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Let (R, \mathfrak{m}, k) be a strictly local normal k -domain of positive characteristic and P a prime divisor on $X = \text{Spec } R$. We study the Galois category of finite covers over X that are at worst tamely ramified over P in the sense of Grothendieck–Murre. Assuming that (X, P) is a purely F -regular pair, our main result is that every Galois cover $f : Y \rightarrow X$ in that Galois category satisfies that $(f^{-1}(P))_{\text{red}}$ is a prime divisor. We shall explain why this should be thought as a (partial) generalization of a classical theorem due to S.S. Abhyankar regarding the étale-local structure of tamely ramified covers between normal schemes with respect to a divisor with normal crossings. Additionally, we investigate the formal consequences this result has on the structure of the fundamental group representing the Galois category. We also obtain a characteristic zero analog by reduction to positive characteristics following Bhatt–Gabber–Olsson's methods.

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1. Introduction

Carvajal-Rojas and Stäbler [2023] studied the behavior of *pure F -regularity* under finite covers; see [Carvajal-Rojas and Stäbler 2023, Sections 4 and 5]. In the present work, we shall deepen into the consequences of [loc. cit., Theorems 4.8 and 5.12], which explain the behavior of splitting primes, splitting ratios, and test ideals along closed subvarieties under finite covers. In the spirit of [Carvajal-Rojas et al. 2018; Carvajal-Rojas 2022; Jeffries and Smirnov 2022], we shall do so by studying the conditions

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they impose on the structure of covers over purely F -regular singularities that are at worst tamely ramified over the minimal center of F -purity divisor. To this end, consider the following setup.

Setup 1.1. Let $(R, \mathfrak{m}, \ell, K)$ be a strictly local normal ℓ -domain of (equi-)characteristic $p \geq 0$ and dimension ≥ 2 .¹ Set $X := \text{Spec } R$, let $Z \subset X$ be a closed subscheme of codimension ≥ 2 , and set $X^\circ = X \setminus Z$. Let P be a prime divisor on X , whose restriction to X° we denote by P as well. Consider the Galois category $\text{Rev}^P(X^\circ)$ of finite covers over X° that are at worst tamely ramified over P and denote by $\pi_1^{t,P}(X^\circ)$ the corresponding fundamental group; see Section 3.

Terminology 1.2 (local pure log pairs). With notation as in Setup 1.1, we say that (X, P) is a *pure pair* if either $p > 0$ and (X, P) is purely F -regular, or $p = 0$ and $(X, P + \Delta)$ is a purely log terminal pair for some (auxiliary) effective divisor Δ on X with coefficients strictly less than 1; see Section 2 for more details on these definitions.

In positive characteristic, our main result is the following.

Theorem A (Theorem 4.7, Proposition 5.2). *Work in Setup 1.1. If (X, P) is a pure pair, then every connected cover $f: Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$ satisfies that $Q := (f^{-1}(P))_{\text{red}}$ is a prime divisor on Y° and (Y, Q) is a pure pair.*

The proof of this result in positive characteristic is inspired by our previous work [Carvajal-Rojas and Stabler 2023]. The analog in characteristic zero is well-known to experts; see Section 5. In a nutshell, we use [loc. cit., Theorem 4.8] and the symmetry induced by the Galois action to prove that there is only one point of Y lying over the generic point of P . Indeed, any such a point must correspond to the splitting prime of the pair (Y, Q) . Then, one may use [loc. cit., Theorem 5.12] to prove that (Y, Q) is a pure pair. In fact, one may do this quantitatively by means of the transformation rule for splitting ratios in [loc. cit., Theorem 4.8]. Theorem A, in combination with finiteness of local fundamental groups [Carvajal-Rojas et al. 2018; Xu 2014], has very strong consequences on the structure of $\pi_1^{t,P}(X^\circ)$. Concretely:

Theorem B (Theorem 4.12). *Work in Setup 1.1. Suppose that (X, P) is a pure pair of characteristic $p > 0$. Then, there exists an exact sequence of topological groups*

$$\hat{\mathbb{Z}}^{(p)} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

where $\pi_{1,\text{ét}}^P(X^\circ)$ is the fundamental group corresponding to the Galois subcategory of covers that are étale over P . The group $\pi_{1,\text{ét}}^P(X^\circ)$ is finite with order prime-to- p and no more than $\min\{1/r(R, P), 1/s(R)\}$, where $r(R, P)$ is the splitting ratio of (R, P) and $s(R)$ is the F -signature of R . Furthermore:

- (1) *The homomorphism $\hat{\mathbb{Z}}^{(p)} \rightarrow \pi_1^{t,P}(X^\circ)$ is injective if the divisor class of P is a prime-to- p torsion element of $\text{Cl } R$. In this case, the short exact sequence*

$$1 \rightarrow \hat{\mathbb{Z}}^{(p)} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1$$

splits (as topological groups) if and only if the divisor class of P is trivial.

¹Let us recall that a strictly local ring is a henselian local ring with separably closed residue field.

(2) If P is a nontorsion element of $\text{Cl } X$, we have a short exact sequence

$$0 \rightarrow \varinjlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

where $N^P(X^\circ) \subset \mathbb{N}$ is the set of prime-to- p positive integers $n \in \mathbb{N}$ for which there is a divisor D on X such that $P - n \cdot D \in \text{Cl } X$ has prime-to- p torsion and $D|_U$ is Cartier, where $U := X^\circ \setminus Z$. The sequence is split if and only if for every $n \in N^P(X^\circ)$ there are divisors D_n with $D_n|_U$ Cartier such that $P = nD_n \in \text{Cl } X$ which are compatible in the sense that $mD_{nm} = D_n$ in $\text{Cl } X$ for all $n, m \in N^P(X^\circ)$.

Remark 1.3. By [Taylor 2019, Corollary 1.2], we expect that $\min\{1/r(R, P), 1/s(R)\} = 1/s(R)$ in Theorem B. Indeed, Taylor's result establishes that this is the case when P has a prime-to- p torsion divisor class.

Over the complex numbers, we obtain the following analog.

Theorem C (Theorem 5.1). *Work in Setup 1.1. Suppose that (X, P) is a pure pair over \mathbb{C} . Then, there is an exact sequence of topological groups*

$$\hat{\mathbb{Z}} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

where $\pi_{1,\text{ét}}^P(X^\circ)$ is finite; it is the fundamental group corresponding to the Galois subcategory of covers which are étale over P . Additionally:

(1) The homomorphism $\hat{\mathbb{Z}} \rightarrow \pi_1^{t,P}(X^\circ)$ is injective if the divisor class of P is a torsion element of $\text{Cl } R$. In this case, the short exact sequence

$$1 \rightarrow \hat{\mathbb{Z}} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

splits (as topological groups) if and only if the divisor class of P is trivial.

(2) If P is a nontorsion element of $\text{Cl } X$, then we have a short exact sequence

$$0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1.$$

The sequence is split if and only if there is a divisor D with $D|_U$ Cartier such that $P = nD \in \text{Cl } X$.

We shall prove Theorems B and C as formal consequences of the following two statements; see Section 3D and especially Theorem 3.29 for further details:

- Every connected cover $f : Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$ satisfies that $(f^{-1}(P))_{\text{red}}$ is a prime divisor on Y° .
- There exists a universal étale-over- P cover $\tilde{X}^\circ \rightarrow X^\circ$.

In positive characteristic, we give direct proofs of these statements in Section 4B.

1A. Abhyankar’s lemma. We briefly mention here why the results in this work should be thought of as partial generalizations to Abhyankar’s lemma; see Section 3 for further details.

Abhyankar’s lemma [SGA 1 1971, Exposé XIII, §5] is a theorem on the local structure, from the point of view of the étale topology, of finite covers between normal integral schemes that are tamely ramified with respect to a divisor with normal crossings (on the base). It establishes that, locally in the étale topology, any such cover is a quotient of a (generalized) Kummer cover; see [Grothendieck and Murre 1971; Stacks 2005–, Tag 0EYG]. In a sense, Abhyankar’s lemma is a *purity* theorem for Kummer covers. Indeed, by definition and Theorem 3.5, a tamely ramified cover with respect to a divisor is a cover that is Kummer at the codimension 1 étale-germs. Assuming the divisor has normal crossings; which is a regularity condition, Abhyankar’s lemma establishes that such a cover is Kummer at all étale germs.

Let us understand Abhyankar’s lemma with a simple but already fundamental example. With notation as in Setup 1.1, assume that R is regular (or just pure in the sense of [Cutkosky 1995]) and $P = \operatorname{div} f$. A finite cover $R \subset S$ with S a normal local domain is tamely ramified with respect to P if $R_f \subset S_f$ is étale and the generic field extension $K(S)/K(R)$ is tamely ramified with respect to the DVR $R_{(f)}$. An example of such an extension is a Kummer cover: $S = R[T]/(T^n - f)$ with n prime to the characteristic. However, there may exist several non-Kummer tamely ramified covers; see Example 3.14. In general, the connected components of the pullback of a tamely ramified cover $R \subset S$ to $R_{(f)}^{\text{sh}}$ must be Kummer covers and the converse holds provided that $R_f \subset S_f$ is étale; see Theorem 3.5 and Lemma 3.3.

In the above setup, Abhyankar’s lemma establishes that if R/f is regular, then any Galois tamely ramified cover of R with respect to the prime divisor $\operatorname{div} f$ is necessarily Kummer. One may then wonder for what singularities of R/f Abhyankar’s lemma hold. We shall see that in the situation of Theorems B and C, if R/f is either KLT in characteristic zero or strongly F -regular in positive characteristic, then the statement of Abhyankar’s lemma hold. A simpler version of our partial generalization of Abhyankar’s lemma is the following. For the more general statement see Lemma 3.34 (keeping in mind Examples 2.10 and 2.24).

Theorem D (Lemma 3.34, Corollaries 4.17 and 5.4). *With notation as in Setup 1.1, suppose that X is regular and $P = \operatorname{div} f$. If (X, P) is a pure pair, then every Galois tamely ramified cover over X with respect to P is of the form $\operatorname{Spec} R[T]/(T^n - f) \rightarrow X$ for n prime to the characteristic.*

Convention 1.4. If a scheme X or ring R is defined over \mathbb{F}_p , then we denote the e -th iterate of the Frobenius endomorphism by $F^e: X \rightarrow X$, or by $R \rightarrow F_*^e R$. We use the shorthand notation $q := p^e$ to denote the e -th power of the prime p , for instance $F^e: r \mapsto r^q$. We assume all our schemes and rings to be locally noetherian. In positive characteristic we also assume that they are F -finite and hence excellent.

2. Preliminaries on pure log pairs

In this preliminary section, we review the definitions and main aspects of *pure log pairs*. By this, we mean log pairs (X, Δ) that are purely F -regular if defined over a positive characteristic field, or purely log terminal if defined over a characteristic zero field.

2A. Pure F -regularity. Consider $X = \text{Spec } R$ where R is an F -finite normal κ -domain of positive characteristic p and let \mathcal{C} be a Cartier algebra acting on R ; see [Carvajal-Rojas and Stabler 2023, Section 2] for the relevant notions of Cartier algebras and modules in the way we employ them here. Following Schwede [2010], a *center of F -purity (or F -pure center) for (R, \mathcal{C})* is an integral closed subscheme $P = V(\mathfrak{p}) \subset X$ such that \mathfrak{p} is a \mathcal{C} -submodule of R . We say that P is a *minimal center of F -purity for (R, \mathcal{C})* if \mathfrak{p} is a maximal proper \mathcal{C} -submodule. Given a closed point $x \in \text{Spec } R$, we call P a *minimal center of F -purity through x* if $x \in P$.

Following Smolkin [2019, Section 3.1; 2020, Section 4], one defines $\tau_{\mathfrak{p}}(R, \mathcal{C})$ to be the smallest Cartier \mathcal{C} -submodule of R not contained in \mathfrak{p} , which exists provided that $\mathcal{C}_e(R) \not\subset \mathfrak{p}$ for some $e > 0$ (this condition is referred to as *nondegeneracy*); see [Takagi 2008; 2010], compare to [Carvajal-Rojas and Stabler 2023, Section 5.2]. By [Smolkin 2019, Proposition 3.1.14], we see that P is a minimal center of F -purity for (R, \mathcal{C}) if and only if $\tau_{\mathfrak{p}}(R, \mathcal{C}) + \mathfrak{p} = R$. When $\tau_{\mathfrak{p}}(R, \mathcal{C}) = R$, one says that (R, \mathcal{C}) is *purely F -regular along P* . For the generalization to the case \mathfrak{p} is radical, see [Carvajal-Rojas and Stabler 2023, Lemma 5.11].

In the local case, minimal centers of F -purity exist, are unique, and admit a simpler description. Indeed, if (R, \mathfrak{m}) is local then *the* minimal F -pure center of (R, \mathcal{C}) is given by the closed subscheme cut out by the splitting prime $p(R, \mathcal{C})$; see [Schwede 2010, Remark 4.4]. Further, we see that $\tau_{\mathfrak{p}}(R, \mathcal{C}) = R$ if $\mathfrak{p} = p(R, \mathcal{C})$. In other words, in the local case, (R, \mathcal{C}) is always purely F -regular along its (unique) minimal F -pure center. We are implicitly assuming that $p(R, \mathcal{C})$ is a proper ideal of R (i.e., (R, \mathcal{C}) is F -pure).

Still assuming R is local, let $P = V(\mathfrak{p}) \subset X$ be the closed subscheme cut out by a prime ideal $\mathfrak{p} \subset R$. Let $\mathcal{C}_R^{[P]} \subset \mathcal{C}_R$ be the Cartier subalgebra consisting of P -compatible p^{-e} -linear maps. In other words, $\varphi \in \mathcal{C}_{e,R}$ belongs to $\mathcal{C}_{e,R}^{[P]}$ if and only if $\varphi(F_*^e \mathfrak{p}) \subset \mathfrak{p}$. Since the splitting prime $p(R, \mathcal{C}_R^{[P]})$ is the unique largest prime ideal compatible with all the p^{-e} -linear maps in $\mathcal{C}_R^{[P]}$, we have an inclusion

$$\mathfrak{p} \subset p(R, \mathcal{C}_R^{[P]}).$$

This inclusion is an equality exactly when P is the minimal F -pure center of $\mathcal{C}_R^{[P]}$. In particular, we may say that P is a *minimal F -pure center of X* (with no explicit reference to a Cartier algebra) to say that it corresponds to the splitting prime of some Cartier algebra — necessarily $\mathcal{C}_R^{[P]}$. In that case, $(R, \mathcal{C}_R^{[P]})$ is purely F -regular along P .

In this paper, we are interested in minimal F -pure center divisors. In this case, we have:

Proposition 2.1. *Let R be a local normal κ -domain with $P = V(\mathfrak{p})$ a prime divisor on $\text{Spec } R$. Then, we have $\mathcal{C}_R^{[P]} = \mathcal{C}_R^P$, where \mathcal{C}_R^P is the Cartier algebra corresponding to the divisor $\Delta = P$; see [Schwede 2009].²*

Proof. Observe that membership in these Cartier algebras can be checked (by localizing) at \mathfrak{p} , where these Cartier algebras are obviously the same. \square

²It also coincides with $\mathcal{C}_R^{\mathfrak{p}}$ as in [Blickle 2013, Section 3.3].

Notation 2.2. With notation as in Proposition 2.1, we write $p(X, P)$ and $r(X, P)$ (or with X replaced by R) to denote the splitting prime and splitting ratio of the pair $(R, \mathcal{C}_R^{[P]})$. Moreover, we shall write \mathcal{C}_R^P instead of $\mathcal{C}_R^{[P]}$.

Definition 2.3 (purely F -regular local pair). With notation as in Proposition 2.1, we say that the pair (X, P) (or with R in place of X) is *purely F -regular* if P is a minimal F -pure center prime divisor on X .³

Remark 2.4. With notation as in Proposition 2.1, notice that (X, P) is a purely F -regular pair if and only $\tau_p(R, P) = R$, i.e., if (X, P) is purely F -regular along P .

We observe that X must have “mild” singularities to admit a purely F -regular divisor.

Proposition 2.5. *Let (X, P) be a purely F -regular local pair, then R (or X) is strongly F -regular (with respect to its full Cartier algebra \mathcal{C}_R). More generally, if A is a local domain with an action by some Cartier algebra $\mathcal{A} \subset \mathcal{C}_A$, and $C = V(\mathfrak{c})$ a minimal F -pure center prime divisor for $\mathcal{A}^C := \mathcal{A} \cap \mathcal{C}_A^C$ and (A, \mathcal{A}) is F -regular at the generic point of C , then (A, \mathcal{A}) is F -regular.*

Proof. Since $\mathcal{A}^C \subset \mathcal{A}$, we have that $p(A, \mathcal{A}) \subset p(A, \mathcal{A}^C) = \mathfrak{c}$, where the equality follows from \mathfrak{c} being a prime maximal center of F -purity. Since \mathfrak{c} has height 1, $p(\mathcal{A})$ is either 0 or \mathfrak{c} . If $p(\mathcal{A}) = 0$, we are done. If $p(\mathcal{A}) = \mathfrak{c}$, then (A, \mathcal{A}) is not F -regular at the generic point of C , contradicting our hypothesis. To see the first statement follows from the last one, notice that, since R is normal, (R, \mathcal{C}_R) is F -regular at the generic point of P . \square

Remark 2.6. In Proposition 2.5, the normality hypothesis on R is necessary. Indeed, we may consider the Whitney’s umbrella singularity as a counterexample; see [Blickle et al. 2012, Section 4.3.2].

Finally, we recall the global-to-local passage for F -pure centers.

Proposition 2.7. *Let X be the spectrum of a normal \mathcal{k} -domain and let \mathcal{C} be a Cartier algebra on X . Let $P = V(\mathfrak{p})$ be a minimal center of F -purity passing through a geometric point $\bar{x} \rightarrow X$, then $\mathfrak{p} \cdot \mathcal{O}_{X, \bar{x}}^{\text{sh}}$ is the splitting prime of the Cartier $\mathcal{O}_{X, \bar{x}}^{\text{sh}}$ -algebra $\mathcal{O}_{X, \bar{x}}^{\text{sh}} \otimes \mathcal{C}$.*

Proof. See Proposition A.3 in the Appendix. \square

2A1. *Some examples of purely F -regular pairs.* We provide next some examples of purely F -regular pairs. Our method to prove that a given pair is purely F -regular is the following.

Lemma 2.8. *Let R be a normal local domain, $\mathfrak{p} \subset R$ be a prime ideal (not necessarily of height 1), and set $P = V(\mathfrak{p})$. Then, \mathfrak{p} is the splitting prime of $\mathcal{C}_R^{[P]}$ if and only if R/\mathfrak{p} is F -regular with respect to the induced action of $\mathcal{C}_R^{[P]}$. In that case, the splitting ratio of (R, P) is the F -signature of R/\mathfrak{p} with respect to $\mathcal{C}_R^{[P]}$.*

³Note that this is called *divisorially F -regular* in [Hara and Watanabe 2002]. However, we use the *purely F -regular* terminology to emphasize the connections with purely log terminal (PLT) singularities and avoid confusions with *divisorially log terminal* singularities (DLT).

Proof. Note that \mathfrak{p} is the splitting prime of $\mathcal{C}_R^{[P]}$ if and only if R/\mathfrak{p} viewed as an $\mathcal{C}_R^{[P]}$ -module is simple. However, we may equivalently view R/\mathfrak{p} as an $\overline{\mathcal{C}_R^{[P]}}$ -module; compare to [Blickle 2013, discussion before Lemma 2.20]. See [Blickle et al. 2012, Lemma 2.13]. \square

We shall also need the following observation.

Remark 2.9. Consider the category of finite type κ -algebras for some F -finite field κ . Fix an isomorphism $\lambda: \kappa \rightarrow F^1\kappa$ with adjoint $\kappa: F_*\kappa \rightarrow \kappa$. If we have two Cartier linear maps $\Phi, \Psi: F_*^e R \rightarrow R$ for some finite type κ -algebra R , then, by choosing a presentation $S = \kappa[x_1, \dots, x_n] \rightarrow R$ and via [Fedder 1983], we reduce the problem of whether $\Phi = \Psi$ to a computation in the polynomial ring S . Indeed, note that λ induces an isomorphism $f^1\kappa = \omega_S \rightarrow F^1\omega_S$, where $f: \text{Spec } S \rightarrow \text{Spec } \kappa$ is the structural map. Identifying ω_S with S , we obtain an isomorphism $\Sigma: S \rightarrow F^1S$. By [Stäbler 2017, Lemma 4.1], the adjoint of Σ is given by

$$\xi x_1^{i_1} \cdots x_n^{i_n} \mapsto \kappa(\xi) x_1^{(i_1+1)/q} \cdots x_n^{(i_n+1)/q},$$

with the usual convention that $x_i^{a/b}$ is zero whenever the exponent is not an integer. Now, by adjunction $\text{Hom}_R(F_*^e R, R) = \text{Hom}_R(R, F^{e^1} R)$ and by our choice of isomorphism Σ , we have that $\text{Hom}_R(R, F^{e^1} R) \cong \text{Hom}_R(R, R)$, and, by making this identification, Σ induces the identity so that the adjoint of Σ is a generator of $\text{Hom}_R(F_*^e R, R)$.

In this way, if we want to check that two Cartier linear maps of a finite type κ -algebra R coincide, we may reduce, via a choice of presentation and Fedder's criterion to a comparison of two Cartier linear maps in a polynomial ring. For those to coincide in turn, we choose any basis B of $F_*\kappa$ as a κ -module and then just need to check that they agree on $b \cdot x_1^{i_1} \cdots x_n^{i_n}$, where $b \in B$ and $0 \leq i_j \leq q - 1$.

This line of reasoning is also preserved if we pass to completions. Indeed, by [Stacks 2005–, Lemma 0394], we may identify $(F_*R)^\wedge$ with F_*R^\wedge . Since both are finite free modules, the claim is clear.

Example 2.10 (purely F -regular pairs on a regular ambient). Let R be a *regular* local ring. Recall that regular local rings are UFD; see [Stacks 2005–, Lemma 0AG0]. In particular, any prime divisor on $\text{Spec } R$ is principal [Matsumura 1980, Section 19, Theorem 47]. Let $\mathfrak{p} = (f)$ be a height 1 prime ideal of R with corresponding prime divisor P . As an immediate application of Lemma 2.8 and Fedder's criterion [1983], we see that (R, P) is purely F -regular if and only if R/f is a strongly F -regular ring. Moreover, in this case, one has $r(R, P) = s(R/f)$.

Example 2.11 (graded hypersurfaces). Let $R = \kappa[[z, x_0, x_1, \dots, x_d]]/(z^n - x_0h)$ be a normal hypersurface over a perfect field κ , where h is an irreducible weighted polynomial in the variables x_1, \dots, x_d ; see [Singh and Spiroff 2007]. Then, $\text{Cl } R \cong \mathbb{Z}/n\mathbb{Z}$ and the divisor class of (z, h) is a generator for $\text{Cl } R$; see [loc. cit., Corollary 3.4]. Letting P be the prime divisor corresponding to (z, h) , we claim the following.

Claim 2.12. *The pair (R, P) is purely F -regular if $A := \kappa[[x_1, \dots, x_d]]/h$ is strongly F -regular. In that case, $r(R, P) \geq s(A)/n$, and $r(R, P) = 1/n$ if A is regular.*

Proof of claim. Let $S := \kappa[[z, x_0, x_1, \dots, x_d]]$ and $f := z^n - x_0h$. By Fedder's criterion [1983], we have that $\mathcal{C}_{e,R}$ is generated by the reduction of $\Phi^e \cdot f^{q-1} \in \mathcal{C}_{e,S}$ to R , where Φ denotes the Frobenius trace on S . In other words, $\mathcal{C}_R = \overline{\mathcal{C}_R^\phi}$ where $\phi := \overline{\Phi \cdot f^{p-1}}$. Having Proposition 2.1 in mind, we recall that \mathcal{C}_R^P is given, in degree e , by all maps $\phi^e \cdot g$ such that $\text{val}_{\mathfrak{p}} g \geq q - 1$. Note that z is a uniformizer for $R_{\mathfrak{p}}$ and $\text{val}_{\mathfrak{p}} h = n$. In particular, \mathcal{C}_R^P contains the maps $\phi^e \cdot z^i h^j$ where $i + nj = q - 1$. Therefore, the reduction of \mathcal{C}_R^P to $R/\mathfrak{p} \cong \kappa[[x_0, x_1, \dots, x_d]]/h$ contains the maps $\overline{\phi^e \cdot z^i h^j}$ with $i + nj = q - 1$. However, these maps are the reductions of $\Phi^e \cdot z^i h^j f^{q-1}$ to $R/\mathfrak{p} = S/(z, h)$, and we have that

$$z^i h^j f^{q-1} \equiv (-1)^{q-1-j} \binom{q-1}{j} z^{q-1} x_0^{q-1-j} h^{q-1} \pmod{(z^q, h^q)}$$

for all $i + nj = q - 1$. In other words, the reduction of \mathcal{C}_R^P to $R/\mathfrak{p} \cong \kappa[[x_0, x_1, \dots, x_d]]/h \cong S/(z, h)$ contains, in degree e , the reductions of $\Phi^e \cdot z^{q-1} x_0^{q-1-j} h^{q-1}$ for all $j \leq (q-1)/n$. Alternatively, if Ψ is the Frobenius trace for $\kappa[[x_0, x_1, \dots, x_d]]$, we have that the reduction of \mathcal{C}_R^P to $\kappa[[x_0, x_1, \dots, x_d]]/h$ contains the reductions of $\Psi^e \cdot x_0^{q-1-j} h^{q-1}$. Therefore, we have

$$s(R/\mathfrak{p}, \overline{\mathcal{C}_R^P}) \geq \frac{1}{n} \cdot s(A).^4$$

Consequently, by applying Lemma 2.8, we conclude that $\mathfrak{p} = p(R, \mathcal{C}_R^P)$, and furthermore

$$r(R, P) = s(R/\mathfrak{p}, \overline{\mathcal{C}_R^P}) \geq \frac{1}{n} \cdot s(A).$$

To see this is an equality if A is regular, we may use the transformation rule for splitting ratios [Carvajal-Rojas and Stabler 2023, Theorem 4.8]. Indeed, suppose for sake of contradiction that the inequality is strict, and let \tilde{R} be the Veronese-type cyclic cover given by P . That is, $\tilde{R} = \bigoplus_{i=0}^{n-1} \mathfrak{p}^{(i)}$. It is not difficult to see that $\tilde{R} = R[x_0^{1/n}, h^{1/n}]$ and the only prime in \tilde{R} lying over \mathfrak{p} is $\tilde{\mathfrak{p}} = (h^{1/n})$; whose corresponding prime divisor we denote by \tilde{P} . Therefore, $\tilde{\mathfrak{p}}$ must be the splitting prime of the pullback of \mathcal{C}_R^P along the cover $R \subset \tilde{R}$. Hence, the transformation rule for splitting ratios yields $r(\tilde{R}, \tilde{P}) = n \cdot r(R, P) > s(A) = 1$, which is a contradiction. \square

Example 2.13. Let $R = \kappa[[x, y, z, w]]/(xy - zw)$. Recall that the divisor class group of R is free of rank 1; see [Hartshorne 1977, II, Exercise 6.5]. Moreover, the divisor class of the height-1 prime ideal $\mathfrak{p} = (x, z)$ is a generator of $\text{Cl } R$. We claim that $P = V(\mathfrak{p})$ is a minimal F -pure center.

Claim 2.14. *The pair (R, P) is purely F -regular and $r(R, P) \geq \frac{1}{2}$.*

Proof of claim. Let $S = \kappa[[x, y, z, w]]$ and $f = xy - zw$. We use Fedder's criterion [1983] to conclude that $\mathcal{C}_{e,R}$ is generated by the reduction of $\Phi^e \cdot f^{q-1} \in \mathcal{C}_{e,S}$ to R ; where Φ denotes the Frobenius trace on S . That is, $\mathcal{C}_R = \overline{\mathcal{C}_R^\phi}$ where $\phi := \overline{\Phi \cdot f^{p-1}}$. With Proposition 2.1 in mind, recall that \mathcal{C}_R^P is given, in degree e , by all maps $\phi^e \cdot g$ such that $\text{val}_{\mathfrak{p}} g \geq q - 1$. In particular, we have that $\mathcal{C}_{e,R}^P$ contains all the maps $\phi^e \cdot x^i z^j$ such that $i + j = q - 1$. Thus, the reduction of \mathcal{C}_R^P to $R/\mathfrak{p} = \kappa[[y, w]]$ contains, in degree e , the

⁴To see this, note that we may work in the polynomial case as completions have no bearing on the value of F -signatures; see [Yao 2006] or [Carvajal-Rojas et al. 2021, Section 3]. Then, the result follows from the behavior of F -signatures with respect to tensor products; see [Carvajal-Rojas and Smolkin 2020, Proposition 5.5] for instance.

maps $\bar{\phi}^e \cdot x^i y^j$ such that $i + j = q - 1$. Notice that these maps are, respectively, the reductions of the map $\Phi \cdot x^i y^j f^{q-1}$. Nonetheless, one readily sees that

$$x^i y^j f^{p-1} \equiv (-1)^i \binom{p-1}{j} (xz)^{q-1} y^j w^i \pmod{(x^p, z^p)}.$$

Therefore, $\bar{\phi}^e \cdot x^i y^j$, $i + j = q - 1$, is, up to premultiplication by units in κ , the dual map of $F_*^e y^i w^j$ with respect to the free basis of $F_*^e R/\mathfrak{p}$ over R/\mathfrak{p} given by $\{F_*^e y^k w^l \mid 0 \leq k, l \leq q - 1\}$. That is, $\bar{\phi}^e \cdot x^i z^j = \Psi^e \cdot y^j w^i$ where Ψ denotes the Frobenius trace of $R/\mathfrak{p} = \kappa[[y, w]]$. Hence,

$$s(\kappa[[y, w]], \overline{\mathcal{C}_R^P}) \geq \text{area}([0, 1]^{\times 2} \cap \{(y, w) \in \mathbb{R}^2 \mid y + w \geq 1\}) = \frac{1}{2}.$$

This proves the claim by Lemma 2.8. \square

Example 2.15. Let $A := \kappa[[u, v, w, x, y, z]]$ and $I := (\Delta_1, \Delta_2, \Delta_3)$ where $\Delta_1 := vz - wy$, $\Delta_2 := wx - uz$, and $\Delta_3 := uy - vx$. Let $R = A/I$. We claim that the prime divisor P defined by the height-1 prime ideal $\mathfrak{p} := (u, v, w)$ is a minimal F -pure center, and moreover $r(R, P) \geq \frac{1}{6}$. We use Lemma 2.8. To this end, we recall that $\mathcal{C}_{e,R}$ was explicitly computed in [Katzman et al. 2014, Proposition 5.1]. Indeed, for nonnegative integers s, t such that $s + t \leq q - 1$, one writes

$$y^s z^t (\Delta_2 \Delta_3)^{q-1} \equiv x^{s+t} f_{s,t} \pmod{I^{[q]}},$$

for some $f_{s,t}$, which is well-defined mod $I^{[q]}$. Then,

$$I^{[q]} : I = I^{[q]} + (f_{s,t} \mid s, t \geq 0, s + t \leq q - 1).$$

Thus, by Fedder's criterion [1983], $\mathcal{C}_{e,R}$ is generated by $\Phi^e \cdot f_{s,t}$, where Φ is a Frobenius trace associated to A . We choose $f_{0,0}$ to be $(\Delta_2 \Delta_3)^{q-1}$. In fact, we have that $I^{2(q-1)} \subset I^{[q]} : I$. In particular, we have the following relations:

$$y^s z^t f_{0,0} \equiv x^{s+t} f_{s,t} \pmod{I^{[q]}}. \quad (2.15.1)$$

Let $\phi_{s,t}^e$ be the map in $\mathcal{C}_{e,R}$ induced by $\Phi^e \cdot f_{s,t}$ for $s + t \leq q - 1$.

Claim 2.16. \mathcal{C}_R^P contains the maps $\{\phi_{s,t}^e \cdot u^l v^m w^n \mid l + m + n = q - 1, s + t \leq q - 1\}$.

Proof of claim. Observe that, up to premultiplications by units, all the maps $\phi_{s,t}^e$ induce the same map after we localize at $\mathfrak{p} = (u, v, w)$ by (2.15.1). Note that $R_{\mathfrak{p}}$ is a DVR so that $\mathcal{C}_{e,R_{\mathfrak{p}}}$ is principally generated. As the $\phi_{s,t}^e$ generate $\mathcal{C}_{e,R}$, any $\phi_{s,t}^e$ is a generator of $\mathcal{C}_{e,R_{\mathfrak{p}}}$. Now, any element u, v, w is a uniformizer in $R_{\mathfrak{p}}$. To verify the claim, we may localize at \mathfrak{p} , but then $\phi_{s,t}^e u^l v^m w^n$ is of the form $\kappa \cdot t^{l+m+n}$ where κ is a generator of $\text{Hom}(F_*^e R_{\mathfrak{p}}, R_{\mathfrak{p}})$ and t is a uniformizer. This map is \mathfrak{p} -compatible if and only if $m + l + n \geq q - 1$. \square

Next, observe that $R/\mathfrak{p} \cong \kappa[[x, y, z]]$, with Frobenius trace denoted by Ψ . By Remark 2.9, we may choose Ψ in such a way that $\phi_{s,t}^e$ and Ψ^e are induced by the same map $\kappa : F_* \kappa \rightarrow \kappa$. Thus, for all $s + t \leq q - 1$ and all $l + n + m = q - 1$, we have that $\phi_{s,t}^e \cdot u^l v^m w^n$ restricts to a map in $\mathcal{C}_{e,R/\mathfrak{p}}$; say

$\varphi_{s,t}^e \cdot u^l v^m w^n$. Hence, we have an equality $\varphi_{s,t}^e \cdot u^l v^m w^n = \Psi^e \cdot a_{s,t;l,m,n}$ for a uniquely determined $a_{s,t;l,m,n} \in \mathcal{K}[[x, y, z]]$, which are explicitly described as follows:

Claim 2.17. *Let $l, m, n; s, t$ be nonnegative integers such that $l + m + n = q - 1$, $s + t \leq q - 1$. Let us set $q - 1 - s - t =: r \geq 0$, so that $r + s + t = q - 1$. Then, we have that $a_{s,t;l,m,n} = 0$ unless one of the following four triples $(l + r, m + s, n + t)$, $(l + r - q, m + s, n + t)$, $(l + r, m + s - q, n + t)$, $(l + r, m + s, n + t - q)$ belongs to $\{0, \dots, q - 1\}^{\times 3}$, in which case*

$$a_{s,t;l,m,n} = \xi \cdot x^{l+r} y^{m+s} z^{n+t}$$

for some unit $\xi \in \mathbb{F}_p^\times \subset \mathcal{K}^\times$.

Proof of claim. First of all, note that

$$\begin{aligned} f_{0,0} &= \Delta_2^{q-1} \Delta_3^{q-1} = \left(\sum_{a+b=q-1} (-1)^b \binom{q-1}{a} (wx)^a (uz)^b \right) \left(\sum_{c+d=q-1} (-1)^d \binom{q-1}{c} (uy)^c (vx)^d \right) \\ &= \sum_{\substack{a+b=q-1 \\ c+d=q-1}} (-1)^{b+d} \binom{q-1}{a} \binom{q-1}{c} u^{b+c} v^d w^a x^{a+d} y^c z^b. \end{aligned}$$

Therefore,

$$u^l v^m w^n f_{0,0} \equiv - \binom{q-1}{m} \binom{q-1}{n} u^{q-1} v^{q-1} w^{q-1} x^{l+q-1} y^m z^n \pmod{\mathfrak{p}^{[q]}}. \quad (2.17.1)$$

Indeed, after multiplying by $u^l v^m w^n$, every summand vanishes modulo $\mathfrak{p}^{[q]}$ except for the summands where simultaneously $l + b + c \leq q - 1$, $m + d \leq q - 1$, and $n + a \leq q - 1$. However, given the constraints $a + b = q - 1$ and $c + d = q - 1$, we have that

$$(l + b + c) + (m + d) + (n + a) = 3(q - 1).$$

Hence, $l + b + c, m + d, n + a = q - 1$, and also $a + d = l + q - 1$. In particular, $m = c$ and $n = b$. Set $\xi := - \binom{q-1}{m} \binom{q-1}{n} \in \mathcal{K}^\times$.

On the other hand, for $0 \leq i, j, l \leq q - 1$ we have that

$$\begin{aligned} u^l v^m w^n x^i y^j z^k f_{s,t} &= \frac{1}{x^q} u^l v^m w^n x^{i+q-s-t} y^j z^k x^{s+t} f_{s,t} \\ &= \frac{1}{x^q} u^l v^m w^n x^{i+r+1} y^{j+s} z^{k+t} f_{0,0} \\ &\equiv \frac{\xi}{x^q} u^{q-1} v^{q-1} w^{q-1} x^{q+i+r+l} y^{j+s+m} z^{k+t+n} \pmod{\mathfrak{p}^{[q]}} \\ &\equiv \xi u^{q-1} v^{q-1} w^{q-1} x^{i+r+l} y^{j+s+m} z^{k+t+n} \pmod{\mathfrak{p}^{[q]}}. \end{aligned}$$

Therefore,

$$\Phi^e(u^l v^m w^n x^i y^j z^k f_{s,t}) \equiv \xi \Phi^e(u^{q-1} v^{q-1} w^{q-1} x^{i+r+l} y^{j+s+m} z^{k+t+n}) \pmod{\mathfrak{p}}$$

Next, we observe that this element is $0 \pmod{\mathfrak{p}}$ unless

$$i + r + l, j + s + m, k + t + n \equiv q - 1 \pmod{q}.$$

Since all these three sums are at most $3(q-1)$, we then have

$$\begin{cases} i+r+l = q-1 + \alpha q, \\ j+s+m = q-1 + \beta q, \\ k+t+n = q-1 + \gamma q, \end{cases}$$

for some $\alpha, \beta, \gamma \in \{0, 1\}$. However, if we add these equations together, we obtain

$$i+j+k+2(q-1) = 3(q-1) + (\alpha + \beta + \gamma)q.$$

Equivalently,

$$i+j+k = q-1 + (\alpha + \beta + \gamma)q.$$

Being $i+j+k$ at most $3(q-1)$, this forces $\alpha + \beta + \gamma \in \{0, 1\}$. Hence, α, β, γ are either all 0 or one of them is 1 while the other two are 0. In the first case, we then have

$$i+r+l, j+s+m, k+t+n = q-1.$$

Therefore, in this case, we have that $\Phi^e(u^l v^m w^n x^i y^j z^k f_{s,t}) \equiv 0 \pmod{\mathfrak{p}}$ unless

$$i = q-1 - (r+l), j = q-1 - (s+m), k = q-1 - (t+n) \geq 0.$$

In that case, $\Phi^e(u^l v^m w^n x^i y^j z^k f_{s,t}) \equiv \xi \pmod{\mathfrak{p}}$, and so $a_{s,t;l,m,n} = \xi x^{r+l} y^{s+m} z^{t+n}$ (whenever $r+l, s+m, t+n \geq q-1$).

Let us consider now the remaining three cases, i.e., $(\alpha, \beta, \gamma) \in \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. By symmetry, it suffices to consider $(\alpha, \beta, \gamma) = (1, 0, 0)$. In this case, we have that the element $\Phi^e(u^l v^m w^n x^i y^j z^k f_{s,t})$ vanishes modulo \mathfrak{p} unless

$$q-1 \geq i = q-1 + q - (r+l) \geq 0 \quad \text{and} \quad j = q-1 - (s+m), k = q-1 - (t+n) \geq 0,$$

equivalently

$$0 \leq (r+l) - q \leq q-1 \quad \text{and} \quad j = q-1 - (s+m), k = q-1 - (t+n) \geq 0,$$

which implies $\Phi^e(u^l v^m w^n x^i y^j z^k f_{s,t}) \equiv \xi x \pmod{\mathfrak{p}}$. In this case, $a_{s,t;l,m,n} = \xi x^{r+l} y^{s+m} z^{t+n}$. \square

Let us analyze which maps the first case $(l+r, m+s, n+t) \in \{0, \dots, q-1\}^{\times 3}$ of Claim 2.17 yields. Note that the map from the set

$$\{(l, m, n; r, s, t) \in \{0, \dots, q-1\}^{\times 6} \mid l+m+n, r+s+t = q-1 \text{ and } l+r, m+s, n+t \leq q-1\}$$

to the set

$$\{(i, j, k) \in \{0, \dots, q-1\}^{\times 3} \mid i+j+k = 2(q-1)\}$$

defined by

$$(l, m, n; r, s, t) \mapsto (l+r, m+s, n+t)$$

is surjective. Indeed, taking $l = s = 0$ and given $0 \leq r, m \leq q - 1$, we obtain $2(q - 1) = r + t + m + n$ or, put differently, $2(q - 1) - r - m = t + n$. Thus, we see that this case yields the maps $\Psi^e \cdot x^i y^j z^k$ with $i + j + k = 2(q - 1)$. In other words, we obtain the Cartier algebra given by the pair $(\kappa[[x, y, z]], (x, y, z)^2)$.

For the remaining three cases of Claim 2.17, we obtain the maps

$$x \cdot \Psi^e \cdot x^{r+l-q} y^{s+m} z^{t+n}, \quad y \cdot \Psi^e \cdot x^{r+l} y^{s+m-q} z^{t+n}, \quad z \cdot \Psi^e \cdot x^{r+l} y^{s+m} z^{t+n-q}$$

where, respectively, $(r + l - q, s + m, t + n) \in \{0, \dots, q - 1\}$, $(r + l, s + m - q, t + n) \in \{0, \dots, q - 1\}$, $(r + l, s + m, t + n - q) \in \{0, \dots, q - 1\}$. However, these are all nonsurjective maps.

In conclusion, we obtain

$$s(R/\mathfrak{p}, \overline{\mathcal{C}}_R^P) \geq \text{volume}([0, 1]^{\times 3} \cap \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \geq 2\}) = \frac{1}{6} > 0,$$

where we use [Blickle et al. 2012, Theorem 4.20] for the inequality. Hence, $r(R, P) \geq \frac{1}{6}$ and (R, P) is purely F -regular.

Remark 2.18. In Example 2.15, it would be interesting to fully compute \mathcal{C}_R^P to check whether or not $r(R, P) = 1/6$. The issue is that we cannot apply Fedder's criterion for R since R is not regular. One may apply Fedder's criterion to $I + \mathfrak{p}$ in A to work around this.

Question 2.19. Let $C_{r,s}$ be the cone singularity given by the Segre embedding of $\mathbb{P}_{\kappa}^r \times \mathbb{P}_{\kappa}^s$. The F -signatures of these toric rings were computed in [Singh 2005] and it is well-known that $\text{Cl } C_{r,s} \cong \mathbb{Z}$. In fact, $C_{r,s}$ is a determinantal ring. Let $S = \kappa[[x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]]$ be the power series ring in the $m \times n$ matrix of variables $(x_{i,j})$, and let I_t be the ideal generated by the $t \times t$ minors of $(x_{i,j})$ ($2 \leq t \leq \min\{m, n\}$). The quotient ring $R = R(m, n, t) = S/I_t$ is called a determinantal ring. We observe that $C_{r,s}$ is none other than $R(r + 1, s + 1, 2)$. Moreover, if P is the prime divisor on $\text{Spec } C_{r,s}$ given by $\mathfrak{p} = (x_{1,1}, \dots, x_{1,s+1})$,⁵ then the divisor class of P is a free generator of $\text{Cl } C_{r,s}$. Based on the previous examples, it is natural to ask whether the pair $(C_{r,s}, P)$ is purely F -regular and if so what its splitting ratio is. More generally, if R is an arbitrary determinantal ring, we have that $\text{Cl } R$ is freely generated by P the divisor class of the height-1 prime ideal \mathfrak{p} generated by the $t - 1$ size minors of any set of $t - 1$ rows (or columns); see [Bruns and Vetter 1988, Corollary 8.4]. We ask the same question as before for the pair (R, P) . Note that in order to answer to these questions along the same ideas we had for $C_{1,1}$ and $C_{1,2}$, a good understanding of the colon ideal $I_t^{[q]} : I_t$ is needed. Nonetheless, to the best of the authors' knowledge, very little is known about this. The authors believe a different approach is required.

2B. Purely log terminal pairs. We refer the reader to [Kollar and Mori 1998] for a detailed exposition on log canonical singularities and to [Ambro 1999] for the notion of (minimal) log canonical centers. We will, however, briefly review these notions here. Let (X, Δ) be a log pair defined over an algebraically closed field of characteristic zero. Fix a log resolution $\pi : Y \rightarrow (X, \Delta)$ and write $\Delta_Y = \pi^*(K_X + \Delta) - K_Y$. The pair (X, Δ) is *log canonical* (LC) if the coefficients of Δ_Y are ≤ 1 . The pair (X, Δ) is called *purely*

⁵In fact, any ideal generated by either a fixed column or row of variables.

log terminal (PLT) if it is LC and the exceptional components of Δ_Y have coefficients < 1 . We say that (X, Δ) is *Kawamata log terminal* (KLT) if all coefficients of Δ_Y are < 1 . A prime divisor P on X is called an *LC center* if the coefficient a_P of the strict transform of P in Δ_Y is ≥ 1 . Since the multiplier ideal $\mathcal{J}(X, \Delta)$ is given by $\pi_* \mathcal{O}_Y(K_Y - \lfloor \pi^*(K_X + \Delta) \rfloor)$, this is equivalent to $\mathcal{O}_X(-P) \supset \mathcal{J}(X, \Delta)$.

In analogy to Proposition 2.7, we recall the global-to-local passage is for LC centers. Let (X, Δ) be a log canonical pair, $\dim X \geq 2$. Let P be an LC center going through a closed point $x \in X$. In studying $\mathcal{O}_{X,x}^{\text{sh}}$, we are free to replace X by any open neighborhood U of x and Δ by Δ_U . In particular, we may assume that (X, Δ) is purely log terminal. Indeed, we may write $\Delta_Y = E_1 + \cdots + E_n + \sum_E a_E E$, for some n and such that $a_E < 1$. Note that one of the E_i , say E_1 is P . By the assumption that P is a divisor and the minimal LC center through x , the other divisors E_i do not contain x . Hence, replacing X by a suitable neighborhood U of x , we may assume that (X, Δ) is PLT and moreover $\lfloor \Delta \rfloor = P$ is a prime divisor going through x .⁶ Thus, we may work in the following setup.

Setup 2.20. Let (X, Δ) be a PLT log pair of dimension at least 2, such that $\lfloor \Delta \rfloor = P$ is a prime divisor going through a closed point $x \in X$. We set $X_x^\circ = \text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \setminus Z$, where Z is some closed subset of codimension ≥ 2 . We denote by P the pullback of P to U .

The following is analogous to Proposition 2.5 and well-known to experts; see [Kollár and Mori 1998, Proposition 2.43].

Proposition 2.21. *Let $(X, \Delta = \sum a_i \Delta_i)$ be a PLT pair with X quasiprojective and $0 \leq a_i \leq 1$. Then there is a \mathbb{Q} -Cartier \mathbb{Q} -divisor Δ' such that the pair $(X, \Delta + \varepsilon \Delta')$ is KLT for all rational $0 < \varepsilon \ll 1$*

Proof. Let $m > 0$ be so that $m\Delta$ is integral. Since X is quasiprojective, there is an ample divisor H . Choose $n \gg 0$ so that $\mathcal{O}_X(nH + m\Delta)$ is globally generated. As the base locus of the linear system $|nH + m\Delta|$ is empty, we find an element D of this linear system having no component in common with Δ . Set $\Delta' = \frac{1}{m}D - \Delta$. Then, $K_X + \Delta + \varepsilon \Delta'$ is \mathbb{Q} -Cartier since Δ' is so: $m \cdot \Delta' = D - m\Delta \sim nH$. Note that $\Delta + \varepsilon \Delta' \geq 0$ for all rational $0 \leq \varepsilon \ll 1$. Since D and Δ share no components, $\lfloor \Delta + \varepsilon \Delta' \rfloor = 0$. As $a(E, X, \Delta + \varepsilon \Delta') \rightarrow a(E, X, \Delta)$ for $\varepsilon \rightarrow 0$, there is ε so that $(X, \Delta + \varepsilon \Delta')$ is KLT. \square

We make precise the connection between purely F -regular pairs and PLT pairs.

Theorem 2.22 [Takagi 2008, Corollary 5.4]. *Let (X, Δ) be a log pair. Spread (X, Δ) out over some finitely generated \mathbb{Z} -algebra A . Then, (X, Δ) is PLT if and only if there is a dense open $U \subset \text{Spec } A$ such that the reduction (X_a, Δ_a) is purely F -regular for all $a \in U$.*

Theorem 2.23. *Let (X, Δ) be an affine PLT pair. Assume that $\lfloor \Delta \rfloor = P$ is an minimal LC center for some closed point $x \in P$. Spread (X, Δ) , P , and x out over some finitely generated \mathbb{Z} -algebra A . Then, for all $a \in U$, where U is a dense open subset of $\text{Spec } A$, the divisor P_a is the minimal F -pure center through x_a . In this situation, a minimal LC center is normal. Conversely, if P is not the minimal LC center through x , then P_a is not the minimal F -pure center for x_a for all closed points in a dense open set.*

⁶That is, the generic point of P is the only codimension 1 point in the non-KLT locus of (X, Δ) .

Proof. See [Schwede 2010, Theorem 6.8]. Schwede’s argument immediately also gives the converse statement: If there is some smaller LC center Q passing through x , after reduction, we obtain an F -compatible ideal q_a strictly containing p_a . Thus, P_a cannot be the minimal F -pure center through x_a . For normality of the minimal LC center see [Fujino and Gongyo 2012, Theorem 7.2]. \square

By Theorems 2.22 and 2.23, examples in Section 2A1 are examples of PLT pairs when we let κ have characteristic zero. However, we need to sharpen our hypothesis for the analog of Example 2.10.

Example 2.24. Let R be regular, local and essentially of finite type over an algebraically closed field of characteristic zero, and let $(f) \subset R$ be a prime ideal. Then, the pair $(R, \operatorname{div} f)$ is a PLT pair if and only if R/f is a (Gorenstein) KLT singularity.

3. Digression on local tame fundamental groups

The objective in this section is twofold. First, we overview all the necessary material regarding tame fundamental groups that we need to establish our results. Second, we prove the theorem establishing that Theorems B and C are formal consequences of structural properties of the Galois category being studied. We start off with our first goal.

3A. Tame ramification, cohomological tameness, and Abhyankar’s lemma. We commence by recalling some standard definitions in [Grothendieck and Murre 1971].

Definition 3.1 (tamely ramified field extensions with respect to a DVR). Let K be a field with a discrete valuation ring (DVR) $(A, (u), \kappa)$. One says that a finite separable field extension L/K is *tamely ramified with respect to A* if for all (the finitely many) discrete valuation rings $(B, (v), \ell)$ of L lying over A , we have that $\kappa \subset \ell$ is separable and $\operatorname{Char} \kappa = p$ does not divide the ramification index of the extension $A \subset B$.⁷ If the extensions $A \subset B$ are étale, we say L/K is *étale with respect to A* .

Definition 3.2 (tamely ramified covers with respect to a divisor). Let X be a connected normal scheme and let $D = \sum_i P_i$ be a reduced effective divisor on X with prime components P_i . One says that a finite cover $Y \rightarrow X$ is *tamely ramified with respect to D* (or *simply over D*) if Y is normal and every connected component $Y' \rightarrow Y \rightarrow X$ of Y is a finite cover X that is étale away from D , and $K(Y')/K(X)$ is tamely ramified with respect to the DVRs \mathcal{O}_{X, η_i} , where η_i is the generic point of P_i .

The following lemma will be important in our forthcoming discussions.

Lemma 3.3 [Grothendieck and Murre 1971, Section 2, Lemma 2.2.8]. *Let $f: Y \rightarrow X$ be a finite cover between connected normal schemes and let $D = \sum_i P_i$ be a reduced divisor on X with prime components P_i . Suppose that $f: Y \rightarrow X$ is étale over the complement of D . The following statements are equivalent:*

⁷The ramification index e is characterized by the equality $u = b \cdot v^e$ with b a unit in B .

- (1) f is a tamely ramified cover with respect to D ,
- (2) For all $x \in D$, the pullback of f along $g: \text{Spec } \mathcal{O}_{X,x} \rightarrow X$ is a tamely ramified cover with respect to g^*D .
- (3) For all $x \in D$, the pullback of f along $g: \text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \rightarrow X$ is a tamely ramified cover with respect to g^*D .
- (4) For all $x \in D$ of codimension 1 (in X), the pullback of f along $g: \text{Spec } \mathcal{O}_{X,x} \rightarrow X$ is a tamely ramified cover with respect to g^*D .
- (5) For all $x \in D$ of codimension 1 (in X), the pullback of f along $g: \text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \rightarrow X$ is a tamely ramified cover with respect to g^*D .

Definition-Proposition 3.4 (Kummer-type cyclic covers, see [Grothendieck and Murre 1971, Example 2.2.4]). *Let (X, D) be as in Definition 3.2 and defined over $\mathbb{Z}[1/n][\zeta]$ where ζ is a primitive n -th root of unity, which means that $n \in \Gamma(X, \mathcal{O}_X)$ is invertible and $\Gamma(X, \mathcal{O}_X)$ contains a primitive n -th root of unity (e.g., X may be defined over a separably closed field of characteristic prime to n). Suppose that $D = n \cdot E$ in $\text{Cl } X$ and E is Cartier away from D . Write $\text{div}_X \kappa + n \cdot E = D$ for some $\kappa \in K(X)^\times$. Then, the finite cover $f: Y \rightarrow X$ determined by the \mathcal{O}_X -algebra*

$$\mathcal{O}_X \xrightarrow{\subset} \bigoplus_{i=0}^{n-1} \mathcal{O}_X(-i \cdot E), \quad \cdot \kappa: \mathcal{O}_X(-n \cdot E) \rightarrow \mathcal{O}_X(-D)$$

is a connected tamely ramified cover over D that is generically cyclic of degree n . We refer to these covers as **Kummer-type cyclic covers** or simply as **Kummer covers** when E and so D are principal divisors. We allow $n = 0$ to include the trivial cover.

Proof. Note that, over $U := X \setminus D$, the cover $Y \rightarrow X$ is the element of $H^1(U, \mu_n)$ corresponding to $\text{div}_U \kappa + n \cdot E|_U = 0$ as $E|_U \in \text{Pic } U$. In particular, $Y|_U \rightarrow U$ is a $\mathbb{Z}/n\mathbb{Z}$ -torsor (as $\zeta \in \Gamma(X, \mathcal{O}_X)$) and in particular étale; see [Milne 1980, III, Section 4, pages 125–126]. Next, we explain why Y is normal. Note that, since f is affine, \mathcal{O}_Y satisfies the (\mathbf{S}_2) condition as so does the \mathcal{O}_X -module $f_*\mathcal{O}_Y = \bigoplus_{i=0}^{n-1} \mathcal{O}_X(-i \cdot E)$. To see why Y satisfies (\mathbf{R}_1) , it suffices to look at those codimension 1 points not lying over (the generic point of) the P_i 's as f is étale away from D . That is, it suffices to check that the \mathcal{O}_{X,P_i} -algebras $\mathcal{O}_{X,P_i} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y$ are regular for all i . Observe that $\mathcal{O}_{X,P_i} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \cong \mathcal{O}_{X,P_i}[T]/(T^n - t)$ where t is a uniformizer of \mathcal{O}_{X,P_i} , and further $\mathcal{O}_{X,P_i}[T]/(T^n - t)$ is local with maximal ideal $(t) \oplus \bigoplus_{i=1}^{n-1} \mathcal{O}_{X,P_i} \cdot T = (T)$ and so regular. This computation further shows that $\mathcal{O}_{X,P_i} \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y$ is an extension of DVRs with ramification index n , which is prime to all residual characteristics of X as $1/n \in \Gamma(X, \mathcal{O}_X)$. This proves that Y is normal.

It remains to prove that Y is connected/integral, for which it suffices to show that $K(X) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y$ is a field. Notice that, $K(X) \otimes_{\mathcal{O}_X} f_*\mathcal{O}_Y \cong K(X)[T]/(T^n - \kappa)$. Suppose, for the sake of contradiction, that $K(X)[T]/(T^n - \kappa)$ is not a field. Then, there is $\kappa' \in K(X)^\times$ such that $\text{div } \kappa = \text{div } \kappa'^m$ for some $1 < m \mid n$ (using [Lang 2002, VI, Section 6, Theorem 9.1]); see [Tomari and Watanabe 1992, Corollary 1.9]. Then, $m \cdot (\text{div}_X \kappa' + (n/m)D) = P$, which violates the reducedness of P . \square

Given the equivalence between (a) and (e) in Lemma 3.3, it is of fundamental importance to understand the tamely ramified covers over a strictly local DVR with respect to its uniformizer. In this regard, the following result together with Lemma 3.3 imply that tamely ramified covers are Kummer over the étale-germs at the generic points of the divisor D .

Theorem 3.5 [Serre 1979]. *Let K be a field with a **strictly local** DVR $(A, (u), \mathfrak{k})$ with $\text{Char } \mathfrak{k} = p \geq 0$. Then, every Galois field extension L/K that is tamely ramified with respect to A is Kummer, i.e., $L = K(u^{1/n})$ for some n prime to p , and in particular cyclic. In other words, every Galois tamely ramified cover over $X = \text{Spec } A$ with respect to $\text{div } u$ is Kummer.*

Proof. See [Serre 1979, Chapter IV, Section 2, Proposition 8] for the case $p = 0$. For characteristic $p > 0$, note that, by tameness and \mathfrak{k} being separably closed, we have $p \nmid [L : K]$. One simply replaces the use of Corollary 2 in [loc. cit.] with Corollary IV, Sections 2 and 3. \square

Remark 3.6. The intuition behind Theorem 3.5 is the following; see [Milne 1980, I, Example 5.2(e)]. We think of $\text{Spec } K \cong \text{Spec } A \setminus \{(u)\}$ as an algebraic analog of the punctured disc in the plane, then this result says that $\pi_1^{\text{ét}}(\text{Spec } K)$ is isomorphic to $\hat{\mathbb{Z}}$ — the profinite completion of \mathbb{Z} — at least if the residual characteristic is 0 else what we can say is $\pi_1^{\text{ét}}(\text{Spec } K) \cong \hat{\mathbb{Z}}^{(p)}$.

As mentioned before, Theorem 3.5 tells us that tamely ramified covers over a reduced effective divisor are of a very special type étale-locally around the generic points of the divisor. In case the divisor D in Definition 3.2 has *normal crossings* [Grothendieck and Murre 1971, Section 1.8], *Abhyankar’s lemma* establishes that the same hold at all *special* points in the support of the divisor; see [Grothendieck and Murre 1971, Section 2.3; SGA 1 1971, Exposé XIII, Section 5]. More precisely:

Theorem 3.7 (Abhyankar’s lemma). *With notation as in Definition 3.2, suppose additionally that (X, D) has normal crossings in the sense of [Grothendieck and Murre 1971, Section 1.8]. Then, the connected components of the pullback of $Y \rightarrow X$ along $\text{Spec } \mathcal{O}_{X, \bar{x}}^{\text{sh}} \rightarrow X$ are (quotients) of Kummer covers for all geometric points $\bar{x} \rightarrow X$.*

We are interested in studying tame cover with respect to divisors that may not have normal crossings. Fortunately, our efforts will lead to a generalization of this result when the divisor D is irreducible yet singular; see Lemma 3.34. Following [Kerz and Schmidt 2010; Chinburg et al. 1996], we have a stronger notion of tameness.

Definition 3.8 (cohomological tameness). Let U be a normal connected scheme equipped with a dense open embedding $U \rightarrow X$ with X normal and connected.⁸ We say that a finite Galois cover $V \rightarrow U$ is *cohomologically tamely ramified with respect to X* if its integral closure $f : Y \rightarrow X$ is so that the trace map $\text{Tr}_{Y/X} : f_* \mathcal{O}_Y \rightarrow \mathcal{O}_X$ is surjective. A finite étale cover $V \rightarrow U$ is cohomologically tamely ramified if it can be dominated by a Galois one.

⁸Unlike [Kerz and Schmidt 2010], we do not require X to be proper over some field.

3B. Tame Galois categories and their fundamental groups. Consider the setup:

Setup 3.9. Let $(R, \mathfrak{m}, \kappa, K)$ be a strictly local normal domain of dimension at least 2. Let Z be a closed subscheme of $X := \text{Spec } R$ of codimension at least 2. We consider a prime Weil divisor P on $X^\circ := X \setminus Z$, which extends to a unique prime divisor on X that we also denote by P , then P corresponds to a unique height-1 prime ideal $\mathfrak{p} \subset R$. We set $U := X^\circ \setminus P$. We further set $p := \text{Char } \kappa \geq 0$ and assume that $p \in \mathfrak{p} \subset \mathfrak{m}$, so that $\text{Char } K(P) = p = \text{Char } \kappa$ (here, $K(P)$ is the function field of P , i.e., the residue field of $R_{\mathfrak{p}}$). In particular, after choosing an embedding $\overline{\mathbb{F}}_p \subset \kappa$, we have that $(R, \mathfrak{m}, \kappa, K)$ is a local algebra over \mathbb{Z}_p^{sh} — the maximal unramified extension of \mathbb{Z}_p , which is given by adjoining all prime-to- p roots of unity to \mathbb{Z}_p . Of course, this is just a fancy way to say that R contains all the n -th roots of unity if $p \nmid n$. In particular, we allow $\mathbb{Z}_p^{\text{sh}} \rightarrow R$ to be injective, i.e., R may be of mixed characteristic in this section. However, we are assuming that $R_{\mathfrak{p}}$ has the same (mixed or not) characteristic as R .

We study two types of tame Galois categories in this paper which we introduce next. We invite the reader to consult [Murre 1967; Cadoret 2013] for a thorough exposition on Galois categories and fundamental groups, or the classic, original reference [SGA 1 1971, Exposé V].

3B1. The cohomologically tame Galois category. Working in Setup 3.9, the first tame fundamental group of interest is the fundamental group $\pi_1^{\text{t},X}(X^\circ)$ classifying the Galois category $\text{F}\acute{\text{E}}\text{t}^{\text{t},X}(X^\circ)$ of covers over X° that are cohomologically tamely ramified with respect to X . The minimal (or connected) objects of this Galois category are the local finite extensions $(R, \mathfrak{m}, \kappa, K) \subset (S, \mathfrak{n}, \ell, L)$ such that S is a normal domain, $\text{Tr}_{S/R}: S \rightarrow R$ is surjective, and $R \subset S$ is étale over X° . Thus, $\pi_1^{\text{t},X}(X^\circ) = \varprojlim \text{Gal}(L/K)$ where the limit runs over all Galois extensions L/K inside a fixed separable closure of K such that the integral closure of R in L ; say R^L/R , is étale over X° and $\text{Tr}: R^L \rightarrow R$ is surjective; see [Carvajal-Rojas et al. 2018, Section 2.4].

3B2. The tame Galois category of a prime divisor. In this section, the perspective is quite different from the one above. Working in Setup 3.9, consider the Galois category $\text{Rev}^P(X^\circ)$ of finite covers over X° that are tamely ramified with respect to P . The corresponding fundamental group is denoted by $\pi_1^{\text{t},P}(X^\circ)$ (we choose a geometric generic point as our base point, which is suppressed from the notation). As before, we may restrict ourselves to a local algebra setup as the following remark explains.

Remark 3.10 (reduction to local algebra). Since R is a strictly local normal domain, the Galois objects of the category $\text{Rev}^P(X^\circ)$ are the (generically) Galois local finite extensions of normal domains $(R, \mathfrak{m}, \kappa, K) \subset (S, \mathfrak{n}, \ell, L)$ that are étale over U but tamely ramified over P (i.e., L/K is tamely ramified with respect to $R_{\mathfrak{p}}$). In this way,

$$\pi_1^{\text{t},P}(X^\circ) = \varprojlim \text{Gal}(L/K)$$

where the limit runs over all finite Galois extensions L/K inside some fixed separable closure of K such that the integral closure of R in L is tamely ramified over X° with respect to P . When we refer to a cover $Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$, we mean that $Y = \text{Spec } S$ with S as above, and $Y^\circ = Y \setminus f^{-1}(Z)$, where $f: \text{Spec } S \rightarrow \text{Spec } R$ is the corresponding morphism.

Example 3.11 (Kummer-type cyclic covers). Suppose that there is a divisor D on X such that $P = n \cdot D$ in $\text{Cl } X$ and so that $D|_U$ is Cartier. Writing, $\text{div } \kappa + nD = P$, let $S = \Gamma(Y, \mathcal{O}_Y) = \bigoplus_{i=0}^{n-1} R(-iD)$ with $f : Y \rightarrow X$ as in Definition-Proposition 3.4. Let $\mathfrak{n} := \mathfrak{m} \oplus \bigoplus_{i=1}^{n-1} R(-iD)$ and $\mathfrak{q} := \mathfrak{p} \oplus \bigoplus_{i=1}^{n-1} R(-iD)$. One readily sees that these two are ideals of S . In fact, $S/\mathfrak{n} = R/\mathfrak{m} = \kappa$ and $S/\mathfrak{q} = R/\mathfrak{p}$, thereby \mathfrak{n} is maximal and \mathfrak{q} is prime. Moreover, $\mathfrak{n} \cap R = \mathfrak{m}$, $\mathfrak{q} \cap R = \mathfrak{p}$, and $\mathfrak{n}, \mathfrak{q}$ are; respectively, the only primes of S with such property. Further, we have that $\text{ht } \mathfrak{q} = 1$. Indeed, $\text{ht } \mathfrak{q} \leq \text{ht}(\mathfrak{q} \cap R) = \text{ht } \mathfrak{p} = 1$ from integrality of S/R (going-up theorem) and S being a domain rules out the possibility $\text{ht } \mathfrak{q} = 0$ (as clearly $\mathfrak{q} \neq 0$). Thus, $Q = V(\mathfrak{q})$ is a prime divisor on Y . Putting everything together, $(R, \mathfrak{m}, \kappa, K, \mathfrak{p}) \subset (S, \mathfrak{n}, \kappa, K(\kappa^{1/n}), \mathfrak{q})$ defines a cyclic cover in $\text{Rev}^P(X^\circ)$ whose pullback to $R_{\mathfrak{p}}$ is Kummer; see the proof of Definition-Proposition 3.4.

If $\mathfrak{p} = (r)$ is principal, then $\text{div } \kappa + nD = \text{div } r$ and so D is torsion, say of index $m \mid n$. In particular, we may use this to define a connected quasiétale cover $g : W \rightarrow X$ of degree m trivializing D . Then, it follows that the base change $f_W : Y_W \rightarrow W$ is a Kummer cover of the form $\text{Spec } \mathcal{O}_W[T]/(T^n - r) \rightarrow W$. Indeed, after trivializing D , say $D = \text{div } s$, the equality $\text{div } \kappa + n \text{div } s = \text{div } r$ says that $r = us^n \kappa$ for some unit $u \in \Gamma(W, \mathcal{O}_W)$. Then, since $\Gamma(W, \mathcal{O}_W)$ is also strictly henselian, we may say that $u = 1$ by replacing s by $u^{-1/n}s$. Thus, $L(r^{1/n}) = L(\kappa^{1/n})$, where L is the function field of W .

3C. Some examples of tamely ramified covers. In this section, we provide some examples illustrating what may go wrong in Abhyankar’s lemma if the divisor in question is too singular. Additionally, we consider instructive to have some examples at hand that we may use across the forthcoming sections to highlight particular features of our results. We will employ the following useful fact throughout.

Proposition 3.12 [Stacks 2005–, Lemma 09EB]. *Let R be a normal domain with fraction field K . Let L/K be a finite Galois extension of degree d , and let S be the integral closure of R in L . Fix a height 1 prime ideal $\mathfrak{p} \subset R$, and let $\mathfrak{q}_1, \dots, \mathfrak{q}_n \subset S$ be the list of distinct prime ideals of S lying over \mathfrak{p} . Then, all the DVR extensions $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}_i}$ share the same ramification index e and residual degree i . Moreover, the formula $d = n \cdot e \cdot i$ holds.*

Terminology 3.13. We shall often refer to i in Proposition 3.12 as the *inertial degree*.

Example 3.14 (the cusp). Let $(R, \mathfrak{m}, \kappa, K)$ be a regular local ring with regular system of parameters $\mathfrak{m} = (x, y)$. We assume $\text{Char } \kappa \neq 2, 3$. Let L be the splitting field of $T^3 + xT + y \in K[T]$. This polynomial is irreducible.⁹ Let $t_1, t_2, t_3 \in L$ be the distinct roots of $T^3 + xT + y$. Setting,

$$\delta := (t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$$

we have that $\delta^2 = -4x^3 - 27y^2 =: \Delta$. In particular, $\delta \notin K$ for Δ is irreducible in R . Therefore, L/K is a Galois extension of degree 6 with $\text{Gal}(L/K) \cong S_3$ — the symmetric group; see [Roman 2006, Section 7.5] or [Lang 2002, VI, Section 2]. In fact, $K(\delta)$ is the fixed field of the (cyclic) alternating group $A_3 \subset S_3$.

⁹Indeed, if it were reducible, it would admit a root in K and further in R by normality of R . In that case, $y = t(t^2 + x)$ for some $t \in R$. Since R is a UFD and y is an irreducible element, this implies that either t or $t^2 + x$ is a unit, and *a fortiori* both are units implying further that y is a unit, which is a contradiction.

Thus, $L = K(\delta, t_1)$ and

$$L = K(\delta)[T]/(T^3 + xT + y).$$

In fact, a direct computation shows that if t is one of the roots then the remaining roots are:

$$\frac{-t}{2} \pm \frac{\delta}{2(3t^2 + x)},$$

where $3t^2 + x \neq 0$ as the minimal polynomial of t over K has degree 3.¹⁰

We set $t = t_1$, and set t_2 to be the root with the positive sign in the above expression. Let S be the integral closure of R in L . Of course, $S \ni \delta, t_1, t_2, t_3$. Then, we have:

Claim 3.15. *$R \subset S$ is a tamely ramified extension with respect to the prime divisor $D = \text{div } \Delta$. Moreover, there are exactly three prime divisors of S lying over (Δ) , with ramification index $e = 2$ and inertial degree $i = 1$.*

Proof of claim. By Definition-Proposition 3.4, the integral closure of R in $K(\delta)$ is $R[\delta]$, so that S is the integral closure of $R[\delta]$ in L . On the other hand, we may consider the flat extension of degree 3

$$R[\delta] \subset R[\delta, t] \cong R[\delta][T]/(T^3 + xT + y).$$

Notice that the discriminant ideal of this extension is (Δ) whereas the different ideal is $(3t^2 + x)$. Therefore, $R \subset R[\delta, t]$ is étale away from D and so $R[\delta, t]_{\delta} = R[\delta, \delta^{-1}, t]$ is normal. In particular, the extension $R[\delta, t] \subset S$ is an equality after localizing at δ (or well at $3t^2 + x$). Thus, the extension $R_{\Delta} \subset S_{\Delta}$ is étale. By Lemma 3.3, we are left with showing $R_{(\Delta)} \rightarrow S_{(\Delta)}$ is a tamely ramified extension. To this end, observe that

$$\Delta = \delta^2 = (t_1 - t_2)^2(t_2 - t_3)^2(t_1 - t_3)^2.$$

Now, let $\mathfrak{q} \subset S$ be a prime ideal lying over (Δ) . It must then contain at least one of the elements $t_1 - t_2, t_2 - t_3, t_1 - t_3$. We argue next it can contain only one of them. Indeed, if it contains two of them it must contain the third one and thus all of them.¹¹ In particular, the ramification index of $R_{(\Delta)} \rightarrow S_{\mathfrak{q}}$ is at least 6 and by applying Proposition 3.12 we conclude that $n = 1, e = 6$, and $i = 1$ (with notation as in Proposition 3.12). In particular, \mathfrak{q} is generated by either of these elements. On the other hand, we have that

$$t_1 - t_2 = \frac{3t(3t^2 + x) - \delta}{2(3t^2 + x)}, \quad t_1 - t_3 = \frac{3t(3t^2 + x) + \delta}{2(3t^2 + x)}, \quad t_2 - t_3 = \frac{2\delta}{2(3t^2 + x)}. \quad (3.15.1)$$

From this, we conclude that all the displayed numerators belong to \mathfrak{q} and so does $6t(3t^2 + x)$. Nonetheless, $\mathfrak{q} \not\ni t$ as otherwise $y = -t(t^2 + x) \in \mathfrak{q} \cap R = (\Delta)$, which is not the case. In this way, our conclusion must

¹⁰In case the reader wants to corroborate this assertion by hand, notice that $T^3 + xT + y = (T - t)(T^2 + tT + t^2 + x)$. Hence, it suffices to verify that these are roots of $T^2 + tT + t^2 + x = (T + t/2)^2 + (3t^2 + 4x)/4$, which in turn boils down to checking $\delta^2 + (3t^2 + 4x)(3t^2 + x)^2 = 0$, which is a straightforward computation.

¹¹For instance, $t_1 - t_3 = (t_1 - t_2) + (t_2 - t_3)$.

be that $3t^2 + x \in \mathfrak{q} \cap R[\delta, t] = \sqrt{(\delta)}$.¹² This, however, is a contradiction. Indeed, we have that $R[\delta, t]$ is a rank-6 free R module and moreover

$$R[\delta, t] = R \cdot 1 \oplus R \cdot t \oplus R \cdot t^2 \oplus R \cdot \delta \oplus R \cdot \delta t \oplus R \cdot \delta t^2 = \langle 1, t, t^2 \rangle_R \oplus \langle \delta, \delta t, \delta t^2 \rangle_R,$$

whence one sees that any power of $3t^2 + x$ is going to belong to the direct summand $\langle 1, t, t^2 \rangle_R$ whereas

$$(\delta) \subset ((\Delta) \cdot \langle 1, t, t^2 \rangle_R) \oplus \langle \delta, \delta t, \delta t^2 \rangle_R.$$

Additionally, an inductive argument readily shows that the constant coefficient of $(3t^2 + x)^n$ is x^n for every exponent n . Putting everything together, we see that $3t^2 + x \in \sqrt{(\delta)}$ yields that $x^n \in (\Delta)$ for some n and so $x \in (\Delta)$, which is the sought contradiction.

In conclusion, the principal ideals $(t_1 - t_2)$, $(t_2 - t_3)$, $(t_1 - t_3) \subset S$ share no minimal prime. By using Proposition 3.12, we conclude that these are (the) prime ideals of S lying over $(\Delta) \subset R$, with ramification index $e = 2$ and inertial degree $i = 1$. This proves the claim. \square

Example 3.16 (Whitney’s umbrella). Let $(R, \mathfrak{m}, \kappa, K)$ be a regular local ring with κ of *odd characteristic*, and let $f := x^2 - y^2z$ where $\mathfrak{m} = (x, y, z)$ is a regular system of parameters. The polynomial expression $x^2 - y^2z$ plays a fundamental role in the description of degree 4 Galois extensions; see [Lang 2002, VI, Example 4]. Thus, we start off by considering the degree 2 Galois extension $E = K(\sqrt{f})$. Next, we consider the tower of degree 2 Galois extensions

$$E \subset E(\sqrt{z}) \subset E(\sqrt{z})(\sqrt{x + y\sqrt{z}}).$$

Set $\alpha = x + y\sqrt{z}$, $\alpha' = x - y\sqrt{z}$, and $\beta = \sqrt{\alpha}$. The above tower is $E \subset E(\alpha) \subset E(\beta)$. By [loc. cit.], $E(\beta)/E$ is a noncyclic degree 4 Galois extension as $\alpha\alpha' = f$ is a square in E . In fact, $E(\beta)/E$ is the splitting field of $T^4 - 2xT^2 + f \in E[T]$:

$$T^4 - 2xT^2 + f = (T^2 - \alpha)(T^2 - \alpha') = (T - \beta)(T + \beta)(T - \sqrt{f}/\beta)(T + \sqrt{f}/\beta).$$

Moreover, $E(\beta)/K$ is a degree 8 noncyclic Galois extension, for it is the splitting field of $T^4 - 2xT^2 + f \in K[T]$. In fact, setting $\beta' := \sqrt{f}/\beta$, we see that $\text{Gal}(E(\beta)/E)$ is generated by the transpositions $\tau: \beta \mapsto -\beta$ and $\sigma: \beta \mapsto \beta'$. Moreover, in $\text{Gal}(E(\beta)/K)$ we have $\rho: \beta \mapsto \beta$, $\sqrt{f} \mapsto -\sqrt{f}$. In this way, $\pi := \sigma\rho: \beta \mapsto \beta'$, $\sqrt{f} \mapsto -\sqrt{f}$ is an element of order 4 whose square and cube are; respectively, τ and $\rho\sigma$. That is, $\text{Gal}(E(\beta)/K)$ is generated by two elements σ and π satisfying relations $\sigma^2 = 1$, $\pi^4 = 1$, and $\sigma\pi = \pi^3\sigma$. In other words, $\text{Gal}(E(\beta)/K)$ is isomorphic the dihedral group — the symmetries of the square. Thus,

$$\text{Gal}(E(\beta)/K) = \{1, \sigma, \rho, \tau, \pi, \sigma\rho, \pi\sigma, \sigma\tau\}.$$

Let S be the integral closure of R in $E(\beta)$. Next, we claim the following.

Claim 3.17. $S = R[\sqrt{z}, \beta, \beta']$

¹²To see the equality, consider $\mathfrak{t} \subset R[\delta, t]$ to be a minimal prime of (δ) . Since $R[t, \delta] \subset S$ is integral, there is at least one prime ideal of S lying over \mathfrak{t} . However, any such a prime must lie over $(\Delta) \subset R$ and so must do \mathfrak{q} .

Proof of claim. By Definition-Proposition 3.4, $R[\sqrt{f}, \sqrt{z}]$ is normal and so it is the integral closure of R in $E(\alpha)$ — the fixed field of $\langle \tau \rangle$. Thus, we just need to prove that S is the integral closure of $R[\sqrt{f}, \sqrt{z}]$ in $E(\beta)$. To this end, we prove that any element $\gamma \in S$ is an $R[\sqrt{f}, \sqrt{z}]$ -linear combination of $1, \beta$, and β' . We know that $\gamma = a + b\beta \in E(\beta)$ for some (uniquely determined) $a, b \in E(\alpha)$. Since $E(\beta)/E(\alpha)$ is a quadratic extension, the minimal polynomial of γ is described in terms of its trace and norm as follows:

$$T^2 + \operatorname{Tr}_{E(\beta)/E(\alpha)}(\gamma)T + \operatorname{N}_{E(\beta)/E(\alpha)}(\gamma).$$

Observe that $\operatorname{Tr}_{E(\beta)/E(\alpha)}(\gamma) = 2a$ and $\operatorname{N}_{E(\beta)/E(\alpha)}(\gamma) = a^2 - b^2\alpha$. Therefore, $\gamma \in S$ if and only if both $2a$ and $a^2 - b^2\alpha$ belong to $R[\sqrt{f}, \sqrt{z}]$, which is equivalent to $a, b^2\alpha \in R[\sqrt{f}, \sqrt{z}]$.

Now, since $b^2\alpha$ belongs to $R[\sqrt{f}, \sqrt{z}]$ so does $b^2f = b^2\alpha\alpha'$. That is, since $b^2\alpha \in R[\sqrt{f}, \sqrt{z}]$, then b^2f belongs to the ideal $(\alpha') \subset R[\sqrt{f}, \sqrt{z}]$. Since $b^2f = (b\sqrt{f})^2$, this is to say that $b\sqrt{f} \in E(\alpha)$ is integral over $(\alpha') \subset R[\sqrt{f}, \sqrt{z}]$. Given that $R[\sqrt{f}, \sqrt{z}]$ is integrally closed in $E(\alpha)$, we conclude that $b\sqrt{f} \in \sqrt{(\alpha')} \subset R[\sqrt{f}, \sqrt{z}]$; see [Kunz 2013, Chapter 2, Corollary 2.6]. The result then follows once we have shown that

$$\sqrt{(\alpha')} = (\alpha', \sqrt{f}), \quad \text{in } R[\sqrt{f}, \sqrt{z}]. \quad (3.17.1)$$

Indeed, granted (3.17.1), we would have that

$$\gamma = a + b\beta = a + (b\sqrt{f})\frac{\beta}{\sqrt{f}} = a + (c\alpha' + d\sqrt{f})\frac{\beta}{\sqrt{f}} = a + d\beta + c\frac{\alpha'}{\beta'} = a + d\beta + c\beta',$$

for some $d, c \in R[\sqrt{f}, \sqrt{z}]$ — we saw before that $a \in R[\sqrt{f}, \sqrt{z}]$.

To prove (3.17.1), observe that the containment from right to left is clear, for $\sqrt{f}^2 = \alpha\alpha'$. For the converse containment, observe that $R[\sqrt{f}, \sqrt{z}]$ is free over R with basis $1, \sqrt{z}, \sqrt{f}, \sqrt{z}\sqrt{f}$. In particular, if an element in $R[\sqrt{f}, \sqrt{z}]$ belongs to $\sqrt{(\alpha')}$ then so does the summand in the R -span of 1 and \sqrt{z} . Thus, it is enough to prove that $r + s\sqrt{z} \in \sqrt{(\alpha')}$; with $r, s \in R$, belongs to (α') . That is, it suffices to explain why the contraction of $\sqrt{(\alpha')} \subset R[\sqrt{f}, \sqrt{z}]$ to $R[\sqrt{z}]$ is the ideal (α') . This, however, follows from observing that $(\alpha') \subset R[\sqrt{z}]$ is a prime ideal. Indeed, observe that $R[\sqrt{z}]$ is a regular local ring (and so an UFD) as its maximal ideal is given by $\mathfrak{m} \oplus R \cdot \sqrt{z} = (x, y, z, \sqrt{z}) = (x, y, \sqrt{z})$. On the other hand, the extension of the prime ideal $(f) \subset R$ to $R[\sqrt{z}]$ splits as $(f) = (\alpha)(\alpha')$. Since there cannot be more than two prime ideals of $R[\sqrt{z}]$ lying over $(f) \subset R$, we conclude that these are (α) and (α') . \square

Claim 3.18. *The extension $R \subset S$ is tamely ramified with respect to the reduced divisor $D = \operatorname{div} z + \operatorname{div} f$. Moreover, for both prime divisors $(z), (f) \subset R$, there are exactly two prime ideals of S lying over with ramification index 2 and inertial degree 2.*

Proof of claim. We begin by proving that $R_{zf} \subset S_{zf}$ is étale. Indeed, we have a tower $R \subset R[\sqrt{f}, \sqrt{z}] \subset S$ where the bottom extension is flat of degree 4. One readily verifies that the discriminant ideal of the bottom extension is (zf) , so it is étale over R_{zf} . It suffices to check that $R[\sqrt{f}, \sqrt{z}]_{zf} \subset S_{zf}$ is étale. To this end, notice that by inverting f we invert α and α' in $R[\sqrt{f}, \sqrt{z}]$ and β and β' in S , for we have the relation

$\alpha\alpha' = f = \beta^2\beta'^2$. In particular, $1/\alpha \in R[\sqrt{f}, \sqrt{z}]_{zf}$ and $1/\beta \in S_{zf}$. Therefore, $R[\sqrt{f}, \sqrt{z}]_{zf} \subset S_{zf}$ is free with basis $1, \beta$ (as $\beta' = \sqrt{f}/\beta = \sqrt{f}/\alpha\beta$) and its discriminant ideal is generated by

$$\begin{vmatrix} 1 & 1 \\ \beta & -\beta \end{vmatrix}^2 = (-2\beta)^2 = 4\alpha,$$

which is a unit and consequently the extension is étale; as needed.

It remains to prove that $E(\beta)/K$ is tamely ramified with respect to both DVRs $R_{(z)}$ and $R_{(f)}$. Nevertheless, this follows from simple characteristic considerations. Indeed, since the extensions are Galois, we know in each case that $8 = n \cdot e \cdot i$ with n being the number of primes lying over, e the ramification indexes, and i the residual degrees; as in Proposition 3.12. Then, e and i are necessarily prime to characteristic, which was assumed odd from the beginning. Recall that $f = \beta^2\beta'^2$. Using Proposition 3.12, this implies that $(\beta), (\beta') \subset S$ are (the) prime ideals of S lying over $(f) \subset R$, and the ramification index is 2 as well as the residual degree.¹³ Similarly, we have that $2y\sqrt{z} = (\beta - \beta')(\beta + \beta')$, so that $4y^2z = (\beta - \beta')^2(\beta + \beta')^2$. Therefore, $(\beta - \beta'), (\beta + \beta') \subset S$ are (the) two prime ideals of S lying over $(z) \subset R$, with ramification index and inertial degree equal to 2.¹⁴ □

Example 3.19. We may specialize Example 3.16 by setting $y = 1$. More precisely, we may consider $(R, \mathfrak{m}, \zeta, K)$ to be a regular local ring of odd residual characteristic with regular system of parameters $\mathfrak{m} = (x, z)$ and set $f := x^2 - z$. Letting L/K be the splitting prime of $T^4 - 2xT^2 + f \in K[T]$, the same arguments *mutatis mutandis* as in Example 3.16 show that $S := R^L = R[\sqrt{z}, \sqrt{x \pm \sqrt{z}}]$ and moreover that $R \subset S$ is a degree 8 Galois tamely ramified extension over $D = \text{div } z + \text{div } f$. Further, for both (regular) prime divisors $(z), (f) \subset R$ there are exactly two prime of S lying over with ramification and inertial indexes equal to 2.

Remark 3.20 (failure of Abhyankar’s lemma for divisors without normal crossings). Observe that Examples 3.14, 3.16 and 3.19 are counterexamples for Abhyankar’s lemma if no regularity condition is imposed on the divisor. Indeed, in each case, we may consider R to be additionally strictly local, then it admits a tamely ramified cover (e.g., S) that is not Kummer (for it is not cyclic). In the cusp case, the divisor D has not normal crossings for it is cut out by a singular (irreducible) equation. In the Whitney’s umbrella case, the divisor has not normal crossings because f is not a regular element in the ring $R_{(x,y)}^{\text{sh}}$ as $f = x^2 - y^2z = (x - y\sqrt{z})(x + y\sqrt{z})$ in this ring. In the case of Example 3.19, we have that z and f are both regular elements yet $R/(z, f)$ is not regular as $(z, f) = (z, x^2)$.

3D. Main formal theorems. Next, we explain why our main results on $\pi_1^{t,P}(X^\circ)$ can be seen as formal consequences of some interesting properties of the Galois category $\text{Rev}^P(X^\circ)$. With notation as in Setup 3.9, to a Galois object $f: Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$ of degree d_f we may associate three positive integers n_f, e_f , and i_f which are subject to the relation $d_f = n_f \cdot e_f \cdot i_f$; see Proposition 3.12. With this in mind:

¹³To see that these two ideals are different, note that otherwise would imply that $(\alpha) = (\alpha')$ in $R[\sqrt{z}]$, which is tantamount to say that $(f) \in \text{Spec } R$ is a branch point of $R \subset R[\sqrt{z}]$. This, however, is not the case.

¹⁴Notice that $(\beta - \beta') \neq (\beta + \beta')$ in S as otherwise this would yield that the common ideal contains both (β) and (β') , which is absurd as then they are all the same ideal.

Terminology 3.21. Working in Setup 3.9, we consider the following properties on $\text{Rev}^P(X^\circ)$:

- (1) *P-irreducibility*: Every connected cover $f : Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$ satisfies that $Q := (f^{-1}(P))_{\text{red}}$ is a prime divisor on Y° . In other words, with notation as in Remark 3.10, there is exactly one prime, say \mathfrak{q} , lying over \mathfrak{p} in the extension $R \subset S$. If f is Galois, this means $n_f = 1$.
- (2) *inertial boundedness*: There exists $N \in \mathbb{N}$ such that $i_f \leq N$ for all Galois objects $f : Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$.
- (3) *inertial tameness*: The inertial degree i_f is prime-to- p for all Galois objects $f : Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$.
- (4) *inertial decantation*: Assuming the P -irreducibility of $\text{Rev}^P(X^\circ)$, inertial decantation means that every Galois cover $f : Y^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$ dominates a quasiétale Galois cover $Y'^\circ \rightarrow X^\circ$ in $\text{Rev}^P(X^\circ)$ whose generic degree is the generic degree of $Q := (f^{-1}(P))_{\text{red}} \rightarrow P$. Equivalently, with notation as in Remark 3.10, if $\mathfrak{q} \subset S$ is the only prime lying over \mathfrak{p} , there is a factorization $(R, \mathfrak{m}, \zeta, K; \mathfrak{p}) \subset (S', \mathfrak{n}', \ell', L'; \mathfrak{q}') \subset (S, \mathfrak{n}, \ell, L; \mathfrak{q})$ such that the bottom extension induces an étale-over- P (i.e., quasiétale) cover in $\text{Rev}^P(X^\circ)$ and $[\kappa(\mathfrak{q}') : \kappa(\mathfrak{p})] = [L' : K] = [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})]$. When the latter degree is 1, we say that f is *totally ramified*.

Remark 3.22. If $\text{Rev}^P(X^\circ)$ is P -irreducible, we may think of the covers in $\text{Rev}^P(X^\circ)$ as local extensions $(R, \mathfrak{m}, \zeta, K; \mathfrak{p}) \subset (S, \mathfrak{n}, \ell, L; \mathfrak{q})$; as in Remark 3.10, where \mathfrak{q} is the only (height 1) prime ideal of S lying over \mathfrak{p} . We follow the convention to denote the prime divisor corresponding to \mathfrak{q} by Q and so on. Note that, if $f : Y^\circ \rightarrow X^\circ$ is a connected cover in $\text{Rev}^P(X^\circ)$, then the category $\text{Rev}^Q(Y^\circ)$ is Q -irreducible and $\text{Rev}^Q(Y^\circ)$ is the Galois category given by the objects of $\text{Rev}^P(X^\circ)$ that lie over (or dominate) the object (Y°, Q) . If f is further Galois, then $\text{Rev}^Q(Y^\circ)$ is inertially bounded (resp. tame) if so is $\text{Rev}^P(X^\circ)$.

Lemma 3.23. *P-irreducibility implies inertial decantation.*

Proof. With notation as in Remark 3.10, since \mathfrak{q} is the only prime lying over \mathfrak{p} , its decomposition group $D := \{\sigma \in \text{Gal}(L/K) \mid \sigma(\mathfrak{q}) = \mathfrak{q}\}$ is the whole Galois group $\text{Gal}(L/K)$. Therefore, its inertia group I sits as the kernel in the following short exact sequence of groups

$$1 \rightarrow I \rightarrow \text{Gal}(L/K) \rightarrow \text{Aut}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) \rightarrow 1.$$

By the tameness of the ramification, $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is a finite separable extension and so Galois by [Stacks 2005–, Lemma 09ED]. Thus, we have

$$1 \rightarrow I \rightarrow \text{Gal}(L/K) \rightarrow \text{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) \rightarrow 1.$$

We may use the Galois correspondence to obtain a factorization

$$(R, \mathfrak{m}, \zeta, K, P) \subset (S^I, \mathfrak{n}^I, \ell^I, L^I, Q^I) \subset (S, \mathfrak{n}, \ell, L, Q)$$

where both covers are Galois and the upper script I denotes the invariant or fixed elements under the action of I . Moreover, $\text{Gal}(L^I/K) = \text{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))$ and the bottom extension is étale at \mathfrak{q}^I ; see [Stacks

2005–, Lemma 09EH], and so quasiétale. Furthermore, $[\kappa(q^I) : \kappa(\mathfrak{p})] = [L^I : K] = [\kappa(q) : \kappa(\mathfrak{p})]$. This proves the lemma. \square

Definition-Proposition 3.24. *In the situation of Setup 3.9, the full subcategory $\text{Rev}_{1,\text{ét}}^P(X^\circ)$ of $\text{Rev}^P(X^\circ)$ consisting of those $Z^\circ \rightarrow X^\circ$ that are étale-over- P is a Galois subcategory. We denote the corresponding fundamental group by $\pi_{1,\text{ét}}^P(X^\circ)$.*

Proof. We follow the proof of [Grothendieck and Murre 1971, Theorem 2.4.2] and only need to verify the conditions G1, G2 and G3 of [SGA 1 1971, Exposé V, 4]. Clearly, X° itself is a final object. For the existence of fiber products, take $Y^\circ \rightarrow Z^\circ, W^\circ \rightarrow Z^\circ$ in $\text{Rev}(X^\circ)$ and consider the following diagram:

$$\begin{array}{ccccc} (Y^\circ \times_{Z^\circ} W^\circ)_{\text{nor}} & \longrightarrow & Y^\circ \times_{Z^\circ} W^\circ & \longrightarrow & Y^\circ \\ & & \downarrow & & \downarrow \\ & & W^\circ & \longrightarrow & Z^\circ \end{array}$$

where the normalization is taken with respect to the total ring of fractions of $Y^\circ \times_{Z^\circ} W^\circ$. By [loc. cit.], this is the fiber product in $\text{Rev}^P(X^\circ)$. Note that $Y^\circ \times_{Z^\circ} W^\circ$ is étale over P since étale morphisms are stable under base change. Moreover, as étale morphisms preserve normality (and X° is normal), we conclude that $Y^\circ \times_{Z^\circ} W^\circ \rightarrow X^\circ$ is normal at P and thus the normalization is an isomorphism at P . The existence of direct sums is clear. Consider now $Y^\circ \rightarrow X^\circ$ a morphism in $\text{Rev}^P(X^\circ)$, G a finite subgroup of $\text{Aut}(Y^\circ)$. Then,

$$\begin{array}{ccc} Y^\circ & \longrightarrow & Y^\circ/G \\ f \downarrow & \swarrow u & \\ X^\circ & & \end{array}$$

is a commutative diagram in $\text{Rev}^P(X^\circ)$. Assume that f is étale over P . If Q is a point in Y° lying over P with image Q' in Y°/G , we have inclusions of DVRs $\mathcal{O}_{X^\circ,P} \subset \mathcal{O}_{Y^\circ/G,Q'} \subset \mathcal{O}_{Y^\circ,Q}$. Since the inclusion $\mathcal{O}_{X^\circ,P} \subset \mathcal{O}_{Y^\circ,Q}$ is unramified, the first extension is also unramified. Hence, u is étale at P . Condition G3 follows just as in [loc. cit.] \square

Lemma 3.25. *Work in Setup 3.9. Suppose that $\text{Rev}^P(X^\circ)$ is P -irreducible. Then, $\text{Rev}^P(X^\circ)$ inertially bounded if and only if $\text{Rev}_{1,\text{ét}}^P(X^\circ)$ has a universal cover, i.e., $\pi_{1,\text{ét}}^P(X^\circ)$ is finite. If $\tilde{f}: \tilde{X}^\circ \rightarrow X^\circ$ is such universal cover, then $\text{Rev}^{\tilde{P}}(\tilde{X}^\circ)$ is so that its Galois objects have inertial degree equal to 1. If $\text{Rev}^P(X^\circ)$ is further inertially tame, then the order of $\pi_{1,\text{ét}}^P(X^\circ)$ is prime-to- p .*

Proof. Since $\text{Rev}^P(X^\circ)$ is P -irreducible, the degree of a Galois object in $\text{Rev}_{1,\text{ét}}^P(X^\circ)$ coincides with its inertial degree. The first and third statements then follow. The second statement follows from the inertial decantation on $\text{Rev}^{\tilde{P}}(\tilde{X}^\circ)$; see Lemma 3.23 and Remark 3.22. \square

By purity of the branch locus, we only need to check inertial boundedness and/or tameness on the regular locus of X , i.e., $X^\circ = X_{\text{reg}}$. This will play a crucial role in Section 5. We make this precise next but it can be skipped for now.

Proposition 3.26. *Work in Setup 3.9. There is a fully faithful functor between Galois categories $\text{Rev}_{1,\acute{\text{e}}\text{t}}^P(X^\circ) \rightarrow \text{F}\acute{\text{E}}\text{t}(X_{\text{reg}})$, which induces a surjective homomorphism between the corresponding fundamental groups. Moreover, this functor induces an isomorphism between fundamental groups if Z cuts out the singular locus of X .*

Proof. Recall that $\text{F}\acute{\text{E}}\text{t}(X_{\text{reg}})$ is equivalent to the Galois subcategory of the absolute Galois category of K given by finite separable extensions $K \subset L \subset K^{\text{sep}}$ such that the integral closure of $R \subset R^L$ in L is étale over X_{reg} ; see [Carvajal-Rojas et al. 2018, Section 2.4]. On the other hand, as mentioned before in Remark 3.10, $\text{Rev}^P(X^\circ)$ corresponds the Galois subcategory given by field extensions where $R \subset R^L$ is étale over U and L/K is tamely ramified with respect to $R_{\mathfrak{p}}$, whereas $\text{Rev}_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$ is the one in which $R \subset R^L$ is étale over U and L/K is étale with respect to $R_{\mathfrak{p}}$; see Definition 3.1. In particular, $\text{Rev}_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$ is (or can be identified with) a full Galois subcategory of $\text{F}\acute{\text{E}}\text{t}(X_{\text{reg}})$. Indeed, if L/K is in $\text{Rev}_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$ then $R \subset R^L$ is quasi-étale, and so induces an étale cover over X_{reg} by Zariski–Nagata–Auslander purity of the branch locus for regular schemes [Stacks 2005–, Lemma 0BMB]; see [Zariski 1958; Nagata 1958; 1959; Auslander 1962]. Moreover, if $X^\circ = X_{\text{reg}}$ (i.e., Z cuts out the singular locus), we have the same categories as in that case $U \subset X_{\text{reg}}$ and X_{reg} contains the regular point of P . It is worth noticing that the normality of X is essential through the previous arguments. Finally, observe that the remaining statements are formal consequences of the just proven; see [Murre 1967, Chapter 5]. \square

Corollary 3.27. *Work in Setup 3.9. Suppose that $\text{Rev}_{1,\acute{\text{e}}\text{t}}^P(X_{\text{reg}})$ has a universal cover (of prime-to- p degree). Then, $\text{Rev}_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$ has a universal cover (of prime-to- p degree).*

Proof. Proposition 3.26 can be summarized as follows: $\pi_{1,1,\acute{\text{e}}\text{t}}^P(X_{\text{reg}}) \cong \pi_1^{1,\acute{\text{e}}\text{t}}(X_{\text{reg}}) \twoheadrightarrow \pi_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$. Thus, finiteness/tameness on the left-hand side group implies finiteness/tameness on the right-hand one. \square

Definition 3.28. Work in Setup 3.9. Define $N^P(X^\circ) \subset \mathbb{N}$ as the set of prime-to- p positive integers $n \in \mathbb{N}$ for which there is a divisor D on X such that $P - n \cdot D \in \text{Cl } X$ has prime-to- p torsion and $D|_U$ is Cartier. Likewise, define $M^P(X^\circ) \subset \mathbb{N}$ as the set of prime-to- p positive integers $n \in \mathbb{N}$ for which there is a divisor D on X such that $P = n \cdot D \in \text{Cl } X$ and $D|_U$ is Cartier. Note that $1 \in M^P(X^\circ) \subset N^P(X^\circ)$.

With the above in place, we have the following theorem.

Theorem 3.29. *Work in Setup 3.9. Suppose that $\text{Rev}^P(X^\circ)$ is P -irreducible and has bounded inertia. Then, there exists a short exact sequence of topological groups*

$$\hat{\mathbb{Z}}^{(p)} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\acute{\text{e}}\text{t}}^P(X^\circ) \rightarrow 1 \quad (3.29.1)$$

where $\pi_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$ is finite and $\hat{\mathbb{Z}}^{(p)}$ is the prime-to- p part of the profinite completion of \mathbb{Z} (if $p = 0$ we shall agree upon $\hat{\mathbb{Z}}^{(p)} := \hat{\mathbb{Z}}$). If $\text{Rev}^P(X^\circ)$ also has tame inertia, then the order of $\pi_{1,\acute{\text{e}}\text{t}}^P(X^\circ)$ is prime-to- p and the following two statements hold:

- If $P \in \text{Cl } X$ has prime-to- p torsion, (3.29.1) yields a short exact sequence

$$0 \rightarrow \hat{\mathbb{Z}}^{(p)} \rightarrow \pi_1^{\text{t},P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1 \quad (3.29.2)$$

which splits (in the category of topological groups) if and only if $M^P(X^\circ)$ equals the set of prime-to- p positive integers. Further, the sequence is split if and only if $P = 0 \in \text{Cl } X$.

- If $P \in \text{Cl } X$ is nontorsion, (3.29.1) yields a short exact sequence

$$0 \rightarrow \varprojlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{\text{t},P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1 \quad (3.29.3)$$

which is split (in the category of topological groups) if and only if $N^P(X^\circ) \supset M^P(X^\circ)$ is an equality and there is a compatible system $\{\frac{1}{n}P \in \text{Cl } X\}_{n \in M^P(X^\circ)}$ of factors of P meaning that $m \cdot (\frac{1}{mn}P) = \frac{1}{n}P$ in $\text{Cl } X$ and $\frac{1}{1}P = P$ (e.g., if $\text{Cl } X$ has no prime-to- p torsion). In particular, if $N^P(X^\circ)$ is finite (e.g., $\text{Cl } X$ modulo prime-to- p torsion is finitely generated), there is a short exact sequence

$$0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{\text{t},P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1 \quad (3.29.4)$$

and so $\pi_1^{\text{t},P}(X^\circ)$ is finite of order prime-to- p . Likewise, (3.29.4) is split if and only if $P \in \text{Cl } X$ is n -divisible with $\frac{1}{n}P \in \text{Pic } U$.

Remark 3.30. The limit in the kernel of (3.29.3) makes sense because, if $m \cdot M = P = n \cdot N$ in $\text{Cl } X$ modulo prime-to- p torsion (with $p \nmid m, n$ and $M|_U, N|_U \in \text{Pic } U$), then $P = l \cdot (a \cdot N + b \cdot M)$ in $\text{Cl } X$ modulo prime-to- p torsion, where: $(m) \cap (n) = (l)$, $am + bn = k$ and $(m, n) = (k)$. In other words, if m, n belong to $N^P(X^\circ)$ then so does their least common multiple l .

The following two lemmas are well-known to experts but are included for lack of a reference.

Lemma 3.31. *Let $\phi: (R, \mathfrak{m}) \rightarrow S$ be a finite extension of normal domains. Denote by S^{sh} the strict henselization of S with respect to a prime \mathfrak{n} lying over \mathfrak{m} . Then, the canonical morphism $\text{Spec } S^{\text{sh}} \rightarrow \text{Spec } S \otimes_R R^{\text{sh}}$ is a connected component, in particular a clopen (i.e., closed and open) immersion. Furthermore, assume that:*

- (1) (R, \mathfrak{m}) is a DVR,
- (2) ϕ is a generically étale extension, and
- (3) $\phi_{\mathfrak{n}}: R \rightarrow S_{\mathfrak{n}}$ has trivial residue field extension for all maximal ideals \mathfrak{n} lying over \mathfrak{m} .

Then, all the connected components of $\text{Spec } S \otimes_R R^{\text{sh}}$ arise in this way and then are in bijective correspondence with the prime ideals of S lying over \mathfrak{m} .

Proof. By [Stacks 2005–, Lemma 05WR], S^{sh} is obtained as the localization of a prime ideal of $S \otimes_R R^{\text{sh}}$ lying above \mathfrak{n} and \mathfrak{m}^{sh} . Since S is normal and $\text{Spec } S \otimes_R R^{\text{sh}} \rightarrow \text{Spec } S$ is a colimit of étale morphisms, $S \otimes_R R^{\text{sh}}$ is normal by [Stacks 2005–, Lemmas 033C and 037D]. Hence, it is a product of normal domains.

Moreover, $S \otimes_R R^{\text{sh}}$ is a finite algebra over the henselian local ring R^{sh} and thus by [Stacks 2005–, Lemma 04GG (10)] we have

$$S \otimes_R R^{\text{sh}} = \prod_{i=1}^m (S \otimes_R R^{\text{sh}})_{\mathfrak{m}_i}$$

where $\mathfrak{m}_1, \dots, \mathfrak{m}_m$ are the maximal ideals of $S \otimes_R R^{\text{sh}}$ lying over \mathfrak{m}^{sh} . We conclude that $S \otimes_R R^{\text{sh}}$ is a finite product of normal local domains. Since any prime of $S \otimes_R R^{\text{sh}}$ lying above \mathfrak{n} and \mathfrak{m}^{sh} is necessarily maximal, $\text{Spec } S^{\text{sh}} \rightarrow \text{Spec } S \otimes_R R^{\text{sh}}$ is a clopen immersion.

Finally, we discuss the statement regarding the case (R, \mathfrak{m}) is a DVR. Set $(u) = \mathfrak{m}$. In this case, S is a semilocal Dedekind domain and in particular a PID; let $\mathfrak{n}_1, \dots, \mathfrak{n}_n$ be the maximal ideals of S lying over \mathfrak{m} . Let K be the function field of R , so that $\text{Spec } K \rightarrow \text{Spec } R$ defines the open immersion given by the principal open $D(u) = \text{Spec } R_u$. Observe that $(R^{\text{sh}}, \theta(u))$ is a (strictly henselian) DVR as well, where θ is the canonical homomorphism $R \rightarrow R^{\text{sh}}$. With this being said, we see that pullback of the cartesian square

$$\begin{array}{ccc} \text{Spec } S & \longleftarrow & \text{Spec } S \otimes_R R^{\text{sh}} \\ \downarrow & & \downarrow \\ \text{Spec } R & \longleftarrow & \text{Spec } R^{\text{sh}} \end{array}$$

to the Zariski open $\text{Spec } K \rightarrow \text{Spec } R$ is given by the cartesian square

$$\begin{array}{ccc} \text{Spec } L & \longleftarrow & \text{Spec } L \otimes_K K(R^{\text{sh}}) \\ \downarrow & & \downarrow \\ \text{Spec } K & \longleftarrow & \text{Spec } K(R^{\text{sh}}) \end{array}$$

where L denotes the fraction field of S . In particular, the generic rank of the finite R^{sh} -algebra $S \otimes_R R^{\text{sh}}$ is equal to $[L : K]$ —the generic rank of S over R . In particular, we have that

$$[L : K] = \sum_{i=1}^m [K((S \otimes_R R^{\text{sh}})_{\mathfrak{m}_i}) : K(R^{\text{sh}})] = \sum_{i=1}^n [K(S_{\mathfrak{n}_i}^{\text{sh}}) : K(R^{\text{sh}})] + \Sigma,$$

where Σ is the remaining summands, i.e., the sum corresponding to the (*a priori* possible) connected components that are not isomorphic to strict henselizations of S at some of its maximal ideals. Our goal is to prove that $\Sigma = 0$ (i.e., it is an empty summation). To this end, observe that, by combining assumptions (a) and (c) with [Stacks 2005–, Remark 09E8], we have $[L : K] = \sum_{i=1}^n e_i$ where e_i is the ramification index of the extension of DVRs $\phi_{\mathfrak{n}_i} : R \rightarrow S_{\mathfrak{n}_i}$. Hence, it suffices to prove $[K(S_{\mathfrak{n}_i}^{\text{sh}}) : K(R^{\text{sh}})] = e_i$. Observe that the ramification index of $R^{\text{sh}} \rightarrow S_{\mathfrak{n}_i}^{\text{sh}}$ is exactly e_i and its residue field extension is trivial (it is tacitly assumed here that the residue field of both is the same separable closure of $R/\mathfrak{m} = S_{\mathfrak{n}_i}/\mathfrak{n}_i S_{\mathfrak{n}_i}$). The result then follows from [Stacks 2005–, Remark 09E8]. \square

Example 3.32. We may use Lemma 3.31 to argue the part in the proof of Claim 3.15 where we explain why there cannot be only one prime of S lying over (Δ) . Indeed, if there were only one such a prime $\mathfrak{q} \subset S$, we saw that the degree 6 extension of DVRs $R_{(\Delta)} \rightarrow S_{\mathfrak{q}}$ has ramification index 6 and Galois group isomorphic to S_3 . However, when we apply Lemma 3.31 and its proof we obtain that $R_{(\Delta)}^{\text{sh}} \rightarrow S_{\mathfrak{q}}^{\text{sh}}$ is a degree 6 extension with Galois group S_3 . Nevertheless, this contradicts Theorem 3.5 as it states that the Galois group must be cyclic.

Either directly or indirectly, we know that there must be three primes $\mathfrak{q}_1 = (t_2 - t_3)$, $\mathfrak{q}_2 = (t_1 - t_3)$, and $\mathfrak{q}_3 = (t_1 - t_2)$ of S lying over $(\Delta) \subset R$, all of them with ramification index 2 and inertial degree 1. As predicted by Lemma 3.31, we can see directly that

$$S \otimes_R R_{(\Delta)}^{\text{sh}} \cong S_{\mathfrak{q}_1}^{\text{sh}} \times S_{\mathfrak{q}_2}^{\text{sh}} \times S_{\mathfrak{q}_3}^{\text{sh}},$$

where each extension $R_{(\Delta)}^{\text{sh}} \subset S_{\mathfrak{q}_i}^{\text{sh}}$ is a degree 2 Kummer extension of strictly local DVRs. Indeed, denoting $\varpi_i := 3t_i^2 + x$, we have from Example 3.14 that $R[\delta, t_i]_{\varpi_i} = S_{\varpi_i}$, and moreover $\text{Spec } S = \bigcup_{i=1}^3 \text{Spec } S_{\varpi_i}$ where $\mathfrak{q}_i \in \text{Spec } S_{\varpi_j}$ if and only if $i = j$ (this follows from (3.15.1) and the argument in the succeeding paragraph). This is nothing but an open covering of $\text{Spec } S$ by standard étale morphisms over $\text{Spec } R[\delta]$ see [Milne 1980, I, Theorem 3.14]. In fact, the morphisms $\text{Spec } S_{\varpi_i} \rightarrow \text{Spec } R[\delta]$ are étale neighborhoods of $(\delta) \subset \text{Spec } R[\delta]$. In particular, the canonical homomorphism $S_{\varpi_i} \rightarrow S_{\mathfrak{q}_i}$ is an isomorphism when twisted by $R[\delta]_{(\delta)}^{\text{sh}}$ — the strict henselization of $R[\delta]$ at (δ) — which is then canonically isomorphic to each of $S_{\mathfrak{q}_i}^{\text{sh}}$. Finally, one verifies directly that the canonical homomorphism $R[\delta] \otimes_R R_{(\Delta)}^{\text{sh}} \rightarrow R[\delta]_{(\delta)}^{\text{sh}}$ is an isomorphism.

Finally, we point out that hypothesis (c) in Lemma 3.31 is (trivially) crucial for the proposition to hold. Indeed, suppose that $R \rightarrow S$ is a finite étale extension of DVRs (i.e., $n, e = 1$ in Proposition 3.12). Then, the generic and inertial degrees coincide; denote them by d . However, $S \otimes_R R^{\text{sh}}$ is product of d copies of R^{sh} . Roughly speaking, we get d connected components of $\text{Spec } S \otimes_R R^{\text{sh}}$ out of just one prime lying over the maximal ideal of R (both are a degree 2 Kummer cover $R_{(\delta)}^{\text{sh}}$ with respect to $\text{div } \Delta$). This concludes the example.

Lemma 3.33. *Let $f: Y \rightarrow X$ be a degree d finite cover of normal integral schemes. Suppose that $f_*\mathcal{O}_Y$ is locally free on some big open $U \subset X$ (i.e., $X \setminus U$ has codimension ≥ 2). Then, the kernel of $f^*: \text{Pic } X \rightarrow \text{Pic } Y$ is d -torsion. In particular, f^* maps nontorsion elements into nontorsion elements.*

Proof. Let \mathcal{L} be an invertible sheaf on X such that $f^*\mathcal{L} \cong \mathcal{O}_Y$. Then, $f_*f^*\mathcal{L} \cong f_*\mathcal{O}_Y$. Nonetheless, $f_*f^*\mathcal{L} \cong f_*(\mathcal{O}_Y \otimes f^*\mathcal{L}) \cong \mathcal{L} \otimes f_*\mathcal{O}_Y$ by the projection formula. Hence, we have an isomorphism $\mathcal{L} \otimes f_*\mathcal{O}_Y \cong f_*\mathcal{O}_Y$. Note that the rank of $f_*\mathcal{O}_Y$ is d . By letting $V = f^{-1}(U)$ and taking determinants we have $\det f_*\mathcal{O}_V \cong \det(\mathcal{L}_V \otimes f_*\mathcal{O}_V) = \mathcal{L}_V^d \otimes \det f_*\mathcal{O}_V$. Therefore, $\mathcal{L}_V^d \cong \mathcal{O}_V$. Since X is normal and $\text{codim } X \setminus U \geq 2$, we conclude that $\mathcal{L}^d \cong \mathcal{O}_X$. □

The following observation plays a crucial role in our main theorem. It can be thought of as a singular Abhyankar’s lemma for prime divisors.

Lemma 3.34. *Work in Setup 3.9 and suppose $\text{Rev}^P(X^\circ)$ to be P -irreducible. Then, every totally ramified cyclic Galois object in $\text{Rev}^P(X^\circ)$ of prime-to- p degree n is (up to isomorphism) a Kummer-type cyclic cover (as in Example 3.11) of degree $n \in M^P(X^\circ)$. In particular, if $P = 0 \in \text{Cl } X$; say $\mathfrak{p} = (f)$, and $\text{Cl } X$ has no prime-to- p torsion, then every totally ramified Galois cover in $\text{Rev}^P(X^\circ)$ is isomorphic to a Kummer cover of the form $\text{Spec } \mathcal{O}_{X^\circ}[T]/(T^n - f) \rightarrow X^\circ$ (with n prime to the characteristic).*

Proof. Let $(R, \mathfrak{m}, \ell, K, P) \subset (S, \mathfrak{n}, \ell, L, Q)$ in $\text{Rev}^P(X^\circ)$ be a totally ramified cyclic Galois cover of degree n with $p \nmid n$. Its pullback to U induces a connected $\mathbb{Z}/n\mathbb{Z}$ -torsor $V \rightarrow U$ (where $V = Y^\circ \setminus Q$ and $U = X^\circ \setminus P$). Therefore, by Kummer theory [Milne 1980, III, Section 4] and using that R contains a primitive n -th root of unity, there exists an n -torsion Cartier divisor D on U such that $f_U: V \rightarrow U$ is the cyclic cover defined by the spectrum of the finite \mathcal{O}_U -algebra $\mathcal{A} := \bigoplus_{i=0}^{n-1} \mathcal{O}_U(-iD)$ defined via a global section $\mathcal{O}_U \xrightarrow{\cong} \mathcal{O}_U(nD)$, say $\kappa \in K^\times$ such that $\text{div}_U \kappa + nD = 0$ (and so $\cdot \kappa: \mathcal{O}_U(-nD) \rightarrow \mathcal{O}_U$). We write $L = \mathcal{A}_K = K \otimes \mathcal{A} = K(\kappa^{1/n})$. Let \bar{D} be the closure of D in X . Thus, \bar{D} is a Weil divisor on X such that $\bar{D}|_U = D$ and $\text{div}_X \kappa + n\bar{D} = e \cdot P$ where $e = \text{val}_P \kappa$. We may assume that $e \geq 0$ (by replacing both κ and D by their respective inverses if necessary). That is, we may assume $\kappa \in R_{\mathfrak{p}}$. Note that $\text{div}_V \kappa^{1/n} + f_U^* D = 0$ on V , which is obtained by dividing by n the pullback of $\text{div}_U \kappa + nD = 0$. Further, $n \cdot \text{val}_Q \kappa^{1/n} = \text{val}_Q \kappa = n \cdot \text{val}_P \kappa$ (using the hypothesis that the extension is totally ramified) and so $\text{val}_Q \kappa^{1/n} = e \geq 0$. This lets us conclude the following.

Claim 3.35. $(n, e) = 1$.

Proof of claim. Consider the subextension $R_{\mathfrak{p}} \subset R_{\mathfrak{p}}[\kappa^{1/n}] \subset S_{\mathfrak{p}} = S_{\mathfrak{q}}$. Note that $R_{\mathfrak{p}} \subset R_{\mathfrak{p}}[\kappa^{1/n}]$ is a free local extension of rank n , where the maximal ideal of $R_{\mathfrak{p}}[\kappa^{1/n}]$ is $(t, \kappa^{1/n})$ with t a uniformizer of $R_{\mathfrak{p}}$. Also, note that $R_{\mathfrak{p}}[\kappa^{1/n}] \subset S_{\mathfrak{q}}$ is birational. In particular, $S_{\mathfrak{q}}$ is the normalization of $R_{\mathfrak{p}}[\kappa^{1/n}]$. However, this can only happen if $(n, e) = 1$. Indeed, we may base change $R_{\mathfrak{p}} \subset R_{\mathfrak{p}}[\kappa^{1/n}] \subset S_{\mathfrak{q}}$ by $R_{\mathfrak{p}}^{\text{sh}}$ to obtain $R_{\mathfrak{p}}^{\text{sh}} \subset R_{\mathfrak{p}}^{\text{sh}}[\kappa^{1/n}] \subset S_{\mathfrak{q}}^{\text{sh}}$ using Lemma 3.31(c). Nonetheless,

$$R_{\mathfrak{p}}^{\text{sh}}[\kappa^{1/n}] \rightarrow (R[\kappa^{1/n}])_{(t, \kappa^{1/n})}^{\text{sh}}$$

is an isomorphism as $R_{\mathfrak{p}}^{\text{sh}}[\kappa^{1/n}]$ is a strictly local algebra over $R[\kappa^{1/n}]$ (use [Milne 1980, I, Corollary 4.3]) with the same residue field as $R_{\mathfrak{p}}^{\text{sh}}$. Therefore, $R_{\mathfrak{p}}^{\text{sh}}[\kappa^{1/n}] \subset S_{\mathfrak{q}}^{\text{sh}}$ is a normalization by [Stacks 2005–, Tag 0CBM]. However, $R_{\mathfrak{p}}^{\text{sh}}[\kappa^{1/n}] \cong R_{\mathfrak{p}}^{\text{sh}}[T]/(T^n - \kappa) \cong R_{\mathfrak{p}}^{\text{sh}}[T]/(T^n - t^e)$ using that $\kappa = u \cdot t^e$ for some unit $u \in R_{\mathfrak{p}}$ (the latter isomorphism is of course $T \leftrightarrow T/u^{1/n}$). Hence, $(n, e) = 1$ for $R_{\mathfrak{p}}^{\text{sh}}[\kappa^{1/n}]$ is a domain. \square

Thus, there are $a, b \in \mathbb{Z}$ such that $1 = an + be$ and so

$$P = (an + be) \cdot P = n(a \cdot P + b \cdot \bar{D}) + \text{div}_X \kappa^b.$$

Further, $(a \cdot P + b \cdot \bar{D})|_U = b \cdot D \in \text{Pic } U$ and so $n \in M^P(X^\circ)$. Now, the above establishes that $K(\kappa^{b/n})/K$ defines (after taking integral closure) an object in $\text{Rev}^P(X^\circ)$ that is a cyclic cover of Kummer-type. However, $L = K(\kappa^{1/n}) = K(\kappa^{b/n})$ as $(b, n) = 1$ (for $1 = an + be$). Then, S/R in $\text{Rev}^P(X^\circ)$ is a cyclic cover of Kummer-type and $n \in M^P(X^\circ)$, as required.

For the last statement when $\mathfrak{p} = (f)$, see Example 3.11. □

Proof of Theorem 3.29. We will subdivide the proof in two parts. First, we prove the statements of Theorem 3.29 except for those establishing when the short exact sequences split. Once this is done, we proceed to show the statements of Theorem 3.29 characterizing when the given short exact sequences split.

We start with the first part now. By formal properties of Galois categories, we obtain from Lemma 3.25 a short exact sequence of topological groups

$$1 \rightarrow \pi_1^{t, \tilde{P}}(\tilde{X}^\circ) \rightarrow \pi_1^{t, P}(X^\circ) \rightarrow \text{Gal}(\tilde{X}^\circ/X^\circ) \rightarrow 1$$

where $G := \pi_{1, \text{ét}}^P(X^\circ) = \text{Gal}(\tilde{X}^\circ/X^\circ)$. Let d be its order.

Claim 3.36. *By replacing X by \tilde{X} , we may assume that: G is trivial, Galois objects have inertial degree equal to 1, if P is prime-to- p torsion then it is trivial, and further $N^P(X^\circ) = M^P(X^\circ)$.*

Proof of claim. Consider the induced homomorphism $\tilde{f}^*: \text{Cl } X \rightarrow \text{Cl } \tilde{X}$. Then, $\tilde{f}^*: P \mapsto \tilde{P}$. Since \tilde{f} is quasiétale, its restriction to X_{reg}° is a Galois étale cover by the purity of the branch locus. By Lemma 3.33, $\ker \tilde{f}^*$ is d -torsion and \tilde{P} is nontorsion if so is P (as $\text{Cl } X$ is the same as $\text{Pic } U$ for any regular big open $U \subset X$).

Let $\phi: \text{Cl } \tilde{X} \rightarrow \text{Cl } \tilde{U}$ be the restriction homomorphism and set $\Lambda := \phi^{-1}(\text{Pic } \tilde{U})$ (where $\tilde{U} = \tilde{X}^\circ \setminus \tilde{P}$ and so on). We observe that Λ has no prime-to- p torsion. Indeed, let D be a divisor on \tilde{X} such that $D \in \Lambda$ and $D \in \text{Cl } \tilde{X}$ has prime-to- p torsion, say of index $p \nmid n > 1$. Then, the corresponding Veronese-type cyclic cover $\tilde{R} \rightarrow \bigoplus_{i=0}^{n-1} \tilde{R}(-iD)$ induces a quasiétale degree n Galois object in $\text{Rev}^{\tilde{P}}(\tilde{X}^\circ)$, which violates the universality of \tilde{f} .

Now, if $P \in \text{Cl } X$ is torsion, then so is $\tilde{P} \in \text{Cl } \tilde{X}$ and its order divides the one of P . Thus, if $P \in \text{Cl } X$ is prime-to- p torsion then $\tilde{P} \in \text{Cl } \tilde{X}$ is trivial as $\tilde{P} \in \Lambda$.

Finally, we must explain why

$$N^P(X^\circ) = \{n \mid p \nmid n, \tilde{P} = n \cdot N \text{ with } N \in \Lambda\} = N^{\tilde{P}}(\tilde{X}^\circ)$$

The second equality follows from Λ having no prime-to- p torsion as $\tilde{P} - n \cdot N \in \Lambda$ if $N \in \Lambda$. The first equality is obtained as follows. The inclusion “ \subset ” is clear. Indeed, if $P = n \cdot M + T$ in $\text{Cl } X$ where $T \in \text{Cl } X$ has prime-to- p torsion and $M|_U \in \text{Pic } U$. Then, $\tilde{P} = n \cdot \tilde{f}^*M + \tilde{f}^*T$ where $\tilde{f}^*M \in \Lambda$ and $\tilde{f}^*T = 0$ as it is a prime-to- p torsion element of Λ . Conversely, suppose that $\tilde{P} = n \cdot N$ in $\text{Cl } \tilde{X}$ for some $N \in \Lambda$. Let us pullback everything to $W := X_{\text{reg}}^\circ$ whose inverse image under \tilde{f} we denote by \tilde{W} , which is a regular big open of \tilde{X} . Then, by using the Hochschild–Serre spectral sequence [Milne 1980, III, Theorem 2.20], the image of $\tilde{f}^*: \text{Pic } \tilde{W} \rightarrow \text{Pic } W$ lies inside $(\text{Pic } \tilde{W})^G$. As in Lemma 3.33, we may consider the norm homomorphism $N_{\tilde{f}}: \text{Pic } \tilde{W} \rightarrow \text{Pic } W$, which is obtained by applying H^1 to the norm morphism of multiplicative groups $f_*\mathcal{O}_{\tilde{W}}^\times \rightarrow \mathcal{O}_W^\times$ (as subsheaves of $\tilde{K} \supset K$ respectively); see [Stacks 2005–, Tag 0BCX] for details. The key property is that the composition

$$\text{Pic } W \xrightarrow{\tilde{f}^*} \text{Pic } \tilde{W} \xrightarrow{N_{\tilde{f}}} \text{Pic } W$$

is multiplication-by- d . However, since \tilde{f} is Galois, $N_{\tilde{f}}((\text{Pic } \tilde{W})^G) \subset d \cdot \text{Pic } W$ by the same principle and the same for any open of X in place of W . Treating the equality $\tilde{P} = n \cdot N$ in $\text{Pic } \tilde{W}$, on the left hand side we have an element of $(\text{Pic } \tilde{W})^G$ as $\tilde{P} = \tilde{f}^* P$. Since, Λ has no prime-to- p torsion and n is prime-to- p , this implies that $N \in (\text{Pic } \tilde{W})^G$. Therefore, by taking norms, we get $d \cdot P = d \cdot n \cdot M$ in $\text{Pic } W$ where $d \cdot M = N_{\tilde{f}}(N)$, which implies that $M|_U \in \text{Pic } U$ as $N|_{\tilde{U}} \in \text{Pic } \tilde{U}$. Thus, $P = n \cdot M + T$ where T is d -torsion and $M|_U \in \text{Pic } U$. \square

With the above reductions in place, we let $X' := \text{Spec } \mathcal{O}_{X,P}^{\text{sh}}$ be the étale germ of X at (the generic point of) P . Note that $\mathcal{O}_{X,P}^{\text{sh}}$ is none other than the strict henselization of $\mathcal{O}_{X,P} = R_{\mathfrak{p}}$ at its maximal ideal. We argue next that the canonical morphism $X' \rightarrow X$ induces a surjection of fundamental groups

$$\eta: \pi_1^{t,P'}(X'^{\circ}) \rightarrow \pi_1^{t,P}(X^{\circ})$$

where P' is the divisor on X' corresponding to its codimension 1 closed point and X'° is the inverse image of X° along $X' \rightarrow X$.

Claim 3.37. *The pullback functor $\text{Rev}^P(X^{\circ}) \rightarrow \text{Rev}^{P'}(X'^{\circ})$ induces a surjective homomorphism of topological groups $\eta: \pi_1^{t,P'}(X'^{\circ}) \rightarrow \pi_1^{t,P}(X^{\circ})$.*

Proof of claim. Note that the pullback functor is well-defined by Lemma 3.3. By the abstract nonsense regarding Galois categories, the first statement amounts to proving the compatibility between the fiber or fundamental functors; see [Murre 1967, Chapter 5]. Recall that, implicitly, we always take our base point to be some fixed separable closure K^{sep} of K . We are going to choose the base point of $\text{Rev}^{P'}(X'^{\circ})$ compatibly, i.e., so that we have a commutative diagram:

$$\begin{array}{ccccc} R_{\mathfrak{p}}^{\text{sh}} & \longrightarrow & K(R_{\mathfrak{p}}^{\text{sh}}) & \longrightarrow & K(R_{\mathfrak{p}}^{\text{sh}})^{\text{sep}} \\ \theta \uparrow & & \uparrow \theta_K & & \uparrow \theta_K^{\text{sep}} \\ R & \longrightarrow & K & \longrightarrow & K^{\text{sep}} \end{array}$$

Equivalently, we choose K^{sep} to be the subfield of $K(R_{\mathfrak{p}}^{\text{sh}})^{\text{sep}}$ of elements that are algebraic separable over K . To simplify notation, we denote the rings on the top of the diagram from left to right; respectively, by R' , K' , and K'^{sep} . Recall that the fiber functor $\mathcal{F}: \text{Rev}^P(X^{\circ}) \rightarrow \text{FSet}$ is given by

$$\mathcal{F}(S/R) = \text{Hom}_{R\text{-alg}}(S, K) = \text{Hom}_{K\text{-alg}}(L, K^{\text{sep}})$$

for all S/R connected in $\text{Rev}^P(X^{\circ})$. Of course, the same definition applies to the fiber functor $\mathcal{F}': \text{Rev}^{P'}(X'^{\circ}) \rightarrow \text{FSet}$ with K'^{sep} in place of K^{sep} and so on. We need to verify the commutativity of the following diagram of functors:

$$\begin{array}{ccc} \text{Rev}^P(X^{\circ}) & \longrightarrow & \text{Rev}^{P'}(X'^{\circ}) \\ & \searrow \mathcal{F} & \swarrow \mathcal{F}' \\ & \text{Set} & \end{array}$$

where the horizontal arrow is the pullback functor; see [Murre 1967, Section 5.1, Example]. To this end, we perform the following computation with S/R a connected object of $\text{Rev}^P(X^\circ)$:

$$\begin{aligned} \mathcal{F}'(S \otimes_R R') &= \text{Hom}_{R'\text{-alg}}(S \otimes_R R', K'^{\text{sep}}) = \text{Hom}_{R\text{-alg}}(S, K'^{\text{sep}}) = \text{Hom}_{K\text{-alg}}(L, K'^{\text{sep}}) \\ &= \text{Hom}_{K\text{-alg}}(L, K^{\text{sep}}) \\ &= \mathcal{F}(S). \end{aligned}$$

where the penultimate equality follows from the compatibility in our choices of base points. Indeed, since L/K is a finite separable extension, any K -embedding of L into K'^{sep} is going to be contained in K^{sep} — the subfield of K'^{sep} of separable elements over K .

Finally, we explain why η is surjective. According to the abstract nonsense [Murre 1967, Section 5.2.1], η is surjective if and only if the pullback of connected objects is connected. Hence, the surjectivity of η is a simple consequence of the equality $S \otimes_R R' = S_q^{\text{sh}}$ provided by Lemma 3.31 — once we know there is only one prime lying over with trivial inertial degree. \square

As a direct application of Theorem 3.5; see [Milne 1980, I, Section 5, Remark 5.1(e)], we have that

$$\pi_1^{t,P'}(X'^\circ) = \varprojlim_{p \nmid n} \mu_n(K') \xrightarrow{\cong} \varprojlim_{p \nmid n} \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}^{(p)},$$

where it is worth noting that the isomorphism is not canonical as it depends on choices of compatible primitive roots of unity of K' in K'^{sep} . We have constructed a (noncanonical) surjective homomorphism of topological groups

$$\eta': \hat{\mathbb{Z}}^{(p)} \xrightarrow{\cong} \pi_1^{t,P'}(X'^\circ) \twoheadrightarrow \pi_1^{t,P}(X^\circ).$$

which explains (3.29.1). To describe $\ker \eta'$ (resp. $\text{Image } \eta'$), consider the following assertion.

Claim 3.38. *There is a factorization of continuous homomorphisms*

$$\begin{array}{ccc} \hat{\mathbb{Z}}^{(p)} & \xrightarrow{\eta'} & \pi_1^{t,P}(X^\circ) \\ \text{can} \downarrow & \nearrow \eta'' & \\ \varprojlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z} & & \end{array}$$

where $N^P(X^\circ) = \{n \in \mathbb{N} \mid p \nmid n, P = n \cdot D \text{ with } D|_U \in \text{Pic } U\}$.

Proof of claim. We may assume that $P \in \text{Cl } X$ is nontrivial and so nontorsion. Since η' is surjective, a Galois object $(R, \mathfrak{m}, \zeta, K, P) \subset (S, \mathfrak{n}, \zeta, L, Q)$ in $\text{Rev}^P(X^\circ)$ is cyclic, i.e., $\text{Gal}(L/K) \cong \mathbb{Z}/n\mathbb{Z}$. Then, Claim 3.38 amounts to $n \in N^P(X^\circ)$ whenever such cover exists and so the claim follows from Lemma 3.34. \square

Claim 3.39. *The homomorphism η'' in Claim 3.38 is injective.*

Proof of claim. By [Murre 1967, Section 5.2.4], η'' is injective if and only if for all $n \in N^P(X^\circ)$ there exists a cover in $\text{Rev}^P(X^\circ)$ whose pullback to X' (has a connected component that) is a Kummer cover $\mathcal{O}_{X,P}^{\text{sh}} \subset \mathcal{O}_{X,P}^{\text{sh}}[t^{1/n}]$. To this end, for $n \in N^P(X^\circ)$, let us set $\text{div}_X \kappa + n \cdot D = P$ so that $D|_U \in \text{Pic } U$. We

then invoke Example 3.11, Definition-Proposition 3.4. That is, we embed $K(\kappa^{1/n})$ in K^{sep} and note that $K(\kappa^{1/n})/K \in \text{Rev}^P(X^\circ)$ has the required property. \square

With the above in place, we explain next why the short exact sequence of topological groups

$$0 \rightarrow \varinjlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow G \rightarrow 1$$

splits if and only if the containment $N^P(X^\circ) \supset M^P(X^\circ)$ is an equality and there are divisors $\frac{1}{n}P$ on X for all $n \in M^P(X^\circ)$ such that: $\frac{1}{1}P = P$ and $m(\frac{1}{mn}P) = \frac{1}{n}P$ in $\text{Cl } X$ for all $m, n \in \mathbb{N}$ so that $mn \in M^P(X^\circ)$. Note that $M^P(X^\circ)$ is an (inversely) directed subset $N^P(X^\circ)$ (with respect to divisibility); see Remark 3.30.

Suppose that $N^P(X^\circ) = M^P(X^\circ)$ and the existence of a compatible system $\{\frac{1}{n}P\}_{n \in M^P(X^\circ)}$ of quotients of P . Recall that $\tilde{f}: \tilde{X} \rightarrow X$ denotes the universal cover of $\text{Rev}_{1,\text{ét}}^P(X^\circ)$. The equality $N^P(X^\circ) = M^P(X^\circ)$ means that for every $n \in N^P(X^\circ)$ there exists $\kappa_n \in K^\times$ such that $\text{div}_{\tilde{X}} \kappa_n + n \cdot \tilde{f}^* D_n = \tilde{P}$ where D_n is a divisor on X such that $D_n|_U$ is Cartier. Although the choice of κ_n is not unique, the cyclic field extension $\tilde{K} \subset \tilde{K}(\kappa_n^{1/n}) \subset K^{\text{sep}}$ is. That is, $\tilde{K}_n := \tilde{K}(\kappa_n^{1/n})$ is independent of the choice of κ_n (which is not necessarily true for $K(\kappa_n^{1/n})$). In fact, these are precisely the Galois objects of $\text{Rev}^{\tilde{P}}(\tilde{X}^\circ)$ with field of fractions inside K^{sep} . Consider the field $\tilde{K}_\infty := \bigcup_{n \in N^P(X^\circ)} \tilde{K}_n \subset K^{\text{sep}}$ (so $\pi_1^{t,P}(X^\circ) = \text{Gal}(\tilde{K}_\infty/K)$). It is worth observing that $K(\kappa_n^{1/n}) \cdot \tilde{K} = \tilde{K}_n$ and $K(\kappa_n^{1/n}) \cap \tilde{K} = K$ (as the normalization of R in $K(\kappa_n^{1/n})$ is totally ramified whereas in \tilde{K} it is quasiétale). By Galois theory [Lang 2002, VI, Section 1, Theorem 1.12], this implies that $\tilde{K}_n/K(\kappa_n^{1/n})$ is Galois and the homomorphism

$$\text{Gal}(\tilde{K}_n/K(\kappa_n^{1/n})) \rightarrow \text{Gal}(\tilde{K}/K) = G, \quad \sigma \mapsto \sigma|_{\tilde{K}}$$

is an isomorphism. In other words, there is an action of G on \tilde{K}_n by K -automorphisms so that $\tilde{K}_n^G = K(\kappa_n^{1/n})$. By the same token, since $K(\kappa_n^{1/n}/K)$ is Galois, the exact sequence

$$1 \rightarrow \text{Gal}(\tilde{K}_n/\tilde{K}) \rightarrow \text{Gal}(\tilde{K}_n/K) \rightarrow G \rightarrow 1$$

then splits (as a direct product) for every n ; see [Lang 2002, VI, Section 1, Theorem 1.14]. Roughly speaking, with the equality $N^P(X^\circ) = M^P(X^\circ)$, we can split the quotient $\text{Gal}(\tilde{K}_\infty/K) \twoheadrightarrow G$ at each finite level quotient $\text{Gal}(\tilde{K}_n/K) \twoheadrightarrow G$. To do this globally, we need these splittings to be compatible under divisibility in $M^P(X^\circ)$. This is precisely what the additional hypothesis regarding the existence of $\{\frac{1}{n}P\}_{n \in M^P(X^\circ)}$ accomplishes. Indeed, by setting $D_n = \frac{1}{n}P$ and $K_n := K(\kappa_n^{1/n})$, the field extensions $\{K_n/K\}_{n \in N^P(X^\circ)}$ yield a projective system in $\text{Rev}^P(X^\circ)$ and we may then set the field $K_\infty := \bigcup_{n \in N^P(X^\circ)} K_n \subset K^{\text{sep}}$. Then, K_∞/K is Galois, $K_\infty \cdot \tilde{K} = \tilde{K}_\infty$, and $K_\infty \cap \tilde{K} = K$ (as $K(\kappa_n^{1/n}) \cap \tilde{K} = K$). As before, this implies that $\tilde{K}_\infty/K_\infty$ is Galois and the homomorphism

$$\text{Gal}(\tilde{K}_\infty/K_\infty) \rightarrow \text{Gal}(\tilde{K}/K) = G, \quad \sigma \mapsto \sigma|_{\tilde{K}}$$

is an isomorphism and the exact sequence

$$1 \rightarrow \text{Gal}(\tilde{K}_\infty/\tilde{K}) \rightarrow \text{Gal}(\tilde{K}_\infty/K) \rightarrow G \rightarrow 1$$

splits, as required. This shows the direction “ \longleftarrow ”.

Conversely, suppose that there is a projection $\pi_1^{t,P}(X^\circ) \rightarrow \varprojlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z}$ splitting the inclusion $\varprojlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z} \hookrightarrow \pi_1^{t,P}(X^\circ)$. In particular, for each $n \in N^P(X^\circ)$, there is a cyclic Galois cover $f_n: Y_n^\circ \rightarrow X_n^\circ$ whose ‘‘Galois-theoretic’’ pullback to $\text{Rev}^{\tilde{P}}(\tilde{X}^\circ)$ is a cyclic Galois cover. Thus, f_n must be totally ramified (as can be seen by using the inertial decantation property). Therefore, each f_n is a cyclic cover of Kummer-type by Lemma 3.34. In particular, $N^P(X^\circ) \subset M^P(X^\circ)$. It remains to explain why we have a compatible system of elements $\{\frac{1}{n}P\}$ in $\text{Cl } X$. To this end, assume that f_n is given by $\text{div } \kappa_n + nD_n = P$ (this for each n). We have an inclusion $K(\kappa_n^{1/n}) \subset K(\kappa_{nm}^{1/nm})$. Since the latter extension is cyclic, $K(\kappa_{nm}^{1/n}) = K(\kappa_n^{1/n})$. Hence, by Kummer theory [Milne 1980, see page 126], $\kappa_n/\kappa_{nm} \in (K^\times)^n$; say $\kappa_n/\kappa_{nm} = x^n$. From this we deduce that $n \text{div } x + n(D_n - mD_{nm}) = 0$ and so $\text{div } x + (D_n - mD_{nm}) = 0$, as equalities in $\text{Div } X$. Hence, $D_n = mD_{nm}$ in $\text{Cl } X$. In particular, we may take $\frac{1}{n}P := D_n$ for all n .

Finally, assuming $P \in \text{Cl } X$ is prime-to- p torsion, we still need to show that $M^P(X^\circ) = \mathbb{Z}_{>0}^{(p)}$ if and only if $P = 0$ in $\text{Cl } X$. The ‘‘if’’ direction is clear. Conversely, assume to the contrary that P is nontrivial but $M^P(X^\circ) = \mathbb{Z}_{>0}^{(p)}$. Let ℓ be the order/index of P in $\text{Cl } X$. Since $\ell^a \in M^P(X^\circ)$ for any positive integer a , we find a divisor D_a which is Cartier on U such that $P = \ell^a D_a \in \text{Cl } X$. Let o_a be the order of D_a . In particular, $o_a \mid \ell^{a+1}$ but $o_a \nmid \ell^a$. Let $\ell = \ell_1^{s_1} \cdots \ell_r^{s_r}$ be the prime factorization of ℓ ($\ell_i \neq p$). Then, there is some index i (depending on a) such that $\ell_i^{s_i(a+1)} \mid o_a$. Since a is arbitrary, $\ell_i \geq 2$, and $s_i \geq 1$, we conclude that o_a is arbitrarily large. On the other hand, we may consider the quasiétale Veronese-type cyclic cover defined via $\text{div}_X \kappa + o_a \cdot D_a = 0$. These then yield objects of $\text{Rev}_{1,\text{ét}}^P(X^\circ)$ of arbitrarily large degree, which contradicts the already proven finiteness of $\pi_{1,\text{ét}}^P(X^\circ)$.

This demonstrates Theorem 3.29. □

Remark 3.40. The homomorphism η in Claim 3.37 can be defined more succinctly as follows. Recall that $\pi_1^{t,P}(X^\circ)$ is the limit $\varprojlim \text{Gal}(L/K)$ traversing all the finite Galois extensions $K \subset L \subset K^{\text{sep}}$ so that the integral closure S/R of R in L is tamely ramified with respect to P , and *verbatim* for $\pi_1^{t,P'}(X'^\circ)$, where we have fixed $K^{\text{sep}} \subset K'^{\text{sep}}$. For any such a L/K , we must define compatible homomorphisms $\pi_1^{t,P'}(X'^\circ) \rightarrow \text{Gal}(L/K)$. Since L/K is Galois, $\text{Gal}(L/K)$ acts transitively and faithfully on $\mathcal{F}(S) = \text{Hom}_{K\text{-alg}}(L, K^{\text{sep}})$. Nonetheless, as noticed in Claim 3.37, this set is the same set as

$$\mathcal{F}'(S \otimes_R R') = \text{Hom}_{R'\text{-alg}}(S \otimes_R R', K'^{\text{sep}}) = \coprod_i \text{Hom}_{R'\text{-alg}}(S_i, K'^{\text{sep}}) = \coprod_i \text{Hom}_{K'\text{-alg}}(K(S_i), K'^{\text{sep}})$$

where $S \otimes_R R' = \prod_i S_i$ is the decomposition of $S \otimes_R R'$ as a finite product of normal, local, and finite R' -algebras; see the proof of Lemma 3.31. Of course, the given inclusion $K \subset L \subset K^{\text{sep}}$ is an element of this set, say ξ . Therefore, ξ is contained in one and only one of the displayed disjoint sets; let i_0 denote the corresponding index. Letting L_{i_0} be the Galois closure of $K(S_{i_0})$ in K'^{sep} , we have that $\text{Gal}(L_{i_0}/K')$ surjects onto $\text{Aut}_{K'}(K(S_{i_0}))$. On the other hand, we define the homomorphism of groups $\varphi: \text{Aut}_{K'}(K(S_{i_0})) \rightarrow \text{Gal}(L/K)$ by declaring $\varphi(h)$ to be the only element of $\text{Gal}(L/K)$ that when acts on ξ yields $\xi \circ h$. In this way, we have

$$\pi_1^{t,P'}(X'^\circ) \xrightarrow{\text{can}} \text{Gal}(L_{i_0}/K') \twoheadrightarrow \text{Aut}_{K'}(K(S_{i_0})) \xrightarrow{\varphi} \text{Gal}(L/K).$$

The limit over these defines η . Observe that η is surjective if and only if these homomorphisms are all surjective, which is equivalent to the surjectivity of φ for all L/K . However, it is not difficult to see that φ is surjective if and only if $S \otimes_R R'$ is connected.

We illustrate with an example the failure of η being surjective if there were more than one prime lying over. In Example 3.32, we had a canonical isomorphism of R' -algebras

$$S \otimes_R R' = S \otimes_R R_{(\Delta)}^{\text{sh}} \xrightarrow{\cong} S_{q_1}^{\text{sh}} \times S_{q_2}^{\text{sh}} \times S_{q_3}^{\text{sh}}.$$

Note that a K -embedding of L into K'^{sep} is the same as a choice of a square root of Δ ; which in our case it was δ , and the choice of a t_i . For instance, when we chose t_1 to be our “ t ” in Example 3.14, we were choosing the R' -embedding

$$S \otimes_R R' \rightarrow S_{q_1}^{\text{sh}} \times S_{q_2}^{\text{sh}} \times S_{q_3}^{\text{sh}} \rightarrow S_{q_1}^{\text{sh}} \rightarrow K'^{\text{sep}},$$

for this is the one in which L is realized as the field of fractions of $R[\delta, t_1]_{\mathfrak{w}_1} \rightarrow S_{\mathfrak{w}_1}$. This specific embedding was our ξ all along. Now, $S_{q_1}^{\text{sh}}$ is a degree 2 Kummer cover over R' , so its Galois group is cyclic of order 2 with generator $\tau: \delta \mapsto -\delta$. On the other hand, under the canonical bijection $\mathcal{F}(S) = \mathcal{F}'(S \otimes_R R')$, we see that $\xi \circ \tau$ correspond to the K -embedding $L \xrightarrow{(23)} L \rightarrow K^{\text{sep}}$ where $(23) \in \text{Gal}(L/K) \cong S_3$ is the transposition switching t_2 and t_3 (leaving t_1 intact). In other words, we have the following commutative diagram of groups:

$$\begin{array}{ccccc} \pi_1^{t,P}(X^\circ) & \longrightarrow & \text{Gal}(L/K) & \xrightarrow{\cong} & S_3 \\ \downarrow \eta & & \uparrow \varphi & & \uparrow 1 \mapsto (23) \\ \pi_1^{t,P'}(X'^\circ) & \longrightarrow & \text{Gal}(K(S_{q_1}^{\text{sh}})/K') & \xrightarrow{\cong} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

so that η cannot be surjective. This finishes our remarks.

4. Tame fundamental groups: Positive characteristic

We proceed to our study of tame Galois categories in positive characteristic.

4A. Cohomologically tame Galois category of an F -pure singularity. We start by making a simple observation about the cohomologically tame Galois category of an F -pure singularity. This is an application of [Carvajal-Rojas and Stabler 2023, Theorem 4.8] following [Carvajal-Rojas et al. 2018].

Theorem 4.1. *Working in Setup 3.9, suppose that X is F -pure. There exists a cover $\tilde{X}^\circ \rightarrow X^\circ$ in $\text{F}\acute{\text{E}}\text{t}^{t,X}(X^\circ)$ such that, for any cover $V \rightarrow \tilde{X}^\circ$ in $\text{F}\acute{\text{E}}\text{t}^{t,\tilde{X}}(\tilde{X}^\circ)$, its integral closure $\text{Spec } S \rightarrow \text{Spec } \tilde{R}$ satisfies that its restriction $V(p(S)) \rightarrow V(p(\tilde{R}))$ is trivial.*

Proof. Note that [Carvajal-Rojas and Stabler 2023, Theorem 4.8] implies that for all connected cover $Y^\circ \rightarrow X^\circ$ in $\text{F}\acute{\text{E}}\text{t}^{t,X}(X^\circ)$ with integral closure $R \subset S$ we must have $1 \geq r(S) = [\kappa(p(S)) : \kappa(p(R))] \cdot r(R)$. In particular, we have that the generic degree of $V(p(S)) \rightarrow V(p(R))$ is no more than $1/r(R)$. Here,

we use that R is F -pure to say $1/r(R) < \infty$. By formal properties of Galois categories (just as in [Carvajal-Rojas et al. 2018]), there exists a universal cover with the required property after we notice that if the generic degree of $V(p(S)) \rightarrow V(p(R))$ is trivial then the map itself is trivial for both $R/p(R)$ and $S/p(S)$ are strongly F -regular and so normal. \square

Remark 4.2. In Theorem 4.1, notice that if $p(R) \neq 0$ then $p(R) \supset \tau(R)$. Hence, $p(R)$ corresponds to a singular point of X , for $\tau(R)$ cuts out the not strongly- F -regular locus of X . In particular, since X is normal, $\text{ht } p(R) \geq 2$. In other words, either $V(p(R))$ has codimension at least 2 or X is strongly F -regular. Hence, if X is not strongly F -regular, $V(p(R)) \subset X$ has codimension ≥ 2 . Thus, Theorem 4.1 is only interesting in higher dimensions if X is a non- F -regular F -pure singularity. In a sense, this justifies next section.

4B. Tame fundamental group of a purely F -regular local pair. In this section, we provide a study of the Galois category $\text{Rev}^P(X^\circ)$ for a purely F -regular pair (X, P) that will lead to a verification of the hypothesis in Theorem 3.29. To this end, the following three fundamental observations Proposition 4.4, Proposition 4.6, and Theorem 4.7 about covers $(R, \mathfrak{m}, \kappa, K) \subset (S, \mathfrak{n}, \ell, L)$ in $\text{Rev}^P(X^\circ)$; as in Remark 3.10, are in order.

4B1. Three fundamental properties. Consider the following setup.

Setup 4.3. Let $(R, \mathfrak{m}, \kappa, K) \subset (S, \mathfrak{n}, \ell, L)$ be a local finite extension of normal local domains with corresponding morphism of schemes $f: Y \rightarrow X$. Let $X^\circ \subset X$ be a big open (i.e., X° contains every codimension 1 point of X). Assume $f: f^{-1}(X^\circ) \rightarrow X^\circ$ to be tamely ramified with respect to a reduced divisor $D = P_1 + \cdots + P_k$ with prime components $P_i = V(\mathfrak{p}_i)$.¹⁵

We invite the reader to look at [Carvajal-Rojas and Stabler 2023, Section 3] for further details regarding transposability.

Proposition 4.4 (transposability). *Work in Setup 4.3. Then, R is a $\text{Tr}_{S/R}$ -transposable Cartier \mathcal{C}_R^D -module, where $\text{Tr}_{S/R}: S \rightarrow R$ is the (generically induced) nonzero trace map. Moreover, $f^*\mathcal{C}_R^D \subset \mathcal{C}_S^E$ where E is the reduced and effective divisor on Y supported on the prime divisors whose generic point lies over the generic point of some of the P_i .*

Proof. Since X and Y are normal, we must prove that $f^*D - \text{Ram}$ is effective; see [Carvajal-Rojas and Stabler 2023, Section 3] and [Schwede and Tucker 2014, Theorem 5.7]. Let $\mathfrak{q}_{i,1}, \dots, \mathfrak{q}_{i,n_i}$ be the prime ideals of S lying over \mathfrak{p}_i . Then,

$$f^*P_i = e_{i,1} \cdot Q_{i,1} + \cdots + e_{i,n_i} \cdot Q_{i,n_i}$$

where $Q_{i,j}$ is the divisor on Y corresponding to $\mathfrak{q}_{i,j}$, and $e_{i,j}$ is the ramification index of f along $\mathfrak{q}_{i,j}$.¹⁶ Since $\mathfrak{p}_1, \dots, \mathfrak{p}_k \in X$ are the only codimension 1 branch points, the ramification divisor Ram is supported

¹⁵It does not matter whether we think of the divisors involved as divisors on X° or on X .

¹⁶That is, $e_{i,j}$ is the order of the uniformizer of $R_{\mathfrak{p}_i}$ in the DVR $S_{\mathfrak{q}_{i,j}}$; see [Schwede and Tucker 2014, Section 2.2].

on the primes divisor $Q_{i,j}$. Moreover, since the extension is tamely ramified (over X°) with respect to D , we have that

$$\text{Ram} = \sum_{i=1}^k \sum_{j=1}^{n_i} (e_{i,j} - 1) \cdot Q_{i,j},$$

using the same computation as in [Hartshorne 1977, IV, Proposition 2.2]; see [Carvajal-Rojas 2018, Remark 2.9], compare to [Schwede and Tucker 2014, Remark 4.6]. In this way, it clearly follows that

$$D^* := f^*D - \text{Ram} = \sum_{i=1}^k f^*P_i - \text{Ram} = \sum_{i=1}^k \sum_{j=1}^{n_i} Q_{i,j} =: E \geq 0, \quad (4.4.1)$$

as required. The last statement follows by recalling $f^*\mathcal{C}_R^D \subset \mathcal{C}_S^{D^*}$ and $D^* = E$. \square

Remark 4.5. The importance of Proposition 4.4 is that we may apply [Carvajal-Rojas and Stabler 2023, Theorem 5.12 and Remark 5.13] to the pair (R, D) along the map f . In particular, we have the equality

$$\text{Tr}_{S/R}(f_*\tau_{S(-E)}(S, f^*\mathcal{C}_R^D)) = \tau_{R(-D)}(R, D)$$

Recall that $R(-D) = \bigcap_i R(-P_i) = \bigcap_i \mathfrak{p}_i$ and similarly for $S(-E)$. Using this together with Remark 2.4, we may obtain direct proofs of Proposition 4.6 and Theorem 4.7 below. Indeed, if (R, D) is purely F -regular along $R(-D)$, then $\tau_{R(-D)}(R, D) = R$ and so $\text{Tr}_{S/R}$ is surjective. Since $\text{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$, see [Carvajal-Rojas et al. 2018, Lemma 2.10; Speyer 2020, Lemma 9], we have that in that case $S = \tau_{S(-E)}(S, f^*\mathcal{C}_R^D) \subset \tau_{S(-E)}(S, E)$. In other words, the pair (S, E) is purely F -regular along E . In particular, the reduced scheme supporting E must have strongly F -regular singularities and so must be normal. Therefore, the irreducible components Q_1, \dots, Q_k cannot intersect pairwise. Nevertheless, S being local, these components intersect at the closed point. Consequently, E must have exactly one irreducible component $E = Q$. Nonetheless, we will provide below proofs for these statement using splitting primes. These proofs are more elementary than [Carvajal-Rojas and Stabler 2023, Theorem 5.12] and the authors believe this approach is valuable in its own right.

Proposition 4.6 (cohomological tameness, see [Kerz and Schmidt 2010]). *Work in Setup 4.3. Suppose that $k = 1$ and $(X, D = P)$ is purely F -regular. Then, the extension $R \subset S$ is cohomologically tame, i.e., $\text{Tr}_{S/R}: S \rightarrow R$ is surjective. Thus, ℓ/κ is separable and $[\ell : \kappa]$ divides $[L : K]$. Furthermore, if ℓ/κ is trivial, then p does not divide $[L : K]$.*

Proof. Let $E = Q_1 + \dots + Q_n$ as in Proposition 4.4 and recall that $E = f^*D - \text{Ram} \geq 0$. Notice that S is $f^*\mathcal{C}_R^P$ -compatible, so $\text{Tr}_{S/R}(S)$ is a nonzero \mathcal{C}_R^P -compatible ideal as $\varphi \circ F_* \text{Tr}_{S/R} = \text{Tr}_{S/R} \circ \varphi^\top$ for all $\varphi \in \mathcal{C}_{e,R}^P$. If $\text{Tr}_{S/R}(S) \subsetneq R$, then $\text{Tr}_{S/R}(S)$ must be contained in the ideal $R(-P)$ — the splitting prime of \mathcal{C}_R^P by hypothesis. In other words,

$$\text{Tr}_{S/R} \in \text{Hom}_R(S, R(-P)) = \text{Hom}_R(S \otimes_R R(P), R) = \text{Hom}_R(S(f^*P), R),$$

which implies $S(f^*P) \subset S(\text{Ram})$ and so $\text{Ram} - f^*P \geq 0$. Thus, $E = 0$, which is absurd.

For the statements regarding $\kappa \subset \ell$, use [Carvajal-Rojas 2018, Proposition 3.17], see [Carvajal-Rojas et al. 2018, Lemma 2.15], with $\Delta = \Delta_\varphi$ and where φ is taken to be any surjective map in $\mathcal{C}_{e,R}^P = (\mathcal{C}_{e,R}^P)^\top$ for $e \gg 0$. Note such a map φ exists because $\mathfrak{p} \neq R$, i.e., (R, P) is F -pure. Notice $\mathcal{C}_{e,R}^P = (\mathcal{C}_{e,R}^P)^\top$ was demonstrated in Proposition 4.4.

For the final statement, since $\text{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$, there is an induced κ -linear map $\overline{\text{Tr}}: \ell \rightarrow \kappa$, which is nonzero if $\text{Tr}: S \rightarrow R$ is surjective. However, if ℓ/κ is trivial then $\overline{\text{Tr}}$ corresponds to the κ -linear map $\kappa \rightarrow \kappa$ given by multiplication-by- $[L : K]$. In other words, $[L : K]$ is not zero as an element of κ and so it is prime to p as a natural number. \square

Theorem 4.7 (only one prime lying over). *Work in Setup 4.3. Suppose that $k = 1$ and $(X, D = P)$ is purely F -regular. The splitting prime $\mathfrak{q} := \mathfrak{p}(S, f^*\mathcal{C}_R^P)$ is the one and only one prime of S lying over $\mathfrak{p} := R(-P)$. Moreover, (Y, Q) is purely F -regular where $Q = V(\mathfrak{p})$.*

Proof. First, \mathfrak{q} is well-defined by [Carvajal-Rojas and Stäbler 2023, Theorem 4.8] and Propositions 4.4 and 4.6.¹⁷ Next, we prove \mathfrak{q} is unique in lying over \mathfrak{p} . We may pass to a cover of S in proving this and may therefore assume that f is generically Galois by [Grothendieck and Murre 1971, Lemma 2.2.6]. Let \mathfrak{q}' be a prime of S lying over \mathfrak{p} , i.e., $\mathfrak{q}' \cap R = \mathfrak{p}$. It suffices to prove $\mathfrak{q}' \subset \mathfrak{q}$; see [Atiyah and Macdonald 1969, Corollary 5.9]. For this, we use that \mathfrak{q} is a splitting prime ideal and the corresponding definition; see [Carvajal-Rojas and Stäbler 2023, Section 2.3.2] for the definition. It suffices to show $\varphi^\top(F_*^e \mathfrak{q}') \subset \mathfrak{n}$ for all $\varphi \in \mathcal{C}_{e,R}^D$ and $e > 0$ as the right S -span of $\{\varphi^\top \mid \varphi \in \mathcal{C}_{e,R}^D\}$ is $f^*\mathcal{C}_{e,R}^D$; see [Carvajal-Rojas and Stäbler 2023, Remark 2.15(c)].

Claim 4.8. $\text{Tr}_{S/R}(\mathfrak{q}') \subset \mathfrak{p}$.

Proof of claim. This has been shown for \mathfrak{q} in the proof of [Carvajal-Rojas and Stäbler 2023, Theorem 4.8]; see [Carvajal-Rojas and Stäbler 2023, Equation (4.8.2)]. We use the symmetry imposed by the Galois condition to induce this property to the other (possible) primes lying over \mathfrak{p} . Concretely, we have that $\text{Gal}(L/K)$ acts transitively on the set of primes lying over \mathfrak{p} [Stacks 2005–, Lemma 09EA or 0BRI]—although faithfulness might be lost due to ramification. Hence, if a prime is mapped into \mathfrak{p} by $\text{Tr}_{S/R}$ then so are its Galois conjugates, for $\text{Tr}_{S/R}(x) = \sum_{\sigma \in \text{Gal}(L/K)} \sigma(x)$ for all $x \in S$. \square

For all $\varphi \in \mathcal{C}_{e,R}^D$, it follows that

$$\text{Tr}_{S/R}(\varphi^\top(F_*^e \mathfrak{q}')) = \varphi(F_*^e \text{Tr}_{S/R}(\mathfrak{q}')) \subset \varphi(F_*^e \mathfrak{p}) \subset \mathfrak{p},$$

where the last containment follows from \mathfrak{p} being the splitting prime of \mathcal{C}_R^D . In other words, $\varphi^\top(F_*^e \mathfrak{q}') \subset \text{Tr}_{S/R}^{-1}(\mathfrak{p}) \subsetneq S$ (as $\text{Tr}_{S/R}$ is surjective by Proposition 4.6). Since $\varphi^\top(F_*^e \mathfrak{q}')$ is an S -module, it must be contained in \mathfrak{n} , which was to be shown. Finally, the pair (Y, Q) is purely F -regular by [Carvajal-Rojas and Stäbler 2023, Theorem 6.12, Remark 5.15] and Remark 2.4. \square

¹⁷The other two conditions are always satisfied. The S -linear map $S \rightarrow \omega_{S/R}; 1 \mapsto \text{Tr}_{S/R}$, is generically an isomorphism as L/K is separable. The condition $\text{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$ holds as in [Carvajal-Rojas et al. 2018, Lemma 2.10].

Example 4.9. Let R and f be as in Example 2.10. A direct consequence of Theorem 4.7 establishes that if L is a finite separable extension over K — the fraction field of R — then there is one and only one DVR of L lying over $R_{(f)}$ if: $R \subset R^L$ is tamely ramified with respect to $\operatorname{div} f$ and R/f is strongly F -regular. This does not hold without assuming R/f is strongly F -regular (i.e., $(R, \operatorname{div} f)$ is purely F -regular). Indeed, consider the cusp Example 3.14. In this case, the singularities of R/f are not even F -pure. One may still wonder if F -purity of R/f may suffice. To see this is not the case, we may get back to the Whitney's umbrella Example 3.16. Indeed; with notation as in Example 3.16, we specialize to $R = \kappa[[x, y, z]]$ with κ an algebraically closed field of odd characteristic. In [Blickle et al. 2012, Section 4.3.3], it is shown that R/f is F -pure yet not strongly F -regular. In fact, it is proven that $p(R, \operatorname{div} f) = (x, y) \not\supseteq (f)$.¹⁸ Considering $R' := R_{(x,y)}$, (R', f) is a counterexample where R'/f is a (nonnormal) F -pure ring. The authors are unaware of a counterexample (R, f) where R/f is normal and F -pure.

Example 4.10. There are interesting instances of multiple components pairs $(X, P_1 + \dots + P_k)$ where (X, P_i) are all purely F -regular. For example, we may consider $X = \operatorname{Spec} R$ with R as in either Example 2.13 or 2.15. With R as in Example 2.13, we may let $R(-Q) = \mathfrak{q} = (x, w)$. By symmetry on the variables, (X, Q) is a purely F -regular pair as well and moreover $\operatorname{div} x = P + Q$. We may also consider $\mathfrak{p}' = (y, z)$, $\mathfrak{q}' = (y, w)$, $P' = V(\mathfrak{p}')$, and $Q' = V(\mathfrak{q}')$, which all define purely F -regular pairs on X as well. In fact, $\operatorname{div} xy = P + Q + P' + Q' = \operatorname{div} zw$. Thus, $(X, \operatorname{div} x)$ or $(X, \operatorname{div} xy)$ are example where the aforementioned setup holds. Similarly, we may let $X = \operatorname{Spec} R$ with R as in Example 2.15. Then, if we consider $R(-Q) = \mathfrak{q} := (x, y, z)$, by symmetry, (X, Q) is a purely F -regular pair as well and $P + Q = \operatorname{div} ux = \operatorname{div} vy = \operatorname{div} wz$.¹⁹ Thus, $(X, \operatorname{div} ux)$ is another example. In any of these examples $(X, \operatorname{div} f)$, we wonder what the structure of $\operatorname{Rev}^{\operatorname{div} f}(X)$ is.

4B2. Main theorem. With Section 4B1 in place, we are ready to establish our main result. First, we make the following observation.

Remark 4.11. An interesting, conceptual consequence of Propositions 4.4 and 4.6, and Theorem 4.7 is that we may think of the objects in the Galois category $\operatorname{Rev}^P(X^\circ)$ as quintuples $(S, \mathfrak{n}, \ell, L, Q)$ where Q is a prime divisor minimal center of F -purity, which corresponds to the only height 1 prime divisor lying over \mathfrak{p} ; namely, the splitting prime of both $f^*\mathcal{C}_R^P \subset \mathcal{C}_S^Q$, say \mathfrak{q} . In particular, (S, Q) is a purely F -regular pair. Applying [Carvajal-Rojas and Stäbler 2023, Theorem 4.8] (note that its assumptions are verified by Proposition 4.6), we have

$$1 \geq r(S, Q) = [\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] \cdot r(R, P) > 0.$$

Hence, $[\kappa(\mathfrak{q}) : \kappa(\mathfrak{p})] \leq 1/r(R, P)$. In retrospective, we also see that Q happens to be the divisor $P^* = f^*P - \operatorname{Ram}$ in (4.4.1).

¹⁸In particular, $(R/f, (x, y)/f)$ is a purely F -regular pair where R/f is not normal — this is the counter Remark 2.6 is referring to.

¹⁹Indeed, one verifies that the ideal of R generated by u is the quotient of the ideal $(u, \Delta_1, \Delta_2, \Delta_1) = (u, vx, wx, \Delta_1) = (u, v, w) \cap (u, x, \Delta_1)$ of A , where the latter is a prime decomposition.

Theorem 4.12. *Work in Setup 3.9 and suppose that (X, P) is a purely F -regular pair. Then, $\text{Rev}^P(X^\circ)$ is P -irreducible and has both inertial boundedness and tameness. In particular, there exists an exact sequence of topological groups*

$$\hat{\mathbb{Z}}^{(P)} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

where $\pi_{1,\text{ét}}^P(X^\circ)$ is a finite group of order at most $\min\{1/r(R, P), 1/s(R)\}$ and prime-to- p . Furthermore, if P is a prime-to- p torsion element of $\text{Cl } X$, the homomorphism $\hat{\mathbb{Z}}^{(P)} \rightarrow \pi_1^{t,P}(X^\circ)$ is injective and so we have a short exact sequence

$$0 \rightarrow \hat{\mathbb{Z}}^{(P)} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

which splits if and only if P is the trivial element of $\text{Cl } X$. If P is nontorsion, we have a short exact sequence

$$0 \rightarrow \varprojlim_{n \in N^P(X^\circ)} \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1$$

which splits if and only if $N^P(X^\circ) = M^P(X^\circ)$ and there is a compatible system $\{\frac{1}{n}P \in \text{Cl } X\}_{n \in M^P(X^\circ)}$ of factors of P .

Proof. This is an application of Theorem 3.29 and Section 4B1; see Remark 4.11. Indeed, P -irreducibility holds by Theorem 4.7. Inertial boundedness was explained in Remark 4.11 whereas inertial tameness follows from Proposition 4.6. For the statements regarding the order of G , recall that it is realized as the Galois group of a universal étale-over- P cover $\tilde{X}^\circ \rightarrow X^\circ$. In particular, its generic degree equals $[\kappa(\tilde{\mathfrak{p}}) : \kappa(\mathfrak{p})]$, which is bounded by both $1/r(R, P)$ and $1/s(R)$; for the latter bound simply use [Carvajal-Rojas et al. 2018, Theorem 3.11]. □

Remark 4.13. It is an important folklore conjecture that the divisor class group of a strongly F -regular singularity is finitely generated. We were taught about this question by Karl Schwede but believe that it was originally raised by Melvin Hochster. For more, see [Polstra 2022]. It is known that the torsion subgroup is finite in this case; see [Polstra 2022; Martin 2022]. Finite generation of the class group is known to be true in dimension ≤ 3 ; see [Carvajal-Rojas et al. 2020]. Whenever finite generation of the class group is known, we may improve upon Theorem 4.12 to say that $\pi_1^{t,P}(X^\circ) \in \text{Ext}(G, \mathbb{Z}/n\mathbb{Z})$ for some $n \gg 0$ (in particular finite) and it is a trivial extension if P is n -divisible in $\text{Cl } X$ with $\frac{1}{n} \cdot P$ Cartier on U .

Remark 4.14. In Theorem 4.12, if $X^\circ = X_{\text{reg}}$, we may replace $\pi_{1,\text{ét}}^P(X^\circ)$ with $G = \pi_1^{\text{ét}}(X_{\text{reg}})$ by Proposition 3.26. By [Taylor 2019, Corollary 1.2], $\min\{1/r(R, P), 1/s(R)\} = 1/s(R)$ if P is prime-to- p torsion in $\text{Cl } X$.

Corollary 4.15. *Let $f: Y \rightarrow X$ be a quasiétale cover. If there is a divisor Δ on X such that (X, Δ) is purely F -regular and $r(\mathcal{O}_{X,x}, \Delta) > \frac{1}{2}$ for all $x \in X$, then f is étale.*

Proof. The proof is *mutatis mutandis* the same as in [Carvajal-Rojas et al. 2018, Corollary 3.3]. □

Remark 4.16. In light of [Taylor 2019, Corollary 1.2], it is unclear to the authors whether there are cases where Corollary 4.15 improves upon [Carvajal-Rojas et al. 2018, Corollary 3.3]. One potential candidate for such examples would be determinantal singularities. In [Carvajal-Rojas 2018, Example 4.12], the first named author proved; based on [Cutkosky 1995], that determinantal singularities satisfy purity of the branch locus. With notation as in Question 2.19, it is known that the F -signature of $C_{1,2}$ is $\frac{11}{24} = \frac{1}{2} - \frac{1}{24}$; see [Singh 2005]. On the other hand, we have estimated that $r(C_{1,2}, P) \geq \frac{1}{6}$ in Example 2.15. Nonetheless, our methods were not sufficient to prove (nor disprove) that $r(C_{1,2}, P) > \frac{1}{2}$.

Corollary 4.17. *In the setup of Theorem 4.12, the conclusion of Lemma 3.34 holds.*

Example 4.18 (determinantal singularities). Let R be a determinantal singularity with P a prime divisor generating $\text{Cl } R$; see Question 2.19. Then, $\pi_1^{\text{ét}}(X^\circ)$ is trivial for all Z by [Carvajal-Rojas 2018, Example 4.12]. Therefore, if (R, P) is a purely F -regular pair; see Question 2.19, then $\pi_1^{t,P}(X^\circ)$ is trivial too as P is not divisible in $\text{Cl } X$.

Question 4.19. Let $(X, \text{div } f)$ be any of the examples in Example 4.10. Does Abhyankar's lemma hold for $(X, \text{div } f)$?

Example 4.20 (graded hypersurfaces). With notation as in Example 2.11, suppose that A is strongly F -regular. If n is prime-to- p , then $\pi_1^{t,P}(X_{\text{reg}})$ is a nontrivial element of $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, \hat{\mathbb{Z}}^{(p)})$. Indeed, the corresponding degree n cyclic cover is its universal étale-over- P cover. If n is a p -power — R might be referred to as a Zariski hypersurface — its étale-over- P universal cover is trivial; see [Murre 1967, Proposition 7.2.2]. Therefore, all we can say is that there is a surjection $\hat{\mathbb{Z}}^{(p)} \rightarrow \pi_1^{t,P}(X^\circ)$. Determining the kernel of this surjection may require obtaining an analog of [loc. cit., Proposition 7.2.2] for the category $\text{Rev}^P(X^\circ)$.

5. Tame fundamental groups: Characteristic zero

The goal of this section is to prove the following by reduction to positive characteristics.

Theorem 5.1. *Let (X, Δ) be a log canonical pair, $\dim X \geq 2$, $x \in X$ be a closed point, and $Z \subset X$ be a closed subscheme of codimension ≥ 2 . Denote by P the minimal LC center through x which we assume to be a divisor. Write $X^\circ = \text{Spec } \mathcal{O}_{X,\bar{x}}^{\text{sh}} \setminus Z$ and denote by Δ and P the pullback of Δ and P to X° . Then, $\text{Rev}^P(X^\circ)$ is P -irreducible and has inertial boundedness. In particular, there exists an exact sequence of topological groups*

$$\hat{\mathbb{Z}} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

where $\pi_{1,\text{ét}}^P(X^\circ)$ is finite. Furthermore, if P is a torsion element of $\text{Cl } X$, the homomorphism $\hat{\mathbb{Z}} \rightarrow \pi_1^{t,P}(X^\circ)$ is injective and so we have a short exact sequence

$$0 \rightarrow \hat{\mathbb{Z}} \rightarrow \pi_1^{t,P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1,$$

which splits if and only if P is the trivial element of $\text{Cl } X$. If $P \in \text{Cl } X$ is nontorsion, we have a short exact sequence

$$0 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow \pi_1^{\text{t},P}(X^\circ) \rightarrow \pi_{1,\text{ét}}^P(X^\circ) \rightarrow 1$$

which splits if and only if there is a divisor D such that $P = nD$ in $\text{Cl } X$ and $D|_U$ is Cartier.

Proof. In order to prove Theorem 5.1, recall that we may work in Setup 2.20. We want to use Theorem 3.29, and thus need to verify that P -irreducibility and inertial boundedness 3.21 hold for the PLT pair $(R = \mathcal{O}_{X,\bar{x}}^{\text{sh}}, \Delta)$. This will be proven below in Proposition 5.2 and Proposition 5.3 respectively.

We still need to explain why, if P is nontorsion, the set $N^P(X^\circ)$ is finite so that (3.29.4) holds. This however is a direct consequence of [Bingener and Flenner 1984, Theorem 6.1]. \square

Proposition 5.2. *Work in Setup 2.20. Then $\text{Rev}^P(X^\circ)$ satisfies P -irreducibility.*

Proposition 5.3. *Work in Setup 2.20. Then $\text{Rev}^P(X^\circ)$ satisfies inertial boundedness.*

We shall see that inertial boundedness follows from minor modifications of the arguments in [Bhatt et al. 2017]; see Section 5B below. Thus, we prove inertial boundedness by spreading out. While a prove of P -irreducibility is also possible via spreading out, there is a direct proof in characteristic zero which we give below. We are thankful to Karl Schwede for pointing this out to us.

Corollary 5.4. *In the setup of Theorem 5.1, the conclusion of Lemma 3.34 holds.*

5A. P -irreducibility in characteristic zero. We need some preparatory lemmata. Recall that an *étale neighborhood* of a geometric point $\bar{x} \rightarrow \text{Spec } R$ is a factorization through an étale morphism $\text{Spec } R' \rightarrow \text{Spec } R$.

Lemma 5.5. *Let R be normal domain and R^{sh} be its strict henselization at a closed point $x \in \text{Spec } R$. Let $f : \text{Spec } S \rightarrow \text{Spec } R^{\text{sh}}$ be a finite dominant morphism. Then, there exists a connected étale neighborhood $\text{Spec } R'$ of \bar{x} and a cartesian square*

$$\begin{array}{ccc} \text{Spec } S & \xrightarrow{f} & \text{Spec } R^{\text{sh}} \\ \downarrow g & & \downarrow h \\ \text{Spec } S' & \xrightarrow{f'} & \text{Spec } R' \end{array} \tag{5.5.1}$$

with f' finite. Furthermore, if $\mathfrak{p} \subset R$ is a height 1 prime such that $\mathfrak{p}R^{\text{sh}}$ is prime, then $h(\mathfrak{p}R^{\text{sh}})$ is a height-1 prime of R and $h^{-1}(h(\mathfrak{p}R^{\text{sh}})) = \mathfrak{p}R^{\text{sh}}$. Finally, if R is local then R' is normal and S' is normal if and only if S is normal.

Proof. Fix generators (a_1, \dots, a_m) of \mathfrak{p} and write $S = R^{\text{sh}}[b_1, \dots, b_e]$. As $R^{\text{sh}} \rightarrow S$ is finite, there are monic polynomials $f_i \in R^{\text{sh}}[T]$ with $f_i(b_i) = 0$. We denote the coefficients of these f_i by c_{ij} . Since R^{sh} is obtained as a filtered colimit of connected étale neighborhoods $R \rightarrow R'$ of \bar{x} , there is some $R \rightarrow R'$ in the colimit system such that R' contains all the a_i and c_{ij} . By construction, $R \rightarrow R'$ is étale and setting $S' = R'[b_1, \dots, b_e]$ one readily checks that the above diagram is a pullback square. In particular, f' is

finite by construction. Since the fibers of $R \rightarrow R'$ are of dimension zero, $h(\mathfrak{p}R^{\text{sh}}) = \mathfrak{p}R'$ is of height 1. Clearly, $h^{-1}(h(\mathfrak{p}R^{\text{sh}})) = \mathfrak{p}R^{\text{sh}}$.

For the final assertion, note that the weakly étale homomorphism $R' \rightarrow R^{\text{sh}}$ is faithfully flat since $\mathfrak{m}R^{\text{sh}}$ is the maximal ideal of R^{sh} . Since (5.5.1) is a pullback square, $S' \rightarrow S$ is a weakly étale faithfully flat homomorphism too. Thus S' is normal if and only if S is and similarly for R' and R^{sh} by [Stacks 2005–, Lemma 033G and Tag 0950]. \square

Remark 5.6. If $R \rightarrow R^{\text{sh}}$ is the strict henselization with respect to some maximal ideal \mathfrak{m} then, given any ideal $\mathfrak{a} \subset \mathfrak{m}$ such that R/\mathfrak{a} is normal, the extension $\mathfrak{a}R^{\text{sh}}$ is prime. Indeed, we may localize R at \mathfrak{m} and thus assume that R is a local ring. Then, the assertion follows from $R^{\text{sh}} \otimes R/I = (R/I)^{\text{sh}}$ for any ideal $I \subset R$ [Stacks 2005–, Lemma 05WS] and the fact that S^{sh} is a normal domain if and only if S is a normal domain [Stacks 2005–, Lemma 033G].

Lemma 5.7. *Let $g: \text{Spec } T \rightarrow \text{Spec } R$ be a surjective étale morphism or a surjective pro-étale morphism. Let $f: \text{Spec } S \rightarrow \text{Spec } R$ be a morphism. Consider the base change diagram:*

$$\begin{array}{ccc} \text{Spec } S \otimes_R T & \xrightarrow{f'} & \text{Spec } T \\ \downarrow g' & & \downarrow g \\ \text{Spec } S & \xrightarrow{f} & \text{Spec } R \end{array}$$

The morphism f is tame with respect to D , if and only if f' is tame with respect to $g^{-1}(D)$.

Proof. The “only if” implication follows from [Grothendieck and Murre 1971, Lemma 2.2.7]. For the converse, by [Stacks 2005–, Lemmas 033C and 033G], R is normal if and only if T is normal. Likewise, S is normal if and only if $T \otimes_R S$ is normal. Thus, it makes sense to talk about tame morphisms. The remaining assertion is a consequence of [Grothendieck and Murre 1971, Proposition 2.2.9]. \square

Proposition 5.8. *Let (X, Δ) be an affine PLT pair where $\lfloor \Delta \rfloor = P$ is a prime divisor. If $g: Y \rightarrow X$ is a tamely ramified cover over P , then $(g^{-1}(P))_{\text{red}}$ is a normal divisor*

Proof. Write $\Delta = P + \Delta_1$. By [Kollár 2013, Corollary 2.43, (2.41.4)] the pair (Y, Δ') is PLT, where $\Delta' = (g^{-1}(P))_{\text{red}} + g^*\Delta_1$, and $K_{Y'} + \Delta' \sim_{\mathbb{Q}} g^*(K_X + \Delta)$. Note that $\lfloor g^*\Delta_1 \rfloor = 0$. Indeed, since (X, Δ) is PLT, Δ_1 and P have no components in common. Since g is étale over $X \setminus P$ the assertion follows. In this way, we see that $(g^{-1}(P))_{\text{red}}$ is a minimal LC center for some closed point $y \in Y$. Hence, by [Fujino and Gongyo 2012, Theorem 7.2] $(g^{-1}(P))_{\text{red}}$ is normal. \square

Proof of Proposition 5.2. We use the notation of Setup 2.20 and write R^{sh} for $\mathcal{O}_{X,x}^{\text{sh}}$. Let $V \rightarrow \text{Spec } R^{\text{sh}} \setminus Z$ be a cover in $\text{Rev}^P(\text{Spec } R^{\text{sh}} \setminus Z)$. Denote the integral closure of R^{sh} inside $\mathcal{O}_V(V)$ by S . Using Lemma 5.5,

we obtain a cartesian square:

$$\begin{array}{ccc}
 \text{Spec } S & \xrightarrow{f} & \text{Spec } R^{\text{sh}} \\
 \downarrow g & & \downarrow h \\
 \text{Spec } S' & \xrightarrow{f'} & \text{Spec } R'
 \end{array}$$

with f' finite and R' a connected étale neighborhood of $\bar{x} \rightarrow W \subseteq X$, where W is some Zariski open neighborhood of $x \in X$. As usual, we write \mathfrak{p} for the prime corresponding to our fixed prime divisor P on X . As f is finite, S is also strictly henselian [Stacks 2005–, Tag 04GH] and S is the strict henselization of S' with respect to some ideal \mathfrak{n} lying over x .

Since R/\mathfrak{p} is normal as a minimal LC center Theorem 2.23, we deduce from Remark 5.6 that $\mathfrak{p}R^{\text{sh}}$ is prime. Write $\mathfrak{p}' = h(\mathfrak{p}R^{\text{sh}})$. Using Lemma 5.5 again, we have that $h^{-1}(\mathfrak{p}') = \mathfrak{p}R^{\text{sh}}$. Note that f' is tamely ramified with respect to P' by Lemma 5.7. Since $\text{Spec } R'$ is an étale neighborhood of $\bar{x} \rightarrow X$, say $\varphi: \text{Spec } R' \rightarrow X$, we conclude that $(\text{Spec } R', \varphi^*(\Delta))$ is PLT with $[\Delta] = P'$. Thus we can apply Proposition 5.8 to conclude that $Q' = (f'^{-1}(P'))_{\text{red}}$ is normal. We denote the corresponding ideal by \mathfrak{q}' and note that $\mathfrak{q}' \subseteq \mathfrak{n}$. Using Remark 5.6 we see that $\mathfrak{q} := \mathfrak{q}'S$ is prime. In other words, there is only one prime in S' lying over \mathfrak{p}' and contained in \mathfrak{n} . Assume now that $\mathfrak{a} \in f^{-1}(\mathfrak{p})$. Then $h(f(\mathfrak{a})) = \mathfrak{p}'$ and hence $f'(g(\mathfrak{a})) = \mathfrak{p}'$. In particular, $g(\mathfrak{a}) \in f'^{-1}(\mathfrak{p}')$. But clearly, $g(\mathfrak{a}) \subseteq \mathfrak{n}$. Hence, $\mathfrak{a} = \mathfrak{q}$ as desired. \square

5B. Inertial boundedness via spread-out. In the situation of Setup 2.20, write $Y = \text{Spec } \mathcal{O}_{X,x}^{\text{sh}}$ and Y_{reg} for its regular locus. By Corollary 3.27, it suffices to show that inertial boundedness holds for Y_{reg} . To this end, we use the result of [Bhatt et al. 2017, Theorem 1.1], where finiteness of $\pi_1(Y \setminus \{x\})$ is proved via reduction mod p . The proof of [loc. cit., Theorem 1.1] is a combination of Theorem 4.1 and Propositions 5.1 and 6.4 in [loc. cit.]. We can directly apply the latter two in our situation. The argument of Theorem 4.1 needs to be modified slightly. We record this below for completeness.

We will use the following notation for spreading out: If R is a κ -algebra, $A \subset \kappa$ a finitely generated \mathbb{Z} -algebra, then we will write R_A for any fixed finite type A algebra whose base changed generic fiber $R_A \otimes_A \text{Frac}(A) \otimes_{\text{Frac}(A)} \kappa$ recovers R . If $s \in \text{Spec } A$ is a point, then we will write R_s for the corresponding fiber of R_A . We will use similar notation for schemes.

Theorem 5.9. *Let A be a finitely generated \mathbb{Z} -algebra equipped with an embedding $A \rightarrow \mathbb{C}$. Fix an affine finite type scheme X_A over $\text{Spec } A$, a closed point $x_a \in X_A$, and a closed subset $Z_A \subset X_A$ of codimension ≥ 2 . Let us denote by X, Z , and x the base changes to $\text{Spec } \mathbb{C}$. Let us furthermore assume that X is normal. Then, there is a dense open $V \subset \text{Spec } A$ such that for every morphism $\text{Spec } \kappa \rightarrow V$ with κ an algebraically closed field of characteristic p there is a canonical isomorphism*

$$\pi_1(\text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \setminus Z)^{(p)} \cong \pi_1(\text{Spec } \mathcal{O}_{X_\kappa, x_\kappa}^{\text{sh}} \setminus Z_\kappa)^{(p)},$$

where by abuse of notation we write Z for $\alpha^{-1}(Z)$ where $\alpha: \text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \rightarrow \text{Spec } \mathcal{O}_X$ is the canonical morphism and similarly for Z_κ .

Proof. Using resolution of singularities, we may choose a truncated proper hypercover $f : Y_\bullet \rightarrow X$ indexed by $\bullet \in \Delta_{\leq 2}^{\text{op}}$ with Y_i smooth and $D_\bullet := f^{-1}(Z)_{\text{red}} \subset Y_\bullet$ giving an SNC divisor at each level. Moreover, by first blowing up x and then Z , so that both are Cartier divisors, we have that $E_\bullet := f^{-1}(x)_{\text{red}} \subset Y_\bullet$ also yields an SNC divisor at each level. Denoting by I_\bullet the finite index set of components of D_\bullet , each $D_{\bullet,i}$ is smooth over \mathbb{C} and proper over Z . Denoting by J_\bullet the subset of I_\bullet that yields the components of E_\bullet , we also obtain that the $E_{\bullet,j}$ are smooth and proper varieties over \mathbb{C} . We write $U_\bullet := Y_\bullet \setminus D_\bullet$. Let $\mathcal{Y}_{\bullet,\ell} \rightarrow Y_\bullet$ be the ℓ -th root stack associated to the divisors in D_\bullet and let $\mathcal{E}_{\bullet,\ell} \rightarrow E_\bullet$ be its pullback to E_\bullet . Now, we base change everything along $\alpha : \text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \rightarrow \text{Spec } \mathcal{O}_X$ adding a superscript sh for base changes, e.g., $U_\bullet^{\text{sh}} := U_\bullet \times_X \text{Spec } \mathcal{O}_{X,x}^{\text{sh}}$. The appropriate pullback maps induce equivalences

$$\begin{aligned} \text{F}\acute{\text{E}}\text{t}(\text{Spec } \mathcal{O}_{X,x}^{\text{sh}} \setminus Z) &\rightarrow \lim_{\bullet \in \Delta_{\leq 2}} \text{F}\acute{\text{E}}\text{t}(U_\bullet^{\text{sh}}) \leftarrow \lim_{\ell} \text{colim}_{\ell} \text{F}\acute{\text{E}}\text{t}(\mathcal{Y}_{\bullet,\ell}^{\text{sh}}) \\ &\rightarrow \lim_{\bullet \in \Delta_{\leq 2}} \text{colim}_{\ell} \text{F}\acute{\text{E}}\text{t}(\mathcal{E}_{\bullet,\ell}) \cong \text{colim}_{\ell} \lim_{\bullet \in \Delta_{\leq 2}} \text{F}\acute{\text{E}}\text{t}(\mathcal{E}_{\bullet,\ell}). \end{aligned}$$

From left to right, these equivalences are given by Lemmas 2.1, 2.8(2) and 2.2 in [Bhatt et al. 2017], and the isomorphism is due to the fact that filtered colimits commute with finite limits. Having made these minor changes, the rest of the argument now proceeds exactly as [loc. cit., Theorem 4.1]. \square

Proof of Proposition 5.3. Let us write $Y = \text{Spec } \mathcal{O}_{X,x}^{\text{sh}}$. By Corollary 3.27 (and its proof), it suffices to show that $\pi_1(Y_{\text{reg}})$ is finite. Using Proposition 2.21, we may perturb Δ to Δ' so that (X, Δ') is KLT. The nonregular locus of Y is cut out by a radical ideal I and likewise the closed subset Z is also given by some radical ideal J . Passing to a connected étale neighborhood $f : \text{Spec } R' \rightarrow \text{Spec } R$ of \bar{x} ; where $\text{Spec } R$ is some Zariski neighborhood of x , we may assume that $I, J \subset R'$. Note that $(\text{Spec } R', f^*\Delta')$ is also KLT; see [Kollár 2013, 2.14(2)].

Spreading out over some finitely generated \mathbb{Z} -algebra A and passing to closed fibers, we obtain pairs $(\text{Spec } R'_s, f^*\Delta'_s)$ that are F -regular for all s in a dense open of $\text{Spec } A$ (by [Takagi 2004, Corollary 3.4]). By the Nullstellensatz applied to the Jacobson ring A , $\kappa(s)$ is finite and its algebraic closure $\kappa(\bar{s})$ is a separable. Hence, $(\text{Spec } R'_s, f^*\Delta'_s)$ is also F -regular. Write $Y_{\bar{s}}$ for the spectrum of a strict henselization of R'_s at $x_{\bar{s}}$ and $W_{\bar{s}}$ for the closed subset defined by $I_{\bar{s}}$. Applying [Carvajal-Rojas et al. 2018, Theorem 5.1] we get $\pi_1(Y_{\bar{s}} \setminus W_{\bar{s}}) \leq 1/s(Y_{\bar{s}})$. Apply [Bhatt et al. 2017, Propositions 6.4 and 5.1] and Theorem 5.9 to conclude that $\pi_1(X_{\text{reg}})$ is finite. \square

Appendix: Splitting primes under strict henselizations

The goal of this appendix is to show that taking the splitting prime commutes with strict henselization. That is, if $p(\mathcal{C}) = p(R, \mathcal{C}) \subset R$ is the splitting prime for some Cartier algebra \mathcal{C} acting on $(R, \mathfrak{m}, \kappa)$ and $f : \text{Spec } R^{\text{sh}} \rightarrow \text{Spec } R$ is the strict henselization with respect to \mathfrak{m} , then $p(\mathcal{C})R^{\text{sh}} = p(f^*\mathcal{C})$. To make sense of this we first need to explain the notion $f^*\mathcal{C}$.

Lemma A.1. *Let R be a noetherian ring. Consider a colimit over a directed system of ring homomorphisms $\{f_{ij} : S_i \rightarrow S_j\}$ of R -algebras, a Cartier R -algebra \mathcal{C} and a \mathcal{C} -module M . Assume that all of the*

morphisms f_{ij} and one structural map $R \rightarrow S_i$ are either finite, étale, or smooth. Then, $\mathcal{D} = \operatorname{colim} g_{ij}^* \mathcal{C}$ exists and $\mathcal{M} = \operatorname{colim} g_{ij}^! M$ is naturally a \mathcal{D} -module, where we denote by $g_{ij}: \operatorname{Spec} S_j \rightarrow \operatorname{Spec} S_i$ the map corresponding to f_{ij} .

Proof. If $g: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ then by definition $g^* \mathcal{C} = \mathcal{C} \otimes_R S$ so that we obtain a corresponding directed system of Cartier algebras; see [Blickle and Stäbler 2019, Proposition 5.3]. We thus need to verify that the directed system of modules from which we construct \mathcal{M} are Cartier modules. This is true by [loc. cit., Theorem 5.5]. □

Lemma A.2. *Let $f: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$ be a surjective (essentially of finite type) étale morphism of F -finite local rings. Then $p(f^* \mathcal{C}) = p(\mathcal{C})S$.*

Proof. By [Blickle and Stäbler 2019, Theorem 6.5],²⁰ R is F -regular if and only if S is so. Similarly, by [loc. cit., Proposition 5.13, Lemma 6.1], R is F -pure if and only if S is so. Therefore, we may assume that both splitting primes are nontrivial. Consider the following diagram:

$$\begin{array}{ccc} \operatorname{Spec} S & \xrightarrow{f} & \operatorname{Spec} R \\ \uparrow & & \uparrow \\ \operatorname{Spec} S/p(\mathcal{C})S & \xrightarrow{f'} & \operatorname{Spec} R/p(\mathcal{C}) \end{array}$$

By Lemma 2.8, $R/p(\mathcal{C})$ is F -regular. Since f' is étale, $S/p(\mathcal{C})S$ is also F -regular (note that since f is surjective the fiber is nonempty). Since the minimal primes of $p(\mathcal{C})$ are $f^* \mathcal{C}$ -submodules (see [Schwede 2010, Corollary 4.8]), any minimal prime of $p(\mathcal{C})S$ is a maximal proper $f^* \mathcal{C}$ -submodule. However, $p(f^* \mathcal{C})$ is unique and (since S is F -pure) the maximal proper $f^* \mathcal{C}$ -submodule. Thus, there is only one prime lying over $p(\mathcal{C})$. Since $R/p(\mathcal{C})$ is reduced and f' is étale, $\sqrt{p(\mathcal{C})S} = p(\mathcal{C})S$ is prime and so coincides with $p(f^* \mathcal{C})$. □

Proposition A.3. *Let $(R, \mathfrak{m}, \kappa, K)$ be a normal local domain and \mathcal{C} be a Cartier R -algebra. Denote by R^{sh} the strict henselization of (R, \mathfrak{m}) and by \mathcal{D} the R^{sh} -Cartier algebra obtained as a colimit over the corresponding system of étale algebras. Then, (R, \mathcal{C}) is F -pure if and only if $(R^{\text{sh}}, \mathcal{D})$ is so. Moreover, if $p(\mathcal{C})$ is the splitting prime of (R, \mathcal{C}) then $p(\mathcal{C})R^{\text{sh}} = p(\mathcal{D})$, where $p(\mathcal{D})$ is the splitting prime of $(R^{\text{sh}}, \mathcal{D})$. Conversely, $p(\mathcal{D}) \cap R = p(\mathcal{C})$.*

Proof. Strict henselizations are obtained by a filtered colimit. By [Stacks 2005–, Lemma 0032], we may even obtain it by a *small* filtered colimit. Moreover, having constructed R^{sh} via the usual direct limit of triples, we may *a posteriori* also obtain it as a filtered colimit of a system of étale maps by viewing everything as embedded in R^{sh} . Using Lemma A.1, we obtain \mathcal{D} .

If (R, \mathcal{C}) is F -pure, then also is $(R^{\text{sh}}, \mathcal{D})$ as well as $(S, \varphi^* \mathcal{C})$ for any (essentially) étale morphism $\varphi: \operatorname{Spec} S \rightarrow \operatorname{Spec} R$. Indeed, this follows from [Blickle and Stäbler 2019, Proposition 5.13] in the latter

²⁰Since we are only dealing with Cartier modules that do not have nonminimal associated primes, we may use test element theory in the sense of [Blickle 2013, Theorem 3.11]—thus we may weaken the assumption that the base is essentially of finite type over an F -finite field to F -finite.

case and the former case follows from the latter. Conversely, if $(R^{\text{sh}}, \mathcal{D})$ is not F -pure, say $x \notin \mathcal{D}_+ R^{\text{sh}}$, then we find a surjective étale morphism $\varphi: \text{Spec } S \rightarrow \text{Spec } R$ for which $x \in S$. Thus S is not F -pure but then by faithful flatness R is also not F -pure (using [loc. cit., Lemma 6.1]). In particular, if (R, \mathcal{C}) or $(R^{\text{sh}}, \mathcal{D})$ is not F -pure, then the statement about splitting primes is trivially true.

Assume that both (R, \mathcal{C}) and $(R^{\text{sh}}, \mathcal{D})$ are F -pure. Let $\varphi: \text{Spec } S \rightarrow \text{Spec } R$ be an étale morphism occurring in the colimit and $\mathfrak{n} \subset S$ a prime above \mathfrak{m} . As in the proof of [Stacks 2005–, Lemma 04GN], we may assume that $\mathfrak{m}S = \mathfrak{n}$. Since R^{sh} is local with maximal ideal $\mathfrak{m}R^{\text{sh}}$, the map $S \rightarrow R^{\text{sh}}$ factors through the localization $S_{\mathfrak{n}}$. Thus, $\varphi': \text{Spec } S_{\mathfrak{n}} \rightarrow \text{Spec } R$ is an essentially étale surjective homomorphism. We apply Lemma A.2 to conclude that the splitting prime $p(\varphi'^{\flat}\mathcal{C})$ of $S_{\mathfrak{n}}$ is $p(\mathcal{C})S_{\mathfrak{n}}$. Next, note that any homogeneous element of $\varphi'^{\flat}\mathcal{C}$ is of the form $\kappa \otimes s^q$ with $\kappa \in \mathcal{C}_e$, which acts on $x = r \otimes t \in R \otimes_R S_{\mathfrak{n}}$ as $\kappa \otimes s^q(r \otimes t) = \kappa(r) \otimes st$; see [Blickle and Stäbler 2019, Theorem 5.5]. Use now the well-known isomorphism $F_*^e R \otimes_R S_{\mathfrak{n}} \rightarrow F_*^e S_{\mathfrak{n}}$, $r \otimes s \mapsto rs^q$.

We now prove $p(\mathcal{C})R^{\text{sh}} \subset p(\mathcal{D})$. If $x \in p(\mathcal{C})R^{\text{sh}}$ and $\kappa \in \mathcal{D}_e$, there is an essentially étale morphism $\varphi': \text{Spec } S_{\mathfrak{n}} \rightarrow \text{Spec } R$ as above so that $x \in p(\mathcal{C})S_{\mathfrak{n}} = p(\varphi'^{\flat}\mathcal{C})$ and $\kappa \in \varphi'^{\flat}\mathcal{C}$. Then, since $x \in p(\varphi'^{\flat}\mathcal{C})$, we have $\kappa(x) \in \mathfrak{n} = \mathfrak{m}S \subset \mathfrak{m}R^{\text{sh}}$ and so $x \in p(\mathcal{D})$. Conversely, let $x \in p(\mathcal{D})$. Then, we find $\varphi': \text{Spec } S \rightarrow \text{Spec } R$ as above such that $x \in S$. Since $\kappa(x) \in \mathfrak{m}R^{\text{sh}}$ for all $\kappa \in \mathcal{D}$, we also have $\kappa(x) \in \mathfrak{m}R^{\text{sh}} \cap S = \mathfrak{n}$ for all $\kappa \in \varphi'^{\flat}\mathcal{C}$. Hence, $x \in p(\varphi'^{\flat}\mathcal{C}) = p(\mathcal{C})S \subset p(\mathcal{C})R^{\text{sh}}$ as desired.

We now show $p(\mathcal{D}) \cap R = p(\mathcal{C})$. If $x \in p(\mathcal{D}) \cap R$, for all $\kappa \in \mathcal{D}$ we have $\kappa(x) \in \mathfrak{m}R^{\text{sh}}$ and so $\kappa(x) \in \mathfrak{m}$ for all $\kappa \in \mathcal{C}$. If $x \in p(\mathcal{C})$ then $x \in p(\mathcal{C})R^{\text{sh}} = p(\mathcal{D})$ by the above. \square

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