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Geometric properties of the Kazhdan–Lusztig Schubert basis

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We study classes determined by the Kazhdan–Lusztig basis of the Hecke algebra in the K-theory and hyperbolic cohomology theory of flag varieties. We first show that, in K-theory, the two different choices of Kazhdan–Lusztig bases produce dual bases, one of which can be interpreted as characteristic classes of the intersection homology mixed Hodge modules. In equivariant hyperbolic cohomology, we show that if the Schubert variety is smooth, then the class it determines coincides with the class of the Kazhdan–Lusztig basis; this property was known as the smoothness conjecture. For Grassmannians, we prove that the classes of the Kazhdan–Lusztig basis coincide with the classes determined by Zelevinsky's small resolutions. These properties of the so-called KL Schubert basis show that it is the closest existing analogue to the Schubert basis for hyperbolic cohomology; the latter is a very useful testbed for more general elliptic cohomologies.

1. Introduction

Let G be a split semisimple linear algebraic group with a fixed Borel subgroup B and a maximal torus $T \subset B$. Let P be a parabolic subgroup containing the Borel subgroup B. The varieties G/P and G/B are called flag varieties, and they are among the most concrete objects in algebraic geometry, because of the Bruhat decompositions. For instance, the equivariant cohomology (Chow group) of flag varieties is freely spanned by the classes of Schubert varieties $X(w)$. Similarly, the equivariant K-theory of flag varieties is spanned by the structure sheaves of Schubert varieties. The field of studying intersection theory of these classes is called Schubert calculus, and is related to combinatorics, representation theory, and enumerative geometry.

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Due to the failure of Schubert varieties being smooth, the present paper deals with two different directions in generalizing classical Schubert calculus. The first one is concerned with the Chern classes. Although the classical Chern class theory does not work for the singular Schubert varieties, there are generalizations to this case, which are called Chern–Schwartz–MacPherson (CSM) classes [MacPherson 1974; Schwartz 1965a; 1965b] in homology and motivic Chern (MC) classes in K-theory [Brasselet et al. 2010; Aluffi et al. 2019; Fehér et al. 2021]. These generalized Chern classes of Schubert cells are closely related to the corresponding stable bases of the cotangent bundle T^*G/B, defined by Maulik and Okounkov [2019; 2017] in their study of quantum cohomology/K-theory of Nakajima quiver varieties. These classes are permuted by various Demazure–Lusztig operators [Aluffi and Mihalcea 2016; Aluffi et al. 2017; 2019; Su 2017; Su et al. 2020; Mihalcea et al. 2022], and are related to unramified principal series representations of the Langlands dual group over a nonarchimedean local field [Su et al. 2020; Aluffi et al. 2019].

We focus on the Kazhdan–Lusztig bases of the Hecke algebra, which are related to the intersection cohomology of Schubert varieties. Classically, there are two choices of Kazhdan–Lusztig bases. In this paper, we consider the K-theory classes determined by these two collections of Kazhdan–Lusztig bases. The cohomology case is studied in [Mihalcea and Singh 2020]. We first show that they are dual to each other in Theorems 13 and 22. These dualities are closely related to the characteristic classes of mixed Hodge modules, studied by J. Schürmann and his collaborators [Schürmann 2011; 2017; Brasselet et al. 2010]. Moreover, we interpret one collection of these classes as the motivic Hodge Chern classes of the intersection homology mixed Hodge modules of the Schubert varieties, which immediately implies that they are invariant under the Serre–Grothendieck duality; see Proposition 17 and Corollary 19.

The other direction is to look at more general cohomology theories, namely the equivariant oriented cohomology theories of Levine and Morel. They are those contravariant functors $\boldsymbol{h}_T$ from the category of smooth (quasi)projective varieties to the category of commutative rings such that for any proper map of varieties, a pushforward of the cohomology groups is defined. One can then define Chern classes, where the first Chern class of the tensor product of line bundles determines a one-dimensional commutative formal group law. The structure of the equivariant oriented cohomology of flag varieties is studied in [Calmès et al. 2016; 2019; 2015; Lenart et al. 2020]. Roughly speaking, there is an algebra generated by push–pull operators between $\boldsymbol{h}_T(G/B)$ and $\boldsymbol{h}_T(G/P)$, called the formal affine Demazure algebra $\boldsymbol{D}_F$, whose dual $\boldsymbol{D}_F^*$ is isomorphic to $\boldsymbol{h}_T(G/B)$.

Since Schubert varieties are not smooth in general, their fundamental classes are not defined beyond the Chow group and K-theory. To resolve the singularities of a Schubert variety $X(w)$, one often uses the Bott–Samelson resolution, which is defined by fixing a reduced decomposition of the Weyl group element w. For an oriented cohomology beyond singular cohomology/K-theory, the classes determined by such resolutions depend on the choice of the reduced decomposition. This corresponds to the fact that, for general $\boldsymbol{h}_T$, the push–pull operators do not satisfy the braid relations [Hoffnung et al. 2014]. Because of this fact, there are no canonically defined Schubert classes.

Aiming for the definition of Schubert classes, in [Lenart and Zainoulline 2017; Lenart et al. 2020], the authors consider the so-called hyperbolic cohomology, denoted by $\mathfrak{h}$. This corresponds to a 2-parameter

Todd genus, and is the first interesting case after K-theory in terms of complexity. A Riemann–Roch type map is defined from K-theory to the hyperbolic cohomology theory, which induces an action of the Hecke algebra (considered on the K-theory side) on the hyperbolic cohomology of G/B. In this way, the action of the Kazhdan–Lusztig basis defines classes KL_w in $\mathfrak{h}_T(G/B)$, called KL Schubert classes. In [Lenart and Zainoulline 2017; Lenart et al. 2020], there is a conjecture stating that, if the Schubert variety $X(w)$ is smooth, then its fundamental class coincides with the class KL_w. This conjecture is proved in those works in some special cases. Our first main result proves this conjecture in full generality:

Theorem 28. *If the Schubert variety $X(w)$ is smooth, then the class determined by $X(w)$ in $\mathfrak{h}_T(G/B)$ coincides with the KL Schubert class* KL_w.

The idea of the proof is as follows: if $X(w)$ is smooth, then all the Kazhdan–Lusztig polynomials $P_{y,w}$ for any $y \leq w$ are equal to 1, so the Kazhdan–Lusztig basis for w is the sum of the Demazure–Lusztig operators. As mentioned above, the MC classes of Schubert cells in K-theory are permuted by the Demazure–Lusztig operators. So the MC class of $X(w)$ coincides with the KL class in K-theory, and the restriction formula for the former is obtained in [Aluffi et al. 2019] by generalizing a result of Kumar [1996]. By using the Riemann–Roch type map from K-theory to hyperbolic cohomology, we compare the restriction formulas of the class KL_w and of the class of the smooth Schubert variety $X(w)$, and prove the smoothness conjecture (Theorem 28). For partial flag varieties, a similar property is also proved.

As mentioned above, the Kazhdan–Lusztig basis defines classes in the K-theory of flag varieties, but they do not coincide with the fundamental classes of Schubert varieties, whether smooth or not. However, in $\mathfrak{h}_T(G/B)$, our Theorem 28 shows that, for smooth Schubert varieties, their fundamental classes coincide with the classes defined by the Kazhdan–Lusztig basis. It is unclear to us why such phenomena appear, and we hope to explore this in a future project.

Restricting to type A Grassmannians, we prove more geometric and combinatorial properties. For example, Zelevinsky constructed small resolutions of all Schubert varieties [Zelevinskiĭ 1983]. Our second main result is the following:

Theorem 42. *The KL Schubert classes for the Grassmannian coincide with the hyperbolic cohomology classes of the corresponding Zelevinsky resolutions.*

To prove this theorem, note that Zelevinsky's small resolutions are similar to the Bott–Samelson resolutions, except that, instead of using minimal parabolic subgroups, one considers more general parabolic subgroups. So the small resolution classes can be computed by using relative push–pull operators between hyperbolic cohomology of G/P and G/Q. These operators were studied in [Calmès et al. 2019]. On the other hand, in [Kirillov and Lascoux 2000], a factorization of the Kazhdan–Lusztig basis elements for Grassmannians is exhibited. By carefully transforming this factorization, one can write the Kazhdan–Lusztig basis elements as products of "relative" Kazhdan–Lusztig elements. Finally, by identifying the latter with the relative push–pull operators, one proves Theorem 42. By the uniqueness of the Kazhdan–Lusztig basis, it follows that all small resolution classes are the same.

There have been important developments in Schubert calculus for general cohomology theories. More specifically, for elliptic cohomology, a stable basis in the cotangent bundle T^*G/B was defined (see [Aganagic and Okounkov 2021; Okounkov 2021], which generalizes stable bases for cohomology and K-theory), and canonical classes were associated with Bott–Samelson resolutions of Schubert varieties [Rimányi and Weber 2020; Kumar et al. 2020]. The elliptic cohomology used in the latter papers can be considered as the oriented cohomology theory associated with a certain elliptic formal group law determined by the Jacobi theta functions; meanwhile, the mentioned cohomology classes are elliptic analogues of the CSM classes in ordinary cohomology and the MC classes in K-theory. On the other hand, the hyperbolic formal group law we consider here comes from a singular cubic curve (in Weierstrass form), so it is a singular elliptic formal group law; see [Buchstaber and Bunkova 2010]. The properties of the KL Schubert basis proved in this paper (namely, the smoothness conjecture and the interpretation in terms of the Zelevinsky small resolutions) show that this basis is the closest existing analogue to the Schubert basis for hyperbolic cohomology. Furthermore, the latter is a very useful testbed for more general elliptic cohomologies.

The paper is organized as follows. In Section 2, we recall the algebraic construction of the equivariant oriented cohomology of flag varieties. In Section 3, we recall basic facts about the Hecke algebra, MC classes, and the smoothness criterion. In Section 4, we use Kazhdan–Lusztig bases to define the two collections of KL classes in $K_T(G/B)$ and $K_T(G/P)$, and show that they are dual to each other. We also give a geometric interpretation for one of them using mixed Hodge modules. In Section 5, we recall the definition of KL Schubert classes in hyperbolic cohomology, and prove the smoothness conjecture. In Section 6, we prove Theorem 42, which connects small resolutions for Grassmannians with the corresponding KL Schubert classes.

2. Formal affine Demazure algebra and its dual

We recall the definition of the formal affine Demazure algebra and its relation with equivariant generalized (oriented) cohomology of flag varieties following [Hoffnung et al. 2014; Calmès et al. 2016; 2019] and especially the paper [Calmès et al. 2015].

Notation. Let G be a semisimple simply connected linear algebraic group over $\mathbb{C}$, and fix B a Borel subgroup with a maximal torus $T \subset B$. Let $X^*(T)$ denote the character lattice of T. Let $W = N_G(T)/T$ be the Weyl group.

Let Σ denote the set of associated roots and let Σ^+ denote the subset of roots in B. For any root α, let $\alpha > 0$ (resp. $\alpha < 0$) denote $\alpha \in \Sigma^+$ (resp. $-\alpha \in \Sigma^+$).

Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ denote the set of simple roots. Let $\ell \colon W \to \mathbb{Z}$ denote the length function. For any $J \subset \Pi$, denote by W_J the parabolic subgroup corresponding to J, by w_J its longest element, and by W^J (resp. ${}^JW$) the set of minimal length representatives of left (resp. right) cosets W/W_J (resp. $W_J\backslash W$). Specifically, $w_0 := w_\Pi \in W$ is the longest element. More generally, if $J' \subset J \subset \Pi$, write $w_{J/J'} := w_J w_{J'} \in W^{J'}$ (resp. $w_{J'\backslash J} := w_{J'} w_J$), that is, $w_{J/J'}$ (resp. $w_{J'\backslash J}$) is the maximal element (in terms of the Bruhat order) in the set $W_J \cap W^{J'}$ (resp. $W_J \cap {}^{J'}W$). Write $\Sigma_J := \{\alpha \in \Sigma \mid s_\alpha \in W_J\}$ and $\Sigma_J^{\pm} := \Sigma_J \cap \Sigma^{\pm}$. Throughout the paper, we use the notation \ for right cosets, not set difference, which is denoted by $-$.

Formal group algebra. Let F be a one-dimensional formal group law over a commutative unital ring R. The formal group algebra $R[\![X^*(T)]\!]_F$ is defined to be the quotient of the completion

$$R[\![x_\lambda \mid \lambda \in X^*(T)]\!]/\mathcal{J}_F,$$

where $\mathcal{J}_F$ is the closure of the ideal generated by $\langle x_0, F(x_\lambda, x_\mu) - x_{\lambda+\mu} \mid \lambda, \mu \in X^*(T)\rangle$. For simplicity it will be denoted by S. It can be shown that if $\{\omega_1, \dots, \omega_n\}$ is a basis of $X^*(T)$, then S is (noncanonically) isomorphic to $R[\![\omega_1, \dots, \omega_n]\!]$.

Localized twisted group ring. Let $Q = S[(1/x_\alpha) \mid \alpha > 0]$, and $Q_W = Q \otimes_R R[W]$. Denote the canonical left Q-basis of Q_W by δ_w, $w \in W$, and define a product on Q_W by

$$(p\delta_w) \cdot (p'\delta_{w'}) := pw(p')\delta_{ww'} \quad \text{for } p, p' \in Q,\ w, w' \in W.$$

In particular, we have $\delta_v p = v(p)\delta_v$ for $p \in Q$.

Push-pull elements. For each root α, define the formal push–pull element

$$Y_\alpha := (1 + \delta_{s_\alpha})\frac{1}{x_{-\alpha}} \in Q_W.$$

For any reduced word $w = s_{i_1} \cdots s_{i_k}$, where s_i is the simple reflection corresponding to the i-th simple root in Π, define $I_w = (i_1, \dots, i_k)$, and $Y_{I_w} = Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_k}}$. The product Y_{I_w} depends on the choice of the reduced sequence, unless the formal group law F is of the form $x + y + \beta xy$ with $\beta \in R$. For simplicity, write $\delta_i := \delta_{s_i}$, $Y_i := Y_{\alpha_i}$ and $x_{\pm i} := x_{\pm\alpha_i}$.

Formal affine Demazure algebra. Let $\boldsymbol{D}_F$ be the subring of Q_W generated by elements of S and push–pull elements Y_i for $i = 1, \dots, n$. This is called the formal affine Demazure algebra. It is proved in [Calmès et al. 2016] that $\boldsymbol{D}_F$ is a free left S-module with basis $\{Y_{I_w} \mid w \in W\}$.

Example 1. If $R = \mathbb{Z}$ and $F_m(x, y) = x + y - xy$ (multiplicative formal group law), then

$$S \cong \mathbb{Z}[X^*(T)]^\wedge, \quad x_\alpha \mapsto 1 - e^{-\alpha},$$

where the completion is taken with respect to the kernel of the augmentation map $e^\lambda \mapsto 1$. The ring $\boldsymbol{D}_F$ is then isomorphic to the (completed) affine 0-Hecke algebra.

For $J' \subset J \subseteq \Pi$, write

$$x_{J/J'} := \prod_{\alpha \in \Sigma_J^- - \Sigma_{J'}^-} x_\alpha, \quad x_J := x_{J/\varnothing}.$$

Fixing a set of left coset representatives $W_{J/J'}$ of $W_J/W_{J'}$, we define a push–pull element

$$Y_{J/J'} := \Bigl(\sum_{w \in W_{J/J'}} \delta_w\Bigr)\frac{1}{x_{J/J'}} \in Q_W, \quad Y_J := Y_{J/\varnothing} = \Bigl(\sum_{w \in W_J} \delta_w\Bigr)\frac{1}{x_J}. \tag{1}$$

Note that the definition of $Y_{J/J'}$ depends on the choice of $W_{J/J'}$, and in general $Y_{J/J'}$ might not be in $\boldsymbol{D}_F$. Similarly, fixing a set of right coset representatives $W_{J'\backslash J}$ of $W_{J'}\backslash W_J$, one can define $Y_{J'\backslash J}$. If $J=\Pi$, x_Π and Y_Π are correspondingly defined. For instance, if $J=\{i\}$, then $Y_{\{i\}}=Y_{\alpha_i}$. Note that in general $Y_{J/J'}\in Q_W$, but $Y_J\in \boldsymbol{D}_F$. We have

$$Y_{J/J'}Y_{J'}=Y_J=Y_{J'}Y_{J'\backslash J}. \tag{2}$$

There is an anti-involution ι of $\boldsymbol{D}_F$, defined by

$$\iota(p\delta_v):=\delta_{v^{-1}}p\frac{v(x_\Pi)}{x_\Pi}=v^{-1}(p)\frac{x_\Pi}{v^{-1}(x_\Pi)}\delta_v \quad \text{for } p\in Q,\ v\in W. \tag{3}$$

For example, it is easy to prove that $\iota(Y_J)=Y_J$, and

$$\iota(Y_{I_w})=Y_{I_w^{-1}}, \tag{4}$$

if I_w^{-1} is the sequence obtained from I_w by reversing the order.

Dual of the Demazure algebra. Let $\boldsymbol{D}_F^*$ denote the S-linear dual $\mathrm{Hom}_S(\boldsymbol{D}_F, S)$ with dual basis $Y_{I_w}^*$, $w\in W$. One can also consider the Q-linear dual $Q_W^*=\mathrm{Hom}_Q(Q_W, Q)$, which is isomorphic to the set-theoretic $\mathrm{Hom}(W, Q)$. There is the dual basis f_w, $w\in W$ of Q_W^* such that $f_w(\delta_v)=\delta_{w,v}^{\mathrm{Kr}}$ and $f_w\cdot f_v=\delta_{w,v}^{\mathrm{Kr}}f_w$, where $\delta_{w,v}^{\mathrm{Kr}}$ is the Kronecker symbol. It turns Q_W^* into a commutative ring with identity $\mathbf{1}=\sum_w f_w$. By definition, we have $\boldsymbol{D}_F^*\subset Q_W^*$ (where the former is a S-module, and the latter is considered as a Q-module), and the product on Q_W^* restricts to the product on $\boldsymbol{D}_F^*$.

Two actions on the dual. There are actions denoted by $\bullet$ and $\odot$ of the ring Q_W on its Q-linear dual Q_W^* defined by

$$(p\delta_v)\bullet(qf_w):=qwv^{-1}(p)f_{wv^{-1}} \quad \text{and} \quad (p\delta_v)\odot(qf_w):=pv(q)f_{vw} \qquad \text{for } v,w\in W,\ p,q\in Q. \tag{5}$$

It follows from [Lenart et al. 2020, § 3] that the $\bullet$-action is Q-linear, while the $\odot$-action is not, and the two actions commute. We also have $z\bullet\mathrm{pt}_e=\iota(z)\odot\mathrm{pt}_e$. Moreover, the two actions induce (via the embeddings $\boldsymbol{D}_F\subset Q_W$ and $\boldsymbol{D}_F^*\subset Q_W^*$) corresponding actions of $\boldsymbol{D}_F$ on $\boldsymbol{D}_F^*$. For homology and K-theory, the $\bullet$ and $\odot$ actions correspond to the right and left actions considered in [Mihalcea et al. 2022].

The class of a point. For each $w\in W$ define the element

$$\mathrm{pt}_w:=x_\Pi\bullet f_w=w(x_\Pi)f_w,$$

and call it the class of a point. From the definition, we have $z\bullet\mathrm{pt}_e=\iota(z)\odot\mathrm{pt}_e$ for $z\in Q_W$, where $e\in W$ denotes the identity element.

Generalized (oriented) cohomology. Given a formal group law F over R, let $\boldsymbol{h}$ be the corresponding free algebraic generalized (oriented) cohomology theory obtained from the algebraic cobordism Ω of Levine and Morel [2007] by tensoring with F, i.e.,

$$\boldsymbol{h}(-):=\Omega(-)\otimes_{\Omega(\mathrm{pt})}R.$$

Here $\Omega(\text{pt})$ is the Lazard ring, the coefficient ring of universal formal group law, and $\Omega(\text{pt}) \to R$ is the evaluation map defining F. Note that such theories are different from the usual generalized cohomology theories from algebraic topology, since the formal group laws do not need to be Landweber exact (since the localization sequences are only right exact; see [Levine and Morel 2007, § 3.2]). We refer to [Levine and Morel 2007] for all the properties of such theories.

In particular, for the additive formal group law $F_a(x, y) = x + y$ one obtains the Chow ring and for the multiplicative group law F_m one gets the usual K-theory.

Equivariant generalized cohomology. Let $\boldsymbol{h}_T$ be the respective T-equivariant generalized (oriented) cohomology theory of [Calmès et al. 2015, § 2]. Replacing $\boldsymbol{h}_T$ if necessary by its characteristic completion (see Section 3 there), the main result of [Calmès et al. 2015] says that the formal affine Demazure algebra $\boldsymbol{D}_F$ and its dual $\boldsymbol{D}_F^*$ are related to generalized cohomology $\boldsymbol{h}_T(G/B)$ and $\boldsymbol{h}_T(G/P_J)$ as follows:

(1) There is an isomorphism $\boldsymbol{D}_F^* \cong \boldsymbol{h}_T(G/B)$, which maps the element $Y_{I_w^{-1}} \bullet \text{pt}_e = Y_{I_w} \odot \text{pt}_e$ to the Bott–Samelson class determined by the sequence I_w.

(2) Via the above isomorphism, the map $Y_\Pi \bullet \_ : \boldsymbol{D}_F^* \to (\boldsymbol{D}_F^*)^W \cong S$ coincides with the map $\boldsymbol{h}_T(G/B) \to \boldsymbol{h}_T(\text{Spec}(k))$.

(3) The group W acts on $\boldsymbol{D}_F^*$ by restriction of the $\bullet$-action via the embedding $W \subset \boldsymbol{D}_F$. For any subset $J \subset \Pi$, one has an isomorphism $(\boldsymbol{D}_F^*)^{W_J} \cong \boldsymbol{h}_T(G/P_J)$, and the map $Y_J : \boldsymbol{D}_F^* \to (\boldsymbol{D}_F^*)^{W_J}$ coincides with the pushforward map $\boldsymbol{h}_T(G/B) \to \boldsymbol{h}_T(G/P_J)$. More generally, the map $Y_{J/J'} \bullet \_ : Q_W^* \to Q_W^*$ restricts to a map $(\boldsymbol{D}_F^*)^{W_{J'}} \to (\boldsymbol{D}_F^*)^{W_J}$, which corresponds to $\boldsymbol{h}_T(G/P_{J'}) \to \boldsymbol{h}_T(G/P_J)$.

(4) The embedding $\boldsymbol{D}_F^* \to Q_W^*$ coincides with the restriction to T-fixed points map $\boldsymbol{h}_T(G/B) \to Q \otimes_S \boldsymbol{h}_T(W)$, and the element pt_w is mapped to the class $\mathfrak{e}_w$ of T-fixed points of G/B.

Remark 2. Observe that the localization axiom [Calmès et al. 2015, A3] used to prove the above properties can be replaced by a weaker CD-property of [Neshitov et al. 2018, Defintion 3.3] which holds for any $\boldsymbol{h}_T$ defined using the Borel construction (see [Neshitov et al. 2018, Example 3.6]).

Generalized Bott–Samelson varieties. Let P_i, Q_i, for $i = 1, \dots, m$, be a collection of parabolic subgroups such that $Q_i \subset P_i \cap P_{i+1}$ and $Q_m := B$. Define

$$Z = P_1 \times^{Q_1} P_2 \times^{Q_2} \times \cdots \times^{Q_{m-1}} P_m.$$

There is a canonical map

$$\pi : Z/Q_m \to G/Q_m, \quad (p_1, \dots, p_m) \mapsto p_1 p_2 \cdots p_m.$$

The following lemma will be used in Section 6 in identifying the small resolution of Zelevinsky with the factorization of Grassmannian Kazhdan–Lusztig basis of Kirillov and Lascoux.

Lemma 3. *Under the isomorphism $\boldsymbol{h}_T(G/B) \cong \boldsymbol{D}_F^*$ and viewing $\boldsymbol{h}_T(G/P) \cong (\boldsymbol{D}_F^*)^{W_P}$, we have*

$$\pi_*(1) = (Y_{P_m/Q_{m-1}} Y_{P_{m-1}/Q_{m-2}} \cdots Y_{P_2/Q_1} Y_{P_1}) \bullet \text{pt}_e\,.$$

Proof. We use induction on m. If $m=1$, then the map is $\pi : P_1/Q_1 \to G/Q_1$. We have the following commutative diagram:

$$\begin{array}{ccc} P_1/Q_1 & \xrightarrow{\pi} & G/Q_1 \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p_{P_1/Q_1}} \\ \mathrm{pt} & \xrightarrow{i} & G/P_1 \end{array}$$

Here i is the embedding of the identity point. Then

$$\pi_*(1) = \pi_* q_*(1) = (p_{P_1/Q_1})^* i_*(1).$$

According to [Calmès et al. 2015, Lemma 8.8], we see that $i_*(1) = Y_{P_1} \bullet \mathrm{pt}_e$, and $p^*_{P_1/\varnothing}$ is the embedding $(\boldsymbol{D}_F^*)^{W_{P_1}} \hookrightarrow (\boldsymbol{D}_F^*)^{W_{Q_1}} \subset \boldsymbol{D}_F^*$. So it holds when $m=1$.

Now write $Z' = P_1 \times^{Q_1} P_2 \times^{Q_2} \cdots \times^{P_{m-2}} Q_{m-1}$. We then have the commutative diagram

$$\begin{array}{ccc} Z' \times^{Q_{m-1}} P_m/Q_m & \xrightarrow{\quad\pi\quad} & G/Q_m \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p_{P_m/Q_m}} \\ Z'/Q_{m-1} & \xrightarrow{p_{P_m/Q_{m-1}} \circ \pi'} & G/P_m \end{array}$$

where $\pi' : Z'/Q_{m-1} \to G/Q_{m-1}$ is the map multiplying all components together. Then

$$\pi_*(1) = \pi_* q_*(1) = (p_{P_m/Q_m})^* (p_{P_m/Q_{m-1}})_* \pi'_*(1).$$

From [Calmès et al. 2015, p. 137], we see that $(p_{P_m/Q_{m-1}})_*$ corresponds to $Y_{P_m/Q_{m-1}} \bullet \_$, and $(p_{P_m/Q_m})^*$ is just the embedding $(\boldsymbol{D}_F)^{W_{P_m}} \hookrightarrow (\boldsymbol{D}_F)^{W_{Q_m}}$. The conclusion then follows from induction. □

Corollary 4. *Via the isomorphism $\boldsymbol{h}_T(G/B) \cong \boldsymbol{D}_F^*$, we have*

$$\pi_*(1) = (Y_{P_1/Q_1} \cdots Y_{P_{m-1}/Q_{m-1}} Y_{P_m}) \odot \mathrm{pt}_e\,.$$

Proof. Note $Y_{P/Q} Y_Q = Y_P$ for any $P \supset Q$, and $Y_P \bullet \mathrm{pt}_e = Y_P \odot \mathrm{pt}_e$ (see [Lenart et al. 2020, (3.5), (3.8)]). If $m=2$, we have

$$\begin{aligned} \pi_*(1) &= (Y_{P_2/Q_1} Y_{P_1}) \bullet \mathrm{pt}_e = Y_{P_2/Q_1} \bullet Y_{P_1} \odot \mathrm{pt}_e = Y_{P_2/Q_1} \bullet Y_{P_1/Q_1} \odot Y_{Q_1} \odot \mathrm{pt}_e \\ &= Y_{P_2/Q_1} \bullet Y_{P_1/Q_1} \odot Y_{Q_1} \bullet \mathrm{pt}_e = Y_{P_1/Q_1} \odot (Y_{P_2/Q_1} Y_{Q_1}) \bullet \mathrm{pt}_e = Y_{P_1/Q_1} \odot Y_{P_2} \odot \mathrm{pt}_e\,. \end{aligned}$$

The general case then follows similarly. □

We prove a lemma that will be used later in Section 6:

Lemma 5. *We have*

$$Y_{P_1/Q_1} Y_{P_2/Q_2} \cdots Y_{P_{m-1}/Q_{m-1}} Y_{P_m} = Y_{P_1} Y_{Q_1 \backslash P_2} \cdots Y_{Q_{m-1} \backslash P_m}.$$

Proof. This follows from recursive use of the identities (2) and the assumption that $Q_i \subset P_i \cap P_{i+1}$. For example, one has

$$Y_{P_{m-1}/Q_{m-1}} Y_{P_m} = Y_{P_{m-1}/Q_{m-1}} Y_{Q_{m-1}} Y_{Q_{m-1} \backslash P_m} = Y_{P_{m-1}} Y_{Q_{m-1} \backslash P_m}.$$

By induction, the formula holds. □

3. Hecke algebra, motivic Chern class, and the smoothness criterion

In this section, we recall the definition of the Kazhdan–Lusztig basis and the motivic Chern (MC) classes.

The multiplicative case. Set $R = \mathbb{Z}[t, t^{-1}, (t+t^{-1})^{-1}]$, where t is a parameter. Definitions of Section 2 applied to the multiplicative formal group law F_m over R give the respective formal group algebra and its localization,

$$S_m := R[\![X^*(T)]\!]_{F_m} \quad \text{and} \quad Q_m := S_m\Big[\frac{1}{x_\alpha} \mid \alpha > 0\Big],$$

the localized twisted group algebra and the formal affine Demazure algebra,

$$Q_{m,W} := Q_m \otimes_R R[W] \quad \text{and} \quad \boldsymbol{D}_m := \langle S_m, Y_1, \dots, Y_n\rangle \subset Q_{m,W}.$$

The Demazure–Lusztig elements. Define the Demazure–Lusztig elements in $Q_{m,W}$ as

$$\tau_i := Y_i^m(t - t^{-1}e^{\alpha_i}) - t = \frac{t^{-1}-t}{1-e^{-\alpha_i}} + \frac{t - t^{-1}e^{-\alpha_i}}{1-e^{-\alpha_i}}\delta_i^m.$$

It can be shown that $\tau_i \in \boldsymbol{D}_m$ for $i = 1, \dots, n$ satisfy the standard quadratic relation $\tau_i^2 = (t^{-1}-t)\tau_i + 1$, and the braid relations. So they generate the Hecke algebra H over R.

Remark 6. Let $y = -t^{-2}$. On $\boldsymbol{D}_m^* \cong R \otimes_{\mathbb{Z}} K_T(G/B)$, as operators, $t^{-1}\tau_i \odot\_$ agrees with $\mathcal{T}_i^L$, and $t^{-1}\tau_i \bullet\_$ agrees with $\mathcal{T}_i^{R,\vee}$, respectively, where the latter are notions from [Mihalcea et al. 2022, Section 5.3] and [Aluffi et al. 2019].

The Kazhdan–Lusztig basis. Consider the involution of the Hecke algebra $H \to H$, $z \mapsto \bar{z}$ such that

$$\bar{t} = t^{-1}, \quad \overline{\tau_i} = \tau_i^{-1}. \tag{6}$$

There is a basis of H over R denoted by $\{\gamma_w\}_{w\in W}$ and called the Kazhdan–Lusztig basis. It is invariant under this involution and satisfies

$$\gamma_w \in \tau_w + \sum_{v<w} t\mathbb{Z}[t]\tau_v.$$

We set $t_w = t^{\ell(w)}$ and

$$\gamma_w = \sum_{v\le w} t_w t_v^{-1} P_{v,w}(t^{-2})\tau_v,$$

where $P_{v,w}$ are the Kazhdan–Lusztig polynomials. In addition to this, there is another canonical basis defined by (see [Kazhdan and Lusztig 1979]),

$$\widetilde{\gamma}_w := \sum_{v\in W} \epsilon_w\epsilon_v t_w^{-1} t_v P_{v,w}(t^2)\tau_v \in \tau_w + \sum_{v<w} t^{-1}\mathbb{Z}[t^{-1}]\tau_v,$$

where ϵ_w is $(-1)^{\ell(w)}$. Since the Schubert variety $X(w_J) \subset G/B$ is smooth, the Kazhdan–Lusztig polynomials satisfy $P_{v,w_J} = 1$ for any $v \le w_J$. Thus, $\gamma_{w_J} = \sum_{v\le w_J} t_{w_J} t_v^{-1}\tau_v$.

More generally, for $J' \subset J \subseteq \Pi$, write

$$\gamma_J := \gamma_{w_J}, \quad \gamma_{J/J'} := \sum_{v\in W_J\cap W^{J'}} t_{w_{J/J'}} t_v^{-1}\tau_v, \quad \gamma_{J'\backslash J} := \sum_{v\in W_J\cap {}^{J'}W} t_{w_{J'\backslash J}} t_v^{-1}\tau_v. \tag{7}$$

It is not difficult to see that

$$\gamma_J = \gamma_{J/J'}\gamma_{J'} = \gamma_{J'}\gamma_{J'\backslash J}. \tag{8}$$

If $Q \subset P$ are the parabolic subgroups corresponding to $J' \subset J$, respectively, write $\gamma_{P/Q} = \gamma_{J/J'}$. For $\gamma_{J/J'}$ and $\gamma_{J'\backslash J}$, the analogue of Lemma 5 holds. It will be used in considering KL Schubert classes in hyperbolic cohomology of partial flag varieties below.

Motivic Chern classes. We recall the definition of the motivic Chern classes, following [Brasselet et al. 2010; Fehér et al. 2021; Aluffi et al. 2019]. Let X be a nonsingular quasiprojective complex algebraic variety with an action of the torus T. Let $G_0^T(\mathrm{var}/X)$ be the (relative) Grothendieck group of varieties over X. By definition, it is the free abelian group generated by isomorphism classes $[f : Z \to X]$ where Z is a quasiprojective T-variety and f is a T-equivariant morphism modulo the usual additivity relations

$$[f : Z \to X] = [f : U \to X] + [f : (Z - U) \to X],$$

for any T-invariant open subvariety $U \subset Z$.

Theorem 7. *There exists a unique natural transformation* $\mathrm{MC}_{-t^{-2}} : G_0^T(\mathrm{var}/X) \to K_T(X)[t^{-2}]$ *satisfying the following properties*:

(1) *It is functorial with respect to T-equivariant proper morphisms of nonsingular*, *quasiprojective varieties.*

(2) *It satisfies the normalization condition*

$$\mathrm{MC}_{-t^{-2}}[\mathrm{id}_X : X \to X] = \sum (-1)^i t^{-2i} \big[\textstyle\bigwedge^i T_X^*\big] =: \lambda_{-t^{-2}}(T_X^*) \in K_T(X)[t^{-2}].$$

The nonequivariant case is proved in [Brasselet et al. 2010], and the equivariant case is shown in [Aluffi et al. 2019; Fehér et al. 2021].

Let

$$\mathcal{D}(-) := (-1)^{\dim X}\, \mathrm{RHom}_{\mathcal{O}_X}(-, \omega_X)$$

be the Serre–Grothendieck duality functor on $K_T(X)$, where $\omega_X := \bigwedge^{\dim X} T_X^*$ is the canonical bundle of X. Extend it to $K_T(X)[t^{\pm 1}]$ by setting $\mathcal{D}(t^i) = t^{-i}$.

Definition 8. Let $Z \subset X$ be a T-invariant subvariety.

(1) Define the motivic Chern class of Z to be

$$\mathrm{MC}_{-t^{-2}}(Z) := \mathrm{MC}_{-t^{-2}}([Z \hookrightarrow X]).$$

(2) Further assume that Z is pure-dimensional. Define the Segre motivic Chern class of Z as follows (see [Mihalcea et al. 2022, Definition 6.2]):

$$\mathrm{SMC}_{-t^{-2}}(Z) := t^{-2\dim Z} \cdot \frac{\mathcal{D}(\mathrm{MC}_{-t^{-2}}(Z))}{\lambda_{-t^{-2}}(T_X^*)}.$$

Smoothness of Schubert varieties. Consider the variety of complete flags G/B. Let $X(w)^\circ := BwB/B$ and $Y(w)^\circ := B^- wB/B$ be the Schubert cells. The closures $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ are the Schubert varieties. Observe that $u \le v$ with respect to the Bruhat order if and only if $X(u) \subset X(v)$. Let $\mathrm{pt}^m_w = w(x_\Pi) f^m_w \in Q^*_{m,W}$ denote the class of the T-fixed point $\mathfrak{e}_w$ corresponding to $w \in W$. Note that here f^m_w is the standard basis in $Q^*_{m,W}$ defined in Section 2, and the superscript m is to indicate the multiplicative formal group law.

The key property of the motivic Chern classes of the Schubert cells that we need are listed below.

Theorem 9. (1) [Mihalcea et al. 2022, Theorem 7.6] *For any $w \in W$, we have*

$$\mathrm{MC}_{-t^{-2}}(X(w)^\circ) = t_w^{-1} \tau_w \odot \mathrm{pt}^m_e .$$

(2) [Aluffi et al. 2019, Theorem 9.1] *For any $u \le w \in W$, the Schubert variety $X(w)$ is smooth at $\mathfrak{e}_u$ if and only if*

$$\mathrm{MC}_{-t^{-2}}(X(w))|_u = \prod_{\alpha>0,\, us_\alpha \nleq w} (1 - e^{u\alpha}) \prod_{\alpha>0,\, us_\alpha \le w} (1 - t^{-2} e^{u\alpha}),$$

where $\mathrm{MC}_{-t^{-2}}(X(w))|_u$ denotes the pullback of $\mathrm{MC}_{-t^{-2}}(X(w))$ to the fixed point $\mathfrak{e}_u$.

Remark 10. (1) This theorem is used to prove the Bump, Nakasuji and Naruse's conjectures about Casselman bases in unramified principal series representations; see [Bump and Nakasuji 2011; 2019; Naruse 2014; Aluffi et al. 2019; Su 2019].

(2) The "only if" direction of part (2) follows directly from basic properties of motivic Chern classes, and it holds in a much more general setting; see [Aluffi et al. 2019, § 9.1].

Proof. The first part follows from the reference mentioned. The second one follows from the fact $\delta_{w_0} \odot \big(\mathrm{MC}_{-t^{-2}}(Y(w))\big) = \mathrm{MC}_{-t^{-2}}(X(w_0 w))$. □

Given $w \in W$, define the coefficients $a_{w,u} \in Q_m$ by the formulas

$$\Gamma_w := \sum_{v \le w} t_v^{-1} \tau_v = \sum_{u \le w} a_{w,u} \delta^m_u \in Q_{m,W}. \tag{9}$$

Note that if the Schubert variety $X(w)$ is smooth, then $P_{v,w} = 1$ for all $v \le w$, so $\Gamma_w = t_w^{-1} \gamma_w$. It is immediate to get the following corollary from Theorem 9.

Corollary 11. *For any $u \le w \in W$, the Schubert variety $X(w)$ is smooth at the fixed point $\mathfrak{e}_u$ if and only if*

$$a_{w,u} = \prod_{\alpha>0,\, us_\alpha \le w} \frac{1 - t^{-2} e^{u\alpha}}{1 - e^{u\alpha}}.$$

Proof. By Theorem 9(1) and Equation (9), we have

$$\begin{aligned}\mathrm{MC}_{-t^{-2}}(X(w)) &= \sum_{v \le w} \mathrm{MC}_{-t^{-2}}(X(v)^\circ) = \sum_{v \le w} t_v^{-1} \tau_v \odot \mathrm{pt}^m_e \\ &= \sum_{v \le w} a_{w,v} \delta^m_v \odot \mathrm{pt}^m_e = \sum_{v \le w} a_{w,v} \prod_{\alpha>0} (1 - e^{v\alpha}) f_v.\end{aligned}$$

Thus, we have

$$\mathrm{MC}_{-t^{-2}}(X(w))|_u = a_{w,u}\prod_{\alpha>0}(1-e^{u\alpha}).$$

The corollary follows from this and Theorem 9(2). □

4. Dual bases in K-theory and characteristic classes of mixed Hodge modules

In this section, we use the two Kazhdan–Lusztig bases of the Hecke algebra to define two collections of classes in K-theory, and show that they are actually dual to each other. We also give a geometric interpretation of one of these collections using the intersection homology mixed Hodge modules. These are also generalized to the partial flag variety case.

K-theory KL classes.

Definition 12. We define two collections of classes (called KL classes) in $K_T(G/B)[t^{\pm 1}]$ as follows:

$$C_w := \gamma_w \odot \mathrm{pt}_e^m \quad \text{and} \quad \widetilde{C}_w := \widetilde{\gamma}_{w^{-1}w_0} \bullet \mathrm{pt}_{w_0}^m .$$

They form a basis of the localized K-theory $Q_m \otimes_{S_m} K_T(G/B)$.

Let $\langle -, - \rangle$ denote the usual nondegenerate tensor product pairing on $K_T(G/B)[t^{\pm 1}]$, i.e., $\langle f, g\rangle = Y_\Pi^m \bullet (f \cdot g)$ for $f, g \in K_T(G/B)[t^{\pm 1}]$. The first result of this section is the following.

Theorem 13. *For any $w, v \in W$, we have*

$$\langle C_w, \widetilde{C}_v\rangle = \delta_{w,v}^{\mathrm{Kr}}\prod_{\alpha>0}(t-t^{-1}e^{-\alpha}).$$

We first recall that the Segre motivic Chern classes of Schubert cells enjoy the following properties.

Lemma 14. (1) *For any $v \in W$, we have*

$$(\tau_{w_0 v})^{-1} \bullet \mathrm{pt}_{w_0}^m = t_{w_0 v}\prod_{\alpha>0}(1-t^{-2}e^{-\alpha})\,\mathrm{SMC}_{-t^{-2}}(Y(v)^\circ).$$

(2) *For any $u, v \in W$, we have*

$$\big\langle \mathrm{MC}_{-t^{-2}}(X(u)^\circ), \mathrm{SMC}_{-t^{-2}}(Y(v)^\circ)\big\rangle = \delta_{u,v}^{\mathrm{Kr}}.$$

Proof. The first part follows from Remark 6 and [Mihalcea et al. 2022, Theorem 7.4], while the second one follows from [loc. cit., Theorem 7.1]. □

Remark 15. By definition, $(t^{-1}\tau_i)|_{t=\infty} = Y_i^m - 1$. Thus, from Theorem 9(1), we get

$$\mathrm{MC}_{-t^{-2}}(X(w)^\circ)|_{t=\infty} = t_w^{-a}\tau_w \odot \mathrm{pt}_e^m\,|_{t=\infty} = [\mathcal{O}_{X(w)}(-\partial X(w))] =: \mathcal{I}_w,$$

where $\partial X(w) = \bigcup_{v<w} X(v)$ is the boundary of the Schubert variety $X(w)$, and $\mathcal{I}_w$ denotes its ideal sheaf. On the other hand, $(t^{-1}\tau_i^{-1})|_{t=\infty} = Y_i^m$. Thus, the first part of the lemma gives

$$\mathrm{SMC}_{-t^{-2}}(Y(v)^\circ)|_{t=\infty} = (t_{w_0 v}\tau_{w_0 v})^{-1} \bullet \mathrm{pt}_{w_0}^m\,|_{t=\infty} = [\mathcal{O}_{Y(v)}].$$

Therefore, setting $t=\infty$ in the second part of the lemma, we get the classical fact

$$\langle \mathcal{I}_w, [\mathcal{O}_{Y(v)}]\rangle = \delta^{\mathrm{Kr}}_{u,v}.$$

Proof of Theorem 13. First of all, we have the following inversion formula for the Kazhdan–Lusztig polynomials (see [Kazhdan and Lusztig 1979, Theorem 3.1]):

$$\sum_z \epsilon_y\epsilon_z P_{x,z}P_{w_0y,w_0z} = \delta^{\mathrm{Kr}}_{x,y}.$$

Therefore,

$$\sum_z \epsilon_x\epsilon_z P_{w_0z,w_0x}P_{z,y} = \delta^{\mathrm{Kr}}_{x,y}. \tag{10}$$

By definition and Theorem 9(1),

$$C_w = \sum_{u\le w} t_w t_u^{-1} P_{u,w}(t^{-2})\tau_u \odot \mathrm{pt}^m_e = \sum_{u\le w} t_w P_{u,w}(t^{-2})\,\mathrm{MC}_{-t^{-2}}(X(u)^\circ). \tag{11}$$

On the other hand, since $\widetilde{\gamma}_w$ is invariant under the involution, we get

$$\widetilde{\gamma}_w = \sum_{v\in W} \epsilon_w\epsilon_v t_w t_v^{-1} P_{v,w}(t^{-2})\tau^{-1}_{v^{-1}}.$$

Thus,

$$\begin{aligned}
\widetilde{C}_w &= \widetilde{\gamma}_{w^{-1}w_0}\bullet \mathrm{pt}^m_{w_0}\\
&= \sum_{v\ge w} \epsilon_w\epsilon_v t_{w^{-1}w_0} t^{-1}_{v^{-1}w_0} P_{v^{-1}w_0,w^{-1}w_0}(t^{-2})\tau^{-1}_{w_0v}\bullet \mathrm{pt}^m_{w_0}\\
&= \prod_{\alpha>0}(1-t^{-2}e^{-\alpha}) \sum_{v\ge w} \epsilon_w\epsilon_v t_{w^{-1}w_0} P_{v^{-1}w_0,w^{-1}w_0}(t^{-2})\,\mathrm{SMC}_{-t^{-2}}(Y(v)^\circ),
\end{aligned} \tag{12}$$

where the last step follows from Lemma 14(1).

Therefore, we have

$$\begin{aligned}
\langle C_w, \widetilde{C}_y\rangle &= \prod_{\alpha>0}(1-t^{-2}e^{-\alpha}) t_w t_{y^{-1}w_0} \sum_u P_{u,w} \sum_v \epsilon_v\epsilon_y P_{v^{-1}w_0,y^{-1}w_0}\delta^{\mathrm{Kr}}_{u,v}\\
&= \prod_{\alpha>0}(1-t^{-2}e^{-\alpha}) t_w t_{y^{-1}w_0} \sum_u P_{u,w}\epsilon_u\epsilon_y P_{w_0u,w_0y}\\
&= \prod_{\alpha>0}(t-t^{-1}e^{-\alpha})\delta^{\mathrm{Kr}}_{w,y},
\end{aligned}$$

where the first equality follows from Lemma 14(2), the second follows from $P_{u,v}=P_{u^{-1},v^{-1}}$, and the third one follows from (10).

An immediate corollary of the proof is the following.

Corollary 16. *If the Schubert variety $X(w)$ is smooth, then*

$$C_w = \sum_{u\le w} t_w\,\mathrm{MC}_{-t^{-2}}(X(u)^\circ) = t_w\,\mathrm{MC}_{-t^{-2}}(X(w)) \in K_T(G/B)[t^{\pm 1}].$$

Proof. It follows directly from (11) and the fact $P_{u,w}=1$ for all $u\le w$. ☐

Characteristic classes of mixed Hodge modules. For any parabolic subgroup P_J, let $K^0(\mathrm{MHM}(G/P_J, B))$ denote its Grothendieck group of B-equivariant mixed Hodge modules. Recall there is a motivic Hodge Chern transformation (see [Schürmann 2011, Definition 5.3 and Remark 5.5])

$$\mathrm{MHC}_{-t^{-2}} : K^0(\mathrm{MHM}(G/P_J, B)) \to K_B(G/P_J)[t^{\pm 1}] \simeq K_T(G/P_J)[t^{\pm 1}]$$

such that for any $[f : Z \to G/P_J] \in G_0^B(\mathrm{var}/(G/P_J))$,

$$\mathrm{MC}_{-t^{-2}}([f : Z \to G/P_J]) = \mathrm{MHC}_{-t^{-2}}([f_! \mathbb{Q}_Z^H]), \tag{13}$$

where $[\mathbb{Q}_Z^H] := [k^* \mathbb{Q}_{\mathrm{pt}}^H] \in K^0(\mathrm{MHM}(Z, B))$ and $k : Z \to \mathrm{pt}$ is the structure morphism. The construction also works for B^--equivariant mixed Hodge modules, where B^- is the opposite Borel subgroup. The natural transformation $\mathrm{MC}_{-t^{-2}}$ commutes with the Serre–Grothendieck dual as follows [loc. cit., Corollary 5.19]:

$$\mathrm{MHC}_{-t^{-2}} \circ \mathcal{D} = \mathcal{D} \circ \mathrm{MHC}_{-t^{-2}}. \tag{14}$$

Here the first $\mathcal{D}$ is the dual of the mixed Hodge modules, and the second one is the Serre–Grothendieck dual. Both are denoted by $\mathcal{D}$, if no confusion is possible.

For any $u \in W$, let $i_u : X(u)^\circ \hookrightarrow G/B$ and $j_u : Y(u)^\circ \hookrightarrow G/B$ be the inclusions. Then, by (13),

$$\mathrm{MC}_{-t^{-2}}(X(u)^\circ) = \mathrm{MHC}_{-t^{-2}}([i_{u!} \mathbb{Q}_{X(u)^\circ}^H]),$$

where $\mathbb{Q}_{X(u)^\circ}^H$ is the constant mixed Hodge module on the Schubert cell $X(u)^\circ$. Since $\mathcal{D} \circ j_{v!} = j_{v*} \circ \mathcal{D}$, and

$$\mathcal{D}(\mathbb{Q}_{Y(v)^\circ}^H) = \mathbb{Q}_{Y(v)^\circ}^H[2 \dim Y(v)^\circ](\dim Y(v)^\circ),$$

where $[2 \dim Y(v)^\circ]$ means shift by $2 \dim Y(v)^\circ$ and $(\dim Y(v)^\circ)$ denotes the twist by the Tate Hodge module $\mathbb{Q}^H(1)^{\otimes \dim Y(v)^\circ}$, Equation (14) gives

$$\mathrm{SMC}_{-t^{-2}}(Y(v)^\circ) = \frac{\mathrm{MHC}_{-t^{-2}}([j_{v*} \mathbb{Q}_{Y(v)^\circ}^H])}{\lambda_{-t^{-2}}(T_{G/B}^*)}.$$

Using these, Lemma 14(2) can also be proved using mixed Hodge modules, by Schürmann. For the analogue in equivariant homology, see [Schürmann 2017, Theorem 1.2].

For any Schubert variety $X(w)$, let $[\mathrm{IC}_{X(w)}^H] \in K^0(\mathrm{MHM}(G/B, B))$ denote the intersection homology Hodge module on $X(w)$. Then it is well known that (see [Kazhdan and Lusztig 1980; Tanisaki 1987; Kashiwara and Tanisaki 2002])

$$[\mathrm{IC}_{X(w)}^H] = \sum_{u \le w} \epsilon_w P_{u,w}(t^{-2}) [i_{u!} \mathbb{Q}_{X(u)^\circ}^H].$$

Thus,

$$\mathrm{MHC}_{-t^{-2}}([\mathrm{IC}_{X(w)}^H]) = \sum_{u \le w} \epsilon_w P_{u,w}(t^{-2}) \, \mathrm{MC}_{-t^{-2}}(X(u)^\circ).$$

Comparing with (11), we get the following geometric interpretation of the KL classes C_w in Definition 12.

Proposition 17. *For any $w \in W$,*

$$C_w = t_w \epsilon_w \, \mathrm{MHC}_{-t^{-2}}([\mathrm{IC}^H_{X(w)}]) \in K_T(G/B)[t^{\pm 1}].$$

Remark 18. If $X(w)$ is smooth or rationally smooth (i.e., $[\mathrm{IC}^H_{X(w)}] = \mathbb{Q}^H_{X(w)}[\dim X(w)]$), then

$$C_w = t_w \epsilon_w \, \mathrm{MHC}_{-t^{-2}}([\mathrm{IC}^H_{X(w)}]) = t_w \, \mathrm{MC}_{-t^{-2}}(X(w)).$$

This is compatible with Corollary 16.

An immediate corollary is the following.

Corollary 19. *The canonical basis C_w is invariant under the Serre–Grothendieck duality, i.e.,*

$$\mathcal{D}(C_w) = C_w \in K_T(G/B)[t^{\pm 1}].$$

Proof. Since

$$\mathcal{D}(\mathrm{IC}^H_{X(w)}) = \mathrm{IC}^H_{X(w)}(\dim X(w)),$$

Equation (14) and Proposition 17 give

$$\mathcal{D}(C_w) = \mathcal{D}\big(t_w \epsilon_w \, \mathrm{MHC}_{-t^{-2}}([\mathrm{IC}^H_{X(w)}])\big) = t_w^{-1} \epsilon_w \, \mathrm{MHC}_{-t^{-2}}\big(\mathcal{D}([\mathrm{IC}^H_{X(w)}])\big) = C_w. \qquad \square$$

Parabolic case. In this subsection, we generalize the above results to the parabolic case. Let $J \subset \Pi$ be a subset of simple roots, with corresponding parabolic subgroup P_J. Schubert cells and varieties and opposite Schubert cells and varieties of G/P_J are indicated by subscripts J. Recall there exist parabolic Kazhdan–Lusztig polynomials (see [Deodhar 1987; Kashiwara and Tanisaki 2002]), denoted by $P^J_{v,w} \in \mathbb{Z}[t^{-2}]$, where $v, w \in W^J$. Here our $P^J_{v,w}$ is the $u = -1$ parabolic KL polynomials in [Deodhar 1987], which are also denoted by $P^{J,q}_{v,w}$ in [Kashiwara and Tanisaki 2002, Remark 2.1]. We have the following property, which generalizes [Deodhar 1987, Proposition 3.4].

Lemma 20 [Lenart et al. 2020, Proposition 5.19]. *For any $w, v \in W^J$ and $u \in W_J$,*

$$P_{vu, ww_J} = P^J_{v,w}.$$

Let $Q_{u,w} := P_{w_0 w, w_0 u}$ denote the usual inverse KL polynomials, which satisfy

$$\sum_w \epsilon_u \epsilon_w Q_{u,w} P_{w,v} = \delta^{\mathrm{Kr}}_{u,v}.$$

For any $u, w \in W^J$, let $Q^J_{u,w} \in \mathbb{Z}[t^{-2}]$ denote the inverse parabolic KL polynomial (see [Kashiwara and Tanisaki 2002]).[1] Then

$$\sum_{w \in W^J} \epsilon_u \epsilon_w Q^J_{u,w} P^J_{w,v} = \delta^{\mathrm{Kr}}_{u,v}. \tag{15}$$

Moreover, it is related to the usual $Q_{u,w}$ as follows (see [Kashiwara and Tanisaki 2002, Proposition 2.6] or [Soergel 1997]):

$$Q^J_{u,w} = \sum_{v \in W_J} \epsilon_v \epsilon_{w_J} Q_{uw_J, wv}.$$

[1] Our $Q^J_{u,w}$ is denoted by $Q^{J,q}_{u,w}$ in [Kashiwara and Tanisaki 2002].

Following (11) and (12), we define the parabolic canonical bases in $K_T(G/P_J)[t^{\pm 1}]$ as follows.

Definition 21. For any $w \in W^J$, let

$$C_w^J := \sum_{u \in W^J, u \le w} t_w P_{u,w}^J(t^{-2}) \, \mathrm{MC}_{-t^{-2}}(X(u)_J^\circ),$$

$$\widetilde{C}_w^J := \prod_{\alpha \in \Sigma^+ - \Sigma_J^+} (1 - t^{-2} e^{-\alpha}) \sum_{v \in W^J, v \ge w} \epsilon_w \epsilon_v t_{w_J w^{-1} w_0} Q_{w,v}^J(t^{-2}) \, \mathrm{SMC}_{-t^{-2}}(Y(v)_J^\circ).$$

If $J = \varnothing$, then $C_w^\varnothing = C_w$, and $\widetilde{C}_w^\varnothing = \widetilde{C}_w$, as defined before.

Let $\langle -, - \rangle_J$ denote the nondegenerate tensor product pairing on $K_T(G/P_J)$. The parabolic analogue of Lemma 14(2) also holds (see [Mihalcea et al. 2022, Theorem 7.2]): for any $u, v \in W^J$,

$$\big\langle \mathrm{MC}_{-t^{-2}}(X(u)_J^\circ), \mathrm{SMC}_{-t^{-2}}(Y(v)_J^\circ) \big\rangle_J = \delta_{u,v}^{\mathrm{Kr}}.$$

Combining this with (15), we immediately get the following generalization of Theorem 13.

Theorem 22. *For any $u, w \in W^J$,*

$$\langle C_w^J, \widetilde{C}_u^J \rangle_J = \delta_{u,w}^{\mathrm{Kr}} \prod_{\alpha \in \Sigma^+ - \Sigma_J^+} (t - t^{-1} e^{-\alpha}).$$

We now investigate the relation between KL classes of G/B and G/P_J. For any $w \in W^J$, let us still use i_u to denote the inclusion $X(u)_J^\circ \hookrightarrow G/P_J$. Then the following identity holds in $K^0(\mathrm{MHM}(G/P_J, B))$ (see [Kashiwara and Tanisaki 2002, Corollary 5.1]):

$$[\mathrm{IC}_{X(w)_J}^H] = \sum_{u \in W^J, u \le w} \epsilon_w P_{u,w}^J [i_{u!} \mathbb{Q}_{X(u)_J^\circ}^H].$$

Thus, we get the following parabolic analogue of Proposition 17 and Corollary 19.

Proposition 23. *For any $w \in W^J$,*

$$C_w^J = t_w \epsilon_w \, \mathrm{MHC}_{-t^{-2}}([\mathrm{IC}_{X(w)_J}^H]).$$

Moreover, let $\mathcal{D}_J$ denote the Serre–Grothendieck duality functor on G/P_J. Then

$$\mathcal{D}_J(C_w^J) = C_w^J.$$

Recall $\pi_J : G/B \to G/P_J$ denotes the natural projection. The relation between C_w and C_w^J is given by the following proposition.

Proposition 24. *Let $\mathcal{P}_J(t) = \sum_{v \in W_J} t_v$ be the Poincaré polynomial of W_J. Then for any $w \in W^J$,*

$$\pi_{J*}(C_{ww_J}) = t_{w_J}^{-1} \mathcal{P}_J(t^2) C_w^J \in K_T(G/P_J)[t^{\pm 1}].$$

Proof. By [Aluffi et al. 2019, Remark 5.5], for any $u \in W^J$ and $v \in W_J$,

$$\pi_{J*}(\mathrm{MC}_{-t^{-2}})(X(uv)^\circ) = t_v^{-2} \, \mathrm{MC}_{-t^{-2}}(X(u)_J^\circ),$$

which also follows directly from the following identity about mixed Hodge modules:

$$\pi_{J!}(i_{uv!}\mathbb{Q}^H_{X(uv)^\circ}) = \mathbb{Q}^H_{X(u)^\circ_J}[-2\ell(v)](-\ell(v)).$$

Thus,

$$\begin{aligned}\pi_{J*}(C_{ww_J}) &= \sum_{u\in W^J, u\le w}\sum_{v\in W_J} t_w t_{w_J} P_{uv,ww_J}\pi_{J*}\mathrm{MC}_{-t^{-2}}(X(uv)^\circ)\\ &= \sum_{u\in W^J, u\le w} t_w t_{w_J} P^J_{u,w}\,\mathrm{MC}_{-t^{-2}}(X(u)^\circ_J)\sum_{v\in W_J} t_v^{-2}\\ &= C^J_w\sum_{v\in W_J} t_v^{-2}t_{w_J} = C^J_w\sum_{v\in W_J} t^{-1}_{w_J}t^2_{w_J}t_v^{-2} = C^J_w\sum_{v\in W_J} t^{-1}_{w_J}t^2_{vw_J} = C^J_w t^{-1}_{w_J}\mathcal{P}_J(t^2),\end{aligned}$$

where the second equality follows from Lemma 20. □

5. The smoothness conjecture for hyperbolic cohomology

In this section, we use the smoothness criterion to prove the smoothness conjecture. Since we will be working with multiplicative and hyperbolic formal group laws at the same time, we add superscripts or subscripts m (resp. t) in the multiplicative case (resp. hyperbolic case).

The hyperbolic case. Consider the hyperbolic formal group law over $R = \mathbb{Z}[t, t^{-1}, \mu^{-1}]$

$$F_t(x, y) := \frac{x + y - xy}{1 - \mu^{-2}xy},$$

where $\mu = t + t^{-1}$. Note that R depends on only one parameter t. The definitions of Section 2 applied to F_t give the respective rings

$$S_t,\quad Q_t,\quad Q_{t,W},\quad \boldsymbol{D}_t.$$

Consider a map of formal group laws

$$g\colon F_t \to F_m,\qquad g(x) = \frac{(1-t^2)x}{x-(t^2+1)},$$

so that $F_m(g(x), g(y)) = g(F_t(x, y))$. It induces ring embeddings

$$\psi\colon S_m \hookrightarrow S_t,\qquad \psi(f(x_\lambda)) = f(g(x_\lambda))\quad\text{for } f(x)\in R[\![x]\!],$$

and

$$\psi\colon Q_m \hookrightarrow R\left[\frac{1}{1-t^2}\right]\otimes Q_t. \tag{16}$$

Consequently, we have a ring embedding

$$\psi : Q_{m,W} \to R\left[\frac{1}{1-t^2}\right]\otimes_R Q_{t,W},\qquad \psi(p\delta^m_w) = \psi(p)\delta^t_w\quad\text{for } p\in Q_m, w\in W.$$

It can be shown that

$$\psi(\tau_i) = \mu Y^t_i - t \in \boldsymbol{D}_t \subset Q_{t,W}. \tag{17}$$

Note that in (16), for the target, we have to invert $t^2 - 1$, but for the one in (17), it is not necessary.

One of the most interesting properties of ψ is the following (see [Lenart et al. 2020, Corollary 5.5 (2)]):

$$\mu^{-\ell(w_{J/J'})}\psi(\gamma_{J/J'})Y^t_{J'} = Y^t_J. \tag{18}$$

In other words, $\psi(\gamma_{J/J'})$ behaves like a replacement of $Y_{J/J'}$; see [Lenart et al. 2020, Remark 5.6]. In particular, letting $J' = \varnothing$, one then has

$$\mu^{-\ell(w_J)}\psi(\gamma_{w_J}) = Y^t_J.$$

Let $\mathfrak{h}$ denote the respective oriented cohomology theory for the hyperbolic formal group law F_t.

Definition 25. Define the KL Schubert class for $w \in W^J$ to be

$$\mathrm{KL}^J_w := \mu^{-\ell(ww_J)}\psi(\gamma_{ww_J}) \odot \mathrm{pt}^t_e \in (\boldsymbol{D}^*_t)^{W_J} \cong \mathfrak{h}_T(G/P_J).$$

Remark 26. Following [Lenart et al. 2020], one can define a certain involution on some subset $\mathcal{N}_J := \psi(H) \odot \mathrm{pt}^t_e \subset \boldsymbol{D}^*_t$ so that KL^J_v is invariant under such an involution, similar to the parabolic Kazhdan–Lusztig basis of Deodhar.

Writing the Kazhdan–Lusztig basis as $\gamma_w = \sum_{v \le w} b_{w,v}\delta_v$, $b_{w,v} \in S_m$, we then have in $K_T(G/B)$

$$C_w = \gamma_w \odot \mathrm{pt}^m_e = \sum_{v \le w} b_{w,v}\delta_v \odot \left(\prod_{\alpha>0}(1-e^{\alpha}) f^m_e\right) = \sum_{v \le w} b_{w,v} v\left(\prod_{\alpha>0}(1-e^{\alpha})\right) f^m_v.$$

On the other hand, inside $\mathfrak{h}_T(G/B)$, we have

$$\mathrm{KL}_w = \mu^{-\ell(w)}\psi(\gamma_w) \odot \mathrm{pt}^t_e = \mu^{-\ell(w)} \sum_{v \le w} \psi(b_{w,v}) v\left(\prod_{\alpha>0} x_{-\alpha}\right) f^t_v.$$

Here $x_\alpha \in S_t$. It would be interesting to compare the two classes in different cohomology theories. Here is an example.

Example 27. We consider the SL_3 case, so there are two simple roots α_1, α_2. Recall that in S_m, we have $x_\lambda = 1 - e^{-\lambda}$. Write $\hat{x}_\lambda = t - t^{-1}e^{-\lambda}$. For simplicity, write $x_{\pm i \pm j} := x_{\pm\alpha_i \pm\alpha_j}$ and $\hat{x}_{\pm i \pm j} = \hat{x}_{\pm\alpha_i \pm\alpha_j}$. Inside $H \subset Q_{m,W}$, we have

$$\begin{aligned}
\gamma_{s_i} &= (\delta_{s_i} + 1)\frac{\hat{x}_{-i}}{x_{-i}},\\
\gamma_{s_1s_2} &= (\delta_{s_1s_2} + \delta_{s_2})\frac{\hat{x}_{-1-2}\hat{x}_{-2}}{x_{-1-2}x_{-2}} + (\delta_{s_1} + 1)\frac{\hat{x}_{-1}\hat{x}_{-2}}{x_{-1}x_{-2}},\\
\gamma_{s_1s_2s_1} &= (\delta_{s_1s_2s_1} + \delta_{s_1s_2} + \delta_{s_2s_1} + \delta_{s_1} + \delta_{s_2} + 1)\frac{\hat{x}_{-1}\hat{x}_{-2}\hat{x}_{-1-2}}{x_{-1}x_{-2}x_{-1-2}}.
\end{aligned}$$

Recall that $\mathrm{pt}^m_e = x_{-1}x_{-2}x_{-1-2} f^m_e \in \boldsymbol{D}^*_m$. So inside $\boldsymbol{D}^*_m \cong K_T(G/B) \otimes_{\mathbb{Z}} R$, we have

$$\begin{aligned}
C_e &= \mathrm{pt}^m_e,\\
C_{s_1} &= \hat{x}_{-1}x_{-2}x_{-1-2} f^m_e + \hat{x}_1 x_{-2}x_{-1-2} f^m_{s_1},\\
C_{s_1s_2} &= \hat{x}_{-1}\hat{x}_{-2}x_{-1-2} f^m_e + \hat{x}_1\hat{x}_{-1-2}x_{-2} f^m_{s_1} + \hat{x}_{-1}\hat{x}_2 x_{-1-2} f^m_{s_2} + \hat{x}_1\hat{x}_{1+2}x_{-2} f^m_{12},
\end{aligned}$$

$$C_{s_1s_2s_1} = \hat{x}_{-1}\hat{x}_{-2}\hat{x}_{-1-2}f_e^m + \hat{x}_1\hat{x}_{-2}\hat{x}_{-1-2}f_{s_1}^m + \hat{x}_{-1}\hat{x}_2\hat{x}_{-1-2}f_{s_2}^m + \hat{x}_1\hat{x}_{1+2}\hat{x}_{-1-2}f_{s_1s_2}^m + \hat{x}_2\hat{x}_{1+2}\hat{x}_{-1}f_{s_2s_1}^m + \hat{x}_1\hat{x}_2\hat{x}_{1+2}f_{s_1s_2s_1}^m.$$

Note that so far in this example all notation is in S_m, $Q_{m,W}$ or $\boldsymbol{D}_m^*$.

On the other hand, one can compute $\mathrm{KL}_w \in \mathfrak{h}_T(G/B)$ as follows: Note that $\psi(\hat{x}_i/x_i) = \mu/x_i$ (where the first x_i is in S_m and the second x_i is in S_t). Then

$$\mathrm{KL}_e = \mathrm{pt}_e^t, \qquad \mathrm{KL}_{s_1s_2} = x_{-1-2}f_e^t + x_{-2}f_{s_1}^t + x_{-1-2}f_{s_2}^t + x_{-2}f_{s_1s_2}^t,$$
$$\mathrm{KL}_{s_1} = x_{-1}x_{-1-2}f_e^t + x_{-1-2}x_{-2}f_{s_1}^t, \quad \mathrm{KL}_{s_1s_2s_1} = f_e^t + f_{s_1}^t + f_{s_2}^t + f_{s_1s_2}^t + f_{s_2s_1}^t + f_{s_1s_2s_1}^t.$$

In this case, all Schubert varieties are smooth, and it is easy to verify that the classes KL_w coincide with the Schubert classes.

We now prove the smoothness conjecture [Lenart et al. 2020, Conjecture 5.14]. Several special cases were proved in [Lenart and Zainoulline 2017; Lenart et al. 2020], such as the case of $w = w_{J/J'}$ for $J' \subset J \subseteq \Pi$ (i.e., w has "relative" maximal length), and that of Schubert varieties in complex projective spaces.

Theorem 28. *If the Schubert variety $X(w)$ is smooth, then the class determined by $X(w)$ in $\mathfrak{h}_T(G/B)$ coincides with the KL Schubert class KL_w.*

Proof. Since $X(w)$ is smooth, $P_{v,w} = 1$ for any $v \le w$; see [Billey and Lakshmibai 2000, 6.1.19]. Therefore,

$$\gamma_w = \sum_{v\le w} t_w t_v^{-1}\tau_v = t_w \sum_{v\le w} t_v^{-1}\tau_v = t_w\Gamma_w = t_w \sum_{v\le w} a_{w,v}\delta_v^m.$$

From the definition of ψ, it is easy to verify that

$$\psi\left(\frac{1-t^{-2}e^\alpha}{1-e^\alpha}\right) = \frac{t^{-1}\mu}{x_{-\alpha}}. \tag{19}$$

Then for any $w \in W$, we have

$$\begin{aligned}
\mathrm{KL}_w &= \mu^{-\ell(w)}\psi(\gamma_w)\odot \mathrm{pt}_e^t\\
&= \mu^{-\ell(w)}\psi\left(t_w\sum_{v\le w} a_{w,v}\delta_v^m\right)\odot \mathrm{pt}_e^t\\
&= \mu^{-\ell(w)} t_w \sum_{v\le w}\psi\left(\prod_{\alpha>0,\, vs_\alpha\le w}\frac{1-t^{-2}e^{u\alpha}}{1-e^{u\alpha}}\right)\delta_v^t\odot \mathrm{pt}_e^t \quad \text{(by Corollary 11)}\\
&= \mu^{-\ell(w)} t_w \sum_{v\le w}\left(\prod_{\alpha>0,\, vs_\alpha\le w}\frac{t^{-1}\mu}{x_{-v\alpha}}\right)\cdot v(x_\Pi^t)f_v^t \qquad \text{(by (5) and (19))}\\
&= \sum_{v\le w} v\left(\frac{\prod_{\alpha<0} x_\alpha}{\prod_{\alpha<0,\, vs_\alpha\le w} x_\alpha}\right)f_v^t\\
&= \sum_{v\le w}\frac{\prod_{\alpha>0} x_{-\alpha}}{\prod_{\alpha>0,\, s_\alpha v\le w} x_{-\alpha}}f_v^t.
\end{aligned}$$

Here the fifth identity follows from the well-known fact that

$$\text{if } X(w) \text{ is smooth, then } |\{\alpha > 0 \mid s_\alpha v \leq w\}| = \ell(w) \text{ for any } v \leq w \in W,$$

and the last one is proved as follows: for any $v \leq w \in W$,

$$\begin{aligned}\frac{\prod_{\alpha<0} x_{v\alpha}}{\prod_{\alpha<0,\, vs_\alpha \leq w} x_{v\alpha}} &= \frac{\prod_{\alpha>0,\, s_\alpha v<v} x_\alpha \cdot \prod_{\alpha>0,\, v<s_\alpha v} x_{-\alpha}}{\prod_{\alpha>0,\, s_\alpha v<v} x_\alpha \cdot \prod_{\alpha>0,\, v<s_\alpha v \leq w} x_{-\alpha}} \\ &= \frac{\prod_{\alpha>0,\, s_\alpha v<v} x_{-\alpha} \cdot \prod_{\alpha>0,\, v<s_\alpha v} x_{-\alpha}}{\prod_{\alpha>0,\, s_\alpha v<v} x_{-\alpha} \cdot \prod_{\alpha>0,\, v<s_\alpha v \leq w} x_{-\alpha}} \\ &= \frac{\prod_{\alpha>0} x_{-\alpha}}{\prod_{\alpha>0,\, s_\alpha v \leq w} x_{-\alpha}}.\end{aligned}$$

Comparing with the restriction formula of $[X(w)]$ in [Lenart et al. 2020, (5.6)], we see that $\mathrm{KL}_w = [X(w)]$. The proof is finished. □

We now look at the case of partial flag varieties. Let P_J be the parabolic subgroup with the projection map $\pi_J : G/B \to G/P_J$. Let w_J be the longest element in the subgroup W_J of W determined by J, and $W^J \subset W$ be the set of minimal length representatives of W/W_J. Recall $X(w)_J$ denotes the Schubert variety of G/P_J determined by $w \in W^J$.

For G/P_J, the definition of the KL Schubert class KL^J_w corresponding to $w \in W^J$ is defined by using the so-called parabolic Kazhdan–Lusztig basis. According to the paragraph right after [Lenart et al. 2020, Definition 5.9], via the embedding $\pi_J^* : \mathfrak{h}_T(G/P_J) \to \mathfrak{h}_T(G/B)$, we have

$$\pi_J^*(\mathrm{KL}^J_w) = \mathrm{KL}_{ww_J}\,.$$

Corollary 29. *Conjecture 5.14 of* [Lenart et al. 2020] *holds for any partial flag variety G/P_J; that is, if the Schubert variety $X(w)_J$ of G/P_J is smooth for $w \in W^J$, then the KL Schubert class KL^J_w of w coincides with the fundamental class $[X(w)_J]$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_J^{-1}(X(w)_J) & \xrightarrow{i'} & G/B \\ \big\downarrow{\scriptstyle \pi_J} & & \big\downarrow{\scriptstyle \pi_J} \\ X(w)_J & \xrightarrow{i} & G/P_J \end{array}$$

Moreover, $\pi_J^{-1}(X(w)_J) = X(ww_J)$. Since $X(w)_J$ is smooth, $X(ww_J)$ is also smooth. Thus, Theorem 28 implies $[X(ww_J)] = \mathrm{KL}_{ww_J}$. On the other hand, by proper base change, we obtain

$$\pi_J^*[X(w)_J] = \pi_J^* i_*[1_{X(w)_J}] = i'_*\pi_J^*[1_{X(w)_J}] = i'_*[1_{X(ww_J)}] = [X(ww_J)],$$

where the third equality follows from the fact that the pullback π_J^* preserves identity. Since $\pi_J^*(\mathrm{KL}^J_w) = \mathrm{KL}_{ww_J}$, and π_J^* is injective, we get $\mathrm{KL}^J_w = [X(ww_J)] \in \mathfrak{h}_T(G/P_J)$. □

6. KL Schubert classes and small resolutions

In this section, we give a geometric interpretation of the KL Schubert classes (for hyperbolic cohomology) in the case of type A Grassmannians.

For subsets $J' \subset J \subseteq \Pi$, for hyperbolic cohomology, we will use relative push–pull elements $Y^t_{J/J'}$ defined in (1). For simplicity, we will skip the superscript t. Moreover, if $Q \subset P$ are the parabolic subgroups corresponding to $J' \subset J$, respectively, we will write $Y_{P/Q} = Y_{J/J'}$.

Consider the Grassmannian $\mathrm{Gr}_d(\mathbb{C}^{n-d}) = \mathrm{SL}_n/P_J$, where the set of simple roots Π is identified with $\{1, \dots, n-1\}$ and $J := \Pi - \{d\}$. Fix a Schubert variety $X(\lambda)$ of it, which is indexed by a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l > 0)$ contained inside the $d \times (n-d)$ rectangle; here we mean that λ is identified with a Young diagram (in English notation), whose top left box is placed on the top left box of the mentioned rectangle.

Alternatively, the Schubert variety $X(\lambda)$ is indexed by a d-subset I_λ of $[n] := \{1, \dots, n\}$, which is constructed as follows. Place the above $d \times (n-d)$ rectangle inside the first quadrant of the xy-plane so that its southwest corner is the origin. Label each horizontal (resp. vertical) unit segment whose left (resp. bottom) endpoint is a lattice point (x, y) by $x + y + 1$. Consider the lattice path from $(0, 0)$ to $(n-d, d)$ defining the southeast boundary of the Young diagram λ when embedded into the $d \times (n-d)$ rectangle as stated above. Then I_λ consists of the labels on the vertical steps of this path.

Yet another indexing of the Schubert variety $X(\lambda)$ is by a *Grassmannian permutation* w_λ in the symmetric group $W = S_n$, which has its unique descent in position d. Written in one-line notation, w_λ consists of the entries in I_λ followed by the entries in $[n] - I_\lambda$, where both sets of entries are ordered increasingly. Here we use $-$ for set difference. Thus, w_λ belongs to the set W^J of lowest coset representatives modulo the parabolic subgroup W_J. Moreover, it has the reduced decomposition

$$w_\lambda = \overrightarrow{\prod_{(i,j)\in\lambda}} s_{d+j-i}, \tag{20}$$

where (i, j) is the box of the Young diagram λ in row i and column j, while in the product we scan the rows of λ from bottom to top, and each row from right to left.

Example 30. We use as a running example the same one as in [Billey and Lakshmibai 2000, Example 9.1.11], namely $n = 10$, $d = 5$, $\lambda = (5, 5, 3, 2, 2)$, $I_\lambda = \{3, 4, 6, 9, 10\}$. In order to illustrate (20), we place the number $d + j - i$ in the box (i, j) of λ, as follows:

5	6	7	8	9
4	5	6	7	8
3	4	5		
2	3			
1	2			

(21)

Thus, we have

$$w_\lambda = [3, 4, 6, 9, 10, 1, 2, 5, 7, 8] = (s_2 s_1)(s_3 s_2)(s_5 s_4 s_3)(s_8 s_7 s_6 s_5 s_4)(s_9 s_8 s_7 s_6 s_5). \tag{22}$$

In [Billey and Lakshmibai 2000, Section 9.1], the permutation w_λ is identified with the d-subset I_λ, and they are encoded into a $2 \times m$ matrix

$$\begin{pmatrix} k_1 \cdots k_m \\ a_1 \cdots a_m \end{pmatrix}, \tag{23}$$

which can be read off from the above lattice path as follows. The entries $0 < k_1 < \cdots < k_m \le n$ are the labels of the last steps in consecutive sequences of vertical (unit) steps. The entries $a_1, \ldots, a_m$ are the lengths of these sequences. The numbers $b_0, \ldots, b_{m-1}$ calculated in [Billey and Lakshmibai 2000] are the lengths of the sequences of horizontal steps, where we set $b_0 := 0$ if $l < d$ (i.e., if the lattice path starts with a vertical step). Recall that we also set $a_0 = b_m := \infty$.

Recall that the Schubert variety $X(\lambda)$ has *small resolutions*, which were defined by Zelevinsky [1983]. We briefly recall their construction following [Billey and Lakshmibai 2000, Section 9.1]. This construction starts with the choice of an index i, with $0 \le i < m$, such that $b_i \le a_i$ and $a_{i+1} \le b_{i+1}$ (any such choice can be made). While it is clear that such an index always exists, we avoid the choice of $i = 0$ if $l < d$. Then, a new permutation w^2 is obtained from $w^1 := w_\lambda$ via a certain procedure, which can be rephrased as follows. Consider the i-th outer corner of λ (counting from 0), from southwest to northeast, where the origin is an outer corner if and only if $l < d$. Consider the rectangle R_1 (inside λ) whose southeast vertex is the mentioned outer corner, and which is maximal in that its removal from λ still leaves a Young diagram. It is clear that the size of R_1 is $b_i \times a_{i+1}$. Then w^2 is the Grassmannian permutation corresponding to the Young diagram $\lambda - R_1$.

The above procedure is then iterated. We thus tile the Young diagram λ with rectangles $R_1, \ldots, R_r$. Let us denote by p_i and q_i the height and width of R_i, respectively. We also define the sequence of Grassmannian permutations $w^1, \ldots, w^r$ so that the Young diagram of w^i is $\lambda^i := \lambda - \rho^{i-1}$, where $\rho^j := R_1 \cup \cdots \cup R_j$. In particular, the Young diagram of w^r is R_r, and the Schubert variety $X(w^r)$ is smooth. Note that $r = m$ if $l = d$, and $r = m - 1$ if $l < d$.

Example 31. We continue Example 30. The encoding of w_λ by the $2 \times m$ matrix (23) and the successive choices of w^1, w^2, w^3 based on it are described in detail in [Billey and Lakshmibai 2000]. In our setup, the tiling of λ with the corresponding rectangles R_1, R_2, R_3 is illustrated below (the number in a box is the index of the rectangle to which that box belongs).

3	3	2	2	2
3	3	2	2	2
3	3	1		
3	3			
3	3			

In order to complete the construction of the Zelevinsky resolution, following [Billey and Lakshmibai 2000, Section 9.1], we need the stabilizer P_{w_λ} of the Schubert variety $X(\lambda) = X(w_\lambda)$. This is the parabolic subgroup corresponding to the subset $\Pi - \{k_1, \ldots, k_m\}$; compare (23). More generally, consider the stabilizers $P_i := P_{w^i}$, for $i = 1, \dots, r$, and $P_{r+1} := P_J$; for simplicity, we use the same notation for the corresponding subsets of Π. Also let $Q_i := P_i \cap P_{i+1}$, for $i = 1, \dots, r$, both as parabolic subgroups and subsets of Π. Then the Zelevinsky resolution of $X(w)$ is expressed as follows:

$$P_1 \times^{Q_1} P_2 \times \cdots \times^{Q_{r-2}} P_{r-1} \times^{Q_{r-1}} X(w^r) =: \widetilde{X}(w_\lambda) \to X(w_\lambda). \tag{24}$$

Therefore, by Corollaries 4 and 29, the pushforward of the fundamental class of $\widetilde{X}(w_\lambda)$ inside $\mathfrak{h}_T(G/B)$ is the element

$$Y_{P_1/Q_1} \cdots Y_{P_r/Q_r} Y_J \odot \mathrm{pt}_e^t \,. \tag{25}$$

Example 32. Continuing Example 31, the operator in (25) is written explicitly as

$$Y_{(\Pi-\{4,6\})/(\Pi-\{4,5,6\})}\, Y_{(\Pi-\{5\})/(\Pi-\{5,7\})}\, Y_{(\Pi-\{7\})/(\Pi-\{5,7\})}\, Y_{\Pi-\{5\}}.$$

Indeed, the parabolic subsets P_i for these examples were exhibited in [Billey and Lakshmibai 2000], while they can also be read off from the Young diagram of $\lambda = (5, 5, 3, 2, 2)$ as indicated above.

We will now state the main technical result of this section, Theorem 34, which is interesting itself, and is needed to make the connection with the KL Schubert classes for the Grassmannian; compare [Lenart et al. 2020]. To this end, we introduce more notation in the above setup. Given the rectangle R_i, with its embedding into the Young diagram of λ and the first quadrant, let C_i and D_i be the sets of labels on its left vertical side and its top horizontal side, respectively. Let

$$c_i := \min C_i, \quad d_i := \max D_i = c_i + p_i + q_i - 1, \quad C_i' := C_i - \{\max C_i\}, \quad D_i' := D_i - \{d_i\}.$$

Finally, let $J_i := C_i \sqcup D_i'$ and $J_i' := C_i' \sqcup D_i'$.

We also need to define the subsets $K_i' \subsetneq K_i$ of Π for $i = 1, \dots, r$. First recall that above we defined the shape ρ^i as the union of the rectangles $R_1, \dots, R_i$. It is not hard to see that ρ^i is a union of completely disjoint Young diagrams (i.e., they do not share even a single point), aligned from southwest to northeast. Let $\mathcal{C}_i$ be set of indices $j \in \{1, \dots, i\}$ such that the left side of R_j is contained in the left boundary of a component of ρ^i. Similarly, let $\mathcal{D}_i$ be set of indices $k \in \{1, \dots, i\}$ such that the top side of R_k is contained in the top boundary of a component of ρ^i. We now define

$$K_i' := \Bigl(\bigsqcup_{j \in \mathcal{C}_i} C_j'\Bigr) \sqcup \Bigl(\bigsqcup_{k \in \mathcal{D}_i} D_k'\Bigr), \quad K_i := K_i' \sqcup \{\max C_i\}.$$

Note that $J_i \subseteq K_i$ and $J_i' \subseteq K_i'$.

Example 33. Continuing Example 32, we have

$$K_1' = J_1' = \varnothing \subsetneq K_1 = J_1 = \{5\}, \qquad K_3' = \{1, 2, 3, 4, 6, 8, 9\} \subsetneq K_3 = \{1, 2, 3, 4, 5, 6, 8, 9\},$$
$$K_2' = J_2' = \{6, 8, 9\} \subsetneq K_2 = J_2 = \{6, 7, 8, 9\}, \qquad J_3' = \{1, 2, 3, 4, 6\} \subsetneq J_3 = \{1, 2, 3, 4, 5, 6\}.$$

As indicated above, all this information is easily read off from the Young diagram of $\lambda = (5, 5, 3, 2, 2)$.

Theorem 34. *In $H \subset Q_{m,W}$, we have*

$$\gamma_{w_\lambda w_J} = \gamma_{J_1/J_1'} \cdots \gamma_{J_r/J_r'} \gamma_J = \gamma_{K_1/K_1'} \cdots \gamma_{K_r/K_r'} \gamma_J. \tag{26}$$

In order to prove Theorem 34, we start by recalling some results from [Kirillov and Lascoux 2000], related to the factorization of Kazhdan–Lusztig elements for the Grassmannian. This paper introduces an element Z_{w_λ} of the Hecke algebra, defined as a product of linear factors in the generators, which are associated with the boxes of the Young diagram λ. Instead of recalling the precise definition, which is not needed here, we will state a weaker form of the factorization, which turns out to be related to factorizations in (26). We will use notation introduced above.

The rectangle R_i corresponds to the Grassmannian permutation

$$v^i := (s_{c_i+q_i-1} \cdots s_{c_i})(s_{c_i+q_i} \cdots s_{c_i+1}) \cdots (s_{c_i+p_i+q_i-2} \cdots s_{c_i+p_i-1});$$

compare (20) and Example 30. It is not hard to see that we have the following factorization of w_λ, which corresponds to a reduced decomposition of w_λ obtained from (20) only by commuting simple reflections:

$$w_\lambda = v^1 \cdots v^r. \tag{27}$$

Example 35. In our running example, the reduced decomposition corresponding to (27) (to be compared with (22) and also (21)) is

$$w_\lambda = [3, 4, 6, 9, 10, 1, 2, 5, 7, 8] = \underbrace{(s_5)}_{v^1} \underbrace{((s_8 s_7 s_6)(s_9 s_8 s_7))}_{v^2} \underbrace{((s_2 s_1)(s_3 s_2)(s_4 s_3)(s_5 s_4)(s_6 s_5))}_{v^3}.$$

The factorization of Z_{w_λ} needed here is the following one, which corresponds to the factorization (27) of w_λ:

$$Z_{w_\lambda} = Z_{v^1} Z_{w^2} = Z_{v^1} \cdots Z_{v^r}. \tag{28}$$

See the proof of [Kirillov and Lascoux 2000, Theorem 3] for details.

The connection between the element Z_{w_λ} and the corresponding parabolic Kazhdan–Lusztig basis element is made in [Kirillov and Lascoux 2000, Theorem 3].

Theorem 36 [Kirillov and Lascoux 2000]. *In $H \subset Q_{m,W}$, we have*

$$Z_{w_\lambda} \gamma_J = \gamma_{w_\lambda w_J}.$$

The proof of Theorem 34 also relies on the following lemmas.

Lemma 37. *Consider $J' \subset J \subseteq \Pi$, and assume that $J \subset [a, b]$ with $a, b \in \Pi$. If $A \subseteq \Pi - [a-1, b+1]$, then we have*

$$\gamma_{J/J'} = \gamma_{J \sqcup A/J' \sqcup A} \in Q_{m,W}, \quad Y_{J/J'} = Y_{J \sqcup A/J' \sqcup A} \in Q_{t,W}.$$

Proof. As the sets of simple roots corresponding to J and A are orthogonal to each other, we have $\Sigma^-_{J \sqcup A} = \Sigma^-_J \sqcup \Sigma^-_A$ and $W_{J \sqcup A} = W_J \times W_A$, and similarly for J replaced by J'. Therefore, we have

$$w_{J/J'} := w_J w_{J'} = w_J w_A w_{J'} w_A =: w_{J \sqcup A/J' \sqcup A}, \quad x_{J/J'} = x_{J \sqcup A/J' \sqcup A}, \tag{29}$$

and $W_J/W_{J'}$ is in a natural bijection with $W_{J\sqcup A}/W_{J'\sqcup A}$. The stated equalities follow by plugging these facts into (7) and the definition (1) of the relative push–pull operator. □

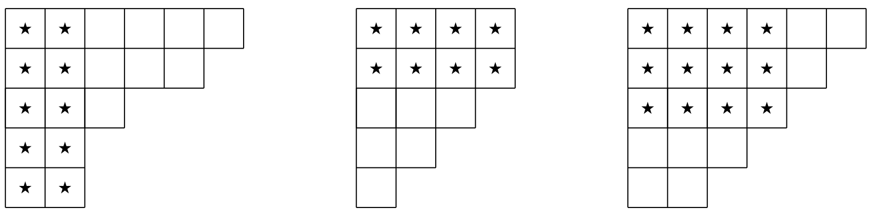
Lemma 38. (1) *We have*

$$K_1 = J_1 \supsetneq K_1' = J_1' \subsetneq K_2 \supsetneq K_2' \subsetneq \cdots \subsetneq K_r \supsetneq K_r' \subseteq J.$$

(2) *For every $i = 1, \ldots, r$, we have*

$$\gamma_{J_i/J_i'} = \gamma_{K_i/K_i'} \in Q_{m,W}, \quad \gamma_{J_i'\backslash J_i} = \gamma_{K_i'\backslash K_i} \in Q_{m,W},$$
$$Y_{J_i/J_i'} = Y_{K_i/K_i'} \in Q_{t,W}, \quad Y_{J_i'\backslash J_i} = Y_{K_i'\backslash K_i} \in Q_{t,W}.$$

Proof. It is clear that $K_r' \subseteq J$. Thus, in order to complete the first part, we need to prove $K_{i-1}' \subsetneq K_i$, for $i = 2, \ldots, r$. This is obvious if the rectangle R_i is, by itself, a connected component of the shape ρ^i. Other than this, there are three ways in which R_i can be attached to ρ^{i-1}, which are indicated below; the boxes of R_i are marked with $\star$, and the empty boxes form the relevant component(s) of ρ^{i-1}.



(30)

Note that the height (respectively width) of R_i is strictly greater than the number of rows (respectively columns) of the relevant Young diagram to its right (respectively at the bottom). It is also useful to observe that all unit segments with the same label form a northwest to southeast staircase shape, and the labels increase by 1 as we move northeast.

Let B denote the set of labels on the boundary of the rectangle R_i. Using the above notation, in all three cases in (30), we have

$$B = C_i \sqcup D_i = \{c_i, \ldots, d_i\}, \quad K_i - B = K_{i-1}' - B, \quad K_i \cap B = C_i \sqcup D_i' = B - \{d_i\} =: J_i.$$

On the other hand, we have $d_i \notin K_{i-1}'$; indeed, in the first and last case in (30), the label d_i is on the left side of a rectangle R_j with $j \in \mathcal{C}_{i-1}$, but $d_i \notin C_j'$, because it is the top label on the mentioned side. We conclude that $K_{i-1}' \subseteq K_i$. In fact, the inclusion is strict because we also have $c_i + q_i - 1 \in (K_i \cap B) - K_{i-1}'$.

For the second part, we note that, in addition to the above facts, we have $K_i' \cap B = C_i' \sqcup D_i' =: J_i'$ and $c_i - 1 \notin K_i$. For the latter part, note that, in the last two cases in (30), the label $c_i - 1$ is on the left side of a rectangle R_j with $j \in \mathcal{C}_i$ and $j \neq i$, but $c_i - 1 \notin C_j'$, because it is the top label on the mentioned side. The proof is concluded by applying Lemma 37. □

Proof of Theorem 34. By using the analogue of Lemma 5 for γ, we have

$$\gamma_{K_2/K_2'} \cdots \gamma_{K_r/K_r'} \gamma_J = \gamma_{K_2}\, \gamma_{K_2'\backslash K_3} \cdots \gamma_{K_{r-1}'\backslash K_r}\, \gamma_{K_r'\backslash J} = \gamma_{K_1'} \gamma_{K_1'\backslash K_2}\, \gamma_{K_2'\backslash K_3} \cdots \gamma_{K_{r-1}'\backslash K_r}\, \gamma_{K_r'\backslash J}. \quad (31)$$

We now prove the theorem using induction on r, with base case $r=0$, which is trivial. We have

$$\begin{aligned}\gamma_{w_\lambda w_J} &\overset{\sharp_1}{=} Z_{w_\lambda}\gamma_J \overset{\sharp_2}{=} Z_{v^1}Z_{w^2}\gamma_J \overset{\sharp_3}{=} Z_{v^1}\gamma_{w^2 w_J}\\ &\overset{\sharp_4}{=} Z_{v^1}\gamma_{J_2/J'_2}\cdots\gamma_{J_r/J'_r}\gamma_J \overset{\sharp_5}{=} Z_{v^1}\gamma_{K_2/K'_2}\cdots\gamma_{K_r/K'_r}\gamma_J\\ &\overset{\sharp_6}{=} Z_{v^1}\gamma_{K'_1}\gamma_{K'_1\setminus K_2}\cdots\gamma_{K'_{r-1}\setminus K_r}\gamma_{K'_r\setminus J}\\ &\overset{\sharp_7}{=} \gamma_{K_1}\gamma_{K'_1\setminus K_2}\cdots\gamma_{K'_{r-1}\setminus K_r}\gamma_{K'_r\setminus J}\\ &\overset{\sharp_8}{=} \gamma_{K_1/K'_1}\gamma_{K_2/K'_2}\cdots\gamma_{K_r/K'_r}\gamma_J \overset{\sharp_9}{=} \gamma_{J_1/J'_1}\gamma_{J_2/J'_2}\cdots\gamma_{J_r/J'_r}\gamma_J.\end{aligned}$$

Here $\sharp_1$, $\sharp_3$, and $\sharp_7$ are based on Theorem 36, $\sharp_2$ on (28), $\sharp_4$ on the induction hypothesis, $\sharp_5$ and $\sharp_9$ on Lemma 38(2), $\sharp_6$ and $\sharp_8$ on (31), and $\sharp_8$ on (8); additionally, in $\sharp_7$ we use the fact that

$$K_1 = J_1 = C_1 \sqcup D'_1 = \{c_1, \ldots, d_1 - 1\}, \qquad K'_1 = J'_1 = C'_1 \sqcup D'_1 = K_1 - \{\max C_1\},$$

and thus we have $v^1 w_{K'_1} = w_{K_1}$. □

Remark 39. We could not have carried out the above proof using only one of the pairs (J_i, J'_i) and (K_i, K'_i). Indeed, the first pair does not satisfy the property in Lemma 38(1), which is crucial in the proof. On the other hand, the induction procedure cannot be applied based on the second pair because the respective sets for $\lambda^1 = \lambda$ and λ^2 (corresponding to w^2) are different.

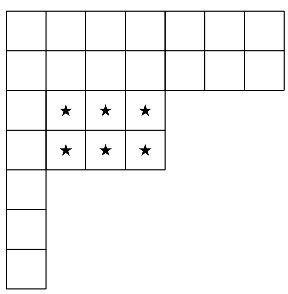
In order to relate Theorem 34 to the Zelevinsky resolution, and more specifically to the operator (25), we need the following result.

Lemma 40. *For every $i = 1, \ldots, r$, we have*

$$Y_{J_i/J'_i} = Y_{K_i/K'_i} = Y_{P_i/Q_i}.$$

Proof. By using Lemma 38(2), it suffices to prove $Y_{J_i/J'_i} = Y_{P_i/Q_i}$. Moreover, it suffices to consider $i=1$, as we can just replace the partition $\lambda^1 = \lambda$ with λ^i. Recall that P_1 is obtained by considering the lattice path from $(0,0)$ to $(n-d, d)$ defining the southeast boundary of λ^1, and by excluding from Π the last label in each sequence of vertical steps. Similarly, P_2 corresponds to $\lambda^2 := \lambda - R_1$.

Let B denote the set of labels on the boundary of the rectangle R_1; see the diagram below, where the boxes of R_1 are marked with $\star$.



Using the above notation, we have $B = C_1 \sqcup D_1 = \{c_1, \dots, d_1\}$. Based on the above interpretation of P_1 and P_2, we deduce

$$P_1 \cap B = C_1 \sqcup D_1' =: J_1 = B - \{d_1\},$$
$$P_2 \cap B = C_1' \sqcup D_1, \text{ which implies } Q_1 \cap B = C_1' \sqcup D_1' =: J_1',$$
$$P_1 - B \subset P_2 - B, \text{ which implies } P_1 - B = Q_1 - B.$$

Moreover, we have $c_1 - 1 \notin P_1$ and $d_1 \notin P_1$. Thus, we are under the hypotheses of Lemma 37, so the conclusion follows. □

We now rephrase Theorem 34 as follows, via the map ψ.

Corollary 41. *We have*

$$\mu^{-\ell(w_\lambda w_J)} \psi(\gamma_{w_\lambda w_J}) = Y_{P_1/Q_1} \cdots Y_{P_r/Q_r} Y_J \in \boldsymbol{D}_t. \tag{32}$$

Proof. We start by observing that

$$w_{K_i/K_i'} = w_{J_i/J_i'} = v^i \quad \Rightarrow \quad \ell(w_{K_i/K_i'}) = p_i q_i = |R_i|, \tag{33}$$

where $|R_i|$ denotes the number of boxes of the rectangle R_i. Here the first equality is based on (29) and the fact that this result can be applied to the pairs (J_i, J_i') and (K_i, K_i'), as discussed in the proof of Lemma 38; the second equality is clear by the definition of v^i.

We now apply $\mu^{-\ell(w_\lambda w_J)} \psi(\,\cdot\,)$ to the first and last part of (26). After doing this, the latter can be written as follows:

$$\begin{aligned}
&\mu^{-\ell(w_\lambda w_J)} \psi(\gamma_{K_1/K_1'}) \cdots \psi(\gamma_{K_r/K_r'}) \psi(\gamma_J) \\
&\qquad \overset{\sharp_1}{=} \big(\mu^{-\ell(w_{K_1/K_1'})} \psi(\gamma_{K_1/K_1'})\big) \cdots \big(\mu^{-\ell(w_{K_r/K_r'})} \psi(\gamma_{K_r/K_r'})\big) (\mu^{-\ell(w_J)} \psi(\gamma_J)) \\
&\qquad \overset{\sharp_2}{=} \big(\mu^{-\ell(w_{K_1/K_1'})} \psi(\gamma_{K_1/K_1'})\big) \cdots \big(\mu^{-\ell(w_{K_r/K_r'})} \psi(\gamma_{K_r/K_r'})\big) Y_J \\
&\qquad \overset{\sharp_3}{=} \big(\mu^{-\ell(w_{K_1/K_1'})} \psi(\gamma_{K_1/K_1'})\big) \cdots \big(\mu^{-\ell(w_{K_r/K_r'})} \psi(\gamma_{K_r/K_r'})\big) Y_{K_r'} Y_{K_r' \setminus J} \\
&\qquad \overset{\sharp_4}{=} \big(\mu^{-\ell(w_{K_1/K_1'})} \psi(\gamma_{K_1/K_1'})\big) \cdots \big(\mu^{-\ell(w_{K_{r-1}/K_{r-1}'})} \psi(\gamma_{K_{r-1}/K_{r-1}'})\big) Y_{K_r} Y_{K_r' \setminus J} \\
&\qquad = \cdots \overset{\sharp_5}{=} Y_{K_1} Y_{K_1' \setminus K_2} \cdots Y_{K_r' \setminus J} \\
&\qquad \overset{\sharp_6}{=} Y_{K_1/K_1'} \cdots Y_{K_r/K_r'} Y_J \overset{\sharp_7}{=} Y_{P_1/Q_1} \cdots Y_{P_r/Q_r} Y_J.
\end{aligned}$$

Here $\sharp_1$ is based on (33) and the fact that $\ell(w_\lambda) = \sum_i |R_i|$. Equalities $\sharp_2$ and $\sharp_4$ are based on (18), $\sharp_3$ on (2), $\sharp_5$ on the repeated use of an argument similar to $\sharp_3$ followed by $\sharp_4$, $\sharp_6$ on Lemma 5 and $\sharp_7$ on Lemma 40. □

We now state the main result of this section.

Theorem 42. *The KL Schubert classes for the Grassmannian coincide with the hyperbolic cohomology classes of the corresponding Zelevinsky resolutions.*

Proof. The result is now immediate by comparing the left- and right-hand sides of (32) with Definition 25 and (25), respectively. □

Remark 43. Theorem 42 implies that all the Zelevinsky resolutions of a Schubert variety in the Grassmannian have the same class in hyperbolic cohomology (i.e., the corresponding KL Schubert class). This agrees with a result of Totaro [2000], which says that the algebraic theories in a larger class (defined by Krichever [Buchstaber and Bunkova 2010]), which includes hyperbolic cohomology, are invariant under small resolutions.

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