Multiplicative preprojective algebras are 2-Calabi–Yau

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Multiplicative preprojective algebras are 2-Calabi–Yau

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We prove that multiplicative preprojective algebras, defined by Crawley-Boevey and Shaw, are 2-Calabi–Yau algebras, in the case of quivers containing unoriented cycles. If the quiver is not itself a cycle, we show that the center is trivial, and hence the Calabi–Yau structure is unique. If the quiver is a cycle, we show that the algebra is a noncommutative crepant resolution of its center, the ring of functions on the corresponding multiplicative quiver variety with a type A surface singularity. We also prove that the dg versions of these algebras (arising as certain Fukaya categories) are formal. We conjecture that the same properties hold for all non-Dynkin quivers, with respect to any extended Dynkin subquiver (note that the cycle is the type A case). Finally, we prove that multiplicative quiver varieties — for all quivers — are formally locally isomorphic to ordinary quiver varieties. In particular, they are all symplectic singularities (which implies they are normal and have rational Gorenstein singularities). This includes character varieties of Riemann surfaces with punctures and monodromy conditions. We deduce this from a more general statement about 2-Calabi–Yau algebras (following Bocklandt, Galluzzi, and Vaccarino).

1. Introduction

Multiplicative preprojective algebras have recently gained attention in geometry and topology. These algebras appear in the study of certain wrapped Fukaya categories [29; 30], in the study of microlocal sheaves on rational curves [12], and in the study of generalized affine Hecke algebras [33, Appendix 1]. Their moduli spaces of representations are called multiplicative quiver varieties, and are analogs of Nakajima's quiver varieties. These include character varieties of rank n local systems on closed Riemann surfaces, or on open Riemann surfaces with punctures and monodromy conditions [25; 55; 61]. Multiplicative quiver

varieties have also been studied from various viewpoints in [9; 13; 20; 22]. A quantization was defined in [39] and further studied in [35].

Historically, Crawley-Boevey and Shaw [25] defined the multiplicative preprojective algebra to view solutions of the Deligne–Simpson problem as irreducible representations of multiplicative preprojective algebras of certain star-shaped quivers. Their paper establishes the foundations for much of this work. For a fixed field k and a quiver Q with vertex set Q_0 and arrow set Q_1 and $q \in (k^\times)^{Q_0}$, Crawley-Boevey and Shaw define

$$\Lambda^q(Q) := \frac{L_Q}{J_Q} := \frac{k\overline{Q}[(1+aa^*)^{-1}]_{a\in\overline{Q}}}{\left(r := \prod_{a\in Q_1}(1+aa^*)(1+a^*a)^{-1} - q\right)},$$

a quotient of the localized path algebra of the double quiver, L_Q, by the two-sided ideal J_Q generated by the single relation, r.

Many of the desirable properties of the (additive) preprojective algebra seem to hold for the multiplicative preprojective algebra. But establishing this rigorously is difficult, as most proof techniques in the additive case (employing the grading on the algebra) are not available in the multiplicative case. In particular, the multiplicative preprojective algebra is not in general a deformation of the ordinary one, nor does it have a useful Hilbert series for a filtration (due to the localization).

In this paper we overcome these difficulties when the quiver contains a cycle, and formulate the general expectations. This is sufficient for applications to multiplicative quiver varieties for *every* quiver.

The main statement is the following:

Conjecture 1.1. $\Lambda^q(Q)$ *is 2-Calabi–Yau for all* $q \in (k^\times)^{Q_0}$ *and all* Q *connected and not Dynkin; moreover, it is a prime ring, and the family* $\Lambda^q(Q)$ *is flat in* q.[1] *If* Q *is furthermore not extended Dynkin, then* $Z(\Lambda^q(Q)) = k$, *and the Calabi–Yau structure is unique.*

Here (extended) Dynkin refers to the underlying unoriented graph being of types A, D, or E. We explain how one can reduce the conjecture to the case where Q is extended Dynkin in Section 7D. We carry out this procedure for $Q = \widetilde{A}_n$ and thereby prove the conjecture for all connected quivers containing it.

Theorem 1.2. $\Lambda^q(Q)$ *is 2-Calabi–Yau and prime for any* $q \in (k^\times)^{Q_0}$ *and any* k *a field, and* Q *connected and containing an unoriented cycle. The family of algebras is flat in* q. *If the quiver properly contains a cycle, then* $Z(\Lambda^q(Q)) = k$, *and the Calabi–Yau structure is unique.*

This theorem is established in Corollaries 3.20 and 7.12, and the results of Section 8: Propositions 8.5 and 8.6, and Corollary 8.7. Each relies on technical results proven in Section 7. Before outlining the proof techniques, we give four different perspectives on this work:

(I) *Symplectic topology: wrapped Fukaya categories.* Multiplicative preprojective algebras arise from studying certain wrapped Fukaya categories. Let X_Γ be the Weinstein manifold formed by plumbing cotangent bundles of 2-spheres according to the graph Γ. Ekholm and Lekili [28] and Etgü and Lekili

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[1]A prime ring is a noncommutative analog of an integral domain, being a ring R in which $aRb = 0$ implies $a = 0$ or $b = 0$.

[29; 30] produced quasiisomorphisms,

$$\mathcal{W}(X_\Gamma) \xrightarrow[\cong]{[28]} \mathscr{B}_\Gamma \xrightarrow[\cong]{[29; 30]} \mathscr{L}_\Gamma$$

where $\mathcal{W}(X_\Gamma)$ is the partially wrapped Fukaya category of X_Γ, $\mathscr{B}_\Gamma$ is the Chekanov–Eliashberg dg-algebra and $\mathscr{L}_\Gamma$ is the dg multiplicative preprojective algebra following [36], with $q = 1$; see Definition 4.3.

Since X_Γ is a Liouville manifold, [34, Theorem 1.3] shows that $\mathcal{W}(X_\Gamma)$ is a 2-Calabi–Yau category and hence $\mathscr{L}_\Gamma$ is 2-Calabi–Yau, as a dg-algebra. We establish this result purely algebraically, in the case Γ contains a cycle. In particular, we show in this case that

$$\Lambda^1(\Gamma) = H^0(\mathscr{L}_\Gamma) \xleftarrow[=]{\text{Proposition 4.4}} H^*(\mathscr{L}_\Gamma)$$

and hence $\mathscr{L}_\Gamma$ is formal. By Theorem 1.2, the dg multiplicative preprojective algebra $\mathscr{L}_\Gamma$ is formal.[2]

Consequently, deformations of the wrapped Fukaya category, $\mathcal{W}(X_\Gamma)$, as an A_∞-category (respectively Calabi–Yau A_∞-category) over a degree zero base, are given by deformations of $\Lambda^1(\Gamma)$ as an associative algebra (respectively Calabi–Yau algebra). The infinitesimal deformations can be identified with $HH^2(\Lambda^1(\Gamma))$. Thanks to Theorem 1.2, $\Lambda^1(\Gamma)$ is 2-Calabi–Yau. Hence, Van den Bergh duality identifies $HH^2(\Lambda^1(\Gamma))$ with $HH_0(\Lambda^1(\Gamma))$. The techniques in [54] can likely be adapted to compute the latter using the explicit basis for $\Lambda^q(Q)$ computed here. Furthermore, by the 2-Calabi–Yau property, $HH^3(\Lambda^q(Q)) = 0$, so there are no obstructions to extending to infinite order deformations.

We conjecture that the same holds for every connected, non-Dynkin quiver. More precisely, in addition to Conjecture 1.1, we expect the following:

Conjecture 1.3. *If Q is connected and not Dynkin, then the dg multiplicative preprojective algebra $\Lambda^{dg,q}(Q)$ is quasiisomorphic to $\Lambda^q(Q)$, in degree zero.*

We give the precise definitions and details, as well as proof in the case Q contains a cycle, in Section 4.

(II) Quiver varieties: local structure of multiplicative quiver varieties and moduli spaces attached to 2-Calabi–Yau algebras. Given a dimension vector $\alpha \in \mathbb{N}^{Q_0}$, the affine multiplicative quiver variety is defined as the (coarse) moduli scheme of representations of $\Lambda^q(Q)$. Explicitly, it is the geometric invariant theory quotient of the space of all representations

$$\Lambda^q(Q) \to \bigoplus_i \text{Mat}_{\alpha_i}(k)$$

by the action of $\prod_i \text{GL}(\alpha_i)$ by change of basis. See Section 5 for more details (where we also recall a version incorporating a stability condition).

Properties of multiplicative preprojective algebras determine properties of the corresponding multiplicative quiver varieties. For instance, in Section 7.5 of [55], Tirelli and the second author observe, following [14], that the 2-Calabi–Yau property determines the (formal) local structure of the moduli space of representations. Namely, any formal neighborhood can be identified with the formal neighborhood of

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[2]Additionally, since submission of this article, the dg multiplicative preprojective algebra was shown to be 2-Calabi–Yau for all q and all Γ in [15].

the zero representation of the moduli space of representations of some (additive) preprojective algebra. This is proved in more detail here, in Theorem 5.3 (expanding on [14, Section 6], where a similar result is given). Among other applications mentioned in [55], it follows that, when k has characteristic zero, the corresponding multiplicative quiver varieties are normal and are symplectic singularities in the sense of Beauville [3] (in particular, they are normal and have rational Gorenstein singularities); see Corollary 5.5.

This includes (as an open subset) character varieties of Riemann surfaces of positive genus with punctures and prescribed monodromy conditions, as explained in [55, Section 3] (following [25; 61]). (In the case of closed Riemann surfaces, as pointed out in [4], this statement does not require our result, since the group algebra $k[\pi_1(\Sigma)]$ is well-known to be 2-Calabi–Yau.)

One subtle point is that we can describe the local structure of multiplicative quiver varieties for *all* quivers despite the fact that we only prove the 2-Calabi–Yau property for quivers containing a cycle (Theorem 5.4). The key idea is that any quiver can be embedded into a quiver containing a loop and hence any representation of a quiver can be viewed as a representation of a quiver with a cycle. Therefore, its formal neighborhood can be identified with a formal neighborhood of the zero representation of an (additive) preprojective algebra. For detailed definitions, statements, and proofs see Section 5.

(III) Noncommutative algebraic geometry: noncommutative resolutions. Although in the non-Dynkin, nonextended Dynkin case, the center is expected to be trivial (Conjecture 1.1, proved when the quiver contains a cycle), this is far from true in the extended Dynkin case. Indeed, ordinary preprojective algebras of extended Dynkin quivers have a large center, the spectrum of which is a du Val singularity. The algebra itself is a noncommutative crepant resolution of this center. Moreover, this center is the algebra of functions on a natural quiver variety. So it is reasonable to ask if multiplicative preprojective algebras also resolve the corresponding multiplicative quiver variety.

In Shaw's thesis [57], he makes progress towards this question by showing that, for an extended Dynkin quiver Q with extended vertex v, the subalgebra $e_v \Lambda^1(Q) e_v$ is commutative of dimension 2, with a unique singularity at the origin; he expects that (for k of characteristic zero) the singularity there has the corresponding du Val type (see Remark 6.5).

In further analogy to the additive case, it is reasonable to pose the following conjecture:

Conjecture 1.4. *Let Q be extended Dynkin. The algebra $\Lambda^1(Q)$ is a 2-dimensional noncommutative crepant resolution (NCCR) of its center, which is the ring of functions on the associated multiplicative quiver variety $\mathcal{M}_{1,0}(Q, \delta)$. The Satake map $Z(\Lambda^1(Q)) \to e_v \Lambda^1(Q) e_v$ defined by $z \mapsto e_v z$ is an isomorphism.*[3]

See Section 5 for the precise definition of the multiplicative quiver variety. Thanks to our aforementioned results on its local structure, the conjecture implies Shaw's expectation that the singularity of $e_v \Lambda^1(Q) e_v$ is du Val of the corresponding type.

For $Q = \widetilde{A}_n$, we prove the conjecture in Section 6B. In the process, we obtain an explicit description

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[3]We use the terminology "Satake" following the analogous one for symplectic reflection algebras at $t = 0$ of Etingof–Ginzburg, itself coming from the map for affine Hecke algebras proposed by Lusztig.

of the center, $Z(\Lambda^1(\widetilde{A}_n))$, which may be of independent interest.

(IV) Representation theory: Kontsevich–Rosenberg principle. A final perspective on this work involves the Kontsevich–Rosenberg Principle which says: a noncommutative geometric structure on an associative algebra A should induce a geometric structure on the representation spaces $\mathrm{Rep}_n(A)$, for all $n \geq 1$. This principle needs adjusting for structures living in the *derived* category of A-modules, as the representation functor is not exact. For a d-Calabi–Yau structure on A, it is shown in [17] and [62] that the derived moduli stack of perfect complexes of A-modules, $\mathbb{R}\,\mathrm{Perf}(A)$, has a canonical $(2-d)$-*shifted* symplectic structure. Since the dg multiplicative preprojective algebra is 2-Calabi–Yau, this implies that its moduli stack of representations has a 0-shifted symplectic structure. By Conjecture 1.3, it is the same as the moduli stack of representations of $\Lambda^q(Q)$ itself. Note that the multiplicative quiver variety can be viewed as a coarse moduli space of semistable representations; so the aforementioned result that this variety locally has the structure of an ordinary quiver variety is a singular analog of the statement on the moduli stack.

We now give a brief overview of the proof of Conjectures 1.1 and 1.3 for quivers containing a cycle. We prove Theorem 1.2 using a complex

$$P_\bullet := \Lambda^q(Q) \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda^q(Q) \xrightarrow{\alpha} \Lambda^q(Q) \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda^q(Q) \xrightarrow{\beta} \Lambda^q(Q) \otimes_{kQ_0} \Lambda^q(Q)$$

defined originally in [25] (following [56, Theorem 10.3] and [26, Corollary 2.11]) and shown to resolve $\Lambda^q(Q)$, *except* for the injectivity of the map α. We show α is injective and then show the dual complex $P_\bullet^\vee$ is a resolution of $\Lambda^q(Q)[-2]$, which implies $\Lambda^q(Q)$ is 2-Calabi–Yau.

First, we establish a chain of implications to reduce the proof to a presentation of the localization L_Q that we call the *strong free product property*, established in Theorem 3.7, see Definition 3.5 or see below for a rough definition. The strong free product property is a version of Anick's weak summand property in the ungraded case; see [1].

To prove the 2-Calabi–Yau property from the strong free product property we show these implications:

Strong free product property for Q :
$$\exists \sigma' : \Lambda^q(Q) *_{kQ_0} kQ_0[t, (q+t)^{-1}] \to L_Q \text{ a linear isomorphism}$$

$\Downarrow$ Section 3A

Weak free product property for Q : $\mathrm{gr}(\sigma') : \Lambda^q(Q) *_{kQ_0} kQ_0[t] \to \mathrm{gr}(L_Q)$ is an algebra isomorphism

$\Downarrow$ Proposition 3.12

$\mathrm{gr}(\sigma')_1 : \Lambda^q(Q) \otimes_{kQ_0} kQ_0[t] \otimes_{kQ_0} \Lambda^q(Q) \to J_Q/J_Q^2$ is an isomorphism of $\Lambda^q(Q)$-bimodules

$\Downarrow$ Propositions 3.11 and 3.12

$P_\bullet$ is a length two projective $\Lambda^q(Q)$-bimodule resolution of $\Lambda^q(Q)$

$\Downarrow$ Theorem 3.17

$\Lambda^q(Q)$ is 2-Calabi–Yau.

Here the isomorphism σ' is determined by a choice of kQ_0-bimodule section $\Lambda^q(Q) \to L_Q$ of the quotient map $L_Q \twoheadrightarrow \Lambda^q(Q)$, but $\mathrm{gr}(\sigma')$ is independent of this choice. The element t maps to the relation, and the filtrations used are the t-adic one on the source and the J_Q-adic filtration on L_Q.

In Section 4, we show that the strong free product property implies that the dg multiplicative preprojective algebra $\Lambda^{dg,q}(Q)$ is formal. Therefore, by the results of this paper, both Conjectures 1.1 and 1.3 would follow from the following more general statement (see Section 7 for precise details).

Conjecture 1.5. *If Q is a connected, non-Dynkin quiver, then σ' as above is a linear isomorphism:* $(L_Q, r, \sigma, kQ_0[t, (t+q)^{-1}])$ *satisfies the strong free product property.*

Proposition 7.11 proves the conjecture for quivers containing a cycle. This is the technical heart of the paper. Our main technique involves reduction systems over the localized ring $kQ_0[t, (t+q)^{-1}]$. Using the diamond lemma [11], we show these give unique reductions of elements of L_Q to basis elements of the given free product. As a consequence, $\Lambda^q(Q)$ itself obtains the module structure of a free product of the cycle part and the rest of the quiver; see (7-12) for a precise statement.

Remark 1.6. After submission of this article Crawley-Boevey and Yuta Kimura [24, Theorem 1.1] proved that a related, more well-studied algebra, the *deformed preprojective algebra* [23], is 2-Calabi–Yau, in the case that the quiver is connected and non-Dynkin. This algebra is a deformation of the usual (additive) preprojective algebra, given as the quotient of the path algebra of the double quiver by the single relation $\sum_{a \in Q_1} aa^* - a^*a - \sum_{i \in Q_0} \lambda_i e_i$, the case $\lambda_i = 0$ returning the original preprojective algebra. This can also be proved via the techniques of this article, by deducing the strong free product property from the known one for the additive preprojective algebra for non-Dynkin quivers.

Namely, the latter are noncommutative complete intersections [1; 32], shown in [54, Proposition 5.2.1] to be equivalent, in the context of graded algebras, to the (strong or weak) free product property. More generally, let $A = TV/(r)$ be a graded algebra satisfying the free product property. Briefly, this means that we have a section $\sigma : A \to TV$, which we can take to be graded, so that the induced linear map $\sigma' : A *_k k[t] \to TV$ sending t to r is a linear isomorphism. Then for every $\lambda \in k$, σ' also defines a linear isomorphism by the same formula except sending t to $r + \lambda$. This is because, taking a homogeneous basis of A with degree nondecreasing, we obtain a homogeneous basis of TV via the free product, and the substitution $r \mapsto r + \lambda$ is a strictly triangular change of basis. Thus, the algebra $TV/(r+\lambda)$ also satisfies the strong free product property. Note that the same argument given here applies if we replace r by any filtered deformation $r + r'$, with r' in degrees strictly lower than r. They also apply to the quiver context, replacing k by kQ_0, hence imply that the deformed preprojective algebra satisfies the strong free product property. It seems likely this argument can apply to many other interesting algebras.

An outline of the paper is as follows: In Section 2, we give elementary background information on multiplicative preprojective algebras and produce an alternative generating set crucial for our approach to the 2-Calabi–Yau property. In Section 3, we prove the 2-Calabi–Yau property for $\Lambda^q(Q)$ assuming the strong free product property. In Section 4, using the strong free product property, we show that the dg multiplicative preprojective algebra has homology $\Lambda^q(Q)$, concentrated in degree zero. In Section 5, we

use the 2-Calabi–Yau property to describe the formal neighborhoods of multiplicative quiver varieties as formal neighborhoods of the zero representation in certain quiver varieties. In Section 6 we use the 2-Calabi–Yau and prime properties in the cycle case, together with work of Shaw, to show the multiplicative preprojective algebra is a noncommutative resolution over its center. In Section 7, we prove the strong free product property first for multiplicative preprojective algebras of cycles, then for partial multiplicative preprojective algebras. Putting the two together, we deduce the strong free product property for connected quivers containing cycles. The key point of the argument, of independent interest, is a construction of bases of these algebras. Finally, in Section 8, we establish the prime property of $\Lambda^q(Q)$ using our explicit bases. We furthermore show that $Z(\Lambda^q(Q)) = k$ for Q connected and properly containing a cycle. This shows that the Calabi–Yau structure in these cases are unique, up to scaling.

2. The multiplicative preprojective algebra

2A. *Definitions.* Throughout the paper we fix an arbitrary field k. For each quiver (i.e., directed graph) Q, let Q_0 be the vertex set, Q_1 be the arrow set, and $h, t : Q_1 \to Q_0$ the head and tail maps, respectively. We will assume that Q_0 and Q_1 are finite for convenience, but really only need finitely many arrows incident to each vertex.

Let Q^{op} denote the quiver with the same underlying graph of vertices and edges, but with every arrow in the opposite direction. $\bar{Q}$ denotes the quiver with the same vertex set as Q and Q^{op} and with arrow set $Q_1 \sqcup Q_1^{\mathrm{op}}$. For each arrow $a \in Q_1$, we write a^* for the corresponding arrow in Q_1^{op}, and vice versa. In $\bar{Q}$ we distinguish between arrows in Q and Q^{op} using a function

$$\epsilon : \bar{Q}_1 \to \{\pm 1\}, \quad \epsilon(a) := \begin{cases} 1 & \text{if } a \in Q_1, \\ -1 & \text{if } a \in Q_1^{\mathrm{op}}. \end{cases}$$

For a quiver Q, we denote the path algebra by kQ and follow the convention that paths are concatenated from left to right. We have an inclusion $e_{(-)} : Q_0 \to kQ$ in order to view a vertex $i \in Q_0$ as a length zero path e_i.

For $a \in \bar{Q}_1$, define $g_a := 1 + aa^* \in k\bar{Q}$. Consider the localization $L_Q := k\bar{Q}[g_a^{-1}]_{a \in \bar{Q}_1}$. We write $L := L_Q$, when the quiver is clear from context. Notice, for all $a \in \bar{Q}_1$,

$$g_a a = a + aa^* a = a g_{a^*}. \tag{2-1}$$

This implies

$$g_{a^*} a^* = a^* g_a, \quad g_a^{-1} a = a g_{a^*}^{-1}, \quad g_{a^*}^{-1} a^* = a^* g_a^{-1}.$$

Fixing a total ordering $\leq$ on the set of arrows $\bar{Q}_1$, one can make sense of a product over (subsets of) the arrow set. Using $\leq$ and ϵ we define

$$\rho := \prod_{a \in \bar{Q}_1} g_a^{\epsilon(a)}, \quad l_a := \prod_{b \in \bar{Q}_1, b < a} g_b^{\epsilon(b)}, \quad r_a := \prod_{b \in \bar{Q}_1, b > a} g_b^{\epsilon(b)}.$$

When we need to make the role of the total ordering $\le$ more explicit, we write $\rho_\le$ (respectively $l_{a,\le}$ and $r_{a,\le}$) for ρ (respectively l_a and r_a). By definition, l_a and r_a are the subproduct of ρ to the left and right of a, respectively. Therefore,

$$\rho = l_a g_a^{\epsilon(a)} r_a \tag{2-2}$$

for all $a \in \bar{Q}_1$.

Definition 2.1. Fix a quiver Q and $q \in (k^\times)^{Q_0}$. Consider $\bar{Q}$ and fix an ordering $\le$ on the arrows and a map ϵ as defined above. The *multiplicative preprojective algebra*, $\Lambda^q(Q)$, is defined to be

$$\Lambda^q(Q) := L/J$$

where $L = k\bar{Q}[g_a^{-1}]_{a \in \bar{Q}_1}$ is the localization and J is the two-sided ideal generated by the element $\rho - q$.

Note that q is viewed as an element of $k\bar{Q}$ via $\sum_{i \in Q_0} q_i e_i \in kQ_0 \subset k\bar{Q}$, and as ρ is invertible we need $q_i \ne 0$ so $e_i \Lambda^q(Q) \ne 0$, for all i.

Remark 2.2. The isomorphism class of $\Lambda^q(Q)$ is independent of both the orientation of the quiver and the choice of an ordering on the arrows, by Section 2 in [25].

In the multiplicative preprojective algebra, (2-2) becomes the identity

$$l_a g_a^{\epsilon(a)} r_a = q.$$

Hence

$$r_a l_a = q g_a^{-\epsilon(a)}. \tag{2-3}$$

As mentioned in the introduction, we say that a quiver is (extended) Dynkin, we mean that the underlying unoriented graph is an (extended) type ADE Dynkin diagram. We don't consider nonsimply laced types because, given a quiver, the associated Cartan matrix is $2I - A$ where A is the adjacency matrix of the underlying unoriented graph, which is symmetric.

Example 2.3 (Dynkin case). Let Q be a Dynkin quiver and let R be a commutative ring. Note that the definitions of (multiplicative) preprojective algebra make sense over R. In [42, Section 5], the first named author constructed explicit isomorphisms

$$\Lambda^1(Q) \cong \Pi^0(Q) := R\bar{Q} \Big/ \Big(\sum_{a \in Q_1} [a, a^*] \Big)$$

if 2, 3, and 5 are invertible in R; see also the earlier works [57, Lemma 5.2.1], [22, Corollary 1], [29, Theorem 13], and [47, Section 5]. In particular, for a field k of characteristic zero, we can work over $k[[\hbar]]$ and set $q = e^\hbar$. Then Λ^q is a formal deformation of Λ^1. Hence, by [31, Proposition 5.0.2], there exists some $\lambda \in k[[\hbar]]$ such that there is a $k[[\hbar]]$-linear algebra isomorphism

$$\Lambda^q(Q) \cong \Pi^\lambda(Q) := k\bar{Q} \Big/ \Big(\sum_{a \in Q_1} [a, a^*] - \sum_{i \in Q_0} \lambda_i e_i \Big).$$

In A types, such an isomorphism holds over any field k and for any actual parameter q. Namely, identifying $(A_n)_0 = \{1, 2, \ldots, n\}$ and $(A_n)_1 = \{a_1, a_2, \ldots, a_{n-1}\}$ with tail $t(a_i) = i$, the isomorphism $\Lambda^q(A_n) \cong \Pi^\lambda(A_n)$ is given by

$$e_i \mapsto e_i, \quad a_i \mapsto a_i, \quad a_i^* \mapsto \left(\prod_{j>i} q_j \right) a_i^*,$$

where $\lambda_i := (q_i - 1) \prod_{j>i} q_j$. Since $\Pi^\lambda(A_n)$ is nonzero if and only if there exists i, j with $i < j$ and $\sum_{\ell=i}^{j} \lambda_\ell = 0$, it follows that $\Lambda^q(A_n)$ is nonzero if and only if $\prod_{\ell=i}^{j} q_\ell = 1$.

2B. *The map θ.* In [25], the important map $\theta : \bar{Q}_1 \to \Lambda^q(Q)$ is defined by $\theta(a) = q^{-1} l_a a r_{a^*}$ and extended to $k\bar{Q}$ by the identity on Q_0 and by requiring θ to be an algebra map. Then Lemma 3.3 in [25] shows that

$$\theta(g_a) = l_a g_a l_a^{-1} \tag{2-4}$$
$$= r_a^{-1} g_a r_a, \tag{2-5}$$

so $\theta(g_a)$ is invertible. Hence θ factors through the localization $L := k\bar{Q}[g_a^{-1}]_{a \in \bar{Q}_1}$. We will show θ descends to the quotient $\Lambda^q(Q)$, with the ordering of the arrows reversed, using the following result.

Lemma 2.4. *Let $\leq$ denote a total order on $\bar{Q}_1$ and let $\geq$ denote its opposite ordering, i.e., $a \geq b$ if $b \leq a$. Such an order fixes a bijection $\bar{Q}_1 \cong \{a_1, a_2, \ldots, a_{|\bar{Q}_1|}\}$. Then*

$$\theta(r_{a_j, \geq}) = l_{a_j, \leq} =: l_{a_j} \quad and \quad \theta(l_{a_j, \geq}) = r_{a_j, \leq} =: r_{a_j}$$

for any $a_j \in \bar{Q}_1$.

Proof. We prove $\theta(r_{a_j, \geq}) = l_{a_j}$ by induction on j, where $j = 1$ is the identity $\theta(1) = 1$. Then,

$$\theta(r_{a_{j+1}, \geq}) = \theta(g_{a_j}^{\epsilon(a_j)}) \theta(r_{a_j, \geq}) \stackrel{\text{(IH)}}{=} \theta(g_{a_j}^{\epsilon(a_j)}) l_{a_j} \stackrel{\text{(2-4)}}{=} l_{a_j} g_{a_j}^{\epsilon(a_j)} l_{a_j}^{-1} l_{a_j} = l_{a_j} g_{a_j}^{\epsilon(a_j)} = l_{a_{j+1}}.$$

The second identity is similar and one can formally obtain a proof from the above by exchanging the symbols r and l, the identity (2-4) for (2-5), and the order of the multiplication. $\square$

Corollary 2.5. $$\theta(\rho_\geq) = \rho.$$

This corollary implies θ descends to a map $\Lambda^q(Q, \geq) \to \Lambda^q(Q, \leq)$. Notice that we can similarly define $\theta_\geq : \Lambda^q(Q, \leq) \to \Lambda^q(Q, \geq)$.

Proposition 2.6. $$\theta_\geq \circ \theta = \mathrm{Id}_{\Lambda^q(Q, \leq)}.$$

Proof. It suffices to check $\theta_\geq \circ \theta$ is the identity on arrows in Q_1. We have

$$\theta_\geq(\theta(a)) = \theta_\geq(q^{-1} l_a a r_{a^*}) = q^{-1} \theta_\geq(l_a) \theta_\geq(a) \theta_\geq(r_{a^*}),$$

which by Lemma 2.4 equals

$$= q^{-1} r_a \theta_\geq(a) l_{a^*} = q^{-1} r_a (q^{-1} l_a a r_{a^*}) l_{a^*} \stackrel{\text{(2-3)}}{=} g_a^{-\epsilon(a)} a g_{a^*}^{-\epsilon(a^*)} \stackrel{\text{(2-1)}}{=} a g_{a^*}^{-\epsilon(a)} g_{a^*}^{-\epsilon(a^*)} = a. \qquad \square$$

3. Calabi–Yau and free product properties

The goal of this section is to prove that $\Lambda^q(Q)$ is 2-Calabi–Yau for Q containing an unoriented cycle. We do so by exhibiting a length two, projective, $\Lambda^q(Q)$-bimodule resolution $P_\bullet$ of $\Lambda^q(Q)$, whose bimodule dual complex $P_\bullet^\vee$ is quasiisomorphic to $\Lambda^q(Q)$. This resolution is due to Crawley-Boevey and Shaw, but they don't state nor prove that it is exact. The main new ingredient we provide is the injectivity of α which relies on the weak free product property. Hence we begin this section with a short digression explaining the strong and weak free product properties.

3A. *Free product (complete intersection) properties.* Recall that, if R is a commutative ring and $r \in R$ an element, then there is a dg analog of the quotient $R/(r)$: the Koszul complex $(R[s]/(s^2), d)$ with $d|_R = 0$ and $ds = r$, here $|s| = -1$. Note here that, in spite of the notation, $R[s]/(s^2)$ is the graded-commutative algebra freely generated by R and a single generator s in degree -1. The quotient map $(R[s]/(s^2), d) \to R/(r)$ is a quasiisomorphism if and only if r is a nonzerodivisor.

Thus, in the commutative setting, the nonzerodivisor condition is the correct one for which the Koszul complex (derived imposition of $r = 0$) is equivalent to the quotient algebra.

Now let us pass to the noncommutative setting. If A is an algebra over a ring S and $J = (r)$ an ideal generated by a single relation r, we can form a canonical algebra map,

$$\Phi : A/J *_S S[t] \to \mathrm{gr}_J A, \quad \Phi|_{A/J} = \mathrm{Id}, \Phi(t) = r, \tag{3-1}$$

where gr_J means the associated graded algebra with respect to the J-adic filtration.

Definition 3.1. The pair (A, r) satisfies the *weak free product property* if Φ is an isomorphism.

Remark 3.2. This condition is significantly more subtle than in the commutative case. In particular, it is insufficient for r to be a nonzerodivisor. For example, if $A = k[x]$ and $r = x^2$, then we have $H^1(A * k[s], d) \ni [xs - sx] \neq 0$. (Here A is actually commutative, but we take the noncommutative construction; for a noncommutative example, simply replace A with $k\langle x, y \rangle$.)

The weak free product property is an analog of a *noncommutative complete intersection* (*NCCI*) [32], and closely matches the *weak summand* property from [1] (considered in the graded setting). We have chosen this terminology to make the algebraic property we are using more evident.

When the context is clear, we will sometimes abuse notion and say the quotient A/J, for $J = (r)$, satisfies the weak free product property, even though the choice of A and $r \in A$ is important.

Given an S-bimodule section $\sigma : A/J \to A$ of the quotient map $\pi : A \to A/J$, we can form an associated linear map,

$$\tilde{\sigma} : A/J *_S S[t] \to A, \quad (a_0 t^{m_1} a_1 t^{m_2} \cdots t^{m_n} a_n) \mapsto \sigma(a_0) r^{m_1} \sigma(a_1) r^{m_2} \cdots r^{m_n} \sigma(a_n), \tag{3-2}$$

for $m_i > 0$, for all i. The existence of such a σ (and hence $\tilde{\sigma}$) is automatic if S is separable, as is the case when $S = kQ_0$ below.

By construction, this is $S[t]$-bilinear, where t acts on A by multiplication by r. It also reduces to the identity modulo (t) on the source and J on the target. If (A, r) satisfies the weak free product property, then moreover the completion

$$\widehat{\tilde{\sigma}} : \widehat{A/J *_S S[t]} \to \widehat{A}, \tag{3-3}$$

with respect to the t-adic and J-adic filtrations, is a linear isomorphism.

The goal of the strong free product property is to find a description of A itself as a free product. The first version is the following:

Definition 3.3. The triple (A, r, σ) satisfies the *strong free product property* if $\tilde{\sigma} : A/J *_S S[t] \to A$ is an S-bimodule isomorphism.

Remark 3.4. The choice of σ is important. Let $A = k\langle x, y \rangle$ and $J = (y)$ so $A/J \cong k[x]$. Here $k[t]$ acts on A via $tf := yf$. Consider two different choices

$$\sigma_1, \sigma_2 : k[x] \to k\langle x, y \rangle, \quad \sigma_1(x + (y)) = x, \quad \sigma_2(x + (y)) = x - xy.$$

Then $\tilde{\sigma}_1$ is a linear isomorphism, while $\tilde{\sigma}_2$ is not surjective as

$$x = \sigma_2(x + (y))(1 - y)^{-1} = \sigma_2(x + (y)) \sum_{i \geq 0} y^i \notin \tilde{\sigma}_2(k[x] *_k k[t]).$$

This property is too much to expect in many situations, such as in the presence of rational functions in t. To fix this, let $B = S[t, f^{-1}]$ be a localization of $S[t]$ obtained by inverting some $f \in S^\times + (t)$, such that the map $S[t] \to A$ extends to an algebra map $\tau : B \to A$ (such an extension is necessarily unique). Let $\bar{B} := tB$, so that we have an S-bimodule decomposition $B = S \oplus \bar{B}$. Then $\tilde{\sigma}$ extends to a map $\sigma' : A/J *_S B \to A$, which has the form

$$a_0 b_1 a_1 \cdots b_n a_n \mapsto \sigma(a_0)\tau(b_1) \cdots \tau(b_n)\sigma(a_n), \quad a_i \in A/J, b_i \in \bar{B}. \tag{3-4}$$

Definition 3.5. The quadruple (A, r, σ, B) satisfies the *strong free product property* if σ' is a linear isomorphism.

This definition reduces to the previous definition in the case $B = S[t]$, $\tau(t) = r$.

In this case, it follows by taking associated graded algebras that (A, r) satisfies the weak free product property. Moreover, A is Hausdorff in the J-adic filtration (because the source of σ' is Hausdorff in the t-adic filtration), and σ' is indeed a restriction of $\widehat{\tilde{\sigma}}$.

Remark 3.6. It is important in the definition of σ' to use the natural bimodule complement $\bar{B} = tB$. Here is an example to show why (see also Remark 7.2 for another one, which we naively ran into before realizing our mistake). Let $A = k\langle x, y, z \rangle/(xyz - xz)$, $r = y$, so that $A/J \cong k\langle x, z \rangle/(xz)$. A basis for A/J is given by $\{z^i x^j\}_{i,j \geq 0}$. Let $\sigma : A/J \to A$ be the section preserving this, i.e., $\sigma(z^i x^j + (y, xyz - xz)) = z^i x^j + (xyz - xz)$. Set $B := k[t]$. Then $\sigma' = \tilde{\sigma} : A/J *_k k[t] \to A$ is a linear isomorphism, so (A, r, σ) is a strong free product. However, if we were to instead choose a complement $\bar{B} = (t - 1)B$, then we now have $\sigma'(x(t - 1)z) = 0$, so the map $\sigma' : A/J *_k B \to A$ defined using $\bar{B}$ is not injective. (It is also

not surjective, as xz is not in the image.) On the other hand, for general weak free products, using the correct choice $\bar{B} = tB$, σ' is always injective.

Now we return to the setup of Q a connected, non-Dynkin quiver and $q = (k^\times)^{Q_0}$. Let $B := kQ_0[t, (q + t)^{-1}]$ and $\bar{B} := tB = \mathrm{Span}(t^m, (t')^m \mid m \geq 1)$, for $t' := (q + t)^{-1} - q^{-1}$. We conjecture that, for every such Q, there exists σ such that the quadruple (L_Q, r, σ, B) satisfies the strong free product property. Moreover, in Section 7 we prove this conjecture in the case of quivers containing a cycle.

Theorem 3.7 (Proposition 7.11). *Let Q be a connected quiver containing an unoriented cycle. Let $B = kQ_0[t, (q + t)^{-1}]$ and let r denote the multiplicative preprojective relation. There exists a section σ such that (L_Q, r, σ, B) satisfies the strong free product property.*

The proof of this theorem is technical and uses combinatorial algebraic techniques. Therefore we delay its proof until Section 7, which does not result in circular logic as that section does not depend on results after Section 2.

Remark 3.8. The connectedness assumption can be weakened as follows. If $Q = Q' \sqcup Q''$ then $L_Q = L_{Q'} \oplus L_{Q''}$ and $\Lambda^q(Q) = \Lambda^q(Q') \oplus \Lambda^q(Q'')$ and so by adding sections, the strong free product property for

$$(L_{Q'} \oplus L_{Q''}, r' + r'', \sigma' + \sigma'', B' \oplus B'') = (L_Q, r, \sigma, B)$$

follows from the strong free product properties for $(L_{Q'}, r', \sigma', B')$ and $(L_{Q''}, r'', \sigma'', B'')$. So one only needs the weaker assumption that Q is a quiver with *each* component containing an unoriented cycle. But we state results in the connected setting to simplify the hypotheses.

Corollary 3.9. *Let Q be a connected quiver containing an unoriented cycle. Then $\Lambda^q(Q)$ satisfies the weak free product property. In particular, there exists an isomorphism of graded algebras*

$$\sum_i \varphi_i : \mathrm{gr}(\Lambda^q(Q) *_{kQ_0} kQ_0[t]) \to \mathrm{gr}(L_Q)$$

where the associated graded algebras are taken with respect to the t-adic and J_Q-adic filtrations on $\Lambda^q(Q) *_{kQ_0} kQ_0[t]$ and L_Q respectively.

Remark 3.10. Note that for ordinary preprojective algebras, the free product property was observed in [54, Propositions 5.1.9 and 5.2.1]. In fact, as these algebras are nonnegatively graded with finite-dimensional subspaces in each degree, and one-dimensional in degree zero (connected), the strong and weak free product properties are equivalent (and independent of the choice of graded section σ), as was already observed by Anick [1].[4] Moreover, if A has global dimension at most two, then these conditions are also equivalent to the condition that $A/(r)$ also has global dimension at most two. In the case A is a tensor or path algebra, such algebras were called *noncommutative complete intersections* in [32] due to their close relationship to the condition that representation varieties be complete intersections. For a

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[4]Anick works in the graded context over a field rather than kQ_0, but his results generalize to this setting; see [32; 54].

nuanced discussion of this relationship, including sufficient conditions for the representation variety to be a complete intersection, see the introduction and Theorem 24 in [6].

However, in the ungraded case, we only have the implication that we need, that the free product property implies the existence of a length-two projective bimodule resolution. Indeed, the latter property only depends on a piece of the associated graded algebra with respect to the (r)-adic filtration, and in the ungraded case this filtration need not even be Hausdorff. In contrast, the strong free product property implies the Hausdorff condition and gives information about the algebra itself.

Motivated by this, we believe that the strong free product property can be viewed as an ungraded analog of the noncommutative complete intersection property. It is an interesting question to investigate when their representation varieties are complete intersections.

3B. *A bimodule resolution of Λ.* In this subsection, we show that for *any* quiver, the weak free product property for $\Lambda^q(Q)$ implies $\Lambda^q(Q)$ has a length two projective bimodule resolution. Consequently, since we establish the weak free product property for connected quivers containing a cycle, we prove $\Lambda^q(Q)$ has Hochschild dimension two for connected quivers containing a cycle. For ease of notation, write $\Lambda := \Lambda^q(Q)$.

Crawley-Boevey and Shaw build a chain complex of Λ-bimodules $P_\bullet = P_2 \xrightarrow{\alpha} P_1 \xrightarrow{\beta} P_0$ where,

$$P_2 = P_0 := \Lambda \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda = \langle \eta_v \rangle_{v \in Q_0}, \quad P_1 := \Lambda \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda = \langle \eta_a \rangle_{a \in \overline{Q}_1}$$

and

$$\alpha(\eta_v) := \sum_{a \in \overline{Q}_1 : t(a) = v} l_a \Delta_a r_a \quad \text{where} \quad \Delta_a = \begin{cases} \eta_a a^* + a \eta_{a^*} & \text{if } a \in Q_1, \\ -g_a^{-1}(\eta_a a^* + a \eta_{a^*}) g_a^{-1} & \text{if } a \in Q_1^{\text{op}}, \end{cases}$$

$$\beta(\eta_a) := a \eta_{t(a)} - \eta_{h(a)} a.$$

We claim that it is a resolution of Λ. To see this, following [25], we first write down an explicit chain map of Λ-bimodule complexes $\psi : P_\bullet \to Q_\bullet$, where $Q_\bullet$ is quasiisomorphic to Λ; we then prove it is an isomorphism. $Q_\bullet$ is the cotangent exact sequence in Corollary 2.11 of [26], but in this context it was defined earlier (and shown quasiisomorphic to Λ) by Schofield [56]. So we have the maps

$$P_\bullet \xrightarrow[{[25]}]{\psi} Q_\bullet \xrightarrow[{[56]}]{\text{quasiiso}} \Lambda.$$

Proposition 3.11 [25, Lemma 3.1]. *For **any** quiver Q, the following diagram commutes*:

$$
\begin{array}{ccccccc}
P_0 & \xrightarrow{\alpha} & P_1 & \xrightarrow{\beta} & P_0 & \xrightarrow{\gamma} & \Lambda \\
\downarrow{\psi_2} & & \cong\downarrow{\psi_1} & & \cong\downarrow{\psi_0} & & =\downarrow{\text{id}} \\
J/J^2 & \xrightarrow{\kappa} & \Lambda \otimes_L \Omega_{kQ_0}(L) \otimes_L \Lambda & \xrightarrow{\lambda} & \Lambda \otimes_{kQ_0} \Lambda & \xrightarrow{\mu} & \Lambda
\end{array}
$$

Where the vertical maps are Λ-bimodule maps defined on generators by,

$$\psi_2(\eta_v) := \rho e_v - q e_v, \quad \psi_1(\eta_a) := 1 \otimes_L [a \otimes_{kQ_0} 1 - 1 \otimes_{kQ_0} a] \otimes_L 1, \quad \psi_0(\eta_v) = e_v \otimes e_v.$$

Here α and β are as defined above and $\gamma(\eta_v) := e_v$. The horizontal maps are defined by

$$\kappa(x+J^2) := 1 \otimes_L \delta(x) \otimes 1 \qquad\qquad \text{where } \delta(x) := x \otimes 1 - 1 \otimes x \text{ for } x \in J,$$

$$\lambda(1 \otimes_L [ab \otimes_{kQ_0} c - a \otimes_{kQ_0} bc] \otimes_L 1) := ab \otimes_{kQ_0} c - a \otimes_{kQ_0} bc \quad \text{for } a, b, c \in L,$$

$$\mu(a \otimes b) := ab \qquad\qquad\qquad \text{for } a, b \in \Lambda.$$

Since ψ_0 and ψ_1 are Λ-bimodule isomorphisms, it remains to show ψ_2 is a Λ-bimodule isomorphism. We show this using the weak free product property.

Proposition 3.12. *Suppose Λ satisfies the weak free product property. Then $P_\bullet$ is a bimodule resolution of Λ.*

Proof. Taking the $i = 1$ piece of the graded isomorphism

$$\mathrm{gr}(\varphi) = \sum_i \varphi_i : \mathrm{gr}(\Lambda *_{kQ_0} kQ_0[t, (t+q)^{-1}]) \longrightarrow \mathrm{gr}(L_Q)$$

gives an isomorphism of Λ-bimodules

$$\varphi_1 : \Lambda \otimes_{kQ_0} kQ_0 \cdot t \otimes_{kQ_0} \Lambda \to J_Q / J_Q^2.$$

Since φ_1 sends $t \mapsto r$, it sends $te_v \mapsto re_v = (\rho - q)e_v$ and hence $\varphi_1 = \psi_2$. We conclude that ψ_2 is an isomorphism of Λ-bimodules and hence $\psi_\bullet : P_\bullet \to Q_\bullet$ is an isomorphism of Λ-bimodule complexes. In particular, $P_\bullet$ is a resolution since $Q_\bullet$ is a resolution. $\qquad\square$

For a complex $C_\bullet$ concentrated in nonnegative degrees, define the length by

$$\mathrm{len}(C_\bullet) := \sup\{i \in \mathbb{N} \mid C_i \neq 0\}.$$

For an algebra A, the Hochschild dimension of A is $\mathrm{HH.dim}(A) := \mathrm{len}(\mathrm{HH}_\bullet(A))$ and the global dimension of A, is $\mathrm{gl.dim}(A) := \sup_{M \in A\text{-mod}} \inf_{P_\bullet} \{\mathrm{len}(P_\bullet)\}$ where the infimum is taken over all projective A-module resolutions of M.

Corollary 3.13. *Let Q be a connected quiver containing a cycle. Then*

$$\mathrm{gl.dim}(\Lambda) \leq \mathrm{HH.dim}(\Lambda) = 2.$$

Proof. Use $P_\bullet$ to compute $\mathrm{HH}_i(\Lambda)$; $\mathrm{HH}_i(\Lambda) = 0$ for $i > 2$ while $\mathrm{HH}_2(\Lambda) \neq 0$. Therefore $\mathrm{HH.dim}(\Lambda) = 2$. Every left Λ-module, M, has a length two projective left Λ-module resolution $P_\bullet \otimes_\Lambda M$, and hence Λ has global dimension at most two. $\qquad\square$

Example 3.14. Note that the inequality in Corollary 3.13 may be strict. If Q is the Jordan quiver (i.e., the quiver with one vertex and one loop) then

$$\Lambda^q(Q) \cong k\langle a, a^* \rangle[(1 + a^* a)^{-1}]/(aa^* - qa^* a - (q-1)).$$

The change of variables $x := a$ and $y := a^*/(q-1)$ when $q \neq 1$, identifies $\Lambda^q(Q)$ with a localization of the first quantum Weyl algebra, $k\langle x, y \rangle/(xy - qyx - 1)$, which has global dimension one.

3C. _The dual complex._ In this subsection, we show for _any_ quiver that if $P_\bullet$ is a resolution of $\Lambda^q(Q)$, then $\Lambda^q(Q)$ is 2-Calabi–Yau. Combining this with the previous subsection, we get that if $\Lambda^q(Q)$ satisfies the weak free product property then $\Lambda^q(Q)$ is 2-Calabi–Yau. In particular, this shows that $\Lambda^q(Q)$ is 2-Calabi–Yau for connected quivers containing a cycle.

First we recall the notion of d-Calabi–Yau algebras [36].

Definition 3.15. A is d-Calabi–Yau if (a) A has finite projective dimension as an A-bimodule; (b) $\mathrm{Ext}^i(A, A \otimes A) = 0$ for $i \neq d$; and (c) there exists an A-bimodule isomorphism

$$\eta : \mathrm{Ext}^d_{A-\mathrm{bimod}}(A, A \otimes A) \to A.$$

The map η is called a d-Calabi–Yau structure.

Remark 3.16. For perfect A-modules, M and N, one has a quasiisomorphism,

$$\mathrm{RHom}_{A-\mathrm{bimod}}(M, N) \xrightarrow{\cong} \mathrm{Hom}_{A-\mathrm{bimod}}(M, A \otimes A) \otimes^{\mathbb{L}}_{A \otimes A^{\mathrm{op}}} N.$$

Taking $M = A^\vee$ and $N = A$ gives $\mathrm{RHom}_{A-\mathrm{bimod}}(A^\vee, A) \cong A \otimes^{\mathbb{L}}_{A \otimes A^{\mathrm{op}}} A$. The isomorphism on the level of d-th homology realizes

$$\eta \in \mathrm{Hom}_{A-\mathrm{bimod}}(A^\vee, A[-d]) =: \mathrm{Ext}^{-d}_{A-\mathrm{bimod}}(A^\vee, A) \cong \mathrm{HH}_d(A)$$

as a class in d-th Hochschild homology.

For dg-algebras, one further equips this structure with a class in negative cyclic homology that lifts the Hochschild homology class of the isomorphism. But, as shown in Proposition 5.7 and explained in Definition 5.9 of [60], for ordinary algebras this additional structure exists uniquely.

We have established $P_\bullet$ as a Λ-bimodule resolution of Λ, if Q is connected and contains a cycle. To show Λ is 2-Calabi–Yau, it suffices to show that its dual complex

$$\mathrm{RHom}_{\Lambda-\mathrm{bimod}}(\Lambda, \Lambda \otimes \Lambda) := \mathrm{Hom}_{\Lambda-\mathrm{bimod}}(P_\bullet, \Lambda \otimes \Lambda) =: P_\bullet^\vee$$

is quasiisomorphic to $\Lambda[-2]$.

Define $\eta_v^\vee \in \mathrm{Hom}_{\Lambda-\mathrm{bimod}}(P_0, \Lambda \otimes \Lambda)$ and $\eta_a^\vee \in \mathrm{Hom}_{\Lambda-\mathrm{bimod}}(P_1, \Lambda \otimes \Lambda)$ by,

$$\eta_v^\vee(\eta_w) := \begin{cases} e_v \otimes e_v & \text{if } v = w, \\ 0 & \text{otherwise}, \end{cases} \qquad \eta_a^\vee(\eta_b) := \begin{cases} e_{t(a)} \otimes e_{h(a)} & \text{if } b = a^*, \\ 0 & \text{otherwise}. \end{cases}$$

These are generators of $P_0^\vee$ and $P_1^\vee$ respectively and give isomorphisms,

$$P_0^\vee \cong \Lambda \otimes_{kQ_0} kQ_0 \otimes_{kQ_0} \Lambda = \langle \eta_v^\vee \rangle, \qquad P_1^\vee \cong \Lambda \otimes_{kQ_0} k\overline{Q}_1 \otimes_{kQ_0} \Lambda = \langle \eta_a^\vee \rangle.$$

Rather than directly study the dual complex $P_\bullet^\vee$, we modify the formulas for $\alpha^\vee$ and $\beta^\vee$ using the map θ, in a way that doesn't affect the homology of the complex. Namely, after choosing generators $\{\xi_v\}$ for $P_0^\vee$ and $\{\xi_a\}$ for $P_1^\vee$, defined below, one can expand

$$\alpha^\vee(\xi_a) = \sum_{v \in Q_0} a_v' \xi_v a_v'', \qquad \beta^\vee(\xi_v) = \sum_{a \in \overline{Q}_1} b_a' \xi_a b_a'',$$

for some $a_v', a_v'', b_a', b_a'' \in \Lambda$ and then define

$$\alpha_\theta^\vee(\xi_a) := \sum_{v \in Q_0} \theta(a_v')\xi_v\theta(a_v''), \quad \beta_\theta^\vee(\xi_v) := \sum_{a \in \overline{Q}_1} \theta(b_a')\xi_a\theta(b_a'').$$

It suffices to show that

$$(P_\bullet^\vee)_\theta := P_0^\vee \xrightarrow{-\beta_\theta^\vee} P_1^\vee \xrightarrow{\alpha_\theta^\vee} P_0^\vee$$

is quasiisomorphic to $\Lambda[-2]$.

We prove this by establishing an isomorphism of Λ-bimodule complexes $\varphi_\bullet : P_\bullet[2] \to (P_\bullet^\vee)_\theta$ following Crawley-Boevey and Shaw, so

$$(P_\bullet^\vee)_\theta \xrightarrow{\varphi_\bullet^{-1}} P_\bullet[2] \xrightarrow{\psi_\bullet[2]} Q_\bullet[2] \xrightarrow{\text{quasiiso}} \Lambda[2].$$

Theorem 3.17. *The following diagram commutes*:

$$
\begin{array}{ccccccc}
P_0^\vee & \xrightarrow{-\beta_\theta^\vee} & P_1^\vee & \xrightarrow{\alpha_\theta^\vee} & P_0^\vee & \xrightarrow{\gamma \circ \varphi_0^{-1}} & \Lambda \\
\varphi_0 \Big\uparrow \cong & (\text{II}) & \varphi_1 \Big\uparrow \cong & (\text{I}) & \varphi_0 \Big\uparrow \cong & \text{id} \Big\Vert = \\
P_0 & \xrightarrow{\alpha} & P_1 & \xrightarrow{\beta} & P_0 & \xrightarrow{\gamma} & \Lambda
\end{array}
$$

Where the vertical maps are Λ-bimodule isomorphisms defined on generators by,

$$\varphi_0(\eta_v) := \xi_v := q\eta_v^\vee, \quad \varphi_1(\eta_a) := \xi_{a^*} := \begin{cases} l_a\eta_{a^*}^\vee l_{a^*}^{-1} & \text{if } a \in Q_1^{\text{op}}, \\ -r_{a^*}^{-1}\eta_{a^*}^\vee r_a & \text{if } a \in Q_1. \end{cases}$$

Note that φ_1 is an invertible map since r_a and l_a are invertible elements of Λ for all $a \in \overline{Q}_1$. The commuting of (I) becomes clear once we compute the maps $\alpha^\vee$, the content of the next lemma.

Lemma 3.18 [25, Lemma 3.2].

$$\alpha^\vee(\eta_a^\vee) = \begin{cases} a^*r_a\eta_{h(a)}^\vee l_a - g_{a^*}^{-1}r_{a^*}\eta_{t(a)}^\vee l_{a^*}g_{a^*}^{-1}a^* & \text{if } a \in Q_1, \\ r_{a^*}\eta_{t(a)}^\vee l_{a^*}a^* - a^*g_a^{-1}r_a\eta_{h(a)}^\vee l_a g_a^{-1} & \text{if } a \in Q_1^{\text{op}}. \end{cases}$$
$$\alpha^\vee(\xi_a) = \theta(a^*)\xi_{t(a^*)} - \xi_{h(a^*)}\theta(a^*).$$
$$\alpha_\theta^\vee(\xi_{a^*}) = a\xi_{t(a)} - \xi_{h(a)}a.$$
$$\beta = \varphi_0^{-1} \circ \alpha_\theta^\vee \circ \varphi_1.$$

So square (I) *in Theorem 3.17 commutes.*

Proof. The first two equalities are shown directly in [25] and the last two are clear from the definitions together with Proposition 2.6. □

Proof of Theorem 3.17. By Lemma 3.18, it suffices to show that (II) commutes. While one can similarly compute $\beta^\vee$ directly, such a calculation is unnecessary as the commuting of (II) follows from that of (I).

Indeed, dualizing and applying $(-)_\theta$ to the maps in (I), produces a still commuting diagram:

$$
\begin{array}{ccc}
P_1 \xleftarrow{\;(\alpha_\theta^\vee)_\theta^\vee\;} P_0 & & P_1 \xleftarrow{\;\alpha\;} P_0 \\
(\varphi_1)_\theta^\vee \downarrow \quad (\mathrm{I})_\theta^\vee \quad \downarrow (\varphi_0)_\theta^\vee & = & -\varphi_1 \downarrow \quad (\mathrm{I})_\theta^\vee \quad \downarrow \varphi_0 \\
P_1^\vee \xleftarrow{\;\beta_\theta^\vee\;} P_0^\vee & & P_1^\vee \xleftarrow{\;\beta_\theta^\vee\;} P_0^\vee
\end{array}
$$

which shows $\varphi_1 \circ \alpha = -\beta_\theta^\vee \circ \varphi_0$, i.e., (II) commutes.

The equality of maps $(\alpha_\theta^\vee)_\theta^\vee = \alpha$ follows from Proposition 2.6, and $(\varphi_0)_\theta^\vee = \varphi_0^\vee = \varphi_0$ follows from the definitions. For $(\varphi_1)_\theta^\vee = -\varphi_1$, observe that it suffices to show $(\varphi_1)_\theta = -(\varphi_1)^\vee$ and indeed,

$$
\begin{aligned}
(\varphi_1)_\theta(\eta_{a^*}) = (\xi_a)_\theta &= \begin{cases} \theta(l_{a^*})\eta_a^\vee \theta(l_a^{-1}) & \text{if } \epsilon(a) = 1, \\ -\theta(r_{a^*}^{-1})\eta_a^\vee \theta(r_a) & \text{if } \epsilon(a) = 1, \end{cases} \\
&= \begin{cases} r_{a^*}\eta_a^\vee r_a^{-1} & \text{if } \epsilon(a) = 1, \\ -l_{a^*}^{-1}\eta_a^\vee l_a & \text{if } \epsilon(a) = -1, \end{cases} = -(\varphi_1)^\vee(\eta_{a^*}). \qquad \square
\end{aligned}
$$

So without conditions on the quiver, we have established:

Corollary 3.19. *If $P_\bullet \to \Lambda$ is exact then $(P_\bullet^\vee)_\theta \to \Lambda[-2]$ is exact and $P_\bullet^\vee \to \Lambda[-2]$ is exact.*

Therefore, the 2-Calabi–Yau property for Λ follows from the a priori weaker Hochschild dimension two property. In the previous subsection, we showed that Λ has Hochschild dimension two for Q connected and containing a cycle.

Corollary 3.20. *If Q is connected and contains a cycle then $\Lambda^q(Q)$ is 2-Calabi–Yau.*

4. Formality of dg multiplicative preprojective algebras

In this section we show that if Q satisfies the strong free product property, then the dg multiplicative preprojective algebra is formal. In particular this proves Conjecture 1.3 in the case Q is connected and contains a cycle. Moreover, it reduces Conjecture 1.3 to the remaining extended Dynkin cases and Conjecture 1.5.

If one views the dg multiplicative preprojective algebra as the central object of study, as in [29; 30], then we are showing one can formally replace it by the non-dg version.

We begin with an elementary lemma. It is not strictly required, but it demonstrates more transparently the construction we will use.

Lemma 4.1. *Let K be a commutative ring. Let A be the dg-algebra defined as a graded algebra to be $K[r] * K[s]$ with $|r| = 0$ and $|s| = -1$, product given by concatenation of words, and differential extended as a derivation from the generators $d(s) = r$ and $d(r) = 0$. Then A is quasiisomorphic to its cohomology $H^*(A) = K$ concentrated in degree zero. In fact, the identity map is chain homotopic to the augmentation map $A \to K$.*

Proof. Let $h : A \to A[-1]$ be the homotopy with the property $h(rf) = sf$ and $h(sf) = 0$ for all $f \in A$, and $h(K) = 0$. Then $h \circ d + d \circ h - 1_A$ is the projection with kernel K to the augmentation ideal of A. Therefore, it defines a contracting homotopy from A to K. $\qquad\square$

In other words, the lemma is observing that A, as the tensor algebra on an acyclic complex $Kr \oplus Ks$, is itself quasiisomorphic to K.

Lemma 4.2. *The dg-algebra A given by*

$$\Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}] *_{kQ_0} kQ_0[s], \quad \text{with } |r| = 0 \text{ and } |s| = -1$$

and with differential determined by $d(s) = r$ is quasiisomorphic to $\Lambda^q(Q)$ concentrated in degree zero.

Proof. Extending the preceding construction, define a homotopy $h : A \to A[-1]$ by

$$h(frg) = fsg, \quad h(fsg) = 0, \quad h(f(r+q)^{-1}g) = q^{-1}h(fg) - q^{-1}fs(r+q)^{-1}g$$

for $f \in \Lambda^q(Q)$ and $g \in A$. The definition of $h(f(r+q)^{-1}g)$ is chosen to match the formula for $h(frg)$ in the r-adic completion. There is an augmentation $A \twoheadrightarrow \Lambda^q(Q)$ with kernel $(r, s, r' := (r+q)^{-1} - q^{-1})$. Notice that $h \circ d + d \circ h$ is a homotopy from the identity on A to the augmentation $\Lambda^q(Q)$, as it annihilates $\Lambda^q(Q)$ and is the identity on s, r, and r'. $\qquad\square$

Definition 4.3. The *dg multiplicative preprojective algebra* is a dg-algebra over kQ_0 defined as a graded algebra by

$$\Lambda^{dg,q}(Q) := L_Q *_{kQ_0} kQ_0[s], \quad |s| = -1, |\alpha| = 0 \text{ for } \alpha \in L_Q.$$

The differential, d, is defined by $d(s) = \rho - q$, $d(L_Q) \equiv 0$, and extended as a kQ_0-linear derivation to $L_Q *_{kQ_0} kQ_0[s]$.

Proposition 4.4. *If $\Lambda^q(Q)$ satisfies the strong free product property,[5] then*

$$H_*(\Lambda^{dg,q}(Q)) = H_0(\Lambda^{dg,q}(Q)) \cong \Lambda^q(Q)$$

so in particular $\Lambda^{dg,q}(Q)$ is formal.

By Theorem 3.7, the proposition holds in particular if Q contains a cycle.

Remark 4.5. Note that for the ordinary preprojective algebra, $\Pi(Q)$, the Ginzburg dg-algebra has homology concentrated in degree zero for any non-Dynkin quiver: $\Pi(Q)$ has a length two bimodule resolution (see [50; 18] for the characteristic zero case, and [31] in general) which [1, Theorems 2.6 and 2.9] shows is equivalent for *graded* connected algebras, and [32] observes this extends to the quiver case.

Proof. The strong free product property yields an isomorphism of graded vector spaces,

$$L_Q \cong \Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}].$$

---

[5]Meaning $(L_Q, r, \sigma, kQ_0[t, (t+q)^{-1}])$ satisfies the strong free product property for some choice of σ.

Hence, as complexes,

$$\Lambda^{dg,q}(Q) \cong L_Q *_{kQ_0} kQ_0[s] \cong \Lambda^q(Q) *_{kQ_0} kQ_0[r, (r+q)^{-1}] *_{kQ_0} kQ_0[s],$$

which by Lemma 4.2 is quasiisomorphic to $\Lambda^q(Q)$, concentrated in degree zero. It follows that

$$\Lambda^{dg,q}(Q) \cong H_*(\Lambda^{dg,q}(Q)) \cong H_0(\Lambda^{dg,q}(Q)) \cong \Lambda^q(Q)$$

as dg-algebras. $\qquad\square$

Remark 4.6. In the presence of Conjecture 1.1, formality of $\Lambda^{dg,q}(Q)$ implies $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau. Hence by Theorem 1.2, we have shown that $\Lambda^{dg,q}(Q)$ is 2-Calabi–Yau, when Q is connected and contains a cycle. One may be able to adapt the techniques in Section 3 to prove that $\Lambda^{dq,q}(Q)$ is 2-Calabi–Yau, in general. In more detail, writing $\Lambda^{dg} := \Lambda^{dg,q}(Q)$, the role of the $\Lambda^q(Q)$-bimodule resolution, $P_\bullet$, should now be played by the Λ^{dg}-dg-bimodule given by the total complex of

$$\Lambda^{dg} \otimes_{kQ_0} kQ_0 \cdot s \otimes_{kQ_0} \Lambda^{dg} \xrightarrow{\alpha^{dg}} \Lambda^{dg} \otimes_{kQ_0} kQ_1 \otimes_{kQ_0} \Lambda^{dg} \xrightarrow[\beta_1^{dg}]{\beta_0^{dg}} \Lambda^{dg} \otimes_{kQ_0} \Lambda^{dg},$$

where $\beta_1^{dg}(a \otimes s \otimes b) = as \otimes b - a \otimes sb$ and α^{dg} (respectively β_0^{dg}) has the same formula as α (respectively β).

Remark 4.7. We are grateful to Georgios Dimitroglou Rizell, who pointed out that our definition differs from that arising in symplectic geometry. Indeed in the derived multiplicative preprojective algebra, $\mathscr{L}_\Gamma$, [30, page 779] define additional variables z_a, ζ_a with z_a invertible and $d(\zeta_a) = z_a - (1 + a^*a)$, and hence $(1 + a^*a)$ is invertible only after taking homology. In contrast, we invert $(1 + a^*a)$ on the chain level in $\Lambda^{dg,q}(Q)$. However, for our main result, Proposition 4.4, this distinction is irrelevant, as we now explain.

We claim that the dg algebra map $\alpha : \mathscr{L}_\Gamma \to \Lambda^{dg,q}(Q)$, given by $\alpha(z_a) = (1 + a^*a)$, $\alpha(\zeta_a) = 0$, and taking arrows to arrows, is a quasiisomorphism. To see this, note that $\mathscr{L}_\Gamma$ can be viewed as a bigraded dg algebra with two differentials: set $|\zeta_a| = (-1, 0)$ and $|s| = (0, -1)$, with horizontal differential $d_H(\zeta_a) = z_a - (1 + a^*a)$, $d_H(s) = 0$, and vertical differential $d_V(\zeta_a) = 0$, $d_V(s) = \rho - q$. We will show in the next paragraph that the map α induces an isomorphism on horizontal cohomology, that is, $\alpha : (\mathscr{L}_\Gamma, d_H) \to (\Lambda^{dg,q}(Q), 0)$ is a quasiisomorphism. Therefore, α is a morphism of bicomplexes (placing the target in horizontal degree zero), that induces an isomorphism on the first page of the associated spectral sequences. These sequences collapse on the second page. They collapse to the cohomology, since both sequences were third-quadrant (cohomologically) and hence convergent. This proves the claim.

It remains to show that α is an isomorphism on horizontal cohomology. More generally, let A be a graded path algebra on the quiver Q (arrows can be assigned any degrees), and let $S \subseteq A$ be a subset of homogeneous elements; in the case above we have $A := k\overline{Q} *_{kQ_0} kQ_0[s]$ and $S := \{1 + a^*a\}_{a \in \overline{Q}_1}$. We wish to compare two localizations. The first is the naive one, $A[f^{-1}]_{f \in S}$. The second is given by replacing A by the quasiisomorphic algebra $\widetilde{A} := A\langle z_f, \zeta_f \rangle_{f \in S}$, with differential $d(\zeta_f) = z_f - f$, $d(z_f) = 0$, $d(A) \equiv 0$. We then consider $\widetilde{A}[z_f^{-1}]_{f \in S}$. To compare these we use the technique of *derived localization*, following [16]. Since A is hereditary with zero differential, by [16, Corollary 4.20, Theorem 5.1], its derived localization by

S is $A[S^{-1}]$ (i.e., it is underived). On the other hand, $\widetilde{A}$ is cofibrant in the category of dg algebras equipped with a morphism from $kQ_0\langle z_f\rangle_{f\in S}$, as it is given by cell attachment (although with nonzero differential). So $\widetilde{A}[z_f^{-1}] = \widetilde{A} *_{kQ_0\langle z_f\rangle} kQ_0\langle z_f, z_f^{-1}\rangle$ is also its derived localization. Now the quasiisomorphism $\widetilde{A} \to A$ is compatible with the morphisms from the path algebra $kQ_0\langle z_f\rangle_{f\in S}$, sending z_f to z_f and to f, respectively. Thus the map $\widetilde{A}[z_f^{-1}]_{f\in S} \to A[f^{-1}]_{f\in S}$ is a quasiisomorphism of derived localizations of A at S.

Note that by combining the two preceding paragraphs, in general, the quasiisomorphism

$$A\langle z_f, z_f^{-1}, \zeta_f\rangle_{f\in S} \to A[f^{-1}]_{f\in S}$$

induces a quasiisomorphism

$$A\langle z_f, z_f^{-1}, \zeta_f, s_i\rangle \to A[f^{-1}]\langle s_i\rangle$$

for any additional arrows s_i and differential $d(s_i)$ compatible with the morphism (only assuming that A is a graded path algebra with S a collection of homogeneous elements). The same is true replacing $A\langle z_f, z_f^{-1}, \zeta_f\rangle$ by any other model of the derived localization of A at S.

Remark 4.8. The dg multiplicative preprojective algebra is called the Legendrian cohomology dg algebra in [29, 3.2] where they establish that it is a multiplicative analog of Ginzburg's dg algebra for a quiver with zero potential defined in [36, 1.4]. It is called a capped Chekanov–Eliashberg algebra in [53, Section 3.2] where they independently prove formality in the case Q is the Jordan quiver and $q = 1$ in [53, Theorem 3.13].

5. Local structure of multiplicative quiver varieties and moduli spaces attached to 2-Calabi–Yau algebras

In this section, we will assume that k is an algebraically closed field of characteristic zero.

We will use our main result to prove, as anticipated in [55, Section 7.5], that multiplicative quiver varieties are étale-locally (or formally locally) isomorphic to ordinary quiver varieties. Our proof uses (a generalization of) a result of Bocklandt, Galluzzi, and Vaccarino in [14] for 2-Calabi–Yau algebras. While our main result is only proved for quivers *with* cycles, we are able to prove this result for *all* quivers. The key idea is to embed any quiver into one containing a new vertex with a cycle, and put the zero vector space at this new vertex. This identifies every multiplicative quiver variety with one for a quiver containing a cycle.

We recall the definition of multiplicative quiver varieties [25; 61; 55], beginning with King's notion of (semi)stability. First, by an algebra over kQ_0, we mean a k-algebra which contains kQ_0 as a subalgebra. Given a module M over such an algebra A, its dimension vector is $\alpha \in \mathbb{N}^{Q_0}$ given by $\alpha_i = \dim e_i M$, $i \in Q_0$. Given a kQ_0-module V, let $\mathrm{Rep}(A, V) := \mathrm{Hom}_{kQ_0\text{-alg}}(A, \mathrm{End}_k(V))$ be the set of A-module structures on V. Let $\mathrm{Rep}_\alpha(A) := \mathrm{Rep}(A, V)$ for $V := \bigoplus_{i\in Q_0} k^{\alpha_i}$, called the representation space of dimension α.

Definition 5.1 [45]. Let Q be a finite quiver. Let A be an algebra over kQ_0, $\theta \in \mathbb{Z}^{Q_0}$ a parameter, $\alpha \in \mathbb{N}^{Q_0}$ a dimension vector. Assume that $\theta \cdot \alpha = 0$. Then an A-module M of dimension vector α is said to be

θ-*semistable* if, for every submodule $N := \{N_i\}_{i \in Q_0}$, with dimension vector $\beta \in \mathbb{N}^{Q_0}$, we have $\beta \cdot \theta \leq 0$. Furthermore M is θ-*stable* if $\beta \cdot \theta < 0$ for all nonzero, proper submodules. Let $Rep^{\theta\text{-ss}}(A, V) \subseteq \mathrm{Rep}(A, V)$ be the subset of θ-semistable module structures, and denote this by $\mathrm{Rep}_\alpha^{\theta\text{-ss}}(A)$ when $V := \bigoplus_{i \in Q_0} k^{\alpha_i}$.

Definition 5.2 [45]. Let Q, α, and θ be as in the definition above, and let A be an algebra over $k Q_0$. Then the corresponding (semistable) moduli space is

$$\mathcal{M}_\theta(A, \alpha) := \mathrm{Rep}_\alpha^{\theta\text{-ss}}(A) /\!/ \mathrm{GL}(\alpha). \tag{5-1}$$

In the case $A = \Lambda^q$, this is called a *multiplicative quiver variety*, denoted $\mathcal{M}_{q,\theta}(Q, \alpha)$. In the case $A = \Pi^\lambda$ is a (deformed) preprojective algebra, it is called an ordinary quiver variety, denoted $\mathcal{M}_{\lambda,\theta}^{\mathrm{add}}(Q, \alpha)$.

The main results of this section are the following:

Theorem 5.3. *Let A be a 2-Calabi–Yau algebra over $k Q_0$, and let ρ be a θ-semistable representation of A of dimension α. Then there exists Q', α' such that the formal neighborhood of ρ of the moduli space $\mathcal{M}_\theta(A, \alpha)$ is isomorphic to the formal neighborhood of $\mathcal{M}_{0,0}^{\mathrm{add}}(Q', \alpha')$ at the zero representation.*

Theorem 5.4. *At every point of a multiplicative quiver variety, a formal neighborhood is isomorphic to the formal neighborhood of zero of an ordinary quiver variety.*

In the case where the quiver contains an oriented cycle, Theorem 5.4 follows immediately from Theorem 5.3 and our main result; in general, we need to enlarge the quiver; see Section 5C. Note that the corresponding result for formal neighborhoods of ordinary quiver varieties is known; see [5, Corollary 3.4].

By Artin's approximation theorem [2, Corollary 2.6], we can replace "formal neighborhoods" in the preceding theorems by étale neighborhoods, since we are in the setting of varieties (by which we always mean of finite type) over a field.

Corollary 5.5. *Let A be a 2-Calabi–Yau algebra over $k Q_0$. Then, all moduli spaces $\mathcal{M}_\theta(A, \alpha)$ are symplectic singularities. In particular, they are normal and have rational Gorenstein singularities. The same holds for all multiplicative quiver varieties.*

The proofs of these results are given in the final subsection.

5A. *Generalities on completions of 2-Calabi–Yau algebras.* To prove Theorem 5.3, we will need the following results about the local structure of n-Calabi–Yau algebras at modules M, adapted from [10].

Definition 5.6. Let A and B be A_∞-algebras. B is *minimal* if $m_1^B = 0$. B is further a *minimal model* for A if there exists an A_∞-quasiisomorphism $B \to A$ lifting the identity. We make the same definitions for L_∞ algebras.

In particular, if B is a minimal model for A then $(B, m_2^B) \cong H^*(A)$ as graded algebras. Kadeishvili showed [40] that every A_∞-algebra has a minimal model.

Theorem 5.7 (minimal model theorem). *Let A be an augmented A_∞-algebra over a semisimple k-algebra S. Then, A admits an augmented A_∞-algebra isomorphism $H^*(A)' \oplus C \to A$, where $H^*(A)'$ is an A_∞-algebra which, as a dg algebra, is the cohomology $H^*(A)$ with zero differential, and C is a contractible complex such that all ≥ 2-ary multiplications involving it are zero.*

Similarly, if $\mathfrak{g}$ is an L_∞-algebra over k, then there is an L_∞-isomorphism $H^*(\mathfrak{g})' \oplus \mathfrak{c} \to \mathfrak{g}$, where the underlying ordinary dg Lie algebra of $H^*(\mathfrak{g})'$ is the cohomology of $\mathfrak{g}$ with zero differential, and $\mathfrak{c}$ is a contractible complex such that all ≥ 2-ary multiplications involving it are zero.

Here, an A_∞-algebra A is *augmented* over S if it is of the form $S \oplus \bar{A}$ where S is a subalgebra and $\bar{A}$ is a strict ideal, i.e., all multiplications with $\bar{A}$ as an input land in $\bar{A}$; moreover, we assume that the only nonzero multiplication between S and $\bar{A}$ are the binary operations (i.e., the S-bimodule structure). An *augmented A_∞-morphism* is an A_∞-morphism which is the identity on S, preserves strictly the augmentation ideals, and all higher A_∞-structure maps vanish when one of the inputs is in S.

Remark 5.8. The map $A \to \bar{A}$ gives an equivalence between augmented A_∞-algebras and nonunital A_∞-algebras in the category of S-bimodules. This makes the statements for A_∞ and L_∞- algebras more symmetric. There are also L_∞ analogs of working over a semisimple algebra; for example, we may work with representations of a reductive group. Given an augmented A_∞-algebra over a matrix algebra, the augmentation ideal has an associated L_∞-algebra which is a representation of the general linear group.

Kadeishvili's approach is direct and explicit: he constructs both the A_∞-structure on $H^*(A)'$ and the A_∞-algebra isomorphism $A \to H^*(A)' \oplus C$. For more conceptual treatments, see e.g., Theorem 5.4 of [41] and Remark 4.18 in [21]. For a sketch in the context of L_∞-algebras see Lemma 4.9 of [46].

Remark 5.9. Note that the minimal model theorem is usually stated in the literature for fields, but it is known that the statement and proof generalizes to the case of semisimple algebras over a field.

Definition 5.10. Let A be an A_∞-algebra. We say A is *formal* if there is an augmented A_∞-isomorphism $H^*(A)' \to H^*(A)$, where $H^*(A)$ has zero ℓ-ary multiplication for $\ell \geq 3$.

Definition 5.11. Given a dg associative algebra A with module M, define the *derived Koszul dual algebra with respect to M* to be $E_M(A) := \mathrm{REnd}_A(M)$.

This is only defined up to quasiisomorphism, but it will not matter to us which model is chosen. Note that if A is a Koszul algebra over S, with S the augmentation module, then up to degree conventions, $E_S(A)$ is the completion of the Koszul dual algebra, $A^!$, with respect to the filtration by powers of the augmentation ideal. In this case, A and $A^!$ have an additional weight grading, and $(A^!)^! \cong A$.

Recall that, if A is an n-Calabi–Yau algebra and M a finite-dimensional module, then there is a trace $\lambda : \mathrm{Ext}^n(M, M) \to k$ such that the composition

$$(-, -) : \mathrm{Ext}^i(M, M) \times \mathrm{Ext}^{n-i}(M, M) \xrightarrow{\circ} \mathrm{Ext}^n(M, M) \xrightarrow{\lambda} k$$

is a graded symmetric perfect pairing [44, Lemma 3.4]. Since it is also graded commutative, this says that $\mathrm{Ext}^\bullet(M, M)$ is a *symmetric dg Frobenius* algebra. In the case that $R := \mathrm{End}_A(M)$ is semisimple, this says

that $\mathrm{Ext}^n(M, M) \cong R$ as R-modules. Moreover, if we realize R as endomorphisms of a kQ'-representation (i.e., $R \cong \prod_{i \in Q'_0} \mathrm{End}_k(k^{\alpha'_i})$ for some finite set Q'_0 and dimension vector $\alpha' \in \mathbb{N}^{Q'_0}$) then we can write, for $V_i := k^{\alpha'_i}$,

$$\mathrm{Ext}^m(M, M) \cong \bigoplus_{i,j \in Q'_0} \mathrm{Hom}_k(V_i, V_j)^{c^m_{i,j}},$$

for some $c^m_{i,j} \in \mathbb{N}$.

Moreover, in the case of interest, $n = 2$, we only need to consider $m = 1$. Then the pairing on $\mathrm{Ext}^1(M, M)$ is symplectic. By picking an appropriate symplectic basis on $\mathrm{Ext}^1(M, M)$, we can write

$$\mathrm{Ext}^1(M, M) \cong T^*\left(\bigoplus_{a \in Q'_1} \mathrm{Hom}_k(V_{t(a)}, V_{h(a)})\right),$$

with the standard symplectic structure on the cotangent bundle, for some set Q'_1 of arrows with vertex set Q'_0 (i.e., extending Q'_0 to a quiver $Q' = (Q'_0, Q'_1)$). It turns out that the symplectic pairing on $\mathrm{Ext}^1(M, M)$, and hence the quiver data (Q', α'), completely determines the dg algebra $\mathrm{REnd}(M)$ up to A_∞-isomorphism.

Continue to assume that $R := \mathrm{End}_A(M)$ is semisimple. In this case, the image, call it S, of the action homomorphism $\rho_M : A \to \mathrm{End}_k(M)$ is also semisimple. We could complete A at M, meaning the completion with respect to the filtration by powers of $\ker \rho_M$. This is not necessarily a quasiisomorphism invariant, however. A better way to take the completion is by a double Koszul duality, as $E_M(E_M A)$, where M is viewed as an $E_M A$-module via the augmentation map $\mathrm{REnd}(M) \to \mathrm{End}_A(M)$. The result is certainly complete, and in certain cases it is indeed the completion of A (e.g., for $A = k[x]$ with $M = k$, one obtains $k[\![x]\!]$; see the proof of the next theorem for more cases).

Since S is semisimple, it is Morita equivalent to a direct sum of copies of k, namely kQ'_0 for Q'_0 the set of isomorphism classes of indecomposable summands of M. Then, we can replace the aforementioned "completion" of A by a completed quiver algebra, by replacing M by M', the direct sum of one copy of each nonisomorphic indecomposable summand of M. Then $E_{M'} E_M A$ is augmented over kQ'_0 and is Morita equivalent to the completion of A at M. More precisely, if $V_i = k^{\alpha'_i}$ as before, so that $\mathrm{End}_A(M) = \bigoplus_i \mathrm{End}_k(V_i)$, then $M' = \bigoplus_i V_i$, viewed as an $E_M A$ module via the augmentation $E_M A \to \mathrm{End}_A(M)$.

Theorem 5.12. *Let A be a 2-Calabi–Yau algebra over kQ_0 and M a finite-dimensional module such that $\mathrm{End}_A(M)$ is semisimple. Then $E_M A$ is formal.*

Proof. We deduce this result from [10, Theorem 11.2.1, Corollary 9.3] as follows. The latter gives a formal local characterization of n-Calabi–Yau algebras (more generally for dg exact Calabi–Yau algebras concentrated in nonpositive degrees) for $n \geq 3$. The proof there is valid also in the case $n = 2$, where it yields that the following are equivalent for a *complete augmented* algebra A over kQ_0:

(a) A is a 2-Calabi–Yau algebra.

(b) $E_{kQ_0} A$ is formal and has a nondegenerate trace of degree -2.

In this case, A itself is isomorphic to $E_{kQ_0} \mathrm{REnd}_A(kQ_0)$.

Now, let A be an *ordinary* 2-Calabi–Yau algebra and M a finite-dimensional module with $\mathrm{End}_A(M)$ semisimple. Then $E_M A = \mathrm{REnd}_A(M)$ has a nondegenerate trace of degree -2. We can now apply the aforementioned result to the dg algebra $A' := E_{M'} E_M A$, which is complete and augmented over kQ_0. By construction, $E_{kQ_0} A' \cong E_M A$ (formally, this is because $B := E_M A$ is its own double Koszul dual, as it is augmented, finite-dimensional, and concentrated in positive degrees [10, Proposition A.5.4]). Thus $E_M A$ is formal. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.13. In fact, the proof shows that the following statements are equivalent for an ordinary algebra A and module M with $\mathrm{End}_A(M)$ semisimple:

 (a) $E_M A$ is formal and has a nondegenerate trace of degree -2.

 (a') $E_M A$ has a nondegenerate trace of degree -2.

 (b) The double dual $E_M E_M A$ is 2-Calabi–Yau.

Since the double Koszul dual is Morita equivalent to the completed dg quiver algebra $E_{M'} E_M A$, these statements are also equivalent to this latter algebra being 2-Calabi–Yau.

Remark 5.14. As stated, [10] actually deals with the case of Calabi–Yau dimension $n \geq 3$. In this case, one can also state a version of the theorem: instead of yielding that $E_M A$ is formal, one can only kill the higher A_∞-structures of $\mathrm{Ext}^\bullet(M, M)$ which land in top degree n. The main result of [10] can then be stated as saying that the remaining structure of $E_M A$ is governed by a single cyclically symmetric element called the superpotential.

Remark 5.15. Since submission of this article, Ben Davison [27, Theorem 1.2] has proved a more general formality result for 2-Calabi–Yau categories. As he explains, the reason for the formality is quite simple: the Koszul dual $E_M A$ can be taken to be a cyclic A_∞-algebra which is augmented over $\mathrm{End}_A(M)$. This means that, for $s \in \mathrm{End}_A(M)$, $\langle m_n(a_1, \ldots, a_n), s \rangle = \langle a_1, m_n(a_2, \ldots, a_n, s) \rangle = 0$. This shows that all A_∞-structures landing in top degree (here, degree two) vanish.

Theorem 5.12 implies that the formal moduli problem, based at M, of modules over a 2-Calabi–Yau algebra A is equivalent to that of a dg preprojective algebra. Indeed, using the bar construction, one can realize $E_{M'} \mathrm{Ext}^\bullet(M, M)$ as the completed dg preprojective algebra of the quiver Q', for M' as above. Note that the module M' is a zero representation of this preprojective algebra: all arrows act by zero.

5B. *The representation and moduli schemes.* We are interested rather in the ordinary representation moduli scheme of A, possibly using a nonzero stability condition. In this case, Theorem 5.12 will imply that, when A is 2-Calabi–Yau the formal neighborhood of this scheme at M will be isomorphic to that of the corresponding quiver variety.

To prove this, we use the following generalization of [14, Theorem 6.3], describing the general structure of these schemes whenever A is an algebra with $\mathrm{End}_A(M)$ semisimple. Given (formal) schemes X, Y with actions by a group G, write $X \times^G Y := (X \times Y)/\!/G$ using the diagonal action.

Theorem 5.16. *Let A be an algebra over kQ_0, and $\alpha \in \mathbb{N}^{Q_0}$ a dimension vector. Suppose that $M \in$ $\mathrm{Rep}_\alpha(A)$ is a representation whose GL_α-orbit is closed in some GL_α-stable affine open subset U of $\mathrm{Rep}_\alpha(A)$. Let $A' := H^0 E_{M'} E_M A$ (for M' as above). Then:*

(1) $\mathrm{End}_A(M)$ *is semisimple.*

(2) *There is a GL_α-equivariant isomorphism $\widehat{\mathrm{Rep}_\alpha(A)}_{\mathrm{GL}_\alpha \cdot M} \cong \widehat{\mathrm{Rep}_{\alpha'}(A')}_{M'} \times^{\mathrm{GL}_{\alpha'}} \mathrm{GL}_\alpha$.*

(3) *The formal neighborhood of $[M]$ in $U /\!/ \mathrm{GL}_\alpha$ is isomorphic to the formal neighborhood of $[M']$ in $\mathrm{Rep}_{\alpha'}(A') /\!/ \mathrm{GL}_{\alpha'}$.*

Before we begin the proof of the theorem, as in [14, Section 6], we need to recall some of the formalism of Maurer–Cartan loci. Let $\mathfrak{g}$ be a dg associative or Lie algebra. Then the Maurer–Cartan locus is

$$\mathrm{MC}(\mathfrak{g}) := \left\{ a \in \mathfrak{g}^1 \mid da + \tfrac{1}{2}[a, a] = 0 \right\}.$$

Let $\widehat{\mathrm{MC}}(\mathfrak{g})$ be its formal completion at $0 \in \mathfrak{g}^1$. More generally, given an A_∞ or L_∞-algebra, we can define

$$\widehat{\mathrm{MC}}(\mathfrak{g}) := Z\left(a \mapsto da + \tfrac{1}{2!}[a, a] + \tfrac{1}{3!}[a, a, a] + \cdots \right) \subseteq \widehat{\mathfrak{g}^1},$$

the formal subscheme of $\widehat{\mathfrak{g}^1}$ cut out by the Maurer–Cartan equation (now a power series).

The algebra of functions on this formal scheme is the zeroth Lie algebra cohomology of $\mathfrak{g}^{>0}$, $H^0 \mathrm{CE}(\mathfrak{g}^{>0}) = H^0(\mathrm{CE}(\mathfrak{g})/((\mathfrak{g}^0)^*))$. Here, the Chevalley–Eilenberg cochain complex is the completed dg symmetric algebra, $\mathrm{CE}(\mathfrak{g}) = (\widehat{\mathrm{Sym}}(\mathfrak{g}^*[-1]), d_{\mathrm{CE}})$, equipped with the Chevalley–Eilenberg differential. For algebras $\mathfrak{g}$ concentrated in positive degrees, this does not depend on A_∞ or L_∞-quasiisomorphisms.

The Maurer–Cartan formal scheme has an infinitesimal action by the Lie algebra $\mathfrak{g}^0$, via gauge equivalence. The gauge action of an element $\xi \in \mathfrak{g}^0$ is recorded by applying the differential and contracting with ξ. The categorical quotient of the Maurer–Cartan formal scheme by this action is defined, on the level of functions, by passing to $\mathfrak{g}^0$-invariant functions. The algebra of functions here is $H^0 \mathrm{CE}(\mathfrak{g}^{\geq 0})/((H^0\mathfrak{g})^*[-1])$. For algebras $\mathfrak{g}$ concentrated in nonnegative degrees, this quotient does not depend on A_∞ or L_∞-quasiisomorphisms.

Now let A be a kQ_0-algebra and M a module. Consider the nonnegatively graded dg associative algebra of kQ_0-bilinear Hochschild cochains,

$$\mathfrak{g} := \mathrm{HC}_{kQ_0}(A, \mathrm{End}_k(M)) := \bigoplus_{i \geq 0} \mathrm{Hom}_{kQ_0-\mathrm{bimod}}(A^{\otimes_{kQ_0} i}, M),$$

equipped with the usual differential and cup product structure. (We remark that this is well known to be quasiisomorphic to the usual algebra of k-linear Hochschild cochains, since kQ_0 is semisimple.)

Given $a \in \mathfrak{g}^1 = \mathrm{Hom}_{kQ_0-\mathrm{bimod}}(A, \mathrm{End}_k(M))$, we can consider the deformation $(\rho_M + a): A \to \mathrm{End}_k(M)$, with ρ_M the original module structure. The condition for $\rho_M + a$ to be a module structure is the Maurer–Cartan equation, $da + a^2 = 0$. Hence $\mathrm{MC}(\mathfrak{g}) = \mathrm{Rep}(A, M)$, with zero corresponding to M. Thus $\widehat{\mathrm{MC}}(\mathfrak{g}) = \widehat{\mathrm{Rep}(A, M)}_M$.

Proof of Theorem 5.16. First, to show $\mathrm{End}_A(M)$ is semisimple, we will use Matsushima's criterion [51]:

> If G is a reductive group acting on an affine variety X, then the stabilizer of a point in a closed orbit is reductive.

In the case at hand, $G = \mathrm{GL}_\alpha$ is acting on $X = U$, so the stabilizer $G_M \cong \mathrm{Aut}(M)$ is reductive. So any element $x \in N(\mathrm{End}_A(M))$, the nilradical of $\mathrm{End}_A(M)$, gives rise to an element $1 + x$ in the unipotent radical, which is $\{1\}$ as $\mathrm{Aut}(M)$ is reductive. So $N(\mathrm{End}_A(M)) = 0$, which implies, as $\mathrm{End}_A(M)$ is finite-dimensional, that the Jacobson radical $J(\mathrm{End}_A(M)) = 0$. We conclude that $\mathrm{End}_A(M)$ is semisimple, being Artinian with vanishing Jacobson radical.

To obtain (2), let $\mathfrak{g}$ be the dg algebra of kQ_0-bilinear cochains, $\mathrm{HC}_{kQ_0}(A, \mathrm{End}_k(M))$ as before the proof. As we explained, the completed Maurer–Cartan subscheme $\widehat{\mathrm{MC}}(\mathfrak{g}) = \widehat{\mathrm{MC}}(\mathfrak{g}^{>0})$ is the same as for the minimal model $H^*(\mathfrak{g}^{>0})$ of $\mathfrak{g}^{>0}$ (as these are concentrated in positive degrees). Next, let $\mathfrak{h} := Z^0(\mathfrak{g}) \cong \mathrm{End}_A(M)$, the zero-cycles of $\mathfrak{g}$, which is a reductive Lie subalgebra of $\mathfrak{g}^0$. Its action integrates to the reductive group $H = \mathrm{Aut}_A(M) \cong \mathrm{GL}_{\alpha'}$, so it acts semisimply. Now, we apply Lemma 5.19 below, to obtain a quasiisomorphic L_∞-algebra (in fact A_∞-algebra, see Remark 5.20) $\mathfrak{g}' := \mathfrak{g}^0 \oplus Z^1(\mathfrak{g}) \oplus H^{>1}(\mathfrak{g})$.

Define $\tilde{H}^1(\mathfrak{g})$ to be an H-invariant complement to the one-coboundaries $B^1(\mathfrak{g})$ in $Z^1(\mathfrak{g})$. The L_∞-structure maps $\tilde{H}^1(\mathfrak{g})^m \to H^2(\mathfrak{g})$ in $\mathfrak{g}'$ are the same as the ones on any minimal model $H^*(\mathfrak{g}')$ induced by transfer (as in the proof of Lemma 5.19 below). This gives an embedding of the Maurer–Cartan locus $\widehat{\mathrm{MC}}(H^*\mathfrak{g}) = \widehat{\mathrm{MC}}(H^{>1}\mathfrak{g})$ of the cohomology into the Maurer–Cartan locus of $\mathfrak{g}'$. By Lemma 5.19, this inclusion is compatible with the H-action, which is linear. It is also a formal slice to the infinitesimal $\mathfrak{g}^0$ action on $\widehat{\mathrm{MC}}(\mathfrak{g}')$: the tangent space to this action is $B^1(\mathfrak{g})$, whereas the tangent space to $\widehat{\mathrm{MC}}(\mathfrak{g}')$ is $Z^1(\mathfrak{g})$.

Next let us turn from the formal neighborhood of M in $\mathrm{Rep}_\alpha(A)$ to a formal neighborhood of its GL_α orbit. Luna's slice theorem [48] implies that there is a $(\mathrm{GL}_\alpha)_M = \mathrm{Aut}_A(M) = H$-stable affine subset $V \subseteq U$, such that the action map $\phi : \mathrm{GL}_\alpha \times^H V \to \mathrm{Rep}_\alpha(A)$ induces a GL_α-equivariant isomorphism onto an étale neighborhood of the orbit $\mathrm{GL}_\alpha \cdot M$. Using the fact that $\mathrm{Aut}(M)$ is connected, we have the following identifications. For ease of reading let $\mathrm{FN}(X, Y) := \widehat{Y}_X$ denote the formal neighborhood of X in Y:

$$\mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U) \cong \mathrm{FN}(\mathrm{GL}_\alpha \times^H \{M\}, \mathrm{GL}_\alpha \times^H V) \cong \mathrm{GL}_\alpha \times^H \mathrm{FN}(\{M\}, V).$$

Finally, we showed above that the slice V can be taken to be the Maurer–Cartan locus of $H^{>0}(\mathfrak{g})$. This identifies with $\widehat{\mathrm{Rep}_{\alpha'}(A')}_{M'}$, since the latter is isomorphic to the Maurer–Cartan locus of the minimal model $H^*(\mathfrak{g})$. (Explicitly, since the augmentation ideal of A' acts by zero on M', $\mathrm{End}_{kQ_0}(M') = \mathrm{End}_{A'}(M')$ is the degree zero part of the Hochschild cochain complex of M' with zero differential, so $\mathrm{Ext}_{A'}^{>0}(M', M')$ is quasiisomorphic to $H^{>0}(\mathfrak{g})$.) This completes the proof of (2), as H is identified with $\mathrm{GL}_{\alpha'}$ by definition of α'.

It remains to deduce (3) from (2). First note that, since GL_α is reductive and the orbit $\mathrm{GL}_\alpha \cdot M \subseteq U$ is closed, by Hilbert's theorem, the ideal of $[M]$ in $\mathcal{O}(U /\!/ \mathrm{GL}_\alpha) = \mathcal{O}(U)^{\mathrm{GL}_\alpha}$ is the set of GL_α-invariant

functions in the ideal of $\mathrm{GL}_\alpha \cdot M$ in $\mathcal{O}(U)$. Therefore, functions on $\mathrm{FN}([M], U /\!/ \mathrm{GL}_\alpha)$ are the GL_α-invariant functions in the completion of $\mathcal{O}(U)$ at the fiber $F \subseteq U$ of the projection $U \to U /\!/ \mathrm{GL}_\alpha$:

$$\mathrm{FN}([M], U /\!/ \mathrm{GL}_\alpha) \cong \mathrm{FN}(F, U) /\!/ \mathrm{GL}_\alpha .$$

Note that $\mathrm{GL}_\alpha \cdot M \subseteq F$, so we get a further map $\mathrm{FN}(F, U) /\!/ \mathrm{GL}_\alpha \to \mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U) /\!/ \mathrm{GL}_\alpha$. We claim that this is an isomorphism. Indeed, let $I_{\mathrm{GL}_\alpha \cdot M} \supseteq I_F$ be the ideals. Then we are considering two different completions of $\mathcal{O}(U /\!/ \mathrm{GL}_\alpha)$ concentrated at $[M]$, by the systems $\{I_{\mathrm{GL}_\alpha \cdot M}^{\mathrm{GL}_\alpha}\}$ and $\{I_F^{\mathrm{GL}_\alpha}\}$. Since U is irreducible, by Krull's intersection theorem, $\bigcap_{n \geq 0} I_{\mathrm{GL}_\alpha \cdot M}^n = 0$. Hence the systems are both exhaustive. Since $I_{[M]}^n / I_{[M]}^{n+1}$ is finite-dimensional for all n, both systems must yield the $I_{[M]}$-adic completion (equivalently, the completion by all finite-dimensional quotients supported at $[M]$). We deduce

$$\mathrm{FN}([M], U /\!/ \mathrm{GL}_\alpha) \cong \mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U) /\!/ \mathrm{GL}_\alpha . \tag{5-2}$$

Applying (2), we have

$$\mathrm{FN}(\mathrm{GL}_\alpha \cdot M, U) /\!/ \mathrm{GL}_\alpha \cong \mathrm{FN}(\mathrm{GL}_{\alpha'} \cdot M', \mathrm{Rep}_{\alpha'}(A')) /\!/ \mathrm{GL}_{\alpha'} .$$

By (5-2) applied to the first and last terms, we obtain finally the desired isomorphism. $\qquad\square$

Remark 5.17. Part of the proof is actually showing is that the derived formal moduli stack at $[M]$ of representations of A is identified with the same for the dg algebra $E_{M'} E_M A$ at the zero representation $[M']$. This is true more generally, but under our hypotheses this implies the stated result by taking a truncation and applying Luna's slice theorem.

Remark 5.18. The second statement of the theorem is a strengthened version of the statement in [14] that a formal neighborhood of $[M]$ in $\mathrm{Rep}_\alpha(A)$ identifies with that of $[M']$ in $\mathrm{Rep}_{\alpha'}(A')$ times a formal disc of dimension $\dim \mathrm{GL}_\alpha - \dim \mathrm{GL}_{\alpha'}$. This is because GL_α is smooth, and taking the formal completion at the identity, the product construction here is multiplying by such a formal disc.

The theorem above uses the following lemma:

Lemma 5.19. *Suppose that $\mathfrak{h} \subseteq Z(\mathfrak{g}^0)$ acts on a dg Lie algebra $\mathfrak{g}$ concentrated in nonnegative degrees. Suppose that all $\mathfrak{h}$-subrepresentations have complements (e.g., this is true if the $\mathfrak{h}$ action integrates to an action of a connected reductive group H with Lie algebra $\mathfrak{h}$). Then there is an L_∞-quasiisomorphism*

$$\phi : \mathfrak{g}' := \mathfrak{g}^0 \oplus Z^1(\mathfrak{g}) \oplus H^{>1}(\mathfrak{g}) \to \mathfrak{g},$$

where on the source, all higher brackets

$$\mathfrak{h} \times (\mathfrak{g}')^{\geq 2} \to \mathfrak{g}' \tag{5-3}$$

vanish. The bracket $\mathfrak{g}^0 \times \mathfrak{g}^0 \to \mathfrak{g}^0$ is the original one. Moreover, the linear part $\phi^1 : \mathfrak{g}' \to \mathfrak{g}$ is $\mathfrak{h}$-linear and induces the identity on $\mathfrak{g}^0 \oplus Z^1(\mathfrak{g})$, as well as on cohomology. Finally, $\phi^{\geq 2}$ vanishes on $\mathfrak{h} \times (\mathfrak{g}')^{\geq 1}$.

Proof. We apply the homotopy transfer formulae from [49] (stated for A_∞-algebras but easily adapted to the L_∞ setting). To do this, for each i we pick a decomposition $\mathfrak{g}^i = B^i(\mathfrak{g}) \oplus \tilde{H}^i(\mathfrak{g}) \oplus \mathfrak{q}^i$, with $B^i(\mathfrak{g})$ the i-coboundaries, $\tilde{H}^i(\mathfrak{g})$ an $\mathfrak{h}$-linear complement to $B^i(\mathfrak{g})$ in the i-cocycles $Z^i(\mathfrak{g})$, and $\mathfrak{q}^i$ a $\mathfrak{h}$-linear complement to $Z^i(\mathfrak{g})$ in $\mathfrak{g}^i$. We then define a homotopy $h : \mathfrak{g}^{>1} \to \mathfrak{g}^{>0}$ via the projection $\mathfrak{g}^i \to B^i(\mathfrak{g})$ followed by a $\mathfrak{h}$-linear isomorphism $B^i(\mathfrak{g}) \to \mathfrak{q}^{i-1}$, for $i > 1$, setting $h|_{\mathfrak{g}^{\le 1}} = 0$.

The resulting homotopy is $\mathfrak{h}$-linear and has the property that $t := \mathrm{Id} - (dh + hd)$ is a projection onto the subcomplex $\mathfrak{g}^0 \oplus Z^1(\mathfrak{g}) \oplus \tilde{H}^{>1}(\mathfrak{g})$, which is an $\mathfrak{h}$-subrepresentation. Call this subcomplex $\mathfrak{g}'$. We have an $\mathfrak{h}$-linear decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{c}$ as complexes, with $\mathfrak{c} = \mathrm{im}(dh + hd)$ a contractible subcomplex (and $\mathfrak{h}$-subrepresentation).

Now use h on all of $\mathfrak{g}$, as in the proof of Theorem 5.7 (see the references above). We obtain a new L_∞-structure on $\mathfrak{g}$, which is L_∞-isomorphic to the original one (with linear part the identity), so that all structures vanish on $\mathfrak{c}$ aside from the differential. The L_∞-structures on $\mathfrak{g}'$ are linear combinations of expressions such as

$$t[a_1, h[a_2, [h[a_3, a_4], h[a_5, a_6]]]],$$

given by iteratively bracketing and applying h, except at the end where t is applied.

By $\mathfrak{h}$-linearity of h, if $x \in \mathfrak{h}$ and $a \in \mathfrak{g}' = \mathrm{im}\, t$, then $h[x, a] = [x, ha] = 0$. Similarly, $t[x, ha] = th[x, a] = 0$. Hence, all contributions to higher brackets $\mathfrak{h} \times \mathfrak{g}^{>1} \to \mathfrak{g}$ vanish. Similarly, $\phi^{>1}$ vanishes on $\mathfrak{h}$ (since $h[x, a] = 0$). By construction ϕ is the identity on $\mathfrak{g}^0 \oplus Z^1(\mathfrak{g})$ and on cohomology. $\square$

Remark 5.20. The lemma has an associative analog with the same proof: let $\mathfrak{g}$ be a dg associative algebra and $\mathfrak{h}$ is a subalgebra for which every $\mathfrak{h}$-subbimodule of $\mathfrak{g}$ admits an $\mathfrak{h}$-complement (e.g., $\mathfrak{g}$ is augmented over a semisimple algebra $\mathfrak{h}$). Then we obtain the same result with an A_∞-quasiisomorphism with higher order parts vanishing on $\mathfrak{h}$, and with higher multiplications on $\mathfrak{g}'$ vanishing on $\mathfrak{h}$. This applies to the situation at hand, so that we could use an A_∞-quasiisomorphism in the proof of Theorem 5.16. However, it makes no difference for the Maurer–Cartan locus. (Actually, this says that the decomposition in Theorem 5.16 enhances to a decomposition of *noncommutative* representation schemes, meaning it describes representations with coefficients in noncommutative Artinian rings.)

5C. *Proof of main results.* In the case where A is 2-Calabi–Yau, we can use Theorem 5.12 (which applies because of part one in Theorem 5.16) and the discussion following it, to refine part three of Theorem 5.16. Namely, we can identify the formal neighborhood of $[M']$ in $\mathrm{Rep}_\alpha(A') /\!/ \mathrm{GL}_{\alpha'}$ with a formal neighborhood of the zero representation in a quiver variety.

Corollary 5.21. *Let A be a 2-Calabi–Yau algebra over kQ_0 for a quiver Q. Let $\alpha \in \mathbb{N}^{Q_0}$ and let $M \in \mathrm{Rep}_\alpha(A)$, such that $\mathrm{GL}_\alpha \cdot M$ is closed in some GL_α-stable open affine subset, U. Then a formal neighborhood of $[M]$ in $U /\!/ \mathrm{GL}_\alpha$ is isomorphic to the formal neighborhood $\widehat{\mathcal{M}^{\mathrm{add}}_{0,0}(Q', \alpha')}_0$ of the zero representation in a quiver variety.*

Pick a stability parameter θ. If $M \in \mathrm{Rep}_\alpha(A)$ is θ-semistable, one has the open set $\mathrm{Rep}_\alpha(A)^{\theta\text{-ss}}$, which is a union of GL_α-stable affine open subsets. As M lies in one such affine open subset, M satisfies the hypotheses of Theorem 5.16 and Corollary 5.21. This implies the following corollary:

Corollary 5.22. *Let Q, α, A be as in Theorem 5.16, let $\theta \in \mathbb{Z}^{Q_0}$. Then for every $M \in \mathrm{Rep}_\alpha(A)^{\theta\text{-ss}}$, the conditions of Theorem 5.16 are satisfied. So, the formal neighborhood of $[M]$ in $\mathcal{M}_\theta(A, \alpha)$ is isomorphic to that of zero in $\mathrm{Rep}_{\alpha'}(A') /\!/ \mathrm{GL}_{\alpha'}$, for A' as in the theorem.*

Proof of Theorem 5.3. Let Q, α, A be as in Corollary 5.21, let $\theta \in \mathbb{Z}^{Q_0}$, and $V := \mathrm{Rep}_\alpha(A)^{\theta\text{-ss}}$. For every $M \in V$ the conditions of Corollary 5.21 are satisfied. So, the formal neighborhood of $[M]$ in $V /\!/ \mathrm{GL}_\alpha$ is isomorphic to the formal neighborhood $\widehat{\mathcal{M}^{\mathrm{add}}_{0,0}(Q', \alpha')}_0$ of the zero representation in a quiver variety. $\square$

Proof of Theorem 5.4. If the quiver Q contains a cycle, then Theorem 5.4 follows immediately from Theorem 5.3 since $\Lambda^q(Q)$ is 2-Calabi–Yau, by Theorem 1.2.

If Q does not contain a cycle, then build $\widetilde{Q}$ from Q by adding a new vertex i_0, an arrow from i_0 to itself, and an arrow from i_0 to any vertex of Q. If $\alpha \in \mathbb{N}^{Q_0}$ is a dimension vector then define $\tilde{\alpha} \in \mathbb{N}^{\widetilde{Q}_0}$ such that $\tilde{\alpha}|_{Q_0} = \alpha$ and $\tilde{\alpha}_{i_0} = 0$. Note that $\mathrm{Rep}_\alpha(\Lambda^q(Q)) = \mathrm{Rep}_{\tilde{\alpha}}(\Lambda^{\tilde{q}}(\widetilde{Q}))$ where $\tilde{q}$ is similarly such that $\tilde{q}|_{Q_0} = q$ and $\tilde{q}_{i_0} = 1$.

Under this identification, the $\mathrm{GL}_{\tilde{\alpha}} = \mathrm{GL}_\alpha \times \mathrm{GL}_1$ action factors through the projection to GL_α, which identifies the actions on the two varieties. For every $\theta \in \mathbb{Z}^{Q_0}$, extending by zero to $\tilde{\theta}$, one also identifies θ-semistable representations of $\Lambda^q(Q)$ of dimension α with $\tilde{\theta}$-semistable representations of $\Lambda^{\tilde{q}}(\widetilde{Q})$ of dimension $\tilde{\alpha}$. Therefore, $\mathcal{M}_{q,\theta}(Q, \alpha) = \mathcal{M}_{\tilde{q},\tilde{\theta}}(\widetilde{Q}, \tilde{\alpha})$, i.e., the semistable moduli spaces in question are identical. So the result follows in general from the specific case where Q contains a cycle. $\square$

Proof of Corollary 5.5. By [3], a (normal) symplectic singularity is rational Gorenstein. The latter is a formal local property. By [5, Theorem 1.2], ordinary quiver varieties are symplectic singularities. Thus, the moduli spaces in question have rational Gorenstein singularities, and in particular are normal.

Next, thanks to Namikawa [52, Theorem 4], the property of being a (normal) symplectic singularity is a equivalent to having rational Gorenstein singularities and having a symplectic form on the smooth locus. It remains to check the last property. (Note that this property is certainly known for many multiplicative quiver varieties: For instance, Yamakawa [61, Theorem 3.4] showed that the stable locus is smooth symplectic, and this is often the entire smooth locus. For another example, character varieties of Riemann surfaces of genus ≥ 1 (and many of genus zero) have symplectic smooth locus by [55, Section 1.2].)

To see that the smooth locus is symplectic in general, first we can assume that we are in the situation of a 2-Calabi–Yau algebra A (in the case of multiplicative quiver varieties, the proof of Theorem 5.4 in Section 5C identifies the moduli space with one for a 2-Calabi–Yau algebra obtained by enlarging the quiver). At a smooth point of the moduli space, Theorem 5.3 endows the formal neighborhood of the point with a symplectic form, given by the canonical symplectic pairing $\mathrm{Ext}^1(M, M) \times \mathrm{Ext}^1(M, M) \to \mathrm{Ext}^2(M, M) \overset{\mathrm{tr}}{\cong} k$ coming from the Calabi–Yau structure. This is functorial in the point of the moduli space: the Calabi–Yau structure furnishes a fixed A-bimodule isomorphism $A \cong \mathrm{Ext}^2(A, A^e)$. This induces a

functorial isomorphism

$$\mathrm{Ext}^2(M, M) \cong H^2(\mathrm{RHom}(A, A^e) \otimes_{A^e}^{\mathbb{L}} \mathrm{End}_k(M)) \to A \otimes_{A^e} \mathrm{End}_k(M) = \frac{\mathrm{End}_k(M)}{[A, \mathrm{End}_k(M)]}.$$

Composing this with the trace map we obtain the functorial trace pairing. □

Remark 5.23. Alternatively, one should be able to construct the symplectic structure on the smooth locus because the latter is an open substack of the symplectic derived moduli stack of representations of $\Lambda^q(Q)$, shown to be symplectic in [17].

6. The multiplicative preprojective algebra of the cycle is an NCCR

The purpose of this section is to prove Conjecture 1.4 in the case where Q is a cycle. We begin with the necessary definitions. Throughout this section, Q denotes an extended Dynkin quiver (not necessarily a cycle).

According to the conjecture, the center of the multiplicative preprojective algebra is the ring of functions on the multiplicative quiver variety $\mathcal{M}_{1,0}(Q, \delta)$. Here δ is the primitive positive imaginary root. In terms of the McKay correspondence, Q is the McKay graph of a finite subgroup $\Gamma < \mathrm{SL}_2(\mathbb{C})$, which means that the vertices are labeled by the irreducible representations of Γ. In these terms, δ_v is the dimension of the irreducible representation of Γ_Q attached to the vertex v. In particular, for the cycle with n vertices, $\Gamma = \mathbb{Z}/n\mathbb{Z}$, and $\delta = (1, \ldots, 1)$ is the all ones vector.

We next recall the notion of an NCCR. Van den Bergh [8, Appendix A] originally defined these in to give an alternate proof of Bridgeland's theorem that a flop of three-dimensional smooth varieties induces an equivalence of their bounded derived categories. Van den Bergh later simplified and generalized the definition to the following:

Definition 6.1 [7, Definition 4.1 and Lemma 4.2]. Let R be an Gorenstein commutative integral domain. An algebra A is an *NCCR* over R if:

(1) A is (maximal) Cohen–Macaulay.

(2) A has finite global dimension.

(3) $A \cong \mathrm{End}_R(M)$ for some reflexive module M.[6]

Note that if A is derived equivalent to a *commutative* crepant resolution of $\mathrm{Spec}(R)$, then it will have to satisfy these conditions by [38, Corollary 4.15]. (However, in general, R could admit a commutative crepant resolution but not a noncommutative one, and vice-versa).

In our case, with $\dim R = 2$, it is convenient to observe that we don't have to check the Cohen–Macaulay condition:

Lemma 6.2. *Let R be a normal Noetherian domain of dimension 2 over k. Let M be a finitely generated, reflexive R-module. Then $A := \mathrm{End}_R(M)$ is Cohen–Macaulay.*

---

[6]Recall an R-module M is *reflexive* if the natural map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ sending $m \in M$ to evaluation on m (i.e., $m \mapsto [\varphi \in \mathrm{Hom}_R(M, R) \mapsto \varphi(m) \in R]$) is an isomorphism.

Proof. Since R is Noetherian and M is finitely generated and reflexive, [58, Lemma 15.23.8] implies that A is reflexive. Since R is 2-dimensional and normal [19, Corollary 3.9] implies that A is Cohen–Macaulay. $\square$

Remark 6.3. Note that, in higher dimensions, while the Cohen–Macaulay property for A is not automatic, it nevertheless can be deduced from the Calabi–Yau property thanks to [37, Theorem 3.2(3)]. This gives an alternative way to handle condition (2) in our situation.

6A. *Shaw's results on the center.* While the center of the multiplicative preprojective algebra is in general unknown, in Shaw's thesis [57], he proves the following. Let v be an extending vertex.

Theorem 6.4 [57, Theorem 4.1.1]. $e_v \Lambda^1(Q) e_v \cong k[X, Y, Z]/(f(X, Y, Z))$ *where f has isolated singularity at the origin. Explicitly,*

$$f(X, Y, Z) = \begin{cases} Z^{n+1} + XY + XYZ & \text{if } Q = \widetilde{A}_n, n \geq 1, \\ Z^2 - p_{n-4}(X)XZ + p_{n-5}(X)X^2Y - XY^2 - XYZ & \text{if } Q = \widetilde{D}_n, n \geq 4, \\ Z^2 + X^2Z + Y^3 - XYZ & \text{if } Q = \widetilde{E}_6, \\ Z^2 + Y^3 + X^3Y - XYZ & \text{if } Q = \widetilde{E}_7, \\ Z^2 - Y^3 - X^5 + XYZ & \text{if } Q = \widetilde{E}_8, \end{cases}$$

where $p_{-1}(X) := -1$, $p_0(X) := 0$, and $p_{i+1}(X) := X(p_{i-1}(X) + p_i(X))$ for $i \geq 1$.

Remark 6.5. Shaw expected that the singularities at the origin have the du Val type corresponding to the quiver. Over a field of characteristic zero, Michael Wemyss checked this in E types via Magma. It is also clear that in A types, the singularity is du Val of the same type as the quiver, by the rational substitution $y \mapsto y/(1+z)$. Presumably it can be checked that in type D (over characteristic not equal to two) the singularity also is the corresponding du Val one.

Note that having du Val singularities is equivalent to the statement that the minimal commutative resolution is symplectic, i.e., 2-Calabi–Yau. Thanks to [59], it is also true that if a Gorenstein surface admits an NCCR, then it has du Val singularities. This is another reason to believe Shaw's expectation.

Remark 6.6. Suppose as expected that the singularities are du Val. Then, as in [43], one may construct an NCCR from the minimal resolution. It seems an interesting question to show that this is Morita equivalent to $\Lambda^1(Q)$.

This motivates the final statement in Conjecture 1.4, that the Satake map, $Z(\Lambda^1(Q)) \to e_v \Lambda^1(Q) e_v$, given by $z \mapsto e_v z$, is an isomorphism. With this in place, the above translates into an explicit description of the center.

6B. *Proof of Conjecture 1.4 for a cycle.* Fix $n \geq 1$. In the remainder of this section we prove Conjecture 1.4 for $Q = \widetilde{A}_n$. As a consequence, using Shaw's result, we conclude:

Corollary 6.7. *The center of $\Lambda^1(\widetilde{A}_n)$ is isomorphic to $k[X, Y, Z]/(Z^{n+1} + XY + XYZ)$.*

The steps of the proof of Conjecture 1.4 for $\widetilde{A}_n$ are as follows:

(1) First we show that $\Lambda^1(\widetilde{A}_n)$ is isomorphic to an NCCR over $e_0 \Lambda^1(\widetilde{A}_n) e_0$.

(2) Then we use the preceding result to establish that the Satake map $Z(\Lambda^1(\widetilde{A}_n)) \to e_0 \Lambda^1(\widetilde{A}_n) e_0$, is an isomorphism.

(3) To complete the proof we consider the canonical map $Z(\Lambda^1(\widetilde{A}_n)) \to k[\mathcal{M}_{0,1}(\widetilde{A}_n, \delta)]$, given by associating to a central element and a simple representation the scalar by which the element acts in the representation. We show that this is an isomorphism.

We carry out these steps in the next subsections.

In the first step, we will make use of the prime property for $\Lambda^1(\widetilde{A}_n)$. We state the prime property now, but defer the proof until Section 7, as our proof uses an explicit basis produced in Proposition 7.1.

Remark 6.8. Note that there is no circular logic in the paper, as Section 7 does not rely on any results after Section 2, and hence could instead fit logically between Sections 3A and 3B, whereby every result would be proven in order. We decided that, due to the technical nature of Section 7, whose methods are not used in the preceding material, it would be better to use its results as a black box in Sections 3B–6.

Definition 6.9. Let R be a ring. We say R is *prime* if $r R r' = 0$ implies $r = 0$ or $r' = 0$, for all $r, r' \in R$.

For a commutative ring, this recovers the usual notion of an integral domain, i.e., that the zero ideal is a prime ideal.

Example 6.10. For a nonexample, take $B = \oplus_{n \in \mathbb{N}} B_n$ to be a finite-dimensional $\mathbb{N}$-graded algebra not concentrated in degree zero. Then there exists $N \in \mathbb{N}$ such that $B_m = 0$ for all $m > N$ but $B_N \neq 0$. Pick $b \in B_N$ nonzero and notice that $bBb \in \oplus_{m \geq 2N} B_m = \{0\}$ since $2N > N$. Hence B is not prime.

In particular for Q Dynkin and $k = \mathbb{C}$, $\Lambda^1(Q) \cong \Pi^0(Q)$ is a finite-dimensional $\mathbb{N}$-graded algebra and therefore not prime. However, for $Q = A_2$ and $q = (1/2, 2) \neq (1, 1)$, then $\Lambda^q(A_2) \cong \Pi^{(-1,1)}(A_2) \cong \text{Mat}_{2 \times 2}(k)$ is prime.

Proposition 6.11 (Proposition 7.3). $\Lambda^q(\widetilde{A}_n)$ *is prime for all* $n \geq 0$ *and all* $q \in (k^\times)^{n+1}$.

6B1. *The NCCR property.* We first show that the multiplicative preprojective algebra is an NCCR (Step 1).

Proposition 6.12. $\Lambda^1(\widetilde{A}_n)$ *is isomorphic to an NCCR over* $e_0 \Lambda^1(\widetilde{A}_n) e_0$.

Proof. Define $\Lambda := \Lambda^1(\widetilde{A}_n)$ for ease of notation. Write the vertex set as $\{0, 1, \ldots, n\}$ and the arrow set $\{a_0, a_0^*, a_1, a_1^*, \ldots, a_n, a_n^*\}$, with $t(a_i) = i = h(a_i^*)$ for $i < n$ but $t(a_n) = 0 = h(a_n^*)$. So the multiplicative preprojective relation at each vertex is

$$e_i(\rho - 1) = \begin{cases} a_0 a_0^* + a_n a_n^* + a_0 a_0^* a_n a_n^* & \text{if } i = 0, \\ a_n^* a_n + a_{n-1}^* a_{n-1} + a_n^* a_n a_{n-1}^* a_{n-1} & \text{if } i = n, \\ a_i a_i^* - a_{i-1}^* a_{i-1} & \text{otherwise.} \end{cases}$$

Shaw's isomorphism in Theorem 6.4 takes the form

$$a_0 a_0^* \mapsto Z \quad a_0 a_1 \cdots a_{n-1} a_n^* \mapsto X \quad a_n a_{n-1}^* a_{n-2}^* \cdots a_0^* \mapsto Y.$$
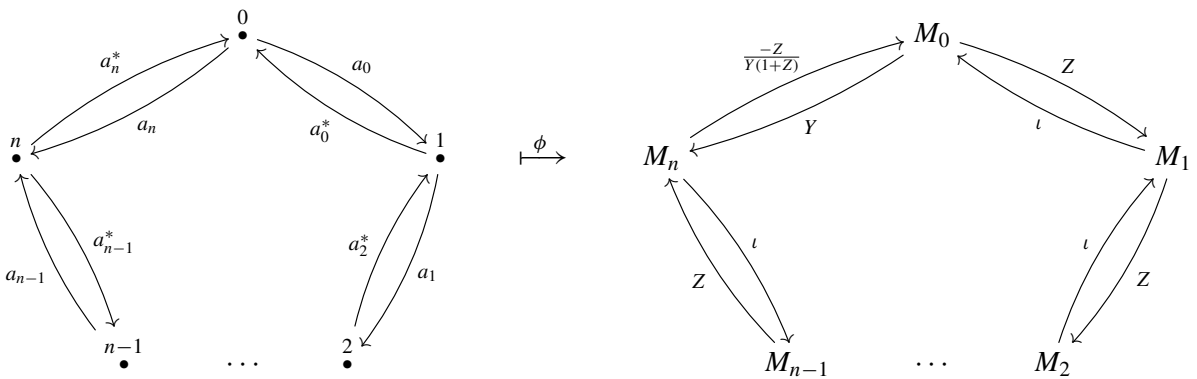
Define $M := e_0\Lambda$ and note that $M = \oplus_{i=0}^n M_i$ where $M_i := e_0\Lambda e_i$. Observe that $M_i \cong (Z^i, Y)$, the two-sided ideal generated by Z^i and Y in Λ, as $e_0\Lambda e_0$-modules via a map,

$$a_0 a_1 \cdots a_{i-1} \mapsto a_0 a_1 \cdots a_{i-1} a_{i-1}^* a_{i-2}^* \cdots a_0^* = (a_0 a_0^*)^i = Z^i$$

$$a_n a_{n-1}^* \cdots a_i^* \mapsto a_n a_{n-1}^* \cdots a_0^* = Y.$$

Define the map

$$\Lambda \xrightarrow{\ \phi\ } \mathrm{End}_{e_0\Lambda e_0}(M),$$

on generators by sending the idempotent e_i at vertex i to the projection map $M \to M_i$, and sending the arrows as follows:

where ι denotes the inclusion map. This map is well-defined at vertex 0 and n since

$$Z + \frac{-YZ}{Y(1+Z)} + \frac{-YZ^2}{Y(1+Z)} = Z + \frac{-Z(1+Z)}{(1+Z)} = Z - Z = 0$$

and at vertex $i \neq 0, n$ since $Z - Z = 0$.

The surjectivity of ϕ follows from the observation that every $e_0\Lambda e_0$-module map of ideals is given by left multiplication by some element of the field of fractions of $e_0\Lambda e_0$. The injectivity follows from the fact that Λ is prime (Proposition 6.11) and injectivity on $e_0\Lambda e_0$, as we now explain.

By definition of primality, for any $a, c \in \Lambda$ both nonzero, there exists $b \in \Lambda$ such that $abc \neq 0$. Fix $\gamma \in \Lambda$ nonzero and take $a = e_0$ and $c = \gamma$ to get a nonzero path $\gamma' \in e_0\Lambda$ containing γ as a subpath. Then take $a = \gamma'$ and $c = e_0$ to get a nonzero path $\gamma'' \in e_0\Lambda e_0$ containing γ as a subpath. Since ϕ is injective on $e_0\Lambda e_0$, $\phi(\gamma'') \neq 0$. Hence $\phi(\gamma) \neq 0$ and ϕ is injective.

To complete the proof that Λ is an NCCR, we need to show that the module $M = e_0\Lambda$ is a reflexive $e_0\Lambda e_0$-module. The computation above shows that

$$\mathrm{Hom}_{e_0\Lambda e_0}(M_i, M_j) \cong e_i\Lambda e_j \cong M_{j-i}$$

as a module over $e_0\Lambda e_0 \cong e_i\Lambda e_i$, so in particular $\mathrm{Hom}_{e_0\Lambda e_0}(M, e_0\Lambda e_0) \cong \oplus_i e_i\Lambda e_0 \cong M$. So M is self-dual and hence reflexive as a $e_0\Lambda e_0$-module. $\qquad\square$

6B2. *The center.* Observe, if A is an NCCR over some ring R, then the center $Z(A)$ is an R-algebra. Under suitable hypotheses, they are actually isomorphic. For example, this holds if R an integrally closed Noetherian domain, by Zariski's main theorem (as $\operatorname{Spec} Z(A) \to \operatorname{Spec} R$ is finite and birational).

Instead of using this to establish our isomorphism, we consider an explicit map in the other direction. More generally, suppose A is a ring, $e \in A$ is an idempotent, and $R := eAe$. Then we have a canonical map

$$Z(A) \to R = eAe, \quad z \mapsto ez. \tag{6-1}$$

We call this the "Satake map" following the terminology for Hecke algebras, symplectic reflection algebras, etc.

Under natural conditions, the Satake map is well known to be an isomorphism. Namely, note that eA is an $(eAe) - A$ bimodule, and $\operatorname{End}_{A^{\mathrm{op}}}(eA) = eAe$. Then we have a natural map $A^{\mathrm{op}} \to \operatorname{End}_{eAe}(eA)$.

Lemma 6.13. *Suppose that (I) the natural map $A^{\mathrm{op}} \to \operatorname{End}_{eAe}(eA)$ is an isomorphism, and (II) eAe is commutative. Then the Satake map (6-1) is an isomorphism.*

Proof. We have an identification

$$Z(eAe) \cong \operatorname{End}_{eAe \otimes A^{\mathrm{op}}}(eA) \cong Z(A) \quad z \mapsto ez.$$

Since eAe is commutative, $Z(A) \cong eAe$, via the Satake map. $\square$

Corollary 6.14. *The Satake map (6-1) is an isomorphism for $A = \Lambda^1(\widetilde{A}_n)$ and $e = e_v$, the idempotent at any vertex.*

Proof. This is a direct consequence of Lemma 6.13, once we check hypotheses (I) and (II). Thanks to Proposition 6.12, $A \cong \operatorname{End}_{eAe}(eA)$ so (I) follows from $A \cong A^{\mathrm{op}}$, a consequence of the independence of orientation established in [25, Theorem 1.4]. By Shaw's Theorem 6.4, (II) holds (alternatively, the commutativity of the generators can be checked directly). $\square$

Corollary 6.15. $\Lambda^1(\widetilde{A}_n)$ *is an NCCR over its center.*

Proof. This follows immediately, provided we identify the $Z(\Lambda^1(\widetilde{A}_n))$-module structure on $\Lambda^1(\widetilde{A}_n)$ with left multiplication. Indeed, given $z \in Z(\Lambda^1(\widetilde{A}_n))$ (by tracing through the above maps) its action on $\operatorname{End}_{e\Lambda^1(\widetilde{A}_n)e}(M)$ via the Satake map is multiplication by ez. $\square$

Note that Corollary 6.14 and Theorem 6.4 immediately imply Corollary 6.7.

6B3. *The center as functions on a quiver variety.* It remains to identify the center with the algebra of functions on the multiplicative quiver variety.

In general, given a kQ_0-algebra A and a finite-dimensional kQ_0-module V, we have a canonical algebra homomorphism $\operatorname{ev} : A \to k[\operatorname{Rep}(A, V)] \otimes \operatorname{End}_k(V)$, called "evaluation": $\operatorname{ev}(a)(\rho) = \rho(a)$.

Suppose that $\rho : A \to \operatorname{End}(V)$ is an irreducible representation. Consider $z \in Z(A)$. If k is algebraically closed, then by Schur's Lemma, $\rho(z) = \lambda_{\rho,z} \operatorname{Id}_V$ for some scalar $\lambda_{\rho,z}$. However, we don't assume here

that k is algebraically closed. We could fix this by passing to the algebraic closure, but this turns out to be unnecessary as follows.

Lemma 6.16. *Suppose that $v \in Q_0$ is a vertex with* $\dim V_v = 1$. *Suppose ρ is an irreducible representation. Then* $\mathrm{End}(\rho) = k \cdot \mathrm{Id}_V$.

Proof. If $\phi \in \mathrm{End}(\rho)$, then $\rho(e_v)\phi = \phi\rho(e_v)$. Therefore, ϕ preserves $\rho(e_v)V = V_v$. As this has dimension one, we have $\phi|_{V_v} = \lambda \mathrm{Id}_{V_v}$. Now, $\phi - \lambda \mathrm{Id}_V$ is not invertible. By Schur's lemma over a general field, this implies that $\phi - \lambda \mathrm{Id}_V$ is zero. So $\phi = \lambda \mathrm{Id}_V$. $\square$

Corollary 6.17. *Let Q_0, A, V, v and ρ be as in* Lemma 6.16. *If $z \in Z(A)$, then $\rho(z) \in \mathrm{End}(V)$ is a scalar.*

Proof. Note that $\rho(z) \in \mathrm{End}(\rho)$. Then apply the lemma. $\square$

Corollary 6.18. *Suppose that for some vertex v, we have $V_v = 1$, and moreover that there exists an irreducible representation $A \to \mathrm{End}(V)$. Then the restriction $\mathrm{ev}|_{Z(A)}$ is an algebra map $Z(A) \to k[\mathrm{Rep}(A, V)] \cdot \mathrm{Id}_V$.*

Proof. Let $U \subseteq \mathrm{Rep}(A, V)$ be the locus of representations ρ such that $\mathrm{End}(\rho) = k \cdot \mathrm{Id}_V$. This is a Zariski open subset, since $k \cdot \mathrm{Id}_V$ is always contained in $\mathrm{End}(\rho)$. If $\rho \in \mathrm{Rep}(A, V)$ is irreducible, then by Lemma 6.16, $\rho \in U$. Thus, by our assumptions, U is nonempty. Since $\mathrm{Rep}(A, V)$ is a vector space, it is irreducible. We conclude that U is Zariski dense.

Now, for every $z \in Z(A)$, $\mathrm{ev}(z) : \mathrm{Rep}(A, V) \to \mathrm{End}(V)$ is scalar-valued on U. As U is dense, it is a scalar on all of $\mathrm{Rep}(A, V)$. Hence $\mathrm{ev}(z) \in k[\mathrm{Rep}(A, V)] \otimes \mathrm{Id}$. As z was arbitrary, we obtain the result. $\square$

Back to the situation at hand, for convenience let us orient $\widetilde{A}_n$ clockwise (note that the statement does not depend on orientation). We consider the vector space $V = k^{Q_0}$, which has the property $\dim V_v = 1$ for all $v \in Q_0$. Consider the representation on V where each clockwise arrow is the identity (i.e., the one-by-one matrix $[1]$) and each counterclockwise arrow is zero. This defines a representation of the localization L_Q that descends to an irreducible representation of $\Lambda^1(Q)$. Therefore, having satisfied the hypotheses of Corollary 6.18, we obtain a canonical map

$$\mathrm{ev}_Z : Z(\Lambda^1(Q)) \to k[\mathcal{M}_{0,1}(Q, \delta)]. \tag{6-2}$$

Proposition 6.19. *The map ev_Z is an isomorphism.*

Proof. To check surjectivity, let $f \in k[\mathcal{M}_{0,1}(Q, \delta)] = k[\mathrm{Rep}_\alpha(\Lambda^1(Q))]^{\mathrm{GL}_\alpha}$. We wish to show that $f \in \mathrm{ev}_Z(Z(\Lambda^1(Q)))$. Note that f is a polynomial in the matrix coefficient functions of the arrows (these are one by one matrices). To be invariant under GL_α, the polynomial must in fact be a polynomial in the functions defined by closed paths in the quiver: each such closed path is canonically a scalar, as it is an endomorphism of a one-dimensional vector space. Thus it suffices to assume that there is a single closed path $a \in e_v \Lambda^1(Q)e_v$ such that $\rho(a) = f(\rho) \cdot \mathrm{Id}_{V_v}$ for all ρ. As the Satake map is an isomorphism (Corollary 6.14), we must have $a = e_v z$ for some $z \in Z(\Lambda^1(Q))$. Then, $\rho(a) = \mathrm{ev}_Z(z) \cdot \mathrm{Id}_{V_v}$. Hence $f(\rho) = \mathrm{ev}_Z(z)$ for all $\rho \in \mathrm{Rep}_\alpha(\Lambda^1(Q))$. This shows that ev_Z is surjective.

By [Corollary 6.7](#) the source is an integral domain. Since we already proved surjectivity, injectivity will follow provided that the target also has dimension at least two. This can be seen by constructing a two-parameter family of representations, e.g., we can take the representations with all clockwise arrows a matrix (a) and all counterclockwise arrows a matrix (b), with $ab \neq -1$. Alternatively, this statement follows from [Theorem 5.4](#).   □

7. The strong free product property

In this section, we prove the strong free product property for connected quivers containing a cycle. We first establish the strong free product property for the quivers $\widetilde{A}_n$ for $n \geq 0$ using the diamond lemma to build a section of the quotient map $\pi : L \to \Lambda^q(\widetilde{A}_n)$. Then we establish the more general result using the corresponding result for *partial* multiplicative preprojective algebras; see [Section 3A](#) for the prerequisite definitions.

As results in Sections [3B](#), [3C](#), [4](#), [5](#), [6](#) rely on results established in this section, the reader should note that we do not use any results beyond [Section 3A](#); see [Remark 6.8](#).

7A. *The case of cycles.* Consider the quiver $\widetilde{A}_{n-1}$ with vertex set $(\widetilde{A}_{n-1})_0 := \{0, 1, \ldots, n-1\}$ and arrow set $(\widetilde{A}_{n-1})_1 = \{a_0, a_0^*, a_1, a_1^*, \ldots, a_{n-1}, a_{n-1}^*\}$ with $t(a_i) = i$ and $h(a_i) = i+1 \pmod{n} i < a_{i+1} < a_j^* < a_{j+1}^*$ for all $i, j \in \{0, 1, \ldots, n-2\}$. The multiplicative preprojective algebra for this quiver, with respect to the ordering, is defined to be

$$\Lambda^q(\widetilde{A}_{n-1}) := \frac{k\overline{\widetilde{A}}_{n-1}[(1+a_i a_i^*)^{-1}, (1+a_i^* a_i)^{-1}]_{i=0,\ldots,n-1}}{\langle \prod_{i=0}^{n-1}(1+a_i a_i^*) \prod_{i=0}^{n-1}(1+a_i^* a_i)^{-1} - \sum_{i=1}^{n} q_i e_i \rangle} =: \frac{L}{J}.$$

Writing $a := \sum_i a_i$, $a^* := \sum_i a_i^*$, and $q = \sum_i q_i e_i$ since

$$1 + aa^* = 1 + \sum_i^{n-1} a_i a_i^* = \prod_{i=0}^{n-1}(1+a_i a_i^*), \quad 1 + a^* a = 1 + \sum_i^{n-1} a_i^* a_i = \prod_{i=0}^{n-1}(1+a_i^* a_i)$$

we have

$$\Lambda^q(\widetilde{A}_{n-1}) := \frac{k\overline{\widetilde{A}}_{n-1}[(1+aa^*)^{-1}, (1+a^*a)^{-1}]}{\langle (1+aa^*)(1+a^*a)^{-1} - q \rangle}.$$

We write $r := (1+aa^*)(1+a^*a)^{-1} - q$ for this relation, S for the degree zero piece $k(\widetilde{A}_{n-1})_0$ of $\Lambda^q(\widetilde{A}_{n-1})$. As in [Section 3A](#), let $B := S[t, (q+t)^{-1}]$ and $\overline{B} = tB$, spanned over S by t^m, $(t')^m$, $m \geq 1$, for $t' := (q+t)^{-1} - q^{-1}$. Let $r' := (q+r)^{-1} - q^{-1}$.

We construct $\sigma : L/(r) *_S B \to L$ so that (L, r, σ', B) satisfies the strong free product property using an explicit basis.

Proposition 7.1. *L is a free left S-module with basis consisting of 1 together with all alternating products of elements of the following two sets, for $x := (1+aa^*)$:*

$$\mathfrak{B} := \{x^m a^\ell, x^m (a^*)^\ell \mid m \in \mathbb{Z}, \ell \in \mathbb{N}\}. \quad \mathfrak{R} := \{r^m, (r')^m \mid m \in \mathbb{N}\}.$$

In particular, $\mathfrak{B}$ *forms a basis for* $\Lambda^q(\widetilde{A}_{n-1}) = L/(r)$, *and* (L, r, σ, B) *satisfies the strong free product property, with* σ *induced from the inclusion of* $\mathfrak{B}$ *into* L.

Proof. Note that, for every vertex i, we have $e_i a = a e_j$ for a unique j, and similarly for the elements $a^*, x, y := 1 + a^* a, x^{-1}, y^{-1}$, and by definition, $e_i r = r e_i$. Therefore L is spanned as a *left* S-module by noncommutative monomials in $a, a^*, x, y, x^{-1}, y^{-1}, r$, and r'. Define $\mathcal{M} := \langle a, a^*, x, y, x^{-1}, y^{-1}, r, r' \rangle$ the set of monomials and $\mathcal{P} := S\langle a, a^*, x, y, x^{-1}, y^{-1}, r, r' \rangle$ the set of noncommutative polynomials with coefficients in S.

The set of relations, R, is the two-sided ideal generated by

$$xx^{-1} = 1 = x^{-1}x, \quad yy^{-1} = 1 = y^{-1}y, \quad x = 1 + aa^*, \quad y = 1 + a^*a, \tag{7-1}$$

$$r = xy^{-1} - q, \quad r' = yx^{-1} - q^{-1}. \tag{7-2}$$

So we have the presentation $L \cong \mathcal{P}/R$ and hence $\Lambda^q(\widetilde{A}_{n-1}) \cong \mathcal{P}/(R, r)$.

The idea of the proof is to produce a basis of the *quotient* $L = \mathcal{P}/R$ by realizing it as an S-module subspace $\mathcal{P}_{\mathrm{irr}} \subset \mathcal{P}$ spanned by *irreducible* monomials, defined below.

That is, we define an ordering, $\leq$, on the set $\mathcal{M}$. Then we use this ordering to build a system of reductions $\{r_i\}$ from R by reading each relation $R_i \in R$ as an S-module map, r_i, taking the leading term $\mathrm{lt}(R_i)$ to the smaller term $\mathrm{lt}(R_i) - R_i$. We extend r_i to $\mathcal{M}$ via $a\,\mathrm{lt}(R_i)b \mapsto a(\mathrm{lt}(R_i) - R_i)b$ for $a, b \in \mathcal{M}$. We say $m \in \mathcal{M}$ is *irreducible* (or in normal form) if every reduction is the identity on m or, equivalently, if m doesn't contain the leading term of any relation as a submonomial.

We will show that every $m \in \mathcal{M}$, reduces *uniquely* to normal form, $m' \in \mathcal{P}_{\mathrm{irr}}$, after applying *finitely* many reductions. This implies the S-module map $\mathrm{r} : \mathcal{P} \to \mathcal{P}_{\mathrm{irr}}$ given by S-linear extension of $m \mapsto m'$ is well-defined. Hence r splits the inclusion map $\mathcal{P}_{\mathrm{irr}} \to \mathcal{P}$. As $\ker(\mathrm{r}) = R$, we conclude that r induces an S-module isomorphism $L \cong \mathcal{P}_{\mathrm{irr}}$ and the set of irreducible monomials gives our desired basis.

First we equip $\mathcal{M}$ with an ordering. Fix $w, z, z' \in \mathcal{M}$ and subsets $Z, Z' \subset \mathcal{M}$. Define

$$n_z(w) := \text{ the number of occurrences of } z \text{ in } w, \tag{7-3}$$

$$n_{z,z'}(w) := \text{ the number of occurrences of } z \text{ and } z' \text{ in } w \text{ with } z \text{ appearing before } z', \tag{7-4}$$

$$n_Z(w) := \sum_{z \in Z} n_z(w) \quad \text{and} \quad n_{Z,Z'} := \sum_{z \in Z, z' \in Z'} n_{z,z'}. \tag{7-5}$$

Define a function $N : \mathcal{M} \to \mathbb{N}^5$ taking w to

$$N(w) := (n_a(w), n_{\{a,a^*\},\{x,x^{-1},y,y^{-1}\}}(w), n_{\{ax,ax^{-1}\}}(w), n_{\{y,y^{-1}\}}(w), n_{\{r,r'\}}(w)) \in \mathbb{N}^5. \tag{7-6}$$

Define the ordering $w' \leq w$ in $\mathcal{M}$ if $N(w') \leq N(w)$ in the lexicographical ordering on $\mathbb{N}^5$. This induces an ordering on $\mathcal{P}$, by extending N to $\mathcal{P}$, via $N\left(\sum_i m_i\right) := \max_i\{N(m_i)\}$.

Next, using this ordering, we define a system of reductions from the relations in (7-1), (7-2):

Inverse reductions: $xx^{-1}, x^{-1}x, yy^{-1}, y^{-1}y \mapsto 1$.

Short cycle reductions: $aa^* \mapsto x - 1, a^*a \mapsto y - 1$.

Reordering reductions: $a^* x^{\pm 1} \mapsto y^{\pm 1} a^*$, $ay^{\pm 1} \mapsto x^{\pm 1} a$.

Substitution reductions: $y^{-1} \mapsto x^{-1}(r + q)$, $y \mapsto (r' + q^{-1})x$ (if not preceded by a); $ax \mapsto a(r + q)y$,
$$ax^{-1} \mapsto ay^{-1}(r' + q^{-1}).$$

Reductions in B: rr', $r'r \mapsto -qr' - q^{-1}r$.

By design, if w' is obtained from w by applying a reduction, then $N(w') < N(w)$. This implies that any sequence of reductions terminates in finitely many steps, by the descending chain condition for the lexicographical ordering on $\mathbb{N}^5$.

Next observe that under this reduction system $m \in \mathcal{M}$ is in normal form (or irreducible) if and only if it is alternating in $\mathfrak{B}$ and $\mathfrak{R}$. Therefore, the set of alternating words in $\mathfrak{B}$ and $\mathfrak{R}$ is a spanning set. It remains to show that $m \in \mathcal{M}$ reduces uniquely to normal form, which establishes linear independence.

To prove uniqueness, we need to show whenever w reduces to $r_1(w)$ and $r_2(w)$ that each further reduces to the same irreducible w'. Bergman's diamond lemma says to show uniqueness for all monomials w it suffices to show uniqueness for specific $w = xyz$ where xy and yz are both leading terms for a relation in (7-1), (7-2) [11, Theorem 1.2]. These w are called *overlap ambiguities*. If the two reduced expressions of $w = xyz$ (i.e., $r_1(xy)z$ and $xr_2(yz)$) both further reduce to the same w', we say the overlap ambiguity *resolves*. To complete the proof it suffices to show all overlap ambiguities resolve.

Next, notice that any unresolvable ambiguity involving $y^{\pm 1}$ gives rise to an unresolvable ambiguity not involving $y^{\pm 1}$ by applying the substitution or reordering reductions. So it suffices to check ambiguities in the following smaller system of reductions:

Inverse reductions:

(1) $xx^{-1} \xmapsto{r_1} 1$. 　　　　　　　　　　　　　　　(2) $x^{-1}x \xmapsto{r_2} 1$.

Short cycle reductions:

(3) $aa^* \xmapsto{r_3} x - 1$. 　　　　　　　　　　　　　(4) $a^*a \xmapsto{r_4} (r' + q^{-1})x - 1$.

Reordering reductions:

(5) $a^*x \xmapsto{r_5} (r' + q^{-1})xa^*$. 　　　　　　　　　(6) $ax \xmapsto{r_6} qxa - qar'x$.

(7) $ax^{-1} \xmapsto{r_7} x^{-1}a(r' + q^{-1})$. 　　　　　　　(8) $a^*x^{-1} \xmapsto{r_8} x^{-1}(r + q)a^*$.

Substitution reductions:

(9) $y^{-1} \mapsto x^{-1}(r + q)$. 　　　　　　　　　　　　(10) $y \mapsto (r' + q^{-1})x$.

Reductions in B:

(11) $rr' \mapsto -qr' - q^{-1}r$. 　　　　　　　　　　　　(12) $r'r \mapsto -qr' - q^{-1}r$.

The substitution reductions and reductions in B don't overlap with any others, so the only overlap ambiguities are amongst the (1)–(8), involving the generators a, a^*, x, x^{-1} only. The inverse, short cycle,

and reordering reductions are quadratic in these generators giving rise to the following 12 cubic overlap ambiguities:

(I) $xx^{-1}x$ (IV) a^*aa^* (VII) $a^*x^{-1}x$ (X) a^*ax

(II) $x^{-1}xx^{-1}$ (V) a^*xx^{-1} (VIII) $ax^{-1}x$ (XI) aa^*x^{-1}

(III) aa^*a (VI) axx^{-1} (IX) aa^*x (XII) a^*ax^{-1}

The resolutions of (I) and (II) are immediate (and are completely general, having to do with a basis for $k[x, x^{-1}]$). Here is a summary of the remaining resolutions of ambiguities:

$$\text{(III)} \quad (r_3 - r_6 \circ r_4)(aa^*a) = 0 \qquad \text{(VIII)} \quad (r_8 \circ r_7 - r_2)(ax^{-1}x) = 0$$

$$\text{(IV)} \quad (r_4 - r_5 \circ r_3)(a^*aa^*) = 0 \qquad \text{(IX)} \quad (r_3 - r_3 \circ r_6 \circ r_5)(aa^*x) = 0$$

$$\text{(V)} \quad (r_8 \circ r_5 - r_1)(a^*xx^{-1}) = 0 \qquad \text{(X)} \quad (r_4 - r_4 \circ r_4 \circ r_5 \circ r_6)(a^*ax) = 0$$

$$\text{(VI)} \quad (r_7 \circ r_6 - r_1)(axx^{-1}) = 0 \qquad \text{(XI)} \quad (r_3 - r_3 \circ r_7 \circ r_8)(aa^*x^{-1}) = 0$$

$$\text{(VII)} \quad (r_5 \circ r_8 - r_2)(a^*x^{-1}x) = 0 \qquad \text{(XII)} \quad (r_4 - r_4 \circ r_8 \circ r_7)(a^*ax^{-1}) = 0$$

We explicitly demonstrate (X), one of the more involved resolutions:

$$a^*ax = (a^*a)x \xmapsto{r_4} [(r' + q^{-1})x - 1]x = (r' + q^{-1})x^2 - x,$$

$$a^*ax = a^*(ax) \xmapsto{r_6} a^*(qxa - qar'x)$$

$$\xmapsto{r_4 \circ r_5} q(r' + q^{-1})xa^*a - q((r' + q^{-1})x - 1)r'x$$

$$\xmapsto{r_4} q(r' + q^{-1})x((r' + q^{-1})x - 1) - q((r' + q^{-1})x - 1)r'x$$

$$= q(r' + q^{-1})x(q^{-1}x - 1) + qr'x = (r' + q^{-1})x^2 - x. \quad \square$$

Remark 7.2. The choice of $\bar{B}$ was important here. If we instead had defined it so that $(q + t)^{-1} \in \bar{B}$, i.e., if we replace $r' = (q + r)^{-1} - q^{-1} \in \mathfrak{R}$ by $(q + r)^{-1}$, then our desired basis would no longer be linearly independent. Indeed, reducing aa^*a one way, we get $(x - 1)a = xa - a$, which is irreducible, whereas the other way we get $a(y - 1) = a(q + r)^{-1}x - a$, also irreducible. That is, $xa = a(q + r)^{-1}x$, an equality of two distinct irreducible elements.

Proposition 7.3. $\Lambda^q(\widetilde{A}_n)$ is prime for all $n \geq 0$ and all $q \in (k^\times)^{n+1}$.

Proof. We need to show, for every pair $f, g \in \Lambda^q(\widetilde{A}_n)$, both nonzero, there exists some $h \in \Lambda^q(\widetilde{A}_n)$ such that $fhg \neq 0$. It suffices to show that there exists vertices i, j and h such that $e_i fe_jhg \neq 0$, and hence we can take f to be a linear combination of basis elements that all begin at i and end j. By right multiplication by a^{n-j} or $(a^*)^j$, one can take f to be a linear combination of basis elements ending at vertex 0. By left multiplication by a^i or $(a^*)^{n-i}$ and then applying reordering reductions — the q-commutator, $ax - qxa$ is zero, for instance, in $\Lambda^q(\widetilde{A}_n)$ — one can take f to be a linear combination of basis elements starting and ending at vertex 0. In fact, f is of the form $e_0 f_1(x, x^{-1}) f_2(a^{n+1})$, where $f_1 \neq 0$ and f_2 has nonzero constant term. And similarly, we can take $g = e_0 g_1(x, x^{-1}) g_2(a^{n+1})$. Then their product has nonzero term $e_0 f_1(x, x^{-1}) f_2(a^{n+1})(0) g_1(x, x^{-1}) g_2(a^{n+1})(0)$ and hence is nonzero. $\square$

7B. *Partial multiplicative preprojective algebras.* First we define a *partial multiplicative* preprojective algebra following the definition of a partial preprojective algebra by [31, Definition 3.1.1].

Definition 7.4. Fix a quiver Q and $q \in (k^*)^{Q_0}$. Define a partition of the vertex set $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ into a set $\mathcal{B}$ of *black* vertices and a set $\mathcal{W}$ of *white* vertices. The *partial multiplicative preprojective algebra* of $(Q, \mathcal{W})$ is

$$\Lambda^q(Q, \mathcal{W}) := L/(r_\mathcal{B}), \quad \text{where} \quad r_\mathcal{B} := 1_\mathcal{B} r 1_\mathcal{B}, \quad \text{for} \quad 1_\mathcal{B} := \sum_{j \in \mathcal{B}} e_j.$$

In words, we don't enforce the relations at the white vertices. Hence this algebra interpolates between $\Lambda^q(Q, Q_0) = L$ and $\Lambda^q(Q, \varnothing) = \Lambda^q(Q)$.

Definition 7.5. Let Q be a quiver and let Γ be its underlying graph. Fix $\mathcal{R} \subset Q_0$:

- A subgraph $T \subset \Gamma$ is a *tree* if it is connected and acyclic.

- A tree $T \subset \Gamma$ is *rooted in* $\mathcal{R}$ if it has a single vertex, called the root, in $\mathcal{R}$.

- A *forest rooted in* $\mathcal{R}$ is a disjoint union of trees *rooted in* $\mathcal{R}$.

- A subgraph $S \subset \Gamma$ is *spanning* if the vertex set of S is Q_0.

Notice that every doubled quiver $\overline{Q}$ with $\mathcal{W} \subset Q_0$ nonempty has a spanning forest, F, rooted in $\mathcal{W}$. We view such an F as a subquiver of $\overline{Q}$ by orienting the arrows *towards* the roots, see Figure 1. Since the isomorphism class of $\Lambda^q(Q)$ is independent of the orientation of Q, see Remark 2.2, we can assume that $F_1 \subset Q_1$.

Let $B := \mathcal{B}[t, (t + q)^{-1}]$. Each choice of spanning forest of $\overline{Q}$ rooted at $\mathcal{W}$ gives rise to a linear isomorphism $\sigma' : \Lambda^q(Q, \mathcal{W}) *_{k Q_0} B \to L$ and hence a basis for $\Lambda^q(Q, \mathcal{W}) = L/(r_\mathcal{B})$.

Proposition 7.6. *Let Q be a connected quiver and $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ a decomposition into black and white vertices with $\mathcal{W} \neq \varnothing$. Then $(L, r_\mathcal{B}, \sigma, B)$ satisfies the strong free product property for some choice of σ.*

In more detail, let $F \subset \overline{Q}$ be a spanning forest rooted in $\mathcal{W}$ with arrows $F_1 \subset Q_1$ directed towards the roots.

A basis for L is given by concatenable words in the set,

$$\{a, \ x_a, \ x_a^{-1} \mid a \in \overline{Q}_1\} \cup \{r_\mathcal{B}, \ r_\mathcal{B}' := (q + r_\mathcal{B})^{-1} - q^{-1}\},$$

such that the following subwords do not occur:

$$x_a x_a^{-1}, \ x_a^{-1} x_a, \ aa^*, \ ax_{a^*}^{\pm 1} \ \text{for } a \in \overline{Q}_1, \quad x_a^{\pm 1}, \ x_{a^*}^{-1}, \ x_{a^*} a^*, \ x_{a^*}^2 \ \text{for } a \in F_1, \quad r_\mathcal{B} r_\mathcal{B}', \ r_\mathcal{B}' r_\mathcal{B}.$$

The words in which $r_\mathcal{B}$ and $r_\mathcal{B}'$ do not occur form a basis for $\Lambda^q(Q, \mathcal{W}) = L/(r_\mathcal{B})$, and the section σ is given by the inclusion of these elements.

Proof. The proof parallels that of Proposition 7.1. Write $r := r_\mathcal{B}$ and $r' := r_\mathcal{B}'$.
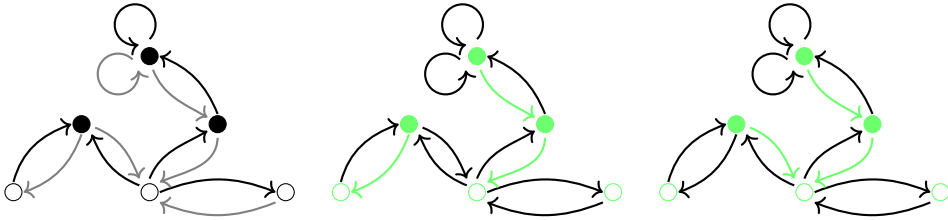
Figure 1. The quiver on the left is a doubled quiver, obtained by adding the gray arrows. It has three white vertices and three black vertices. The middle and right diagrams show two inequivalent spanning forests, in light green, with roots at the white vertices.

Note that L is spanned by the set, $\mathcal{M}$, of concatenable words in $\{a, x_a, x_a^{-1}, r, r' \mid a \in \bar{Q}_1\}$. These words are subject to the following relations, depending on a choice of ordering $\leq$ on the arrows $a \in \bar{Q}_1$:

$$x_a x_a^{-1} = 1 = x_a^{-1} x_a, \quad x_a = 1 + aa^*, \tag{7-7}$$

$$r = \prod_{\substack{a \in \bar{Q} \\ t(a) \in \mathcal{B}}} x_a^{\epsilon(a)} - q, \quad r' = \prod_{\substack{a \in \bar{Q} \\ t(a) \in \mathcal{B}}} x_a^{-\epsilon(a)} - q^{-1}, \tag{7-8}$$

$$rr' = r'r = -qr' - q^{-1}r, \tag{7-9}$$

where recall we write $t(a)$ for the *tail* or source of a, not the target. Define

$$l_a := \prod_{\substack{b \in \bar{Q} \\ b < a,\ t(b) \in \mathcal{B}}} x_b^{\epsilon(b)} \quad \text{and} \quad r_a := \prod_{\substack{b \in \bar{Q} \\ b > a,\ t(b) \in \mathcal{B}}} x_b^{\epsilon(b)}.$$

So for $a \in \bar{Q}_1$ with $t(a) \in \mathcal{B}$ we have the relation

$$l_a (1 + aa^*)^{\epsilon(a)} r_a = (r + q) e_{t(a)} \implies x_a^{\epsilon(a)} = l_a^{-1}(r + q)(e_{t(a)}) r_a^{-1}.$$

Hence in L, define $\mathrm{red}_a^{\epsilon(a)} := l_a^{-1}(r + q)(e_{t(a)}) r_a^{-1}$.

We implement the above relations with the following reductions:

Inverse reductions: $x_a x_a^{-1}, x_a^{-1} x_a \mapsto 1$ for $a \in \bar{Q}_1$.

Short cycle reductions: $aa^* \mapsto x_a - 1$ for $a \in \bar{Q}_1$.

Reordering reductions: $a^* x_a^{\pm} \mapsto x_{a^*}^{\pm} a^*$ for $a \in \bar{Q}_1$.

Substitution reductions: $x_a^{\pm} \mapsto \mathrm{red}_a^{\pm},\ x_{a^*}^{-1} \mapsto 1 - a^* \mathrm{red}_a^{-1} a,\ x_{a^*}^2 \mapsto x_{a^*} + a^* \mathrm{red}_a a,\ x_{a^*} a^* \mapsto a^* \mathrm{red}_a$, for $a \in F_1$

Reductions in B: $rr', r'r \mapsto -qr' - q^{-1}r$.

For each word $w \in \mathcal{M}$, use the definition in (7-4) to define a weighted size,

$$\varphi_a(w) := n_{\{a, a^*\}}(w) + \tfrac{3}{2} n_{\{x_a, x_{a^*}\}}(w) + 3 n_{\{x_a^{-1}, x_{a^*}^{-1}\}}(w)$$

for each $a \in \bar{Q}_1$. Define a total ordering on the arrows $(\bar{Q}_1, \prec)$ such that,

$$a \prec a' \text{ if } a \in F_1, a' \in \bar{Q}_1 \setminus F_1, \text{ or if } a, a' \in F_1 \text{ with } a' \text{ disconnected from } \mathcal{W} \text{ in } F_1 \setminus \{a\}.$$

Intuitively, we are saying that arrows in the spanning forest come before the rest in the ordering, with arrows closer to the white vertices coming first. Using $\prec$, φ_a, and (7-4), (7-5) define

$$N' : \mathcal{M} \to \mathbb{N}^{(\bar{Q}_1, \prec)} \times \mathbb{N}^2, \quad w \mapsto (2\varphi_a(w), n_{\{a|a\in\bar{Q}_1\},\{x_a|a\in\bar{Q}_1\}}(w), n_{\{r,r'\}}(w)),$$

from which we say $w \leq w'$ if $N'(w) \leq N'(w')$ in the lexicographical ordering on $\mathbb{N}^{|\bar{Q}_1|+2}$.

Notice, as in Proposition 7.1, that $N'(r_i(w)) < N'(w)$ for any word w and reduction r_i with $r_i(w) \neq w$. First notice that, by design, φ_a decreases under the following reductions:

$\underline{\text{Inverse reductions:}}$ $\varphi_a(x_a x_a^{-1}) = \varphi_a(x_a^{-1} x_a) = 3 + \frac{3}{2} > 0 = \varphi_a(1).$

$\underline{\text{Short cycle reductions:}}$ $\varphi_a(aa^*) = 2 > \frac{3}{2} = \varphi_a(x_a).$

$\underline{\text{Substitution reductions:}}$ $\varphi_a(x_a) = \frac{3}{2} > 0 = \varphi_a(\text{red}_a),\ \varphi_a(x_a^{-1}) = 3 > 0 = \varphi_a(\text{red}_a^{-1}),\ \varphi(x_{a^*}^{-1}) = 3 > 2 = \varphi_a(a^* \text{red}_a^{-1} a),\ \varphi(x_{a^*}^2) = 3 > 2 = \varphi_a(a^* \text{red}_a a)$ and $\varphi(x_{a^*}^2) = 3 > \frac{3}{2} = \varphi_a(x_{a^*}),$ $\varphi_a(x_{a^*}a^*) = \frac{5}{2} > 1 = \varphi_a(a^* \text{red}_a).$

For the substitution reductions observe that red_a for $a \in F_1$ has subwords $x_b^{\pm 1}, x_{b^*}^{\pm 1}$ for only $b \in F_1$ which are necessarily *farther* from the root than a, and the remaining arrows are not in the spanning forest. Consequently, φ_a decreasing — despite φ_b increasing for some $b \succ a$ — implies that N' decreases. The reordering reductions preserve all φ_a but decrease $n_{\{a|a\in\bar{Q}_1\},\{x_a|a\in\bar{Q}_1\}}$ by definition, and hence decrease N'. The reductions in B preserve all φ_a and $n_{\{a|a\in\bar{Q}_1\},\{x_a|a\in\bar{Q}_1\}}$ but decrease $n_{\{r,r'\}}$, hence N'.

We conclude that every $w \in \mathcal{M}$ reduces to a kQ_0-linear combination of words without subwords in the leading terms of the reductions:

$$\{x_a x_a^{-1}, x_a^{-1} x_a, aa^*, ax_{a^*}, ax_{a^*}^{-1} \mid a \in \bar{Q}_1\} \cup \{x_{a^*}^{-1}, x_{a^*}a, x_{a^*}^2 \mid a \in F_1\}$$

after applying *finitely* many reductions.

Note that some generators are nonreduced: x_a, x_a^{-1}, and $x_{a^*}^{-1}$ for $a \in F_1$. Therefore, we can put in reductions for each of these and throw out all other reductions involving these generators, provided we check that all the defining relations still reduce to zero. We have the reductions:

(1) $x_a x_a^{-1} \overset{r_1}{\longmapsto} 1$ for $a \in \bar{Q}_1$. (5) $ax_{a^*}^{\pm} \overset{r_5}{\longmapsto} x_a^{\pm} a$ for $a \notin \bar{F}_1$.

(2) $x_a^{-1} x_a \overset{r_2}{\longmapsto} 1$ for $a \in \bar{Q}_1$. (6) $ax_{a^*} \overset{r_6}{\longmapsto} \text{red}_a a$ for $a \in F_1$.

(3) $aa^* \overset{r_3}{\longmapsto} \text{red}_a - 1$ for $a \in F_1$. (7) $x_{a^*}^2 \overset{r_7}{\longmapsto} x_{a^*} + a^* \text{red}_a a$ for $a \in F_1$.

(4) $aa^* \overset{r_4}{\longmapsto} x_a - 1$ for $a \notin F_1$. (8) $x_{a^*}a^* \overset{r_8}{\longmapsto} a^* \text{red}_a$ for $a \in F_1$.

Which don't overlap with the remaining reductions:

Substitution reductions: $x_a^{\pm 1} \mapsto \text{red}_a^{\pm 1},\ x_{a^*}^{-1} \mapsto 1 - a^* x_a^{-1} a,\ a \in F_1.$

Reductions in B: $rr', r'r \mapsto -qr' - q^{-1}r.$

As before, reductions (3) and (4) imply the relations $x_a = 1 + aa^*$, whereas the Substitution Reductions imply the defining relations for r, r'. So this is a valid reduction system.

This reduction system has thirteen ambiguities:

(I) $x_a x_a^{-1} x_a$ for $a \notin \overline{F}_1$. (VI) $x_{a^*}^2 a^*$ for $a \in F_1$. (XI) $aa^* a$ for $a \in \overline{Q}_1 \backslash \overline{F}_1$.

(II) $x_a^{-1} x_a x_a^{-1}$ for $a \notin \overline{F}_1$. (VII) $x_{a^*} a^* a$ for $a \in F_1$. (XII) $aa^* a$ for $a \in F_1$.

(III) $a x_{a^*} x_{a^*}^{-1}$ for $a \notin \overline{F}_1$. (VIII) $a x_{a^*} a^*$ for $a \in F_1$. (XIII) $aa^* a$ for $a^* \in F_1$.

(IV) $a x_{a^*}^{-1} x_{a^*}$ for $a \notin \overline{F}_1$. (IX) $a^* a x_{a^*}$ for $a \in F_1$.

(V) $a x_{a^*}^2$ for $a \in F_1$. (X) $a^* a x_{a^*}$ for $a \in \overline{Q} \backslash F_1$.

Which all resolve by the resolutions:

(I) $x_a x_a^{-1} x_a$ for $a \notin \overline{F}_1$.   (VIII) $(r_3 \circ r_6 - r_3 \circ r_8)(a x_{a^*} a^*) = 0$.

(II) $(r_2 - r_1)(x_a^{-1} x_a x_a^{-1}) = 0$.   (IX) $(r_7 \circ r_4 - r_6)(a^* a x_{a^*}) = 0$.

(III) $(r_1 - r_1 \circ r_5 \circ r_5)(a x_{a^*} x_{a^*}^{-1}) = 0$.   (X) $(r_4 \circ r_5 \circ r_5 - r_4)(a^* a x_{a^*}) = 0$.

(IV) $(r_2 - r_2 \circ r_5 \circ r_5)(a x_{a^*}^{-1} x_{a^*}) = 0$.   (XI) $(r_4 - r_5 \circ r_4)(aa^* a) = 0$.

(V) $(r_3 \circ r_6 \circ r_7 - r_6 \circ r_6)(a x_{a^*}^2) = 0$.   (XII) $(r_6 \circ r_4 - r_3)(aa^* a) = 0$.

(VI) $(r_3 \circ r_8 \circ r_7 - r_8 \circ r_8)(x_{a^*}^2 a^*) = 0$.   (XIII) $(r_8 \circ r_4 - r_3)(aa^* a) = 0$.

(VII) $(r_7 \circ r_4 - r_8)(x_{a^*} a^* a) = 0$.

The resolutions of the ambiguities (I)–(IV) and (X)–(XIII) are quick, leaving the computational heart of the calculations with the five resolutions (V)–(IX). Note that the resolutions for (V) and (VI) are identical after swapping the roles of reductions r_6 and r_8, and similarly for (IX) and (VII), leaving three calculations: (V), (VIII), and (IX). These ambiguities express the overlap of r_6 with r_7, r_8, and r_4 respectively and further reduce uniquely to $\mathrm{red}_a^2 \, a$, $\mathrm{red}_a (\mathrm{red}_a - 1)$, and $a^* \, \mathrm{red}_a \, a$. $\qquad\square$

7C. A convenient substitution. It will be convenient for us to make the substitutions

$$\overline{x_a^{\pm}} := x_a^{\pm 1} - 1, \tag{7-10}$$

motivated as follows.

Let $A \cong \Lambda^q(Q, \mathcal{W})$ for Q connected, and $\mathcal{W}$ possibly empty. Let I be the ideal generated by all paths beginning and ending at vertices having either $q = 1$ or in $\mathcal{W}$ (if nonempty). Then A/I is nonzero, and we can make use of the I-adic filtration. The modified generators $\overline{x_a^{\pm}}$, for a an arrow in I, have the advantage of lying in the ideal I. As we will show, in the cases Q contains a cycle and $\mathcal{W} \neq \varnothing$, the I-adic filtration is Hausdorff.

Thus, we get an embedding of A into the completion $\widehat{A}_I$, realizing $\overline{x_a^{\pm}}$ as power series with zero constant term. In the special case where $q = 1$ at all black vertices, this embedding sends every modified generator, $\overline{x_a^{\pm}}$, to a noncommutative power series in arrows *with* zero constant term. This completion

is closely related to the completion of (partial) additive preprojective algebras with $\lambda = 0$ at all black vertices.

Practically speaking, we only require the above substitution at white vertices to obtain a basis for quivers containing cycles, see Section 7D. But theoretically, we advocate for this substitution at any vertex where we think of q as a deformation parameter based at $q = 1$.

Let us explain how this substitution works in the case of the cycle $\widetilde{A}_n$ (although we do not strictly need it in that case). We formally set $\overline{x^{\pm}} := x^{\pm 1} - 1$ and $\overline{y^{\pm}} := y^{\pm 1} - 1$; then the modified reductions from Section 7A are the following ones:

Inverse reductions: $\overline{x^+}\overline{x^-}, \overline{x^-}\overline{x^+} \mapsto -\overline{x^+} - \overline{x^-}$ and $\overline{y^+}\overline{y^-}, \overline{y^-}\overline{y^+} \mapsto -\overline{y^+} - \overline{y^-}$.

Short cycle reductions: $aa^* \mapsto \overline{x^+}, \quad a^*a \mapsto \overline{y^+}$.

Reordering reductions: $a^*\overline{x^{\pm}} \mapsto \overline{x^{\pm}}a^*, \quad a\overline{y^{\pm}} \mapsto \overline{y^{\pm}}a$.

Substitution reductions: $\overline{y^-} \mapsto \overline{x^-}(r + q) + r + (q - 1)$, (if not preceded by a); $\overline{y^+} \mapsto (r' + q^{-1})\overline{x^+} + r' + (q^{-1} - 1)$ (if not preceded by a); $a\overline{x^+} \mapsto a(r + q)\overline{y^+} + ar + (q - 1)a$; $a\overline{x^-} \mapsto a\overline{y^-}(r' + q^{-1}) + ar' + (q^{-1} - 1)a$.

This produces the same ambiguities as before, which resolve in the same way after eliminating the nonreduced generators $\overline{y^{\pm}}$ (another way to say this is that the reductions are the same up to the change of variables, so ambiguities resolve if and only if they did before). The modified ordering function,

$$N^z(w) := (n_a(w), n_{\{a,a^*\},\{\overline{x^+},\overline{x^-},\overline{y^+},\overline{y^-}\}}(w), n_{\{a\overline{x^+},a\overline{x^-}\}}(w), n_{\{\overline{y^+},\overline{y^-}\}}(w)),$$

is strictly decreasing under applications of reductions and hence every term reduces after applying finitely many reductions. So we have proven the following variant of Proposition 7.1:

Proposition 7.7. *Let $Q \cong \widetilde{A}_n$ be a cycle. Then L_Q is a free left kQ_0-module with basis given by alternating words in $\mathfrak{R}$ and $\mathfrak{B}' := \{(\overline{x^{\pm}})^m a^{\ell}, (\overline{x^{\pm}})^m (a^*)^{\ell} \mid m \in \mathbb{N}, \ell \in \mathbb{N}\}$. Hence $\mathfrak{B}'$ is a basis for $\Lambda^q(Q)$.*

In the case of the partial multiplicative preprojective algebra, the modified reductions are as follows:

Inverse reductions: $\overline{x_a^+}\,\overline{x_a^-}, \overline{x_a^-}\,\overline{x_a^+} \mapsto -\overline{x_a^+} - \overline{x_a^-}$ for $a \in \overline{Q}_1$.

Short cycle reductions: $aa^* \mapsto \overline{x_a^+}$ for $a \in \overline{Q}_1$.

Reordering reductions: $a^*\overline{x_a^{\pm}} \mapsto \overline{x_{a^*}^{\pm}}a^*$ for $a \in \overline{Q}_1$.

Substitution reductions: $\overline{x_a^{\pm}} \mapsto \mathrm{red}_a^{\pm} - 1, \quad \overline{x_{a^*}^-} \mapsto -a^* \mathrm{red}_a^{-1} a, \quad \overline{x_{a^*}^{+2}} \mapsto -\overline{x_{a^*}^+} + a^* \mathrm{red}_a\, a, \quad \overline{x_{a^*}^+}a^* \mapsto a^*(\mathrm{red}_a - 1)$, for $a \in F_1$.

Again, the same ordering function applies here and strictly decreases under these reductions. The ambiguities must resolve since they did before.

Proposition 7.8. *Let $Q, \mathcal{B}, \mathcal{W}$ be as in Proposition 7.6. Then L_Q is a free left kQ_0-module with basis given by concatenable words in the set,*

$$\{a, \overline{x_a^+}, \overline{x_a^-} \mid a \in \overline{Q}_1\} \cup \{r_{\mathcal{B}}, r'_{\mathcal{B}}\},$$

such that the following subwords do not occur:

$$\overline{x_a^+}\,\overline{x_a^-},\ \overline{x_a^-}\,\overline{x_a^+},\ aa^*,\ a\overline{x_{a^*}^{\pm}}\ \text{for } a \in \overline{Q}_1, \quad \overline{x_a^{\pm}},\ \overline{x_{a^*}^-},\ \overline{x_{a^*}^+}a^*,\ \overline{x_{a^*}^{+2}}\ \text{for } a \in F_1, \quad r_{\mathcal{B}}r_{\mathcal{B}}',\ r_{\mathcal{B}}'r_{\mathcal{B}}.$$

A basis for Λ_Q^q *as a free* kQ_0-*module is given by those words above not containing* $r_{\mathcal{B}}, r_{\mathcal{B}}'$.

Remark 7.9. Note that, for the following subsection, we only require the substitutions $\overline{x_a}$ in the case where the arrow a begins at a white vertex (which in particular implies that $a \notin F_1$, although it could be that $a^* \in F_1$). If we only make these substitutions, it is similarly easy to write the above reductions in the case where for certain arrows $\overline{x_a^{\pm}}$ appears and for others $x_a^{\pm 1}$ appears; we leave this to the reader.

The only thing that we require from the above in the next subsection is the following observation:

Reductions on $1_{\mathcal{W}}L_Q1_{\mathcal{W}}$ *preserve the augmentation ideal,* $\ker(\Lambda^q(Q, \mathcal{W}) \to k\mathcal{W})$. (7-11)

In other words, any monomial of positive length beginning and ending at white vertices reduces to a linear combination of other such monomials. This was not true with the original generators (e.g., looking at the inverse reductions).

7D. *Quivers containing cycles.* In this section, we prove the strong free product property for a connected quiver containing a cycle, along with providing a natural decomposition and basis for its multiplicative preprojective algebra. In more detail, the multiplicative preprojective algebra decomposes (as a vector space) into a free product of the multiplicative preprojective algebra for the cycle and a *partial* multiplicative preprojective algebra for the complement of the cycle. This technique should extend to the case of general extended Dynkin quivers, hence reducing Conjecture 1.1 to the extended Dynkin case.

Let Q be a connected quiver containing a cycle Q_E, with complement $Q' := Q \setminus Q_E$. Let $\mathcal{W} := (Q_E)_0$, so the vertices of the cycle are white. Fix $q \in (k^*)^{Q_0}$ and a decomposition $q = (q_E, q')$. There is a linear isomorphism

$$\Psi : \Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^{q'}(Q', \mathcal{W}) \to \Lambda^q(Q). \tag{7-12}$$

We prove this by producing a basis of $\Lambda^q(Q)$ of alternating words in $\Lambda^{q_E}(Q_E)$ and $\Lambda^{q'}(Q', \mathcal{W})$.

Remark 7.10. For the (deformed) *additive* preprojective algebra, the analogous map,

$$\Psi_{\text{add}} : \Pi^{\lambda_E}(Q_E) *_{kQ_0} \Pi^{\lambda'}(Q', \mathcal{W}) \to \Pi^{\lambda}(Q),$$

is an isomorphism for all connected quivers Q containing an extended Dynkin quiver Q_E. This follows from the proof of [31, Theorem 3.4.2]; see also [54, Section 5], particularly Corollary 5.2.9(ii).

As before, let $B := kQ_0[t, (q+t)^{-1}]$ and $\overline{B} = tB$, which is spanned by elements $\{t^m, (t')^m \mid m \geq 1\}$ where $t' := (q+t)^{-1} - q^{-1}$.

Proposition 7.11. *Let Q be a connected quiver containing a cycle $Q_E \subseteq Q$ ($Q_E \cong \widetilde{A}_{n-1}$). Then there exists a section $\sigma : \Lambda^q(Q) \to L$ such that (L, r, σ, B) satisfies the strong free product property.*

In more detail, L_Q is a free left kQ_0-module with basis given by concatenable alternating products in the bases of $\Lambda^{q_E}(Q_E)$ given by Proposition 7.1 or 7.7, of $\Lambda^{q'}(Q', \mathcal{W})$ given by Proposition 7.8, and $r^m, (r')^m$ ($m \geq 1$).

Corollary 7.12. *Let Q be as in Proposition 7.11. A basis for $\Lambda^q(Q)$ is given by concatenable alternating words in the mentioned bases of $\Lambda^{q_E}(Q_E)$ and $\Lambda^{q'}(Q', \mathcal{W})$. In particular, the family $\Lambda^q(Q)$ defines a free $k[q_i, q_i^{-1}]_{i \in Q_0}$-module, and hence is flat over $(k^\times)^{Q_0}$.*

Remark 7.13. Note in Proposition 7.11 that we only need to replace $x_a^{\pm 1}$ for $\overline{x_a^{\pm}}$ if $a \in \overline{Q}'_1$ begins at a vertex of Q_E. Moreover, making this change to the statement does not affect the proof. On the other hand, we could freely replace $x_a^{\pm 1}$ by $\overline{x_a^{\pm}}$ for *all* arrow in $\overline{Q}_1$, again without changing the proof.

Proof of Proposition 7.11. First we will establish that our proposed basis for L implies the strong free product property. To see this, observe that the set of subwords not containing r_i, r_i' for $i \in Q_0$ form a basis for $\Lambda^q(Q)$. The inclusion of basis elements $\Lambda^q(Q) \to L$ defines a section σ. Using σ define $\sigma' : \Lambda^q(Q) *_{kQ_0} kQ_0[t, (q+t)^{-1}] \to L$ to be the extension of the map taking $t \mapsto r, (q+t)^{-1} \mapsto r' + q$, and $p \mapsto \sigma(p)$ for $p \in \Lambda^q(Q)$. Then σ' is clearly a kQ_0-linear isomorphism, and hence (L, r, σ, B) satisfies the strong free product property.

Next we will show that the proposed basis for L implies that there exists a kQ_0-linear isomorphism: $\Psi : \Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^q(Q', \mathcal{W}) \to \Lambda^q(Q)$. For this, identify:

- $\Lambda^q(Q)$ as the span of words in L without the subwords r_i, r_i'.
- $\Lambda^{q_E}(Q_E)$ as the span of words in $\Lambda^q(Q)$ without the subwords $a, \overline{x_a^{\pm}}$ for $a \in \overline{Q}'_1$.
- $\Lambda^q(Q', \mathcal{W})$ as the span of words in $\Lambda^q(Q)$ without the subwords $b_i, x_{b_i}^{\pm 1}$ for $b_i \in \overline{Q_{E1}}$.

Hence there exists kQ_0-linear maps $\iota_1, \iota_2 : \Lambda^{q_E}(Q_E), \Lambda^q(Q', \mathcal{W}) \to \Lambda^q(Q)$ defined by the inclusion of basis elements. These maps determine a unique injective kQ_0-linear map $\Psi := \iota_1 *_{kQ_0} \iota_2 : \Lambda^{q_E}(Q_E) *_{kQ_0} \Lambda^q(Q', \mathcal{W}) \to \Lambda^q(Q)$, which is clearly surjective, hence an isomorphism.

It remains to establish that the given set is indeed a basis for L. By Proposition 7.1 we have a basis $\mathfrak{B}_{Q_E}$ for L_{Q_E} and by Proposition 7.8 we have bases $\mathfrak{B}_{Q'}$ for $L_{Q'}$. Therefore we have a basis of alternating words in $\mathfrak{B}_{Q_E}$ and $\mathfrak{B}_{Q'}$ for $L = L_{Q_E} *_{kQ_0} L_{Q'}$.

However this basis gives rise to a basis for the quotient $L/(\rho_{Q_E} + \rho_{Q'} 1_B - q)$. So we need to show that $L/(\rho_{Q_E} + \rho_{Q'} 1_B - q)$ is isomorphic to $L/(\rho_{Q_E}\rho_{Q'} - q) =: \Lambda^q(Q)$ as kQ_0-modules. Hence we consider the system of reductions combining the systems of reductions from Propositions 7.1 and 7.8. Crucially, we perturb the system of reductions by perturbing the relation $r^{\mathrm{pre}} := \rho_{Q_E} + \rho_{Q'} 1_B - q$ to $r = \rho_{Q_E}\rho_{Q'} - q$.

First observe that this change does nothing to the reductions for $L_{Q'}$, since the transformation is the identity on black vertices. That is, $r^{\mathrm{pre}} 1_B = \rho_{Q'} 1_B - q 1_B = r 1_B$.

For L_{Q_E}, notice $r^{\mathrm{pre}} 1_{\mathcal{W}} = \rho_{Q_E} - q_E$ while $r 1_{\mathcal{W}} = \rho_{Q_E}\rho_{Q'} - q_E$. So we alter each reduction involving $\rho_{Q_E}^{\pm}$ by the transformation

$$\rho_{Q_E} \mapsto \rho_{Q_E}\rho_{Q'} = \rho_{Q_E}(\rho_{Q'} - 1) + \rho_{Q_E}, \quad \rho_{Q_E}^{-1} \mapsto \rho_{Q'}^{-1}\rho_{Q_E}^{-1} = (\rho_{Q'}^{-1} - 1)\rho_{Q_E}^{-1} + \rho_{Q_E}^{-1}.$$

Note that we choose this form for the transformation to emphasize that the new relation splits as a sum of (I) the old relation and (II) a piece in the ideal generated by $x_a^{\mp}$ for $a \in \overline{Q}'_1$. This transformation only effects the substitution reductions in the original reduction system for the cycle; see the proof of Proposition 7.1. The substitution reductions become (after applying a reordering reduction) the following:

$$y^{-1} \mapsto x^{-1}(\rho_{Q'} - 1)(r + q) + x^{-1}(r + q), \quad \text{(if not preceded by } a),$$
$$y \mapsto (r' + q^{-1})(\rho_{Q'}^{-1} - 1)x + (r' + q^{-1})x \quad \text{(if not preceded by } a),$$
$$ax \mapsto a(\rho_{Q'} - 1)(r + q)y + ary + qya,$$
$$ax^{-1} \mapsto ay^{-1}(r' + q^{-1})(\rho_{Q'}^{-1} - 1) + ay^{-1}r' + q^{-1}y^{-1}a.$$

Order monomials in L lexicographically in the orderings N and N' of Propositions 7.1 and 7.6. Then the above reductions strictly decrease the ordering. Here we are using (7-11) from the previous subsection to deduce that the ideal of positive-length monomials beginning and ending at vertices of Q_E is preserved under reductions.

All ambiguities lie either entirely in L_{Q_E} or entirely in $L_{Q'}$. Hence the ambiguities in $L_{Q'}$ resolve as before. The ambiguities in L_{Q_E} still resolve using the same reductions as before perturbing. To see this, note that we have replaced the formal variables $\rho_{Q_E}^{\pm 1}$ (which do not interact with $a, a^*, x^{\pm 1}, y^{\pm 1}$) with the new formal variables $(\rho_{Q_E}\rho_{Q'})^{\pm 1}$.

Since the perturbed system of reductions has all the same leading coefficients as the original, we conclude that L has the desired basis. $\qquad\square$

8. The center and primality of multiplicative preprojective algebras

Let Q be a connected quiver strictly containing a cycle. The goal of this section is to complete the proof of Theorem 1.2 by first establishing that $\Lambda^q(Q)$ is prime and then that $Z(\Lambda^q(Q)) = k$ and hence the Calabi–Yau structure is unique up to rescaling.

8A. *Primality of multiplicative preprojective algebras.* We will show $\Lambda^q(Q)$ is prime by first showing that left multiplication by certain elements is injective on the subspace of concatenable elements.

Lemma 8.1. *Let a denote the sum of all the positively oriented arrows of the cycle in $\overline{Q}_1$. Then left multiplication by a, $L_a : 1_{\mathcal{W}}\Lambda^q(Q) \to 1_{\mathcal{W}}\Lambda^q(Q)$, is injective.*

Proof. Decompose the vertices $Q_0 = \mathcal{B} \sqcup \mathcal{W}$ where the white vertices are in the cycle. Decompose the arrows in $\overline{Q}_1 = \overline{Q_{E1}} \sqcup \overline{Q}'_1$. Define

$$A_+ := \ker(\epsilon_A : \Lambda^{q_E}(Q_E) \to k\mathcal{W}), \quad B_+ := \ker(\epsilon_B : \Lambda^q(Q', \mathcal{W}) \to k\mathcal{W}).$$

Then one can define a descending filtration by $\mathcal{F}_0 = \Lambda^q(Q)$ and $\mathcal{F}_m := \mathrm{Span}(B_+(A_+B_+)^{\geq m}))$ for $m > 0$. Notice $a \in \mathcal{F}_m, b \in \mathcal{F}_\ell$ implies $ab \in \mathcal{F}_{m+\ell}$, so this is an *algebra* filtration.

Consider the exact sequence $B_+ \overset{\iota}{\hookrightarrow} \Lambda^q(Q) \overset{\pi}{\twoheadrightarrow} \Lambda^{q_E}(Q_E)$. The basis of Proposition 7.11 realizes an inclusion $i : \Lambda^{q_E}(Q_E) \to \Lambda^q(Q)$, a kQ_0-module splitting. So for $\alpha, \beta \in \Lambda^{q_E}(Q_E)$, $i(\alpha) \cdot i(\beta) \equiv \alpha \cdot \beta$

modulo the two-sided ideal generated by B_+. Therefore, in the associated graded algebra $gr_{\mathcal{F}}(\Lambda^q(Q)) := \bigoplus_{m=0} \mathcal{F}_m/\mathcal{F}_{m+1}$,

$$i(\alpha) \cdot i(\beta) = \alpha \cdot \beta + \alpha \cdot \beta b'' + \alpha b' \beta + b\alpha \cdot \beta$$

for $b, b', b'' \in B_+$. Therefore, for $b_1 \in B_+$, there exists $b_2 \in B_+$ such that

$$i(\alpha) \cdot i(\beta)b_1 = \alpha \cdot \beta(1 + b_2)b_1.$$

Recall A_+ has kQ_0-module basis given by $\{x_a^p, x_a^m a^\ell, x_a^m(a^*)^\ell \mid m, \ell, p \in \mathbb{Z},\ p \neq 0,\ \ell > 0\}$ by Proposition 7.1. In the associated graded algebra $gr_{\mathcal{F}}(\Lambda^q(Q))$, L_a acts on A_+B_+ as follows, for $b \in B_+$:

$$a(x_a^m a^\ell)b = q^m x_a^m a^{\ell+1}b.$$
$$a(x_a^m(a^*)^\ell)b = q^m x_a^m(x_a - 1)(a^*)^{\ell-1}b.$$
$$a(x_a^p)b = q^p x_a^p(x_a - 1)(\rho_{Q'}^{-1})b.$$

Since L_a is injective on A_+, by Proposition 7.3, we conclude that L_a is injective on the right ideal generated by A_+.

Consider the basis of Proposition 7.11, and write $b \in 1_{\mathcal{W}}B_+$ in this basis. Then ab is again a basis element, and hence L_a takes basis elements injectively to basis elements. We conclude that L_a is injective on the right ideal generated by $1_{\mathcal{W}}B_+$, and therefore on all of $1_{\mathcal{W}}\Lambda^q(Q)$.   □

Lemma 8.2. *Right multiplication by a, $R_a : \Lambda^q(Q)1_{\mathcal{W}} \to \Lambda^q(Q)1_{\mathcal{W}}$ is injective.*

The proof is completely analogous, using the same filtration, together with the calculations:

$$b(x_a^m a^\ell)a = bx_a^m a^{\ell+1}.$$
$$b(x_a^m(a^*)^\ell)a = bq^{-\ell+1}x_a^{m+1}(a^*)^{\ell-1} - bx_a^m(a^*)^{\ell-1}.$$
$$b(x_a^p)a = bx_a^p a.$$

Lemma 8.3. *Let $v \in Q_0$. There is unique path $\gamma_{v,w}$ in the spanning forest from v to a white vertex $w \in \mathcal{W}$. Right multiplication by $\gamma_{v,w}$, $R_{\gamma_{v,w}} : \Lambda^q(Q)e_v \to \Lambda^q(Q)e_w$, is injective.*

Proof. We need to show $\alpha\gamma_{h(\alpha),w} \neq 0$ for $\alpha \neq 0$. Consider the basis in Proposition 7.8, consisting of words in $a, x_a^\pm$ for a an arrow, without certain disallowed subwords, e.g., aa^* for $a \in \bar{Q}_1$. Note that $\gamma_{h(\alpha),w}$ is a basis element as aa^* cannot appear in a shortest path. Write α as a linear combination of basis elements. Notice $\alpha\gamma_{h(\alpha),w}$ is a linear combination of basis elements *unless* the disallowed subword a^*a is created for some arrow $a \in F_1$. This disallowed subword reduces to $\overline{x_{a^*}^+}$ (which is not itself disallowed since $a^* \notin F_1$, as $a \in F_1$.) Furthermore, the appearance of $\overline{x_{a^*}^+}$ for $a \in F_1$ cannot create the disallowed subwords

$$\text{(I)}\ \overline{x_{a^*}^+}\overline{x_{a^*}^-}, \quad \text{(II)}\ \overline{x_{a^*}^-}\overline{x_{a^*}^+}, \quad \text{(III)}\ a\overline{x_{a^*}^\pm}, \quad \text{(IV)}\ \overline{x_{a^*}^+}^2, \quad \text{(V)}\ \overline{x_{a^*}^+}a^*,$$

for $a \in F_1$, as in each case α or $\gamma_{h(\alpha),w}$ would itself contain a disallowed subword

$$\text{(I)}\ \overline{x_{a^*}^-}, \quad \text{(II)}\ \overline{x_{a^*}^-}, \quad \text{(III)}\ aa^*, \quad \text{(IV)}\ a\overline{x_{a^*}^+}\ \text{or}\ \overline{x_{a^*}^+}a^*, \quad \text{(V)}\ aa^*,$$

each a contradiction. We conclude that right multiplication by $\gamma_{h(\alpha),w}$ takes basis elements injectively to basis elements and hence is injective. □

Lemma 8.4. *Let* $v \in Q_0$. *There is unique path* $\gamma_{w,v}$ *in the opposite of the spanning forest,* F_1^{op}, *from* $w \in \mathcal{W}$ *to* v. *Left multiplication by* $\gamma_{w,v}$, $L_{\gamma_{w,v}} : e_v \Lambda^q(Q) \to e_w \Lambda^q(Q)$, *is injective.*

The proof is identical, and follows from the isomorphism $\Lambda^q(Q) \cong \Lambda^q(Q)^{\mathrm{op}}$.

Proposition 8.5. $\Lambda^q(Q)$ *is prime, for* Q *connected and containing a cycle.*

Proof. Let $\alpha, \beta \in \Lambda^q(Q)$ be nonzero. We will show $\alpha \Lambda^q(Q)\beta \neq 0$ by building an explicit element $\gamma = \gamma_1 \gamma_2$ so that $\alpha \gamma \beta \neq 0$. That is, define

$$\gamma_1 := \gamma_{h(\alpha),w} x_a^M a^N, \quad \gamma_2 := a^{N'} x_a^{M'} \gamma_{w,t(\beta)}$$

where $M, M', N, N' \in \mathbb{N}$ are sufficiently large (depending on α and β) and where $\gamma_{h(\alpha),w}$ and $\gamma_{w,t(\beta)}$ are as defined in Lemmas 8.3 and 8.4, respectively.

We will first show that right multiplication by γ_1 is injective on concatenable paths to conclude $\alpha \gamma_1 \neq 0$. Then we will argue that left multiplication by γ_2 is injective on concatenable paths to conclude $\gamma_2 \beta \neq 0$. Finally, we will show that $\alpha \gamma_1 \gamma_2 \beta \neq 0$.

To show $R_{\gamma_1} : \Lambda^q(Q)e_{h(\alpha)} \to \Lambda^q(Q)e_{h(\gamma_1)}$ is injective, it suffices to show that right multiplication by each piece, $\gamma_{h(\alpha),w}$, x_a^M, and a^N, is injective. $R_{\gamma_{h(\alpha),w}}$ is injective by Lemma 8.3, $R_{x_a^M}$ is injective since x_a is invertible, and R_{a^N} is injective by Lemma 8.2.

Similarly, $L_{\gamma_2} : e_{t(\beta)}\Lambda^q(Q) \to e_{t(\gamma_2)}\Lambda^q(Q)$ is injective since L_a, L_{x_a}, and $L_{\gamma_{w,t(\beta)}}$ are injective by Lemma 8.1, invertibility of x_a, and Lemma 8.4, respectively.

Finally notice that $\alpha \gamma_1 \neq 0$ and $\gamma_2 \beta \neq 0$ implies $\alpha \gamma_1 \gamma_2 \beta \neq 0$. To see this, consider the filtration $\mathcal{F}$ defined in the proof of Lemma 8.1. It suffices to show $\alpha \gamma_1 \gamma_2 \beta \neq 0$ in $\mathrm{gr}_{\mathcal{F}}(\Lambda^q(Q))$. Write $\alpha \gamma_1$ and $\gamma_2 \beta$ in the basis of Proposition 7.11 (see the basis in Proposition 7.1). By design $\alpha \gamma_1$ ends with a basis element of the form $x_a^m a^n$ for $m, n > 0$ and $\gamma_2 \beta$ begins with a basis element of the form $a^{n'} x_a^{m'}$ for $m', n' > 0$. Their product in $\mathrm{gr}_{\mathcal{F}}(\Lambda^q(Q))$ is the scaled basis element $q^{nm'} x_a^{m+m'} a^{n+n'}$. So $\alpha \gamma_1 \gamma_2 \beta \neq 0$ in $\mathrm{gr}_{\mathcal{F}}(\Lambda^q(Q))$ and hence in $\Lambda^q(Q)$, completing the proof. □

8B. *The center of multiplicative preprojective algebras.* The center of $\Lambda^1(Q)$ depends dramatically on the taxonomy of quiver Q into Dynkin, extended Dynkin, and others:

- For Q Dynkin and k characteristic not 2, 3, or 5, one can compute the center using the isomorphism $\Lambda^1(Q) \cong \Pi(Q)$; see Example 2.3.

- For Q extended Dynkin, Conjecture 1.4 predicts $Z(\Lambda^1(Q)) \cong e_v \Lambda^1(Q)e_v$, which is proven in Section 6B in the case $Q = \widetilde{A}_n$.

- In the remaining cases, Conjecture 1.1 predicts $Z(\Lambda^q(Q)) = k$, for any $q \in (k^*)^{Q_0}$.

The goal of this section is to establish the conjecture in the case Q contains a cycle.

Proposition 8.6. *Let Q be a connected quiver strictly containing a cycle and fix $q \in (k^\times)^{Q_0}$. Then $Z(\Lambda^q(Q)) = k$.*

Proof. Let $z \in Z(\Lambda^q(Q))$. Decompose $z = z_0 + z_+$ into a sum of length zero and positive length paths. First suppose that $z_+ = 0$. Then $z = \sum_{i \in Q_0} c_i e_i$. Note that every individual arrow forms a basis element of Proposition 7.11. Then $za = az$ for every arrow implies that all c_i are equal, as Q is connected.

Now assume $z_+ \neq 0$. Expanding z_+ in the basis of Proposition 7.11, we write $z_+ = \sum_i c_i z_i$, where each z_i is a positive-length alternating word in the cycle and the complement. We claim that each z_i has an arrow not in the cycle. Suppose, by contradiction, there exists j such that z_j consists of only arrows in the cycle. Since Q strictly contains the cycle, there exists an arrow $b \in Q_1$ not in the cycle. And as z_+ commutes with each arrow a_i in the cycle, there exists l such that z_l consists of only arrows in the cycle that ends at $t(b)$. Then $z_+ b = b z_+$. But $z_+ b$ contains a term beginning with $x_a^m a^j$ for some m, j with $(m, j) \neq (0, 0)$. However, $b z_+$ has no term beginning $x_a^m a^j$ unless $(m, j) = (0, 0)$. This contradicts the existence of z_j consisting of only arrows in the cycle.

Since $z_+ \neq 0$, thanks to Lemma 8.3, there exists a vertex i and a path $b = \gamma_{h(z_+),i}$ such that $z_+ b e_i \neq 0$. Therefore also $b z_+ e_i \neq 0$, so $z_+ e_i \neq 0$. By Lemma 8.2, we then have $z_+ a^n \neq 0$ for all n. Hence also $a^n z_+ \neq 0$. Now, for sufficiently large $N \gg 0$, $a^N z_+$ contains basis elements beginning with an arbitrarily high power of the cycle. However, terms of $z_+ a^N$ begin only with powers of the cycle appearing in z_+, since every z_j has a term not in the cycle. These powers are bounded, so this contradicts the assumption that $z_+ \neq 0$. We conclude that z is a scalar multiple of the identity. $\qquad\square$

Corollary 8.7. *If Q is connected and properly contains a nonoriented cycle, then $\Lambda^q(Q)$ has a unique, up to scaling, Calabi–Yau structure.*

Proof. Write $\Lambda := \Lambda^q(Q)$. Any two Calabi–Yau structures differ by an invertible map in $\mathrm{Hom}_{\Lambda-\mathrm{bimod}}(\Lambda, \Lambda)$, which is determined by the image of the unit, a central invertible element. So the set of Calabi–Yau structures on Λ, when nonempty, is a $Z(\Lambda)^\times$-torsor. By Proposition 8.6, $Z(\Lambda)^\times = k^\times$, so any two Calabi–Yau structures differ by an invertible scalar. $\qquad\square$

This completes the proof of Theorem 1.2.

Acknowledgements

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