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# Axiomatizing the existential theory of $\mathbb{F}_q((t))$

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We study the existential theory of equicharacteristic henselian valued fields with a distinguished uniformizer. In particular, assuming a weak consequence of resolution of singularities, we obtain an axiomatization of — and therefore an algorithm to decide — the existential theory relative to the existential theory of the residue field. This is both more general and works under weaker resolution hypotheses than the algorithm of Denef and Schoutens, which we also discuss in detail. In fact, the consequence of resolution of singularities our results are conditional on is the weakest under which they hold true.

## 1. Introduction

Hilbert's tenth problem asks for an algorithm to decide whether a given polynomial over  $\mathbb{Z}$  has a solution in  $\mathbb{Z}$ , which was shown to be impossible by work of Davis, Putnam, Robinson and Matiyasevich. Analogues for various other rings and fields have been studied since; see [Poonen 2003; Shlapentokh 2007; Koenigsmann 2014]. For local fields of positive characteristic, i.e., Laurent series fields  $\mathbb{F}_q((t))$  over a finite field with  $q = p^n$  elements, there are two results in the literature. The first one works with polynomials over  $\mathbb{F}_p(t)$ , which is arguably the correct analogue of  $\mathbb{Z}$  in this setting, but needs to assume the truth of a deep and unresolved conjecture from algebraic geometry, resolution of singularities (see Section 2 for precise statements and discussion):

**Theorem 1.1** [Denef and Schoutens 2003, Theorem 4.3]. *Assume that resolution of singularities holds in characteristic  $p$ . There exists an algorithm that decides whether a given system of polynomial equations over  $\mathbb{F}_p(t)$  has a solution in  $\mathbb{F}_q((t))$ .*

The second one, which is proved by completely different methods, needs no such hypothesis but allows for polynomials only over  $\mathbb{F}_p$ :

**Theorem 1.2** [Ancombe and Fehm 2016, Corollary 7.7]. *There exists an algorithm that decides whether a given system of polynomial equations over  $\mathbb{F}_p$  has a solution in  $\mathbb{F}_q((t))$ .*

Both results have recently found various applications; see, for example, [Onay 2018; Martínez-Ranero et al. 2022; Kartas 2021a; 2021b; 2023b]. In fact, both results are more general than stated here, and these more general results are most naturally phrased in the language of the model theory of valued fields:

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**Definition 1.3.** We denote by  $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$  the language of rings, by  $\mathcal{L}_{\text{val}} = \mathcal{L}_{\text{ring}} \cup \{\mathcal{O}\}$  the language of valued fields, where  $\mathcal{O}$  is a unary predicate symbol, and by  $\mathcal{L}_{\text{ring}}(\varpi)$  and  $\mathcal{L}_{\text{val}}(\varpi)$  the respective language expanded by a constant symbol  $\varpi$ . We view a valued field  $(K, v)$  as an  $\mathcal{L}_{\text{val}}$ -structure by interpreting  $\mathcal{O}$  as the valuation ring  $\mathcal{O}_v$  of  $v$ . A *uniformizer* of a valued field  $(K, v)$  is an element  $\pi \in K$  of smallest positive value. Given a distinguished uniformizer  $\pi$  of  $(K, v)$ , we view  $(K, v, \pi)$  as an  $\mathcal{L}_{\text{val}}(\varpi)$ -structure by interpreting  $\varpi$  as  $\pi$ .

So, for example,  $\mathbb{F}_q((t))$  carries the natural equicharacteristic henselian valuation  $v_t$  (the  $t$ -adic valuation), and if we view  $(\mathbb{F}_q((t)), v_t, t)$  as an  $\mathcal{L}_{\text{val}}(\varpi)$ -structure, the fact that a given system of polynomial equations over  $\mathbb{F}_p(t)$  has a solution in  $\mathbb{F}_q((t))$  can be expressed as the truth in  $\mathbb{F}_q((t))$  of a particular existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentence (by existential we mean of the form  $\exists x_1 \dots \exists x_n \psi(\underline{x})$  with a quantifier-free formula  $\psi(\underline{x})$ ). The general result of Denef and Schoutens, which we revisit and discuss in Section 3, can now be phrased as follows:

**Theorem 1.4** [Denef and Schoutens 2003, Theorem 4.3]. *Let  $(K, v)$  be an equicharacteristic henselian valued field with distinguished uniformizer  $\pi$ . Assume that*

- (1) *resolution of singularities holds in characteristic  $p$ ,*
- (2)  *$\mathcal{O}_v$  is a discrete valuation ring, and*
- (3)  *$\mathcal{O}_v$  is excellent.*

*Then the existential  $\mathcal{L}_{\text{val}}(\varpi)$ -theory  $\text{Th}_{\exists}(K, v, \pi)$  of  $(K, v, \pi)$  is decidable relative to the existential  $\mathcal{L}_{\text{ring}}$ -theory  $\text{Th}_{\exists}(Kv)$  of the residue field  $Kv$  (in the sense of Turing reduction; see Remark 3.3).*

The more general form of Theorem 1.2 [Ancombe and Fehm 2016, Theorem 6.5] gives an axiomatization of the existential theory of an equicharacteristic henselian valued field relative to the existential theory of the residue field, which in particular implies a relative decidability result; it is also interesting in its own right, and useful, for example, when questions of uniformity are concerned.

The goal of this work is to show that the approach from [Ancombe and Fehm 2016] can be extended to the setting of Theorem 1.4 as well, thereby combining the best of both worlds. Besides obtaining an axiomatization of the existential theory, we weaken the assumption (1) to a local version, called local uniformization (see Section 2 for definitions and an extensive discussion), and completely eliminate the assumptions (2) and (3). In Section 4 on page 2028 we prove:

**Theorem 1.5.** *Assume that consequence (R4) (see page 2016) of local uniformization holds. Let  $(K, v)$  be an equicharacteristic henselian valued field with distinguished uniformizer  $\pi$ . Then the universal/existential  $\mathcal{L}_{\text{val}}(\varpi)$ -theory of  $(K, v, \pi)$  is entailed by*

- (i)  *$\mathcal{L}_{\text{val}}$ -axioms for equicharacteristic henselian valued fields,*
- (ii) *the  $\mathcal{L}_{\text{val}}(\varpi)$ -axiom expressing that  $\pi$  has smallest positive value, and*
- (iii)  *$\mathcal{L}_{\text{val}}$ -axioms expressing that the residue field models the universal/existential  $\mathcal{L}_{\text{ring}}$ -theory of  $Kv$ .*

In particular, the existential  $\mathcal{L}_{\text{val}}(\varpi)$ -theory  $\text{Th}_{\exists}(K, v, \pi)$  of  $(K, v, \pi)$  is decidable relative to the existential  $\mathcal{L}_{\text{ring}}$ -theory  $\text{Th}_{\exists}(Kv)$  of  $Kv$ .

Here, by universal/existential we mean sentences that are either universal or existential. In fact, in Theorem 4.12 we prove this with parameters from a base field more general than  $\mathbb{F}_p(\pi)$ . In this strong form, this axiomatization statement for universal/existential theories is in fact equivalent to (R4) (see Remark 4.17) so our hypothesis cannot be weakened further.

Our results in particular give a new proof of Theorem 1.1 which works under this weaker hypothesis:

**Corollary 1.6.** *Assume that consequence (R4) of local uniformization holds. There exists an algorithm that decides whether a given system of polynomial equations over  $\mathbb{F}_q(t)$  has a solution in  $\mathbb{F}_q((t))$ .*

Recent work of Kartas includes related results; see Remark 2.6.

## 2. Resolution of singularities and local uniformization

In this section we discuss several versions of resolution of singularities and some of its consequences.

**Definition 2.1.** Let  $K$  be a field. A  $K$ -variety is an integral separated  $K$ -scheme of finite type, and a morphism of  $K$ -varieties  $Y \rightarrow X$  is simply a morphism of  $K$ -schemes.

Given a  $K$ -variety  $X$ , a *resolution of singularities* (or for short *resolution*) of  $X$  is a proper birational morphism  $Y \rightarrow X$  with  $Y$  a regular  $K$ -variety. A *blowing-up resolution* is a morphism  $Y \rightarrow X$  arising as the blowing-up along some closed proper subscheme  $Z \subset X$  such that  $Y$  is regular.

By a valuation  $v$  on a field extension  $F/K$  we mean a valuation on  $F$  which is trivial on  $K$ . Given a finitely generated field extension  $F/K$  and a valuation  $v$  on  $F/K$ , a *local uniformization* of  $v$  consists of a  $K$ -variety  $Y$  and an isomorphism  $F \cong_K K(Y)$  such that, under the identification of  $F$  with  $K(Y)$ ,  $v$  is centered on a regular point  $y$  of  $Y$  (i.e.,  $\mathcal{O}_v$  dominates the regular local ring  $\mathcal{O}_{Y,y}$ ).

**Lemma 2.2.** *Let  $K$  be a field.*

- (1) *Every blowing-up resolution of a  $K$ -variety  $X$  is a resolution of  $X$ .*
- (2) *If a proper  $K$ -variety  $X$  has a resolution, then any valuation  $v$  on  $K(X)/K$  has a local uniformization.*
- (3) *Let  $F/K$  be a finitely generated field extension and  $v$  a valuation on  $F/K$  which has residue field  $Fv = K$ . If  $v$  has a local uniformization, then there exists a  $K$ -embedding of  $F$  into  $K((t))$ .*

*Proof.* Every blowing-up morphism along a closed proper subscheme is proper and birational [Liu 2002, Proposition 8.1.12(d)], implying (1). For (2), let  $Y \rightarrow X$  be a resolution of  $X$ . Every valuation  $v$  on the function field  $K(Y)/K$  is centered at some point of  $Y$  by the valuative criterion of properness; see the discussion in [Liu 2002, Definition 8.3.17]. In particular, every such  $v$  has a local uniformization, and the same holds for valuations on  $K(X) \cong_K K(Y)$ . In the situation of (3), the field extension  $F/K$  is necessarily separable [Fried and Jarden 2008, Lemma 2.6.9], so the statement follows from [Fehm 2011, Lemma 9]; alternatively, it can be deduced from [Kuhlmann 2004, Theorem 13].  $\square$

The condition that a finitely generated field extension  $F/K$  can be embedded over  $K$  into  $K((t))$  is especially interesting when  $K$  is a *large* (or *ample*) field in the sense of [Pop 1996], that is,  $K$  is existentially closed in  $K((t))$  in  $\mathcal{L}_{\text{ring}}$ . Apart from the definition the only important facts for us are that large fields form an elementary class (see [Pop 1996, Remark 1.3]) and that henselian nontrivially valued fields are large (see Proposition 1 on page 41 and part 1) of the remarks at the bottom of page 43 of [Ershov 1967]). See [Jarden 2011, Example 5.6.2; Pop 2010] for more modern accounts that prove this (using an equivalent definition of largeness), and see [Bary-Soroker and Fehm 2013] for more background on large fields.

We are led to consider the following hypotheses:

- (R0) For every field  $K$  and every  $K$ -variety  $X$  there exists a blowing-up resolution  $Y \rightarrow X$ .
- (R1) For every field  $K$  and every  $K$ -variety  $X$  there exists a resolution  $Y \rightarrow X$ .
- (R2) For every field  $K$ , every finitely generated field extension  $F/K$  and every valuation  $v$  on  $F/K$  there exists a local uniformization of  $v$ .
- (R3) For every field  $K$  and every nontrivial finitely generated extension  $F/K$  such that there exists a valuation  $v$  on  $F/K$  with residue field  $Fv = K$ , there also exists a valuation with value group  $\mathbb{Z}$  which has that property.
- (R4) Every large field  $K$  is existentially closed in every extension  $F/K$  for which there exists a valuation  $v$  on  $F/K$  with residue field  $Fv = K$ .

**Proposition 2.3.** *The following implications hold true: (R0)  $\Rightarrow$  (R1)  $\Rightarrow$  (R2)  $\Rightarrow$  (R3)  $\iff$  (R4).*

*Proof.* The implication (R0)  $\Rightarrow$  (R1) follow from Lemma 2.2(1). For (R1)  $\Rightarrow$  (R2) first choose any proper  $K$ -variety  $X$  with  $K(X) \cong_K F$ , and then apply Lemma 2.2(2). By Lemma 2.2(3), (R2) implies that every  $F/K$  as in (R3) can be embedded into  $K((t))$ , and the restriction of the  $t$ -adic valuation to  $F$  then satisfies  $Fv = K$  and  $vF \cong \mathbb{Z}$ , which shows that (R2)  $\Rightarrow$  (R3). If  $F/K$  is as in (R4), then (R3) gives a valuation  $v$  on  $F/K$  with  $Fv = K$  and  $vF = \mathbb{Z}$ . The completion  $(\hat{F}, \hat{v})$  of  $(F, v)$  is then isomorphic over  $K$  to  $(K((t)), v_t)$ , see, for example, Proposition 4.2 below, so the large field  $K$  is existentially closed in  $\hat{F}$  and therefore in particular in  $F$ , which shows that (R3)  $\Rightarrow$  (R4). Finally, we have that (R4)  $\Rightarrow$  (R3) by [Fehm 2011, Corollary 10].  $\square$

The resolution of singularities that Denef and Schoutens assume is (R0); see Section 3. In our main theorem (Theorem 4.12), we will work under the weaker condition (R4).

**Remark 2.4.** The terminology on resolution of singularities is not entirely standardized. Depending on the author, either a resolution or a blowing-up resolution in our terminology might be called a resolution, a desingularization or a weak desingularization. For a resolution  $Y \rightarrow X$ , it may additionally be demanded that it is an isomorphism over the regular locus of  $X$ . For varieties  $X$  over a field  $K$  of characteristic zero, the existence of a blowing-up resolution in this strong sense was established in [Hironaka 1964]. Therefore all five hypotheses above are valid when restricted to fields  $K$  of characteristic zero.

In general, it is known that every variety of dimension three or less has a resolution of singularities (see [Cossart and Piltant 2019]), and it follows that (R2), (R3) and (R4) hold restricted to field extensions  $F/K$  of transcendence degree at most three.

Local uniformization is occasionally stated in a stronger version (see, for instance, the introduction to [Temkin 2013]), where one demands that for any given  $K$ -variety  $X$  with function field  $K(X) \cong_K F$  on which  $v$  is centered, the uniformizing  $Y$  can be chosen to dominate  $X$ , i.e., to come with a birational morphism  $Y \rightarrow X$  compatible with the isomorphism  $K(X) \cong F \cong K(Y)$ . The best result known here seems to be that (R2) holds up to replacing  $F$  by a purely inseparable finite extension; see [Temkin 2013, Section 1.3]. It follows that (R3) and (R4) hold when restricted to perfect base fields  $K$ . In fact, this restricted form of (R4) was already proven earlier in [Kuhlmann 2004, Theorem 17]. Less deep is the fact (immediate from [Fried and Jarden 2008, Lemma 2.6.9(b), Proposition 11.3.5] and [Fehm 2011, Lemma 9]) that (R3) and (R4) also hold when restricted to *pseudoalgebraically closed* base fields  $K$ , i.e., fields  $K$  for which every geometrically integral  $K$ -variety has a  $K$ -rational point.

**Remark 2.5.** Desingularization problems are also studied for more general schemes, rather than only varieties over fields. For instance, it is conjectured that for any reduced quasiexcellent scheme  $X$  there exists a proper birational morphism  $Y \rightarrow X$  with  $Y$  regular [EGA IV<sub>2</sub> 1965, Remarque (7.9.6)]. This is a generalization of (R1) since any variety over a field is quasiexcellent. Similarly, local uniformization is also studied in this more general setting; see [Temkin 2013]. The condition of excellence also appears in our own results below, but for reasons unrelated to resolution of singularities — indeed, we only need resolution in the shape of the weak condition (R4).

**Remark 2.6.** Other variations of resolutions exist, including the desingularization of a pair of a reduced quasiexcellent scheme  $X$  and a nowhere dense closed subscheme  $Z$ ; see [Temkin 2008, Introduction].

A resolution hypothesis of this type is used in [Kartas 2023a] to conditionally obtain an axiomatization for existential theories of certain, not necessarily discrete, valuation rings. The approach there is based on the geometric method of Denef–Schoutens, which we discuss in Section 3.

The first available preprint of [Kartas 2023a] in Section 5 also announces (with sketch proofs) results of this kind conditional only on variants of (R2). These do not appear in the published version but are included in [Kartas 2022].

### 3. The Denef–Schoutens algorithm

We now discuss the algorithm given by Denef and Schoutens, with a view towards improving the statements given in [Denef and Schoutens 2003], addressing some subtle points of the algorithm, and avoiding scheme-theoretic language at least to some extent. Our own technique is entirely different, so the reader interested in new results can skip ahead to Section 4.

Throughout,  $R$  is a fixed henselian discrete valuation ring with uniformizer  $\pi$  and fraction field  $K$ ,  $p > 0$  the characteristic of  $K$ , and  $\kappa = R/(\pi)$  the residue field of  $R$ . Note that  $\pi$  is necessarily transcendental over the prime field  $\mathbb{F}_p$  of  $K$ . We view  $R$  as an  $\mathcal{L}_{\text{ring}}(\varpi)$ -structure  $(R, \pi)$  by interpreting  $\varpi$  as  $\pi$ .

**Overview.** Theorem 1.4, concerning the decidability of the existential  $\mathcal{L}_{\text{val}}(\varpi)$ -theory of  $(K, v, \pi)$ , is equivalent to the decidability of the existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -theory of  $(R, \pi)$  under the hypothesis (R0), and the latter is what Proposition 3.9 below proves. To decide the truth of existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences in  $R$ , one proceeds in three steps. Deciding the truth of a *positive* existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentence in  $R$  means determining whether a system of polynomial equations has a solution in  $R$ . By a variant of a theorem of Greenberg, such a solution exists if and only if there is a solution in the residue ring  $R/(\pi^N)$  to the appropriately reduced system, for a sufficiently high (computable) power  $\pi^N$ . This reduces the original question about satisfaction of positive existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences in  $R$  to a question about the existential  $\mathcal{L}_{\text{ring}}$ -theory of the residue field  $\kappa$ .

For the second step, one considers existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences (that is, systems of equations and inequations), where the underlying system of equations describes a *regular* affine variety. A version of the implicit function theorem then shows that any solution in  $R$  to the system of equation gives rise to a wealth of other solutions, by perturbing some coordinates. This then implies that if there is any solution to the equations (which is a computable condition by the first step), there will also be one satisfying the inequations.

In the third step, one considers systems of equations and inequations in full generality; only here does resolution of singularities enter. Assuming that the system of equations (or rather, the affine scheme described thereby) has a resolution of singularities, one can in fact effectively find such a resolution, simply by enumerating candidates. In an inductive procedure, one can then relate the existence of solutions in  $R$  to the original system of equations and inequations, and solutions to related systems without singularities, which can be handled by the second step.

**Step 1: Positive existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences in  $R$ .** Here we slightly rephrase the results of [Denef and Schoutens 2003, Section 3].

**Lemma 3.1.** *Let  $f_1, \dots, f_n \in \mathbb{F}_p[t][X_1, \dots, X_m]$  be polynomials over the ring  $\mathbb{F}_p[t]$ . The following are equivalent:*

- (1) *The system of equations  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$  has a solution in  $\kappa[[t]]$ .*
- (2) *The system of equations  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$  has a solution in  $\kappa[t]^h$ , the henselization of  $\kappa[t]$  at the ideal  $(t)$  (i.e., the valuation ring of the henselization  $\kappa(t)^h$  of  $\kappa(t)$  with respect to the  $t$ -adic valuation).*
- (3) *For every  $N > 0$ , there is a solution in  $\kappa[t]/(t^N)$  to the system of equations  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$  (more precisely, to the reduction mod  $t^N$ ).*

*It suffices to check (3) for one specific large  $N$ , which can be computed from  $n, m$  and the total degree of the  $f_i$  (in the variables  $t, X_1, \dots, X_m$ ).*

*Proof.* It is clear that (2)  $\Rightarrow$  (1) (since  $\kappa[t]^h \subseteq \kappa[[t]]$ ) and (1)  $\Rightarrow$  (3) (by reduction modulo  $t^N$ ). The implication (3)  $\Rightarrow$  (2) is Greenberg's result [1966, Corollary 2]. The reduction to one specific large  $N$  comes down to an *effective* version of Greenberg's theorem, which is a special case of the effective Artin approximation given in [Becker et al. 1979, Theorem 6.1] (take  $n = 1$  and  $\alpha = 0$  in the notation there).  $\square$



**Lemma 3.2.** *Given finitely many polynomials  $f_1, \dots, f_n \in \mathbb{F}_p[T, X_1, \dots, X_m]$ , one can effectively find finitely many polynomials  $g_1, \dots, g_{n'} \in \mathbb{F}_p[Y_1, \dots, Y_{m'}]$  such that the system of equations  $\bigwedge_{i=1}^n f_i(\pi, \underline{X}) = 0$  has a solution in  $R$  if and only if the system of equations  $\bigwedge_{j=1}^{n'} g_j(\underline{Y}) = 0$  has a solution in  $\kappa$ . In particular, the positive existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -theory of  $(R, \pi)$  is decidable relative to the existential  $\mathcal{L}_{\text{ring}}$ -theory of  $\kappa$ .*

*Proof.* The ring  $R$  embeds into its completion  $\widehat{R}$ . By the Cohen structure theorem (see also Propositions 4.2 and 4.3 below) there is an isomorphism  $\varphi : \widehat{R} \rightarrow \kappa[[t]]$  that sends  $\pi$  to  $t$ . For each  $N > 0$ ,  $\varphi$  induces an isomorphism

$$\varphi_N : R/(\pi^N) \cong \widehat{R}/(\pi^N) \rightarrow \kappa[[t]]/(t^N) \cong \kappa[t]/(t^N)$$

that maps  $\pi + (\pi^N)$  to  $t + (t^N)$ .

Let  $N > 0$  be the specific  $N$  given in the last point of Lemma 3.1, computed from the polynomials  $f_i(t, \underline{X}) \in \mathbb{F}_p[t][X_1, \dots, X_m]$ . We claim that the system of equations (I) :  $\bigwedge_{i=1}^n f_i(\pi, \underline{X}) = 0$  has a solution in  $R$  if and only if the reduction of (II) :  $\bigwedge_{i=1}^n f_i(t, \underline{X}) = 0$  has a solution in  $\kappa[t]/(t^N)$ .

One direction is trivial: if there is a solution to (I) in  $R$  then there is a solution to the reduction of (I) in  $R/(\pi^N)$ , and via the isomorphism  $\varphi_N$  there is a solution to the reduction of (II) in  $\kappa[t]/(t^N)$ . Conversely, suppose a solution to the reduction of (II) exists in  $\kappa[t]/(t^N)$ . By our choice of  $N$ , there is a solution to (II) in  $\kappa[t]^h$ . Since  $\kappa[t]^h$  is the union of  $\kappa_0[t]^h$  where  $\kappa_0$  ranges over the finitely generated subfields of  $\kappa$ , we have a solution to (II) in  $\kappa_0[t]^h$  for some  $\kappa_0$ . By [Anscombe and Fehm 2016, Lemma 2.3], there exists a partial section  $\kappa_0 \rightarrow R$  of the residue map  $R \rightarrow \kappa$  (see Definition 4.1). This extends to an embedding  $\kappa_0[t]^h \rightarrow R$ , sending  $t$  to  $\pi$ . Thus we have a solution to (I) in  $R$ , proving the claim.

The condition that the reduction of (II) has a solution in  $\kappa[t]/(t^N)$  is easily translated into the solvability in  $\kappa$  of a system of finitely many polynomial equations over  $\mathbb{F}_p$  (the  $g_j$  of the statement) by a standard interpretation (or Weil restriction) argument, since  $\kappa[t]/(t^N)$  is an algebra of finite dimension over  $\kappa$ .  $\square$

**Remark 3.3.** Let us briefly describe the formal meaning of “one can effectively find” and “relatively decidable” in the statement of Lemma 3.2.

Define an injection of the set of  $\mathcal{L}_{\text{ring}}(\varpi)$ -terms and  $\mathcal{L}_{\text{ring}}(\varpi)$ -formulas into  $\mathbb{N}$  by a standard Gödel coding. Every  $\mathcal{L}_{\text{ring}}(\varpi)$ -term with free variables  $X_1, \dots, X_n$  induces a polynomial in  $\mathbb{F}_p[T, X_1, \dots, X_n]$  in the natural way, where the indeterminate  $T$  takes the place of the constant symbol  $\varpi$ . We now obtain a Gödel numbering of  $\mathbb{F}_p[T, X_1, \dots, X_n]$  by assigning to every polynomial the minimal Gödel number of an  $\mathcal{L}_{\text{ring}}(\varpi)$ -term which induces it. The set of Gödel numbers of polynomials in  $\mathbb{F}_p[T, X_1, \dots, X_n]$  is a decidable set of natural numbers, and mapping the Gödel number of two polynomials in  $\mathbb{F}_p[T, X_1, \dots, X_n]$  to the Gödel number of their sum (or their product) is a computable function. (This is essentially equivalent to asserting that the numbering gives an explicit representation of the ring  $\mathbb{F}_p[T, X_1, \dots, X_n]$  in the sense of [Fröhlich and Shepherdson 1956, §2].)

The first part of Lemma 3.2 now means that there exists a Turing machine which takes as input the codes of finitely many polynomials  $f_1, \dots, f_n$ , and produces as output the codes of finitely many polynomials  $g_1, \dots, g_{n'}$  with the given property. The second part of the lemma asserts that there exists a Turing machine that takes as input the code of a positive existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentence  $\varphi$  and outputs yes

or no according to whether  $(R, \pi) \models \varphi$ , using an oracle (as described in [Shoenfield 1971, Chapter 4]) which for the code of an existential  $\mathcal{L}_{\text{ring}}$ -sentence  $\psi$  decides whether  $\kappa \models \psi$ . In other words, the set of codes of positive existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences satisfied by  $(R, \pi)$  is Turing reducible to the set of codes of sentences in  $\text{Th}_{\exists}(\kappa)$ .

**Remark 3.4.** Although in this section we have fixed a ring  $R$ , we note that the algorithm of Lemma 3.2 to produce the polynomials  $g_j$  from the  $f_i$  does not depend on  $R$ . This is because the interpretation (or Weil restriction) argument used is uniform in  $\kappa$ , since  $\kappa[t]/(t^N)$  is an  $N$ -dimensional  $\kappa$ -algebra with multiplication defined by structure constants in  $\mathbb{F}_p$  that do not themselves depend on  $\kappa$ . In particular,  $\kappa[t]/(t^N) = \mathbb{F}_p[t]/(t^N) \otimes_{\mathbb{F}_p} \kappa$  is an extension of scalars.

**Remark 3.5.** As remarked in [Becker et al. 1979, Section 6], the effective version of Greenberg’s theorem used in the reduction in Lemma 3.2 is hopeless in practice since the computability of the quantity  $N$  comes from an argument involving an enumeration of proofs in some proof calculus. The same holds true for the algorithm that we obtain in Theorem 4.12.

**Remark 3.6.** The version of Lemma 3.2 given in [Denef and Schoutens 2003, Proposition 3.5] is significantly stronger, being phrased for an arbitrary equicharacteristic excellent henselian local domain instead of an equicharacteristic henselian discrete valuation ring, and allowing finitely many parameters. In the proof, this comes down to replacing Greenberg’s theorem by a more general version of Artin approximation.

On the other hand, note that we can dispense with the excellence condition on the valuation ring  $R$ . This is essentially due to us only allowing parameters from  $\mathbb{F}_p[\pi]$ .

**Step 2: Existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences in  $R$ , with a regularity condition.** For this step, let us fix  $f_1, \dots, f_n \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$  such that the system of equations  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$  describes an affine  $\mathbb{F}_p(\pi)$ -variety  $V$ , that is, the ideal

$$\mathfrak{f} := (f_1, \dots, f_n) \trianglelefteq \mathbb{F}_p(\pi)[X_1, \dots, X_m]$$

is prime. Let  $d$  be the dimension of  $V$ .

Recall that a solution  $\underline{x}$  to the system of equations in some extension field  $F/\mathbb{F}_p(\pi)$  is called *smooth* if the Jacobian condition is satisfied, i.e., the matrix  $((\partial f_i / \partial X_j)(\underline{x}))_{ij}$  has rank  $m - d$ .

**Lemma 3.7.** *Let  $g \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$  be a polynomial which is not in the ideal  $\mathfrak{f}$ , so that  $g$  does not vanish identically on the variety  $V$ . If there is a solution  $\underline{x} \in R^m$  to the system of equations  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$  such that the  $K$ -point described by  $\underline{x}$  is smooth, then there exists such a solution with  $g(\underline{x}) \neq 0$ .*

*Proof.* This is [Denef and Schoutens 2003, Theorem 2.4], or at least the special case of it which we will need below. We only briefly describe the idea of the proof.

The statement comes down to an elaborate application of Hensel’s lemma, or the implicit function theorem in henselian fields. Starting with the given solution  $\underline{x} \in R^m$ , perturb  $d$  variables by a small amount (in the valuation topology) and solve for the remaining variables (which is where smoothness is required).

By wisely choosing the initial perturbation, one can ensure that the inequation  $g(\underline{X}) \neq 0$  becomes satisfied, since  $g$  does not vanish identically on  $V$ . (For instance, one may reduce to the case where the  $f_i$  describe an affine curve over  $K$  by a generic hyperplane section argument, so the condition  $g(\underline{X}) \neq 0$  is satisfied by all but finitely many points on the curve, in which case almost all small perturbations will be as required.)  $\square$

Below we want to work in a context where the affine  $\mathbb{F}_p(\pi)$ -variety  $V$  is not necessarily smooth (i.e., not all solutions to the equation system in all extension fields satisfy the Jacobian condition), but instead satisfies the condition of regularity, i.e., the coordinate ring  $\mathbb{F}_p(\pi)[X_1, \dots, X_m]/\mathfrak{f}$  is a regular ring in the sense of commutative algebra. In general, smooth varieties are regular, and over a perfect field the converse holds [Liu 2002, Corollaries 4.3.32 and 4.3.33]. In our situation we still have the following:

**Lemma 3.8.** *Suppose that  $V$  is regular. Then any solution  $\underline{x} \in K^m$  to the system of equations  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0$  is smooth, that is, satisfies the Jacobian condition.*

*Proof.* The field extension  $K/\mathbb{F}_p(\pi)$  is separable as  $\pi$  does not have a  $p$ -th root in  $K$  and hence the fields  $K$  and  $\mathbb{F}_p(\pi)^{1/p} = \mathbb{F}_p(\pi^{1/p})$  are linearly disjoint over  $\mathbb{F}_p(\pi)$ . The tuple  $\underline{x}$  describes an element of  $V(K)$ , i.e., a point  $P$  of the scheme  $V$  together with an  $\mathbb{F}_p(\pi)$ -embedding of the residue field of  $P$  into  $K$ . Thus the residue field of  $P$  is separable over  $\mathbb{F}_p(\pi)$ . As  $P$  is regular by assumption, [EGA IV<sub>4</sub> 1967, Proposition (17.15.1)] forces the point  $P$  to be smooth, so the Jacobian condition holds.  $\square$

Now suppose that  $V$  is regular, and  $g \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$  is not in  $\mathfrak{f}$ . In this situation, satisfaction of the statement  $\exists \underline{X} (\bigwedge_{i=1}^n f_i(\underline{X}) = 0 \wedge g(\underline{X}) \neq 0)$  in  $R$  can now be effectively reduced to an existential statement about the residue field  $\kappa$ : by Lemmas 3.7 and 3.8, the condition  $g(\underline{X}) \neq 0$  can be dropped, and the remaining formula is handled by the previous step (Lemma 3.2).

### **Step 3: Arbitrary existential $\mathcal{L}_{\text{ring}}(\varpi)$ -sentences in $R$ , using resolution of singularities.**

Let  $f_1, \dots, f_n, g \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$ . We wish to determine whether there exists  $\underline{x} \in R^m$  with  $f_i(\underline{x}) = 0$  for all  $i = 1, \dots, n$ , and  $g(\underline{x}) \neq 0$ .

We may assume that the ideal  $\mathfrak{f} = (f_1, \dots, f_n) \trianglelefteq \mathbb{F}_p(\pi)[X_1, \dots, X_m]$  is prime, i.e., that the  $f_i$  define an affine  $\mathbb{F}_p(\pi)$ -variety  $V$ : indeed, by primary decomposition there exist ideals  $I_1, \dots, I_k \trianglelefteq \mathbb{F}_p(\pi)[X_1, \dots, X_m]$  which are primary, so in particular the radicals  $\sqrt{I_1}, \dots, \sqrt{I_k}$  are prime ideals, and  $\mathfrak{f} = I_1 \cap \dots \cap I_k$ . Then a tuple  $\underline{x} \in R^m$  lies in the vanishing locus of  $f_1, \dots, f_n$  if and only if it lies in the vanishing locus of  $\sqrt{I_j}$  for some  $j = 1, \dots, k$ , so we may replace the  $f_i$  by a collection of generators for each of the  $\sqrt{I_j}$ , running all subsequent steps for all  $j = 1, \dots, k$ . The computation of the ideals  $\sqrt{I_j}$  (or rather, generating sets thereof) is completely explicit [Seidenberg 1974].

We can check algorithmically whether the  $\mathbb{F}_p(\pi)$ -variety  $V$  is regular: for this we investigate the  $\mathbb{F}_p$ -scheme described by the  $f_i$  (that is, we “spread out”  $V$  to a scheme over  $\mathbb{F}_p[\pi]$  and hence a scheme over  $\mathbb{F}_p$ , by interpreting  $\pi$  as another indeterminate) and check whether the nonregular locus intersects the generic fiber of the map to  $\text{Spec}(\mathbb{F}_p[\pi])$ . The regular locus can be computed explicitly since it coincides with the smooth locus over the perfect field  $\mathbb{F}_p$ , which is given by explicit polynomial inequations (once the dimension of the scheme is computed).

Suppose that  $V$  is regular. We can check algorithmically whether  $g$  vanishes identically on the zero locus of the  $f_i$ , i.e., whether  $g$  is contained in  $\mathfrak{f}$ . If yes, then there certainly is no solution to  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0 \wedge g(\underline{X}) \neq 0$ . Otherwise, step 2 is applicable. In other words, we can reduce to  $\kappa$  the question of whether there is a tuple in  $R$  as required.

Suppose now that  $V$  defined by the  $f_i$  is not regular. Assuming hypothesis (R0), there exists a morphism  $V' \rightarrow V$  which is a blowing-up along some closed proper subscheme, and where  $V'$  is a regular  $\mathbb{F}_p(\pi)$ -variety. The morphism  $V' \rightarrow V$  is projective [Liu 2002, Proposition 8.1.22], and so we can see  $V'$  as a closed subscheme of  $\mathbb{P}_V^{m'}$  for suitable  $m'$ . Thus  $V'$  (usually not affine) is described by finitely many polynomials  $h_1, \dots, h_{n'} \in \mathbb{F}_p(\pi)[X_1, \dots, X_m, Y_0, \dots, Y_{m'}]$  which are homogeneous in the variables  $Y_j$  (but not usually in the  $X_i$ ), and include the polynomials  $f_1, \dots, f_n$  defining  $V$ . We may assume that the defining polynomials  $h_i$  have coefficients in  $\mathbb{F}_p[\pi]$  by scaling suitably. A point of  $V'$  (in some field containing  $\mathbb{F}_p(\pi)$ ) is given by a zero  $(\underline{x}, \underline{y})$  of the  $h_i$ , where  $\underline{y}$  is not the zero tuple, and we identify solution tuples that differ only by a scaling of  $\underline{y}$ . The morphism  $V' \rightarrow V$  is given on points by simply dropping the variables  $Y_0, \dots, Y_{m'}$ .

We can cover  $V'$  by affine open subvarieties  $V'_0, \dots, V'_{m'}$ , where  $V'_i$  is obtained from the description of  $V'$  by setting  $Y_i$  to 1, i.e., dehomogenizing at  $Y_i$ . This eliminates the need to work with nonaffine varieties.

The morphism  $V' \rightarrow V$  is birational as a blowing-up along a closed proper subscheme; thus it is an isomorphism above some dense open  $U \subseteq V$ . Concretely, let us suppose that  $V'_0 \neq \emptyset$  by permuting the variables  $Y_i$  if necessary, so that  $V'_0 \subseteq V'$  is a dense open subvariety. Then  $V' \rightarrow V$  being birational means that there is  $u \in \mathbb{F}_p[\pi][X_1, \dots, X_m]$ , not in  $\mathfrak{f}$ , such that  $V'_0 \rightarrow V$  is an isomorphism above the open set  $U \subseteq V$  defined by  $u(\underline{X}) \neq 0$ , that is, the ring homomorphism

$$\begin{aligned} \varphi_u : \mathbb{F}_p(\pi)[X_1, \dots, X_m, 1/u(\underline{X})]/(f_1, \dots, f_n) \\ \rightarrow \mathbb{F}_p(\pi)[X_1, \dots, X_m, Y_0, \dots, Y_{m'}, 1/u(\underline{X})]/(h_1, \dots, h_{n'}, Y_0 - 1) \end{aligned}$$

is an isomorphism.

Assuming the existence of  $V' \rightarrow V$  as above, i.e., projective and birational with  $V'$  regular, such a morphism can in fact be found algorithmically: to do this, one simply searches exhaustively for all the defining data ( $m'$ , polynomials  $h_1, \dots, h_{n'}$ ,  $u$ , and an inverse  $\psi_u$  to the ring homomorphism  $\varphi_u$  above), and checks that these data satisfy the conditions, that is, the  $V'_i$  are all regular and  $\psi_u$  is indeed an inverse to  $\varphi_u$ .

Let  $S$  be the set of tuples  $\underline{x} \in R^m$  satisfying  $f_i(\underline{x}) = 0$  for all  $i$  and  $g(\underline{x}) \neq 0$ . To check whether  $S = \emptyset$ , we now proceed as follows. First check whether there is some tuple  $(\underline{x}, \underline{y}) \in R^m \times R^{m'}$  which gives a  $K$ -point of one of  $V'_0, \dots, V'_{m'}$  and satisfies  $g(\underline{x}) \neq 0$ . This can be accomplished using the work already done, since each  $V'_i$  is a regular affine  $\mathbb{F}_p(\pi)$ -variety. If such a tuple  $(\underline{x}, \underline{y})$  exists, then  $\underline{x} \in S$ , and we are done.

Hence let us suppose that no such tuple  $(\underline{x}, \underline{y})$  exists. Then there exists no  $\underline{x} \in R^m$  with  $g(\underline{x}) \neq 0$  which gives a  $K$ -point of  $U$ : indeed, if there were such  $\underline{x}$  which was additionally a  $K$ -point of  $U$ , then we could lift to a  $K$ -point  $(\underline{x}, \underline{y}) \in K^{m+m'+1}$  of  $V'$ , and by scaling we could take  $\underline{y}$  to be in  $R^{m'+1}$  with one coordinate equal to 1, i.e.,  $(\underline{x}, \underline{y})$  would describe a point on some  $V'_i$ .

Thus any  $\underline{x} \in S$  must lie in the lower-dimensional  $V \setminus U$ . We continue the algorithm by replacing  $V$  with the algebraic set  $V \setminus U$ , i.e., by adding  $u$  to the list of polynomials  $f_1, \dots, f_n$ . Since  $V \setminus U$  has lower dimension than  $V$ , this procedure eventually terminates.

We have shown how to determine whether the system  $\bigwedge_{i=1}^n f_i(\underline{X}) = 0 \wedge g(\underline{X}) \neq 0$  has a solution in  $R$ . The question of whether any given existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -sentence holds in  $R$  can be effectively reduced to a disjunction of finitely many systems of this form, using the disjunctive normal form and replacing conjunctions of inequations  $g_1(\underline{X}) \neq 0 \wedge g_2(\underline{X}) \neq 0$  by a single inequation  $(g_1 \cdot g_2)(\underline{X}) \neq 0$ . We have proven:

**Proposition 3.9** [Denef and Schoutens 2003, Theorem 4.3]. *Assume (R0). The existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -theory of  $(R, \pi)$  is decidable relative to the existential  $\mathcal{L}_{\text{ring}}$ -theory of the residue field  $\kappa$ .*

Theorem 1.4 follows since the  $\mathcal{L}_{\text{val}}(\varpi)$ -structure  $(K, v, \pi)$ , where  $v$  is the valuation with valuation ring  $R$ , is quantifier-freely interpretable in the  $\mathcal{L}_{\text{ring}}(\varpi)$ -structure  $(R, \pi)$ .

**Remark 3.10.** Comparing with the phrasing in [Denef and Schoutens 2003, Theorem 4.3], it is clear in our presentation that the assumption of excellence of  $R$  may be dropped. The phrasing in [Denef and Schoutens 2003, Theorem 4.3] also only states that the  $\mathcal{L}_{\text{ring}}(\varpi)$ -theory of  $(R, \pi)$  is decidable “relative to the existential theory of  $\kappa$  and the  $[\mathcal{L}_{\text{ring}}(\varpi)]$ -diagram of  $R$ ”. However, it is unclear what is meant by this diagram. In particular, if  $R$  is uncountable, then the diagram of  $R$  in its usual model-theoretic meaning is an uncountable object, and hence cannot be made available to any algorithm as an oracle. In any case, the presentation above shows that nothing is required besides the existential  $\mathcal{L}_{\text{ring}}$ -theory of  $\kappa$ .

**Remark 3.11.** Above we have taken care to perform all computation in polynomial rings over the concrete field  $\mathbb{F}_p(\pi)$  (or over the ring  $\mathbb{F}_p[\pi]$ ). Denef and Schoutens are not always clear on this point. For instance, [Denef and Schoutens 2003, Lemma 4.2] makes a claim about an algorithm computing a reduction over the ring  $R$ , but it is unclear what this means if  $R$  is not explicitly presented (for example, uncountable).

While it may appear that such problems can be easily avoided by working over finitely generated subfields (and indeed this is suggested in [Denef and Schoutens 2003, Remark 4.1]), this leads to rather subtle issues in general, since neither reducedness nor regularity is preserved under inseparable base change: if  $K_0 \subseteq K$  is a subfield over which  $K$  is not separable, then a regular variety over  $K_0$  may become nonreduced (thus in particular nonregular) over  $K$ . Similarly, one needs to be careful in general to distinguish regularity from smoothness, because these disagree for varieties over imperfect base fields, and so regularity cannot be checked using the Jacobian criterion as suggested in [Denef and Schoutens 2003, Remark 4.1].

We can avoid these issues above because the extension  $K/\mathbb{F}_p(\pi)$  is separable and therefore none of the pathologies mentioned can arise.

**Remark 3.12.** Above we have in fact only used that there exists a resolution of the affine variety  $V$  by a projective birational morphism, rather than a blowing-up. This looks superficially weaker than hypothesis (R0) or the equivalent [Denef and Schoutens 2003, Section 4, Conjecture 1], but in fact any projective birational morphism to a quasiprojective variety is a blowing-up [Liu 2002, Theorem 8.1.24]. (Of course, it suffices for us to demand (R0) for affine varieties.)

In fact, the argument above can be modified to make do with (R1) in place of (R0). To do this, consider the  $\mathbb{F}_p[\pi]$ -scheme  $\mathcal{V} = \text{Spec}(\mathbb{F}_p[\pi][X_1, \dots, X_m]/(f_1, \dots, f_n))$ , which has  $V$  as above as its generic fiber. Then a birational proper morphism  $V' \rightarrow V$  with  $V'$  regular (which exists under the assumption of (R1)) can be extended to a proper morphism  $\mathcal{V}' \rightarrow \mathcal{V}$ , with an  $\mathbb{F}_p[\pi]$ -scheme  $\mathcal{V}'$  which has  $V'$  as its generic fiber. (This step seems to require Nagata's compactification theorem; see Section (12.15) in [Görtz and Wedhorn 2010].) Covering  $\mathcal{V}'$  by affine  $\mathbb{F}_p[\pi]$ -schemes  $\mathcal{V}'_0, \dots, \mathcal{V}'_{m'}$ , one can then proceed as above. We omit the details.

**Remark 3.13.** In spite of the technical superiority of scheme-theoretic algebraic geometry over more naive conceptions of varieties, we find that one really can work in the naive language almost throughout, excepting some of the more subtle points surrounding smoothness and regularity which cannot be handled adequately in this way.

Denef and Schoutens largely use scheme-theoretic language, but not entirely correctly. In particular, by an open  $W$  of a finite-type  $R$ -scheme  $X$  as in [Denef and Schoutens 2003, Theorem 2.4 and proof of Theorem 4.3] they seem to mean a subset of the  $R$ -points of  $X$  defined by finitely many inequations, which is not the same as an open subscheme (which, after all, would itself be a finite-type  $R$ -scheme and could simply replace  $X$ ), but rather an open subscheme of the generic fiber  $X_K$ . An  $R$ -rational point of  $W$  in their language is then an  $R$ -point of  $X$  such that the associated  $K$ -point of  $X_K$  lies on  $W$ . This issue also seems to have been noted independently in [Kartas 2023a, proof of Theorem A].

#### 4. Axiomatization of the existential theory

In this final section we prove Theorem 4.12, which is a strengthening of Theorem 1.5 to allow parameters from a  $\mathbb{Z}$ -valued base field  $(C, u)$ . Here, by  $\mathbb{Z}$ -valued we mean that  $uC \cong \mathbb{Z}$  (sometimes called *discrete*). A major role in our proof is played by partial sections of residue maps:

**Definition 4.1.** Let  $(K, v)$  be a valued field. We denote by  $\text{res}_v : \mathcal{O}_v \rightarrow Kv$  the residue map. A *partial section* of  $\text{res}_v$  is a ring homomorphism  $\zeta : k \rightarrow \mathcal{O}_v$  with  $\text{res}_v \circ \zeta = \text{id}_k$  for some subfield  $k$  of  $Kv$ . A *section* of  $\text{res}_v$  is a partial section  $\zeta : Kv \rightarrow \mathcal{O}_v$  defined on the full residue field  $Kv$ .

We will repeatedly have to extend such partial sections. There are two main settings where this is possible. The first one is in the context of complete discrete valuation rings:

**Proposition 4.2.** *Let  $(K, v)$  be an equicharacteristic complete  $\mathbb{Z}$ -valued field with uniformizer  $\pi$ . Then every partial section  $\zeta : k \rightarrow K$  of  $\text{res}_v$  has a unique extension to an embedding of valued fields  $\alpha : (k((t)), v_t) \rightarrow (K, v)$  with  $\alpha(t) = \pi$ , which is an isomorphism if  $k = Kv$ .*

*Proof.* The partial section  $\zeta$  extends uniquely to an embedding  $\alpha_0 : (k(t), v_t) \rightarrow (K, v)$  with  $\alpha_0(t) = \pi$ , and then the existence and uniqueness of the completion give a unique extension  $\alpha$  of  $\alpha_0$  from the completion  $(k((t)), v_t)$  of  $(k(t), v_t)$  into the complete valued field  $(K, v)$ . If  $\zeta$  is a section, every element of  $K$  has a power series expansion in  $t$  with coefficients in  $\zeta(K)$  (see, for example, [Serre 1979, Chapter II §4]); hence  $\alpha$  is surjective.  $\square$

**Proposition 4.3.** *Let  $(K, v)$  be an equicharacteristic complete  $\mathbb{Z}$ -valued field. Every partial section  $\zeta : k \rightarrow \mathcal{O}_v$  of  $\text{res}_v$  with  $Kv/k$  separable extends to a section of  $\text{res}_v$ . In particular,  $\text{res}_v$  has a section.*

*Proof.* This follows from the structure theory of complete discrete valuation rings; for example, apply [Roquette 1959, Satz 1a] with  $E_0 = \zeta(k)$  in the notation there. The “in particular” follows by applying the first part to the partial section defined on the prime field.  $\square$

Secondly, we can always extend partial sections after passage to a suitable elementary extension:

**Lemma 4.4.** *Let  $(K, v)$  be a henselian valued field. For every partial section  $\zeta_0 : k_0 \rightarrow K$  of  $\text{res}_v$  with  $Kv/k_0$  separable there exists an elementary extension  $(K, v) \prec (K^*, v^*)$  with a partial section  $\zeta : Kv \rightarrow K^*$  of  $\text{res}_{v^*}$  that extends  $\zeta_0$ .*

*Proof.* For each subfield  $E \subseteq Kv$ , let  $\mathcal{L}_{\text{val}}(E, Kv)$  be the language expanding  $\mathcal{L}_{\text{val}}$  by new constant symbols  $\{c_x \mid x \in E\}$  and  $\{d_x \mid x \in Kv\}$ . Let  $\rho(a, b)$  be an  $\mathcal{L}_{\text{val}}$ -formula expressing that  $a$  and  $b$  have the same residue and consider the  $\mathcal{L}_{\text{val}}(E, Kv)$ -theory

$$T_E := \{d_x \in \mathcal{O} \mid x \in Kv\} \cup \{c_x \in \mathcal{O} \wedge \rho(c_x, d_x) \mid x \in E\} \cup \{c_x + c_y = c_{x+y} \wedge c_x c_y = c_{xy} \mid x, y \in E\}.$$

If we expand  $(K, v)$  to an  $\mathcal{L}_{\text{val}}(k_0, Kv)$ -structure  $\mathbb{K}_{\zeta_0}$  by interpreting  $c_x^{\mathbb{K}_{\zeta_0}} = \zeta_0(x)$  for  $x \in k_0$  and  $d_x^{\mathbb{K}_{\zeta_0}}$  any choice of element in the valuation ring of  $v$  with residue  $x$ , for each  $x \in Kv$ , then  $\mathbb{K}_{\zeta_0} \models T_{k_0}$ .

Let  $\Delta$  be a finite subset of  $T_{Kv}$ . Then there exists a finitely generated subextension  $E/k_0$  of  $Kv/k_0$  such that  $\Delta$  is an  $\mathcal{L}_{\text{val}}(E, Kv)$ -theory. Since  $Kv/k_0$  is separable,  $E/k_0$  is separably generated. Therefore, by [Anscombe and Fehm 2016, Lemma 2.3],  $\zeta_0$  extends to a partial section  $\hat{\zeta} : E \rightarrow K$  of  $\text{res}_v$ . Equivalently, there is an  $\mathcal{L}_{\text{val}}(E, Kv)$ -expansion  $\mathbb{K}_{\hat{\zeta}}$  of  $\mathbb{K}_{\zeta_0}$  to a model of  $T_E$ . In particular,  $\mathbb{K}_{\hat{\zeta}}$  models  $\Delta$ .

Therefore,  $T_{Kv}$  is consistent with the elementary diagram of  $\mathbb{K}_{\zeta_0}$ ; hence by the compactness theorem there exists an  $\mathcal{L}_{\text{val}}(Kv, Kv)$ -structure  $\mathbb{K}^* \models T_{Kv}$  whose  $\mathcal{L}_{\text{val}}(k_0, Kv)$ -reduct is an elementary extension of  $\mathbb{K}_{\zeta_0}$ . In particular, its  $\mathcal{L}_{\text{val}}$ -reduct  $(K^*, v^*)$  is an elementary extension of  $(K, v)$ , and  $\zeta : Kv \rightarrow K^*$  defined by  $x \mapsto c_x^{\mathbb{K}^*}$  is a partial section of  $\text{res}_{v^*}$  that extends  $\zeta_0$ .  $\square$

**Proposition 4.5.** *Let  $(K, v)$  be a henselian valued field. For every partial section  $\zeta_0 : k_0 \rightarrow K$  of  $\text{res}_v$  with  $Kv/k_0$  separable there exists an elementary extension  $(K, v) \prec (K^*, v^*)$  with a section  $\zeta : K^*v^* \rightarrow K^*$  of  $\text{res}_{v^*}$  that extends  $\zeta_0$ .*

*Proof.* By Lemma 4.4 there exists an elementary extension  $\mathbb{K}_0 := (K, v) \prec \mathbb{K}_1 = (K_1, v_1)$  with a partial section  $\zeta_1 : Kv \rightarrow K_1$  of  $\text{res}_{v_1}$  extending  $\zeta_0$ . Since  $Kv \prec K_1v_1$ , the extension  $K_1v_1/Kv$  is in particular separable (see, for example, [Ershov 2001, Corollary 3.1.3]), and so we can iterate Lemma 4.4 to obtain a chain of elementary extensions  $\mathbb{K} = \mathbb{K}_0 \prec \mathbb{K}_1 \prec \dots$  and for each  $i > 0$  a partial section  $\zeta_i : K_{i-1}v_{i-1} \rightarrow K_i$  of  $\text{res}_{v_i}$  extending  $\zeta_{i-1}$ . The direct limit  $\mathbb{K}^* = (K^*, v^*) := \varinjlim_i \mathbb{K}_i$  is then an elementary extension of  $\mathbb{K}_0$  with a partial section  $K^*v^* = \varinjlim_i K_{i-1}v_{i-1} \rightarrow \varinjlim_i K_i = K^*$  extending  $\zeta_0$ .  $\square$

Given a field extension  $K/C$  we denote by  $(K, C)$  the  $\mathcal{L}_{\text{ring}}(C)$ -structure expanding  $K$  in the usual way: the constant symbol  $c_x$  is interpreted by  $x$  itself, for each  $x \in C$ . Analogously,  $(K, v, C)$  denotes the

$\mathcal{L}_{\text{val}}(C)$ -structure similarly expanding  $(K, v)$ , and  $(K, v, \pi, C)$  denotes the  $\mathcal{L}_{\text{val}}(\varpi, C)$ -structure similarly expanding  $(K, v, \pi)$ .

**Corollary 4.6.** *Assume (R4). Let  $(K, v)$  be an equicharacteristic henselian valued field with  $Kv$  large, and assume that  $v$  is trivial on a subfield  $C$  of  $K$ . We identify  $C$  with its image under  $\text{res}_v$  and assume that the extension  $Kv/C$  is separable. Then the existential  $\mathcal{L}_{\text{ring}}(C)$ -theories of  $K$  and  $Kv$  coincide.*

*Proof.* By Proposition 4.5 there is an elementary extension  $(K, v) \prec (K^*, v^*)$  with a section  $\zeta : K^*v^* \rightarrow K^*$  of  $\text{res}_{v^*}$  that extends the partial section  $\text{id}_C$  of  $\text{res}_v$ . Since  $Kv$  is large, so is its elementary extension  $K^*v^*$ ; hence  $\zeta(K^*v^*) \prec_{\exists} K^*$  by (R4). So for the existential  $\mathcal{L}_{\text{ring}}(C)$ -theories we get  $\text{Th}_{\exists}(K) = \text{Th}_{\exists}(K^*) = \text{Th}_{\exists}(\zeta(K^*v^*)) = \text{Th}_{\exists}(K^*v^*) = \text{Th}_{\exists}(Kv)$ .  $\square$

This concludes our discussion on extensions of partial sections.

**Definition 4.7.** We denote by  $T$  the  $\mathcal{L}_{\text{val}}(\varpi)$ -theory of equicharacteristic henselian valued fields with distinguished uniformizer.

**Remark 4.8.** The class of equicharacteristic henselian valued fields with distinguished uniformizer is elementary. Also,  $T$  admits a recursive axiomatization; for example, see [Kuhlmann 2016, Section 4] for an explicit axiomatization of henselianity.

**Lemma 4.9.** *Every existential  $\mathcal{L}_{\text{val}}(\varpi)$ -formula is equivalent modulo  $T$  to an existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -formula.*

*Proof.* This follows since the existential  $\mathcal{L}_{\text{ring}}(\varpi)$ -formula  $\varphi(x)$  given by  $\exists y y^2 + y = \varpi x^2$  defines the valuation ring  $\mathcal{O}_v$  in every  $(K, v, \pi) \models T$ , and therefore so does the universal formula  $\neg\varphi((\varpi x)^{-1})$  given by  $\forall y \varpi x^2(y^2 + y) \neq 1$ . These formulas basically go back to [Robinson 1965]; see [Fehm and Jahnke 2017] for more on definable valuations.  $\square$

Let  $R$  be a discrete valuation ring with field of fractions  $C$ ,  $u$  the valuation on  $C$  with  $\mathcal{O}_u = R$ , and  $(\hat{C}, \hat{u})$  the completion of  $(C, u)$ . Then  $R$  is *excellent* if and only if  $\hat{C}/C$  is separable. This is compatible with the more general definition of excellent rings in commutative algebra (see Corollary 8.2.40(b) in [Liu 2002]) but for discrete valuation rings we may as well take it as our definition.

**Remark 4.10.** Although we will not need it below, we note that  $R$  is excellent if and only if it is defectless in the sense of valuation theory, i.e., the fundamental inequality is an equality for all finite extensions of  $C$ ; see [Kuhlmann 2016, (1.5)]. Indeed,  $\hat{C}/C$  is separable if and only if  $R$  is a Japanese ring by [EGA IV<sub>2</sub> 1965, Corollaire (7.6.6)], which in turn is the case if and only if  $R$  is defectless by [Bourbaki 2006, Chapitre VI §8 No 5 Théorème 2].

**Proposition 4.11.** *Assume (R4). Let  $(C, u)$  be an equicharacteristic  $\mathbb{Z}$ -valued field with distinguished uniformizer  $\pi$  such that  $\mathcal{O}_u$  is excellent. Let  $(K, v, \pi), (L, w, \pi) \models T$  be extensions of  $(C, u, \pi)$  such that  $Kv/Cu$  and  $Lw/Cu$  are separable. If  $\text{Th}_{\exists}(Kv, Cu) \subseteq \text{Th}_{\exists}(Lw, Cu)$ , then  $\text{Th}_{\exists}(K, v, \pi, C) \subseteq \text{Th}_{\exists}(L, w, \pi, C)$ .*



*Proof.* Let  $\varphi$  be an existential  $\mathcal{L}_{\text{val}}(\varpi, C)$ -sentence with  $(K, v, \pi, C) \models \varphi$ . We have to show that  $(L, w, \pi, C) \models \varphi$ , and for this we are allowed to replace both  $(K, v, \pi, C)$  and  $(L, w, \pi, C)$  by arbitrary elementary extensions, so assume without loss of generality that  $(K, v, \pi, C)$  is  $\aleph_1$ -saturated and  $(L, w, \pi, C)$  is  $|K|^+$ -saturated. Since  $Lw$  is then at least  $|Kv|^+$ -saturated, and  $(Lw, Cu) \models \text{Th}_{\exists}(Kv, Cu)$ , there is a  $Cu$ -embedding  $f : Kv \rightarrow Lw$ ; see [Chang and Keisler 1990, Lemma 5.2.1]. By Lemma 4.9 we can assume without loss of generality that  $\varphi$  is an existential  $\mathcal{L}_{\text{ring}}(\varpi, C)$ -sentence. Furthermore, we can replace each occurrence of  $\varpi$  by the constant symbol  $c_{\pi}$  to assume without loss of generality that  $\varphi$  is an existential  $\mathcal{L}_{\text{ring}}(C)$ -sentence.

Since  $\pi$  is a uniformizer of  $v$ ,  $v(\pi)$  generates a convex subgroup of  $vK$  isomorphic to  $\mathbb{Z}$ . In particular,  $v$  has a finest proper coarsening  $v^+$ . As  $v^+$  is trivial on  $C$ , we can view  $Kv^+$  as an extension of  $C$ . On  $Kv^+$ ,  $v$  induces a henselian valuation  $\bar{v}$  extending  $u$  with  $\bar{v}(Kv^+) \cong \mathbb{Z}$  and with uniformizer  $\pi$ . Since  $(K, v)$  is  $\aleph_1$ -saturated,  $(Kv^+, \bar{v})$  is complete; see [Anscombe and Kuhlmann 2016, Claim on page 411] and note that maximal  $\mathbb{Z}$ -valued fields are complete. Given that  $\mathcal{O}_u$  is excellent, that  $(Kv^+, \bar{v})$  and  $(C, u)$  have a common uniformizer, and that the residue extension  $Kv/Cu$  of  $(Kv^+, \bar{v})/(C, u)$  is separable, we have that  $Kv^+/C$  is separable by [Bosch et al. 1990, proof of Lemma 3.6/2]<sup>1</sup>. Since  $Kv^+$  is large, because it admits the nontrivial henselian valuation  $\bar{v}$ , Corollary 4.6 (which works under (R4)) shows that  $\text{Th}_{\exists}(K, C) = \text{Th}_{\exists}(Kv^+, C)$ , and the analogous argument (note that  $|K|^+ \geq \aleph_1$ ) gives that  $\text{Th}_{\exists}(L, C) = \text{Th}_{\exists}(Lw^+, C)$ .

Since we assumed that  $\varphi \in \text{Th}_{\exists}(K, C)$  and want to show that  $\varphi \in \text{Th}_{\exists}(L, C)$ , we can now replace  $(K, v, \pi, C)$  by  $(Kv^+, \bar{v}, \pi, C)$  and  $(L, w, \pi, C)$  by  $(Lw^+, \bar{w}, \pi, C)$ , so that both  $(K, v)$  and  $(L, w)$  are complete  $\mathbb{Z}$ -valued fields with uniformizer  $\pi$  — note however that we may no longer assume any saturation for either  $(K, v)$  or  $(L, w)$ . We may then also replace  $C$  by  $\hat{C}$  to assume that  $(C, u)$  is complete. By Proposition 4.3 there exists a section  $\zeta : Cu \rightarrow C$  of  $\text{res}_u$ , which we can see as a partial section of  $\text{res}_v$  and  $\text{res}_w$ . Since  $Kv/Cu$  and  $Lw/Cu$  are separable, applying Proposition 4.3 again yields that  $\zeta$  extends to a section  $\xi_K : Kv \rightarrow K$  of  $\text{res}_v$  and to a section  $\xi_L : Lw \rightarrow L$  of  $\text{res}_w$ . By Proposition 4.2,  $\zeta, \xi_K$  and  $\xi_L$  extend to isomorphisms  $\gamma : (Cu((t)), v_t) \rightarrow (C, u)$ ,  $\alpha : (Kv((t)), v_t) \rightarrow (K, v)$  and  $\beta : (Lw((t)), v_t) \rightarrow (L, w)$  with  $\gamma(t) = \pi, \alpha(t) = \pi$  and  $\beta(t) = \pi$ , and  $\alpha|_{Cu((t))} = \beta|_{Cu((t))} = \gamma$  by the uniqueness assertion thereof. Also, the  $Cu$ -embedding  $f : Kv \rightarrow Lw$  extends uniquely to an embedding  $\iota : (Kv((t)), v_t) \rightarrow (Lw((t)), v_t)$  with  $\iota(t) = t$  that is the identity on  $Cu((t))$ . Therefore,  $\beta \circ \iota \circ \alpha^{-1} : (K, v) \rightarrow (L, w)$  is an embedding which restricted to  $C$  is  $\gamma \circ \text{id}_{Cu((t))} \circ \gamma^{-1} = \text{id}_C$ :

$$\begin{array}{ccccccc}
 K & \xleftarrow{\alpha} & Kv((t)) & \xhookrightarrow{\iota} & Lw((t)) & \xrightarrow{\beta} & L \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 C & \xleftarrow{\gamma} & Cu((t)) & \xrightarrow{\text{id}} & Cu((t)) & \xrightarrow{\gamma} & C
 \end{array}$$

So  $(K, C) \models \varphi$  implies that  $(L, C) \models \varphi$ , which concludes the proof. □

<sup>1</sup>The “in particular” of the statement of the lemma is not correct, but this does not concern us. Alternatively, the separability of  $Kv^+/C$  follows from applying [Ershov 1967, page 44, Lemma] to the henselization of  $(C, u)$ .

**Theorem 4.12.** *Assume (R4). Let  $(C, u)$  be an equicharacteristic  $\mathbb{Z}$ -valued field with uniformizer  $\pi$  such that  $\mathcal{O}_u$  is excellent, and let  $(K, v)$  be a henselian extension of  $(C, u)$  for which  $\pi$  is a uniformizer and  $Kv/Cu$  is separable. Then the universal/existential  $\mathcal{L}_{\text{val}}(\varpi, C)$ -theory of  $(K, v, \pi, C)$  is entailed by*

- (i) *the quantifier-free diagram of the  $\mathcal{L}_{\text{val}}(\varpi)$ -structure  $(C, u, \pi)$ ,*
- (ii) *the  $\mathcal{L}_{\text{val}}$ -axioms for equicharacteristic henselian valued fields,*
- (iii) *the  $\mathcal{L}_{\text{val}}(\varpi)$ -axiom expressing that  $\pi$  has smallest positive value, and*
- (iv)  *$\mathcal{L}_{\text{val}}(C)$ -axioms expressing that the residue field models the universal/existential  $\mathcal{L}_{\text{ring}}(Cu)$ -theory of  $Kv$ .*

*In particular, if  $C$  is countable and we fix a surjection  $\rho : \mathbb{N} \rightarrow C$ , the existential  $\mathcal{L}_{\text{val}}(\varpi, C)$ -theory of  $(K, v, \pi, C)$  is decidable relative to the existential  $\mathcal{L}_{\text{ring}}(Cu)$ -theory of  $(Kv, Cu)$  and the quantifier-free diagram of the  $\mathcal{L}_{\text{val}}(\varpi)$ -structure  $(C, u, \pi)$ .*

*Proof.* Note that  $(K, v, \pi)$  is a model of the theory  $T$  from Definition 4.7. Every model of (i)–(iii) can be viewed as a henselian valued field extending  $(C, u)$  with uniformizer  $\pi$ , and, in particular, as a model of  $T$ . Let  $(L, w, \pi, C)$  be a model of (i)–(iv). Since  $Kv/Cu$  is separable, the universal  $\mathcal{L}_{\text{ring}}(Cu)$ -theory of  $Kv$  contains, for every finite  $p$ -independent tuple  $\underline{x}$  from  $Cu$ , the statement that  $\underline{x}$  remains  $p$ -independent over  $Kv$ . Therefore, since  $(L, w)$  satisfies (iv),  $Lw/Cu$  is also separable. Since (iv) for  $(L, w)$  also implies that  $\text{Th}_{\exists}(Lw, Cu) = \text{Th}_{\exists}(Kv, Cu)$ , Proposition 4.11 gives that  $\text{Th}_{\exists}(L, w, \pi, C) = \text{Th}_{\exists}(K, v, \pi, C)$ . This shows that every universal or existential sentence true in  $(K, v, \pi, C)$  is entailed by (i)–(iv).

The list of axioms (ii) is recursive, and (iii) is a single axiom. Therefore, with an oracle for (i) and an oracle for (iv), the sentences deducible from (i)–(iv) can be recursively enumerated. Since the negation of an existential sentence is universal, and every existential or universal sentence true in  $(K, v, \pi, C)$  is entailed by (i)–(iv), the completeness theorem then gives a decision procedure for the existential theory of  $(K, v, \pi, C)$  modulo these two oracles.  $\square$

**Remark 4.13.** The notion of relative decidability used here is the obvious modification of Remark 3.3. We define an injection of the set of  $\mathcal{L}_{\text{val}}(\varpi, C)$ -sentences into  $\mathbb{N}$  by a standard Gödel coding using the map  $C \rightarrow \mathbb{N}, c \mapsto \min \rho^{-1}(c)$ . Similarly, we define an injection of the set of  $\mathcal{L}_{\text{ring}}(Cu)$ -sentences into  $\mathbb{N}$  using in addition the map  $Cu \rightarrow \mathbb{N}, d \mapsto \min \rho^{-1}(\text{res}_u^{-1}(d))$ . Relative decidability then means that there exists a Turing machine that takes as input the code of an existential  $\mathcal{L}_{\text{val}}(\varpi, C)$ -sentence  $\varphi$  and outputs yes or no according to whether  $(K, v, \pi, C) \models \varphi$ , and uses an oracle for the existential  $\mathcal{L}_{\text{ring}}(Cu)$ -theory of  $(Kv, Cu)$  and another oracle for the quantifier-free  $\mathcal{L}_{\text{val}}(\varpi, C)$ -theory of  $(C, u, \pi, C)$ .

*Proof of Theorem 1.5.* Let  $F$  be the prime subfield of  $K$ . Since  $v(\pi) \neq 0$  and  $(K, v)$  is equicharacteristic,  $\pi$  is transcendental over  $F$ . Since  $v(\pi) > 0$  and  $v$  restricted to  $F$  is trivial, the restriction of  $v$  to  $F(\pi)$  is the  $\pi$ -adic valuation  $v_{\pi}$ . Let  $(C, u) = (F(\pi), v_{\pi})$ . Then  $(K, v)$  is an extension of  $(C, u)$ . Since  $Cu = F$  is perfect,  $Kv/Cu$  is separable. The valuation ring  $\mathcal{O}_u$  is excellent since  $F((\pi))/F(\pi)$  is separable (see the proof of Lemma 3.8).

Let  $(L, w, \pi')$  be a model of Theorem 1.5(i)–(iii). Then  $(L, w, \pi')$  is a model of Theorem 4.12(ii)–(iii). In particular,  $(L, w, \pi')$  is equicharacteristic, and by Theorem 1.5(iii),  $Lw$  has the same characteristic as  $Kv$ . Thus, without loss of generality,  $L$  is an extension of  $F$ . By the same argument as above for  $(K, v)$ ,  $(L, w)$  is an extension of  $(F(\pi'), v_{\pi'})$ . The latter is isomorphic to  $(F(\pi), v_{\pi})$ ; thus  $(L, w, \pi')$  admits a unique expansion to an  $\mathcal{L}_{\text{val}}(\varpi, C)$ -structure  $\mathbb{L}$  that models Theorem 4.12(i) for  $\mathbb{K} := (K, v, \pi, C)$ . Since  $Lw$  models the universal/existential theory of  $Kv$ , and  $Cu = F$  is a prime field,  $\mathbb{L}$  models Theorem 4.12(iv) for  $\mathbb{K}$ .

By Theorem 4.12,  $\mathbb{L}$  models the universal/existential theory of  $\mathbb{K}$ . In particular,  $(L, w, \pi')$  models the universal/existential theory of  $(K, v, \pi)$ , which proves the first part of the theorem. The final sentence of the theorem follows, just as the final sentence of Theorem 4.12.  $\square$

*Proof of Corollary 1.6.* This follows from Theorem 4.12 with  $(C, u) = (\mathbb{F}_q(t), v_t)$  and  $(K, v) = (\mathbb{F}_q((t)), v_t)$ , for which we verify the prerequisites. The existential  $\mathcal{L}_{\text{ring}}(\mathbb{F}_q)$ -theory of  $\mathbb{F}_q$  is decidable. The same holds for the quantifier-free diagram of  $(\mathbb{F}_q(t), v_t, t)$ , since elements of  $\mathbb{F}_q(t)$  can be explicitly represented as quotients of polynomials in such a way as to allow concrete computation of sums, products and the valuation under  $v_t$ . More formally, the ring  $\mathbb{F}_q(t)$  has an *explicit representation* in the sense of [Fröhlich and Shepherdson 1956, §2] by Theorem 3.4 there, and this representation in particular fixes a surjection  $\rho : \mathbb{N} \rightarrow \mathbb{F}_q(t)$  with respect to which the quantifier-free diagram of  $(\mathbb{F}_q(t), t)$  is decidable in the sense of Remark 4.13. Since elements of  $\mathbb{F}_q(t)$  are represented by quotients  $f/g$  with  $f, g \in \mathbb{F}_q[t]$  with  $g \neq 0$  in the proof of [Fröhlich and Shepherdson 1956, Theorem 3.4], and one can determine  $v_t(f)$  and  $v_t(g)$  for explicitly given  $f$  and  $g$ , it is decidable whether a given element of  $\mathbb{F}_q(t)$  lies in the valuation ring of  $v_t$ .  $\square$

**Corollary 4.14.** *Assume (R4). Let  $(C, u) \subseteq (K, v)$  be an extension of equicharacteristic henselian valued fields. Suppose that  $uC = \mathbb{Z}$  and that  $\mathcal{O}_u$  is excellent. If  $uC \prec_{\exists} vK$  and  $Cu \prec_{\exists} Kv$ , then  $(C, u) \prec_{\exists} (K, v)$ .*

*Proof.* The assumption  $\mathbb{Z} = uC \prec_{\exists} vK$  implies that a uniformizer of  $u$  is also a uniformizer of  $v$ . The assumption  $Cu \prec_{\exists} Kv$  implies that  $Kv/Cu$  is separable (see again [Ershov 2001, Corollary 3.1.3]) and that  $Cu$  and  $Kv$  have the same existential  $\mathcal{L}_{\text{ring}}(Cu)$ -theory. Theorem 4.12 then gives that  $(C, u)$  and  $(K, v)$  have the same existential  $\mathcal{L}_{\text{val}}(C)$ -theory.  $\square$

**Remark 4.15.** The assumption that  $\mathcal{O}_u$  is excellent is necessary in all these statements. For example, in Corollary 4.14, if  $(C, u)$  is existentially closed in its completion  $(K, v) = (\hat{C}, \hat{u})$ , then in particular  $\hat{C}/C$  is separable.

As a consequence we also obtain the following variant of [Anscombe and Fehm 2016, Theorem 7.1] and its corollaries:

**Corollary 4.16.** *Assume (R4). Let  $(K, v)$  and  $(L, w)$  be equicharacteristic nontrivially valued henselian fields with the common trivially valued subfield  $C$ . We identify  $C$  with its image under  $\text{res}_v$  and under  $\text{res}_w$  and assume that  $Kv/C$  and  $Lw/C$  are separable. If  $\text{Th}_{\exists}(Kv, C) \subseteq \text{Th}_{\exists}(Lw, C)$ , then  $\text{Th}_{\exists}(K, v, C) \subseteq \text{Th}_{\exists}(L, w, C)$ .*

*Proof.* Let  $(K', v')$  be a henselian extension of  $(K, v)$  that has a uniformizer  $\pi$  (say a fresh indeterminate) and satisfies  $Kv = K'v'$  — for example, let  $(K', \tilde{v})$  be a henselian extension of  $(K, v)$  with  $K'\tilde{v} = Kv((t))$  as in [Ancombe and Fehm 2017, Lemma 2.1], and let  $v'$  be the composed valuation  $v_t \circ \tilde{v}$ . Then both  $(K', v', \pi)$  and  $(Lw(\pi)^h, v_\pi, \pi)$  are extensions of the equicharacteristic  $\mathbb{Z}$ -valued field  $(C(\pi), v_\pi)$ . Also, the extension  $K'v'/C(\pi)v_\pi$  is equal to  $Kv/C$ , and  $(Lw(\pi)^h)v_\pi/C(\pi)v_\pi$  is equal to  $Lw/C$ , so we may apply Proposition 4.11 to obtain  $\text{Th}_\exists(K', v', \pi, C(\pi)) \subseteq \text{Th}_\exists(Lw(\pi)^h, v_\pi, \pi, C(\pi))$ . Taking reducts yields  $\text{Th}_\exists(K, v, C) \subseteq \text{Th}_\exists(K', v', C) \subseteq \text{Th}_\exists(Lw(\pi)^h, v_\pi, C)$ .

By Lemma 4.4 there is an elementary extension  $(L, w) \prec (L^*, w^*)$  with a partial section  $\zeta : Lw \rightarrow L^*$  of  $\text{res}_{w^*}$  over  $C$ , which then extends to an embedding  $(Lw(\pi)^h, v_\pi) \rightarrow (L^*, w^*)$  over  $C$  by sending  $\pi$  to any nonzero element of positive value. Hence we obtain the inclusion  $\text{Th}_\exists(Lw(\pi)^h, v_\pi, C) \subseteq \text{Th}_\exists(L^*, w^*, C) = \text{Th}_\exists(L, w, C)$ .  $\square$

**Remark 4.17.** We observe that the assumption (R4) is necessary in Corollary 4.16 (and hence in Proposition 4.11 and also, less immediately, in Theorem 4.12): If  $C$  is large and  $K/C$  an extension with a valuation  $v$  on  $K/C$  with  $Kv = C$ , we apply Corollary 4.16 with  $(L, w) = (C((t)), v_t)$ ; since  $Kv = C = Lw$ , we conclude that  $\text{Th}_\exists(K, v, C) = \text{Th}_\exists(C((t)), v_t, C)$ , so  $C \prec_{\exists} C((t))$  implies that  $C \prec_{\exists} K$ ; hence (R4) must hold.

**Remark 4.18.** As noted in Remark 2.4, the statement of (R4) is known to hold at least for certain base fields  $K$ , and inspection of our proofs shows that the assumption that (R4) holds can be weakened to require only that it holds when restricted to specific base fields: In Corollary 4.6 it suffices to assume that (R4) holds when restricted to elementary extensions of  $Kv$ . In Proposition 4.11 it therefore suffices to assume that (R4) holds when restricted to residue fields of proper coarsenings of elementary extensions of  $(K, v)$  and  $(L, w)$ . In particular, both here and similarly in Theorem 4.12 and Corollaries 4.14 and 4.16 it suffices to assume that (R4) holds when restricted to extensions of  $C$ .

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### References

- [Ancombe and Fehm 2016] S. Ancombe and A. Fehm, “The existential theory of equicharacteristic henselian valued fields”, *Algebra Number Theory* **10**:3 (2016), 665–683. MR Zbl
- [Ancombe and Fehm 2017] S. Ancombe and A. Fehm, “Characterizing diophantine henselian valuation rings and valuation ideals”, *Proc. Lond. Math. Soc.* (3) **115**:2 (2017), 293–322. MR Zbl
- [Ancombe and Kuhlmann 2016] S. Ancombe and F.-V. Kuhlmann, “Notes on extremal and tame valued fields”, *J. Symb. Log.* **81**:2 (2016), 400–416. MR Zbl
- [Bary-Soroker and Fehm 2013] L. Bary-Soroker and A. Fehm, “Open problems in the theory of ample fields”, pp. 1–11 in *Geometric and differential Galois theories*, edited by D. Bertrand et al., Sémin. Congr. **27**, Soc. Math. France, Paris, 2013. MR
- [Becker et al. 1979] J. Becker, J. Denef, L. Lipshitz, and L. van den Dries, “Ultraproducts and approximations in local rings, I”, *Invent. Math.* **51**:2 (1979), 189–203. MR

- [Bosch et al. 1990] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, *Ergebnisse der Math. (3)* **21**, Springer, 1990. MR Zbl
- [Bourbaki 2006] N. Bourbaki, *Algèbre commutative: chapitres 5 à 7*, Springer, 2006.
- [Chang and Keisler 1990] C. C. Chang and H. J. Keisler, *Model theory*, 3rd ed., *Studies in Logic and the Foundations of Mathematics* **73**, North-Holland, Amsterdam, 1990. MR
- [Cossart and Piltant 2019] V. Cossart and O. Piltant, “Resolution of singularities of arithmetical threefolds”, *J. Algebra* **529** (2019), 268–535. MR Zbl
- [Denef and Schoutens 2003] J. Denef and H. Schoutens, “On the decidability of the existential theory of  $\mathbb{F}_p[[t]]$ ”, pp. 43–60 in *Valuation theory and its applications* (Saskatoon, 1999), vol. II, edited by F.-V. Kuhlmann et al., *Fields Inst. Commun.* **33**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl
- [EGA IV<sub>2</sub> 1965] A. Grothendieck, “Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, II”, *Inst. Hautes Études Sci. Publ. Math.* **24** (1965), 5–231. MR Zbl
- [EGA IV<sub>4</sub> 1967] A. Grothendieck, “Éléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, IV”, *Inst. Hautes Études Sci. Publ. Math.* **32** (1967), 5–361. MR Zbl
- [Ershov 1967] Y. L. Ershov, “Rational points over Hensel fields”, *Algebra i Logika Sem.* **6:3** (1967), 39–49. MR
- [Ershov 2001] Y. L. Ershov, *Multi-valued fields*, Consultants Bureau, New York, 2001. MR
- [Fehm 2011] A. Fehm, “Embeddings of function fields into ample fields”, *Manuscripta Math.* **134:3-4** (2011), 533–544. MR Zbl
- [Fehm and Jahnke 2017] A. Fehm and F. Jahnke, “Recent progress on definability of Henselian valuations”, pp. 135–143 in *Ordered algebraic structures and related topics*, edited by F. Broglia et al., *Contemp. Math.* **697**, Amer. Math. Soc., Providence, RI, 2017. MR Zbl
- [Fried and Jarden 2008] M. D. Fried and M. Jarden, *Field arithmetic*, 3rd ed., *Ergebnisse der Math. (3)* **11**, Springer, 2008. MR Zbl
- [Fröhlich and Shepherdson 1956] A. Fröhlich and J. C. Shepherdson, “Effective procedures in field theory”, *Philos. Trans. Roy. Soc. London Ser. A* **248** (1956), 407–432. MR Zbl
- [Görtz and Wedhorn 2010] U. Görtz and T. Wedhorn, *Algebraic geometry, I: Schemes with examples and exercises*, Vieweg Teubner, Wiesbaden, Germany, 2010. MR Zbl
- [Greenberg 1966] M. J. Greenberg, “Rational points in Henselian discrete valuation rings”, *Inst. Hautes Études Sci. Publ. Math.* **31** (1966), 59–64. MR Zbl
- [Hironaka 1964] H. Hironaka, “Resolution of singularities of an algebraic variety over a field of characteristic zero, I”, *Ann. of Math. (2)* **79** (1964), 109–203. MR Zbl
- [Jarden 2011] M. Jarden, *Algebraic patching*, Springer, 2011. MR Zbl
- [Kartas 2021a] K. Kartas, “Decidability via the tilting correspondence”, preprint, 2021. arXiv 2001.04424v4
- [Kartas 2021b] K. Kartas, “Diophantine problems over  $\mathbb{Z}^{ab}$  modulo prime numbers”, preprint, 2021. arXiv 2104.06741
- [Kartas 2022] K. Kartas, *Contributions to the model theory of henselian fields*, Ph.D. thesis, University of Oxford, 2022, available at <https://ora.ox.ac.uk/objects/uuid:02e054fc-eb58-48aa-80de-c66692aa5101>.
- [Kartas 2023a] K. Kartas, “Diophantine problems over tamely ramified fields”, *J. Algebra* **617** (2023), 127–159. MR Zbl
- [Kartas 2023b] K. Kartas, “An undecidability result for the asymptotic theory of  $p$ -adic fields”, *Ann. Pure Appl. Logic* **174:2** (2023), 103203. MR Zbl
- [Koenigsmann 2014] J. Koenigsmann, “Undecidability in number theory”, pp. 159–195 in *Model theory in algebra, analysis and arithmetic*, edited by D. Macpherson and C. Toffalori, *Lecture Notes in Math.* **2111**, Springer, 2014. MR Zbl
- [Kuhlmann 2004] F.-V. Kuhlmann, “Places of algebraic function fields in arbitrary characteristic”, *Adv. Math.* **188:2** (2004), 399–424. MR Zbl
- [Kuhlmann 2016] F.-V. Kuhlmann, “The algebra and model theory of tame valued fields”, *J. Reine Angew. Math.* **719** (2016), 1–43. MR Zbl

- [Liu 2002] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics **6**, Oxford University Press, 2002. MR Zbl
- [Martínez-Ranero et al. 2022] C. Martínez-Ranero, J. Utreras, and X. Vidaux, “Existential decidability for addition and divisibility in holomorphy subrings of global fields”, *J. Number Theory* **241** (2022), 504–530. MR Zbl
- [Onay 2018] G. Onay, “ $\mathbb{F}_p((X))$  is decidable as a module over the ring of additive polynomials”, preprint, 2018. arXiv 1806.03123
- [Poonen 2003] B. Poonen, “Hilbert’s tenth problem over rings of number theoretic interest”, lecture notes, 2003, available at <http://www-math.mit.edu/~poonen/papers/aws2003.pdf>.
- [Pop 1996] F. Pop, “Embedding problems over large fields”, *Ann. of Math. (2)* **144**:1 (1996), 1–34. MR Zbl
- [Pop 2010] F. Pop, “Henselian implies large”, *Ann. of Math. (2)* **172**:3 (2010), 2183–2195. MR Zbl
- [Robinson 1965] J. Robinson, “The decision problem for fields”, pp. 299–311 in *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, North-Holland, Amsterdam, 1965. MR Zbl
- [Roquette 1959] P. Roquette, “Abspaltung des Radikals in vollständigen lokalen Ringen”, *Abh. Math. Sem. Univ. Hamburg* **23** (1959), 75–113. MR Zbl
- [Seidenberg 1974] A. Seidenberg, “Constructions in algebra”, *Trans. Amer. Math. Soc.* **197** (1974), 273–313. MR Zbl
- [Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, 1979. MR Zbl
- [Shlapentokh 2007] A. Shlapentokh, *Hilbert’s tenth problem: diophantine classes and extensions to global fields*, New Mathematical Monographs **7**, Cambridge University Press, 2007. MR Zbl
- [Shoenfield 1971] J. R. Shoenfield, *Degrees of unsolvability*, North-Holland Mathematics Studies **2**, North-Holland, Amsterdam, 1971. MR Zbl
- [Temkin 2008] M. Temkin, “Desingularization of quasi-excellent schemes in characteristic zero”, *Adv. Math.* **219**:2 (2008), 488–522. MR Zbl
- [Temkin 2013] M. Temkin, “Inseparable local uniformization”, *J. Algebra* **373** (2013), 65–119. MR Zbl

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