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One-level density estimates for Dirichlet L-functions with extended support

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We estimate the 1-level density of low-lying zeros of $L(s,\chi)$ with χ ranging over primitive Dirichlet characters of conductor in $\left[\frac{1}{2}Q, Q\right]$ and for test functions whose Fourier transform is supported in $\left(-2-\frac{50}{1093}, 2+\frac{50}{1093}\right)$. Previously, any extension of the support past the range $(-2, 2)$ was only known conditionally on deep conjectures about the distribution of primes in arithmetic progressions, beyond the reach of the generalized Riemann hypothesis (e.g., Montgomery's conjecture). Our work provides the first example of a family of L-functions in which the support is unconditionally extended past the "diagonal range" that follows from a straightforward application of the underlying trace formula (in this case orthogonality of characters). We also highlight consequences for nonvanishing of $L(s,\chi)$.

1. Introduction

Motivated by the problem of establishing the nonexistence of Siegel zeros (see [Conrey and Iwaniec 2002] for details), Montgomery [1973] investigated the *vertical* distribution of the zeros of the Riemann zeta function. He showed that under the assumption of the Riemann hypothesis, for any smooth function f with $\operatorname{supp}\hat{f} \subset (-1, 1)$,

$$\lim_{T\to\infty}\frac{1}{N(T)}\sum_{T\le\gamma,\gamma'\le 2T} f\left(\frac{\log T}{2\pi}\cdot(\gamma-\gamma')\right)=\int_{\mathbb{R}} f(u)\cdot\left(\delta(u)+1-\left(\frac{\sin 2\pi u}{2\pi u}\right)^2\right)du, \tag{1}$$

where $N(T)$ denotes the number of zeros of the Riemann zeta function up to height T, and γ,γ' are ordinates of the zeros of the Riemann zeta function, and $\delta(u)$ is a Dirac mass at 0. Dyson famously observed that the right-hand side coincides with the pair correlation function of eigenvalues of a random Hermitian matrix.

Dyson's observation leads one to conjecture that the spacings between the zeros of the Riemann zeta function are distributed in the same way as spacings between eigenvalues of a large random Hermitian matrix. Subsequent work of Rudnick and Sarnak [1994] provided strong evidence towards this conjecture by computing (under increasingly restrictive conditions) the n-correlations of the zeros of any given automorphic L-function. Importantly, the work of Rudnick and Sarnak [1996] suggested that the distribution of the zeros of an automorphic L-function is universal and independent of the distribution of its coefficients.

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For number-theoretic applications, the distribution of the so-called "low-lying zeros", that is, zeros close to the central point, is particularly interesting (see, e.g., [Heath-Brown 2004; Young 2006] for various applications; see also [Granville and Soundararajan 2018] and [Watkins 2021], for instance, for results in a different direction). Following the work of Katz and Sarnak [1999] and Iwaniec, Luo and Sarnak [Iwaniec et al. 2000], we believe that the distribution of these low-lying zeros is also universal and predicted by only a few random matrix ensembles (which are either symplectic, orthogonal or unitary).

Specifically, the work of Katz and Sarnak suggests that for any smooth function ϕ and any natural "family" of automorphic objects $\mathcal{F}$,

$$\frac{1}{\#\mathcal{F}} \sum_{\pi \in \mathcal{F}} \sum_{\gamma_\pi} \phi\left(\frac{\log \mathfrak{c}_\pi}{2\pi} \cdot \gamma_\pi\right) \xrightarrow[\#\mathcal{F}\to\infty]{} \int_{\mathbb{R}} \phi(x) K_{\mathcal{F}}(x)\, dx, \tag{2}$$

where γ_π are ordinates of the zeros of the L-function attached to π, $\mathfrak{c}_\pi$ is the analytic conductor of π, and $K_{\mathcal{F}}(x)$ is a function depending only on the "symmetry type" of $\mathcal{F}$. One may wish to consult [Iwaniec et al. 2000] and [Sarnak et al. 2016] for a more detailed discussion.

There is a vast literature providing evidence for (2) (see [Mackall et al. 2016]). Similarly to Montgomery's result (1), all of the results in the literature place a restriction on the support of the Fourier transform of ϕ. This restriction arises from the limitations of the relevant trace formula (in some families it is not always readily apparent what this relevant trace formula is). In practice, an application of the trace formula gives rise to so-called "diagonal" and "off-diagonal" terms. Trivially bounding the off-diagonal terms corresponds to what we call a "straightforward" application of the trace formula.

A central yet extremely difficult problem is to extend the support of $\hat{\phi}$ beyond what a "straightforward" application of the trace formula gives. In fact most works in which the support of $\hat{\phi}$ has been extended further rely on the assumption of various deep hypotheses about primes that sometimes lie beyond the reach of the generalized Riemann hypothesis (GRH).

For example, Iwaniec, Luo and Sarnak show that in the case of holomorphic forms of even weight $\leq K$ one obtains unconditionally a result for $\hat{\phi}$ supported in $(-1, 1)$ and that under the assumption of the generalized Riemann hypothesis this can be enlarged to $(-2, 2)$ (it is observed in [Devin et al. 2022] that assuming GRH only for Dirichlet L-functions is sufficient). Iwaniec, Luo and Sarnak also show that this range can be pushed further to $\operatorname{supp} \hat{\phi} \subset \left(-\frac{22}{9}, \frac{22}{9}\right)$ under the additional assumption that, for any $c \geq 1$, $(a, c) = 1$ and $\varepsilon > 0$,

$$\sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\, c)}} e(2\sqrt{p}/c) \ll_\varepsilon x^{\frac{1}{2}+\varepsilon}.$$

A similar behaviour is observed on low-lying zeros of dihedral L-functions associated to an imaginary quadratic field [Fouvry and Iwaniec 2003], where an extension of the support is shown to be equivalent to an asymptotic formula on primes with a certain splitting behaviour.

Assuming GRH, Brumer [1992] studied the one-level density of the family of elliptic curves and proved a result for test functions supported in $\left(-\frac{5}{9}, \frac{5}{9}\right)$; this corresponds to the "diagonal" range for this family. Heath-Brown [2004] improved this range to $\left(-\frac{2}{3}, \frac{2}{3}\right)$, and Young [2006] pushed the support to $\left(-\frac{7}{9}, \frac{7}{9}\right)$. One-level density estimates for this family have deep implications for average ranks of elliptic curves. In particular, the work of Young was the first to show that, under some reasonable conjectures, a positive proportion of elliptic curves have rank 0 or 1 and thus satisfy the rank part of the Birch and Swinnerton–Dyer conjecture.[1]

As another example, it follows for instance from minor modifications of [Hughes and Rudnick 2003; Chandee et al. 2014] that in the family of primitive Dirichlet characters of modulus $\le Q$ one can estimate 1-level densities unconditionally for ϕ with $\hat{\phi}$ supported in $(-2, 2)$.[2] As a byproduct of work of Fiorilli and Miller [2015, Theorem 2.8], it follows that for any $\delta \in (0, 2)$, this support can be enlarged to $(-2-\delta, 2+\delta)$ under the following "de-averaging hypothesis":

$$\sum_{\frac{1}{2}Q \le q \le Q} \left| \sum_{\substack{p \le x \\ p \equiv 1 \ (\mathrm{mod}\, q)}} \log p - \frac{x}{\varphi(q)} \right|^2 \ll Q^{-\frac{1}{2}\delta} \sum_{\frac{1}{2}Q \le q \le Q} \sum_{(a,q)=1} \left| \sum_{\substack{p \le x \\ p \equiv a \ (\mathrm{mod}\, q)}} \log p - \frac{x}{\varphi(q)} \right|^2. \tag{3}$$

In this paper we give a first example of a family of L-functions in which we can *unconditionally* enlarge the support past the "diagonal" range, which would follow from a straightforward application of the trace formula (in this case orthogonality of characters).

Theorem 1. *Let Φ be a smooth function compactly supported in $\left[\frac{1}{2}, 3\right]$, and ϕ be a smooth function such that supp $\hat{\phi} \subset \left(-2 - \frac{50}{1093}, 2 + \frac{50}{1093}\right)$. Then, as $Q \to \infty$,*

$$\sum_q \Phi\left(\frac{q}{Q}\right) \sum_{\substack{\chi \ (\mathrm{mod}\, q) \\ \text{primitive}}} \sum_{\gamma_\chi} \phi\left(\frac{\log Q}{2\pi}\gamma_\chi\right) = \hat{\phi}(0) \sum_q \Phi\left(\frac{q}{Q}\right) \sum_{\substack{\chi \ (\mathrm{mod}\, q) \\ \text{primitive}}} 1 + o(Q^2). \tag{4}$$

Here $\frac{1}{2} + i\gamma_\chi$ correspond to nontrivial zeros of $L(s, \chi)$ and since we do not assume the generalized Riemann hypothesis we allow the γ_χ to be complex.

Remark. In stating the theorem we have, for technical simplicity, made a suitable approximation to the conductor $\mathfrak{c}_\pi$ appearing in (2).

Note that ϕ, initially defined on $\mathbb{R}$, is analytically continued to $\mathbb{C}$ by compactness of supp $\hat{\phi}$. Our arguments can be adapted to show that if supp $\hat{\phi} \subset \left(-2 - \frac{50}{1093} + \varepsilon, 2 + \frac{50}{1093} - \varepsilon\right)$ for some $\varepsilon > 0$, then the error term in (4) is $O(Q^{2-\delta})$ with $\delta = \delta(\varepsilon)$, up to altering slightly the main terms: after applying the explicit formula as in Section 2.2, include the terms of order $\asymp Q^2/\log Q$ into the main term instead of treating them as error terms.

We remark that we make no progress on the "de-averaging hypothesis" (3) of Fiorilli and Miller, which remains a difficult open problem. We estimate the original sum over primes in arithmetic progressions, on average over moduli, by a variant of an argument of Fouvry [1985] and Bombieri, Friedlander and

[1] A stronger conclusion was later reached unconditionally by Bhargava and Shankar [2015] through other methods.

[2] This is in fact the GL(1) analogue of the result of Iwaniec, Luo and Sarnak for holomorphic forms.

Iwaniec [Bombieri et al. 1986] which is based on Linnik's dispersion method. The GRH will be dispensed with by working throughout, as in [Drappeau 2015], with characters of large conductors.

The asymptotic formula (4) is expected to hold true without the extra averaging over q. This extra averaging over q and the cancellation of arguments which comes along play an important role in our arguments.

If the GRH is true for Dirichlet L-functions, then let any $0 < \kappa < \frac{50}{1093}$ be fixed, and let $\lambda > 1$ be small enough that $\kappa' := 2(\lambda - 1) + \lambda\kappa \in \left(0, \frac{50}{1093}\right)$ as well. Defining

$$\tilde{\phi}(x) = \lambda\left(\frac{\sin \pi(2+\kappa)x}{\pi(2+\kappa)x}\right)^2, \quad \phi = \tilde{\phi} * u,$$

where u is a smooth, positive approximation of unity such that $\phi(0) \geq \lambda^{-1}\tilde{\phi}(0) = 1$, and using the inequality

$$1 - \sum_{\gamma_\chi} \phi\left(\frac{\log Q}{2\pi}\gamma_\chi\right) \leq \mathbf{1}\left(L\left(\tfrac{1}{2}, \chi\right) \neq 0\right),$$

we deduce from Theorem 1 that the proportion of nonvanishing $L\left(\frac{1}{2}, \chi\right)$ with χ ranging over primitive characters of conductor in $\left[\frac{1}{2}Q, Q\right]$ is at least $1 - \lambda(2+\kappa')^{-1} = 1 - (2+\kappa)^{-1}$ for any $\kappa < \frac{50}{1093}$.

Corollary 2. *Let $\varepsilon \in (0, 10^{-7})$. Assume the generalized Riemann hypothesis for Dirichlet L-functions. Then for all Q large enough, the proportion of primitive characters χ with modulus in $\left[\frac{1}{2}Q, Q\right]$ for which*

$$L\left(\tfrac{1}{2}, \chi\right) \neq 0$$

is at least

$$\tfrac{1}{2} + \tfrac{25}{2236} - \varepsilon > 0.51118.$$

Corollary 2 is related to a recent result of Pratt [2019], who showed unconditionally that the proportion of nonvanishing in this family is at least 0.50073. We note that both the arguments of [Pratt 2019] and those presented here eventually rely on bounds of Deshouillers and Iwaniec [1982] on cancellation in sums of Kloosterman sums.

Notation. We call a map $f : \mathbb{R}_+ \to \mathbb{C}$ a *test function* if f is smooth and supported inside $\left[\frac{1}{2}, 3\right]$.

For $w \in \mathbb{N}$, $n \in \mathbb{Z}$ and $R \geq 1$, we let

$$\mathfrak{u}_R(n, w) := \mathbf{1}_{n \equiv 1 \ (\mathrm{mod}\ w)} - \frac{1}{\varphi(w)} \sum_{\substack{\chi \ (\mathrm{mod}\ w) \\ \mathrm{cond}(\chi) \leq R}} \chi(n).$$

Note the trivial bound

$$|\mathfrak{u}_R(n, w)| \ll \mathbf{1}_{n \equiv 1 \ (\mathrm{mod}\ w)} + \frac{R\tau(w)}{\varphi(w)}. \tag{5}$$

The symbol $n \sim N$ in a summation means $n \in [N, 2N) \cap \mathbb{Z}$. We say that a sequence $(\alpha_n)_n$ is *supported at scale N* if $\alpha_n = 0$ unless $n \sim N$.

The letter ε will denote an arbitrarily small number, whose value may differ at each occurrence. The implied constants will be allowed to depend on ε.

2. Proof of Theorem 1

2.1. *Lemmas on primes in arithmetic progressions.* We will require two results about primes in arithmetic progressions. The first is a standard estimate, obtained from an application of the large sieve.

Lemma 3. *Let $A > 0$, X, Q, $R \geq 2$ satisfy $1 \leq R \leq Q$ and $X \geq Q^2/(\log Q)^A$, and let f be a test function with $\|f^{(j)}\|_\infty \ll_j 1$. Then*

$$\sum_{q \leq Q} \left| \sum_{n \in \mathbb{N}} f\left(\frac{n}{X}\right) \Lambda(n) \mathrm{u}_R(n, q) \right| \ll Q (\log Q)^{O(1)} \sqrt{X} \left(1 + \frac{\sqrt{X}}{RQ} + \frac{X^{\frac{3}{8}}}{Q}\right). \tag{6}$$

The implied constant depends at most on A and the implied constants in the hypothesis.

Proof. By Heath-Brown's combinatorial formula for primes [Iwaniec and Kowalski 2004, Proposition 13.3] (with $K = 2$), we restrict to proving the bound with $\Lambda(n)$ replaced by convolutions of types I and II, of the shape

$$\sum_{\substack{n = m\ell \\ m \sim M}} \sum \alpha_m \quad (M \ll X^{\frac{1}{4}}), \qquad \sum_{\substack{n = m\ell \\ m \sim M}} \sum \alpha_m \beta_\ell \quad (X^{\frac{1}{4}} \ll M \ll X^{\frac{3}{4}}),$$

where $|\alpha_m| \ll (\log X)\tau_4(m)$ and the analogous bound holds for β_ℓ; here we noted that if $m_1 \leq m_2 \leq \sqrt{X}$ and $m_1 m_2 > X^{\frac{1}{4}}$, then either $X^{\frac{1}{4}} < m_1 m_2 \leq X^{\frac{3}{4}}$ or $X^{\frac{1}{4}} \leq m_1 \ll X^{\frac{1}{2}}$. We treat the type I case by the Pólya–Vinogradov inequality [Iwaniec and Kowalski 2004, Theorem 12.5], getting a bound $O(MR^{\frac{3}{2}}(\log Q)^{O(1)})$. We treat the type II case by the large sieve [Iwaniec and Kowalski 2004, Theorem 17.4], getting a contribution $O\big(\sqrt{X}(\log Q)^{O(1)}(Q + \sqrt{M} + \sqrt{X/M} + \sqrt{X}R^{-1})\big)$. □

The second estimate is substantially deeper and we defer its proof to Section 4.

Proposition 4. *Let $\kappa \in \big(0, \frac{50}{1093}\big)$ and $\varepsilon > 0$. Let Ψ and f be test functions, $A > 0$, X, Q, W, $R \geq 1$, and $b \in \mathbb{N}$. Assume that*

$$\frac{Q^2}{(\log Q)^A} \ll X \ll Q^{2+\kappa}, \quad X^{\frac{11}{20}} Q^{-1} \leq R \leq Q^{\frac{2}{3}} X^{-\frac{2}{9}}, \quad b \leq Q^\varepsilon, \quad Q^{1-\varepsilon} \ll W \ll Q,$$

and that $\|f^{(j)}\|_\infty, \|\Psi^{(j)}\|_\infty \ll_j 1$. Then, if $\varepsilon > 0$ is small enough in terms of κ, we have

$$\sum_{w \in \mathbb{N}} \Psi\left(\frac{w}{W}\right) \sum_{n \in \mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) \mathrm{u}_R(n, bw) \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends at most on κ, A, and the implied constants in the hypotheses.

Proof. See Section 4. □

2.2. *Explicit formula.* We let $\kappa \in \big(0, \frac{50}{1093}\big)$ be such that $\operatorname{supp} \hat{\phi} \subset (-2-\kappa, 2+\kappa)$.

We rewrite the left-hand side of (4) by applying the explicit formula, e.g., [Sica 1998, Theorem 2.2], where the quantity $\Phi(\rho)$ there (not to be confused with our test function) is replaced by $\phi\big(\frac{1}{2\pi i}\big(\rho - \frac{1}{2}\big)\log Q\big)$, so that $F(x) = (1/\log Q)\hat{\phi}(x/\log Q)$. For $q > 1$ and $\chi \pmod q$ primitive, we obtain

$$\sum_{\substack{\rho\in\mathbb{C}\\ \mathrm{Re}(\rho)\in(0,1)\\ L(\rho,\chi)=0}} \phi\left(\frac{\left(\rho-\frac{1}{2}\right)\log Q}{2\pi i}\right)=O\left(\frac{1}{\log Q}\right)+\hat{\phi}(0)\frac{\log q}{\log Q}-\frac{1}{\log Q}\sum_{n\geq 1}(\chi(n)+\bar{\chi}(n))\frac{\Lambda(n)}{\sqrt{n}}\hat{\phi}\left(\frac{\log n}{\log Q}\right), \tag{7}$$

since the terms I, J appearing in [Sica 1998, Theorem 2.2] satisfy $\left|I\left(\frac{1}{2},b\right)\right| + \left|J\left(\frac{1}{2},b\right)\right| \ll (\log Q)^{-1}$ for $b \in \left\{0, \frac{1}{2}\right\}$ by reasoning similarly as in [Sica 1998, Lemma 3.1]. Let $\Psi(x) = \Phi(x)x$. Summing (7) over χ and q, we see that to conclude it remains to show that

$$S_\phi(Q) := \sum_{q\in\mathbb{N}} \frac{1}{q}\Psi\left(\frac{q}{Q}\right) \sum_{\substack{\chi(q)\\ \text{primitive}}} \frac{1}{\log Q}\sum_{n\geq 1}(\chi(n)+\bar{\chi}(n))\frac{\Lambda(n)}{\sqrt{n}}\hat{\phi}\left(\frac{\log n}{\log Q}\right) = o(Q). \tag{8}$$

We will in fact obtain the following slightly stronger result.

Proposition 5. *Let $\kappa \in \left(0, \frac{50}{1093}\right)$. For all Q large enough and $\varepsilon > 0$ small enough in terms of κ, we have*

$$S_\phi(Q) = O\left(\frac{Q}{\log Q}\right).$$

The implied constant depends on ϕ and ε at most.

We break down the proof of Proposition 5 into the following three sections.

2.3. *Orthogonality and partition of unity.* Applying character orthogonality for primitive characters (see the third display in the proof of Lemma 4.1 of [Bui and Milinovich 2011]), we get

$$S_\phi(Q) = \frac{2}{\log Q}\sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\sum_{n\equiv 1 \,(\mathrm{mod}\, w)}\frac{\Lambda(n)}{\sqrt{n}}\hat{\phi}\left(\frac{\log n}{\log Q}\right). \tag{9}$$

Let V be any test function generating the partition of unity

$$\sum_{j\in\mathbb{Z}} V\left(\frac{x}{2^j}\right) = 1$$

for all $x > 0$. Inserting this in (9), we obtain

$$S_\phi(Q) = \frac{2}{\log Q}\sum_{\substack{j\in\mathbb{Z}\\ \frac{1}{2}\leq X:=2^j\leq 2Q^{2+\kappa}}} \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\sum_{n\equiv 1 \,(\mathrm{mod}\, w)}\frac{\Lambda(n)}{\sqrt{n}}V\left(\frac{n}{X}\right)\hat{\phi}\left(\frac{\log n}{\log Q}\right).$$

Set $f_j(x) = x^{-\frac{1}{2}}V(x)\hat{\phi}(\log(2^j x)/\log Q)$ for $\frac{1}{2} \leq 2^j \leq 2Q^{2+\kappa}$. Differentiating the product, we have that for all $k \geq 0$, there exists $C_{\phi,k} \geq 0$ such that $\|f_j^{(k)}\|_\infty \leq C_{\phi,k}$ for all j. We deduce

$$S_\phi(Q) \ll \sup_{1\ll X\ll Q^{2+\kappa}} X^{-\frac{1}{2}} \sup_f |T(Q,X)|,$$

where f varies among test functions subject to $\|f^{(k)}\|_\infty \leq C_{\phi,k}$, and

$$T(Q,X) := \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\sum_{n\equiv 1 \,(\mathrm{mod}\, w)}\Lambda(n) f\left(\frac{n}{X}\right).$$

We handle the very small values of X by the trivial bound

$$\sum_{n\equiv 1\ (\mathrm{mod}\ w)} \Lambda(n) f\left(\frac{n}{X}\right) \ll \log Q \sum_{\substack{\frac{1}{2}X<n<3X \\ n\neq 1, n\equiv 1\ (\mathrm{mod}\ w)}} 1 \ll \frac{X\log Q}{w},$$

which implies

$$T(Q,X) \ll \frac{X\log Q}{Q} \sum_{vw\asymp Q}\sum 1 \ll X(\log Q)^2.$$

It will therefore suffice to show that for

$$Q^2/(\log Q)^6 \ll X \ll Q^{2+\kappa},$$

we have

$$T(Q,X) \ll \frac{\sqrt{X}Q}{\log Q}.$$

2.4. ***Subtracting the main term.*** We insert the coprimality condition $(n, v) = 1$. Since

$$\sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w} \sum_{\substack{n\equiv 1\ (\mathrm{mod}\ w) \\ (n,v)>1}} \Lambda(n) f\left(\frac{n}{X}\right) \ll \sum_{v\ll Q} v^{-1} \sum_{\substack{p\mid v \\ 1\leq k\ll \log X}} \left((\log p)\sum_{w\mid p^k-1} 1\right) \ll Q^{1+\varepsilon},$$

we obtain

$$T(Q,X) = \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w} \sum_{\substack{n\equiv 1\ (\mathrm{mod}\ w) \\ (n,v)=1}} \Lambda(n) f\left(\frac{n}{X}\right) + O(Q^{1+\varepsilon}).$$

Let $1 \leq R < \frac{1}{2}Q$ so that $R < vw$ for any v, w appearing in the sum. We replace the condition $n \equiv 1\ (\mathrm{mod}\ w)$ by $\mathfrak{u}_R(n, w)$. The difference is

$$\sum_q \frac{1}{q}\Psi\left(\frac{q}{Q}\right) \sum_{\substack{\chi\ (\mathrm{mod}\ q) \\ r=\mathrm{cond}(\chi)\leq R \\ r\mid q}} \sum_{(n,q)=1} \Lambda(n) f\left(\frac{n}{X}\right)\chi(n) \sum_{v\mid q/r} \mu(v) = 0$$

since $r < q$ by our choice of R, so that

$$T(Q,X) = \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w} \sum_{(n,v)=1} \Lambda(n) f\left(\frac{n}{X}\right)\mathfrak{u}_R(n,w) + O(Q^{1+\varepsilon}).$$

We next remove the coprimality condition on n, using the trivial bound (5). For the first term $\mathbf{1}_{n\equiv 1\ (\mathrm{mod}\ w)}$ in $\mathfrak{u}_R(n, w)$, this was already justified above. For the second term, we get

$$\ll RQ^{-1+\varepsilon} \sum_{\substack{v,w \\ vw\asymp Q}}\sum \sum_{p\mid v} \log p \ll RQ^{\varepsilon}.$$

Since $R \ll Q$, both error terms are acceptable. We get

$$T(Q,X) = T(Q,X,R) + O(Q^{1+\varepsilon}),$$

where

$$T(Q,X,R) := \sum_{v,w}\sum \Psi\left(\frac{vw}{Q}\right)\frac{\mu(v)}{v}\frac{\varphi(w)}{w}\Delta(w),$$

$$\Delta(w) := \sum_n \Lambda(n) f\left(\frac{n}{X}\right)\mathfrak{u}_R(n,w). \tag{10}$$

We are required to show that

$$T(Q,X,R) \ll \frac{\sqrt{X}Q}{\log Q}. \tag{11}$$

2.5. *Reduction to the critical range.* We now impose the additional conditions

$$Q^{\frac{1}{2}\kappa+\varepsilon} \le R \le Q^{\frac{1}{2}} \quad \text{and} \quad \kappa < \tfrac{2}{3}. \tag{12}$$

Observe that this κ is the same as that appearing in the statement of Proposition 4. The condition $\kappa < \frac{2}{3}$ is convenient for applying (6) below, but is rather loose since κ is ultimately required to be much smaller than $\frac{2}{3}$.

Let $B \in [1, Q^{\frac{1}{2}}]$ be a parameter. In $T(Q,X,R)$, we write $\varphi(w)/w = \sum_{b|w}\mu(b)/b$ and exchange summation, so that

$$T(Q,X,R) \le \sum_{b,v}\frac{1}{bv}\left|\sum_w \Psi\Big(\frac{bvw}{Q}\Big)\Delta(bw)\right| \ll (\log B)^2 \sup_{b,v\le B}\left|\sum_w \Psi\Big(\frac{bvw}{Q}\Big)\Delta(bw)\right| + E_1 + E_2,$$

where E_1 (resp. E_2) corresponds to the sum over b, v restricted to $b > B$ (resp. $v > B$). We recall that $\operatorname{supp}\Psi \subset \left[\frac{1}{2}, 3\right]$ by hypothesis. On the one hand, we have

$$E_1 \ll \sum_{\substack{b,w \\ bw\le 3Q \\ b>B}}\sum \frac{1}{b}|\Delta(bw)| \ll Q^{\frac{1}{2}\varepsilon}B^{-1}\sum_{q\le 3Q}|\Delta(q)| \ll Q^{1+\frac{1}{2}\varepsilon}\sqrt{X}B^{-1},$$

using (6) along with our hypotheses (12). On the other hand, we have

$$E_2 \ll \sum_{\substack{b,w \\ bw\le 3Q/B}}\sum \frac{1}{b}|\Delta(bw)| \ll Q^{\frac{1}{2}\varepsilon}\sum_{q\le 3Q/B}|\Delta(q)| \ll Q\sqrt{X}(Q^{\frac{1}{2}\varepsilon}B^{-1} + Q^{-\varepsilon})$$

again by (12) and (6); we have used the bounds $Q^{-1+\varepsilon}\sqrt{X}R^{-1} \ll Q^{-\varepsilon}$ and $Q^{-1+\varepsilon}X^{\frac{3}{8}} \ll Q^{-\varepsilon}$, which follow from $Q^{\frac{1}{2}\kappa+\varepsilon} \le R$ and $\kappa < \frac{2}{3}$ respectively upon reinterpreting ε.

Grouping the above, it will suffice to show that

$$\sum_w \Psi\Big(\frac{bvw}{Q}\Big)\Delta(bw) \ll Q^{1-\varepsilon}\sqrt{X}$$

uniformly for $b, v \le Q^{\varepsilon}$ and test functions Ψ and f. Assume now $\kappa \in \left(0, \frac{50}{1093}\right)$. Then the conditions on R in (12) and in Proposition 4 overlap, so that we may apply Proposition 4 with $W = Q/(bv)$. This gives the above bound, and completes the proof of (11), hence of Proposition 5.

3. Exponential sums estimates

In this section, we work out the modifications to be made to the arguments underlying [Deshouillers and Iwaniec 1982] in order to exploit current knowledge on the spectral gap of the Laplacian on congruence surfaces [Kim and Sarnak 2003]. We will follow the setting in Theorem 2.1 of [Drappeau 2017], since we will need to keep track of the uniformity in q_0. We also take the opportunity to implement the recently described correction to [Bombieri et al. 1986].

Let $\theta \geq 0$ be a bound towards the Petersson–Ramanujan conjecture, in the sense of [Drappeau 2017, (4.6)]. Selberg's $\frac{3}{16}$ theorem corresponds to $\theta \leq \frac{1}{4}$, and the Kim–Sarnak bound [2003] asserts that $\theta \leq \frac{7}{64}$.

Proposition 6. *Let the notation and hypotheses be as in* [Drappeau 2017, Theorem 2.1]. *Then*

$$\sum_{c}\sum_{d}\sum_{n}\sum_{r}\sum_{\substack{s \\ c\equiv c_0 \text{ and } d\equiv d_0 \,(\mathrm{mod}\, q) \\ (qrd,sc)=1}} b_{n,r,s}\, g(c,d,n,r,s)\,\mathrm{e}\Bigl(n\frac{\overline{rd}}{sc}\Bigr) \ll_{\varepsilon,\varepsilon_0} (qCDNRS)^{\varepsilon+O(\varepsilon_0)} q^{\frac{3}{2}} K(C,D,N,R,S)\|b_{N,R,S}\|_2,$$

where $\|b_{N,R,S}\|_2^2 = \sum_{n,r,s}|b_{n,r,s}|^2$, *and here*

$$K(C,D,N,R,S)^2 = qCS(RS+N)(C+RD) + C^{1+4\theta}DS((RS+N)R)^{1-2\theta}\Bigl(1+\frac{qC}{RD}\Bigr)^{1-4\theta} + D^2NR. \tag{13}$$

Remark 7. The bound of Proposition 6 is monotonically stronger as θ decreases, since the first term is larger than $CS(RS+N)(RD+qC)$. Under the Petersson–Ramanujan conjecture for Maass forms, which predicts that $\theta = 0$ is admissible, the second term in (13) is smaller than the first.

Proof. The proof of the proposition, as with all results of this type, relies on the Kuznetsov formula and large sieve inequalities for coefficients of automorphic forms. The application of the Kuznetsov formula requires one to understand the contribution of holomorphic forms, Eisenstein series, and Maass forms (whether the holomorphic forms appear depends on the sign of the variables inside the Kloosterman sum). We divide these forms into the *exceptional spectrum* and the *regular spectrum*. The exceptional spectrum consists of those (conjecturally nonexistent) Maass forms whose eigenvalues $t_f = \frac{1}{2} + it_f$ have $t_f \in i\mathbb{R}$. By the definition of θ above we have that $|t_f| \leq \theta$ for all f in the exceptional spectrum. The regular spectrum consists of everything that is not exceptional. The contribution of the regular spectrum is handled as in [Drappeau 2017], and does not require any modification here. We improve upon the analysis there in handling the exceptional spectrum by keeping track of the dependence on θ (see the remark made in [Drappeau 2017, p. 703]). The statements which are affected are [Drappeau 2017, Lemma 4.10, Proposition 4.12, Proposition 4.13 and the proof of Theorem 2.1]. The treatment of the exceptional spectrum rests upon a weighted large sieve inequality. These weighted large sieve inequalities are proved, following [Deshouillers and Iwaniec 1982], by an iterative procedure.

With the notation of [Drappeau 2017], the changes to be made are as follows:

- Lemma 4.10 bounds sums of the form

$$\sum_{\substack{q\le Q\\ q_0\mid q}} \sum_{\substack{f\in\mathcal{B}(q,\chi)\\ t_f\in i\mathbb{R}}} Y^{2|t_f|}\Bigl|\sum_{N<n\le 2N} n^{\frac{1}{2}}\rho_{f\infty}(n)\Bigr|^2,$$

and serves to control the first step of the recursion. The bound

$$\sum_{\substack{q\le Q\\ q_0\mid q}} \sum_{\substack{f\in\mathcal{B}(q,\chi)\\ t_f\in i\mathbb{R}}} Y^{2|t_f|}\Bigl|\sum_{N<n\le 2N} n^{\frac{1}{2}}\rho_{f\infty}(n)\Bigr|^2 \ll (QN)^{\varepsilon}(Qq_0^{-1}+N+(NY)^{\frac{1}{2}})N$$

may be replaced by the bound

$$\ll (QN)^{\varepsilon}\bigl(Qq_0^{-1}+N+(NY)^{2\theta}(Q^{1-4\theta}+N^{1-4\theta})\bigr)N.$$

This does not require any change in the recursion argument, but merely the use of the bound $|t_f|\le\theta$ in the very last step [Deshouillers and Iwaniec 1982, page 278], whereby $\sqrt{Y/Y_1}$ is replaced by $(Y/Y_1)^{2\theta}$.

- In Proposition 4.12 one bounds sums of the form

$$\sum_{\substack{m,n,r,s\\ (s,rq)=1}} a_m b_{n,r,s} \sum_{c\in\mathcal{C}(\infty,1/s)} \frac{1}{c}\phi\Bigl(\frac{4\pi\sqrt{mn}}{c}\Bigr) S_{\infty,1/s}(m,\pm n;c)$$

in terms of quantities L_{reg} and L_{exc}. In place of

$$L_{\mathrm{exc}} = \Bigl(1+\sqrt{\frac{N}{RS}}\Bigr)\sqrt{\frac{1+X^{-1}}{RS}}\Bigl(\frac{MN}{RS+N}\Bigr)^{\frac{1}{4}}\frac{\sqrt{RS}}{1+X}\sqrt{M}\|b_{N,R,S}\|_2,$$

we claim the improved

$$L_{\mathrm{exc}} = q_0^{\frac{1}{2}-2\theta}\Bigl(1+\sqrt{\frac{N}{RS}}\Bigr)\Bigl(\frac{1+X^{-1}}{RS}\Bigr)^{2\theta}\Bigl(\frac{MN}{RS+N}\Bigr)^{\theta}\Bigl(1+\frac{M}{RS}\Bigr)^{\frac{1}{2}-2\theta}\frac{\sqrt{RS}}{1+X}\sqrt{M}\|b_{N,R,S}\|_2.$$

To obtain this bound one uses the new bound for Lemma 4.10 and follows the arguments of [Deshouillers and Iwaniec 1982, Section 9.1].

- In Proposition 4.13, one bounds

$$\sum_{\substack{c,m,n,r,s\\ (sc,rq)=1}} b_{n,r,s}\bar{\chi}(c)g(c,m,n,r,s)e(mt)S(n\bar{r},\pm m\bar{q};sc)$$

in terms of quantities K_{reg} and K_{exc}. The term

$$K_{\mathrm{exc}}^2 = C^3S^2\sqrt{R(N+RS)}$$

can be replaced by

$$K_{\mathrm{exc}}^2 = C^{2+4\theta}S^2(R(N+RS))^{1-2\theta}\Bigl(1+\frac{M}{RS}\Bigr)^{1-4\theta}.$$

This is seen by using the new definition on L_{exc} in Proposition 4.12, and by keeping track of a factor $q^{-1+2\theta}$ coming from the term $(1+X^{-1})^{2\theta}/(1+X)$.

• Finally, we modify the proof of Theorem 2.1 at two places. First, the bound for $\mathcal{A}_0$ on page 706, as explained in the correction to [Bombieri et al. 1986], is wrong unless further hypotheses on $(b_{n,r,s})$ are imposed. The correct bound in general is

$$\mathcal{A}_0 \ll q^{-2}(\log S)^2 D(NR)^{\frac{1}{2}} \|b_{N,R,S}\|_2,$$

and this yields the term D^2NR instead of D^2NRS^{-1}. Second, our new bound for K_{exc} in Proposition 4.13 gives a contribution $C^{2+4\theta}S^2(R(RS+N))^{1-2\theta}(1+M_1/(RS))^{1-4\theta}$ instead of $C^3S^2\sqrt{R(RS+N)}$ in the definition of L^2_{exc} and $L^*(M_1)^2$ on page 707 of [Drappeau 2017]. This yields a term $C^{1+4\theta}DS\times((N+RS)R)^{1-2\theta}(1+qC/(RD))^{1-4\theta}$ instead of $C^2DS\sqrt{(N+RS)R}$ in (4.39) of [Drappeau 2017], and by following the rest of the arguments we deduce our claimed bound. □

4. Primes in arithmetic progressions: proof of Proposition 4

The proof of Theorem 1 relies on Proposition 4, which for the convenience of the reader we recall below.

Proposition 4. *Let $\kappa \in \left(0, \frac{50}{1093}\right)$ and $\varepsilon > 0$. Let Ψ and f be test functions, $A > 0$, $X, Q, W, R \geq 1$, and $b \in \mathbb{N}$. Assume that*

$$\frac{Q^2}{(\log Q)^A} \ll X \ll Q^{2+\kappa}, \quad X^{\frac{11}{20}}Q^{-1} \leq R \leq Q^{\frac{2}{3}}X^{-\frac{2}{9}}, \quad b \leq Q^{\varepsilon}, \quad Q^{1-\varepsilon} \ll W \ll Q,$$

and that $\|f^{(j)}\|_\infty, \|\Psi^{(j)}\|_\infty \ll_j 1$. Then, if $\varepsilon > 0$ is small enough in terms of κ, we have

$$\sum_{w\in\mathbb{N}} \Psi\left(\frac{w}{W}\right) \sum_{n\in\mathbb{N}} \Lambda(n) f\left(\frac{n}{X}\right) \mathfrak{u}_R(n, bw) \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends at most on κ, A, and the implied constants in the hypotheses.

Remark 8. What is crucial in our statement is the size of the upper bound, which should be negligible with respect to $Q\sqrt{X}$. On the other hand, we are only interested in values of X larger than Q^2. This is in contrast with most works on primes in arithmetic progressions [Fouvry and Iwaniec 1983; Bombieri et al. 1986; Zhang 2014], where the main challenge is to work with values of X much smaller than Q^2, while only aiming at an error term which is negligible with respect to X. The main point is that in both cases, the large sieve yields an error term which is always too large (see [Iwaniec and Kowalski 2004, Theorem 17.4]), an obstacle which the dispersion method is designed to handle.

In what follows, we will systematically write

$$X = Q^{2+\varpi},$$

so that $-o(1) \leq \varpi \leq \kappa + o(1)$ as $Q \to \infty$.

4.1. ***Combinatorial identity.*** We perform a combinatorial decomposition of the von Mangoldt function into sums of different shapes: type d_1 sums have a long smooth variable, type d_2 sums have two long smooth variables, and type II sums have two rough variables that are neither too small nor too large. We accomplish this decomposition with the Heath-Brown identity and the following combinatorial lemma.

Lemma 9. *Let $\{t_j\}_{1\le j\le J} \in \mathbb{R}$ be nonnegative real numbers such that $\sum_j t_j = 1$. Let $\lambda, \sigma, \delta \ge 0$ be real numbers such that*

- $\delta < \frac{1}{12}$,
- $\sigma \le \frac{1}{6} - \frac{1}{2}\delta$,
- $2\lambda + \sigma < \frac{1}{3}$.

Then at least one of the following must occur:

- Type d_1: *There exists t_j with $t_j \ge \frac{1}{3} + \lambda$.*
- Type d_2: *There exist i, j, k such that $\frac{1}{3} - \delta < t_i, t_j, t_k < \frac{1}{3} + \lambda$, and*

$$\sum_{t_j^* \notin \{t_i, t_j, t_k\}} t_j^* < \sigma.$$

- Type II: *There exists $S \subset \{1, \dots, J\}$ such that*

$$\sigma \le \sum_{j\in S} t_j \le \frac{1}{3} - \delta.$$

Proof. Assume that the type d_1 case and the type II case both fail. Then for every j we have $t_j < \frac{1}{3} + \lambda$, and for every subset S of $\{1, \dots, J\}$ we have either

$$\sum_{j\in S} t_j < \sigma \quad \text{or} \quad \sum_{j\in S} t_j > \tfrac{1}{3} - \delta.$$

Let $s_1, \dots, s_K$ denote those t_j with $\frac{1}{3} - \delta < t_j < \frac{1}{3} + \lambda$. We will show that $K = 3$. Let t_j^* be any other t_j, so that $t_j^* \le \frac{1}{3} - \delta$, and therefore $t_j^* < \sigma$. We claim that

$$\sum_j t_j^* < \sigma.$$

If not, then $\sum_j t_j^* > \frac{1}{3} - \delta$. By a greedy algorithm we can find some subcollection S^* of the t_j^* such that

$$\sigma < \sum_{j\in S^*} t_j^* \le 2\sigma.$$

Since $2\sigma \le \frac{1}{3} - \delta$ this subcollection satisfies the type II condition, in contradiction to our assumption.

Now we show that $K = 3$. Observe that $K \ge 3$, since if $K \le 2$ we have

$$1 = \sum_j t_j = \sum_{i=1}^{K} s_i + \sum_j t_j^* < 2\bigl(\tfrac{1}{3} + \lambda\bigr) + \sigma < 1.$$

Furthermore, we must have $K \le 3$, since if $K \ge 4$ we have

$$1 = \sum_j t_j \ge \sum_{i=1}^{K} s_i > 4\bigl(\tfrac{1}{3} - \delta\bigr) > 1.$$

This completes the proof. □

Using, e.g., the combinatorial identity of Heath-Brown [1982], we deduce the following.

Corollary 10. *Let f be a test function, $u:\mathbb{N}\to\mathbb{C}$ be any map, and $X\geq 1$. Then there exists a sequence $(C_j)_{j\geq 0}$ of positive numbers, depending only on f, such that we have*

$$\left|\sum_{n\in\mathbb{N}}\Lambda(n)f\left(\frac{n}{X}\right)u(n)\right| \ll (\log X)^8(T_1+T_2+T_{\mathrm{II}}), \tag{14}$$

where

$$T_1=\sup_{\substack{N\gg X^{1/3+\lambda}\\ MN\asymp X}}\ \sup_{\substack{g\in\mathcal{G}\\ \beta\in\mathcal{S}}}\left|\sum_{\substack{n\in\mathbb{N}\\ m\sim M}}\sum g\left(\frac{n}{N}\right)\beta_m u(mn)\right|, \tag{15}$$

$$T_2=\sup_{\substack{X^{1/3-\delta}\ll N_2\leq N_1\ll X^{1/3+\lambda}\\ MN_1N_2\asymp X}}\ \sup_{\substack{g_1,g_2\in\mathcal{G}\\ \beta\in\mathcal{S}}}\left|\sum\sum_{\substack{n_1,n_2\in\mathbb{N}\\ m\sim M}}\sum g_1\left(\frac{n_1}{N_1}\right)g_2\left(\frac{n_2}{N_2}\right)\beta_m u(mn_1n_2)\right|, \tag{16}$$

$$T_{\mathrm{II}}=\sup_{\substack{X^{\sigma}\ll N\ll X^{1/3-\delta}\\ MN\asymp X}}\ \sup_{\alpha,\beta\in\mathcal{S}}\left|\sum_{\substack{n\sim N\\ m\sim M}}\sum \alpha_m\beta_n u(mn)\right|, \tag{17}$$

where the implied constants are absolute, $\mathcal{G}$ is the set of test functions g satisfying $\|g^{(j)}\|_\infty\leq C_j$, and $\mathcal{S}$ is the set of sequences (β_n) satisfying $|\beta_n|\leq d(n)^8$.

Proof. By the Heath-Brown identity [Iwaniec and Kowalski 2004, Proposition 13.3], there exists bounded coefficients $(c_J)_{1\leq J\leq 4}$ such that

$$\Lambda(n)=\sum_{J=1}^{4}c_J\sum_{\substack{m_1,\dots,m_J\\ n_1,\dots,n_J\\ n=m_1\cdots m_Jn_1\cdots n_J\\ m_j\leq(3X)^{1/4}}}\log(n_1)\prod_j\mu(m_j)$$

for any n involved in the left-hand side of (14). Let ψ be a test function inducing a partition of unity in the sense that $\sum_{j\in\mathbb{Z}}\psi(x/2^j)=1$ for all $x>0$. Then we have

$$\sum_{n\in\mathbb{N}}\Lambda(n)f\left(\frac{n}{X}\right)u(n)=\sum_{J=1}^{4}c_J\sum_{(M_1,\dots,M_J,N_1,\dots,N_J)\in U_J}S(M_1,\dots,M_J,N_1,\dots,N_J),$$

$$S(M_1,\dots,N_J)=\sum_{m_1,\dots,n_J\in\mathbb{N}}\log(n_1)\left(\prod_j\psi\left(\frac{n_j}{N_j}\right)\right)\left(\prod_j\mu^*(m_j)\psi\left(\frac{m_j}{M_j}\right)\right)f\left(\frac{m_1\cdots n_J}{X}\right)u(m_1\cdots n_J),$$

where U_J is the set of $2J$-tuples of powers of 2 such that $\frac{1}{6}X\leq M_1\cdots M_JN_1\cdots N_J\leq 6X$, and $\mu^*(m)=\mu(m)$ if $m\leq(3X)^{\frac14}$ and 0 otherwise. We abbreviated $m_1\cdots n_J=m_1\cdots m_Jn_1\cdots n_J$. The set U_J has at most $O((\log X)^{2J-1})$ elements. By Lemma 9, for each choice of J and $(M_1,\dots,N_J)\in U_J$ we have either $N\geq\frac16 X^{\frac13+\lambda}$ for some $N\in\{N_j\}$, or $\frac16 X^{\frac13-\delta}\leq N',N''\leq 6X^{\frac13+\lambda}$ for some $N',N''\in\{N_j\}$, or $\frac16 X^{\sigma}\leq N\leq 6X^{\frac13-\delta}$ for some subproduct N of N_j and M_j (here we used that for X large enough, we have $(3X)^{\frac{1}{4}} < \frac{1}{6}X^{\frac{1}{3}-\delta}$). Sorting the sum over J and $(M_1, \dots, N_J)$ according to this trichotomy, and writing $\log(n_1) = \log N_1 + \log(n_1/N_1)$, the above is bounded in absolute values by

$$\ll (\log X)^8 (T_1^* + T_2^* + T_{\mathrm{II}}^*),$$

where

$$T_1^* = \sup_{\substack{\frac{1}{6}X \le MN \le 6X \\ \frac{1}{6}X^{1/3+\lambda} \le N \\ |r| \le 8}} \sup_{\substack{g \in \{\psi, \psi \log\} \\ \beta \in \mathcal{S}}} \left| \sum_{\substack{n \in \mathbb{N} \\ m \sim M}} g\left(\frac{n}{N}\right) \beta_m f\left(\frac{mn}{2^r MN}\right) u(mn) \right|,$$

$$T_2^* = \sup_{\substack{\frac{1}{6}X \le N_1 N_2 M \le 6X \\ \frac{1}{6}X^{1/3-\delta} \le N_1, N_2 \le 6X^{1/3+\lambda} \\ |r| \le 8}} \sup_{\substack{g_1, g_2 \in \{\psi, \psi \log\} \\ \beta \in \mathcal{S}}} \left| \sum_{\substack{n_1, n_2 \in \mathbb{N} \\ m \sim M}} g_1\left(\frac{n_1}{N_1}\right) g_2\left(\frac{n_2}{N_2}\right) \beta_m f\left(\frac{n_1 n_2 m}{2^r N_1 N_2 M}\right) u(n_1 n_2 m) \right|,$$

$$T_{\mathrm{II}}^* = \sup_{\substack{\frac{1}{6}X \le NM \le 6X \\ \frac{1}{6}X^{\sigma} \le N \le 6X^{1/3-\delta} \\ |r| \le 8}} \sup_{\alpha, \beta \in \mathcal{S}} \left| \sum_{\substack{m \sim M \\ n \sim N}} \alpha_m \beta_n f\left(\frac{mn}{2^r MN}\right) u(mn) \right|.$$

Here the conditions $m \sim M$ and $n \sim N$ in the sums were added by an additional bounded dichotomy (which is the reason for the presence of the sup over r). Finally, letting $\check{f}$ be the Mellin transform of f, we have by Mellin inversion $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \check{f}(it) x^{-it} \, \mathrm{d}t$, and the map $t \mapsto \check{f}(it)$ is of Schwartz class on $\mathbb{R}$. In particular, for M, N, r, g, β as in T_1^* we have

$$\left| \sum_{n \in \mathbb{N}, m \sim M} g\left(\frac{n}{N}\right) \beta_m f\left(\frac{mn}{2^r MN}\right) u(mn) \right| \ll \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathbb{N}, m \sim M} g_t\left(\frac{n}{N}\right) \beta_{m,t} u(mn) \right|,$$

where $g_t(x) = (1+t^2)\check{f}(it) x^{-it} g(x)$ (the factor $1+t^2$ being included so that we could write a supremum) and $\beta_{m,t} = m^{-it}\beta_m \in \mathcal{S}$. We note that g_t is a test function satisfying $\|g_t^{(j)}\|_\infty \ll C_j$, where $C_j := \sup_{0 \le k, \ell, m \le j+2} \|t^k \check{f}(it)\|_\infty \|x^{-\ell} g^{(m)}(x)\|_\infty$ can be bounded in terms of f only. This yields the contribution of T_1 in our claim. The contributions of T_2 and T_{II} are obtained in the same way. □

In what follows, we successively consider T_1, T_2 and T_{II}, which we specialize at

$$u(n) := \sum_{w \in \mathbb{N}} \Psi\left(\frac{w}{W}\right) \mathfrak{u}_R(n, bw),$$

and we will write

$$R = X^\rho.$$

4.2. ***Type d_1 sums.*** We suppose M and N are given as in (15). The quantity we wish to bound is

$$T_1(M,N) = \sum_w \Psi\left(\frac{w}{W}\right) \sum_{\substack{m \sim M \\ (m,bw)=1}} \beta_m \left(\sum_{\substack{n \in \mathbb{N} \\ mn \equiv 1 \ (\mathrm{mod}\, bw)}} g\left(\frac{n}{N}\right) - \frac{1}{\varphi(bw)} \sum_{\substack{\chi \ (\mathrm{mod}\, bw) \\ \mathrm{cond}(\chi) \le R}} \chi(m) \sum_{(n,bw)=1} \chi(n) g\left(\frac{n}{N}\right) \right).$$

By Poisson summation and the classical bound on Gauss sums [Iwaniec and Kowalski 2004, Lemma 3.2],

$$\sum_{n\equiv \bar{m} \ (\mathrm{mod}\, bw)} g\Big(\frac{n}{N}\Big) = \frac{N}{bw}\hat{g}(0) + \frac{N}{bw}\sum_{0<|h|\le W^{1+\varepsilon}/N} \hat{g}\Big(\frac{Nh}{bw}\Big)e\Big(\frac{\bar{m}h}{bw}\Big) + O(Q^{-A}),$$

$$\frac{1}{\varphi(bw)}\sum_{(c,bw)=1} \chi(c) g\Big(\frac{c}{N}\Big) = \frac{N}{bw}\hat{g}(0)\mathbf{1}(\chi=\chi_0) + O\Big(\frac{Q^{\varepsilon}R^{\frac{1}{2}}}{W}\Big).$$

Therefore,

$$T_1(M,N) = \frac{N}{b}\sum_{w}\frac{1}{w}\Psi\Big(\frac{w}{W}\Big)\sum_{\substack{(m,bw)=1\\ m\sim M}} \beta_m \sum_{0<|h|\le W^{1+\varepsilon}/N} \hat{g}\Big(\frac{Nh}{bw}\Big)e\Big(\frac{\bar{m}h}{bw}\Big) + O(MR^{\frac{3}{2}}Q^{\varepsilon}).$$

Our goal is to get cancellation in the exponential phases by summing over the smooth variable w. We apply the reciprocity formula

$$\frac{\bar{m}h}{bw} \equiv -\frac{\overline{bw}h}{m} + \frac{h}{mbw} \pmod 1,$$

which implies

$$T_1(M,N) = \frac{N}{b}\sum_{w}\frac{1}{w}\Psi\Big(\frac{w}{W}\Big)\sum_{\substack{(m,bw)=1\\ m\sim M}} \beta_m \sum_{0<|h|\le W^{1+\varepsilon}/N} \hat{g}\Big(\frac{Nh}{bw}\Big)e\Big(\frac{\overline{bw}h}{m}\Big) + O(MR^{\frac{3}{2}}Q^{\varepsilon} + Q^{1+\varepsilon}N^{-1}).$$

We rearrange the sum as

$$\frac{N}{bW}\sum_{\substack{(m,b)=1\\ m\sim M}} \beta_m \sum_{0<|h|\le W^{1+\varepsilon}/N}\ \sum_{(w,m)=1} \hat{g}\Big(\frac{Nh}{bw}\Big)\frac{W}{w}\Psi\Big(\frac{w}{W}\Big)e\Big(\frac{\overline{bw}h}{m}\Big).$$

By partial summation and a variant of the Weil bound [Drappeau 2015, (2.4)], the sum on w is

$$\ll ((h,m)WM^{-1} + \sqrt{(h,m)}\sqrt{M})Q^{\varepsilon}.$$

Summing over h and m, we obtain a bound

$$T_1(M,N) \ll Q^{1+\varepsilon} + M^{\frac{3}{2}}Q^{\varepsilon} + MR^{\frac{3}{2}}Q^{\varepsilon}.$$

This bound is acceptably small provided

$$N \gg \Big(\frac{X}{Q}\Big)^{\frac{2}{3}+\varepsilon} = X^{\frac{1}{3}+\frac{1}{3}\varpi/(2+\varpi)+\varepsilon(1+\varpi)/(2+\varpi)} \quad \text{and} \quad N \gg \frac{X^{\frac{1}{2}}R^{\frac{3}{2}}}{Q^{1-2\varepsilon}} = X^{\frac{1}{2}\varpi/(2+\varpi)+\frac{3}{2}\rho+2\varepsilon/(2+\varpi)}.$$

These inequalities are satisfied, for all sufficiently small $\varepsilon > 0$, under the assumptions

$$\lambda > \frac{\varpi}{3(2+\varpi)} \quad \text{and} \quad \rho < \frac{4+\varpi}{9(2+\varpi)}. \tag{18}$$

We have proved the following.

Lemma 11. *Under the notation and hypotheses of Corollary 10, and assuming (18), we have*

$$T_1 \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends on λ, ρ and ϖ.

4.3. ***Type d_2 sums.*** The treatment of the type d_2 sums (16) is nearly identical to [Bombieri et al. 1986, Section 14]. For convenience, we rename (N_1, N_2, M) into (M, N, L) so that we have $MNL \asymp X$. We wish to bound

$$T_2(M,N,L) = \sum_{\ell\sim L} \beta_\ell \sum_{(w,\ell)=1} \Psi\Big(\frac{w}{W}\Big)\Big(\sum_{\substack{m,n \\ \ell mn \equiv 1 \,(\mathrm{mod}\, bw)}}\sum g_1\Big(\frac{m}{M}\Big) g_2\Big(\frac{n}{N}\Big) \\ - \frac{1}{\varphi(bw)} \sum_{\substack{\chi \,(\mathrm{mod}\, bw) \\ \mathrm{cond}(\chi)\le R}} \chi(\ell) \sum_{(mn,bw)=1}\sum g_1\Big(\frac{m}{M}\Big) g_2\Big(\frac{n}{N}\Big)\chi(mn)\Big).$$

We perform Poisson summation on the m-sums to get

$$\sum_{m\equiv \overline{\ell n} \,(\mathrm{mod}\, bw)} g_1\Big(\frac{m}{M}\Big) = \frac{M}{bw} \sum_{|h|\le H} \hat{g}_1\Big(\frac{Mh}{bw}\Big) e\Big(\frac{\overline{\ell n}h}{bw}\Big) + O(Q^{-A}),$$
$$\sum_{(m,bw)=1} \chi(m) g_1\Big(\frac{m}{M}\Big) = \frac{\varphi(bw)}{bw} M \hat{g}_1(0)\mathbf{1}(\chi=\chi_0) + O(Q^{\varepsilon} R^{\frac{1}{2}}),$$

where $H = W^{1+\varepsilon} M^{-1}$. The contribution of the error terms is

$$\ll LNR^{\frac{3}{2}} Q^{\varepsilon}.$$

The zero frequency of Poisson summation cancels out. For the nonzero frequencies we employ reciprocity in the form

$$e\Big(\frac{\overline{\ell n}h}{bw}\Big) = e\Big(-\frac{\overline{bw}h}{\ell n}\Big) + O\Big(\frac{H}{LNW}\Big),$$

and the error term contributes a quantity of size $O(Q^{1+\varepsilon})$. We therefore have

$$T_2(M,N,L) = \frac{M}{b} \sum_{\substack{\ell\sim L \\ (\ell,b)=1}} \beta_\ell \sum_{(w,\ell)=1} \frac{1}{w}\Psi\Big(\frac{w}{W}\Big) \sum_{(n,bw)=1} g_2\Big(\frac{n}{N}\Big) \sum_{0<|h|\le H} \hat{g}_1\Big(\frac{Mh}{bw}\Big) e\Big(-\frac{\overline{bw}h}{\ell n}\Big) \\ + O(Q^{1+\varepsilon} + LNR^{\frac{3}{2}} Q^{\varepsilon}). \tag{19}$$

We next separate the variables h and w. We change variables to write

$$\hat{g}_1\Big(\frac{Mh}{bw}\Big) = \frac{w}{M} \int_{\mathbb{R}} g_1\Big(\frac{wy}{M}\Big) e\Big(-\frac{hy}{b}\Big)\, dy.$$

Since g_1 and Ψ are test functions, the integral is restricted to $y \asymp M/W$. We move the integral to the outside to write the first term of the right-hand side of (19) as

$$\ll \frac{M}{bW} \sup_{y\asymp M/W} \Big| \sum_{\ell} \beta_\ell \sum_{0<|h|\le H} \mathrm{e}\Big(-\frac{hy}{b}\Big) \sum_{w}\sum_{n} \Psi\Big(\frac{w}{W}\Big) g_1\Big(\frac{wy}{M}\Big) g_2\Big(\frac{n}{N}\Big) \mathrm{e}\Big(-\frac{\overline{bw}h}{\ell n}\Big)\Big|. \tag{20}$$

We then use [Deshouillers and Iwaniec 1982, Theorem 12], amended as described in the correction to [Bombieri et al. 1986], more specifically, with the dictionary (the bold symbols denote the variables names from [Deshouillers and Iwaniec 1982])

$$\begin{aligned} \boldsymbol{c}, \boldsymbol{C} &\leftrightarrow n, N, & \boldsymbol{d}, \boldsymbol{D} &\leftrightarrow w, W, \\ \boldsymbol{n}, \boldsymbol{N} &\leftrightarrow h, H, & \boldsymbol{r}, \boldsymbol{R} &\leftrightarrow b', b, \\ \boldsymbol{s}, \boldsymbol{S} &\leftrightarrow \ell, L, & \boldsymbol{b}_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}} &\leftrightarrow \mathbf{1}_{b'=b} \mathrm{e}(-hy/b)\beta_\ell. \end{aligned}$$

Since $\lambda < \frac{1}{6}$, we have $H \ll L$ if ε is sufficiently small. Thus, with the same notation, we find the bounds

$$\boldsymbol{K}(\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{N}, \boldsymbol{R}, \boldsymbol{S}) \ll b(NL^2(N+W) + N^2WL^{\frac{3}{2}} + W^2H)^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{b}_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2 \ll L^\varepsilon (HL)^{\frac{1}{2}}.$$

It will also be easier to sum up the bounds if we assume

$$N \ll W^{1+\varepsilon}. \tag{21}$$

We find

$$T_2(M, N, L) \ll LNR^{\frac{3}{2}} Q^\varepsilon + Q^\varepsilon(\sqrt{X}L + \sqrt{M}NL^{\frac{5}{4}} + L^{\frac{1}{2}}W) \ll LNR^{\frac{3}{2}} Q^\varepsilon + Q^\varepsilon(\sqrt{X}L + \sqrt{M}NL^{\frac{5}{4}}),$$

the second inequality following since $L^{\frac{1}{2}}W \ll X^{\frac{1}{2}}L$. This contribution is acceptable provided

$$M \gg X^{\frac{1}{2}\varpi/(2+\varpi)+\frac{3}{2}\rho+\varepsilon}, \quad MN \gg X^{\frac{1}{2}+\frac{1}{2}\varpi/(2+\varpi)+\varepsilon}, \tag{22}$$

$$M^{\frac{3}{2}}N^{\frac{1}{2}} \gg X^{\frac{1}{2}+\varpi/(2+\varpi)+2\varepsilon}. \tag{23}$$

The bounds (21)–(23) are satisfied if

$$\delta < \frac{1}{12} - \frac{\varpi}{2(2+\varpi)}, \quad \lambda < \frac{1}{6} - \frac{\varpi}{2(2+\varpi)}, \quad \rho < \frac{1}{6}. \tag{24}$$

We therefore conclude the following.

Lemma 12. *Under the notation and hypotheses of Corollary 10, and assuming (24), we have*

$$T_2 \ll Q^{1-\varepsilon}\sqrt{X}.$$

The implied constant depends on λ, δ, ρ and ϖ.

4.4. *Type II sums.* In the type II case (17), we wish to prove the bound

$$T_{\mathrm{II}}(M, N) := \sum_w \Psi\Big(\frac{w}{W}\Big) \sum_{m,n}\sum \alpha_m \beta_n \mathfrak{u}_R(mn, bw) \ll \sqrt{X} Q^{1-\varepsilon},$$

where α is supported at scale M, β is supported at scale N, $MN \asymp X$, and $X^\sigma \ll N \ll X^{\frac{1}{3}-\delta}$. We have $|\alpha_m| \le \tau(m)^{O(1)}$, and similarly for β. We use the dispersion method of Linnik [1963], following closely [Fouvry 1985]; see also [Bombieri et al. 1986, Section 10].

We interchange the order of summation and apply the triangle inequality, writing our sum as

$$|T_{\mathrm{II}}(M, N)| \le \sum_m \Big| \sum_w \sum_n \Big|.$$

Applying the Cauchy–Schwarz inequality, we arrive at

$$T_{\mathrm{II}}(M,N)^2 \ll M(\log M)^{O(1)}\mathcal{D}, \tag{25}$$

where

$$\mathcal{D} = \sum_m f\Big(\frac{m}{M}\Big)\Bigg|\sum_{\substack{n,w\\ mn\equiv 1 \,(\mathrm{mod}\, bw)}}\sum \Psi\Big(\frac{w}{W}\Big)\beta_n - \frac{1}{\varphi(bw)}\sum_{\substack{\chi \,(\mathrm{mod}\, bw)\\ \mathrm{cond}(\chi)\le R}}\sum_{\substack{n,w\\ (mn,bw)=1}}\sum \Psi\Big(\frac{w}{W}\Big)\beta_n\chi(mn)\Bigg|^2.$$

Here f is some fixed, nonnegative test function majorizing $\mathbf{1}_{[1,2]}$. It suffices to show that

$$\mathcal{D} \ll NQ^{2-\varepsilon}.$$

We open the square and arrive at

$$\mathcal{D} = \mathcal{D}_1 - 2\operatorname{Re}\mathcal{D}_2 + \mathcal{D}_3, \tag{26}$$

say. We treat each sum $\mathcal{D}_i$ in turn.

4.4.1. *Evaluation of $\mathcal{D}_3$.* By definition we have

$$\mathcal{D}_3 := \sum_m f\Big(\frac{m}{M}\Big)\sum_{\substack{w_1,w_2,n_1,n_2\\ (mn_1,bw_1)=1\\ (mn_2,bw_2)=1}}\sum\sum\sum\sum_{\substack{\chi_1,\chi_2\\ \chi_j \,(\mathrm{mod}\, bw_j)\\ \mathrm{cond}(\chi_j)\le R}}\sum \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}}\frac{\chi_1(mn_1)\overline{\chi_2(mn_2)}}{\varphi(bw_1)\varphi(bw_2)}.$$

The computations in [Drappeau 2017, pp. 711–712] can be directly quoted, putting formally

$$\gamma(q) = \mathbf{1}(b \mid q)\Psi(q/(bW)), \tag{27}$$

with the modification that $\mathrm{cond}(\chi_1\overline{\chi_2}) \le R^2$ (instead of R, as stated incorrectly in [Drappeau 2017]). Writing $H = Q^\varepsilon b[w_1,w_2]M^{-1}$, we get

$$\begin{aligned}\mathcal{D}_3 &= \mathcal{M}_3 + O\Bigg(Q^\varepsilon \sum_{\substack{w_1,w_2\asymp W\\ n_1,n_2\asymp N}}\frac{1}{\varphi(bw_1)\varphi(bw_2)}\sum_{\substack{\chi_1,\chi_2\\ \mathrm{cond}(\chi_j)\le R}}\frac{M}{b[w_1,w_2]}\sum_{0<|h|\le H} R \sum_{d\mid(h,b[w_1,w_2])} d\Bigg)\\ &= \mathcal{M}_3 + O(Q^\varepsilon N^2R^5),\end{aligned}$$

where the main term is computed as in [Drappeau 2017, p. 712] to be

$$\mathcal{M}_3 := M\hat{f}(0)\sum_{\substack{w_1,w_2,n_1,n_2\\ (n_j,bw_j)=1}}\sum\sum\sum\sum_{\substack{\chi \text{ primitive}\\ \mathrm{cond}(\chi)\le R\\ \mathrm{cond}(\chi)\mid b(w_1,w_2)}}\Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}}\chi(n_1\overline{n_2})\frac{\varphi(bw_1w_2)}{bw_1w_2\varphi(bw_1)\varphi(bw_2)}.$$

The error term is acceptable provided

$$NR^5 \ll Q^{2-\varepsilon}.$$

Since $N \ll X^{\frac{1}{3}}$ this is acceptable provided

$$\rho < \frac{4-\varpi}{15(2+\varpi)}. \tag{28}$$

4.4.2. *Evaluation of* $\mathcal{D}_2$. We have

$$\mathcal{D}_2 := \sum\sum\sum\sum_{\substack{w_1,w_2,n_1,n_2\\(n_j,bw_j)=1}} \sum_{\substack{\chi\ (\mathrm{mod}\, bw_2)\\ \mathrm{cond}(\chi)\le R}} \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\overline{\beta_{n_1}}\beta_{n_2}\frac{\chi(n_2)}{\varphi(bw_2)} \sum_{\substack{mn_1\equiv 1 (bw_1)\\(m,w_2)=1}} \chi(m) f\Big(\frac{m}{M}\Big).$$

The computations in [Drappeau 2017, pp. 712–713] can be also quoted directly with the identification (27). We obtain

$$\mathcal{D}_2 = \mathcal{M}_3 + O(R^{\frac{3}{2}} N^2 Q^{1+\varepsilon}).$$

This is acceptable if

$$\rho < \frac{2}{3}\lambda + \frac{2(1-\varpi)}{9(2+\varpi)}. \tag{29}$$

4.4.3. *Evaluation of* $\mathcal{D}_1$. We have

$$\mathcal{D}_1 := \sum\sum\sum\sum_{\substack{w_1,w_2,n_1,n_2\\(n_j,bw_j)=1\\ n_1\equiv n_2\ (\mathrm{mod}\, b)}} \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}} \sum_{mn_j\equiv 1\ (\mathrm{mod}\, bw_j)} f\Big(\frac{m}{M}\Big).$$

We need to separate the variables w_1, w_2, n_1, n_2 from each other, and this requires a subdivision of the variables. We decompose these variables uniquely, following [Fouvry and Radziwiłł 2018], as follows:

$$\begin{cases} d = (n_1, n_2), & n_1 = dd_1\nu_1 \text{ with } d_1 \mid d^\infty \text{ and } (d,\nu_1)=1, \quad n_2 = d\nu_2, \\ q_0 = (w_1, w_2), & w_i = q_0 q_i \text{ for } i\in\{1,2\}. \end{cases}$$

The summation conditions imply

$$(dd_1\nu_1, q_0q_1) = (d\nu_2, q_0q_2) = 1.$$

We therefore have

$$\mathcal{D}_1 = \sum_{(d,b)=1}\sum_{d_1\mid d^\infty}\sum_{(q_0,d)=1} \mathcal{D}_1(d,d_1,q_0),$$

$$\mathcal{D}_1(\cdots) = \sum\sum\sum\sum_{\substack{q_1,q_2,\nu_1,\nu_2\\ (d\nu_1,\nu_2)=(q_1,q_2)=1\\ (q_1q_2,d)=(\nu_1,d)=1\\ (\nu_1,q_1)=(\nu_2,q_2)=(\nu_1\nu_2,bq_0)=1\\ d_1\nu_1\equiv\nu_2\ (\mathrm{mod}\, bq_0)}} \Psi\Big(\frac{q_0q_1}{W}\Big)\Psi\Big(\frac{q_0q_2}{W}\Big)\beta_{dd_1\nu_1}\overline{\beta_{d\nu_2}} \sum_{\substack{mdd_1\nu_1\equiv 1\ (\mathrm{mod}\, bq_0q_1)\\ md\nu_2\equiv 1\ (\mathrm{mod}\, bq_0q_2)}} f\Big(\frac{m}{M}\Big).$$

Using smooth partitions of unity we break the variables into dyadic ranges: $d\asymp D$, $d_1\asymp D_1$, $q_0\asymp Q_0$. The contribution from $d\asymp D$ and $d_1\asymp D_1$ is

$$\ll Q^\varepsilon M \sum_{d\asymp D}\sum_{\substack{d_1\mid d^\infty\\ d_1\asymp D_1}}\sum_{\nu_1\asymp N/dd_1}\sum_{\nu_2\asymp N/d} |\beta_{dd_1\nu_1}||\beta_{d\nu_2}| \ll Q^\varepsilon M N^2 \sum_{d\asymp D}\frac{1}{d^2}\sum_{d_1\mid d^\infty}\frac{\tau(d_1)^{O(1)}}{d_1}\Big(\frac{d_1}{D_1}\Big)^{1-\varepsilon^2}$$
$$\ll Q^\varepsilon M N^2 D_1^{-1+\varepsilon^2} D^{-1},$$

where the sum over q_0 and q_1 was bounded by $O(\tau_3(|mdd_1\nu_1-1|))=O(Q^\varepsilon)$, likewise for the sum over q_2 (note that $md\nu_2\neq 1$ and $mdd_1\nu_1\neq 1$). This bound is acceptable provided

$$DD_1 \gg \frac{X}{Q^{2-\varepsilon}}, \tag{30}$$

so we may henceforth assume $DD_1 \ll XQ^{-2+\varepsilon}$.

The contribution from $q_0 \asymp Q_0$ is

$$\begin{aligned}&\ll Q^\varepsilon \sum_{q_0\asymp Q_0}\sum_{q_1\asymp Q/q_0}\sum\sum_{\substack{n_1\equiv n_2 \,(\mathrm{mod}\, q_0)\\ n_j\asymp N}}\sum_{\substack{m\asymp M\\ m\equiv \overline{n_1}\,(\mathrm{mod}\, q_0q_1)}} 1\\ &\ll Q^\varepsilon M \sum_{q_0\asymp Q_0}\sum_{q_1\asymp Q/q_0}\frac{1}{q_0q_1}\sum\sum_{\substack{n_1\equiv n_2 \,(\mathrm{mod}\, q_0)\\ n_j\asymp N}} 1\\ &\ll Q^\varepsilon(MN^2Q_0^{-1}+MN),\end{aligned}$$

where in the first line the sum over q_2 was again bounded by $\tau(|md\nu_2-1|)$. This is acceptable provided

$$N \gg \frac{X}{Q^{2-\varepsilon}} \quad \text{and} \quad Q_0 \gg \frac{X}{Q^{2-\varepsilon}}, \tag{31}$$

so we may henceforth assume $Q_0 \ll XQ^{-2+\varepsilon}$.

We use Poisson summation, following [Drappeau 2017, pp. 714–716]. Let

$$\tilde{q} = bq_0q_1q_2 \quad \text{and} \quad \mu \equiv \begin{cases}\overline{dd_1\nu_1} & (\mathrm{mod}\, bq_0q_1),\\ \overline{d\nu_2} & (\mathrm{mod}\, bq_0q_2).\end{cases}$$

Note that $\tilde{q} \geq \frac{1}{2}W \gg Q^{1-\varepsilon}$. With $H=\tilde{q}^{1+\varepsilon}M^{-1} \ll Q^{2+\varepsilon}/(q_0M)$, we get, for any fixed $A>0$,

$$\sum_{m\equiv\mu\,(\mathrm{mod}\,\tilde{q})} f\Big(\frac{m}{M}\Big) = \frac{M}{\tilde{q}}\sum_{|h|\leq H}\hat{f}\Big(\frac{hM}{\tilde{q}}\Big)\mathrm{e}\Big(\frac{\mu h}{\tilde{q}}\Big)+O(Q^{-A}). \tag{32}$$

The zero frequency in (32) contributes the main term, which, after summing over d, d_1 and q_0 (and reintegrating the values DD_1 and Q_0 larger than $XQ^{-2+\varepsilon}$ which were discarded earlier), is given by

$$\mathcal{M}_1 := \frac{M}{b}\hat{f}(0)\sum\sum\sum\sum_{\substack{w_1,w_2,n_1,n_2\\ (n_j,bw_j)=1\\ n_1\equiv n_2\,(\mathrm{mod}\, b(w_1,w_2))}} \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\beta_{n_1}\overline{\beta_{n_2}}\frac{1}{[w_1,w_2]}.$$

The error term in (32) induces in $\mathcal{D}_1(d,d_1,q_0)$ a contribution

$$\ll Q^{-10}N^2,$$

and therefore in $\mathcal{D}_1$ a contribution $O(1)$, which is acceptable.

We solve the congruence conditions on μ by writing

$$d_1\nu_1-\nu_2 = bq_0t, \quad \mu dd_1\nu_1 = 1+bq_0q_1\ell, \quad \mu d\nu_2 = 1+bq_0q_2m,$$

with $t, \ell, m \in \mathbb{Z}$. We deduce

$$\mu d t = q_1 \ell - q_2 m, \quad t = q_1 \nu_2 \ell - q_2 d_1 \nu_1 m.$$

Then we have the equalities, modulo $\mathbb{Z}$,

$$\begin{aligned}
\frac{\mu}{\tilde{q}} = \frac{\mu}{b q_0 q_1 q_2} &= \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{\ell}{d d_1 \nu_1 q_2} \\
&\equiv \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{\ell \overline{d d_1}}{\nu_1 q_2} + \frac{\ell \overline{\nu_1 q_2}}{d d_1} \\
&\equiv \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{t \overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{\overline{b q_0 q_1 \nu_1 q_2}}{d d_1} \\
&\equiv \frac{1}{d d_1 \nu_1 b q_0 q_1 q_2} + \frac{d_1 \nu_1 - \nu_2}{b q_0} \frac{\overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{\overline{b q_0 q_1 \nu_1 q_2}}{d d_1}.
\end{aligned}$$

By estimating trivially the first term, we have

$$\mathrm{e}\left(\frac{h\mu}{\tilde{q}}\right) = \mathrm{e}\left(h \frac{d_1 \nu_1 - \nu_2}{b q_0} \frac{\overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{h \overline{b q_0 q_1 \nu_1 q_2}}{d d_1}\right) + O\left(\frac{H q_0}{N W^2}\right). \tag{33}$$

The error term here is $\ll Q^{\varepsilon} X^{-1}$, which contributes to $\mathcal{D}_1(d, d_1, q_0)$ a quantity

$$\frac{Q^{2+\varepsilon} N}{X q_0^2 d d_1} \left(1 + \frac{N}{d}\right),$$

and upon summing over (d, d_1, q_0), this contributes to $\mathcal{D}_1$ a quantity $O(Q^{2+\varepsilon} N^2 X^{-1})$. This error is acceptable if

$$N \ll Q^{2-\varepsilon}. \tag{34}$$

Then we insert the first term of (33) in (32), and insert the Fourier integral. The nonzero frequencies contribute a term

$$\begin{aligned}
\mathcal{R}_1(d, d_1, q_0) := \frac{M q_0}{b W^2} \int \sum_{\substack{q_1, q_2, \nu_1, \nu_2 \\ (d\nu_1, \nu_2) = (q_1, q_2) = 1 \\ (q_1 q_2, d) = (\nu_1, d) = 1 \\ (\nu_1, q_1) = (\nu_2, q_2) = (\nu_1 \nu_2, b q_0) = 1 \\ d_1 \nu_1 \equiv \nu_2 \, (\mathrm{mod}\, b q_0)}} \sum\sum\sum \sum_{0 < |h| \le H} \Psi\left(\frac{q_0 q_1}{W}\right) \Psi\left(\frac{q_0 q_2}{W}\right) \beta_{d d_1 \nu_1} \overline{\beta_{d \nu_2}} \\
\times f\left(t \frac{q_0^2 q_1 q_2}{W^2}\right) \mathrm{e}\left(h \frac{d_1 \nu_1 - \nu_2}{b q_0} \frac{\overline{q_1 \nu_2 d d_1}}{\nu_1 q_2} - \frac{h \overline{b q_0 q_1 \nu_1 q_2}}{d d_1}\right) \mathrm{e}\left(\frac{-h t M q_0}{b W^2}\right) \mathrm{d}t.
\end{aligned}$$

So far, we have obtained under the conditions (31) and (34) the bound

$$\mathcal{D}_1 = \mathcal{M}_1 + \mathcal{R}_1 + O(N Q^{2-\varepsilon}), \quad \text{where } \mathcal{R}_1 := \sum_{\substack{Q_0, D D 1 \ll X Q^{-2+\varepsilon} \\ Q, D, D_1 \text{ dyadic}}} \sum_{\substack{d \asymp D \\ d_1 \asymp D_1 \\ q_0 \asymp Q_0}} \mathcal{R}(d, d_1, q_0).$$

We now restrict the summation over q_1 and q_2 in residue classes modulo dd_1, to account for the oscillatory factors. Let $\lambda_1, \lambda_2 \in (\mathbb{Z}/dd_1\mathbb{Z})^\times$, and

$$b_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}} = \sum_{\substack{\nu_1 \\ \nu_1=\boldsymbol{s}}}\sum_{\substack{\nu_2 \\ \nu_2 dd_1=\boldsymbol{r} \\ (d\nu_1,\nu_2)=(\nu_1\nu_2,bq_0)=1 \\ (\nu_1,d)=1 \\ d_1\nu_1\equiv\nu_2 \ (\mathrm{mod}\, bq_0)}} \sum_{\substack{0<|h|\le H \\ h(d_1\nu_1-\nu_2)=bq_0\boldsymbol{n}}} \beta_{dd_1\nu_1}\overline{\beta_{d\nu_2}}\,\mathrm{e}\Big(-\frac{h\overline{bq_0\lambda_1\nu_1\lambda_2}}{dd_1} - \frac{htMq_0}{bW^2}\Big),$$

$$g(\boldsymbol{c},\boldsymbol{d},\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}) = \Psi\Big(\frac{q_0\boldsymbol{c}}{W}\Big)\Psi\Big(\frac{q_0\boldsymbol{d}}{W}\Big) f\Big(\frac{tq_0^2\boldsymbol{cd}}{W^2}\Big).$$

Then

$$\mathcal{R}_1(d,d_1,q_0) = \frac{Mq_0}{bW^2}\int_{t\asymp_f 1} \sum_{\lambda_1,\lambda_2 \ (\mathrm{mod}\, dd_1)^*} \tilde{\mathcal{R}}_1(t,(\lambda_j))\,\mathrm{d}t,$$

$$\tilde{\mathcal{R}}_1(t,(\lambda_j)) = \sum_{\substack{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s},\boldsymbol{c},\boldsymbol{d} \\ \boldsymbol{c}\equiv\lambda_1,\boldsymbol{d}\equiv\lambda_2 \ (\mathrm{mod}\, dd_1) \\ (\boldsymbol{sc},\boldsymbol{rd}bdd_1)=1}} b_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}}\, g(\boldsymbol{c},\boldsymbol{d},\boldsymbol{n},\boldsymbol{r},\boldsymbol{s})\,\mathrm{e}\Big(\frac{\boldsymbol{n}\boldsymbol{r}\overline{\boldsymbol{d}}}{\boldsymbol{sc}}\Big).$$

We apply Proposition 6, with sizes given by

$$\boldsymbol{C} = \boldsymbol{D} = \frac{W}{q_0}, \quad \boldsymbol{S} = \frac{N}{dd_1}, \quad \boldsymbol{R} = Nd_1, \quad \boldsymbol{N} = \frac{HN}{dbq_0}.$$

Let $X = Q^2 Y$. Then $Y = Q^\varpi$. Note that

$$\boldsymbol{RS} \asymp N^2 D^{-1}, \quad \boldsymbol{N} \ll Q^\varepsilon N^2 Y^{-1} D^{-1} Q_0^{-2} \ll Q^\varepsilon \boldsymbol{RS}, \quad \boldsymbol{C} \ll Q^\varepsilon \boldsymbol{RD}.$$

We get

$$\tilde{\mathcal{R}}_1(t,\lambda_j) \ll Q^\varepsilon (DD_1)^{\frac{3}{2}} K \|b_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2,$$

where

$$Q^{-\varepsilon}K^2 \ll Q^2N^4D^{-1}D_1Q_0^{-2} + Q^{2+4\theta}N^{4-6\theta}D^{-2+2\theta}D_1^{-2\theta}Q_0^{-2-4\theta}\Big(1+\frac{D}{N}\Big)^{1-4\theta} + Q^2N^3Y^{-1}D^{-1}D_1Q_0^{-4}.$$

To bound the term $\|b_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2$, we assume

$$XQ^{-2+\varepsilon} = o(N), \tag{35}$$

so that $D = o(N)$ by virtue of the line below (30), and the case $d_1\nu_1 = \nu_2$ never occurs in $b_{\boldsymbol{n},\boldsymbol{r},\boldsymbol{s}}$. Then

$$\|b_{\boldsymbol{N},\boldsymbol{R},\boldsymbol{S}}\|_2^2 \le \sum_{\substack{\nu_1,\nu_2,h \\ d_1\nu_1\equiv\nu_2 \ (\mathrm{mod}\, q_0) \\ 0<|h|<H}} |\beta_{dd_1\nu_1}\beta_{d\nu_2}|^2 \ll \frac{Q^{2+\varepsilon}}{Q_0 M}\frac{N}{DD_1}\Big(\frac{N}{DQ_0}+1\Big)$$
$$\ll Q^\varepsilon(N^3Y^{-1}D^{-2}D_1^{-1}Q_0^{-2} + N^2Y^{-1}D^{-1}D_1^{-1}Q_0^{-1}).$$

We deduce

$$\tilde{\mathcal{R}}_1(t,(\lambda_j)) \ll Q^{\varepsilon} \sum_{k=1}^{6} Q^{\eta_{k,1}} N^{\eta_{k,2}} Y^{\eta_{k,3}} D^{\eta_{k,4}} D_1^{\eta_{k,5}} Q_0^{\eta_{k,6}},$$

where, for each k, $\eta_k = (\eta_{k,\ell})_{1\le \ell \le 6}$ is given by

$$\{\eta_k\} = \left\{ \begin{pmatrix} 1 \\ 3 \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -2 \end{pmatrix}, \begin{pmatrix} 2\theta+1 \\ 3-3\theta \\ -\frac{1}{2} \\ \theta \\ 1-\theta \\ -2\theta-\frac{3}{2} \end{pmatrix}, \begin{pmatrix} 2\theta+1 \\ \frac{7}{2}-3\theta \\ -\frac{1}{2} \\ \theta-\frac{1}{2} \\ 1-\theta \\ -2\theta-2 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{5}{2} \\ -1 \\ \frac{1}{2} \\ \frac{3}{2} \\ -\frac{5}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \\ \frac{3}{2} \\ -3 \end{pmatrix} \right\}.$$

Summing over λ_j, integrating over t, and multiplying by $Mq_0/(bW^2) \ll Q^{\varepsilon} N^{-1} Y Q_0$, we get

$$\mathcal{R}_1(d,d_1,q_0) \ll Q^{\varepsilon} \sum_{k=1}^{6} Q^{\eta_{k,1}} N^{\eta_{k,2}-1} Y^{\eta_{k,3}+1} D^{\eta_{k,4}+2} D_1^{\eta_{k,5}+2} Q_0^{\eta_{k,6}+1}.$$

We sum over d, d_1 and q_0 in dyadic intervals of lengths D, D_1 and Q_0, obtaining

$$\sum_{\substack{d \asymp D \\ d_1 \asymp D_1,\, d_1 | d^{\infty} \\ q_0 \asymp Q_0 \\ (d,b)=(q_0,d)=1}} \mathcal{R}_1(d,d_1,q_0) \ll Q^{\varepsilon} \sum_{k=1}^{6} Q^{\eta_{k,1}} N^{\eta_{k,2}-1} Y^{\eta_{k,3}+1} D^{\eta_{k,4}+3} D_1^{\eta_{k,5}+2} Q_0^{\eta_{k,6}+2}.$$

Finally, we sum this dyadically over Q_0, D and D_1 subject to $Q_0 + DD_1 \ll YQ^{\varepsilon}$. We get

$$\mathcal{R}_1 \ll Q^{\varepsilon} \sum_{k=1}^{6} Q^{\eta_{k,1}} N^{\eta_{k,2}-1} Y^{\eta_{k,3}+1+\max(0,\eta_{k,6}+2)+\max(0,\eta_{k,4}+3,\eta_{k,5}+2)}.$$

Here, the terms for $k=5,6$ are majorized by the term $k=1$, so

$$\mathcal{R}_1 \ll Q^{\varepsilon} \sum_{k=1}^{4} Q^{\theta_{k,1}} N^{\theta_{k,2}} Y^{\theta_{k,3}},$$

where

$$\{\theta_k\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ \frac{9}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{5}{2} \\ 4 \end{pmatrix}, \begin{pmatrix} 1+2\theta \\ 2-3\theta \\ 4-\theta \end{pmatrix}, \begin{pmatrix} 1+2\theta \\ \frac{5}{2}-3\theta \\ \frac{7}{2}-\theta \end{pmatrix} \right\}.$$

We conclude that

$$\mathcal{D}_1 = \mathcal{M}_1 + O(Q^{2-\varepsilon} N)$$

on the condition that $N \ll Q^{-\varepsilon} \min(QY^{-\frac{9}{2}}, Q^{\frac{2}{3}} Y^{-\frac{8}{3}}, Q^{(1-2\theta)/(1-3\theta)} Y^{-(4-\theta)/(1-3\theta)}, Q^{\frac{2}{3}} Y^{-\frac{1}{3}(7-2\theta)/(1-2\theta)})$.

Upon using $\theta \le \frac{7}{64}$, these conditions are implied by

$$N \ll X^{-\varepsilon} \min(X^{\frac{1}{2}(2-9\varpi)/(2+\varpi)}, X^{\frac{1}{43}(50-249\varpi)/(2+\varpi)}, X^{\frac{1}{75}(50-217\varpi)/(2+\varpi)}), \tag{36}$$

and hypotheses (31), (34) and (35).

4.4.4. *Main terms.* The main terms $\mathcal{M}_1$ and $\mathcal{M}_3$, which are real numbers by the symmetry $n_1 \leftrightarrow n_2$, combine to form

$$\mathcal{M}_1 - \mathcal{M}_3$$
$$= M\hat{f}(0) \sum\sum_{w_1,w_2} \Psi\Big(\frac{w_1}{W}\Big)\Psi\Big(\frac{w_2}{W}\Big)\frac{1}{b[w_1,w_2]\varphi(b(w_1,w_2))} \times \sum_{\substack{\chi \text{ prim} \\ \text{cond}(\chi)>R \\ \text{cond}(\chi)\mid b(w_1,w_2)}} \sum\sum_{\substack{n_1,n_2 \\ (n_j,bw_j)=1}} \beta_{n_1}\overline{\beta_{n_2}}\chi(n_1)\overline{\chi(n_2)}.$$

We may quote the computations in [Drappeau 2017, p. 717], again with the identification (27), to obtain

$$|\mathcal{M}_3 - \mathcal{M}_1| \ll Q^{\varepsilon} M(N + N^2R^{-2}) \ll Q^{\varepsilon}(X + NXR^{-2}).$$

This is acceptable provided

$$N \gg Q^{\varpi+\varepsilon} \quad \text{and} \quad R \gg Q^{\frac{1}{2}\varpi+\varepsilon}. \tag{37}$$

4.4.5. *Conclusion.* Hypotheses (28), (29), (31), (34), (35), (36) and (37) are all satisfied if

$$\varpi < \frac{1}{8}, \quad \varpi < \sigma < \frac{1}{3} - \delta < \frac{1}{3} - \frac{242\varpi}{75(2+\varpi)}, \quad \frac{\varpi}{2(2+\varpi)} < \rho < \frac{1}{9} - \frac{\varpi}{3(2+\varpi)}. \tag{38}$$

We therefore conclude the following.

Lemma 13. *Under the notation and hypotheses of Corollary 10, assuming (38), we have*

$$T_{\mathrm{II}} \ll \sqrt{X}Q^{1-\varepsilon}.$$

4.5. ***Proof of Proposition 4.*** We combine Lemmas 11, 12, 13 and 9. Setting $\sigma = \varpi + \varepsilon$ and recalling that $\varpi < \frac{1}{8}$, we obtain the conditions

$$\frac{\varpi}{3(2+\varpi)} < \lambda < \frac{1}{6} - \frac{\varpi}{2}, \quad \frac{242\varpi}{75(2+\varpi)} < \delta < \frac{1}{12} - \frac{\varpi}{2(2+\varpi)}, \quad \frac{\varpi}{2(2+\varpi)} < \rho < \frac{1}{9} - \frac{\varpi}{3(2+\varpi)}.$$

The third is implied by our hypothesis on R. The first two can be satisfied whenever $-o(1) \le \varpi < \frac{50}{1093} - o(1)$. This proves Proposition 4.

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