. $\square$

Now we are ready to prove the main theorem of the section: Theorem 5.1. As above, we will write F_i and $H_{i,\boldsymbol{m}}$ for F_{σ_i} and $H_{\sigma_i,\boldsymbol{m}}$, respectively.

Proof of Theorem 5.1. Suppose first that R is Gorenstein. Then ω_R is principal, say $\omega_R=\langle t^{\boldsymbol{c}}\rangle$. Now by Proposition 5.5, we obtain $\omega_R D(R)=D(R,\omega_R)$.

Conversely assume that R is not Gorenstein, that is, ω_R is not principal. Since $D(R,\omega_R)$ and $\omega_R D(R)$ are both multigraded and it is always true that $\omega_R D(R)\subseteq D(R,\omega_R)$, we may prove that they are not equal by identifying their difference at $\boldsymbol{m}=\boldsymbol{0}$.

As mentioned above, $D(R,\omega_R)_{\boldsymbol{m}}$ is generated by a single element $t^{\boldsymbol{m}}\big\langle G_{\boldsymbol{m}}(\theta)\prod_{i=1}^{r}H_{i,\boldsymbol{m}}(\theta)\big\rangle$. Thus

$$D(R,\omega_R)_{\boldsymbol{0}}=\Big\langle\prod_{i=1}^{r}H_{i,\boldsymbol{0}}(\theta)\Big\rangle=\langle F_1(\theta)\cdots F_r(\theta)\rangle.$$

Let ω_R be minimally generated by $t^{\boldsymbol{c}_1},t^{\boldsymbol{c}_2},\ldots,t^{\boldsymbol{c}_h}$. Then

$$(\omega D(R))_{\boldsymbol{m}}=\sum_{j=1}^{h}t^{\boldsymbol{c}_j}t^{\boldsymbol{m}-\boldsymbol{c}_j}\cdot D(R)_{\boldsymbol{m}-\boldsymbol{c}_j}=\sum_{j=1}^{h}t^{\boldsymbol{m}}\cdot\langle G_{\boldsymbol{m}-\boldsymbol{c}_j}(\theta)\rangle=t^{\boldsymbol{m}}\cdot\langle G_{\boldsymbol{m}-\boldsymbol{c}_j}(\theta)\mid j=1,2,\ldots,h\rangle.$$

In particular, when $\boldsymbol{m}=\boldsymbol{0}$,

$$(\omega D(R))_{\boldsymbol{0}}=\langle G_{-\boldsymbol{c}_j}(\theta)\mid j=1,2,\ldots,h\rangle.$$

We know from Lemma 5.4 that some $F_{\sigma_i}(\boldsymbol{c}_j)\neq 1$. Therefore

$$\begin{aligned}(\omega D(R))_{\boldsymbol{0}}&=\sum_{j=1}^{h}\Big\langle\prod_{i=1}^{r}F_i(\theta)(F_i(\theta)-1)\cdots(F_i(\theta)-(F_i(c_j)-1))\Big\rangle &(5.6)\\ &=\langle F_1(\theta)F_2(\theta)\cdots F_r(\theta)\rangle\cap\sum_{j=1}^{h}\Big\langle\prod_{i=1}^{r}(F_i(\theta)-1)\cdots(F_i(\theta)-(F_i(\boldsymbol{c}_j)-1))\Big\rangle &(5.7)\end{aligned}$$

We claim that the ideal

$$\sum_{j=1}^{h}\Big\langle\prod_{i=1}^{r}(F_i(\theta)-1)\cdots(F_i(\theta)-(F_i(\boldsymbol{c}_j)-1))\Big\rangle \tag{5.8}$$

from (5.6) is a proper ideal of $\Bbbk[\theta]$. Once this is established, it will follow from (5.6) that the inclusion

$$(\omega_R D(R))_{\boldsymbol{0}}\subseteq D(R,\omega_R)_{\boldsymbol{0}}=\langle F_1(\theta)\cdots F_r(\theta)\rangle,$$

is proper, which will conclude the proof.

To see that (5.8) is a proper ideal of $\Bbbk[\theta]$, we may assume after reordering that $\boldsymbol{c}_1$ generates a ray of the cone

$$\mathscr{C}:=\mathbb{R}_{\geq 0}\{\boldsymbol{c}_1,\boldsymbol{c}_2,\ldots,\boldsymbol{c}_h\}=\{\boldsymbol{m}\in\mathbb{N}A\mid F_1(\boldsymbol{m})\geq 1,\ldots,F_r(\boldsymbol{m})\geq 1\}\subseteq\mathbb{R}^d$$

over the exponents of the minimal generators of ω_R and, in light of Lemma 5.4, that there are $d-1$ linearly independent primitive integral support functions, which after relabeling can be taken as $F_1,\ldots,F_{d-1}$, such that

$$F_1(\boldsymbol{c}_1)=F_2(\boldsymbol{c}_1)=\cdots=F_{d-1}(\boldsymbol{c}_1)=1\quad\text{and}\quad F_d(\boldsymbol{c}_1)>1.$$

(There may be more i such that $F_i(\boldsymbol{c}_1)=1$, but we need to make use of only a linearly independent set of them, which is necessarily of size $d-1$, since $\boldsymbol{c}_1$ generates a ray of $\mathscr{C}$.)

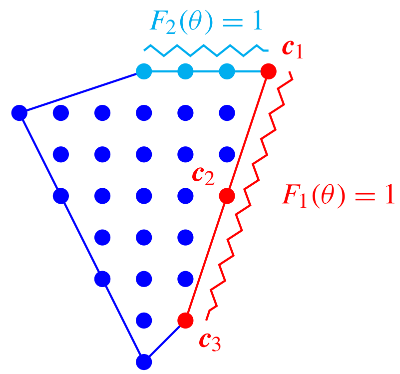

Figure 1. Hypothetical generators of ω_R in dimension 3.

Example 5.9. We pause our proof to give an example illustrating the newly introduced notation. Consider the case $d=3$ and refer to Figure 1 as a visualization. Fix $\boldsymbol{c}_1$ to be a vertex of the convex hull of the $\boldsymbol{c}_i$, or, rather, a generator of a ray of the Newton polyhedron of ω_R. Without loss of generality,

$$F_1(\boldsymbol{c}_1)=1=F_2(\boldsymbol{c}_1) \quad \text{and} \quad F_3(\boldsymbol{c}_1)>1.$$

Further, there exists an r such that

$$F_1(\boldsymbol{c}_i)>1 \quad \text{for all } r+1\le i\le h \quad \text{and} \quad F_1(\boldsymbol{c}_i)=1 \quad \text{for all } 1\le i\le r.$$

Note that in Figure 1, we have chosen $r=3$. Now, since $\boldsymbol{c}_1$ is a vertex, it must be that

$$F_2(\boldsymbol{c}_i)>1 \quad \text{for all } 2\le i\le r.$$

Hence, $\langle F_1(\theta)-1,\, F_2(\theta)-1,\, F_3(\theta)-1\rangle$ is a primary component of (5.8). □

We resume our proof in full generality. Given that $\boldsymbol{c}_1$ is a ray of the cone $\mathscr{C}$ with $F_i(\boldsymbol{c}_1)=1$ for every i with $1\le i\le d-1$, it follows that, for every $j>1$, there is an i_j with $1\le i_j\le d-1$ such that $F_{i_j}(\boldsymbol{c}_j)>1$. Now set $i_1=d$ because $F_d(\boldsymbol{c}_1)>1$. Then since $1\le i_j\le d$ for all j and the hyperplanes defined by $\{F_i(\theta)=0\}_{i=1}^{d}$ are necessarily nonparallel, the ideal

$$\langle F_{i_1}(\theta)-1,\, F_{i_2}(\theta)-1,\, \dots,\, F_{i_d}(\theta)-1\rangle$$

is a proper, primary (in fact, prime) component of (5.8). Thus (5.8) is a proper ideal of $\Bbbk[\theta]$, as desired to establish the final needed claim. □

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Bézoutians and the $\mathbb{A}^1$-degree

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We prove that both the local and global $\mathbb{A}^1$-degree of an endomorphism of affine space can be computed in terms of the multivariate Bézoutian. In particular, we show that the Bézoutian bilinear form, the Scheja–Storch form, and the $\mathbb{A}^1$-degree for complete intersections are isomorphic. Our global theorem generalizes Cazanave's theorem in the univariate case, and our local theorem generalizes Kass–Wickelgren's theorem on EKL forms and the local degree. This result provides an algebraic formula for local and global degrees in motivic homotopy theory.

1. Introduction

Morel's $\mathbb{A}^1$-Brouwer degree [25] assigns a bilinear form-valued invariant to a given endomorphism of affine space. However, Morel's construction is not explicit. In order to make computations and applications, we would like algebraic formulas for the $\mathbb{A}^1$-degree. Such formulas were constructed by Cazanave for the global $\mathbb{A}^1$-degree in dimension 1 [9], Kass and Wickelgren for the local $\mathbb{A}^1$-degree at rational points and étale points [16], and Brazelton, Burklund, McKean, Montoro and Opie for the local $\mathbb{A}^1$-degree at separable points [7]. In this paper, we give a general algebraic formula for the $\mathbb{A}^1$-degree in both the global and local cases. In the global case, we remove Cazanave's dimension restriction, while in the local case, we remove previous restrictions on the residue field of the point at which the local $\mathbb{A}^1$-degree is taken.

Let k be a field, and let $f = (f_1, \dots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of affine space with isolated zeros, so that $Q := k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ is a complete intersection. We now recall the definition of the Bézoutian of f, as well as a special bilinear form determined by the Bézoutian. Introduce new variables $X := (X_1, \dots, X_n)$ and $Y := (Y_1, \dots, Y_n)$. For each $1 \le i, j \le n$, define the quantity

$$\Delta_{ij} := \frac{f_i(Y_1, \dots, Y_{j-1}, X_j, \dots, X_n) - f_i(Y_1, \dots, Y_j, X_{j+1}, \dots, X_n)}{X_j - Y_j}.$$

Definition 1.1. The *Bézoutian* of f is the image $\text{Béz}(f_1, \dots, f_n)$ of the determinant $\det(\Delta_{ij})$ in $k[X, Y]/(f(X), f(Y))$. Given a basis $\{a_1, \dots, a_m\}$ of Q as a k-vector space, there exist scalars $B_{i,j}$ for which

$$\text{Béz}(f_1, \dots, f_n) = \sum_{i,j=1}^{m} B_{i,j} a_i(X) a_j(Y).$$

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We define the *Bézoutian form* of f to be the class β_f in the Grothendieck–Witt ring $\mathrm{GW}(k)$ determined by the bilinear form $Q \times Q \to k$ with Gram matrix $(B_{i,j})$.

For any isolated zero of f corresponding to a maximal ideal $\mathfrak{m}$, there is an analogous bilinear form $\beta_{f,\mathfrak{m}}$ on the local algebra $Q_{\mathfrak{m}}$. We refer to $\beta_{f,\mathfrak{m}}$ as the *local Bézoutian form* of f at $\mathfrak{m}$. We will demonstrate that both β_f and $\beta_{f,\mathfrak{m}}$ yield well-defined classes in $\mathrm{GW}(k)$. Our main theorem is that the Bézoutian form of f agrees with the $\mathbb{A}^1$-degree in both the local and global contexts.

Theorem 1.2. *Let* $\operatorname{char} k \neq 2$. *Let* $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ *have an isolated zero at a closed point* $\mathfrak{m}$. *Then* $\beta_{f,\mathfrak{m}}$ *is isomorphic to the local* $\mathbb{A}^1$*-degree of* f *at* $\mathfrak{m}$. *If we further assume that all the zeros of* f *are isolated*, *then* β_f *is isomorphic to the global* $\mathbb{A}^1$*-degree of* f.

Because the Bézoutian form can be explicitly computed using commutative algebraic tools, Theorem 1.2 provides a tractable formula for $\mathbb{A}^1$-degrees and Euler classes in motivic homotopy theory. Using the Bézoutian formula for the $\mathbb{A}^1$-degree, we are able to deduce several computational rules for the degree. We also provide a Sage implementation for calculating local and global $\mathbb{A}^1$-degrees via the Bézoutian at [8].

Remark 1.3. The key contribution of this article is computability. Building on the work of Kass and Wickelgren [16], Bachmann and Wickelgren [2] show that the $\mathbb{A}^1$-degree agrees with the Scheja–Storch form as elements of $\mathrm{KO}^0(k)$. In Theorem 5.1, we show how this immediately implies that the $\mathbb{A}^1$-degree and Scheja–Storch form determine the same element of $\mathrm{GW}(k)$. Scheja and Storch [30] showed that their form is a Bézoutian bilinear form (in the sense of Definition 3.8; see also Lemma 4.4 and Remark 4.8), which was further explored by Becker, Cardinal, Roy and Szafraniec [4]. Putting these results together shows that the isomorphism class of the Bézoutian bilinear form is the $\mathbb{A}^1$-degree.

In dimension 1, Cazanave [9] gives a simple formula for computing the $\mathbb{A}^1$-degree as a Bézoutian bilinear form in the global setting. However, it is not immediately clear how to adapt this to higher dimensions or the local setting. Becker, Cardinal, Roy and Szafraniec show how to compute Bézoutian bilinear forms in terms of "dualizing forms," but this method is computationally analogous to using the Eisenbud–Khimshiashvili–Levine form to compute the $\mathbb{A}^1$-degree [16]. In the proof of Theorem 1.2 (found in Section 5), we show that our two notions of Bézoutian bilinear forms (Definitions 1.1 and 3.8) agree up to isomorphism. Since Definition 1.1 is the desired generalization of Cazanave's formula, this enables us to calculate $\mathbb{A}^1$-degrees in full generality.

1A. ***Outline.*** Before proving Theorem 1.2, we recall some classical results on Bézoutians (following [4]) in Section 3, as well as the work of Scheja and Storch on residue pairings [30] in Section 4. We then discuss a local decomposition procedure for the Scheja–Storch form and show that the global Scheja–Storch form is isomorphic to the Bézoutian form in Section 4A. In Section 5, we complete the proof of Theorem 1.2 by applying the work of Kass and Wickelgren [16] and Bachmann and Wickelgren [2] on the local $\mathbb{A}^1$-degree and the Scheja–Storch form. Using Theorem 1.2, we give an algorithm for computing the local and global $\mathbb{A}^1$-degree at the end of Section 5A, available at [8]. In Section 6, we establish some basic properties for computing degrees. In Section 7, we provide a step-by-step illustration of our ideas by working through some explicit examples. Finally, we implement our code to compute some examples of $\mathbb{A}^1$-Euler characteristics of Grassmannians in Section 8. We check our computations by proving a general formula for the $\mathbb{A}^1$-Euler characteristic of a Grassmannian in Theorem 8.4. The $\mathbb{A}^1$-Euler characteristic of Grassmannians is essentially a folklore result that follows from the work of Hoyois, Levine, and Bachmann and Wickelgren.

1B. ***Background.*** Let $\mathrm{GW}(k)$ denote the Grothendieck–Witt group of isomorphism classes of symmetric, nondegenerate bilinear forms over a field k. Morel's $\mathbb{A}^1$-Brouwer degree [25, Corollary 1.24]

$$\deg : [\mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \mathbb{P}^n_k/\mathbb{P}^{n-1}_k]_{\mathbb{A}^1} \to \mathrm{GW}(k),$$

which is a group isomorphism (in fact, a ring isomorphism [24, Lemma 6.3.8]) for $n \geq 2$, demonstrates that bilinear forms play a critical role in motivic homotopy theory. However, Morel's $\mathbb{A}^1$-degree is nonconstructive. Kass and Wickelgren addressed this problem by expressing the $\mathbb{A}^1$-degree as a sum of local degrees [17, Lemma 19] and providing an explicit formula (building on the work of Eisenbud and Levine [11] and Khimshiashvili [13]) for the local $\mathbb{A}^1$-degree [16] at rational points and étale points. This explicit formula can also be used to compute the local $\mathbb{A}^1$-degree at points with separable residue field by [7]. Together, these results allow one to compute the global $\mathbb{A}^1$-degree of a morphism $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with only isolated zeros by computing the local $\mathbb{A}^1$-degrees of f over its zero locus, so long as the residue field of each point in the zero locus is separable over the base field. In the local case, Theorem 1.2 gives a commutative algebraic formula for the local $\mathbb{A}^1$-degree at any closed point.

Cazanave showed that the Bézoutian gives a formula for the global $\mathbb{A}^1$-degree of any endomorphism of $\mathbb{P}^1_k$ [9]. An advantage to Cazanave's formula is that one does not need to determine the zero locus or other local information about f. We extend Cazanave's formula for morphisms $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zeros. The work of Scheja and Storch on global complete intersections [30] is central to both [16] and our result. We also rely on the work of Becker, Cardinal, Roy and Szafraniec [4], who describe a procedure for recovering the global version of the Scheja–Storch form.

Theorem 1.2 has applications wherever Morel's $\mathbb{A}^1$-degree is used. One particularly successful application of the $\mathbb{A}^1$-degree has been the $\mathbb{A}^1$-enumerative geometry program. The goal of this program is to enrich enumerative problems over arbitrary fields by producing $\mathrm{GW}(k)$-valued enumerative equations and interpreting them geometrically over various fields. Notable results in this direction include Srinivasan and Wickelgren's count of lines meeting four lines in three-space [31], Larson and Vogt's count of bitangents to a smooth plane quartic [19], and Bethea, Kass, and Wickelgren's enriched Riemann–Hurwitz formula [5]. See [22; 26] for other related works. For a more detailed account of recent developments in $\mathbb{A}^1$-enumerative geometry; see [6; 28].

2. Notation and conventions

In this section, we fix some standard terminology and notation. Let k denote an arbitrary field. We will always use $f = (f_1, \dots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ to denote an endomorphism of affine space, assumed to have isolated zeros when we work with it in the global context. We denote by Q the global algebra associated to this endomorphism

$$Q := \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)}.$$

The maximal ideals of Q correspond to the maximal ideals of $k[x_1, \ldots, x_n]$ on which f vanishes. For any maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ on which f vanishes, we denote by $Q_{\mathfrak{m}}$ the local algebra

$$Q_{\mathfrak{m}} := \frac{k[x_1, \ldots, x_n]_{\mathfrak{m}}}{(f_1, \ldots, f_n)}.$$

If $\lambda : V \to k$ is a k-linear form on any k-algebra, we will denote by Φ_λ the associated bilinear form given by

$$\begin{aligned} \Phi_\lambda : V \times V &\to k \\ (a, b) &\mapsto \lambda(ab). \end{aligned}$$

Definition 2.1. We say that λ is a *dualizing linear form* if Φ_λ is nondegenerate as a symmetric bilinear form [4, 2.1]. If λ is dualizing, then we say that two vector space bases $\{a_i\}$ and $\{b_i\}$ of V are *dual with respect to* λ if

$$\lambda(a_i b_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. We show in Remark 3.6 that if $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ, then λ is a dualizing linear form.

More notation will be introduced as we provide an overview of Bézoutians and the Scheja–Storch bilinear form. We will borrow and clarify notation from both [30] and [4].

3. Bézoutians

We first provide an overview of the construction of the *Bézoutian*, following [4]. Given one of our n polynomials f_i, we introduce two sets of auxiliary indeterminants and study how f_i changes when we incrementally exchange one set of indeterminants for the other. Explicitly, consider variables $X := (X_1, \ldots, X_n)$ and $Y := (Y_1, \ldots, Y_n)$. For any $1 \leq i, j \leq n$, we denote by Δ_{ij} the quantity

$$\Delta_{ij} := \frac{f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) - f_i(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n)}{X_j - Y_j}.$$

Note that Δ_{ij} is a multivariate polynomial. Indeed,

$$f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) \quad \text{and} \quad f_i(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n)$$

differ only in the terms in which X_j or Y_j appear, so we can expand the difference

$$f_i(Y_1, \ldots, Y_{j-1}, X_j, \ldots, X_n) - f_i(Y_1, \ldots, Y_j, X_{j+1}, \ldots, X_n) = \sum_{\ell \geq 1} g_\ell \cdot (X_j - Y_j)^\ell,$$

where $g_\ell \in k[Y_1, \ldots, Y_{j-1}, X_{j+1}, \ldots, X_n]$. In this notation, $\Delta_{ij} = \sum_{\ell \geq 1} g_\ell \cdot (X_j - Y_j)^{\ell - 1}$.

We view Δ_{ij} as living in the tensor product ring $Q \otimes_k Q$, under the isomorphism

$$\varepsilon : \frac{k[X, Y]}{(f(X), f(Y))} \xrightarrow{\cong} Q \otimes_k Q,$$

given by sending X_i to $x_i \otimes 1$, and Y_i to $1 \otimes x_i$.

Definition 3.1. We define the *Bézoutian* of the polynomials $f_1, \dots, f_n$ to be the image $\text{Béz}(f_1, \dots, f_n)$ of the determinant $\det(\Delta_{ij})$ in $Q \otimes_k Q$.

Example 3.2. Let $(f_1, f_2, f_3) = (x_1^2, x_2^2, x_3^2)$. Then we have that

$$\begin{aligned}\text{Béz}(f_1, f_2, f_3) &= \varepsilon\left(\det\begin{pmatrix} X_1+Y_1 & 0 & 0 \\ 0 & X_2+Y_2 & 0 \\ 0 & 0 & X_3+Y_3\end{pmatrix}\right) \\ &= \varepsilon((X_1+Y_1)(X_2+Y_2)(X_3+Y_3)) \\ &= x_1x_2x_3 \otimes 1 + x_1x_2 \otimes x_3 + x_1x_3 \otimes x_2 + x_2x_3 \otimes x_1 \\ &\qquad + x_1 \otimes x_2x_3 + x_2 \otimes x_1x_3 + x_3 \otimes x_1x_2 + 1 \otimes x_1x_2x_3.\end{aligned}$$

There is a natural multiplication map $\delta : Q \otimes_k Q \to Q$, defined by $\delta(a \otimes b) = ab$, that sends the Bézoutian of f to the image of the Jacobian of f in Q.

Proposition 3.3. *Let* $\text{Jac}(f_1, \dots, f_n)$ *be the image of the Jacobian determinant* $\det\left(\frac{\partial f_i}{\partial x_j}\right)$ *in Q. Then*

$$\delta(\text{Béz}(f_1, \dots, f_n)) = \text{Jac}(f_1, \dots, f_n) \in Q.$$

Proof. Note that $(\delta \circ \varepsilon)(a(X, Y)) = a(x, x)$ and $\delta \circ \varepsilon$ is an algebra homomorphism. In particular, $\delta \circ \varepsilon$ preserves the multiplication and addition occurring in the determinant which defines $\text{Béz}(f_1, \dots, f_n)$. Therefore it suffices for us to verify that

$$(\delta \circ \varepsilon)(\Delta_{ij}) = \frac{\partial f_i}{\partial x_j}.$$

Recall that

$$\Delta_{ij} = \frac{f_i(Y_1, \dots, Y_{j-1}, X_j, \dots, X_n) - f_i(Y_1, \dots, Y_j, X_{j+1}, \dots, X_n)}{X_j - Y_j}.$$

Taking the x_j-Taylor expansion of $f(x_1, \dots, x_n)$ about Y_j gives us

$$f_i(x_1, \dots, x_n) = f_i(x_1, \dots, Y_j, \dots, x_n) + \sum_{\ell \geq 1} \frac{\partial^\ell f_i}{\partial x_j^\ell} \cdot (x_j - Y_j)^\ell.$$

We now subtract $f_i(x_1, \dots, Y_j, \dots, x_n)$ from both sides, evaluate $x_j \mapsto X_j$, and divide by $X_j - Y_j$ to deduce

$$\frac{f_i(x_1, \dots, X_j, \dots, x_n) - f_i(x_1, \dots, Y_j, \dots, x_n)}{(X_j - Y_j)} = \frac{\partial f_i}{\partial x_j} + \sum_{\ell \geq 2} \frac{\partial^\ell f_i}{\partial x_j^\ell} \cdot (X_j - Y_j)^{\ell - 1}.$$

Finally, evaluating $X_j \mapsto x_j$ and $Y_j \mapsto x_j$ gives us $(\delta \circ \varepsilon)(\Delta_{ij}) = \frac{\partial f_i}{\partial x_j}$, as desired. ☐

Lemma 3.4. *Let $a_1, \dots, a_m$ be any vector space basis for Q, and write the Bézoutian as*

$$\text{Béz}(f_1, \dots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i$$

for some $b_1, \dots, b_n \in Q$. Then $\{b_i\}_{i=1}^m$ is a basis for Q.

Proof. This is [4, 2.10(iii)]. □

This allows us to associate to the Bézoutian a pair of vector space bases for Q. Given any such pair of bases, we will construct a unique linear form for which the bases are dual. Before doing so, we establish some equivalent conditions for the duality of a linear form given a pair of bases.

Proposition 3.5. *Let $\{a_i\}$ and $\{b_i\}$ be a pair of bases for B. Consider the induced k-linear isomorphism*

$$\begin{aligned}\Theta : \text{Hom}_k(Q, k) &\to Q \\ \varphi &\mapsto \sum_i \varphi(a_i) b_i.\end{aligned}$$

Given a linear form $\lambda : Q \to k$, the following are equivalent:

(1) *We have that $\Theta(\lambda) = \sum_i \lambda(a_i) b_i = 1$.*

(2) *For any $a \in Q$, we have $a = \sum_i \lambda(a a_i) b_i$.*

(3) *We have that $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ.*

Proof. Note that (2) implies (1) by setting $a = 1$. Next, we remark that Θ is a Q-module isomorphism by [30, 3.3 Satz], where the Q-module structure on $\text{Hom}_k(Q, k)$ is given by $a \cdot \varphi = \varphi(a \cdot -)$. This allows us to conclude that $a \cdot \Theta(\lambda) = \Theta(a \cdot \lambda)$ for any linear form λ. In particular, we have

$$a \sum_i \lambda(a_i) b_i = \sum_i \lambda(a a_i) b_i.$$

It follows from this identity that (1) implies (2). Now suppose that (2) holds. By setting $a = b_j$ for some j, we have

$$\sum_i \lambda(a_i b_j) b_i = b_j.$$

Since $\{b_i\}$ is a basis, it follows that $\lambda(a_i b_j) = \delta_{ij}$. Thus the bases $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ. Finally, suppose that (3) holds, so that $\lambda(a_i b_j) = \delta_{ij}$. For any $a \in Q$, write a as $a := \sum_j c_j b_j$ for some scalars c_j. Then

$$\sum_i \lambda(a a_i) b_i = \sum_i \lambda\Big(a_i \sum_j c_j b_j\Big) b_i = \sum_i \Big(\sum_j c_j \lambda(a_i b_j)\Big) b_i = \sum_i c_i b_i = a.$$

Thus (3) implies (2). □

Remark 3.6. If $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ, then λ is a dualizing form. Indeed, suppose there exists $x \in Q$ such that $\Phi_\lambda(x, y) = 0$ for all $y \in Q$. Write $x = \sum_i x_i a_i$ with $x_i \in k$. Then

$$0 = \lambda(xb_j) = \lambda\biggl(\sum_i x_i a_i b_j\biggr) = \sum_i x_i \lambda(a_i b_j) = x_j$$

for all j, so $x = 0$.

Corollary 3.7. *Let $\{a_i\}$ and $\{b_i\}$ be two k-vector space bases for Q. Then there exists a unique dualizing linear form $\lambda : Q \to k$ such that $\{a_i\}$ and $\{b_i\}$ are dual with respect to λ.*

Proof. As Θ is a k-algebra isomorphism, it admits a unique preimage of 1. Thus, given any pair of bases $\{a_i\}$ and $\{b_i\}$ of Q, there is a unique dualizing linear form with respect to which $\{a_i\}$ and $\{b_i\}$ are dual. □

Definition 3.8. We call Φ_λ a *Bézoutian bilinear form* if $\lambda : Q \to k$ is a dualizing linear form such that

$$\text{Béz}(f_1, \dots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i,$$

where $\{a_i\}$ and $\{b_i\}$ are dual bases with respect to λ.

A priori this is different than the Bézoutian form detailed in Definition 1.1, although we will prove that they define the same class in $\mathrm{GW}(k)$ in Section 5A.

Proposition 3.9. *Given a function $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ with isolated zeros, its Bézoutian bilinear form is a well-defined class in* $\mathrm{GW}(k)$.

Proof. Let Φ_λ be a Bézoutian bilinear form for f. Recall that $\Phi_\lambda : Q \times Q \to k$ is defined by $\Phi_\lambda(a, b) = \lambda(ab)$. Since λ is a dualizing linear form, Φ_λ is nondegenerate and as Q is commutative, Φ_λ is symmetric. Lemma 3.4 implies that given a basis $a_1, \dots, a_m$ for Q, we can write

$$\text{Béz}(f_1, \dots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i,$$

and obtain a second basis $b_1, \dots, b_m$ for Q. By Corollary 3.7, there is a dualizing linear form for the two bases $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$. It remains to show that if

$$\text{Béz}(f_1, \dots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i = \sum_{i=1}^{m} a'_i \otimes b'_i,$$

for some bases $\{a_i\}$, $\{b_i\}$ dual with respect to λ and $\{a'_i\}$, $\{b'_i\}$ dual with respect to λ', then Φ_λ and $\Phi_{\lambda'}$ are isomorphic. We will in fact show that $\lambda = \lambda'$, so that $\Phi_\lambda = \Phi_{\lambda'}$. Write $a_i = \sum_s \alpha_{is} a'_s$ and $b_i = \sum_s \beta_{is} b'_s$. Then

$$\sum_{i=1}^{m} a'_i \otimes b'_i = \sum_{i=1}^{m} a_i \otimes b_i = \sum_i \biggl(\sum_s \alpha_{is} a'_s\biggr) \otimes \biggl(\sum_t \beta_{it} b'_t\biggr) = \sum_{s,t} \biggl(\sum_i \alpha_{is} \beta_{it}\biggr) a'_s \otimes b'_t.$$

Since $\{a'_s \otimes b'_t\}$ is a basis for $Q \otimes_k Q$, we conclude that $\sum_i \alpha_{is}\beta_{it} = \delta_{st}$. In particular, $(\alpha_{ij})^{-1} = (\beta_{ij})^T$, so $(\beta_{ij})(\alpha_{ij})^T$ is the identity matrix. Thus $\sum_j \alpha_{sj}\beta_{tj} = \delta_{st}$.

Now given $g = \sum_i c_i a_i = \sum_i c'_i a'_i \in Q$ and $1 = \sum_i d_i b_i = \sum_i d'_i b'_i$, we have that

$$\lambda(g) = \lambda\biggl(\sum_i (c_i a_i) \cdot \sum_j d_j b_j\biggr) = \sum_{i,j} c_i d_j \lambda(a_i b_j) = \sum_i c_i d_i.$$

Similarly, we have $\lambda'(g) = \sum_i c'_i d'_i$. By our change of bases, we have $c'_j = \sum_i c_j \alpha_{ij}$ and $d'_j = \sum_i d_i \beta_{ij}$. Thus

$$\lambda'(g) = \sum_j c'_j d'_j = \sum_j \biggl(\sum_s c_s \alpha_{sj}\biggr)\biggl(\sum_t d_t \beta_{tj}\biggr) = \sum_{s,t} c_s d_t \biggl(\sum_j \alpha_{sj}\beta_{tj}\biggr) = \sum_s c_s d_s = \lambda(g).$$

Therefore $\lambda = \lambda'$, as desired. $\square$

Example 3.10. Continuing Example 3.2, let $f = (x_1^2, x_2^2, x_3^2)$, so that

$$\begin{aligned}\varepsilon^{-1}(\text{Béz}(f_1, f_2, f_3)) &= (X_1+Y_1)(X_2+Y_2)(X_3+Y_3)\\ &= X_1X_2X_3+X_1X_2Y_3+X_1Y_2X_3+X_1Y_2Y_3+Y_1X_2X_3+Y_1X_2Y_3+Y_1Y_2X_3+Y_1Y_2Y_3.\end{aligned}$$

We give two bases for $k[Z_1, Z_2, Z_3]/(Z_1^2, Z_2^2, Z_3^2)$ in the following table, where we replace Z by either X or Y. We pair off these bases in a convenient way.

i	a_i	b_i
1	1	$Y_1Y_2Y_3$
2	X_1	Y_2Y_3
3	X_2	Y_1Y_3
4	X_3	Y_1Y_2
5	X_1X_2	Y_3
6	X_1X_3	Y_2
7	X_2X_3	Y_1
8	$X_1X_2X_3$	1

The Bézoutian we computed is in the desired form $\sum_{i=1}^{8} a_i \otimes b_i$, so we now need to compute the dualizing linear form λ for $\{a_i\}$ and $\{b_i\}$. Since $1 = 1 \cdot b_8 + \sum_{i=1}^{7} 0 \cdot b_i$, we define λ by $\lambda(a_i) = 0$ for $1 \le i \le 7$ and $\lambda(a_8) = \lambda(X_1X_2X_3) = 1$. Now let $g \in k[X_1, X_2, X_3]/(X_1^2, X_2^2, X_3^2)$ be arbitrary. We can write g as

$$g = c_1 + c_2X_1 + c_3X_2 + c_4X_3 + c_5X_1X_2 + c_6X_1X_3 + c_7X_2X_3 + c_8X_1X_2X_3.$$

Then λ is the dualizing linear form sending

$$\begin{aligned}\lambda : \frac{k[X_1, X_2, X_3]}{(X_1^2, X_2^2, X_3^2)} &\to k\\ g &\mapsto c_8.\end{aligned}$$

Finally we can compute the Gram matrix of Φ_λ in the basis $\{a_i\}$. Note that $a_i a_j$ is a scalar multiple of $X_1X_2X_3$ if and only if $i+j-1=8$. Thus the Gram matrix is

$$\Phi_\lambda = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix} \cong \bigoplus_{i=1}^{4} \begin{pmatrix} 1&0\\ 0&-1 \end{pmatrix}.$$

4. The Scheja–Storch bilinear form

Associated to any polynomial with an isolated zero, Eisenbud and Levine [11] and Khimshiashvili [13] used the Scheja–Storch construction [30] to produce a bilinear form on the local algebra $Q_{\mathfrak{m}}$. Kass and Wickelgren proved that this Eisenbud–Khimshiashvili–Levine bilinear form computes the local $\mathbb{A}^1$-degree [16]. The machinery of Scheja and Storch works in great generality; in particular, one may produce a Scheja–Storch bilinear form on the global algebra Q as well as the local algebras $Q_{\mathfrak{m}}$. We will provide a brief account of the Scheja–Storch construction before comparing it with the Bézoutian.

In [30], $k\langle X\rangle := k\langle X_1, \dots, X_n\rangle$ denotes either a polynomial ring $k[X_1, \dots, X_n]$ or a power series ring $k[\![X_1, \dots, X_n]\!]$. We will also use this notation, although we will focus on the situation where $k\langle X\rangle$ is a polynomial ring. Let $\rho : k\langle X\rangle \to Q$ denote the map obtained by quotienting out by the ideal $(f_1, \dots, f_n)$, let $\mu_1 : k\langle X\rangle \otimes_k k\langle X\rangle \to k\langle X\rangle$ denote the multiplication map, and let $\mu : Q \otimes_k Q \to Q$ denote the multiplication map on the global algebra, fitting into a commutative diagram:

$$\begin{tikzcd} k\langle X\rangle \otimes_k k\langle X\rangle \arrow[r, "\mu_1"] \arrow[d, "\rho\otimes\rho"'] & k\langle X\rangle \arrow[d, "\rho"] \\ Q \otimes_k Q \arrow[r, "\mu"'] & Q \end{tikzcd}$$

We remark that $f_j \otimes 1 - 1 \otimes f_j$ lies in $\ker(\mu_1)$, and that $\ker(\mu_1)$ is generated by elements of the form $X_i \otimes 1 - 1 \otimes X_i$. Thus for any j, there are elements $a_{ij} \in k\langle X\rangle \otimes_k k\langle X\rangle$ such that

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^{n} a_{ij}(X_i \otimes 1 - 1 \otimes X_i). \tag{4-1}$$

We denote by Δ the following distinguished element in the tensor algebra $Q \otimes_k Q$

$$\Delta := (\rho \otimes \rho)(\det(a_{ij})),$$

which corresponds to the Bézoutian which we will later demonstrate. It is true that Δ is independent of the choice of a_{ij}, as shown by Scheja and Storch [30, 3.1 Satz]. We now define an important isomorphism

χ of k-algebras used in the Scheja–Storch construction. However, we will phrase this more categorically than in [30], as it will benefit us later.

Proposition 4.1. *Consider two endofunctors $F, G : \mathtt{Alg}_k^{f.g.} \to \mathtt{Alg}_k^{f.g.}$ on the category of finitely generated k-algebras, where $F(A) = A \otimes_k A$ and $G(A) = \mathrm{Hom}_k(\mathrm{Hom}_k(A, k), A)$. Then there is a natural isomorphism $\chi : F \to G$ whose component at a k-algebra A is*

$$\begin{aligned} \chi_A : A \otimes_k A &\to \mathrm{Hom}_k(\mathrm{Hom}_k(A, k), A) \\ b \otimes c &\mapsto [\varphi \mapsto \varphi(b)c]. \end{aligned}$$

Proof. This canonical isomorphism is given in [30, page 181], so it will suffice for us to verify naturality. Let $g : A \to B$ be any morphism of k-algebras. Consider the induced maps $g \otimes g : A \otimes_k A \to B \otimes_k B$ and

$$\begin{aligned} g_* : \mathrm{Hom}_k(\mathrm{Hom}_k(A, k), A) &\to \mathrm{Hom}_k(\mathrm{Hom}_k(B, k), B) \\ \psi &\mapsto [\epsilon \mapsto g \circ \psi(\epsilon \circ g)]. \end{aligned}$$

It remains to show that the following diagram commutes:

$$\begin{tikzcd} A \otimes_k A \arrow[r, "\chi_A"] \arrow[d, "g\otimes g"'] & \mathrm{Hom}_k(\mathrm{Hom}_k(A, k), A) \arrow[d, "g_*"] \\ B \otimes_k B \arrow[r, "\chi_B"'] & \mathrm{Hom}_k(\mathrm{Hom}_k(B, k), B) \end{tikzcd}$$

To see this, we compute $g_* \circ \chi_A = [b \otimes c \mapsto [\epsilon \mapsto g((\epsilon \circ g)(b) \cdot c)]]$. Note that $\epsilon \circ g : B \to k$, so $(\epsilon \circ g)(b) \in k$. Since g is k-linear, we have $g((\epsilon \circ g)(b) \cdot c) = \epsilon(g(b)) \cdot g(c)$. Next, we compute $\chi_B \circ (g \otimes g) = [b \otimes c \mapsto [\epsilon \mapsto \epsilon(g(b)) \cdot g(c)]]$. Thus $g_* \circ \chi_A = \chi_B \circ (g \otimes g)$, so the diagram commutes. □

We now let $\Theta := \chi_Q(\Delta)$ denote the image of Δ under the component of this natural isomorphism at the global algebra Q. We have that Θ is a k-linear map $\Theta : \mathrm{Hom}_k(Q, k) \to Q$. Letting η denote $\Theta^{-1}(1)$, we obtain a well-defined linear form $\eta : Q \to k$ by [30, 3.3 Satz].

Definition 4.2. We refer to $\Phi_\eta : Q \times Q \to k$ as the *global Scheja–Storch bilinear form.*

The Bézoutian gives us an explicit formula for Δ. As a result, the global Scheja–Storch form agrees with the Bézoutian form.

Proposition 4.3. *In $Q \otimes_k Q$, we have $\Delta = \mathrm{Béz}(f_1, \ldots, f_n)$.*

Proof. We first compute

$$\begin{aligned} \sum_{i=1}^n \Delta_{ji}(X_i - Y_i) &= \sum_{i=1}^n \frac{f_j(Y_1, \ldots, Y_{i-1}, X_i, \ldots, X_n) - f_j(Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_n)}{(X_i - Y_i)} \cdot (X_i - Y_i) \\ &= \sum_{i=1}^n f_j(Y_1, \ldots, Y_{i-1}, X_i, \ldots, X_n) - f_j(Y_1, \ldots, Y_i, X_{i+1}, \ldots, X_n) \\ &= f_j(X_1, \ldots, X_n) - f_j(Y_1, \ldots, Y_n). \end{aligned}$$

Let $\varphi : k\langle X\rangle \otimes_k k\langle X\rangle \xrightarrow{\sim} k\langle X, Y\rangle$ be the ring isomorphism given by $\varphi(b \otimes c) = b(X)c(Y)$. Note that $\varphi(x_i \otimes 1) = X_i$ and $\varphi(1 \otimes x_i) = Y_i$, so the inverse of φ is characterized by $\varphi^{-1}(X_i) = x_i \otimes 1$ and $\varphi^{-1}(Y_i) = 1 \otimes x_i$. It follows that

$$f_j \otimes 1 - 1 \otimes f_j = \varphi^{-1}(f_j(X) - f_j(Y)) = \sum_{i=1}^{n} \varphi^{-1}(\Delta_{ji}(X_i - Y_i)) = \sum_{i=1}^{n} \varphi^{-1}(\Delta_{ji})(x_i \otimes 1 - 1 \otimes x_i).$$

We may thus set $a_{ij} = \varphi^{-1}(\Delta_{ji})$, and [30, 3.1 Satz] implies that $\Delta = (\rho \otimes \rho)(\det(a_{ij}))$. On the other hand, $(\rho \otimes \rho)(\varphi^{-1}(\det(\Delta_{ji}))) = \text{Béz}(f_1, \dots, f_n)$ by Definition 3.1. □

Lemma 4.4. *The Bézoutian bilinear form and the global Scheja–Storch bilinear form are identical.*

Proof. We showed in Proposition 4.3 that Δ is the Bézoutian in $Q \otimes_k Q$. We now show that the associated forms are identical. Pick bases $\{a_i\}$ and $\{b_i\}$ of Q such that

$$\Delta = \text{Béz}(f_1, \dots, f_n) = \sum_{i=1}^{m} a_i \otimes b_i.$$

Since the natural isomorphism χ has k-linear components, Δ is mapped to

$$\Theta := \chi_Q(\Delta) = \left[\varphi \mapsto \sum_{i=1}^{m} \varphi(a_i)b_i\right].$$

Thus $\eta := \Theta^{-1}(1)$ is the linear form $\eta : Q \to k$ satisfying $\sum_{i=1}^{m} \eta(a_i)b_i = 1$. By Proposition 3.5, this implies that η is the form for which $\{a_i\}$ and $\{b_i\}$ are dual bases. As in Definition 3.8, this tells us that η is the linear form producing the Bézoutian bilinear form. □

4A. ***Local decomposition.*** While our discussion of the Scheja–Storch form in the previous section was global, it is perfectly valid to localize at a maximal ideal and repeat the story again [30, pages 180–181]. The fact that Q is an Artinian ring then gives a convenient way to relate the global version of η to the local version of η. This local decomposition has been utilized previously, for example in [16].

Let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \dots, x_n]$ at which the morphism $f = (f_1, \dots, f_n)$ has an isolated root. Letting $\rho_{\mathfrak{m}}$ denote the quotient map $k\langle X\rangle_{\mathfrak{m}} \to Q_{\mathfrak{m}}$, we have a commutative diagram:

$$\begin{tikzcd} k\langle X\rangle_{\mathfrak{m}} \otimes_k k\langle X\rangle_{\mathfrak{m}} \arrow[r, "\mu_1'"] \arrow[d, "\rho_{\mathfrak{m}}\otimes\rho_{\mathfrak{m}}"'] & k\langle X\rangle_{\mathfrak{m}} \arrow[d, "\rho_{\mathfrak{m}}"] \\ Q_{\mathfrak{m}} \otimes_k Q_{\mathfrak{m}} \arrow[r, "\mu'"'] & Q_{\mathfrak{m}} \end{tikzcd}$$

In $k\langle X\rangle_{\mathfrak{m}} \otimes_k k\langle X\rangle_{\mathfrak{m}}$, we can again write

$$f_j \otimes 1 - 1 \otimes f_j = \sum_{i=1}^{n} \tilde{a}_{ij}(X_i \otimes 1 - 1 \otimes X_i)$$

to obtain the local Bézoutian $\Delta_{\mathfrak{m}} := (\rho_{\mathfrak{m}} \otimes \rho_{\mathfrak{m}})(\det(\tilde{a}_{ij})) \in Q_{\mathfrak{m}} \otimes_k Q_{\mathfrak{m}}$. Let $\lambda_{\mathfrak{m}} : Q \to Q_{\mathfrak{m}}$ be the localization map. From [30, page 181] we have $(\lambda_{\mathfrak{m}} \otimes \lambda_{\mathfrak{m}})(\Delta) = \Delta_{\mathfrak{m}}$. Via the natural isomorphism χ in Proposition 4.1, we have a commutative diagram of the form:

$$\begin{tikzcd} Q \otimes_k Q \arrow[r, "\chi_Q"] \arrow[d, "\lambda_{\mathfrak{m}} \otimes \lambda_{\mathfrak{m}}"'] & \mathrm{Hom}_k(\mathrm{Hom}_k(Q, k), Q) \arrow[d, "\lambda_{\mathfrak{m}*}"] \\ Q_{\mathfrak{m}} \otimes_k Q_{\mathfrak{m}} \arrow[r, "\chi_{Q_{\mathfrak{m}}}"'] & \mathrm{Hom}_k(\mathrm{Hom}_k(Q_{\mathfrak{m}}, k), Q_{\mathfrak{m}}) \end{tikzcd}$$

Tracing Δ through this diagram, we see that

$$\begin{tikzcd} \Delta \arrow[r, mapsto] \arrow[d, mapsto] & \Theta \arrow[d, mapsto] \\ \Delta_{\mathfrak{m}} \arrow[r, mapsto] & \Theta_{\mathfrak{m}} \end{tikzcd}$$

where $\Theta_{\mathfrak{m}} = \chi_{Q_{\mathfrak{m}}}(\Delta_{\mathfrak{m}})$. Unwinding $\Theta_{\mathfrak{m}} = \lambda_{\mathfrak{m}*}(\Theta)$, we find that $\Theta_{\mathfrak{m}}$ is the map

$$\begin{aligned} \Theta_{\mathfrak{m}} : \mathrm{Hom}_k(Q_{\mathfrak{m}}, k) &\to Q_{\mathfrak{m}} \\ \psi &\mapsto \lambda_{\mathfrak{m}} \circ \Theta(\psi \circ \lambda_{\mathfrak{m}}). \end{aligned}$$

Recall that as Q is a zero-dimensional Noetherian commutative k-algebra, the localization maps induce a k-algebra isomorphism:[1]

$$(\lambda_{\mathfrak{m}})_{\mathfrak{m}} : Q \xrightarrow{\sim} \prod_{\mathfrak{m}} Q_{\mathfrak{m}}.$$

This is reflected by an internal decomposition of Q in terms of orthogonal idempotents [4, 2.13], which we now describe; see also [32, Lemma 00JA]. By the Chinese remainder theorem, we may pick a collection of pairwise orthogonal idempotents $\{e_{\mathfrak{m}}\}_{\mathfrak{m}}$ such that $\sum_{\mathfrak{m}} e_{\mathfrak{m}} = 1$. The internal decomposition of Q is then

$$Q = \bigoplus_{\mathfrak{m}} Q \cdot e_{\mathfrak{m}},$$

and the localization maps restrict to isomorphisms $\lambda_{\mathfrak{m}}|_{Q \cdot e_{\mathfrak{m}}} : Q \cdot e_{\mathfrak{m}} \xrightarrow{\sim} Q_{\mathfrak{m}}$ with $\lambda_{\mathfrak{m}}(e_{\mathfrak{m}}) = 1$. Moreover, $\lambda_{\mathfrak{m}}(Q \cdot e_{\mathfrak{n}}) = 0$ for any $\mathfrak{n} \neq \mathfrak{m}$.

Proposition 4.5. *Suppose $\ell : Q \to k$ is a linear form which factors through the localization $\lambda_{\mathfrak{m}} : Q \to Q_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$. Then $\Theta(\ell)$ lies in $Q \cdot e_{\mathfrak{m}}$.*

Proof. Recall that $\lambda_{\mathfrak{m}}|_{Q \cdot e_{\mathfrak{n}}} = 0$ for $\mathfrak{n} \neq \mathfrak{m}$. Since $e_{\mathfrak{m}} \cdot e_{\mathfrak{n}} = 0$ for $\mathfrak{n} \neq \mathfrak{m}$ and $e_{\mathfrak{m}}$ is idempotent, the localization $\lambda_{\mathfrak{m}} : Q \to Q_{\mathfrak{m}}$ can be written as the following composition:

$$\lambda_{\mathfrak{m}} : Q \xrightarrow{- \cdot e_{\mathfrak{m}}} Q \xrightarrow{\lambda_{\mathfrak{m}}} Q_{\mathfrak{m}}.$$

Since ℓ factors through the localization, it can be written as a composite

$$\ell : Q \xrightarrow{- \cdot e_{\mathfrak{m}}} Q \xrightarrow{\lambda_{\mathfrak{m}}} Q_{\mathfrak{m}} \xrightarrow{\ell_{\mathfrak{m}}} k.$$

[1] Q is Artinian by [32, Lemma 00KH], so the claimed isomorphism exists by [32, Lemma 00JA].

Thus $\Theta(\ell) = \Theta(\ell_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}} \circ (e_{\mathfrak{m}} \cdot -))$. Scheja–Storch proved that Θ respects the Q-module structure on $\mathrm{Hom}_k(Q, k)$ given by $a \cdot \sigma = \sigma(a \cdot -)$ [30, 3.3 Satz]. That is, $\Theta(\sigma(a \cdot -)) = \Theta(a \cdot \sigma) = a\Theta(\sigma)$ for any $a \in Q$ and $\sigma \in \mathrm{Hom}_k(Q, k)$. Thus

$$\Theta(\ell) = e_{\mathfrak{m}} \cdot \Theta(\ell_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}),$$

so $\Theta(\ell) \in Q \cdot e_{\mathfrak{m}}$. □

Returning to the Scheja–Storch form, we have the following commutative diagram relating $\Theta_{\mathfrak{m}}$ and Θ:

$$\begin{tikzcd}
\mathrm{Hom}_k(Q, k) \arrow[r, "\Theta"] & Q \arrow[d, "\lambda_{\mathfrak{m}}"] \\
\mathrm{Hom}_k(Q_{\mathfrak{m}}, k) \arrow[u, "- \circ \lambda_{\mathfrak{m}}"] \arrow[r, "\Theta_{\mathfrak{m}}"'] & Q_{\mathfrak{m}}
\end{tikzcd}$$

This coherence between Θ and $\Theta_{\mathfrak{m}}$ allows us to relate the local linear forms $\eta_{\mathfrak{m}} := \Theta_{\mathfrak{m}}^{-1}(1)$ to the global linear form $\eta := \Theta^{-1}(1)$ in the following way.

Proposition 4.6. *For each maximal ideal* $\mathfrak{m}$ *of* Q, *let* $\eta_{\mathfrak{m}} := \Theta_{\mathfrak{m}}^{-1}(1) : Q_{\mathfrak{m}} \to k$, *and let* $\eta := \Theta^{-1}(1) : Q \to k$. *Then* $\eta = \sum_{\mathfrak{m}} \eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}$.

Proof. It suffices to show that $\Theta(\sum_{\mathfrak{m}} \eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}) = 1$. Since $\eta_{\mathfrak{m}} = \Theta_{\mathfrak{m}}^{-1}(1)$ by definition, we have $1 = \Theta_{\mathfrak{m}}(\eta_{\mathfrak{m}}) := \lambda_{\mathfrak{m}}(\Theta(\eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}))$. By Proposition 4.5, we have $\Theta(\eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}) \in Q \cdot e_{\mathfrak{m}}$. Since $\lambda_{\mathfrak{m}}(\Theta(\eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}})) = 1$ and $\lambda_{\mathfrak{m}}|_{Q \cdot e_{\mathfrak{m}}}$ is an isomorphism sending $e_{\mathfrak{m}}$ to 1, it follows that $\Theta(\eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}) = e_{\mathfrak{m}}$. Finally, since Θ is k-linear, we have

$$\Theta\bigg(\sum_{\mathfrak{m}} \eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}\bigg) = \sum_{\mathfrak{m}} \Theta(\eta_{\mathfrak{m}} \circ \lambda_{\mathfrak{m}}) = \sum_{\mathfrak{m}} e_{\mathfrak{m}} = 1. \qquad \square$$

Using this local decomposition procedure for the linear forms $\eta_{\mathfrak{m}}$ and η, we obtain a local decomposition for Scheja–Storch bilinear forms.

Lemma 4.7. **(*Local decomposition of Scheja–Storch forms*)** *Let* η *and* $\eta_{\mathfrak{m}}$ *be as in Proposition 4.6. Then* $\Phi_\eta = \bigoplus_{\mathfrak{m}} \Phi_{\eta_{\mathfrak{m}}}$. *In particular, the global Scheja–Storch form is a sum over local Scheja–Storch forms*

$$\mathrm{SS}(f) = \sum_{\mathfrak{m}} \mathrm{SS}_{\mathfrak{m}}(f).$$

Proof. For each maximal ideal $\mathfrak{m}$, let $\{w_{\mathfrak{m},i}\}_i$ be a k-vector space basis for $Q_{\mathfrak{m}}$. Let $\{v_{\mathfrak{m},i}\}_{\mathfrak{m},i}$ (ranging over all i and all maximal ideals) be a basis of Q such that $\lambda_{\mathfrak{m}}(v_{\mathfrak{m},i}) = w_{\mathfrak{m},i}$ for each i and $\mathfrak{m}$, and $\lambda_{\mathfrak{m}}(v_{\mathfrak{n},i}) = 0$ for $\mathfrak{m} \neq \mathfrak{n}$. We now compare the Gram matrix for $\eta : Q \to k$ and the Gram matrices for $\eta_{\mathfrak{m}} : Q_{\mathfrak{m}} \to k$ in these bases. Via the internal decomposition consisting of pairwise orthogonal idempotents, we have $v_{\mathfrak{m},i} \cdot v_{\mathfrak{n},j} = 0$ if $\mathfrak{m} \neq \mathfrak{n}$. Thus

$$\eta(v_{\mathfrak{m},i} \cdot v_{\mathfrak{n},j}) = 0,$$

so the Gram matrix for Φ_η will be a block sum indexed over the maximal ideals. If $\mathfrak{m} = \mathfrak{n}$, then Proposition 4.6 implies

$$\eta(v_{\mathfrak{m},i} \cdot v_{\mathfrak{m},j}) = \sum_{\mathfrak{n}} \eta_{\mathfrak{n}}(\lambda_{\mathfrak{n}}(v_{\mathfrak{m},i} \cdot v_{\mathfrak{m},j})) = \eta_{\mathfrak{m}}(\lambda_{\mathfrak{m}}(v_{\mathfrak{m},i} \cdot v_{\mathfrak{m},j}) = \eta_{\mathfrak{m}}(w_{\mathfrak{m},i} \cdot w_{\mathfrak{m},j}).$$

Thus the Gram matrices of Φ_η and $\bigoplus_{\mathfrak{m}} \Phi_{\eta_{\mathfrak{m}}}$ are equal, so $\Phi_\eta = \bigoplus_{\mathfrak{m}} \Phi_{\eta_{\mathfrak{m}}}$. □

Remark 4.8. The local Scheja–Storch bilinear form is given by $\Phi_{\eta_{\mathfrak{m}}} : Q_{\mathfrak{m}} \times Q_{\mathfrak{m}} \to k$. Given a basis $\{a_1, \dots, a_m\}$ of $Q_{\mathfrak{m}}$, we may write $\Delta_{\mathfrak{m}} = \sum a_i \otimes b_i$ and define the local Bézoutian bilinear form as a suitable dualizing form. Replacing Q, Δ, Θ, and η with $Q_{\mathfrak{m}}$, $\Delta_{\mathfrak{m}}$, $\Theta_{\mathfrak{m}}$, and $\eta_{\mathfrak{m}}$, the results of Sections 3 and 4 also hold for local Bézoutians and the local Scheja–Storch form. In particular, the local analog of Lemma 4.4 implies that the local Scheja–Storch form is equal to the local Bézoutian form.

5. Proof of Theorem 1.2

We now relate the Scheja–Storch form to the $\mathbb{A}^1$-degree. The following theorem was first proven in the case where p is a rational zero by Kass and Wickelgren [16], and then in the case where p has finite separable residue field over the ground field in [7, Corollary 1.4]. Recent work of Bachmann and Wickelgren [2] gives a general result about the relation between local $\mathbb{A}^1$-degrees and Scheja–Storch forms.

Theorem 5.1. *Let* $\operatorname{char} k \neq 2$. *Let* $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ *be an endomorphism of affine space with an isolated zero at a closed point* p. *Then we have that the local* $\mathbb{A}^1$*-degree of* f *at* p *and the Scheja–Storch form of* f *at* p *coincide as elements of* $\mathrm{GW}(k)$:

$$\deg_p^{\mathbb{A}^1}(f) = \mathrm{SS}_p(f).$$

Proof. We may rewrite f as a section of the trivial rank n bundle over affine space $\mathcal{O}^n_{\mathbb{A}^n_k} \to \mathbb{A}^n_k$. Under the hypothesis that p is isolated, we may find a neighborhood $X \subseteq \mathbb{A}^n_k$ of p where the section f is nondegenerate (meaning it is cut out by a regular sequence). By [2, Corollary 8.2], the local index of f at p with the trivial orientation, corresponding to the representable Hermitian K-theory spectrum KO, agrees with the local Scheja–Storch form as elements of $\mathrm{KO}^0(k)$:

$$\mathrm{ind}_p(f, \rho_{\mathrm{triv}}, \mathrm{KO}) = \mathrm{SS}_p(f). \tag{5-1}$$

Let $\mathbb{S}$ denote the sphere spectrum in the stable motivic homotopy category $\mathcal{SH}(k)$. It is a well-known fact that Hermitian K-theory receives a map from the sphere spectrum, inducing an isomorphism $\pi_0(\mathbb{S}) \xrightarrow{\sim} \pi_0(\mathrm{KO})$ if $\operatorname{char} k \neq 2$ (see for example [14, 6.9] for more detail); this is the only place where we use the assumption that $\operatorname{char} k \neq 2$. Combining this with the fact that $\pi_0(\mathbb{S}) = \mathrm{GW}(k)$ under Morel's degree isomorphism, we observe that (5-1) is really an equality in $\mathrm{GW}(k)$. By [2, Theorem 7.6, Example 7.7], the local index associated to the representable theory agrees with the local $\mathbb{A}^1$-degree:

$$\mathrm{ind}_p(f, \rho_{\mathrm{triv}}, \mathrm{KO}) = \deg_p^{\mathbb{A}^1}(f).$$

Combining these equalities gives the desired equality in $\mathrm{GW}(k)$. □

Remark 5.2. Bachmann and Wickelgren in fact show that $\deg_Z^{\mathbb{A}^1}(f) = \mathrm{SS}_Z(f)$ for any isolated zero locus Z of f [2, Corollary 8.2]. This gives an alternate viewpoint on the local decomposition described in Lemma 4.7

Corollary 5.3. *Let* $\operatorname{char} k \neq 2$. *The local Bézoutian bilinear form is the local* $\mathbb{A}^1$*-degree.*

Proof. As discussed in Remark 4.8, we can modify Lemma 4.4 to the local case by replacing Q, Δ, Θ, and η with $Q_{\mathfrak{m}}$, $\Delta_{\mathfrak{m}}$, $\Theta_{\mathfrak{m}}$, and $\eta_{\mathfrak{m}}$. The local Bézoutian form is thus equal to the local Scheja–Storch form, which is equal to the local $\mathbb{A}^1$-degree by Theorem 5.1. □

In contrast to previous techniques for computing the local $\mathbb{A}^1$-degree at rational or separable points, Corollary 5.3 gives an algebraic formula for the local $\mathbb{A}^1$-degree at any closed point.

As a result of the local decomposition of Scheja–Storch forms, the Bézoutian form agrees with the $\mathbb{A}^1$-degree globally as well.

Corollary 5.4. *Let* $\operatorname{char} k \neq 2$. *The Bézoutian bilinear form is the global* $\mathbb{A}^1$*-degree.*

Proof. Let Φ_η denote the Bézoutian bilinear form, which is equal to the global Scheja–Storch bilinear form by Lemma 4.4. By Lemma 4.7, the global Scheja–Storch form decomposes as a block sum of local Scheja–Storch forms. By Theorem 5.1, the local Scheja–Storch bilinear form agrees with the local $\mathbb{A}^1$-degree. Finally, we have that the sum of local $\mathbb{A}^1$-degrees is the global $\mathbb{A}^1$-degree. Putting this all together, we have

$$\Phi_\eta = \mathrm{SS}(f) = \sum_{\mathfrak{m}} \mathrm{SS}_{\mathfrak{m}}(f) = \sum_{\mathfrak{m}} \deg_{\mathfrak{m}}^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}(f).$$ □

Remark 5.5. It is not known if GW is represented by KO over fields of characteristic 2, which is the source of our assumption that $\operatorname{char} k \neq 2$. If this problem is resolved, one can remove any characteristic restrictions from our results. Alternately, Lemma 4.7 implies Corollaries 5.3 and 5.4 if all roots of f satisfy $\deg_p^{\mathbb{A}^1}(f) = \mathrm{SS}_p(f)$. By [16], [7], and [17, Proposition 34], Corollaries 5.3 and 5.4 are true in any characteristic if all roots of f are rational, étale, or separable.

5A. ***Computing the Bézoutian bilinear form.*** We now prove Theorem 1.2 by describing a method for computing the class in $\mathrm{GW}(k)$ of the Bézoutian bilinear form in terms of the Bézoutian.

Proof of Theorem 1.2. Let R denote either a global algebra Q or a local algebra $Q_{\mathfrak{m}}$. Let $\{\alpha_i\}$ be any basis for R, and express

$$\mathrm{B\acute{e}z}(f_1, \dots, f_n) = \sum_{i,j} B_{i,j}\alpha_i \otimes \alpha_j. \tag{5-2}$$

Rewriting this, we have

$$\mathrm{B\acute{e}z}(f_1, \dots, f_n) = \sum_i \alpha_i \otimes \Big(\sum_j B_{i,j}\alpha_j\Big).$$

Let $\beta_i := \sum_j B_{i,j}\alpha_j$, so that $\{\alpha_i\}$ and $\{\beta_i\}$ are dual bases. Then for any linear form $\lambda : R \to k$ for which $\{\alpha_i\}$ and $\{\beta_i\}$ are dual, we will have that Φ_λ agrees with the global or local $\mathbb{A}^1$-degree (depending on our choice of R) by Corollaries 5.3 and 5.4. Let λ be such a form. The product of α_i and β_j is given by

$$\alpha_i\beta_j = \alpha_i \cdot \sum_s B_{j,s}\alpha_s.$$

Applying λ to each side, we get an indicator function

$$\delta_{ij} = \lambda(\alpha_i\beta_j) = \lambda\Big(\alpha_i \sum_s B_{j,s}\alpha_s\Big) = \sum_s B_{j,s}\lambda(\alpha_i\alpha_s).$$

Varying over all i, j, s, this equation above tells us that the identity matrix is equal to the product of the matrix $(B_{j,s})$ and the matrix $(\lambda(\alpha_i\alpha_s)) = (\lambda(\alpha_s\alpha_i))$. Explicitly, we have that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,m} \end{pmatrix} \begin{pmatrix} \lambda(\alpha_1^2) & \lambda(\alpha_1\alpha_2) & \cdots & \lambda(\alpha_1\alpha_m) \\ \lambda(\alpha_2\alpha_1) & \lambda(\alpha_2^2) & \cdots & \lambda(\alpha_2\alpha_m) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda(\alpha_m\alpha_1) & \lambda(\alpha_m\alpha_2) & \cdots & \lambda(\alpha_m^2) \end{pmatrix}.$$

Thus the Gram matrix for Φ_λ in the basis $\{\alpha_i\}$ is $(B_{i,j})^{-1}$. We conclude by proving that $(B_{i,j})$ and $(B_{i,j})^{-1}$ represent the same element of $\mathrm{GW}(k)$. Since any symmetric bilinear form can be diagonalized, there is an invertible $m \times m$ matrix S such that $S^T \cdot (B_{i,j}) \cdot S$ is diagonal. Since $(S^T \cdot (B_{i,j}) \cdot S) \cdot (S^{-1} \cdot (B_{i,j})^{-1} \cdot (S^{-1})^T)$ is equal to the identity matrix, it follows that $S^{-1} \cdot (\lambda(\alpha_i\alpha_j)) \cdot (S^{-1})^T$ is diagonal with entries inverse to the diagonal entries of $S^T \cdot (B_{i,j}) \cdot S$. Applying the equality $\langle a\rangle = \langle 1/a\rangle$ along the diagonals, it follows that $(B_{i,j})^{-1}$ and $(B_{i,j})$ define the same element in $\mathrm{GW}(k)$. Theorem 1.2 now follows from Corollaries 5.3 and 5.4. □

The following tables describe algorithms for computing the global and local $\mathbb{A}^1$-degrees in terms of the Bézoutian bilinear form. A Sage implementation of these algorithms is available at [8].

Computing the global $\mathbb{A}^1$-degree via the Bézoutian

(1) Compute the Δ_{ij} and the image of their determinant $\text{Béz}(f) = \det(\Delta_{ij})$ in $k[X, Y]/(f(X), f(Y))$.

(2) Pick a k-vector space basis $a_1, \ldots, a_m$ of $Q = k[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. Find $B_{i,j} \in k$ such that

$$\text{Béz}(f) = \sum_{i=1}^{m} B_{i,j} a_i(X) a_j(Y).$$

(3) The matrix $B = (B_{i,j})$ represents $\deg^{\mathbb{A}^1}(f)$. Diagonalize B to write its class in $\mathrm{GW}(k)$.

Computing the local $\mathbb{A}^1$-degree via the Bézoutian

(1) Compute the Δ_{ij} and the image of their determinant $\text{Béz}(f) = \det(\Delta_{ij})$ in $k[X, Y]/(f(X), f(Y))$.

(2) Pick a k-vector space basis $a_1, \ldots, a_m$ of $Q_\mathfrak{m} = k[X_1, \ldots, X_n]_\mathfrak{m}/(f_1, \ldots, f_n)$. Find $B_{i,j} \in k$ such that

$$\text{Béz}(f) = \sum_{i=1}^{m} B_{i,j} a_i(X) a_j(Y).$$

(3) The matrix $B = (B_{i,j})$ represents $\deg_\mathfrak{m}^{\mathbb{A}^1}(f)$. Diagonalize B to write its class in $\mathrm{GW}(k)$.

6. Calculation rules

Using the Bézoutian characterization of the $\mathbb{A}^1$-degree, we are able to establish various calculation rules for local and global $\mathbb{A}^1$-degrees. See [18; 29] for related results in the local case.

Our ultimate goal in this section is the product rule for the $\mathbb{A}^1$-degree (see Proposition 6.5), which was already known by the work of Morel. See the paragraph preceding Proposition 6.5 for a more detailed discussion.

Proposition 6.1. *Suppose that $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$ are endomorphisms of affine space that generate the same ideal*

$$I = (f_1, \dots, f_n) = (g_1, \dots, g_n) \lhd k[x_1, \dots, x_n].$$

If $\text{Béz}(f) = \text{Béz}(g)$ *in* $k[X, Y]$, *then* $\deg^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}(g)$, *and* $\deg_p^{\mathbb{A}^1}(f) = \deg_p^{\mathbb{A}^1}(g)$ *for all* p.

Proof. We may choose the same basis for $Q = k[x_1, \dots, x_n]/I$ (or Q_p in the local case) in our computation for the degrees of f and g. The Bézoutians $\text{Béz}(f) = \text{Béz}(g)$ will have the same coefficients in this basis, so their Gram matrices will coincide. □

The following result is the global analogue of [29, Lemma 14].

Lemma 6.2. *Let $f = (f_1, \dots, f_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be an endomorphism of $\mathbb{A}^n_k$ with only isolated zeros. Let $A \in k^{n\times n}$ be an invertible matrix. Then*

$$\langle \det A\rangle \cdot \deg^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}(A \circ f)$$

as elements of $\text{GW}(k)$.

Proof. Write $A = (a_{ij})$ and

$$\Delta^g_{ij} = \frac{g_i(X_1, \dots, X_j, Y_{j+1}, \dots, Y_n) - g_i(X_1, \dots, X_{j-1}, Y_j, \dots, Y_n)}{X_j - Y_j},$$

where g is either f or $A \circ f$. Then $\Delta^{A\circ f}_{ij} = \sum_{l=1}^n a_{il}\Delta^f_{lj}$, and thus $(\Delta^{A\circ f}_{ij}) = A \cdot (\Delta^f_{ij})$ as matrices over $k[X, Y]$. The ideals generated by $A \circ (f_1, \dots, f_n)$ and $(f_1, \dots, f_n)$ are equal, and the images in $Q \otimes_k Q$ of $\det(\Delta^{A\circ f}_{ij})$ and $\det A \cdot \det(\Delta f_{ij})$ are equal. Thus the Gram matrix of the Bézoutian bilinear form for $A \circ f$ is $\det A$ times the Gram matrix of the Bézoutian bilinear form for f. Proposition 6.1 then proves the claim. □

Example 6.3. We may apply Lemma 6.2 in the case where A is a permutation matrix associated to some permutation $\sigma \in \Sigma_n$. Letting $f_\sigma := (f_{\sigma(1)}, \dots, f_{\sigma(n)})$, we observe that

$$\deg_p^{\mathbb{A}^1}(f_\sigma) = \langle \text{sgn}(\sigma)\rangle \cdot \deg_p^{\mathbb{A}^1}(f)$$

at any isolated zero p of f, and an analogous statement is true for global degrees as well.

Next, we prove a lemma inspired by [18, Lemma 12].

Lemma 6.4. *Let $f, g : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be two endomorphisms of $\mathbb{A}^n_k$. Assume that f and g are quasifinite. Let $L \in M_n(k)$ be an invertible $n \times n$ matrix, which defines a morphism $L : \mathbb{A}^n_k \to \mathbb{A}^n_k$ given by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n) \cdot L^T$. Let I_n denote the $n \times n$ identity matrix, and assume that* $\det(I_n + t(L - I_n)) \in k[t]$ *is in fact an element of $k^\times$. Then* $\deg^{\mathbb{A}^1}(f \circ g) = \deg^{\mathbb{A}^1}(f \circ L \circ g)$.

Proof. Quasifinite morphisms have isolated zero loci by [32, Definition 01TD (3)]. The composition of quasifinite morphisms is again quasifinite [32, Lemma 01TL], so $f \circ g$ has isolated zero locus.

Next, we show that L is also quasifinite. We will actually prove a stronger statement. Let $A_t \in M_n(k[t])$ be an invertible $n \times n$ matrix, which implies that $\det A_t \in k[t]^\times = k^\times$. This matrix determines a family of morphisms $A_t : \mathbb{A}^n_k \times \mathbb{A}^1_k \to \mathbb{A}^n_k$ by $(x_1, \dots, x_n, t) \mapsto (x_1, \dots, x_n) \cdot A_t^T$. Given $t_0 \in \mathbb{A}^1_k$, the morphism A_{t_0} has Jacobian determinant $\det\big(\frac{\partial (A_{t_0})_i}{\partial x_j}\big) = \det A_t$, which is a unit. In particular, A_{t_0} is unramified for each $t_0 \in \mathbb{A}^1_k$. Thus A_{t_0} is locally quasifinite [32, Lemma 02VF]. Since $\mathbb{A}^n_k$ is Noetherian, $A_{t_0} : \mathbb{A}^n_k \to \mathbb{A}^n_k$ is quasicompact. Quasicompact and locally quasifinite morphisms are quasifinite [32, Lemma 01TJ], so we conclude that A_{t_0} is quasifinite for each $t_0 \in \mathbb{A}^1_k$.

Just as in [18, Lemma 12], we now define $L_t = I_n + t \cdot (L - I_n)$. Our assumption on $\det(I_n + t(L - I_n))$ implies that L_t is invertible. Thus L_t is quasifinite, so $f \circ L_t \circ g$ is quasifinite and hence only has isolated zeros for all t. Set

$$\widetilde{Q} = \frac{k[t][x_1, \dots, x_n]}{(f \circ L_t \circ g)}.$$

Then [30, page 182] gives us a Scheja–Storch form $\tilde{\eta} : \widetilde{Q} \to k[t]$ such that the bilinear form $\Phi_{\tilde{\eta}} : \widetilde{Q} \times \widetilde{Q} \to k[t]$ is symmetric and nondegenerate. By Harder's theorem [16, Lemma 30], the stable isomorphism class of $\Phi_{\tilde{\eta}} \otimes_k k(t_0) \in \mathrm{GW}(k)$ is independent of $t_0 \in \mathbb{A}^1_k(k)$. In particular, the Scheja–Storch bilinear forms of $f \circ L_0 \circ g = f \circ g$ and $f \circ L_1 \circ g = f \circ L \circ g$ are isomorphic. □

The following product rule is a consequence of Morel's proof that the $\mathbb{A}^1$-degree is a ring isomorphism [24, Lemma 6.3.8]. We give a more hands-on proof of this product rule. See [18, Theorem 13] and [29, Theorem 26] for an analogous proof of the product rule for local degrees at rational points.

Proposition 6.5 (product rule). *Let $f, g : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be two quasifinite endomorphisms of $\mathbb{A}^n_k$. Then* $\deg^{\mathbb{A}^1}(f \circ g) = \deg^{\mathbb{A}^1}(f) \cdot \deg^{\mathbb{A}^1}(g)$.

Proof. We follow the proofs of [18, Theorem 13] and [29, Theorem 26]. The general idea is to mimic the Eckmann–Hilton argument [10]. Let $x := (x_1, \dots, x_n)$ and $y := (y_1, \dots, y_n)$. Define $\tilde{f}, \tilde{g} : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ by $\tilde{f}(x, y) = (f(x), y)$ and $\tilde{g}(x, y) = (g(x), y)$, and note that $\tilde{f}$ and $\tilde{g}$ are both quasifinite because f and g are quasifinite. Since $(f \circ g, y)$ and $\tilde{f} \circ \tilde{g}$ define the same ideal in $k[x, y]$ and have the same Bézoutian, we have $\deg^{\mathbb{A}^1}(f \circ g) = \deg^{\mathbb{A}^1}(\tilde{f} \circ \tilde{g})$ by Proposition 6.1.

Let $g \times f : \mathbb{A}^n_k \times \mathbb{A}^n_k \to \mathbb{A}^n_k \times \mathbb{A}^n_k$ be given by $(g \times f)(x, y) = (g(x), f(y))$. Using Lemma 6.4 repeatedly, we will show that $\deg^{\mathbb{A}^1}(\tilde{f} \circ \tilde{g}) = \deg^{\mathbb{A}^1}(g \times f)$. Let I_n be the $n \times n$ identity matrix, and let

$$L_1 = \begin{pmatrix} I_n & 0 \\ -I_n & I_n \end{pmatrix}, \quad L_2 = \begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

By construction, $\det(I_{2n}+t(L_1-I_{2n}))=\det(I_{2n}+t(L_2-I_{2n}))=1$, so Lemma 6.4 implies that

$$\deg^{\mathbb{A}^1}(\tilde{f}\circ\tilde{g})=\deg^{\mathbb{A}^1}(\tilde{f}\circ L_1\circ\tilde{g})=\deg^{\mathbb{A}^1}(\tilde{f}\circ L_2\circ(L_1\circ\tilde{g}))=\deg^{\mathbb{A}^1}(\tilde{f}\circ L_1\circ(L_2\circ L_1\circ\tilde{g})).$$

One can check that $A\circ\tilde{f}\circ L_1\circ L_2\circ L_1\circ\tilde{g}=g\times f$. By Lemma 6.2, we have

$$\langle\det A\rangle\cdot\deg^{\mathbb{A}^1}(\tilde{f}\circ\tilde{g})=\langle\det A\rangle\cdot\deg^{\mathbb{A}^1}(\tilde{f}\circ L_1\circ L_2\circ L_1\circ\tilde{g})=\deg^{\mathbb{A}^1}(g\times f).$$

Since $\det A=1$, it just remains to show that $\deg^{\mathbb{A}^1}(g\times f)=\deg^{\mathbb{A}^1}(g)\cdot\deg^{\mathbb{A}^1}(f)$. Let $a_1,\dots,a_m$ be a basis for $k[x_1,\dots,x_n]/(g_1,\dots,g_n)$ and $a'_1,\dots,a'_{m'}$ be a basis for $k[y_1,\dots,y_n]/(f_1,\dots,f_n)$. Write $\text{Béz}(g)=\sum_{i,j=1}^m B_{ij}a_i\otimes a_j$ and $\text{Béz}(f)=\sum_{i,j=1}^{m'} B'_{ij}a'_i\otimes a'_j$. By Theorem 1.2, (B_{ij}) and (B'_{ij}) are the Gram matrices for $\deg^{\mathbb{A}^1}(g)$ and $\deg^{\mathbb{A}^1}(f)$, respectively. Next, we have $\text{Béz}(g\times f)=\text{Béz}(g)\cdot\text{Béz}(f)$, since

$$(\Delta_{ij}^{g\times f})=\begin{pmatrix}(\Delta_{ij}^{g}) & 0\\ 0 & (\Delta_{ij}^{f})\end{pmatrix}.$$

Note that $\{a_i(x)a'_{i'}(y)\}_{i,i'=1}^{m,m'}$ is a basis of $k[x_1,\dots,x_n,y_1,\dots,y_n]/(g_1(x),\dots,g_n(x),f_1(y),\dots,f_n(y))$. In this basis, we have

$$\text{Béz}(g)\cdot\text{Béz}(f)=\sum_{i,j=1}^{m}\sum_{i',j'=1}^{m'}B_{ij}B'_{i'j'}a_ia'_{i'}\otimes a_ja'_{j'},$$

so the Gram matrix of $\deg^{\mathbb{A}^1}(g\times f)$ is the tensor product $(B_{ij})\otimes(B'_{ij})$. We thus we have an equality $\deg^{\mathbb{A}^1}(g\times f)=\deg^{\mathbb{A}^1}(g)\cdot\deg^{\mathbb{A}^1}(f)$ in $\mathrm{GW}(k)$. □

7. Examples

We now give a few remarks and examples about computing the Bézoutian.

Remark 7.1. It is not always the case that the determinant $\det(\Delta_{ij})\in k[X,Y]$ is symmetric. For example, consider the morphism $f:\mathbb{A}^2_k\to\mathbb{A}^2_k$ sending $(x_1,x_2)\mapsto(x_1x_2,x_1+x_2)$. Then the Bézoutian is given by

$$\text{Béz}(f)=\det\begin{pmatrix}X_2 & Y_1\\ 1 & 1\end{pmatrix}=X_2-Y_1.$$

However, the Bézoutian is symmetric once we pass to the quotient $k[X,Y]/(f(X),f(Y))$ [4, 2.12]. Continuing the present example, let $\{1,x_2\}$ be a basis for the algebra $Q=k[x_1,x_2]/(x_1x_2,x_1+x_2)$. Then we have that

$$\text{Béz}(f)=X_2-Y_1=X_2+Y_2,$$

which is symmetric. Moreover, the Bézoutian bilinear form is represented by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, so $\deg^{\mathbb{A}^1}(f)=\mathbb{H}$.

Example 7.2. Let $k=\mathbb{F}_p(t)$, where p is an odd prime, and consider the endomorphism of the affine plane given by

$$\begin{aligned}f:\operatorname{Spec}\mathbb{F}_p(t)[x_1,x_2]&\to\operatorname{Spec}\mathbb{F}_p(t)[x_1,x_2]\\(x_1,x_2)&\mapsto(x_1^p-t,x_1x_2).\end{aligned}$$

As the residue field of the zero of f is not separable over k, existing strategies for computing the local $\mathbb{A}^1$-degree are insufficient. Our results allow us to compute this $\mathbb{A}^1$-degree. The Bézoutian is given by

$$\begin{aligned}\mathrm{Béz}(f) &= \det\begin{pmatrix}\frac{X_1^p-Y_1^p}{X_1-Y_1} & 0\\ X_2 & Y_1\end{pmatrix}\\ &= X_1^{p-1}Y_1 + X_1^{p-2}Y_1^2 + \ldots + X_1Y_1^{p-1} + Y_1^p\\ &= X_1^{p-1}Y_1 + X_1^{p-2}Y_1^2 + \ldots + X_1Y_1^{p-1} + t.\end{aligned}$$

In the basis $\{1, x_1, \ldots, x_1^{p-1}\}$ of Q, the Bézoutian bilinear form consists of a t in the upper left corner and a 1 in each entry just below the antidiagonal. Thus

$$\deg^{\mathbb{A}^1}(f) = \deg^{\mathbb{A}^1}_{(t^{1/p},0)}(f) = \langle t\rangle + \tfrac{1}{2}(p-1)\mathbb{H}.$$

Example 7.3. Let $f_1 = (x_1-1)x_1x_2$ and $f_2 = (ax_1^2 - bx_2^2)$ for some $a, b \in k^\times$ with $\frac{a}{b}$ not a square in k. Then $f = (f_1, f_2)$ has isolated zeros at $\mathfrak{m} := (x_1 - 0, x_2 - 0)$ and $\mathfrak{n} := (x_1 - 1, x_2^2 - a/b)$. We will use Bézoutians to compute the local degrees $\deg^{\mathbb{A}^1}_{\mathfrak{m}}(f)$ and $\deg^{\mathbb{A}^1}_{\mathfrak{n}}(f)$, as well as the global degree $\deg^{\mathbb{A}^1}(f)$. Let

$$Q = \frac{k[x_1, x_2]}{((x_1-1)x_1x_2,\, ax_1^2 - bx_2^2)}.$$

We first compute the global Bézoutian as

$$\begin{aligned}\mathrm{Béz}(f) &= \det\begin{pmatrix}(X_1+Y_1-1)X_2 & a(X_1+Y_1)\\ Y_1^2-Y_1 & -b(X_2+Y_2)\end{pmatrix}\\ &= -a(X_1Y_1^2 - X_1Y_1 + Y_1^3 - Y_1^2) - b(X_1X_2^2 + X_2^2Y_1 - X_2^2 + X_1X_2Y_2 + X_2Y_1Y_2 - X_2Y_2).\end{aligned}$$

In the basis $\{1, x_1, x_2, x_1^2, x_1x_2, x_1^3\}$ of Q, the Bézoutian is given by

$$\mathrm{Béz}(f) = -a(X_1Y_1^2 - X_1Y_1 + Y_1^3 - Y_1^2 + X_1^3 + X_1^2Y_1 - X_1^2) - b(X_1X_2Y_2 + X_2Y_1Y_2 - X_2Y_2).$$

We now write the Bézoutian matrix given by the coefficients of $\mathrm{Béz}(f)$:

	1	X_1	X_2	X_1^2	X_1X_2	X_1^3
1	0	0	0	a	0	$-a$
Y_1	0	a	0	$-a$	0	0
Y_2	0	0	b	0	$-b$	0
Y_1^2	a	$-a$	0	0	0	0
Y_1Y_2	0	0	$-b$	0	0	0
Y_1^3	$-a$	0	0	0	0	0

One may check (e.g., with a computer) that this is equal to $3\mathbb{H}$ in $\mathrm{GW}(k)$.

In $Q_{\mathfrak{m}}$, we have that $x_1^2x_2 = x_1x_2 = 0$ and $x_1^3 = \frac{b}{a}x_1x_2^2 = 0$. In the basis $\{1, x_1, x_2, x_1^2\}$ of $Q_{\mathfrak{m}}$, the global Bézoutian reduces to

$$\mathrm{Béz}(f) = -a(X_1Y_1^2 - X_1Y_1 - Y_1^2 + X_1^2Y_1 - X_1^2) + bX_2Y_2.$$

We thus get the Bézoutian matrix at $\mathfrak{m}$:

$$\begin{array}{c|cccc} & 1 & X_1 & X_2 & X_1^2 \\ \hline 1 & 0 & 0 & 0 & a \\ Y_1 & 0 & a & 0 & -a \\ Y_2 & 0 & 0 & b & 0 \\ Y_1^2 & a & -a & 0 & 0 \end{array}$$

This is $\mathbb{H} + \langle a, b\rangle$ in $\mathrm{GW}(k)$.

In $Q_{\mathfrak{n}}$, we have $x_1 = 1$. In the basis $\{1, x_2\}$ for $Q_{\mathfrak{n}}$, the Bézoutian reduces to

$$\mathrm{Béz}(f) = -a - bX_2Y_2.$$

We can then write the Bézoutian matrix at $\mathfrak{n}$:

$$\begin{array}{c|cc} & 1 & X_2 \\ \hline 1 & -a & 0 \\ Y_2 & 0 & -b \end{array}$$

This is $\langle -a, -b\rangle$ in $\mathrm{GW}(k)$. Note that $\langle -a, -b\rangle$ need not be equal to $\mathbb{H}$. However, this does not contradict [29, Theorem 2], since $\mathfrak{n}$ is a nonrational point.

Putting these computations together, we see that

$$\deg_{\mathfrak{m}}^{\mathbb{A}^1}(f) + \deg_{\mathfrak{n}}^{\mathbb{A}^1}(f) = \mathbb{H} + \langle a, b\rangle + \langle -a, -b\rangle = 3\mathbb{H} = \deg^{\mathbb{A}^1}(f).$$

8. Application: The $\mathbb{A}^1$-Euler characteristic of Grassmannians

As an application of Theorem 1.2, we compute the $\mathbb{A}^1$-Euler characteristic of various low-dimensional Grassmannians in Example 8.2 and Figure 1. These computations suggest a recursive formula for the $\mathbb{A}^1$-Euler characteristic of an arbitrary Grassmannian, which we prove in Theorem 8.4. This formula is analogous to the recursive formulas for the Euler characteristics of complex and real Grassmannians. Theorem 8.4 is probably well-known, and the proof is essentially a combination of results of Hoyois, Levine, and Bachmann–Wickelgren.

8A. ***The $\mathbb{A}^1$-Euler characteristic.*** Let X be a smooth, proper k-variety of dimension n with structure map $\pi : X \to \operatorname{Spec} k$. Let $p : T_X \to X$ denote the tangent bundle of X. The $\mathbb{A}^1$-Euler characteristic $\chi^{\mathbb{A}^1}(X) \in \mathrm{GW}(k)$ is a refinement of the classical Euler characteristic. In particular, if $k = \mathbb{R}$, then $\operatorname{rank} \chi^{\mathbb{A}^1}(X) = \chi(X(\mathbb{C}))$ and $\operatorname{sgn} \chi^{\mathbb{A}^1}(X) = \chi(X(\mathbb{R}))$. There exist several equivalent definitions of the $\mathbb{A}^1$-Euler characteristic [20; 21; 1]. For example, we may define $\chi^{\mathbb{A}^1}(X)$ to be the π-pushforward of the $\mathbb{A}^1$-Euler class

$$e(T_X) := z^* z_* 1_X \in \widetilde{\mathrm{CH}}^n(X, \omega_{X/k}),$$

of the tangent bundle [20], where $z : X \to T_X$ is the zero section and $\widetilde{\mathrm{CH}}^d(X, \omega_{X/k})$ is the Chow–Witt group defined by Barge and Morel [3; 12]. That is,

$$\chi^{\mathbb{A}^1}(X) := \pi_*(e(T_X)) \in \widetilde{\mathrm{CH}}^0(\operatorname{Spec} k) = \mathrm{GW}(k).$$

Analogous to the classical case [23], the $\mathbb{A}^1$-Euler characteristic can be computed as the sum of local $\mathbb{A}^1$-degrees at the zeros of a general section of the tangent bundle using the work of Kass and Wickelgren [2; 17; 20]. We now describe this process. Let σ be a section of T_X which only has isolated zeros. For a zero x of σ, choose Nisnevich coordinates $\psi : U \to \mathbb{A}^n_k$ around x.[2] Since ψ is étale, it induces an isomorphism of tangent spaces and thus yields local coordinates around x. Shrinking U if necessary, we can trivialize $T_X|_U \cong U \times \mathbb{A}^n_k$. The chosen Nisnevich coordinates (ψ, U) and trivialization $\tau : T_X|_U \cong U \times \mathbb{A}^n_k$ each define distinguished elements $d_\psi, d_\tau \in \det T_X|_U$. In turn, this yields a distinguished section d of $\mathcal{H}om(\det T_X|_U, \det T_X|_U)$, which is defined by $d_\psi \mapsto d_\tau$. We say that a trivialization τ is *compatible* with the chosen coordinates (ψ, U) if the image of the distinguished section d under the canonical isomorphism $\rho : \mathcal{H}om(\det T_X|_U, \det T_X|_U) \cong \mathcal{O}_U$ is a square [17, Definition 21].

Given a compatible trivialization $\tau : T_X|_U \cong U \times \mathbb{A}^n_k$, the section σ trivializes to $\sigma : U \to \mathbb{A}^n_k$. We can then define the *local index* $\operatorname{ind}_x \sigma$ at x to be the $\mathbb{A}^1$-degree of the composite

$$\frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k} \to \frac{\mathbb{P}^n_k}{\mathbb{P}^n_k \backslash \{\psi(x)\}} \cong \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \backslash \{\psi(x)\}} \cong \frac{U}{U \backslash \{x\}} \xrightarrow{\sigma} \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k \backslash \{0\}} \cong \frac{\mathbb{P}^n_k}{\mathbb{P}^{n-1}_k}.$$

Here, the first map is the collapse map, the second map is excision, the third map is induced by the Nisnevich coordinates (ψ, U), and the fifth map is purity; see e.g., [2, Definition 7.1]. By [17, Theorem 3], the $\mathbb{A}^1$-Euler characteristic is then the sum of local indices

$$\chi^{\mathbb{A}^1}(X) = \sum_{x \in \sigma^{-1}(0)} \operatorname{ind}_x \sigma \in \mathrm{GW}(k).$$

By Theorem 1.2, we may thus compute the $\mathbb{A}^1$-Euler characteristic by computing the global Bézoutian bilinear form of an appropriate map $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$.

Remark 8.1. If all the zeros of $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ are simple, then each local ring $Q_\mathfrak{m}$ in the decomposition of $Q = k[x_1, \dots, x_n]/(f_1, \dots, f_n) = Q_{\mathfrak{m}_1} \times \dots \times Q_{\mathfrak{m}_s}$ is equal to the residue field of the corresponding zero. If each residue field $Q_{\mathfrak{m}_i}$ is a separable extension of k, then the $\mathbb{A}^1$-degree of f is equal to sum of the scaled trace forms $\mathrm{Tr}_{Q_{\mathfrak{m}_i}/k}(\langle J(f)|_{\mathfrak{m}_i}\rangle)$ (see e.g., [7, Definition 1.2]), where $J(f)|_{\mathfrak{m}_i}$ is the determinant of the Jacobian of f evaluated at the point $\mathfrak{m}_i$. In [27] the last named author uses the scaled trace form for several $\mathbb{A}^1$-Euler number computations. However, Theorem 1.2 yields a formula for $\deg^{\mathbb{A}^1}(f)$ for any f with only isolated zeros and without any restriction on the residue field of each zero. Moreover, we can even compute $\deg^{\mathbb{A}^1}(f)$ without solving for the zero locus of f.

8B. ***The $\mathbb{A}^1$-Euler characteristic of Grassmannians.*** Let $G := \mathrm{Gr}_k(r, n)$ be the Grassmannian of r-planes in k^n. In order to compute $\chi^{\mathbb{A}^1}(G)$, we first need to describe Nisnevich coordinates and compatible trivializations for G and T_G. We then need to choose a convenient section of T_G and describe the resulting endomorphism $\mathbb{A}^{r(n-r)}_k$. The tangent bundle $T_G \to G$ is isomorphic to $p : \mathcal{H}om(\mathcal{S}, \mathcal{Q}) \to G$, where $\mathcal{S} \to G$ and $\mathcal{Q} \to G$ are the universal sub- and quotient bundles.

[2] *Nisnevich coordinates* consist of an open neighborhood U of x and an étale map $\psi : U \to \mathbb{A}^n_k$ that induces an isomorphism of residue fields $k(x) \cong k(\psi(x))$ [17, Definition 18].

We now describe Nisnevich coordinates on G and a compatible trivialization of T_G, following [31]. Let $d = r(n-r)$ be the dimension of G, and let $\{e_1, \dots, e_n\}$ be the standard basis of k^n. Let $\mathbb{A}^d_k = \operatorname{Spec} k[\{x_{i,j}\}_{i,j=1}^{r,n-r}] \cong U \subset G$ be the open affine subset consisting of the r-planes

$$H(\{x_{i,j}\}_{i,j=1}^{r,n-r}) := \operatorname{span}\left\{e_{n-r+i} + \sum_{j=1}^{n-r} x_{i,j} e_j\right\}_{i=1}^{r}.$$

The map $\psi : U \to \mathbb{A}^d_k$ given by $\psi(H(\{x_{i,j}\}_{i,j=1}^{r,n-r})) = (\{x_{i,j}\}_{i,j=1}^{n-r,r})$ yields Nisnevich coordinates (ψ, U) centered at $\psi(\operatorname{span}\{e_{n-r+1}, \dots, e_n\}) = (0, \dots, 0)$. For the trivialization of $T_G|_U$, let

$$\tilde{e}_i = \begin{cases} e_i & i \leq n-r, \\ e_i + \sum_{j=1}^{n-r} x_{i-(n-r),j} e_j & i \geq n-r+1. \end{cases}$$

Then $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ is a basis for k^n, and we denote the dual basis by $\{\tilde{\phi}_1, \dots, \tilde{\phi}_n\}$. Over U, the bundles $\mathcal{S}^*$ and $\mathcal{Q}$ are trivialized by $\{\tilde{\phi}_{n-r+1}, \dots, \tilde{\phi}_n\}$ and $\{\tilde{e}_1, \dots, \tilde{e}_{n-r}\}$, respectively. Since

$$T_G \cong \mathcal{H}om(\mathcal{S}, \mathcal{Q}) \cong \mathcal{S}^* \otimes \mathcal{Q},$$

we get a trivialization of $T_G|_U$ given by $\{\tilde{\phi}_{n-r+i} \otimes \tilde{e}_j\}_{i,j=1}^{r,n-r}$. By construction, our Nisnevich coordinates (ψ, U) induce this local trivialization of T_G. It follows that the distinguished element of $\operatorname{Hom}(\det T_G|_U, \det T_G|_U)$ sending the distinguished element of $\det T_G|_U$ (determined by the Nisnevich coordinates) to the distinguished element of $T_G|_U$ (determined by our local trivialization) is just the identity, which is a square.

Next, we describe sections of $T_G \to G$ and the resulting endomorphisms $\mathbb{A}^d_k \to \mathbb{A}^d_k$. Let $\{\phi_1, \dots, \phi_n\}$ be the dual basis of the standard basis $\{e_1, \dots, e_n\}$ of k^n. A homogeneous degree 1 polynomial $\alpha \in k[\phi_1, \dots, \phi_n]$ gives rise to a section s of $\mathcal{S}^*$, defined by evaluating α. In particular, given a vector $t = \sum_{i=1}^n t_i \tilde{e}_i$ in $H(\{x_{i,j}\}_{i,j=1}^{r,n-r})$, we use the dual change of basis

$$\phi_j = \begin{cases} \tilde{\phi}_j + \sum_{i=1}^r x_{i,j} \tilde{\phi}_{n-r+i} & j \leq n-r, \\ \tilde{\phi}_j & j \geq n-r+1 \end{cases}$$

to set

$$s(t) = \alpha\left(t_1 + \sum_{i=1}^r x_{i,1} t_{n-r+i}, \dots, t_{n-r} + \sum_{i=1}^r x_{i,n-r} t_{n-r+i}, t_{n-r+1}, \dots, t_n\right).$$

Note that $t_1 = \cdots = t_{n-r} = 0$ if and only if $t \in H(\{x_{i,j}\}_{i,j=1}^{r,n-r})$, so $s(t) \in k[t_{n-r+1}, \dots, t_n]$. Taking n sections $s_1, \dots, s_n$ of $\mathcal{S}^*$, we get a section of $T_G \cong \mathcal{H}om(\mathcal{S}, \mathcal{Q})$ given by

$$\mathcal{S} \xrightarrow{(s_1, \dots, s_n)} \mathbb{A}^n_k \to \mathcal{Q},$$

where the second map is quotienting by $\{\tilde{e}_{n-r+1}, \dots, \tilde{e}_n\}$. We obtain our map $\mathbb{A}^d_k \to \mathbb{A}^d_k$ by applying the trivializations $\{\tilde{\phi}_{n-r+i} \otimes \tilde{e}_j\}_{i,j=1}^{r,n-r}$ of T_G. Explicitly, take n sections $s_1, \dots, s_n$ of $\mathcal{S}^*$. Since $e_i =$

$\tilde{e}_i - \sum_{j=1}^{n-r} x_{i-(n-r),j} e_j$ for $i > n-r$, we have

$$s_j e_j \equiv s_j e_j - \sum_{i=1}^{r} x_{i,j} s_{n-r+i} e_j \mod (\tilde{e}_{n-r+1}, \ldots, \tilde{e}_n),$$

for all $j \leq n-r$. Recall that $e_j = \tilde{e}_j$ for $j \leq n-r$. The coordinate of $\mathbb{A}_k^d \to \mathbb{A}_k^d$ corresponding to $\tilde{\phi}_{n-r+i} \otimes \tilde{e}_j$ is thus the coefficient of t_{n-r+i} in $s_j(t) - \sum_{\ell=1}^{r} x_{\ell,j} s_{n-r+\ell}(t)$.

For a general section σ of $p : T_G \to G$, the finitely many zeros of σ will all lie in U. In this case, the $\mathbb{A}^1$-Euler characteristic of G is equal to the global $\mathbb{A}^1$-degree of the resulting map $\mathbb{A}_k^d \to \mathbb{A}_k^d$, which can computed using the Bézoutian.

Example 8.2 ($\mathrm{Gr}_k(2, 4)$). Let

$$\begin{aligned}
\alpha_1 &= \phi_2 = \tilde{\phi}_2 + x_{1,2}\tilde{\phi}_3 + x_{2,2}\tilde{\phi}_4, \\
\alpha_2 &= \phi_3 = \tilde{\phi}_3, \\
\alpha_3 &= \phi_4 = \tilde{\phi}_4, \\
\alpha_4 &= \phi_1 = \tilde{\phi}_1 + x_{1,1}\tilde{\phi}_3 + x_{2,1}\tilde{\phi}_4.
\end{aligned}$$

Evaluating at $t = (0, 0, t_3, t_4)$ in the basis $\{\tilde{e}_i\}$, we have

$$\begin{aligned}
s_1 &= x_{1,2}t_3 + x_{2,2}t_4, \\
s_2 &= t_3, \\
s_3 &= t_4, \\
s_4 &= x_{1,1}t_3 + x_{2,1}t_4.
\end{aligned}$$

It remains to read off the coefficients of t_3 and t_4 of

$$\begin{aligned}
s_1 - x_{1,1}s_3 - x_{2,1}s_4 &= (x_{1,2} - x_{1,1}x_{2,1})t_3 + (x_{2,2} - x_{1,1} - x_{2,1}^2)t_4, \\
s_2 - x_{1,2}s_3 - x_{2,2}s_4 &= (1 - x_{1,1}x_{2,2})t_3 + (-x_{1,2} - x_{2,1}x_{2,2})t_4.
\end{aligned}$$

We thus have our endomorphism $\sigma : \mathbb{A}_k^4 \to \mathbb{A}_k^4$ defined by

$$\sigma = (x_{1,2} - x_{1,1}x_{2,1},\ x_{2,2} - x_{1,1} - x_{2,1}^2,\ 1 - x_{1,1}x_{2,2},\ -x_{1,2} - x_{2,1}x_{2,2}).$$

Using the Sage implementation of the Bézoutian formula for the $\mathbb{A}^1$-degree [8], we can calculate $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(2, 4)) = \deg^{\mathbb{A}^1}(\sigma) = 2\mathbb{H} + \langle 1, 1\rangle$.

Using a computer, we performed computations analogous to Example 8.2 for $r \leq 5$ and $n \leq 7$. These $\mathbb{A}^1$-Euler characteristics of Grassmannians are recorded in Figure 1.

Recall that the Euler characteristics of real and complex Grassmannians are given by binomial coefficients. In particular, these Euler characteristics satisfy certain recurrence relations related to Pascal's rule. The computations in Figure 1 indicate that an analogous recurrence relation is true for the $\mathbb{A}^1$-Euler characteristic of Grassmannians over an arbitrary field. In fact, this recurrence relation is a direct consequence of a result of Levine [20].

n	$r=$ 1	2	3	4	5
2	$\mathbb{H}$	$\langle 1\rangle$			
3	$\mathbb{H}+\langle 1\rangle$	$\mathbb{H}+\langle 1\rangle$	$\langle 1\rangle$		
4	$2\mathbb{H}$	$2\mathbb{H}+\langle 1,1\rangle$	$2\mathbb{H}$	$\langle 1\rangle$	
5	$2\mathbb{H}+\langle 1\rangle$	$4\mathbb{H}+\langle 1,1\rangle$	$4\mathbb{H}+\langle 1,1\rangle$	$2\mathbb{H}+\langle 1\rangle$	$\langle 1\rangle$
6	$3\mathbb{H}$	$6\mathbb{H}+\langle 1,1,1\rangle$	$10\mathbb{H}$	$6\mathbb{H}+\langle 1,1,1\rangle$	$3\mathbb{H}$
7	$3\mathbb{H}+\langle 1\rangle$	$9\mathbb{H}+\langle 1,1,1\rangle$	$16\mathbb{H}+\langle 1,1,1\rangle$	$16\mathbb{H}+\langle 1,1,1\rangle$	$9\mathbb{H}+\langle 1,1,1\rangle$

Figure 1. More examples of $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$.

Proposition 8.3. *Let* $1\le r<n$ *be integers. Then*

$$\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))=\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r-1,n-1))+\langle -1\rangle^r\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n-1)).$$

Proof. Fix a line L in k^n. Let Z be the closed subvariety consisting of all r-planes containing L (which is isomorphic to $\mathrm{Gr}_k(r-1,n-1)$), and let U be its open complement (which is isomorphic to an affine rank r bundle over $\mathrm{Gr}_k(r,n-1)$). We then get a decomposition $\mathrm{Gr}_k(r,n)=Z\cup U$. Since $\mathrm{Gr}_k(l,m)\cong\mathrm{Gr}_k(m-l,m)$, we have $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(l,m))=\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(m-l,m))$. We can thus apply [20, Proposition 1.4(3)] to obtain

$$\begin{aligned}\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))&=\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(n-r,n))\\&=\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(n-r,n-1))+\langle -1\rangle^r\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(n-r-1,n-1))\\&=\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r-1,n-1))+\langle -1\rangle^r\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n-1)).\end{aligned}$$ □

We can now apply a theorem of Bachmann and Wickelgren [2] to completely characterize $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$.

Theorem 8.4. *Let k be field of characteristic not equal to 2. Let $n_{\mathbb{C}}:=\binom{n}{r}$, and let $n_{\mathbb{R}}:=\binom{\lfloor n/2\rfloor}{\lfloor r/2\rfloor}$. Then*

$$\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))=\frac{n_{\mathbb{C}}+n_{\mathbb{R}}}{2}\langle 1\rangle+\frac{n_{\mathbb{C}}-n_{\mathbb{R}}}{2}\langle -1\rangle.$$

Proof. By [2, Theorem 5.8], we can restrict this computation to two different possibilities. We will prove by induction that $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$ mod $\mathbb{H}$ has no $\langle 2\rangle$ summand. The desired result will then follow from [2, Theorem 5.8] by noting that $n_{\mathbb{C}}$ and $n_{\mathbb{R}}$ are the Euler characteristics of $\mathrm{Gr}_{\mathbb{C}}(r,n)$ and $\mathrm{Gr}_{\mathbb{R}}(r,n)$, respectively.

Since $\mathbb{A}^n_k$ is $\mathbb{A}^1$-homotopic to $\operatorname{Spec} k$, we have $\chi^{\mathbb{A}^1}(\mathbb{A}^n_k)=\chi^{\mathbb{A}^1}(\operatorname{Spec} k)=\langle 1\rangle$. Using this observation and the decomposition $\mathbb{P}^n_k=\bigcup_{i=0}^n\mathbb{A}^i_k$ (and a result analogous to [20, Proposition 1.4(3)]), Hoyois computed the $\mathbb{A}^1$-Euler characteristic of projective space [15, Example 1.7]:

$$\chi^{\mathbb{A}^1}(\mathbb{P}^n_k)=\begin{cases}\frac{n}{2}\mathbb{H}+\langle 1\rangle & n \text{ is even},\\ \frac{n+1}{2}\mathbb{H} & n\text{ is odd}.\end{cases}$$

Note that $\mathrm{Gr}_k(0,n)\cong\mathrm{Gr}_k(n,n)\cong\operatorname{Spec} k$ and $\mathrm{Gr}_k(1,n)\cong\mathrm{Gr}_k(n-1,n)\cong\mathbb{P}^{n-1}_k$. In particular, $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(i,n))$ mod $\mathbb{H}$ is either trivial or $\langle 1\rangle$ for $i=0,1,n-1$, or n. This forms the base case of our

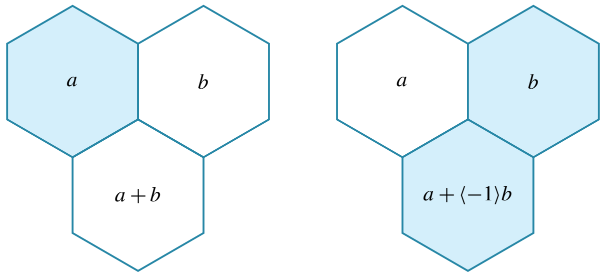

Figure 2. Addition rules for modified Pascal's triangle.

induction, with the inductive step given by Proposition 8.3 — namely, if $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r-1, n-1)) \bmod \mathbb{H}$ and $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r, n-1)) \bmod \mathbb{H}$ only have $\langle 1\rangle$ and $\langle -1\rangle$ summands, then

$$(\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r-1, n-1)) + \langle -1\rangle^r \chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r, n-1))) \bmod \mathbb{H}$$

only has $\langle 1\rangle$ and $\langle -1\rangle$ summands. □

8C. ***Modified Pascal's triangle for $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r, n))$.*** Pascal's triangle gives a mnemonic device for binomial coefficients and hence for the Euler characteristics of complex and real Grassmannians. The recurrence relation of Proposition 8.3 indicates that a modification of Pascal's triangle can also be used to calculate the $\mathbb{A}^1$-Euler characteristics of Grassmannians. Explicitly, each entry in the modified Pascal's triangle is an element of $\mathrm{GW}(k)$. The two diagonal edges of this triangle correspond to $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(0, n)) = \langle 1\rangle$ and $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(n, n)) = \langle 1\rangle$. Elements of each row of the modified Pascal's triangle are obtained from the previous row by the addition rule illustrated in Figure 2.

We rewrite the data recorded in Figure 1 in a modified Pascal's triangle in Figure 3. The rows correspond to the dimension n of the ambient affine space k^n, while the southwest-to-northeast diagonals correspond to the dimension r of the planes k^r in the ambient space.

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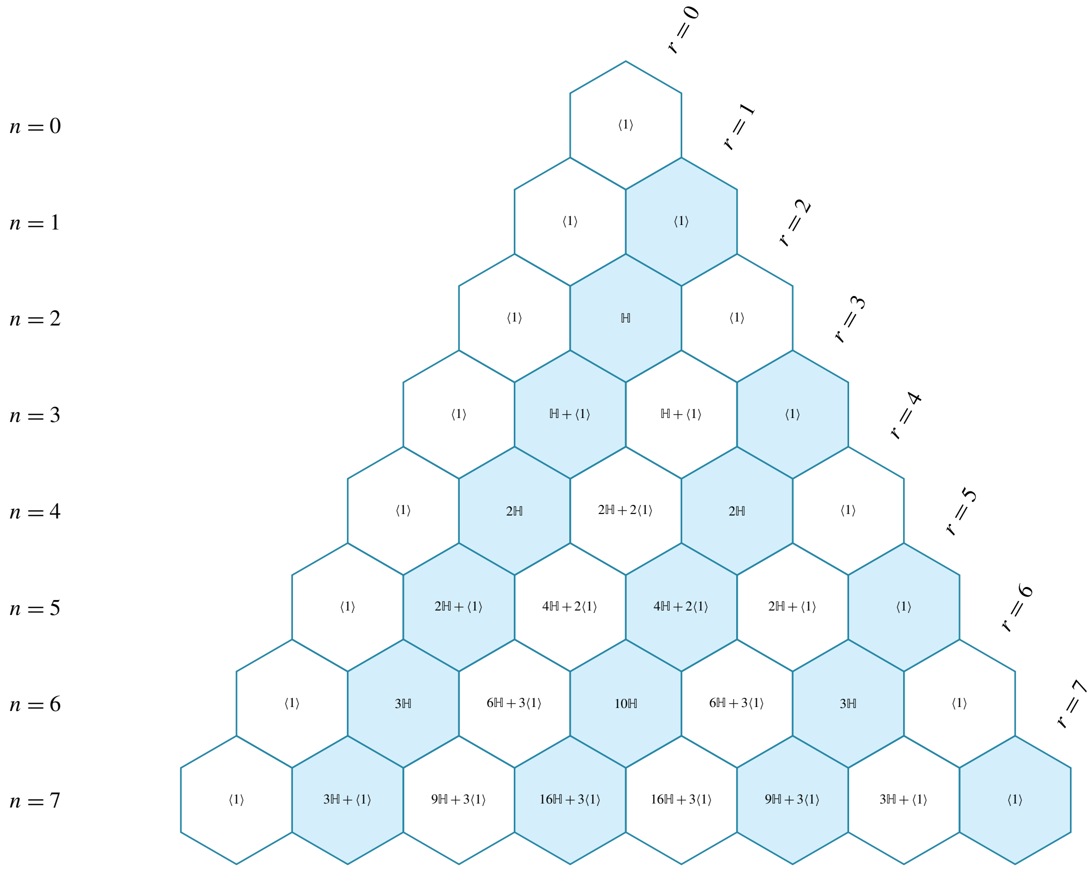

Figure 3. Modified Pascal's triangle for $\chi^{\mathbb{A}^1}(\mathrm{Gr}_k(r,n))$ (see Section 8C).

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Axiomatizing the existential theory of $\mathbb{F}_q((t))$

Sylvy Anscombe, Philip Dittmann and Arno Fehm

We study the existential theory of equicharacteristic henselian valued fields with a distinguished uniformizer. In particular, assuming a weak consequence of resolution of singularities, we obtain an axiomatization of — and therefore an algorithm to decide — the existential theory relative to the existential theory of the residue field. This is both more general and works under weaker resolution hypotheses than the algorithm of Denef and Schoutens, which we also discuss in detail. In fact, the consequence of resolution of singularities our results are conditional on is the weakest under which they hold true.

1. Introduction

Hilbert's tenth problem asks for an algorithm to decide whether a given polynomial over $\mathbb{Z}$ has a solution in $\mathbb{Z}$, which was shown to be impossible by work of Davis, Putnam, Robinson and Matiyasevich. Analogues for various other rings and fields have been studied since; see [Poonen 2003; Shlapentokh 2007; Koenigsmann 2014]. For local fields of positive characteristic, i.e., Laurent series fields $\mathbb{F}_q((t))$ over a finite field with $q = p^n$ elements, there are two results in the literature. The first one works with polynomials over $\mathbb{F}_p(t)$, which is arguably the correct analogue of $\mathbb{Z}$ in this setting, but needs to assume the truth of a deep and unresolved conjecture from algebraic geometry, resolution of singularities (see Section 2 for precise statements and discussion):

Theorem 1.1 [Denef and Schoutens 2003, Theorem 4.3]. *Assume that resolution of singularities holds in characteristic p. There exists an algorithm that decides whether a given system of polynomial equations over $\mathbb{F}_p(t)$ has a solution in $\mathbb{F}_q((t))$.*

The second one, which is proved by completely different methods, needs no such hypothesis but allows for polynomials only over $\mathbb{F}_p$:

Theorem 1.2 [Anscombe and Fehm 2016, Corollary 7.7]. *There exists an algorithm that decides whether a given system of polynomial equations over $\mathbb{F}_p$ has a solution in $\mathbb{F}_q((t))$.*

Both results have recently found various applications; see, for example, [Onay 2018; Martínez-Ranero et al. 2022; Kartas 2021a; 2021b; 2023b]. In fact, both results are more general than stated here, and these more general results are most naturally phrased in the language of the model theory of valued fields:

MSC2020: 03C60, 11D88, 11G25, 12L05.
Keywords: local fields, positive characteristic, henselian valued field, existential theory, decision algorithm, resolution of singularities, local uniformization.

Definition 1.3. We denote by $\mathcal{L}_{\mathrm{ring}} = \{+, -, \cdot, 0, 1\}$ the language of rings, by $\mathcal{L}_{\mathrm{val}} = \mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ the language of valued fields, where $\mathcal{O}$ is a unary predicate symbol, and by $\mathcal{L}_{\mathrm{ring}}(\varpi)$ and $\mathcal{L}_{\mathrm{val}}(\varpi)$ the respective language expanded by a constant symbol ϖ. We view a valued field (K, v) as an $\mathcal{L}_{\mathrm{val}}$-structure by interpreting $\mathcal{O}$ as the valuation ring $\mathcal{O}_v$ of v. A *uniformizer* of a valued field (K, v) is an element $\pi \in K$ of smallest positive value. Given a distinguished uniformizer π of (K, v), we view (K, v, π) as an $\mathcal{L}_{\mathrm{val}}(\varpi)$-structure by interpreting ϖ as π.

So, for example, $\mathbb{F}_q((t))$ carries the natural equicharacteristic henselian valuation v_t (the t-adic valuation), and if we view $(\mathbb{F}_q((t)), v_t, t)$ as an $\mathcal{L}_{\mathrm{val}}(\varpi)$-structure, the fact that a given system of polynomial equations over $\mathbb{F}_p(t)$ has a solution in $\mathbb{F}_q((t))$ can be expressed as the truth in $\mathbb{F}_q((t))$ of a particular existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-sentence (by existential we mean of the form $\exists x_1 \dots \exists x_n \psi(\underline{x})$ with a quantifier-free formula $\psi(\underline{x})$). The general result of Denef and Schoutens, which we revisit and discuss in Section 3, can now be phrased as follows:

Theorem 1.4 [Denef and Schoutens 2003, Theorem 4.3]. *Let (K, v) be an equicharacteristic henselian valued field with distinguished uniformizer π. Assume that*

(1) *resolution of singularities holds in characteristic p,*

(2) *$\mathcal{O}_v$ is a discrete valuation ring, and*

(3) *$\mathcal{O}_v$ is excellent.*

Then the existential $\mathcal{L}_{\mathrm{val}}(\varpi)$-theory $\mathrm{Th}_\exists(K, v, \pi)$ of (K, v, π) is decidable relative to the existential $\mathcal{L}_{\mathrm{ring}}$-theory $\mathrm{Th}_\exists(Kv)$ of the residue field Kv (in the sense of Turing reduction; see Remark 3.3).

The more general form of Theorem 1.2 [Anscombe and Fehm 2016, Theorem 6.5] gives an axiomatization of the existential theory of an equicharacteristic henselian valued field relative to the existential theory of the residue field, which in particular implies a relative decidability result; it is also interesting in its own right, and useful, for example, when questions of uniformity are concerned.

The goal of this work is to show that the approach from [Anscombe and Fehm 2016] can be extended to the setting of Theorem 1.4 as well, thereby combining the best of both worlds. Besides obtaining an axiomatization of the existential theory, we weaken the assumption (1) to a local version, called local uniformization (see Section 2 for definitions and an extensive discussion), and completely eliminate the assumptions (2) and (3). In Section 4 on page 2028 we prove:

Theorem 1.5. *Assume that consequence* (R4) *(see page 2016) of local uniformization holds. Let (K, v) be an equicharacteristic henselian valued field with distinguished uniformizer π. Then the universal/existential $\mathcal{L}_{\mathrm{val}}(\varpi)$-theory of (K, v, π) is entailed by*

(i) *$\mathcal{L}_{\mathrm{val}}$-axioms for equicharacteristic henselian valued fields,*

(ii) *the $\mathcal{L}_{\mathrm{val}}(\varpi)$-axiom expressing that π has smallest positive value, and*

(iii) *$\mathcal{L}_{\mathrm{val}}$-axioms expressing that the residue field models the universal/existential $\mathcal{L}_{\mathrm{ring}}$-theory of Kv.*

In particular, the existential $\mathcal{L}_{\mathrm{val}}(\varpi)$*-theory* $\mathrm{Th}_\exists(K, v, \pi)$ *of* (K, v, π) *is decidable relative to the existential* $\mathcal{L}_{\mathrm{ring}}$*-theory* $\mathrm{Th}_\exists(Kv)$ *of* Kv.

Here, by universal/existential we mean sentences that are either universal or existential. In fact, in Theorem 4.12 we prove this with parameters from a base field more general than $\mathbb{F}_p(\pi)$. In this strong form, this axiomatization statement for universal/existential theories is in fact equivalent to (R4) (see Remark 4.17) so our hypothesis cannot be weakened further.

Our results in particular give a new proof of Theorem 1.1 which works under this weaker hypothesis:

Corollary 1.6. *Assume that consequence* (R4) *of local uniformization holds. There exists an algorithm that decides whether a given system of polynomial equations over* $\mathbb{F}_q(t)$ *has a solution in* $\mathbb{F}_q((t))$.

Recent work of Kartas includes related results; see Remark 2.6.

2. Resolution of singularities and local uniformization

In this section we discuss several versions of resolution of singularities and some of its consequences.

Definition 2.1. Let K be a field. A K*-variety* is an integral separated K-scheme of finite type, and a morphism of K-varieties $Y \to X$ is simply a morphism of K-schemes.

Given a K-variety X, a *resolution of singularities* (or for short *resolution*) of X is a proper birational morphism $Y \to X$ with Y a regular K-variety. A *blowing-up resolution* is a morphism $Y \to X$ arising as the blowing-up along some closed proper subscheme $Z \subset X$ such that Y is regular.

By a valuation v on a field extension F/K we mean a valuation on F which is trivial on K. Given a finitely generated field extension F/K and a valuation v on F/K, a *local uniformization* of v consists of a K-variety Y and an isomorphism $F \cong_K K(Y)$ such that, under the identification of F with $K(Y)$, v is centered on a regular point y of Y (i.e., $\mathcal{O}_v$ dominates the regular local ring $\mathcal{O}_{Y,y}$).

Lemma 2.2. *Let* K *be a field.*

(1) *Every blowing-up resolution of a* K*-variety* X *is a resolution of* X.

(2) *If a proper* K*-variety* X *has a resolution, then any valuation* v *on* $K(X)/K$ *has a local uniformization.*

(3) *Let* F/K *be a finitely generated field extension and* v *a valuation on* F/K *which has residue field* $Fv = K$. *If* v *has a local uniformization, then there exists a* K*-embedding of* F *into* $K((t))$.

Proof. Every blowing-up morphism along a closed proper subscheme is proper and birational [Liu 2002, Proposition 8.1.12(d)], implying (1). For (2), let $Y \to X$ be a resolution of X. Every valuation v on the function field $K(Y)/K$ is centered at some point of Y by the valuative criterion of properness; see the discussion in [Liu 2002, Definition 8.3.17]. In particular, every such v has a local uniformization, and the same holds for valuations on $K(X) \cong_K K(Y)$. In the situation of (3), the field extension F/K is necessarily separable [Fried and Jarden 2008, Lemma 2.6.9], so the statement follows from [Fehm 2011, Lemma 9]; alternatively, it can be deduced from [Kuhlmann 2004, Theorem 13]. ☐

The condition that a finitely generated field extension F/K can be embedded over K into $K((t))$ is especially interesting when K is a *large* (or *ample*) field in the sense of [Pop 1996], that is, K is existentially closed in $K((t))$ in $\mathcal{L}_{\text{ring}}$. Apart from the definition the only important facts for us are that large fields form an elementary class (see [Pop 1996, Remark 1.3]) and that henselian nontrivially valued fields are large (see Proposition 1 on page 41 and part 1) of the remarks at the bottom of page 43 of [Ershov 1967]). See [Jarden 2011, Example 5.6.2; Pop 2010] for more modern accounts that prove this (using an equivalent definition of largeness), and see [Bary-Soroker and Fehm 2013] for more background on large fields.

We are led to consider the following hypotheses:

(R0) For every field K and every K-variety X there exists a blowing-up resolution $Y \to X$.

(R1) For every field K and every K-variety X there exists a resolution $Y \to X$.

(R2) For every field K, every finitely generated field extension F/K and every valuation v on F/K there exists a local uniformization of v.

(R3) For every field K and every nontrivial finitely generated extension F/K such that there exists a valuation v on F/K with residue field $Fv = K$, there also exists a valuation with value group $\mathbb{Z}$ which has that property.

(R4) Every large field K is existentially closed in every extension F/K for which there exists a valuation v on F/K with residue field $Fv = K$.

Proposition 2.3. *The following implications hold true*: (R0) $\Rightarrow$ (R1) $\Rightarrow$ (R2) $\Rightarrow$ (R3) $\Longleftrightarrow$ (R4).

Proof. The implication (R0) $\Rightarrow$ (R1) follow from Lemma 2.2(1). For (R1) $\Rightarrow$ (R2) first choose any proper K-variety X with $K(X) \cong_K F$, and then apply Lemma 2.2(2). By Lemma 2.2(3), (R2) implies that every F/K as in (R3) can be embedded into $K((t))$, and the restriction of the t-adic valuation to F then satisfies $Fv = K$ and $vF \cong \mathbb{Z}$, which shows that (R2) $\Rightarrow$ (R3). If F/K is as in (R4), then (R3) gives a valuation v on F/K with $Fv = K$ and $vF = \mathbb{Z}$. The completion $(\hat{F}, \hat{v})$ of (F, v) is then isomorphic over K to $(K((t)), v_t)$, see, for example, Proposition 4.2 below, so the large field K is existentially closed in $\hat{F}$ and therefore in particular in F, which shows that (R3) $\Rightarrow$ (R4). Finally, we have that (R4) $\Rightarrow$ (R3) by [Fehm 2011, Corollary 10]. □

The resolution of singularities that Denef and Schoutens assume is (R0); see Section 3. In our main theorem (Theorem 4.12), we will work under the weaker condition (R4).

Remark 2.4. The terminology on resolution of singularities is not entirely standardized. Depending on the author, either a resolution or a blowing-up resolution in our terminology might be called a resolution, a desingularization or a weak desingularization. For a resolution $Y \to X$, it may additionally be demanded that it is an isomorphism over the regular locus of X. For varieties X over a field K of characteristic zero, the existence of a blowing-up resolution in this strong sense was established in [Hironaka 1964]. Therefore all five hypotheses above are valid when restricted to fields K of characteristic zero.

In general, it is known that every variety of dimension three or less has a resolution of singularities (see [Cossart and Piltant 2019]), and it follows that (R2), (R3) and (R4) hold restricted to field extensions F/K of transcendence degree at most three.

Local uniformization is occasionally stated in a stronger version (see, for instance, the introduction to [Temkin 2013]), where one demands that for any given K-variety X with function field $K(X) \cong_K F$ on which v is centered, the uniformizing Y can be chosen to dominate X, i.e., to come with a birational morphism $Y \to X$ compatible with the isomorphism $K(X) \cong F \cong K(Y)$. The best result known here seems to be that (R2) holds up to replacing F by a purely inseparable finite extension; see [Temkin 2013, Section 1.3]. It follows that (R3) and (R4) hold when restricted to perfect base fields K. In fact, this restricted form of (R4) was already proven earlier in [Kuhlmann 2004, Theorem 17]. Less deep is the fact (immediate from [Fried and Jarden 2008, Lemma 2.6.9(b), Proposition 11.3.5] and [Fehm 2011, Lemma 9]) that (R3) and (R4) also hold when restricted to *pseudoalgebraically closed* base fields K, i.e., fields K for which every geometrically integral K-variety has a K-rational point.

Remark 2.5. Desingularization problems are also studied for more general schemes, rather than only varieties over fields. For instance, it is conjectured that for any reduced quasiexcellent scheme X there exists a proper birational morphism $Y \to X$ with Y regular [EGA IV_2 1965, Remarque (7.9.6)]. This is a generalization of (R1) since any variety over a field is quasiexcellent. Similarly, local uniformization is also studied in this more general setting; see [Temkin 2013]. The condition of excellence also appears in our own results below, but for reasons unrelated to resolution of singularities — indeed, we only need resolution in the shape of the weak condition (R4).

Remark 2.6. Other variations of resolutions exist, including the desingularization of a pair of a reduced quasiexcellent scheme X and a nowhere dense closed subscheme Z; see [Temkin 2008, Introduction].

A resolution hypothesis of this type is used in [Kartas 2023a] to conditionally obtain an axiomatization for existential theories of certain, not necessarily discrete, valuation rings. The approach there is based on the geometric method of Denef–Schoutens, which we discuss in Section 3.

The first available preprint of [Kartas 2023a] in Section 5 also announces (with sketch proofs) results of this kind conditional only on variants of (R2). These do not appear in the published version but are included in [Kartas 2022].

3. The Denef–Schoutens algorithm

We now discuss the algorithm given by Denef and Schoutens, with a view towards improving the statements given in [Denef and Schoutens 2003], addressing some subtle points of the algorithm, and avoiding scheme-theoretic language at least to some extent. Our own technique is entirely different, so the reader interested in new results can skip ahead to Section 4.

Throughout, R is a fixed henselian discrete valuation ring with uniformizer π and fraction field K, $p > 0$ the characteristic of K, and $\kappa = R/(\pi)$ the residue field of R. Note that π is necessarily transcendental over the prime field $\mathbb{F}_p$ of K. We view R as an $\mathcal{L}_{\mathrm{ring}}(\varpi)$-structure (R, π) by interpreting ϖ as π.

Overview. Theorem 1.4, concerning the decidability of the existential $\mathcal{L}_{\text{val}}(\varpi)$-theory of (K, v, π), is equivalent to the decidability of the existential $\mathcal{L}_{\text{ring}}(\varpi)$-theory of (R, π) under the hypothesis (R0), and the latter is what Proposition 3.9 below proves. To decide the truth of existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentences in R, one proceeds in three steps. Deciding the truth of a *positive* existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentence in R means determining whether a system of polynomial equations has a solution in R. By a variant of a theorem of Greenberg, such a solution exists if and only if there is a solution in the residue ring $R/(\pi^N)$ to the appropriately reduced system, for a sufficiently high (computable) power π^N. This reduces the original question about satisfaction of positive existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentences in R to a question about the existential $\mathcal{L}_{\text{ring}}$-theory of the residue field κ.

For the second step, one considers existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentences (that is, systems of equations and inequations), where the underlying system of equations describes a *regular* affine variety. A version of the implicit function theorem then shows that any solution in R to the system of equation gives rise to a wealth of other solutions, by perturbing some coordinates. This then implies that if there is any solution to the equations (which is a computable condition by the first step), there will also be one satisfying the inequations.

In the third step, one considers systems of equations and inequations in full generality; only here does resolution of singularities enter. Assuming that the system of equations (or rather, the affine scheme described thereby) has a resolution of singularities, one can in fact effectively find such a resolution, simply by enumerating candidates. In an inductive procedure, one can then relate the existence of solutions in R to the original system of equations and inequations, and solutions to related systems without singularities, which can be handled by the second step.

Step 1: Positive existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentences in R. Here we slightly rephrase the results of [Denef and Schoutens 2003, Section 3].

Lemma 3.1. *Let $f_1, \dots, f_n \in \mathbb{F}_p[t][X_1, \dots, X_m]$ be polynomials over the ring $\mathbb{F}_p[t]$. The following are equivalent*:

(1) *The system of equations $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$ has a solution in $\kappa[\![t]\!]$.*

(2) *The system of equations $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$ has a solution in $\kappa[t]^h$, the henselization of $\kappa[t]$ at the ideal (t) (i.e., the valuation ring of the henselization $\kappa(t)^h$ of $\kappa(t)$ with respect to the t-adic valuation).*

(3) *For every $N > 0$, there is a solution in $\kappa[t]/(t^N)$ to the system of equations $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$ (more precisely, to the reduction mod t^N).*

It suffices to check (3) *for one specific large N, which can be computed from n, m and the total degree of the f_i (in the variables $t, X_1, \dots, X_m$).*

Proof. It is clear that (2) $\Rightarrow$ (1) (since $\kappa[t]^h \subseteq \kappa[\![t]\!]$) and (1) $\Rightarrow$ (3) (by reduction modulo t^N). The implication (3) $\Rightarrow$ (2) is Greenberg's result [1966, Corollary 2]. The reduction to one specific large N comes down to an *effective* version of Greenberg's theorem, which is a special case of the effective Artin approximation given in [Becker et al. 1979, Theorem 6.1] (take $n = 1$ and $\alpha = 0$ in the notation there). □

Lemma 3.2. *Given finitely many polynomials $f_1, \dots, f_n \in \mathbb{F}_p[T, X_1, \dots, X_m]$, one can effectively find finitely many polynomials $g_1, \dots, g_{n'} \in \mathbb{F}_p[Y_1, \dots, Y_{m'}]$ such that the system of equations $\bigwedge_{i=1}^n f_i(\pi, \underline{X}) = 0$ has a solution in R if and only if the system of equations $\bigwedge_{j=1}^{n'} g_j(\underline{Y}) = 0$ has a solution in κ. In particular, the positive existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-theory of (R, π) is decidable relative to the existential $\mathcal{L}_{\mathrm{ring}}$-theory of κ.*

Proof. The ring R embeds into its completion $\widehat{R}$. By the Cohen structure theorem (see also Propositions 4.2 and 4.3 below) there is an isomorphism $\varphi : \widehat{R} \to \kappa[[t]]$ that sends π to t. For each $N > 0$, φ induces an isomorphism

$$\varphi_N : R/(\pi^N) \cong \hat{R}/(\pi^N) \to \kappa[[t]]/(t^N) \cong \kappa[t]/(t^N)$$

that maps $\pi + (\pi^N)$ to $t + (t^N)$.

Let $N > 0$ be the specific N given in the last point of Lemma 3.1, computed from the polynomials $f_i(t, \underline{X}) \in \mathbb{F}_p[t][X_1, \dots, X_m]$. We claim that the system of equations (I) : $\bigwedge_{i=1}^n f_i(\pi, \underline{X}) = 0$ has a solution in R if and only if the reduction of (II) : $\bigwedge_{i=1}^n f_i(t, \underline{X}) = 0$ has a solution in $\kappa[t]/(t^N)$.

One direction is trivial: if there is a solution to (I) in R then there is a solution to the reduction of (I) in $R/(\pi^N)$, and via the isomorphism φ_N there is a solution to the reduction of (II) in $\kappa[t]/(t^N)$. Conversely, suppose a solution to the reduction of (II) exists in $\kappa[t]/(t^N)$. By our choice of N, there is a solution to (II) in $\kappa[t]^h$. Since $\kappa[t]^h$ is the union of $\kappa_0[t]^h$ where κ_0 ranges over the finitely generated subfields of κ, we have a solution to (II) in $\kappa_0[t]^h$ for some κ_0. By [Anscombe and Fehm 2016, Lemma 2.3], there exists a partial section $\kappa_0 \to R$ of the residue map $R \to \kappa$ (see Definition 4.1). This extends to an embedding $\kappa_0[t]^h \to R$, sending t to π. Thus we have a solution to (I) in R, proving the claim.

The condition that the reduction of (II) has a solution in $\kappa[t]/(t^N)$ is easily translated into the solvability in κ of a system of finitely many polynomial equations over $\mathbb{F}_p$ (the g_j of the statement) by a standard interpretation (or Weil restriction) argument, since $\kappa[t]/(t^N)$ is an algebra of finite dimension over κ. $\square$

Remark 3.3. Let us briefly describe the formal meaning of "one can effectively find" and "relatively decidable" in the statement of Lemma 3.2.

Define an injection of the set of $\mathcal{L}_{\mathrm{ring}}(\varpi)$-terms and $\mathcal{L}_{\mathrm{ring}}(\varpi)$-formulas into $\mathbb{N}$ by a standard Gödel coding. Every $\mathcal{L}_{\mathrm{ring}}(\varpi)$-term with free variables $X_1, \dots, X_n$ induces a polynomial in $\mathbb{F}_p[T, X_1, \dots, X_n]$ in the natural way, where the indeterminate T takes the place of the constant symbol ϖ. We now obtain a Gödel numbering of $\mathbb{F}_p[T, X_1, \dots, X_n]$ by assigning to every polynomial the minimal Gödel number of an $\mathcal{L}_{\mathrm{ring}}(\varpi)$-term which induces it. The set of Gödel numbers of polynomials in $\mathbb{F}_p[T, X_1, \dots, X_n]$ is a decidable set of natural numbers, and mapping the Gödel number of two polynomials in $\mathbb{F}_p[T, X_1, \dots, X_n]$ to the Gödel number of their sum (or their product) is a computable function. (This is essentially equivalent to asserting that the numbering gives an explicit representation of the ring $\mathbb{F}_p[T, X_1, \dots, X_n]$ in the sense of [Fröhlich and Shepherdson 1956, §2].)

The first part of Lemma 3.2 now means that there exists a Turing machine which takes as input the codes of finitely many polynomials $f_1, \dots, f_n$, and produces as output the codes of finitely many polynomials $g_1, \dots, g_{n'}$ with the given property. The second part of the lemma asserts that there exists a Turing machine that takes as input the code of a positive existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-sentence φ and outputs yes or no according to whether $(R, \pi) \models \varphi$, using an oracle (as described in [Shoenfield 1971, Chapter 4]) which for the code of an existential $\mathcal{L}_{\text{ring}}$-sentence ψ decides whether $\kappa \models \psi$. In other words, the set of codes of positive existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentences satisfied by (R, π) is Turing reducible to the set of codes of sentences in $\text{Th}_{\exists}(\kappa)$.

Remark 3.4. Although in this section we have fixed a ring R, we note that the algorithm of Lemma 3.2 to produce the polynomials g_j from the f_i does not depend on R. This is because the interpretation (or Weil restriction) argument used is uniform in κ, since $\kappa[t]/(t^N)$ is an N-dimensional κ-algebra with multiplication defined by structure constants in $\mathbb{F}_p$ that do not themselves depend on κ. In particular, $\kappa[t]/(t^N) = \mathbb{F}_p[t]/(t^N) \otimes_{\mathbb{F}_p} \kappa$ is an extension of scalars.

Remark 3.5. As remarked in [Becker et al. 1979, Section 6], the effective version of Greenberg's theorem used in the reduction in Lemma 3.2 is hopeless in practice since the computability of the quantity N comes from an argument involving an enumeration of proofs in some proof calculus. The same holds true for the algorithm that we obtain in Theorem 4.12.

Remark 3.6. The version of Lemma 3.2 given in [Denef and Schoutens 2003, Proposition 3.5] is significantly stronger, being phrased for an arbitrary equicharacteristic excellent henselian local domain instead of an equicharacteristic henselian discrete valuation ring, and allowing finitely many parameters. In the proof, this comes down to replacing Greenberg's theorem by a more general version of Artin approximation.

On the other hand, note that we can dispense with the excellence condition on the valuation ring R. This is essentially due to us only allowing parameters from $\mathbb{F}_p[\pi]$.

Step 2: Existential $\mathcal{L}_{\text{ring}}(\varpi)$-sentences in R, with a regularity condition. For this step, let us fix $f_1, \ldots, f_n \in \mathbb{F}_p[\pi][X_1, \ldots, X_m]$ such that the system of equations $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$ describes an affine $\mathbb{F}_p(\pi)$-variety V, that is, the ideal

$$\mathfrak{f} := (f_1, \ldots, f_n) \trianglelefteq \mathbb{F}_p(\pi)[X_1, \ldots, X_m]$$

is prime. Let d be the dimension of V.

Recall that a solution $\underline{x}$ to the system of equations in some extension field $F/\mathbb{F}_p(\pi)$ is called *smooth* if the Jacobian condition is satisfied, i.e., the matrix $((\partial f_i/\partial X_j)(\underline{x}))_{ij}$ has rank $m - d$.

Lemma 3.7. *Let $g \in \mathbb{F}_p[\pi][X_1, \ldots, X_m]$ be a polynomial which is not in the ideal $\mathfrak{f}$, so that g does not vanish identically on the variety V. If there is a solution $\underline{x} \in R^m$ to the system of equations $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$ such that the K-point described by $\underline{x}$ is smooth, then there exists such a solution with $g(\underline{x}) \neq 0$.*

Proof. This is [Denef and Schoutens 2003, Theorem 2.4], or at least the special case of it which we will need below. We only briefly describe the idea of the proof.

The statement comes down to an elaborate application of Hensel's lemma, or the implicit function theorem in henselian fields. Starting with the given solution $\underline{x} \in R^m$, perturb d variables by a small amount (in the valuation topology) and solve for the remaining variables (which is where smoothness is required).

By wisely choosing the initial perturbation, one can ensure that the inequation $g(\underline{X}) \neq 0$ becomes satisfied, since g does not vanish identically on V. (For instance, one may reduce to the case where the f_i describe an affine curve over K by a generic hyperplane section argument, so the condition $g(\underline{X}) \neq 0$ is satisfied by all but finitely many points on the curve, in which case almost all small perturbations will be as required.) □

Below we want to work in a context where the affine $\mathbb{F}_p(\pi)$-variety V is not necessarily smooth (i.e., not all solutions to the equation system in all extension fields satisfy the Jacobian condition), but instead satisfies the condition of regularity, i.e., the coordinate ring $\mathbb{F}_p(\pi)[X_1, \dots, X_m]/\mathfrak{f}$ is a regular ring in the sense of commutative algebra. In general, smooth varieties are regular, and over a perfect field the converse holds [Liu 2002, Corollaries 4.3.32 and 4.3.33]. In our situation we still have the following:

Lemma 3.8. *Suppose that V is regular. Then any solution $\underline{x} \in K^m$ to the system of equations $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$ is smooth, that is, satisfies the Jacobian condition.*

Proof. The field extension $K/\mathbb{F}_p(\pi)$ is separable as π does not have a p-th root in K and hence the fields K and $\mathbb{F}_p(\pi)^{1/p} = \mathbb{F}_p(\pi^{1/p})$ are linearly disjoint over $\mathbb{F}_p(\pi)$. The tuple $\underline{x}$ describes an element of $V(K)$, i.e., a point P of the scheme V together with an $\mathbb{F}_p(\pi)$-embedding of the residue field of P into K. Thus the residue field of P is separable over $\mathbb{F}_p(\pi)$. As P is regular by assumption, [EGA IV$_4$ 1967, Proposition (17.15.1)] forces the point P to be smooth, so the Jacobian condition holds. □

Now suppose that V is regular, and $g \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$ is not in $\mathfrak{f}$. In this situation, satisfaction of the statement $\exists \underline{X}\big(\bigwedge_{i=1}^n f_i(\underline{X}) = 0 \wedge g(\underline{X}) \neq 0\big)$ in R can now be effectively reduced to an existential statement about the residue field κ: by Lemmas 3.7 and 3.8, the condition $g(\underline{X}) \neq 0$ can be dropped, and the remaining formula is handled by the previous step (Lemma 3.2).

Step 3: Arbitrary existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-sentences in R, using resolution of singularities.

Let $f_1, \dots, f_n, g \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$. We wish to determine whether there exists $\underline{x} \in R^m$ with $f_i(\underline{x}) = 0$ for all $i = 1, \dots, n$, and $g(\underline{x}) \neq 0$.

We may assume that the ideal $\mathfrak{f} = (f_1, \dots, f_n) \trianglelefteq \mathbb{F}_p(\pi)[X_1, \dots, X_m]$ is prime, i.e., that the f_i define an affine $\mathbb{F}_p(\pi)$-variety V: indeed, by primary decomposition there exist ideals $I_1, \dots, I_k \trianglelefteq \mathbb{F}_p(\pi)[X_1, \dots, X_m]$ which are primary, so in particular the radicals $\sqrt{I_1}, \dots, \sqrt{I_k}$ are prime ideals, and $\mathfrak{f} = I_1 \cap \cdots \cap I_k$. Then a tuple $\underline{x} \in R^m$ lies in the vanishing locus of $f_1, \dots, f_n$ if and only if it lies in the vanishing locus of $\sqrt{I_j}$ for some $j = 1, \dots, k$, so we may replace the f_i by a collection of generators for each of the $\sqrt{I_j}$, running all subsequent steps for all $j = 1, \dots, k$. The computation of the ideals $\sqrt{I_j}$ (or rather, generating sets thereof) is completely explicit [Seidenberg 1974].

We can check algorithmically whether the $\mathbb{F}_p(\pi)$-variety V is regular: for this we investigate the $\mathbb{F}_p$-scheme described by the f_i (that is, we "spread out" V to a scheme over $\mathbb{F}_p[\pi]$ and hence a scheme over $\mathbb{F}_p$, by interpreting π as another indeterminate) and check whether the nonregular locus intersects the generic fiber of the map to $\mathrm{Spec}(\mathbb{F}_p[\pi])$. The regular locus can be computed explicitly since it coincides with the smooth locus over the perfect field $\mathbb{F}_p$, which is given by explicit polynomial inequations (once the dimension of the scheme is computed).

Suppose that V is regular. We can check algorithmically whether g vanishes identically on the zero locus of the f_i, i.e., whether g is contained in $\mathfrak{f}$. If yes, then there certainly is no solution to $\bigwedge_{i=1}^n f_i(\underline{X}) = 0 \wedge g(\underline{X}) \neq 0$. Otherwise, step 2 is applicable. In other words, we can reduce to κ the question of whether there is a tuple in R as required.

Suppose now that V defined by the f_i is not regular. Assuming hypothesis (R0), there exists a morphism $V' \to V$ which is a blowing-up along some closed proper subscheme, and where V' is a regular $\mathbb{F}_p(\pi)$-variety. The morphism $V' \to V$ is projective [Liu 2002, Proposition 8.1.22], and so we can see V' as a closed subscheme of $\mathbb{P}_V^{m'}$ for suitable m'. Thus V' (usually not affine) is described by finitely many polynomials $h_1, \dots, h_{n'} \in \mathbb{F}_p(\pi)[X_1, \dots, X_m, Y_0, \dots, Y_{m'}]$ which are homogeneous in the variables Y_j (but not usually in the X_i), and include the polynomials $f_1, \dots, f_n$ defining V. We may assume that the defining polynomials h_i have coefficients in $\mathbb{F}_p[\pi]$ by scaling suitably. A point of V' (in some field containing $\mathbb{F}_p(\pi)$) is given by a zero $(\underline{x}, \underline{y})$ of the h_i, where $\underline{y}$ is not the zero tuple, and we identify solution tuples that differ only by a scaling of $\underline{y}$. The morphism $V' \to V$ is given on points by simply dropping the variables $Y_0, \dots, Y_{m'}$.

We can cover V' by affine open subvarieties $V'_0, \dots, V'_{m'}$, where V'_i is obtained from the description of V' by setting Y_i to 1, i.e., dehomogenizing at Y_i. This eliminates the need to work with nonaffine varieties.

The morphism $V' \to V$ is birational as a blowing-up along a closed proper subscheme; thus it is an isomorphism above some dense open $U \subseteq V$. Concretely, let us suppose that $V'_0 \neq \varnothing$ by permuting the variables Y_i if necessary, so that $V'_0 \subseteq V'$ is a dense open subvariety. Then $V' \to V$ being birational means that there is $u \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$, not in $\mathfrak{f}$, such that $V'_0 \to V$ is an isomorphism above the open set $U \subseteq V$ defined by $u(\underline{X}) \neq 0$, that is, the ring homomorphism

$$\begin{aligned}\varphi_u : \mathbb{F}_p(\pi)[X_1, \dots, X_m, 1/u(\underline{X})]/(f_1, \dots, f_n)& \\ \to \mathbb{F}_p(\pi)[X_1, \dots, X_m, Y_0, \dots, Y_{m'}, 1/u(\underline{X})]/(h_1, \dots, h_{n'}, Y_0 - 1)&\end{aligned}$$

is an isomorphism.

Assuming the existence of $V' \to V$ as above, i.e., projective and birational with V' regular, such a morphism can in fact be found algorithmically: to do this, one simply searches exhaustively for all the defining data (m', polynomials $h_1, \dots, h_{n'}$, u, and an inverse ψ_u to the ring homomorphism φ_u above), and checks that these data satisfy the conditions, that is, the V'_i are all regular and ψ_u is indeed an inverse to φ_u.

Let S be the set of tuples $\underline{x} \in R^m$ satisfying $f_i(\underline{x}) = 0$ for all i and $g(\underline{x}) \neq 0$. To check whether $S = \varnothing$, we now proceed as follows. First check whether there is some tuple $(\underline{x}, \underline{y}) \in R^m \times R^{m'}$ which gives a K-point of one of $V'_0, \dots, V'_{m'}$ and satisfies $g(\underline{x}) \neq 0$. This can be accomplished using the work already done, since each V'_i is a regular affine $\mathbb{F}_p(\pi)$-variety. If such a tuple $(\underline{x}, \underline{y})$ exists, then $\underline{x} \in S$, and we are done.

Hence let us suppose that no such tuple $(\underline{x}, \underline{y})$ exists. Then there exists no $\underline{x} \in R^m$ with $g(\underline{x}) \neq 0$ which gives a K-point of U: indeed, if there were such $\underline{x}$ which was additionally a K-point of U, then we could lift to a K-point $(\underline{x}, \underline{y}) \in K^{m+m'+1}$ of V', and by scaling we could take $\underline{y}$ to be in $R^{m'+1}$ with one coordinate equal to 1, i.e., $(\underline{x}, \underline{y})$ would describe a point on some V'_i.

Thus any $\underline{x} \in S$ must lie in the lower-dimensional $V \setminus U$. We continue the algorithm by replacing V with the algebraic set $V \setminus U$, i.e., by adding u to the list of polynomials $f_1, \ldots, f_n$. Since $V \setminus U$ has lower dimension than V, this procedure eventually terminates.

We have shown how to determine whether the system $\bigwedge_{i=1}^n f_i(\underline{X}) = 0 \wedge g(\underline{X}) \neq 0$ has a solution in R. The question of whether any given existential $\mathcal{L}_{\mathrm{ring}}(\pi)$-sentence holds in R can be effectively reduced to a disjunction of finitely many systems of this form, using the disjunctive normal form and replacing conjunctions of inequations $g_1(\underline{X}) \neq 0 \wedge g_2(\underline{X}) \neq 0$ by a single inequation $(g_1 \cdot g_2)(\underline{X}) \neq 0$. We have proven:

Proposition 3.9 [Denef and Schoutens 2003, Theorem 4.3]. *Assume* (R0). *The existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-theory of (R, π) is decidable relative to the existential $\mathcal{L}_{\mathrm{ring}}$-theory of the residue field κ.*

Theorem 1.4 follows since the $\mathcal{L}_{\mathrm{val}}(\varpi)$-structure (K, v, π), where v is the valuation with valuation ring R, is quantifier-freely interpretable in the $\mathcal{L}_{\mathrm{ring}}(\varpi)$-structure (R, π).

Remark 3.10. Comparing with the phrasing in [Denef and Schoutens 2003, Theorem 4.3], it is clear in our presentation that the assumption of excellence of R may be dropped. The phrasing in [Denef and Schoutens 2003, Theorem 4.3] also only states that the $\mathcal{L}_{\mathrm{ring}}(\varpi)$-theory of (R, π) is decidable "relative to the existential theory of κ and the [$\mathcal{L}_{\mathrm{ring}}(\varpi)$]-diagram of R". However, it is unclear what is meant by this diagram. In particular, if R is uncountable, then the diagram of R in its usual model-theoretic meaning is an uncountable object, and hence cannot be made available to any algorithm as an oracle. In any case, the presentation above shows that nothing is required besides the existential $\mathcal{L}_{\mathrm{ring}}$-theory of κ.

Remark 3.11. Above we have taken care to perform all computation in polynomial rings over the concrete field $\mathbb{F}_p(\pi)$ (or over the ring $\mathbb{F}_p[\pi]$). Denef and Schoutens are not always clear on this point. For instance, [Denef and Schoutens 2003, Lemma 4.2] makes a claim about an algorithm computing a reduction over the ring R, but it is unclear what this means if R is not explicitly presented (for example, uncountable).

While it may appear that such problems can be easily avoided by working over finitely generated subfields (and indeed this is suggested in [Denef and Schoutens 2003, Remark 4.1]), this leads to rather subtle issues in general, since neither reducedness nor regularity is preserved under inseparable base change: if $K_0 \subseteq K$ is a subfield over which K is not separable, then a regular variety over K_0 may become nonreduced (thus in particular nonregular) over K. Similarly, one needs to be careful in general to distinguish regularity from smoothness, because these disagree for varieties over imperfect base fields, and so regularity cannot be checked using the Jacobian criterion as suggested in [Denef and Schoutens 2003, Remark 4.1].

We can avoid these issues above because the extension $K/\mathbb{F}_p(\pi)$ is separable and therefore none of the pathologies mentioned can arise.

Remark 3.12. Above we have in fact only used that there exists a resolution of the affine variety V by a projective birational morphism, rather than a blowing-up. This looks superficially weaker than hypothesis (R0) or the equivalent [Denef and Schoutens 2003, Section 4, Conjecture 1], but in fact any projective birational morphism to a quasiprojective variety is a blowing-up [Liu 2002, Theorem 8.1.24]. (Of course, it suffices for us to demand (R0) for affine varieties.)

In fact, the argument above can be modified to make do with (R1) in place of (R0). To do this, consider the $\mathbb{F}_p[\pi]$-scheme $\mathcal{V} = \operatorname{Spec}(\mathbb{F}_p[\pi][X_1, \dots, X_m]/(f_1, \dots, f_n))$, which has V as above as its generic fiber. Then a birational proper morphism $V' \to V$ with V' regular (which exists under the assumption of (R1)) can be extended to a proper morphism $\mathcal{V}' \to \mathcal{V}$, with an $\mathbb{F}_p[\pi]$-scheme $\mathcal{V}'$ which has V' as its generic fiber. (This step seems to require Nagata's compactification theorem; see Section (12.15) in [Görtz and Wedhorn 2010].) Covering $\mathcal{V}'$ by affine $\mathbb{F}_p[\pi]$-schemes $\mathcal{V}'_0, \dots, \mathcal{V}'_{m'}$, one can then proceed as above. We omit the details.

Remark 3.13. In spite of the technical superiority of scheme-theoretic algebraic geometry over more naive conceptions of varieties, we find that one really can work in the naive language almost throughout, excepting some of the more subtle points surrounding smoothness and regularity which cannot be handled adequately in this way.

Denef and Schoutens largely use scheme-theoretic language, but not entirely correctly. In particular, by an open W of a finite-type R-scheme X as in [Denef and Schoutens 2003, Theorem 2.4 and proof of Theorem 4.3] they seem to mean a subset of the R-points of X defined by finitely many inequations, which is not the same as an open subscheme (which, after all, would itself be a finite-type R-scheme and could simply replace X), but rather an open subscheme of the generic fiber X_K. An R-rational point of W in their language is then an R-point of X such that the associated K-point of X_K lies on W. This issue also seems to have been noted independently in [Kartas 2023a, proof of Theorem A].

4. Axiomatization of the existential theory

In this final section we prove Theorem 4.12, which is a strengthening of Theorem 1.5 to allow parameters from a $\mathbb{Z}$-valued base field (C, u). Here, by $\mathbb{Z}$-valued we mean that $uC \cong \mathbb{Z}$ (sometimes called *discrete*). A major role in our proof is played by partial sections of residue maps:

Definition 4.1. Let (K, v) be a valued field. We denote by $\operatorname{res}_v : \mathcal{O}_v \to Kv$ the residue map. A *partial section* of res_v is a ring homomorphism $\zeta : k \to \mathcal{O}_v$ with $\operatorname{res}_v \circ \zeta = \operatorname{id}_k$ for some subfield k of Kv. A *section* of res_v is a partial section $\zeta : Kv \to \mathcal{O}_v$ defined on the full residue field Kv.

We will repeatedly have to extend such partial sections. There are two main settings where this is possible. The first one is in the context of complete discrete valuation rings:

Proposition 4.2. *Let (K, v) be an equicharacteristic complete $\mathbb{Z}$-valued field with uniformizer π. Then every partial section $\zeta : k \to K$ of res_v has a unique extension to an embedding of valued fields $\alpha : (k((t)), v_t) \to (K, v)$ with $\alpha(t) = \pi$, which is an isomorphism if $k = Kv$.*

Proof. The partial section ζ extends uniquely to an embedding $\alpha_0 : (k(t), v_t) \to (K, v)$ with $\alpha_0(t) = \pi$, and then the existence and uniqueness of the completion give a unique extension α of α_0 from the completion $(k((t)), v_t)$ of $(k(t), v_t)$ into the complete valued field (K, v). If ζ is a section, every element of K has a power series expansion in t with coefficients in $\zeta(K)$ (see, for example, [Serre 1979, Chapter II §4]); hence α is surjective. □

Proposition 4.3. *Let (K, v) be an equicharacteristic complete $\mathbb{Z}$-valued field. Every partial section $\zeta : k \to \mathcal{O}_v$ of res_v with Kv/k separable extends to a section of res_v. In particular, res_v has a section.*

Proof. This follows from the structure theory of complete discrete valuation rings; for example, apply [Roquette 1959, Satz 1a] with $E_0 = \zeta(k)$ in the notation there. The "in particular" follows by applying the first part to the partial section defined on the prime field. □

Secondly, we can always extend partial sections after passage to a suitable elementary extension:

Lemma 4.4. *Let (K, v) be a henselian valued field. For every partial section $\zeta_0 : k_0 \to K$ of res_v with Kv/k_0 separable there exists an elementary extension $(K, v) \prec (K^*, v^*)$ with a partial section $\zeta : Kv \to K^*$ of res_{v^*} that extends ζ_0.*

Proof. For each subfield $E \subseteq Kv$, let $\mathcal{L}_{\mathrm{val}}(E, Kv)$ be the language expanding $\mathcal{L}_{\mathrm{val}}$ by new constant symbols $\{c_x \mid x \in E\}$ and $\{d_x \mid x \in Kv\}$. Let $\rho(a, b)$ be an $\mathcal{L}_{\mathrm{val}}$-formula expressing that a and b have the same residue and consider the $\mathcal{L}_{\mathrm{val}}(E, Kv)$-theory

$$T_E := \{d_x \in \mathcal{O} \mid x \in Kv\} \cup \{c_x \in \mathcal{O} \wedge \rho(c_x, d_x) \mid x \in E\} \cup \{c_x + c_y = c_{x+y} \wedge c_x c_y = c_{xy} \mid x, y \in E\}.$$

If we expand (K, v) to an $\mathcal{L}_{\mathrm{val}}(k_0, Kv)$-structure $\mathbb{K}_{\zeta_0}$ by interpreting $c_x^{\mathbb{K}_{\zeta_0}} = \zeta_0(x)$ for $x \in k_0$ and $d_x^{\mathbb{K}_{\zeta_0}}$ any choice of element in the valuation ring of v with residue x, for each $x \in Kv$, then $\mathbb{K}_{\zeta_0} \models T_{k_0}$.

Let Δ be a finite subset of T_{Kv}. Then there exists a finitely generated subextension E/k_0 of Kv/k_0 such that Δ is an $\mathcal{L}_{\mathrm{val}}(E, Kv)$-theory. Since Kv/k_0 is separable, E/k_0 is separably generated. Therefore, by [Anscombe and Fehm 2016, Lemma 2.3], ζ_0 extends to a partial section $\hat{\zeta} : E \to K$ of res_v. Equivalently, there is an $\mathcal{L}_{\mathrm{val}}(E, Kv)$-expansion $\mathbb{K}_{\hat{\zeta}}$ of $\mathbb{K}_{\zeta_0}$ to a model of T_E. In particular, $\mathbb{K}_{\hat{\zeta}}$ models Δ.

Therefore, T_{Kv} is consistent with the elementary diagram of $\mathbb{K}_{\zeta_0}$; hence by the compactness theorem there exists an $\mathcal{L}_{\mathrm{val}}(Kv, Kv)$-structure $\mathbb{K}^* \models T_{Kv}$ whose $\mathcal{L}_{\mathrm{val}}(k_0, Kv)$-reduct is an elementary extension of $\mathbb{K}_{\zeta_0}$. In particular, its $\mathcal{L}_{\mathrm{val}}$-reduct (K^*, v^*) is an elementary extension of (K, v), and $\zeta : Kv \to K^*$ defined by $x \mapsto c_x^{\mathbb{K}^*}$ is a partial section of res_{v^*} that extends ζ_0. □

Proposition 4.5. *Let (K, v) be a henselian valued field. For every partial section $\zeta_0 : k_0 \to K$ of res_v with Kv/k_0 separable there exists an elementary extension $(K, v) \prec (K^*, v^*)$ with a section $\zeta : K^*v^* \to K^*$ of res_{v^*} that extends ζ_0.*

Proof. By Lemma 4.4 there exists an elementary extension $\mathbb{K}_0 := (K, v) \prec \mathbb{K}_1 = (K_1, v_1)$ with a partial section $\zeta_1 : Kv \to K_1$ of res_{v_1} extending ζ_0. Since $Kv \prec K_1v_1$, the extension K_1v_1/Kv is in particular separable (see, for example, [Ershov 2001, Corollary 3.1.3]), and so we can iterate Lemma 4.4 to obtain a chain of elementary extensions $\mathbb{K} = \mathbb{K}_0 \prec \mathbb{K}_1 \prec \cdots$ and for each $i > 0$ a partial section $\zeta_i : K_{i-1}v_{i-1} \to K_i$ of res_{v_i} extending ζ_{i-1}. The direct limit $\mathbb{K}^* = (K^*, v^*) := \varinjlim_i \mathbb{K}_i$ is then an elementary extension of $\mathbb{K}_0$ with a partial section $K^*v^* = \varinjlim_i K_{i-1}v_{i-1} \to \varinjlim_i K_i = K^*$ extending ζ_0. □

Given a field extension K/C we denote by (K, C) the $\mathcal{L}_{\mathrm{ring}}(C)$-structure expanding K in the usual way: the constant symbol c_x is interpreted by x itself, for each $x \in C$. Analogously, (K, v, C) denotes the

$\mathcal{L}_{\mathrm{val}}(C)$-structure similarly expanding (K, v), and (K, v, π, C) denotes the $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-structure similarly expanding (K, v, π).

Corollary 4.6. *Assume* (R4). *Let (K, v) be an equicharacteristic henselian valued field with Kv large, and assume that v is trivial on a subfield C of K. We identify C with its image under* res_v *and assume that the extension Kv/C is separable. Then the existential $\mathcal{L}_{\mathrm{ring}}(C)$-theories of K and Kv coincide.*

Proof. By Proposition 4.5 there is an elementary extension $(K, v) \prec (K^*, v^*)$ with a section $\zeta : K^*v^* \to K^*$ of res_{v^*} that extends the partial section id_C of res_v. Since Kv is large, so is its elementary extension K^*v^*; hence $\zeta(K^*v^*) \prec_\exists K^*$ by (R4). So for the existential $\mathcal{L}_{\mathrm{ring}}(C)$-theories we get $\mathrm{Th}_\exists(K) = \mathrm{Th}_\exists(K^*) = \mathrm{Th}_\exists(\zeta(K^*v^*)) = \mathrm{Th}_\exists(K^*v^*) = \mathrm{Th}_\exists(Kv)$. □

This concludes our discussion on extensions of partial sections.

Definition 4.7. We denote by T the $\mathcal{L}_{\mathrm{val}}(\varpi)$-theory of equicharacteristic henselian valued fields with distinguished uniformizer.

Remark 4.8. The class of equicharacteristic henselian valued fields with distinguished uniformizer is elementary. Also, T admits a recursive axiomatization; for example, see [Kuhlmann 2016, Section 4] for an explicit axiomatization of henselianity.

Lemma 4.9. *Every existential $\mathcal{L}_{\mathrm{val}}(\varpi)$-formula is equivalent modulo T to an existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-formula.*

Proof. This follows since the existential $\mathcal{L}_{\mathrm{ring}}(\varpi)$-formula $\varphi(x)$ given by $\exists y\ y^2 + y = \varpi x^2$ defines the valuation ring $\mathcal{O}_v$ in every $(K, v, \pi) \models T$, and therefore so does the universal formula $\neg\varphi((\varpi x)^{-1})$ given by $\forall y\ \varpi x^2(y^2 + y) \neq 1$. These formulas basically go back to [Robinson 1965]; see [Fehm and Jahnke 2017] for more on definable valuations. □

Let R be a discrete valuation ring with field of fractions C, u the valuation on C with $\mathcal{O}_u = R$, and $(\hat{C}, \hat{u})$ the completion of (C, u). Then R is *excellent* if and only if $\hat{C}/C$ is separable. This is compatible with the more general definition of excellent rings in commutative algebra (see Corollary 8.2.40(b) in [Liu 2002]) but for discrete valuation rings we may as well take it as our definition.

Remark 4.10. Although we will not need it below, we note that R is excellent if and only if it is defectless in the sense of valuation theory, i.e., the fundamental inequality is an equality for all finite extensions of C; see [Kuhlmann 2016, (1.5)]. Indeed, $\hat{C}/C$ is separable if and only if R is a Japanese ring by [EGA IV_2 1965, Corollaire (7.6.6)], which in turn is the case if and only if R is defectless by [Bourbaki 2006, Chapitre VI §8 No 5 Théorème 2].

Proposition 4.11. *Assume* (R4). *Let (C, u) be an equicharacteristic $\mathbb{Z}$-valued field with distinguished uniformizer π such that $\mathcal{O}_u$ is excellent. Let $(K, v, \pi), (L, w, \pi) \models T$ be extensions of (C, u, π) such that Kv/Cu and Lw/Cu are separable. If* $\mathrm{Th}_\exists(Kv, Cu) \subseteq \mathrm{Th}_\exists(Lw, Cu)$, *then* $\mathrm{Th}_\exists(K, v, \pi, C) \subseteq \mathrm{Th}_\exists(L, w, \pi, C)$.

Proof. Let φ be an existential $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-sentence with $(K, v, \pi, C) \models \varphi$. We have to show that $(L, w, \pi, C) \models \varphi$, and for this we are allowed to replace both (K, v, π, C) and (L, w, π, C) by arbitrary elementary extensions, so assume without loss of generality that (K, v, π, C) is $\aleph_1$-saturated and (L, w, π, C) is $|K|^+$-saturated. Since Lw is then at least $|Kv|^+$-saturated, and $(Lw, Cu) \models \mathrm{Th}_\exists(Kv, Cu)$, there is a Cu-embedding $f : Kv \to Lw$; see [Chang and Keisler 1990, Lemma 5.2.1]. By Lemma 4.9 we can assume without loss of generality that φ is an existential $\mathcal{L}_{\mathrm{ring}}(\varpi, C)$-sentence. Furthermore, we can replace each occurrence of ϖ by the constant symbol c_π to assume without loss of generality that φ is an existential $\mathcal{L}_{\mathrm{ring}}(C)$-sentence.

Since π is a uniformizer of v, $v(\pi)$ generates a convex subgroup of vK isomorphic to $\mathbb{Z}$. In particular, v has a finest proper coarsening v^+. As v^+ is trivial on C, we can view Kv^+ as an extension of C. On Kv^+, v induces a henselian valuation $\bar{v}$ extending u with $\bar{v}(Kv^+) \cong \mathbb{Z}$ and with uniformizer π. Since (K, v) is $\aleph_1$-saturated, $(Kv^+, \bar{v})$ is complete; see [Anscombe and Kuhlmann 2016, Claim on page 411] and note that maximal $\mathbb{Z}$-valued fields are complete. Given that $\mathcal{O}_u$ is excellent, that $(Kv^+, \bar{v})$ and (C, u) have a common uniformizer, and that the residue extension Kv/Cu of $(Kv^+, \bar{v})/(C, u)$ is separable, we have that Kv^+/C is separable by [Bosch et al. 1990, proof of Lemma 3.6/2][1]. Since Kv^+ is large, because it admits the nontrivial henselian valuation $\bar{v}$, Corollary 4.6 (which works under (R4)) shows that $\mathrm{Th}_\exists(K, C) = \mathrm{Th}_\exists(Kv^+, C)$, and the analogous argument (note that $|K|^+ \geq \aleph_1$) gives that $\mathrm{Th}_\exists(L, C) = \mathrm{Th}_\exists(Lw^+, C)$.

Since we assumed that $\varphi \in \mathrm{Th}_\exists(K, C)$ and want to show that $\varphi \in \mathrm{Th}_\exists(L, C)$, we can now replace (K, v, π, C) by $(Kv^+, \bar{v}, \pi, C)$ and (L, w, π, C) by $(Lw^+, \bar{w}, \pi, C)$, so that both (K, v) and (L, w) are complete $\mathbb{Z}$-valued fields with uniformizer π — note however that we may no longer assume any saturation for either (K, v) or (L, w). We may then also replace C by $\hat{C}$ to assume that (C, u) is complete. By Proposition 4.3 there exists a section $\zeta : Cu \to C$ of res_u, which we can see as a partial section of res_v and res_w. Since Kv/Cu and Lw/Cu are separable, applying Proposition 4.3 again yields that ζ extends to a section $\xi_K : Kv \to K$ of res_v and to a section $\xi_L : Lw \to L$ of res_w. By Proposition 4.2, ζ, ξ_K and ξ_L extend to isomorphisms $\gamma : (Cu((t)), v_t) \to (C, u)$, $\alpha : (Kv((t)), v_t) \to (K, v)$ and $\beta : (Lw((t)), v_t) \to (L, w)$ with $\gamma(t) = \pi$, $\alpha(t) = \pi$ and $\beta(t) = \pi$, and $\alpha|_{Cu((t))} = \beta|_{Cu((t))} = \gamma$ by the uniqueness assertion thereof. Also, the Cu-embedding $f : Kv \to Lw$ extends uniquely to an embedding $\iota : (Kv((t)), v_t) \to (Lw((t)), v_t)$ with $\iota(t) = t$ that is the identity on $Cu((t))$. Therefore, $\beta \circ \iota \circ \alpha^{-1} : (K, v) \to (L, w)$ is an embedding which restricted to C is $\gamma \circ \mathrm{id}_{Cu((t))} \circ \gamma^{-1} = \mathrm{id}_C$:

$$\begin{tikzcd}
K & Kv((t)) \arrow[l, "\alpha"'] \arrow[r, hook, "\iota"] & Lw((t)) \arrow[r, "\beta"] & L \\
C \arrow[u, dash] & Cu((t)) \arrow[u, dash] \arrow[l, "\gamma"'] \arrow[r, "\mathrm{id}"] & Cu((t)) \arrow[u, dash] \arrow[r, "\gamma"] & C \arrow[u, dash]
\end{tikzcd}$$

So $(K, C) \models \varphi$ implies that $(L, C) \models \varphi$, which concludes the proof. □

[1]The "in particular" of the statement of the lemma is not correct, but this does not concern us. Alternatively, the separability of Kv^+/C follows from applying [Ershov 1967, page 44, Lemma] to the henselization of (C, u).

Theorem 4.12. *Assume* (R4). *Let (C, u) be an equicharacteristic $\mathbb{Z}$-valued field with uniformizer π such that $\mathcal{O}_u$ is excellent, and let (K, v) be a henselian extension of (C, u) for which π is a uniformizer and Kv/Cu is separable. Then the universal/existential $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-theory of (K, v, π, C) is entailed by*

(i) *the quantifier-free diagram of the $\mathcal{L}_{\mathrm{val}}(\varpi)$-structure (C, u, π),*

(ii) *the $\mathcal{L}_{\mathrm{val}}$-axioms for equicharacteristic henselian valued fields,*

(iii) *the $\mathcal{L}_{\mathrm{val}}(\varpi)$-axiom expressing that π has smallest positive value, and*

(iv) *$\mathcal{L}_{\mathrm{val}}(C)$-axioms expressing that the residue field models the universal/existential $\mathcal{L}_{\mathrm{ring}}(Cu)$-theory of Kv.*

In particular, if C is countable and we fix a surjection $\rho : \mathbb{N} \to C$, the existential $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-theory of (K, v, π, C) is decidable relative to the existential $\mathcal{L}_{\mathrm{ring}}(Cu)$-theory of (Kv, Cu) and the quantifier-free diagram of the $\mathcal{L}_{\mathrm{val}}(\varpi)$-structure (C, u, π).

Proof. Note that (K, v, π) is a model of the theory T from Definition 4.7. Every model of (i)–(iii) can be viewed as a henselian valued field extending (C, u) with uniformizer π, and, in particular, as a model of T. Let (L, w, π, C) be a model of (i)–(iv). Since Kv/Cu is separable, the universal $\mathcal{L}_{\mathrm{ring}}(Cu)$-theory of Kv contains, for every finite p-independent tuple $\underline{x}$ from Cu, the statement that $\underline{x}$ remains p-independent over Kv. Therefore, since (L, w) satisfies (iv), Lw/Cu is also separable. Since (iv) for (L, w) also implies that $\mathrm{Th}_\exists(Lw, Cu) = \mathrm{Th}_\exists(Kv, Cu)$, Proposition 4.11 gives that $\mathrm{Th}_\exists(L, w, \pi, C) = \mathrm{Th}_\exists(K, v, \pi, C)$. This shows that every universal or existential sentence true in (K, v, π, C) is entailed by (i)–(iv).

The list of axioms (ii) is recursive, and (iii) is a single axiom. Therefore, with an oracle for (i) and an oracle for (iv), the sentences deducible from (i)–(iv) can be recursively enumerated. Since the negation of an existential sentence is universal, and every existential or universal sentence true in (K, v, π, C) is entailed by (i)–(iv), the completeness theorem then gives a decision procedure for the existential theory of (K, v, π, C) modulo these two oracles. □

Remark 4.13. The notion of relative decidability used here is the obvious modification of Remark 3.3. We define an injection of the set of $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-sentences into $\mathbb{N}$ by a standard Gödel coding using the map $C \to \mathbb{N}$, $c \mapsto \min \rho^{-1}(c)$. Similarly, we define an injection of the set of $\mathcal{L}_{\mathrm{ring}}(Cu)$-sentences into $\mathbb{N}$ using in addition the map $Cu \to \mathbb{N}$, $d \mapsto \min \rho^{-1}(\mathrm{res}_u^{-1}(d))$. Relative decidability then means that there exists a Turing machine that takes as input the code of an existential $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-sentence φ and outputs yes or no according to whether $(K, v, \pi, C) \models \varphi$, and uses an oracle for the existential $\mathcal{L}_{\mathrm{ring}}(Cu)$-theory of (Kv, Cu) and another oracle for the quantifier-free $\mathcal{L}_{\mathrm{val}}(\varpi, C)$-theory of (C, u, π, C).

Proof of Theorem 1.5. Let F be the prime subfield of K. Since $v(\pi) \neq 0$ and (K, v) is equicharacteristic, π is transcendental over F. Since $v(\pi) > 0$ and v restricted to F is trivial, the restriction of v to $F(\pi)$ is the π-adic valuation v_π. Let $(C, u) = (F(\pi), v_\pi)$. Then (K, v) is an extension of (C, u). Since $Cu = F$ is perfect, Kv/Cu is separable. The valuation ring $\mathcal{O}_u$ is excellent since $F(\!(\pi)\!)/F(\pi)$ is separable (see the proof of Lemma 3.8).

Let (L, w, π') be a model of Theorem 1.5(i)–(iii). Then (L, w, π') is a model of Theorem 4.12(ii)–(iii). In particular, (L, w, π') is equicharacteristic, and by Theorem 1.5(iii), Lw has the same characteristic as Kv. Thus, without loss of generality, L is an extension of F. By the same argument as above for (K, v), (L, w) is an extension of $(F(\pi'), v_{\pi'})$. The latter is isomorphic to $(F(\pi), v_\pi)$; thus (L, w, π') admits a unique expansion to an $\mathcal{L}_{\text{val}}(\varpi, C)$-structure $\mathbb{L}$ that models Theorem 4.12(i) for $\mathbb{K} := (K, v, \pi, C)$. Since Lw models the universal/existential theory of Kv, and $Cu = F$ is a prime field, $\mathbb{L}$ models Theorem 4.12(iv) for $\mathbb{K}$.

By Theorem 4.12, $\mathbb{L}$ models the universal/existential theory of $\mathbb{K}$. In particular, (L, w, π') models the universal/existential theory of (K, v, π), which proves the first part of the theorem. The final sentence of the theorem follows, just as the final sentence of Theorem 4.12. □

Proof of Corollary 1.6. This follow from Theorem 4.12 with $(C, u) = (\mathbb{F}_q(t), v_t)$ and $(K, v) = (\mathbb{F}_q((t)), v_t)$, for which we verify the prerequisites. The existential $\mathcal{L}_{\text{ring}}(\mathbb{F}_q)$-theory of $\mathbb{F}_q$ is decidable. The same holds for the quantifier-free diagram of $(\mathbb{F}_q(t), v_t, t)$, since elements of $\mathbb{F}_q(t)$ can be explicitly represented as quotients of polynomials in such a way as to allow concrete computation of sums, products and the valuation under v_t. More formally, the ring $\mathbb{F}_q(t)$ has an *explicit representation* in the sense of [Fröhlich and Shepherdson 1956, §2] by Theorem 3.4 there, and this representation in particular fixes a surjection $\rho : \mathbb{N} \to \mathbb{F}_q(t)$ with respect to which the quantifier-free diagram of $(\mathbb{F}_q(t), t)$ is decidable in the sense of Remark 4.13. Since elements of $\mathbb{F}_q(t)$ are represented by quotients f/g with $f, g \in \mathbb{F}_q[t]$ with $g \neq 0$ in the proof of [Fröhlich and Shepherdson 1956, Theorem 3.4], and one can determine $v_t(f)$ and $v_t(g)$ for explicitly given f and g, it is decidable whether a given element of $\mathbb{F}_q(t)$ lies in the valuation ring of v_t. □

Corollary 4.14. *Assume* (R4). *Let $(C, u) \subseteq (K, v)$ be an extension of equicharacteristic henselian valued fields. Suppose that $uC = \mathbb{Z}$ and that $\mathcal{O}_u$ is excellent. If $uC \prec_\exists vK$ and $Cu \prec_\exists Kv$, then $(C, u) \prec_\exists (K, v)$.*

Proof. The assumption $\mathbb{Z} = uC \prec_\exists vK$ implies that a uniformizer of u is also a uniformizer of v. The assumption $Cu \prec_\exists Kv$ implies that Kv/Cu is separable (see again [Ershov 2001, Corollary 3.1.3]) and that Cu and Kv have the same existential $\mathcal{L}_{\text{ring}}(Cu)$-theory. Theorem 4.12 then gives that (C, u) and (K, v) have the same existential $\mathcal{L}_{\text{val}}(C)$-theory. □

Remark 4.15. The assumption that $\mathcal{O}_u$ is excellent is necessary in all these statements. For example, in Corollary 4.14, if (C, u) is existentially closed in its completion $(K, v) = (\hat{C}, \hat{u})$, then in particular $\hat{C}/C$ is separable.

As a consequence we also obtain the following variant of [Anscombe and Fehm 2016, Theorem 7.1] and its corollaries:

Corollary 4.16. *Assume* (R4). *Let (K, v) and (L, w) be equicharacteristic nontrivially valued henselian fields with the common trivially valued subfield C. We identify C with its image under* res_v *and under* res_w *and assume that Kv/C and Lw/C are separable. If* $\text{Th}_\exists(Kv, C) \subseteq \text{Th}_\exists(Lw, C)$, *then* $\text{Th}_\exists(K, v, C) \subseteq \text{Th}_\exists(L, w, C)$.

Proof. Let (K', v') be a henselian extension of (K, v) that has a uniformizer π (say a fresh indeterminate) and satisfies $Kv = K'v'$ — for example, let $(K', \tilde{v})$ be a henselian extension of (K, v) with $K'\tilde{v} = Kv((t))$ as in [Anscombe and Fehm 2017, Lemma 2.1], and let v' be the composed valuation $v_t \circ \tilde{v}$. Then both (K', v', π) and $(Lw(\pi)^h, v_\pi, \pi)$ are extensions of the equicharacteristic $\mathbb{Z}$-valued field $(C(\pi), v_\pi)$. Also, the extension $K'v'/C(\pi)v_\pi$ is equal to Kv/C, and $(Lw(\pi)^h)v_\pi/C(\pi)v_\pi$ is equal to Lw/C, so we may apply Proposition 4.11 to obtain $\mathrm{Th}_\exists(K', v', \pi, C(\pi)) \subseteq \mathrm{Th}_\exists(Lw(\pi)^h, v_\pi, \pi, C(\pi))$. Taking reducts yields $\mathrm{Th}_\exists(K, v, C) \subseteq \mathrm{Th}_\exists(K', v', C) \subseteq \mathrm{Th}_\exists(Lw(\pi)^h, v_\pi, C)$.

By Lemma 4.4 there is an elementary extension $(L, w) \prec (L^*, w^*)$ with a partial section $\zeta : Lw \to L^*$ of res_{w^*} over C, which then extends to an embedding $(Lw(\pi)^h, v_\pi) \to (L^*, w^*)$ over C by sending π to any nonzero element of positive value. Hence we obtain the inclusion $\mathrm{Th}_\exists(Lw(\pi)^h, v_\pi, C) \subseteq \mathrm{Th}_\exists(L^*, w^*, C) = \mathrm{Th}_\exists(L, w, C)$. □

Remark 4.17. We observe that the assumption (R4) is necessary in Corollary 4.16 (and hence in Proposition 4.11 and also, less immediately, in Theorem 4.12): If C is large and K/C an extension with a valuation v on K/C with $Kv = C$, we apply Corollary 4.16 with $(L, w) = (C((t)), v_t)$; since $Kv = C = Lw$, we conclude that $\mathrm{Th}_\exists(K, v, C) = \mathrm{Th}_\exists(C((t)), v_t, C)$, so $C \prec_\exists C((t))$ implies that $C \prec_\exists K$; hence (R4) must hold.

Remark 4.18. As noted in Remark 2.4, the statement of (R4) is known to hold at least for certain base fields K, and inspection of our proofs shows that the assumption that (R4) holds can be weakened to require only that it holds when restricted to specific base fields: In Corollary 4.6 it suffices to assume that (R4) holds when restricted to elementary extensions of Kv. In Proposition 4.11 it therefore suffices to assume that (R4) holds when restricted to residue fields of proper coarsenings of elementary extensions of (K, v) and (L, w). In particular, both here and similarly in Theorem 4.12 and Corollaries 4.14 and 4.16 it suffices to assume that (R4) holds when restricted to extensions of C.

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The diagonal coinvariant ring of a complex reflection group

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For an irreducible complex reflection group W of rank n containing N reflections, we put $g = 2N/n$ and construct a $(g+1)^n$-dimensional irreducible representation of the Cherednik algebra which is (as a vector space) a quotient of the diagonal coinvariant ring of W. We propose that this representation of the Cherednik algebra is the single largest representation bearing this relationship to the diagonal coinvariant ring, and that further corrections to this estimate of the dimension of the diagonal coinvariant ring by $(g+1)^n$ should be orders of magnitude smaller. A crucial ingredient in the construction is the existence of a dot action of a certain product of symmetric groups (the Namikawa–Weyl group) acting on the parameter space of the rational Cherednik algebra and leaving invariant both the finite Hecke algebra and the spherical subalgebra; this fact is a consequence of ideas of Berest and Chalykh on the relationship between the Cherednik algebra and quasiinvariants.

1. Introduction

1A. ***Coinvariant rings.*** Given a finite linear group $W \subseteq \mathrm{GL}(\mathfrak{h})$, where $\mathfrak{h}$ is a finite-dimensional complex vector space, the group W acts by automorphisms on the ring $\mathbf{C}[\mathfrak{h}]$ of polynomial functions on $\mathfrak{h}$. It is well known that the quotient variety $\mathfrak{h}/W$ is smooth precisely when the ring $\mathbf{C}[\mathfrak{h}]^W$ is isomorphic to a polynomial ring, which happens exactly when the group W is generated by reflections. In this case, letting J_W be the ideal of $\mathbf{C}[\mathfrak{h}]$ generated by the positive-degree W-invariant polynomials, the *coinvariant ring* of W is the ring $\mathbf{C}[\mathfrak{h}]/J_W$, which might be thought of as the ring of functions on the scheme-theoretic fiber over 0 of the quotient map $\mathfrak{h} \to \mathfrak{h}/W$, and is isomorphic to the regular representation of W. In fact, it is a graded W-module, and the *exponents* of a given irreducible representation E of $\mathbf{C}W$ are the degrees in which it occurs in this graded module. For reflection groups such as the symmetric group with combinatorial structure encoded via partitions and various sorts of tableaux, these exponents may be calculated via combinatorial statistics.

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1B. ***Diagonal coinvariant rings.*** The group W also acts on $\mathbf{C}[\mathfrak{h}^*\oplus\mathfrak{h}]$ by automorphisms, and, by analogy with the preceding one, we may consider the quotient variety $(\mathfrak{h}^*\times\mathfrak{h})/W$, which is an example of a symplectic singularity. Its ring of functions is the invariant ring $\mathbf{C}[\mathfrak{h}^*\oplus\mathfrak{h}]^W$, which in the case where W is a reflection group, has a more interesting structure than $\mathbf{C}[\mathfrak{h}]^W$. Likewise, letting I_W be the ideal of $\mathbf{C}[\mathfrak{h}^*\oplus\mathfrak{h}]$ generated by the positive degree elements of $\mathbf{C}[\mathfrak{h}^*\oplus\mathfrak{h}]^W$, the *diagonal coinvariant ring* is the quotient

$$R_W=\mathbf{C}[\mathfrak{h}^*\oplus\mathfrak{h}]/I_W.$$

It may be thought of as the ring of functions on the scheme-theoretic fiber over 0 of the quotient map

$$\mathfrak{h}^*\times\mathfrak{h}\to(\mathfrak{h}^*\times\mathfrak{h})/W$$

(for this reason, and to avoid confusion with the coinvariants of a group action, perhaps the names *zero-fiber ring* and *diagonal zero-fiber ring* are more suitable). The ring R_W carries a bigrading, and in analogy with the case of the coinvariant ring, one may ask for the bigraded character of this ring. However, the answer to this question is known explicitly only for two classes of examples: the symmetric groups and the dihedral groups. For most reflection groups W, we do not even have a conjectural formula for the dimension of R_W.

However, for real reflection groups, Haiman [1994] conjectured, and Gordon [2003] proved, that there is a quotient of R_W of dimension $(h+1)^n$, where h is the Coxeter number of W. Gordon predicted that this phenomenon generalizes at least to the complex reflection groups of type $G(\ell,m,n)$, and Vale [2007a] and the author [Griffeth 2010b] proved this. Later, Gordon and the author [Gordon and Griffeth 2012] showed (assuming the freeness conjecture for Hecke algebras, which is now known) that a similar technique, based on Rouquier's theorem on the uniqueness of highest weight covers, would produce a quotient ring of R_W of dimension $(h+1)^n$, where now we define the Coxeter number h of an irreducible complex reflection group by

$$h=\frac{N+N^*}{n},$$

where N is the number of reflections in W, N^* is the number of reflecting hyperplanes, and $n=\dim(\mathfrak{h})$ is the rank. Here we point out that, while these quotients are natural from the point of view of Catalan combinatorics (as predicted in [Bessis and Reiner 2011]), they should not be regarded as the best approximations available to the full diagonal coinvariant ring in the case where the group W contains reflections of order greater than 2.

1C. ***Lower bounds via representation theory.*** Meanwhile, together with Ajila [Ajila and Griffeth 2021], we have very recently observed that a more delicate application of the same techniques can be used to improve the lower bound $\dim(R_W)\ge(h+1)^n$ for the type B Weyl groups. However, this improvement is orders of magnitude smaller than $(h+1)^n$, which we argue should perhaps be regarded as the principal term in an approximation of $\dim(R_W)$. Thus the first question to be answered is, do we already know the analogous principal term for an irreducible complex reflection group?

Our main purpose here is to observe that for complex groups containing reflections of order greater than 2, the approximation by $(h+1)^n$ should *not* be regarded as the principal term. Rather, we have:

Theorem 1.1. *Let W be an irreducible complex reflection group of rank n containing N reflections. There is a quotient of the diagonal coinvariant ring R_W of dimension $(g+1)^n$, where $g = 2N/n$.*

As usual, we prove this bound by exhibiting an irreducible representation $L = L_c(\mathrm{triv})$ of the rational Cherednik algebra of dimension $(g+1)^n$ in which the determinant appears exactly once (see Lemma 3.1; in [Ajila and Griffeth 2021], we have called such representations *coinvariant type*). We do this in three ways, one of which is conjectural: firstly, we prove it in general using the philosophy from [Gordon and Griffeth 2012]. The technical details must be modified substantially, due to the fact that $g+1$ is not prime to h in general. This proof requires as input some striking coincidences from the numerology of complex reflection groups. Secondly, for the infinite family $G(\ell, m, n)$, we use the tools developed in [Griffeth 2010a; 2010b; 2018; Fishel et al. 2021]. This method gives the most information; it gives explicit bases and a practical graded character formula. For future work improving the bound in the case where $W = G(\ell, m, n)$, this construction is likely to be the most useful of the three. Finally, we sketch a construction of the required representation that depends on an elaboration of the beautiful observations of Berest and Chalykh [2011] linking the Cherednik algebra to quasiinvariants; our proof that this actually works depends, however, on calculations with Schur elements and thereby on the widely believed symmetrizing trace conjecture for the Hecke algebra, which is currently known to hold only for the infinite family, real groups, and a few of the exceptional complex reflection groups, as well as a conjecture about how Heckman–Opdam shift functors interact with standard modules. The numerological coincidences mentioned above would follow as corollaries to this method (assuming the needed conjectures are established) rather than appearing as miraculously convenient ingredients in the proof.

1D. ***KZ twists and the duality in the exponents.*** The parameter space $\mathscr{C}$ for the rational Cherednik algebra consists of W-invariant tuples of numbers $c = (c_{H,j})_{H \in \mathscr{A},\, 0 \le j \le n_H - 1}$ indexed by pairs (H, j) consisting of a reflecting hyperplane $H \in \mathscr{A}$ for W and an integer $0 \le j \le n_H - 1$. The corresponding finite Hecke algebra is the quotient of the group algebra of the braid group of W by relations of the form

$$\prod_{j=0}^{n_H-1} (T_H - e^{2\pi i (c_{H,j} + j/n_H)}) = 0,$$

where $H \in \mathscr{A}$ is a reflecting hyperplane, T_H is a generator of monodromy around H, and n_H is the order of the cyclic reflection subgroup of W fixing H pointwise. This quotient is invariant by the group $\mathscr{C}_{\mathbf{Z}}$ of translations by integer parameters, as well as by the group G_W of permutations of the parameters $c_{H,j} + j/n_H$. Thus, the parameter space for the Hecke algebra is effectively the quotient $\mathscr{C}/(\mathscr{C}_{\mathbf{Z}} \rtimes G_W)$, and this together with the relationship between the Hecke algebra and the Cherednik algebra implies that the group $\mathscr{C}_{\mathbf{Z}} \rtimes G_W$ acts by permutations (*KZ twists*) on the set of irreducible representations of W. Let σ_0 be the longest element of the subgroup of G_W fixing the indices 0, and let τ be the translation by 1 of all $c_{H,j}$, with $j \neq 0$. The composite $\sigma = \tau\sigma_0$ thus induces a permutation κ of the irreducible representations of W (see Section 3K for the precise definition).

We should remark that the group G_W is the *Namikawa–Weyl group* of the symplectic singularity $(\mathfrak{h}^* \times \mathfrak{h})/W$ (see [Namikawa 2010; 2011; 2015] for the general theory and Lemma 4.1 of [Bellamy et al. 2018] for the agreement with G_W in our case). It is, therefore, tempting to refer to $\mathscr{C}_{\mathbf{Z}} \rtimes G_W$ as the *affine Namikawa–Weyl group of W*.

In the first construction proving Theorem 1.1 above, a certain duality for the exponents plays a key role. Since it may be of independent interest, we state the result here:

Theorem 1.2. *Let κ be the permutation of the irreducible representations of W induced by σ. Then*

$$e_i(\mathfrak{h}) + e_{n-i+1}(\kappa(\mathfrak{h}^*)) = g \quad \text{for all } 1 \le i \le n,$$

where $e_i(E)$ are the exponents of the irreducible representation E of W, which are the degrees in which it appears in the ordinary coinvariant ring. In particular, $g = 2N/n$ is an integer.

We note that when the group is real, $\kappa(\mathfrak{h}^*) = \mathfrak{h}$, $h = g$, and this duality reduces to the usual one. On the other hand, when W is one of the groups $G(\ell, 1, n)$ or is a primitive group containing reflections of order greater than 2, we also have $\kappa(\mathfrak{h}^*) = \mathfrak{h}$, and the duality becomes

$$d_i + d_{n-i+1} = g + 2.$$

We obtain Theorem 1.2 via the technique of [Berest and Chalykh 2011, Section 8.3], as explained below.

In the conjectural construction of the principal coinvariant type representation, a version of a result of Berest and Chalykh [2011] plays a key role. Because it will be of use in future work on related problems and the study of the Cherednik algebra itself, we state it separately here (in the body of the paper, it is Theorem 3.6). We write $D(\mathfrak{h}^\circ)$ for the algebra of polynomial coefficient differential operators on the complement $\mathfrak{h}^\circ$ to the set of reflecting hyperplanes for W, and $D(\mathfrak{h}^\circ) \rtimes W$ for the algebra of operators generated by it and W. We recall that the Cherednik algebra H_c and its spherical subalgebra eH_ce, where e is the symmetrizing idempotent of W, are both subalgebras (the latter nonunital) of $D(\mathfrak{h}^\circ) \rtimes W$. The following theorem appears as Theorem 3.6 below, where precise definitions and conventions are specified:

Theorem 1.3. *For all $c \in \mathscr{C}$ and $g \in G_W$, we have an equality of (nonunital) subalgebras of the algebra $D(\mathfrak{h}^\circ) \rtimes W$:*

$$eH_ce = eH_{g\cdot c}e.$$

Thus the Namikawa–Weyl group G_W preserves not only the finite Hecke algebra, but the spherical subalgebra as well. In Proposition 5.4 of [Berest and Chalykh 2011], this same equality is proved for g in a certain cyclic subgroup of G_W and is the key point in their construction of Heckman–Opdam shift functors. We emphasize that no new ideas beyond what is contained in [Berest and Chalykh 2011] are necessary for the proof of this more general theorem (just very careful bookkeeping), and that another proof using different ideas recently appeared — this is Corollary 2.22 of [Bellamy et al. 2021] (see also Theorem 3.4 of [Losev 2022], which replaces equality with isomorphism but works in more generality).

1E. ***More mysterious numerology.*** There is a connection here to a conjecture of Stump [2010], that the number of occurrences of the determinant representation of W in the diagonal coinvariant ring for a well-generated complex reflection group W is given by the W-Catalan number

$$\mathrm{Cat}(W)=\prod_{i=1}^{n}\frac{h+d_i}{d_i}.$$

For the groups $G(\ell,1,n)$ and the primitive groups containing reflections of order greater than 2, our results imply that the number of occurrences of the determinant in the diagonal coinvariant ring is *at least*

$$\prod_{i=1}^{n}\frac{g+d_i^*+1}{d_i},$$

where d_i^* are the *codegrees* of W. But it turns out that, thanks to another instance of mysteriously favorable numerology, we actually have

$$g+d_i^*+1=h+d_i \quad \text{for all } 1\le i\le n$$

for such groups. This coincidence deserves further thought and gives a bit of evidence for Stump's conjecture: for although we have improved our estimation of the diagonal coinvariant ring, the number of occurrences of the determinant representation we have discovered has not increased.

1F. ***An asymptotic version of the $(n+1)^{n-1}$ conjecture.*** In order to make more concrete our hope that the number $(g+1)^n$ is the principal term in an approximation to $\dim(R_W)$, we state the result for the monomial group $W=G(\ell,m,n)$, for which

$$g=\ell(n-1)+2\Big(\frac{\ell}{m}-1\Big),$$

more explicitly:

Theorem 1.4. *Let ℓ, m, and n be positive integers with m dividing ℓ, and let $W=G(\ell,m,n)$ be the group of $n\times n$ matrices with entries that are either 0 or ℓ-th roots of 1, so that each row and each column has precisely one nonzero entry, and so that the product of the nonzero entries is an (ℓ/m)-th root of 1. Then*

$$\dim(R_W)\ge\left(\ell(n-1)+\frac{2\ell}{m}-1\right)^n.$$

In fact, we will give a construction of the relevant representation $L_c(\mathrm{triv})$ for these groups $G(\ell,m,n)$ which depends on the techniques from [Griffeth 2010b] and gives somewhat more detailed information on its graded character.

Now we can make more precise the hope that $(g+1)^n$ is almost the dimension of R_W.

Conjecture 1.1. *Suppose ℓ and m are positive integers with $\ell\ge 2$ and m dividing ℓ. Then*

$$\lim_{n\to\infty}\frac{\dim(R_{G(\ell,m,n)})}{\big(\ell(n-1)+2\ell/m-1\big)^n}=1.$$

Admittedly, the evidence for the conjecture is rather thin: it consists solely of the fact that we have so far been unable to improve the lower bound by anything of the same order of magnitude.

There is another sort of limit one can take to obtain reasonable combinatorics: as suggested by Bergeron [2013], one might work with the analog of the diagonal coinvariant ring for the product of m copies of the reflection representation $\mathfrak{h}^{\times m}$ and let m tend to infinity. But as far as we know, there is no connection between the two.

2. Quotients of $\mathbf{C}[\mathfrak{h}]$ by systems of parameters

2A. ***Reflection groups.*** Throughout this paper, we will write $W \subseteq \mathrm{GL}(\mathfrak{h})$ for an irreducible complex reflection group acting on an n-dimensional vector space $\mathfrak{h}$, $R \subseteq W$ for the set of reflections in W, and $\mathscr{A}$ for the set of reflecting hyperplanes of W. Given $H \in \mathscr{A}$, we let W_H be the pointwise stabilizer of H, which is a cyclic reflection subgroup of W. We write $\det : W \to \mathbf{C}^\times$ for the determinant character of W, which is the restriction of the determinant on $\mathrm{GL}(\mathfrak{h})$ to W. Putting $n_H = |W_H|$, we write

$$e_{H,j} = \frac{1}{n_H} \sum_{w \in W_H} \det{}^{-j}(w) w$$

for the primitive idempotents of the group algebra $\mathbf{C}W_H$. For a $\mathbf{C}W$-module E, we put

$$E_{H,j} = \dim_{\mathbf{C}}(e_{H,j} E), \quad \text{for } H \in \mathscr{A} \text{ and } 0 \le j \le n_H - 1,$$

and call the collection $E_{H,j}$ of numbers the *local data* of E. The *Coxeter number* h of W is

$$h = \frac{N + N^*}{n}, \quad \text{where } N = |R| \text{ and } N^* = |\mathscr{A}|,$$

and we also define the number $g = 2N/n$ as in Section 1. We will see that g is, in fact, an integer (when we prove Theorem 1.2). This also follows from Corollary 6.98 of [Orlik and Terao 1992], and the same number appears in Remark 8.10 of [Chapuy and Douvropoulos 2023] in the context of reflection factorizations of Coxeter elements (see Definition 3.1 from [Chapuy and Douvropoulos 2022]); I thank Theo Douvropoulos for pointing me to these references.

2B. ***W-equivariant homogeneous systems of parameters and the Koszul complex.*** Suppose $E \subseteq \mathbf{C}[\mathfrak{h}]^d$ is an n-dimensional W-submodule in the degree d piece $\mathbf{C}[\mathfrak{h}]^d$ of $\mathbf{C}[\mathfrak{h}]$ such that the quotient $\mathbf{C}[\mathfrak{h}]/\mathbf{C}[\mathfrak{h}]E$ by the ideal generated by E is finite-dimensional. That is, a basis for E is a homogeneous system of parameters in $\mathbf{C}[\mathfrak{h}]$. In this case, it follows that the Koszul complex

$$0 \to \mathbf{C}[\mathfrak{h}] \otimes \Lambda^n E \to \mathbf{C}[\mathfrak{h}] \otimes \Lambda^{n-1} E \to \cdots \to \mathbf{C}[\mathfrak{h}] \otimes E \to \mathbf{C}[\mathfrak{h}] \to \mathbf{C}[\mathfrak{h}]/\mathbf{C}[\mathfrak{h}]E \to 0$$

is exact, where the map $\mathbf{C}[\mathfrak{h}] \otimes \Lambda^k E \to \mathbf{C}[\mathfrak{h}] \otimes \Lambda^{k-1} E$ is given by the formula

$$f \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_k \mapsto \sum_{j=1}^{k} (-1)^{j-1} e_j f \otimes e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_k \quad \text{for } f \in \mathbf{C}[\mathfrak{h}] \text{ and } e_1, \ldots, e_k \in E,$$

in which the hat over a factor in a product indicates, as usual, that the factor is to be omitted. Evidently these are maps of graded $\mathbf{C}W$-modules, provided that we equip $\mathbf{C}[\mathfrak{h}] \otimes \Lambda^k E$ with the grading for which the degree of $f \otimes e_1 \wedge \cdots \wedge e_k$ is

$$\deg(f \otimes e_1 \wedge \cdots \wedge e_k) = \deg(f) + kd.$$

2C. *Graded traces.* Suppose $w \in W$ and we fixed eigenbases $x_1, \ldots, x_n$ of $\mathfrak{h}^*$ and $e_1, \ldots, e_n$ of E for the w-action, with $wx_i = \zeta_i x_i$ and $we_i = \mu_i e_i$ for certain roots of unity ζ_i and μ_i. Now the expressions $x_{i_1} x_{i_2} \cdots x_{i_k} \otimes e_{j_1} \wedge \cdots \wedge e_{j_m}$ for weakly increasing $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$ and strictly increasing $1 \leq j_1 < j_2 < \cdots j_m \leq n$ are a basis of $\mathbf{C}[\mathfrak{h}]^k \otimes \Lambda^m E$. The trace of w on $\mathbf{C}[\mathfrak{h}]^k \otimes \Lambda^m E$ is therefore,

$$\operatorname{tr}(w, \mathbf{C}[\mathfrak{h}]^k \otimes \Lambda^m E) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n \\ 1 \leq j_1 < j_2 < \cdots < j_m \leq n}} \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_k} \mu_{j_1} \mu_{j_2} \cdots \mu_{j_m},$$

which is the coefficient c_{km} of $t^k q^m$ in the expansion

$$\frac{\det(1 + qw)}{\det(1 - tw)} = \sum_{\substack{0 \leq k < \infty \\ 0 \leq m \leq n}} c_{km} t^k q^m.$$

2D. *Reflection representations and amenable representations of W.* Let E be an irreducible $\mathbf{C}W$-module of dimension m. We say that E is a *reflection representation* of W if each $r \in R$ acts as a reflection on E. For $H \in \mathcal{A}$, we put

$$C(H, E) = \sum_{j=0}^{n_H - 1} j E_{H,j}, \quad \text{where we recall that } E_{H,j} = \dim(e_{H,j} E).$$

Following [Lehrer and Taylor 2009, Definition 10.14 and Lemma 10.15], we say E is *amenable* if

$$C(H, E) \leq n_H - 1 \quad \text{for all } H \in \mathcal{A}.$$

It is immediate (as in Corollary 10.16 of [Lehrer and Taylor 2009]) that if E is a reflection representation, then E and E^* are amenable. The important point for us is the following, which is Theorem 10.18 of [Lehrer and Taylor 2009]:

Theorem 2.1. *Let E be an amenable $\mathbf{C}W$-module of dimension m with exponents $e_1, \ldots, e_m$. Then there are homogeneous elements $\omega_1, \ldots, \omega_m \in (\mathbf{C}[\mathfrak{h}] \otimes E^*)^W$ of degrees $e_1, \ldots, e_m$ such that*

$$(\mathbf{C}[\mathfrak{h}] \otimes \Lambda^\bullet E^*)^W = \bigoplus_{\substack{0 \leq p \leq m \\ 1 \leq i_1 < i_2 < \cdots < i_p \leq m}} \mathbf{C}[\mathfrak{h}]^W \omega_{i_1} \omega_{i_2} \cdots \omega_{i_p},$$

where, as usual, a product with zero factors should be interpreted as a 1.

2E. ***The determinant appears exactly once.*** We suppose we have an occurrence of an n-dimensional representation E in degree $g+1$ of $\mathbf{C}[\mathfrak{h}]$, with the property that the quotient $L = \mathbf{C}[\mathfrak{h}]/E\mathbf{C}[\mathfrak{h}]$ of $\mathbf{C}[\mathfrak{h}]$ by the ideal generated by E is finite-dimensional. We further suppose that E is an n-dimensional irreducible reflection representation of W satisfying

$$g+1 = d_i + e_{n-i+1} \quad \text{for } 1 \le i \le n, \text{ where } e_1 \le \cdots \le e_n \text{ are the exponents of } E.$$

Then using the previous material on the Koszul resolution and arguing as in Theorem 3.2 of [Griffeth 2010b] shows that there is a single occurrence of the determinant representation of W in L, which occurs in degree $e_1 + e_2 + \cdots + e_n$. Later, we will use this to establish the hypotheses of Lemma 3.1 below.

3. Proof of Theorem 1.1: the Cherednik algebra, the Hecke algebra, and KZ twists

3A. ***Outline.*** In this section, we first give the definitions of Cherednik and Hecke algebras corresponding to W, and then present what we believe to be the natural level of generality for the beautiful constructions of Berest and Chalykh [2011]. In the level of generality we need, this latter material is technically new but requires no new ideas, so we omit the proofs whenever they are completely parallel to those of Berest and Chalykh. We finish by deducing Theorem 1.1 from these ingredients.

3B. ***The parameter space.*** We write $\mathscr{C}$ for the set of tuples $c = (c_{H,j})_{H \in \mathscr{A},\, 0 \le j \le n_H - 1}$ of complex numbers $c_{H,j} \in \mathbf{C}$ indexed by pairs (H, j) consisting of a reflecting hyperplane H for W and an integer $0 \le j \le n_H - 1$, subject to the condition

$$c_{H,j} = c_{w(H),j} \quad \text{for all } w \in W,\ H \in \mathscr{A}, \text{ and } 0 \le j \le n_H - 1.$$

Thus, $\mathscr{C}$ is a finite-dimensional $\mathbf{C}$-vector space. We put

$$\mathscr{C}_{\mathbf{Z}} = \{c \in \mathscr{C} \mid c_{H,j} \in \mathbf{Z} \text{ for all } H \in \mathscr{A} \text{ and } 0 \le j \le n_H - 1\}.$$

3C. ***The Dunkl operators.*** Given $c \in \mathscr{C}$ and $y \in \mathfrak{h}$, we define the *Dunkl operator* $y_c \in D(\mathfrak{h}^\circ) \rtimes W$ by

$$y_c(f) = \partial_y(f) - \sum_{H \in \mathscr{A}} \frac{\alpha_H(y)}{\alpha_H} \sum_{j=0}^{n_H - 1} n_H c_{H,j} e_{H,j},$$

where we have fixed $\alpha_H \in \mathfrak{h}^*$ with zero set equal to H. We note that since we do not require $c_{H,0} = 0$, these Dunkl operators do not necessarily preserve the space of polynomial functions. They do, however, commute with one another, and preserve the space $\mathbf{C}[\mathfrak{h}^\circ]$ of polynomial functions on $\mathfrak{h}^\circ$.

3D. ***The rational Cherednik algebra.*** Given $c \in \mathscr{C}$, the *rational Cherednik algebra* H_c is the subalgebra of $D(\mathfrak{h}^\circ) \rtimes W$ generated by $\mathbf{C}[\mathfrak{h}]$, the group W, and the Dunkl operators y_c for all $y \in \mathfrak{h}$. It has a triangular decomposition

$$H_c \cong \mathbf{C}[\mathfrak{h}] \otimes \mathbf{C}W \otimes \mathbf{C}[\mathfrak{h}^*],$$

where we identify $\mathbf{C}[\mathfrak{h}^*]$ with the subalgebra of H_c generated by the Dunkl operators. Taking $\delta = \prod \alpha_r$, and adjoining the inverse of δ to H_c, gives the algebra $H_c[\delta^{-1}] = D(\mathfrak{h}^\circ) \rtimes W$, independent of $c \in \mathscr{C}$.

3E. ***Category*** $\mathbb{O}_c$**.** The category $\mathbb{O}_c$ is the full subcategory of H_c-mod consisting of finitely generated H_c-modules on which each Dunkl operator y_c acts locally nilpotently. Among the objects of $\mathbb{O}_c$ are the *standard modules* $\Delta_c(E)$, indexed by $E \in \mathrm{Irr}(\mathbf{C}W)$ and defined by

$$\Delta_c(E) = \mathrm{Ind}_{\mathbf{C}[\mathfrak{h}^*]\rtimes W}^{H_c}(E),$$

where $\mathbf{C}[\mathfrak{h}^*] \rtimes W$ is the subalgebra of H_c generated by W and the Dunkl operators, which act on E by 0.

3F. ***The Euler element and the c-function.*** Fix dual bases $x_1, \dots, x_n$ and $y_1, \dots, y_n$ of $\mathfrak{h}^*$ and $\mathfrak{h}$. A short calculation shows that the Euler vector field eu on $\mathfrak{h}$ may be written in terms of the Dunkl operators as

$$\mathrm{eu} = \sum_{i=1}^{n} x_i \partial_{y_i} = \sum x_i (y_i)_c + \sum_{\substack{H \in \mathcal{A} \\ 0 \le j \le n_H - 1}} n_H c_{H,j} e_{H,j}.$$

In particular, its action on the standard module $\Delta_c(E)$ is easy to describe: it acts by the scalar $d + c_E$ on the polynomial degree d piece $\mathbf{C}[\mathfrak{h}]^d \otimes E$ of $\Delta_c(E)$, where

$$c_E = \frac{1}{\dim(E)} \sum_{\substack{H \in \mathcal{A} \\ 0 \le j \le n_H - 1}} n_H c_{H,j} E_{H,j}$$

depends on the parameter c and the local data $E_{H,j} = \dim(e_{H,j} E)$ of E.

3G. ***Coinvariant type representations.*** We recall from [Ajila and Griffeth 2021] that a *coinvariant type* representation of H_c is an irreducible H_c-module L such that upon restricting L to the group algebra $\mathbf{C}W$, the determinant representation of W occurs with multiplicity one in L. Each such representation carries a canonical filtration: take the filtration on H_c defined by placing $\mathfrak{h}^*$ and $\mathfrak{h}$ in degree 1 and W in degree 0, and define

$$L^{\le d} = H_c^{\le d} L^{\det},$$

where $L^{\det}$ is the isotypic component of L for the determinant representation. The following lemma is the key point relating coinvariant type representations of H_c to the diagonal coinvariant ring R_W. The proof is straightforward, but we include it because of the central role it plays in all that follows. We define a somewhat unusual bigrading on R_W as follows: take f to be homogeneous of bidegree (a, b) if it is of total degree a in the x's and y's and if its x degree minus its y degree is b (this second grading is compatible with the Euler grading on H_c and L).

Lemma 3.1. *Let L be a coinvariant type representation of H_c, and let δ be a basis element of $L^{\det}$. The map $\mathrm{gr}(H_c) \to \mathrm{gr}(L)$ defined by $f \mapsto f \cdot \delta$ induces a surjective map of bigraded $\mathbf{C}W$-modules $R_W \otimes \det \to \mathrm{gr}(L)$.*

Proof. Since $\dim(L^{\det}) = 1$ is equal to the multiplicity of the determinant representation in $L^{\le 0} = \mathbf{C}\delta$, it follows that det does not occur in $L^{\le d}/L^{\le d-1}$ for any $d > 0$. Hence, if $f \in \mathbf{C}[\mathfrak{h}^* \oplus \mathfrak{h}]^W$ is homogeneous of positive degree, then working in $\mathrm{gr}(L)$, we have $f \cdot \delta = 0$. Therefore, the map $\mathbf{C}[\mathfrak{h}^* \oplus \mathfrak{h}] \to \mathrm{gr}(L)$ defined by $f \mapsto f \cdot \delta$ factors through R_W. Since L is irreducible and $\mathbf{C}W \cdot \delta = \mathbf{C}\delta$, we have

$$L = H_c \cdot \delta = \mathbf{C}[\mathfrak{h}]\mathbf{C}[\mathfrak{h}^*]\mathbf{C}W \cdot \delta = \mathbf{C}[\mathfrak{h}]\mathbf{C}[\mathfrak{h}^*] \cdot \delta,$$

which implies that the map $f \mapsto f \cdot \delta$ is surjective. Tensoring R_W by det makes it W-equivariant, and observing that δ is homogeneous (of a certain degree k) for the Euler grading on L implies that it is a bigraded map sending f of bidegree (a, b) to an element of bidegree $(a, b+k)$. □

3H. ***The fiber functors and the braid group.*** Given an object $M \in \mathcal{O}_c$ and a point $p \in \mathfrak{h}$, we define the *fiber* of M at p to be the vector space

$$M(p) = M/I(p)M,$$

where $I(p) \subseteq \mathbf{C}[\mathfrak{h}]$ is the ideal of functions vanishing at p. In fact, $M(p)$ is a finite-dimensional $\mathbf{C}W_p$-module. In general the functor $M \mapsto M(p)$ is only right-exact.

Writing $\delta = \prod_{r \in R} \alpha_r$, we put

$$M^\circ = \mathbf{C}[\mathfrak{h}][\delta^{-1}] \otimes_{\mathbf{C}[\mathfrak{h}]} M.$$

The functor $M \mapsto M^\circ$ is exact, and in fact M° is a $H_c[\delta^{-1}] = D(\mathfrak{h}^\circ) \rtimes W$-module which is finitely generated as a $\mathbf{C}[\mathfrak{h}^\circ] = \mathbf{C}[\mathfrak{h}][\delta^{-1}]$-module. That is, M° is a W-equivariant vector bundle on $\mathfrak{h}^\circ$ equipped with a W-equivariant flat connection. Since $M(p) = M^\circ(p)$ for $p \in \mathfrak{h}^\circ$, it therefore follows that the fiber functor $M \mapsto M(p)$ is exact for $p \in \mathfrak{h}^\circ$, and the braid group $B_W = \pi_1(\mathfrak{h}^\circ/W, p)$ acts by automorphisms on this fiber functor.

3I. ***The Hecke algebra and the KZ functor.*** In fact, the braid group action factors through the Hecke algebra $\mathcal{H}_c$, which is the quotient of the group algebra $\mathbf{C}B_W$ by the relations

$$0 = \prod_{j=0}^{n_H - 1} (T_H - e^{2\pi i (j/n_H + c_{H,j})}) \quad \text{for all } H \in \mathcal{A},$$

where T_H is a generator of monodromy around H. We will write $\mathrm{KZ}(M) = M(p)$ for the fiber $M(p)$ regarded as an $\mathcal{H}_c$ module, and refer to $M \mapsto \mathrm{KZ}(M)$ as the *Knizhnik–Zamolodchikov functor* or *KZ functor* for short. Vale [2007b] proved (see also [Berest and Chalykh 2011, Theorem 6.6]) that $\mathcal{H}_c$ is semisimple if and only if $\mathcal{O}_c$ is a semisimple category, which happens exactly when each standard module $\Delta_c(E)$ is irreducible.

3J. ***The group G_W.*** It follows from the definition of $\mathcal{H}_c$ that if c is a parameter such that for each $H \in A$, the numbers $j/n_H + c_{H,j}$ are a permutation of the numbers j/n_H (for $0 \le j \le n_H - 1$) modulo $\mathbf{Z}$, then $\mathcal{H}_c \cong \mathbf{C}W$ is isomorphic to the group algebra of W. More generally, two parameters c and c' give the same Hecke algebra provided the multisets

$$\{j/n_H + c_{H,j} \bmod \mathbf{Z} \mid 0 \le j \le n_H - 1\} \quad \text{and} \quad \{j/n_H + c'_{H,j} \bmod \mathbf{Z} \mid 0 \le j \le n_H - 1\}$$

are equal for all $H \in \mathcal{A}$.

This may be rephrased as follows: define $\rho \in \mathscr{C}$ by

$$\rho_{H,j} = j/n_H$$

and the group G_W by

$$G_W = \left\{ (s_H)_{H\in\mathscr{A}} \in \prod_{H\in\mathscr{A}} \mathrm{Sym}(\{0, 1, \ldots, n_H - 1\}) \mid s_H = s_{w(H)} \text{ for all } H \in \mathscr{A} \text{ and } w \in W \right\}.$$

Thus, an element of G_W may be regarded as a list of permutations s_H of $\{0, 1, 2, \ldots, n_H - 1\}$, one for each W-orbit on $\mathscr{A}$. By construction G_W acts on $\mathscr{C}$, but the interesting action for us is the *dot action* of G_W on $\mathscr{C}$, which is defined by the formula

$$s \cdot c = s(c + \rho) - \rho \quad \text{for } s \in G_W \text{ and } c \in \mathscr{C}.$$

Recalling the lattice $\mathscr{C}_{\mathbf{Z}}$ of integral parameters, the semidirect product group $\mathscr{C}_{\mathbf{Z}} \rtimes G_W$ acts on $\mathscr{C}$, and the quotients $\mathscr{H}_c$ and $\mathscr{H}_{g(c)}$ are equal for all $c \in \mathscr{C}$ and $g \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$. As we will never again use any other action of G_W on $\mathscr{C}$, in all formulas below we will drop the dot.

3K. *The KZ twists.* By the preceding observations, the Hecke algebras $\mathscr{H}_{g(0)}$ for $g \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$ are all equal to $\mathbf{C}W$, and the KZ functor gives an equivalence $\mathrm{KZ}_{g(0)} : \mathbb{O}_c \to \mathbf{C}W$-mod for all such g. We obtain:

Lemma 3.2. *Let $g \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$. Then there is a unique permutation κ_g^{-1} of $\mathrm{Irr}(\mathbf{C}W)$ such that*

$$\mathrm{KZ}_{g(0)}(\Delta_{g(0)}(E)) \cong \kappa_g^{-1}(E) \quad \textit{in } \mathbf{C}W\textit{-mod for all } E \in \mathrm{Irr}(\mathbf{C}W).$$

We refer to κ_g^{-1} as the *KZ twist* associated with g. The particular case in which $g \in \mathscr{C}_{\mathbf{Z}}$ is simply a translation by an element of the lattice $\mathscr{C}_{\mathbf{Z}}$ has been studied previously by Opdam [1998] and Berest and Chalykh [2011]. So the added generality here is allowing an additional permutation of the indices by some $\sigma \in G_W$. Just as in [Berest and Chalykh 2011, Corollary 7.12], the map $g \mapsto \kappa_g$ is a homomorphism from $\mathscr{C}_{\mathbf{Z}} \rtimes G_W$ to the group of permutations of $\mathrm{Irr}(\mathbf{C}W)$ (here we note that the inverse appears in κ_g^{-1} in order that this defines a homomorphism), and as in [Berest and Chalykh 2011, Theorem 7.11] (defining the set $\mathscr{C}^\circ$ of *regular* parameters to be those for which $\mathbb{O}_c$ is semisimple), we have more generally:

Lemma 3.3. *For $c \in \mathscr{C}^\circ$ regular and $g \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$, we have*

$$\mathrm{KZ}_{g(c)}(\Delta_{g(c)}(E)) \cong \mathrm{KZ}_c\big(\Delta_c(\kappa_g^{-1}(E))\big).$$

3L. *KZ twists preserve local data.* Corollary 7.18 of [Berest and Chalykh 2011] shows that for $\tau \in \mathscr{C}_{\mathbf{Z}}$, the KZ twist by τ preserves local data,

$$E_{H,j} = \kappa_\tau(E)_{H,j} \quad \text{for all } H, j.$$

Let $\sigma \in G_W$, and let $c \in \mathscr{C}^\circ$ be regular. By taking $g = \sigma$ and replacing E by $\kappa_\sigma(E)$ in Lemma 3.3, we obtain $\Delta_c(E)^\circ \cong \Delta_{\sigma(c)}(\kappa_\sigma(E))^\circ$, so there is a space of isotype $\kappa_\sigma(E)$ singular vectors for the $\sigma(c)$-Dunkl operators in $\Delta_c(E)^\circ = \mathbf{C}[\mathfrak{h}^\circ] \otimes E$. As in the proof of [Berest and Chalykh 2011, Corollary 7.18], its homogeneous degree m does not vary with c, implying that

$$m = \sigma(c)_{\kappa_\sigma(E)} - c_E = \frac{1}{\dim(E)} \sum_{H,j} \big(n_H c_{H,\sigma^{-1}(j)} + \sigma^{-1}(j) - j\big)\kappa_\sigma(E)_{H,j} - n_H c_{H,j} E_{H,j}$$

is constant. This implies

$$\kappa_\sigma(E)_{H,\sigma(j)} = E_{H,j},$$

which is the sense in which κ_σ preserves local data.

3M. ***A particular KZ twist we will use.*** In order to apply the preceding material to the proof of Theorem 1.1, we must choose a particular element of $\mathscr{C}_{\mathbf{Z}} \rtimes G_W$. There are many possible choices that would work for us. Here we fix one.

For $c \in \mathscr{C}$, put

$$\sigma(c)_{H,j} = \begin{cases} c_{H,0}, & \text{if } j = 0, \\ c_{H,n_H - j} + 2(n_H - j)/n_H, & \text{if } j \neq 0. \end{cases}$$

This is the product $\sigma = \tau\sigma_0$ of the longest element σ_0 of G_W with the transformation τ defined by

$$\tau(c)_{H,j} = c_{H,j-1} + \frac{n_H - 1}{n_H}.$$

Alternatively, it is the product of the longest element of the subgroup of G_W fixing $(H, 0)$ for all H with the translation

$$c_{H,j} \mapsto \begin{cases} c_{H,0}, & \text{if } j = 0, \\ c_{H,j} + 1, & \text{if } j \neq 0. \end{cases}$$

Thus, in the case where all reflections have order 2 and $c_{H,0} = 0$, this σ is simply the translation $c \mapsto c+1$. But it is more complicated in general, and the extra complication is definitely necessary for the proof of Theorem 1.1.

3N. ***Preservation of c-order.*** Consider the hyperplane of parameters $c \in \mathscr{C}$ satisfying the condition

$$c_{\mathfrak{h}^*} = 1.$$

There is at least one c on this hyperplane such that, in addition, there is a positive real number c_0 with

$$c_{H,j} = 2jc_0 \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \leq j \leq n_H - 1.$$

Fix such a choice of c and suppose E, F are irreducible W-modules with $c_E - c_F > 0$. We then have

$$\begin{aligned} 0 < c_E - c_F &= \frac{1}{\dim(E)} \sum n_H c_{H,j} E_{H,j} - \frac{1}{\dim(F)} \sum n_H c_{H,j} F_{H,j} \\ &= c_0 \bigg(\frac{1}{\dim(E)} \sum n_H 2j E_{H,j} - \frac{1}{\dim(F)} \sum n_H 2j F_{H,j} \bigg). \end{aligned}$$

Hence, by using Section 3L,

$$
\begin{aligned}
\sigma(c)_{\kappa_\sigma(E)} - \sigma(c)_{\kappa_\sigma(F)} \\
&= \frac{1}{\dim(E)} \sum_{j\neq 0} \bigl(n_H c_{H,n_H-j} + 2(n_H-i)\bigr) E_{n_H-i} - \frac{1}{\dim(F)} \sum_{j\neq 0} \bigl(n_H c_{H,n_H-j} + 2(n_H-i)\bigr) F_{n_H-i} \\
&= c_E - c_F + \frac{1}{\dim(E)} \sum n_H 2j E_{H,j} - \frac{1}{\dim(F)} \sum n_H 2j F_{H,j} > 0.
\end{aligned}
$$

It follows that the bijection κ_σ intertwines the c-order on $\mathrm{Irr}(\mathbf{C}W)$ with the $\sigma(c)$-order. Moreover, the same is true for any parameter c on the hyperplane $c_{\mathfrak{h}^*} = 1$ sufficiently close to such a choice.

3O. *Quasiinvariants.* A *multiplicity function* is a collection $m = (m_{H,j})_{H\in\mathcal{A}, 0\le j\le n_H-1}$ of integers indexed by pairs $H \in \mathcal{A}$ and $0 \le j \le n_H - 1$ with the property that $m_{H,j} = m_{w(H),j}$ for all $w \in W$, $H \in \mathcal{A}$, and $0 \le j \le n_H - 1$. Given a multiplicity function m and a $\mathbf{C}W$-module E, we define the space $Q_m(E)$ of *E-valued quasiinvariants* as in Berest and Chalykh [2011, (3.12)] to be the space of $f \in \mathbf{C}[\mathfrak{h}^\circ]\otimes E$ such that

$$v_H(1\otimes e_{H,i}\cdot f) \ge m_{H,i} \quad \text{for all } H \in \mathcal{A} \text{ and } 0 \le i \le n_H - 1,$$

where v_H is the valuation on $\mathbf{C}[\mathfrak{h}^\circ]\otimes E$ that gives the order of vanishing along H (and where, e.g., $v_H(f) \ge -2$ means f has at most a pole of order 2 along H). Given a multiplicity function m and a parameter $c \in \mathscr{C}$, we say that m and c are *compatible* if

$$n_H c_{H,i-m_{H,i}} = m_{H,i} \quad \text{for all } H \in \mathcal{A} \text{ and } 0 \le i \le n_H - 1.$$

This relationship may seem complicated, but we note that if $c \in \mathscr{C}_{\mathbf{Z}}$ then defining $m_{H,i} = n_H c_{H,i}$, we have m compatible with c, and if $g \in G_W$ and $c \in \mathscr{C}_{\mathbf{Z}}$ then by defining

$$m_{H,i} = n_H c_{H,i} + i - g(i),$$

we have m compatible with $g\cdot c$. Thus, each element of the orbit $\mathscr{C}_{\mathbf{Z}} \rtimes G_W(0)$ is compatible with a (unique) multiplicity function m.

Just as in [Berest and Chalykh 2011, Proposition 3.10] , one checks:

Lemma 3.4. *If m and c are compatible, then $Q_m(E)$ is a H_c-submodule of $\mathbf{C}[\mathfrak{h}^\circ]\otimes E$, where H_c acts on $\mathbf{C}[\mathfrak{h}^\circ]\otimes E$ via the inclusion $H_c \subseteq D(\mathfrak{h}^\circ)\rtimes W$.*

In fact, when m is the unique multiplicity function compatible with a parameter $g(0)$ with $g \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$, the module $Q_m(E)$ of E-valued quasiinvariants is an irreducible object of category $\mathcal{O}_{g(0)}$ with localization $Q_m(E)[\delta^{-1}]$ equal to $\mathbf{C}[\mathfrak{h}^\circ]\otimes E$ as $D(\mathfrak{h}^\circ)\rtimes W$-modules. It follows that

$$\mathrm{KZ}_{g(0)}(Q_m(E)) \cong E \quad \Longrightarrow \quad Q_m(E) \cong \Delta_{g(0)}(\kappa_g(E)).$$

We define the space of *quasiinvariants* $Q_m \subseteq \mathbf{C}[\mathfrak{h}^\circ]$ by $f \in Q_m$ if and only if

$$v_H(e_{H,-i} f) \ge m_{H,i} \quad \text{for all } H \in \mathcal{A} \text{ and } 0 \le i \le n_H - 1.$$

Note that Q_m is different from $Q_m(\mathrm{triv})$. As in [Berest and Chalykh 2011, Theorem 3.4], the relationship is:

Theorem 3.5. *We have*

$$e(Q_m \otimes 1) = eQ_m(\mathbf{C}W)$$

as subsets of $\mathbf{C}[\mathfrak{h}^\circ] \otimes \mathbf{C}W$, *and hence* Q_m *is a* eH_ce*-module if* c *and* m *are compatible. Moreover,* eH_ce *is equal to the algebra* $D(Q_m)^W e$ *of* W*-invariant differential operators on* Q_m $\big($*both regarded as subalgebras of* $e(D(\mathfrak{h}^\circ) \rtimes W)e\big)$.

Finally, by noting that $Q_m = Q_k$ if m and k are multiplicity functions satisfying

$$\left\lceil \frac{m_{H,i} - i}{n_H} \right\rceil = \left\lceil \frac{k_{H,i} - i}{n_H} \right\rceil \quad \text{for all } H \in \mathcal{A} \text{ and } 0 \le i \le n_H - 1, \tag{3-1}$$

one checks that for $c \in \mathscr{C}_{\mathbf{Z}}$ and $g \in G_W$, the multiplicity functions compatible with c and with $g(c)$ produce the same space of quasiinvariants, implying $eH_ce = eH_{g(c)}e$. Now, applying a density argument as in [Berest and Chalykh 2011, Proposition 5.4] gives the version that we will use (as mentioned in the introduction, this is Corollary 2.22 from [Bellamy et al. 2021], who give a completely different proof):

Theorem 3.6. *For* $g \in G_W$ *and* $c \in \mathscr{C}$, *we have*

$$eH_ce = eH_{g(c)}e \quad \textit{and} \quad f(y_c)e = f(y_{g(c)})e \quad \textit{for all } f \in \mathbf{C}[\mathfrak{h}^*]^W.$$

This last inequality is to be interpreted as follows: for a symmetric polynomial f, evaluating f on the Dunkl operators y_c and then multiplying by e gives the same result as evaluating f on the Dunkl operators $y_{g(c)}$ and then multiplying by e.

In fact, for multiplicity functions m and k satisfying (3-1), we have

$$eQ_m(E) = eQ_k(E) \quad \text{for all } E \in \mathbf{C}W\text{-mod}. \tag{3-2}$$

Just as in [Berest and Chalykh 2011], this produces a host of consequences for the numerology of the fake degrees of complex reflection groups, some of which are intimately related to the numerology of diagonal coinvariants we are exploring here. We record the most general version of this now.

3P. ***Symmetries of the exponents.*** Here we record the version of the symmetries of the exponents (referred to as symmetries of the fake degrees in [Berest and Chalykh 2011] and [Opdam 1998]) we will need.

Theorem 3.7. *Let* $\kappa = \kappa_\sigma$ *be the KZ twist associated with the element* $\sigma \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$ *defined above and as above write* $e_1(E), e_2(E), \dots$ *for the exponents of an irreducible* W*-module* E. *Then*

$$e_i(\mathfrak{h}) + e_{n-i+1}(\kappa(\mathfrak{h}^*)) = g \quad \textit{for all } 1 \le i \le n.$$

We note that for real reflection groups, we always have $g = h = d_n$, where d_n is the largest degree and $\kappa(\mathfrak{h}^*) = \mathfrak{h}^*$, so that this reduces to the classical symmetry $d_i + d_{n-i+1} = d_n + 2$.

Proof. We now prove Theorem 3.7. For each W-orbit $S \subseteq \mathcal{A}$ of hyperplanes, we put

$$\delta_S = \prod_{H \in S} \alpha_H$$

and we fix an integer m_S. Defining the multiplicity function m by $m_{H,i} = m_S$ for all $H \in S$, we have

$$Q_m(E) = \prod_{S \in \mathscr{A}/W} \delta_S^{m_S}(\mathbf{C}[\mathfrak{h}] \otimes E).$$

Let $c \in \mathscr{C}$ be compatible with m, let $\sigma_0 \in G_W$, and let k be the multiplicity function compatible with $\sigma_0 \cdot c$. Let $\tau \in \mathscr{C}_{\mathbf{Z}} \rtimes G_W$ be determined by $\tau(0) = \sigma_0(c)$. We note that

$$c_{H,i} = m_H/n_H \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \le i \le n_H - 1.$$

Then

$$eQ_m(E) = eQ_k(E).$$

We compute the graded character of the space in two ways by means of this equality: with $M = \deg\left(\prod \delta_S^{m_S}\right)$, the graded character of $eQ_m(E)$ is

$$\operatorname{ch}(eQ_m(E)) = \operatorname{ch}\left(\prod_{S \in \mathscr{A}/W} \delta_S^{m_S}(\mathbf{C}[\mathfrak{h}] \otimes E)\right)^W = t^M \sum_{i=1}^{\dim(E)} t^{e_i} \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}},$$

where $e_1 \le e_2 \le \cdots$ are the exponents of the representation $(\chi \otimes E)^*$, with χ the linear character of W afforded by $\prod_{S \in \mathscr{A}/W} \delta_S^{m_S}$.

On the other hand, we have $eQ_k(E) \cong \left(\Delta_{\tau(0)}(\kappa_\tau(E))\right)^W$. This has graded character

$$t^{\tau(0)_{\kappa_\tau(E)}} \operatorname{ch}\left((\mathbf{C}[\mathfrak{h}] \otimes \kappa_\tau(E))^W\right) = t^{\tau(0)_{\kappa_\tau(E)}} \sum_{i=1}^{\dim(E)} t^{e_i'} \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}},$$

where $e_1' \le e_2' \le \cdots$ are the exponents of $\kappa_\tau(E)^*$. We conclude

$$\tau(0)_{\kappa_\tau(E)} + e_i' = M + e_i \quad \text{for } 1 \le i \le \dim(E). \tag{3-3}$$

The special case in which $m_S = -(n_H - 1)$ for $H \in S$ is especially interesting: here $-M = N$ is the number of reflections in W, χ is the inverse determinant representation of W, and the exponents of $\chi \otimes E$ may be related to those of E^* as follows: there is a $\mathbf{C}$-linear isomorphism $\mathbf{C}[\mathfrak{h}^*]_W$ onto $\mathbf{C}[\mathfrak{h}]_W$ given by

$$f \mapsto f(\partial) \cdot \delta.$$

Moreover, there is a W-equivariant nondegenerate pairing of $\mathbf{C}[\mathfrak{h}^*]_W$ with $\mathbf{C}[\mathfrak{h}]_W$ given by

$$(f, g) = f(\partial)(g)(0),$$

which implies that the occurrences of F in $\mathbf{C}[\mathfrak{h}]_W$ are in the same degrees as the occurrences of F^* in $\mathbf{C}[\mathfrak{h}^*]_W$. Putting this together implies that $E^* \otimes \det$ occurs in $\mathbf{C}[\mathfrak{h}]_W$ in a degree d each time E^* occurs in $\mathbf{C}[\mathfrak{h}^*]_W$ in degree $N - d$, which happens when E occurs in $\mathbf{C}[\mathfrak{h}]_W$ in degree $N - d$. So the occurrences of E in degree $N - d$ are in bijection with the occurrences of $E^* \otimes \det$ in degree d, and the exponents e_i above are given by

$$e_i = N - e_{\dim(E) - i + 1}(E).$$

Thus, (3-3) becomes

$$e_i(\kappa_\tau(E)^*) + e_{\dim(E)-i+1}(E) = -\tau(0)_{\kappa_\tau^{-1}(E)} \quad \text{for } 1 \le i \le \dim(E). \tag{3-4}$$

We take $E = \kappa_\tau^{-1}(\mathfrak{h}^*)$ here to obtain

$$e_i(\mathfrak{h}) + e_{n-i+1}(\kappa_\tau^{-1}(\mathfrak{h}^*)) = -\tau(0)_{\mathfrak{h}^*}. \tag{3-5}$$

Finally, observing that taking the longest element in G_W produces $\tau = \sigma^{-1}$, a calculation then shows that the right-hand side is g. □

3Q. ***Rouquier's theorem and the BMR freeness conjecture.*** In the proof of Theorem 1.1, we will apply [Rouquier 2008, Theorem 4.49] to produce an equivalence $\mathcal{O}_c \to \mathcal{O}_{\sigma(c)}$. To be able to apply this theorem to $\mathcal{O}_c$ regarded as a highest weight cover of $\mathcal{H}_c$-mod, we must use the fact that $\mathcal{H}_c$ is of dimension $|W|$, which appeared as a hypothesis in [Gordon and Griffeth 2012], but which is now known in general (see [Etingof 2017] for an overview).

3R. ***Proof of Theorem 1.1.*** Finally we complete the proof of Theorem 1.1. By Lemma 3.1, it suffices to construct a coinvariant type representation L of dimension $(g+1)^n$. We choose a parameter c as in Section 3N, and let σ be as defined in Section 3M; Proposition 4.1 of [Etingof and Stoica 2009] shows that with this choice of c, we have a map $\Delta_c(\mathfrak{h}^*) \to \Delta_c(\mathrm{triv})$ with cokernel $L_c(\mathrm{triv})$ of dimension 1. Using Section 3N, together with Rouquier's theorem 4.49 [2008] as in the proof of Theorem 2.7 from [Gordon and Griffeth 2012], shows that there is an equivalence of highest weight categories $\mathcal{O}_c \to \mathcal{O}_{\sigma(c)}$ sending $\Delta_c(E)$ to $\Delta_{\sigma(c)}(\kappa_\sigma(E))$ for all $E \in \mathrm{Irr}(\mathbf{C}W)$ (here, we note first that we may choose c as above so that, in addition, the rank one Hecke subalgebras are semisimple; we note second that there are regular parameters c' arbitrarily close to c, so that we may apply Lemma 3.3 to see that κ_σ is the correct bijection). By using Section 3L, it follows that this particular choice of σ has $\kappa_\sigma(\mathrm{triv}) = \mathrm{triv}$. We obtain a short exact sequence

$$\Delta_{\sigma(c)}(\kappa_\sigma(\mathfrak{h}^*)) \to \Delta_{\sigma(c)}(\mathrm{triv}) \to L_{\sigma(c)}(\mathrm{triv}) \to 0.$$

By [Ginzburg et al. 2003, Corollary 4.14], the module $L_{\sigma(c)}(\mathrm{triv})$ is finite-dimensional (this corollary implies that finite-dimensionality is invariant by highest-weight equivalences between different categories $\mathcal{O}$).

Now a calculation using Section 3L gives $\sigma(c)_{\kappa_\sigma(\mathfrak{h}^*)} = g+1$, so that the image of $\kappa_\sigma(\mathfrak{h}^*)$ in $\mathbf{C}[\mathfrak{h}] = \Delta_{\sigma(c)}(\mathrm{triv})$ is a homogeneous sequence of parameters in degree $g+1$. The symmetry of the exponents from Theorem 3.7, together with Section 2E, implies that the determinant appears exactly once in $L_{\sigma(c)}(\mathrm{triv})$, which is of dimension $(g+1)^n$ as required. This proves Theorem 1.1.

4. Two more constructions: Heckman–Opdam shift functors and representation-valued Jack polynomials

4A. ***Outline.*** In this section, we give two more constructions of the coinvariant-type representation L of dimension $(g+1)^n$. First, as in [Gordon 2003] for the case of a real group W and [Vale 2007a] for the groups $W = G(\ell, m, n)$, we construct it as a tensor product

$$L = H_{\sigma(c)} f \otimes_{eH_c e} \mathbf{C},$$

where $\mathbf{C} = eL_c(\mathrm{triv})$ is a one-dimensional representation of $eH_c e$ and f is the determinant idempotent of W. We note that Theorem 4.1 below allows us to regard $H_{\sigma(c)} f$ as an $eH_c e$-module. Then we give a construction similar to that of [Griffeth 2010b], based on the techniques from [Griffeth 2010a; 2010b; 2018; Fishel et al. 2021].

4B. ***Heckman–Opdam shift functors.***

Theorem 4.1. *Let $c \in \mathscr{C}$ and recall that $\delta = \prod_{r \in R} \alpha_r$. As* (*nonunital*) *subalgebras of $D(\mathfrak{h}^\circ) \rtimes W$, we have*

$$\delta e H_c e \delta^{-1} = f H_{\sigma(c)} f,$$

where, *as in Section 3M*, *the shifted parameter $\sigma(c)$ is defined by $\sigma(c)_{H,0} = 0$ and*

$$\sigma(c)_{H,i} = c_{H,n_H - i} + \frac{2(n_H - i)}{n_H} \quad \text{for } i \neq 0.$$

Proof. This is obtained from Theorem 3.6 by taking $g_0 \in G_W$ to be the longest element. $\square$

This equality allows us to define a functor F from H_c-mod to $H_{\sigma(c)}$-mod by

$$F(M) = H_{\sigma(c)} f \otimes_{eH_c e} eM. \tag{4-1}$$

Following the terminology from [Berest and Chalykh 2011], we refer to F as the *Heckman–Opdam shift functor*. In fact, F preserves category $\mathcal{O}$'s, and therefore induces a functor, which we also denote by F, from $\mathcal{O}_c$ to $\mathcal{O}_{\sigma(c)}$. The functor F is similar to the Heckman–Opdam shift functor employed by Gordon for real reflection groups, but in the generality in which we require it, belongs purely to the world of complex reflection groups and has no direct real analog.

4C. ***The symmetrizing trace conjecture.*** Below we will use the Schur elements as stored by the computer algebra package GAP. In order to justify the conclusions we draw from this, we need to know that the symmetrizing trace conjecture holds for the Hecke algebra $\mathscr{H}_c$. For very recent work in this direction and further references, see [Boura et al. 2020a; 2020b].

4D. ***Equivalences.*** The following theorem of Etingof (obtained by twisting [Etingof 2012, Theorem 5.5] by a linear character) makes our lives easier:

Theorem 4.2. *Let $e \in \mathbf{C}W$ be the idempotent for a linear character of W. The functor $M \mapsto eM$ from H_c-mod to $eH_c e$-mod is an equivalence if and only if $eH_c e$ is of finite global dimension.*

When e is the trivial idempotent for W, we call c *aspherical* if the functor $M \mapsto eM$ is not an equivalence and *spherical* if it is. Combining this theorem with Theorem 3.6 shows that the set of aspherical (respectively, spherical) parameters c is stable by the dot action of G_W on $\mathscr{C}$.

Furthermore, by Theorem 4.1 from [Bezrukavnikov and Etingof 2009], whether or not $M \mapsto eM$ is an equivalence can be checked on the category $\mathcal{O}_c$:

Theorem 4.3. *For an idempotent $e \in \mathbf{C}W$ of a linear character of W, the functor $M \mapsto eM$ is not an equivalence if and only if there exists $L \in \operatorname{Irr}(\mathcal{O}_c)$ with $eL = 0$.*

The next lemma is a key technical point, and the only place we will appeal to the classification of irreducible complex reflection groups and the hypothesis that the Hecke algebra is symmetric.

Lemma 4.4. *For a parameter $c \in \mathscr{C}$ subject to $c_{\mathfrak{h}^*} = 1$ but otherwise generic, the functor $M \mapsto eM$ is an equivalence from H_c-mod to eH_ce-mod.*

Proof. For the groups in the infinite family $G(\ell, m, n)$, this follows from the main theorem of [Dunkl and Griffeth 2010], upon observing that the equation $c_{\mathfrak{h}^*} = 1$ is

$$d_0 - d_{\ell-1} + \ell(n-1)c_0 = 1$$

in the coordinates for $\mathscr{C}$ used there. For the exceptional groups, one checks using GAP that the Schur elements for the exterior powers of $\mathfrak{h}^*$ are the only ones which are zero when the parameters c are chosen with $c_{H,0} = 0$ and $c_{H,i} = 1/h$ for $i \neq 0$, and moreover, that in this case, the Schur elements for the exterior powers vanish to order one. This implies that we are in the block of defect one case studied by Rouquier, and hence that every irreducible object of $\mathcal{O}_c$ other than $L_c(\mathrm{triv})$ is fully supported. Thus, c is a spherical parameter, and one checks that it belongs to the hyperplane $c_{\mathfrak{h}^*} = 1$. $\square$

Fixing a parameter $c \in \mathscr{C}$ subject to $c_{\mathfrak{h}^*} = 1$ but otherwise generic, we define the Heckman–Opdam equivalence $F : \mathcal{O}_c \to \mathcal{O}_{\sigma(c)}$ as above by

$$F(M) = H_{\sigma(c)} f \otimes_{eH_ce} eM,$$

where we view $H_{\sigma(c)} f$ as a right eH_ce-module via the isomorphism $eH_ce \cong f H_{\sigma(c)} f$ from Theorem 4.1, sending $ehe \in eH_ce$ to $\delta ehe \delta^{-1} \in f H_{\sigma(c)} f$.

4E. *Shifting and KZ.* The results just summarized imply that if c is a spherical value, then F defines an equivalence from $\mathcal{O}_c$ to $\mathcal{O}_{\sigma(c)}$; and if regular, then so is $\sigma(c)$.

Lemma 4.5. *If F is an equivalence, the functor F commutes with the KZ functor: there is an isomorphism $\mathrm{KZ}_{\sigma(c)} \circ F \cong \mathrm{KZ}_c$ for all spherical parameters $c \in \mathscr{C}$.*

Proof. We first observe that the dimension of the generic fiber of $F(M)$ is equal to the dimension of the generic fiber of M. This follows from the fact that for $p \in \mathfrak{h}^\circ$, we have

$$\dim(M(p)) = \mathrm{rk}_{\mathbf{C}[\mathfrak{h}]}(M) = \mathrm{rk}_{\mathbf{C}[\mathfrak{h}]^W}(eM) = \mathrm{rk}_{\mathbf{C}[\mathfrak{h}]^W}(f F(M)) = \mathrm{rk}_{\mathbf{C}[\mathfrak{h}]} F(M) = \dim(F(M)(p)),$$

since $f F(M) \cong eM$ as $\mathbf{C}[\mathfrak{h}]^W$-modules and $\mathrm{rk}_{\mathbf{C}[\mathfrak{h}]^W} f F(M) = \mathrm{rk}_{\mathbf{C}[\mathfrak{h}]}(F(M))$. Now since F is an equivalence, it takes the indecomposable projective objects of $\mathcal{O}_c$ to the indecomposable projective objects of $\mathcal{O}_{\sigma(c)}$. The KZ functor is represented by the projective object

$$P_{\mathrm{KZ},c} = \oplus P_c(E)^{\oplus d_E}, \quad \text{where } d_E = \dim(L_c(E)(p)),$$

and hence,

$$F(P_{\mathrm{KZ},c}) \cong \oplus F(P_c(E))^{\oplus d_E},$$

where d_E is the dimension of $L_c(E)(p)$, which is, by the preceding argument, the dimension of $F(L_c(E))(p) = \dim\bigl(\mathrm{top}\bigl(F(P_c(E))\bigr)(p)\bigr)$. It follows that $F(P_{\mathrm{KZ},c}) \cong P_{\mathrm{KZ},\sigma(c)}$, and the lemma follows from this. □

This lemma should also follow from the ideas in [Simental 2017] (see, especially, the proof of Lemma 4.9). The following conjecture is then the final ingredient for this approach:

Conjecture 4.1. *If F is an equivalence, then it is an equivalence of highest weight categories with $F(\Delta_c(E)) \cong \Delta_{s(c)}(\kappa_\sigma(E))$.*

Given the conjecture, we have a short exact sequence

$$\Delta_{\sigma(c)}(\kappa_\sigma(\mathfrak{h}^*)) \to \Delta_{\sigma(c)}(\mathrm{triv}) \to L_{\sigma(c)}(\mathrm{triv}) = F(\mathbf{C}) \to 0,$$

where $\mathbf{C} = L_c(\mathrm{triv})$ is actually one-dimensional thanks to our condition $c_{\mathfrak{h}^*} = 1$, and the determinant appears exactly once in $F(\mathbf{C}) = L_{\sigma(c)}(\mathrm{triv})$ by construction. The calculation of the c-function implies that the image of $\kappa_\sigma(\mathfrak{h}^*)$ lies in degree $g+1$, which implies that the dimension of $F(\mathbf{C})$ is $(g+1)^n$, just as before. José Simental has pointed out that the conjecture will follow if one checks that F and its inverse preserve the class of standardly filtered modules, which can also be characterized as those objects of $\mathcal{O}_c$ that are free as modules over the polynomial ring $\mathbf{C}[\mathfrak{h}]$. It seems likely to me that this observation can be turned into a proof.

4F. ***The classical groups.*** Finally, we present our third construction of the module approximating R_W in the case where $W = G(\ell, m, n)$. For these groups, we will use the coordinates $(d_0, d_1, \dots, d_{\ell-1}, c_0)$ on $\mathscr{C}$, as in [Griffeth 2010a]. We define the set $\Gamma(\mathrm{triv})$ as in [Griffeth 2018]: it consists of pairs (P, Q), where P is a bijection from the boxes of the trivial partition (n) to the integers $\{1, 2, \dots, n\}$, Q is a function from the boxes of (n) to the nonnegative integers which is weakly increasing from left to right, and whenever b_1 and b_2 are boxes with b_1 appearing to the left of b_2 and $Q(b_1) = Q(b_2)$, then we have $P(b_1) > P(b_2)$ (thus, for instance, if Q is the zero function then P is strictly decreasing from left to right).

4G. ***The principal coinvariant type representation.*** We take parameters $c = (c_0, d_0, d_1, \dots, d_{\ell-1})$ generic, subject only to the condition

$$d_0 - d_{1-2\ell/m} + \ell(n-1)c_0 = \ell(n-1) + \frac{2\ell}{m} - 1.$$

Then by [Griffeth 2018, Theorem 1.1], the module $L = L_c(\mathrm{triv})$ has basis $f_{P,Q}$ indexed by those pairs $(P, Q) \in \Gamma(\mathrm{triv})$, with

$$Q(b) \le \ell(n-1) + \frac{2\ell}{m} - 2 \quad \text{for all } b \in \mathrm{triv}.$$

Here, as explained in [Griffeth 2018, Section 2.13], instead of using $\Gamma(\mathrm{triv})$, the basis elements $f_{P,Q}$ may alternatively be indexed by $\mu \in \mathbf{Z}_{\ge 0}$, and the condition is simply

$$\mu_i \le \ell(n-1) + \frac{2\ell}{m} - 2 \quad \text{for all } 1 \le i \le n.$$

Hence, L has basis

$$L = \mathbf{C}\Big\{ f_\mu \mid \mu_i \leq \ell(n-1) + \frac{2\ell}{m} - 2,\ \forall 1 \leq i \leq n \Big\},$$

where f_μ are the nonsymmetric Jack polynomials of type $G(\ell, 1, n)$. In particular, the dimension of L is

$$\dim(L) = \Big(\ell(n-1) + \frac{2\ell}{m} - 1\Big)^n.$$

By using the machinery from [Fishel et al. 2021], we can compute its graded W-character; for the moment, we will just note that, as in [Ajila and Griffeth 2021], copies of the determinant representation in L are in bijection with the set of Q's appearing in some pair (P, Q) as above, with Q strictly increasing from left to right, $Q(b) = Q(b')$ mod ℓ for all b, b', and with $Q(b) = \ell/m - 1$ modulo ℓ/m.

4H. ***Proof that L is of $G(\ell, m, n)$-coinvariant type.*** For the unique Q with these properties (in [Ajila and Griffeth 2021], we use the notation $Q \in \mathrm{Tab}_c(\mathrm{triv})$), which produces a copy of the determinant representation of $G(\ell, m, n)$ given by the Q with sequence

$$\frac{\ell}{m} - 1, \quad \ell + \frac{\ell}{m} - 1, \quad 2\ell + \frac{\ell}{m} - 1, \ldots, (n-1)\ell + \frac{\ell}{m} - 1.$$

Thus, using the character formula from [Fishel et al. 2021], as in [Ajila and Griffeth 2021], shows that the determinant appears exactly once in L.

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