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The Manin–Mumford conjecture and the Tate–Voloch conjecture for a product of Siegel moduli spaces

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We use perfectoid spaces associated to abelian varieties and Siegel moduli spaces to study torsion points and ordinary CM points. We reprove the Manin–Mumford conjecture, i.e., Raynaud's theorem. We also prove the Tate–Voloch conjecture for a product of Siegel moduli spaces, namely ordinary CM points outside a closed subvariety can not be p-adically too close to it.

1. Introduction

We use the theory of perfectoid spaces to study torsion points in abelian varieties and ordinary CM points in Siegel moduli spaces. The use of perfectoid spaces is inspired by Xie's recent work [2018].

Tate–Voloch conjecture. Our main new result is about ordinary CM points. Let p be a prime number, L the complete maximal unramified extension of $\mathbb{Q}_p$. Let X be a product of Siegel moduli spaces over L with arbitrary level structures.

Theorem 1.1. *Let Z be a closed subvariety of $X_{\bar{L}}$. There exists a constant $c > 0$ such that for every ordinary CM point $x \in X(\bar{L})$, if the distance $d(x, Z)$ from x to Z satisfies $d(x, Z) \leq c$, then $x \in Z$.*

The distance $d(x, Z)$ is defined as follows. Let $\|\cdot\|$ be a p-adic norm on $\bar{L}$. Let $\mathfrak{X}$ be an integral model of X over $\mathcal{O}_{\bar{L}}$. Let $\{U_1, \ldots, U_n\}$ be a finite open cover of $\mathfrak{X}$ by affine schemes flat over $\mathcal{O}_{\bar{L}}$. Define $d(x, Z)$ to be the supremum of the $\|f(x)\|$ where U_i contains x and $f \in \mathcal{O}_{\mathfrak{X}}(U_i)$ vanishing on $Z \bigcap U_i$. The definition of $d(x, Z)$ depends on the choices of the integral model and the cover. However, the truth

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of Theorem 1.1 does not depend on these choices; see page 986. Moreover, we show that Theorem 1.1 holds for formal subschemes of $\mathfrak{X}$ (with maximal level at p); see Theorem 6.8. For CM points which are canonical liftings, we prove an "almost effective" version; see Theorem 6.13.

It is clear that the same statement in Theorem 1.1 is true replacing $X_{\bar{L}}$ by a closed subvariety. In particular, Theorem 1.1 is in fact equivalent the same statement for X being a single Siegel moduli space, by embedding a product of Siegel moduli spaces into a larger Siegel moduli space.

Remark 1.2. (1) For a power of the modular curve without level structure, Theorem 1.1 was proved by Habegger [2014] by a different method. However, Habegger's proof relies on a result of Pila [2014] (see also [Habegger 2014, Theorem 8]) concerning Zariski closure of a Hecke orbit. As far as we know, it is not available for Siegel moduli spaces yet. Moreover, Habegger's method seems not applicable to formal schemes.

(2) Habegger [2014] also showed that the ordinary condition is necessary.

(3) The original Tate–Voloch conjecture [Tate and Voloch 1996] states that in a semiabelian variety, torsion points outside a closed subvariety can not be p-adically too close to it. This conjecture was proved by Scanlon [1998; 1999] when the semiabelian variety is defined over $\overline{\mathbb{Q}}_p$. Xie [2018] proved dynamical analogs of Tate–Voloch conjecture for projective spaces.

Idea of the proof of Theorem 1.1. It is not hard to reduce Theorem 1.1 to the case that X has maximal level at p; see Lemma 2.15. We sketch the proof of Theorem 1.1 in this case. Relative to the canonical lifting of an ordinary point x in the reduction of X, ordinary CM points in X with reduction x are like p-primary roots of unity relative to 1 in the open unit disc around 1; see Proposition 5.1. This is the Serre–Tate theorem. If we only consider one such disc, Theorem 1.1 follows from a result of Serban [2018]. In general, we need to study all infinitely many Serre–Tate deformation spaces together. In characteristic p, this can be achieved by Chai's [2003] global Serre–Tate theorem; see Section 5. To prove Theorem 1.1, we at first prove a Tate–Voloch type result in a family characteristic p; see Section 6. Then we use the ordinary perfectoid Siegel space associated to X and the perfectoid universal covers of Serre–Tate deformation spaces to translate this result to the desired Theorem 1.1.

Possible generalizations. For Shimura varieties of Hodge type, the ordinary locus in the usual sense could be empty. In this case, we consider the notion of μ-ordinariness; see [Wedhorn 1999]. Then following our strategy, we need three ingredients. At first, a theory of Serre–Tate coordinates for μ-ordinary CM points. For Shimura varieties of Hodge type; see [Hong 2019; Shankar and Zhou 2016]. Secondly, a global theory of Serre–Tate coordinates in characteristic p. For Shimura varieties of PEL type, such results should be known to experts. Thirdly, μ-ordinary perfectoid Shimura varieties. Following Scholze [2015], certain perfectoid Shimura varieties of abelian type are constructed in [Shen 2017]. For universal abelian varieties over Shimura varieties of PEL type, we expect a Tate–Voloch type result for torsion points in fibers over μ-ordinary CM points. Still, we need analogs of the above three ingredients.

Manin–Mumford conjecture. For torsion points in abelian varieties, we reprove Raynaud's theorem, which is also known as the Manin–Mumford conjecture.

Theorem 1.3 [Raynaud 1983b]. *Let F be a number field. Let A be an abelian variety over F and V a closed subvariety of A. If V contains a dense subset of torsion points of A, then V is the translate of an abelian subvariety of A by a torsion point.*

Idea of the proof of Theorem 1.3. We simply consider the case when V does not contain any translate of a nontrivial abelian subvariety. Suppose that A has good reduction at a place of F unramified over a prime number p. Let $[p] : A \to A$ be the morphism multiplication by p. Let Λ_n be a suitable set of reductions of torsions in $[p^n]^{-1}(V)$, and Λ_n^{Zar} its Zariski closure in the base change to $\overline{\mathbb{F}}_p$ of the reduction of A. Use the p-adic perfectoid universal cover of A to lift Λ_n^{Zar} to A. A variant of Scholze's approximation lemma [2012] shows that as n get larger, the liftings are closer to V; see Proposition 2.23. A result of Scanlon [1998] on the Tate–Voloch conjecture for prime-to-p torsions implies that the prime-to-p torsions of these points are in V for n large enough; see Proposition 4.8. Assume that Λ_n is infinite and we deduce a contradiction as follows. A result of Poonen [2005] (see Theorem 4.1) shows that the size of the set of prime-to-p torsions in Λ_n^{Zar} is not small. Then the liftings give a lower bound on the size of the set of prime-to-p torsions in V; see Proposition 4.9. Now consider the l-adic perfectoid space associated to A. By the same approach, we can repeatedly improve such lower bounds. Finally we get a contradiction as A is of finite dimensional.

Remark 1.4. The proofs of Poonen's result and Scanlon's result are independent of Theorem 1.3.

Organization of the Paper. The preliminaries on adic spaces and perfectoid spaces are given in Section 2. We introduce the perfectoid universal cover of an abelian scheme in Section 3. The reader may skip these materials and only come back for references. We set up notations for the proof of Theorem 1.3 in Section 3, then prove Theorem 1.3 in Section 4. We introduce the ordinary perfectoid Siegel space and set up notations for the proof of Theorem 1.1 in Section 5. Then we prove Theorem 1.1 in Section 6.

2. Adic spaces and perfectoid spaces

We briefly recall the theory of adic spaces due to Huber [1993a; 1993b; 1994; 1996], and the generalization by Scholze and Weinstein [2013]. Then we define tube neighborhoods in adic spaces and distance functions. Finally we recall the theory of perfectoid spaces of Scholze [2012] and an approximation lemma due to Scholze.

Let K be a nonarchimedean field, i.e., a complete nondiscrete topological field whose topology is induced by a nonarchimedean norm $\|\cdot\|_K$ ($\|\cdot\|$ for short). Define

$$K^\circ = \{x \in K : \|x\| \leq 1\} \quad \text{and} \quad K^{\circ\circ} = \{x \in K : \|x\| < 1\}.$$

Let $\varpi \in K^{\circ\circ} - \{0\}$.

Adic generic fibers of certain formal schemes.

Adic spaces. Let R be a complete Tate K-algebra, i.e., a complete topological K-algebra with a subring $R_0 \subset R$ such that $\{aR_0 : a \in K^\times\}$ forms a basis of open neighborhoods of 0. A subset of R is called bounded if it is contained in a certain aR_0. Let R° be the subring of power bounded elements, i.e., $x \in R^\circ$ if and only if the set of all powers of x form a bounded subset of R. Let $R^+ \subset R^\circ$ be an open integrally closed subring. Such a pair (R, R^+) is called an affinoid K-algebra. Let $\mathrm{Spa}(R, R^+)$ be the topological space whose underlying set is the set of equivalent classes of continuous valuations $|\cdot(x)|$ on R such that $|f(x)| \le 1$ for every $f \in R^+$ and topology is generated by the subsets of the form

$$U\left(\frac{f_1, \dots, f_n}{g}\right) := \{x \in \mathrm{Spa}(R, R^+) : \forall i\, |f_i(x)| \le |g(x)|\}$$

such that $(f_1, \dots, f_n) = R$. There is a natural presheaf on $\mathrm{Spa}(R, R^+)$; see [Huber 1994, page 519]. If this presheaf is a sheaf, then the affinoid K-algebra (R, R^+) is called sheafy, and $\mathrm{Spa}(R, R^+)$ is called an affinoid adic space over K.

Assumption 2.1. If $K^\circ \subset R^+$, for every $x \in \mathrm{Spa}(R, R^+)$, we always choose a representative $|\cdot(x)|$ in the equivalence class of x such that $|f(x)| = \|f\|_K$ for every $f \in K$.

Define a category (V) as in [Scholze 2012, Definition 2.7]. Objects in (V) are triples $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}, \{|\cdot(x)| : x \in \mathcal{X}\})$ where $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a locally ringed topological space whose structure sheaf is a sheaf of complete topological K-algebras, and $|\cdot(x)|$ is an equivalence class of continuous valuations on the stalk of $\mathcal{O}_{\mathcal{X}}$ at x. Morphisms in (V) are morphisms of locally ringed topological spaces which are continuous K-algebra morphisms on the structure sheaves, and compatible with the valuations on the stalks in the obvious sense.

Definition 2.2. An adic space $\mathcal{X}$ over K is an object in (V) which is locally on $\mathcal{X}$ an affinoid adic space over K. An adic space over $\mathrm{Spa}(K, K^\circ)$ is an adic space over K with a morphism to $\mathrm{Spa}(K, K^\circ)$. A morphism between two adic spaces over $\mathrm{Spa}(K, K^\circ)$ is a morphism in (V) compatible the morphisms to $\mathrm{Spa}(K, K^\circ)$. The set of morphisms $\mathrm{Spa}(K, K^\circ) \to \mathcal{X}$ is denoted by $\mathcal{X}(K, K^\circ)$.

There is a natural inclusion $\mathcal{X}(K, K^\circ) \hookrightarrow \mathcal{X}$ by mapping a morphism $\mathrm{Spa}(K, K^\circ) \to \mathcal{X}$ to its image. We always identify $\mathcal{X}(K, K^\circ)$ as a subset of $\mathcal{X}$ by this inclusion.

Adic generic fibers of certain formal schemes. A Tate K-algebra R is called of topologically finite type (tft for short) if R is a quotient of $K\langle T_1, T_2, \dots, T_n\rangle$. In particular, it is equipped with the ϖ-adic topology. Similarly define K°-algebras of tft. By [Bosch et al. 1984, 5.2.6, Theorem 1] and [Huber 1994, Theorem 2.5], if R is of tft, then an affinoid K-algebra (R, R^+) is sheafy. Similar to the rigid analytic generic fibers of formal schemes over K° [Bosch 2014, 7.4], we naturally have a functor from the category of formal schemes over K° locally of tft to adic spaces over $\mathrm{Spa}(K, K^\circ)$ such that the image of $\mathrm{Spf}\, A$ is $\mathrm{Spa}\big(A\big[\frac{1}{\varpi}\big], A^c\big)$ where A^c is the integral closure of A in $A\big[\frac{1}{\varpi}\big]$. The image of a formal scheme under this functor is called its adic generic fiber.

We are interested in certain infinite covers of abelian schemes and Siegel moduli spaces. They are not of tft. We need to generalize the adic generic fiber functor. In [Scholze and Weinstein 2013], the category of adic spaces over $\mathrm{Spa}(K, K^\circ)$ is enlarged in a sheaf-theoretical way. Moreover, the adic generic fiber functor extends to the category of formal schemes over K° locally admitting a finitely generated ideal of definition.

For our purpose, we only need the following special case. Let $\mathfrak{X}$ be a formal K°-scheme which is covered by affine open formal subschemes $\{\mathrm{Spf}\, A_i : i \in I\}$, where I is an index set, such that each affinoid K-algebra $\left(A_i\left[\frac{1}{\varpi}\right], A_i^c\right)$ is sheafy. Then the adic generic fiber $\mathcal{X}$ of $\mathfrak{X}$ is an adic space over $\mathrm{Spa}(K, K^\circ)$ in the sense of Definition 2.2. Indeed, $\mathcal{X}$ is the obtained by gluing the affinoid adic spaces $\mathrm{Spa}\left(A_i\left[\frac{1}{\varpi}\right], A_i^c\right)$ in the obvious way. We have an easy consequence.

Lemma 2.3. *Let $\mathcal{X}$ be the adic generic fiber $\mathfrak{X}$. Then there is a natural bijection $\mathfrak{X}(K^\circ) \simeq \mathcal{X}(K, K^\circ)$.*

Tube neighborhoods and distance functions.

Tube neighborhoods. Let $\mathfrak{X} = \mathrm{Spf}\, B$, where B is a flat K°-algebra of tft. Let $\mathfrak{Z}$ be a closed formal subscheme defined by a closed ideal I. Let $\mathcal{X}$ be the adic generic fiber of $\mathfrak{X}$. Then $\mathcal{X} = \mathrm{Spa}(R, R^+)$ where $R = B\left[\frac{1}{\varpi}\right]$ and R^+ is the integral closure of B in R.

Definition 2.4. For $\epsilon \in K^\times$, the ϵ-neighborhood of $\mathfrak{Z}$ in $\mathcal{X}$ is defined to be the subset

$$\mathcal{Z}_\epsilon := \{x \in \mathcal{X} : |f(x)| \le |\epsilon(x)| \text{ for every } f \in I\}.$$

Remark 2.5. Note that $\mathcal{Z}_\epsilon$ may not be open in $\mathcal{X}$. If I is generated by $\{f_1, \dots, f_n\}$, then $\mathcal{Z}_\epsilon = U((f_1, \dots, f_n, \epsilon)/\epsilon)$ is naturally an open adic subspace of $\mathcal{X}$. In fact, for our applications, we only use this case.

Definition 2.4 immediately implies the following lemmas.

Lemma 2.6. *Let $\mathfrak{Z} = \bigcap_{i=1}^m \mathfrak{Z}_i$, where each $\mathfrak{Z}_i$ is a closed formal subschemes of $\mathfrak{X}$. For $\epsilon \in K^\times$, let $\mathcal{Z}_{i,\epsilon}$ be the ϵ-neighborhood of $\mathfrak{Z}_i$. Then $\mathcal{Z}_\epsilon = \bigcap_{i=1}^m \mathcal{Z}_{i,\epsilon}$.*

Lemma 2.7. *Let $\mathfrak{Z} = \mathfrak{Z}_1 \bigcup \mathfrak{Z}_2$, where $\mathfrak{Z}_1, \mathfrak{Z}_2$ are closed formal subschemes of $\mathfrak{X}$:*

(1) *Then $\mathcal{Z}_{1,\epsilon} \subset \mathcal{Z}_\epsilon$.*

(2) *Suppose that there exists $\delta \in K^{\circ\circ} - \{0\}$ which vanishes on $\mathfrak{Z}_2$. Then $\mathcal{Z}_\epsilon \subset \mathcal{Z}_{1,\epsilon/\delta}$.*

Let X be a K°-scheme locally of finite type, and $\mathfrak{X}$ the ϖ-adic formal completion of X. Let $\mathcal{X}$ be the adic generic fiber of $\mathfrak{X}$. We also call $\mathcal{X}$ the adic generic fiber of X. Let Z be a closed subscheme of X_K. We define tube neighborhoods of Z in $\mathcal{X}$ as follows; see also [Scholze 2012, Proposition 8.7].

Suppose that X is affine. Let $\mathfrak{Z} \subset \mathfrak{X}$ be the closed formal subscheme associated to the schematic closure of Z.

Definition 2.8. For $\epsilon \in K^\times$, the ϵ-neighborhood of Z in $\mathcal{X}$ is defined to be the ϵ-neighborhood of $\mathfrak{Z}$ in $\mathcal{X}$.

Remark 2.9. If the schematic closure of Z has empty special fiber, then $\mathcal{Z}_\epsilon$ is empty.

To define tube neighborhoods in general, we need to glue affinoid pieces. We consider the following relative situation. Let Y be another affine K°-scheme of finite type, and $\Phi : Y \to X$ a K°-morphism. Let W be the preimage of Z which is a closed subscheme of Y_K, and $\mathcal{W}_\epsilon$ its ϵ-neighborhood. By the functoriality of formal completion and taking adic generic fibers, we have an induced morphism $\Psi : \mathcal{Y} \to \mathcal{X}$. From the fact that schematic image is compatible with flat base change (see [Bosch et al. 1990, 2.5, Proposition 2]), we easily deduce the following lemma.

Lemma 2.10. *If $\Phi : Y \to X$ is flat, then $\Psi^{-1}(\mathcal{Z}_\epsilon) = \mathcal{W}_\epsilon$. In particular, if $Y \subset X$ is an open K°-subscheme, $\mathcal{W}_\epsilon = \mathcal{Z}_\epsilon \bigcap \mathcal{Y}$ under the natural inclusion $\mathcal{Y} \hookrightarrow \mathcal{X}$.*

Now we turn to the general case. Let X be an K°-scheme locally of finite type. For an open subscheme $U \subset X$, let Z_U be the restriction of Z to U. Let $S = \{U_i : i \in I\}$ be an affine open cover of X, where I is an index set and each U_i is of finite type over K°. Let $\mathcal{Z}_{U_i,\epsilon}$ be the ϵ-neighborhood of Z_{U_i} in the adic generic fiber $\mathcal{U}_i$ of U_i. Note that each $\mathcal{U}_i$ is naturally an open adic subspace of $\mathcal{X}$.

Definition 2.11. Define the ϵ-neighborhood of Z in $\mathcal{X}$ by $\mathcal{Z}_\epsilon := \bigcup_{U \in S} \mathcal{Z}_{U,\epsilon}$.

As a corollary of Lemma 2.10, this definition is independent of the choice of the cover $\mathcal{U}$.

Distance functions. Let U be an affine open subset of X which is flat over K°. Let I be an ideal of the coordinate ring of U. For $x \in U(K)$, define $d_U(x, I) := \sup\{\|f(x)\| : f \in I\}$. Let $\mathcal{I}$ be the ideal sheaf of the schematic closure of Z in X.

Assume that X is of finite type over K°. Let $\mathcal{U} := \{U_1, \dots, U_n\}$ be a finite affine open cover of X such that each U_i is flat over K°. For $x \in X(K)$, define $d^{\mathcal{U}}(x, Z)$ to be the maximum of $d_{U_i}(x, I)$ over all i such that $x \in U_i$.

Let $x^\circ \in X(K^\circ)$ and x the generic point of x°. Regard x as a point in $\mathcal{X}(K, K^\circ)$ via Lemma 2.3. Let U be an affine open subset of X flat over K° such that $x^\circ \in U(K^\circ)$. We have a tautological relation between the distance function and tube neighborhoods.

Lemma 2.12. *Let $\epsilon \in K^\times$. Then $x \in \mathcal{Z}_{U,\epsilon}$ if and only if $d_U(x, \mathcal{I}(U)) \le \|\epsilon\|$.*

By Lemma 2.10, the number $d_U(x, \mathcal{I}(U))$ does not depend on the choice of U. Define

$$d(x, Z) := d_U(x, \mathcal{I}(U)).$$

Then $d(x, Z) = d^{\mathcal{U}}(x, Z)$ for every finite affine open cover $\mathcal{U}$ of X. Our distance function coincides with the one in the end of [Scanlon 1998, Section 1], which is defined globally.

A finite extension of K has a natural structure of a nonarchimedean field; see [Bosch et al. 1984]. Let $\overline{K}$ be an algebraic closure of K. The above discussion is naturally generalized to $x \in X(\overline{K})$ and $Z \subset X_{\overline{K}}$.

Tate–Voloch type sets. Let X be of finite type over K°.

Definition 2.13. Fix an arbitrary finite affine open cover $\mathcal{U}$ of X by subschemes flat over K°. A set $T \subset X(\overline{K})$ is of Tate–Voloch type if for every closed subscheme Z of $X_{\overline{K}}$, there exists a constant $c > 0$ such that for every $x \in T$, if $d^{\mathcal{U}}(x, Z) \le c$, then $x \in Z(\overline{K})$.

Remark 2.14. Is there always a set of Tate–Voloch type? Let $C \subset X$ be irreducible and flat over K° of relative dimension 1. Choose one point in each residue disk in C. Easy to check that this set of points of X is of Tate–Voloch type. Moreover, we can choose points in residue disks in C chose degrees are unbounded. The following questions are more meaningful. Is there always a Tate–Voloch type set which is Zariski dense in X? Can the points in this set have unbounded the degrees over K? Indeed, the Tate–Voloch type sets in Theorem 1.1 and in the results of Habegger, Scanlon and Xie give positive answers to these two questions.

Let Y be a K°-scheme of finite type, and $\pi : Y \to X$ a finite schematically dominant morphism.

Lemma 2.15. *Let $T \subset X(\overline{K})$ be of Tate–Voloch type and $T' = \pi^{-1}(T) \subset Y(\overline{K})$. Then T' is of Tate–Voloch type.*

Proof. We may assume that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ where A is a subring of B. Let L be a finite extension of K. Let Z' be a closed subscheme of Y_L. We need to show that $d(x', Z')$ has a positive lower bound for $x' \in T' - Z'(\overline{K})$. Define the dimension of Z' to be the maximal dimension of the irreducible components of Z'. We allow Z' to be empty, in which case we define its dimension to be -1. We do induction on the dimension of Z'. Then the dimension -1 case is trivial. Now we consider the general case with the hypothesis that the lemma holds for all lower dimensions.

Suppose such a lower bound does not exists, then there exists a sequence of $x'_n \in T' - Z'(\overline{K})$ such that $d(x'_n, Z') \to 0$ as $n \to \infty$. We will find a contradiction. Let Z be the schematic image of Z' by π, $x_n = \pi(x'_n)$. Let the schematic closure of Z in X_{L°(resp. Z' in Y_{L°) be defined by an ideal $J \subset A \otimes L^\circ$ (resp. $I \subset B \otimes L^\circ$). Then $I \otimes L \supset JB \otimes L$. Since $JB \otimes L^\circ$ is finitely generated, there exists a positive integer r such that $I \supset \varpi^r JB \otimes L^\circ$. Thus $d(x_n, Z) \to 0$ as $n \to \infty$. Since T is of Tate–Voloch type, $x_n \in Z(\overline{K})$ for n large enough. We may assume that every $x_n \in Z(\overline{K})$. Since $x'_n \notin Z'$, $\pi^{-1}(Z) = Z' \bigcap Z_1$ where Z_1 is a closed subscheme of Y_L not containing Z' but containing all x'_n. Claim: $d\big(x'_n, Z' \bigcap Z_1\big) \to 0$ as $n \to \infty$. This contradicts the induction hypothesis. Thus $d(x', Z')$ has a positive lower bound for $x' \in T' - Z'(K)$. Now we prove the claim. Let the schematic closure of Z_1 in Y_{L° be defined by an ideal $I_1 \subset B \otimes L^\circ$. Then the schematic closure of $Z' \bigcap Z_1$ is defined by the following ideal of $B \otimes L^\circ$:

$$I_2 := (I_1 \otimes L + I' \otimes L) \bigcap B \otimes L^\circ = (I_1 + I') \otimes L \bigcap B \otimes L^\circ,$$

which is finitely generated. Thus there exists a positive integer s such that $(I_1 + I') \supset \varpi^s I_2$. Now the claim follows from that $d(x'_n, Z') \to 0$ and $x'_n \in Z_1$. □

Perfectoid spaces.

Two perfectoid fields. Instead of recalling the definition of perfectoid fields (see [Scholze 2012, Definition 3.1]), we consider two examples and use them through out this paper.

Let $k = \overline{\mathbb{F}}_p$, $W = W(k)$ the ring of Witt vectors, and $L = W\left[\frac{1}{p}\right]$. For each integer $n \geq 0$, let μ_{p^n} be a primitive p^n-th root of unity in $\bar{L}$ such that $\mu_{p^{n+1}}^p = \mu_{p^n}$. Let

$$L^{\mathrm{cycl}} := \bigcup_{n=1}^{\infty} L(\mu_{p^n}).$$

Let $\varpi = \mu_p - 1$, and K the ϖ-adic completion of L^{cycl}. Then K is a perfectoid field in the sense that

$$K^\circ/\varpi \to K^\circ/\varpi, \quad x \mapsto x^p$$

is surjective; see [Scholze 2012, Definition 3.1]. Let

$$K^\flat = k((t^{1/p^\infty}/))$$

be the t-adic completion of $\bigcup_{n=1}^{\infty} k((t))(t^{1/p^n})$. Then $K^\flat$ is a perfectoid field. Let $\varpi^\flat = t^{1/p}$. Equip $K^\flat$ with the nonarchimedean norm $\|\cdot\|_{K^\flat}$ such that $\|\varpi^\flat\|_{K^\flat} = \|\varpi\|_K$. Consider the morphism

$$K^\circ/\varpi \to K^{\flat\circ}/\varpi^\flat, \quad \mu_{p^n} - 1 \mapsto t^{1/p^n}. \tag{2-1}$$

This morphism is well-defined since

$$(\mu_{p^n} - 1)^{p^m} \simeq \mu_{p^{n-m}} - 1 \pmod{\varpi}$$

for $m < n$. Easy to check this morphism is an isomorphism. We call $K^\flat$ the tilt of K.

Perfectoid spaces. The most important property of a perfectoid K-algebra R is that

$$R^\circ/\varpi \to R^\circ/\varpi, \quad x \mapsto x^p$$

is surjective; see [Scholze 2012, Definition 5.1]. An affinoid K-algebra (R, R^+) is called perfectoid if R is perfectoid. By [loc. cit., Theorem 6.3], an affinoid K-algebra (R, R^+) is sheafy. Define a perfectoid space over K to be an adic space over K locally isomorphic to $\mathrm{Spa}(R, R^+)$, where (R, R^+) is a perfectoid affinoid K-algebra.

By [loc. cit., Theorem 5.2], there is an equivalence between the categories of perfectoid K-algebras and perfectoid $K^\flat$-algebras. By [loc. cit., Lemma 6.2 and Proposition 6.17], this category equivalence induces an equivalence between the categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras, as well as an equivalence between the categories of perfectoid spaces over K and perfectoid spaces over $K^\flat$.

The image of an object or a morphism in the category of perfectoid K-algebras, perfectoid affinoid K-algebras, or perfectoid spaces over K is called its tilt.

Two important maps $\sharp$ and ρ. Let R be perfectoid K-algebra and $R^\flat$ its tilt. By [loc. cit., Proposition 5.17], there is a multiplicative homeomorphism $R^\flat \simeq \varprojlim_{x \mapsto x^p} R$. Denote the projection to the first component by

$$R^\flat \to R, \quad f \mapsto f^\sharp.$$

Let (R, R^+) be perfectoid affinoid K-algebra and $(R^\flat, R^{\flat+})$ its tilt. For $x \in \mathrm{Spa}(R, R^+)$, let $\rho(x) \in \mathrm{Spa}(R^\flat, R^{\flat+})$ be the valuation $|f(\rho(x))| = |f^\sharp(x)|$ for $f \in R^\flat$. This defines a map between sets

$$\rho : \mathrm{Spa}(R, R^+) \mapsto \mathrm{Spa}(R^\flat, R^{\flat+}).$$

Note that $\mathrm{Spa}(R^\flat, R^{\flat+})$ is the tilt of $\mathrm{Spa}(R, R^+)$. The definition of ρ glues and we have a map

$$\rho_{\mathcal{X}} : |\mathcal{X}| \simeq |\mathcal{X}^\flat|$$

between the underlying sets of a perfectoid space $\mathcal{X}$ over K and its tilt $\mathcal{X}^\flat$.

Lemma 2.16. (1) *Let $\phi : R \to S$ be a morphism between perfectoid K-algebras, and $\phi^\flat : R^\flat \to S^\flat$ its tilt. Then for every $f \in R^\flat$, we have $\phi^\flat(f)^\sharp = \phi(f^\sharp)$.*

(2) *Let $\Phi : \mathcal{X} \to \mathcal{Y}$ be a morphism between perfectoid spaces over K and $\Phi^\flat$ its tilt. Then as maps between topological spaces, we have*

$$\rho_{\mathcal{Y}} \circ \Phi = \Phi^\flat \circ \rho_{\mathcal{X}}.$$

Proof. (1) follows from the definition of the $\sharp$-map and [Scholze 2012, Theorem 5.2]. (2) follows from (1). □

By (2), the restriction of $\rho_{\mathcal{X}}$ to $\mathcal{X}(K, K^\circ)$ gives the functorial bijection $\mathcal{X}(K, K^\circ) \simeq \mathcal{X}^\flat(K^\flat, K^{\flat\circ})$, which we also denote by $\rho_{\mathcal{X}}$. In the next two paragraphs, we compute $\rho_{\mathcal{X}}$ in two cases.

Tilting and reduction. Let (R, R^+) be a perfectoid affinoid K-algebra and $(R^\flat, R^{\flat+})$ its tilt. Suppose there exists a flat W-algebra S such that:

(1) R^+ is the ϖ-adic completion of $S \otimes_W K^\circ$,

(2) $R^{\flat+}$ is the $\varpi^\flat$-adic completion of $S_k \otimes_k K^{\flat\circ}$.

Let $\phi : S \to W$ be a W-algebra morphism, $\phi_k : S_k \to k$ be its base change. Then ϕ induces a map $\psi : R^+ \to K^\circ$ which further induces a point x of $\mathrm{Spa}(R, R^+)$. Similarly, ϕ_k induces a map $\psi' : R^{\flat+} \to K^{\flat\circ}$ which further induces a point x' of $\mathrm{Spa}(R^\flat, R^{\flat+})$. Then $\psi/\varpi = \psi'/\varpi^\flat$ under the isomorphism $R^+/\varpi \simeq R^{\flat+}/\varpi^\flat$. By [Scholze 2012, Theorem 5.2], ϕ' is the tilt of ϕ and thus we have the following lemma.

Lemma 2.17. *We have $\rho_{\mathrm{Spa}(R,R^+)}(x) = x'$.*

An example: the perfectoid closed unit disc. Let $R = K\langle T^{1/p^\infty}, T^{-1/p^\infty}\rangle$, the $\varpi^\flat$-adic completion of $\bigcup_{r\in\mathbb{Z}_{\geq 0}} K[T^{1/p^r}, T^{-1/p^r}]$. Then R is perfectoid. The tilt $R^\flat$ of R is $K^\flat\langle T^{1/p^\infty}, T^{-1/p^\infty}\rangle$. Let $\mathcal{G}^{\mathrm{perf}} := \mathrm{Spa}(R, R^\circ)$. Then $\mathcal{G}^{\mathrm{perf}}$ is a perfectoid space over $\mathrm{Spa}(K, K^\circ)$, and $\mathcal{G}^{\mathrm{perf},\flat} := \mathrm{Spa}(R^\flat, R^{\flat\circ})$ is its tilt.

Let $c \in \mathbb{Z}_p$, and $m \in \mathbb{Z}_{\geq 0}$. The K°-morphism $R^\circ \to K^\circ$ defined by

$$T^{1/p^n} \to \mu^c_{p^{m+n}}$$

gives a point $x \in \mathcal{G}^{\mathrm{perf}}(K, K^\circ)$. The $K^{\flat\circ}$-morphism $R^{\mathrm{perf}} \to K^\flat$ defined by

$$T^{1/p^n} \to (1 + t^{1/p^{m+n}})^c$$

gives a point $x' \in \mathcal{G}^{\mathrm{perf},\flat}(K^\flat, K^{\flat\circ})$. The following lemma follows from (2-1) and [Scholze 2012, Theorem 5.2].

Lemma 2.18. *We have $\rho_{\mathcal{G}^{\mathrm{perf}}}(x) = x'$.*

Similar result holds for $\mathcal{G}^{l,\mathrm{perf}} = \mathrm{Spa}(R, R^\circ)$ where

$$R = K\langle T_1^{1/p^\infty}, T_1^{-1/p^\infty}, \dots, T_l^{1/p^\infty}, T_l^{-1/p^\infty}\rangle,$$

and its tilt $\mathcal{G}^{l,\mathrm{perf}\,\flat} = \mathrm{Spa}(R^\flat, R^{\flat\circ})$ where

$$R^\flat = K^\flat\langle T_1^{1/p^\infty}, T_1^{-1/p^\infty}, \dots, T_l^{1/p^\infty}, T_l^{-1/p^\infty}\rangle.$$

A variant of Scholze's approximation lemma. The perfectoid fields K, $K^\flat$ and related notations are as on page 987. Let (R, R^+) be a perfectoid affinoid (K, K°)-algebra with tilt $(R^\flat, R^{\flat+})$. Let $\mathcal{X} = \mathrm{Spa}(R, R^+)$ with tilt $\mathcal{X}^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$. For $f, g \in R$, define $|f(x) - g(x)|$ to be $|(f-g)(x)|$. The following approximation lemma plays an important role in Scholze's work [2012].

Lemma 2.19 [Scholze 2012, Corollary 6.7(1)]. *Let $f \in R^+$. Then for every $c \geq 0$, there exists $g \in R^{\flat+}$ such that for every $x \in \mathcal{X}$, we have*

$$|f(x) - g^\sharp(x)| \leq \|\varpi\|^{1/p} \max\{|f(x)|, \|\varpi\|^c\} = \|\varpi\|^{1/p} \max\{|g^\sharp(x)|, \|\varpi\|^c\}. \tag{2-2}$$

Here the map $\sharp$ is as on page 988 (i.e., $|g(\rho(x))| = |g^\sharp(x)|$), and we use $\|\cdot\|$ to denote $\|\cdot\|_K$.

Recall that $k = \overline{\mathbb{F}}_p$. Assume that there exists a k-algebra S, such that $R^{\flat+}$ is the $\varpi^\flat$-adic completion of $S \otimes K^{\flat\circ}$. Then we have natural maps

$$\mathrm{Hom}_k(S, k) \hookrightarrow \mathrm{Hom}_{K^{\flat\circ}}(S \otimes K^{\flat\circ}, K^{\flat\circ}) \simeq \mathcal{X}^\flat(K^\flat, K^{\flat\circ}).$$

Thus we regard $(\mathrm{Spec}\, S)(k)$ as a subset of $\mathcal{X}^\flat$.

Lemma 2.20. *Continue to use the notations in Lemma 2.19. Assume that $c \in \mathbb{Z}\left[\frac{1}{p}\right]$. There exists a finite sum*

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0},\\ i < 1/p + c}} g_{c,i} \cdot (\varpi^\flat)^i$$

with $g_{c,i} \in S$ and only finitely many $g_{c,i} \neq 0$, such that

$$g - g_c \in (\varpi^\flat)^{1/p+c} R^{\flat+}. \tag{2-3}$$

Proof. There exists a *finite sum* $g' = \sum s_j a_j \in S \otimes K^{\flat\circ}$, where $s_j \in S$ and $a_j \in K^{\flat\circ}$, such that $g - g' \in (\varpi^\flat)^{1/p+c} R^{\flat+}$.

Claim. *Let $a \in K^{\flat\circ}$, then there exists a positive integer N such that*

$$a - \sum_{\substack{h \in (\mathbb{Z}/p^N)_{\geq 0},\\ h < 1/p + c}} \alpha_h \cdot (\varpi^\flat)^h \in (\varpi^\flat)^{1/p+c} K^{\flat\circ}$$

for certain $\alpha_h \in k$.

Indeed, the claim follows from that $K^{\flat\circ}$ is the $\varpi^\flat$-adic completion of $\bigcup_{n=1}^{\infty} k[\![t]\!][(\varpi^\flat)^{1/p^n}]$. Note that $\{h \in (\mathbb{Z}/p^N)_{\geq 0}, h < 1/p + c\}$ is finite set. So there exists a finite sum

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0},\\ i < 1/p+c}} g_{c,i} \cdot (\varpi^\flat)^i$$

with $g_{c,i} \in S$ such that $g' - g_c \in (\varpi^\flat)^{1/p+c} R^{\flat +}$. Then $g - g_c \in (\varpi^\flat)^{1/p+c} R^{\flat +}$. □

Lemma 2.21. *Let g_c be as in Lemma 2.20 and $x \in (\operatorname{Spec} S)(k)$. Regarding $x \in \mathcal{X}^\flat(K^\flat, K^{\flat\circ})$ via the inclusion above. If $|g_c(x)| \leq \|\varpi\|^{1/p+c}$, then $g_{c,i}(x) = 0$ for all i.*

Proof. Since $x \in (\operatorname{Spec} S)(k)$, if $g_{c,i}(x) \neq 0$, then $|g_{c,i}(x)| = 1$. Let $i_0 < 1/p + c$ be the minimal i such that $|g_{c,i}(x)| = 1$. Then $|g_c(x)| = \|\varpi^\flat\|_{K^\flat}^{i_0} > \|\varpi\|^{1/p+c}$, a contradiction. □

Profinite setting. Impose the following assumption.

Assumption 2.22. There are k-algebras $S_0 \subset S_1 \subset \dots$ such that $S = \bigcup S_n$.

Let $\mathcal{X}_n$ is the adic generic fiber of Spec $S_n \otimes K^{\flat\circ}$. Then we have a natural morphism

$$\pi_n : \mathcal{X}^\flat \to \mathcal{X}_n.$$

We also use π_n to denote the morphism $(\operatorname{Spec} S)(k) \to (\operatorname{Spec} S_n)(k)$. We have natural maps

$$(\operatorname{Spec} S_n)(k) \hookrightarrow \operatorname{Hom}_{K^{\flat\circ}}(S_n \otimes K^{\flat\circ}, K^{\flat\circ}) \simeq \mathcal{X}_n(K^\flat, K^{\flat\circ})$$

by which we regard $(\operatorname{Spec} S_n)(k)$ as a subset of $\mathcal{X}_n$. For each n, let $\Lambda_n \subset (\operatorname{Spec} S_n)(k)$ be a set of k-points, and Λ_n^{Zar} the Zariski closure of Λ_n in Spec S_n. We have the following maps and inclusions between sets:

$$|\mathcal{X}| \xrightarrow{\rho} |\mathcal{X}^\flat| \xrightarrow{\pi_n} |\mathcal{X}_n| \supset \Lambda_n^{\mathrm{Zar}}(k) \supset \Lambda_n,$$

where ρ is as on page 988.

Let $f \in R^+$, and $\Xi := \{x \in \mathcal{X} : |f(x)| = 0\}$. We have the following variant of Lemma 2.19.

Proposition 2.23. *Assume that $\Lambda_n \subset \pi_n(\rho(\Xi))$ for each n. Then for each $\epsilon \in K^\times$, there exists a positive integer n such that $|f(x)| \leq \|\epsilon\|_K$ for every $x \in (\pi_n \circ \rho)^{-1}(\Lambda_n^{\mathrm{Zar}}(k))$.*

Proof. Choose $c \in \mathbb{Z}_{\geq 0}$ large enough such that $\|\varpi\|_K^{1/p+c} \leq \|\epsilon\|_K$, choose g as in Lemma 2.19 and choose a finite sum

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0},\\ i < 1/p+c}} g_{c,i} \cdot (\varpi^\flat)^i$$

as in Lemma 2.20 where $g_{c,i} \in S$ for all i. There exists a positive integer $n(c)$ such that $g_{c,i} \in S_{n(c)}$ for all i by the finiteness of the sum. By the assumption, every element $x \in \Lambda_{n(c)}$ can be written as $\pi_{n(c)} \circ \rho(y)$ where $y \in \Xi$. By (2-2) and (2-3), $|g_c(\rho(y))| \leq \|\varpi\|^{1/p+c}$. Then by Lemma 2.21 and that $\rho(y) \in (\operatorname{Spec} S)(k)$, $g_{c,i}(\rho(y)) = 0$. Since $g_{c,i} \in S_{n(c)}$, $g_{c,i}(x) = 0$. Thus $g_{c,i}$ lies in the ideal defining $\Lambda_{n(c)}^{\mathrm{Zar}}$.

So $g_{c,i}(x)=0$, and thus $g_c(x)=0$, for every $x\in\Lambda^{\mathrm{Zar}}_{n(c)}(k)$. By (2-2) and (2-3), for every $x\in\Lambda^{\mathrm{Zar}}_{n(c)}(k)$, we have

$$|f(\rho^{-1}(\pi^{-1}_{n(c)}(x)))|\leq\|\varpi\|^{1/p+c}\leq\|\epsilon\|. \qquad \square$$

3. Perfectoid universal cover of an abelian scheme

Let K be the perfectoid field on page 987 and $K^\flat$ its tilt. Let $\mathfrak{A}$ be a formal abelian scheme over K°. We first recall the perfectoid universal cover of $\mathfrak{A}$ and its tilt constructed in [Pilloni and Stroh 2016, Lemme A.16]. Then we study the relation between tilting and reduction.

Perfectoid universal cover of an abelian scheme. Let $\mathfrak{A}'$ be a formal abelian scheme over $\operatorname{Spf} K^{\flat\circ}$. Assume that there is an isomorphism

$$\mathfrak{A}\otimes K^\circ/\varpi\simeq\mathfrak{A}'\otimes K^{\flat\circ}/\varpi^\flat \tag{3-1}$$

of abelian schemes over $K^\circ/\varpi\simeq K^{\flat\circ}/\varpi^\flat$. Let

$$\tilde{\mathfrak{A}}:=\varprojlim_{[p]}\mathfrak{A},\quad \tilde{\mathfrak{A}}':=\varprojlim_{[p]}\mathfrak{A}'.$$

Here the transition maps $[p]$ are the morphism multiplication by p and inverse limits exist in the categories of ϖ-adic and $\varpi^\flat$-adic formal schemes; see [Pilloni and Stroh 2016, Lemme A.15]. Index the inverse systems by $\mathbb{Z}_{\geq 0}$. Let $\operatorname{Spf} R_0^+\subset\mathfrak{A}$ be an affine open formal subscheme. Let R_i^+ be the coordinate ring of $([p]^i)^{-1}\operatorname{Spf} R_0^+$, in other words, $\operatorname{Spf} R_i^+=([p]^i)^{-1}\operatorname{Spf} R_0^+$. Let $R_i=R_i^+\big[\frac{1}{\varpi}\big]$, then R_i^+ is integrally closed in R_i. Let R^+ be the ϖ-adic completion of $\bigcup_{i=0}^\infty R_i^+$, $R=R^+\big[\frac{1}{\varpi}\big]$. Let $\operatorname{Spf} R_0'^+\subset\mathfrak{A}'$ be an affine open formal subscheme such that the restriction of (3-1) to $\operatorname{Spf} R_0^+\otimes K^\circ/\varpi$ is an isomorphism to $\operatorname{Spf} R_0'^+\otimes K^{\flat\circ}/\varpi^\flat$. We similarly define $R_i'^+$, R'^+ and R'.

Lemma 3.1 [Pilloni and Stroh 2016, Lemme A.16]. *The affinoid $K^\flat$-algebra (R',R'^+) is perfectoid. So is (R,R^+). Moreover, (R',R'^+) is the tilt of (R,R^+).*

Thus the adic generic fiber $\mathcal{A}^{\mathrm{perf}}$ (resp. $\mathcal{A}'^{\mathrm{perf}}$) of $\tilde{\mathfrak{A}}$ (resp. $\tilde{\mathfrak{A}}'$) is a perfectoid space. Moreover, $\mathcal{A}'^{\mathrm{perf}}$ is the tilt of $\mathcal{A}^{\mathrm{perf}}$. Thus we use $\mathcal{A}^{\mathrm{perf}\flat}$ to denote $\mathcal{A}'^{\mathrm{perf}}$. We call $\mathcal{A}^{\mathrm{perf}}$ (resp. $\mathcal{A}^{\mathrm{perf}\flat}$) the perfectoid universal cover of $\mathfrak{A}$ (resp. $\mathfrak{A}'$). By Lemma 2.3, there are natural bijections

$$\tilde{\mathfrak{A}}(K^\circ)\simeq\mathcal{A}^{\mathrm{perf}}(K,K^\circ),\quad \tilde{\mathfrak{A}}'(K^{\flat\circ})\simeq\mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ}).$$

Let $\mathcal{A}$ (resp. $\mathcal{A}'$) be the adic generic fiber of $\mathfrak{A}$ (resp. $\mathfrak{A}'$). By Lemma 2.3, we have natural bijections

$$\mathfrak{A}(K^\circ)\simeq\mathcal{A}(K,K^\circ),\quad \mathfrak{A}'(K^{\flat\circ})\simeq\mathcal{A}'(K^\flat,K^{\flat\circ}).$$

Definition 3.2. The group structures on $\mathcal{A}(K,K^\circ)$, $\mathcal{A}^{\mathrm{perf}}(K,K^\circ)$, $\mathcal{A}'(K^\flat,K^{\flat\circ})$ and $\mathcal{A}^{\mathrm{perf}\flat}(K^\flat,K^{\flat\circ})$ are defined to be the ones induced from the natural bijections above.

By the functoriality of taking adic generic fibers, we have morphisms

$$\pi_n:\mathcal{A}^{\mathrm{perf}}\to\mathcal{A},\quad \pi_n':\mathcal{A}^{\mathrm{perf}\flat}\to\mathcal{A}'$$

for $n \in \mathbb{Z}_{\geq 0}$, and morphisms

$$[p] : \mathcal{A} \to \mathcal{A}, \quad [p] : \mathcal{A}' \to \mathcal{A}'.$$

Consider the following commutative diagram

$$\begin{tikzcd} \tilde{\mathfrak{A}}(K^\circ) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \varprojlim_{[p]} \mathfrak{A}(K^\circ) \arrow[d, "\simeq"] \\ \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \arrow[r] & \varprojlim_{[p]} \mathcal{A}(K, K^\circ) \end{tikzcd} \tag{3-2}$$

where the bottom map is given by the π_n. We immediately have the following lemma.

Lemma 3.3. *The bottom map in (3-2) is a group isomorphism.*

Remark 3.4. Indeed, $\mathcal{A}^{\mathrm{perf}}$ serves as certain "limit" of the inverse system $\varprojlim \mathcal{A}$ in the sense of [Scholze and Weinstein 2013, Definition 2.4.1] by [loc. cit., Proposition 2.4.2]. Then Lemma 3.3 also follows from [loc. cit., Proposition 2.4.5].

Now we study torsion points in the inverse limit. We set up some group theoretical convention once for all. Let G be an abelian group. We denote by $G[n]$ the subgroup of elements of orders dividing n and by G_{tor} the subgroup of torsion elements. For a prime p, we use $G[p^\infty]$ to denote the subgroup of p-primary torsion points, and $G_{p'\text{-tor}}$ to denote the subgroup of prime-to-p torsion points. If H is a *subset* of G, H_{tor} and $H_{p'-\mathrm{tor}}$ to denote the subset $H \bigcap G_{\mathrm{tor}}$ and $H \bigcap G_{p'-\mathrm{tor}}$ when both the definitions of H and G are clear from the context. The following lemma is elementary.

Lemma 3.5. *Let G be an abelian group, then*

$$(\varprojlim_{[p]} G)_{p'-\mathrm{tor}} \simeq \varprojlim_{[p]} G_{p'-\mathrm{tor}}.$$

Lemma 3.6. *There are group isomorphisms*

$$\mathcal{A}^{\mathrm{perf}}(K, K^\circ)_{p'-\mathrm{tor}} \simeq \varprojlim_{[p]} \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \simeq \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}}$$

where the second isomorphism is the restriction of π_n. Similar result holds for $\mathcal{A}'$ and $\mathcal{A}^{\mathrm{perf}\flat}$.

Proof. The first isomorphism is from Lemmas 3.3 and 3.5. Since $\mathcal{A}(K, K^\circ)[n] \simeq \mathfrak{A}(K^\circ)[n]$ is a finite group, $[p]$ is an isomorphism on $\mathcal{A}(K, K^\circ)[n]$ for every natural number n coprime to p. The second isomorphism follows. □

Proposition 3.7. *The functorial bijection*

$$\rho = \rho_{\mathcal{A}^{\mathrm{perf}}} : \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \simeq \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})$$

(*see page 988*) *is a group isomorphism.*

Proof. We only show the compatibility of ρ with the multiplication maps, i.e., we show that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \times \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho\times\rho"] \arrow[d] & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) \times \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) \arrow[d] \\
\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho"] & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})
\end{tikzcd}$$

Here the vertical maps are the multiplication maps on corresponding groups.

Consider the formal abelian schemes $\mathfrak{B} = \mathfrak{A} \times \mathfrak{A}$ and $\mathfrak{B}' = \mathfrak{A}' \times \mathfrak{A}'$. We do the same construction to get their perfectoid universal covers $\mathcal{B}^{\mathrm{perf}}$ and $\mathcal{B}^{\mathrm{perf}\flat}$. The multiplication morphism $\mathfrak{B} \to \mathfrak{A}$ induces $m : \mathcal{B}^{\mathrm{perf}} \to \mathcal{A}^{\mathrm{perf}}$. The multiplication morphism $\mathfrak{B}' \to \mathfrak{A}'$ induces $m' : \mathcal{B}^{\mathrm{perf}\flat} \to \mathcal{A}^{\mathrm{perf}\flat}$. By (3-1) and [Scholze 2012, Theorem 5.2], $m' = m^\flat$. By functoriality, we have a commutative diagram:

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho_{\mathcal{B}^{\mathrm{perf}}}"] \arrow[d, "m"] & \mathcal{B}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) \arrow[d, "m^\flat"] \\
\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho_{\mathcal{A}^{\mathrm{perf}}}"] & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})
\end{tikzcd}$$

We only need to show that this diagram can be identified with the diagram we want. For example we show that the top horizontal maps in the two diagrams coincide, i.e., a commutative diagram:

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho_{\mathcal{B}^{\mathrm{perf}}}"] \arrow[d, "\simeq"] & \mathcal{B}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) \arrow[d, "\simeq"] \\
\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \times \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho\times\rho"] & \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \times \mathcal{A}^{\mathrm{perf}}(K, K^\circ)
\end{tikzcd}$$

The projection $\mathfrak{B} = \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ to the i-th component, $i = 1, 2$, induces $p_i : \mathcal{B}^{\mathrm{perf}} \to \mathcal{A}^{\mathrm{perf}}$. Easy to check that

$$p_1 \times p_2 : \mathcal{B}^{\mathrm{perf}}(K, K^\circ) \to \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \times \mathcal{A}^{\mathrm{perf}}(K, K^\circ)$$

is a group isomorphism by passing to formal schemes. Similarly we have an isomorphism

$$p_1' \times p_2' : \mathcal{B}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) \to \mathcal{A}^{\mathrm{perf}}(K, K^\circ) \times \mathcal{A}^{\mathrm{perf}}(K, K^\circ).$$

The commutativity is implied by that $p_i' = p_i^\flat$, which is from (3-1) and [Scholze 2012, Theorem 5.2]. $\square$

Tilting and reduction. Let $k = \overline{\mathbb{F}}_p$ and let $W = W(k)$ be the ring of Witt vectors. Let A be an abelian scheme over W, A_{K° be its base change to K°, $\mathcal{A}$ be the adic generic fiber of A_{K°. Let A_k be the special fiber of A, and A' be the base change $A_k \otimes K^{\flat\circ}$ with adic generic fiber $\mathcal{A}'$. Since

$$A_{K^\circ} \otimes (K^\circ/\varpi) \simeq A \otimes_W k \otimes_k (K^\circ/\varpi) \simeq A' \otimes_{K^{\flat\circ}} (K^{\flat\circ}/\varpi^\flat),$$

we can apply the construction in Lemma 3.1 to the formal completions of $A_k \otimes_k K^\circ$ and A'. Then we have the perfectoid universal cover $\mathcal{A}^{\mathrm{perf}}$ of the ϖ-adic formal completion of A_{K°, the perfectoid universal cover $\mathcal{A}^{\mathrm{perf}\flat}$ of the $\varpi^\flat$-adic formal completion of $A_{K^{\flat\circ}}$, and the morphisms $\pi_n : \mathcal{A}^{\mathrm{perf}} \to \mathcal{A}$, $\pi'_n : \mathcal{A}^{\mathrm{perf}\flat} \to \mathcal{A}'$ for each $n \in \mathbb{Z}_{\geq 0}$. The following well-known results can be deduced from [Serre and Tate 1968].

Lemma 3.8. (1) *The inclusion $A(W) \hookrightarrow A(K^\circ)$ gives an isomorphism $A(W)_{p'-\mathrm{tor}} \simeq A(K^\circ)_{p'-\mathrm{tor}}$.*

(2) *The reduction map gives an isomorphism*

$$\mathrm{red} : A(W)_{p'-\mathrm{tor}} \simeq A(k)_{p'-\mathrm{tor}}.$$

(3) *The natural inclusion $A(k) \hookrightarrow A_{K^{\flat\circ}}(K^{\flat\circ})$ gives an isomorphism $A(k)_{p'-\mathrm{tor}} \simeq A_{K^{\flat\circ}}(K^{\flat\circ})_{p'-\mathrm{tor}}$.*

Now we relate reduction and tilting.

Lemma 3.9. *Let the unindexed maps in the following diagram be the naturals ones*:

$$\begin{tikzcd}
\mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} & \mathcal{A}^{\mathrm{perf}}(K, K^\circ)_{p'-\mathrm{tor}} \arrow[l, "\pi_n"'] \arrow[r, "\rho"] & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})_{p'-\mathrm{tor}} \arrow[r, "\pi'_n"] & \mathcal{A}'(K^\flat, K^{\flat\circ})_{p'-\mathrm{tor}} \\
A_{K^\circ}(K^\circ)_{p'-\mathrm{tor}} \arrow[u] & A(W)_{p'-\mathrm{tor}} \arrow[l] \arrow[r, "\mathrm{red}"] & A(k)_{p'-\mathrm{tor}} \arrow[r] & A_{K^{\flat\circ}}(K^{\flat\circ})_{p'-\mathrm{tor}} \arrow[u]
\end{tikzcd}$$

Then each map is a group isomorphism, and the diagram is commutative (*up to inverting the arrows*).

Proof. We may assume $n = 0$. Definition 3.2, Lemma 3.6, Proposition 3.7 and Lemma 3.8 give the isomorphisms. We only need to check the commutativity. And we only need to check the two maps from $A(W)_{p'-\mathrm{tor}}$ to $\mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})_{p'-\mathrm{tor}}$ are the same. This follows from Lemma 2.17. $\square$

Similarly, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \arrow[r, "\rho"] & \mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ}) \arrow[dr, "\simeq"] & \\
\varprojlim_{[p]} A(W) \arrow[u, hook, "\iota"] \arrow[r, "\mathrm{red}"] \arrow[d, "\pi_0"] & \varprojlim_{[p]} A(k) \arrow[r, hook] \arrow[d, "\pi'_n"] & \varprojlim_{[p]} \mathcal{A}'(K^\flat, K^{\flat\circ}) \arrow[d, "\pi'_n"] \\
\bigcap_{i=0}^{\infty} p^i A(W) \arrow[d] & A(k) \arrow[r, hook] \arrow[d, "{[p^n]}"] & \mathcal{A}'(K^\flat, K^{\flat\circ}) \arrow[d, "{[p^n]}"] \\
A(W) \arrow[r, "\mathrm{red}"] & A(k) \arrow[r, hook] & \mathcal{A}'(K^\flat, K^{\flat\circ})
\end{tikzcd} \tag{3-3}$$

Here ι is induced from the inclusion $A(W) \hookrightarrow \mathcal{A}^{\mathrm{perf}}(K, K^\circ)$ and the isomorphism $\mathcal{A}^{\mathrm{perf}}(K, K^\circ) \simeq \varprojlim_{[p]} \mathcal{A}(K, K^\circ)$ (see Lemma 3.3). Here and from now on we regard $\varprojlim_{[p]} A(W)$ as a subset of $\mathcal{A}^{\mathrm{perf}}(K, K^\circ)$ via ι, $A(k)$ as a subset $\mathcal{A}'(K^\flat, K^{\flat\circ})$, and $\varprojlim_{[p]} A(k)$ as a subset of $\mathcal{A}^{\mathrm{perf}\flat}(K^\flat, K^{\flat\circ})$.

4. Proof of Theorem 1.3

In this section, we at first prove a lower bound on prime-to-p torsion points in a subvariety. Then we prove Theorem 1.3. Let $k = \overline{\mathbb{F}}_p$, $W = W(k)$ the ring of Witt vectors, and $L = W\left[\frac{1}{p}\right]$.

Results of Poonen, Raynaud and Scanlon.

Theorem 4.1 [Poonen 2005]. *Let B be an abelian variety defined over k, and V an irreducible closed subvariety of B. Let S be a finite set of primes. Suppose that V generates B, then the composition of*

$$V(k) \hookrightarrow B(k) \xrightarrow{\bigoplus_{l\in S} \mathrm{pr}_l} \bigoplus_{l\in S} B[l^\infty]$$

is surjective, where pr_l is the projection to the l-primary component.

Let A be an abelian scheme over W. Let $T = \bigcap_{n=0}^\infty p^n(A(L)[p^\infty])$, the maximal divisible subgroup of $A(L)[p^\infty]$. Though not needed, as an illustration, we note that by [Raynaud 1983a, Exemples 5.2.3], $T = 0$ if the p-rank of A_k is 0 or if A is a "general ordinary abelian variety", and $T = A(L)[p^\infty] \simeq L/\mathbb{Z}_p^{\dim A_L}$ if A is the canonical lifting in Serre–Tate theory; see Section 5.

Lemma 4.2 [Raynaud 1983a, Lemma 5.2.1]. (1) *Let T_o be the subgroup of $A(\bar{L})[p^\infty]$ coming from the connected component of the p-divisible group of A, then $T_o \bigcap T = 0$.*

(2) *As a subgroup of $A(\bar{L})[p^\infty]$, T is a $\mathrm{Gal}(\bar{L}/L)$-direct summand.*

Note that

$$\bigcap_{n=0}^{\infty} p^n(A(W)_{\mathrm{tor}}) = A(W)_{p'-\mathrm{tor}} \bigoplus \bigcap_{n=0}^{\infty} p^n(A(W)[p^\infty]). \tag{4-1}$$

Corollary 4.3. *The following reduction map is injective*

$$\mathrm{red} : \bigcap_{n=0}^{\infty} p^n(A(W)_{\mathrm{tor}}) \to A(k).$$

Let $Z \subset A_L$ be a closed subvariety.

Lemma 4.4 [Raynaud 1983b, 8.2]. *Let T' be a $\mathrm{Gal}(\bar{L}/L)$-direct summand such that as $\mathrm{Gal}(\bar{L}/L)$-modules*

$$A(\bar{L})_{\mathrm{tor}} = A(\bar{L})_{p'-\mathrm{tor}} \bigoplus T \bigoplus T'.$$

If Z does not contain any translate of a nontrivial abelian subvariety of A_L, there exists a positive integer N such that the order of the T'-component of every element in $Z(\bar{L})_{\mathrm{tor}}$ divides p^N.

Remark 4.5. Lemma 4.4 is used by Raynaud [1983b] to reduce the Manin–Mumford conjecture to a theorem (see [loc. cit., Theorem 3.5.1]) obtained by studying p-adic rigid analytic properties of universal vector extension of an abelian variety.

Let K and $K^\flat$ be the perfectoid fields on page 987. Let $\mathcal{A}$ be the adic generic fiber of A_{K°. Let Z^{Zar} be the Zariski closure of Z in A, and $\mathcal{Z}$ the adic generic fiber of $Z^{\mathrm{Zar}}_{K^\circ}$. For $\epsilon \in K^\times$, let $\mathcal{Z}_\epsilon$ be the ϵ-neighborhood of Z_K in $\mathcal{A}$ as in Definition 2.11. By Lemma 2.12, a result of Scanlon [1998] on the Tate–Voloch conjecture implies the following lemma.

Lemma 4.6 [Scanlon 1998]. *There exists $\epsilon \in K^\times$, such that $\mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \bigcap \mathcal{Z}_\epsilon \subset \mathcal{Z}$.*

Remark 4.7. The proofs of Poonen's result and Scanlon's result are independent of Theorem 1.3.

A lower bound. Define

$$\Lambda := Z^{\mathrm{Zar}}(W) \bigcap \bigcap_{n=0}^{\infty} p^n(A(W)_{\mathrm{tor}}), \quad \Lambda_\infty := \iota(\pi_0^{-1}(\Lambda)), \tag{4-2}$$

where π_0 and ι are as in the left column of diagram (3-3). Then $\rho(\Lambda_\infty)$ is contained in (the image of) $\varprojlim_{[p]} A(k)$ by diagram (3-3). Now let $\Lambda_n := \pi'_n(\rho(\Lambda_\infty))$. Then Λ_n is contained in (the image of) $A(k)$. Let Λ_n^{Zar} be the Zariski closure of Λ_n in A_k.

Proposition 4.8. *There exists a positive integer n such that*

$$\pi_0(\rho^{-1}(\pi_n'^{-1}(\Lambda_n^{\mathrm{Zar}}(k)_{p'-\mathrm{tor}}))) \bigcap \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \subset \mathcal{Z}.$$

Proof. Let $\mathcal{U}$ be a finite affine open cover of A by affine open subschemes flat over W. Let $U \in \mathcal{U}$. The restriction of $\mathcal{A}^{\mathrm{perf}}$ over the adic generic fiber of U_{K° is a perfectoid space $\mathcal{X} = \mathrm{Spa}(R, R^+)$ whose tilt satisfies Assumption 2.22 (see Lemma 3.1 and the discussion above it). Let $\mathcal{I}$ be the ideal sheaf of Z^{Zar}. Let $f \in \mathcal{I}(U)$. Regard f as in R. By definition of Λ_n, we can apply Proposition 2.23 to f and Λ_n. Varying U in $\mathcal{U}$ and varying f in a finite set of generators of $\mathcal{I}(U)$, Proposition 2.23 implies that for every $\epsilon \in K^\times$, there exists a positive integer n such that

$$\pi_0(\rho^{-1}(\pi_n'^{-1}(\Lambda_n^{\mathrm{Zar}}(k)))) \subset \mathcal{Z}_\epsilon.$$

Then Proposition 4.8 follows from Lemma 4.6. $\square$

Our lower bound on the size of the set of prime-to-p torsions in Z is as follows.

Proposition 4.9. *Let $p > 2$. Assume that Z contains the unit $0 \in A_L$:*

(1) *Assume Λ is infinite. For every prime number $l \neq p$, the image of the composition of*

$$Z^{\mathrm{Zar}}(W)_{p'-\mathrm{tor}} \hookrightarrow A(W)_{p'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} A(W)[l^\infty] \tag{4-3}$$

contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least 2. Here the map pr_l is the projection to the l-primary component.

(2) *Assume that the image of the composition of*

$$\Lambda \hookrightarrow A(W)_{\mathrm{tor}} \xrightarrow{\mathrm{pr}_p} A(W)[p^\infty]$$

contains a translate of a free $L/\mathbb{Z}_p$-submodule of rank r. For every prime number $l \neq p$, the image of the composition of (4-3) contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank $2r$.

Proof. Fix a large n such that

$$\pi_0(\rho^{-1}(\pi_n'^{-1}(\Lambda_n^{\mathrm{Zar}}(k)_{p'-\mathrm{tor}}))) \bigcap \mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \subset \mathcal{Z}(K, K^\circ)_{p'-\mathrm{tor}} \tag{4-4}$$

as in Proposition 4.8. Let X be the image of the left hand side of (4-4) via the composition of

$$\mathcal{A}(K, K^\circ)_{p'-\mathrm{tor}} \simeq A(W)_{p'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} A(W)[l^\infty].$$

Then X is contained in the image of the composition of (4-3).

To prove (1), we only need to prove the following claim.

Claim. *X contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least 2 for every l.*

By diagram (3-3), we have $\Lambda_0 = \mathrm{red}(\Lambda)$. Since $p > 2$, by Corollary 4.3, Λ_0 is infinite. Since $\Lambda_0 = [p]^n(\Lambda_n)$, Λ_n is infinite. There exists $a \in A(k)$ such that an irreducible component of $\Lambda_n^{\mathrm{Zar}} + a$ (is contained and) generates a nontrivial abelian subvariety A' of A_k. Since Z contains the unit $0 \in A_L$, Λ_n^{Zar} contains the unit $0 \in A_k$ and A' contains a. Let a_p be the p-primary part of a and $a_{p'} = a - a_p$. By Theorem 4.1 (for $\Lambda_n^{\mathrm{Zar}} + a \subset A'$ and $S = \{p, l\}$), the image of

$$\Lambda_n^{\mathrm{Zar}}(k) + a \xrightarrow{\mathrm{pr}_l \bigoplus \mathrm{pr}_p} A(k)[l^\infty] \bigoplus A(k)[p^\infty] \tag{4-5}$$

contains $M \bigoplus \{a_p\}$, where M is a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(k)[l^\infty]$ of rank at least 2. Thus

$$(\mathrm{pr}_l \bigoplus \mathrm{pr}_p)(\Lambda_n^{\mathrm{Zar}}(k) + a_{p'}) \supset M \bigoplus \{0\}.$$

We claim

$$\mathrm{pr}_l((\Lambda_n^{\mathrm{Zar}}(k) + a_{p'})_{p'-\mathrm{tor}}) \supset M.$$

Indeed, write $b \in \Lambda_n^{\mathrm{Zar}}(k) + a_{p'}$ as the sum $b_p + b_{p'}$ of p-primary part and prime-to-p part. Then $(\mathrm{pr}_l \bigoplus \mathrm{pr}_p) b = \mathrm{pr}_l(b_{p'}) + b_p$. If this is $x + 0 \in M \bigoplus \{0\}$, then $b_p = 0$, and $b = b_{p'}$. Thus $\mathrm{pr}_l(b) = x \in M$. The claim is proved. By the claim,

$$M - \mathrm{pr}_l(a_{p'}) \subset Y := \mathrm{pr}_l(\Lambda_n^{\mathrm{Zar}}(k)_{p'-\mathrm{tor}}).$$

By Lemma 3.9, X contains the preimage of $[p]^n(Y)$ under the isomorphism $\mathrm{red} : A(W)_{p'-\mathrm{tor}} \simeq A(k)_{p'-\mathrm{tor}}$. Thus we proved the claim above.

To prove (2), we only need to prove the following claim.

Claim. *X contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(W)[l^\infty]$ of rank at least $2r$ for every l.*

By diagram (3-3), we have $\Lambda_0 = \mathrm{red}(\Lambda)$. Since $p > 2$, by Corollary 4.3 and the assumption on Λ, $\mathrm{pr}_p(\Lambda_0)$ contains a translate of a free $L/\mathbb{Z}_p$-submodule of rank r. Let $V_1, \dots, V_m$ be the irreducible components of Λ_0^{Zar}. Let A_i be the minimal abelian subvariety of A_k such that a certain translate of A_i contains V_i. Since the p-rank of A_i is at most its dimension, at least one A_i is of dimension at least r. Since $\Lambda_0 = [p]^n(\Lambda_n)$, there exists $a \in A(k)$ such that an irreducible component of $\Lambda_n^{\mathrm{Zar}} + a$ generates an abelian subvariety of A_k of dimension at least r. Then we prove (2) by copying the proof of (1) above, starting from the sentence containing (4-5). The only modification needed is that the rank of M should be at least $2r$. $\square$

The proof of Theorem 1.3. Now we prove Theorem 1.3. By the argument in [Pila and Zannier 2008], we only need to prove the following weaker theorem. We save the symbol A for the proof.

Theorem 4.10. *Let F be number field. Let B be an abelian variety over F and V a closed subvariety of B. If V does not contain any translate of an abelian subvariety of B of positive dimension, then V contains only finitely many torsion points of B.*

Proof. We only need to prove the theorem up to replacing V by a multiple.

Let v be a place of F unramified over a prime number $p > 2$ such that B has good reduction. Let A be the base change to W of the integral smooth model of B over $\mathcal{O}_{F_v}$. Let $Z = V_L \subset A$. By (4-1) and Lemma 4.4, up to replacing V by $[p^N]V$ for N large enough, we may assume that $Z^{\mathrm{Zar}}(W)_{\mathrm{tor}} \subset \bigcap_{n=0}^{\infty} p^n(A(W)_{\mathrm{tor}})$. Thus $\Lambda = Z^{\mathrm{Zar}}(W)_{\mathrm{tor}}$, where Λ is defined as in (4-2). Suppose that V contains infinitely many torsion points. Then Λ is infinite. Up to replacing V by $[p^N]V$, we may assume that Z contains the unit $0 \in A_L$. Now we want to find a contradiction. By Proposition 4.9(1), for every prime number $l \neq p$, the composition of

$$Z(L)_{p'-\mathrm{tor}} \hookrightarrow A(L)_{p'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} A(L)[l^\infty]$$

contains a translate of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $A(L)[l^\infty]$ of rank 2.

Let u be another place of F, unramified over an odd prime number $l \neq p$, such that B has good reduction at u. Let B_o be the reduction. Let M be the completion of the maximal unramified extension of F_u and $\overline{M}$ its algebraic closure. Then the composition

$$V(\overline{M})_{\mathrm{tor}} \hookrightarrow B(\overline{M})_{\mathrm{tor}} \xrightarrow{\mathrm{pr}_l} B(\overline{M})[l^\infty]$$

contains a translate G of a free $\mathbb{Q}_l/\mathbb{Z}_l$-submodule of $B(\overline{M})[l^\infty]$ of rank 2. Let $T = \bigcap_{n=0}^{\infty} l^n(B(M)[l^\infty])$. By Lemma 4.4 (applied to l, M instead of p, L), up to replacing V by $[l^N]V$ for N large enough, G is contained in T. By (4-1) (applied to l, M° instead of p, W), the image of the composition of

$$V(M) \bigcap \bigcap_{n=0}^{\infty} l^n(B(M)_{\mathrm{tor}}) \hookrightarrow B(M) \xrightarrow{\mathrm{pr}_l} B(M)[l^\infty]$$

contains G. By Proposition 4.9(2) (applied to l, M° instead of p, W), for every prime number $q \neq l$, the composition

$$V(M)_{l'-\mathrm{tor}} \hookrightarrow B(M)_{l'-\mathrm{tor}} \xrightarrow{\mathrm{pr}_q} B(M)[q^\infty]$$

contains a translate of a free $\mathbb{Q}_q/\mathbb{Z}_q$-submodule of rank 4. Repeating this process (use more places or only work at v and u), we get a contradiction as A is of finite dimension. $\square$

5. Ordinary perfectoid Siegel space and Serre–Tate theory

Let $\mathbb{A}_f$ be the ring of finite adeles of $\mathbb{Q}$, $U^p \subset \mathrm{GSp}_{2g}(\mathbb{A}_f^p)$ an open compact subgroup contained in the congruence subgroup of level-N for some $N \geq 3$ prime to p. Let $X = X_{g,U^p}$ over $\mathbb{Z}_p$ be the Siegel moduli space of principally polarized g-dimensional abelian varieties over $\mathbb{Z}_p$-schemes with level U^p structure. Let X_o be special fiber of X. We will use the perfectoid fields defined on page 987. We briefly recall some notations. Let $k = \overline{\mathbb{F}}_p$, $W = W(k)$ the ring of Witt vectors, L the fraction field of W, and L^{cycl} the field extension of L by adjoining all p-power-th roots of unity. Let K be the p-adic completion of L^{cycl} which is a perfectoid field. Then $K^\flat = k((t^{1/p^\infty}))$ is the tilt of K. Fix a primitive p^n-th root of unity μ_{p^n} for every positive integer n such that $\mu_{p^{n+1}}^p = \mu_{p^n}$.

Ordinary perfectoid Siegel space. Let $X_o(0) \subset X_o$ be the ordinary locus. Let $\mathfrak{X}(0)$ over $\mathbb{Z}_p$ be the open formal subscheme of the formal completion of X along X_o defined by the condition that every local lifting of the Hasse invariant is invertible; see [Scholze 2015, Definition 3.2.12, Lemma 3.2.13]. Then $\mathfrak{X}(0)/p = X_o(0)$; see [loc. cit., Lemma 3.2.5]. Let $\widehat{X_o(0)_{K^{\flat\circ}}}$ be the $\varpi^\flat$-adic formal completion of $X_o(0)_{K^{\flat\circ}}$. Let $\mathcal{X}(0)$ and $\mathcal{X}'(0)$ be the adic generic fibers of $\mathfrak{X}(0)_{K^\circ}$ and $\widehat{X_o(0)_{K^{\flat\circ}}}$ respectively.

Let $\mathrm{Fr} : X_o(0) \to X_o(0)$ be the (relative) Frobenius morphism (note that $X_o(0)$ is defined over $\mathbb{F}_p$). Let $\mathrm{Fr}^{\mathrm{can}} : \mathfrak{X}(0) \to \mathfrak{X}(0)$ be given by the functor sending an abelian scheme A to its quotient by the connected subgroup scheme of $A[p]$. Then $\mathrm{Fr}^{\mathrm{can}}/p = \mathrm{Fr}$. We also use $\mathrm{Fr}^{\mathrm{can}}$ and Fr to denote their base changes to K° and $K^{\flat\circ}$ respectively. Let

$$\tilde{\mathfrak{X}}(0) := \varprojlim_{\mathrm{Fr}^{\mathrm{can}}} \mathfrak{X}(0)_{K^\circ}, \qquad \tilde{\mathfrak{X}}'(0) := \varprojlim_{\mathrm{Fr}} \widehat{X_o(0)_{K^{\flat\circ}}},$$

where the inverse limits are taken in the categories of ϖ-adic and $\varpi^\flat$-adic formal schemes respectively. Here $\varpi = \mu_p - 1$ and $\varpi^\flat = t^{1/p}$. By [Scholze 2015, Corollary 3.2.19], the corresponding adic generic fibers $\mathcal{X}(0)^{\mathrm{perf}}$ and $\mathcal{X}'(0)^{\mathrm{perf}}$ of $\tilde{\mathfrak{X}}(0)$ and $\tilde{\mathfrak{X}}'(0)$ are perfectoid spaces. Moreover, $\mathcal{X}'(0)^{\mathrm{perf}} = \mathcal{X}(0)^{\mathrm{perf},\flat}$, the tilt of $\mathcal{X}(0)^{\mathrm{perf}}$. Then we have the natural projections

$$\pi : \mathcal{X}(0)^{\mathrm{perf}} \to \mathcal{X}(0), \qquad \pi' : \mathcal{X}(0)^{\mathrm{perf},\flat} \to \mathcal{X}'(0). \tag{5-1}$$

We also have a natural map between the underlying sets defined on page 988

$$\rho_{\mathcal{X}(0)^{\mathrm{perf}}} : |\mathcal{X}(0)^{\mathrm{perf}}| \to |\mathcal{X}(0)^{\mathrm{perf},\flat}|. \tag{5-2}$$

(The map $\rho_{\mathcal{X}(0)^{\mathrm{perf}}}$ is in fact a homeomorphism and we do not need this fact.)

Classical Serre–Tate theory. We use the adjective "classical" to indicate the Serre–Tate theory [Katz 1981] discussed in this subsection, compared with Chai's global Serre–Tate theory to be discussed in Section 5.

Let R be an Artinian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $A/\operatorname{Spec} R$ be an abelian scheme with ordinary special fiber A_k. Let $A_k^\vee$ be the dual abelian variety of A_k. There is a $\mathbb{Z}_p$-module morphism from the product of Tate-modules $T_pA_k \otimes T_pA_k^\vee$ to $1 + \mathfrak{m}$ constructed by Katz [1981]. We call this morphism the classical Serre–Tate coordinate system for $A/\operatorname{Spec} R$. If $A/\operatorname{Spec} R$ is moreover a principally polarized abelian scheme, the Serre–Tate coordinate system for $A/\operatorname{Spec} R$ is a $\mathbb{Z}_p$-module morphism

$$q_{A/\operatorname{Spec} R} : \operatorname{Sym}^2(T_p A_k) \to 1 + \mathfrak{m}. \tag{5-3}$$

Let $x \in X_o(0)(k)$, and let A_x be the corresponding principally polarized abelian variety. Let $\mathfrak{M}_x$ be the formal completion of X at x, and $\mathfrak{A}/\mathfrak{M}_x$ the formal universal deformation of A_x. Then as part of the construction of $q_{A/\operatorname{Spec} R}$, there is an isomorphism of formal schemes over W

$$\mathfrak{M}_x \simeq \operatorname{Hom}_{\mathbb{Z}_p}(\operatorname{Sym}^2(T_p A_x), \hat{\mathbb{G}}_m), \tag{5-4}$$

where $\hat{\mathbb{G}}_m$ is the formal completion of the multiplicative group scheme over W along the unit section. In particular, $\mathfrak{M}_x$ has a formal torus structure. Moreover, if $A_k \simeq A_x$ in (5-3), then (5-3) is the value of (5-4) at the morphism $\operatorname{Spec} R \to \mathfrak{M}_x$ induced by A. Let $\mathcal{O}(\mathfrak{M}_x)$ be the coordinate ring of $\mathfrak{M}_x$, and let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}(\mathfrak{M}_x)$. From (5-4), we have a morphism of $\mathbb{Z}_p$-modules

$$q = q_{\mathfrak{A}/\mathfrak{M}_x} : \operatorname{Sym}^2(T_p A_x) \to 1 + \mathfrak{m}_x.$$

Fix a basis $\xi_1, \ldots, \xi_{g(g+1)/2}$ of $\operatorname{Sym}^2(T_p A_x)$.

Proposition 5.1 [de Jong and Noot 1991, 3.2]. *Let F be a finite extension of L with ring of integers F°. Let $y^\circ \in X(F^\circ)$ with generic fiber y. Suppose that $y^\circ \in \mathfrak{M}_x(F^\circ)$. Then y is a CM point if and only if $q(\xi_i)(y^\circ)$ is a p-primary root of unity for $i = 1, \ldots, g(g+1)/2$.*

Thus every ordinary CM point is contained in $X(L^{\mathrm{cycl}})$. For an ordinary CM point $y \in X(L^{\mathrm{cycl}})$. there is a unique $y^\circ \in X(K^\circ)$ whose generic fiber is $y_K \in X(K)$. We regard y° as a point in $\mathfrak{X}(0)(K^\circ)$ and y_K as a point in $\mathcal{X}(K, K^\circ)$ via Lemma 2.3.

Definition 5.2. Let $a = (a^{(1)}, \ldots, a^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$:

(1) An ordinary CM point $y \in X(L^{\mathrm{cycl}})$ with reduction x is called of order p^a with respect to the basis $\xi_1, \ldots, \xi_{g(g+1)/2}$ if $q(\xi_i)(y^\circ)$ is a primitive $p^{a^{(i)}}$-th root of unity for each $i = 1, \ldots, g(g+1)/2$. If moreover $q(\xi_i)(y^\circ) = \mu_{p^{a^{(i)}}}$, y is called a μ-generator with respect to the basis $\xi_1, \ldots, \xi_{g(g+1)/2}$.

(2) Assume that a is nonincreasing so that $q(\xi_{i+1})(y^\circ)$ is an $r^{(i)}$-th power of $q(\xi_i)(y^\circ)$ for some (nonunique) $r^{(i)} \in \mathbb{Z}_p$, $i = 1, \ldots, g(g+1)/2 - 1$. We call $(r^{(1)}, \ldots, r^{(g(g+1)/2-1)}) \in \mathbb{Z}_p^{g(g+1)/2-1}$ a ratio of y with respect to the basis $\xi_1, \ldots, \xi_{g(g+1)/2}$.

It is clear that if a is nonincreasing, then the usual p-adic absolute value $|r^{(i)}|_p = p^{a^{(i+1)} - a^{(i)}}$.

Let $T_i = q(\xi_i) - 1 \in \mathfrak{m}_x$. Then we have an isomorphism

$$\mathcal{O}(\mathfrak{M}_x) \simeq W[\![T_1, \ldots, T_{g(g+1)/2}]\!]. \tag{5-5}$$

Let $\widehat{X_o(0)}_{/x}$ be the formal completion of $X_o(0)$ at x. Restricted to $\widehat{X_o(0)}_{/x}$, (5-5) gives an isomorphism

$$\mathcal{O}(\widehat{X_o(0)}_{/x}) \simeq k[\![T_1, \ldots, T_{g(g+1)/2}]\!]. \tag{5-6}$$

Let $U \to X_o(0)$ be an étale morphism, $z \in U(k)$ with image x. Then (5-6) gives an isomorphism

$$\mathcal{O}(\widehat{U}_{/z}) \simeq k[\![T_1, \dots, T_{g(g+1)/2}]\!]. \tag{5-7}$$

Let A_z be the pullback of A_x at z. Then we naturally have $T_pA_z \simeq T_pA_x$. Thus we also regard $\xi_1, \dots, \xi_{g(g+1)/2}$ as a basis of $\mathrm{Sym}^2(T_pA_z)$.

Definition 5.3. We call (5-7) the realization of the classical Serre–Tate coordinate system of $\widehat{U}_{/z}$ at the basis $\xi_1, \dots, \xi_{g(g+1)/2}$ of $\mathrm{Sym}^2(T_pA_z)$.

We have another description of (5-7). Let I_n be a descending sequence of open ideals of $\mathcal{O}(\widehat{U}_{/z})$ defining the topology of $\mathcal{O}(\widehat{U}_{/z})$. Let $R_n := \mathcal{O}(\widehat{U}_{/z})/I_n$, let A_n be the pullback of the formal universal principally polarized abelian scheme over $\mathfrak{M}_x$ to $\mathrm{Spec}\, R_n$ with special fiber A_z. Let

$$q_{A_n/\mathrm{Spec}\, R_n} : \mathrm{Sym}^2(T_pA_x^{\mathrm{univ}}) \to R_n^\times$$

be the classical Serre–Tate coordinate system of A_n/R_n. Then $q_{A_n/\mathrm{Spec}\, R_n}(\xi_i) - 1 = T_i \pmod{I_n}$. Thus the sequence $\{q_{A_n/\mathrm{Spec}\, R_n}(\xi_i) - 1\}_n$ gives an element in $\mathcal{O}(\widehat{U}_{/z}) \simeq \varprojlim_n R_n$, which equals T_i.

Tilts of ordinary CM points. Let $\widehat{X_o(0)_{K^{\flat\circ}}/x}$ be the formal completion of $X_o(0)_{K^{\flat\circ}}$ at x. By (5-6), we have

$$\mathcal{O}(\widehat{X_o(0)_{K^{\flat\circ}}/x}) \simeq K^{\flat\circ}[\![T_1, \dots, T_{g(g+1)/2}]\!]. \tag{5-8}$$

Let $\mathcal{D}_x$ be the adic generic fiber of $\widehat{X_o(0)_{K^{\flat\circ}}/x}$. Then $\mathcal{D}_x$ is an adic subspace of $\mathcal{X}'(0)$ in the sense of Definition 2.2. Moreover, (5-8) and Lemma 2.3 imply an isomorphism

$$\mathcal{D}_x(K^\flat, K^{\flat\circ}) \simeq K^{\flat\circ\circ, g(g+1)/2}. \tag{5-9}$$

Lemma 5.4. *Let $y \in X(L^{\mathrm{cycl}})$ be an ordinary CM point with reduction x:*

(1) *For every $\tilde{y} \in \pi^{-1}(y_K) \subset \mathcal{X}(0)^{\mathrm{perf}}$, we have*

$$\pi' \circ \rho_{\mathcal{X}(0)^{\mathrm{perf}}}(\tilde{y}) \in \mathcal{D}_x.$$

(2) *Let $a = (a^{(1)}, \dots, a^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$ and $I \subset \{1, 2, \dots, g(g+1)/2\}$ the subset of the i such that $a^{(i)} = 0$. Let y be a μ-generator of order p^a with respect to the basis $\xi_1, \dots, \xi_{g(g+1)/2}$ (see Definition 5.2). There exists $\tilde{y} \in \pi^{-1}(y_K)$ such that via the isomorphism (5-9), the i-th coordinate of $\pi' \circ \rho_{\mathcal{X}(0)^{\mathrm{perf}}}(\tilde{y})$ is 0 for $i \in I$ and is $t^{1/p^{a^{(i)}}}$ for $i \notin I$.*

Proof. We recall the effect of $\mathrm{Fr}^{\mathrm{can}}$ on $\mathfrak{M}_x$ (see [Katz 1981, 4.1]). Denote $\mathfrak{M}_x$ by $\mathfrak{M}_{A_x}$. Let $\sigma \in \mathrm{Aut}(k)$ be the Frobenius. Let $A_x^{(\sigma)} := A_x \otimes_{k,\sigma} k$ be the base change by σ. Then $\mathrm{Fr}^{\mathrm{can}}$, restricted to $\mathfrak{M}_{A_x}$ gives a morphism $\mathrm{Fr}^{\mathrm{can}} : \mathfrak{M}_{A_x} \to \mathfrak{M}_{A_x^{(\sigma)}}$ over W [loc. cit., page 171]. Let $\sigma(\xi_1), \dots, \sigma(\xi_{g(g+1)/2})$ be the induced basis of $\mathrm{Sym}^2(T_pA_x^{(\sigma)}[p^\infty])$. Then [loc. cit., Lemma 4.1.2] implies that

$$\mathrm{Fr}^{\mathrm{can},*}(q(\sigma(\xi_i))) = q(\xi_i)^p. \tag{5-10}$$

We associate a perfectoid space to $\mathfrak{M}_x$. Let

$$\tilde{\mathfrak{M}}_x := \varprojlim_{\mathrm{Fr}^{\mathrm{can}}} \mathfrak{M}_{A_x^{(\sigma^{-n})}}.$$

By a similar (and easier) proof as the one for [Scholze 2015, Corollary 3.2.19], the adic generic fiber $\mathcal{M}_x^{\mathrm{perf}}$ of $\tilde{\mathfrak{M}}_{x,K^\circ}$ is a perfectoid space. Moreover, let $\mathcal{M}_x'^{\mathrm{perf}}$ be the adic generic fiber of $\varprojlim_{\mathrm{Fr}} \widehat{X_o(0)_{K^{\flat\circ}}/x}$. Then $\mathcal{M}_x'^{\mathrm{perf}}$ is the tilt of $\mathcal{M}_x^{\mathrm{perf}}$. By Lemma 2.16, the tilting process commutes with restriction to an open subspace. Thus to prove Lemma 5.4, we only need to consider the tilting between $\mathcal{M}_x^{\mathrm{perf}}$ and $\mathcal{M}_x'^{\mathrm{perf}}$. Then Lemma 5.4 follows from the cases $c = 0$ and $c = 1$ of Lemma 2.18 (which deals with closed units discs while here we are dealing with open unit discs so that we apply Lemma 2.16 again). $\square$

Global Serre–Tate theory.

The algebraic and geometric formulations. Now we review Chai's [2003] globalization of Serre–Tate coordinate system in characteristic p. Let U be a $\mathbb{F}_p$-scheme. Let A/U be an abelian scheme whose relative dimensions on connected components of U are the same. Define

$$\nu_U = \varprojlim_n \mathrm{Coker}([p^n] : \mathbb{G}_m \to \mathbb{G}_m),$$

which is a $\mathbb{Z}_p$-sheaf on U_{et}.

Example 5.5. (1) Let $m \geq n$ be positive integers, and $U_0 = \mathrm{Spec}\, k[T]/T^{p^n}$. Then the p^m-th power of an element in $(k[T]/T^{p^n})^\times$ with constant term b is b^{p^m}. Thus

$$\nu_{U_0}(U_0) = (k[T]/T^{p^n})^\times/k^\times \simeq 1 + T(k[T]/T^{p^n}). \tag{5-11}$$

(2) Let B be an $\mathbb{F}_p$-algebra, $U = \mathrm{Spec}\, B$ and $U' = \mathrm{Spec}\, B[T]/T^{p^n}$. For $m \geq n$, consider the map

$$B^\times/(B^\times)^{p^m} \bigoplus (1 + TB[T]/T^{p^n}) \to (B[T]/T^{p^n})^\times/((B[T]/T^{p^n})^\times)^{p^m}$$

defined by $(a, f) \mapsto af$. Easy to check that this is a group isomorphism. In particular,

$$\nu_{U'}(U') \simeq \nu_U(U) \bigoplus (1 + TB[T]/T^{p^n}). \tag{5-12}$$

(3) For every $z \in U(k)$, $\{z\} \times_U U' \simeq U_0$. Then the restriction of the isomorphism (5-12) at z is the isomorphism (5-11).

Suppose A/U is ordinary. Let $T_pA[p^\infty]^{\mathrm{et}}$ be the Tate module attached to the maximal étale quotient of the p-divisible group $A[p^\infty]$. The global Serre–Tate coordinate system for A/U is a homomorphism of $\mathbb{Z}_p$-sheaves

$$q_{A/U} : T_pA[p^\infty]^{\mathrm{et}} \otimes T_pA^\vee[p^\infty]^{\mathrm{et}} \to \nu_U$$

constructed by Chai [2003, 2.5]. Let $U_0 = \mathrm{Spec}\, k[T]/T^{p^n}$. Let A/U_0 be an ordinary abelian scheme, and A_k the special fiber of A. Then

$$T_pA[p^\infty]^{\mathrm{et}} \otimes T_pA^\vee[p^\infty]^{\mathrm{et}} \simeq T_pA_k[p^\infty] \otimes T_pA_k^\vee[p^\infty], \tag{5-13}$$

where the right-hand side is regarded as a constant sheaf.

Lemma 5.6 [Chai 2003, (2.5.1)]. *The morphism of $\mathbb{Z}_p$-modules*

$$T_pA_k[p^\infty]\otimes T_pA_k^\vee[p^\infty]\to \nu_U(U)\simeq 1+T(k[T]/T^{p^n})$$

induced from q_{A/U_0} via (5-13) coincides with the classical Serre–Tate coordinate system; *see (5-3).*

The geometric formulation of global Serre–Tate coordinate system is as follows. Let A^{univ} be the universal principally polarized abelian scheme over $X_o(0)$, and $\hat{A}^{\mathrm{univ}}$ the formal completion of A^{univ} along the zero section which is a formal torus over $X_o(0)$. Then the sheaf of polarization-preserving $\mathbb{Z}_p$-homomorphisms between $T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}}$ and $\hat{A}^{\mathrm{univ}}$ is a formal torus over $X_o(0)$ of dimension $g(g+1)/2$. Let us call it $\mathfrak{T}_1$. Let Δ be the diagonal embedding of $X_o(0)$ into $X_o(0)\times X_o(0)$, and let $\mathfrak{T}_2$ be the formal completion of $X_o(0)\times X_o(0)$ along this embedding.

Proposition 5.7 [Chai 2003, Proposition 5.4]. *There is a canonical isomorphism $\mathfrak{T}_1\simeq\mathfrak{T}_2$. In particular, $\mathfrak{T}_2$ has a formal torus structure over the first $X_o(0)$.*

Igusa tower. In order to have sections of the étale $\mathbb{Z}_p$-sheaf $T_pA[p^\infty]^{\mathrm{et}}$ over U, or equivalently to trivialize the formal torus, we need to pass to the Igusa tower, defined as follow. For $n=0,1,\ldots,\infty$, let $\mathfrak{I}_n$ be the functor assigning to every k-algebra R the set of isomorphism classes of pairs

$$\{(A,\varepsilon): A\in X_o(0)(R),\quad \varepsilon: A[p^n]\simeq\hat{\mathbb{G}}_{m,R}^g[p^n]\}.$$

By [Hida 2004, 8.1.1], for $n<\infty$ (resp. $n=\infty$) the functor $\mathfrak{I}_n$ is represented by a k-scheme (which we still denote by $\mathfrak{I}_n$) finite (resp. profinite) Galois over $X_o(0)$ with Galois group $\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})$ (resp. $\mathrm{GL}_g(\mathbb{Z}_p)$). And $\mathfrak{I}_n$ is known as the Igusa scheme of level n.

Realization of the global Serre–Tate coordinate system at a basis. Let U_0 be an affine open subscheme of $X_o(0)$. Let $U=\operatorname{Spec}B:=\mathfrak{I}_\infty|_{U_0}$. Let Δ be the diagonal of $U\times U$. We have two projection maps $\mathrm{pr}_1,\mathrm{pr}_2$ from $\widehat{U\times U}_{/\Delta}$ to the first and second U. For $z\in U(k)$, the restriction of pr_2 induces

$$\mathrm{pr}_1^{-1}(\{z\})\overset{\mathrm{pr}_2}{\simeq}\widehat{U}_{/z}. \tag{5-14}$$

Let $\mathcal{O}(\widehat{U\times U}_{/\Delta})$ be the coordinate ring of $\widehat{U\times U}_{/\Delta}$. Endow $\mathcal{O}(\widehat{U\times U}_{/\Delta})$ a B-algebra structure via pr_1. By Proposition 5.7, we have a (nonunique) B-algebra isomorphism

$$\mathcal{O}(\widehat{U\times U}_{/\Delta})\simeq B[\![T_1,\ldots,T_{g(g+1)/2}]\!]. \tag{5-15}$$

Let $A/\widehat{U\times U}_{/\Delta}$ be the pullback of $A^{\mathrm{univ}}|_{U_0}$. Assume that $\mathrm{Sym}^2(T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ is a free $\mathbb{Z}_p$-modules of rank $g(g+1)/2$. Let $\xi_1,\ldots,\xi_{g(g+1)/2}$ be a basis of $\mathrm{Sym}^2(T_pA^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ (whose existence follows from the definition of $\mathfrak{I}_\infty$ and the polarization). The realization of the global Serre–Tate coordinate system of $A/\widehat{U\times U}_{/\Delta}$ at the basis $\xi_1,\ldots,\xi_{g(g+1)/2}$ is a construction of an isomorphism (5-15) as follows.

For the simplicity of notations, let us assume $g=1$. The general case can be dealt in the same way. Let $\xi=\xi_1$ and $T=T_1$. Let

$$U'=\widehat{U\times U}_{/\Delta}/T^{p^n}\simeq\operatorname{Spec}B[\![T]\!]/T^{p^n}$$

and let A_n be the restriction of A to U'. The global Serre–Tate coordinate system of A_n/U' is a homomorphism of $\mathbb{Z}_p$-sheaves over $U_{2,\mathrm{et}}$

$$q_{A_n/U'} : \mathrm{Sym}^2(T_p A_n[p^\infty]^{\mathrm{et}}) \to \nu_{U'}.$$

Note that ξ gives a basis ξ_n of $\mathrm{Sym}^2(T_p A_n[p^\infty]^{\mathrm{et}})(U')$. Then we have

$$q_{A_n/U'}(\xi_n) \in \nu(U') \simeq \nu_U(U) \bigoplus (1 + TB[T]/T^{p^n})$$

where the second isomorphism is (5-12). Consider the morphism

$$\phi_n : \nu(U') \to (1 + TB[T]/T^{p^n}) \hookrightarrow B[T]/T^{p^n}$$

where the first map is the projection and second map is the natural inclusion. Let $T_n^{\mathrm{ST}} \in B[T]/T^{p^n}$ be $\phi_n(q_{A_n/U'}(\xi_n)) - 1$. As n varies, the T_n^{ST} give an element

$$T^{\mathrm{ST}} \in \mathcal{O}(\widehat{U \times U}_{/\Delta}) \simeq \varprojlim_n B[T]/T^{p^n}.$$

We compare the above construction with the realization of the classical Serre–Tate coordinate system. Let $z \in U(k)$. The restriction of A to $\mathrm{pr}_1^{-1}(\{z\})$ is pullback $A^{\mathrm{univ}}|_{\widehat{U}_{/z}}$ of $A^{\mathrm{univ}}|_{U_0}$ to $\widehat{U}_{/z}$ via (5-14). (Thus we may regard A as the family $\{A^{\mathrm{univ}}|_{\widehat{U}_{/z}} : z \in U(k)\}$.) The realization of the classical Serre–Tate coordinate system of $\widehat{U}_{/z}$ at ξ_z (the restriction of ξ at z) gives an element $T_z^{c\,\mathrm{ST}} \in \widehat{U}_{/z}$ and an isomorphism $\widehat{U}_{/z} \simeq \mathrm{Spf}\, k[\![T_z^{c\,\mathrm{ST}}]\!]$ (see Definition 5.3). Here and below, the superscript c indicates "classical".

Lemma 5.8. *The restriction of T^{ST} to $\mathrm{pr}_1^{-1}(\{z\}) \simeq \widehat{U}_{/z}$ is $T_z^{c\,\mathrm{ST}}$. In particular,*

$$\mathcal{O}(\widehat{U \times U}_{/\Delta}) = B[\![T^{\mathrm{ST}}]\!].$$

Proof. The restriction of (5-15) to $\mathrm{pr}_1^{-1}(\{z\}) \simeq \widehat{U}_{/z}$ via (5-14) gives an isomorphism $\mathcal{O}(\widehat{U}_{/z}) \simeq k[\![T]\!]$. Let

$$q_n^c = q^c_{A^{\mathrm{univ}}|_{\widehat{U}_{/z}}/T^{p^n}} : \mathrm{Sym}^2(T_p A_z^{\mathrm{univ}}[p^\infty]) \to 1 + Tk[\![T]\!]/T^{p^n}$$

be the classical Serre–Tate coordinate system of $A^{\mathrm{univ}}|_{\widehat{U}_{/z}}/T^{p^n}$ (see (5-3)). Then the image of $T_z^{c\,\mathrm{ST}}$ in $k[\![T]\!]/T^{p^n}$ is $q_n^c(\xi_z) - 1$. By Example 5.5(3) and Lemma 5.6, $q_n^c(\xi_z)$ equals the restriction of $\phi_n(q_{A_n/U'}(\xi_n))$ at z. Thus the first statement follows. The second statement follows from the first one. □

6. Proof of Theorem 1.1

In this section, we at first prove a Tate–Voloch type result in a family in characteristic p. Combined with the results in Section 5, we prove Theorem 1.1. We continue to use the notations in Section 5.

Tate–Voloch type result in a family in characteristic p. Recall that $k = \overline{\mathbb{F}}_p$ and $K^\flat = k((t^{1/p^\infty}))$. In the proof of Lemma 2.21, we used the following simple fact: let S a k-algebra, $g \in S$ and $x \in (\mathrm{Spec}\, S)(k)$, then $g(x) = 0$ or $|g(x)|_k = 1$ where the valuation $|\cdot|_k$ on k takes value 0 on $0 \in k$ and 1 on $k^\times$. This fact can be naively regarded as an analog of the Tate–Voloch conjecture over k. We want to consider this analog in a family. We need some notations.

Let l be a positive integer. For $d = (d^{(1)}, \dots, d^{(l)}) \in (p^{\mathbb{Z}_{<0}})^l$, define $t^d := (t^{d^{(1)}}, \dots, t^{d^{(l)}}) \in (K^{\flat\circ})^l$. For $c = (c^{(1)}, \dots, c^{(l)}) \in (\mathbb{Z}_p{}^\times)^l$, define

$$(1+t^d)^c - 1 := ((1+t^{d^{(1)}})^{c^{(1)}} - 1, \dots, (1+t^{d^{(l)}})^{c^{(l)}} - 1) \in (K^{\flat\circ})^l.$$

Fix a sequence $\{d_n\}_{n=1}^\infty$ of elements in $(p^{\mathbb{Z}_{<0}})^l$ and a sequence $\{c_n\}_{n=1}^\infty$ of elements in $(\mathbb{Z}_p^\times)^l$. Let

$$y_n = (1+t^{d_n})^{c_n} - 1 \in (K^{\flat\circ})^l \subset \operatorname{Spec} K^{\flat\circ}[\![T_1, \dots, T_l]\!]. \tag{6-1}$$

Let $\mathbb{N} = \{1, 2, \dots\}$ the sequence of positive integers. For $\delta \in (0, 1)$ and the given sequence $\{d_n\}_{n=1}^\infty$, let

$$\mathbb{N}(\delta) = \{n \in \mathbb{N} : d_n^{(i)}/d_n^{(i+1)} < \delta\}.$$

If $l = 1$, we understand $\mathbb{N}(\delta)$ as $\mathbb{N}$.

Proposition 6.1. *Let A be a reduced k-algebra and $V = \operatorname{Spec} A$. Let $\{z_n\}_{n=1}^\infty$ be a sequence of (not necessarily distinct) points in $V(k)$. Let $f \in A[\![T_1, \dots, T_l]\!]$ and let $f_{z_n} \in k[\![T_1, \dots, T_l]\!]$ be the restriction of f at z_n. Assume that*

$$\text{for every infinite subset } \mathbb{N}' \subset \mathbb{N}, \text{ the set } \{z_n : n \in \mathbb{N}'\} \text{ is Zariski dense in } V. \tag{$\star$}$$

If $f \neq 0$, then there exists $D_0 \in \mathbb{R}_{>0}$ and $\delta_0 \in (0, 1)$ such that for every $D \geq D_0$ and $\delta \leq \delta_0$, the following set is finite

$$\{n \in \mathbb{N}(\delta) : \|f_{z_n}(y_n)\| < \|T_l(y_n)\|^D\}. \tag{6-2}$$

Here $T_l(y_n)$ is, by definition, the l-th coordinate of y_n.

Proof. We do induction on l.

The case $l = 1$ is proved as follows. Let $f = \sum_{m \geq 0} a_m T^m$ where $a_m \in A$. Regard a_m as a function on V so that $a_m(z_n) \in k$. Claim: there exists some m such that $a_m(z_n) \neq 0$ for n large enough. Let m_0 be the smallest such m. Then

$$\|f_{z_n}(y_n)\| = \|t^{d_n m_0}\|$$

for n large enough. Let $D_0 = m_0$ and we are done. Now we prove the claim by contradiction. Assume that for every m, $a_m(z_n) = 0$ for infinitely many n. By assumption ($\star$) and the reducedness of A, $a_m = 0$. Thus $f = 0$. This is a contradiction.

Now we do the induction. Let $l > 1$. We prepare some notations. Let d_n', y_n' be the first $l-1$ components of d_n, y_n respectively. For $\delta \in (0, 1)$, we have a subsequence $\mathbb{N}(\delta)' \subset \mathbb{N}$ defined using the sequence $\{d_n'\}_{i=1}^\infty$. Then $\mathbb{N}(\delta)' \supset \mathbb{N}(\delta)$.

Assume that $f \neq 0$. Write $f = T_l^{m_1}(g_1 + f_1)$ where $g_1 \in A[\![T_1, \dots, T_{l-1}]\!] \backslash \{0\}$ and $f_1 \in T_l A[\![T_1, \dots, T_l]\!]$. Below, to lighten notation, we abbreviate the subscript z_n. Then for n in the set (6-2), with D and δ to be determined, we have

$$\|g_1(y_n)+f_1(y_n)\| = \|T_l(y_n)\|^{-m_1}\|f(y_n)\| < \|T_l(y_n)\|^{D-m_1}.$$

If $D \geq m_1+1$, then

$$\|g_1(y_n)\| \leq \|g_1(y_n)+f_1(y_n)\| + \|f_1(y_n)\| \leq \|T_l(y_n)\|.$$

Since $\|T_l(y_n)\| < \|T_{l-1}(y_n')\|^{1/\delta}$ and $\|g_1(y_n')\| = \|g_1(y_n)\|$, we have

$$\|g_1(y_n')\| < \|T_{l-1}(y_n')\|^{1/\delta}. \tag{6-3}$$

By the induction hypothesis, there exists $D' > 0$ and $\delta_0' \in (0,1)$ such that if $\delta \leq 1/D'$ and $\delta \leq \delta_0'$, $\{n \in \mathbb{N}(\delta)' : \text{(6-3) holds}\}$ is finite. Then (6-2) is finite by choosing $\delta_0 = \min\{1/D', \delta_0'\}$. □

Remark 6.2. (1) D_0 and δ_0 are uniform for all choices of $\{c_n\}_{n=1}^{\infty}$. We do not need this fact later.

(2) The proposition is inspired by [Serban 2018, Lemma 2.10]. In the proof of that result, there is a minor imprecision. The following modification is suggested by Serban. Define T_δ in [loc. cit., Lemma 2.10] to be the first set in the intersection but not the entire intersection, so that the statement (2) in loc. cit. is about $T_\delta \cap S_\phi(q^{-1-c})$. The 3rd displayed formula in the proof of [loc. cit., Lemma 2.10] should be removed. Then, on can still get the 5th displayed formula in that proof with slightly more effort.

Closure and limit. We show that assumption ($\star$) in Proposition 6.1 holds in some situations.

Lemma 6.3. *Let $\{B_i\}_{i=0}^{\infty}$ be a system of rings and $B = \varinjlim_i B_i$. Let $f_i : \operatorname{Spec} B \to \operatorname{Spec} B_i$ be the natural morphism. Let $\Lambda \subset \operatorname{Spec} B$ be a subset and $\Lambda_i = f_i(\Lambda) \subset \operatorname{Spec} B_i$. We have the following relation between Zariski closures*:

$$\Lambda^{\mathrm{Zar}} = \bigcap_{i=0}^{\infty} f_i^{-1}(\Lambda_i^{\mathrm{Zar}}). \tag{6-4}$$

Proof. The ideal $I \subset B$ defining Λ^{Zar}, with reduced induced structure as a closed subscheme, is generated by the union of the images I_i in B, where $I_i \subset B_i$ is the ideal of elements whose image in B vanishes on Λ^{Zar}. By the definition of Λ_i, I_i is the ideal defining Λ_i^{Zar}. Then (6-4) follows. □

Let $f : U \to U_0$ be a surjective morphism of schemes. Let $\Lambda_0 \subset U_0$ be a subset with Zariski closure Λ_0^{Zar} in U_0. For $s \in \Lambda_0$, choose $z_s \in f^{-1}(s)$. Let $\Lambda = \{z_s : s \in \Lambda_0\}$ with Zariski closure Λ^{Zar} in U.

Lemma 6.4. *Assume that f is closed*:

(1) *The image of Λ^{Zar} in U_0 is Λ_0^{Zar}.*

(2) *Assume that Λ_0^{Zar} is irreducible and U is noetherian. There exists a choice of Λ such that Λ^{Zar} is irreducible.*

(3) *In (2), further assume that f is finite and the Zariski closure of every infinite subset of Λ_0 is Λ_0^{Zar}. Then the Zariski closure of every infinite subset of Λ is Λ^{Zar}.*

Proof. (1) is easy and the proof is omitted.

(2) For every member of the finitely many irreducible (so closed) components of $f^{-1}(\Lambda_0^{\mathrm{Zar}})$, its image in Λ_0^{Zar} is a closed subscheme. By the irreducibility of Λ_0^{Zar}, some irreducible component of $f^{-1}(\Lambda_0^{\mathrm{Zar}})$ is surjective to Λ_0^{Zar}. We choose all the z_s in this component.

(3) Note that a finite surjective morphism preserves dimension, and a proper closed subscheme of a noetherian irreducible scheme has a strictly smaller dimension. Then (3) follows from (1) and counting dimensions. □

The last two lemmas imply the following corollary.

Corollary 6.5. *Let the B, B_i be as in Lemma 6.3. Let $U = \operatorname{Spec} B$ (not necessary noetherian), $U_0 = \operatorname{Spec} B_0$ and $f = f_0$. Assume that each B_i is noetherian and the transition morphisms $\operatorname{Spec} B_j \to \operatorname{Spec} B_i$ are finite surjective. Assume that the Zariski closure of every infinite subset of Λ_0 is Λ_0^{Zar}. There exists a choice of Λ such that the Zariski closure of every infinite subset of Λ is Λ^{Zar}.*

Later, to fulfill the second assumption of the corollary, we will use the following lemma.

Lemma 6.6. *Let U_0 be a noetherian scheme. For every infinite subset $Y \subset U_0$, there is an infinite subset $\Lambda_0 \subset Y$ such that the Zariski closure of every infinite subset of Λ_0 is Λ_0^{Zar}.*

Proof. By the noetherianness of U_0, there exists a closed subscheme V of U_0 containing an infinite subset Λ_0 of Y such that every proper closed subscheme of V only contains finitely many elements in Y. □

Proof of Theorem 1.1. Let X be a product of Siegel moduli spaces over $\mathbb{Z}_p$ with certain level structures away from p. By Lemma 2.15, Theorem 1.1 follows from the following theorem.

Theorem 6.7. *Let Z be a closed subvariety of $X_{\bar{L}}$. There exists a constant $c > 0$ such that for every ordinary CM point $x \in X(L^{\mathrm{cyc}})$, if $d(x, Z) \le c$, then $x \in Z$.*

Here the distance function $d(x, Z)$ is defined as on page 986 using the integral model X

Proof. We prove Theorem 6.7 when X is a single Siegel moduli space. The general case is proved in the same way or by embedding a product of Siegel moduli spaces into a bigger one. We continue to use the notations in Section 5. In particular, the fields L, L^{cyc}, K and $K^\flat$ below are as in the beginning of Section 5; the formal scheme $\mathfrak{X}(0)$, the adic locus $\mathcal{X}(0)$, the perfectoid spaces $\mathcal{X}(0)^{\mathrm{perf}}$, $\mathcal{X}(0)^{\mathrm{perf}\flat}$ and Frobenius morphism $\mathrm{Fr}^{\mathrm{can}}$ below are as in Section 5. For an ordinary CM point $x \in X(L^{\mathrm{cyc}})$, we the same notation x to denote its base change in $X(K)$. Let x° be the unique K°-point in X whose generic fiber is x.

Suppose that Z is defined over a finite Galois extension F of L. Let $\mathcal{I}$ be the ideal sheaf of the schematic closure of Z in X_{F°. Let $\mathcal{U}$ be an affine open subscheme of X_W, of finite type over W. (This is the only use of a calligraphic font not representing an adic space in this paper.) We only need to find a constant c such that, if an ordinary CM point $x \in X(L^{\mathrm{cyc}})$ satisfies $x^\circ \in \mathcal{U}(K^\circ)$ and $d_{\mathcal{U}_{K^\circ}}(x_K, I) < c$, then $x \in Z$. Here the distance function is as on page 986.

We at first have the following simplification on F. Let $K' = FK$. Suppose $\mathcal{I}(\mathcal{U}_{F^\circ})$ is generated by f_i, $i = 1, \dots, n$. For $\sigma \in G := \mathrm{Gal}(K'/K)$, f_i^σ is in the coordinate ring of $\mathcal{U}_{K'^\circ}$ and $\|f_i(x_{K'})\| = \|f_i^\sigma(x_{K'})\|$. Let I be the ideal of the coordinate ring of $\mathcal{U}_{K^\circ}$ generated by $\prod_{\sigma \in G} f_i^\sigma$, $i = 1, \dots, n$. Then

$$d_{\mathcal{U}_{K^\circ}}(x_K, I) = d_{\mathcal{U}_{K'^\circ}}(x_{K'}, \mathcal{I}_{K'^\circ}(\mathcal{U}_{K'^\circ}))^{|G|}.$$

Thus we may assume that $F \subset K$. Equivalently, $F \subset L^{\mathrm{cycl}}$.

Now we reduce Theorem 6.7 to Theorem 6.8 below, which is formulated with affine formal schemes. For an ordinary CM point $x \in X(K)$, we also use x to denote the corresponding point $\mathcal{X}(0)(K, K^\circ)$. Let $\mathfrak{U}$ be the restriction of the ϖ-adic formal completion of $\mathcal{U}$ to $\mathfrak{X}(0)$. By Lemma 2.12, Theorem 6.7 is deduced from Theorem 6.8. $\square$

Theorem 6.8. *Let $\mathfrak{Z}$ be an irreducible closed formal subscheme of $\mathfrak{U}_{F^\circ}$. For a sequence $\{x_n\}_{n=1}^\infty$ of ordinary CM points such that x_n is in the ϵ_n-neighborhood of $\mathfrak{Z}$ and with $\|\epsilon_n\| \to 0$, we have $x_n \in \mathcal{Z}$ for infinitely many n.*

The proof of Theorem 6.8 consists of two bulks: one involves perfectoid spaces and one does not. The perfectoid one is more technical and proves results to be used in the second one. The nonperfectoid one concludes Theorem 6.8. We will present the non-perfectoid one first, on pages 1009 and 1010.

A canonical lifting is an ordinary CM points of order 1 with respect to a (equivalently every) basis, see Definition 5.2(1). The following lemma will be proved in Theorem 6.13 using perfectoid spaces.

Lemma 6.9. *Theorem 6.8 holds if we replace "ordinary CM points" by "canonical liftings".*

Global Serre–Tate coordinate. Before we proceed to the proof of Theorem 6.8, let us recall the realization of the global Serre–Tate coordinate system on page 1004.

Let U_0 be the special fiber of $\mathfrak{U}$. Let $U = \mathrm{Spec}\, B$ be the profinite Galois cover of U_0 defined on page 1004 (and coming from the infinite level Igusa scheme) such that $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ is a free $\mathbb{Z}_p$-modules of rank $g(g+1)/2$. Let Δ be the diagonal of $U \times U$. Then by Lemma 5.8, for a basis

$$\xi_1, \dots, \xi_{g(g+1)/2} \tag{6-5}$$

of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$, we have the realization of the global Serre–Tate coordinate system

$$\mathcal{O}(\widehat{U \times U}_{/\Delta}) = B[\![T_1^{\mathrm{ST}}, \dots, T_{g(g+1)/2}^{\mathrm{ST}}]\!],$$

which has the following property. For every $z \in U(k)$, we have an isomorphism

$$\mathrm{pr}_1^{-1}(\{z\}) \overset{\mathrm{pr}_2}{\simeq} \widehat{U}_{/z} \tag{6-6}$$

as in (5-14), and the corresponding isomorphism

$$\widehat{U}_{/z} \simeq \mathrm{Spf}\, k[\![T_{1,z}^{\mathrm{ST}}, \dots, T_{g(g+1)/2,z}^{\mathrm{ST}}]\!]. \tag{6-7}$$

Let $T_{i,z}^{\mathrm{ST}}$ be the restriction of T_i^{ST} to $\mathrm{pr}_1^{-1}(\{z\}) \simeq \widehat{U}_{/z}$. Let

$$\xi_{z,1}, \dots, \xi_{z,g(g+1)/2} \tag{6-8}$$

be the restriction of $\xi_1, \dots, \xi_{g(g+1)/2}$. Then (6-7) coincides with the realization of the classical Serre–Tate coordinate system of $\widehat{U}_{/z}$ at $\xi_{z,1}, \dots, \xi_{z,g(g+1)/2}$; see Definition 5.3.

Proof of Theorem 6.8. After passing to an infinite subsequence, we may assume that $\{\mathrm{red}(x_n)\}_{n=1}^{\infty}$ is a sequence of the same point or pairwise different points. Let $z_n \in U(k)$ be over $\mathrm{red}(x_n) \in U_0(k)$. By Corollary 6.5 and Lemma 6.6, after passing to an infinite subsequence, we may assume the following.

Assumption 6.10. For every infinite subset $\mathbb{N}' \subset \mathbb{N}$, the Zariski closure of the set $\{z_n : n \in \mathbb{N}'\}$ in U is the Zariski closure of the set $\{z_n : n \in \mathbb{N}\}$.

We regarded the basis (6-8) for $z = z_n$ as a basis of $\mathrm{Sym}^2(T_p A_{x_n})$ naturally. Let x_n be of order p^{a_n} with respect to (6-8) (see Definition 5.2(1)) where $a_n = (a_n^{(1)}, \dots, a_n^{(g(g+1)/2)}) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$. After passing to an infinite subsequence and permuting the basis (6-5) of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$, we may assume that every a_n is nonincreasing; see Definition 5.2(2). Let $l \leq g(g+1)/2$ be a nonnegative integer such that for every n, if $i > l$, then $a_n^{(i)} = 0$. For example, if $l = g(g+1)/2$, the assumption automatically holds; if $l = 0$, we are in the situation of Lemma 6.11.

We will reduce Theorem 6.8 to the case $l = 0$ by using Lemma 6.11 below. We need the "upper triangular change of variables" argument following [Serban 2018]. By "upper triangular change of variables", we indeed mean changing the first l-element of the basis (6-5) of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$ via an upper triangular matrix as follows. For $C \in \mathrm{GL}_l(\mathbb{Z}_p)$, $(\xi_1, \dots, \xi_l)C$ combined with $(\xi_{l+1}, \dots, \xi_{(g(g+1)/2)})$ gives a new basis of $\mathrm{Sym}^2(T_p A^{\mathrm{univ}}[p^\infty]^{\mathrm{et}})(U)$. Thus by restriction as in (6-8), we have a new basis of $\mathrm{Sym}^2(T_p A_{x_n})$ for every n. Let x_n be of order $p^{a_n(C)}$ with respect to this new basis, where $a_n(C) \in \mathbb{Z}_{\geq 0}^{g(g+1)/2}$. Then for C upper triangular, $a_n(C)$ is still nonincreasing.

Lemma 6.11. *Assume Assumption 6.10. Assume that for every upper triangular matrix $C \in \mathrm{GL}_l(\mathbb{Z}_p)$, the l-th component (so the i-th component for $i = 1, \dots, l$ as well) of $a_n(C)$ goes to ∞ as $n \to \infty$. Then $x_n \in \mathcal{Z}$ for all $n \in \mathbb{N}$.*

We postpone the proof of Lemma 6.11.

We finish the proof of Theorem 6.8 by induction on the dimension of $\mathfrak{Z}$. If $\mathfrak{Z}$ is empty, define its dimension to be -1. When $\mathfrak{Z}$ is of dimension -1, the theorem is trivial. The induction hypothesis is that the theorem holds for lower dimensions, and it will only be used in the proof of Lemma 6.12(2) below.

By Lemma 6.11 and passing to an infinite subsequence, we may assume that for an upper triangular matrix $C \in \mathrm{GL}_l(\mathbb{Z}_p)$, the l-th component of $a_n(C)$ is bounded. Replacing the basis (6-5) by the new basis that is $(\xi_1, \dots, \xi_l)C$ combined with $(\xi_{l+1}, \dots, \xi_{(g(g+1)/2)})$, we may assume that there is a nonnegative integer m such that for every n, $a_n^{(l)} \leq p^m$. The fact that $a_n^{(i)} = 0$ for $i > l$ does not change.

Lemma 6.12. *Let m be a nonnegative integer. Then the following hold*:

(1) *The adic generic fiber of $(\mathrm{Fr}^{\mathrm{can}})^m(x_n^\circ)$ is in the ϵ_n-neighborhood of the scheme theoretic image $(\mathrm{Fr}^{\mathrm{can}})^m(\mathfrak{Z})$; see* [Kappen 2013, 2.3].

(2) *Assume that $(\mathrm{Fr}^{\mathrm{can}})^m(x_n^\circ) \in (\mathrm{Fr}^{\mathrm{can}})^m(\mathfrak{Z}(K^\circ))$ for infinitely many n, then $x_n^\circ \in \mathfrak{Z}(K^\circ)$ for infinitely many n.*

Proof. To lighten the notations, assume that $m = 1$.

Consider the closed formal subscheme $(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z}))$ of $\mathfrak{U}$ which contains $\mathfrak{Z}$. Then x_n° is contained in the ϵ_n-neighborhood of $(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z}))$ by Lemma 2.7(1). Then (1) follows from the analog of Lemma 2.10 for formal schemes (which directly follows from Definition 2.4).

For (2), we prove it by contradiction. Let $\{n_i\} \subset \mathbb{N}$ be an infinite subsequence such that $\mathrm{Fr}^{\mathrm{can}}(x_{n_i}^\circ) \in \mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z}(K^\circ))$ and $x_{n_i}^\circ \notin \mathfrak{Z}(K^\circ)$ for n_i large enough. In particular, $(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z})) \neq \mathfrak{Z}$. Thus by [Kappen 2013, Proposition 2.10], it is not hard to show that

$$(\mathrm{Fr}^{\mathrm{can}})^{-1}(\mathrm{Fr}^{\mathrm{can}}(\mathfrak{Z})) = \mathfrak{Z} \bigcup \mathfrak{Z}'$$

such that $\mathfrak{Z}'$ does not contain $\mathfrak{Z}$ and $x_{n_i}^\circ \in \mathfrak{Z}'(K^\circ)$. By Lemma 2.6, every x_{n_i} is contained in the ϵ_{n_i}-neighborhood of $\mathfrak{Z} \bigcap \mathfrak{Z}'$. Let $\mathfrak{Z}_1$ be the union of irreducible components of $\mathfrak{Z} \bigcap \mathfrak{Z}'$ which dominate $\mathrm{Spf}\, F^\circ$. By Lemma 2.7(2), there exists $\delta \in K^\circ - \{0\}$ such that every x_{n_i} is contained in the ϵ_{n_i}/δ-neighborhood of $\mathfrak{Z}_1$. Since every irreducible component of $\mathfrak{Z}_1$ has dimension less than the dimension of $\mathfrak{Z}$, by the induction hypothesis, we have $x_{n_i} \in \mathfrak{Z}_1(K^\circ) \subset \mathfrak{Z}(K^\circ)$. This is a contradiction. $\square$

By (5-10) and Lemma 6.12, after passing to an infinite subsequence, we may assume that for every n, if $i \geq l$, then $a_n^{(i)} = 0$, i.e., we may replace l by $l - 1$. Continue this process, we may assume that $l = 0$, i.e., $a_n^{(i)} = 0$ for every n and i. Now Theorem 6.8 follows from Lemma 6.11.

Canonical liftings and perfectoid strategy. Now our remaining tasks are the proofs of Lemmas 6.9 and 6.11. For Lemma 6.9, we prove an "almost effective" version of Theorem 6.8 for canonical liftings. In the proof, we use the ordinary perfectoid Siegel space and Scholze's approximation lemma, following a strategy of Xie [2018]. Our later proof of Lemma 6.11 involves a more complicated version of this proof (which in particular uses the global Serre–Tate coordinate).

Let $\mathcal{X}$ be the restriction of $\mathcal{X}(0)^{\mathrm{perf}}$ to the adic generic fiber of $\mathfrak{U}_{K^\circ}$. Then $\mathcal{X} = \mathrm{Spa}(R, R^+)$ where (R, R^+) is a perfectoid affinoid (K, K°)-algebra (there is no need to specify R though it is easy to do so). The restriction of $\mathcal{X}(0)^{\mathrm{perf},\flat}$ to the adic generic fiber of $U_0 \otimes K^{\flat\circ}$ is $\mathcal{X}^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$, the tilt of $\mathcal{X}$. More concretely, it is given as follows: let S_m be the coordinate ring of $(\mathrm{Fr}^m)^{-1}(U_0)$ with the natural inclusion $S_{m-1} \hookrightarrow S_m$, and $S = \bigcup_m S_m$, then $R^{\flat+}$ is the $\varpi^\flat$-adic completion of $S \otimes K^{\flat\circ}$. Let $\mathcal{X}_m$ be the adic generic fiber of $\mathrm{Spec}\, S_m \otimes K^{\flat\circ}$ and

$$\pi_m : \mathcal{X}^\flat \to \mathcal{X}_m$$

the natural projection. Recall π and π' as defined in (5-1). Then $\pi_0 = \pi'|_{\mathcal{X}^\flat}$ (which has image in $\mathcal{X}_0$). We abbreviate $\pi|_{\mathcal{X}}$ as π (which has image in the adic generic fiber of $\mathfrak{U}_{K^\circ}$.) Let ρ be the restriction of $\rho_{\mathcal{X}(0)^{\mathrm{perf}}}$ (see (5-2)) to $\mathcal{X}$.

For $f \in \mathcal{O}(\mathfrak{U})$ in the defining ideal of $\mathfrak{Z}$, regard f as an element of R^+ by the inclusion $\mathcal{O}(\mathfrak{U}) \subset R^+$. For $c \in \mathbb{Z}_{>0}$, choose g as in Lemma 2.19 (with respect to f) and choose a finite sum

$$g_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0},\\ i < 1/p + c}} g_{c,i} \cdot (\varpi^\flat)^i$$

as in Lemma 2.20 where $g_{c,i} \in S$ for all i. There exists a positive integer $m(c)$ such that $g_{c,i} \in S_{m(c)}$ for all i by the finiteness of the sum. Let $G_c := g_c^{p^{m(c)}}$. Then we have the finite sum

$$G_c = \sum_{\substack{i \in \mathbb{Z}[1/p]_{\geq 0}, \\ i < 1/p+c}} G_{c,i} \cdot (\varpi^\flat)^{p^{m(c)} i}, \tag{6-9}$$

where $G_{c,i} = g_{c,i}^{p^{m(c)}}$. By the construction of the S_n, we have $G_{c,i} \in S_0$. Let I_c be the ideal of S_0 generated by $\{G_{c,i} : i \in \mathbb{Z}[1/p]_{\geq 0}, i < 1/p + c\}$. By the noetherianness of S_0, there exists a positive integer M such that

$$\sum_{c=1}^{\infty} I_c = \sum_{c=1}^{M} I_c. \tag{6-10}$$

For $y \in \mathfrak{X}(0)$ and $\tilde{y} \in \pi^{-1}(y) \subset \mathcal{X}$, $|f(\tilde{y})| = \|f(y)\|$. If $\|f(y)\| \leq \|\varpi\|^{1/p+M}$, by (2-2) and (2-3), we have $|g_c(\pi_{m(c)}(\rho(\tilde{y})))| \leq \|\varpi\|^{1/p+c}$ for $c = 1, \ldots, M$. So for $c = 1, \ldots, M$, we have

$$|G_c(\pi_0(\rho(\tilde{y})))| = |G_c(\pi_{m(c)}(\rho(\tilde{y})))| \leq \|\varpi\|^{(1/p+c)p^{m(c)}}. \tag{6-11}$$

Theorem 6.13. *Assume that $\{f_1, \ldots, f_r\} \subset \mathcal{O}(\mathfrak{U})$ generates the ideal defining $\mathfrak{Z}$. For each f_j, let M_j be the M as in (6-10) with f replaced by f_j. Let $\mathbb{M} = \max\{M_j : j = 1, \ldots, t\}$. Let y be a canonical lifting in the $\varpi^{1/p+\mathbb{M}}$-neighborhood of $\mathfrak{Z}$. Then $y \in \mathcal{Z}$.*

Proof. Apply Lemma 5.4(2) to y with $a = 1$. Choose $\tilde{y} \in \pi^{-1}(y)$ to be as in Lemma 5.4(2). Then $\pi_0(\rho(\tilde{y})) = \mathrm{red}(y) \in U_0(k)$, where we understand $U_0(k)$ as a subset of $\mathfrak{X}(K^\flat, K^{\flat\circ})$ naturally. Let $f = f_j$ and $M = M_j$ for some j. Then $|f(\tilde{y})| \leq \|\varpi\|^{1/p+M}$ and thus we have (6-11). Similar to Lemma 2.21, by (6-11) and (6-9), we have $G_{c,i}(\pi_0(\rho(\tilde{y}))) = 0$ for every $c = 1, \ldots, M$ and the corresponding i. By (6-10), $I_c(\pi_0(\rho(\tilde{y}))) = \{0\}$ for every $c \in \mathbb{Z}_{>0}$. So

$$G_c(\pi_{m(c)}(\rho(\tilde{y}))) = G_c(\pi_0(\rho(\tilde{y}))) = 0$$

for every $c \in \mathbb{Z}_{>0}$. Thus $g_c(\pi_{m(c)}(\rho(\tilde{y}))) = 0$. By (2-2) and (2-3), $|f(\tilde{y})| \leq \|\varpi\|^{1/p+c}$ for every $c \in \mathbb{Z}_{>0}$. Thus $|f(\tilde{y})| = 0$. $\square$

Remark 6.14. The effectivity of $\mathbb{M}$ is essentially determined by the effectivity of the determination of the approximating function g in Lemma 2.19. However, Scholze's proof of Lemma 2.19 uses "almost ring theory" and is not effective. It is meaningful to ask if Lemma 2.19 can be made effective.

Toward the proof of Lemma 6.11. This paragraph closely mimics the proof of Theorem 6.13. Let notations be as above Theorem 6.13 and let $y = x_n$. For every $c = 1, \ldots, M$ and a corresponding i, we want to show that $G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = 0$. Then by (6-10), $I_c(\pi_0(\rho(\tilde{x}_n))) = \{0\}$ for every $c \in \mathbb{Z}_{>0}$. So $G_c(\pi_{m(c)}(\rho(\tilde{x}_n))) = G_c(\pi_0(\rho(\tilde{x}_n))) = 0$ for every $c \in \mathbb{Z}_{>0}$. Thus $g_c(\pi_{m(c)}(\rho(\tilde{x}_n))) = 0$. By (2-2) and (2-3), $|f(\tilde{x}_n)| \leq \|\varpi\|^{1/p+c}$ for every $c \in \mathbb{Z}_{>0}$. Thus o $|f(\tilde{x}_n)| = 0$. Let f run over a finite set of generators of the defining ideal of $\mathfrak{Z}$ and choose infinite subsequences successively, we have $x_n \in \mathcal{Z}$ for infinitely many n.

Spaces. For $x \in U_0(k)$ (resp. $U(k)$), let $\mathcal{D}_x$ be the adic generic fiber of the formal completion of $U_0 \otimes K^{\flat\circ}$ (resp. $U \otimes K^{\flat\circ}$) at x. (This coincides with the definition in Section 5.) Equivalently, $\mathcal{D}_x$ is the adic generic fiber of the formal completion of $\widehat{U_0}_{/x} \otimes K^{\flat\circ}$ (resp. $\widehat{U}_{/x} \otimes K^{\flat\circ}$). The following two diagrams summarize the adic spaces/k-schemes and morphisms between them that we use:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\rho"] \arrow[d, "\pi"] & \mathcal{X}^\flat \arrow[d, "\pi_0"] & & \\ \mathcal{X}(0) & \mathcal{X}_0 & \coprod_{x \in U_0(k)} \mathcal{D}_x \arrow[l, "(1)"'] & \coprod_{z \in U(k)} \mathcal{D}_z \arrow[l, "(2)"'] \end{tikzcd} \tag{6-12}$$

$$U_0 \xleftarrow{(1')} \coprod_{x \in U_0(k)} \widehat{U_0}_{/x} \xleftarrow{(2')} \coprod_{z \in U(k)} \widehat{U}_{/z} \xleftarrow{(6\text{-}6)} \coprod_{z \in U(k)} \mathrm{pr}_1^{-1}(\{z\}) \xrightarrow{(3)} \widehat{U \times U}_{/\Delta}$$

Here the morphisms (1), (1') and (3) are the natural inclusions. And the morphism (2), when restricted to $\mathcal{D}_z$, $z \in U(k)$, is the natural isomorphism $\mathcal{D}_z \simeq \mathcal{D}_x$ where $x \in U_0(k)$ is the image of z. We have the parallel statement for (2').

Functions. Let $H_{c,i}$ be the image of $G_{c,i}$ in B under the morphism $S_0 = \mathcal{O}(U_0) \to B = \mathcal{O}(U)$, and $H_{c,i,z_n} \in \mathcal{O}(\widehat{U}_{/z_n})$ the image of $H_{c,i}$ under the morphism $B = \mathcal{O}(U) \to \mathcal{O}(\widehat{U}_{/z_n})$.

For $\tilde{x}_n \in \pi^{-1}(x_n) \subset \mathcal{X}$, by Lemma 5.4(1), $\pi_0(\rho(\tilde{x}_n)) \in \mathcal{D}_{\mathrm{red}(x_n)}$. Let y_n be the preimage of $\pi_0(\rho(\tilde{x}_n))$ in $\mathcal{D}_{z_n}$ via the natural isomorphism $\mathcal{D}_{z_n} \simeq \mathcal{D}_{\mathrm{red}(x_n)}$. Then as elements in $K^{\flat\circ}$, we have

$$H_{c,i,z_n}(y_n) = H_{c,i}(y_n) = G_{c,i}(\pi_0(\rho(\tilde{x}_n))).$$

Lemma 6.15. *There is a constant $h_{c,i} < 1$ such that $\|H_{c,i,z_{n_m}}(y_{n_m})\| < h_{c,i}$.*

Proof. If the lemma is not true, let i_0 be the smallest i appearing in the finite sum (6-9) such that $\|H_{c,i,z_{n_m}}(y_{n_m})\| \to 1$ for a subsequence $\{n_m\}_{m=1}^{\infty} \subset \mathbb{N}$. Then (6-9) implies that $\|G_c(\pi_0(\rho(\tilde{x}_{n_m})))\| \to \|\varpi\|^{i_0 p^{m(c)}}$, which contradicts (6-11). $\square$

Let ϕ be the composition of

$$\phi : B = \mathcal{O}(U) \to B \otimes B \to \mathcal{O}(\widehat{U \times U}_{/\Delta}) = B[[T_1^{\mathrm{ST}}, \dots, T_{g(g+1)/2}^{\mathrm{ST}}]]$$

where the first morphism is $b \mapsto 1 \otimes b$. i.e., ϕ gives the projection $\mathrm{pr}_2 : \widehat{U \times U}_{/\Delta}$ to the second U. Tracking the second diagram of (6-12), we have the following lemma.

Lemma 6.16. *The restriction of $\phi(H_{c,i})$ to $\mathrm{pr}_1^{-1}(\{z_n\}) \overset{\mathrm{pr}_2}{\simeq} \widehat{U_1}_{/z_n}$ in (6-6) is $H_{c,i,z_n} \in \mathcal{O}(\widehat{U_1}_{/z_n})$.*

Proof of Lemma 6.11. We need some notations. For an open subset $O \subset \mathbb{Z}_p^{l-1}$, let $\mathbb{N}(O) \subset \mathbb{N}$ be the subsequence such that the first $l-1$ components of a ratio of x_n with respect to this basis (see Definition 5.2) is in O. If $l = 1$, we understand $\mathbb{N}(O)$ as the whole $\mathbb{N}$ (and we will not need the case $l = 0$). For $r \in \mathbb{Z}_p^{l-1}$ and $\delta \in (0, 1)$, let $\mathbb{N}(r, \delta) = \mathbb{N}(O(r, \delta))$ where $O(r, \delta)$ is the p-adic closed disc centered at r of radius δ.

Now we start to prove Lemma 6.11. By the discussion on page 1012, we only need to prove that for every $n \in \mathbb{N}$, $G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = 0$. Let Spec $A \subset$ Spec B be the Zariski closure of the set $\{z_n : n \in \mathbb{N}\}$. Let f be the image of $\phi(H_{c,i})$ under $B[\![T_1^{\mathrm{ST}}, \dots, T_{g(g+1)/2}^{\mathrm{ST}}]\!] \to A[\![T_1^{\mathrm{ST}}, \dots, T_{g(g+1)/2}^{\mathrm{ST}}]\!]$. By Lemma 6.16, we have

$$G_{c,i}(\pi_0(\rho(\tilde{x}_n))) = H_{c,i,z_n}(y_n) = f(y_n). \tag{6-13}$$

We prove the stronger result $f = 0$ by contradiction.

Assume that $f \neq 0$. We want to apply Proposition 6.1 to f and the y_n. We check the conditions in Proposition 6.1. First, by the compatibility between the Global and classical Serre–Tate coordinates as in the end of page 1009, we use Lemma 5.4(2) to conclude that y_n are as in (6-1) above Proposition 6.1. Second, the assumption $(\star)$ in Proposition 6.1 holds by Assumption 6.10. By the assumption that a_n goes to ∞ as $n \to \infty$ in Lemma 6.11, Lemma 6.15 and the second "=" of (6-13), for n large enough, n satisfies the inequality in (6-2) of Proposition 6.1 (for every D). Then by Proposition 6.1, there exists $\delta_0 \in (0, 1)$ such that $\mathbb{N}(0, \delta_0)$ is finite. For a general $r \in \mathbb{Z}_p^{l-1}$, by [Serban 2018, Lemma 2.7], after an "upper triangular change of variables" (as defined above Lemma 6.11), we may use the same proof for $r = 0$ to conclude that there exists $\delta_r \in (0, 1)$ such that $\mathbb{N}(r, \delta_r)$ is finite. By its compactness, $\mathbb{Z}_p^{l-1}$ is the union of p-adic closed discs centered at r of radius δ_r for finitely many r. Then the infinite set $\mathbb{N}$ is the union of the finite sets $\mathbb{N}(r, \delta_r)$ for these finitely many r. This is a contradiction.

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Discriminant groups of wild cyclic quotient singularities

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Let p be prime. We describe explicitly the resolution of singularities of several families of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension two, including families that generalize the quotient singularities of type E_6, E_7, and E_8 from $p=2$ to arbitrary characteristics. We prove that for p odd, any power of p can appear as the determinant of the intersection matrix of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. We also provide evidence towards the conjecture that in this situation one may choose the wild action to be ramified precisely at the origin.

Introduction

The goal of this paper is to study *wild quotient singularities* which arise from actions of $G := \mathbb{Z}/p\mathbb{Z}$ on the formal power series ring $A := k[[u, v]]$ when k is an algebraically closed field of characteristic $p > 0$. Here the term "wild" refers to the fact that the order of the group G is not coprime to the characteristic exponent of the ground field k. The resulting quotient singularity is the ring of invariants A^G or, more precisely, the closed point of $\operatorname{Spec}(A^G)$.

Let $X \to \operatorname{Spec}(A^G)$ be a resolution of the singularity. Let C_i, $i = 1, \dots, r$, denote the irreducible components of the exceptional divisor, and form the *intersection matrix*

$$N := ((C_i \cdot C_j)_X)_{1\le i,j\le r} \in \operatorname{Mat}_r(\mathbb{Z}).$$

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This matrix is negative-definite. The *discriminant group* $\Phi_N := \mathbb{Z}^r/N\mathbb{Z}^r$ attached to N is a finite group of order $|\det(N)|$, independent of the resolution. The group Φ_N appears as a natural quotient of the class group $\mathrm{Cl}(A^G)$; see Remark 5.7. Attached to the resolution is its *dual graph* Γ_N, with vertices $v_1, \dots, v_r$, where v_i and v_j are linked by $(C_i \cdot C_j)_X$ distinct edges when $i \neq j$. Our ultimate, long term, goal is to characterize the intersection matrices N, discriminant groups Φ_N, and dual graphs Γ_N, that can arise from such wild quotient singularities.

The *fixed point scheme* of the action of G on Spec A is defined by the ideal $I := (\sigma(a)-a \mid a \in A, \sigma \in G)$. We say that the action is *ramified precisely at the origin* if the radical of I is the maximal ideal (u, v); in this case, the closed point of $\mathrm{Spec}(A^G)$ is singular. Otherwise, we say that the action is *ramified in codimension* 1. When I is principal, A^G is regular [Kiràly and Lütkebohmert 2013, Theorem 2], and when A^G is regular, I is conjectured to be principal [loc. cit., Conjecture 9].

It is known that when the exceptional divisor has smooth components with normal crossings, all components C_i are smooth projective lines and the dual graph Γ_N is a tree [Lorenzini 2013, Theorem 2.8]. It is also known that the discriminant group Φ_N is an elementary abelian p-group [loc. cit., Theorem 2.6], so that in particular we may write

$$|\Phi_N| = |\det(N)| = p^s$$

for some integer $s \geq 0$. In this article, we consider which exponents $s \geq 0$ can arise in this way. By studying diagonal actions on products of curves, the first author [Lorenzini 2018, Theorem 3.15] produced wild quotient singularities with $|\Phi_N| = p^s$ for all exponents $s \geq 2$ with $s \not\equiv 1$ modulo p. Mitsui [2021] later explicitly resolved all wild quotient singularities arising from product of curves, and showed that the previous list is the complete list of exponents arising from product of curves. The *missing exponents* are then $s = 0$, as well as all s with $s \equiv 1 \bmod p$.

Conjecture 0.1. *We conjecture that for p odd, all exponents $s \geq 0$ arise in this way from wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities associated with an action that is ramified precisely at the origin.*

In this article, we prove this conjecture for $s = 0$ and $s = 1$ by explicitly resolving certain wild quotient singularities of independent interest. We also exhibit singularities as in the conjecture that are likely to produce a group Φ_N with $|\Phi_N| = p^s$ for all other missing values $s > 1$ (see Section 0.3). When the condition that the action be ramified precisely at the origin is relaxed, we can prove the following result.

Theorem (see Theorem 5.5). *For p odd, all missing exponents $s \geq 0$ arise from wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities associated with an action that is ramified in codimension* 1.

Let $c, d, e \geq 2$ be integers. Recall that the equation $x^c + y^d + z^e = 0$ is said to define a *Brieskorn surface singularity*. The missing exponents s are exhibited to arise from wild quotient singularities with the help of well-chosen Brieskorn singularities, as in our next theorem.

Theorem (see Theorems 5.1 and 5.3). *Let $B := k[[x, y, z]]/(z^p + x^c + y^d)$. Assume that p does not divide cd. Let $g := \gcd(c, d)$. Any resolution of* Spec B *has an intersection matrix whose associated discriminant group has order p^{g-1} and is killed by p. When $c = pm + 1$ and $d = pn + 1$ for some $m, n \geq 1$, then* Spec B *is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.*

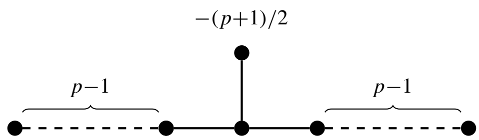
The resolutions of the Brieskorn singularities in the previous theorem are found in Theorem 5.1, and coincide with the known resolutions in characteristic 0 [Hirzebruch and Jänich 1969, Theorem, page 232; Orlik and Wagreich 1971a]. The theorem is valid when $p = 2$, but in this case, the order p^{g-1} is always an even power of 2, and thus provides no examples of missing odd exponents. The theorem shows that when $p = 2$ and $\gcd(p, cd) = 1$, all singularities $z^p + x^c + y^d = 0$ are wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities. It would be of interest to determine whether this fails to be the case when $p > 2$.

Let now C_n denote the n-th Catalan number, and let $p \geq 3$. To produce singularities associated with an action that is ramified precisely at the origin and which have $|\Phi_N| = p$, we expand on the work of Peskin [1983] and consider the ring $B_\mu := k[[x, y, z]]/(h)$, where $\mu \in k[y]$ and

$$h := z^p + 2y^{p+1} - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} (\mu y)^{2p-2n} z^n.$$

When $\mu = 1$, this equation defines a wild quotient singularity that can be regarded as an analogue of the E_6^1-singularity (notation as in Artin's classification [1977]). We compute explicitly its resolution in our next theorem. When $p = 3$, the graph Γ_N below reduces to the Dynkin diagram E_6. When drawing a dual graph, we adopt in this article the usual convention that a vertex is adorned with the associated self-intersection number, unless this self-intersection number is -2, in which case it is suppressed.

Theorem (see Theorem 6.3). *Let p be an odd prime. Let B_μ be as above. Then* Spec B_μ *has a resolution of singularities with dual graph Γ_N independent of μ of the following form:*



The associated discriminant group Φ_N has order p.

0.2. To treat the case where Φ_N is the trivial group in Conjecture 0.1, we use a family of hypersurface singularities introduced in [Lorenzini and Schröer 2020] and which is of independent interest. Fix a system of parameters a, b in $k[[x, y]]$. Let $\mu \in k[[x, y]]$, and consider the equation

$$z^p - (\mu ab)^{p-1} z - a^p y + b^p x = 0, \tag{0-1}$$

and the associated ring

$$B_\mu = B := k[[x, y, z]]/(z^p - (\mu ab)^{p-1} z - a^p y + b^p x).$$

(a) *Assume that μ is a unit in $k[\![x,y]\!]$*. It is shown in [loc. cit., 7.1], that B is isomorphic to the ring of invariants A^G of an explicit wild action of $\mathbb{Z}/p\mathbb{Z}$ on $A := k[\![u,v]\!]$ ramified precisely at the origin. More precisely, after identifying A with the ring

$$k[\![x,y]\!][u,v]/(u^p-(\mu a)^{p-1}u-x,\, v^p-(\mu b)^{p-1}v-y),$$

the action is determined by the automorphism σ with $\sigma(u)=u+\mu a$ and $\sigma(v)=v+\mu b$. The morphism $\operatorname{Spec} A\to\operatorname{Spec} A^G$ is ramified only at the maximal ideal $\mathfrak{m}$, and we find that the étale fundamental group $\pi_1^{\mathrm{loc}}(A^G)$ of the punctured spectrum $U:=\operatorname{Spec} A^G\setminus\{\mathfrak{m}\}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Such actions are called *moderately ramified* in [loc. cit.], and we refer the reader to that article for further information on these actions.

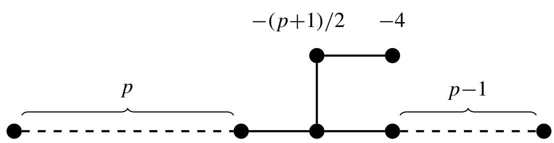
(b) *Assume that μ is not a unit in $k[\![x,y]\!]$, that $\mu\neq 0$, and that it is coprime to both a and b*. Then B is again isomorphic to the ring of invariants A^G for the action on $A:=k[\![u,v]\!]$ described above. However, in this case the morphism $\operatorname{Spec} A\to\operatorname{Spec} A^G$ is ramified in codimension 1 and the group $\pi_1^{\mathrm{loc}}(A^G)$ is trivial.

We restrict our attention to the case where $a=y^n$ and $b=x^m$. The case $\mu=0$ is then also of interest.

(c) *Assume that $\mu=0$, with $a=y^n$ and $b=x^m$*. The resulting hypersurface is a Brieskorn singularity of type $z^p-y^{pn+1}+x^{pm+1}$.

In the specialized case where $a=y^n$ and $b=x^m$, preliminary computations with Magma [Bosma et al. 1997] and Singular [Decker et al. 2022] suggest that the resolution of singularities in all three cases above might have the same combinatorial type, independent of μ. We prove that this is indeed the case in two instances in this article, when $a=y$ and $b=x$ in Theorem 9.2, and when $a=y^2$ and $b=x$ in Theorem 7.1. In the latter case, Artin [1977] (see also [Peskin 1980]) shows when $p=2$ that the values $\mu=0$, $\mu=1$, and $\mu=y$ produce the rational double points E_8^0, E_8^2, and E_8^1, respectively. These singularities are not isomorphic but have the same resolution graph, the Dynkin diagram E_8. Our generalization of these singularities to any odd prime p has a resolution with the following dual graph.

Theorem (see Theorem 7.1). *Let p be an odd prime. Let B_μ be as in Section 0.2. Assume that $a=y^2$ and $b=x$. Then* $\operatorname{Spec} B_\mu$ *has a resolution of singularities with dual graph Γ_N independent of μ of the following form*:



The associated discriminant group Φ_N is trivial.

0.3. Let p be odd. Recall that when $\mu=1$, the associated quotient singularity $\operatorname{Spec} B_{\mu=1}$ is induced by an action that is ramified precisely at the origin. It is likely that by varying the exponents m and n in $a=y^n$ and $b=x^m$, one will obtain examples of resolutions of $\operatorname{Spec} B_{\mu=1}$ with associated discriminant group Φ_N of order p^s for any power s with $s \not\equiv -1 \bmod p$. In particular, we exhibit in Lemma 5.6 the appropriate exponents m and n that would cover all remaining open cases in our Conjecture 0.1 (that is, all values of s with $s \equiv 1 \bmod p$).

Peskin's singularity with $\mu = 1$ introduced above, and all the singularities considered in [Lorenzini 2018] or [Mitsui 2021], are also induced by an action that is ramified precisely at the origin. When $p = 2$, none of the known explicit resolutions for examples in these classes of singularities produce an associated discriminant group Φ_N with order 2^s and s *odd*. This lack of examples might indicate that there is a serious obstruction to exhibiting such examples. It is natural to wonder whether such examples in fact do not exist for actions ramified precisely at the origin.

Let $p = 2$. The Dynkin diagram E_7, with discriminant group Φ_{E_7} of order 2, might be the most ubiquitous graph with discriminant group of order 2^s with s odd. Many other such examples are exhibited in Example 8.2. Artin [1977] showed that there exists a wild $\mathbb{Z}/2\mathbb{Z}$-action on $A := k[[u, v]]$, ramified in codimension 1, such that $\operatorname{Spec} A^{\mathbb{Z}/2\mathbb{Z}}$ is a rational double point of type E_7. He also showed that any such surface singularity must have a trivial local fundamental group. In other words, there cannot exist a wild $\mathbb{Z}/2\mathbb{Z}$-action on $A = k[[u, v]]$, ramified precisely at the origin, such that $\operatorname{Spec} A^{\mathbb{Z}/2\mathbb{Z}}$ has a resolution of combinatorial type E_7.

Inspired by Artin's considerations, we define in Section 8 some explicit wild $\mathbb{Z}/p\mathbb{Z}$-actions on $A = k[[u, v]]$ ramified in codimension 1. When $p = 2$, we exhibit for each s *odd* an explicit example conjectured to have discriminant group of order 2^s. In Section 9, for any prime p, we exhibit a wild $\mathbb{Z}/p\mathbb{Z}$-action on $A = k[[u, v]]$ ramified in codimension 1 which results in an A_{p-1}-singularity.

Theorem (see Theorem 9.4). *Let k be a field of characteristic $p > 0$. Let $A := k[[u, v]]$. Then there exists an automorphism $\sigma : A \to A$ of order p such that* $\operatorname{Spec} A^{\langle\sigma\rangle}$ *is a rational double point of type A_{p-1}, which has discriminant group $\Phi_{A_{p-1}}$ of order p. Any such automorphism induces a morphism* $\operatorname{Spec} A \to \operatorname{Spec} A^{\langle\sigma\rangle}$ *that must be ramified in codimension* 1.

It is natural to wonder whether the same result holds for any *Hirzebruch–Jung chain* whose discriminant group has order p (definition recalled in Section 1.1). The last statement in the above theorem follows from a result of Ito and Schröer [2015], which states that if the action is ramified precisely at the origin, then the resolution of the resulting quotient singularity has a dual graph Γ_N which must have a vertex of valency at least 3.

Artin [1975] showed that in characteristic $p = 2$, all wild quotient singularities A^G with $\operatorname{Spec} A \to \operatorname{Spec} A^G$ ramified precisely at the origin can be described by an equation of the form (0-1) with $\mu = 1$. In particular, all such singularities are complete intersection. We show in Proposition 10.1 that when $p = 2$, any wild quotient singularity A^G is a complete intersection, even when $\operatorname{Spec} A \to \operatorname{Spec} A^G$ ramifies in codimension 1. When A^G is a complete intersection, it is then also Gorenstein, with an intersection matrix which is *numerically Gorenstein*. The purely linear algebraic definition of numerically Gorenstein is recalled in Section 10.2, and it is natural to wonder whether this condition imposes a new restriction on intersection matrices associated with $\mathbb{Z}/2\mathbb{Z}$-quotient singularities. The answer to this question is negative, and we show in Proposition 10.5 that any intersection matrix N such that Φ_N is killed by 2 is always numerically Gorenstein.

The paper is organized as follows. Section 1 contains several useful facts concerning the linear algebra of intersection matrices N, in particular formulas for the order of Φ_N when the dual graph Γ_N is star-shaped. Sections 2 and 3 are preparatory sections, where we recall basic facts regarding how to compute self-intersection numbers on a resolution of a singularity using data coming from intermediate blow-ups. This will be applied in later sections to the resolution of Spec B_μ, where we found it useful, instead of starting the resolution process by blowing up the maximal ideal, to first blow up an ideal naturally related to the ideal defining the fixed scheme of the action. We provide in Section 4 the explicit resolution of certain weighted homogeneous singularities of the form

$$W^q - U^a V^b (V^d - U^c) = 0,$$

with p, q, a, b, c, d subject to certain mild restrictions. Over $\mathbb{C}$, such resolution has already been obtained by Orlik and Wagreich [1971a; 1971b; 1977] in full generality. The proofs of the theorems presented in this introduction are found in Sections 5–10.

1. Intersection matrices

Let B be a complete noetherian local ring that is two-dimensional and normal. Let C_i, $i = 1, \dots, n$, denote the irreducible components of the exceptional divisor of a resolution of singularities of Spec B, with associated intersection matrix $N := ((C_i \cdot C_j))_{1 \le i, j \le n}$. This section collects some facts that depend only on the linear algebra of the matrix N and which are used in later sections.

An $n \times n$ *intersection matrix* $N = (c_{ij})$ is a symmetric negative-definite integer matrix with negative coefficients on the diagonal, and nonnegative coefficients off the diagonal. The *discriminant group* $\Phi = \Phi_N$ is defined as the finite abelian group $\mathbb{Z}^n / N\mathbb{Z}^n$, which has order $|\det(N)|$. The associated *graph* $\Gamma = \Gamma_N$ arises as follows: Introduce vertices $v_1, \dots, v_n$ corresponding to the standard basis vectors in $\mathbb{Z}^n$. Two vertices $v_i \ne v_j$ are linked by exactly $c_{ij} \ge 0$ edges. If not stated otherwise, we tacitly assume that Γ is connected.

The *degree* or *valency* of a vertex $v \in \Gamma$ is the number of edges attached to v. A vertex v with valency at least three is called a *node*, and a vertex v with valency one is called *terminal*. A graph is a *chain* if it is connected and does not contain any node. It is called *star-shaped* if it is a tree with a unique node. Given a star-shaped graph Γ with node v_0, we can consider the subgraph $\Gamma \smallsetminus \{v_0\}$ obtained by removing the vertex v_0 and all the edges containing v_0. This complement is the disjoint union of $m \ge 3$ chains $\Delta_1, \dots, \Delta_m$ that we call the *terminal chains* of Γ.

1.1. Suppose that N is an intersection matrix whose graph Γ_N is a chain, with $\ell \ge 1$ consecutive vertices $v_1, \dots, v_\ell$. For convenience, we label the diagonal entries of N by $c_{ii} = -s_i$, and we assume below that $s_i \ge 2$ for $i = 1, \dots, \ell$, unless $\ell = 1$, in which case we also allow $s_1 = 1$. We associate to N with this ordering of the vertices a unique sequence of positive integers $1 = r_\ell < \cdots < r_1 < r_0$ such that the following matrix equality holds, where the square matrix on the left is N:

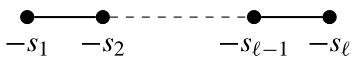
$$\begin{pmatrix} -s_1 & 1 & & \\ 1 & -s_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -s_\ell \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_{\ell-1} \\ r_\ell \end{pmatrix} = \begin{pmatrix} -r_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

When needed, we will denote $R = R_N$ the transpose of the vector $(r_1, \dots, r_\ell)$, so that NR is the transpose of $(-r_0, 0, \dots, 0)$. It is known that $|\det(N)| = r_0$, and that Φ_N is cyclic of order r_0 [Lorenzini 2013, 3.13]. To be able to refer to r_0 and r_1 without indices, we will relabel them as $r_0 = a$ and $r_1 = b$. Note that by construction, $\gcd(a, b) = 1$, and that we can express the reduced fraction a/b completely in terms of $s_1, \dots, s_\ell$ as a continued fraction:

$$\frac{a}{b} = [s_1, s_2, \dots, s_\ell] := s_1 - \cfrac{1}{s_2 - \cfrac{1}{\ddots - \cfrac{1}{s_\ell}}} \tag{1-1}$$

Clearly, any reduced fraction a/b with $a > b$ determines an intersection matrix N as above. The reduced fraction $a/b = 1/1$ determines the matrix $N = (-1)$. We note that $-a/b = \det(N)/\det(N')$, where N' is obtained from N by removing its first line and first column (recall that the determinant of the empty matrix is 1 by convention).

As is customary, the vertices of the graph Γ_N of an intersection matrix $N = (c_{ij})$ are labeled with the *self-intersection numbers* $-s_i := c_{ii}$, and self-intersection numbers $-s_i = -2$ are usually omitted. For a chain Γ_N as above, we get the following drawing:



We call such chain a *Hirzebruch–Jung chain*. Recall that $p/(p-1) = [2, \dots, 2]$ and that the corresponding intersection matrix of size $p-1$ and determinant $(-1)^{p-1}p$ is denoted by A_{p-1}. This intersection matrix will be shown to arise in the context of $\mathbb{Z}/p\mathbb{Z}$-singularities in Theorem 9.4.

1.2. Let $m \geq 3$. Let $a_1/b_1, \dots, a_m/b_m$ be reduced fractions with $a_i/b_i \geq 1$ for $i = 1, \dots, m$. Let $s_0 \geq 1$ be any integer. We denote by $N = N(s_0 \mid a_1/b_1, \dots, a_m/b_m)$ the following matrix. Its graph $\Gamma = \Gamma_N = \Gamma(s_0 \mid a_1/b_1, \dots, a_m/b_m)$ is star-shaped with m terminal chains attached to a central node v_0 having self-intersection number $-s_0$. Let $\Delta_1, \dots, \Delta_m$ be the Hirzebruch–Jung chains determined by the fractions $a_1/b_1, \dots, a_m/b_m$. The graph Γ is obtained by attaching to v_0 with a single edge the initial vertex of each chain Δ_i. In this article, when referring to a matrix of the form $N = N(s_0 \mid a_1/b_1, \dots, a_m/b_m)$, we will always assume that it is an intersection matrix, i.e., that N is negative-definite.

Proposition 1.3. *Let $N = N(s_0 \mid a_1/b_1, \dots, a_m/b_m)$ be an $n \times n$ intersection matrix as above, with star-shaped graph Γ_N. Then $s_0 > \sum_{j=1}^m b_j/a_j$, and the following hold:*

(i) *We have* $\det(N) = (-1)^n \big(\prod_j a_j\big)\big(s_0 - \sum_j b_j/a_j\big)$. *In particular, there is an integer factorization*

$$|\det(N)| = \left(\frac{\prod_j a_j}{\operatorname{lcm}(a_1, \dots, a_m)}\right)\left(\operatorname{lcm}(a_1, \dots, a_m)(s_0 - \sum_j b_j/a_j)\right).$$

(ii) *In the discriminant group Φ_N, the class of the standard basis vector $e_{v_0} \in \mathbb{Z}^n$ corresponding to the central node v_0 has order* $\operatorname{lcm}(a_1, \dots, a_m)\big(s_0 - \sum_j b_j/a_j\big)$.

(iii) *Let w_j denote the terminal vertex in Γ_N of the chain Δ_j. Then Φ_N is generated by the classes of e_{w_j}, $j = 1, \dots, m$. Moreover, the class of e_{v_0} is equal to the class of $a_j e_{w_j}$, and the group Φ_N is killed by* $\operatorname{lcm}(a_1, \dots, a_m)^2\big(s_0 - \sum_j b_j/a_j\big)$.

(iv) *If a_j is a prime p for all j and $ps_0 - \sum_j b_j = 1$, then Φ_N is killed by p and has order p^{m-1}.*

(v) *Assume that Φ_N is killed by a prime p. If p divides a_j for some j, then the class of e_{v_0} is trivial in Φ_N.*

Proof. Without loss of generality, we may assume that N equals the block matrix

$$N = \begin{pmatrix} -s_0 & * & \cdots & * \\ * & N_1 & & \\ \vdots & & \ddots & \\ * & & & N_m \end{pmatrix} \in \operatorname{Mat}_n(\mathbb{Z}),$$

where N_i is the intersection matrix with graph Δ_i, with vertices numbered consecutively starting from the vertex adjacent to the node v_0. The $*$'s in the above matrix stand for sequences of appropriate size, starting with 1 followed by zeros. Let R_i denote the positive integer vector associated to N_i, such that

$$N_i R_i = {}^t(-a_i, 0, \dots, 0).$$

Form the block column integer vector R in $\mathbb{Z}^n$ given as

$$R := \operatorname{lcm}(a_1, \dots, a_m)\ {}^t(1, {}^tR_1/a_1, \dots, {}^tR_m/a_m).$$

By construction, the greatest common divisor of the entries in R is 1, since, given a prime p such that p^s exactly divides $\operatorname{lcm}(a_1, \dots, a_m)$, there exists at least one index i such that a_i is exactly divisible by p^s. In particular, the coefficient of R corresponding to the last vertex on the chain Δ_i is coprime to p. Let $x := s_0 - \sum_j b_j/a_j$. Then

$$NR = \operatorname{lcm}(a_1, \dots, a_m)\ {}^t(-x, 0, \dots, 0).$$

Note that $x > 0$, because N is negative-definite, so the integer ${}^tRNR$ must be negative. By negative-definiteness, we also know that $\det(N)$ has sign $(-1)^n$. Using [Lorenzini 2013, Theorem 3.14], with the matrix N and the vector R, we get

$$\det(N) = (-1)^n \Big(s_0 - \sum_j b_j/a_j \Big) \cdot \Big(\prod_j a_j \Big)$$

and the assertion (i) follows. The assertion in (ii) follows immediately from the equality

$$NR = \operatorname{lcm}(a_1, \ldots, a_m)\ {}^t(-x, 0, \ldots, 0)$$

and the fact that the greatest common divisor of the coefficients of R is 1. For (iii), to show that $e_{v_0} - a_j e_{w_j}$ is in the image of N, consider the unique positive vector S_j whose first component is 1 and such that $N_j S_j$ is equal to the transpose of $(0, \ldots, 0, -a_j)$. Extend this vector to a vector $\bar{S}_j \in \mathbb{Z}^n$ by setting all other components to 0. Then $N\bar{S}_j = e_{v_0} - a_j e_{w_j}$. The proof that for any vertex w on the chain Δ_j, there exists an integer c_w such that $e_w - c_w e_{w_j}$ is in the image of N is similar, and is left to the reader. Using (ii) to find the order of the class of e_{v_0}, it follows immediately that the class of e_{w_j} is killed by $\operatorname{lcm}(a_1, \ldots, a_m)^2 \big(s_0 - \sum_i b_i/a_i\big)$, for all j. Part (iv) is immediate from (i) and (iii). In Part (v), assume that p divides a_j. As the class of e_{w_j} is killed by p by hypothesis, we find from (iii) that the class of e_{v_0} is trivial. $\square$

2. Computation of self-intersections

Let B be a complete local noetherian ring that is two-dimensional and normal. It is known that a resolution of singularities $X \to \operatorname{Spec}(B)$ exists, and that it can be obtained from the sequence

$$X = Y_t \to Y_{t-1} \to \cdots \to Y_1 \to Y_0 = \operatorname{Spec}(B),$$

where each $Y_i \to Y_{i-1}$ is the normalization of the blow-up of the finitely many singular points of Y_{i-1}; see, e.g., [Lipman 1978, Theorem on page 151 and Remark B on page 155]. In this section we develop a method for computing the self-intersection of particular irreducible components of the exceptional divisor on X. This information is needed in the proofs of each of our explicit computation of resolutions in Theorems 4.4, 6.3, 7.1, and 9.2. For the sake of exposition, we assume that the residue field $k = B/\mathfrak{m}_B$ is algebraically closed.

Note that the process described above usually does not produce the minimal desingularization, as some irreducible components of the exceptional divisor on X might be (-1)-curves, and thus contract to smaller resolutions of singularities. This may even happen for the strict transforms of the exceptional divisors on the first blow-up Y_1; see Example in [Lipman 1969, page 205].

2.1. Let $X \to \operatorname{Spec}(B)$ be any resolution of singularities, and write $C_1, \ldots, C_n$ for the irreducible components of the exceptional divisor. We then have intersection numbers

$$c_{ij} = (C_i \cdot C_j)_X := \chi(\mathscr{O}_{C_j}(C_i)) - \chi(\mathscr{O}_{C_j}) = \deg(\mathscr{O}_{C_j}(C_i)),$$

and can form the resulting intersection matrix $N = (c_{ij})_{1\leq i,j\leq n}$. Associated with N is the connected graph $\Gamma = \Gamma_N$ with vertices $v_1, \dots, v_n$, and a pair of vertices $v_i \neq v_j$ is linked by exactly c_{ij} edges. We call Γ the *resolution graph* or the *dual graph* attached to $X \to \operatorname{Spec} B$.

Now consider a factorization $X \to Y \to \operatorname{Spec} B$, where $\pi : X \to Y$ is the contraction of certain exceptional curves, say $C_{s+1} \cup \dots \cup C_n$. We regard the induced morphism $Y \to \operatorname{Spec}(B)$ as a partial resolution of singularities, and by definition of contraction, Y is *normal*. Write $D_1, \dots, D_s \subset Y$ for the images in Y of the noncontracted curves $C_1, \dots, C_s \subset X$. These images are Weil divisors which are not necessarily Cartier. Following Mumford [1961, page 17] (see also [Fulton 1984, 7.1.16] or [Schröer 2019, Theorem 1.2]) one has *rational intersection numbers* $(D_i \cdot D_j)_Y \in \mathbb{Q}$ obtained as follows: First define the rational pull-back $\pi^*(D_i) := C_i + \sum_{k>s} \lambda_k C_k$, where $\lambda_{s+1}, \dots, \lambda_n \in \mathbb{Q}$ are the fractions uniquely determined by the conditions $(\pi^*(D_i) \cdot C_k)_X = 0$ for all $s < k \leq n$. One then sets

$$(D_i \cdot D_j)_Y := (\pi^*(D_i) \cdot C_j)_X = (\pi^*(D_i) \cdot \pi^*(D_j))_X.$$

These numbers actually do not depend on the choice of resolution $\pi : X \to Y$.

Suppose now that $\pi : X \to Y$ is the contraction of all but the first curve C_1. Assume furthermore that Γ is a tree. Let v be the vertex corresponding to C_1, and consider the graph $\Gamma \smallsetminus \{v\}$ obtained from Γ by removing the vertex v and all the edges attached to v. The graph $\Gamma \smallsetminus \{v\}$ decomposes into connected components $\Gamma \smallsetminus \{v\} = \Delta_1 \cup \dots \cup \Delta_r$, with corresponding intersection matrices $N_1, \dots, N_r$ for each component. Since Γ is a tree, there exists a unique vertex $w_i \in \Delta_i$ which is adjacent to v in Γ. Define $\Delta_i' := \Delta_i \smallsetminus \{w_i\}$, with intersection matrix N_i'. We call

$$\delta_i := -\frac{\det(N_i')}{\det(N_i)} \in \mathbb{Q}_{>0}$$

the *correction term* at w_i (recall that the determinant of the empty matrix is 1, and we use this convention if Δ_i is reduced to the single vertex w_i). The correction terms δ_i are indeed positive, since the signs of $\det(N_i)$ and $\det(N_i')$ are given by $(-1)^{r_i}$ and $(-1)^{r_i-1}$, where r_i is the number of vertices of Δ_i. When Δ_i is a chain as in Section 1.1 corresponding to a fraction a_i/b_i, we have $\delta_i = b_i/a_i$. The geometric meaning of the correction terms is as follows:

Proposition 2.2. *In the above situation, the integral self-intersection and the rational self-intersection are related by the formula*

$$(C_1 \cdot C_1)_X = (D_1 \cdot D_1)_Y - \sum_{i=1}^{r} \delta_i.$$

Proof. For ease of notation, we let in this proof $C = C_1$ and $D = D_1$. Let N_0 denote the lower-right principal submatrix of N. Recall from our earlier description that N_0 is a block diagonal matrix with $\det(N_0) = \prod_{i=1}^{r} \det(N_i)$. Then

$$(\lambda_2, \dots, \lambda_n) = -((C \cdot C_2)_X, \dots, (C \cdot C_n)_X) N_0^{-1}.$$

It follows that

$$(D\cdot D)_Y = (\pi^*(D)\cdot C)_X = (C\cdot C)_X + \sum_{j=2}^{n} \lambda_j (C_j\cdot C)_X.$$

Since Γ_N is a tree, we find that if $(C\cdot C_j)_X \neq 0$, then $(C\cdot C_j)_X = 1$. We only need to compute explicitly λ_j when $(C\cdot C_j)_X \neq 0$. According to our definitions, there are r such indices j and, renumbering the components if necessary, we find that in each case, the coefficient λ_j is the top left corner of the corresponding matrix N_j^{-1}, that is, $\det(N_j')/\det(N_j)$, as desired. □

We will use Proposition 2.2 in the following situation. Let $\mathfrak{b}$ be an ideal in B, and let $Z \to \operatorname{Spec} B$ denote the blowing-up with center $V(\mathfrak{b})$. Denote by $E \subset Z$ the schematic preimage of the center. Let $\nu : Y \to Z$ be the normalization map and denote by $D = \nu^{-1}(E)$ the schematic preimage of E. *Assume that D, and hence E, are irreducible.* Let D_{red} denote the support of D endowed with its induced reduced structure. Letting D_{red} play the role of D_1 in Proposition 2.2, we find a formula for the rational intersection number $(D_{\mathrm{red}}\cdot D_{\mathrm{red}})_Y$ in term of data from a resolution $X \to Y$. Our next proposition shows how to obtain $(D_{\mathrm{red}}\cdot D_{\mathrm{red}})_Y$ from data associated with the blowing-up $Z \to \operatorname{Spec} B$.

The exceptional divisor $E \subset Z$ is given by the sheaf of ideals $\mathscr{O}_Z(1) \subset \mathscr{O}_Z$. The reduction E_{red} is a projective curve over the residue field k, allowing us to define the integral intersection number

$$(E\cdot E_{\mathrm{red}})_Z := \chi(\mathscr{O}_{E_{\mathrm{red}}}(E)) - \chi(\mathscr{O}_{E_{\mathrm{red}}}) = \deg \mathscr{O}_{E_{\mathrm{red}}}(-1).$$

In practice, $(E\cdot E_{\mathrm{red}})_Z$ can often be computed, and such computation is done for instance in Proposition 3.6.

Let η denote the generic point of E, and set $m := \operatorname{length}(\mathcal{O}_{E,\eta})$. When Z is normal, we have the equality of Weil divisors $E = mE_{\mathrm{red}}$. When Z is not normal, the abuse of notation $E = mE_{\mathrm{red}}$ should be interpreted to mean that the length of the local ring $\mathcal{O}_{E,\eta}$ is m.

Proposition 2.3. *In the above situation where D, and hence E, are assumed irreducible, let $m := \operatorname{length}(\mathcal{O}_{E,\eta})$ and let $d \geq 1$ be the degree of the induced map $\nu : D_{\mathrm{red}} \to E_{\mathrm{red}}$. Then we have*

$$(D_{\mathrm{red}}\cdot D_{\mathrm{red}})_Y = \frac{d^2}{m}(E\cdot E_{\mathrm{red}})_Z.$$

Proof. First, we check that $(D\cdot \nu^{-1}(F))_Y = (E\cdot F)_Z$ for every effective Cartier divisor $F \subset Z$ that does not contain the support of E. The two intersection numbers are the k-degrees of the finite schemes $D \cap \nu^{-1}(F)$ and $E \cap F$, respectively. Fix a point $z \in E \cap F$, consider the local ring $A := \mathscr{O}_{F,z}$ and choose an element $t \in \mathfrak{m}_A$ defining $F \cap E \subset F$ locally. Then A is a local noetherian ring of dimension one without embedded components, and $M := \mathscr{O}_{\nu^{-1}(F),z}$ is a finite A-module of rank one for which the multiplication map $t : M \to M$ is injective. According to [EGA IV$_1$ 1964, Chapter IV, Lemma 21.10.13], the modules A/tA and M/tM have the same A-length, hence also the same k-vector space dimension. Applying this with a difference $F - F'$ of effective Cartier divisors that are linearly equivalent to E, we conclude $(D\cdot D)_Y = (E\cdot E)_Z$.

To simplify notation write $E' = E_{\mathrm{red}}$ and $D' = D_{\mathrm{red}}$. Since Y us normal, we can write $D = hD'$ for some $h \geq 1$, and we get

$$h^2(D' \cdot D')_Y = (D \cdot D)_Y = (E \cdot E)_Z = m(E \cdot E')_Z. \tag{2-1}$$

We now use Kleiman's theory of rational degrees $\deg(V'/V) \in \mathbb{Q}_{\geq 0}$ for morphisms $V' \to V$ between irreducible proper schemes that are not necessarily integral [Kleiman 1966, Definition on page 277]. According to [Kleiman 1966, Lemma 2], the commutative diagram

$$\begin{array}{ccc} D' & \longrightarrow & D \\ \downarrow & & \downarrow \\ E' & \longrightarrow & E \end{array}$$

gives the equation $\deg(D'/E') \cdot \deg(E'/E) = \deg(D'/D) \cdot \deg(D/E)$, and furthermore we have $\deg(E'/E) = 1/m$ and $\deg(D'/D) = 1/h$. Thus $\deg(D'/E') = m/h$. Inserting this into (2-1) yields the assertion. □

3. Blowing up nonreduced centers

We begin this section with some general facts on the computation of blowing-ups, needed for instance to fully justify the explicit computations done in Proposition 3.6. Let B be a noetherian ring, and let $\mathfrak{b} \subset B$ be an ideal. Endow the associated *Rees ring*

$$B[\mathfrak{b}T] := B \oplus \mathfrak{b}T \oplus \mathfrak{b}^2T^2 \oplus \cdots \subset B[T]$$

with the grading induced by the standard grading on $B[T]$. The morphism $\mathrm{Proj}(B[\mathfrak{b}T]) \to \mathrm{Spec}\, B$ is called the *blowing-up* of $\mathrm{Spec}(B)$ with center $\mathrm{Spec}(B/\mathfrak{b})$. We denote $\mathrm{Proj}(B[\mathfrak{b}T])$ by $\mathrm{Bl}_{\mathfrak{b}}(B)$ or, when no confusion may ensue, simply by Z. Let E denote the schematic preimage in Z of the center of the blowing-up.

Assume now that R is a noetherian ring with a surjection $R \to B$. Let $\mathfrak{a}$ denote the preimage in R of the ideal $\mathfrak{b}$. Consider the blowing-up $Z' := \mathrm{Bl}_{\mathfrak{a}}(R)$ with center $V(\mathfrak{a})$, and the commutative diagram induced by the surjection $R[\mathfrak{a}T] \to B[\mathfrak{b}T]$ of Rees rings:

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, R \end{array}$$

The horizontal morphisms are closed immersions.

Recall that an element $f \in R$ is called *regular* if multiplication by f on R is an injective map. Assume now that the kernel of $R \to B$ is generated by a regular element $f \in R$. Then $\mathrm{Spec}(B)$ is an effective Cartier divisor in $\mathrm{Spec}(R)$, and our next proposition provides a criterion for checking whether the closed subscheme Z is an effective Cartier divisor in Z', when Z' and $V(\mathfrak{a})$ are "nice". This criterion is explicit and in general not very difficult to verify.

Each element $g \in \mathfrak{a}$ defines a basic open set $D_+(g) := \operatorname{Spec} R[\mathfrak{a}T]_{(gT)}$ of Z' called the *g-chart*. When $\mathfrak{a} = (g_1, \ldots, g_r)$, the union $\bigcup_{i=1}^r D_+(g_i)$ is an affine open cover of Z'.

Proposition 3.1. *Let R be a noetherian ring, locally of complete intersection.*[1] *Let $g_1, \ldots, g_r \in R$ be a regular sequence, and set $\mathfrak{a} := (g_1, \ldots, g_r)$. Let $f \in R$ be a regular element contained in $\mathfrak{a}$, and set $B := R/(f)$ and $\mathfrak{b} := \mathfrak{a}B$. Consider as above the blowing-ups $Z \to \operatorname{Spec} B$ and $Z' \to \operatorname{Spec} R$.*

For each $i = 1, \ldots, r$, choose a factorization $f/1 = (g_i/1)^{s_i} h_i$ in $R[\mathfrak{a}T]_{(g_iT)}$, with $s_i \geq 0$ and $h_i \in R[\mathfrak{a}T]_{(g_iT)}$. Assume that for each i, the closed subscheme $V(h_i, g_i/1)$ of $D_+(g_i)$ has codimension two in $D_+(g_i)$. Then:

(a) *The closed subscheme Z of Z' is an effective Cartier divisor. Its restriction to the g_i-chart $D_+(g_i)$ is the closed subscheme $V(h_i)$.*

(b) *The scheme Z is locally of complete intersection.*

Proof. Part (a) follows from Proposition 3.2. Part (b) follows from Proposition 3.4. □

Proposition 3.2. *Keep the notation introduced at the beginning of this section. Let $g \in \mathfrak{a}$. Suppose that we have a factorization $f/1 = (g/1)^s h$ in $R[\mathfrak{a}T]_{(gT)}$, for some $s \geq 0$ and some element $h \in R[\mathfrak{a}T]_{(gT)}$. Suppose also that the following two assumptions hold*:

(i) *The closed subscheme $V(h, g/1)$ of $D_+(g)$ has codimension at least two.*

(ii) *The basic open set $D_+(g) \subset Z'$ satisfies Serre's condition (S_2).*

Then $Z \cap D_+(g) = V(h)$ as closed subschemes of the g-chart $D_+(g)$.

Proof. By hypothesis, $g/1$ and h define two closed subschemes $V(g/1)$ and $V(h)$ in $D_+(g)$. All schemes below are viewed as subschemes in $Z' := \mathrm{Bl}_{\mathfrak{a}}(R)$. The conclusion of the proposition is implied by the following two claims:

(a) The subsets $D_+(g) \cap (Z \setminus E)$ and $V(h) \setminus V(g/1)$, which are open in Z, are equal.

(b) The subscheme $V(h) \cap V(g/1)$ is an effective Cartier divisor on $V(h)$.

Indeed, on one hand the schematic closure of the inclusion $D_+(g) \cap (Z \setminus E) \to D_+(g) \cap Z$ is equal to $D_+(g) \cap Z$ by Lemma 3.3, and on the other hand the schematic closure of the inclusion $V(h) \cap V(g/1) \to V(h)$ is equal to $V(h)$, also by Lemma 3.3.

We leave it to the reader to verify (a). To prove (b), note that since f is regular in R, the element $f/1$ is regular in $R[\mathfrak{a}T]_{(gT)}$. Thus $V(h)$ and $V(g/1)$ are two Cartier divisors in $D_+(g)$. We need to show that the image of $g/1$ is not a zero-divisor in $R[\mathfrak{a}T]_{(gT)}/(h)$. Assumption (ii) implies that any effective Cartier divisor on $D_+(g)$ satisfies Serre's condition (S_1). In particular, the ring $R[\mathfrak{a}T]_{(gT)}/(h)$ has no embedded primes, and thus the zero divisors in $R[\mathfrak{a}T]_{(gT)}/(h)$ are contained in the minimal primes ideals. Krull's

[1] Recall that $g_1, \ldots, g_d \in R$ is called a *regular sequence* if the class of g_i is a regular element in the ring $R/(g_1, \ldots, g_{i-1})$, for each $1 \leq i \leq d$. The ring R is called *locally of complete intersection* if for each $\mathfrak{p} \in \operatorname{Spec} R$, the completion of $R_{\mathfrak{p}}$ is isomorphic to a ring of the form $A/(a_1, \ldots, a_s)$, where A is a regular complete local ring, and $a_1, \ldots, a_s$ is a regular sequence.

principal ideal theorem shows the irreducible components of $V(h)$ all have codimension one in $D_+(g)$. Assumption (i) implies then that $g/1$ cannot be contained in a minimal prime ideal of $R[\mathfrak{a}T]_{(gT)}/(h)$. Thus $g/1$ is regular in $R[\mathfrak{a}T]_{(gT)}/(h)$. $\square$

Lemma 3.3. *Let V be the complement of an effective Cartier divisor F on a noetherian scheme Y. Then the schematic image in Y of the open embedding $V \to Y$ coincides with Y.*

Proof. The assertion is local, so we may assume that $Y = \operatorname{Spec}(A)$ and $F = V(g)$, where $g \in A$ is a regular element. The schematic image is defined by the kernel of the localization map $A \to A_g$, with $a \mapsto a/1$. Since g is regular, this kernel is the zero ideal. $\square$

In the context of Proposition 3.2, we say that the equation $h = 0$ is the *strict transform* of $f = 0$ on the g-chart. One easily sees that condition (i) ensures that the exponent $s \geq 0$ is the maximal exponent. Note that in any case there is a factorization $f/1 = (g/1)^s h$ with maximal $s \geq 0$, by Krull's intersection theorem, and the resulting factor h is unique because $g/1$ is regular. In light of Krull's principal ideal theorem, when $V(h, g/1)$ has codimension at least two in $D_+(g)$, it has codimension exactly two. This condition depends only on the radical ideal $\sqrt{(h, g/1)}$, a remark which usually substantially simplifies the computations.

Proposition 3.4. *Suppose that the ideal $\mathfrak{a} \subset R$ is generated by a regular sequence $g_1, \ldots, g_d \in R$. If the scheme $S := \operatorname{Spec}(R)$ satisfies Serre's condition (S_m), or is locally of complete intersection, the same holds for the blowing-up $\operatorname{Bl}_{\mathfrak{a}}(R)$.*

Proof. The canonical module surjection $R^{\oplus d} \to \mathfrak{a}$ coming from the regular sequence yields a closed embedding $\operatorname{Bl}_{\mathfrak{a}}(R) \subset \mathbb{P}^{d-1}_R$. Consider the short exact sequence

$$0 \to \mathscr{F} \to \mathscr{O}_P^{\oplus d} \xrightarrow{(g_i T)} \mathscr{O}_P(1) \to 0$$

of locally free sheaves on $P := \mathbb{P}^r_R$. The kernel has $\operatorname{rank}(\mathscr{F}) = d - 1$. Let $\mathscr{F} \to \mathscr{O}_P$ be the composition of the inclusion $\mathscr{F} \subset \mathscr{O}_P^{\otimes d}$ followed by $\mathscr{O}_P^{\oplus d} \xrightarrow{(g_i)} \mathscr{O}_P$. According to [SGA 6 1971, Exposé VII, Proposition 1.8], the image is the quasicoherent ideal corresponding to the closed subscheme $X := \operatorname{Bl}_{\mathfrak{a}}(R)$. Moreover, for each point $x \in X$, the image of any basis in $\mathscr{F}_x$ in the local ring $\mathscr{O}_{P,x}$ is a regular sequence contained in the maximal ideal $\mathfrak{m}_x$. More explicitly, we have

$$R[\mathfrak{a}T]_{(Tg_j)} = R[S_1, \ldots, S_d]/(S_1 g_j - g_1, \ldots, S_d g_j - g_d), \tag{3-1}$$

where the identification is given by $S_i = g_i T / g_j T$, and the generators in the above ideal form a regular sequence in the polynomial ring. This result is due to Micali [1964, Theorem 1]. It follows that the scheme $\operatorname{Bl}_{\mathfrak{a}}(R)$ is locally of complete intersection if this holds for the ring R.

Note that the relation $S_j g_j - g_j = 0$ is equivalent to $S_j = 1$, because g_j is regular. In other words, in (3-1) one may simply omit the indeterminate S_j. Also note that if R is integral, so is the Rees ring, and we may regard (3-1) as the R-subalgebra in $\operatorname{Frac}(R)$ generated by the fractions $g_1/g_j, \ldots, g_d/g_j$.

Fix a point $x \in X$ and consider the local ring $A := \mathscr{O}_{X,x}$. It remains to show that $\operatorname{depth}(A) \geq m$ or $\operatorname{depth}(A) = \dim(A) < m$. For this we may assume that $S = \operatorname{Spec}(R)$ is local, and that x lies over the closed point $s \in S$. Set $c := d-1$. The local ring $A' := \mathscr{O}_{P,x}$ has $\dim(A') = \dim(R) + c$ and $\operatorname{depth}(A') = \operatorname{depth}(R) + c$. Moreover, the residue class ring A has $\dim(A) = \dim(A') - c$ and $\operatorname{depth}(A) = \operatorname{depth}(A') - c$, the former by Krull's principal ideal theorem, the latter by [EGA IV$_1$ 1964, Chapter 0, Proposition 16.4.6]. The assertion on the Serre condition is immediate. $\square$

3.5. Let us return now to the wild quotient singularities recalled in Section 0.2. Let $R := k[[x, y, z]]$ be a formal power series ring over a field k of characteristic $p > 0$, and consider the element

$$f := z^p - (\mu ab)^{p-1} z - a^p y + b^p x.$$

Here $a, b \in k[[x, y]]$ is a system of parameters, and $\mu \in k[[x, y]]$. Let $B := R/(f)$.

Let $\mathfrak{a} := (a, b, z) \subset R$. We call $Z := \mathrm{Bl}_{\mathfrak{a}B}(B) \to \operatorname{Spec} B$ the *initial blowing-up*. In Theorem 7.1 and Theorem 9.2, we will later compute a complete resolution $X \to Z \to \operatorname{Spec} B$ of this initial blowing-up in two special cases. Recall that the exceptional divisor $E \subset Z$ is given by the sheaf of ideals $\mathscr{O}_Z(1) \subset \mathscr{O}_Z$. Our next proposition computes the term $(E \cdot E_{\mathrm{red}})_Z$, needed for instance when applying Proposition 2.3.

Proposition 3.6. *Keep the assumptions of Section 3.5. Then the following hold*:

(i) *The reduction E_{red} is isomorphic to the projective line $\mathbb{P}^1_k$.*

(ii) *The z-chart on Z is disjoint from the exceptional divisor, and thus is regular.*

(iii) *The scheme Z is locally of complete intersection.*

(iv) *We have $(E \cdot E_{\mathrm{red}})_Z = -1$.*

(v) *The local ring $\mathscr{O}_{E,\eta}$ at the generic point η of E has length $p \cdot \dim_k k[[x, y]]/(a, b)$.*

Proof. The blowing-up $\mathrm{Bl}_{\mathfrak{a}}(R)$ is covered by the a-chart, the b-chart and the z-chart. We start by examining the a-chart, which is the spectrum of the ring

$$R[\mathfrak{a}T]_{(aT)} = R[b/a, z/a]/(b/a \cdot a - b, z/a \cdot a - z).$$

Consider the factorization $f = a^p h$ with

$$h := \left(\frac{z}{a}\right)^p - \mu^{p-1} a^{p-1} \left(\frac{b}{a}\right)^{p-1} \left(\frac{z}{a}\right) - y + \left(\frac{b}{a}\right)^p x.$$

The radical J of the ideal generated by h and a in $R[\mathfrak{a}T]_{(aT)}$ clearly contains b. It thus also contains x and y, because a, b is a system of parameters in $k[[x, y]]$. Hence, J also contains z/a and z. It follows that the subscheme $V(h, a)$ of the a-chart is one-dimensional. According to Proposition 3.1, the scheme $\mathrm{Bl}_{\mathfrak{a}B}(B)$ coincides on the a-chart with the effective Cartier divisor defined by the equation $h = 0$. The exceptional divisor is given by the additional equation $a = 0$, and thus equals $\operatorname{Spec} A$, where A is the quotient of $k[[x, y, z]][b/a, z/a]$ modulo the ideal generated by a, b, z, and $(z/a)^p - y + (b/a)^p x$. Let $Q := (x, y, z/a) \subset A$. Since the classes of $x, y, z/a$ are nilpotent, and since the quotient A/Q is isomorphic to the domain $k[b/a]$, we find that Q is the minimal prime ideal of A.

One easily sees that the z-chart on $\mathrm{Bl}_{\mathfrak{a}}(R)$ is disjoint from the exceptional divisor. The situation for the b-chart is similar to the a-chart, and it follows that $\mathrm{Bl}_{\mathfrak{a}B}(B)$ is locally of complete intersection. Moreover, the reduced exceptional divisor $E_{\mathrm{red}} = \operatorname{Spec} k[b/a] \cup \operatorname{Spec} k[a/b]$ is a copy of $\mathbb{P}^1_k$.

The restriction to E_{red} of the invertible sheaf $\mathscr{O}_Z(1) = \mathscr{O}_Z(-E)$ is generated by the elements $aT/1$ and $bT/1$ on the two charts, respectively. Viewing $a/b \in k[a/b, b/a]^\times$ as a cocycle, one deduces that $\mathscr{O}_Z(1)$ has degree 1 on E_{red}, so that $(E \cdot E_{\mathrm{red}})_Z = -1$.

It remains to compute the length of $\mathscr{O}_{E,\eta}$. The coordinate ring of the exceptional divisor E on the a-chart is given by

$$R[b/a, z/a]/(b/a \cdot a - b, z/a \cdot a - z, h, a).$$

Clearly, the ideal on the right is also generated by b, z, h, a. In turn, the above ring is isomorphic to $k[x, y, b/a, z/a]/(a, b, h)$. Regard the latter as $\Lambda[z/a]/(h)$, where Λ is the polynomial ring in the indeterminate b/a over the local Artin ring $k[x, y]/(a, b)$. The ring extension $\Lambda \subset \Lambda[z/a]/(h)$ is finite and free, because h is a monic in z/a. All coefficients of h except the leading one are nilpotent in Λ, consequently z/a becomes nilpotent modulo h. It follows that $\Lambda \subset \Lambda[z/a]/(h)$ induces bijections on all residue fields. Clearly, the minimal prime $\mathfrak{p} \subset \Lambda$ is generated by x and y. In turn, the local Artin ring $\Lambda_{\mathfrak{p}}$ has length $\dim_k k[x, y]/(a, b)$, whereas the local Artin ring $\mathscr{O}_{E,\eta} = \Lambda_{\mathfrak{p}}[z/a]/(h)$ has length $\deg(h) \cdot \operatorname{length}(\Lambda_{\mathfrak{p}}) = p \cdot \dim_k k[x, y]/(a, b)$. $\square$

Remark 3.7. Keep the notation recalled in Section 3.5. Let $\mu \in k[[x, y]]$ and assume that it is a unit, or that it is nonzero and coprime to both a and b. The ring $B = k[[x, y, z]]/(f)$ can be identified with the ring of invariants A^G for an action of the group $G := \mathbb{Z}/p\mathbb{Z}$ on the ring $A := k[[u, v]]$, as recalled in Section 0.2, where the generator acts via $u \mapsto u + \mu a$ and $v \mapsto v + \mu b$. Under this identification, the element z corresponds to $ub - va$. We note below that the initial blowing-up $\mathrm{Bl}_{\mathfrak{a}B}(B) \to \operatorname{Spec}(B)$ considered in Proposition 3.6 is canonically associated to the action.

Indeed, the *fixed scheme* of the action is by definition the largest closed subscheme of $\operatorname{Spec} A$ on which the action is trivial, and we find that for the above action it corresponds to the ideal $I := (\sigma(u) - u, \sigma(v) - v) = (\mu a, \mu b)$ in A. Under the above identification $B = A^G$ we have $z = ub - va$, and therefore $\mu z \in I$. We find that $(\mu a, \mu b, \mu z) \subseteq I \cap B$. The reverse inclusion also holds since A is flat over $k[[x, y]]$ (same proof as in [Schröer 2009, Lemma 1.5], when $p = 2$ and a similar choice of initial blow-up was also used). Thus the ideals $I \cap B$ and $\mathfrak{a}B = (a, b, z)$ coincide up to the factor μ and, hence, the total spaces of the resulting blowing-ups coincide.

4. Some weighted homogeneous singularities

Let k be an algebraically closed field of characteristic exponent $p \geq 1$. The goal of this section is to describe a resolution of the singularity at the origin on the hypersurface given by the equation

$$W^q - U^a V^b (V^d - U^c) = 0$$

when the integers $p, q, a, b, c, d \geq 1$ are subject to certain mild restrictions. This is achieved in Theorem 4.4. Note that this singularity is not necessarily isolated. The above polynomial is weighted homogeneous, and resolutions of such singularities were already studied by Orlik and Wagreich [1971a; 1971b; 1977], exploiting $\mathbb{G}_m$-actions corresponding to the weights. The former two papers rely on transcendental methods, and the latter mainly treats the case of isolated singularities. Our method is completely algebraic, and relies on the theory of toric varieties and Hirzebruch–Jung singularities.

To compute a resolution of our surface singularity, we first make an initial blow-up that separates the irreducible components of the plane curve $U^a V^b (V^d - U^c) = 0$. We then pass to certain nicer subrings of the charts, and identify their formal completions with suitable monoid rings. This necessitates taking roots of power series along the way, requiring some restrictions on the integers p, q, a, b, c, d as in Section 4.3.

Let us start with a brief review of the theory of *Hirzebruch–Jung singularities*. Suppose that $t, r \geq 1$ and $s \geq 0$ are integers such that $\rho := \gcd(t, r, s)$ is prime to p. Consider the ring

$$R := k[U, V, W]/(W^t - U^r V^s).$$

We have a factorization $W^t - U^r V^s = \prod (W^{t/\rho} - \zeta U^{r/\rho} V^{s/\rho})$, where the product runs over the ρ-th roots of unity ζ in k. The corresponding minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_\rho \subset R$ define a partial normalization $R \subset \prod R/\mathfrak{p}_i$, and it usually suffices to understand the rings $R/\mathfrak{p}_i$.

4.1. Assume from now on that $\rho = 1$, so that R is an integral domain. Let R' be its normalization. To describe the resolution of the singularity of Spec R' at the maximal ideal (U, V, W) when Spec R' is singular at this point, it is standard to first express R' as the normalization of a different domain R_0, as we now recall. Given the triple (t, r, s), we describe below its *fraction type*, which can be 0, and when the fraction type is not 0, it is equal to $(t' - s')/t'$, where $(t', 1, s')$ is the unique triple with $0 < s' < t'$ and s' coprime to t' such that R' can be identified with the normalization of the ring $R_0 := k[u, v, w]/(w^{t'} - uv^{s'})$.

Let D_U and D_V denote the preimages in Spec R' of the closed subsets of Spec R defined by $U = W = 0$ and $V = W = 0$, respectively. *The identification of R' as the normalization of R_0 is such that the closed subsets D_U and D_V on* Spec R' *are again equal to the preimages under the new normalization map* Spec $R' \to$ Spec R_0 *of the closed subsets of* Spec R_0 *defined by $u = w = 0$ and $v = w = 0$, respectively.* We leave it to the reader to check this claim, for instance using the explicit description of R_0 recalled below.

Write $r = r_0 + ct$ and $s = s_0 + dt$ for some integers $r_0, s_0, c, d \geq 0$ with $r_0, s_0 < t$. Then the fraction $W/(U^c V^d)$ is integral over R since it satisfies the equation $(W/(U^c V^d))^t = U^{r_0} V^{s_0}$. We can thus replace R by $R[W/(U^c V^d)]$. In particular, if either r or s is divisible by t, then R' is regular above (U, V, W). We define in this case the *fraction type* of R or R' to be 0. If R' is not regular, then upon replacing R with $R[W/(U^c V^d)]$ we may assume that $0 < r, s < t$.

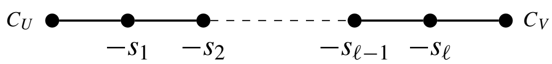
Let $h := \gcd(t, r)$ and $h' := \gcd(t, s)$. Since $\gcd(t, r, s) = 1$, we find that $\gcd(r, h') = \gcd(s, h) = 1$. Thus we can write $ar = 1 + bh'$ and $cs = 1 + dh$ for some nonnegative integers a, b, c, d. Let $U_1 := W^{at/h'}/(U^{(ar-1)/h'} V^{as/h'})$ and $V_1 := W^{ct/h}/(U^{cr/h} V^{(cs-1)/h})$. We find that $U_1^{h'} = U$ and $V_1^h = V$. In the integral extension $R[U_1, V_1]$, we find that $W^{t/(hh')} = U_1^{r/h} V_1^{s/h'}$. If r divides t/h', or if s divides t/h, we find that R' is regular above (U, V, W), and we define again in this case the *fraction type* of R or R' to be 0.

Assume then that R' is not regular. Replacing R with $R[U_1, V_1]$, we may assume now that $h = h' = 1$, and upon replacing R by a larger integral extension if necessary, we can also assume that $0 < r, s < t$. In this process, t has been replaced by t/hh'.

There exists a unique integer e with $0 < e < t$ and $er = s + ct$ for some integer c. Since $s < t$ by assumption, we find that $c \geq 0$. Consider the ring $R_1 := k[U, V, Z]/(Z^t - U^r V^{s+ct})$. We find that this ring has two natural integral extensions. Indeed, $R_1[Z/V^c]$ is isomorphic to the ring R. Writing $r\rho = 1 + ft$ for some integers $\rho, f \geq 0$, we find that $w := Z^\rho/(UV^e)^f$ is such that $w^r = Z$ and $w^t = UV^e$. Thus $R_1[w]$ is integral over R_1 and isomorphic to $R_0 := k[U, V, W]/(W^t - UV^e)$. We define in this case the *fraction type* of R or R' to be $(t-e)/t$, with $0 < (t-e)/t < 1$ and $\gcd(e, t) = 1$. This concludes our description of how to compute the fraction type of the ring R.

Given a resolution of singularities $X \to \operatorname{Spec} R'$, we write $C \subset X$ for the exceptional curve, and C_U and C_V for the strict transforms in X of the Weil divisors D_U and D_V on $\operatorname{Spec} R'$, respectively. We endow all these closed subsets with the induced reduced structure of scheme. The following theorem is well-known (see, e.g., the pictures in [Kempf et al. 1973, page 37] or [Conrad et al. 2003, Theorem 2.4.1]), but we did not find a suitable reference in the literature which also proved the statement regarding the divisors C_U and C_V. We include a sketch of proof below, with references, for the convenience of the reader.

Theorem 4.2. *Let s and t be coprime integers with $0 < s < t$. Let $R := k[U, V, W]/(W^t - UV^s)$ and denote by R' its normalization. There is a resolution of singularities $X \to \operatorname{Spec} R'$ such that $C_U \cup C \cup C_V$ is a divisor with simple normal crossings having the following dual graph*:



The integer $\ell \geq 1$ and the self-intersection numbers $-s_i$ are computed from the continued fraction expansion $t/(t-s) = [s_1, \dots, s_\ell]$ as described in (1-1). Moreover, the irreducible components of C are isomorphic to $\mathbb{P}^1_k$.

Proof. The proof relies on the theory of toric varieties, and we refer the reader to the monographs [Cox et al. 2011; Danilov 1978; Kempf et al. 1973] for the general theory. The book [Cox et al. 2011] assumes from the onset that the characteristic of k is 0, but the proofs of the results quoted below are valid in all characteristics and can be applied to our purposes. We identify $Z := \operatorname{Spec} R$ as an explicit (nonnormal) toric variety, and use the general theory of toric varieties to describe the normalization $Y \to Z$ and the toric resolution $X_\Sigma \to Y$ attached to an explicit fan Σ.

Consider the lattices $N := \mathbb{Z}^2$ and $M := \operatorname{Hom}(N, \mathbb{Z})$. Write $e_1, e_2 \in N$ for the standard basis of N, and $e_1^*, e_2^* \in M$ for the dual basis. Let $\sigma \subset N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ be the closed convex cone generated by the vectors e_2 and $te_1 - (t-s)e_2$. The dual cone $\sigma^\vee \subset M_\mathbb{R}$ is generated by $\alpha := (t-s)e_1^* + te_2^*$ and $\beta := e_1^*$. Let

$\gamma := e_1^* + e_2^*$, and let $S \subset M$ be the submonoid generated by α, β, γ. We have the relation $t\gamma = \alpha + s\beta$, and can identify $k[U, V, W]/(W^t - UV^s)$ with the monoid ring $k[S]$ via $U \mapsto \alpha$, $V \mapsto \beta$, and $W \mapsto \gamma$.

Let $S' := \sigma^\vee \cap M$. Clearly, the abelian group M is generated by β and γ. It follows that $\alpha \in M$ and, hence, $S \subseteq S'$. Since S' is always saturated, S' is equal to the saturation of the monoid S. It follows that the normal toric variety Y attached to N and σ, namely $Y := \operatorname{Spec} k[\sigma^\vee \cap M]$, is the normalization of the nonnormal toric variety $Z := \operatorname{Spec} k[S]$.

The cone σ is in normal form, and since $t > s > 0$, [Cox et al. 2011, Theorem 10.2.3], provides an explicit description of a refinement fan Σ of σ such that the induced morphism $X_\Sigma \to Y$ is a toric resolution of singularities. Using the Hirzebruch–Jung continued fraction $[s_1, \dots, s_\ell]$ of $t/(t-s)$, one constructs a sequence of vectors $u_0 := e_2, u_1, \dots, u_\ell, u_{\ell+1} := te_1 - (t-s)e_2$ such that $\sigma = \bigcup_{i=1}^{\ell+1} \sigma_i$ with σ_i the cone generated by u_{i-1} and u_i. The fan Σ consists of the cones σ_i and their faces.

Using the orbit-cone correspondence [Cox et al. 2011, Theorem 3.2.6], we find that the ray generated by u_i, $i = 0, \dots, \ell+1$, corresponds to a curve C_i on X_Σ. Since Σ is a simplicial fan, the intersection products $(C_i \cdot C_j)_{X_\Sigma}$ with $0 \le i \ne j \le \ell+1$ can be computed as in [loc. cit., Corollary 6.4.3], and are found to equal 1. The self-intersections $(C_i \cdot C_i)_{X_\Sigma}$ for $i = 1, \dots, \ell$ are computed to equal $-s_i$ using Theorem 10.2.5, along with Theorem 10.4.4, of [loc. cit.].

The curve $C_1 \cup \dots \cup C_\ell$ is the exceptional divisor of the toric desingularization $X_\Sigma \to Y$. Using the orbit-cone correspondence for the surface Y, we let D and D' denote the curves on Y corresponding to the rays in the cone σ generated e_2 and $te_1 - (t-s)e_2$, respectively. The natural properties of the map $X_\Sigma \to Y$ implies that D is the image of C_0, and D' is the image of $C_{\ell+1}$. The proof is concluded by the fact that D is the reduced preimage of the Weil divisor $U = W = 0$ on Z, and that similarly, D' is the reduced preimage of the Weil divisor $V = W = 0$ on Z. □

4.3. Let $q, a, b, c, d \ge 1$ be integers. Set

$$m := ad + bc + cd \quad \text{and} \quad g := \gcd(c, d).$$

Noting that m/g is an integer, we further set

$$h := \gcd(q, m/g), \quad h_a := \gcd(q, m/g, a), \quad \text{and} \quad h_b := \gcd(q, m/g, b).$$

In our main result below on the resolution of the hypersurface singularity $W^q - U^a V^b (V^d - U^c) = 0$, we assume that

$$\gcd(a, c/g) = \gcd(b, d/g) = 1 \quad \text{and} \quad \gcd(p, hg) = 1. \tag{4-1}$$

Note that the latter condition automatically holds when $p = 1$. The reader will easily check that the condition $\gcd(a, c/g) = 1$ is equivalent to the condition $\gcd(m/g, c/g) = 1$. Similarly, $\gcd(b, d/g) = 1$ if and only if $\gcd(m/g, d/g) = 1$.

Denote by $\alpha, \beta, \gamma \in \mathbb{Q}_{<1}$ the fraction types (see Section 4.1) of the normal Hirzebruch–Jung singularities associated with the triples (t, r, s) given by

$$\left(\frac{qc}{gh_a}, \frac{m}{gh_a}, \frac{a}{h_a}\right), \quad \left(\frac{qd}{gh_b}, \frac{m}{gh_b}, \frac{b}{h_b}\right), \quad \text{and} \quad \left(q, \frac{m}{g}, 1\right),$$

respectively. Finally, set

$$s_0 := \frac{h^2 g^2}{qcd} + h_a\alpha + h_b\beta + g\gamma. \tag{4-2}$$

We are now ready to state the main result of this section. Three complements to Theorem 4.4 are given in Propositions 4.7–4.9.

Theorem 4.4. *Set $B := k[U, V, W]/(W^q - U^a V^b(V^d - U^c))$, and assume that the conditions (4-1) hold. With the above notation, we have the following*:

(i) *The fraction $s_0 > 0$ is an integer.*

(ii) *The hypersurface singularity has a resolution of singularities $X \to \operatorname{Spec}(B)$ where, using the notation in Section 1.2, the exceptional divisor $C \subset X$ has star-shaped dual graph*

$$\Gamma = \Gamma(s_0 \mid \underbrace{\alpha^{-1}, \ldots, \alpha^{-1}}_{h_a}, \underbrace{\beta^{-1}, \ldots, \beta^{-1}}_{h_b}, \underbrace{\gamma^{-1}, \ldots, \gamma^{-1}}_{g})$$

when $\alpha, \beta, \gamma > 0$. When α (resp. β, resp. γ) equals 0 (e.g., when q divides m/g), the graph Γ is as above except that the corresponding h_a (resp. h_b, resp. g) chains are removed.

(iii) *The curve C has simple normal crossings. All irreducible components of C are copies of $\mathbb{P}^1_k$, except possibly for the central node. When $h = 1$, the central node is also isomorphic to $\mathbb{P}^1_k$.*

Proof. Since our ground field k is algebraically closed, we can rewrite the defining polynomial for our hypersurface singularity as

$$f = W^q - U^a V^b \prod_\zeta (V^{d/g} - \zeta U^{c/g}),$$

where the product runs over the g-th roots of unity $\zeta \in k$. Assumption (4-1) ensures that we have exactly $g \geq 1$ distinct factors in the product.

To construct the desired resolution of singularities $X \to \operatorname{Spec}(B)$, we first make an initial blowing-up $Z := \operatorname{Bl}_{\mathfrak{a}B}(B) \to \operatorname{Spec}(B)$, for the ideal $\mathfrak{a} := (U^{c/g}, V^{d/g})$ in the polynomial ring $R := k[U, V, W]$. The ambient blowing-up $\operatorname{Bl}_{\mathfrak{a}}(R)$ has two charts, the $U^{c/g}$-chart and the $V^{d/g}$-chart. The former is given by four generators $U, V, W, V^{d/g}/U^{c/g}$ subject to the single relation

$$\left(\frac{V^{d/g}}{U^{c/g}}\right) \cdot U^{c/g} = V^{d/g}, \tag{4-3}$$

as recalled in Proposition 3.4. On this chart we rewrite the defining polynomial as

$$f = W^q - U^{a+c} V^b \cdot \prod_\zeta (V^{d/g}/U^{c/g} - \zeta). \tag{4-4}$$

Clearly, the radical of the ideal generated by f and $U^{c/g}$ contains U, V, and W. Hence, its zero-locus is one-dimensional, and according to Proposition 3.1 the blowing-up $Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ on the $U^{c/g}$-chart of $\mathrm{Bl}_{\mathfrak{a}}(R)$ is the effective Cartier divisor with equation $f = 0$. In other words, write A' for the coordinate ring of the blowing-up $Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ on the $U^{c/g}$-chart. Then this ring is generated by four indeterminates $U, V, W, V^{d/g}/U^{c/g}$ subject to the two relations (4-3) and $f = 0$ with f as in (4-4).

4.5. The exceptional divisor $E \subset Z$ is given by $f = U^{c/g} = 0$ on this chart. The reduction E_{red} is defined by $U = V = W = 0$, and $V^{d/g}/U^{c/g}$ can be regarded as a coordinate function. The situation on the $V^{d/g}$-chart is symmetric, and we conclude that $E_{\mathrm{red}} = \mathbb{P}^1_k$ is a projective line. This description also yields the intersection number: Recall that the ambient $\mathrm{Bl}_{\mathfrak{a}}(R)$ is the homogeneous spectrum of the Rees ring $R[\mathfrak{a}T]$, so the invertible sheaf $\mathscr{O}_Z(1)$ is generated by $TU^{c/g}$ and $TV^{d/g}$ on our two charts. In turn, the restriction to $E_{\mathrm{red}} = \mathbb{P}^1_k$ is given by the cocycle $U^{c/g}/V^{d/g}$, and it follows that $(E \cdot E_{\mathrm{red}})_Z = -1$.

4.6. Let us note here also that the multiplicity of E is qcd/g^2. This can be seen as follows. On the $U^{c/g}$-chart, the scheme E_{red} is defined by the ideal $Q := (U, V, W)$. Thus the multiplicity of E can be computed as the length of the ring $(A'/(U^{c/g}))_Q$. It is easy to verify that the ring $A'/(U^{c/g})$ is k-isomorphic to the ring $(k[U, V, W]/(U^{c/g}, V^{d/g}, W^q))[V^{d/g}/U^{c/g}]$, and the claim follows.

The ring A' is locally of complete intersection, but usually fails to be normal. Let $\nu : Y \to Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ denote the normalization morphism. To understand the normalization and minimal resolution of the singularities of the chart $\operatorname{Spec} A'$ of Z, we pass to a subring A of A' with only three generators and one relation that has the same normalization as A'. It turns out that on formal completions, the resolution of singularities of A is given by the theory of toric surface (i.e., Hirzebruch–Jung) singularities. This formal passage to toric varieties requires the existence of certain roots of formal power series. When $p > 1$, their existence follows from Hensel's lemma together with the conditions (4-1), which imply that $\gcd(m/g, c/g)$ and $\gcd(m/g, d/g)$ are coprime to p.

We proceed as follows: Let A be the k-subalgebra of A' generated by the three elements U, W, and $V^{d/g}/U^{c/g}$. The ring extension $A \subset A'$ is finite, because $A' = A[V]$ and the generator V satisfies the integral equation $V^{d/g} - U^{c/g}(V^{d/g}/U^{c/g}) = 0$ in (4-3). Clearly, $V^{d/g} \in A$, and the relation (4-4) shows that $V^b \in \operatorname{Frac}(A)$. Since we assume that $\gcd(b, d/g) = 1$ in (4-1), we find that V can be written as rational function in V^b and $V^{d/g}$ and, hence, $V \in \operatorname{Frac}(A)$. It follows that the rings A and A' have the same integral closure in $\operatorname{Frac}(A)$. The reduced exceptional divisor on $\operatorname{Spec}(A')$ is defined by the ideal (U, V, W), and the restriction of $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ to it is a closed embedding, because $V^{d/g}/U^{c/g} \in A$ and thus the map $A \to A'/(U, V, W) = k[V^{d/g}/U^{c/g}]$ is surjective.

It turns out that the subring A has a much nicer description than A', in particular when passing to formal completions along the exceptional divisor. Recall that $m := ad + cd + bc$. Taking the d/g-power of (4-4) and using (4-3) we get a single relation

$$W^{qd/g} = U^{m/g}\left(\frac{V^{d/g}}{U^{c/g}}\right)^b \cdot \prod_{\zeta}(V^{d/g}/U^{c/g} - \zeta)^{d/g}. \tag{4-5}$$

Since b and d/g are coprime by assumption (4-1), we find that $w^{qd/g} = u^{m/g} z^b \prod_\zeta (z-\zeta)^{d/g}$ is an irreducible polynomial in $k[u, w, z]$. By abuse of notation, we will also say that the *equation* (4-5) *is irreducible*. Using Krull's principal ideal theorem, we conclude that the algebra A is generated by $U, W, V^{d/g}/U^{c/g}$ subject to the single relation (4-5).

To understand the normalization of A, we pass to formal completions $\widehat{A_\mathfrak{m}}$ with respect to maximal ideals $\mathfrak{m}$ of the form $(U, W, V^{d/g}/U^{c/g} - \xi)$ for various scalars $\xi \in k$. Note that these maximal ideals correspond to points on the exceptional divisor.

Let us start with the simplest case where ξ is neither zero nor a g-th root of unity; here it turns out that the normalization of $\widehat{A_\mathfrak{m}}$ is regular. Indeed, the relation (4-5) now takes the form

$$W^{qd/g} = U^{m/g} \cdot \delta \tag{4-6}$$

for some unit $\delta \in \widehat{A_\mathfrak{m}}$. To proceed, we first verify that $\gcd(qd/g, m/g, p) = 1$. This is clear when $p = 1$, so let us assume that $p \geq 2$ is prime. Suppose that p divides both qd/g and m/g. Since p does not divide $h = \gcd(q, m/g)$ by hypothesis, we have $p \nmid q$ and, hence, $p \mid d/g$, contradicting $\gcd(d/g, m/g) = 1$, which we also assume in (4-1).

We conclude that there exist positive integers r and s such that $\ell := r(m/g) - s(qd/g)$ is coprime to $p \geq 1$. With Hensel's lemma we find roots $\delta_1 := \delta^{r/\ell}$ and $\delta_2 := \delta^{s/\ell}$ in $\widehat{A_\mathfrak{m}}$, and obtain a factorization $\delta = \delta_1^{m/g}/\delta_2^{qd/g}$. It follows that $\widehat{A_\mathfrak{m}}$ is isomorphic to the complete local ring described by the same three generators, but with a modified relation (4-6) in which $\delta = 1$. This shows that $\widehat{A_\mathfrak{m}}$ is isomorphic to a complete local ring for a point on the product of a plane curve with the affine line. Consequently, the normalization is indeed regular. Note that the plane curve is usually reducible, and the number of irreducible components is our integer $h = \gcd(q, m/g) = \gcd(qd/g, m/g)$.

Next, assume that $\xi = \zeta$ is one of the g-th root of unity. Rewrite (4-5) as

$$W^{qd/g} = U^{m/g} \left(\frac{V^{d/g}}{U^{c/g}} - \zeta \right)^{d/g} \cdot \delta \tag{4-7}$$

for some unit $\delta \in \widehat{A_\mathfrak{m}}$. As in the preceding paragraph, one reduces the situation to $\delta = 1$. Since we noted in Section 4.3 that $\gcd(d/g, m/g) = 1$, the above relation is then irreducible.

Consider the triple $(t, r, s) = (qd/g, m/g, d/g)$. We identify $\widehat{A_\mathfrak{m}}$ with $k[\![u, v, w]\!]/(w^t - u^r v^s)$. Using the results reviewed in Section 4.1 and Theorem 4.2 regarding the desingularization of $\operatorname{Spec} k[u, v, w]/(w^t - u^r v^s)$, we find that the singularity on $\widehat{A_\mathfrak{m}}$ is a Hirzebruch–Jung singularity of fraction type γ.

Finally, assume that $\xi = 0$. Our relation becomes

$$W^{qd/g} = U^{m/g} \left(\frac{V^{d/g}}{U^{c/g}} \right)^{b} \cdot \delta$$

for some unit $\delta \in \widehat{A_\mathfrak{m}}$, and again we reduce to the situation $\delta = 1$. The above equation is usually not irreducible, and the number of irreducible factors is our integer $h_b = \gcd(q, m/g, b)$, which also equals

$\gcd(qd/g, m/g, b)$ since we noted in Section 4.3 that $\gcd(d/g, m/g) = 1$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_{h_b} \subset \widehat{A_\mathfrak{m}}$ be the resulting minimal prime ideals.

Consider the triple $(t, r, s) = (qd/(gh_b), m/(gh_b), b/h_b)$. We identify $\widehat{A_\mathfrak{m}}/\mathfrak{p}_i$ with $k[\![u, v, w]\!]/(w^t - u^r v^s)$. Using the results reviewed in Section 4.1 and Theorem 4.2 regarding the desingularization of $\operatorname{Spec} k[u, v, w]/(w^t - u^r v^s)$, we find that the singularity on $\widehat{A_\mathfrak{m}}/\mathfrak{p}_i$ has the resolution of a Hirzebruch–Jung singularity of fraction type β. The number of such singularities on the normalization of $\widehat{A_\mathfrak{m}}$ is $h_b \geq 1$.

The situation on the $V^{d/g}$-chart is symmetric, where $h_a \geq 1$ Hirzebruch–Jung singularities of fraction type α appear. Summing up, we have described the singularities appearing on the normalization $\nu : Y \to Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$.

Recall from Section 4.5 that the exceptional divisor $E \subset Z$ has reduction $E_{\mathrm{red}} = \mathbb{P}^1_k$, with coordinate rings $k[V^{d/g}/U^{c/g}]$ and $k[U^{c/g}/V^{d/g}]$. Write $D := \nu^{-1}(E)$ for the preimage of the exceptional divisor under the map ν. We now analyze the induced morphism $D_{\mathrm{red}} \to E_{\mathrm{red}}$. This morphism is flat, because E_{red} is regular. The formal description of the normalization $\nu : Y \to Z$ via inclusions $k[\![S]\!] \subset k[\![S']\!]$ of monoid rings shows that D_{red} is regular. Equation (4-6) implies that

$$\deg(D_{\mathrm{red}}/E_{\mathrm{red}}) = \gcd(q, m/g) = h. \tag{4-8}$$

In a similar way, (4-7) tells us that $D_{\mathrm{red}} \to E_{\mathrm{red}}$ is completely ramified over the points where $V^{d/g}/U^{c/g} = \xi$ is a g-th root of unity. Hence, the curve D_{red} is connected. Since it is also regular, it is in fact irreducible. We can then apply Proposition 2.3 along with Sections 4.5 and 4.6 and obtain that

$$(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = \frac{h^2}{(qcd/g^2)} (E \cdot E_{\mathrm{red}})_Z = -h^2 g^2/qcd.$$

Let $X \to Y$ be the resolution of singularities obtained by resolving the Hirzebruch–Jung singularities of fraction types α, β and γ occurring on Y. The resulting dual graph Γ is star-shaped, with the central node corresponding to the strict transform $C_0 \subset X$ of $D_{\mathrm{red}} \subset Y$. When $\gamma > 0$, there are g terminal chains obtained from the continued fraction development of $1/\gamma = [s_1, \dots, s_\ell]$. Using the identification of $\widehat{A_\mathfrak{m}}$ with the completion of $k[u, v, w]/(w^{qd/g} - u^{m/g} v^{d/g})$ at (u, v, w) discussed above, as well as Theorem 4.2 and the identifications reviewed in Section 4.1, one sees that the vertex of the terminal chain adjacent to the central node has self-intersection $-s_1$. The situation for the other Hirzebruch–Jung singularities is similar.

It is now an easy matter to compute the self-intersection $(C_0 \cdot C_0)_X$ using Proposition 2.2, which asserts that $(C_0 \cdot C_0)_X = (D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y - \sum_i \delta_i$. There are h_a correcting terms α, h_b correcting terms β, and g correcting terms γ (see just before Proposition 2.2 for the correcting term of a chain). Hence, $-(C_0 \cdot C_0)_X = s_0$ (see (4-2)), as desired. Since $(C_0 \cdot C_0)_X$ is the self-intersection of a curve on a regular surface, we find that it must be a negative integer, proving (i). To complete the proof of Theorem 4.4 it remains to show in (iii) that the central node C_0 is a rational curve when $h = 1$. This is done using the following proposition. $\square$

Proposition 4.7. *Keep the hypotheses of Theorem 4.4. Let $v_0 \in \Gamma$ be the central node, and let $C_0 \subset X$ be the corresponding curve on the resolution $X \to \operatorname{Spec} B$. We have $h^1(\mathscr{O}_{C_0}) = (g(h-1) + 2 - h_a - h_b)/2$. In particular, when $h = 1$, $h^1(\mathscr{O}_{C_0}) = 0$.*

Proof. Consider the ramified covering $C_0 \to E_{\mathrm{red}} = \mathbb{P}^1_k$ induced from the morphism $X \to Z$. It follows from (4-8) that the degree of this map is h. Assumption (4-1) ensures that this degree is coprime to the characteristic exponent, so that the map is separable. Let us regard the closed points on $\mathbb{P}^1_k$ as elements $\xi \in k \cup \{\infty\}$. The description of the normalization of the rings $\widehat{A_\mathfrak{m}}$ in the preceding proof shows that $C_0 \to \mathbb{P}^1_k$ is totally ramified over each of the g-th roots of unity in k, and therefore the ramification indices are coprime to p. Furthermore, there are h_a points in C_0 over $\xi = 0$ and all these points have the same ramification index h/h_a. Similarly, there are h_b points in C_0 over $\xi = \infty$ with ramification index h/h_b. Applying the Riemann–Hurwitz formula $2h^1(\mathscr{O}_{C_0}) - 2 = h(2h^1(\mathscr{O}_{\mathbb{P}^1_k}) - 2) + \sum_x (e_x - 1)$, we get the desired formula (where e_x denotes the ramification index of the morphism at x). $\square$

Keep the hypotheses of Theorem 4.4. The scheme $\mathrm{Spec}(B)$ contains two copies of the affine line, given by the equations $U = W = 0$ and $V = W = 0$. Write C_U and C_V for their respective strict transforms in X with respect to the resolution $X \to \mathrm{Spec}(B)$. For a later application in Theorem 7.1, we explicitly determine below how these curves intersect the exceptional divisor $C \subset X$ *when* $h = 1$. Under this additional hypothesis, the partial resolution $Y \to \mathrm{Spec}\, B$ contains exactly one Hirzebruch–Jung singularity of fraction type α and one of type β. Let Δ_α and Δ_β be the terminal chains of Γ resulting from resolving these two singularities. Write C_α and C_β for the irreducible components of C corresponding to the terminal vertices of Γ lying on Δ_α and Δ_β, respectively.

Proposition 4.8. *Keep the hypotheses of Theorem 4.4. Assume that $h = 1$. Then the strict transform C_V intersects the exceptional divisor C only in C_β, with intersection number $(C_V \cdot C_\beta)_X = 1$. Likewise, C_U intersects C only in C_α, with $(C_U \cdot C_\alpha)_X = 1$.*

Proof. By symmetry, it suffices to verify the first assertion. Let us first work with the effective Cartier divisor on $\mathrm{Spec}(B)$ given by $V^{d/g} = 0$. Its strict transform $C'_V \subset X$ has the same support as C_V. Using the notation from the proof of Theorem 4.4, we see that its image on $\mathrm{Spec}(A)$ is given by $V^{d/g}/U^{c/d} = 0$. Using Theorem 4.2 one infers that C'_V intersects only C_β, and that its reduction has intersection number $(C_V \cdot C_\beta)_X = 1$. $\square$

Proposition 4.9. *Keep the hypotheses of Theorem 4.4, and suppose furthermore that $p = q$. Set $a_p := 1$ if $p \mid a$, and $a_p := 0$ otherwise. Similarly, set $b_p := 1$ if $p \mid b$, and $b_p := 0$ otherwise. Let N denote the intersection matrix of the resolution of the hypersurface singularity*

$$W^p - U^a V^b (V^d - U^c) = 0$$

described in Theorem 4.4. Then $|\Phi_N| = p^{g+1-a_p-b_p}$, and the group Φ_N is killed by p.

Proof. First note that for $q = p = 1$, the assertion is trivially true, because then our hypersurface singularity is actually regular. So we may assume that $q = p \geq 2$ is a prime number. From our assumptions (4-1), one easily sees that m/g is coprime to p, pc/g and pd/g. In particular, we have $h = h_a = h_b = 1$. The triples (t, r, s) in Section 4.3 specialize to $(pc/g, m/g, a)$, $(pd/g, m/g, b)$, and $(p, m/g, 1)$, respectively. Furthermore, the resulting reduced fractions $\alpha, \beta, \gamma \in \mathbb{Q}$ have as denominators the integers $p^{1-a_p}c/g$,

$p^{1-b_p}d/g$, and p, respectively. According to Theorem 4.4, the graph Γ_N is star-shaped. Thus we may compute the determinant of the intersection matrix with Proposition 1.3 and obtain

$$|\det(N)| = (p^{1-a_p}c/g)(p^{1-b_p}d/g)p^g(s_0 - \alpha - \beta - g\gamma).$$

The last factor is g^2/pcd in light of the formula (4-2) for the self-intersection $-s_0$ of the central node in Theorem 4.4. Thus $|\Phi_N| = |\det(N)| = p^{g+1-a_p-b_p}$.

The group structure of Φ_N can be obtained by computing the Smith normal form of the matrix N, using a row and column reduction of N. Reducing the intersection matrix of each terminal chain as in [Lorenzini 1992, Lemma 2.5], we find that the matrix N is equivalent to a block diagonal matrix with two blocks, a square matrix A of size $(g+3)\times(g+3)$ that we describe below, and an identity matrix:

$$A := \begin{pmatrix} -s_0 & * & * & * & \cdots & * \\ 1 & -p^{1-a_p}c/g & 0 & 0 & & 0 \\ 1 & 0 & -p^{1-b_p}d/g & 0 & & 0 \\ 1 & 0 & 0 & -p & & 0 \\ \vdots & & & & \ddots & \vdots \\ 1 & & & & & -p \end{pmatrix}$$

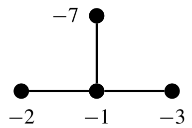
The matrix $A \otimes \mathbb{F}_p$ has $g + a_p + b_p$ rows equal to $(1, 0, \ldots, 0)$, and we see that the rank of A is at most $r := 1 + b_p + a_p + 1$. In turn, the vector space dimension of the cokernel is at least $g+3-r = g+1-a_p-b_p$. It follows that $\Phi_N = \Phi_N \otimes \mathbb{F}_p$. □

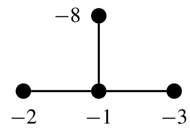
Remark 4.10. The explicit resolution of $W^p - UV(V - U^p) = 0$ is needed in the proof of Theorem 7.1. In this case, the intersection matrix is $N = N(2 \mid p/(p-1),\, p/(p-1),\, p^2/(2p-1))$, with $|\Phi_N| = p^2$. When p is odd, we do not know if this intersection matrix can occur as the intersection matrix of the resolution of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. When $p = 2$, this equation defines the singularity D_6^0 with trivial local fundamental group [Artin 1977]. The singularity D_6^1 is a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity 8.5.

More generally, one might wonder whether every intersection matrix arising in Proposition 4.9 can occur as the intersection matrix of the resolution of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. We discuss the case of $W^p - U^pV^p(V^{pm+1} - U^{pn+1})$ and $W^p - UV(V^{pm-1} - U^{pn-1})$ in Theorem 5.3. We note in Remark 8.5 how the intersection matrix of the resolution of the singularity defined by $W^p - UV(V^{pm} - U^{pn-1}) = 0$ might occur as the intersection matrix of the resolution of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.

Remark 4.11. The resolution $X \to Y \to \operatorname{Spec} B$ provided in Theorem 4.4 is not always minimal. This can be seen already in the case where $q = 1$, in which case $\operatorname{Spec} B$ is regular, but the exceptional divisor C on X is not reduced to a point. The graph Γ consists in this case of a central node of self-intersection -1 with two terminal chains obtained by resolving Hirzebruch–Jung singularities associated with the triples $(c/g, m/g, a)$ and $(d/g, m/g, b)$. The fraction types of these triples are independent of a and b. Indeed, let $\rho, \sigma > 0$ be the unique positive integers such that $\rho(d/g) + \sigma(c/g) = 1 + (c/g)(d/g)$. Then the triple $(c/g, m/g, a)$ reduces to $(c/g, 1, \rho)$, and $(d/g, m/g, b)$ reduces to $(d/g, 1, \sigma)$.

Other examples where the resolution is not minimal can also be obtained when $q > 1$; for instance, when $p = 2$, the singularity $W^2 - U^2V^2(V^7 - U^3) = 0$ (resp. $W^2 - U^2V(V^4 - U^3) = 0$) admits a resolution with smooth rational curves and dual graph drawn on the left below (resp. on the right):





5. Brieskorn singularities

Let k be an algebraically closed field of characteristic exponent $p \geq 1$. Let $q, c, d \geq 2$ be integers, with q coprime to cd. Let

$$B := k[\![x, y, z]\!]/(z^q + x^c + y^d).$$

We study in this section properties of the singularity $\operatorname{Spec} B$. Let $g := \gcd(c, d)$.

Theorem 5.1. *Assume that* $\gcd(p, g) = 1$. *Then* $\operatorname{Spec} B$ *admits a star-shaped resolution of singularities* $X \to \operatorname{Spec} B$ *whose associated intersection matrix is*

$$N = N(s_0 \mid a_1/b_1, a_2/b_2, \underbrace{a_0/b_0, \ldots, a_0/b_0}_{g \text{ entries}}),$$

where N is specified as follows (notation as in Section 1.2). Let

$$a_1 := c/g, \quad a_2 := d/g, \quad \text{and} \quad a_0 := q.$$

Set $\ell_1 := dq/g$, $\ell_2 := cq/g$ and $\ell_0 := cd/g$, and define b_i by $b_i\ell_i \equiv -1 \bmod a_i$ and $0 \leq b_i < a_i$. Finally, set

$$s_0 := g^2/cdq + b_1/a_1 + b_2/a_2 + gb_0/q.$$

In case $a_1 = 1$ (resp. $a_2 = 1$), in which case $b_1 = 0$ (resp. $b_2 = 0$), we remove the term a_1/b_1 (resp. a_2/b_2) from the matrix N.

When $q = p$, the associated discriminant group Φ_N is killed by p and has order p^{g-1}.

Proof. Consider the weighted homogeneous singularity

$$C := k[\![x, Y, Z]\!]/(Z^q - x^qY^q(Y^d - x^c)).$$

Since we assume that $\gcd(p, g) = 1$ and q is coprime to cd, the conditions (4-1) are satisfied, and Theorem 4.4 provides a resolution of $\operatorname{Spec} C$. Since k is algebraically closed, the field k contains an element ζ_{2d} such that $\zeta_{2d}^d = -1$. Let $B := k[\![x, y, z]\!]/(z^q + x^c + y^d)$. The scheme $\operatorname{Spec} C$ is not normal, and the natural map $C \to B$, with $Z \mapsto \zeta_{2d}zxy$ and $Y \mapsto \zeta_{2d}y$, induces a finite birational morphism $\operatorname{Spec} B \to \operatorname{Spec} C$. Hence, $\operatorname{Spec} B$ has the same resolution as $\operatorname{Spec} C$. The reader will check that the matrix N_C associated to the resolution of $\operatorname{Spec} C$ in Theorem 4.4 is the same as the matrix N appearing in the statement of Theorem 5.1. The discriminant group Φ_N is computed in Proposition 4.9. $\square$

Remark 5.2. A resolution of the Brieskorn singularity of the form $x^c + y^d + z^e = 0$ is known over the complex numbers thanks to the work of [Hirzebruch and Jänich 1969, Theorem, page 232], when c, d, and e are pairwise coprime, and [Orlik and Wagreich 1971a] in general. An explicit description for the intersection matrix N and dual graph Γ_N of a resolution is found for instance in [Tomaru 1995, page 284], with a formula giving the self-intersection $-s_0$ of the node given on page 287.

Let now $p > 1$ be prime. When p is coprime to cd, the intersection matrix for the resolution of $z^p + x^c + y^d = 0$ obtained in Theorem 5.1 is the same as the intersection matrix obtained in characteristic 0. Some characteristic $p > 1$ examples appear explicitly already in the literature, such as the case of $z^p + x^2 + y^{p+2} = 0$ when p is odd, treated in [Miyanishi and Russell 1983, Lemma 3.13].

Assume that $p > 1$ is prime and divides cd. The Brieskorn singularity $z^p + x^c + y^d = 0$ has then a resolution in characteristic p which is quite different than in characteristic 0. Indeed, assume that $c = p\gamma$ for some integer γ, and $\gcd(p, d) = 1$. Then in characteristic p, $z^p + x^c + y^d = (z + x^\gamma)^p + y^d$. It follows that the normalization of $k[\![x, y]\!][z]/(z^p + x^c + y^d)$ is regular when $\mathrm{char}(k) = p$. On the other hand, in the case for instance of $z^2 + x^3 + y^6 = 0$ in characteristic 0 (a case which is not covered by Theorem 5.1), the minimal resolution is a smooth elliptic curve of self-intersection -1. This explicit example of a resolution in characteristic 0 (and many others) is found for instance in [Laufer 1977, page 1290].

Theorem 5.3. *Let $B := k[\![x, y, z]\!]/(f)$, where $f(x, y, z)$ is a weighted homogeneous polynomial of the following form, with $n, m \geq 1$:*

(i) $z^p + x^{pm+1} + y^{pn+1}$.

(ii) $z^p + xy(x^{pm-1} - y^{pn-1})$.

(iii) $z^p - x^2 + 2y^{p+1}$ *when* $p \geq 3$.

Then $\operatorname{Spec} B$ *is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. Moreover, the fundamental group of the punctured spectrum* $\operatorname{Spec} B \setminus \{\mathfrak{m}_B\}$ *is trivial.*

Proof. The proof of the theorem is similar for each of the three types of homogeneous polynomials. In each case, there exists a family of rings B_μ, μ homogeneous in $k[x, y]$, such that the ring B can be identified with the ring $B_{\mu=0}$, and such that when $\deg(\mu)$ is large enough, there is an isomorphism between $B_{\mu=0}$ and B_μ. The family B_μ is constructed such that when $\mu \neq 0$ is chosen adequately, the ring B_μ is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.

For the weighted homogeneous form in (iii), we use the family B_μ (with $\mu \in k[y]$) described in Proposition 6.2. For the weighted homogeneous forms in (i) and (ii), we use the families discussed in [Lorenzini and Schröer 2020] and recalled in Section 0.2. More precisely, fix a system of parameters a, b in $k[\![x, y]\!]$. Consider the family of hypersurface singularities $\operatorname{Spec} B_\mu$, $\mu \in k[\![x, y]\!]$, with

$$B_\mu := k[\![x, y, z]\!]/(z^p - (\mu ab)^{p-1} z - a^p y + b^p x).$$

Let $G := \mathbb{Z}/p\mathbb{Z}$. When μ is not a unit, is not zero, and is coprime to a and coprime to b, then B_μ is isomorphic to the ring of invariants A^G of an action of G on $A = k[\![u, v]\!]$, and the morphism

Spec $A \to$ Spec A^G is ramified in codimension 1. Cases (i) and (ii) are obtained when $\mu = 0$ by setting $a = -y^n$ and $b = x^m$, and $a = -x^m$ and $b = -y^n$, respectively.

We now claim that it is possible to find a homogeneous polynomial μ of large enough degree such that $B := k[[x, y, z]]/(f)$ is isomorphic over k to B_μ. In cases (i) and (ii), we note that the homogeneous polynomial $\mu := x^t + y^t$ $(t \geq 1)$, is coprime to both a and b, so that the corresponding Spec B_μ is a quotient singularity associated with an action that is ramified in codimension 1.

To prove the existence of a k-isomorphism from $B := k[[x, y, z]]/(f)$ to B_μ, we use the lemma in [Greuel and Kröning 1990, 2.6, page 345]. For the details of the proof of this lemma, the authors of [loc. cit.] refer the reader to the paper [Bochnak and Łojasiewicz 1971]. Recall that the Tjurina ideal of f is $j(f) := \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, and that there exists an integer $s > 0$ such that $(x, y, z)^s \subseteq j(f)$ if and only if the Tjurina number $\tau := \dim_k(k[[x, y, z]]/j(f))$ is finite. This is indeed the case for all polynomials f in (i), (ii), and (iii). Then the lemma in [Greuel and Kröning 1990, 2.6], implies that if $\deg(\mu g) > 2\tau$ (with $g \in k[[x, y, z]]$), then $B := k[[x, y, z]]/(f)$ is isomorphic over k to $k[[x, y, z]]/(f + \mu g)$.

In each case above, we have shown that Spec B is isomorphic to a quotient singularity Spec B_μ such that B_μ is the ring of invariants of an action of $\mathbb{Z}/p\mathbb{Z}$ on the ring $A := k[[u, v]]$ such that the morphism Spec $A \to$ Spec B_μ is ramified in codimension 1. Corollary 1.2(ii) in [Artin 1977] shows that the fundamental group of the punctured spectrum Spec $B \setminus \{\mathfrak{m}_B\}$ is trivial. □

Remark 5.4. Consider the equation $f := z^q + x^c + y^d$ with q, c, d three distinct primes. Let k be a field of characteristic p. Let $B := k[[x, y, z]]/(f)$. Theorem 5.1 shows that the intersection matrix of the resolution of Spec B is the same in all three characteristics $p = q, c, d$, and has determinant 1. It is natural to wonder whether this matrix can occur in more than one characteristic as the intersection matrix attached to a resolution of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.

Consider the intersection matrix with resolution graph E_8. In Artin's notation [1977], $f := z^2 + x^3 + y^5$ defines the singularity Spec B denoted by E_8^0, with resolution graph E_8. This singularity is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity when $p = 2$; see Theorem 5.3(i). When $p = 5$, a different singularity, denoted by E_8^1 in [Artin 1977], also has resolution graph E_8 and is a wild $\mathbb{Z}/5\mathbb{Z}$-quotient singularity.

Theorem 5.5. *Let p be prime. Let $s \geq 0$:*

(a) *Assume that either $s \not\equiv 1 \bmod p$, or that p is odd and $s = 1$. Then there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity* Spec A^G *with associated action ramified precisely at the origin, and such that the discriminant group of a resolution of the singularity has order p^s.*

(b) *Assume that either p is odd and $s \equiv 1 \bmod p$, or that $p = 2$ and $s = 1$. Then there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity* Spec A^G *with associated action ramified in codimension* 1 *and such that the discriminant group of a resolution of the singularity has order p^s.*

Proof. (a) The cases $s = 0$ and $s = 1$ are covered by Theorem 7.1 and Section 0.2, and Proposition 6.2 and Theorem 6.3, respectively. The cases with $s \geq 2$ and $s \not\equiv 1 \bmod p$ were obtained earlier in the papers [Lorenzini 2018; Mitsui 2021].

(b) When $s \equiv 1 \bmod p$ and $s \geq p+1$, we use the Brieskorn singularities exhibited in Lemma 5.6, and apply Theorem 5.1 and Theorem 5.3. The case $p = 2$ and $s = 1$ was noted by Artin and is discussed in Section 8. The case $s = 1$ is treated in Theorem 9.4. $\square$

Lemma 5.6. *Let p be an odd prime, and r be any positive integer. Then there are integers $m, n > 0$ such that the discriminant group Φ_N of the intersection matrix N associated with the Brieskorn singularity $z^p + x^{pm+1} + y^{pn+1} = 0$ described in Theorem 5.1 is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{pr+1}$.*

Proof. In view of Theorem 5.1, we need to produce integers n and m such that $\gcd(pn+1, pm+1) = pr+2$. For this, it suffices to take $n := (pr + r + 2)/2$, so that $pn + 1 = (pr + 2)(p + 1)/2$, and to set $m := (3pr + r + 6)/2$, so that $m = n + (pr + 2)$. $\square$

Note that not all elementary abelian p-groups appear as discriminant groups Φ_N attached to the intersection matrix N associated with a Brieskorn singularity $z^p + x^{pm+1} + y^{pn+1} = 0$. Indeed, for all $m, n > 0$, the integer $g = \gcd(pm + 1, pn + 1)$ is never divisible by p. Thus in the above setting Φ_N cannot be isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{pr-1}$ for any $r > 0$.

Remark 5.7. Let B be a complete noetherian local ring that is two-dimensional and normal, with algebraically closed residue field. Consider a resolution of singularities $X \to \operatorname{Spec} B$, with associated intersection matrix N. Recall that there is a natural surjection $\operatorname{Cl}(B) \to \Phi_N$; see [Lipman 1969, 14.4]. In particular, when $\det(N) \neq 1$, we obtain a natural nontrivial finite quotient of $\operatorname{Cl}(B)$ from the computation of a resolution of $\operatorname{Spec} B$.

The study of the class group $\operatorname{Cl}(B)$ of $B := k[[x, y]][z]/(z^p - x^c - y^d)$ was initiated by Samuel [1964, Proposition (3) in Section 6]; see also [Fossum 1973, Chapter IV, Section 17]. When $p = 2$, Samuel is able to exhibit by a completely algebraic method a finite quotient of $\operatorname{Cl}(B)$ of order p^{g-1}, where $g := \gcd(c, d)$. Under the hypothesis of Theorem 5.1, p^{g-1} would also be the order of the corresponding group Φ_N.

6. Analogues of the E_6 singularities

Let k be an algebraically closed field of characteristic $p \geq 3$. Let $\mu \in k[y]$, $\mu \neq 0$. Consider the automorphism σ of the polynomial ring $k[u, v, y]$ given by

$$u \mapsto u + \mu v, \quad v \mapsto v + \mu y, \quad \text{and} \quad y \mapsto y.$$

This automorphism has order p. We exclude the case $p = 2$ in this section because when $p = 2$, σ has order 4. Let

$$N_u := \operatorname{Norm}(u) = \prod_{d=0}^{p-1} \sigma^d(u) = \prod_{d=0}^{p-1} \left(u + d\mu v + \frac{d(d-1)}{2} \mu^2 y \right),$$

and

$$x := \operatorname{Norm}(v) = v^p - (\mu y)^{p-1} v.$$

Finally, let

$$z := v^2 - \mu y v - 2yu.$$

Let $G := \mathbb{Z}/p\mathbb{Z}$ act on $k[u, v, y]$ through σ. When $\mu = 1$, the ring of invariants $k[u, v, y]^G$ is known to be generated by x, y, z, and N_u, subject to a single relation; see, e.g., [Campbell and Wehlau 2011, 4.10]. This relation was made explicit by Peskin, who showed [1983, Lemma 5.6], that $h_{\mu=1}(x, y, z, N_u) = 0$, where,

$$h_{\mu=1}(x, y, z, N_u) := z^p + 2y^p N_u - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} y^{2p-2n} z^n.$$

Here $C_{n-1} := (2n-2)!/n!(n-1)!$ are the Catalan numbers.

When $\mu \neq 1$, the above result can be used to show that x, y, z, and N_u are subject to the relation

$$h(x, y, z, N_u) := z^p + 2y^p N_u - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} (\mu y)^{2p-2n} z^n = 0.$$

Indeed, the morphism $k[u, v, y] \to k[U, V, y]$, which sends $u \mapsto \mu^2 U$, $v \mapsto \mu V$, and $y \mapsto y$, is G-equivariant when $k[u, v, y]$ is endowed with the action of σ, and $k[U, V, y]$ is endowed with the action of σ_1, with $\sigma_1(U) = U + V$ and $\sigma_1(V) = V + y$.

For any choice of $c(y) \in yk[y]$, we can consider the ring

$$A_0 := k[u, v, y]/(N_u - c(y)).$$

We will slightly abuse notation and denote again by x, y, z, u, v, the classes of these elements in A_0. Clearly, the automorphism σ fixes the polynomial $N_u - c(y)$, and thus induces an automorphism on A_0, again denoted by σ. This endows A_0 with an action of G. Let A denote the formal completion $\widehat{A_0}$ of the ring A_0 at the maximal ideal (u, v, y).

The fixed scheme of the G-action on $\operatorname{Spec}(A_0)$ is given by the ideal $I := (\mu v, \mu y)$. When $\mu \in k^*$, $I = (v, y) = (u^p, v, y)$, and thus its radical is the maximal ideal (u, v). Hence, the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is ramified precisely at the origin. When $\mu \neq 0$ is not a unit in A, the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is ramified in codimension 1.

The study of the singularities of the rings $\operatorname{Spec} A^G$ when $\mu = 1$ was initiated by Peskin [1980, Chapter III, Section 4; 1983, Section 5]. *In the remainder of this section, we treat the case where $c(y) = y$*, and obtain a family of wild quotient singularities A^G of multiplicity 2 whose discriminant groups have order $|\Phi| = p$. For $p > 3$, these singularities can be viewed as analogues of the rational double point of type E_6^1 in characteristic $p = 3$, which was shown to be a wild $\mathbb{Z}/3\mathbb{Z}$-quotient singularity by Artin [1977].

Proposition 6.1. *Let $c(y) := y$. Let $\mu \in k[y]$. Then the ring A_0 is a domain, the formal completion A is regular, and the canonical map $k[[u, v]] \to A$ is bijective.*

Proof. The expression $f(u, v, y) := N_u - y$ is a monic polynomial of degree p in the variable u over the factorial ring $k[v, y]$, with constant term $f(0, v, y) = -y$. Since f is monic in u, to prove f irreducible in

$k[u, v, y]$, it suffices to prove that $f(u, 0, y)$ is irreducible in $k[u, y]$. The Newton polygon of $f(u, 0, y)$ with respect to the y-adic valuation is the straight line from $(0, 1)$ to $(p, 0)$ in $\mathbb{R}^2$, and we conclude with the Eisenstein–Dumas theorem [Mott 1995] that $f(u, 0, y)$ is irreducible.

The ring A_0 and its formal completion A are thus two-dimensional domains. To see that the local ring A is regular, we have to check that the cotangent space $\mathfrak{m}_A/\mathfrak{m}_A^2$ has vector space dimension at most two. Indeed, this vector space is generated by u, v, y. In light of the relation $N_u - y = 0$, the class of y vanishes. In turn, the canonical map $k[[u, v]] \to A$ between complete local rings induces a bijection on cotangent spaces, and is thus bijective. □

Let $\mu \in k[y]$. Abusing notation slightly, we let $h(x, y, z) \in k[x, y, z]$ be defined as

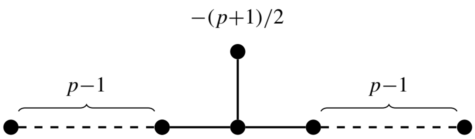
$$h(x, y, z) := z^p + 2y^{p+1} - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} (\mu y)^{2p-2n} z^n. \tag{6-1}$$

We let $B_\mu := k[[x, y, z]]/(h)$.

Proposition 6.2. *Let $c(y) := y$. Let $\mu \in k[y]$, $\mu \neq 0$. Then the canonical map $B_\mu \to A^G$ is bijective. In particular, the wild quotient singularity A^G is a complete intersection of multiplicity two.*

Proof. Both local rings B_μ and A^G are Cohen–Macaulay, and finite $k[[x, y]]$-algebras of rank p. One easily sees that $h(x, y, z) = 0$ defines an isolated singularity, by using the relations $h_x = -2x$ and $2z(\mu + y\mu_y)h_z + \mu y h_y = 2\mu y^{p+1}$ between partial derivatives. It follows that $k[[x, y, z]]/(h)$ is normal, and that the canonical map induces a bijection on the field of fractions. The map in question is thus bijective, by Zariski's main theorem. Clearly, the monomial x^2 is the lowest term in $h(x, y, z)$, and it follows that the complete intersection A^G has multiplicity two. □

Theorem 6.3. *Let $c(y) := y$. Let $\mu \in k[y]$. Let $X \to \operatorname{Spec}(B_\mu)$ be the minimal resolution of singularity, with associated intersection matrix N. Then the dual graph Γ_N is independent of μ, and takes the form*:



The associated discriminant group Φ_N has order p.

Proof. Consider the blow-up $Z \to \operatorname{Spec}(B_\mu)$ of $\operatorname{Spec}(B_\mu)$ with respect to the ideal (x, y, z). Let $Y \to Z$ denote the normalization of Z. Let E denote the exceptional divisor of the blow-up, and let D denote its schematic preimage in Y.

The blow-up Z is covered by three charts that we call the x-chart, y-chart, and z-chart. We consider in detail below the y-chart and show that its normalization contains a unique singular point y_0. Proceeding in an analogous way as for the y-chart, the reader will check that the normalizations of the x-chart and the z-chart are regular.

On the y-chart, the strict transform of $h(x, y, z) = 0$ becomes

$$\left(\frac{z}{y}\right)^p y^{p-2} + 2y^{p-1} - \left(\frac{x}{y}\right)^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} \mu^{2p-2n} y^{2p-n-2} \left(\frac{z}{y}\right)^n = 0.$$

The fraction $x/y^{(p-1)/2}$ satisfies the integral equation

$$\left(\frac{z}{y}\right)^p y + 2y^2 - \left(\frac{x}{y^{(p-1)/2}}\right)^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} \mu^{2p-2n} y^{p-n+1} \left(\frac{z}{y}\right)^n = 0. \tag{6-2}$$

Write $g = (z/y)^p y + 2y^2 - (x/y^{(p-1)/2})^2 + \cdots$ for the polynomial on the left. The radical of the Tjurina ideal associated with g contains y, because y defines the exceptional divisor on the y-chart and there are no singularities outside the exceptional divisor. Obviously the Tjurina ideal also contains $x/y^{(p-1)/2}$ (consider the derivative of g with respect to the variable $x/y^{(p-1)/2}$). Using the partial derivative $g_y = (z/y)^p + \cdots$, we see that the radical of the Tjurina ideal furthermore contains z/y. Thus the normalization of the y-chart is given by the three variables z/y, y, $x/y^{(p-1)/2}$ and the equation $g = 0$.

We claim that D_{red} is a smooth rational curve, and that $(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = -\frac{1}{2}$. For this it suffices to check analogously as in Proposition 3.6 that the curve E_{red} is regular, and that $(E \cdot E_{\mathrm{red}})_Z = -1$. Then one checks that the natural map $D_{\mathrm{red}} \to E_{\mathrm{red}}$ is an isomorphism. Finally, noting that the multiplicity of E is $\ell = 2$, we apply the formula $(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = (E \cdot E_{\mathrm{red}})_Z/\ell$ in Proposition 2.3 to obtain the claim.

Regarded as a formal power series, the initial term of g is the quadratic polynomial $2y^2 - (x/y^{(p-1)/2})^2$, which is thus a product of two linear factors since k is algebraically closed. According to Lemma 6.4 below, the singularity must be a rational double point of type A_m for some integer $m \geq 1$. To determine this integer, we compute the Tjurina number of the singularity, which is the colength of the ideal generated by g and its partial derivatives. Setting $x' = x/y^{(p-1)/2}$ and $z' = z/y$, the partial derivatives take the form

$$g_{x'} = 2x', \quad g_y = z'^p + y \cdot (4 + y \cdot *) \quad \text{and} \quad g_{z'} = \sum_{n=2}^{(p+1)/2} (-1)^n n C_{n-1} \mu^{2p-2n} y^{p-n+1} z'^{n-1}.$$

We now use $g_y = 0$ to substitute for y in the equations $g(0, y, z') = 0$ and $g_{z'}(0, y, z') = 0$, and infer that the Tjurina ideal has colength $\tau = 2p$. The first two summands in $g(0, y, z') = 0$ do not cancel after the substitution.

Recall that the Tjurina number for the A_m-singularity, which is formally isomorphic to $Z^{m+1} - XY = 0$, is given by

$$\tau = \begin{cases} m & \text{if } p \text{ does not divide } m+1; \\ m+1 & \text{else.} \end{cases}$$

It follows that either $m = 2p - 1$ or $m = 2p$, and we shall see below that m is odd.

Write $X \to Y$ for the minimal resolution of singularities of the rational double point, such that the composite map $X \to Y \to \mathrm{Spec}(B_\mu)$ is a resolution of the singularity. The dual graph of this resolution contains a chain $C_1, \ldots, C_m$ of (-2)-curves, together with the strict transform C_0 of the divisor D_{red} on Y.

Suppose that C_0 intersects two distinct exceptional curves $C_i \neq C_j$. Then $\left(\bigcup_{i \geq 1} C_i\right) \cap C_0$ is an Artin scheme of length ≥ 2 on C_0. We claim that this is not possible. Indeed, consider the blow-up $X \to Y$.

The induced morphism $C_0 \to D_{\mathrm{red}}$ is an isomorphism since we have shown above that the point y_0 is a regular point of D_{red}. The scheme $\left(\bigcup_{i\geq 1} C_i\right)$, which is proper, has schematic image in Y the reduced closed point y_0. The same is true for any closed subscheme of the exceptional divisor, including the subscheme $\left(\bigcup_{i\geq 1} C_i\right) \cap C_0$. This is a contradiction since we have on the other hand an isomorphism $C_0 \to D_{\mathrm{red}}$, and a closed subscheme of length bigger than one in the source cannot be sent to a closed subscheme of length 1 in the target. Thus C_0 hits precisely one divisor C_i. If $(C_0 \cdot C_i)_X > 1$, a similar argument leads again to a contradiction, and thus we must have $(C_0 \cdot C_i)_X = 1$.

Consider now the involution on B_μ given by $x \mapsto -x$, $y \mapsto y$ and $z \mapsto z$. This involution fixes Peskin's equation (6-1), and induces an involution on the initial blow-up Z and its normalization Y. There the equation $z/y = 0$ defines an invariant Cartier divisor on the A_m-singularity $\operatorname{Spec} \mathcal{O}_{Y,y_0}$, which is the union of two regular Weil divisors D_1 and D_2, and these divisors are interchanged by the involution. The blow-up $Y' \to Y$ of the singular point $y_0 \in Y$ with reduced structure introduces two exceptional curves F_1 and F_2, and the strict transforms of D_1 and D_2 in Y' are disjoint. The intersection $F_1 \cap F_2$ consists of a single point y_0', and the local ring $\mathcal{O}_{Y',y_0'}$ is a rational double point of type A_{m-2}.

We now show that m is odd. First, suppose that the strict transforms of D_1 and D_2 in Y' do not intersect the same exceptional component of the blow-up $Y' \to Y$. It then follows that the involution acts nontrivially on the dual graph attached to the resolution of singularities $X \to Y$. If $m = 2p$ was even, the curve C_0 would pass through the sole fixed point $C_p \cap C_{p+1}$ of the exceptional divisor, and as we have seen above, this is a contradiction. It follows that $m = 2p - 1$ must be odd in this case, and that $(C_0 \cdot C_p)_X = 1$. The assertion on the dual graph Γ_N follows.

Suppose now that the strict transforms of D_1 and D_2 in Y' intersect the same exceptional component of the blow-up $Y' \to Y$. We are going to show that this case cannot happen. Indeed, then the Weil divisors $D_1, D_2 \subset Y$ define the same class in the class group $\mathrm{Cl}(\mathcal{O}_{Y,y_0}) = \mathbb{Z}/(m+1)\mathbb{Z}$ of the rational double point of type A_m. Since the curves D_i are regular, the divisors $D_i \subset Y$ are not Cartier. It follows that D_i has order two in $\mathrm{Cl}(\mathcal{O}_{Y,y_0})$ since the sum of D_1 and D_2 is a Cartier divisor on Y. On the other hand, the strict transform of D_i in X intersects a terminal vertex of the exceptional divisor of $X \to Y$, and this fact along with a computation using the intersection matrix of the chain of m curves implies that D_i has order $m+1$ in the class group. This gives $m = 1$, contradicting $m \geq 2p - 1 \geq 5$.

To completely determine the intersection matrix N of the resolution $X \to \operatorname{Spec}(B_\mu)$, it remains to compute the self-intersection number $(C_0 \cdot C_0)_X$. We have already observed above that $(D_0 \cdot D_0)_Y = -\frac{1}{2}$, and Proposition 2.2 shows that $(C_0 \cdot C_0)_X = (D_0 \cdot D_0)_Y - \delta$, where the correcting term δ is computed as follows. The determinant of the intersection matrix of the full chain of length $2p-1$ is $-2p$. Removing the vertex adjacent to C_0 from this chain yields two chains of length $p-1$. The determinant of the associated intersection matrix is then p^2. It follows that $\delta = p^2/2p = p/2$. Hence,

$$(C_0 \cdot C_0)_X = -1/2 - p/2 = -(p+1)/2.$$

Proposition 1.3 shows that $|\Phi_N| = p$. $\square$

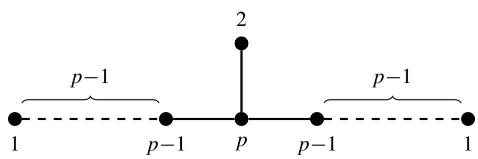
In the course of the proof we have used the following well-known general observation:

Lemma 6.4. *Let $f \in k[\![x, y, z]\!]$ by a power series over an arbitrary field k. Write $f = \sum_{j=0}^{\infty} f^{(j)}$, where $f^{(j)}$ is a homogeneous polynomial of degree j. Suppose that $f^{(0)} = f^{(1)} = 0$, and that f defines an isolated singularity. Assume also that the quadratic part $f^{(2)}$ is the product of two nonassociated linear forms. Then $k[\![x, y, z]\!]/(f)$ is isomorphic to $k[\![x, y, z]\!]/(z^{m+1} - xy)$ for some integer $m \geq 2$. In other words, the singularity in question is a rational double point of type A_m.*

Proof. After a linear change of coordinates, we may assume that $f = xy + O(3)$, where we denote by $O(d)$ an element of $\mathfrak{m}^d$. By induction on $d \geq 3$, one makes further coordinate changes of the form $x' := x + a(x, y, z)$, $y' := y + b(x, y, z)$ with $a, b \in \mathfrak{m}^{d-1}$ sending f to a power series of the form $x'y' + \sum_{i=3}^{d} \lambda_i z^i + O(d+1)$, for some $\lambda_i \in k$. This shows that we may assume $f = xy + \sum_{i=3}^{\infty} \lambda_i z^i$. If all coefficients λ_i vanish, the singularity would not be isolated. Thus our equation is of the form $xy + z^{m+1}\epsilon$ for some $m \geq 2$ and unit ϵ. Multiplying with ϵ^{-1}, we get the equation $(\epsilon^{-1}x)y + z^{m+1}$ for the rational double point of type A_m. □

Recall that the *fundamental cycle* $\mathbf{Z}$ of an intersection matrix N is the minimal positive vector $\mathbf{Z}$ such that $N\mathbf{Z}$ is a nonpositive vector. The *canonical cycle* K of an intersection matrix N is recalled in Section 10.2. The *fundamental genus* $h^1(\mathcal{O}_{\mathbf{Z}})$ can be computed for the hypersurface singularities considered below as $2h^1(\mathcal{O}_{\mathbf{Z}}) - 2 = (K + \mathbf{Z}) \cdot \mathbf{Z}$.

Proposition 6.5. *The multiplicities in the fundamental cycle $\mathbf{Z}$ of the resolution of* Spec B_μ *are indicated below next to the corresponding vertex*:



The canonical cycle is given by $K = -\frac{p-3}{2}\mathbf{Z}$. We have $\mathbf{Z}^2 = -2$, and $h^1(\mathcal{O}_{\mathbf{Z}}) = (p-3)/2$.

Proof. Let us denote by E_0 the node of Γ_N, and by E_1 the pendant vertex of self-intersection $E_1^2 = -(p+1)/2$. To compute $\mathbf{Z}$, we apply Artin's algorithm [1966]: one starts with the cycle C having all coefficients equal to 1, which we will draw pictorially as $\begin{smallmatrix} & & & 1 & & & \\ 1 & 1 & \cdots & 1 & \cdots & 1 & 1 \end{smallmatrix}$. The algorithm updates C by increasing some coefficient of C at each step. We denote by m_0 the multiplicity of E_0 in C. Since $C \cdot E_0 > 0$, the algorithm increases m_0 by 1. The new cycle C has positive intersection number with both vertices adjacent to the node on the two terminal chains of length $p-1$, and one then increases their multiplicities by 1. Proceeding along these terminal chains, one ends with the new cycle C given by $\begin{smallmatrix} & & & 1 & & & \\ 1 & 2 & \cdots & 2 & \cdots & 2 & 1 \end{smallmatrix}$. Now one repeats the process, starting again at the node E_0. After $p-1$ steps, one obtains the cycle $\begin{smallmatrix} & & & 1 & & & \\ 1 & 2 & \cdots & p & \cdots & 2 & 1 \end{smallmatrix}$. This new cycle has positive intersection number with the terminal vertex E_1. Increasing the multiplicity m_1 of E_1 by 1 gives the fundamental cycle: indeed, this new cycle $C = \mathbf{Z}$ now has $(\mathbf{Z} \cdot E_1) = -1$, and all other intersection numbers are 0.

This description of $\mathbf{Z}$ immediately lets us compute that $\mathbf{Z}^2 = -2$. It is easy to check that the canonical cycle is $K = -\frac{p-3}{2}\mathbf{Z}$, and that $(K + \mathbf{Z}) \cdot \mathbf{Z} = p - 5$. □

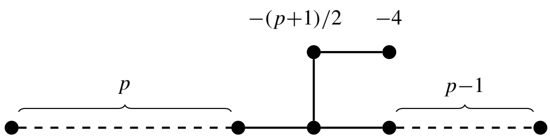
7. Analogues of the E_8 singularities

Let k be an algebraically closed field of characteristic $p > 0$. We compute in this section the resolution of the singularity of $\operatorname{Spec} B_\mu$ introduced in Section 0.2, for any value of the parameter $\mu \in k[[x, y]]$ when $a = -y^2$ and $b = -x$. The ring B_μ is given in this case by

$$B_\mu := k[[x, y]][z]/(z^p - (\mu x y^2)^{p-1} z - x^{p+1} + y^{2p+1}).$$

When $p = 2$, the resolution of $\operatorname{Spec} B_\mu$ is known to have dual graph E_8 when $\mu = 0$, $\mu = 1$ and $\mu = y$: these values produce the rational double points E_8^0, E_8^2, and E_8^1, respectively [Artin 1977]; see also [Peskin 1980]. The index of determinacy of a singularity E_8^r in characteristic 2 is computed to be 5 in [Greuel and Kröning 1990, page 346]. It follows that when $\mu \in (x, y)^2$, then $\operatorname{Spec} B_\mu$ is isomorphic to E_8^0. For $\mu \in k^\times$, we find that B_μ is isomorphic to E_8^2 through the change of variables $X = \mu^{10/7} x$, $Y = \mu^{6/7} y$, and $Z = \mu^{15/7} z$.

Theorem 7.1. *Let $p \geq 3$. Then* $\operatorname{Spec} B_\mu$ *has a resolution of singularities with dual graph Γ_N independent of μ of the following form*:



The associated discriminant group Φ_N is trivial.

Proof. Set $R := k[[x, y, z]]$ and $f := z^p - (\mu x y^2)^{p-1} z - x^{p+1} + y^{2p+1}$, and write $B := R/(f)$. We start with an initial blowing-up $Z := \operatorname{Bl}_{\mathfrak{a}B}(B)$ for the ideal $\mathfrak{a} := (x, y^2, z)$, as in Proposition 3.6. As usual, let $E \subset Z$ denote the exceptional divisor of the blow-up, and E_{red} its reduction. Proposition 3.6 shows that E_{red} is a smooth rational curve, that $E = 2pE_{\mathrm{red}}$, and that $(E \cdot E_{\mathrm{red}})_Z = -1$. One checks that the blow-up is regular on the y^2-chart and the z-chart, and contains a unique singularity, which is located at the origin of the x-chart.

The x-chart is given by four variables $x, y, y^2/x, z/x$ modulo the two relations

$$y^2 = \left(\frac{y^2}{x}\right)x \quad \text{and} \quad \left(\frac{z}{x}\right)^p - \mu^{p-1}x^{p-1}\left(\frac{y^2}{x}\right)^{p-1}\frac{z}{x} - x + \left(\frac{y^2}{x}\right)^p y = 0.$$

The exceptional divisor is given by $x = 0$. Its reduction is defined by $x = y = z/x = 0$. Let us rewrite the second equation above as

$$\left(\frac{z}{x}\right)^p + \left(\frac{y^2}{x}\right)^p y = x\left(\mu^{p-1}x^{p-2}\left(\frac{y^2}{x}\right)^{p-1}\frac{z}{x} + 1\right). \tag{7-1}$$

On the formal completion along the exceptional divisor, $1 + \mu^{p-1}x^{p-2}(y^2/x)^{p-1}(z/x)$ is invertible, and we denote by ϵ its inverse. The unit ϵ admits a $(p+1)$-st root δ (with $\delta^{p+1} = \epsilon$). After extracting an expression for x from (7-1) and substituting it in the expression $y^2 = \frac{y^2}{x}x$, we find that

$$y^2 = \frac{y^2}{x}\left(\left(\frac{z}{x}\right)^p + \left(\frac{y^2}{x}\right)^p y\right)\epsilon.$$

This is formally isomorphic to the equation

$$y^2 - U^{p+1}y - UW^p = 0$$

in the new set of variables y, U, W, via the map given by $y \mapsto y$, $U \mapsto (y^2/x)\delta$ and $W \mapsto (z/x)\delta$. Note that the reduced exceptional divisor is given by $x = y = z/x = 0$ in the old coordinates, and by $y = W = 0$ in the new ones. Let

$$B' := k[\![y, U, W]\!]/(y^2 - U^{p+1}y - UW^p).$$

We now make a second blow-up $Z' \to \operatorname{Spec}(B')$, with nonreduced center given by (y, U, W^p). Let E' denote the exceptional divisor of this blow-up. Using Proposition 3.1, we infer that the U-chart of Z' is described by four variables $U, W, y/U, W^p/U$ and two relations

$$W^p = \left(\frac{W^p}{U}\right)U \quad \text{and} \quad \left(\frac{y}{U}\right)^2 - U^p\left(\frac{y}{U}\right) - \frac{W^p}{U} = 0.$$

Substituting the latter in the former and renaming y/U by V gives

$$W^p = UV(V - U^p). \tag{7-2}$$

The origin (U, V, W) is obviously singular on this chart, and this is a singularity analyzed in Theorem 4.4. The reader will check that Z' has no further singularities on other charts, and that the only singularity on the U-chart is located at the origin. On this chart, the exceptional divisor is given by $U = 0$. Its reduction has $U = W = 0$. The reader will check that the exceptional divisor E' of this blow-up is a smooth projective line. Note also that the strict transform of the exceptional divisor from the initial blow-up is given by $V^2 = 0$ (since $x = V^2U\delta$), with reduction $V = W = 0$, and that this strict transform is also a smooth projective line.

Theorem 4.4 lets us describe explicitly the intersection matrix $N(s_0 \mid \alpha^{-1}, \beta^{-1}, \gamma^{-1})$ of the unique singularity in the U-chart. Using the notation from Section 4.3, we set $q = p$, $a = b = 1$, $c = p$ and $d = 1$, and find that $g := \gcd(c, d) = 1$ and $(ad + bc + cd)/g = 2p + 1$. It follows that

$$\alpha^{-1} = p^2/(2p - 1) \quad \text{and} \quad \beta^{-1} = \gamma^{-1} = p/(p - 1).$$

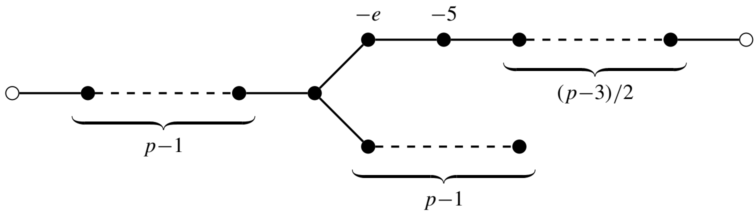
Recall that $p \geq 3$ and set $e := (p + 1)/2$. The reader will check that the continued fraction expansion of $\alpha^{-1} = p^2/(2p - 1)$ is $\alpha^{-1} = [e, 5, 2, \ldots, 2]$ with $2 + (p - 3)/2$ overall entries, starting with the relations

$$p^2 = e(2p - 1) - (p - 1)/2, \quad \text{and} \quad (2p - 1) = 5(p - 1)/2 - (p - 3)/2.$$

The self-intersection $-s_0$ of the node of the star-shaped graph is computed as

$$s_0 = \frac{1}{p^2} + \frac{2p-1}{p^2} + 2\frac{p-1}{p} = 2.$$

Having resolved the singularity (7-2), we get a resolution for our original singularity Spec B_μ with the following resolution graph:



(7-3)

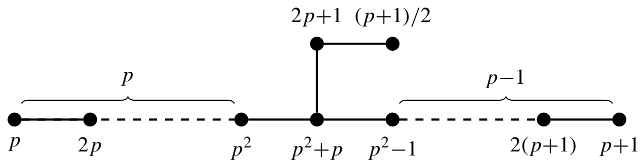
According to Proposition 4.8, the white terminal vertex to the left corresponds to the strict transform of the exceptional divisor on the initial blow-up, whereas the white terminal vertex on the top right corresponds to the strict transform of the exceptional divisor on the second blow-up.

It remains to determine the self-intersection of both of these strict transforms in the resolution of Spec B. Recall that E' is the exceptional divisor for the second blow-up $Z' \to \mathrm{Spec}(B')$. Computing in the affine charts, one sees that E'_{red} is a projective line, with $E' = pE'_{\mathrm{red}}$ and $(E' \cdot E'_{\mathrm{red}})_{Z'} = -2$. Since the U-chart is regular away from the origin, we can conclude using Proposition 2.3 that the self-intersection of the strict transform of E'_{red} in the normalization of Z' is $-2/p$. Proposition 2.2 shows that the strict transform C' of E'_{red} in X has thus $(C' \cdot C')_X = -2/p - \delta$ for some correction term $\delta \in \mathbb{Q}_{>0}$. The term δ is computed as follows. Let Γ_1 be the star-shaped subgraph in (7-3) consisting of all the black vertices, and let $\Gamma'_1 \subset \Gamma_1$ be the star-shaped subgraph obtained from Γ_1 by removing the terminal black vertex in the top right position. Let N_1 and N'_1 be the resulting intersection matrices. According to Proposition 2.2, we have $\delta = -\det(N'_1)/\det(N_1)$. Using Proposition 1.3, we compute that $|\det(N_1)| = p^2$ and $|\det(N'_1)| = p^2 - 2p$. Hence, $\delta = (p^2 - 2p)/p^2 = 1 - 2/p$, and it follows that the white terminal vertex on the top right has self-intersection -1. We can thus contract this divisor. Successively contracting (-1)-curves from the right, we get the desired graph as in the statement of Theorem 7.1 with a terminal vertex of self-intersection number $-4 = -5 + 1$ on the top right.

Recall that we denoted by E the exceptional divisor of $Z \to \mathrm{Spec}\, B$, and determined using Proposition 3.6 that E_{red} is a smooth rational line, that $E = 2pE_{\mathrm{red}}$, and that $(E \cdot E_{\mathrm{red}})_Z = -1$. As above, Proposition 2.2 shows that the strict transform C of E_{red} in X has $(C \cdot C)_X = -1/2p - \delta$ for some correction term $\delta \in \mathbb{Q}_{>0}$. Let Γ_2 be the star-shaped subgraph in (7-3) consisting of all the black vertices and the terminal white vertex (of self-intersection (-1)) in the top right position. Let Γ'_2 be the star-shaped subgraph obtained from Γ_2 by removing the terminal black vertex of Γ_2 attached to the terminal white vertex on the left corresponding to E. Let N_2 and N'_2 be the resulting intersection matrices. According to Proposition 2.2, we have $\delta = -\det(N'_2)/\det(N_2)$. The matrix N_2 has the same determinant as $N(2 \mid p/(p-1),\, p/(p-1),\, (2p+1)/4)$, and N_2' has the same determinant as $N(2 \mid (p-1)/(p-2),\, p/(p-1),\, (2p+1)/4)$. Using Proposition 1.3, we compute that $|\det(N_2)| = 2p$ and $|\det(N_2')| = 4p-1$. Hence, $(C \cdot C)_X = -2$.

Now that the intersection matrix N of the resolution has been determined, with $N = N(2 \mid p/(p-1),\, (p+1)/p,\, (2p+1)/4)$, Proposition 1.3 can be used to show that $|\det(N)| = 1$. $\square$

Proposition 7.2. *Keep the assumptions of Theorem 7.1. The multiplicities in the fundamental cycle* $\mathbf{Z}$ *of the resolution of* $\operatorname{Spec} B_\mu$ *are indicated below next to the corresponding vertex.*



The canonical cycle of the resolution is $K = -(2p-4)\mathbf{Z} + \frac{p-3}{2}E_2$, *where* E_2 *is the terminal vertex on the top right of the above graph. We have* $\mathbf{Z}^2 = -(p+1)/2$, *and* $h^1(\mathcal{O}_{\mathbf{Z}}) = (p^2-p+2)/2$.

Proof. The self-intersection numbers along the three terminal chains in the dual graph Γ_N yield the continued fractions

$$\frac{p+1}{p} = [2, \ldots, 2], \qquad \frac{p}{p-1} = [2, \ldots, 2], \qquad \text{and} \qquad \frac{2p+1}{4} = \left[\frac{(p+1)}{2}, 4\right].$$

Recall that given a fraction r/s, the ceiling $\lceil r/s \rceil$ is the smallest integer larger than or equal to r/s. Write $E_0 \in \Gamma_N$ for the central node. According to [Tomaru 1995, equation (3.4) on page 282], its multiplicity $m_0 \geq 1$ in the fundamental cycle $\mathbf{Z}$ is the smallest integer $m \geq 1$ such that

$$2m - \left\lceil \frac{mp}{p+1} \right\rceil - \left\lceil \frac{m(p-1)}{p} \right\rceil - \left\lceil \frac{4m}{2p+1} \right\rceil \geq 0. \tag{7-4}$$

Let us show that $m_0 = (p+1)p$. First, we claim that when $m = (p+1)p$, then equality holds in (7-4). Indeed, the first two fractions on the left of (7-4) are then the integers p^2 and p^2-1, whereas the last summand becomes

$$\left\lceil \frac{4m}{2p+1} \right\rceil = \left\lceil 2p + \frac{2p}{2p+1} \right\rceil = 2p+1.$$

Assume now $m < (p+1)p$. We claim that in this case (7-4) fails. Indeed, since the fraction $p/(p+1)$ and $(p-1)/p$ are reduced, and one of the integers p or $p+1$ does not divide m, one of the fractions $mp/(p+1)$ and $m(p-1)/p$ is not an integer. Using $1/p > 1/(p+1)$, we obtain

$$\left\lceil \frac{mp}{p+1} \right\rceil + \left\lceil \frac{m(p-1)}{p} \right\rceil \geq \frac{mp}{p+1} + \frac{m(p-1)}{p} + \frac{1}{p+1}.$$

In turn, the left-hand side of (7-4) is bounded above by

$$2m - \frac{mp}{p+1} - \frac{m(p-1)}{p} - \frac{4m}{2p+1} - \frac{1}{p+1} = \frac{m}{p(p+1)(2p+1)} - \frac{1}{p+1}.$$

This is bounded above by $1/(2p+1) - 1/(p+1) < 0$, because $m < p(p+1)$. As desired, the inequality (7-4) fails.

For convenience in this proof, let us denote by Z_0 the vector whose coefficients are given in the proposition. Without loss of generality, we may assume that E_1 and E_2 are the two vertices on the very short chain of the graph, with self-intersection numbers $E_1^2 = -(p+1)/2$ and $E_2^2 = -4$, respectively. It is easy to check that $NZ_0 = -E_2$. Since NZ_0 has nonpositive coefficients, we find that by definition of the fundamental cycle, we must have $\mathbf{Z} \leq Z_0$.

We now determine that the multiplicities of $\mathbf{Z}$ along the two terminal chains comprising only (-2)-curves are the ones indicated in the statement of the proposition. We treat the case of the terminal chain on the right, with $p-1$ vertices. The other chain is treated similarly. Let us denote by $z_{p-1}, \ldots, z_2, z_1$ the multiplicities of $\mathbf{Z}$ along the terminal chain with $p-1$ vertices. The central node has multiplicity denoted for convenience $z_p := p^2 + p$. The fact that $N\mathbf{Z}$ has nonpositive coefficients produces the following inequalities. At the last vertex, we have $-2z_1 + z_2 \leq 0$, and at each other vertex of the chain, we find that $-2z_i + z_{i-1} + z_{i+1} \leq 0$. It follows that

$$z_{i+1} - z_i \leq z_i - z_{i-1}$$

when $2 \leq i \leq p-1$. Since $\mathbf{Z} \leq Z_0$, we have $z_{p-1} \leq p^2 - 1$. Suppose that $z_{p-1} = p^2 - 1 - a$ for some $a \geq 0$. Then

$$(p^2+p) - (p^2 - 1 - a) = p + 1 + a = z_p - z_{p-1} \leq z_{p-1} - z_{p-2} \leq \cdots \leq z_2 - z_1 \leq z_1.$$

Hence,

$$p^2 + p = z_p \geq p(p+1+a).$$

It follows that $a = 0$ and $z_{p-1} = p^2 - 1$. A similar argument shows that $z_{p-i} = (p-i)(p+1)$, as desired.

It remains to determine the coefficients of $\mathbf{Z}$ along the terminal chain of length 2. As above, E_0 is the central node, and we denote by m_0, m_1, m_2, the coefficients of $\mathbf{Z}$ corresponding to E_0, E_1, E_2, respectively. We have shown above that $m_0 = p^2 + p$. We have

$$0 \geq (\mathbf{Z} \cdot E_1) = (m_0 E_0 + m_1 E_1 + m_2 E_2) \cdot E_1 = p^2 + p - m_1(p+1)/2 + m_2.$$
$$0 \geq (\mathbf{Z} \cdot E_2) = (m_1 E_1 + m_2 E_2) \cdot E_2 = m_1 - 4m_2.$$

This gives $m_1 \geq 4(p^2+p)/(2p+1) > 2p$. Since $\mathbf{Z} \leq Z_0$, we have $m_1 \leq 2p+1$ and, thus, $m_1 = 2p+1$. From $m_2 \geq m_1/4$, we conclude that $m_2 \geq (2p+1)/4$, and since m_2 is an integer, we must have $m_2 \geq (p+1)/2$. Again because $\mathbf{Z} \leq Z_0$, we have $m_2 \leq (p+1)/2$ and, hence, $m_2 = (p+1)/2$. Thus, the vector Z_0 described in the proposition is indeed the fundamental vector of N.

It is easy to compute that $\mathbf{Z}^2 = -(p+1)/2$. It is also an easy matter to check that the vector K satisfies the matricial condition defining the canonical cycle recalled in Section 10.2. Similarly, one checks that $(K + \mathbf{Z}) \cdot \mathbf{Z} = p^2 - p$. □

Remark 7.3. Let $G := \mathbb{Z}/p\mathbb{Z}$ and let $\operatorname{Spec} A^G$ be a G-quotient singularity. Let $X \to \operatorname{Spec} A^G$ be a resolution of singularities with an exceptional divisor having smooth components and normal crossings. It is known that the fundamental cycle $\boldsymbol{Z}$ associated with the intersection matrix of the exceptional divisor satisfies $|\boldsymbol{Z}^2| \leq p$; see [Lorenzini 2013, 2.4]. It is not immediate to produce examples of such singularities where $|\boldsymbol{Z}^2| < p$. We note that the singularities exhibited in Theorems 6.3 and 7.1 have $|\boldsymbol{Z}^2| = 2$ and $|\boldsymbol{Z}^2| = (p+1)/2$, respectively.

It is shown in [Lorenzini 2013, Lemma 3.7], that if the discriminant group Φ_N of an intersection matrix N is killed by e, then the fundamental cycle $\boldsymbol{Z}$ associated with N satisfies $|\boldsymbol{Z}^2| \leq ez_{min}$, where z_{min} is the smallest coefficient of $\boldsymbol{Z}$. In the case of the intersection matrix in Theorem 7.1, $z_{min} = (p+1)/2$ and $|\Phi_N| = 1$, showing that the inequality $|\boldsymbol{Z}^2| \leq ez_{min}$ is sharp.

8. Analogues of the E_7 singularities

When $p = 2$, the blow-up at the maximal ideal of the $\mathbb{Z}/2\mathbb{Z}$-quotient singularity E_8^2 given by

$$z^2 + xy^2z + x^3 + y^5 = 0$$

has a new singularity, namely the singularity E_7^1 given by the equation

$$z^2 + xy^2z + yx^3 + y^3 = 0;$$

see for instance [Roczen 1992, 1.1]. The singularity E_8^2 has resolution graph the Dynkin diagram E_8 with trivial discriminant group, while the resolution of E_7^1 has resolution graph E_7 with discriminant group of order 2.

Artin [1977, bottom of page 18] (or Peskin [1980, (2.16), page 104]) shows that the Dynkin diagram E_7 cannot be obtained as the resolution graph of a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity whose associated action is ramified precisely at the origin. He shows however that the singularity E_7^1 does occur as the resolution graph of a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity for an action that is ramified in codimension 1.

When $p = 2$, we have not been able to exhibit any wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity whose action is *ramified precisely at the origin* and whose associated intersection matrix has discriminant group of order 2^s with s odd. We suggest in Example 8.3 for each s odd the existence of explicit examples with group $(\mathbb{Z}/2\mathbb{Z})^s$. In each case, these wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities are associated to actions that are ramified in codimension 1.

The above considerations have analogues for any prime p. Indeed, consider the singularity at the maximal ideal of $\operatorname{Spec} B_n$, where

$$B_n := k[\![x, y, z]\!]/(z^p - (xy^n)^{p-1}z - y^{pn+1} + x^{p+1}).$$

This singularity is a special case of the singularity recalled in Section 0.2, where we have set $\mu = 1$, $a = y^n$, and $b = x$. In particular, this singularity is a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity whose moderately ramified action is ramified precisely at the origin. When $n = p = 2$, this singularity is E_8^2.

Consider the blow-up of Spec B_n at the maximal ideal (x, y, z). Then the y-chart (defined by the variables $y, x/y, z/y$) has a singular point whose local ring is isomorphic to the local ring C_n, where

$$C_n := k[\![x, y, z]\!]/(z^p - (xy^n)^{p-1}z - y^{(n-1)p+1} + yx^{p+1}). \tag{8-1}$$

When $n > 1$, the closed point of Spec C_n is singular, and we show below in Proposition 8.1 that the singularity of Spec C_n is again a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity, but for an action that is ramified in codimension 1. In the examples that we were able to compute, the discriminant groups Φ_{B_n} and Φ_{C_n} of the intersection matrices of the resolutions of Spec B_n and Spec C_n when $n > 1$ satisfy $|\Phi_{C_n}| = p|\Phi_{B_n}|$.

When $n = 2$, the singularity of Spec B_2 is treated in Theorem 7.1 and generalizes the E_8^2-singularity. The singularity of Spec C_2 is the E_7^1-singularity when $p = 2$, and thus Spec C_2 is a natural generalization for all primes p of the E_7^1-singularity. Our educated guess for the resolution of Spec C_2 is discussed in Example 8.4.

We can further generalize the ring C_n as follows. Let $a, b \in k[\![x, y]\!]$, not both 0. Set

$$A_0 := k[\![x, y]\!][U, V]/(U^p - (ay)^{p-1}U - y,\ V^p - (by)^{p-1}V - xy).$$

Let L denote the field of fractions of A_0. The ring A_0 and the field L are endowed with an automorphism σ of order p fixing $k[\![x, y]\!]$ and with

$$\sigma(U) := U + ay, \quad \sigma(V) := V + by.$$

As usual, we set $G := \langle\sigma\rangle$. Let $z := aV - bU$. Then $\sigma(z) = z$, and we find that

$$z^p - (aby)^{p-1}z - a^p xy + b^p y = 0. \tag{8-2}$$

Let B denote the subring $k[\![x, y]\!][z]$ of A_0. Let A denote the subring $A_0\left[\frac{V}{U}\right]$ of L. The group G acts on A, since $\sigma(V/U) = (V/U + by/U)(1 + ay/U)^{-1}$ and $1 + ay/U$ is a unit in A_0.

Proposition 8.1. *Keep the above notation. The ring homomorphism $A \to k[\![u, v]\!]$, which sends U to u and V/U to v, is a k-isomorphism. In the special case where either $a = x^m$ and $b = y^\ell$, or $a = y^\ell$ and $b = x^m$ for some integers $\ell, m \geq 1$, then the ring of invariants A^G is equal to the ring B. In particular,* Spec C_n *is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity when $n > 1$.*

Proof. The equation $U^p - (ay)^{p-1}U - y = 0$ first shows that y/U is in the maximal ideal of A_0, and then that y/U^p is in A_0 and is a unit. The ring A_0 is not integrally closed, since it is clear from the equation $V^p - (by)^{p-1}V - xy = 0$ that

$$\left(\frac{V}{U}\right)^p - \left(\frac{by}{U}\right)^{p-1}\left(\frac{V}{U}\right) - \frac{y}{U^p}x = 0$$

is an integral relation for $\frac{V}{U}$ over A_0. Since x and y can be expressed in terms of U and V/U, we find that $A := A_0\left[\frac{V}{U}\right]$, viewed as a subring of L, is in fact isomorphic to the power series ring $k[\![u, v]\!]$, with $u := U$ and $v := V/U$.

Consider the ring

$$B' := k[\![x, y]\!][Z]/(Z^p - (aby)^{p-1}Z - a^p xy + b^p y)$$

and the natural map $\varphi : B' \to A^G$ which sends Z to z. Assume that either $a = x^\ell$ and $b = y^m$, or that $a = y^\ell$ and $b = x^m$ for some integers $\ell, m \geq 1$. We claim that φ is an isomorphism. One can show that B' is an integral domain, and that its field of fractions injects in $\mathrm{Frac}(A)$, and has image by degree considerations equal to $\mathrm{Frac}(A^G)$. The ring B' is Cohen–Macaulay since it is free as a module over the regular ring $k[\![x, y]\!]$. Thus B' is normal as soon as it is regular in codimension 1. This can be shown, because of the special forms of a and b, by using the Jacobian criterion. Let $f := Z^p - (aby)^{p-1}Z - a^p xy + b^p y$. Then if a prime ideal $\mathfrak{p}$ of B' contains the classes of f, and of the partial derivatives f_x, f_y, f_Z, then $\mathfrak{p}$ contains (x, y, Z).

The reader will check that when $n > 1$, the ring C_n is isomorphic to B when $a = -x$ and $b = -y^{n-1}$. When $p = n = 2$, the proposition is proved in [Peskin 1980, (2.16), page 104]. □

Example 8.2. We show in this example that there are (many) intersection matrices N with Φ_N killed by 2 and of order 2^s with s odd. Since our interest is to provide evidence that there may exist wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities whose resolutions have discriminant groups of order 2^s with s odd, we note that any such resolution must also have an intersection matrix N whose fundamental cycle $\boldsymbol{Z}$ satisfies $|\boldsymbol{Z}^2| \leq 2$ [Lorenzini 2013, 2.4]. This is a nontrivial restriction on the possible matrices N, and we exhibit below matrices that also satisfy this restriction.

Recall that a star-shaped graph with $n \geq 4$ vertices is called a *star*, or the *complete bipartite graph* $K_{1,n-1}$, if it consists of a single node and $n-1$ terminal vertices attached to the node. We write the intersection matrix N of a star on n vertices as $N = N(s_0 \mid s_1/1, \ldots, s_{n-1}/1)$, where $-s_0$ denotes the self-intersection of the node, and $-s_i$ denotes the self-intersection of the i-th terminal vertex when $i > 0$. The Dynkin diagram D_4 is a star on 4 vertices, and so are the two graphs in Remark 4.11.

Consider any intersection matrix $N = N(s_0 \mid s_1/1, \ldots, s_{n-1}/1)$ such that one of the s_j with $j \geq 1$ is even and at most one of the s_j with $j \geq 1$ is divisible by 4. Assume in addition that Φ_N is killed by 2, and that the fundamental cycle $\boldsymbol{Z}$ of N satisfies $|\boldsymbol{Z}^2| \leq 2$. Define the matrix $N_i(s_0 \mid s_1/1, \ldots, s_{n-1}/1, s_n/1)$, $i = 1, 2$, by

$$s_n := i + \left(\prod_{j=1}^{n-1} s_j\right)/|\Phi_N|.$$

We claim that the two intersection matrices N_1 and N_2 have graphs that are stars on $n+1$ vertices with $|\det(N_i)| = i|\det(N)|$. Moreover, both groups Φ_{N_i} are killed by 2, and both fundamental cycles $\boldsymbol{Z}_i$ of N_i satisfy $|\boldsymbol{Z}_i^2| \leq 2$.

Proof. Let $\ell_{n-1} := \mathrm{lcm}(s_1, \ldots, s_{n-1})$. Then the order of the node in Φ_N is equal to $\ell_{n-1}\big(s_0 - \sum_{j=1}^{n-1} 1/s_j\big)$, use Proposition 1.3(ii). This order equals 1 since we assume that one of the s_j is even, use Proposition 1.3(v). It follows that $|\Phi_N| = \big(\prod_{j=1}^{n-1} s_j\big)/\ell_{n-1}$ (use Proposition 1.3(i)). In particular, $\big(\prod_{j=1}^{n-1} s_j\big)/|\Phi_N| = \ell_{n-1}$ is an integer. The equality $|\det(N_i)| = i|\det(N)|$ follows from an easy computation.

We find that $\operatorname{lcm}(s_1, \dots, s_{n-1}, \ell_{n-1}+i) = \operatorname{lcm}(\ell_{n-1}, \ell_{n-1}+i)$, which equals $\ell_{n-1}(\ell_{n-1}+1)$ when $i = 1$, and $\ell_{n-1}(\ell_{n-1}/2+1)$ when $i = 2$. Hence, the node is trivial in Φ_{N_i} since its order is

$$\operatorname{lcm}(s_1, \dots, s_{n-1}, \ell_{n-1}+i)\left(s_0 - \sum_{j=1}^{n-1} 1/s_j - 1/(\ell_{n-1}+i)\right) = 1.$$

Let $R \in \mathbb{Z}^{n+1}$ denote the transpose of the vector $(\ell_{n-1}, \ell_{n-1}/s_1, \dots, \ell_{n-1}/s_{n-1}, 1)$. Then $N_i R = -i e_{n+1}$. Since all coefficients of R are positive and $N_i R$ has nonpositive coefficients, we find that R is an upper bound for the fundamental cycle $\boldsymbol{Z}_i$ of N_i. Then $|\boldsymbol{Z}_i^2| \le |R^2| \le i$, as desired.

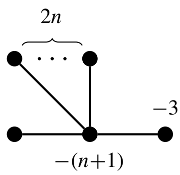
To show that Φ_{N_i} is killed by 2, it suffices to show that the classes of the standard vectors have order 1 or 2 in Φ_{N_i} for each terminal vertex of the graph. This is clear for a terminal vertex v_j with s_j odd or exactly divisible by 2, since the column of N_i corresponding to v_j shows that the class of $s_j v_j$ is equal to the class of the node. We note now that the construction implies that there can be at most one terminal vertex v_j with s_j divisible by 4. If the corresponding class in Φ_{N_i} has order divisible by 4, we would find using the first column of the matrix N_i that this unique class is equal to the sum of classes which all have order 1 or 2, a contradiction. This ends the proof of the claim. □

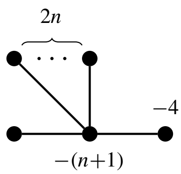
The sequence $\{s_n\}_{n\ge1}$ with $s_1 = 2$ and $s_n := \operatorname{lcm}(s_1, \dots, s_{n-1}) + 1$ is called *Sylvester's sequence* $\{2, 3, 7, 43, \dots\}$ in the literature. It produces the only intersection matrices $N(1 \mid s_1/1, \dots, s_{n-1}/1)$ with trivial group Φ_N in the above construction.

An example of a star with intersection matrix N such that Φ_N is killed by 2 but $|\boldsymbol{Z}^2| > 2$ is given by $N = N(1 \mid 2/1, 3/1, 10/1, 16/1)$, with group $\Phi_N = (\mathbb{Z}/2\mathbb{Z})^2$ and $\boldsymbol{Z} = (30, 15, 10, 3, 2)$, giving $|\boldsymbol{Z}^2| = 4$.

Example 8.3. Let $p = 2$. Fix an integer $n \ge 1$. Consider the star graph with a central node of self-intersection $-(n+1)$ attached to $2n+1$ terminal vertices of self-intersection -2. Denote by N_0 its intersection matrix. Proposition 1.3(iv) shows that $\Phi_{N_0} = (\mathbb{Z}/2\mathbb{Z})^{2n}$. We remark in passing that this matrix does occur as the intersection matrix attached to a quotient singularity (use the equation $z^2 = xy(x^{2n-1} - y^{2n-1})$ and Theorem 5.3(ii)).

Starting with N_0, the construction in Example 8.2 produces two intersection matrices, the matrix $N_1(n) := N(n+1 \mid 2/1, \dots, 2/1, 3/1)$ with group of order 2^{2n} and whose graph is represented on the left below, and the matrix $N_2(n) := N(n+1 \mid 2/1, \dots, 2/1, 4/1)$ with group of order 2^{2n+1} and whose graph is represented below on the right:





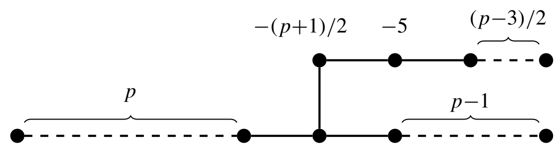
When $n = 1$, the intersection matrices $N_1(n)$ and $N_2(n)$ are the matrices of the resolutions of the wild quotient singularities Spec B_4 and Spec C_4, respectively. This can be verified using the Magma [Bosma et al. 1997] commands *ResolveSingByBlowUp*() and *IntersectionMatrix*().

When $n \geq 1$, consider $f := z^p - (aby)^{p-1}z - a^p xy + b^p y$ introduced in (8-2), and set $a := x^n$ and $b := y^{2n+1}$. Let $B := k[\![x, y, z]\!]/(f)$. Proposition 8.1 shows that the equation $f = 0$ defines a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. We conjecture that Spec B has a resolution $X \to \operatorname{Spec} B$ with a dual graph equal to the dual graph of $N_2(n)$ represented on the right above. The conjecture thus provides examples of wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities with discriminant group of order 2^{2n+1} for all $n \geq 1$. These quotient singularities are associated with actions that are ramified in codimension 1.

Example 8.4 (analogues of E_7). Let p be prime. Computations suggest that the resolution of the wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity Spec C_2 (see (8-1)) has intersection matrix (notation as in Section 1.2)

$$N = N\left(2 \,\middle|\, \frac{p}{p-1}, \frac{p+1}{p}, \frac{p^2}{2p-1}\right)$$

with group $\Phi_N = \mathbb{Z}/p\mathbb{Z}$. When p is odd, the intersection matrix N has the following graph:



The resolution of Spec B_2 is discussed in Theorem 7.1.

Remark 8.5. Consider the equation $z^p - (aby)^{p-1}z - a^p xy + b^p y = 0$ introduced in (8-2), and set $a = y^n$ and $b = x^m$ for some integers $m, n \geq 1$. Proposition 8.1 shows that this equation defines a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. Computations with Magma [Bosma et al. 1997] suggest that for such a and b, the resolution of the singularity at the origin of $z^p - (aby)^{p-1}z - a^p xy + b^p y = 0$ has the same intersection matrix as the resolution of the singularity of $z^p - a^p xy + b^p y = 0$.

When $a = y^n$ and $b = x^m$, this latter singularity has the form $z^p - xy(y^{pn} - x^{pm-1}) = 0$, and Theorem 4.4 provides an explicit resolution for it. When $p = 2$, we find that $g := \gcd(pn, pm-1)$ is always odd, so the discriminant group of this resolution, which has order 2^{g+1} by Proposition 4.9, is always of the form $|\Phi_N| = 2^s$ with s even. Thus the quotient singularity (8-2) in this case is unlikely to provide examples of discriminant groups of order $|\Phi_N| = 2^s$ with s odd.

When $p = 2$, (8-2) in the case $b = x$ and $a = y^n$ gives the equation of the singularity $D^n_{2(2n+1)}$ with resolution graph the Dynkin diagram $D_{2(2n+1)}$; notation as in [Artin 1977, Section 3].

9. D_4 and A_{p-1}

We compute in this section the resolution of the singularity of Spec B_μ introduced in Section 0.2, for any value of the parameter μ when $a = -y$ and $b = -x$. The ring B_μ is given in this case by

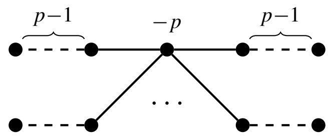
$$B_\mu := k[\![x, y]\!][z]/(z^p - (\mu xy)^{p-1}z - x^{p+1} + y^{p+1}).$$

Let $Z \to \mathrm{Spec}(B_\mu)$ be the blow-up of the ideal $\mathfrak{b} = (x, y, z)$, as in Proposition 3.6. We note in Theorem 9.4 that Z has $p+1$ singularities, each again $\mathbb{Z}/p\mathbb{Z}$-quotient singularities, with resolution graph A_{p-1} and associated discriminant group $\mathbb{Z}/p\mathbb{Z}$.

Remark 9.1. When k contains a third root of unity ζ with $\zeta^2+\zeta+1=0$, the change of variables $X := x+\zeta y$ and $Y := x+\zeta^2 y$ produces $x^3+y^3 = -\zeta XY(X+\zeta Y)$. In that case, for any integer $q \geq 1$, the singularity $z^q - (x^3+y^3) = 0$ is always isomorphic over k to the singularity $z^q - (x^2y - xy^2) = 0$. When in addition $p = 2$, we find that $B_{\mu=0}$ is isomorphic over $\mathbb{F}_4$ to the singularity D_4^0, given by the equation $z^2+x^2y+xy^2 = 0$. The dual graph of its resolution is the Dynkin diagram D_4. The Tjurina number of this singularity is equal to 8.

The resolution of Spec $B_{\mu=1}$ when $p=2$ is also known to have dual graph D_4 over an algebraically closed field. Indeed, the equation when $\mu = 1$ is stated to be equivalent to D_4^1 in [Peskin 1980, page 102], where D_4^1 is given by the equation $z^2+xyz+x^2y+xy^2 = 0$. The quotient singularity Spec $B_{\mu=1}$ when $p > 2$ can thus be considered as a generalization of D_4^1.

Theorem 9.2. *Assume that $p \geq 3$. Then* Spec B_μ *has a resolution of singularities with star-shaped dual graph Γ_N independent of μ having $p+1$ identical terminal chains, each with $p-1$ vertices, as follows*:



The associated discriminant group Φ_N has order p^p.

Proof. Let $Z \to \mathrm{Spec}(B_\mu)$ be the blow-up of the ideal $\mathfrak{a}B = (a, b, z) = (x, y, z)$, as in Proposition 3.6. Let as usual E denote the exceptional divisor. We find from Proposition 3.6 that E_{red} is a smooth rational curve over k, and that $(E \cdot E_{\mathrm{red}})_Z = -1$. In addition, $E = pE_{\mathrm{red}}$, and the z-chart is regular.

The blow-up Z is covered by three affine charts, and we see that the x-chart is generated by the expressions x, y/x, z/x modulo the relation

$$\left(\frac{z}{x}\right)^p - x\left(1+\mu^{p-1}x^{p-2}\left(\frac{y}{x}\right)^{p-1}\frac{z}{x} - \left(\frac{y}{x}\right)^{p+1}\right) = 0. \tag{9-1}$$

Clearly, this chart is regular at the origin. Let $Y \to Z$ denote the normalization of Z. Let D denote as usual the pull-back of the exceptional divisor of Z. It follows from the regularity at the origin that the induced morphism $D_{\mathrm{red}} \to E_{\mathrm{red}}$ is an isomorphism. Hence, we can conclude from Proposition 2.3 that $(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = -1/p$.

Using partial derivatives, one sees that the singular locus on the x-chart is given by $x = z/x = 0$ and $(y/x)^{p+1} = 1$. In particular, the singular locus is finite and, hence, Z is normal since it is Cohen–Macaulay and regular in codimension 1. We thus have $Y = Z$. Let ζ denote a primitive root of the equation $u^{p+1} = 1$. When rewriting the above equation defining the x-chart in terms of the expressions x, $y/x - \zeta^j$, and z/x, we obtain a polynomial of the form $x(y/x - \zeta^j) + O(3)$ when $p \geq 3$. Using the changes of variables discussed in the proof of Lemma 6.4, we find that the singularity is in fact a rational double point. Since (9-1) contains the monomial $(z/x)^p$ and no other monomial $(z/x)^i$ with $i < p$, we find that the rational double point is of type A_{p-1}.

Let $X \to Y$ denote a resolution of the singularities of Y. Let C denote the strict transform of D_{red} in X. It follows from Proposition 2.2 that $(C \cdot C)_X = -1/p - (p+1)\delta$, where δ is the correcting term associated with the rational double point A_{p-1}. As noted in Section 1.1, $\delta = (p-1)/p$, and we find that $(C \cdot C)_X = -p$. The associated discriminant group is computed with Proposition 1.3. □

Let $R := k[[x, y]]$. As recalled in Section 0.2, let

$$A := k[[u, v]] = R[u, v]/(u^p - (\mu y)^{p-1}u - x,\ v^p - (\mu x)^{p-1}v - y),$$

and let σ be the automorphism defined by $\sigma(u) = u + \mu y$ and $\sigma(v) = v + \mu x$. Let $G := \langle\sigma\rangle$. The element $z := xu - yv$ is invariant, and we can identify the ring B_μ with A^G.

Let $Z' \to \mathrm{Spec}(A)$ be the blow-up of the induced ideal $\mathfrak{a}A$, with $\mathfrak{a} = (x, y, z)$. Let $Y' \to Z'$ denote the normalization of Z'. We have the commutative diagram:

$$\begin{tikzcd} & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & \mathrm{Spec}(A^G) \end{tikzcd}$$

Let y_i, $i = 1, \dots, p+1$, denote the rational double points in Y of type A_{p-1}. We show below that these points are in fact $\mathbb{Z}/p\mathbb{Z}$-quotient singularities.

Lemma 9.3. *The scheme Y' is regular, and the morphism $Y' \to \mathrm{Spec}(A)$ coincides with the blow-up of the maximal ideal $\mathfrak{m}_A = (u, v)$.*

Proof. Indeed, using the relations

$$u^p - (\mu y)^{p-1}u = x \quad \text{and} \quad v^p - (\mu x)^{p-1}v = y, \tag{9-2}$$

we get $u^p, v^p \in \mathfrak{a}A$. Since the finite ring extension $R \subset A$ is flat of degree p^2, we must have $\mathfrak{a}A = (u^p, v^p)$. More precisely, substituting the equations (9-2) into each others one obtains

$$x \cdot \mathrm{unit} = u^p - \mu^{p-1}v^{p(p-1)}u \quad \text{and} \quad y \cdot \mathrm{unit} = v^p - \mu^{p-1}u^{p(p-1)}v,$$

showing explicitly that $(x, y)A \subseteq (u^p, v^p)$. Since $z = xu - yv$, we have $(u^p, v^p) = \mathfrak{a}A$.

The blow-up Z' of the ideal (u^p, v^p) in $\mathrm{Spec}(A)$ is covered by two charts. The u^p-chart has generators u, v, and v^p/u^p, so v/u satisfies an obvious integral equation, and we also have $v = v/u \cdot u$. It follows that on the normalization the chart becomes regular. The situation on the v^p-chart is similar, and we see that the scheme Y' is regular. □

Theorem 9.4. *The preimage of each y_i under the map $Y' \to Y$ consists of a single regular point $x_i \in Y'$, and $\mathscr{O}_{Y,y_i} = (\mathscr{O}_{Y',x_i})^G$. Thus y_i is a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity whose resolution has dual graph A_{p-1} and associated discriminant group $\mathbb{Z}/p\mathbb{Z}$. The morphism* $\operatorname{Spec} \mathscr{O}_{Y',x_i} \to \operatorname{Spec}(\mathscr{O}_{Y',x_i})^G$ *is ramified in codimension* 1 *and the punctured spectrum of the rational double point y_i has trivial fundamental group.*

Proof. The G-action on the ring A induces a G-action on the normalized blow-up Y', which on the field of fractions of the u-chart is given by

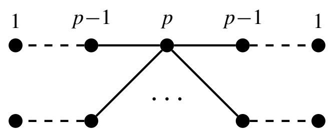
$$u \mapsto u + \mu y \quad \text{and} \quad v/u \mapsto (v + \mu x)/(u + \mu y).$$

Since Y is normal, the induced morphism $Y' \to Y$ yields an identification $Y = Y'/G$.

Let E' denote the exceptional divisor of the blow-up $Y' \to \operatorname{Spec}(A)$ of the maximal ideal. Then the natural map $E' \to D_{\text{red}}$ induced by $Y' \to Y$ is purely inseparable of degree p, and, hence, the morphism $\operatorname{Spec} \mathscr{O}_{Y',x_i} \to \operatorname{Spec}(\mathscr{O}_{Y',x_i})^G$ is ramified at the codimension 1 point corresponding to E'. It follows from [Artin 1977, Corollary 1.2], that the punctured spectrum of the rational double point y_i has trivial fundamental group. $\square$

Remark 9.5. The occurrence of the A_{p-1}-singularities y_i on the quotient $Y = Y'/G$ is caused by points $x_i \in Y'$ where the ideal of the fixed scheme $Y'^G \subset Y'$ is not a Cartier divisor. Indeed, using Theorem 2 in [Király and Lütkebohmert 2013], we find that when the action of σ on the local ring $A = k[[u, v]]$ is such that the ideal $(\sigma(u) - u, \sigma(v) - v)$ of the fixed scheme is principal, then the fixed ring $A^{\langle\sigma\rangle}$ is regular.

Proposition 9.6. *The multiplicities in the fundamental cycle $\mathbf{Z}$ of the resolution of* $\operatorname{Spec} B_\mu$ *are strictly decreasing along each terminal chain, as indicated below next to the corresponding vertex*:



The canonical cycle of the singularity is $K = -(p-2)\mathbf{Z}$. Moreover, $\mathbf{Z}^2 = -p$ and $h^1(\mathcal{O}_{\mathbf{Z}}) = \frac{1}{2}(p-2)(p-1)$.

Proof. For convenience in this proof, let us denote by Z_0 the vector whose coefficients are given in the proposition. Since NZ_0 has nonpositive coefficients, we find that by definition of the fundamental cycle, we must have $\mathbf{Z} \le Z_0$. In particular, the multiplicities in $\mathbf{Z}$ of the terminal vertices of the graph must be all equal to 1. Since the fundamental cycle $\mathbf{Z}$ is unique, it is easy to check that it must be "rotationally symmetric" around the central node. Let us denote by $z_{p-1}, \ldots, z_2, z_1$ the multiplicities of $\mathbf{Z}$ along a terminal chain. The central node has multiplicity denoted z_p, and the fact that $N\mathbf{Z}$ has nonpositive coefficients produces the following inequalities. First at the central node, we find

$$-pz_p + (p+1)z_{p-1} \le 0.$$

It follows that $z_p - z_{p-1} \ge z_{p-1}/p$, so that $z_p - z_{p-1} \ge 1$. At each other vertex, we find that

$$-2z_i + z_{i-1} + z_{i+1} \le 0.$$

It follows that $z_{i+1} - z_i \le z_i - z_{i-1}$. Hence, for each $i = 2, \dots, p$, we have $z_i - z_{i-1} \ge 1$. Since $\mathbf{Z} \le Z_0$, we must have $\mathbf{Z} = Z_0$.

It is easy to compute that $\mathbf{Z}^2 = -p$. It is also an easy matter to check that the vector K satisfies the matricial condition defining the canonical cycle recalled in Section 10.2. Similarly, one checks that $(K + \mathbf{Z}) \cdot \mathbf{Z} = p^2 - 3p$. □

10. Numerically Gorenstein intersection matrices

All wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities resolved in this article are hypersurface singularities. We prove in this section that all wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities are hypersurface singularities. We then recall that the intersection matrix associated with a hypersurface singularity is always numerically Gorenstein. We show in Proposition 10.5 that any intersection matrix N whose discriminant group Φ_N is killed by 2 is automatically numerically Gorenstein. We exhibit in Example 10.7 an example when $p > 2$ of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity which is not numerically Gorenstein.

Proposition 10.1. *Let $p = 2$. Let $A = k[[u, v]]$, endowed with a nontrivial action of $G = \mathbb{Z}/2\mathbb{Z}$. Then there exists a power series ring $R := k[[x, y]]$ such that A^G is k-isomorphic to $R[z]/(z^2 + sz + t)$, with $s, t \in R$.*

Proof. Let σ denote the generator of G. Proposition 2.9 in [Lorenzini and Schröer 2020] allows us, if necessary, to replace the system of parameters (u, v) for A with a new system of parameters (again denoted by (u, v) below) with the following properties (use [loc. cit., Proposition 2.3]): let $x := u\sigma(u)$ and $y := v\sigma(v)$. Let $R := k[[x, y]]$ be the subring of A generated by k, x, and y. Then A is a free R-module of rank 4.

We have the inclusions $R \subset A^G \subset A$, and the fraction field of A^G is then of degree 2 over the fraction field of R. Since R is regular and A^G is Cohen–Macaulay because it is normal of dimension 2, we find that A^G is a free R-module of rank 2. Thus, R is a direct summand of A^G, with quotient A^G/R free of rank 1. We can therefore find an element $z \in A^G$ which generates the quotient A^G/R. It follows that the natural map $R[Z] \to A^G$ with $Z \mapsto z$ is surjective. Since $z \notin R$, it satisfies a quadratic equation $z^2 + sz + t = 0$, with $s, t \in R$ and $Z^2 + sZ + t$ irreducible in $R[Z]$. Since $R[Z]$ is a UFD, we find that $R[Z]/(Z^2 + sZ + t) \to A^G$ is an isomorphism. □

10.2. Let $N = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an intersection matrix. Let $H_0 \in \mathbb{Z}^n$ be the integer vector whose i-th coefficient is $h_i := -c_{ii} - 2$ for $i = 1, \dots, n$. Since N is invertible, there exists a vector $K \in \mathbb{Q}^n$ such that $NK = H_0$. The vector K is called the *canonical cycle* of N. We say that N is *numerically Gorenstein* if $K \in \mathbb{Z}^n$.

When N is the intersection matrix associated with a collection of irreducible curves C_i, $i = 1, \dots, n$ on a surface, each component C_i has an arithmetical genus $p_a(C_i)$. Our definition of numerically Gorenstein coincides with the usual one (see for instance [Popescu-Pampu and Seade 2009, (2.5)]) when all arithmetical genera are equal to 0. When a matrix N is numerically Gorenstein and $\mathbf{Z}$ denotes its fundamental cycle, then $-K \ge \mathbf{Z}$, unless the dual graph of N is the dual graph of a rational double point [Laufer 1987, Proposition 2.1; Popescu-Pampu and Seade 2009, Proposition 2.4].

Lemma 10.3. *Let k be a field of characteristic p. Let B denote a complete local ring of dimension 2, isomorphic to $k[\![x, y, z]\!]/(f)$ for some $f \in (x, y, z)$, and formally smooth outside its closed point. Let $X \to \operatorname{Spec} B$ be a resolution of the singularity, with associated intersection matrix N. Assume that all the irreducible components in the exceptional locus of the resolution are smooth rational curves. Then N is numerically Gorenstein.*

Proof. We first use [Artin 1969, 3.8], to find an algebraic scheme S over k and a point $s \in S$ such that the completion of $\mathcal{O}_{S,s}$ is isomorphic to B. The ring $\mathcal{O}_{S,s}$ is Gorenstein since its completion B is [Eisenbud 1995, 21.18]. Thus there exists an open set U of S, containing s, and such that U is everywhere Gorenstein [Greco and Marinari 1978, 1.5]. It follows that U has a canonical sheaf that is an invertible sheaf. Consider a resolution $\pi : V \to U$ of the singularity $s \in U$. Then the canonical divisor K_V on V is supported on the exceptional divisor of π. The adjunction formula for each irreducible component E_i shows that $(K_V \cdot E_i) + (E_i \cdot E_i) = 2p_a(E_i) - 2$. Since K_V is equal to a linear combination of the E_i, we find that the intersection matrix N of the exceptional locus is numerically Gorenstein. □

Let $N = (c_{ij}) \in \operatorname{Mat}_n(\mathbb{Z})$ be an intersection matrix with discriminant group Φ_N. As usual, denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n$, and let p_i denote the order of the class of e_i in Φ_N. For each $i = 1, \ldots, n$, let $R_i \in \mathbb{Z}^n$ denote the unique positive vector such that $NR_i = -p_i e_i$. Let $(R_i)_j$ denote the j-th coefficient of R_i, and define

$$g_i := \sum_{j=1}^{n} (R_i)_j (|c_{jj}| - 2) = ({}^t R_i) H_0.$$

If the matrix N is such that $c_{jj} \leq -2$ for all $j = 1, \ldots, n$, then $g_i \geq 0$.

Lemma 10.4. *Let N be an intersection matrix. Then ${}^t K = (-g_1/p_1, \ldots, -g_n/p_n)$. In particular, the matrix N is numerically Gorenstein if and only if p_i divides g_i for each $i = 1, \ldots, n$.*

Proof. By hypothesis, we have $NK = H_0$ for some vector $K \in \mathbb{Q}^n$. It follows that $-p_i K_i = {}^t R_i N K = {}^t R_i H_0 = g_i$, and we find that $K_i = -g_i/p_i$. □

Proposition 10.5. *Let $N = (c_{ij}) \in \operatorname{Mat}_n(\mathbb{Z})$ be an intersection matrix with discriminant group Φ_N killed by 2. Then N is numerically Gorenstein.*

Proof. Our hypothesis implies that $p_i = 1$ or 2, for all $i = 1, \ldots, n$. We use the criterion given in Lemma 10.4: To show that N is numerically Gorenstein, it suffices to show, for each i, that the integer g_i is even when $p_i = 2$. Assume then that $p_i = 2$. Then by construction,

$${}^t R_i N R_i = -p_i (R_i)_i.$$

We now compute explicitly the term ${}^t R_i N R_i$ and obtain

$${}^t R_i N R_i = \sum_{j=1}^{n} c_{jj} (R_i)_j^2 + 2 \sum_{j<k} c_{jk} (R_i)_j (R_i)_k.$$

Since p_i is even and $(R_i)_j^2 \equiv (R_i)_j \pmod 2$, we find that $\sum_{j=1}^{n} c_{jj} (R_i)_j$ is even, and so is g_i, as desired. □

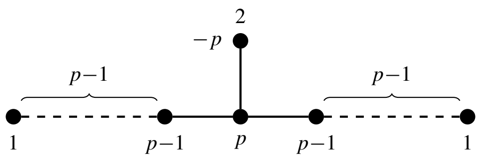
Remark 10.6. Let $N = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an intersection matrix associated with the resolution of a hypersurface singularity, all of whose exceptional components are smooth rational curves. Assume that $c_{ii} \leq -2$ for all $i = 1, \dots, n$. Laufer [1987, 3.7] provides additional constraints on the canonical vector K associated with such N, with an improvement by M. Tomari stated in the addendum on page 496 of [loc. cit.]. A further improvement was found by Yau [1989, Theorems B and C], which show that for such N,

$$g_i/p_i \geq (|\boldsymbol{Z} \cdot \boldsymbol{Z}| - 2)z_i,$$

where ${}^t\boldsymbol{Z} = (z_1, \dots, z_n)$ is the fundamental cycle of N. In other words, we have $-K \geq (|\boldsymbol{Z} \cdot \boldsymbol{Z}| - 2)\boldsymbol{Z}$. Note that the singularity in Proposition 9.6 satisfies $-K = (|\boldsymbol{Z} \cdot \boldsymbol{Z}| - 2)\boldsymbol{Z}$.

In the context of wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities treated in this article, the resolution of such a singularity has intersection matrix N with Φ_N killed by 2 and with $|\boldsymbol{Z} \cdot \boldsymbol{Z}| \leq 2$. Proposition 10.5 shows that any such N is always numerically Gorenstein, and since $|\boldsymbol{Z} \cdot \boldsymbol{Z}| \leq 2$ and $\boldsymbol{Z} > 0$, Laufer's constraints are also automatically satisfied.

Example 10.7. We exhibit below a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity that is not numerically Gorenstein. Let $p > 2$ be prime and consider the wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity in [Lorenzini 2014, 6.8], with resolution graph with $r_1(i) = 1$. This resolution graph has a single vertex of self-intersection different from -2, namely the terminal vertex C with $r_1(i) = 1$ and self-intersection $-p$, represented as the top center vertex in the graph below:



The graph is adorned with the coefficients of an integer vector R, and it is easy to check that the canonical vector K is $-(p-2)R/p$. Since $p > 2$, the vector K is not an integer vector. The fundamental cycle of the singularity is given in [Lorenzini 2018, 4.4], and it is shown in [loc. cit., 4.1], that this singularity is rational.

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Twisted derived equivalences and isogenies between K3 surfaces in positive characteristic

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We study isogenies between K3 surfaces in positive characteristic. Our main result is a characterization of K3 surfaces isogenous to a given K3 surface X in terms of certain integral sublattices of the second rational ℓ-adic and crystalline cohomology groups of X. This is a positive characteristic analog of a result of Huybrechts (*Comment. Math. Helv.* **94**:3 (2019), 445–458), and extends results of Yang (*Int. Math. Res. Not.* **2022**:6 (2022), 4407–4450). We give applications to the reduction types of K3 surfaces and to the surjectivity of the period morphism. To prove these results we describe a theory of B-fields and Mukai lattices in positive characteristic, which may be of independent interest. We also prove some results on lifting twisted Fourier–Mukai equivalences to characteristic 0, generalizing results of Lieblich and Olsson (*Ann. Sci. Éc. Norm. Supér.* (4) **48**:5 (2015), 1001–1033).

1. Introduction

The purpose of this paper is to study twisted Fourier–Mukai partners of K3 surfaces in positive characteristics and to develop an isogeny theory for these surfaces which is analogous to that of abelian varieties.

Let k be an algebraically closed field and p be a prime number. When $\operatorname{char} k = p$, we simply write W for the ring of Witt vectors $W(k)$. Let $\widehat{\boldsymbol{Z}}^p$ denote the prime-to-p part of $\widehat{\boldsymbol{Z}}$. For a variety Y over k, we set $\mathrm{H}^*(Y) := \mathrm{H}^*_{\text{ét}}(Y, \widehat{\boldsymbol{Z}})$ if $\operatorname{char} k = 0$, and $\mathrm{H}^*(Y) := \mathrm{H}^*_{\text{ét}}(Y, \widehat{\boldsymbol{Z}}^p) \times \mathrm{H}^*_{\text{cris}}(Y/W)$ if $\operatorname{char} k = p$, and write $\mathrm{H}^*(Y)_{\boldsymbol{Q}} := \mathrm{H}^*(Y) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$.

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Definition 1.1 (cf. [Yang 2022, Definition 1.1]). Let X and X' be K3 surfaces over k. An isogeny $f : X \rightsquigarrow X'$ is a correspondence, i.e., a $\boldsymbol{Q}$-linear combination of algebraic cycles on $X \times X'$, such that the induced action $\mathrm{H}^2(X')_{\boldsymbol{Q}} \to \mathrm{H}^2(X)_{\boldsymbol{Q}}$ is an isomorphism which preserves the Poincaré pairing. Two isogenies are deemed equivalent if they induce the same map $\mathrm{H}^2(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}^2(X)_{\boldsymbol{Q}}$.

Our main results concern the existence and uniqueness of isogenies with prescribed cohomological action. We begin with the former. A natural source for isogenies between K3 surfaces is provided by twisted Fourier–Mukai equivalences: For a K3 surface X and Brauer class $\alpha \in \mathrm{Br}(X)$, we denote by $D^b(X, \alpha)$ the bounded derived category of α-twisted sheaves. Given another K3 surface X' and Brauer class α', an equivalence $D^b(X, \alpha) \xrightarrow{\sim} D^b(X', \alpha')$ induces, up to some choices, an isogeny $f : X \rightsquigarrow X'$. We call isogenies which arise this way *primitive derived isogenies*, and compositions of such isogenies *derived isogenies*. The precise definitions are given in Section 4. There we also give a motivic reformulation of the above definition, which will be used for the rest of the paper.

To state our theorems, we denote the K3 lattice $\boldsymbol{U}^{\oplus 3} \oplus \boldsymbol{E}_8^{\oplus 2}$ by Λ and recall Ogus's notion of K3 crystals [1979, Definition 3.1]. Here $\boldsymbol{U}$ denotes the standard hyperbolic plane and $\boldsymbol{E}_8$ denotes the unique unimodular even negative definite lattice of rank 8. Our first theorem is an existence result on derived isogenies:

Theorem 1.2. *Assume* $\operatorname{char} k = p \geq 5$. *Let X be a* K3 *surface over k. Endow $\Lambda \otimes W$ with a* K3 *crystal structure and denote it by H_p and let H^p denote $\Lambda \otimes \widehat{\boldsymbol{Z}}^p$.*

Let $\iota : H^p \times H_p \hookrightarrow \mathrm{H}^2(X)_{\boldsymbol{Q}}$ be an isometric embedding which respects the Frobenius actions on H_p and $\mathrm{H}^2_{\mathrm{cris}}(X/W)[1/p]$. There exists a derived isogeny $f : X \rightsquigarrow X'$ to another K3 *surface X' such that $f^*(\mathrm{H}^2(X')) = \mathrm{im}(\iota)$ if and only if ι sends the slope < 1 part of H_p isomorphically onto that of $\mathrm{H}^2_{\mathrm{cris}}(X/W)$.*

We refer the reader to Remark 6.10 for the reason to restrict to $p \geq 5$. The above result is inspired by a theorem of Huybrechts [2019, Theorem 0.1], which can be stated as follows in our terminology:

Theorem 1.3 (Huybrechts). *Let X and X' be two* K3 *surfaces over $\boldsymbol{C}$. Every isomorphism of Hodge structures $\mathrm{H}^2(X', \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{H}^2(X, \boldsymbol{Q})$ which preserves the Poincaré pairings is induced by a derived isogeny $f : X \rightsquigarrow X'$.*

This refines an earlier theorem of Buskin [2019, Theorem 1.1], which affirms a conjecture of Shafarevich. Using the global Torelli theorem and surjectivity of the period map, one checks that Huybrechts' theorem is equivalent to an existence theorem for isogenies: for every K3 surface X over $\boldsymbol{C}$ and every isometric embedding $\iota : \Lambda \hookrightarrow \mathrm{H}^2(X, \boldsymbol{Q})$, there exists another K3 surface X' over $\boldsymbol{C}$ and a derived isogeny $f : X \rightsquigarrow X'$ such that $f^*(\mathrm{H}^2(X', \boldsymbol{Z})) = \mathrm{im}(\iota)$ (see Section 6D). Note that this statement does not involve Hodge structures. Our Theorem 1.2 is a positive characteristic analog for this version of Huybrechts' theorem.

Huybrechts' refinement shows in particular that every isogeny between K3 surfaces over $\boldsymbol{C}$ is equivalent to a derived isogeny. In contrast, the "only if" part of Theorem 1.2 implies that the cohomological actions of derived isogenies in characteristic p obey a certain nontrivial constraint at p. In particular, not every isogeny is equivalent to a derived isogeny. Given this, it is of interest to characterize also the possible cohomological actions of all (not necessarily derived) isogenies. The following result shows that, under some technical assumptions, the "if" part of Theorem 1.2 can be removed for $k = \bar{\boldsymbol{F}}_p$ if one is willing to consider all isogenies:

Theorem 1.4. *Let X, H_p, H^p and ι be as in Theorem 1.2. If $k = \bar{\boldsymbol{F}}_p$ and*

(a) $\mathrm{Pic}(X)$ *has rank ≥ 12 or contains a standard hyperbolic plane, or*

(b) $\mathrm{Pic}(X)$ *contains an ample line bundle L of degree $L^2 < p-4$,*

then there exists another K3 *surface X' over k and an isogeny $f : X \rightsquigarrow X'$ such that $f^*(\mathrm{H}^2(X')) = \mathrm{im}(\iota)$.*

This is a strengthening of [Yang 2022, Theorem 1.4]. We mention that a byproduct in the course of proving the above is a generalization (Theorem 6.18) of Taelman's characterization [2020, Theorem C] of the canonical liftings of ordinary K3 surfaces. Nygaard and Ogus [1985] constructed, for every nonsupersingular K3 surface X, a "section" to the natural morphism $\mathrm{Def}(X) \to \mathrm{Def}(\widehat{\mathrm{Br}}_X)$ from the deformation space of X to that of its formal Brauer group, such that a lifting of $\widehat{\mathrm{Br}}_X$ induces a lifting of X. We call liftings of X which arise this way "Nygaard–Ogus liftings". When X is ordinary, a Nygaard–Ogus lifting is the same as a canonical lifting. Theorem 6.18 gives an integral p-adic Hodge-theoretic characterization of Nygaard–Ogus liftings. See Section 6E for details.

We now describe our uniqueness results. We recall some terminology from [Yang 2022, §6]: an isogeny $f : X \rightsquigarrow X'$ between K3 surfaces is said to be *polarizable* if the induced map $\mathrm{Pic}(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{Pic}(X)_{\boldsymbol{Q}}$ sends an ample class to another ample class, and $\boldsymbol{Z}$*-integral* if the induced isomorphism $\mathrm{H}^2(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}^2(X)_{\boldsymbol{Q}}$ restricts to an isomorphism $\mathrm{H}^2(X') \xrightarrow{\sim} \mathrm{H}^2(X)$. We prove the following Torelli theorem for derived isogenies:

Theorem 1.5. *Assume* $\mathrm{char}\, k \geq 5$. *Let X and X' be* K3 *surfaces over k. A derived isogeny $f : X \rightsquigarrow X'$ is equivalent to the graph of an isomorphism $X' \xrightarrow{\sim} X$, if and only if f is polarizable and $\boldsymbol{Z}$-integral.*

Finally, we remark that Li and Zou [2021] considered derived isogenies and Torelli type theorems for abelian surfaces.

1A. ***Applications to good reductions of* K3 *surfaces.*** We apply our results to study the good reduction conjecture for K3 surfaces:

Conjecture 1.6. Let k be an algebraically closed field of characteristic $p > 0$ and let F be a finite extension of $W[1/p]$. Let X_F be a K3 surface over F such that $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_\ell)$ is unramified for some prime $\ell \neq p$. Then, X_F has potentially good reduction.

This conjecture is a K3 analog of the Néron–Ogg–Shafarevich criterion for abelian varieties. It admits many variants (e.g., ones that concern semistable reductions) and is verified in cases when X_F admits a polarization of low degree (see [Matsumoto 2015] and [Liedtke and Matsumoto 2018]). We prove the following:

Theorem 1.7. *Let X_F be as in Conjecture 1.6. Assume $p > 2$ and X_F admits a line bundle of degree prime to p. Then the* Gal_F*-representation $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ is potentially crystalline. If $p \geq 5$* (*resp. $p > 2$*) *and $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ has potentially good ordinary or* (*resp. supersingular*) *reduction, then so does X_F.*

Roughly speaking, the theorem is saying that if the cohomology of X_F predicts that X_F should have potential ordinary or supersingular reduction, then it does. We derive this as a consequence of a more general result (Theorem 8.10), which essentially reduces Conjecture 1.6 to the Hecke orbit conjecture (see Conjecture 8.2), which is a purely Shimura–theoretic statement. In particular, we prove the following.

Theorem 1.8. *Let X_F be an in Conjecture 1.6. Suppose that $p > 2$ and that X_F admits a line bundle of degree prime to p. Assume the Hecke orbit conjecture* (Conjecture 8.2) *holds for all i. Then, X_F has potentially good reduction.*

Our unconditional Theorem 1.7, in the ordinary case, is then a consequence of recent work of Maulik, Shankar, and Tang [Maulik et al. 2022, Theorem 1.4] proving the Hecke orbit conjecture in certain special cases. The supersingular case will be treated by a slightly different argument. Moreover, it seems very likely that a slight generalization of the conjecture can remove the condition on the existence of a prime-to-p line bundle as well, and hence completely affirms Conjecture 1.6.

We remark that nowhere in the proofs of the above results do we directly analyze a degeneration of K3 surfaces, unlike in [Matsumoto 2015] and [Liedtke and Matsumoto 2018]. In particular, we avoid the use of any techniques from the minimal model program. As far as the authors are aware, our method of proving good reduction results by marrying moduli theory of sheaves with density arguments is new in the literature.

After the paper was accepted for publication, Marco D'Addezio and Pol van Hoften proved the Hecke orbit conjecture for Shimura varieties of Hodge type and in particular proved Conjecture 8.2 under a very minor assumption on p [D'Addezio and van Hoften 2022, Section 7.5].

1B. ***Ideas of proof.*** (1) The "only if" part of Theorem 1.2 follows from the general theory of twisted derived equivalences in positive characteristics. The idea for the "if" part is to construct the desired X' together with the isogeny $f : X \rightsquigarrow X'$ by iteratively taking moduli spaces of twisted sheaves on X. This approach is inspired by that of [Huybrechts 2019, Theorem 1.1]. A key technical tool is the theory of B-fields in ℓ-adic and crystalline cohomology, described in Section 2. This allows us to relate classes in $\mathrm{H}^2(X)_{\mathbf{Q}}$ to the Brauer group, and provides a replacement for the Hodge-theoretic B-fields in Huybrechts' proof, although there are some additional complications at p. There are some further technical difficulties caused by the fact that in positive characteristic the cohomology $\mathrm{H}^2(X)_{\mathbf{Q}}$ can only take on adelic coefficients (i.e., $\mathbf{A}_f^p \times W[1/p]$) instead of $\mathbf{Q}$-coefficients. For instance, the Mukai vector which one must specify in order to form a moduli of sheaves is not an adelic object. That is, unlike Brauer classes, one cannot specify a Mukai vector by prescribing its local factors in $\mathrm{H}^2(X)_{\mathbf{Q}}$. We solve these problems by using local–global type results on quadratic forms (e.g., the strong approximation theorem), and the theory of quadratic forms over local rings.

(2) Theorem 1.4 is obtained by the realizing X' as the reduction of a suitable K3 surface in characteristic zero. This strategy is a simultaneous simplification and strengthening of that of [Yang 2022], with the additional input of Theorem 1.2. The characterization of Nygaard–Ogus liftings (Theorem 6.18) is obtained by applying recent advances on integral p-adic Hodge theory from [Bhatt et al. 2018] and

[Cais and Liu 2019] to study deformations of K3 crystals. These techniques for handling crystalline cohomology were unnecessary in Taelman's case [2020], as the deformation of the formal Brauer group of an ordinary K3 is rigid, which is not true for a general finite-height K3. We remark that here the restriction $p \geq 5$ is mainly due to our usage of the deformation theory of K3 crystals.

(3) Theorem 1.5 is a twisted generalization of the derived Torelli theorem of Lieblich and Olsson [2015, Theorem 6.1]. Just as in loc. cit., we prove this result by using a lifting argument to reduce to the global Torelli theorem over $\boldsymbol{C}$. The main difficulty which arises in our generalization is that instead of considering isogenies which arise directly from a (twisted or untwisted) derived equivalence, we are allowing any finite compositions of such. The derived equivalences involved may not be simultaneously liftable to characteristic zero. To overcome this difficulty, we combine the lifting results on derived equivalences with the Kuga–Satake method. This helps us reduce composing isogenies of K3's to composing isogenies of abelian varieties, which is much better understood. There is a technical problem which arises from the usage of Kuga–Satake. Namely, we need to put the relevant K3 surfaces into the same moduli space. However, the K3 surfaces themselves may not have a quasipolarization of a common degree. To overcome this problem, we pass from K3 surfaces to their Hilbert squares, which are treated in [Yang 2023]. The restriction to $p \geq 5$ is imposed because in loc. cit. the second author only treated $\mathrm{K3}^{[n]}$-type varieties when $p > n+1$ for certain technical reasons.

(4) For Theorem 1.7, we first show that the derived prime-to-p isogeny classes of K3's match up with the notion of prime-to-p Hecke orbit on the period domains of Kuga–Satake morphisms, which are some orthogonal Shimura varieties. It follows from some intermediate steps in the proof of Theorem 1.2 that the property of satisfying Conjecture 1.6 is invariant in a prime-to-p derived isogeny class. On the other hand, any X_F which satisfies the hypothesis of Theorem 1.7 produces a mod p point $x(X_F)$ on the period domain, and the set $\mathscr{L}_{\mathrm{bad}} := \{x(X_F) : X_F$ violates Conjecture 1.6$\}$ is closed.

If we combine the above observations with the Hecke orbit (HO) conjecture (see Conjecture 8.2), we see that if $\mathscr{L}_{\mathrm{bad}}$ intersects any of the height stratum of the period domains, then it must contain the entirety of that stratum, which is false by a deformation argument. Hence the HO conjecture forces $\mathscr{L}_{\mathrm{bad}}$ to be empty. The HO conjecture is now known for the ordinary locus by the recent work of Maulik, Shankar, and Tang [Maulik et al. 2022] and we will verify it in the superspecial locus for cases relevant to us (Theorem 8.6). This gives Theorem 1.7.

1C. *Plan of paper.* In Section 2, we develop the formalism of B-fields and twisted Mukai lattices in positive characteristic. Section 3 concerns the construction of twisted Chern characters, the twisted Néron–Severi lattice, and the action of a twisted derived equivalence on cohomology. In Section 4 we discuss rational Chow motives and isogenies. In Section 5 we prove some lifting results for twisted derived isogenies. In Section 6, we first prove Theorem 1.2. We then revisit Nygaard–Ogus theory for the point of view of integral p-adic Hodge theory and prove Theorem 1.4. In Section 7, we review the basics of Hilbert squares and the Kuga–Satake period morphism, and then prove Theorem 1.5. Finally, in Section 8, we explain the relationship between our isogeny theory and Hecke orbits, and prove Theorem 1.7.

1D. ***Notation.***

- Let p denote a prime. The letter k denotes a perfect base field of characteristic either 0 or p and ℓ denotes a prime not equal to $\operatorname{char} k$. When $\operatorname{char} k = p$, we write W for $W(k)$ and K for $W[1/p]$.
- If Z is a scheme, we write $\mathrm{H}^i(Z, \mu_n)$ for the flat (fppf) cohomology of the sheaf of n-th roots of unity on Z. If n is coprime to the characteristics of all residue fields of Z, this is equal to the étale cohomology of μ_n.
- We normalize our Chern characters so that the mod m Chern character of a line bundle L is equal to the image of the class of L under the boundary map $\mathrm{H}^1(Z, \boldsymbol{G}_m) \to \mathrm{H}^2(Z, \mu_m)$ of the Kummer sequence.
- Suppose k is a perfect field of characteristic p and S is a k-scheme. If $f : X \to S$ is a scheme, we denote by $\mathrm{H}^j_{\mathrm{cris}}(X)$ the sheaf on $\boldsymbol{Cris}(S/W)$ given by $R^j f_{\mathrm{cris}*}\mathscr{O}_{X/W}$ when S is understood.
- For any integral domain R, and R-modules M and N, an isomorphism $f : M_{\boldsymbol{Q}} \xrightarrow{\sim} N_{\boldsymbol{Q}}$ is said to be R-integral if $f(M) = N$.
- In this paper we only make use of singular, de Rham, étale, flat, and crystalline cohomology. We may omit the subscripts cris, fl, or dR when the choice of the relevant Grothendieck topology is clear from the coefficients.
- For a smooth proper variety Y over k, we let $\mathrm{H}^j(Y)$ denote either $\mathrm{H}^j_{\text{ét}}(Y, \widehat{\boldsymbol{Z}})$ if $\operatorname{char} k = 0$ or $\mathrm{H}^j_{\text{ét}}(Y, \widehat{\boldsymbol{Z}}^p) \times \mathrm{H}^j_{\mathrm{cris}}(Y/W)$ if $\operatorname{char} k = p$.
- Let R be a commutative ring. A quadratic lattice M over R is a free R-module of finite rank equipped with a bilinear symmetric pairing $M \times M \to R$. The pairing is said to be nondegenerate (resp. unimodular or perfect) if the induced map $M \to M^\vee$ is an injection (resp. an isomorphism).

2. B-fields and the twisted Mukai lattice in positive characteristic

Let X be a K3 surface over the complex numbers. Associated to X is the *Mukai lattice* $\widetilde{\mathrm{H}}(X, \boldsymbol{Z})$, which is the direct sum of the singular cohomology groups of X equipped with a certain pairing and Hodge structure. Consider a class $\alpha \in \mathrm{Br}(X)$. Huybrechts and Stellari [2005, Remark 1.3] generalized Mukai's construction to the twisted K3 surface (X, α) by defining the *twisted Mukai lattice* $\widetilde{\mathrm{H}}(X, B, \boldsymbol{Z})$. This construction modifies the Hodge structure on the Mukai lattice in a certain way using an auxiliary choice of a *B-field lift* of α, which is a class $B \in \mathrm{H}^2(X, \boldsymbol{Q})$ whose image in $\mathrm{Br}(X)$ under the exponential map is equal to α.

Suppose now that X is a K3 surface defined over an algebraically closed field of characteristic $p > 0$. After [Lieblich and Olsson 2015], we may consider the ℓ-adic and crystalline realizations of the Mukai motive of X. These are respectively a $\boldsymbol{Z}_l$-lattice $\widetilde{\mathrm{H}}(X, \boldsymbol{Z}_l)$ and a W-lattice $\widetilde{\mathrm{H}}(X/W)$, both of rank 24. In the crystalline setting, $\widetilde{\mathrm{H}}(X/W)$ is equipped with a Frobenius action, which makes $\widetilde{\mathrm{H}}(X/W)$ into a K3 crystal in the sense of Ogus [1979, Definition 3.1]. That this construction makes sense integrally is first observed in [Bragg and Lieblich 2018].

Consider a Brauer class $\alpha \in \mathrm{Br}(X)$. We wish to have an analog of Huybrechts and Stellari's construction of the twisted Mukai lattice in both the ℓ-adic and crystalline settings. The main task is to find the appropriate analog of a B-field lift of a Brauer class in ℓ-adic or crystalline cohomology. The ℓ-adic case is considered in [Lieblich et al. 2014] (we remark that the authors also deal with some additional complications coming from working over a field that is not algebraically closed, which we ignore here). The crystalline case is considered in [Bragg 2021, §3] and [Bragg and Lieblich 2018, §3.4], with the restriction that the Brauer class α is killed by p (rather than a power of p).

In this section we make two contributions. First, we complete the crystalline realization by defining crystalline B-field lifts of classes killed by an arbitrary power of p. We then treat the mixed case, considering all primes simultaneously, and define mixed B-field lifts of Brauer classes whose order is divisible by more than one prime. To assist the reader in connecting these constructions in the Hodge, ℓ-adic, and crystalline settings, we have included a brief summary of the Hodge and ℓ-adic realizations. We have tried to present a perspective which emphasizes the unifying features present in the different settings.

2A. ***Hodge realization.*** Let X be a K3 surface over the complex numbers. We have the exponential exact sequence

$$0 \to \boldsymbol{Z} \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 1.$$

Consider the induced map $\mathrm{H}^2(X, \mathscr{O}_X) \xrightarrow{\exp} \mathrm{H}^2(X, \mathscr{O}_X^\times)$, which, because $\mathrm{H}^3(X, \boldsymbol{Z}) = 0$, is a surjection. Given a class $v \in \mathrm{H}^2(X, \mathscr{O}_X)$, we note that $\exp(v)$ is contained in the torsion subgroup $\mathrm{H}^2(X, \mathscr{O}_X^\times)_{\mathrm{tors}} = \mathrm{H}^2(X, \boldsymbol{G}_m) = \mathrm{Br}(X)$ if and only if v is contained in the subgroup $\mathrm{H}^2(X, \boldsymbol{Q}) \subset \mathrm{H}^2(X, \mathscr{O}_X)$. Thus, this map restricts to a surjection

$$\exp : \mathrm{H}^2(X, \boldsymbol{Q}) \to \mathrm{Br}(X), \tag{1}$$

which we denote by $B \mapsto \alpha_B = \exp(B)$. According to [Huybrechts and Stellari 2005], a *B-field lift* of a class $\alpha \in \mathrm{Br}(X)$ is a class $B \in \mathrm{H}^2(X, \boldsymbol{Q})$ such that $\alpha_B = \alpha$.

The relationship between B-fields and the Brauer group is expressed in the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^2(X, \boldsymbol{Z}) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(X, \boldsymbol{Z}) + \mathrm{Pic}(X) \otimes \boldsymbol{Q} \arrow[r] \arrow[d] & \mathrm{Pic}(X) \otimes (\boldsymbol{Q}/\boldsymbol{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^2(X, \boldsymbol{Z}) \arrow[r] & \mathrm{H}^2(X, \boldsymbol{Q}) \arrow[r] \arrow[d] & \mathrm{H}^2(X, \boldsymbol{Z}) \otimes (\boldsymbol{Q}/\boldsymbol{Z}) \arrow[r] \arrow[d] & 0 \\
 & & \dfrac{\mathrm{H}^2(X, \boldsymbol{Q})}{\mathrm{H}^2(X, \boldsymbol{Z}) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}} \arrow[r, "\sim"] \arrow[d] & \mathrm{Br}(X) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{2}$$

with exact rows and columns. In particular, we see that there are two sources of ambiguity in choosing a B-field lift of a Brauer class, namely, integral classes in $\mathrm{H}^2(X, \boldsymbol{Z})$ and rational classes in $\mathrm{H}^{1,1}(X, \boldsymbol{Q}) = \mathrm{Pic}(X) \otimes \boldsymbol{Q} \subset \mathrm{H}^2(X, \boldsymbol{Q})$.

2B. ***ℓ-adic realization.*** Let k be an algebraically closed field of arbitrary characteristic. Fix a prime number ℓ, not equal to the characteristic of k. We review the ℓ-adic B-fields and the ℓ-adic realization of the twisted Mukai motive introduced in [Lieblich et al. 2014].

Let X be a K3 surface over k. By duality in étale cohomology, we have $\mathrm{H}^3(X, \mu_{\ell^n}) = 0$ for all $n \geq 1$. It follows that the natural map

$$\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \to \mathrm{H}^2(X, \mu_{\ell^n}) \tag{3}$$

is surjective, and hence we have an identification

$$\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \otimes \boldsymbol{Z}/\ell^n\boldsymbol{Z} \cong \mathrm{H}^2(X, \mu_{\ell^n}).$$

We consider the composition

$$\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \twoheadrightarrow \mathrm{H}^2(X, \mu_{\ell^n}) \twoheadrightarrow \mathrm{Br}(X)[\ell^n], \tag{4}$$

where the second map is induced by the inclusion $\mu_{\ell^n} \subset \boldsymbol{G}_m$.

Definition 2.1. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class which is killed by a power of ℓ. An *ℓ-adic B-field lift* of α is an element

$$B \in \mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \stackrel{\text{def}}{=} \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \otimes_{\mathbf{Z}_\ell} \boldsymbol{Q}_\ell$$

such that if we write $B = a/\ell^n$ for some $a \in \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1))$, then a maps to α under the composition (4).

We give the following alternative description. Define $\mu_{\ell^\infty} = \bigcup_n \mu_{\ell^n} \subset \boldsymbol{G}_m$. The Picard group of X is torsion-free, which implies the vanishing $\mathrm{H}^1(X, \mu_\ell) = 0$. It follows that the inclusions $\mu_{\ell^n} \subset \mu_{\ell^{n+1}}$ induce injections on H^2, and we have a natural identification $\mathrm{H}^2(X, \mu_{\ell^\infty}) = \bigcup_n \mathrm{H}^2(X, \mu_{\ell^n})$. Moreover, for every n we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \arrow[r, "\cdot\ell^m"] \arrow[d, two heads] & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \arrow[d, two heads] \\ \mathrm{H}^2(X, \mu_{\ell^n}) \arrow[r, hook] & \mathrm{H}^2(X, \mu_{\ell^{n+m}}) \end{tikzcd} \tag{5}$$

Taking the direct limit of the maps (3), we get a map

$$\mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \to \mathrm{H}^2(X, \mu_{\ell^\infty}). \tag{6}$$

This map may be explicitly described as follows: given $B \in \mathrm{H}^2(X, \boldsymbol{Q}_\ell(1))$, choose $n \geq 0$ such that $\ell^n B \in \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1))$, and map B to the image of $\ell^n B$ under the left map of (4). Note that by the commutativity of (5), this association is well defined, independent of our choice of n. Composing (6) with the natural map $\mathrm{H}^2(X, \mu_{\ell^\infty}) \to \mathrm{Br}(X)$, we get a map

$$\mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \to \mathrm{Br}(X). \tag{7}$$

This is the ℓ-adic analog of the exponential map (1). The image of this map is exactly the subgroup $\mathrm{Br}(X)[\ell^\infty] \subset \mathrm{Br}(X)$ consisting of classes killed by some power of ℓ. Furthermore, an ℓ-adic B-field lift of a class $\alpha \in \mathrm{Br}(X)[\ell^\infty]$ (in the sense of Definition 2.1) is exactly a preimage of α under (7). We denote (7) by $B \mapsto \alpha_B$.

The relationship between ℓ-adic B-fields and the Brauer group is expressed by the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}_\ell \arrow[r] \arrow[d] & \mathrm{Pic}(X) \otimes (\boldsymbol{Q}_\ell/\boldsymbol{Z}_\ell) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) \arrow[r] & \mathrm{H}^2(X, \boldsymbol{Q}_\ell(1)) \arrow[r] \arrow[d] & \mathrm{H}^2(X, \mu_{\ell^\infty}) \arrow[r] \arrow[d] & 0 \\
 & & \dfrac{\mathrm{H}^2(X, \boldsymbol{Q}_\ell(1))}{\mathrm{H}^2(X, \boldsymbol{Z}_\ell(1)) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}_\ell} \arrow[r, "\sim"] \arrow[d] & \mathrm{Br}(X)[\ell^\infty] \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{8}$$

with exact rows and columns, where the right-hand column is given by taking the direct limit of the exact sequence induced by the Kummer sequence.

In particular, we have an isomorphism

$$\mathrm{Br}(X)[\ell^\infty] \cong (\boldsymbol{Q}_\ell/\boldsymbol{Z}_\ell)^{\oplus 22-\rho}, \tag{9}$$

where ρ is the Picard rank of X.

2C. ***The twisted ℓ-adic Mukai lattice.*** The *ℓ-adic Mukai lattice* associated to X [Lieblich et al. 2014, Definition 3.3.1] is

$$\widetilde{\mathrm{H}}(X, \boldsymbol{Z}_\ell) = \mathrm{H}^0(X, \boldsymbol{Z}_\ell)(-1) \oplus \mathrm{H}^2(X, \boldsymbol{Z}_\ell) \oplus \mathrm{H}^4(X, \boldsymbol{Z}_\ell)(1),$$

which we equip with the Mukai pairing. Given a class $B \in \mathrm{H}^2(X, \boldsymbol{Q}_\ell)$, we define the associated *twisted ℓ-adic Mukai lattice* to be the submodule

$$\widetilde{\mathrm{H}}(X, \boldsymbol{Z}_\ell, B) = \exp(B)\, \widetilde{\mathrm{H}}(X, \boldsymbol{Z}_\ell) \subset \widetilde{\mathrm{H}}(X, \boldsymbol{Q}_\ell).$$

Here, $\exp(B)$ denotes the isometry $\widetilde{\mathrm{H}}(X, \boldsymbol{Q}_\ell) \to \widetilde{\mathrm{H}}(X, \boldsymbol{Q}_\ell)$ given by

$$(a, b, c) \mapsto \big(a, b + aB, c + b.B + \tfrac{1}{2}aB^2\big). \tag{10}$$

2D. ***Crystalline realization.*** Let k be an algebraically closed field of characteristic $p > 0$ and let X be a K3 surface over k. We will define crystalline B-fields associated to Brauer classes on X whose order is a power of p. There are some new phenomena which present themselves in the crystalline setting that are not present in the Hodge and ℓ-adic theories. In particular, there is a nontrivial interaction between crystalline B-fields and the Frobenius operator on the crystalline cohomology. A related feature is that not every class in rational crystalline cohomology is a crystalline B-field. We give a characterization of which classes are B-fields using only the F–crystal structure on crystalline cohomology in Proposition 2.7. We then construct the crystalline version of the twisted Mukai lattice, and show that this object has a natural structure of a K3 crystal in the sense of Ogus [1979, Definition 3.1]. We conclude with some calculations with the twisted Mukai crystals. In the special case when the Brauer class is killed by p, the results of this section have appeared in [Bragg 2021; Bragg and Lieblich 2018].

Set $W_n = W/p^n W$, so in particular $W_1 = k$. Let $\sigma : k \to k$ be the Frobenius $\lambda \mapsto \lambda^p$. We denote the induced map $\sigma : W \to W$ (abusively) by the same symbol.

2E. *Crystalline B-fields.* We begin by relating the flat cohomology of μ_{p^n} to certain étale cohomology groups. Consider the Kummer sequence

$$1 \to \mu_{p^n} \to \boldsymbol{G}_m \xrightarrow{x \mapsto x^{p^n}} \boldsymbol{G}_m \to 1,$$

which is exact in the fppf topology. Let $\varepsilon : X_{\text{fl}} \to X_{\text{ét}}$ be the natural map from the big fppf site of X to the small étale site of X. By a theorem of Grothendieck, the cohomology of the complex $R\varepsilon_* \boldsymbol{G}_m$ vanishes in all positive degrees. Applying ε_* to the Kummer sequence, we obtain an exact sequence

$$1 \to \boldsymbol{G}_m \xrightarrow{x \mapsto x^{p^n}} \boldsymbol{G}_m \to R^1 \varepsilon_* \mu_{p^n} \to 1$$

of sheaves on the small étale site of X (because X is reduced, the restriction of μ_{p^n} to the small étale site of X is trivial). It follows that

$$R^1 \varepsilon_* \mu_{p^n} = \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n},$$

where the quotient is taken in the étale topology. We therefore obtain isomorphisms

$$\mathrm{H}^i(X_{\text{fl}}, \mu_{p^n}) \xrightarrow{\sim} \mathrm{H}^{i-1}(X_{\text{ét}}, \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n}). \tag{11}$$

We next relate the étale cohomology groups on the right to crystalline cohomology. We consider the map of étale sheaves

$$d\log : \boldsymbol{G}_m \to \mathrm{W}_n \, \Omega^1_X$$

given by $x \mapsto d\underline{x}/\underline{x}$, where $\underline{x} = (x, 0, 0, \ldots)$ is the multiplicative representative of x in $\mathrm{W}_n \, \mathcal{O}_X$. By [Illusie 1971, Proposition I.3.23.2, p. 580] the kernel of $d\log$ is equal to the subsheaf $\boldsymbol{G}_m^{\times p^n} \subset \boldsymbol{G}_m$, so there is an induced injection

$$d\log : \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n} \hookrightarrow \mathrm{W}_n \, \Omega^1_X. \tag{12}$$

As the image of $d\log$ is contained in the kernel of d, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \boldsymbol{G}_m / \boldsymbol{G}_m^{\times p^n} \arrow[r] \arrow[d, "d\log"] & 0 \arrow[d] \\
\mathrm{W}_n \, \mathcal{O}_X \arrow[r, "d"] & \mathrm{W}_n \, \Omega^1_X \arrow[r, "d"] & \mathrm{W}_n \, \Omega^2_X
\end{tikzcd}$$

which we interpret as a map of complexes

$$d\log : \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^n}[-1] \hookrightarrow \mathrm{W}_n\,\Omega_X^\bullet. \tag{13}$$

An important fact is that the de Rham–Witt complex computes crystalline cohomology, in the sense that there is a canonical isomorphism

$$\mathrm{H}^*(X, \mathrm{W}_n\,\Omega_X^\bullet) \xrightarrow{\sim} \mathrm{H}^*(X/W_n) \tag{14}$$

in each degree [Illusie 1971, Théoréme II.1.4, p. 606]. Taking cohomology of (13) and using the identifications (11) and (14), we find a map

$$d\log : \mathrm{H}^2(X, \mu_{p^n}) \to \mathrm{H}^2(X/W_n). \tag{15}$$

Lemma 2.2. *For each $n \geq 1$, the map (15) is injective.*

Proof. We induct on n. By [Illusie 1971, Corollaire 0.2.1.18, p. 517], there is a short exact sequence

$$1 \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p} \xrightarrow{d\log} Z\Omega_X^1 \xrightarrow{W-C} \Omega_{X'}^1 \to 0$$

of étale sheaves, where X' denotes the Frobenius twist of X over k. In particular, from the vanishing of $\mathrm{H}^0(X, \Omega_X^1)$ and the injectivity of $\mathrm{H}^1(X, Z\Omega_X^1) \to \mathrm{H}^2_{\mathrm{dR}}(X/k) = \mathrm{H}^2(X/W_1)$ (a consequence of the degeneration of the Hodge–de Rham spectral sequence) we obtain injectivity of (15) for $n = 1$.

We recall that the crystalline cohomology groups $\mathrm{H}^*(X/W)$ of a K3 surface are torsion-free. This implies in particular that the maps

$$\mathrm{H}^2(X/W) \otimes_{\mathbf{Z}} \boldsymbol{Z}/p^n\boldsymbol{Z} \to \mathrm{H}^2(X/W_n)$$

are isomorphisms. Hence, multiplication by p^n on $\mathrm{H}^2(X/W)$ induces a short exact sequence

$$0 \to \mathrm{H}^2(X/k) \xrightarrow{\cdot p^n} \mathrm{H}^2(X/W_{n+1}) \to \mathrm{H}^2(X/W_n) \to 0.$$

We also have a short exact sequence

$$1 \to \mu_p \to \mu_{p^{n+1}} \xrightarrow{\cdot p} \mu_{p^n} \to 1 \tag{16}$$

of fppf groups. We claim that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2(X, \mu_p) \arrow[r] \arrow[d, hook] & \mathrm{H}^2(X, \mu_{p^{n+1}}) \arrow[r, "\cdot p"] \arrow[d] & \mathrm{H}^2(X, \mu_{p^n}) \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^2(X/k) \arrow[r, "\cdot p^n"] & \mathrm{H}^2(X/W_{n+1}) \arrow[r] & \mathrm{H}^2(X/W_n) \arrow[r] & 0
\end{tikzcd} \tag{17}$$

commutes and has exact rows, where the top horizontal row is given by the second cohomology of (16), and the vertical arrows are (15). The exactness of the top row follows from the vanishing of $\mathrm{H}^1(X, \mu_{p^n})$

(we remark that the top right horizontal arrow is surjective if and only if X has finite height). To see the commutativity, note that applying $R^1\varepsilon_*$ to (16) results in the short exact sequence

$$1 \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p} \xrightarrow{\cdot p^n} \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^{n+1}} \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^n} \to 1$$

of étale sheaves. Using diagram (17), the result follows immediately by induction. ☐

We arrive at a diagram

$$\begin{array}{ccccc} \mathrm{H}^2(X/W) & \xrightarrow{\pi_n} & \mathrm{H}^2(X/W_n) & & \\ & & \cup & & \\ & & \mathrm{H}^2(X, \mu_{p^n}) & \twoheadrightarrow & \mathrm{Br}(X)[p^n] \end{array} \tag{18}$$

where π_n denotes reduction modulo p^n. This is the crystalline analog of (4).

Definition 2.3. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class which is killed by a power of p. A *crystalline B-field lift of* α is an element

$$B \in \mathrm{H}^2(X/K) \stackrel{\text{def}}{=} \mathrm{H}^2(X/W) \otimes_W K$$

such that if we write $B = a/p^n$ for some $a \in \mathrm{H}^2(X/W)$, then $\pi_n(a)$ is equal to $d\log(\alpha')$ for some $\alpha' \in \mathrm{H}^2(X, \mu_{p^n})$ whose image in $\mathrm{Br}(X)$ is equal to α.

From the surjectivity of the horizontal maps in (18), we see that any p-power torsion Brauer class admits a crystalline B-field lift. However, in contrast to the Hodge and ℓ-adic cases, not every element of $\mathrm{H}^2(X/K)$ is a crystalline B-field lift of a Brauer class, because $\mathrm{H}^2(X, \mu_{p^n})$ is only a subgroup of $\mathrm{H}^2(X/W) \otimes \boldsymbol{Z}/p^n\boldsymbol{Z}$.

Definition 2.4. A class $B \in \mathrm{H}^2(X/K)$ is a *crystalline B-field* if it is a B-field lift of some Brauer class. Let $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$ denote the subgroup of crystalline B-fields. Let $\mathscr{B}_n(X) \subset \mathscr{B}(X)$ denote the subgroup of crystalline B-fields B such that $p^n B \in \mathrm{H}^2(X/W)$.

We take the direct limit of the maps $\mathscr{B}_n(X) \to \mathrm{H}^2(X, \mu_{p^n})$ to obtain a map

$$\mathscr{B}(X) \twoheadrightarrow \mathrm{H}^2(X, \mu_{p^\infty}), \tag{19}$$

which may be explicitly described exactly as in the étale case (6): given a class $B \in \mathscr{B}(X)$, we choose $n \geq 0$ such that $p^n B \in \mathrm{H}^2(X/W)$, and then reduce modulo p^n. We compose (19) with the map to the Brauer group to obtain a map

$$\mathscr{B}(X) \to \mathrm{Br}(X), \tag{20}$$

which we denote by $B \mapsto \alpha_B$. This is the crystalline analog of the exponential map (1). As in the ℓ-adic case, the image of this map is $\mathrm{Br}(X)[p^\infty] \subset \mathrm{Br}(X)$, and a crystalline B-field lift of a class $\alpha \in \mathrm{Br}(X)[p^\infty]$

is exactly a preimage of α under (20). We have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^2(X/W) \arrow[r] \arrow[d, equal] & \mathrm{H}^2(X/W)+\mathrm{Pic}(X)\otimes \boldsymbol{Q}_p \arrow[r] \arrow[d] & \mathrm{Pic}(X)\otimes(\boldsymbol{Q}_p/\boldsymbol{Z}_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^2(X/W) \arrow[r] & \mathscr{B}(X) \arrow[r] \arrow[d] & \mathrm{H}^2(X,\mu_{p^\infty}) \arrow[r] \arrow[d] & 0 \\
 & & \dfrac{\mathscr{B}(X)}{\mathrm{H}^2(X/W)+\mathrm{Pic}(X)\otimes \boldsymbol{Q}_p} \arrow[r, "\sim"] \arrow[d] & \mathrm{Br}(X)[p^\infty] \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{21}$$

with exact rows and columns.

2F. ***Description of the group of crystalline B-fields.*** We will now give some results describing the subgroup $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$ more explicitly.

We recall that the *Tate module* of a K3 crystal H (in the sense of Ogus [1979, Definition 3.1]) is the $\boldsymbol{Z}_p$-module $H^{\phi=1} \subset H$ consisting of those elements $h \in H$ satisfying $\phi(h) = h$, where $\phi := p^{-1}\Phi$ and Φ is the Frobenius endomorphism of H. By a result of Illusie [1971, Théorème 5.14, p. 631], if X is a K3 surface then we have an exact sequence

$$0 \to \mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \to \mathrm{H}^2(X/W) \xrightarrow{p-\Phi} \mathrm{H}^2(X/W)$$

identifying $\mathrm{H}^2(X, \boldsymbol{Z}_p(1))$ with the Tate module $\mathrm{H}^2(X/W)^{\phi=1}$ of the K3 crystal $\mathrm{H}^2(X/W)$, where the left inclusion is given by the inverse limit of the inclusions (15). We have inclusions

$$\mathrm{Pic}(X) \otimes \boldsymbol{Q}_p \subset \mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \subset \mathscr{B}(X),$$

where as usual $\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) = \mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \otimes \boldsymbol{Q}_p$.

Remark 2.5. By analogy with the Lefschetz (1,1) theorem, one might imagine that the inclusion $\mathrm{Pic}(X) \otimes \boldsymbol{Q}_p \subset \mathrm{H}^2(X, \boldsymbol{Q}_p(1))$ is an equality. However, this is frequently false, e.g., for a very general ordinary K3 surface. It is true if X is supersingular, as a consequence of the Tate conjecture for supersingular K3 surfaces (of course, the Tate conjecture is known for all K3 surfaces, but it is only in the supersingular case that there is such a consequence for K3 surfaces over general algebraically closed fields).

Proposition 2.6. *Let X be a* K3 *surface.*

(1) *If X has finite height, then* $\mathscr{B}(X) = \mathrm{H}^2(X/W) + \mathrm{H}^2(X, \boldsymbol{Q}_p(1))$.

(2) *If X is supersingular, then* $\mathscr{B}(X) = \mathscr{B}_1(X) + \mathrm{H}^2(X, \boldsymbol{Q}_p(1))$.

Proof. In either case, we have $\mathrm{H}^2(X/W) \subset \mathscr{B}_1(X) \subset \mathscr{B}(X)$ and $\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \subset \mathscr{B}(X)$. It follows that in both cases the right-hand side is contained in $\mathscr{B}(X)$. We prove the reverse containments. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \arrow[r, hook] \arrow[d] & \mathrm{H}^2(X/W) \arrow[d, two heads, "\bmod p^n"] \\ \mathrm{H}^2(X, \mu_{p^n}) \arrow[r, hook] & \mathrm{H}^2(X/W_n) \end{tikzcd} \tag{22}$$

Suppose that X has finite height. Flat duality implies that $\mathrm{H}^3(X, \mu_{p^n}) = 0$ for all $n \geq 1$. Hence, the maps $\mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \to \mathrm{H}^2(X, \mu_{p^n})$ are surjective. It follows that the restriction $\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \to \mathrm{Br}(X)[p^\infty]$ of the exponential map (20) is surjective. This proves (1). We next prove (2). Suppose that X is supersingular. For each n and i we consider the short exact sequence

$$0 \to \mathrm{U}^i(X, \mu_{p^n}) \to \mathrm{H}^i(X, \mu_{p^n}) \to \mathrm{D}^i(X, \mu_{p^n}) \to 0.$$

As $\mathrm{H}^1(X, \mu_{p^n}) = 0$, flat duality shows that $\mathrm{D}^3(X, \mu_{p^n}) = 0$. Hence, the maps $\mathrm{D}^2(X, \mu_{p^{n+1}}) \to \mathrm{D}^2(X, \mu_{p^n})$ induced by the multiplication $p : \mu_{p^{n+1}} \to \mu_{p^n}$ are surjective. Furthermore, the formal group associated to $\mathrm{U}^2(X, \mu_{p^n})$ is isomorphic to $\widehat{\mathrm{Br}}(X) \cong \widehat{\boldsymbol{G}}_a$, so $\mathrm{U}^2(X, \mu_{p^n}) \cong \boldsymbol{G}_a(k)$. In particular, the groups $\mathrm{U}^2(X, \mu_{p^n})$ are p-torsion, and the maps $\mathrm{U}^2(X, \mu_{p^n}) \to \mathrm{U}^2(X, \mu_{p^{n+1}})$ induced by the inclusion $\mu_{p^n} \subset \mu_{p^{n+1}}$ are isomorphisms. Write $\mathrm{U}^2(X, \mu_{p^\infty})$ for the union of the $\mathrm{U}^2(X, \mu_{p^n})$ and $\mathrm{D}^2(X, \mu_{p^\infty})$ for the union of the $\mathrm{D}^2(X, \mu_{p^n})$. It follows that the composition

$$\mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \to \mathrm{H}^2(X, \mu_{p^\infty}) \to \mathrm{D}^2(X, \mu_{p^\infty})$$

is surjective, and that $\mathrm{U}^2(X, \mu_p) = \mathrm{U}^2(X, \mu_{p^\infty})$. Hence, the exponential map (20) restricts to a surjection $\mathscr{B}_1(X) + \mathrm{H}^2(X, \boldsymbol{Q}_p(1)) \to \mathrm{Br}(X)[p^\infty]$, which proves (2). ☐

The following describes the subgroup $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$ in terms of the F-crystal structure on $\mathrm{H}^2(X/W)$, without explicit mention of flat cohomology or the Brauer group. The special case of classes $B \in p^{-1}\,\mathrm{H}^2(X/W)$ is Lemma 3.4.11 of [Bragg and Lieblich 2018].

Proposition 2.7. *A class $B \in \mathrm{H}^2(X/K)$ is a crystalline B-field if and only if*

$$B - \phi(B) \in \mathrm{H}^2(X/W) + \phi(\mathrm{H}^2(X/W)), \tag{23}$$

where $\phi = p^{-1}\Phi$.

Proof. Write $H = \mathrm{H}^2(X/W)$. Suppose that X has finite height. It is immediate from Proposition 2.6(1) that any B-field satisfies the claimed relation. Conversely, suppose that $B = a/p^n$ is an element satisfying (23). Consider the Newton–Hodge decomposition

$$\mathrm{H}^2(X/W) = H_{<1} \oplus H_1 \oplus H_{>1}$$

of $\mathrm{H}^2(X/W)$ into subcrystals with the indicated slopes (see Section 2I below). Write $a = (a_{<1}, a_1, a_{>1})$. We have $pH_{<1} \subset \Phi(H_{<1})$ or, equivalently, $H_{<1} \subset \phi(H_{<1})$ (see for instance [Katz 1979, §1.2]). Consider the map

$$1-\phi : H_{<1} \to \phi(H_{<1}).$$

All slopes of $H_{<1}$ are less than one, so this map is injective. By [Illusie 1971, Lemme II.5.3], it is surjective, and hence an isomorphism. We have $(1-\phi)(a_{<1}) \in p^n \phi(H_{<1})$, so in fact $a_{<1} \in p^n H_{<1}$. We have $\phi(H_{>1}) \subset H_{>1}$. Thus, we have a map

$$1-\phi : H_{>1} \to H_{>1},$$

which as before is both injective and surjective, and hence an isomorphism. We have $(1-\phi)(a_{>1}) \in p^n H_{>1}$, so in fact $a_{>1} \in p^n H_{>1}$. Finally, note that H_1 is a unit root crystal. It follows quickly that $a_1 = p^n h + t$ for some $h \in H_1$ and some t which is fixed by ϕ. We conclude that $B \in \mathscr{B}(X)$. This completes the proof of Proposition 2.7 in the case when X has finite height.

Suppose that X is supersingular. By Lemma 3.4.11 of [Bragg and Lieblich 2018], we have that $\mathscr{B}_1(X)$ consists exactly of those classes $B = a/p$ with $a \in H$ that satisfy (23). By Proposition 2.6(2), any B-field satisfies the claimed relation. We prove the converse. The inclusion of the Tate module is an isogeny, meaning that the map $T \otimes K \to H \otimes K$ is an isomorphism. Thus, the natural map $H \xrightarrow{\sim} H^\vee \to T^\vee \otimes W$ is injective, and we may regard H as a subgroup of the dual lattice $T^\vee \otimes W$. Note that if $h \in H$ and $t \in T$, then $\phi(h).t = \phi(h).\phi(t) = \sigma(h.t)$. It follows that $H + \phi(H) \subset T^\vee \otimes W$. Now, if $B \in \mathrm{H}^2(X/K)$ satisfies the claimed relation, then B is in the kernel of the map $1-\phi : T^\vee \otimes W \to T^\vee \otimes (K/W)$, which is equal to $T^\vee \otimes W + T \otimes \boldsymbol{Q}_p$. We may therefore write $B = B' + t/p^n$ for some $B' \in T^\vee \otimes W$ and some $t \in T$. As t is killed by $1-\phi$, B' also satisfies the relation (23). But by [Ogus 1979, Lemma 3.10], we have $T^\vee \otimes W \subset p^{-1}H$, so $B' \in p^{-1}H$. By Lemma 3.4.11 of [Bragg and Lieblich 2018] we have $B' \in \mathscr{B}(X)$. We also have $t/p^n \in \mathscr{B}(X)$, and we conclude that $B \in \mathscr{B}(X)$, as desired. □

Remark 2.8. One can alternatively prove Proposition 2.7 by generalizing the method of [Bragg and Lieblich 2018, Lemma 3.4.11], which we sketch. This proof has the advantage of avoiding flat duality and being uniform in the height of X. The first step is to understand the cokernel of the map (12). This is described by the short exact sequence [Colliot-Thélène et al. 1983, Lemma 2, p. 779]

$$0 \to \boldsymbol{G}_m/\boldsymbol{G}_m^{\times p^n} \to \mathrm{W}_n\,\Omega^1_X \xrightarrow{1-\mathrm{F}} \mathrm{W}_n\,\Omega^1_X/d(\mathrm{W}_n\,\mathcal{O}_X) \to 0, \tag{24}$$

where 1 denotes the projection and F is the map defined in [Illusie 1971, Proposition II.3.3]. One then proceeds by analyzing the p-adic filtrations on crystalline and de Rham–Witt cohomology.

2G. ***p-primary torsion in the Brauer group.*** We make some observations connecting the group $\mathscr{B}(X)$ of crystalline B-fields to the p-primary torsion in the Brauer group of X. Suppose that X has finite height h. By Proposition 2.6, we have

$$\mathscr{B}(X) = \mathrm{H}^2(X, \boldsymbol{Q}_p(1)) + \mathrm{H}^2(X/W).$$

In particular, (21) induces an isomorphism

$$\frac{\mathrm{H}^2(X, \boldsymbol{Q}_p(1))}{\mathrm{H}^2(X, \boldsymbol{Z}_p(1)) + \mathrm{Pic}(X) \otimes \boldsymbol{Q}_p} \xrightarrow{\sim} \mathrm{Br}(X)[p^\infty]. \tag{25}$$

The slope 1 part of $\mathrm{H}^2(X/W)$ has rank $22-2h$, so we have $\mathrm{H}^2(X, \boldsymbol{Z}_p(1)) \cong \boldsymbol{Z}_p^{\oplus 22-2h}$. Thus, (25) gives an isomorphism

$$\mathrm{Br}(X)[p^\infty] \cong (\boldsymbol{Q}_p/\boldsymbol{Z}_p)^{\oplus 22-\rho-2h}, \tag{26}$$

where ρ is the Picard rank of X. This could also be seen from the fact that, in the finite-height case, the diagram (8) with ℓ replaced by p (and étale cohomology with flat cohomology) still has exact rows and columns.

Remark 2.9. The exponent appearing in the formula (26) for the p-primary torsion of the Brauer group is smaller than that for the l-primary torsion (9) by a factor of $2h$. These "missing" p-primary torsion Brauer classes are the cause of the restriction at p in Theorem 1.2.

We now suppose X is supersingular. By Proposition 2.6, we have

$$\mathscr{B}(X) = \mathscr{B}_1(X) + \mathrm{H}^2(X, \boldsymbol{Q}_p(1)).$$

By the Tate conjecture for supersingular K3 surfaces, the first crystalline Chern character induces an isomorphism $\mathrm{Pic}(X) \otimes \boldsymbol{Z}_p \xrightarrow{\sim} T \otimes \boldsymbol{Z}_p = \mathrm{H}^2(X, \boldsymbol{Z}_p(1))$, and so $\mathrm{H}^2(X, \boldsymbol{Q}_p(1))$ is in the kernel of the crystalline exponential map (20). Write $N = \mathrm{Pic}(X)$. We have $\rho = 22$, so $\mathrm{Br}(X)$ has no prime-to-p torsion (see (9)). We conclude that (20) restricts to a surjection $\mathscr{B}_1(X) \to \mathrm{Br}(X)$. We have a short exact sequence

$$0 \to p^{-1}N/N \to \mathscr{B}_1(X)/H \to \mathrm{Br}(X) \to 0.$$

In particular, $\mathrm{Br}(X)$ is p-torsion. As shown in the proof of Proposition 2.7, we have that $\mathscr{B}_1(X) \subset N^\vee \otimes W + p^{-1}N \subset p^{-1}N \otimes W$, where the latter inclusion holds because discriminant group of N is p-torsion. Let $\mathscr{B}_1(X)^\circ = \mathscr{B}_1(X) \cap (N^\vee \otimes W)$. We have a short exact sequence

$$0 \to N^\vee/N \to \mathscr{B}_1(X)^\circ/H \to \mathrm{Br}(X) \to 0. \tag{27}$$

The subgroup $\mathscr{B}_1(X)^\circ$ can be understood using Ogus's results [1979] on the classification of supersingular K3 crystals. Write $K = H/N \otimes W$ and $V = N^\vee/N \cong \boldsymbol{F}_p^{2\sigma_0}$ (here, σ_0 is the Artin invariant of X). The subspace $K \subset V \otimes k$ is Ogus's *characteristic subspace*, and has dimension σ_0. Let $\phi : V \otimes k \to V \otimes k$ be the map $\phi(v \otimes \lambda) = v \otimes \lambda^p$. Ogus showed that K is totally isotropic and is in a special position with respect to ϕ. Namely, $K + \phi(K)$ has dimension $\sigma_0 + 1$, and $V \otimes k = \sum_i \phi^i(K)$ has dimension $2\sigma_0$. This implies that there exists a *characteristic vector* for K, which is an element $e_1 \in V \otimes k$ such that, writing $e_i = \phi^{i-1}(e_1)$, we have that $\{e_0, \dots, e_{\sigma_0-1}\}$ is a basis for K and $\{e_0, \dots, e_{2\sigma_0-1}\}$ is a basis for $V \otimes k$. We let f_i denote the functional given by pairing with e_i, so that $\{f_0, \dots, f_{\sigma_0-1}\}$ is a basis for $K^\vee = V \otimes k/K$. By Proposition 2.7, the subgroup $\mathscr{B}_1(X)^\circ/H \subset V \otimes k/K$ is the kernel of the map $1-\phi : V \otimes k/K \to V \otimes k/(K + \phi(K))$. It follows that we have

$$\mathscr{B}_1(X)^\circ/H = \{\lambda f_1 + \lambda^p f_2 + \cdots + \lambda^{p^{\sigma_0-1}} f_{\sigma_0-1} \mid \lambda \in k\}.$$

We conclude that $\mathscr{B}_1(X)^\circ/H$ is isomorphic to the underlying additive group $\boldsymbol{G}_a(k)$ of the group field k. The left term of (27) is discrete, and hence there is an isomorphism

$$\mathrm{Br}(X) \cong \boldsymbol{G}_a(k).$$

Remark 2.10. Multiplying by p and then reducing modulo p, the characteristic subspace K is identified with the kernel of the k-linearized first de Rham Chern character $c_1^{\mathrm{dR}} \otimes k : \mathrm{Pic}(X) \otimes k \to \mathrm{H}^2_{\mathrm{dR}}(X/k)$, and the vector space $V \otimes k/K$ is identified with its image. Furthermore, $\mathscr{B}_1(X)/H$ is identified with $\mathrm{H}^2(X, \mu_p)$ (regarded as a subgroup of $\mathrm{H}^2_{\mathrm{dR}}(X/k)$ via the map $d\log$) and $\mathscr{B}_1(X)^\circ/H$ is identified with $\mathrm{U}^2(X, \mu_p)$.

2H. *The twisted Mukai crystal.* We recall the Mukai crystal introduced in [Lieblich and Olsson 2015]. We set

$$\widetilde{\mathrm{H}}(X/W) = \mathrm{H}^0(X/W)(-1) \oplus \mathrm{H}^2(X/W) \oplus \mathrm{H}^4(X/W)(1).$$

As a result of the Tate twists on the first and third factors on the right-hand side, the Frobenius operator $\widetilde{\Phi}$ on $\widetilde{\mathrm{H}}(X/W)$ is given by the formula

$$\widetilde{\Phi}(a, b, c) = (p\sigma(a), \Phi(b), p\sigma(c)),$$

where we have identified H^0 and H^4 with W, and where Φ is the Frobenius operator on $\mathrm{H}^2(X/W)$. We equip $\widetilde{\mathrm{H}}(X/W)$ with the Mukai pairing. It is immediate from the definitions that $\widetilde{\mathrm{H}}(X/W)$ is a K3 crystal of rank 24.

Definition 2.11. Let B be a crystalline B-field. The *twisted Mukai crystal* associated to (X, B) is

$$\widetilde{\mathrm{H}}(X/W, B) = \exp(B)\,\widetilde{\mathrm{H}}(X/W) \subset \widetilde{\mathrm{H}}(X/K).$$

Here, $\exp(B)$ is the isometry of $\widetilde{\mathrm{H}}(X/K)$ defined by the formula (10).

The twisted Mukai crystal has a natural structure of a K3 crystal by the following result.

Theorem 2.12. *Let $B \in \mathscr{B}(X)$ be a crystalline B-field. The endomorphism $\widetilde{\Phi}$ of $\widetilde{\mathrm{H}}(X/K)$ restricts to an endomorphism of $\widetilde{\mathrm{H}}(X/W, B)$. When equipped with the restriction of the Mukai pairing, the twisted Mukai crystal $\widetilde{\mathrm{H}}(X/W, B)$ is a* K3 *crystal of rank* 24.

Proof. When $B \in \mathscr{B}_1(X)$, this is Proposition 3.4.15 of [Bragg and Lieblich 2018]. Using Proposition 2.7, the proof of loc. cit. applies verbatim to give the result for general B-fields as well. $\square$

Note that if $h \in \mathrm{H}^2(X/W)$ then $\exp(h) = \left(1, h, \frac{1}{2}h^2\right) \in \mathrm{H}^*(X/W)$. Thus, as a submodule of $\widetilde{\mathrm{H}}(X/K)$, $\widetilde{\mathrm{H}}(X/W, B)$ depends only on the image of B in $\mathrm{H}^2(X, \mu_{p^n})$. Furthermore, up to isomorphism (of K3 crystals), $\widetilde{\mathrm{H}}(X/W, B)$ only depends on the Brauer class α_B (see [Bragg 2021, Lemma 3.2.4]).

Remark 2.13. For a K3 surface over the complex numbers, Huybrechts and Stellari [2005] define the twisted Mukai lattice $\widetilde{\mathrm{H}}(X, B, \boldsymbol{Z})$ to be equal to the untwisted lattice $\widetilde{\mathrm{H}}(X, \boldsymbol{Z})$ with a modified Hodge structure. This differs from our definition of the twisted Mukai crystal (as well as the twisted ℓ-adic Mukai lattice), as we have defined $\widetilde{\mathrm{H}}(X/W, B)$ by equipping the rational Mukai lattice $\widetilde{\mathrm{H}}(X/K)$ with a nonstandard integral structure, but the same crystal structure. The convention analogous to that of loc. cit. would be to define $\widetilde{\mathrm{H}}(X/W, B)$ to be equal to $\widetilde{\mathrm{H}}(X/W)$ as a W-module, but equipped with the twisted Frobenius operator $\widetilde{\Phi}_B = \exp(-B) \circ \widetilde{\Phi} \circ \exp(B) = \exp(\phi(B) - B)$.

We record the following observation.

Proposition 2.14. *Let X be a* K3 *surface and B be a crystalline B-field. If X has finite height h, then $\widetilde{\mathrm{H}}(X/W, B)$ is a* K3 *crystal of height h and, in particular, is abstractly isomorphic to $\widetilde{\mathrm{H}}(X/W)$. If X is supersingular of Artin invariant σ_0, then $\widetilde{\mathrm{H}}(X/W, B)$ is a supersingular* K3 *crystal whose Artin invariant is equal to either σ_0 if $\alpha_B = 0$ or $\sigma_0 + 1$ if $\alpha_B \neq 0$.*

Proof. Suppose that X has finite height. The defining inclusion $\widetilde{\mathrm{H}}(X/W, B) \subset \widetilde{\mathrm{H}}(X/K)$ is compatible with the pairing and Frobenius. Thus, $\widetilde{\mathrm{H}}(X/W, B)$ and $\widetilde{\mathrm{H}}(X/K)$ are isogenous, and so $\widetilde{\mathrm{H}}(X/W, B)$ has height h. If h is finite, this implies the crystals are isomorphic integrally. Alternatively, we may reason as follows. Because X has finite height, by Proposition 2.6 we may assume B satisfies $B = \phi(B)$. The map $\exp(-B)$ then defines an isomorphism $\widetilde{\mathrm{H}}(X/W, B) \cong \widetilde{\mathrm{H}}(X/W)$ of K3 crystals. If X is supersingular, then the Brauer group of X is p-torsion. As the twisted Mukai crystal depends up to isomorphism only on the class α_B, we may assume that $B \in \mathscr{B}(X)_1$. The result then follows from [Bragg and Lieblich 2018, Corollary 3.4.23]. □

2I. ***The Newton–Hodge decomposition of the twisted Mukai crystal.*** Let H be a K3 crystal. The *Newton–Hodge decomposition* of H is a canonical direct sum decomposition

$$H = H_{<1} \oplus H_1 \oplus H_{>1}$$

with the following properties. If H has finite height h and rank r, then $H_{<1}$ has slope $1 - 1/h$ and rank h, H_1 has slope 1 and rank $r - 2h$, and $H_{>1}$ has slope $1 + 1/h$ and rank h. Furthermore, H_1 is orthogonal to $H_{<1} \oplus H_{>1}$, and under the pairing, $H_{<1}$ and $H_{>1}$ are dual. If H is supersingular, then $H_1 = H$ and $H_{<1} = H_{>1} = 0$.

Let X be a K3 surface and let B be a crystalline B-field. We will relate the Newton–Hodge decompositions of $\widetilde{\mathrm{H}}(X/W, B)$ and $\mathrm{H}^2(X/W)$.

Proposition 2.15. *As submodules of $\widetilde{\mathrm{H}}(X/K)$,*

$$\begin{aligned}
\widetilde{\mathrm{H}}(X/W, B)_{<1} &= \mathrm{H}^2(X/W)_{<1},\\
\widetilde{\mathrm{H}}(X/W, B)_{1} &= \exp(B)\big(\mathrm{H}^0(X/W) \oplus \mathrm{H}^2(X/W)_1 \oplus \mathrm{H}^4(X/W)\big),\\
\widetilde{\mathrm{H}}(X/W, B)_{>1} &= \mathrm{H}^2(X/W)_{>1}.
\end{aligned}$$

Proof. If X is supersingular, then $\widetilde{\mathrm{H}}(X/W, B)$ is also supersingular, and the result is trivial. Suppose X has finite height. Write $\mathrm{H}^i = \mathrm{H}^i(X/W)$. By Proposition 2.6, B is congruent modulo H^2 to a B-field B' satisfying $\phi(B') = B'$. As $\widetilde{\mathrm{H}}(X/W, B) = \widetilde{\mathrm{H}}(X/W, B')$, we may assume without loss of generality that B is fixed by ϕ and, in particular, $B \in (\mathrm{H}^2)_1 \otimes K$. We then have that $\widetilde{\mathrm{H}}(X/W, B)_{<1} = \exp(B)\, \widetilde{\mathrm{H}}(X/W)_{<1}$, and similarly for the slope 1 and > 1 parts. It is immediate that the Newton–Hodge decomposition of $\widetilde{\mathrm{H}}(X/W)$ is given by $\widetilde{\mathrm{H}}(X/W)_{<1} = (\mathrm{H}^2)_{<1}$, $\widetilde{\mathrm{H}}(X/W)_1 = \mathrm{H}^0 \oplus (\mathrm{H}^2)_1 \oplus \mathrm{H}^4$, and $\widetilde{\mathrm{H}}(X/W)_{>1} = (\mathrm{H}^2)_{>1}$. The result follows upon noting that $\mathrm{H}_1 \otimes K$ is orthogonal to $(\mathrm{H}^2)_{<1}$ and $(\mathrm{H}^2)_{>1}$, so $\exp(B)(\mathrm{H}^2)_{<1} = (\mathrm{H}^2)_{<1}$ and $\exp(B)(\mathrm{H}^2)_{>1} = (\mathrm{H}^2)_{>1}$. □

Note that if B is a general B-field, the direct sum decomposition of $\widetilde{\mathrm{H}}(X/W, B)_1$ described in the statement of Proposition 2.15 may *not* be preserved by $\widetilde{\Phi}$.

2J. ***Mixed realization.*** We define B-fields for Brauer classes whose order is not necessarily a prime power. For simplicity we give the definitions only when $\operatorname{char} k = p > 0$.

Definition 2.16. Let $\alpha \in \mathrm{Br}(X)$ be a class of exact order m. Fix a prime q. Let q^n be the largest power of q dividing m, and set $t = m/q^n$. If $q = \ell \neq p$, then an *ℓ-adic B-field lift* of α is an ℓ-adic B-field lift (in the sense of Definition 2.1) of $t\alpha$. Similarly, if $q = p$, then a *crystalline B-field lift* of α is a crystalline B-field lift (in the sense of Definition 2.3) of $t\alpha$.

Definition 2.17. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class. A *mixed B-field lift of* α is a set $\boldsymbol{B} = \{B_\ell\}_{\ell \neq p} \cup \{B_p\}$ consisting of a choice of an ℓ-adic B-field lift B_ℓ of α for each prime $\ell \neq p$ and a crystalline B-field lift B_p of α (in both cases in the sense of Definition 2.16).

Given a mixed B-field $\boldsymbol{B}$, we write $\boldsymbol{B}^p$ for the component in $\mathrm{H}^2(X, \boldsymbol{A}_f^p)$, and $\boldsymbol{B}_p = B_p$ for the component in $\mathscr{B}(X) \subset \mathrm{H}^2(X/K)$.

We say a few words to explain this definition. Let $\mu_* = \bigcup_m \mu_m$ be the subsheaf of torsion sections of $\boldsymbol{G}_m$. Let $p_0 = p$ and let $p_1, p_2, \ldots$ be an enumeration of the remaining primes. We have a canonical isomorphism

$$\mu_{p_0^\infty} \oplus \mu_{p_1^\infty} \oplus \mu_{p_2^\infty} \cdots \cong \mu_*$$

given by multiplication. As described in the introduction, we have

$$\mathrm{H}^2(X, \boldsymbol{A}_f^p) = \prod_{i \geq 1}{}' \mathrm{H}^2(X, \boldsymbol{Q}_{p_i}(1)),$$

where the restricted product on the right-hand side consists of tuples $\{B_i\}$ such that for all but finitely many i we have $B_i \in \mathrm{H}^2(X, \boldsymbol{Z}_{p_i}(1))$. A mixed B-field lift of a class α is a preimage of α under the composition

$$\mathscr{B}(X) \times \mathrm{H}^2(X, \boldsymbol{A}_f^p) \twoheadrightarrow \bigoplus_i \mathrm{H}^2(X, \mu_{p_i^\infty}) \xrightarrow{\sim} \mathrm{H}^2(X, \mu_*) \twoheadrightarrow \mathrm{Br}(X), \tag{28}$$

which we denote by $\boldsymbol{B} \mapsto \alpha_{\boldsymbol{B}}$. Here, the right horizontal map is induced by the inclusion $\mu_* \subset \boldsymbol{G}_m$.

3. Twisted Chern characters and action on cohomology

Let X be a smooth projective variety over a field k and let $\alpha \in \mathrm{Br}(X)$ be a torsion Brauer class. In this section we will define a certain twisted Chern character for α-twisted sheaves on X. This will be a map from the Grothendieck group of coherent α-twisted sheaves on X to the rational Chow group $A^*(X)_{\boldsymbol{Q}}$ of X. There are multiple inequivalent definitions of twisted Chern characters appearing in the literature, several of which are reviewed and compared in [Huybrechts and Stellari 2006, §3]. These all seem to be essentially equivalent in practice. We will use the notion appearing in [Lieblich et al. 2014; Bragg 2021; Bragg and Lieblich 2018], which is also used in [Huybrechts 2019, §2]. This formulation seems to us to be the most flexible, and has a uniform interaction with B-fields in each of the contexts we have considered. We remark that our definition below is described in terms of cocycles in [Bragg 2021, Appendix A.1] and is compared to the twisted Chern characters of Huybrechts and Stellari [2005] in [Bragg 2021, Appendix A.2].

Suppose that $n\alpha = 0$ for some positive integer n. To define our twisted Chern character we will make an auxiliary choice of a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α under the surjection

$$\mathrm{H}^2(X, \mu_n) \twoheadrightarrow \mathrm{Br}(X)[n]$$

induced by the inclusion $\mu_n \subset \boldsymbol{G}_m$.

We choose a $\boldsymbol{G}_m$–gerbe $\pi : \mathscr{X} \to X$ with cohomology class α, and identify the category of α-twisted sheaves on X with the category of coherent sheaves on $\mathscr{X}$ of weight 1. We also choose a μ_n-gerbe $\mathscr{X}' \to X$ with cohomology class α', and an isomorphism $\mathscr{X}' \wedge_{\mu_n} \boldsymbol{G}_m \cong \mathscr{X}$ (see [Olsson 2016, Chapter 12.3]). There is then a canonical n-fold twisted invertible sheaf $\mathscr{L}$ on $\mathscr{X}$. Given a locally free α-twisted sheaf $\mathscr{E}$ of finite rank, we note that $\mathscr{E}^{\otimes n} \otimes \mathscr{L}^\vee$ is a 0-twisted sheaf on $\mathscr{X}$. We define

$$\mathrm{ch}^{\alpha'}(\mathscr{E}) = \sqrt[n]{\mathrm{ch}(\pi_*(\mathscr{E}^{\otimes n} \otimes \mathscr{L}^\vee))},$$

where the n-th root is chosen so that rk is positive. One can check that $\mathrm{ch}^{\alpha'}$ depends only on α', and not on the choice of gerbes or on $\mathscr{L}$. We note that ch_0 and ch_1 are given by

$$\mathrm{ch}^{\alpha'}(\mathscr{E}) = (\mathrm{rk}(\mathscr{E}), \pi_*(\det(\mathscr{E}) \otimes \mathscr{L}^\vee), \dots). \tag{29}$$

Assume that $\mathscr{X}$ has the resolution property, so that every α-twisted sheaf admits a finite resolution by locally free α-twisted sheaves. We then obtain by additivity a map

$$\mathrm{ch}^{\alpha'} : K(X, \alpha) \to A^*(X)_{\boldsymbol{Q}},$$

where $K(X, \alpha)$ denotes the Grothendieck group of the category of α-twisted sheaves. We note that this definition is purely algebraic, and hence makes sense in any characteristic. Furthermore, we did not need α to be topologically trivial, only torsion.

Suppose that X is a K3 surface. We explain the relationship between the choice of α' and the choice of a B-field lift of α. We first observe that, in any of the contexts we have considered, a choice of B-field lift for α determines in particular a choice of preimage of α in $\mathrm{H}^2(X, \mu_n)$. More precisely, a choice of singular B-field lift (if the ground field is the complex numbers) or of a mixed B-field lift determines a preimage in $\mathrm{H}^2(X, \mu_n)$. If α is killed by ℓ^n, then a choice of ℓ-adic B-field lift determines a preimage in $\mathrm{H}^2(X, \mu_{\ell^n})$, and if α is killed by p^n a choice of crystalline B-field lift determines a preimage in $\mathrm{H}^2(X, \mu_{p^n})$. In any of these situations, we write

$$\mathrm{ch}^B(\mathscr{E}) = \mathrm{ch}^{\alpha'}(\mathscr{E}),$$

where α' is the induced preimage. We also set

$$v^B(\mathscr{E}) = \mathrm{ch}^B(\mathscr{E}).\sqrt{\mathrm{td}(X)}.$$

3A. *Twisted Chern characters on twisted* K3 *surfaces.*

Definition 3.1. We assume now that k is an algebraically closed field of characteristic $p > 0$. If X is a K3 surface over k, we define the *extended Néron–Severi group* of X by

$$\widetilde{N}(X) = \langle(1, 0, 0)\rangle \oplus N(X) \oplus \langle(0, 0, 1)\rangle = A^*(X) \subset A^*(X)_{\boldsymbol{Q}}.$$

As the Chern characters of a coherent sheaf on a K3 surface are integral, the extended Néron–Severi group is a natural recipient for the Chern class map, and in fact the Chern class map

$$\mathrm{ch} : K(X) \to \widetilde{N}(X)$$

is an isomorphism. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class. For two α-twisted sheaves $\mathscr{E}$, $\mathscr{F}$, we have the Riemann–Roch formula

$$\chi(\mathscr{E}, \mathscr{F}) = -\langle v^B(\mathscr{E}), v^B(\mathscr{F})\rangle.$$

We will identify a subgroup of $\widetilde{N}(X) \otimes \boldsymbol{Q}$ which contains the image of the twisted Chern class map

$$\mathrm{ch}^B : K(X, \alpha) \to \widetilde{N}(X) \otimes \boldsymbol{Q}.$$

Definition 3.2. If B is an ℓ-adic B-field, we define the *ℓ-adic twisted Néron–Severi group* by

$$\widetilde{N}(X, B_\ell) = (\widetilde{N}(X) \otimes \mathbf{Z}[\ell^{-1}]) \cap \widetilde{\mathrm{H}}(X, \mathbf{Z}_\ell, B_\ell).$$

If $\operatorname{char} k = p$ and B is a crystalline B-field, we define the *crystalline twisted Néron–Severi group* by

$$\widetilde{N}(X, B_p) = (\widetilde{N}(X) \otimes \mathbf{Z}[p^{-1}]) \cap \widetilde{\mathrm{H}}(X/W, B_p),$$

and if $\boldsymbol{B} = \{B_\ell\}_{\ell \neq p} \cup \{B_p\}$ is a mixed B-field, we define the *mixed twisted Néron–Severi group* by

$$\widetilde{N}(X, \boldsymbol{B}) = \Big(\bigcap_{\ell \neq p} \widetilde{N}(X, B_\ell)\Big) \cap \widetilde{N}(X, B_p),$$

where the intersection is taken inside of $\widetilde{N}(X) \otimes \boldsymbol{Q}$. Note that for all but finitely many primes q the B-field B_q is integral. Hence, the intersection defining $\widetilde{N}(X, \boldsymbol{B})$ is finite.

The restriction of the Mukai pairing on $\widetilde{N}(X) \otimes \boldsymbol{Q}$ to the ℓ-adic twisted Néron–Severi group takes values in $\mathbf{Z}[\ell^{-1}] \cap \mathbf{Z}_\ell = \mathbf{Z}$. Similarly, the Mukai pairing restricts to an integral pairing on the crystalline twisted Néron–Severi group. The following is the crucial integrality result for twisted Chern characters, generalizing the fact that the Chern characters of usual sheaves on K3 surfaces are integral.

Proposition 3.3. *Let X be a* K3 *surface and $\boldsymbol{B}$ a mixed* (*resp. ℓ-adic*, *resp. crystalline*) *B-field lift of a Brauer class $\alpha \in \mathrm{Br}(X)$. For any twisted sheaf $\mathscr{E} \in \boldsymbol{Coh}^{(1)}(X, \alpha)$, the twisted Chern character $\mathrm{ch}^{\boldsymbol{B}}(\mathscr{E})$ lies in the mixed* (*resp. ℓ-adic*, *resp. crystalline*) *twisted Néron–Severi group $\widetilde{N}(X, \boldsymbol{B})$.*

Proof. This is proved in Appendix A of [Bragg 2021] (the quoted statement is written for a crystalline B-field of the form $B = a/p$, but the proof applies essentially unchanged). $\square$

Remark 3.4. The analog of Proposition 3.3 for the Hodge realization follows immediately from the existence of an invertible twisted sheaf in the differentiable category (in fact, this existence is used to define twisted Chern characters in [Huybrechts and Stellari 2005]). The ℓ-adic case is proved in [Lieblich et al. 2014, Lemma 3.3.7] by lifting to characteristic 0.

Proposition 3.5. *For any mixed B-field lift of* α, *the twisted Chern character*

$$\mathrm{ch}^{\boldsymbol{B}} : K(X, \alpha) \to \widetilde{N}(X, \boldsymbol{B})$$

is surjective.

Proof. The analogous result over the complex numbers is [Huybrechts and Stellari 2005, Proposition 1.4]. The proof in our case is identical, up to our differences in convention. □

3B. *Action on cohomology.* Let (X, α) and (Y, β) be twisted K3 surfaces over k. Choose mixed B-field lifts $\boldsymbol{B}$ of α and $\boldsymbol{B}'$ of β. As above, we define the twisted Chern character map

$$\mathrm{ch}^{-\boldsymbol{B}\boxplus\boldsymbol{B}'} : K(X \times Y, -\alpha \boxplus \beta) \to \widetilde{N}(X \times Y) \otimes \boldsymbol{Q},$$

and set

$$v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(\_) = \mathrm{ch}^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(\_).\sqrt{\mathrm{td}(X \times Y)}.$$

Let $\Phi_P : D^b(X, \alpha) \to D^b(Y, \beta)$ be a Fourier–Mukai equivalence. We consider the map

$$\Phi_{v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(P)} := \pi_{2*}(\pi_1^*(\_) \cup v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(P)) : \mathrm{H}^*(X)_{\boldsymbol{Q}} \to \mathrm{H}^*(Y)_{\boldsymbol{Q}}, \tag{30}$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the respective projections. Using the same formula, we define maps $\Phi^{\ell}_{v^{-B_\ell \boxplus B'_\ell}(P)}$ on the rational ℓ-adic cohomologies and $\Phi^{\mathrm{cris}}_{v^{-B_p \boxplus B'_p}(P)}$ on rational crystalline cohomology. By definition, these maps are equal to the maps given by restricting (30) to the ℓ-adic and crystalline components of $\mathrm{H}^*_{\boldsymbol{Q}}$.

Theorem 3.6. *Let* $\Phi_P : D^b(X, \alpha) \to D^b(Y, \beta)$ *be a Fourier–Mukai equivalence. The map* (30) *restricts to an isomorphism*

$$\Phi^{\ell}_{v^{-B_\ell \boxplus B'_\ell}(P)} : \widetilde{\mathrm{H}}(X, \mathbf{Z}_\ell, B_\ell) \to \widetilde{\mathrm{H}}(Y, \mathbf{Z}_\ell, B'_\ell) \tag{31}$$

which is compatible with the Mukai pairings for each $\ell \neq p$, *to an isomorphism*

$$\Phi^{\mathrm{cris}}_{v^{-B_p \boxplus B'_p}(P)} : \widetilde{\mathrm{H}}(X/W, B_p) \to \widetilde{\mathrm{H}}(Y/W, B'_p) \tag{32}$$

of K3 *crystals* (*that is*, *an isomorphism of* W*-modules which is compatible with the pairing and Frobenius operators*), *and to an isometry*

$$\Phi_{v^{-\boldsymbol{B}\boxplus\boldsymbol{B}'}(P)} : \widetilde{N}(X, \boldsymbol{B}) \xrightarrow{\sim} \widetilde{N}(Y, \boldsymbol{B}'). \tag{33}$$

Proof. By definition, the map (31) is equal to the correspondence induced by the cycle $v^{-B_\ell \boxplus B'_\ell}(P)$, and the map (32) is equal to the correspondence induced by the cycle $v^{-B_p \boxplus B'_p}(P)$. The compatibility with the pairing, and in the crystalline case, with the Frobenius, is proved exactly as in [Bragg 2021, §3.4]. It remains to show that the correspondences preserve the integral structures. Under the assumption that $p \geq 5$, this is shown in [Bragg 2021, Appendix A]. The result in general can be shown by lifting to characteristic 0, using the techniques of the following section. We omit further details. This proves the claims regarding (31) and (32). To prove the claimed properties of (33), note that the indicated correspondence preserves the subgroups of algebraic cycles, and so restricts to an isomorphism $\widetilde{N}(X) \otimes \boldsymbol{Q} \xrightarrow{\sim} \widetilde{N}(Y) \otimes \boldsymbol{Q}$. The result then follows from the previous claims. $\square$

4. Rational Chow motives and isogenies

Given a smooth proper variety X over and algebraically closed field k, we let $\mathfrak{h}(X)$ denote its rational Chow motive.

Definition 4.1 (cf. [Yang 2022, Definition 1.1]). Let X and X' be K3 surfaces over k. An *isogeny* from X to X' is an isomorphism of motives $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ whose cohomological realization $\mathrm{H}^2(X')_{\boldsymbol{Q}} \to \mathrm{H}^2(X)_{\boldsymbol{Q}}$ preserves the Poincaré pairing. Two isogenies are said to be *equivalent* if they induce the same map $\mathrm{H}^2(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}^2(X)_{\boldsymbol{Q}}$ (see Section 1D).

Recall [Kahn et al. 2007, 14.2.2] that if X is a K3 surface over an algebraically closed field k then there is a canonical decomposition

$$\mathfrak{h}^2(X) = \mathfrak{h}^2_{\mathrm{alg}}(X) \oplus \mathfrak{h}^2_{\mathrm{tr}}(X)$$

of the Chow motive in degree two into an algebraic part and a transcendental part. The algebraic part $\mathfrak{h}^2_{\mathrm{alg}}(X)$ is isomorphic to $\boldsymbol{L} \otimes \mathrm{NS}(X)$, where $\boldsymbol{L}$ stands for the Lefschetz motive. Similarly, $\mathfrak{h}(X)$ decomposes as $\mathfrak{h}_{\mathrm{alg}}(X) \oplus \mathfrak{h}^2_{\mathrm{tr}}(X)$, where $\mathfrak{h}_{\mathrm{alg}} = \boldsymbol{L}^0 \oplus \mathfrak{h}^2_{\mathrm{alg}} \oplus \boldsymbol{L}^2$.

Now suppose (X, α) and (Y, β) are twisted K3 surfaces with mixed B-field lifts $\boldsymbol{B}$ of α and $\boldsymbol{B}'$ of β. Let $\Phi_P : D^b(X, \alpha) \to D^b(Y, \beta)$ be a Fourier–Mukai equivalence. Following Huybrechts [2019, Theorem 2.1], we have that the correspondence $v^{-\boldsymbol{B} \boxplus \boldsymbol{B}'}(P)$ induces an isomorphism $\mathfrak{h}(X) \xrightarrow{\sim} \mathfrak{h}(Y)$, which restricts to isomorphisms $\mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y)$ and $\mathfrak{h}_{\mathrm{alg}}(X) \xrightarrow{\sim} \mathfrak{h}_{\mathrm{alg}}(Y)$. By Witt's cancellation theorem, we can always find some isomorphism $\mathfrak{h}^2_{\mathrm{alg}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{alg}}(Y)$ that preserves the Poincaré pairing. Adding this and the isomorphism $\mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y)$ induced by $v^{-\boldsymbol{B} \boxplus \boldsymbol{B}'}(P)$, we obtain an isogeny $\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$.

Definition 4.2. Let X, Y be K3 surfaces. An isogeny $f : \mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$ is a *primitive derived isogeny* if its restriction $\mathfrak{h}^2_{\mathrm{tr}}(X) \xrightarrow{\sim} \mathfrak{h}^2_{\mathrm{tr}}(Y)$ agrees with the one induced by $v^{-\boldsymbol{B} \boxplus \boldsymbol{B}'}(P)$ for some choices of α, β, $\boldsymbol{B}$, $\boldsymbol{B}'$ and Φ_P as above. A *derived isogeny* is a composition of finitely many primitive derived isogenies.

In particular, note that if there exists a primitive derived isogeny between X and Y, then X and Y are twisted derived equivalent. Twisted derived equivalent K3 surfaces clearly have the same rational Chow motive. In a recent paper, Fu and Vials proved that their motives are moreover isomorphic as Frobenius algebra objects, and over $\boldsymbol{C}$ they also give a motivic characterization of twisted derived equivalent K3's [Fu and Vial 2021, Theorem 1, Corollary 2].

5. Lifting derived isogenies to characteristic 0

The goal of this section is to give some lifting results for primitive derived isogenies. This requires understanding deformations of twisted K3 surfaces and of twisted Fourier–Mukai equivalences in mixed characteristic. Deformations of a twisted K3 surface (X, α) over the complex numbers can be profitably understood in terms of deformations of a pair (X, B), where $B \in \mathrm{H}^2(X, \boldsymbol{Q})$ is a Hodge B-field lift of α (see for instance [Reinecke 2019]). Considering deformations of (X, B) serves two purposes: first, the B-field B allows one to algebraize formal deformations of the Brauer class, and second, B gives a notion of twisted Chern characters in the deformation family. Suppose now that (X, α) is a twisted K3 surface in positive characteristic. To similarly understand deformations of (X, α) over a base of mixed characteristic, we would need a notion of mixed characteristic B-field lift. The ℓ-adic theory works essentially unchanged in this setting, but the analog of the crystalline theory seems more complicated. We will avoid this issue by using instead of a B-field a simpler object, namely a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α under the map $\mathrm{H}^2(X, \mu_n) \to \mathrm{Br}(X)$. The deformation theory of such pairs (X, α') has been considered in [Bragg 2023]: the flat cohomology groups $\mathrm{H}^2(X, \mu_n)$ can be defined relatively in families, and their tangent spaces can be understood in terms of de Rham cohomology. Moreover, it turns out that formal projective deformations of such pairs (X, α') algebraize, and furthermore the class α' can be used to define twisted Chern characters in families. Our approach to the deformation theory of twisted Fourier–Mukai equivalences is based on the techniques of Lieblich and Olsson [2015], which we, in particular, extend to the twisted setting.

Let (X, α) and (Y, β) be twisted K3 surfaces over an algebraically closed field k of characteristic $p > 0$. Let $\Phi_P : D^b(X, \alpha) \cong D^b(Y, \beta)$ be a Fourier–Mukai equivalence induced by a complex $P \in D^b(X \times Y, -\alpha \boxplus \beta)$.

Definition 5.1. The equivalence $\Phi_P : D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$ is *filtered* if there exist preimages $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α and $\beta' \in \mathrm{H}^2(Y, \mu_m)$ of β such that the cohomological transform

$$\Phi_{v^{-\alpha' \boxplus \beta'}(P)} : \widetilde{N}(X)_{\boldsymbol{Q}} \xrightarrow{\sim} \widetilde{N}(Y)_{\boldsymbol{Q}}$$

sends $(0, 0, 1)$ to $(0, 0, 1)$.

Note that the condition for being filtered does not depend on the choices of α' and β', and thus is an intrinsic property of Φ_P. We consider the deformation functor $\mathrm{Def}_{(X,\alpha')}$, whose objects over an Artinian local W-algebra A are isomorphism classes of pairs (X_A, α'_A) where X_A is a flat scheme over Spec A such that $X_A \otimes k \cong X$, and $\alpha'_A \in \mathrm{H}^2(X_A, \mu_n)$ is a cohomology class such that $\alpha'_A|_X = \alpha'$ (see [Bragg 2023, Definition 1.1]).

Proposition 5.2. *Suppose that Φ_P is filtered. Given a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α, there is a canonically induced preimage $\beta' \in \mathrm{H}^2(Y, \mu_n)$ of β and a morphism*

$$\delta_P : \mathrm{Def}_{(Y,\beta')} \to \mathrm{Def}_{(X,\alpha')}$$

of deformation functors over W (depending on P and α').

Proof. Let $\mathscr{X} \to X$ and $\mathscr{Y} \to Y$ be $\boldsymbol{G}_m$-gerbes representing α and β. The chosen preimage α' corresponds to an n-twisted invertible sheaf $\mathscr{L}$ on $\mathscr{X}$. Using Proposition 3.5, we find a complex of twisted sheaves $\mathscr{E}$ on $\mathscr{X}$ with rank n and $\det \mathscr{E} \cong \mathscr{L}$. Using the assumption that Φ_P is filtered, we see that $\Phi_P(\mathscr{E})$ is a complex of twisted sheaves on $\mathscr{Y}$ of rank n. Thus, its determinant $\mathscr{N} = \det(\Phi_P(\mathscr{E}))$ is an invertible n-twisted sheaf on $\mathscr{Y}$. Note that this implies $n\beta = 0$. We let $\beta' \in \mathrm{H}^2(Y, \mu_n)$ be the preimage of β corresponding to $\mathscr{N}$. Note that the class β' does not depend on our choice of $\mathscr{E}$.

Let $\mathscr{X}' \to X$ and $\mathscr{Y}' \to Y$ be μ_n-gerbes corresponding to α' and β'. Suppose given an Artinian local W-algebra A and a deformation of (Y, β') over A. Up to isomorphism, this is the same as giving a pair $(\mathscr{Y}'_A, \varphi)$, where $\mathscr{Y}'_A$ is a μ_n-gerbe equipped with a flat proper map to $\operatorname{Spec} A$ and $\varphi : \mathscr{Y}'_A \otimes k \cong \mathscr{Y}'$ is an isomorphism of gerbes. We let $\mathscr{D}_{\mathscr{Y}'_A/A}$ be the stack of relatively perfect universally glueable simple $\mathscr{Y}'_A$-twisted complexes over $\operatorname{Spec} A$ with twisted Mukai vector $(0, 0, 1)$ (see [Lieblich and Olsson 2015, Section 5]). We let P' be the pullback of P along the product of the maps $\mathscr{X}' \subset \mathscr{X}$ and $\mathscr{Y}' \subset \mathscr{Y}$. Because Φ_P is filtered, the complex P' induces a map $\mathscr{X}' \to \mathscr{D}_{\mathscr{Y}'_A/A} \otimes k$. By reasoning identical to [Lieblich and Olsson 2015, Lemma 5.5], this map is an open immersion. The image of $\mathscr{X}'$ is contained in the smooth locus of the morphism $\pi \otimes k$, so there is a unique open substack $\mathscr{X}'_A \subset \mathscr{D}_{\mathscr{Y}'_A/A}$ which is flat and proper over $\operatorname{Spec} A$ whose restriction to the closed fiber is isomorphic to $\mathscr{X}'$. Via this isomorphism, the stack $\mathscr{X}'_A$ has a canonical structure of μ_n-gerbe. Thus, given a deformation of (Y, β') over A, we have produced (using the complex P) a deformation of (X, α') over A. This defines a morphism $\mathrm{Def}_{(Y,\beta')} \to \mathrm{Def}_{(X,\alpha')}$. ☐

We now assume that Φ_P is a filtered Fourier–Mukai equivalence. We fix a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α. Let $\beta' \in \mathrm{H}^2(Y, \mu_n)$, and let

$$\delta_P : \mathrm{Def}_{(Y,\beta')} \to \mathrm{Def}_{(X,\alpha')} \tag{34}$$

be the preimage and morphism produced by Proposition 5.2. We continue the notation introduced above, so that $\pi_{\mathscr{X}} : \mathscr{X} \to X$ and $\pi_{\mathscr{Y}} : \mathscr{Y} \to Y$ are $\boldsymbol{G}_m$-gerbes corresponding to α and β, $\mathscr{X}'$ and $\mathscr{Y}'$ are μ_n-gerbes corresponding to α' and β', $\mathscr{L}$ and $\mathscr{N}$ are the corresponding n-fold twisted invertible sheaves on $\mathscr{X}'$ and $\mathscr{Y}'$, and P' is the restriction of P to $\mathscr{X}' \times \mathscr{Y}'$. Let $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$ be mixed B-field lifts of α and β such that $n\boldsymbol{B}_\alpha$ and $n\boldsymbol{B}_\beta$ are integral and such that $n\boldsymbol{B}_\alpha \pmod n$ equals α' and $n\boldsymbol{B}_\beta \pmod n$ equals β'. Write Φ for the cohomological transform

$$\Phi = \Phi_{v^{-\alpha' \boxplus \beta'}(P)} : \widetilde{N}(X)_{\boldsymbol{Q}} \to \widetilde{N}(Y)_{\boldsymbol{Q}}.$$

Lemma 5.3. *The transform Φ satisfies $\Phi(0,0,1) = (0,0,1)$ and $\Phi(1,0,0) = (1,0,0)$, and restricts to an isometry $N(X) \xrightarrow{\sim} N(Y)$ of integral Néron–Severi lattices.*

Proof. We are assuming that Φ_P is filtered, so we have $\Phi(0,0,1) = (0,0,1)$. Consider a complex $\mathscr{E}$ of twisted sheaves on $\mathscr{X}$ with rank n and $\det \mathscr{E} \cong \mathscr{L}$. It follows immediately from the definition of the twisted Chern character that $v^{\alpha'}(\mathscr{E}) = (n, 0, s)$ for some integer s. Moreover, we see that the vector

$$\Phi(n, 0, s) = \Phi(v^{\alpha'}(\mathscr{E})) = v^{\beta'}(\Phi_P(\mathscr{E}))$$

has trivial second component. As Φ is an isometry, we conclude that $\Phi(n, 0, s) = (n, 0, s)$. It follows that $\Phi(1, 0, 0) = (1, 0, 0)$, and that Φ restricts to an isometry on the rational Néron–Severi lattices.

We now prove that Φ in fact restricts to an isometry between the integral Néron–Severi lattices. Consider an invertible sheaf L on X. The complex $\Phi_P(\mathscr{E} \otimes \pi^* L)$ of twisted sheaves on $\mathscr{Y}$ has rank n. Let $\mathscr{M}$ be its determinant. Using the formula (29), we see that the pushforward of the (0-twisted) invertible sheaf $\mathscr{M} \otimes \mathscr{N}^\vee$ to Y has cohomology class $\Phi(L)$. In particular, $\Phi(L)$ is in $N(Y)$. $\square$

The following result is our twisted analog of [Lieblich and Olsson 2015, Proposition 6.3].

Proposition 5.4. *The morphism δ_P (34) is an isomorphism, and furthermore has the following properties*:

(1) *For any class $L \in \mathrm{Pic}(X)$, the map δ restricts to an isomorphism*

$$\mathrm{Def}_{(Y,\beta',\Phi(L))} \cong \mathrm{Def}_{(X,\alpha',L)} .$$

(2) *For any augmented Artinian W-algebra A and any lift (X_A, α'_A) of (X, α') over A, there exists a perfect complex $P_A \in D^b(X_A \times_A Y_A, -\alpha_A \boxplus \beta_A)$ lifting P, where $(Y_A, \beta'_A) = \delta^{-1}(X_A, \alpha'_A)$ and α_A and β_A are the Brauer classes associated to α'_A and β'_A.*

Proof. To see that μ_P is an isomorphism, consider the same construction applied to the kernel $Q = P^\vee$ of the inverse Fourier–Mukai transform and the preimage β' of β, which yields a map

$$\mu_Q : \mathrm{Def}_{(X,\alpha')} \xrightarrow{\sim} \mathrm{Def}_{(Y,\beta')} .$$

We claim that μ_P and μ_Q are inverses. This may be verified exactly as in [Lieblich and Olsson 2015, Proposition 6.3]. To see claim (2), note that the restriction along the open immersion

$$\mathscr{X}'_A \times \mathscr{Y}'_A \subset \mathscr{D}_{\mathscr{Y}'_A/A} \times \mathscr{Y}'_A$$

of the universal complex lifts P'. To see (1), suppose that the deformation (X_A, α'_A) is contained in the subfunctor $\mathrm{Def}_{(X,\alpha',L)}$. There is then an invertible sheaf L_A on X_A deforming L. Let $\mathscr{E}_A$ be a relatively perfect complex of α_A-twisted sheaves on X_A with rank n and trivial determinant. Let $\pi_A : \mathscr{X}_A \to X_A$ be the coarse space map. The determinant of the complex $\Phi_{P_A}(\mathscr{E}_A \otimes \pi_A^* L_A)$ is a 0-fold twisted sheaf on $\mathscr{Y}_A$, so its pushforward to Y_A is an invertible sheaf. Moreover, this sheaf has class lifting $\Phi(L)$. $\square$

Definition 5.5. We say that a filtered Fourier–Mukai equivalence Φ_P is *polarized* if there exists B-field lifts $\boldsymbol{B}$ and $\boldsymbol{B}'$ of α and β such that the isometry $\Phi : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ (see Lemma 5.3) sends the ample cone C_X of X to the ample cone C_Y of Y.

One checks that the condition to be polarized is independent of the choice of B-field lifts, in the sense that it is verified for one choice of lifts if and only if it is verified for all choices of lifts.

We now prove our main lifting results. By results in [Bragg 2023], the twisted K3 surface (X, α) can be lifted to characteristic 0. Moreover, we can also compatibly lift the preimage α' of α. As a consequence, Proposition 5.4 shows that given such a lift there is an induced formal lift of (Y, β), together with a lift of β' and of the complex P inducing the equivalence. Under the assumption that Φ_P is polarized, we can even produce a (nonformal) lift. We make this precise in the following result.

Theorem 5.6. *Suppose that Φ_P is a filtered polarized Fourier–Mukai equivalence. Let L be an ample line bundle on X. Suppose we are given a complete DVR V with residue field k and a lift (X_V, α'_V, L_V) of (X, α', L) over V. There exists an ample line bundle M on Y, a lift (Y_V, β'_V, M_V) of (Y, β', M) over V, and a perfect complex $P_V \in D^b(X_V \times_V Y_V, -\alpha_V \boxplus \beta_V)$ (where β_V is the image of β'_V in the Brauer group) which induces a Fourier–Mukai equivalence and whose restriction to $D^b(X \times Y, -\alpha \boxplus \beta)$ is quasi-isomorphic to P.*

Proof. Let M be the line bundle on Y corresponding to $\Phi(L)$. By Proposition 5.4, we find compatible deformations $(Y_{V_n}, \beta'_{V_n}, M_{V_n})$ of (Y, β', M) over $V_n = V/\mathfrak{m}^{n+1}$ for each $n \geq 0$, together with compatible perfect complexes $P_{V_n} \in D^b(X_{V_n} \times_{V_n} Y_{V_n}, -\alpha_{V_n} \boxplus \beta_{V_n})$ deforming P, where β_{V_n} is the image of β'_{V_n} in the Brauer group. As Φ_P is polarized, M is ample, so by the Grothendieck existence theorem, there exists a scheme (Y_V, M_V) over V restricting to the (Y_{V_n}, β'_{V_n}). By [Bragg 2023, Proposition 1.4] there exists a class $\beta'_V \in \mathrm{H}^2(Y_V, \mu_n)$ restricting to the β'_{V_n}. Finally, by the Grothendieck existence theorem for perfect complexes [Lieblich 2006, Proposition 3.6.1], there is a perfect complex $P_V \in D^b(X_V \times_V Y_V, -\alpha_V \boxplus \beta_V)$ whose restriction to V_n is quasi-isomorphic to P_{V_n} for each n. Moreover, arguing as in the proof of Theorem 6.1 of [Lieblich and Olsson 2015], we see that the complex P_V induces a Fourier–Mukai equivalence. □

Definition 5.7. Let $\mathcal{X}$ be a K3 surface over a local ring and X be its special fiber. We say that $\mathcal{X}$ is a *perfect lifting* of X if the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$ is an isomorphism.

We remark that if $\mathcal{X}$ as above is over a DVR, the ample and the big and nef cones of the generic fiber are canonically identified with those of the special fiber.

Theorem 5.8. *Let (X, α) and (Y, β) be twisted* K3 *surfaces over k. Let $\Phi_P : D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$ be a Fourier–Mukai equivalence. There exists*

(a) *an autoequivalence Φ' of $D^b(Y, \beta)$ which is a composition of spherical twists about (-2)-curves,*

(b) *a DVR V whose fraction field has characteristic 0 and with residue field k,*

(c) *projective lifts (X_V, α_V) and (Y_V, β_V) of (X, α) and (Y, β) over V, and*

(d) *a perfect complex $R_V \in D^b(X_V \times_V Y_V, -\alpha_V \boxplus \beta_V)$ which induces a Fourier–Mukai equivalence and whose restriction to $X \times Y$ is quasi-isomorphic to the kernel R of the equivalence $\Phi' \circ \Phi_P$.*

Moreover, if X and Y have finite height, we may choose the above data so that Φ' is the identity and X_V and Y_V are perfect liftings.

Proof. Choose a preimage $\alpha' \in \mathrm{H}^2(X, \mu_n)$ of α. Given a choice of preimage of β, we obtain an isometry

$$\Phi : \widetilde{N}(X)_{\mathbb{Q}} \xrightarrow{\sim} \widetilde{N}(Y)_{\mathbb{Q}}.$$

Consider the class $v = (\Phi)^{-1}(0, 0, 1) \in \widetilde{\mathrm{NS}}(X)_{\mathbb{Q}}$. Note that this class does not depend on the choice of preimage of β. By [Bragg 2023, Theorem 7.3], we may find a DVR V of characteristic 0 and residue field k and a polarized lift (X_V, α'_V) of (X, α') over V over which the class v extends. Let α_V be the image of α'_V in the Brauer group of X_V. Let $\mathscr{M}_V = \mathscr{M}_{(X_V, \alpha_V)}(v)$ be the relative moduli space of H-stable

α_V-twisted sheaves on $X_V \to \operatorname{Spec} V$ with twisted Mukai vector $v^{\alpha'_V} = v$, where H is a v-generic polarization. Let M_V be the coarse space of $\mathscr{M}_V$. The morphism $M_V \to \operatorname{Spec} V$ is a projective family of K3 surfaces, and there is a class $\gamma_V \in \operatorname{Br}(M_V)$ such that the universal complex Q_V induces an equivalence

$$\Phi_{Q_V} : D^b(M_V, \gamma_V) \xrightarrow{\sim} D^b(X_V, \alpha_V).$$

Let $\gamma \in \operatorname{Br}(M)$ be the restriction of γ_V to M, and let Q be the restriction of Q_V. The Fourier–Mukai equivalence

$$\Phi_P \circ \Phi_Q : D^b(M, \gamma) \xrightarrow{\sim} D^b(Y, \beta)$$

is filtered. As in [Lieblich and Olsson 2015, Lemma 6.2], we may find an autoequivalence Φ' as in the statement of the theorem so that $\Phi' \circ \Phi_P \circ \Phi_Q$ is both filtered and polarized. Let R denote its kernel. Choose a preimage $\gamma'_V \in \mathrm{H}^2(M_V, \mu_m)$ of γ_V, and write γ' for the restriction of γ'_V to M. Let β' be the corresponding lift of β produced by Proposition 5.2. By Theorem 5.6, there is a lift (Y_V, β'_V) of (Y, β') and R_V of R over V, corresponding to the lift (M_V, γ'_V) of (M, γ'). Consider the Fourier–Mukai equivalence

$$\Phi_{R_V} \circ \Phi_{Q_V}^{-1} : D^b(X_V, \alpha_V) \to D^b(Y_V, \beta_V).$$

This equivalence restricts over k to $\Phi' \circ \Phi_P$. By the uniqueness of the kernel, we conclude that R_V restricts to the kernel of the equivalence $\Phi' \circ \Phi_P$, as claimed.

Suppose that X and Y have finite height. We modify the above as follows. Choose α' so that p does not divide $n/\operatorname{ord}(\alpha)$. By [Bragg 2023, Theorem 7.3], we may choose the lift X_V so that the restriction map $\operatorname{Pic}(X_V) \to \operatorname{Pic}(X)$ is an isomorphism. It follows that $\operatorname{Pic}(Y_V) \to \operatorname{Pic}(Y)$ is also an isomorphism. In particular, every (-2)-class in $\operatorname{Pic}(Y)$ extends to Y_V. We now compose Φ_{R_V} with an autoequivalence of $D^b(Y, \beta)$ lifting the inverse of Φ'. The kernel of the resulting equivalence then restricts to P, as desired. ☐

6. Existence theorems

The goal of this section is to construct isogenies with prescribed action on cohomology. In particular, we will prove Theorems 1.2 and 1.4.

6A. ***Construction of derived isogenies.*** We begin with Theorem 1.2.

Let R be an integral domain whose field of fractions is of characteristic 0 (we have in mind $R = \mathbf{Z}_\ell$ or $R = W$). Set $R_{\mathbf{Q}} := R \otimes_{\mathbf{Z}} \mathbf{Q}$. Let M be a quadratic lattice such that $2^{-1}m^2 \in R$ for every $m \in M$.

Given an element $b \in M$ such that $\langle b, b\rangle \neq 0$, the reflection in b is the isometry $s_b : M_{\mathbf{Q}} \to M_{\mathbf{Q}}$ defined by

$$s_b(x) = x - \frac{2\langle x, b\rangle}{\langle b, b\rangle} b.$$

Let $\widetilde{H}$ be a lattice of the form $R \oplus M \oplus R$ equipped with the Mukai pairing, i.e.,

$$\langle (r, m, s), (r', m', s')\rangle = \langle m, m'\rangle - rs' - r's,$$

and a multiplicative structure given by

$$(r, m, s) \cdot (r', m', s') = (rr', rm' + r'm, rs' + r's + \langle m, m'\rangle).$$

Lemma 6.1. *Let $b \in M$ be a primitive element such that $\langle b, b\rangle \neq 0$. Set $n := \frac{1}{2}b^2$ and $B := b/n \in M_{\boldsymbol{Q}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Let $B' \in M_{\boldsymbol{Q}}$ be another element. If $\Phi : \widetilde{H}_{\boldsymbol{Q}} \xrightarrow{\sim} \widetilde{H}_{\boldsymbol{Q}}$ satisfies*

(a) $\Phi(1, 0, 0) = (0, 0, 1/n)$ *and* $\Phi(0, 0, 1) = (n, 0, 0)$, *and*

(b) $e^B \Phi e^{-B'}$ *is R-integral (i.e., restricts to an isometry $\widetilde{H} \xrightarrow{\sim} \widetilde{H}$),*

then $\varphi(M) = s_b(M)$, where $s_b \in \operatorname{Aut}(M_{\boldsymbol{Q}})$ is the reflection in b and φ is the restriction of Φ to $M_{\boldsymbol{Q}}$.

Proof. We extend $r_b := -s_b$ to an isometry $\Psi : \widetilde{H}_{\boldsymbol{Q}} \xrightarrow{\sim} \widetilde{H}_{\boldsymbol{Q}}$ by requiring that Ψ satisfies (a). It is straightforward to verify that $e^B \Psi e^{-B}$ is R-integral:

$$\begin{aligned}
e^B\Psi e^{-B}(0,0,1) &= e^B\Psi(0,0,1) = e^B(n,0,0) = (n,b,1),\\
e^B\Psi e^{-B}(1,0,0) &= e^B\Psi(1,-B,1/n) = e^B(1,-B,1/n) = (1,0,0),\\
e^B\Psi e^{-B}(0,m,0) &= e^B\Psi(0,m,-\langle B,m\rangle) = e^B(n\langle -B,m\rangle, r_b(m),0) = (\langle -b,m\rangle,-m,0).
\end{aligned}$$

We now consider the composition $(e^B\Psi e^{-B})^{-1} \circ (e^B \Phi e^{-B'}) = e^B(\Psi^{-1}\circ\Phi)e^{-B'}$, which has to be R-integral. Direct computation shows

$$\begin{aligned}
e^B(\Psi^{-1}\circ\Phi)e^{-B'}(0,m,0) &= e^B(\Psi^{-1}\circ\Phi)(0,m,-\langle B',m\rangle)\\
&= e^B\big(0, r_b^{-1}(\varphi(m)), -\langle B',m\rangle\big)\\
&= \big(0, r_b^{-1}(\varphi(m)), \langle B,\varphi(m)\rangle - \langle B',m\rangle\big).
\end{aligned}$$

As $e^B(\Psi^{-1}\circ\Phi)e^{-B'}$ is R-integral, we deduce that $r_b^{-1}\circ\varphi$ is R-integral, and so $r_b(M) = \varphi(M)$. □

The next result is the key geometric input for the proof of Theorem 1.2. Let k be an algebraically closed field of characteristic p. Given a K3 surface X over k and a class $b \in \mathrm{H}^2(X)$, we let $s_b : \mathrm{H}^2(X)_{\boldsymbol{Q}} \to \mathrm{H}^2(X)_{\boldsymbol{Q}}$ denote the isometry $s_{b_{p_1}} \times s_{b_{p_2}} \times \cdots$. We say that a class $b \in \mathrm{H}^2(X)$ is *primitive* if $nb' = b$ for an integer n and $b' \in \mathrm{H}^2(X)$ implies $n = \pm 1$.

Proposition 6.2. *Let X be a* K3 *surface over k. Let $b \in \mathrm{H}^2(X)$ be a primitive class such that $n := \frac{1}{2}b^2$ is an integer*[1] *and such that b/n is a mixed B-field. There exists a* K3 *surface X' together with a primitive derived isogeny $f : \mathfrak{h}^2(X') \to \mathfrak{h}^2(X)$ such that $f_*(\mathrm{H}^2(X')) = s_b(\mathrm{H}^2(X))$ in $\mathrm{H}^2(X)_{\boldsymbol{Q}}$.*

Proof. Set $\boldsymbol{B} := b/n$ and let $\alpha = \alpha_{\boldsymbol{B}}$ be the Brauer class defined by $\boldsymbol{B}$. Let X' be the moduli space of stable α-twisted sheaves with Mukai vector $v^{\boldsymbol{B}} = (n, 0, 0)$ (where stability is taken with respect to a sufficiently generic polarization). As b is primitive, the class $(n, 0, 0)$ is primitive in $\widetilde{N}(X, \boldsymbol{B})$. Thus, X' is a K3 surface, and there exists a Brauer class $\alpha' \in \mathrm{Br}(X')$ together with an equivalence $\Phi_{\mathscr{E}} : D^b(X', \alpha') \xrightarrow{\sim} D^b(X, \alpha)$. Choose a mixed B-field lift $\boldsymbol{B}'$ of α'. Then the cohomological action $\Phi : \mathrm{H}(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{H}(X)_{\boldsymbol{Q}}$ of the algebraic cycle $v^{-\boldsymbol{B}'\boxplus\boldsymbol{B}}(\mathscr{E})$ sends $(0, 0, 1)$ to $(n, 0, 0)$.

Since Φ is an isometry, the vector $u = (\Phi)^{-1}(0, 0, 1/n)$ satisfies $u^2 = 0$ and $\langle u, (0, 0, 1)\rangle = -1$. Therefore, u is necessarily of the form $e^\delta = \big(1, \delta, \frac{1}{2}\delta^2\big)$ for some $\delta \in \mathrm{H}^2_{\text{ét}}(X')_{\boldsymbol{Q}}$. As $(0, 0, 1)$ is an algebraic class, and Φ is induced by an algebraic cycle, we have $\delta \in \mathrm{NS}(X')_{\boldsymbol{Q}}$. After replacing $\boldsymbol{B}'$ by $\boldsymbol{B}' + \delta$,

[1] That is, n is in the image of the diagonal embedding $\boldsymbol{Z} \hookrightarrow W \times \widehat{\boldsymbol{Z}}^p$.

we may assume that Φ sends $(1, 0, 0)$ to $(0, 0, 1/n)$. Now we may apply Lemma 6.1 to the ℓ-adic part for each $\ell \neq p$ and to the crystalline part. We conclude that the degree 0 part of the correspondence $v^{-\boldsymbol{B}'\boxplus\boldsymbol{B}}(\mathscr{E})$ sends $\mathrm{H}^2(X')$ to $s_b(\mathrm{H}^2(X))$. □

6B. *Cartan–Dieudonné theorems and strong approximation.* To apply Proposition 6.2 towards the proof of Theorem 1.2, we need to show that the reflections s_b about classes $b \in \mathrm{H}^2(X)$ satisfying the conditions of Proposition 6.2 generate a sufficiently large subgroup of isometries of $\mathrm{H}^2(X)_{\boldsymbol{Q}}$. We need two lattice-theoretic inputs. The first is the following generalized Cartan–Dieudonné theorem [Klingenberg 1961, Theorem 2].

Theorem 6.3. *Let R be a local ring with residue characteristic $\neq 2$ and let L be a unimodular quadratic lattice over R. The group $\mathrm{O}(L)$ is generated by the set of reflections s_b, where b ranges over the elements of L such that $b^2 \in R^\times$.*

We also will use the following consequence of the strong approximation theorem. Recall that $\boldsymbol{U}$ denotes the hyperbolic plane, which is a $\boldsymbol{Z}$–lattice of rank 2.

Lemma 6.4. *Let L be a nondegenerate indefinite quadratic lattice over $\boldsymbol{Z}$ of rank ≥ 3. If q is a prime such that $L \otimes \boldsymbol{Z}_q$ contains a copy of $\boldsymbol{U} \otimes \boldsymbol{Z}_q$ as an orthogonal direct summand, then the double quotient*

$$\mathrm{O}(L \otimes \boldsymbol{Q}) \backslash \mathrm{O}(L \otimes \boldsymbol{Q}_q) / \mathrm{O}(L \otimes \boldsymbol{Z}_q)$$

is a singleton.

Proof. This is a slight variant of [Yang 2023, Lemma 2.1.12], whose proof follows from that of [Ogus 1979, Lemma 7.7]. We briefly summarize the argument: Let $\mathsf{K} \subseteq \mathrm{Spin}(L \otimes \boldsymbol{Q}_q)$ be the preimage of $\mathrm{SO}(L \otimes \boldsymbol{Z}_q)$ under the natural map $\mathrm{ad} : \mathrm{Spin} \to \mathrm{SO}$. Using the fact that $L \otimes \boldsymbol{Z}_q$ contains $\boldsymbol{U} \otimes \boldsymbol{Z}_q$ as an orthogonal direct summand, we show that the maps

$$\begin{aligned}\mathrm{Spin}(L \otimes \boldsymbol{Q})\backslash\mathrm{Spin}(L \otimes \boldsymbol{Q}_q)/\mathsf{K} &\to \mathrm{SO}(L \otimes \boldsymbol{Q})\backslash\mathrm{SO}(L \otimes \boldsymbol{Q}_q)/\mathrm{SO}(L \otimes \boldsymbol{Z}_q) \\ &\to \mathrm{O}(L \otimes \boldsymbol{Q}) \backslash \mathrm{O}(L \otimes \boldsymbol{Q}_q) / \mathrm{O}(L \otimes \boldsymbol{Z}_q)\end{aligned}$$

are both surjections. Now we conclude using the fact that the first double quotient is a singleton by the strong approximation theorem. □

We now return to the setting of a K3 surface X over an algebraically closed field k of characteristic p.

Lemma 6.5. *Let X be a* K3 *surface over k, and assume that $p \geq 5$. There exists a $\boldsymbol{Z}$–lattice L of rank* 22 *and a primitive indefinite sublattice $L' \subset L$ such that*

(a) *for each $\ell \neq p$, there exists an isometry $L \otimes \boldsymbol{Z}_\ell \cong \mathrm{H}^2(X, \boldsymbol{Z}_\ell)$,*

(b) *there exists an isometry $L' \otimes \boldsymbol{Z}_p \cong T(X) := \mathrm{H}^2(X/W)^{\varphi=1}$, and*

(c) *the double quotients*

$$\mathrm{O}(L \otimes \boldsymbol{Z}_{(p)}) \backslash \mathrm{O}(L \otimes \boldsymbol{A}_f^p) / \mathrm{O}(L \otimes \widehat{\boldsymbol{Z}}^p) \quad \textit{and} \quad \mathrm{O}(L' \otimes \boldsymbol{Q}) \backslash \mathrm{O}(L' \otimes \boldsymbol{Q}_p) / \mathrm{O}(L' \otimes \boldsymbol{Z}_p)$$

are both singletons.

Proof. Suppose that X has finite height h. We take $L = \Lambda$ to be the K3 lattice. As L contains a copy of $\boldsymbol{U}$ as an orthogonal direct summand, we may apply [Yang 2023, Lemma 2.1.12] to conclude that the indicated double quotient is a singleton. We will now produce L'. Suppose that $h \leq 9$. By [Ito 2019, Theorem 6.4] (which requires $p \geq 5$), there exists a K3 surface Y over $\overline{\boldsymbol{F}}_p$ such that $h(Y) = h$ and $\rho(Y) = 22 - 2h$. Set $L' = \mathrm{Pic}(Y)$. The existence of a perfect lifting of Y to characteristic zero shows that L' admits a primitive embedding into $L = \Lambda$. Condition (a) is immediate. The embedding $L' \to \mathrm{H}^2(Y/W)$ induces an isomorphism $L' \otimes \boldsymbol{Z}_p \cong T(Y) = T(X)$, giving (b). It remains to check that the double quotient involving L' is a singleton. The pairing on H_1 is perfect, and the inclusion $T(X) \subset \mathrm{H}_1$ induces an isomorphism $T(X) \otimes_{\boldsymbol{Z}_p} W \cong \mathrm{H}_1$, so the discriminant of the pairing on $T(X) \cong L' \otimes \boldsymbol{Z}_p$ is a p-adic unit. As L' has rank ≥ 4, the classification of p-adic lattices [Ogus 1979, Lemma 7.5] implies that $L' \otimes \boldsymbol{Z}_p$ contains a copy of $\boldsymbol{U} \otimes \boldsymbol{Z}_p$ as an orthogonal direct summand. By the Hodge index theorem, L' is indefinite. We conclude using Lemma 6.4. Suppose $h = 10$. We take $L' = \boldsymbol{U}$. This is certainly a primitive sublattice of $L = \Lambda$, and the double quotient involving L' is a singleton. It remains to check that $\boldsymbol{U} \otimes \boldsymbol{Z}_p \cong T(X)$. As explained by Ogus [1983, Remark 1.5], the discriminant of the pairing on H_1 is -1. The same is then true for $T(X)$, because $T(X) \otimes_{\boldsymbol{Z}_p} W \cong \mathrm{H}_1$. By the classification of quadratic lattices over $\boldsymbol{Z}_p$, we conclude that $\boldsymbol{U} \otimes \boldsymbol{Z}_p \cong T(X)$.

Suppose that X is supersingular. Let $L' = L = \Lambda_{\sigma_0}$ be the supersingular K3 lattice of Artin invariant $\sigma_0 = \sigma_0(X)$. The discriminant of the pairing on Λ_{σ_0} is equal to $-p^{2\sigma_0}$, which is an ℓ-adic unit for all $\ell \neq p$, and so (a) holds. Condition (b) is immediate. Finally, by [Ogus 1979, Lemma 7.7], condition (c) holds. □

The following results could be phrased purely in terms of (semi)linear algebra, but for clarity we will maintain the geometric notation.

We recall that $\mathrm{O}(\mathrm{H}^2(X, \boldsymbol{A}_f^p))$ is the subgroup of $\prod_{\ell \neq p} \mathrm{O}(\mathrm{H}^2(X, \boldsymbol{Z}_\ell))$ consisting of those tuples Θ such that Θ_ℓ is ℓ–integral for all but finitely many ℓ (here, we say that Θ_ℓ is ℓ-integral if $\Theta_\ell(\mathrm{H}^2(X, \boldsymbol{Z}_\ell)) = \mathrm{H}^2(X, \boldsymbol{Z}_\ell)$). We let $\mathrm{O}_\Phi(\mathrm{H}^2(X/K))$ be the group of automorphisms of $\mathrm{H}^2(X/K)$ which are isometries with respect to the pairing and which commute with Φ. We set $\mathrm{O}_\Phi(\mathrm{H}^2(X)) = \mathrm{O}_\Phi(\mathrm{H}^2(X/K)) \times \mathrm{O}(\mathrm{H}^2(X, \boldsymbol{A}_f^p))$.

Remark 6.6. Giving an isometric embedding ι as in the statement of Theorem 1.2 is equivalent to giving an isometry $\iota_p : \Lambda \otimes W \hookrightarrow \mathrm{H}^2(X/K)$ of W-modules and for each prime $\ell \neq p$ an isometry $\iota_\ell : \Lambda \otimes \boldsymbol{Z}_\ell \hookrightarrow \mathrm{H}^2(X, \boldsymbol{Q}_\ell)$ of $\boldsymbol{Q}_\ell$-modules such that for all but finitely many ℓ we have $\mathrm{im}(\iota_\ell) = \mathrm{H}^2(X, \boldsymbol{Z}_\ell)$. A similar description holds for the isometric embedding in the statement of Theorem 6.13.

Lemma 6.7. *Suppose that $p \geq 5$. If $\Theta^p \in \mathrm{O}(\mathrm{H}^2(X, \boldsymbol{A}_f^p))$ is an isometry, then there exists a sequence $b_1, \ldots, b_r$ of primitive elements of* $\mathrm{H}^2(X)$ *such that*

(1) *for each i, $n_i := \frac{1}{2} b_i^2$ is an integer which is not divisible by p, and*

(2) *the isometry $s := s_{b_1} \circ \cdots \circ s_{b_r}$ satisfies $s(\mathrm{H}^2(X, \widehat{\boldsymbol{Z}}^p)) = \Theta^p(\mathrm{H}^2(X, \widehat{\boldsymbol{Z}}^p))$.*

Proof. Let L be a lattice as in Lemma 6.5, and choose an identification $L \otimes \widehat{\boldsymbol{Z}}^p = \mathrm{H}^2(X, \widehat{\boldsymbol{Z}}^p)$. By Lemma 6.4,

$$\mathrm{O}(L \otimes \boldsymbol{Z}_{(p)}) \backslash \mathrm{O}(L \otimes \boldsymbol{A}_f^p) / \mathrm{O}(L \otimes \widehat{\boldsymbol{Z}}^p)$$

is a singleton. Hence, there exists an isometry $\Psi \in \mathrm{O}(L \otimes \mathbf{Z}_{(p)})$ such that $\Psi(L) \otimes \widehat{\mathbf{Z}}^p = \Theta^p(L \otimes \widehat{\mathbf{Z}}^p)$. We apply Theorem 6.3 with $R = \mathbf{Z}_{(p)}$ to produce a sequence $b_1, \ldots, b_r$ of elements of $L \otimes \mathbf{Z}_{(p)}$ such that $b_i^2 \in \mathbf{Z}_{(p)}^\times$ for each i and $\Psi = s_{b_1} \circ \cdots \circ s_{b_r}$. For each i, we may write $b_i = v/m$ for some primitive $v \in L$ and an integer m which is coprime to p. Note that the integer $\frac{1}{2}v^2 = \frac{1}{2}m^2 b_i^2$ is in $\mathbf{Z}_{(p)}^\times$, and hence is not divisible by p. Moreover, we have $s_{b_i} = s_v$. So, by replacing each b_i with the corresponding v, we may arrange that the b_i satisfy (1). Condition (2) holds by construction. □

Lemma 6.8. *Suppose that $p \geq 5$. Let $\Theta_p \in \mathrm{O}_\Phi(\mathrm{H}^2(X/K))$ be an isometry which restricts to the identity on $\mathrm{H}^2(X/W)_{<1}$. There exists a sequence $b_1, \ldots, b_r$ of primitive elements of $\mathrm{H}^2(X)$ such that*

(1) *for each i, $n_i := \frac{1}{2}b_i^2$ is an integer and $\varphi(b_i) = b_i$, and*

(2) *the isometry $s := s_{b_1} \circ \cdots \circ s_{b_r}$ satisfies $s(\mathrm{H}^2(X/W)) = \Theta_p(\mathrm{H}^2(X/W))$.*

Proof. Write $\mathrm{H} = \mathrm{H}^2(X/W)$, and consider the Newton–Hodge decomposition $\mathrm{H} = \mathrm{H}_{<1} \oplus \mathrm{H}_1 \oplus \mathrm{H}_{>1}$ of H. The first and third factors are dual, and orthogonal to H_1. Because Θ_p restricts to the identity on $\mathrm{H}_{<1}$, it must also restrict to the identity on $\mathrm{H}_{>1}$, and hence Θ_p restricts to an element of $\mathrm{O}_\Phi(\mathrm{H}_1) = \mathrm{O}(T(X))$. We fix lattices L, L' as in Lemma 6.5 and an identification $L' \otimes \mathbf{Z}_p = T(X)$. By Lemma 6.4, we may find $\Psi \in \mathrm{O}(L' \otimes \boldsymbol{Q})$ such that $\Psi(L') \otimes \mathbf{Z}_p = \Theta_p|_{T(X)}(L' \otimes \mathbf{Z}_p)$. By the classical Cartan–Dieudonné theorem, we may find a sequence $b_1, \ldots, b_r$ of elements of $L' \otimes \boldsymbol{Q}$ such that $\Psi = s_p = s_{b_1} \circ \cdots \circ s_{b_r}$. By scaling, we may assume that each b_i is in L' and is primitive. Note that, as $\mathrm{H}_{<1}$ and $\mathrm{H}_{>1}$ are orthogonal to H_1, the reflections s_{b_i} are the identity on $\mathrm{H}_{<1} \oplus \mathrm{H}_{>1}$. If follows that s satisfies condition (2). □

Lemma 6.9. *Suppose that $p \geq 5$. Let $\Theta \in \mathrm{O}_\Phi(\mathrm{H}^2(X)_{\boldsymbol{Q}})$ be an isometry such that Θ_p restricts to the identity on $\mathrm{H}^2(X/W)_{<1}$. There exists a sequence $b_1, \ldots, b_m$ of primitive elements of $\mathrm{H}^2(X)$ such that*

(1) *for each i, $n_i := \frac{1}{2}b_i^2$ is an integer and b_i/n_i is a mixed B-field, and*

(2) *the isometry $s := s_{b_1} \circ \cdots \circ s_{b_m}$ satisfies $s(\mathrm{H}^2(X)) = \Theta(\mathrm{H}^2(X))$.*

Proof. We first choose elements $b_1, \ldots, b_r \in \mathrm{H}^2(X)$ by applying Lemma 6.8 to Θ_p. We set $s = s_{b_1} \circ \cdots \circ s_{b_r}$. We apply Lemma 6.7 to $(s^{-1} \circ \Theta)^p$ to obtain elements $b'_1, \ldots, b'_t \in \mathrm{H}^2(X)$. Set $s' = s_{b'_1} \circ \cdots \circ s_{b'_t}$. We claim that the sequence $b_1, \ldots, b_r, b'_1, \ldots, b'_t \in \mathrm{H}^2(X)$ satisfies the desired conditions. We check (1). We have that $n_i := \frac{1}{2}b_i^2$ and $n'_i := \frac{1}{2}(b'_i)^2$ are integers. We have that $\varphi((b_i)_p) = (b_i)_p$, so by Proposition 2.7 each $(b_i)_p/n_i$ is a crystalline B-field. It follows that b_i/n_i is a mixed B-field. As n'_i is not divisible by p, $(b'_i)_p/n'_i$ is in $\mathrm{H}^2(X/W)$, so $(b'_i)_p/n'_i$ is a crystalline B-field, and b'_i/n'_i is a mixed B-field. We have shown that (1) holds. To check (2), note that by construction, we have

$$(s \circ s')^p(\mathrm{H}^2(X, \widehat{\mathbf{Z}}^p)) = \Theta^p(\mathrm{H}^2(X, \widehat{\mathbf{Z}}^p)).$$

Furthermore, as p does not divide $\frac{1}{2}(b'_i)^2$, we have $s'_p(\mathrm{H}^2(X/W)) = \mathrm{H}^2(X/W)$, and so

$$(s \circ s')_p(\mathrm{H}^2(X/W)) = s_p(\mathrm{H}^2(X/W)) = \Theta_p(\mathrm{H}^2(X/W)).$$ □

Proof of Theorem 1.2. We prove the "only if" direction first. Suppose that $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ is a primitive derived isogeny. We may choose Brauer classes $\alpha \in \mathrm{Br}(X)$ and $\alpha' \in \mathrm{Br}(X')$, a Fourier–Mukai equivalence $\Phi_P : D^b(X', \alpha') \xrightarrow{\sim} D^b(X, \alpha)$, and crystalline B-field lifts B and B' of α and α' such that the cohomological transform $\Phi_{v^{-B'\boxplus B}}(P) : \widetilde{\mathrm{H}}(X'/K) \to \widetilde{\mathrm{H}}(X/K)$ and the cohomological realization $\mathrm{H}^2(X'/K) \xrightarrow{\sim} \mathrm{H}^2(X/K)$ of f restrict to the same map $T^2(X'/K) \xrightarrow{\sim} T^2(X/K)$, where $T^2(X/K)$ denotes the orthogonal complement to $\mathrm{NS}(X) \otimes K$ in $\mathrm{H}^2(X/K)$ (not to be confused with the Tate module of $\mathrm{H}^2(X/W)$). By Theorem 3.6, $\Phi_{v^{-B'\boxplus B}}(P)$ restricts to an isomorphism $\widetilde{\mathrm{H}}(X'/W, B') \xrightarrow{\sim} \widetilde{\mathrm{H}}(X/W, B)$ of crystals. Thus, by Proposition 2.15, it induces an isomorphism

$$\mathrm{H}^2(X'/W)_{<1} = \widetilde{\mathrm{H}}(X'/W, B')_{<1} \xrightarrow{\sim} \widetilde{\mathrm{H}}(X/W, B)_{<1} = \mathrm{H}^2(X/W)_{<1}.$$

The transcendental part $T^2(X/K)$ contains $\mathrm{H}^2(X/W)_{<1}$, so the cohomological realization of f also maps the slope < 1 part to the slope < 1 part. This gives the result.

We now prove the "if" direction. For each $\ell \neq p$ fix an isometry $\mathrm{H}^2(X, \mathbf{Z}_\ell) \cong \Lambda \otimes \mathbf{Z}_\ell$. Assume first that the K3 crystals H_p and $\mathrm{H}^2(X/W)$ are abstractly isomorphic. This is the case, for instance, if X has finite height. We fix an isomorphism $\mathrm{H}^2(X/W) \cong H_p$ of K3 crystals. Composing with the given embedding ι and tensoring with $\boldsymbol{Q}$, we find an isometry $\Theta \in \mathrm{O}_\Phi(\mathrm{H}^2(X/K)) \times \mathrm{O}(\mathrm{H}^2(X, \widehat{\mathbf{Z}}^p))$ which maps $\mathrm{H}^2_{\text{ét}}(X, \mathbf{Z}_\ell)$ to $\iota_l(\Lambda \otimes \mathbf{Z}_\ell)$ and $\mathrm{H}^2(X/W)$ to $\iota_p(\Lambda \otimes W)$. By Lemma 6.9, we may find a sequence $b_1, \ldots, b_m \in \mathrm{H}^2(X)$ of primitive elements such that for every i, $n_i := \frac{1}{2}b_i^2$ is an integer and b_i/n_i is a mixed B-field, and furthermore the isometry $s := s_{b_1} \circ \cdots \circ s_{b_m}$ satisfies $s(\mathrm{H}^2(X)) = \Theta(\mathrm{H}^2(X))$. The result follows by repeatedly applying Proposition 6.2.

We now consider the case when X is supersingular and H_p and $\mathrm{H}^2(X/W)$ are not isomorphic. This can certainly occur: any two supersingular K3 crystals over k of the same rank and discriminant are isogenous, but by results of Ogus [1979], supersingular K3 crystals themselves have nontrivial moduli. We argue as follows. By the global crystalline Torelli theorem [Ogus 1983], there exists a supersingular K3 surface X' such that $\mathrm{H}^2(X'/W)$ is isomorphic as a K3 crystal to H_p. By Theorem 6.11 below, there exists a derived isogeny $\mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$, which induces an isometry $\mathrm{H}^2(X'/K) \cong \mathrm{H}^2(X/K)$. We are now reduced to the previous case, and we conclude the result. □

Remark 6.10. The only place where the assumption $p \geq 5$ is used in the above proof is in applying the result of Ito [2019, Theorem 6.4]. If in Theorem 1.2, $H_p = \mathrm{H}^2_{\mathrm{cris}}(X/W)$, i.e., Θ_p as above can be taken to be the identity, then the assumption $p > 2$ suffices. In this case, in producing X' we only need to iteratively take moduli of sheaves twisted by Brauer classes of prime-to-p order.

6C. *Existence in the supersingular case.* We make a few remarks specific to the supersingular case. Here, very strong cohomological results are available: there is a global Torelli theorem [Ogus 1979; 1983; Bragg and Lieblich 2018], as well as a derived Torelli theorem [Bragg 2021]. Together, these give a picture which closely parallels the case of complex K3 surfaces. We will show that any two supersingular K3 surfaces are derived isogenous. More refined results (along the lines of [Huybrechts 2019, Theorem 0.1]) are possible, but we will omit this discussion here.

Theorem 6.11. *Suppose that $p \geq 3$. Let X and Y be two supersingular* K3 *surfaces over k. There exists a derived isogeny $\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Y)$.*

Proof. We use [Bragg and Lieblich 2018, Proposition 5.2.5]: if X is a supersingular K3 surface, then there exists a sequence $X_0, X_1, \dots, X_n$ of supersingular K3 surfaces together with Brauer classes $\alpha_i \in \mathrm{Br}(X_i)$ such that $X_0 = X$, $D^b(X_i, \alpha_i) \cong D^b(X_{i+1}, \alpha_{i+1})$ for each $0 \le i \le n-1$, and $X_n = Z$ is the unique supersingular K3 surface with Artin invariant 1. Applying this to both X and Y, we find derived isogenies

$$\mathfrak{h}^2(X) \xrightarrow{\sim} \mathfrak{h}^2(Z) \xleftarrow{\sim} \mathfrak{h}^2(Y). \qquad \square$$

Remark 6.12. Shioda [1977, Theorem 1.1] showed that supersingular Kummer surfaces are unirational. By a result of Ogus [1979] and the crystalline Torelli theorem these are exactly the supersingular K3 surfaces with Artin invariant $\sigma_0 \le 2$. The Chow motive of a unirational surface is of Tate type. Combining this with Theorem 6.11 we deduce that for any supersingular K3 surface X we have $\mathfrak{h}(X) = \mathfrak{h}_{\mathrm{alg}}(X) = \boldsymbol{L}^0 \oplus \boldsymbol{L}^{\oplus 22} \oplus \boldsymbol{L}^2$ and $\mathfrak{h}^2_{\mathrm{tr}}(X) = 0$. In particular, we have $\mathrm{CH}^2(X) = \boldsymbol{Z}$. This result was first proved by Fakhruddin [2002], using a related method.

6D. *Existence in characteristic* 0. It is possible to formulate a purely algebraic analog of Huybrechts' Theorem 1.3 along the lines of Theorem 1.2, valid over any algebraically closed field of characteristic 0.

Theorem 6.13. *Let X be a* K3 *surface over an algebraically closed field of characteristic* 0. *Let $H = \Lambda \otimes \widehat{\boldsymbol{Z}}$. Let $\iota : H \hookrightarrow \mathrm{H}^2(X)_{\boldsymbol{Q}}$ be an isometric embedding. There exists a* K3 *surface X' and a derived isogeny $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ such that $f_*(\mathrm{H}^2(X')) = \mathrm{im}(\iota)$.*

Proof. This can be proved purely algebraically along the same lines as our proof of Theorem 1.2 (but avoiding the extra complications at p). Alternatively, it can be deduced directly from Theorem 1.3. We omit further details. $\square$

6E. *Nygaard–Ogus theory revisited.* In preparation for the proof of Theorem 1.4, we briefly recap the deformation theory of K3 crystals and K3 surfaces established in [Nygaard and Ogus 1985, §5]. For the rest of Section 6, assume that k is a perfect field with $\mathrm{char}\, k = p \ge 5$. We refer the reader to the paragraph below the proof of Lemma 4.6 in loc. cit. for this restriction on p. Let $R := k[\varepsilon]/(\varepsilon^e)$ for some e. Recall that a K3 crystal over R is an F-crystal $\boldsymbol{H}$ on $\boldsymbol{Cris}(R/W)$ equipped with a pairing $\boldsymbol{H} \times \boldsymbol{H} \to \mathcal{O}_{R/W}$ and an isotropic line $\mathrm{Fil} \subset \boldsymbol{H}_R$ which satisfy some properties (see Definition 5.1 in loc. cit. for details).[2]

Definition 6.14. Suppose V is a finite flat extension of W such that $V/(p) = R$. A deformation of $\boldsymbol{H}$ to V is a pair $(\boldsymbol{H}, \widetilde{\mathrm{Fil}})$ where $\widetilde{\mathrm{Fil}} \subset \boldsymbol{H}_V$ is an isotropic direct summand which lifts $\mathrm{Fil} \subset \boldsymbol{H}_R$.

Theorem 6.15 (Nygaard and Ogus). *Let X be a* K3 *surface over k and R be as above.*

(a) *The natural map $X_R \mapsto \mathrm{H}^2_{\mathrm{cris}}(X_R)$ defines a bijection between deformations X_R of X to R to deformations of the* K3 *crystal $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ to R, i.e.,* K3 *crystals $\boldsymbol{H}$ over R with $\boldsymbol{H}|_k = \mathrm{H}^2_{\mathrm{cris}}(X/W)$.*

(b) *If X_R is a deformation of X to R, then the map $X_V \mapsto (\mathrm{H}^2_{\mathrm{cris}}(X_R), \mathrm{Fil}^2\, \mathrm{H}^2_{\mathrm{dR}}(X_V/V))$ defines a bijection between deformations X_V of X_R to V and deformations of the* K3 *crystal $\mathrm{H}^2_{\mathrm{cris}}(X_R)$ to V, in the sense of Definition 6.14.*

[2] In fact, [Nygaard and Ogus 1985, Definition 5.1] defined K3 crystals over a more general base which satisfies a technical assumption [Nygaard and Ogus 1985, (4.4.1)]. For our purposes it suffices to consider bases of the form $k[\varepsilon]/(\varepsilon^e)$.

Proof. This follows from [Nygaard and Ogus 1985, Theorem 5.3] and its proof. □

For the rest of Section 6E, X denotes a K3 surface of finite height over k. Recall that there is a canonical slope decomposition (cf. [Nygaard and Ogus 1985, Proposition 5.4])

$$\delta_{\mathrm{can}} : \mathrm{H}^2_{\mathrm{cris}}(X/W) = \mathbb{D}(\widehat{\mathrm{Br}}^*_X) \oplus \mathbb{D}(D^*) \oplus \mathbb{D}(\widehat{\mathrm{Br}}_X)(-1). \tag{35}$$

We define a map $\mathbb{K}$ which sends a deformation of $\widehat{\mathrm{Br}}_X$ to R to a deformation of the K3 crystal $\mathrm{H}^2_{\mathrm{cris}}(X/W)$ to R by setting $\mathbb{K}(G_R) := \mathbb{D}(G^*_R) \oplus \mathbb{D}(D^*_R) \oplus \mathbb{D}(G_R)(-1)$, where D_R denote the canonical lift of D to R. The K3 crystal structure on $\mathbb{K}(G_R)$ is given as follows: Let $\mathscr{P}_{G_R} : \mathbb{D}(G^*_R) \times \mathbb{D}(G_R) \to \mathcal{O}_{R/W}(-1)$ be the canonical pairing and let $\mathscr{P}_{D_R} : \mathbb{D}(D^*_R) \times \mathbb{D}(D^*_R) \to \mathcal{O}_{R/W}(-2)$ be the pairing inherited from that on $\mathbb{D}(D^*)$. The pairing on $\mathbb{K}(G_R)$ is $\mathscr{P}_{G_R}(-1) \oplus \mathscr{P}_{D_R}$. Finally, the isotropic direct summand Fil in $\mathbb{K}(G_R)_R$ is given by $[\mathrm{Fil}^1\, \mathbb{D}(G_R)_R](-1)$. We define a decreasing filtration on $\mathbb{K}(G_R)_R$ by setting

$$0 = \mathrm{Fil}^3 \subset \mathrm{Fil}^2 := \mathrm{Fil} \subset \mathrm{Fil}^1 := (\mathrm{Fil}^2)^\perp \subset \mathrm{Fil}^0 = \mathbb{K}(G_R)_R. \tag{36}$$

If we further lift G_R to a p-divisible group G_V for a finite flat extension V of W with $V/(p) = R$, then we can attach a deformation of $\mathbb{K}(G_R)$ to V by setting $\widetilde{\mathrm{Fil}} = [\mathrm{Fil}^1\, \mathbb{D}(G_V)_V](-1)$, which we denote by $\mathbb{K}(G_V)$. We define a filtration on $\mathbb{K}(G_V)_V$ using (36) with $\mathbb{K}(G_R)_R$ replaced by $\mathbb{K}(G_V)_V$.

Definition 6.16. If X_V is a formal scheme over $\operatorname{Spf} V$ which deforms X, we say X_V is a Nygaard–Ogus lifting if it comes from $\mathbb{K}(G_V)$ for some p-divisible group G_V lifting $\widehat{\mathrm{Br}}_X$ to V via Theorem 6.15. That is, setting $R := V/(p)$, $G_R := (G_V) \otimes R$ and $X_R := (X_V) \otimes R$, we have an isomorphism

$$(\mathrm{H}^2_{\mathrm{cris}}(X_R), \mathrm{Fil}^2\, \mathrm{H}^2_{\mathrm{dR}}(X_V/V)) \xrightarrow{\sim} \mathbb{K}(G_V)$$

lifting δ_{can} in the obvious sense. If X_V is an algebraic space over $\operatorname{Spec} V$ which deforms X, then we say X_V is a Nygaard–Ogus lifting if its formal completion at the special fiber is a Nygaard–Ogus lifting.

Proposition 6.17. *If a formal scheme X_V is a Nygaard–Ogus lifting of X, then the natural map* $\operatorname{Pic}(X_V) \to \operatorname{Pic}(X)$ *is an isomorphism. In particular, X_V is algebraizable.*

Proof. See Proposition 4.5 and Remark 4.6 of [Yang 2022]. □

Using integral p-adic Hodge theory, we can characterize Nygaard–Ogus liftings:

Theorem 6.18. *Let F be a finite extension of K with $V := \mathcal{O}_F$. Let X_V be a formal scheme over* $\operatorname{Spf} V$ *which lifts X and let X_F denote its rigid-analytic generic fiber. Then X_V is a Nygaard–Ogus lifting if and only if there are* Gal_F*-stable $\mathbf{Z}_p$-sublattices T^0, T^1, T^2 in* $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \mathbf{Z}_p)$ *of ranks h, $22-2h$, h respectively, such that, as crystalline* Gal_F*-representations,*

(a) *$T^1(1)$ is unramified,*

(b) *T^0 has Hodge–Tate weight* 1 *with multiplicity $h-1$ and* 0 *with multiplicity* 1,

(c) *$T^2(1)$ has Hodge–Tate weight* 1 *with multiplicity* 1 *and* 0 *with multiplicity $h-1$.*

Proof. We recap in the Appendix the results from the integral p-adic Hodge theory used in this proof.

Let S denote Breuil's S-ring. Using the data $\mathrm{Fil}^\bullet \mathrm{H}^2_{\mathrm{dR}}(X_F/F)$, we equip $\mathrm{H}^2_{\mathrm{cris}}(X/W) \otimes_W S_K$ with the structure of an object in $\mathcal{MF}^{\varphi,N}_{S_K}$. For any G_V which lifts $\widehat{\mathrm{Br}}_X$ to V, we set

$$\mathbb{T}_p(G_V) := T_p G_V \oplus T_p D_V \oplus T_p G_V^*(-1).$$

Suppose first that X_V is Nygaard–Ogus, so that it comes from some G_V lifting $G := \widehat{\mathrm{Br}}_X$. Combining (41) and (45), we obtain isomorphisms

$$\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}} \cong \mathrm{H}^2(X/W) \underset{W}{\otimes} B_{\mathrm{cris}} = \mathbb{K}(G) \underset{W}{\otimes} B_{\mathrm{cris}} \cong \mathbb{T}_p(G_V)(-1) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}},$$

which give rise to a rational isomorphism $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p \xrightarrow{\sim} \mathbb{T}_p(G_V)(-1) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$. We now show that the this restricts to an integral isomorphism

$$\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \cong \mathbb{T}_p(G_V)(-1). \tag{37}$$

It is easy to check that the object $(\mathbb{K}(G)_K, \mathrm{Fil}^\bullet \mathbb{K}(G_V)_F)$ in MF^φ_F admits a decomposition into

$$(\mathbb{D}(\widehat{\mathrm{Br}}^*_X)_K, \mathrm{Fil}^\bullet \mathbb{D}(G_V^*)_F) \oplus (\mathbb{D}(D^*)_K, \mathrm{Fil}^\bullet \mathbb{D}(D_V^*)_F) \oplus (\mathbb{D}(\widehat{\mathrm{Br}}_X)_K, \mathrm{Fil}^\bullet \mathbb{D}(G_V)_F)(-1).$$

By the construction of Nygaard–Ogus liftings, there is an isomorphism

$$\mathbb{D}(G_R^*)_S \oplus \mathbb{D}(D_R^*)_S \oplus \mathbb{D}(G_R)(-1)_S \cong \mathrm{H}^2_{\mathrm{cris}}(X_R)_S$$

of strongly divisible S-modules which is compatible with the isomorphism

$$\mathrm{H}^2(X/W) \underset{W}{\otimes} S_K \cong \mathbb{K}(G) \underset{W}{\otimes} S_K$$

induced by δ_{can}. By applying the functor T_{cris}, we obtain (37), which readily implies the "only if" part of the theorem.

Now we show the "if" part. The proof is essentially a reincarnation of the proof of [Nygaard and Ogus 1985, Proposition 5.5]. The hypothesis implies that there exists an isomorphism $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \cong \mathbb{T}_p(G_V)(-1)$ for some G_V which lifts $\widehat{\mathrm{Br}}_X$ to V. By Theorem A.3, there exists a unique isomorphism

$$\mathbb{D}(G_R^*)_S \oplus \mathbb{D}(D_R^*)_S \oplus \mathbb{D}(G_R)(-1)_S \cong \mathrm{H}^2_{\mathrm{cris}}(X_R)_S \tag{38}$$

which gives this isomorphism, $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p) \cong \mathbb{T}_p(G_V)(-1)$, under T_{cris}.

The only thing we need to check is that this isomorphism of S-modules comes from an isomorphism of F-crystals on $\boldsymbol{Cris}(R/W)$

$$\mathbb{K}(G_R) = \mathbb{D}(G_R^*) \oplus \mathbb{D}(D_R^*) \oplus \mathbb{D}(G_R)(-1) \cong \mathrm{H}^2_{\mathrm{cris}}(X_R) \tag{39}$$

which restricts to δ_{can}.

Let e be the ramification degree of V over W and j be any positive number $\le e$. Set $R_j := R/(\varepsilon^j)$. We claim that there exists a sequence of isomorphisms $\delta_j : \mathbb{K}(G_{R_j}) \cong \mathrm{H}^2_{\mathrm{cris}}(X_R)$ of F-crystals on $\boldsymbol{Cris}(R_j/W)$ such that δ_j is the restriction of δ_{j+1} for each $j < e$ such that $\delta_1 = \delta_{\mathrm{can}}$, and δ_e gives the desired isomorphism (39). Suppose we have constructed δ_j for some $j < e$. Note that (ε^j) is a square-zero ideal in R_{j+1} and we can view R_{j+1} as an object of $\boldsymbol{Cris}(R_j/W)$ by equipping (ε^j) with the trivial PD structure. By [Nygaard and Ogus 1985, Theorem 5.2], to construct δ_{j+1} it suffices to show that $[\mathrm{Fil}^1\,\mathbb{D}(G_{R_{j+1}})_{R_{j+1}}](-1)$ is sent to $\mathrm{Fil}^2\,\mathrm{H}^2_{\mathrm{dR}}(X_{R_{j+1}}/R_{j+1})$ via the composition

$$\mathbb{K}(G_{R_{j+1}})_{R_{j+1}} \cong \mathbb{K}(G_{R_j})_{R_{j+1}} \xrightarrow[(\delta_j)_{R_{j+1}}]{\sim} \mathrm{H}^2_{\mathrm{cris}}(X_{R_j})_{R_{j+1}} \cong \mathrm{H}^2_{\mathrm{dR}}(X_{R_{j+1}}/R_{j+1}).$$

However, this follows directly from the fact that (38) respects the filtrations. Indeed, viewing R_{j+1} as an S-algebra via $S \to \mathcal{O}_F \to R \to R_{j+1}$, we get the above isomorphism by tensoring (38) with R_{j+1}. □

Remark 6.19. When X is ordinary, X_V is Nygaard–Ogus if and only if it is obtained via base change from the canonical lifting, because in this case deformations of $\widehat{\mathrm{Br}}_X$ are completely rigid. Therefore, the above theorem is a generalization of [Taelman 2020, Theorem C] when $p \geq 5$. It also follows from (37) in the above proof that when X_V is a Nygaard–Ogus lifting, for the enlarged formal Brauer group Ψ_{X_V} of X_V, there is a natural injective map of Gal_F-modules

$$T_p\Psi_{X_V} \to \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p(1)),$$

which generalizes [Taelman 2020, Theorem 2.1]. Indeed, we have $\Psi_{X_V} = \widehat{\mathrm{Br}}_{X_V} \oplus D_V$ for a Nygaard–Ogus lifting.

6F. ***Construction of liftable isogenies.*** We now prove Theorem 1.4.

Proof of Theorem 1.4. Again write ι^p and ι_p for the prime-to-p and crystalline component of ι. If a Frobenius-preserving isometric embedding $\iota_p : H_p \hookrightarrow \mathrm{H}^2(X/W)$ as in the hypothesis exists, then the K3 crystal H_p has to be abstractly isomorphic to $\mathrm{H}^2(X/W)$ and hence to $\mathbb{K}(\widehat{\mathrm{Br}}_X)$. We choose an isomorphism $H_p \xrightarrow{\sim} \mathbb{K}(\widehat{\mathrm{Br}}_X)$ and consider $(\iota_p)_K := \iota_p \otimes K$ as an isometric automorphism of the F-isocrystal $\mathbb{K}(\widehat{\mathrm{Br}}_X)_K$. Then $(\iota_p)_K$ determines, and is conversely determined by, a pair (h, g), where $h \in \mathrm{End}(\widehat{\mathrm{Br}}_X)[1/p]$ and $g \in \mathrm{End}(D)[1/p]$. Our goal is to produce an isogeny $f : \mathfrak{h}^2(X') \to \mathfrak{h}^2(X)$ for some other K3 surface X' over $k = \overline{\boldsymbol{F}}_p$ such that $f_*(\mathrm{H}^2_{\text{ét}}(X', \widehat{\boldsymbol{Z}}^p)) = \iota^p(\Lambda \otimes \widehat{\boldsymbol{Z}}^p)$ and $f_*(\mathrm{H}^2(X'/W)) = (\iota_p)_K(\mathbb{K}(\widehat{\mathrm{Br}}_X))$. By Theorem 1.2, we first reduce to the case when $\iota^p(\Lambda \otimes \widehat{\boldsymbol{Z}}^p) = \mathrm{H}^2_{\text{ét}}(X, \widehat{\boldsymbol{Z}}^p)$ and $(\iota_p)_K$ sends the slope 1 part, i.e., $\mathbb{D}(D^*)$, isomorphically onto itself.

By Lubin–Tate theory, for some finite flat extension V of W, there exists a lift G_V of $\widehat{\mathrm{Br}}_X$ to V such that h lifts to $\mathrm{End}(G_V)[1/p]$ [Yang 2022, Lemma 4.8]. Note that $\mathrm{Fil}^\bullet\,\mathbb{K}(G_V)_F$ equips $\mathbb{K}(\widehat{\mathrm{Br}}_X)_K$ with the structure of an object in MF^{φ}_F and $\mathscr{M} := \mathbb{K}(G_V)_S$ defines a strongly divisible S-lattice in the corresponding object $\mathscr{D} := \mathbb{K}(\widehat{\mathrm{Br}}_X) \otimes S_K$ in $\mathcal{MF}^{\varphi,N}_{S_K}$. It is clear that ι_K preserves $\mathrm{Fil}^\bullet\,\mathbb{K}(G_V)_F$ and extends to an automorphism ι_{S_K} of $\mathscr{D}$.

Let X_V be the Nygaard–Ogus lifting of X which corresponds to G_V. We have $T_{\mathrm{cris}}(\mathscr{M}) = \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p)$ inside $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ by the proof of Theorem 6.18, and $V_{\mathrm{cris}}((\iota_p)_K)$ is an automorphism of the Gal_F-module $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Q}_p)$ which preserves the Poincaré pairing. The image of $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p)$ under $V_{\mathrm{cris}}((\iota_p)_K)$ can also be interpreted as $T_{\mathrm{cris}}(\iota_{S_K}(\mathscr{M}))$. Denote this Gal_F-stable $\boldsymbol{Z}_p$-lattice by Λ'_p. By Theorem 6.13, up to replacing F by a finite extension, we can find another K3 surface X' over F with a derived isogeny $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X_F)$ such that

$$f_*(\mathrm{H}^2_{\text{ét}}(X'_{\bar{F}}, \widehat{\boldsymbol{Z}})) = \Lambda'_p \times \prod_{\ell \neq p} (\Lambda'_\ell := \iota^p(\Lambda \otimes \boldsymbol{Z}_\ell) = \mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_\ell)).$$

We argue that f induces an integral isomorphism $\mathrm{Pic}(X'_F) \xrightarrow{\sim} \mathrm{Pic}(X_F)$. Indeed, f induces an isomorphism

$$\mathrm{Pic}(X'_F) \xrightarrow{\sim} \mathrm{Pic}(X_F)_{\boldsymbol{Q}} \cap \prod_{\ell} \Lambda'_\ell(1).$$

However, we know that the image of $\mathrm{Pic}(X'_F)$ lies in the unramified part of $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p(1))$, and the unramified part of $\Lambda'_p(1)$ coincides with that of $\mathrm{H}^2_{\text{ét}}(X_{\bar{F}}, \boldsymbol{Z}_p(1))$. This implies that the target of the above isomorphism is just $\mathrm{Pic}(X_F)$.

It follows that $\mathrm{Pic}(X'_F)$ also satisfies hypothesis (a), (b) or (c) if $\mathrm{Pic}(X_F) \cong \mathrm{Pic}(X)$ does. For (a) and (c) this is clear; for (b) this follows from [Lieblich et al. 2014, Lemma 2.3.2]. In any case, by [Matsumoto 2015, Theorem 1.1; Ito 2019, §2] and Theorem 8.10 to be proved below, X'_F admits potentially good reduction. Up to replacing F by a further extension, we can find a smooth proper algebraic space X'_V over V such that X'_F is the generic fiber of X'_V. The map induced on crystalline cohomology of special fibers is $D_{\text{cris}}(f)$, which sends $\mathrm{H}^2(X'/W)$ onto $(\iota_p)_K(\mathbb{K}(\widehat{\mathrm{Br}}_X))$. □

7. Uniqueness theorems

In this section we prove Theorem 1.5 by lifting to characteristic 0 (as outlined in the introduction).

7A. ***Shimura varieties.*** Let $p > 2$ be a prime and L be any self-dual quadratic lattice over $\boldsymbol{Z}_{(p)}$ of rank $m \geq 5$ and signature $(2+, (m-2)-)$. Set $\widetilde{G} := \mathrm{CSpin}(L_{(p)})$, $G := \mathrm{SO}(L_{(p)})$, $\mathcal{K}_p := \mathrm{CSpin}(L \otimes \boldsymbol{Z}_p)$, $\mathsf{K}_p := \mathrm{SO}(L \otimes \boldsymbol{Z}_p)$ and $\Omega := \{\omega \in \boldsymbol{P}(L \otimes \boldsymbol{C}) : \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}$. Let $\widetilde{\mathscr{S}}_{\mathcal{K}_p}(L)$ (resp. $\mathscr{S}_{\mathsf{K}_p}(L)$) denote the canonical integral model of $\mathrm{Sh}_{\mathcal{K}_p}(\widetilde{G}, \Omega)$ (resp. $\mathrm{Sh}_{\mathsf{K}_p}(G, \Omega)$) over $\boldsymbol{Z}_{(p)}$ given by [Kisin 2010] (see also [Madapusi Pera 2016, §4]). We choose a compact open subgroup $\mathcal{K}^p$ of $\widetilde{G}(\boldsymbol{A}^p_f)$ and set $\mathcal{K} = \mathcal{K}_p \mathcal{K}^p$. Similarly, set K^p to be the image of $\mathcal{K}^p$ and $\mathsf{K} := \mathsf{K}_p \mathsf{K}^p$. Denote by $\widetilde{\mathrm{Sh}}_{\mathcal{K}}(L)$, $\widetilde{\mathscr{S}}_{\mathcal{K}}(L)$, $\mathrm{Sh}_{\mathsf{K}}(L)$, and $\mathscr{S}_{\mathsf{K}}(L)$ the stacky quotients $\widetilde{\mathrm{Sh}}_{\mathcal{K}_p}(L)/\mathcal{K}^p$, $\widetilde{\mathscr{S}}_{\mathcal{K}_p}(L)/\mathcal{K}^p$, $\mathrm{Sh}_{\mathsf{K}_p}(L)/\mathsf{K}^p$, and $\mathscr{S}_{\mathsf{K}_p}(L)/\mathsf{K}^p$ respectively.

The model $\widetilde{\mathscr{S}}_{\mathcal{K}}(L)$ is equipped with a universal abelian scheme $\mathscr{A}$ up to prime-to-p isogeny whose cohomology gives rise to sheaves $\boldsymbol{H}_*$ $(* = B, \mathrm{cris}, \ell, \mathrm{dR})$ on suitable fibers of $\widetilde{\mathscr{S}}(L)$. The abelian scheme $\mathscr{A}$ is equipped with a $\mathrm{Cl}(L)$-action and $\boldsymbol{Z}/2\boldsymbol{Z}$-grading, and the sheaves $\boldsymbol{H}_*$ are equipped with tensors $\boldsymbol{\pi}_* \in \boldsymbol{H}_*^{\otimes(2,2)}$. We call the triple of $\boldsymbol{Z}/2\boldsymbol{Z}$-grading, $\mathrm{Cl}(L)$-action and various realizations of π the *CSpin structures* on $\mathscr{A}$ or $\boldsymbol{H}_*$. The dual of the images of $\boldsymbol{\pi}_*$ are denoted by $\boldsymbol{L}_*$. We refer the reader to [Madapusi Pera 2016, §4] for details of these constructions or [Yang 2022, (3.1.3)] for a quick summary.

Here is another way to view the sheaves $\boldsymbol{L}_*$: On the double quotient $\mathrm{Sh}_{\mathsf{K}}(L)_{\boldsymbol{C}} = G(\boldsymbol{Q}) \backslash \Omega \times G(\boldsymbol{A}_f)/\mathsf{K}$, the standard representation $\mathrm{SO}(L) \to \mathrm{GL}(L)$ produces a variation of $\boldsymbol{Z}$-Hodge structures [Madapusi Pera 2016, §3.3], which is nothing but $(\boldsymbol{L}_B, \boldsymbol{L}_{\mathrm{dR},\boldsymbol{C}} := \boldsymbol{L}_{\mathrm{dR}}|_{\mathrm{Sh}_{\mathsf{K}}(L)_{\boldsymbol{C}}})$. The filtered vector bundle $\boldsymbol{L}_{\mathrm{dR},\boldsymbol{C}}$ is commonly called the automorphic vector bundle associated to this representation, and by the general theory of automorphic vector bundles, we know that it admits a canonical descent to the canonical model $\mathrm{Sh}_{\mathsf{K}}(L)$ over the reflex field $\boldsymbol{Q}$. This canonical descent is nothing but $\boldsymbol{L}_{\mathrm{dR}}$ (when restricted to $\mathrm{Sh}_{\mathsf{K}}(L)$). In fact, the pair $(\boldsymbol{L}_B, \boldsymbol{L}_{\mathrm{dR},\boldsymbol{C}})$ is the variation of $\boldsymbol{Z}$-Hodge structures associated to a family of $\boldsymbol{Z}$-motives $\boldsymbol{L}$ over $\mathrm{Sh}_{\mathsf{K}}(L)_{\boldsymbol{C}}$ in the sense of [Madapusi Pera 2015, §1.4]. This family of motives descend to the canonical model $\mathrm{Sh}_{\mathsf{K}}(L)$, whose ℓ-adic realizations give $\boldsymbol{L}_\ell|_{\mathrm{Sh}_{\mathsf{K}}(L)}$ and whose de Rham realization gives $\boldsymbol{L}_{\mathrm{dR}}|_{\mathrm{Sh}_{\mathsf{K}}(L)}$. Once we extend $\mathrm{Sh}_{\mathsf{K}}(L)$ to $\mathscr{S}_{\mathsf{K}}(L)$ over $\boldsymbol{Z}_{(p)}$, these sheaves arising from cohomological realizations of motives over $\mathrm{Sh}_{\mathsf{K}}(L)$ also extend. This motivic point of view is discussed in more detail in [Madapusi Pera 2015, §4.7].

It is explained in [Yang 2022, (3.1.3)] that the sheaves $\boldsymbol{L}_*$ are equipped with an orientation tensor $\boldsymbol{\delta}_* : \underline{\det(L)} \xrightarrow{\sim} \det(\boldsymbol{L}_*)$ $(* = B,\ \ell \neq p)$. Here $\underline{\det(L)}$ denotes the constant sheaf whose stalks are $\det(L)$ on $\widetilde{\mathscr{S}}(L)$ in the appropriate Grothendieck topology. In short, $\boldsymbol{\delta}_*$'s come up because the adjoint representation of $\widetilde{G}$ on $L_{(p)}$ factors through $\mathrm{SO}(L_{(p)})$, i.e., it preserves a choice of orientation δ on $L_{(p)}$. It is possible to discuss de Rham or crystalline realizations of δ, but for our purposes it suffices to use the 2-adic realization $\boldsymbol{\delta}_2$. The sheaves $\boldsymbol{L}_*$ and the tensors $\boldsymbol{\pi}_*$ and $\boldsymbol{\delta}_*$ descend to $\mathscr{S}(L)$.

We will repeatedly make use of the following key fact about $\boldsymbol{L}_{\mathrm{dR}}$ and $\boldsymbol{H}_{\mathrm{dR}}$:

Proposition 7.1. *Let s be any point on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$. $\mathrm{Fil}^1 \boldsymbol{L}_{\mathrm{dR},s}$ is one-dimensional, and $\mathrm{Fil}^1 \boldsymbol{H}_{\mathrm{dR},s} = \ker(x)$ for any nonzero element $x \in \mathrm{Fil}^1 \boldsymbol{L}_{\mathrm{dR},s}$.*

Proof. If $\mathrm{char}\, k(s) = 0$, we can simply base change to $\boldsymbol{C}$ and apply Hodge theory (see [Yang 2022, p. 8]). If $\mathrm{char}\, k(s) = p$, we can check this by a lifting argument or read it off from [Madapusi Pera 2016, §4.9]. □

We recall the definition of a CSpin-isogeny [Yang 2022, Definition 3.2]:

Definition 7.2. Let κ be a perfect field with algebraic closure $\bar{\kappa}$, and let s, s' be κ-points on $\widetilde{\mathscr{S}}_{K_p}(L)$. We call a quasi-isogeny $\mathscr{A}_s \to \mathscr{A}_{s'}$ a *CSpin-isogeny* if it commutes with the CSpin structures, i.e., it respects the $\boldsymbol{Z}/2\boldsymbol{Z}$-grading, $\mathrm{Cl}(L)$-action and sends $\boldsymbol{\pi}_{\ell,s\otimes\bar{\kappa}}$ to $\boldsymbol{\pi}_{\ell,s'\otimes\bar{\kappa}}$ for every $\ell \neq \mathrm{char}\,\kappa$ and in addition $\boldsymbol{\pi}_{\mathrm{cris},s}$ to $\boldsymbol{\pi}_{\mathrm{cris},s'}$ if $\mathrm{char}\,\kappa = p$.

We remark that CSpin-isogenies are stable under liftings and reductions:

Lemma 7.3. *Let κ be a perfect field of characteristic p, and let s, s' be two k-points on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$. Let K denote $W(\kappa)[1/p]$ and $F \subseteq \overline{K}$ be a finite extension of K, and let s_F, s'_F be F-valued points on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$ which specialize to s, s'. Suppose $\psi_F : \mathscr{A}_{s_F} \to \mathscr{A}_{s'_F}$ is a quasi-isogeny which specializes to $\psi : \mathscr{A}_s \to \mathscr{A}_{s'}$. Then ψ is a CSpin-isogeny if and only if ψ_F is also a CSpin-isogeny.*

Proof. Clearly, ψ respects the $\boldsymbol{Z}/2\boldsymbol{Z}$-grading and the $\mathrm{Cl}(L)$-actions if and only if ψ_F also respects these structures. Let $s_{\overline{K}}$ and $s'_{\overline{K}}$ denote the $\overline{K}$-valued geometric points over s_F and s'_F. To check whether ψ_F sends $\boldsymbol{\pi}_{\ell,s_{\overline{K}}}$ to $\boldsymbol{\pi}_{\ell,s'_{\overline{K}}}$ for every ℓ, it suffices to check this for one ℓ, as one can always take a base change to $\boldsymbol{C}$ and use Betti realizations. Therefore, the only part of the statement which does not follow directly from the smooth and proper base change theorem is that if ψ_F is a CSpin-isogeny, then ψ sends $\boldsymbol{\pi}_{\mathrm{cris},s}$ to $\boldsymbol{\pi}_{\mathrm{cris},s'}$. This follows from [Yang 2022, Remark 3.1]. □

Lemma 7.4. *Let $s_{\boldsymbol{C}}, s'_{\boldsymbol{C}}$ be two $\boldsymbol{C}$-points on $\widetilde{\mathscr{S}}_{\mathsf{K}}(L)$. For every Hodge isometry*

$$g : \boldsymbol{L}_{B,s'_{\boldsymbol{C}}} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{B,s_{\boldsymbol{C}}} \otimes \boldsymbol{Q}$$

which sends $\boldsymbol{\delta}_{2,s_{\boldsymbol{C}}}$ to $\boldsymbol{\delta}_{2,s'_{\boldsymbol{C}}}$, there exists a CSpin-isogeny $\mathscr{A}_{s_{\boldsymbol{C}}} \xrightarrow{\sim} \mathscr{A}_{s'_{\boldsymbol{C}}}$ which induces g by conjugation.

Proof. From the construction of the local system $\boldsymbol{H}_B$ (see [Madapusi Pera 2016, §3.3]) it is clear that there exists an isomorphism of free $\boldsymbol{Z}_{(p)}$-modules $H \xrightarrow{\sim} \boldsymbol{H}_{B,s_{\boldsymbol{C}}}$ which respects the CSpin-structures, i.e., it respects the $\boldsymbol{Z}/2\boldsymbol{Z}$-grading, $\mathrm{Cl}(L)$-action and sends π to $\boldsymbol{\pi}_{B,s_{\boldsymbol{C}}}$. The same is true for $s'_{\boldsymbol{C}}$, so there exists an isomorphism of $\boldsymbol{Z}_{(p)}$-modules $\psi' : \boldsymbol{H}_{B,s_{\boldsymbol{C}}} \xrightarrow{\sim} \boldsymbol{H}_{B,s'_{\boldsymbol{C}}}$ which respects the CSpin structures. The map $g' : \boldsymbol{L}_{B,s_{\boldsymbol{C}}} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{B,s'_{\boldsymbol{C}}} \otimes \boldsymbol{Q}$ induced by ψ' by conjugation sends $\boldsymbol{\delta}_{2,s_{\boldsymbol{C}}}$ to $\boldsymbol{\delta}_{2,s'_{\boldsymbol{C}}}$. Therefore, the composition $g^{-1} \circ g'$ lies in $\mathrm{SO}(\boldsymbol{L}_{B,s_{\boldsymbol{C}}} \otimes \boldsymbol{Q})$. Since the natural morphism $\mathrm{CSpin}(L_{\boldsymbol{Q}}) \to \mathrm{SO}(L_{\boldsymbol{Q}})$ is surjective, we may lift $g^{-1} \circ g'$ to an automorphism of $\boldsymbol{H}_{B,s_{\boldsymbol{C}}}$ which preserves the CSpin structures and use it to adjust ψ' to obtain a morphism ψ which induces g by conjugation. It follows from Proposition 7.1 that f preserves the Hodge structures, so that $\tilde{g}$ comes from a CSpin-isogeny. ☐

7B. *Hilbert squares and period morphisms.* We will apply the period morphism construction to Hilbert squares of K3 surfaces, so we recollect some basic facts and set up some notation here. Let k be any algebraically closed field of characteristic 0 or $p > 2$, X be any K3 surface over k and $Y := X^{[2]}$ be the Hilbert scheme of two points on X. The lemma below implies that Y is a $\mathrm{K3}^{[2]}$-type variety in the sense of [Yang 2023, Definition 1].

Lemma 7.5. *When* $\operatorname{char} k = p > 2$ *or* 0, *Y has the same Hodge numbers as those of a complex* $\mathrm{K3}^{[2]}$*-type variety, and the Hodge–de Rham spectral sequence of Y degenerates at the E_1-page.*

Proof. Let $Y' := \mathrm{Bl}_\Delta(X \times X)$ be the blowup of $X \times X$ along the diagonal $\Delta \subset X \times X$. Let $E \subset Y'$ be the exceptional divisor, which is isomorphic to the projectivization of the tangent bundle of X. There is an action of $\boldsymbol{Z}/2$ on $X \times X$ given by permuting the factors, which lifts to an action on Y' that is trivial on E, and there is a natural map $q : Y' \to Y$ that identifies Y with the quotient $Y'/(\boldsymbol{Z}/2)$. The map q is a double cover branched over the divisor $D = q(E) \subset Y$, which may be described explicitly as the locus of nonreduced subschemes. Using our assumption that 2 is invertible in k, we obtain a canonical direct sum decomposition

$$q_*\mathscr{O}_{Y'} = \mathscr{O}_Y \oplus \mathscr{L},$$

where $\mathscr{L}$ is the cokernel of the pullback map $\mathscr{O}_Y \to q_*\mathscr{O}_{Y'}$. From this and the projection formula we deduce the equality

$$\mathrm{H}^j(Y', q^*\Omega^i_Y) = \mathrm{H}^j(Y, \Omega^i_Y) \oplus \mathrm{H}^j(Y, \Omega^i_Y \otimes \mathscr{L}).$$

All of these data may be defined in a flat family over a flat finite type $\boldsymbol{Z}$-scheme. By semicontinuity, the dimensions of both summands on the right-hand side must be greater than or equal to their corresponding values over the complex numbers. Thus, it will suffice to verify that the groups $\mathrm{H}^j(Y', q^*\Omega^i_Y)$ have the same dimensions as over the complex numbers. This can be done via a direct computation. In more detail, we compute using the identification $q^*\Omega^1_Y \cong \Omega^1_{Y'}(-E)$, which yields isomorphisms

$$q^*\Omega^i_Y \cong \Omega^i_{Y'}(-iE).$$

The cohomology of these sheaves may be related to the Hodge cohomology of X by pushing forward along the blowup morphism $Y' \to X \times X$. The result then follows (eventually) from the fact that the Hodge numbers of X do not depend on the characteristic of the ground field.

The degeneration of the Hodge–de Rham spectral sequence at the E_1-page follows from the fact that $\mathsf{H}^i(Y, \Omega^j_Y) = 0$ for $i + j$ odd. □

Let $\mathsf{H}^*(-)$ be a Weil cohomology with coefficient field $\mathcal{K}$.[3] We will only make use of Betti, ℓ-adic, crystalline, de Rham when appropriate. When there is a specified polarization, let $\mathsf{P}^*(-)$ denote the corresponding primitive cohomology. We will view $\mathrm{NS}(Y)$ as a $\mathbf{Z}$-lattice inside $\mathsf{H}^2(Y)$ via c_1, and will not write c_1 explicitly. $\mathsf{H}^2(Y)$ is equipped with natural *Beauville–Bogomolov forms* (BBF). When $\operatorname{char} k = 0$, these forms are well known. When $\operatorname{char} k = p > n+1$, the étale and crystalline versions of these forms for $\mathrm{K3}^{[n]}$-type varieties were defined in [Yang 2023, §2.1]. Since Y is a Hilbert square on a K3 surface X, as opposed to a general deformation of such a variety, the Beauville–Bogomolov form on Y is easily described by the Poincaré pairing on X: Let δ be the class of the exceptional divisor. Then $\delta^2 = -2$ under the BBF. The incidence correspondence between X and Y embeds $\mathsf{H}^2(X)$ isometrically into $\mathsf{H}^2(Y)$ such that $\mathsf{H}^2(Y)$ admits a natural orthogonal decomposition $\mathsf{H}^2(X) \oplus \mathcal{K}\delta$. Similarly, $\mathrm{NS}(Y)$ decomposes as $\mathrm{NS}(X) \oplus \mathbf{Z}\delta$.

Lemma 7.6. *Let ξ be a polarization on X and ζ be a polarization on Y of the form $m\xi - \delta$. Denote by* $\mathrm{proj}_{\mathsf{P}^2(Y)}\,\delta$ *the projection of δ to $\mathsf{P}^2(Y)$ and by* $\mathrm{Isom}(-,-)$ *the set of isometries between two quadratic lattices. Now let X' be another* K3 *surface over k, take Y', ξ', δ' similarly, and suppose Y' is polarized by $\zeta' := m\xi - \delta'$. There are natural identifications*

$$\begin{aligned}\mathrm{Isom}(\mathsf{P}^2(X), \mathsf{P}^2(X')) &= \{f \in \mathrm{Isom}(\mathsf{H}^2(X), \mathsf{H}^2(X') : f(\xi) = \xi'\} \\ &= \{f \in \mathrm{Isom}(\mathsf{H}^2(Y), \mathsf{H}^2(Y')) : f(\zeta) = \zeta',\ f(\delta) = \delta'\} \\ &= \{f \in \mathrm{Isom}(\mathsf{P}^2(Y), \mathsf{P}^2(Y') : f(\mathrm{proj}_{\mathsf{P}^2(Y)}\,\delta) = \mathrm{proj}_{\mathsf{P}^2(Y')}\,\delta'\}. \end{aligned} \tag{40}$$

Assume now $p \geq 5$ to apply the results of [Yang 2023]. Let X be a K3 surface and $Y := X^{[2]}$. Let ζ be any primitive polarization on Y such that p is prime to the top intersection number ζ^4. Let $\mathrm{Def}(Y; \zeta)$ denote the deformation functor of the pair (Y, ζ), i.e., the functor which sends an Artin W-algebra A to the set of isomorphism classes of the flat deformations of (Y, ζ) over A. We have that $\mathrm{Def}(Y; \zeta)$ is representable by a formal scheme isomorphic to $\mathrm{Spf}(\mathcal{R})$ for $\mathcal{R} := W[\![x_1, \dots, x_{20}]\!]$. Let $(\mathscr{Y}, \boldsymbol{\zeta})$ denote the universal family over $\mathrm{Def}(Y; \zeta)$. Note that $\boldsymbol{\zeta}$ algebraizes $\mathscr{Y}$ into a scheme over $\mathrm{Spec}(\mathcal{R})$. Again we use the symbol $\mathrm{P}^2(-)$ for the primitive cohomologies of (Y, ζ). There are natural pairings on $\mathrm{P}^2(Y, \widehat{\mathbf{Z}}^p)$ and $\mathrm{P}^2(Y/W)$ given by restricting the Beauville–Bogomolov forms (see [Yang 2023, §2.1]).

[3] Here we are using a different font for $\mathsf{H}^*(-)$ to distinguish from the $\mathrm{H}^*(-)$ in Section 1D.

Let $F \subset \overline{K}$ be any finite extension of K and $\tilde{b}$ be any $\mathcal{O}_F$-point on $\mathrm{Def}(X;\zeta)$. Choose an isomorphism $\iota : \overline{K} \xrightarrow{\sim} \boldsymbol{C}$. Let L be the quadratic lattice $\mathrm{P}^2(\mathscr{Y}_{\tilde{b}}(\boldsymbol{C}), \boldsymbol{Z}_{(p)})$, equipped with the restriction of the negative Beauville–Bogomolov form. We remark that since $\mathrm{H}^2(\mathscr{Y}_{\tilde{b}}(\boldsymbol{C}), \boldsymbol{Z})$ is always isomorphic to the lattice $\Lambda^{[2]} := \Lambda \oplus \boldsymbol{Z}(-2)$ and $p \nmid c_1(\boldsymbol{\zeta}_{\tilde{b}_C})^2$, the isomorphism class of L as a quadratic lattice over $\boldsymbol{Z}_{(p)}$ is completely determined by the number $c_1(\boldsymbol{\zeta}_{\tilde{b}_C})^2$ [Milnor and Husemoller 1973, I, Lemma 4.2].

Let b be the closed point of $\mathrm{Def}(X,\zeta)$. We pack the input we need from the Kuga–Satake period morphism into the following proposition:

Proposition 7.7. *Assume $p \geq 5$. There exists a local period morphism $\rho : \mathrm{Spec}\,\mathcal{R} \to \mathscr{S}_{\mathsf{K}_p}(L)$ which identifies $\mathrm{Spec}\,\mathcal{R}$ with the complete local ring $\widehat{\mathcal{O}}_s$ of $s := \rho(b)$ on $\mathscr{S}(L)_W$ such that*:

(a) *There exist an isometry $\alpha_{\mathrm{dR}} : \boldsymbol{P}^2_{\mathrm{dR}} \xrightarrow{\sim} \rho^*\boldsymbol{L}_{\mathrm{dR}}(-1)$ of filtered vector bundles and an isometry $\alpha_{\mathrm{cris}} : \boldsymbol{P}^2_{\mathrm{cris}} \xrightarrow{\sim} \rho^*\boldsymbol{L}_{\mathrm{cris}}(-1)$ of F-crystals that are compatible via the crystalline–de Rham comparison isomorphisms.*

(b) *There is an isometry $\alpha_{\boldsymbol{A}_f,b} : \mathrm{P}^2_{\text{ét}}(\mathscr{Y}_{b,\boldsymbol{A}_f}) \to \boldsymbol{L}_{\boldsymbol{A}_f,b}$ such that for any geometric $\tilde{b}'$ of characteristic zero on $\mathrm{Spec}\,\mathcal{R}$, the pair of isometries $(\alpha_{\boldsymbol{A}_f,\tilde{b}'}, \alpha_{\mathrm{dR},\tilde{b}'})$, where $\alpha_{\boldsymbol{A}_f,\tilde{b}'} : \mathrm{P}^2_{\text{ét}}(\mathscr{Y}_{\tilde{b}'}, \boldsymbol{A}_f) \to \boldsymbol{L}_{\boldsymbol{A}_f,\tilde{b}'}$ is induced by the smooth and proper base change theorem, is absolute Hodge.*

Moreover, for any choice of trivialization $\epsilon_2 : \det(L \otimes \boldsymbol{Q}_2) \xrightarrow{\sim} \det(\mathrm{P}^2_{\text{ét}}(Y, \boldsymbol{Q}_2))$, s can always be chosen such that $\det(\alpha_{2,b})$ sends ϵ_2 to $\boldsymbol{\delta}_{2,s}$.

Proof. See [Yang 2023, §3.3], which is a direct generalization of the results in [Madapusi Pera 2015, §5]. ☐

Remark 7.8. We remark that in order to construct the local period morphism ρ, we actually have to choose an appropriate $\boldsymbol{Z}$-integral structure for the $\boldsymbol{Z}_{(p)}$-lattice L. However, once it is constructed, we are allowed to forget about the $\boldsymbol{Z}$-integral structure, as the integral models of the relevant Shimura varieties only depend on the $\boldsymbol{Z}_{(p)}$-lattice L.

7C. *Twisted derived Torelli theorem.*

Definition 7.9. Let X and X' be K3 surfaces over an algebraically closed field k of characteristic $p > 0$. Let $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ be an isogeny. We say that f is *liftable* if for some finite extension F of K with $V := \mathcal{O}_F$ and projective schemes X_V and X'_V over V which deform X and X' to V, f lifts to an isogeny $f_F : \mathfrak{h}^2(X'_F) \xrightarrow{\sim} \mathfrak{h}^2(X_F)$. If X and X' are nonsupersingular, we say that f is *perfectly liftable* if X_V and X'_V can be chosen to be perfect liftings.

For the rest of Section 7C, let k be an algebraically closed field of $p \geq 5$.

Lemma 7.10. *Let $(X_0,\xi_0),\dots,(X_m,\xi_m)$ be finitely many nonsupersingular polarized* K3 *surfaces over k and let $f_i : \mathfrak{h}^2(X_i) \xrightarrow{\sim} \mathfrak{h}^2(X_{i+1})$ be a perfectly liftable isogeny which sends ξ_i to ξ_{i+1} for $i = 0, 1, \dots, m-1$. If $f := f_{m-1} \circ \cdots \circ f_0 : (\mathfrak{h}^2(X_0), \xi_0) \xrightarrow{\sim} (\mathfrak{h}^2(X_m), \xi_m)$ induces an integral isomorphism $\mathrm{H}^2_{\mathrm{cris}}(X_0/W) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{cris}}(X_m/W)$, then f is perfectly liftable to K up to equivalence.*

Proof. Set $Y_i := X_i^{[2]}$ and let δ_i be the exceptional divisor on Y_i. For some number $N \gg 0$, $\zeta_i := p^N\xi_i - \delta_i$ is a polarization on Y_i for each i. The number $\langle \zeta_i, \zeta_i\rangle$ under the Beauville–Bogomolov form on Y_i

is an integer M which is independent of i. Let L denote a $\boldsymbol{Z}_{(p)}$-lattice which is isomorphic to the orthogonal complement of an element $\lambda \in \Lambda^{[2]} \otimes \boldsymbol{Z}_{(p)}$ with $\langle \lambda, \lambda \rangle = M$. We choose trivializations $\epsilon_i : \det(L \otimes \boldsymbol{Q}_2) \xrightarrow{\sim} \det(\mathrm{P}^2_{\text{ét}}(Y_i, \boldsymbol{Q}_2))$ such that f_i sends ϵ_i to ϵ_{i+1}. Let ρ_i denote a local period morphism obtained by applying Proposition 7.7 to (Y_i, ζ_i) and ϵ_i, and let s_i denote the image of the basepoint under ρ_i. Let $\tilde{s}_i$ be a lift of s_i to $\widetilde{\mathscr{S}}_{\mathcal{K}_p}(L)$.

We claim that there exists a CSpin-isogeny $\psi_i : \mathscr{A}_{\tilde{s}_i} \to \mathscr{A}_{\tilde{s}_{i+1}}$ which induces the same isometries

$$\boldsymbol{L}_{\ell, s_i} \xrightarrow{\sim} \boldsymbol{L}_{\ell, s_{i+1}} \quad \text{and} \quad \boldsymbol{L}_{\text{cris}, s_i} \xrightarrow{\sim} \boldsymbol{L}_{\text{cris}, s_{i+1}}$$

as f_i for each $i = 0, \ldots, m-1$. Indeed, fix an i and let $X_{i,V}, X_{i+1,V}$ be perfect liftings of X_i, X_{i+1} over some finite extension V of W such that f lifts to $f_F : X_{i,F} \xrightarrow{\sim} X_{i+1,F}$, where $F = V[1/p]$. Let $Y_{i,V}$, $Y_{i+1,V}$ be the Hilbert squares of $X_{i,V}$, $X_{i+1,V}$. Note that $Y_{i,V}$ and $Y_{i+1,V}$ carry liftings of ζ_i and ζ_{i+1}, so via the local Torelli morphisms ρ_i and ρ_{i+1}, $X_{i,V}$ and $X_{i+1,V}$ induce V-points $s_{i,V}$, $s_{i+1,V}$ on $\mathscr{S}_{\mathsf{K}}(L)$. Lift these points to V-points $\tilde{s}_{i,V}$, $\tilde{s}_{i+1,V}$ on $\widetilde{\mathscr{S}}_{\mathcal{K}}(L)$, which is étale over $\mathscr{S}_{\mathsf{K}}(L)$. Now choose an isomorphism $\bar{F} \xrightarrow{\sim} \boldsymbol{C}$. The isogeny $f_{i,F}(\boldsymbol{C})$ induces a Hodge isometry $\mathrm{P}^2(X_{i,F}(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{P}^2(X_{i+1,F}(\boldsymbol{C}), \boldsymbol{Q})$, which canonically extends to a Hodge isometry $\mathrm{P}^2(Y_{i,F}(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{P}^2(Y_{i+1,F}(\boldsymbol{C}), \boldsymbol{Q})$ via Lemma 7.6. By Proposition 7.7, the latter can be identified with a Hodge isometry $\boldsymbol{L}_{B, s_{i,F}(\boldsymbol{C})} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{B, s_{i+1,F}(\boldsymbol{C})} \otimes \boldsymbol{Q}$. Note that we have required that f_i send ϵ_i to ϵ_{i+1}. By Lemma 7.4, we obtain a CSpin-isogeny $\psi_{i,\boldsymbol{C}} : \mathscr{A}_{\tilde{s}_{i,F}(\boldsymbol{C})} \xrightarrow{\sim} \mathscr{A}_{\tilde{s}_{i+1,F}(\boldsymbol{C})}$. By Lemma 7.3, $\psi_{i,\boldsymbol{C}}$ specializes to a CSpin-isogeny ψ_i, which can be easily checked to have the desired properties.

By [Lieblich and Maulik 2018, Corollary 4.2], we can find a lifting $X_{0,W}$ of X_0 which also lifts all line bundles on X_0. We transport the induced Hodge filtration on $\mathrm{H}^2_{\text{cris}}(X_0/W)$ to $\mathrm{H}^2_{\text{cris}}(X_m/W)$ using f, which induces a lift $X_{m,W}$ of X_m over W. It is easy to check that $X_{m,W}$ also carries liftings of all line bundles on X_m using [Ogus 1979, Proposition 1.12]. Just as in the previous paragraph, after taking Hilbert squares of the liftings, we obtain via the local period morphisms K-valued points $s_{0,K}$, $s_{m,K}$, $\tilde{s}_{0,K}$, $\tilde{s}_{m,K}$ which lift s_0, s_m, $\tilde{s}_0$, $\tilde{s}_m$. It follows from Proposition 7.1 that the crystalline realization of $\psi := \psi_{m-1} \circ \cdots \circ \psi_0$ preserves the Hodge filtrations of $\mathscr{A}_{s_{0,K}}$ and $\mathscr{A}_{s_{m,K}}$ via the Berthelot–Ogus comparison isomorphisms. By [Berthelot and Ogus 1983, Theorem 3.15], ψ lifts to a CSpin-isogeny $\psi_K : \mathscr{A}_{s_{0,K}} \xrightarrow{\sim} \mathscr{A}_{s_{m,K}}$. Choose an isomorphism $\bar{K} \cong \boldsymbol{C}$. By running the arguments in the preceding paragraph backwards, we obtain a rational Hodge isometry $\mathrm{H}^2(X_{0,K}(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_{m,K}(\boldsymbol{C}), \boldsymbol{Q})$, which by Huybrechts' theorem [2019, Theorem 0.2] is induced by an isogeny $f_{\boldsymbol{C}}$. We get the desired isogeny f by specializing $f_{\boldsymbol{C}}$. □

Proof of Theorem 1.5.. The forward direction is immediate (and does not need the restriction on p). For the converse, suppose that $f : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ is polarizable and $\boldsymbol{Z}$-integral. It is easy to see that if X is supersingular, then so is X'. In this case, the result follows from the crystalline Torelli theorem of Ogus [1983, Theorem II] (cf. [Yang 2022, Theorem 6.5]). Therefore, we reduce to the case when X and X' have finite height. We first remark that f maps $\mathrm{NS}(X')$ isomorphically onto $\mathrm{NS}(X)$, so that by the structure of ample cones of K3 surfaces [Ogus 1983, Proposition 1.10], $f(\xi')$ is ample for any ample ξ. By definition, there exists a sequence of K3 surfaces $X' = X_0, \ldots, X_m = X$ over k and primitive derived isogenies $f_i : \mathfrak{h}^2(X_i) \xrightarrow{\sim} \mathfrak{h}^2(X_{i+1})$ such that $f = f_{m-1} \circ \cdots \circ f_0$.

We now show that there exists a sequence $\delta_i : \mathfrak{h}^2(X_i) \xrightarrow{\sim} \mathfrak{h}^2(X_i)$ given by compositions of reflections in (-2)-curves up to a sign and a sequence of ample class $\xi_i \in \mathrm{NS}(X_i)_{\boldsymbol{Q}}$ such that $(\delta_{i+1} \circ f_i)(\xi_i) = \xi_{i+1}$ for each i. We do this by slightly refining the argument of [Yang 2022, Lemma 6.2]. Set δ_0 to be the identity. Choose any ample class $\zeta_0 \in \mathrm{NS}(X_0)_{\boldsymbol{Q}}$ and $\epsilon_0 > 0$, such that the open ball $B(\zeta_0, \epsilon_0)$ centered at ζ_0 of radius ϵ_0 in $\mathrm{NS}(X_0)_{\boldsymbol{R}}$ lies inside the ample cone. By [Ogus 1979, Lemma 7.9], there exists some δ_1, such that $\zeta_1' := \delta_1 \circ f_0(\zeta_0)$ is big and nef. The image of $B(\zeta_0, \epsilon_0)$ in $\mathrm{NS}(X_1)_{\boldsymbol{R}}$ under $\delta_1 \circ f_0$ is an open neighborhood of ζ_1' which necessarily intersects the ample cone of X_1. Therefore, we may now choose ζ_1 together with $\epsilon_1 > 0$ such that $(\delta_1 \circ f_0)^{-1} B(\zeta_1, \epsilon_1) \subseteq B(\zeta_0, \epsilon_0)$. We iterate this process to obtain a sequence of open balls $B(\zeta_i, \epsilon_i) \subset \mathrm{NS}(X_i)_{\boldsymbol{R}}$ which lie inside the ample cones, and a sequence of δ_i's such that $(\delta_{i+1} \circ f_i)^{-1}(B(\zeta_{i+1}, \epsilon_{i+1})) \subseteq B(\zeta_i, \epsilon_i)$. Now we win by choosing an element $\xi_m \in B(\zeta_m, \epsilon_m)$, and iteratively set $\xi_i := (\delta_{i+1} \circ f_i)^{-1}(\xi_{i+1})$. By clearing denominators we may assume that each ξ_i is integral.

Set $\xi = \xi_m$, $\xi' = \xi_0$, $h_i := \delta_{i+1} \circ f_i$ for each $i < m$, and $f' := h_{m-1} \circ \cdots \circ h_0 = f$. For each i, consider $T(X_i) := \mathrm{NS}(X_i)^{\perp} \subset \mathrm{H}^2(X_i)$. Clearly f_i and h_i induce the same maps on transcendental lattices $\mathrm{T}(X_i)_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{T}(X_{i+1})_{\boldsymbol{Q}}$. Therefore, f and f' induce the same maps $\mathrm{T}(X')_{\boldsymbol{Q}} \xrightarrow{\sim} \mathrm{T}(X)_{\boldsymbol{Q}}$ but their induced maps $\mathrm{NS}(X') \xrightarrow{\sim} \mathrm{NS}(X)$ may differ by an automorphism of $\mathrm{NS}(X)$ which preserves the ample cone. By Theorem 5.8, each h_i is liftable, so that by Lemma 7.10, $f' : \mathfrak{h}^2(X') \xrightarrow{\sim} \mathfrak{h}^2(X)$ admits a perfect lifting $f'_K : \mathfrak{h}^2(X'_K) \xrightarrow{\sim} \mathfrak{h}^2(X_K)$. Therefore, f', and hence f, lifts to a Hodge isometry $\mathrm{H}^2(X'_K(\boldsymbol{C}), \boldsymbol{Q}) \xrightarrow{\sim} \mathrm{H}^2(X_K(\boldsymbol{C}), \boldsymbol{Q})$ for a chosen isomorphism $\bar{K} \cong \boldsymbol{C}$. Using the smooth and proper base change theorem for étale cohomology, we see that this rational Hodge isometry is $\boldsymbol{Z}[1/p]$-integral. Now we show that it is $\boldsymbol{Z}$-integral. Indeed, we first note that f induces isomorphism $f_p : \mathrm{H}^2_{\text{ét}}(X'_{\bar{K}}, \boldsymbol{Q}_p) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \boldsymbol{Q}_p)$ and $f_{\text{cris}} : \mathrm{H}^2_{\text{cris}}(X'/W)[1/p] \xrightarrow{\sim} \mathrm{H}^2_{\text{cris}}(X/W)[1/p]$. We have $f_p \otimes_{\boldsymbol{Z}_p} B_{\text{cris}} = f_{\text{cris}} \otimes_W B_{\text{cris}}$ under the p-adic comparison isomorphism (see (41) in the Appendix) as it is compatible with cycle class maps, Poincaré duality and trace maps [Ito et al. 2018, Corollary 11.6]. Let S be Breuil's S-ring. Then we have an identification $\mathrm{H}^2_{\text{cris}}(X/W) \otimes_W S = \mathrm{H}^2_{\text{cris}}(X/S)$ and a similar one for X'. Now, we are given that $f_{\text{cris}} \otimes_W B_{\text{cris}}$ sends the S-module $\mathrm{H}^2_{\text{cris}}(X'/S)$ isomorphically onto $\mathrm{H}^2_{\text{cris}}(X/S)$. By [Cais and Liu 2019, Theorem 5.2] (see also Theorem A.5 and Remark A.4 below), f_p sends the $\boldsymbol{Z}_p$-lattice $\mathrm{H}^2_{\text{ét}}(X'_{\bar{K}}, \boldsymbol{Z}_p)$ isomorphically onto $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \boldsymbol{Z}_p)$. Therefore, we have shown that f in fact induces an integral Hodge isometry $\mathrm{H}^2(X'_K(\boldsymbol{C}), \boldsymbol{Z}) \xrightarrow{\sim} \mathrm{H}^2(X_K(\boldsymbol{C}), \boldsymbol{Z})$ which preserves the ample cones. We may now conclude using the global Torelli theorem and [Matsusaka and Mumford 1964, Theorem 2]. □

8. Isogenies and Hecke orbits

We briefly recall the definition of prime-to-p Hecke orbit on the orthogonal Shimura varieties. Let Λ be the K3 lattice $\boldsymbol{U}^{\oplus 3} \oplus \boldsymbol{E}_8^{\oplus 2}$, $\lambda \in \Lambda$ be a primitive element with $d := \lambda^2$ and $p > 2$ be a prime such that $p \nmid d$. We shall use the same notation for orthogonal and spinor Shimura varieties as in Section 7A with $L = L_d$ and fix $\mathsf{K}_p = G(\boldsymbol{Z}_p)$. The only difference is that this time L_d has a $\boldsymbol{Z}$-structure, so that the sheaf $\boldsymbol{L}_{\boldsymbol{A}_f^p}$ also has a $\widehat{\boldsymbol{Z}}^p$-structure. Let K_0^p denote the image of $\mathrm{CSpin}(L_d \otimes \widehat{\boldsymbol{Z}}^p)$ in $G(\boldsymbol{A}_f^p)$. More concretely, K_0^p

can be described as the maximal subgroup of $\mathrm{SO}(L_d \otimes \widehat{\boldsymbol{Z}}^p)$ which acts trivially on the discriminant group $\mathrm{disc}(L_d \otimes \widehat{\boldsymbol{Z}}^p) = \mathrm{disc}(L_d)$. A more helpful alternative description for us is that K_0^p can be viewed as the stabilizer of the element $\lambda \otimes 1$ of $\mathrm{SO}(\Lambda \otimes \widehat{\boldsymbol{Z}}^p)$, which can naturally be viewed as a subgroup of $G(\boldsymbol{A}_f^p)$.

The limit $\mathrm{Sh}_{\mathsf{K}_p}(L_d)$ is equipped with a (right) $G(\boldsymbol{A}_f^p)$-action. By the extension property of the canonical integral models, this action extends to $\mathscr{S}_{\mathsf{K}_p}(L_d)$. Recall the complex uniformization of $\mathrm{Sh}_{\mathsf{K}_p}(L_d)$

$$\mathrm{Sh}_{\mathsf{K}_p}(L_d)(\boldsymbol{C}) = G(\boldsymbol{Z}_{(p)})\backslash\Omega \times G(\boldsymbol{A}_f^p),$$

where Ω is the period domain parametrizing Hodge structures of K3 type on L_d [Madapusi Pera 2016, §3.1, 3.2; Yang 2022, Definition 3.1]. Given a point $(\omega, g) \in \Omega \times G(\boldsymbol{A}_f^p)$ and an element $g' \in G(\boldsymbol{A}_f^p)$, g' sends the class of (ω, g) in $\mathrm{Sh}_{\mathsf{K}_p}(L_d)(\boldsymbol{C})$ to that of (ω, gg'). Let k be an algebraically closed field of characteristic 0 or p. Let $\mathsf{M}_{2d,\mathsf{K}_p}$ be the moduli stack over $\boldsymbol{Z}_{(p)}$ of oriented quasipolarized K3 surfaces of degree $2d$ with hyperspecial level structure at p (see [Yang 2022, 3.3.4], where it is denoted by $\widetilde{\mathsf{M}}_{2d,\mathsf{K}_p^{\mathrm{ad}},\mathbb{Z}_{(p)}}$). By the modular interpretation of $\mathsf{M}_{2d,\mathsf{K}_p}$, $\mathsf{M}_{2d,\mathsf{K}_p}(k)$ is in natural bijection with the set of tuples (X, ξ, ϵ, η), where

- (X, ξ) is a quasipolarized K3 surface of degree $2d$ over k,
- ϵ is an isometry
$$\det(L_d \otimes \boldsymbol{Q}_2) \xrightarrow{\sim} \boldsymbol{P}^2_{\text{ét}}(X, \boldsymbol{Q}_2),$$
which naturally extends to an isometry[4]
$$\epsilon^p : \det(L_d \otimes \boldsymbol{A}_f^p) \xrightarrow{\sim} \boldsymbol{P}^2_{\text{ét}}(X, \boldsymbol{A}_f^p),$$
- η is an isometry
$$\Lambda \otimes \widehat{\boldsymbol{Z}}^p \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X, \widehat{\boldsymbol{Z}}^p)$$
which sends $\lambda \otimes 1$ to $c_1(\xi)$ and is compatible with the isometry ϵ^p.

Using these explicit descriptions, it is easy to write down the map $\mathsf{M}_{2d,\mathsf{K}_p}(\boldsymbol{C}) \to \mathrm{Sh}_{\mathsf{K}_p}(L_d)(\boldsymbol{C})$ explicitly: Let (X, ξ, ϵ, η) be the tuple which corresponds to a point $s \in \mathsf{M}_{2d,\mathsf{K}_p}(\boldsymbol{C})$. Choose an isomorphism $\alpha : (\Lambda \otimes \boldsymbol{Z}_{(p)}) \xrightarrow{\sim} (\mathrm{H}^2(X, \boldsymbol{Z}_{(p)}), c_1(\xi))$ which is compatible with ϵ. Then s is sent to the class of $(\omega, \eta^{-1} \circ (\alpha \otimes \boldsymbol{A}_f^p))$, where ω is the Hodge structure on L_d endowed by α. This map is clearly well defined. The integral extension $\mathsf{M}_{2d} \to \mathscr{S}_{\mathsf{K}_0}(L_d)$ is constructed and studied in [Madapusi Pera 2015]. The reader can also look at [Yang 2022, §3.3] for a quick summary of the properties.

Theorem 8.1. *Assume* $\mathrm{char}\, k = p > 2$. *If any point* $x \in \mathscr{S}_{\mathsf{K}_p}(L_d)(k)$ *lies in the image of* $\mathsf{M}_{2d,\mathsf{K}_p}(k)$, *then so does* $x \cdot g$ *for any* $g \in G(\boldsymbol{A}_f^p)$.

Proof. Let $s \in \mathsf{M}_{2d,\mathsf{K}_p}(k)$ be a point such that $x = \rho_{\mathsf{K}}(s)$. Let (X, ξ, η, ϵ) be the tuple which corresponds to s. We view $G(\boldsymbol{A}_f^p)$ as the subgroup of $\mathrm{SO}(\Lambda \otimes \boldsymbol{A}_f^p)$ which fixes $\lambda \otimes 1$.

By Theorem 1.2 and Remark 6.10, there exists a K3 surface X' together with a derived isogeny $f : \mathfrak{h}^2(X') \to \mathfrak{h}^2(X)$ such that $f_*(\mathrm{H}^2(X')) = \mathrm{H}^2_{\mathrm{cris}}(X/W) \times \mathrm{im}(g) \subset \mathrm{H}^2(X)_{\boldsymbol{Q}}$. Moreover, f is a composition

[4] For details on how to obtain this extension, see [Yang 2022, §3.3.3 or Corollary 3.3.7].

of primitive derived isogenies which come from twisted derived equivalences involving Brauer classes of prime-to-p order. Since $f_*(\mathrm{NS}(X')) = f_*(\mathrm{H}^2(X')) \cap \mathrm{NS}(X)_{\boldsymbol{Q}}$, $\xi \in f_*(\mathrm{NS}(X'))$, so that $\mathrm{NS}(X')$ contains a primitive vector of degree $2d$. By [Ogus 1979, Lemma 7.3], we can find a derived auto-isogeny δ on X which is given by reflections in (-2)-curves up to a sign such that $\delta \circ f$ sends ξ to a quasipolarization ξ'. Now we use $\delta \circ f$ to transport (ϵ, η) to similar structures (ϵ', η') on (X', ξ') so that we obtain a point $s' \in \mathsf{M}_{2d,\mathsf{K}^p}(k)$. We claim that $\rho(s') = x \cdot g$. Although $\mathscr{S}_{\mathsf{K}_p}(L_d)$ lacks a direct modular interpretation, we can do this by a lifting argument.

We claim that there exist liftings (X_W, ξ_W) and (X'_W, ξ'_W) of (X, ξ) and (X', ξ') together with an isogeny $(\mathfrak{h}^2(X'_K), \xi'_K) \to (\mathfrak{h}^2(X_K), \xi_K)$ whose étale realization agrees with $\delta \circ f$ via the smooth and proper base change theorem. If X and X' are of finite height, by Theorem 5.8, $\delta \circ f$ can be lifted to an isogeny on the nose. In the supersingular case, we first choose a lifting (X_W, ξ_W). Then X_W induces a Hodge filtration on $\mathrm{H}^2_{\mathrm{cris}}(X/W)$, which can be transported to a filtration on $\mathrm{H}^2_{\mathrm{cris}}(X'/W)$ lifting the one on $\mathrm{H}^2_{\mathrm{dR}}(X'/k)$. By the local Torelli theorem, this defines a lifting X'_W of X'. One easily checks by [Ogus 1979, Proposition 1.12] that ξ' lifts to X'_W. Now we apply [Yang 2023, Lemma 4.3.5] and Theorem 6.13.

Liftings as above induce W-points s_W and s'_W on $\mathsf{M}_{2d,\mathsf{K}_p}$ which lift s and s'. Let $x_W := \rho(s_W)$ and $x'_W := \rho(s'_W)$. Using the $G(\boldsymbol{A}_f^p)$-action, the lifting x_W of x induces a lifting x''_W of $x'' := x \cdot g$. Using the complex uniformization one quickly checks that $x''_W \otimes \boldsymbol{C} = x'_W \otimes \boldsymbol{C}$ for any embedding $K \subset \boldsymbol{C}$. Since $\mathscr{S}_{\mathsf{K}_p}(L_d)$ is a limit of separated schemes, we conclude that $x' = x''$ as desired. □

Choose a small enough compact open $\mathsf{K}^p \subseteq \mathsf{K}_0^p$ such that for $\mathsf{K} := \mathsf{K}_p \mathsf{K}^p$, $\mathscr{S}_{\mathsf{K}}(L_d)$ is a scheme and denote the period morphism $\mathsf{M}_{2d,\mathsf{K}} \to \mathscr{S}_{\mathsf{K}}(L_d)$ by ρ_{K}. For any k-point $x \in \mathscr{S}_{\mathsf{K}}(L_d)$, the image of the $G(\boldsymbol{A}_f^p)$-orbit of a lift $\tilde{x} \in \mathscr{S}_{\mathsf{K}_p}(L_d)(k)$ under the natural projection $\mathscr{S}_{\mathsf{K}_p}(L_d) \to \mathscr{S}_{\mathsf{K}}(L_d)$ is what we call the prime-to-p Hecke orbit of x.

Let $\mathscr{X}$ denote the universal family over $\mathsf{M}_{2d,\mathsf{K}}$. The mod p fiber $\mathsf{M}_{2d,\mathsf{K},\boldsymbol{F}_p}$ (resp. $\mathscr{S}_{\mathsf{K}}(L_d)_{\boldsymbol{F}_p}$) of moduli space $\mathsf{M}_{2d,\mathsf{K}}$ admits a stratification $\mathsf{M}_{2d,\mathsf{K},\boldsymbol{F}_p} = \mathsf{M}^1 \supseteq \mathsf{M}^2 \supseteq \cdots \supseteq \mathsf{M}^{20}$ (resp. $\mathscr{S}_{\mathsf{K}}(L_d)_{\boldsymbol{F}_p} = \mathscr{S}^1 \supseteq \mathscr{S}^2 \supseteq \cdots \supseteq \mathscr{S}^{20}$) such that for $1 \le i \le 10$, a geometric point s lies in M^i (resp. $\mathscr{S}^i$) if and only if $\mathscr{X}_s$ (resp. $\boldsymbol{L}_{\mathrm{cris},s}(-1)$) has height $\ge i$, and for $11 \le i \le 20$, a geometric point s lies in M^i (resp. $\mathscr{S}^i$) if and only if $\mathscr{X}_s$ (resp. $\boldsymbol{L}_{\mathrm{cris},s}(-1)$) is supersingular and has Artin invariant $\le 21 - i$. Set $\mathring{\mathsf{M}}^i := \mathsf{M}^i - \mathsf{M}^{i-1}$ and $\mathring{\mathscr{S}}^i := \mathscr{S}^i - \mathscr{S}^{i-1}$. Heights and Artin invariants are rather classical invariants. For a more modern interpretation in terms of Newton and Ekedahl–Oort (E–O) strata for $\mathscr{S}_{\mathsf{K}}(L_d)_{\boldsymbol{F}_p}$, see for example [Shen 2020, §8.4]. It follows from [Madapusi Pera 2015, Corollary 5.14] that the period morphism respects these stratifications in the sense that $\mathsf{M}^i = \mathscr{S}^i \times_{\mathscr{S}_{\mathsf{K}}(L_d)} \mathsf{M}_{2d,\mathsf{K}}$. We remark that the Zariski closure of the locally closed subscheme $\mathring{\mathscr{S}}^i$ is $\mathscr{S}^i$. By [Shen and Zhang 2022, Corollaries 7.2.2 and 7.3.4], if $1 \le i \le 10$, then $\mathscr{S}^i$ is a central leaf. The locus $\mathscr{S}^{20}$ is the superspecial locus (the unique closed E–O stratum), and is also a central leaf (see [Shen and Zhang 2022, Remark 3.2.2, Examples 6.2.4]).

In our case, the Hecke orbit conjecture predicts the following:

Conjecture 8.2. For $1 \le i \le 10$ or $i = 20$, the prime-to-p Hecke orbit of every $s \in \mathscr{S}^i(\overline{\boldsymbol{F}}_p)$ is Zariski dense in $\mathscr{S}^i$.

We remark that once the above conjecture is known for $\overline{\boldsymbol{F}}_p$, it is automatically true for any algebraically closed field over $\boldsymbol{F}_p$ by a specialization argument. Conjecture 8.2 has been proved by Maulik, Shankar, and Tang [2022, Theorem 1.4] when $i = 1$ and $p \geq 5$. We prove another special case below (Theorem 8.6).

We use N_σ to denote the supersingular lattice of Artin invariant σ. We restrict to considering the $p > 2$ case, when these lattices are characterized by [Huybrechts 2016, §17, Proposition 2.20]. The original reference [Rudakov and Shafarevich 1978] also treated the $p = 2$ case.

Lemma 8.3. *For each $d > 0$ and $i = 0, 1$, there exist a primitive element $\xi \in N_1$ with $\xi^2 = 2d$ and an $\alpha_i \in \mathrm{O}(N_1)$ such that α_i fixes ξ and interchanges the two isotropic lines in $(N_1^\vee/N_1) \otimes \boldsymbol{F}_{p^2}$ and* $\det(\alpha_i) = (-1)^i$.

Proof. The supersingular K3 surface with Artin invariant 1, which is unique up to isomorphism, is given by the desingularization of $A/A[2]$, where $A = E \times E$ for a supersingular elliptic curve E [Ogus 1979, Corollary 7.14]. Since E admits a model over $\boldsymbol{F}_p$, so does X. Let φ be a topological generator of $\mathrm{Gal}_{\boldsymbol{F}_p}$. We fix an isomorphism between N_1 and $\mathrm{NS}(X_{\overline{\boldsymbol{F}}_p})$, so that N is equipped with a $\mathrm{Gal}_{\boldsymbol{F}_p}$-action such that $\mathrm{NS}(X)$ is identified with the φ-invariants N^φ.

Let $\mathrm{NS}(A)(2)$ denote the lattice $\mathrm{NS}(A)$ but with the quadratic form multiplied by a factor of 2. As a result of the Kummer construction, there exist 16 (-2)-curves $\delta_1, \dots, \delta_{16}$ on X and an isometric embedding

$$\mathrm{NS}(A)(2) \oplus \left(\bigoplus_{i=1}^{16} \boldsymbol{Z}\delta_i \right) \hookrightarrow \mathrm{NS}(X).$$

Let $\mu \in \mathrm{NS}(A)(2)$ be a primitive element such that $\mu^2 > 0$. For some coprime numbers a and b, $(a\mu + b\delta_1)^2 = 2d$. The generator φ fixes $\xi := a\mu + b\delta_1$ and interchanges the isotropic lines in $(N_1^\vee/N_1) \otimes \boldsymbol{F}_{p^2}$ (cf. the paragraph below [Liedtke 2016, Examples 4.20]).

Let s_{δ_2} be the reflection in δ_2. Note that s_{δ_2} fixes μ and δ_1, and hence ξ. Moreover, it is not hard to check that s_{δ_2} acts trivially on $N^\vee/N$. Therefore, we can simply set α_0 and α_1 to be φ and $s_{\delta_2} \circ \varphi$, up to permutation. $\square$

Lemma 8.4. $$\mathring{\mathsf{M}}^i \neq \varnothing \quad \textit{for all } i.$$

Proof. Each $\mathsf{M}^{i+1} \subseteq \mathsf{M}^i$ is locally cut out by a single equation. $\mathsf{M}_{2d,\mathsf{K},\boldsymbol{F}_p}$ is smooth of dimension 19, and we know that M^{20} is zero-dimensional (cf. [Artin 1974, §7]). Therefore, it suffices to show that $\mathsf{M}^{20} \neq \varnothing$, i.e., there exists a quasipolarization of degree $2d$ on the superspecial K3 surface, which is unique up to isomorphism. This follows from the preceding lemma and [Ogus 1979, Lemma 7.9]. $\square$

Let $\mathcal{K} \subset \widetilde{G}(\boldsymbol{A}_f^p)$ be the preimage of K. Before proceeding we recall that for any geometric point $t \in \widetilde{\mathscr{S}}_{\mathcal{K}}(L_d)$, there is a distinguished subspace $\mathrm{LEnd}(\mathscr{A}_t)$ of $\mathrm{End}(\mathscr{A}_t)$ which consists of the elements whose cohomological realizations lie in $\boldsymbol{L}_{\boldsymbol{A}_f^p,t}$ and $\boldsymbol{L}_{\mathrm{cris},t}$ ([Yang 2022, Definition 3.10]; cf. [Madapusi Pera 2016, Definition 5.11]). When t is on the supersingular locus, the natural maps $\mathrm{LEnd}(\mathscr{A}_t) \otimes \widehat{\boldsymbol{Z}}^p \to \boldsymbol{L}_{\ell,t}$ and $\mathrm{LEnd}(\mathscr{A}_t) \otimes \boldsymbol{Z}_p \to \boldsymbol{L}_{\mathrm{cris},t}^{F=1}$ are isomorphisms [Yang 2023, Proposition 3.2.3].

Lemma 8.5. *Let k be an algebraically closed field with $\operatorname{char} k = p$. Let x be a k-point on $\mathscr{S}^i$ for some $i \geq 11$ and t be a k-point on $\widetilde{\mathscr{S}}_{\mathbb{K}}(L_d)$ which lifts x, and set $P := \mathrm{LEnd}(\mathscr{A}_t)$. Then there exists a primitive element $\nu \in N_\sigma$ with $\sigma := 21 - i$ and $\nu^2 = 2d$ such that $P \cong \nu^\perp$.*

Proof. Let $\boldsymbol{Z}\nu$ be a quadratic lattice of rank 1 generated by ν with $\nu^2 = 2d$. By the theory of gluing lattices (see [McMullen 2011, §2] for a quick summary), primitive extensions of $P \oplus \boldsymbol{Z}\nu$ corresponds to the data (G_1, G_2, ϕ), where G_1, G_2 are subgroups of $\mathrm{disc}(P)$ and $\mathrm{disc}(\boldsymbol{Z}\nu)$ and ϕ is an isometry $G_1 \xrightarrow{\sim} G_2$. Therefore, constructing N amounts to choosing appropriate (G_1, G_2, ϕ).

In our case, we take G_1 to be the prime-to-p part of $\mathrm{disc}(P)$, i.e., $\mathrm{disc}(P \otimes \widehat{\boldsymbol{Z}}^p)$, and $G_2 = \mathrm{disc}(\boldsymbol{Z}\nu)$, which is isomorphic to $\boldsymbol{Z}/(2d)\boldsymbol{Z}$ as an abelian group. Then we construct ϕ by a lifting argument: Let x_W be a W-point on $\mathscr{S}(L_d)$ which lifts x and let $x_{\boldsymbol{C}}$ be x_W for some embedding $W \hookrightarrow \boldsymbol{C}$. The period morphism ρ_{K} is known to be surjective on $\boldsymbol{C}$-points, so there exists a quasipolarized K3 surface $(X_{\boldsymbol{C}}, \xi_{\boldsymbol{C}})$ such that the $\boldsymbol{Z}$-Hodge structure $\boldsymbol{L}_{B,x_{\boldsymbol{C}}}$ is naturally identified with $\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z})$. We have that the natural map $P \otimes \widehat{\boldsymbol{Z}}^p \to \boldsymbol{L}_{\widehat{\boldsymbol{Z}}^p,x}$ is an isomorphism [Yang 2023, Proposition 3.2.3] and $\boldsymbol{L}_{\widehat{\boldsymbol{Z}}^p,x} \cong \boldsymbol{L}_{B,x_{\boldsymbol{C}}} \otimes \widehat{\boldsymbol{Z}}^p$ by smooth and proper base change and the Artin comparison isomorphisms. Therefore, there is an isomorphism $\beta_1 : G_1 \xrightarrow{\sim} \mathrm{disc}(\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z}) \otimes \widehat{\boldsymbol{Z}}^p) = \mathrm{disc}(\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z}))$. On the other hand, let $\beta_2 : G_2 \xrightarrow{\sim} \mathrm{disc}(\boldsymbol{Z}\xi_{\boldsymbol{C}})$ be the isomorphism given by sending ν to $\xi_{\boldsymbol{C}} \in \mathrm{H}^2(X_{\boldsymbol{C}}, \boldsymbol{Z})$.

We may transport the gluing data given by the primitive embedding $\boldsymbol{P}^2(X_{\boldsymbol{C}}, \boldsymbol{Z}) \oplus \boldsymbol{Z}\xi \subset \mathrm{H}^2(X_{\boldsymbol{C}}, \boldsymbol{Z})$ to a gluing data ϕ for G_1, G_2 via β_1, β_2. Let N be the lattice given by (G_1, G_2, ϕ). We check that it is a supersingular K3 lattice. Clearly, by our construction, $N \otimes \widehat{\boldsymbol{Z}}^p \cong \Lambda \otimes \widehat{\boldsymbol{Z}}^p$. As P is negative definite, N has signature $(1+, 21-)$. Finally, $\mathrm{disc}(N \otimes \boldsymbol{Z}_p) = \mathrm{disc}(P \otimes \boldsymbol{Z}_p) \cong (\boldsymbol{Z}/p\boldsymbol{Z})^{2\sigma}$ as an abelian group. Therefore, $N \cong N_\sigma$. □

We now prove another special case of Conjecture 8.2:

Theorem 8.6. *Conjecture 8.2 holds for $i = 20$.*

Proof. Take two $\overline{\boldsymbol{F}}_p$-points $x, x' \in \mathscr{S}^{20}$. Choose lifts t, t' for x, x' in $\widetilde{\mathscr{S}}_K(L_d)$. We only need to show that there exists a CSpin-isogeny $\mathscr{A}_t \to \mathscr{A}_{t'}$ which is prime to p. Indeed, this follows from an explicit description of the isogeny classes in $\widetilde{\mathscr{S}}_{\mathbb{K}_p}(L_d)(\overline{\boldsymbol{F}}_p)$ and their images on $\mathscr{S}_{\mathsf{K}_p}(L_d)(\overline{\boldsymbol{F}}_p)$ [Yang 2022, §3.2.3]. Let P and P' denote $\mathrm{LEnd}(\mathscr{A}_t)$ and $\mathrm{LEnd}(\mathscr{A}_{t'})$ respectively.

We first show that every isometry $P_{\boldsymbol{Q}} \xrightarrow{\sim} P'_{\boldsymbol{Q}}$ whose induced isomorphism $\boldsymbol{L}_{2,t} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{2,t'} \otimes \boldsymbol{Q}$ sends $\delta_{2,t}$ to $\delta_{2,t'}$ is induced by a CSpin-isogeny $\psi : \mathscr{A}_t \to \mathscr{A}_{t'}$ by conjugation. Indeed, by [Yang 2023, Proposition 3.2.4], there exists some CSpin-isogeny $\psi' : \mathscr{A}_t \to \mathscr{A}_{t'}$, which induces some isomorphism $P_{\boldsymbol{Q}} \xrightarrow{\sim} P'_{\boldsymbol{Q}}$ whose induced isomorphism $\boldsymbol{L}_{2,t} \otimes \boldsymbol{Q} \xrightarrow{\sim} \boldsymbol{L}_{2,t'} \otimes \boldsymbol{Q}$ sends $\delta_{2,t}$ to $\delta_{2,t'}$. The group of CSpin-isogenies from $\mathscr{A}_t$ to itself is identified with $\mathrm{CSpin}(P_{\boldsymbol{Q}})$, which surjects to $\mathrm{SO}(P_{\boldsymbol{Q}})$. By composing ψ' with some CSpin-isogeny $\mathscr{A}_t \to \mathscr{A}_t$, we get the desired ψ.

We only need to show that there exists a CSpin-isogeny $\mathscr{A}_t \to \mathscr{A}_{t'}$ which is prime to p. By a Cartan decomposition trick [Yang 2023, Lemma 3.2.6], we only need to show the following claim:

Claim. *There exists an isometry $P \otimes \boldsymbol{Z}_{(p)} \xrightarrow{\sim} P' \otimes \boldsymbol{Z}_{(p)}$ which sends $\delta_{2,x}$ to $\delta_{2,x'}$ and extends to an isomorphism $\boldsymbol{L}_{\mathrm{cris},x} \xrightarrow{\sim} \boldsymbol{L}_{\mathrm{cris},x'}$.*

By Lemma 8.5, for some primitive vectors ξ, ξ' in N_1 with $\xi^2 = (\xi')^2 = 2d$, we have $P \cong \xi^\perp$ and $P' \cong (\xi')^\perp$. Since some reflection of $N \otimes \boldsymbol{Z}_{(p)}$ takes ξ to ξ' [Milnor and Husemoller 1973, I, Lemma 4.2], $P \otimes \boldsymbol{Z}_{(p)} \cong P' \otimes \boldsymbol{Z}_{(p)}$ as quadratic lattice over $\boldsymbol{Z}_{(p)}$. Now $P \otimes \boldsymbol{Z}_p$ and $P' \otimes \boldsymbol{Z}_p$ are the Tate modules of the supersingular K3 crystals $\boldsymbol{L}_{\mathrm{cris},t}(-1)$ and $\boldsymbol{L}_{\mathrm{cris},t'}(-1)$ respectively. By Ogus's theory of characteristic subspaces [1979, Theorem 3.20], $\boldsymbol{L}_{\mathrm{cris},t}$ (resp. $\boldsymbol{L}_{\mathrm{cris},t'}$) determines an isotropic line of $(P^\vee/P) \otimes \overline{\boldsymbol{F}}_p$ (resp. $((P')^\vee/P') \otimes \overline{\boldsymbol{F}}_p$) and the isomorphism $P \otimes \boldsymbol{Z}_p \to P' \otimes \boldsymbol{Z}_p$ extends to an isomorphism $\boldsymbol{L}_{\mathrm{cris},t} \xrightarrow{\sim} \boldsymbol{L}_{\mathrm{cris},t'}$ if and only these isotropic lines are respected. Now the claim follows from Lemma 8.3. □

Remark 8.7. As the reader can readily tell, the heart of the above theorem is the claim. Here we have proved the claim in a rather ad hoc way. We go through Lemma 8.5 because there does not seem to be a good classification theory for quadratic lattices over $\boldsymbol{Z}_{(p)}$. Moreover, P and P' are negative definite, so one cannot apply, say, Nikulin's theory to generate automorphisms, which only handles indefinite lattices. Luckily, in our special case, there is a geometric way of constructing the automorphisms we need.

Lemma 8.8. *Let k be an algebraically closed field with $\operatorname{char} k = p > 2$. Let R be a DVR over k with fraction field κ and let X_κ be a supersingular* K3 *surface over κ such that $X_{\bar\kappa}$ has Artin invariant σ_0. There exists a DVR S over k with fraction field L, a finite separable map $R \to S$, and an N_{σ_0}-marked supersingular* K3 *surface X_S over S such that $(X_S)_L \cong (X_\kappa)_L$.*

Proof. By a result of Rudakov and Shafarevich (see [Rudakov and Shafarevich 1976, Theorem 50], and [Bragg and Lieblich 2018, Theorem 5.2.1] for $p = 3$) there exists a DVR S, a finite separable map $R \to S$, and a supersingular K3 surface X_S over S such that $(X_S)_L \cong (X_\kappa)_L$. The Picard scheme $\underline{\mathrm{Pic}}_{X_L}$ is formally étale over $\operatorname{Spec} L$. As $\mathrm{Pic}(X_{\bar L})$ is finitely generated, after taking a further finite separable extension we may ensure that the restriction map $\mathrm{Pic}(X_L) \cong \mathrm{Pic}(X_{\bar L})$ is an isomorphism. Thus, X_L admits an N_{σ_0}-marking. As S is a DVR, we have $\mathrm{Pic}(X_L) = \mathrm{Pic}(X)$, so the generic marking extends uniquely to an N_{σ_0}-marking of X_S. □

Theorem 8.9. *If Conjecture 8.2 holds for i, or $i \geq 11$, then $\mathscr{S}^i \subset \mathrm{im}(\rho_{\mathsf{K}})$.*

Proof. If Conjecture 8.2 holds for i then the conclusion is a direct consequence of Theorem 8.1 and the fact that $\mathrm{im}(\rho_{\mathsf{K}})$ is open. Now assume $i \geq 11$ and take $k = \overline{\boldsymbol{F}}_p$. Note that by Theorem 8.6, $\mathscr{S}^{20} \subset \mathrm{im}(\rho_{\mathsf{K}})$. Since the Zariski closure of $\mathring{\mathscr{S}}^i$ is $\mathscr{S}^i$, the intersection $\mathrm{im}(\rho_{\mathsf{K}}) \cap \mathscr{S}^i$ is open and dense in $\mathscr{S}^i$. Take a closed point $x \in \mathring{\mathscr{S}}^i_k$. Let $\mathcal{R}$ be the ring $k[\![t]\!]$ and $\mathcal{F}$ be its fraction field. Choose an $\mathcal{R}$-valued point $\tilde{x}$ which extends x such that $\tilde{x}_{\mathcal{F}}$ lies in $\mathrm{im}(\rho_{\mathsf{K}}) \cap \mathscr{S}^i$. Such an $\tilde{x}$ can always be found: we can always choose a smooth curve which passes through x and whose generic point lies in $\mathrm{im}(\rho_{\mathsf{K}}) \cap \mathscr{S}^i$. Then we simply take the completion of this curve at x. Let $X_{\mathcal{F}}$ be a supersingular K3 surface over the generic point of $\tilde{x}_{\mathcal{F}}$. Note that the geometric fiber of $X_{\mathcal{F}}$ has Artin invariant $\sigma := 21 - i$. By the preceding lemma, there exists a DVR $\mathcal{R}'$ over $\mathcal{R}$, whose fraction field $\mathcal{F}'$ is a finite extension of $\mathcal{F}$, such that there is an N_σ-marked supersingular K3 surface $\mathcal{X}$ over $\mathcal{R}'$.

We argue that the special fiber $\mathcal{X}_k$ of $\mathcal{X}$ has Artin invariant σ. There are two families of supersingular K3 crystals over $\mathcal{R}'$ (see [Ogus 1979, §5] for the definition): One is obtained by pulling back $\boldsymbol{L}_{\mathrm{cris},\tilde{x}}(-1)$

along $\mathcal{R} \to \mathcal{R}'$. The other is given by $\mathrm{H}^2_{\mathrm{cris}}(\mathcal{X}')$. By construction, these two families agree on the generic fiber. By Proposition 4.6 and Theorem 5.3 of [Ogus 1979], there exists a universal family of supersingular K3 crystals over a smooth projective space $\mathcal{M}$ such that these two families are both obtained by pulling back the universal family along morphisms $\mathcal{R} \to \mathcal{M}$. Since $\mathcal{M}$ is in particular separated, these two morphisms have to agree. Therefore, $\mathrm{H}^2_{\mathrm{cris}}(\mathcal{X}'/\mathcal{R}')$ is precisely the pullback of $\boldsymbol{L}_{\mathrm{cris},\tilde{x}}(-1)$. Now we conclude by the hypothesis that $x \in \mathscr{S}^i_k$.

Now we know that $\mathcal{X}_{\overline{\mathcal{F}}} := \mathcal{X} \otimes \overline{\mathcal{F}}$ and $\mathcal{X}_k$ have the same Artin invariant. This guarantees that the specialization map $\mathrm{Pic}(\mathcal{X}_{\mathcal{F}'}) \to \mathrm{Pic}(\mathcal{X}_k)$ must be an isomorphism, and hence must send the ample cone isomorphically onto the ample cone. Since the big and nef cone is the closure of the ample cone, the quasipolarization on $\mathcal{X}_{\mathcal{F}'}$ extends to a quasipolarization on $\mathcal{X}_k$. This shows that $x \in \mathrm{im}(\rho_{\mathsf{K}})$. □

Finally we discuss some implications of the surjectivity of the period morphism to the good reduction theory of K3 surfaces. As Conjecture 8.2 is known for $i = 1$ and $p \geq 5$ (by [Maulik et al. 2022, Theorem 1.4]), the following result in particular implies the unconditional Theorem 1.7.

Theorem 8.10. *Let k be a perfect field of characteristic $p > 2$. Let F be a finite extension of $K = W[1/p]$. Let X_F be a* K3 *surface over F equipped with a quasipolarization ξ of degree $2d$ with $p \nmid d$. Suppose that the* Gal_F*-action on $\boldsymbol{H}^2_{\text{ét}}(X_{\overline{F}}, \boldsymbol{Q}_\ell)$ is potentially unramified for some $\ell \neq p$. Then we have*:

(a) $\mathrm{H}^2_{\text{ét}}(X_{\overline{F}}, \boldsymbol{A}^p_f)$ *and* $\mathrm{H}^2_{\text{ét}}(X_{\overline{F}}, \boldsymbol{Q}_p)$ *are potentially unramified and crystalline respectively.*

(b) *If* $\mathrm{H}^2_{\text{ét}}(X_{\overline{F}}, \boldsymbol{Q}_p)$ *is crystalline, then* $\mathbb{D}_{\mathrm{cris}}(\mathrm{H}^2_{\text{ét}}(X_{\overline{F}}, \boldsymbol{Q}_p))$ *is a* K3 *crystal.*

(c) *Suppose that the hypothesis of* (b) *is satisfied and* $\mathbb{D}_{\mathrm{cris}}(\mathrm{H}^2_{\text{ét}}(X_{\overline{F}}, \boldsymbol{Q}_p))$ *is a* K3 *crystal of height i. If Conjecture 8.2 holds for i or if $i = \infty$, then X_F has potential good reduction.*

We recall that X_F as above is said to have potential good reduction if, up to replacing F by a finite extension, there exists a smooth proper algebraic space $\mathcal{X}$ over $\mathcal{O}_F$, whose special fiber is a K3 surface over k and whose generic fiber is X_F (cf. [Liedtke and Matsumoto 2018, Definition 2.1]).

Proof. (a) and (b) Up to replacing F by a finite extension, we may equip (X, ξ) with a K-level structure and an orientation so that it is given by an F-point s on $\mathsf{M}_{2d,\mathsf{K}}$, and find a lift $t \in \widetilde{\mathscr{S}}_{\mathsf{K}}(L_d)(F)$ of $\rho_{\mathsf{K}}(s)$. Consider the abelian variety $\mathscr{A}_t$. One easily adapts the argument of Deligne [1981, §6.6] to see that, up to replacing F by a further extension, $\mathscr{A}_t$ admits good reduction. By the extension property of the integral models, we can extend t to an $\mathcal{O}_F$-valued point τ on $\mathscr{S}_{\mathsf{K}}(L_d)$. This implies both (a) and (b).

(c) We have $\tau \otimes k \in \mathscr{S}^i$. If the hypothesis is satisfied, then $\mathscr{S}^i \subset \mathrm{im}(\rho_{\mathsf{K}})$. Now we conclude by the étaleness of ρ_{K}. Indeed, the global Torelli theorem implies that if two $\boldsymbol{C}$-points of $\mathsf{M}_{2d,\mathsf{K}}$ are mapped to the same points under ρ_{K}, then the K3 surfaces they correspond to are (noncanonically) isomorphic. If there is a quasipolarized K3 surface over k whose moduli point is sent to $\tau \otimes k$, then the étaleness of ρ_{K} tells us that there exists an F-point s' of $\mathsf{M}_{2d,\mathsf{K}}$ such that $\rho_{\mathsf{K}}(s) = \rho_{\mathsf{K}}(s')$. Up to replacing F by a finite extension, the K3 surfaces defined by s and s' are isomorphic. □

Appendix: Some results from integral p-adic Hodge theory

We review some basic results in p-adic Hodge theory. Let k be a perfect field of characteristic $p > 0$. We write W for $W(k)$ and K_0 for $W[1/p]$. Let K be a totally ramified extension of K_0 and let π be a uniformizer of its ring of integers $\mathcal{O}_K$.[5] Let G_K denote the absolute Galois group Gal_K. Set $R := \mathcal{O}_K/(p)$.

Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be a smooth and proper scheme (more generally, the following discussion applies also when $\mathcal{X}$ is only a formal scheme and $\mathcal{X}_K$ denotes the rigid analytic generic fiber). The subject of p-adic Hodge theory is concerned with how to recover the following tuples of data from one another under suitable assumptions:

(A) The $\boldsymbol{Z}_p$-module $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \boldsymbol{Z}_p)$ equipped with a G_K-action.

(B) The F-crystal $\boldsymbol{R}^i f_{R,\mathrm{cris}*}\mathcal{O}_{X_R}$ over $\boldsymbol{Cris}(R/W)$ together with the filtered $\mathcal{O}_K$-module $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$.

(B′) The F-crystal $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ together with the filtered $\mathcal{O}_K$-module $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$.

Remark A.1. Let e be the ramification degree of $\mathcal{O}_K$ over W. When $e \le p-1$, $R \cong k[\varepsilon]/\varepsilon^e$ has a PD structure, so that the category of crystals of quasicoherent sheaves over $\boldsymbol{Cris}(R/W)$ is equivalent to that over $\boldsymbol{Cris}(k/W)$ [Berthelot and Ogus 1978, Corollary 6.7]. Therefore, under mild torsion-freeness assumptions on various cohomology modules of $\mathcal{X}$, (B) and (B′) are equivalent data. Moreover, as $\mathcal{O}_K$ is a PD thickening of W, the crystalline de Rham comparison theorem gives us a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W) \underset{W}{\otimes} \mathcal{O}_K \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K).$$

If $e > p-1$, then (B) contains strictly more information than (B′). The above isomorphism no longer holds integrally in general. However, there is still a canonical isomorphism after inverting p:

$$\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W) \underset{W}{\otimes} K \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \underset{\mathcal{O}_K}{\otimes} K.$$

This isomorphism is often called the Berthelot–Ogus isomorphism because it was first introduced in [Berthelot and Ogus 1983]. Below we will often make use of this isomorphism implicitly. Note that in the above isomorphisms, the left-hand side only depends on the special fiber $\mathcal{X}_k$, whereas the right-hand side is equipped with the additional data of a Hodge filtration, which in general depends on the lifting $\mathcal{X}$ of $\mathcal{X}_k$.

Here is an overview of the relationship between the above tuples: The classical (rational) p-adic comparison isomorphisms tell us how to recover (A) and (B′) from one another after inverting p. Integral p-adic Hodge theory (e.g., the seminal paper of Bhatt, Morrow, and Scholze [Bhatt et al. 2018]) tells us how to recover (B) from (A). For our purposes, we are mainly concerned with how to recover (A) from (B). Roughly speaking, the way to do this is to evaluate the F-crystal $\boldsymbol{R}^i f_{R,\mathrm{cris}*}\mathcal{O}_{X_R}$ on a certain PD-thickening S of R (S is often called Breuil's S-ring), so that we obtain an S-module. This S-module is equipped with a Frobenius action from the F-crystal structure on $\boldsymbol{R}^i f_{R,\mathrm{cris}*}\mathcal{O}_{X_R}$, and is moreover equipped with a filtration which absorbs the data of the Hodge filtration on $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$. The main result of

[5] This notation is chosen to be in line with most references in p-adic Hodge theory. In the main text, the letters K and F take the roles of K_0 and K respectively. We apologize for this inconsistency of notation.

[Cais and Liu 2019] tells us that by applying a certain functor (denoted by T_{cris} below) to this S-module, we recover (A). Of course, [Bhatt et al. 2018] already treats the relationship between (A) and (B), but the conclusions there are packaged in a more abstract way.

After inverting p. Let $\mathrm{MF}_K^{\varphi,N}$ denote the category of filtered (φ, N)-modules. An object of this category is a K_0-vector space D which is equipped with

- a Frobenius semilinear injection $\varphi : D \to D$;
- a K_0-linear map $N : D \to D$ such that $N\varphi = pN\varphi$;
- a descending filtration on D_K such that $\mathrm{Fil}^i D_K = D_K$ for $i \ll 0$ and $\mathrm{Fil}^i D_K = 0$ for $i \gg 0$.

Let MF_K^{φ} denote the subcategory with $N = 0$. The motivation to consider this category is that the data in (A) is naturally an object in MF_K^{φ} after inverting p, because there is a canonical Berthelot–Ogus isomorphism $\mathrm{H}^i_{\text{cris}}(\mathcal{X}_k/W) \otimes_W K \cong \mathrm{H}^i_{\text{dR}}(\mathcal{X}/\mathcal{O}_K) \otimes_{\mathcal{O}_K} K$. We will use this isomorphism repeatedly without explicitly mentioning it. We remark that in most references the operator N is in $\mathrm{MF}_K^{\varphi,N}$ to treat varieties with semistable reductions. Since we are assuming good reduction, we may restrict to considering the category MF_K^{φ}.

Let Rep_{G_K} denote the category of G_K-representations over $\boldsymbol{Q}_p$ and let $\mathrm{Rep}_{G_K}^{\text{cris}}$ denote the subcategory of crystalline representations. Given an object $Q \in \mathrm{Rep}_{G_K}^{\text{cris}}$, one may define an object in MF_K^{φ} using the (covariant) Fontaine's functors D_{cris} and D_{dR}, which are defined by $D_{\text{cris}}(Q) = (Q \otimes_{\boldsymbol{Q}_p} B_{\text{cris}})^{G_K}$ and $D_{\text{dR}}(Q) = (Q \otimes_{\boldsymbol{Q}_p} B_{\text{dR}})^{G_K}$. The pair $(D_{\text{cris}}(Q), D_{\text{dR}}(Q))$ are equipped with a Frobenius action and filtrations respectively, and hence define an object in MF_K^{φ}. We abusively denote the resulting functor $\mathrm{Rep}_{G_K} \to \mathrm{MF}_K^{\varphi}$ also by D_{cris}. We define a functor from the essential image of D_{cris} to $\mathrm{Rep}_{G_K}^{\text{cris}}$ by $V_{\text{cris}} = \mathrm{Fil}^0(D \otimes_{K_0} B_{\text{cris}})^{\varphi=1}$. There is an equality of $\boldsymbol{Q}_p$-submodules

$$Q = V_{\text{cris}}(D_{\text{cris}}(Q))$$

of $(Q \otimes_{\boldsymbol{Q}_p} B_{\text{cris}}) \otimes_{K_0} B_{\text{cris}}$, which specifies a natural transformation $V_{\text{cris}} \circ D_{\text{cris}} \Rightarrow \mathrm{id}$ on $\mathrm{Rep}_{G_K}^{\text{cris}}$.[6] The reader may look at [Brinon and Conrad 2009, Part I, Sections 8 and 9] for more details about these objects.

By [Bhatt et al. 2018, Proposition 5.1, Theorem 14.6], there is a p-adic comparison isomorphism

$$\mathrm{H}^i_{\text{cris}}(\mathcal{X}_k/W) \underset{W}{\otimes} B_{\text{cris}} \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\bar{K}}, \boldsymbol{Z}_p) \underset{\boldsymbol{Z}_p}{\otimes} B_{\text{cris}} \tag{41}$$

which respects the Gal_F-actions and filtrations. Therefore, we obtain an isomorphism of objects in MF_K^{φ}

$$D_{\text{cris}}(\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\bar{F}}, \boldsymbol{Q}_p)) \xrightarrow{\sim} (\mathrm{H}^i_{\text{cris}}(\mathcal{X}_k/W)[1/p], \mathrm{H}^i_{\text{dR}}(\mathcal{X}_K/K)). \tag{42}$$

There are multiple rational p-adic comparison isomorphisms of the form (41) (e.g., those constructed earlier by Faltings [1999], Tsuji [1999], and others). We choose to use the one from [Bhatt et al. 2018] because this is the one used in [Cais and Liu 2019], to be cited below. Once we fix this choice of rational p-adic comparison isomorphism, then the isomorphism (42) is also fixed.

[6]Note that the natural transformations between two functors between 1-categories (or locally small categories in the usual sense) do form a *set* (as opposed to a groupoid), so it makes sense to specify an element in this set.

Recovering integral lattices. We now explain how to recover the natural integral lattices in the objects of (42) from one another. Let $\mathfrak{S} := W[\![u]\!]$, and let $\theta : \mathfrak{S} \to \mathcal{O}_K$ be the map sending u to π. Let $\mathrm{Rep}^{\mathrm{cris}\circ}_{\mathrm{Gal}_K}$ denote the category of G_K-stable $\boldsymbol{Z}_p$-lattices in objects of $\mathrm{Rep}^{\mathrm{cris}}_{G_K}$. Let $\mathfrak{M}(-)$ be the functor as in [Kisin 2010, Theorem 1.2.1] which sends an object in $\mathrm{Rep}^{\mathrm{cris}\circ}_{G_K}$ to a Breuil–Kisin module in the sense of [Bhatt et al. 2018, Theorem 4.4], so that there exist canonical isomorphisms

$$\varphi^*\mathfrak{M}(T) \underset{\mathfrak{S}}{\otimes} K_0 \xrightarrow{\sim} D_{\mathrm{cris}}(T[1/p]) \quad \text{and} \quad \varphi^*\mathfrak{M}(T) \underset{\mathfrak{S},\theta}{\otimes} K \xrightarrow{\sim} D_{\mathrm{dR}}(T[1/p]) \tag{43}$$

which preserve Frobenius actions and filtrations respectively. Then we have the following result [Bhatt et al. 2018, Theorem 14.6].

Theorem A.2. *Assume that $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ and $\mathrm{H}^{i+1}(\mathcal{X}_k/W)$ are torsion-free. Then for $T = \mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\bar{K}}, \boldsymbol{Z}_p)$ the isomorphisms (43) map $\mathfrak{M}(T) \otimes_{\mathfrak{S}} W$ and $\mathfrak{M}(T) \otimes_{\mathfrak{S},\theta} \mathcal{O}_K$ isomorphically onto $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ and $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ respectively, when composed with the isomorphisms in (41).*

We refer the reader also to [Ito et al. 2018, Theorem 3.2] for an exposition which is closer to ours in notation. The above theorem tells us how to recover (B′) from (A). Under the additional assumption that $i < p-1$, [Cais and Liu 2019, Theorem 5.4] tells us how to recover (A) from (B). Before doing so we need to introduce the intermediate category of Breuil's S-modules, which packages the data of (B) in a different way.

Breuil's S-modules. Let S denote the p-adic completion of the PD envelope of $(\mathfrak{S}, \ker\theta)$. Let S_π denote the ring $W[\![u-\pi]\!]$. Then there is an embedding $\iota : S \hookrightarrow S_\pi$ which sends u to $u-\pi$. Let $f_\pi : S_\pi \to \mathcal{O}_K$ (resp. $f_0 : S \to W$) be the projection which sends $u-\pi$ to 0 (resp. u to 0). Then there is a commutative diagram of W-algebras

$$\begin{array}{ccc} S & \xrightarrow{\ \iota\ } & S_\pi \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_\pi} \\ W & \longrightarrow & \mathcal{O}_K \end{array}$$

In [Breuil 1997], the above ring S is denoted by $S^0_{\min}$. The letter S in loc. cit. denotes a certain extension of $S^0_{\min,K_0}$. For our purposes, one may simply take $S = S^0_{\min,K_0}$ when reading [Breuil 1997]. The letter S in our notation is in line with [Cais and Liu 2019] and [Liu 2008].

Let $\mathcal{MF}^{\varphi,N}_{S_{K_0}}$ denote the category of filtered (φ, N)-modules over S_{K_0}.[7] There is an equivalence of categories

$$\eta : \mathrm{MF}^{\varphi,N}_K \to \mathcal{MF}^{\varphi,N}_{S_{K_0}} \tag{44}$$

which sends $(D, \mathrm{Fil}^\bullet D_K, \varphi, N)$ to an object $(\mathscr{D}, \mathrm{Fil}^\bullet \mathscr{D}, \varphi_{\mathscr{D}}, N_{\mathscr{D}})$ with $\mathscr{D} = D \otimes_W S$ ([Cais and Liu 2019, p. 1215]; see also [Breuil 1997, Theorem 6.1.1]). The quasi-inverse η^{-1} is defined by $(\mathscr{D} \otimes_{f_0} W, \mathscr{D} \otimes_{f_\pi \circ \iota} \mathcal{O}_K)$, for which the Frobenius action and filtration are inherited from those on $\mathscr{D}$. There is a

[7] This is just the category denoted by $\mathcal{MF}(\varphi, N)$ in [Liu 2008, §2.2], except that we have not restricted to positive objects, so that we replace the condition $\mathrm{Fil}^0 \mathscr{D} = \mathscr{D}$ by $\mathrm{Fil}^j \mathscr{D} = \mathscr{D}$ for $j \ll 0$.

canonical natural transformation $\eta^{-1}\circ\eta \Rightarrow \mathrm{id}$ on MF^{φ}_K which underlies the tautological identification of modules

$$(D, D_K) = (D \underset{W}{\otimes} S \underset{f_0}{\otimes} W,\, D_K \underset{W}{\otimes} S \underset{f_\pi\circ\iota}{\otimes} \mathcal{O}_K).$$

A strongly divisible S-lattice (of height r) in an object $\mathscr{D} \in \mathcal{MF}^{\varphi,N}_{S_K}$ with $\mathrm{Fil}^0\mathscr{D} = \mathscr{D}$ is an S-lattice such that $\mathscr{M}[1/p] = \mathscr{D}$, $N_{\mathscr{D}}(\mathscr{M}) \subseteq \mathscr{M}$, and $\varphi_{\mathscr{D}}(\mathrm{Fil}^r\mathscr{M}) \subseteq p^r\mathscr{M}$, where $\mathrm{Fil}^r\mathscr{M} := \mathscr{M}\cap\mathrm{Fil}^r\mathscr{D}$. Let $\mathcal{MF}^{\varphi,N}_S$ denote the category of strongly divisible S-lattices in objects of $\mathcal{MF}^{\varphi,N}_{S_{K_0}}$.

Theorem A.3 (Liu). *Suppose that $Q \in \mathrm{Rep}^{\mathrm{cris}}_{G_K}$ has Hodge–Tate weights in $\{0, 1, \dots, p-2\}$. Let $\mathscr{D}$ denote $\eta(Q)$. The covariant functor $T_{\mathrm{cris}} : \mathscr{M} \mapsto \mathrm{Fil}^0(\mathscr{M}\otimes_S A_{\mathrm{cris}})^{\varphi=1}$ defines a bijection between the set of strongly divisible S-lattices in $\mathscr{D}$ and that of G_K-stable $\mathbf{Z}_p$-lattices in $V_{\mathrm{cris}}(D_{\mathrm{cris}}(Q)) = Q$.*

Proof. Theorem 2.3.5 of [Liu 2008] tells us that the above theorem holds for Breuil's functor T_{st}. The contravariant version of this functor is reviewed in Section 2.2 of loc. cit. If we use the superscript (resp. subscript) $*$ to indicate contravariance (resp. covariance), then $T^*_{\mathrm{st}}(-) = T_{*\mathrm{st}}((-)^\vee)$. Proposition 3.5.1 of loc. cit. tells us that $T_{\mathrm{st}}(\mathscr{M}) = T_{\mathrm{cris}}(\mathscr{M})$ as Q is crystalline. □

Remark A.4. Let $\mathcal{C}$ denote the full subcategory of $\mathcal{MF}^{\varphi,N}_S$ whose image in $\mathcal{MF}^{\varphi,N}_{S_{K_0}}$ lies in the essential image of MF^{φ}_K (as a subcategory of $\mathrm{MF}^{\varphi,N}_K$) under η. To sum up, we now have a commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{cris}}_{G_K} \arrow[r, bend left, "D_{\mathrm{cris}}"] & \mathrm{MF}^{\varphi}_K \arrow[l, bend left, "V_{\mathrm{cris}}"] \arrow[r, hook] & \mathrm{MF}^{\varphi,N}_K \arrow[r, bend left, "\eta"] & \mathcal{MF}^{\varphi,N}_{S_{K_0}} \arrow[l, bend left, "\eta^{-1}"] \\
\mathrm{Rep}^{\mathrm{cris}\circ}_{G_K} \arrow[u] & & \mathcal{C} \arrow[ll, "T_{\mathrm{cris}}"'] \arrow[r, hook] & \mathcal{MF}^{\varphi,N}_S \arrow[u]
\end{tikzcd}$$

in which the vertical arrows are given by inverting p. Moreover, the natural transformations $V_{\mathrm{cris}}\circ D_{\mathrm{cris}} \Rightarrow \mathrm{id}$ and $\eta^{-1}\circ\eta \Rightarrow \mathrm{id}$ are tautological. By the above theorem, T_{cris} is an equivalence of categories. We remark that since A_{cris} is a W-subalgebra of B_{cris} and the inclusion $A_{\mathrm{cris}} \subseteq B_{\mathrm{cris}}$ respects the filtration and Frobenius structures, $T_{\mathrm{cris}}(\mathscr{M})$ is a priori a $\mathbf{Z}_p$-submodule of $V_{\mathrm{cris}}(D_{\mathrm{cris}}(Q))$. The reason that we emphasize the natural transformations used is to decategorify the language, so that T_{cris}, which is often stated as an equivalence of categories, is concretely an equality of sets.

Theorem A.5 (Cais and Liu). *Assume that $H^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ and $H^{i+1}_{\mathrm{cris}}(\mathcal{X}_k/W)$ are torsion-free and $i \le p-2$. Set $\mathscr{M} := \mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_R/S)$. Let $\mathfrak{p} : \mathscr{M} \to \mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W)$ be the canonical projection induced by f_0. Let $\mathscr{D} \in \mathcal{MF}^{\varphi,N}_{S_{K_0}}$ be given by the object $(\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k)_K, \mathrm{Fil}^\bullet\,\mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}_K/K))$ in MF^{φ}_K via η. Then we have:*

(a) *There is a canonical section s to $\mathfrak{p}[1/p]$ such that s is φ-equivariant and $s\otimes_W S$ induces an isomorphism $\mathscr{M}[1/p] \xrightarrow{\sim} \mathscr{D}$.*

(b) *Under the isomorphism in* (a), *$\mathscr{M}$ defines a strongly divisible S-lattice in $\mathscr{D}$ and $T_{\mathrm{cris}}(\mathscr{M}) = \mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{K}}, \mathbf{Z}_p)$.*

Proof. Part (a) is a variant of the Berthelot–Ogus isomorphism [Cais and Liu 2019, Proposition 5.1]. Part (b) follows from [Cais and Liu 2019, Theorem 5.4(2)] and its proof, which proceeds by reducing to proving the equality of two lattices.

Let T be an object of $\mathrm{Rep}_{G_K}^{\mathrm{cris}\circ}$ and let $\mathfrak{M}(T)$ be the Breuil–Kisin module associated to T. Let $M(-)$ be the functor defined by $\varphi^*(\mathfrak{M}(-))$. Then $\underline{\mathscr{M}}(M(T)) := M(T) \otimes_{\mathfrak{S}} S$ can be equipped with additional structures so that it becomes an object in $\mathcal{MF}_S^{\varphi,N}$. The base-change-to-$S$ functor $\underline{\mathscr{M}}$ used here is defined in (3.6) of loc. cit. There is a natural isomorphism $\underline{\mathscr{M}}(M(T))[1/p] \xrightarrow{\sim} \eta(D_{\mathrm{cris}}(T[1/p]))$ which lifts the isomorphism $M(T) \otimes_{\mathfrak{S}} K_0 \xrightarrow{\sim} D_{\mathrm{cris}}(T[1/p])$ in (43). Moreover, T_{cris} sends the strongly divisible S-lattice $\underline{\mathscr{M}}(M(T))$ to T. The reader may also check out the proof of [Snowden 2014, Lemma A.3] for entirely similar considerations.

Now let T be $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{K}}, \boldsymbol{Z}_p)$. Since T_{cris} establishes a bijection between strongly divisible S-lattices in $\mathscr{D}$ and G_K-stable $\boldsymbol{Z}_p$-lattices in $T[1/p]$, one reduces to showing an equality of S-lattices $\mathscr{M} = \underline{\mathscr{M}}(M(T))$ under the isomorphisms

$$\underline{\mathscr{M}}(M(T))[1/p] \cong \mathscr{D} \cong \mathscr{M}[1/p].$$

This is the main step in the proof of [Cais and Liu 2019, Theorem 5.4(2)] (see the second paragraph on page 1226). □

Remark A.6. In the above setting, let $f : \mathcal{X}_R \to \mathrm{Spf}(R)$ be the structure morphism and let $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)$ denote the F-crystal $\boldsymbol{R}^i f_{\mathrm{cris}*}\mathcal{O}_{\mathcal{X}_R}$. Then $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_R/S)$ (resp. $\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_R/\mathcal{O}_K)$) can be viewed as a the S-module given by evaluating $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)$ on the object S (resp. $\mathcal{O}_K$) of $\boldsymbol{Cris}(R/W)$. The morphism $\theta : S \to \mathcal{O}_K$ defines a canonical isomorphism $\theta^*\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_S \xrightarrow{\sim} \boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_{\mathcal{O}_K}$. The lifting $\mathcal{X}$ of $\mathcal{X}_R$ to $\mathcal{O}_K$ endows $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_{\mathcal{O}_K}$ with a Hodge filtration via the crystalline de Rham comparison $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_{\mathcal{O}_K} \cong \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$. The S-module $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_S$, being an object of $\mathcal{MF}_S^{\varphi,N}$, is also equipped with a natural filtration, which maps isomorphically onto the Hodge filtration on $\boldsymbol{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$. However, note that the filtration on $\boldsymbol{H}^i_{\mathrm{cris}}(\mathcal{X}_R)_S$ is defined in a more formal way, with the Hodge filtration on $\boldsymbol{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ being the key input. Namely, one first constructs $\mathscr{D}$ out of $(\mathrm{H}^i_{\mathrm{cris}}(\mathcal{X}_k/W), \mathrm{H}^i_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K))$, and then defines a filtration on $\mathscr{M}$ by intersecting with $\mathrm{Fil}^\bullet \mathscr{D}$ under the isomorphism in part (a) of the above theorem. One naturally wonders whether this filtration has a more direct cohomological construction. This question is addressed in [Cais and Liu 2019, §6.1]. However, we won't make use of this cohomological interpretation.

Remark A.7. If $\mathcal{X}$ is a smooth proper scheme over $\mathcal{O}_K$, or more generally a smooth proper algebraic space over $\mathcal{O}_K$ whose special and generic fibers are schemes, then the above results hold for $\mathcal{X}_K$ interpreted as the generic fiber in the usual sense. The point is that the analytification of the generic fiber is functorially isomorphic to the rigid analytic generic fiber of the formal completion of $\mathcal{X}$ at the special fiber. The reader may look at [Ito et al. 2018, §11.2] for details.

Applications to p-divisible groups. Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$ and assume $p \geq 3$. Let $T_p(-)$ denote the Tate module functor, $\mathbb{D}(-)$ denote the contravariant Dieudonné module functor and $\mathscr{G}^*$ denote the Cartier dual of $\mathscr{G}$. There is a p-adic comparison isomorphism

$$\mathbb{D}(\mathscr{G}_k) \underset{W}{\otimes} B_{\mathrm{cris}} \xrightarrow{\sim} T_p\mathscr{G}^*(-1) \underset{\boldsymbol{Z}_p}{\otimes} B_{\mathrm{cris}} \tag{45}$$

which induces an isomorphism $D_{\mathrm{cris}}(T_p\mathscr{G}^*(-1) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p) \xrightarrow{\sim} \mathbb{D}(\mathscr{G}_k)[1/p]$. $T_{\mathrm{cris}}(\mathbb{D}(\mathscr{G}_R)_S)$ recovers the $\boldsymbol{Z}_p$-lattice $T_p\mathscr{G}^*(-1)$ inside $T_p\mathscr{G}^*(-1) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$ [Kisin 2006, Lemma 2.2.4]. Note that $T_p\mathscr{G}^*(-1)$ is canonically isomorphic to $(T_p\mathscr{G})^\vee$.

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