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On Héthelyi–Külshammer's conjecture for principal blocks

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We prove that the number of irreducible ordinary characters in the principal p-block of a finite group G of order divisible by p is always at least $2\sqrt{p-1}$. This confirms a conjecture of Héthelyi and Külshammer (2000) for principal blocks and provides an affirmative answer to Brauer's problem 21 (1963) for principal blocks of bounded defect. Our proof relies on recent works of Maróti (2016) and Malle and Maróti (2016) on bounding the conjugacy class number and the number of p'-degree irreducible characters of finite groups, earlier works of Broué, Malle and Michel (1993) and Cabanes and Enguehard (2004) on the distribution of characters into unipotent blocks and e-Harish-Chandra series of finite reductive groups, and known cases of the Alperin–McKay conjecture.

1. Introduction

Bounding the number $k(G)$ of conjugacy classes of a finite group G in terms of a certain invariant associated to G is a fundamental problem in group representation theory. An equally important problem in modular representation theory is to bound the number $k(B)$ of ordinary irreducible characters in a block B of G. It is not surprising that these two problems are closely related to each other. For instance, the p-solvable case of the Brauer's celebrated $k(B)$-conjecture [Brauer 1963, Problem 20], which asserts that $k(B)$ is bounded *above* by the order of a defect group for B, was known to be equivalent to the coprime $k(GV)$-problem (by [Nagao 1962]), which in turn was eventually solved in [Gluck et al. 2004]; see also [Schmid 2007]. While there have been a number of results on upper bounds for $k(B)$ [Brauer and Feit 1959; Robinson 2004; Sambale 2017; Malle 2018], not much has been done on lower bounds.

Let p be a prime dividing the order of G. A result of Brauer [1942] on characters and blocks of groups G of order divisible by p but not by p^2 implies that $k(G) \geq 2\sqrt{p-1}$ for those groups, and the bound was later conjectured to be true for all finite groups. After several partial results [Héthelyi and Külshammer 2000; 2003; Malle 2006; Keller 2009; Héthelyi et al. 2011], the conjecture was finally proved by Maróti [2016]. In the proof of the conjecture for solvable groups, Héthelyi and Külshammer [2000] speculated that "*perhaps it is even true that $k(B) \geq 2\sqrt{p-1}$ for every p-block B of positive defect*, where $k(B)$

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denotes the number of irreducible ordinary characters in B". Of course, they were aware of blocks of defect zero, which have a unique irreducible ordinary character (whose degree has the same p-part as the order of the group) and a unique irreducible Brauer character as well; see [Navarro 1998, Theorem 3.18].

The main aim of this paper is to confirm Héthelyi and Külshammer's conjecture for principal blocks. Throughout, we use $B_0(G)$ to denote the principal p-block of G.

Theorem 1.1. *Let G be a finite group and p a prime such that $p \mid |G|$. Then $k(B_0(G)) \geq 2\sqrt{p-1}$.*

Problem 21 in Brauer's famous list [1963] asks whether there exists a function $f(q)$ on prime powers q such that $f(q) \to \infty$ for $q \to \infty$ and that $k(B) \geq f(p^{d(B)})$ for every p-block B of defect $d(B) > 0$. Our Theorem 1.1 provides an affirmative answer to this question for principal blocks of *bounded* defect. See [Külshammer 1990] for more discussion on this problem.

One may expect to improve the bound in Theorem 1.1 when the power of p in $|G|$ is large. Kovács and Leedham-Green [1986] constructed, for each odd prime p, a p-group P of order p^p with $k(P) = (p^3 - p^2 + p + 1)/2$. Therefore, the bound $k(B_0(G)) \geq 2\sqrt{p-1}$ cannot be replaced by $k(B_0(G)) \geq p^3$, for example, even when any given large power of p divided the group order.

Building upon the ideas of Maróti [2016] and the subsequent paper by Malle and Maróti [2016] on bounding the number of p'-degree irreducible characters in a finite group, we observe that Héthelyi and Külshammer's conjecture for principal blocks essentially reduces to bounding the number of irreducible ordinary characters in principal blocks of almost simple groups, as well as bounding the number of orbits of irreducible characters in principal blocks of simple groups under the action of their automorphism groups.

Theorem 1.2. *Let S be a finite nonabelian simple group and p a prime such that $p \mid |S|$. Let G be an almost simple group with socle S such that $p \nmid |G/S|$. Then*:

(i) $k(B_0(G)) \geq 2\sqrt{p-1}$. *Moreover*, $k(B_0(G)) > 2\sqrt{p-1}$ *if S does not have cyclic Sylow p-subgroups.*

(ii) *Assume further that $p \geq 11$ and S does not have cyclic Sylow p-subgroups. Then the number of* $\mathrm{Aut}(S)$*-orbits on* $\mathrm{Irr}(B_0(S))$ *is at least* $2(p-1)^{1/4}$.

As we will explain in the next section, Theorem 1.1 is a consequence of [Maróti 2016] and the well-known Alperin–McKay conjecture, which asserts that the number of irreducible characters of height 0 in a block B of a finite group G coincides with the number of irreducible characters of height 0 in the Brauer correspondent of B of the normalizer of a defect subgroup for B in G. We take advantage of the recent advances on the conjecture in the proof of our results, particularly the fact that Späth's inductive Alperin–McKay conditions hold for all p-blocks with cyclic defect groups [Späth 2013; Koshitani and Späth 2016]. This explains why simple groups with cyclic Sylow p-subgroups are excluded in Theorem 1.2(ii). Additionally, we take advantage of recent results on the possible structure of defect groups of principal blocks with few ordinary characters [Koshitani and Sakurai 2021; Rizo et al. 2021], and this explains why the smaller values of p are excluded in Theorem 1.2(ii).

Theorem 1.2 turns out to be straightforward for alternating groups or groups of Lie type in characteristic p, but highly nontrivial for groups of Lie type in characteristic not equal to p. We make use of Cabanes and Enguehard's results [1994; 2004] on the distribution of characters into unipotent blocks and Broué, Malle and Michel's results [Broué et al. 1993] on the compatibility between the distributions of unipotent characters into unipotent blocks and e-Harish-Chandra series to obtain a general bound for the number of $\mathrm{Aut}(S)$-orbits of characters in $\mathrm{Irr}(B_0(S))$ in terms of certain data associated to S, for S a simple group of Lie type, see Theorem 5.4. We hope this result will be useful in other purposes.

The next result classifies groups for which $k(B_0(G))$ is minimal in the sense of Theorem 1.1.

Theorem 1.3. *Let G be a finite group and p a prime. Let P be a Sylow p-subgroup of G. Then $k(B_0(G)) = 2\sqrt{p-1}$ if and only if $\sqrt{p-1} \in \mathbb{N}$ and $\boldsymbol{N}_G(P)/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P))$ is isomorphic to the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$.*

We remark that, in the situation of Theorem 1.3, the number of p'-degree irreducible characters in $B_0(G)$ is also equal to $2\sqrt{p-1}$. In general, if a p-block B of a finite group has an abelian defect group, then every ordinary irreducible character of B has height zero. This is the "if direction" of Brauer's height-zero conjecture, which is now known to be true, thanks to the work of Kessar and Malle [2013]. Theorem 1.1 therefore implies that if $P \in \mathrm{Syl}_p(G)$ is abelian and nontrivial then $k_0(B_0(G)) \geq 2\sqrt{p-1}$, where $k_0(B)$ denotes the number of height zero ordinary irreducible characters of a block B.

Theorems 1.1 and 1.3 are useful in the study of principal blocks with few height zero ordinary irreducible characters. In fact, using them, we are able to show in [Hung et al. 2023] that $k_0(B_0(G)) = 3$ if and only if $P \cong C_3$, and that $k_0(B_0(G)) = 4$ if and only if $|P/P'| = 4$ or $P \cong C_5$ and $\boldsymbol{N}_G(P)/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P))$ is isomorphic to the dihedral group D_{10}. These results have been known only in the case $p \leq 3$; see [Navarro et al. 2018, Theorems A and C].

The paper is organized as follows. In Section 2, we recall some known results on the Alperin–McKay conjecture and prove that our results follow when all the nonabelian composition factors of G have cyclic Sylow p-subgroups. We also prove Theorem 1.2 for the sporadic simple groups and groups of Lie type defined in characteristic p in Section 2. The alternating groups are treated in Section 3. Section 4 takes care of the case when the Sylow p-subgroups of S are nonabelian. Sections 6, 7, and 8 are devoted to proving Theorem 1.2 for simple groups of Lie type defined in characteristics different from p. To do so, in Section 5, we prove a bound for the number of $\mathrm{Aut}(S)$-orbits of characters in $\mathrm{Irr}(B_0(S))$. Finally, we finish the proofs of Theorems 1.1 and 1.3 in Section 9.

2. Some first observations

In this section we make some observations toward the proofs of the main results.

2A. ***The Alperin–McKay conjecture.*** The well-known Alperin–McKay (AM) conjecture predicts that the number of irreducible characters of height zero in a block B of a finite group G coincides with the number of irreducible characters of height zero in the Brauer correspondent of B of the normalizer of a defect subgroup of B in G. For the principal blocks, the conjecture is equivalent to

$$k_{p'}(B_0(G)) = k_{p'}(B_0(\boldsymbol{N}_G(P))),$$

where P is a Sylow p-subgroup of G and $k_{p'}(B_0(G))$ denotes the number of p'-degree irreducible ordinary characters in $B_0(G)$.

On the other hand, if $p \mid |G|$, we have

$$\begin{aligned} k_{p'}(B_0(\boldsymbol{N}_G(P))) &\geq k_{p'}(B_0(\boldsymbol{N}_G(P)/P')) \\ &= k(B_0(\boldsymbol{N}_G(P)/P')) \\ &= k(B_0((\boldsymbol{N}_G(P)/P')/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P)/P'))) \\ &= k((\boldsymbol{N}_G(P)/P')/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P)/P')) \\ &\geq 2\sqrt{p-1}, \end{aligned}$$

where the first inequality follows from [Navarro 1998, page 137], the first equality follows from the fact that every irreducible ordinary character of $\boldsymbol{N}_G(P)/P'$ has p'-degree, the last two equalities follow from [loc. cit., Theorem 9.9] and Fong's theorem (see [loc. cit., Theorem 10.20]), and the last inequality follows from [Maróti 2016]. Therefore, if the AM conjecture holds for G and p, then the number of p'-degree irreducible ordinary characters in $B_0(G)$ is bounded below by $2\sqrt{p-1}$.

From this, we see that Theorems 1.1 and 1.2(i) hold if the AM conjecture holds for (G, p). We now prove that the same is true for Theorem 1.3. Note that the "if" implication of this theorem is clear. Assume that the AM conjecture holds for $B_0(G)$ and $k(B_0(G)) = 2\sqrt{p-1}$ for some prime p such that $\sqrt{p-1} \in \mathbb{N}$. Then, as seen above, we have

$$2\sqrt{p-1} = k(B_0(G)) \geq k((\boldsymbol{N}_G(P)/P')/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P)/P')) \geq 2\sqrt{p-1},$$

implying

$$k((\boldsymbol{N}_G(P)/P')/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P)/P')) = 2\sqrt{p-1},$$

and thus $(\boldsymbol{N}_G(P)/P')/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P)/P')$ is isomorphic to the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$, by [Maróti 2016, Theorem 1]. In particular, $P/P' \cong C_p$, implying that $P \cong C_p$, and hence it follows that $\boldsymbol{N}_G(P)/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P))$ is isomorphic to the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$, as wanted.

The AM conjecture is known to be true when G has a cyclic Sylow p-subgroup by Dade's theory [1966]. In fact, by [Späth 2013; Koshitani and Späth 2016], the so-called *inductive Alperin–McKay conditions* are satisfied for all blocks with cyclic defect groups. Therefore, we have:

Lemma 2.1 (Koshitani–Späth). *Let p be a prime. Assume that all the composition factors of a finite group G have cyclic Sylow p-subgroups. Then the Alperin–McKay conjecture holds for G and p, and thus Theorems 1.1, 1.2(i), and 1.3 hold for G and p.*

Note that the linear groups $\mathrm{PSL}_2(q)$, the Suzuki groups ${}^2B_2(2^{2f+1})$ and the Ree groups ${}^2G_2(3^{2f+1})$ all have cyclic Sylow p-subgroups for odd p different from the defining characteristic of the group. So

Theorem 1.2 automatically follows from Lemma 2.1 for these groups in characteristic not equal to p, when p is odd.

2B. ***Small blocks.*** Blocks with a small number of ordinary characters have been studied significantly in the literature. In particular, the possible structure of defect groups of principal blocks with at most 5 ordinary irreducible characters are now known; see [Brandt 1982; Belonogov 1990; Koshitani and Sakurai 2021; Rizo et al. 2021]. (B. Sambale informed us that he and S. Koshitani think that Belonogov's work [1990] for the case $k(B_0)=3$ is not conclusive. However, this case has been recently reproved in [Koshitani and Sakurai 2021, Section 3].) Using these results, we can easily confirm our results for $p \leq 7$. For instance, to prove Theorems 1.1 and 1.2 for $p=7$ it is enough to assume that $k(B_0(G)) \leq 4$, but by going through the list of possible defect groups of $B_0(G)$, we then have $\mathrm{Syl}_p(G) \in \{1, C_2, C_3, C_2 \times C_2, C_4, C_5\}$, which cannot happen. To prove Theorem 1.3 for $p<7$ we note that if $p=5$ and $k(B_0(G))=4$ then $P=C_5$; and if $p=2$ and $k(B_0(G))=2$ then $P=C_2$, in both of which cases P is cyclic, and thus the result of Section 2A applies.

Therefore we will assume from now on that $p \geq 11$, unless stated otherwise.

2C. ***Sporadic groups and the Tits group.*** We remark that Theorem 1.2 can be confirmed directly using [Conway et al. 1985; Jansen et al. 1995] or [GAP 2020] for sporadic simple groups and the Tits group. Therefore, we are left with the alternating groups and groups of Lie type, which will be treated in the subsequent sections.

2D. ***Groups of Lie type in characteristic p.*** Let S be a simple group of Lie type defined over the field of $q=p^f$ elements, where p is a prime and f a positive integer. According to results of Dagger and Humphreys on defect groups of finite reductive groups in defining characteristic; see [Cabanes 2018, Proposition 1.18 and Theorem 3.3] for instance, S has only two p-blocks. The only nonprincipal block is a defect-zero block containing only the Steinberg character of S. Therefore,

$$k(B_0(S)) = k(S) - 1.$$

Let $\boldsymbol{G}$ be a simple algebraic group of simply connected type and let F be a Steinberg endomorphism on $\boldsymbol{G}$ such that $S = X/\boldsymbol{Z}(X)$, where $X=\boldsymbol{G}^F$. Assume that the rank of $\boldsymbol{G}$ is r. By a result of Steinberg (see [Fulman and Guralnick 2012, Theorem 3.1]), X has at least q^r semisimple conjugacy classes, and thus $k(X) > q^r$. It follows that

$$k(B_0(S)) > \frac{q^r}{|\boldsymbol{Z}(X)|} - 1,$$

which yields $k(B_0(S)) \geq \lfloor q^r/|\boldsymbol{Z}(X)| \rfloor$. Using the values of $|\boldsymbol{Z}(X)|$ and $|\mathrm{Out}(S)|$ available in [Conway et al. 1985, page xvi], it is straightforward to check that $\lfloor q^r/|\boldsymbol{Z}(X)| \rfloor \geq 2\sqrt{p-1}|\mathrm{Out}(S)|$, proving Theorem 1.2 for the relevant S and p.

3. Alternating groups

In this section we prove Theorem 1.2 for the alternating groups. The background on block theory of symmetric and alternating groups can be found in [Olsson 1993] for instance.

The ordinary irreducible characters of S_n are naturally labeled by partitions of n. Two characters are in the same p-block if and only if their corresponding partitions have the same p-cores, which are obtained from the partitions by successive removals of rim p-hooks until no p-hook is left. Therefore, p-blocks of S_n are in one-to-one correspondence with p-cores of partitions of n.

Let B be a p-block of S_n. The number $k(B)$ of ordinary irreducible characters in B turns out to depend only on p and the so-called *weight* of B, which is defined to be $w(B) := (n - |\mu|)/p$, where μ is the p-core corresponding to B under the aforementioned correspondence. In fact,

$$k(B) = k(p, w(B)) := \Sigma_{(w_0, w_1, \dots, w_{p-1})} \pi(w_0)\pi(w_1) \cdots \pi(w_{p-1}),$$

where $(w_0, w_1, \dots, w_{p-1})$ runs through all p-tuples of nonnegative integers such that $w(B) = \Sigma_{i=0}^{p-1} w_i$ and $\pi(x)$ is the number of partitions of x; see [Olsson 1993, Proposition 11.4]. Note that $k(p, w(B))$ is precisely the number of p-tuples of partitions of $w(B)$.

For the principal block $B_0(\mathsf{S}_n)$ of S_n, we have $w(B_0(\mathsf{S}_n)) = \lfloor n/p \rfloor$, which is at least 1 by the assumption $p \mid |S|$. It follows that

$$k(B_0(\mathsf{S}_n)) \geq k(p, 1) = p \geq 2\sqrt{p-1}.$$

Moreover, according to [Olsson 1992, Proposition 2.8], when p is odd and $\widetilde{B}$ is a block of A_n covered by B, then B and $\widetilde{B}$ have the same number of irreducible ordinary characters (and indeed the same number of irreducible Brauer characters as well). In particular, when p is odd, we have $k(B_0(\mathsf{A}_n)) = k(B_0(\mathsf{S}_n)) \geq 2\sqrt{p-1}$, which proves Theorem 1.2(i) for the alternating groups.

For part (ii) of Theorem 1.2, recall that $p \geq 11$, and thus $n \geq 11$ and $\mathrm{Aut}(S) = \mathsf{S}_n$. The number of S_n-orbits on $\mathrm{Irr}(B_0(\mathsf{A}_n))$ is at least $1 + (k(B_0(\mathsf{A}_n)) - 1)/2$, which in turn is at least

$$1 + \frac{p-1}{2} = \frac{p+1}{2} > 2(p-1)^{1/4},$$

and this proves Theorem 1.2(ii) for the alternating groups.

4. Groups of Lie type: the nonabelian Sylow case

In this section, we let $\boldsymbol{G}$ be a simple algebraic group of adjoint type and F a Steinberg endomorphism on $\boldsymbol{G}$ such that $S \cong [\mathbb{G}, \mathbb{G}]$ where $\mathbb{G} := \boldsymbol{G}^F$. Let ℓ be a prime different from p and assume $q = \ell^f$ is the absolute value of all eigenvalues of F on the character group of an F-stable maximal torus of G. Recall that we are assuming $p \geq 11$.

In this section we prove Theorem 1.2 for those S of Lie type in characteristic different from p such that the Sylow p-subgroups of $\mathbb{G}$ are nonabelian.

In that case, there are then more than one $d \in \mathbb{N}$ such that $p \mid \Phi_d(q)$ with Φ_d dividing the order polynomial of $(\boldsymbol{G}, F)$. Here, as usual, Φ_d denotes the d-th cyclotomic polynomial. In fact, if there a unique such d, then a Sylow p-subgroup of $\mathbb{G}$ is contained in a Sylow d-torus of $\mathbb{G}$, and hence is abelian; see [Malle and Testerman 2011, Theorem 25.14].

Let $e_p(q)$ denote the multiplicative order of q modulo p. Note that, by [Malle and Testerman 2011, Lemma 25.13], $p \mid \Phi_d(q)$ if and only if $d = e_p(q)p^i$ for some $i \geq 0$. Therefore, as there is more than one $d \in \mathbb{N}$ such that $p \mid \Phi_d(q)$, we must have $p \mid d$ for some $d \in \mathbb{N}$ such that Φ_d divides the order polynomial of $(\boldsymbol{G}, F)$. The fact that $p \geq 11$ then rules out the cases when $\boldsymbol{G}$ is of exceptional type and thus we are left with only the classical types. That is, $\mathbb{G} = \mathrm{PGL}_n(q)$, $\mathrm{PGU}_n(q)$, $\mathrm{SO}_{2n+1}(q)$, $\mathrm{PCSp}_{2n}(q)$, or $\mathrm{P}(\mathrm{CO}^{\pm}_{2n}(q))^0$.

For $\mathbb{G} = \mathrm{PGL}_n(q)$ or $\mathrm{PGU}_n(q)$, we define e to be the smallest positive integer such that $p \mid (q^e - (\epsilon)^e)$ ($\epsilon = 1$ for linear groups and $\epsilon = -1$ for unitary groups), so that $e = e_p(q)$ when $\mathbb{G} = \mathrm{PGL}_n(q)$ or $\mathbb{G} = \mathrm{PGU}_n(q)$ and $4 \mid e_p(q)$, $e = e_p(q)/2$ when $\mathbb{G} = \mathrm{PGU}_n(q)$ and $2 \mid e_p(q)$ but $4 \nmid e_p(q)$, and $e = 2e_p(q)$ when $\mathbb{G} = \mathrm{PGU}_n(q)$ and $2 \nmid e_p(q)$. For $\mathbb{G} = \mathrm{SO}_{2n+1}(q)$, $\mathrm{PCSp}_{2n}(q)$, or $\mathrm{P}(\mathrm{CO}^{\pm}_{2n}(q))^0$, we define e to be the smallest positive integer such that $p \mid (q^e \pm 1)$, so that $e = e_p(q)$ when $e_p(q)$ is odd and $e = e_p(q)/2$ when $e_p(q)$ is even.

Let $n = we + m$ where w and m are integers with $0 \leq m < e$. We claim that $p \leq w$. To see this, first assume that $\mathbb{G} = \mathrm{PGL}_n(q)$. Then, as mentioned above, $ep \leq n$, which implies that $ep < (w+1)e$, and thus $p \leq w$. Next, assume that $\mathbb{G} = \mathrm{SO}_{2n+1}(q)$, $\mathrm{PCSp}_{2n}(q)$, or $\mathrm{P}(\mathrm{CO}^{\pm}_{2n}(q))^0$. If $e = e_p(q)$ is odd, then since $p \mid (q^e - 1)$ and $\gcd(q^e - 1, q^i + 1) \leq 2$ for every $i \in \mathbb{N}$, we have $p \mid (q^j - 1)$ for some $e < j \leq n$, and it follows that $ep \leq n$, implying $p \leq w$. On the other hand, if $2e = e_p(q)$ is even then

$$2ep = e_p(q)p \leq 2n < 2(w+1)e,$$

which also implies that $p \leq w$. Finally, assume $\mathbb{G} = \mathrm{PGU}_n(q)$. The case $4 \mid e_p(q)$ is argued as in the case $S = \mathrm{PGL}_n(q)$; the case $2 \mid e_p(q)$ but $4 \nmid e_p(q)$ is argued as in the case $S = \mathrm{SO}_{2n+1}(q)$ and $2 \mid e_p(q)$. For the last case $2 \nmid e_p(q)$, we have $ep/2 = e_p(q)p$, and in order for $\Phi_{e_p(q)p}$ to divide the generic order of $|\mathrm{PGU}_n(q)|$, $e_p(q)p \leq n/2$, and hence it follows that $ep \leq n$, which also implies that $p \leq w$. The claim is fully proved.

Since p is good for $\boldsymbol{G}$, by [Broué et al. 1993, Theorem 3.2] and [Cabanes and Enguehard 1994, main theorem], the number of unipotent characters of $\mathbb{G}$ in the principal block $B_0(\mathbb{G})$ is equal to $k(W_e)$ — the number of irreducible complex characters of the relative Weyl group W_e of a Sylow $e_p(q)$-torus of $\mathbb{G}$. This W_e is the wreath product $C_e \wr \mathsf{S}_w$ when $\boldsymbol{G}$ is of type A and is a subgroup of index 1 or 2 of $C_{2e} \wr \mathsf{S}_w$ when $\boldsymbol{G}$ is of type B, C, or D; see [Broué et al. 1993, Section 3A]. In any case, W_e has a quotient S_w, so we have that the number of unipotent characters in $\mathrm{Irr}(B_0(\mathbb{G}))$ is at least $k(\mathsf{S}_w) = \pi(w)$, which in turns is at least $\pi(p)$ as $p \leq w$. Since every unipotent character of $\mathbb{G}$ restricts irreducibly to S and $B_0(\mathbb{G})$ covers a unique block of S, it follows that the number of unipotent characters in $\mathrm{Irr}(B_0(S))$ is at least $\pi(p)$.

By a result of Lusztig (see [Malle 2008, Theorem 2.5]), every unipotent character of a simple group of Lie type lies in a $\mathrm{Aut}(S)$-orbit of length at most 3. (In fact, every $\mathrm{Aut}(S)$-orbit on unipotent characters of S has length 1 or 2, except when $S = P\Omega^+_8(q)$ whose graph automorphism of order 3 produces two orbits of length 3.) Therefore, together with the conclusion of the previous paragraph, we deduce that the number of $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}(B_0(S))$ is at least $\pi(p)/3$. This bound is greater than $2\sqrt{p-1}$ when $p \geq 11$, as required.

5. A general bound for the number of Aut(S)-orbits on Irr($B_0(S)$)

The aim of this section is to obtain a general bound for the number of $\mathrm{Aut}(S)$-orbits on irreducible ordinary characters in the principal block of S, for S a simple group of Lie type.

5A. ***Semisimple characters.*** Before continuing with our proof of Theorem 1.2 for groups of Lie type, we recall some background on certain characters known as *semisimple characters* and the fact that they fall into the principal block in a certain situation. Background on character theory of finite reductive groups can be found in [Carter 1985; Cabanes and Enguehard 2004; Digne and Michel 1991]. Let $\boldsymbol{G}$ be a connected reductive group defined over $\mathbb{F}_q$ and F an associated Frobenius endomorphism on $\boldsymbol{G}$. Let $\boldsymbol{G}^*$ be an algebraic group with a Frobenius endomorphism which, for simplicity, we denote by the same F, such that $(\boldsymbol{G}, F)$ is in duality to $(\boldsymbol{G}^*, F)$.

Let t be a semisimple element of $(\boldsymbol{G}^*)^F$. The rational Lusztig series $\mathcal{E}(\boldsymbol{G}^F, (t))$ associated to the $(\boldsymbol{G}^*)^F$-conjugacy class (t) of t is defined to be the set of irreducible characters of $\boldsymbol{G}^F$ occurring in some Deligne–Lusztig character $R_{\boldsymbol{T}}^{\boldsymbol{G}}\theta$, where $\boldsymbol{T}$ is an F-stable maximal torus of $\boldsymbol{G}$ and $\theta \in \mathrm{Irr}(\boldsymbol{T}^F)$ such that $(\boldsymbol{T}, \theta)$ corresponds in duality to a pair $(\boldsymbol{T}^*, s)$ with $s \in \boldsymbol{T}^* \cap (t)$. Here we recall from [Digne and Michel 1991, Proposition 13.13] that there is a one-to-one duality correspondence between $\boldsymbol{G}^F$-conjugacy classes of pairs $(\boldsymbol{T}, \theta)$, where $\boldsymbol{T}$ is an F-stable maximal torus of $\boldsymbol{G}$ and $\theta \in \mathrm{Irr}(\boldsymbol{T}^F)$, and the $(\boldsymbol{G}^*)^F$-conjugacy classes of pairs $(\boldsymbol{T}^*, s)$, where $\boldsymbol{T}^*$ is dual to $\boldsymbol{T}$ and $s \in (\boldsymbol{T}^*)^F$.

We continue to let t be a semisimple element of $(\boldsymbol{G}^*)^F$ and assume furthermore that $\boldsymbol{C}_{\boldsymbol{G}^*}(t)$ is a Levi subgroup of $\boldsymbol{G}^*$. Let $\boldsymbol{G}(t)$ be an F-stable Levi subgroup of $\boldsymbol{G}$ in duality with $\boldsymbol{C}_{\boldsymbol{G}^*}(t)$ and $\boldsymbol{P}$ be a parabolic subgroup of $\boldsymbol{G}$ for which $\boldsymbol{G}(t)$ is the Levi complement. The twisted induction $R^{\boldsymbol{G}}_{\boldsymbol{G}(t)\subseteq P}$ and the multiplication by $\hat{t}$, a certain linear character of $\mathrm{Irr}(\boldsymbol{G}(t)^F)$ naturally defined by t (see [Cabanes and Enguehard 2004, (8.19)]), then induce a bijection between the Lusztig series $\mathcal{E}(\boldsymbol{G}(t)^F, 1)$ and $\mathcal{E}(\boldsymbol{G}^F, (t))$; see [Cabanes and Enguehard 2004, Proposition 8.26 and Theorem 8.27]. In fact, for each $\lambda \in \mathcal{E}(\boldsymbol{G}(t)^F, 1)$, one has

$$\varepsilon_{\boldsymbol{G}}\varepsilon_{\boldsymbol{G}(t)} R^{\boldsymbol{G}}_{\boldsymbol{G}(t)\subseteq P}(\hat{t}\lambda) \in \mathcal{E}(\boldsymbol{G}^F, (t)),$$

where $\varepsilon_{\boldsymbol{G}} := (-1)^{\sigma(\boldsymbol{G})}$ with $\sigma(\boldsymbol{G})$ the $\mathbb{F}_q$-rank of $\boldsymbol{G}$. Taking λ to be trivial, we have the character

$$\chi_{(t)} := \varepsilon_{\boldsymbol{G}}\varepsilon_{\boldsymbol{G}(t)} R^{\boldsymbol{G}}_{\boldsymbol{G}(t)\subseteq P}(\hat{t}\mathbf{1}_{\boldsymbol{G}(t)^F}) \in \mathcal{E}(\boldsymbol{G}^F, (t)),$$

which is often referred to as a semisimple character of $\boldsymbol{G}^F$, of degree

$$\chi_{(t)}(1) = |(\boldsymbol{G}^*)^F : \boldsymbol{C}_{\boldsymbol{G}^{*F}}(t)|_{\ell'},$$

where ℓ is the defining characteristic of $\boldsymbol{G}$; see [Digne and Michel 1991, Theorem 13.23].

By [Cabanes and Enguehard 2004, Theorem 9.12], every element of $B_0(\boldsymbol{G}^F)$ lies in a Lusztig series $\mathcal{E}(\boldsymbol{G}^F, (t))$ where t is a p-element of $\boldsymbol{G}^{*F}$. Hence one might ask which such t indeed produce semisimple characters that contribute to the principal block. We will see in the following theorem that in a certain nice situation which is indeed enough for our purpose, the centralizer $\boldsymbol{C}_{\boldsymbol{G}^*}(t)$ is a Levi subgroup of $\boldsymbol{G}^*$, and thus the semisimple character $\chi_{(t)}$ associated to (t) is well-defined and belongs to $B_0(\boldsymbol{G}^F)$.

Recall that a prime p is good for $\boldsymbol{G}$ if it does not divide the coefficients of the highest root of the root system associated to $\boldsymbol{G}$. The following result, mainly due to Hiss [1990, Corollary 3.4] and Cabanes and Enguehard [2004, Theorem 21.13], will be very useful in later sections.

Theorem 5.1. *Let $(\boldsymbol{G}, F)$ be a connected reductive group defined over $\mathbb{F}_q$. Let p be a prime not dividing q. Let t be a p-element of $\boldsymbol{G}^{*F}$. If $\boldsymbol{C}_{\boldsymbol{G}^*}(t)$ is connected and p is good for $\boldsymbol{G}$, then the semisimple character $\chi_{(t)} \in \mathrm{Irr}(\boldsymbol{G}^F)$ belongs to the principal p-block of $\boldsymbol{G}^F$. Also, if $\boldsymbol{Z}(\boldsymbol{G})$ is connected, then $\chi_{(t)}$ belongs to the principal block of $\boldsymbol{G}^F$.*

We note that Theorem 5.1 can also be deduced from [Cabanes and Enguehard 1994, main theorem], but with more restricted conditions on p.

Building on Theorem 5.1, we observe that the principal block of S contains *many* irreducible semisimple characters. By controlling the length of $\mathrm{Aut}(S)$-orbits on these characters, we are able to bound below the number of $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}(B_0(S))$. The bound turns out to be enough to prove Theorem 1.2, at least in the case when the Sylow p-subgroups of the group of inner and diagonal automorphisms of S are abelian but non-cyclic, which is precisely the case we need after Sections 2A and 4.

5B. ***Specific setup for our purpose.*** For the rest of this section, we will work with the following setup: $\boldsymbol{G}$ is a simple algebraic group of adjoint type defined over $\mathbb{F}_q$ and F a Steinberg endomorphism on $\boldsymbol{G}$ such that $S = [\mathbb{G}, \mathbb{G}]$ with $\mathbb{G} = \boldsymbol{G}^F$. Let $(\boldsymbol{G}^*, F^*)$ be the dual pair of $(\boldsymbol{G}, F)$ and for simplicity we will use the same notation F for F^*, and thus $\boldsymbol{G}^*$ is a simple algebraic group of simply connected type and $S = \mathbb{G}^*/\boldsymbol{Z}(\mathbb{G}^*)$, where $\mathbb{G}^* := (\boldsymbol{G}^*)^F$.

Theorem 5.1 has the following consequence.

Lemma 5.2. *Assume the above notation. Let p be a prime not dividing q. For every p-element t of $\mathbb{G}^*$, the semisimple character $\chi_{(t)} \in \mathcal{E}(\mathbb{G}, (t))$ belongs to the principal block of $\mathbb{G}$.*

Proof. Since $\boldsymbol{G}^*$ has connected center, the lemma follows from Theorem 5.1; see also [Bessenrodt et al. 2007, Lemma 3.1]. □

5C. ***Orbits of semisimple characters.*** Knowing that the semisimple characters $\chi_{(t)} \in \mathrm{Irr}(\mathbb{G})$ associated to $\mathbb{G}^*$-conjugacy classes of p-elements all belong to $B_0(\mathbb{G})$, we now wish to control the number of orbits of the action of the automorphism group $\mathrm{Aut}(S)$ on these characters. By a result of Bonnafé [Navarro et al. 2008, Section 2] (see also [Taylor 2018, Section 7]), this action turns out to be well-behaved.

Let $\alpha \in \mathrm{Aut}(\mathbb{G})$, which in our situation will be a product of a field automorphism and a graph automorphism. It is easy to see that α then can be extended to a bijective morphism $\bar{\alpha} : \boldsymbol{G} \to \boldsymbol{G}$ such that $\bar{\alpha}$ commutes with F. This $\bar{\alpha}$ induces a bijective morphism $\bar{\alpha}^* : \boldsymbol{G}^* \to \boldsymbol{G}^*$ which commutes with the dual of F. The restriction of $\bar{\alpha}^*$ to $\mathbb{G}^*$, which we denote by α^*, is now an automorphism of $\mathbb{G}^*$. Recall that $\alpha \in \mathrm{Aut}(\mathbb{G})$ induces a natural action on $\mathrm{Irr}(\mathbb{G})$ by $\chi^\alpha := \chi \circ \alpha^{-1}$. By [Navarro et al. 2008, Section 2], α maps the Lusztig series $\mathcal{E}(\mathbb{G}, (t))$ of $\mathbb{G}$ associated to (t) to the series $\mathcal{E}(\mathbb{G}, (\alpha^*(t)))$ associated to $(\alpha^*(t))$. Consequently,

$$\chi_{(t)}{}^{\alpha} = \chi_{(\alpha^*(t))}, \tag{5.1}$$

which means that an automorphism of $\mathbb{G}$ maps the semisimple character associated to a conjugacy class (t) (of $\mathbb{G}^*$) to the semisimple character associated to $(\alpha^*(t))$. Here we note that if $\boldsymbol{C}_{\boldsymbol{G}^*}(t)$ is connected, then $\boldsymbol{C}_{\boldsymbol{G}^*}(\alpha^*(t))$ is also connected.

Due to Section 4 and Section 2A, we may assume that the Sylow p-subgroups of $\mathbb{G}$ are abelian. Assume for a moment that $\mathbb{G}$ is not of type ${}^2B_2$, ${}^2G_2$, or ${}^2F_4$. Then there is a unique positive integer e such that $p \mid \Phi_e(q)$ and Φ_e divides the generic order of $\mathbb{G}$. (Recall that Φ_e denotes the e-th cyclotomic polynomial.) This e then must be the multiplicative order of q modulo p, which means that $p \mid (q^e - 1)$ but $p \nmid (q^i - 1)$ for every $0 < i < e$. In the case where $\mathbb{G}$ is of type ${}^2B_2$, ${}^2G_2$, or ${}^2F_4$, what we just discussed still holds with slight modification on e, Φ_e, and $\boldsymbol{S}_e$; see [Malle 2007, Section 8] for more details.

Let $\Phi_e(q) = p^a m$ where $\gcd(p, m) = 1$ and $\Phi_e^{k_e}$ the precise power of Φ_e dividing the generic order of $\mathbb{G}$. We will use k for k_e for convenience if e is not specified. A Sylow e-torus of $\mathbb{G}^*$ has order $\Phi_e(q)^k$ and contains a Sylow p-subgroup of $\mathbb{G}^*$. Sylow p-subgroups of $\mathbb{G}^*$ (and $\mathbb{G}$) are then isomorphic to

$$\underbrace{C_{p^a} \times C_{p^a} \times \cdots \times C_{p^a}}_{k \text{ times}};$$

see [Malle and Testerman 2011, Theorem 25.14].

Assume that ℓ is the defining characteristic of S.

Lemma 5.3. *Assume the above notation. Let α be a field automorphism of $\mathbb{G}$. Each α-orbit on semisimple characters $\chi_{(t)} \in \mathrm{Irr}(\mathbb{G})$ associated to conjugacy classes of p-elements ($p \neq \ell$) of $\mathbb{G}^*$ has length at most $p^a - p^{a-1}$.*

Proof. Let α^* be an automorphism of $\mathbb{G}^*$ constructed from α by the process described above. For simplicity we use α for α^*. By (5.1), it is enough to show that each α-orbit on $\mathbb{G}^*$-conjugacy classes of (semisimple) p-elements of $\mathbb{G}^*$ has length at most $p^a - p^{a-1}$.

Let $t \in \mathbb{G}^*$ be a p-element. Note that each element in $\mathbb{G}^*$ conjugate to t under $\boldsymbol{G}^*$ is automatically conjugate to t under $\mathbb{G}^*$, by [Digne and Michel 1991, (3.25)] and the fact that $\boldsymbol{C}_{\boldsymbol{G}^*}(t)$ is connected. Let t be conjugate to $h_{\alpha_1}(\lambda_1) \cdots h_{\alpha_n}(\lambda_n)$, where the h_{α_i} are the coroots corresponding to a set of fundamental roots with respect to a maximal torus $\boldsymbol{T}^*$ of $\boldsymbol{G}^*$ and n is the rank of $\boldsymbol{G}^*$. Since $\boldsymbol{G}^*$ is simply connected, note that $(t_1, \dots, t_n) \mapsto h_{\alpha_1}(t_1) \cdots h_{\alpha_n}(t_n)$ is an isomorphism from $(\bar{\mathbb{F}}_q^\times)^n$ to $\boldsymbol{T}^*$; see [Gorenstein et al. 1994, Theorem 1.12.5].

Now, if $\lambda = \lambda_i$ for some $1 \leq i \leq n$, then $\lambda^{p^a} = 1$, since $|t| \mid p^a$. Recall that $\ell \neq p$, and thus $\ell^{p^a - p^{a-1}} \equiv 1 \pmod{p^a}$ by Euler's totient theorem. It follows that $\lambda^{\ell^{p^a - p^{a-1}}} = \lambda$, which yields that the α-orbit on (t) is contained in $\{(t), (\alpha(t)), \dots, (\alpha^{p^a - p^{a-1} - 1}(t))\}$, as desired. $\square$

5D. ***A bound for the number of* $\mathbf{Aut}(S)$*-orbits on* $\mathbf{Irr}(B_0(S))$.** Let $\boldsymbol{S}_e$ be a Sylow e-torus of $\boldsymbol{G}^*$ and let $P \subseteq \boldsymbol{S}_e$ be Sylow p-subgroup of $\mathbb{G}^*$. Such P exists by [Malle and Testerman 2011, Theorem 25.14]. Let $W(\boldsymbol{L}_e)$ denote the relative Weyl group of the centralizer $\boldsymbol{L}_e := \boldsymbol{C}_{\boldsymbol{G}^*}(\boldsymbol{S}_e)$ of $\boldsymbol{S}_e$ in $\boldsymbol{G}^*$. Here we note that $\boldsymbol{L}_e$ is a minimal e-split Levi subgroup of $\boldsymbol{G}^*$ and $W(\boldsymbol{L}_e) \cong \boldsymbol{N}_{\mathbb{G}^*}(\boldsymbol{S}_e)/\boldsymbol{C}_{\mathbb{G}^*}(\boldsymbol{S}_e)$. By [Malle 2007, Proposition 5.11], $\boldsymbol{N}_{\mathbb{G}^*}(\boldsymbol{S}_e)$ controls $\mathbb{G}^*$-fusion in $\boldsymbol{C}_{\mathbb{G}^*}(\boldsymbol{S}_e)$, and since $P \subseteq \boldsymbol{S}_e$, the number of conjugacy classes of (nontrivial) p-elements of $\mathbb{G}^*$ is at least

$$\frac{|P|-1}{|W(\boldsymbol{L}_e)|} = \frac{p^{ak}-1}{|W(\boldsymbol{L}_e)|}.$$

Note that $\chi_{(t)}$ belongs to the Lusztig series $\mathcal{E}(\mathbb{G}, (t))$ defined by the conjugacy class (t) and the Lusztig series are disjoint, and so two semisimple characters $\chi_{(t)}$ and $\chi_{(t_1)}$ are equal if and only if t and t_1 are conjugate in $\mathbb{G}^*$. Also, since $\mathbb{G}^*$ has abelian Sylow p-subgroup, p is a good prime for $\boldsymbol{G}$, by [Malle 2014, Lemma 2.1]. Therefore, using Lemma 5.2, we deduce that

$$|\mathrm{Irr}_{ss}(B_0(\mathbb{G}))| \geq \frac{p^{ak}-1}{|W(\boldsymbol{L}_e)|}, \tag{5.2}$$

where $\mathrm{Irr}_{ss}(B_0(\mathbb{G}))$ denotes the set of (nontrivial) semisimple characters (associated to p-elements of $\mathbb{G}^*$) in $B_0(\mathbb{G})$. Let $n(X, Y)$ denote the number of X-orbits on a set Y. Using Lemma 5.3, we then have

$$n(\mathrm{Aut}(S), \mathrm{Irr}_{ss}(B_0(\mathbb{G}))) \geq \frac{p^{ak}-1}{g(p^a - p^{a-1})|W(\boldsymbol{L}_e)|} \geq \frac{p^k-1}{g(p-1)|W(\boldsymbol{L}_e)|},$$

where g is the order of the group of graph automorphisms of S. Let $d := |\mathbb{G}/S|$ — the order of the group of diagonal automorphisms of S and viewing the irreducible constituents of the restrictions of semisimple characters of $\mathbb{G}$ to S as semisimple characters of S, we now have

$$n(\mathrm{Aut}(S), \mathrm{Irr}_{ss}(B_0(S))) \geq \frac{p^k-1}{dg(p-1)|W(\boldsymbol{L}_e)|}. \tag{5.3}$$

We note that values of d, f, and g for various families of simple groups are known; see [Conway et al. 1985, page xvi] for instance.

We now turn to unipotent characters in the principal block $B_0(S)$. Broué, Malle and Michel [1993, Theorem 3.2] partitioned the set $\mathcal{E}(\mathbb{G}^*, 1)$ of unipotent characters of $\mathbb{G}^*$ into e-Harish-Chandra series associated to e-cuspidal pairs of $\boldsymbol{G}^*$, and furthermore obtained one-to-one correspondences between e-Harish-Chandra series and the irreducible characters of the relative Weyl groups of the e-cuspidal pairs defining these series. Broué, Malle and Michel [1993, Theorem 5.24] then show that, when the Sylow p-subgroups of $\mathbb{G}^*$ is abelian, the partition of unipotent characters of $\mathbb{G}^*$ by e-Harish-Chandra series is compatible with the partition of unipotent characters by unipotent blocks; see [Cabanes and Enguehard 2004, Theorem 21.7] for a more general result. These results imply that the number of unipotent characters in $B_0(S)$ (and $B_0(\mathbb{G}^*)$ as well) is the same as the number $k(W(\boldsymbol{L}_e))$ of conjugacy classes of the relative Weyl group $W(\boldsymbol{L}_e)$ with $\boldsymbol{L}_e := \boldsymbol{C}_{\boldsymbol{G}^*}(\boldsymbol{S}_e)$, where $\boldsymbol{S}_e$ is a Sylow e-torus of $\boldsymbol{G}^*$, as mentioned above.

By the aforementioned result of Lusztig (see [Malle 2008, Theorem 2.5] and also [Malle 2007, Theorem 3.9] for the corrected version), every unipotent character of a simple group of Lie type lies in an $\mathrm{Aut}(S)$-orbit of length at most 3. In fact, every unipotent character of S is $\mathrm{Aut}(S)$-invariant, except in the following cases:

(1) $S = P\Omega_{2n}^{+}(q)$ (n even), the graph automorphism of order 2 has $o_2(S)$ orbits of length 2, where $o_2(S)$ is the number of degenerate symbols of defect 0 and rank n parametrizing unipotent characters of S; see [Carter 1985, page 471].

(2) $S = P\Omega_8^{+}(q)$, the graph automorphism of order 3 has $o_3(S) = 2$ orbits of length 3, each of which contains one pair of characters parametrized by one degenerate symbol of defect 0 and rank 2 in (1).

(3) $S = \mathrm{Sp}_4(2^f)$, the graph automorphism of order 2 has $o_2(S) = 1$ orbit of length 2.

(4) $S = G_2(3^f)$, the graph automorphism of order 2 has $o_2(S) = 1$ orbit of length 2 on unipotent characters.

(5) $S = F_4(2^f)$, the graph automorphism of order 2 has $o_2(S) = 8$ orbits of length 2 on unipotent characters.

Combining this with the bound (5.3), we obtain:

Theorem 5.4. *Let S be a simple group of Lie type (including Suzuki and Ree groups). Let p be a prime different from the defining characteristic of S. Assume that Sylow p-subgroups of the group of inner and diagonal automorphisms of S are abelian. Let k, d, f, g, and $\boldsymbol{L}_e$ be as above and let $n(S)$ denote the number of* $\mathrm{Aut}(S)$*-orbits on irreducible ordinary characters in $B_0(S)$. Then*

$$n(S) \geq k(W(\boldsymbol{L}_e)) + \frac{p^k - 1}{dg(p-1)|W(\boldsymbol{L}_e)|},$$

except possibly the above cases (1), (3), (4), *and* (5) *in which the bound is lower by the number $o_2(S)$ of orbits of length* 2 *on unipotent characters and case* (2) *in which the bound is lower by* 4.

We remark that when the Sylow p-subgroups of the group of inner and diagonal automorphisms of S are furthermore noncyclic, then $k \geq 2$, and, away from those exceptions, we have a rougher bound

$$n(S) \geq k(W(\boldsymbol{L}_e)) + \frac{p+1}{dg|W(\boldsymbol{L}_e)|}, \tag{5.4}$$

but this turns out to be sufficient for our purpose in most cases.

6. Linear and unitary Groups

In this section, we let $S = \mathrm{PSL}_n^{\epsilon}(q)$, where $p \nmid q$ and $\epsilon \in \{\pm 1\}$. Here $\mathrm{PSL}_n^{\epsilon}(q) := \mathrm{PSL}_n(q)$ in the case $\epsilon = 1$ and $\mathrm{PSU}_n(q)$ in the case $\epsilon = -1$, and analogous for $\mathrm{SL}_n^{\epsilon}(q)$, $\mathrm{GL}_n^{\epsilon}(q)$, and $\mathrm{PGL}_n^{\epsilon}(q)$. We further let $\bar{q} := q$ if $\epsilon = 1$ and $\bar{q} := q^2$ if $\epsilon = -1$. Note that with our notation, $\mathrm{SL}_n^{\epsilon}(q)$ and $\mathrm{GL}_n^{\epsilon}(q)$ are naturally subgroups of $\mathrm{SL}_n(\bar{q})$ and $\mathrm{GL}_n(\bar{q})$, respectively.

We are now ready to prove Theorem 1.2 in the case of linear and unitary groups. Since $k(B_0(A))$ is bounded below by the number of A-orbits on $\mathrm{Irr}(B_0(S))$ for any $S \le A \le \mathrm{Aut}(S)$, our strategy in most cases will be to prove that there are more than $2\sqrt{p-1}$ orbits under $\mathrm{Aut}(S)$ in $\mathrm{Irr}(B_0(S))$, thus proving parts (i) and (ii) of Theorem 1.2 simultaneously.

Proposition 6.1. *Let $S = \mathrm{PSL}_n^\epsilon(q)$ and let $p \nmid q$ be a prime. Then Theorem 1.2 holds for any almost simple group A with socle S and $p \nmid |A/S|$.*

Proof. With the results of the previous sections, we may assume $n \ge 3$ and $p \ge 11$.

Write $S = \mathrm{PSL}_n^\epsilon(q)$, $\mathbb{G} = \mathrm{PGL}_n^\epsilon(q)$, $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$, and $G = \mathrm{SL}_n^\epsilon(q)$. Then we have $G = [\widetilde{G}, \widetilde{G}]$, $S = G/\boldsymbol{Z}(G)$, and $\mathbb{G} = \widetilde{G}/\boldsymbol{Z}(\widetilde{G})$. From Section 4, we may assume that Sylow p-subgroups of $\mathbb{G}$ are abelian, which implies that there is a unique e such that $p \mid \Phi_e(q)$ and Φ_e divides the generic order polynomial of $\mathbb{G}$. Here e must be $e_p(q)$, the multiplicative order of q modulo p. Note that this also forces $p \nmid n$ by again appealing to [Malle and Testerman 2011, Lemma 25.13].

We will further define $\bar{e} := e_p(\bar{q})$ and e' as follows:

$$e' := \begin{cases} \bar{e} & \text{if } \epsilon = 1 \text{ or if } \epsilon = -1 \text{ and } p \mid q^{\bar{e}} - (-1)^{\bar{e}}, \\ 2\bar{e} & \text{if } \epsilon = -1 \text{ and } p \mid q^{\bar{e}} + (-1)^{\bar{e}}. \end{cases}$$

To prove Theorem 1.2, our aim is to show that when a Sylow p-subgroup of S is not cyclic, then the number of $\mathrm{Aut}(S)$-orbits on $\mathrm{Irr}(B_0(S))$ is larger than $2\sqrt{p-1}$.

Note that since $p \nmid \gcd(n, q-\epsilon) = |\boldsymbol{Z}(G)|$, the irreducible characters in the principal block of S are the same as those of G, under inflation; see [Navarro 1998, Theorem 9.9]. Similarly, if $e' > 1$, then $p \nmid (q-\epsilon) = |\boldsymbol{Z}(\widetilde{G})|$ and an analogous statement holds for $\mathbb{G}$ and $\widetilde{G}$. Hence, we begin by studying $B_0(\widetilde{G})$, which will be sufficient for our purposes in the case $e' > 1$.

Let $n = we' + m$ with $0 \le m < e'$. Set $p^a := (\bar{q}^{\bar{e}} - 1)_p \ge p$. The case $p \le w$ was treated in Section 4, so we assume that $p > w$. Note that by [Michler and Olsson 1983, Theorem 1.9], $B_0(\widetilde{G})$ and $B_0(\mathrm{GL}_{we'}^\epsilon(q))$ have the same number of ordinary irreducible characters, so we may assume that $n = we'$. (Note that the action of $\mathrm{Aut}(S)$ is analogous as well.)

Let $\mathcal{F}(p, a)$ denote the set of monic polynomials over $\mathbb{F}_{\bar{q}}$ in the set $\mathscr{F}$ defined in [Fong and Srinivasan 1982] whose roots have p-power order in $\overline{\mathbb{F}}_q^\times$ at most p^a. Note that $|\mathcal{F}(p, a)| = 1 + (p^a - 1)/e'$; see [Michler and Olsson 1983, page 211].

The conjugacy classes $(t) := t^{\widetilde{G}}$ of p-elements in $\widetilde{G}$ are parametrized by p-weight vectors of w, which are functions $\boldsymbol{w} := \boldsymbol{w}_{(t)} : \mathcal{F}(p, a) \to \mathbb{Z}_{\ge 0}$ such that $w = \sum_{g \in \mathcal{F}(p,a)} \boldsymbol{w}(g)$. The characteristic polynomial of elements in (t) is

$$(x-1)^{e'\boldsymbol{w}(x-1)} \prod_{x-1 \ne g \in \mathcal{F}(p,a)} g^{\boldsymbol{w}(g)},$$

and the centralizer of t is

$$\boldsymbol{C}_{\widetilde{G}}(t) = \mathrm{GL}^\epsilon_{e'\boldsymbol{w}(x-1)}(q) \times \prod_{x-1 \ne g \in \mathcal{F}(p,a)} \mathrm{GL}^\eta_{\boldsymbol{w}(g)}(q^{e'})$$

where $\eta = \epsilon$ unless $\epsilon = -1$ and $e' = 2\bar{e}$, in which case $\eta = 1$.

Each character in the Lusztig series $\mathcal{E}(\widetilde{G}, t)$ is labeled by $\chi_{t,\psi}$ where ψ is a unipotent character of $\boldsymbol{C}_{\widetilde{G}}(t)$. So $\psi = \prod_{g\in\mathcal{F}(p,a)} \psi_g$ where ψ_g is a unipotent character of $\mathrm{GL}^{\eta}_{\boldsymbol{w}(g)}(q^{e'})$ if $g \neq x-1$ and of $\mathrm{GL}_{e'\boldsymbol{w}(x-1)}(q)$ if $g = x-1$. Note that there is a canonical correspondence between unipotent characters of $\mathrm{GL}^{\pm}_x(q)$ and partitions of x, so we may view ψ_g as a partition of $\boldsymbol{w}(g)$ when $g \neq x-1$ and of $e'\boldsymbol{w}(x-1)$ when $g = x-1$. Further, by [Fong and Srinivasan 1982, Theorem (7A)], the characters of $B_0(\widetilde{G})$ are exactly those $\chi_{t,\psi}$ satisfying t is a p-element and the partition ψ_{x-1} has trivial e'-core.

By [Olsson 1984, Proposition 6], since $w < p$, we have

$$k(B_0(\widetilde{G})) = k\left(e' + \frac{p^a-1}{e'}, w\right),$$

where $k(x, y)$ is as defined in Section 3 above. This number is at least

$$e' + \frac{p^a-1}{e'} \geq 2\sqrt{p^a-1} \geq 2\sqrt{p-1}. \tag{6.1}$$

But, recall that we wish to show that there are at least $2\sqrt{p-1}$ orbits on $\mathrm{Irr}(B_0(S))$ under $\mathrm{Aut}(S)$.

Now, by taking $t = 1$, the number of unipotent characters in $B_0(\widetilde{G})$ is precisely $k(e', w)$. Note that $k(e', w) \geq k(e', 1) = e'$, and that further $k(e', w) \geq 2e'$ if $w \geq 2$ with strict inequality for $(e', w) \neq (1, 2)$, and each unipotent character is $\mathrm{Aut}(S)$-invariant. So we have at least e' $\mathrm{Aut}(S)$-orbits of unipotent characters in $B_0(\mathbb{G})$, and hence of $B_0(S)$, since restriction yields a bijection between unipotent characters of S and $\mathbb{G}$.

Let $\widetilde{\boldsymbol{G}} := \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ so that $\widetilde{G} = \widetilde{\boldsymbol{G}}^F$. Since $\boldsymbol{Z}(\widetilde{\boldsymbol{G}})$ is connected, [Cabanes and Späth 2013, Theorem 3.1] yields that the "Jordan decomposition" $\psi_{t,\psi} \leftrightarrow (t, \psi)$ can be chosen to be $\mathrm{Aut}(S)$-equivariant. Since ψ is a unipotent character of a product of groups of the form $\mathrm{GL}^{\pm}_x(q^d)$, which are invariant under automorphisms as discussed above, it follows that the orbit of $\chi_{t,\psi}$ is completely determined by the action of $\mathrm{Aut}(S)$ on the class (t).

Now, recall that the $\widetilde{G}$-class of t is completely determined by its eigenvalues. Let $|t| = p^c$ and note that $c \leq a$. By viewing t as an element $1 \times \prod_{x-1\neq g\in\mathcal{F}(p,a)} \zeta_g$ of

$$\boldsymbol{Z}(\boldsymbol{C}_{\widetilde{G}}(t)) \cong C_{q-\epsilon} \times \prod_{x-1\neq g\in\mathcal{F}(p,a)} C_{q^{e'}-\eta}$$

we see that for $\alpha \in \mathrm{Aut}(S)$, the eigenvalues of t^α are those of t raised to some power ηq_0^e for some $\eta \in \{\pm 1\}$ and some q_0 such that $\bar{q}$ is a power of q_0. This implies that the $\mathrm{Aut}(S)$-orbit of (t) has size at most $(p^c-1)/e' \leq (p^a-1)/e'$.

Now, a Sylow p-subgroup P of $\widetilde{G}$ is of the form $C^w_{p^a} \leq (\mathbb{F}^\times_{\bar{q}^{\bar{e}}})^w$. Then if $w = 1$, P is cyclic, and hence we may assume that $w \geq 2$. In this case, we have at least $\frac{1}{2}((p^a-1)/e')^2$ choices for $(t) \neq (1)$, and hence at least $\frac{1}{2}((p^a-1)/e')^2$ nonunipotent characters in $B_0(\widetilde{G})$ by taking ψ_{x-1} to be trivial. This gives at least $(p^a-1)/2e'$ distinct orbits of nonunipotent characters, and hence more than $2\sqrt{p-1}$ orbits of characters in $B_0(\mathbb{G})$ under $\mathrm{Aut}(S)$ when $e' > 1$, by (6.1) with $2e'$ rather than e'. This completes the proof of Theorem 1.2 for S in the case $e' > 1$ by the discussion at the beginning of the section.

Finally suppose $e' = 1$, so $w = n \geq 3$ and we may continue to assume $p > w$. Consider the elements t of $\widetilde{G}$ whose eigenvalues are of the form $\{\zeta, \xi, (\zeta\xi)^{-1}, 1, \dots, 1\}$ with ζ and ξ p-elements of $C_{q-\epsilon} \leq \mathbb{F}_{\bar{q}}^{\times}$. Note that each member of $\mathcal{E}(\widetilde{G}, t)$ lies in the principal block of $\widetilde{G}$, using [Cabanes and Enguehard 2004, Theorem 9.12] and that every unipotent character lies in $B_0(\widetilde{G})$ since $e' = 1$. Further, t lies in $G = [\widetilde{G}, \widetilde{G}]$ and $|\boldsymbol{C}_{\boldsymbol{G}^*}(t)/\boldsymbol{C}^{\circ}_{\boldsymbol{G}^*}(t)| = 1$ since this number must divide both the order of t and $|\boldsymbol{Z}(G)|$, contradicting $p > n$. Hence each character in such a $\mathcal{E}(\widetilde{G}, t)$ is irreducible on restriction to G, yielding at least $(p^a - 1)^2/2$ nonunipotent members of $B_0(G)$. Since the $\mathrm{Aut}(S)$-orbits of such characters are again of size at most $p^a - 1$, this yields at least $2 + (p^a - 1)/2$ distinct orbits, which is larger than $2\sqrt{p-1}$. This completes the proof of Theorem 1.2 in the case that $S = \mathrm{PSL}_n^{\epsilon}(q)$. □

7. Symplectic and orthogonal Groups

In this section, we consider the simple groups coming from orthogonal and symplectic groups. That is, simple groups of Lie type B_n, C_n, D_n, and ${}^2D_n$. We let $\epsilon \in \{\pm\}$, and let $\mathrm{P}\Omega_{2n}^{\epsilon}(q)$ denote the simple group of Lie type $D_n(q)$ for $\epsilon = +$ and of type ${}^2D_n(q)$ for $\epsilon = -$.

Proposition 7.1. *Let q be a power of a prime different from p and let $S = \mathrm{PSp}_{2n}(q)$ with $n \geq 2$, $\mathrm{P}\Omega_{2n+1}(q)$ with $n \geq 3$, or $\mathrm{P}\Omega_{2n}^{\epsilon}(q)$ with $n \geq 4$. Then Theorem 1.2 holds for any almost simple group A with socle S and $p \nmid |A/S|$.*

Proof. With the results of the previous sections, we may again assume that $p \geq 11$ and that a Sylow p-subgroup of S is abelian, but not cyclic.

Let H be the corresponding symplectic or special orthogonal group $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$, or $\mathrm{SO}_{2n}^{\epsilon}(q)$ and let $(\boldsymbol{H}, F)$ be the corresponding simple algebraic group and Frobenius endomorphism so that $H = \boldsymbol{H}^F$. Let $G = \boldsymbol{G}^F$ be the corresponding group of simply connected type, so that $G = H$ in the symplectic case or G is the appropriate spin group in the orthogonal cases. Further, let $(\boldsymbol{H}^*, F)$ and $(\boldsymbol{G}^*, F)$ be dual to $(\boldsymbol{H}, F)$ and $(\boldsymbol{G}, F)$, respectively, and $H^* = \boldsymbol{H}^{*F}$ and $G^* = \boldsymbol{G}^{*F}$.

Define $\overline{H}$ to be the group $\mathrm{GO}_{2n}^{\epsilon}(q)$ in the case $S = \mathrm{P}\Omega_{2n}^{\epsilon}(q)$, and $\overline{H} := H$ otherwise. We also let Ω be the unique subgroup of index 2 in H for the orthogonal cases when q is odd, and let $\Omega = H$ otherwise, so that $\Omega/\boldsymbol{Z}(\Omega) = S = G/\boldsymbol{Z}(G)$ and $\Omega \lhd \overline{H}$. Note that since $p \neq 2$, $B_0(S)$ can be identified with $B_0(\Omega)$ or with $B_0(G)$, by [Navarro 1998, Theorem 9.9].

Now, let $e := e_p(q)/\gcd(e_p(q), 2)$ and write $n = we + m$ with $0 \leq m < e$. From Section 4, we may again assume $w < p$. To obtain our result, we will rely on the case of linear groups and use some of the ideas of the arguments used in [Malle 2018, Propositions 5.4 and 5.5], which provides an analogue in this situation to the results of Michler and Olsson discussed above. Namely, [Malle 2018, Propositions 5.4 and 5.5] tells us

$$k(B_0(\overline{H})) = k\left(2e + \frac{p^a - 1}{2e}, w\right),$$

where $p^a = (q^{2e} - 1)_p$. Note that again, this number is at least $2\sqrt{p-1}$ (with strict inequality when $w \geq 2$), but that we wish to show the inequality for $k(B_0(A))$. In most cases, we will again show that the number of $\mathrm{Aut}(S)$-orbits of characters in $B_0(S)$ is at least $2\sqrt{p-1}$.

If $w = 1$, a Sylow p-subgroup of Ω, G, H, or $\overline{H}$ (recall $p \geq 11$) is cyclic, so we may assume by Lemma 2.1 that $w \geq 2$. Note that the unipotent characters of H are irreducible on restriction to Ω. Assume first that $S \neq D_4(q)$ nor $\mathrm{Sp}_4(2^f)$. By [Malle 2018, Discussion before Propositions 5.4 and 5.5], the number of unipotent characters in $B_0(\overline{H})$ is $k(2e, w)$. If further $S \neq \mathrm{P}\Omega^{\epsilon}_{2n}(q)$, then since all unipotent characters are $\mathrm{Aut}(S)$-invariant, this yields $k(2e, w)$ $\mathrm{Aut}(S)$-orbits of unipotent characters in $B_0(H)$, and hence $B_0(S)$. Note that $k(2e, w) > 4e$ since $w \geq 2$. If $S = \mathrm{P}\Omega^{\epsilon}_{2n}(q)$, then note that $n \geq 4$ forces $e \geq 2$ if $w = 2$. Now, in this case, the proof of [Malle 2018, Lemma 5.6 and Corollary 5.7] yields that the number of $\overline{H}$-orbits of unipotent characters in $B_0(H)$ is at least $k(2e, w)/2$, and that this number is $(k(2e, w) + k(e, w/2))/2$ if w is even. Now, if $w \geq 3$, we have $k(2e, w)/2 > 4e$. If $w = 2$ and $e \geq 2$, we have

$$\frac{k(2e, 2) + k(e, 1)}{2} = e^2 + 2e \geq 4e.$$

Hence in all cases, the number of $\mathrm{Aut}(S)$-orbits of unipotent characters in $B_0(S)$ is at least $4e$, and is strictly greater unless $e = 2 = w$ in the case $S = \mathrm{P}\Omega^{\epsilon}_{2n}(q)$.

The characters in $B_0(H)$ and $B_0(G)$ lie in Lusztig series indexed by p-elements t of H^*, respectively G^*; by [Cabanes and Enguehard 2004, Theorem 9.12]. Note that centralizers of odd-order elements of $\boldsymbol{H}^*$ and of $\boldsymbol{G}^*$ are always connected (see, e.g., [Malle and Testerman 2011, Exercise (20.16)]) and that every odd p is good for $\boldsymbol{H}$ and $\boldsymbol{G}$, so that $\chi_{(t)}$ lies in $B_0(H)$, respectively $B_0(G)$, for every p-element t of H^*, respectively G^*, by Theorem 5.1. Further, note that the action on $\chi_{(t)}$ under a graph-field automorphism of H is determined by the action of a corresponding graph-field automorphism on (t), by [Navarro et al. 2008, Corollary 2.8]; see also (5.1) above.

Now let $\boldsymbol{G} \hookrightarrow \widetilde{\boldsymbol{G}}$ be a regular embedding as in [Cabanes and Enguehard 2004, 15.1] and let $\widetilde{G} := \widetilde{\boldsymbol{G}}^F$. Then the action of $\widetilde{G}$ on G induces all diagonal automorphisms of S. Now, since $C_{\boldsymbol{G}^*}(t)$ is connected for any p-element $t \in G^*$, we have every character in $\mathcal{E}(G, (t))$ extends to a character in $\widetilde{G}$. (Indeed, since $\widetilde{G}/G$ is abelian and restrictions from $\widetilde{G}$ to G are multiplicity-free, the number of characters lying below a given $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{G})$ is the number of $\beta \in \mathrm{Irr}(\widetilde{G}/G)$ such that $\widetilde{\chi}\beta = \widetilde{\chi}$, as noted in [Rizo et al. 2021, Lemma 1.4]. Hence [Bonnafé 2005, Corollary 2.8] and [Schaeffer Fry and Taylor 2023, Proposition 2.6] yields the claim.) Therefore, each member of $B_0(S)$ is invariant under diagonal automorphisms.

First consider the case $H = \mathrm{SO}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$, so $H^* = \mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$, respectively. Note that $\mathrm{Aut}(S)/S$ in this case is generated by field automorphisms, which also act on H, along with a diagonal or graph automorphism of order at most 2.

If $H = \mathrm{SO}_{2n+1}(q)$, then $\mathrm{GL}_n(q)$ may be embedded into $H^* = \mathrm{Sp}_{2n}(q)$ in a natural way (namely, block diagonally as the set of matrices of the form (A, A^{-T}) for $A \in \mathrm{GL}_n(q)$), and the conjugacy class of t is again determined by its eigenvalues. Arguing as in the case of $\mathrm{SL}_n(q)$ above and noting that every eigenvalue of t must have the same multiplicity as its inverse, we then have at least $(p^a - 1)/4e$ distinct orbits of nonunipotent characters in $B_0(H)$ under the field automorphisms, and hence at least $(p^a - 1)/4e$ orbits in $B_0(S)$ under $\mathrm{Aut}(S)$. This gives more than $4e + (p^a - 1)/4e$ orbits in $\mathrm{Irr}(B_0(S))$ under $\mathrm{Aut}(S)$, which proves Theorem 1.2 in this case using (6.1).

If $H = \mathrm{Sp}_{2n}(q)$, by [Geck and Hiss 1991, Theorem 4.2], there is a bijection between classes of p-elements of H and H^*, and we note that field automorphisms act analogously on the p-elements of H and H^*. Then the above again yields the result in this case as long as $S \neq \mathrm{Sp}_4(2^f)$.

If $S = \mathrm{Sp}_4(2^f)$, then we must have $e = 1$ and $w = 2$. Here [Malle 2008, Theorem 2.5] tells us that there is a pair of unipotent characters permuted by the exceptional graph automorphism, leaving $k(2, 2) - 1 = 4$ orbits of unipotent characters in $B_0(S)$ under $\mathrm{Aut}(S)$. In this case, arguing as before and considering the action of the graph automorphism gives at least $4 + (p^a - 1)/8$ orbits in $B_0(S)$ under $\mathrm{Aut}(S)$, which is at least $2(p-1)^{1/4}$. Hence part (ii) of Theorem 1.2 holds. So let $S \leq A \leq \mathrm{Aut}(S)$, and we wish to show that $B_0(A)$ contains more than $2\sqrt{p-1}$ characters. Note that in this case, $\mathrm{Aut}(S)/S$ is cyclic. Let $X := SC_A(P)$ for $P \in \mathrm{Syl}_p(S)$. Then A/X is cyclic, say of size b, and $B_0(A)$ is the unique block covering $B_0(X)$ by [Navarro 1998, (9.19) and (9.20)]. Note that since at least 3 of the unipotent characters of S are A-invariant, we have at least $3b$ characters in $B_0(A)$ lying above unipotent characters. Further, since the automorphisms corresponding to those in X stabilize p-classes in G^*, the arguments above give at least $\frac{1}{2} \cdot ((p^a - 1)/2)^2$ members of $B_0(X)$ lying above semisimple characters of S, and hence there are at least $(p^a - 1)^2/8b$ members of $B_0(A)$ lying above semisimple characters of S. Note then that the size of $B_0(A)$ is at least $3b + (p^a - 1)^2/8b$, which is larger than $2\sqrt{p-1}$, completing the proof in this case.

Now, suppose we are in the case that $\overline{H} = \mathrm{GO}^{\epsilon}_{2n}(q)$. Note that the action of $\overline{H}/H$ induces a graph automorphism of order 2 in the case $\epsilon = 1$, and that $\mathrm{Aut}(S)/S$ is generated by a group of diagonal automorphisms of size at most 4, along with graph and field automorphisms. Further, note that the action of H on Ω induces a diagonal automorphism of order 2 on S. We may embed $\overline{H}$ in $\mathrm{SO}_{2n+1}(q)$, and by [Malle 2018, proof of Proposition 5.5], the classes of p-elements t with Lusztig series contributing to $B_0(\overline{H})$ are parametrized exactly as in the case of $\mathrm{SO}_{2n+1}(q)$ above.

Assume that $(n, \epsilon) \neq (4, 1)$. By again considering semisimple characters $\chi_{(t)}$ of H for p-elements $t \in H^*$, we may conclude that the number of orbits of nonunipotent characters in $B_0(S)$ under $\mathrm{Aut}(S)$ is at least $(p^a - 1)/(4e)$. This yields at least $4e + (p^a - 1)/(4e)$ orbits in $\mathrm{Irr}(B_0(S))$ under $\mathrm{Aut}(S)$, with strict inequality unless $e = 2$. Hence we have the number of $\mathrm{Aut}(S)$ orbits in $B_0(S)$ is strictly larger than $4e + (p^a - 1)/(4e)$, completing Theorem 1.2 again in this case using (6.1), unless possibly if $e = 2$. But in the latter situation, we have $8 + (p^a - 1)/8 > 2\sqrt{p^a - 1}$ unless $8 = p^a - 1$, contradicting $p \geq 11$ and we are again done.

Finally, suppose $S = D_4(q) = \mathrm{P}\Omega^+_8(q)$ so $\overline{H} = \mathrm{GO}^+_8(q)$. In this case, the graph automorphisms generate a group of size 6, and a triality graph automorphism of order 3 permutes two triples of unipotent characters; see [Malle 2008, Theorem 2.5]. Since $w \geq 2$, we have $(e, w) \in \{(1, 4), (2, 2)\}$. The arguments above give at least $\big(\big(k(2e, w) + k(e, w/2)\big)/2\big) - 4 + (p^a - 1)/12e$ distinct $\mathrm{Aut}(S)$-orbits in $\mathrm{Irr}(B_0(S))$. Since $k(2, 4) = 20$, $k(1, 2) = 2 = k(2, 1)$, and $k(4, 2) = 14$, we have

$$\frac{k(2e, w) + k\big(e, \frac{w}{2}\big)}{2} - 4 + \frac{p^a - 1}{12e} > 2(p-1)^{1/4},$$

so Theorem 1.2(ii) is proved in this case.

Now, let $S \leq A \leq \mathrm{Aut}(S)$, let Γ be the subgroup of $\mathrm{Aut}(S)$ generated by inner, diagonal, and graph automorphisms, and let $X := (\Gamma \cap A)C_A(P)$. Then A/X is cyclic, and by [Navarro 1998, (9.19) and (9.20)], $B_0(A)$ is the unique block covering $B_0(X)$. Let $b := |A/X|$. Now, the arguments above give at least $\frac{1}{3} \cdot \frac{1}{2} \cdot ((p-1)/4)^2$ members of $B_0(X)$ lying above semisimple characters of S, since members of $C_A(P)$ correspond to automorphisms stabilizing classes of p-elements of G^*, and hence there are at least $(p-1)^2/(96b)$ members of $B_0(A)$ lying above semisimple characters of S. Further, there are at least 10 characters in $B_0(X)$ lying above unipotent characters in $B_0(S)$. Since unipotent characters extend to their inertia groups and are invariant under field automorphisms (see [Malle 2008, Theorems 2.4 and 2.5]), this gives at least $4b$ elements of $B_0(A)$ lying above unipotent characters of S. Together, this gives $k(B_0(A)) \geq 4b + (p-1)^2/(96b) > 2\sqrt{p-1}$ since $p \geq 11$, proving part (i) of Theorem 1.2. □

8. Groups of exceptional types

In this section we prove Theorem 1.2 for S being of exceptional type. This is achieved by considering each type case by case, with the help of Theorem 5.4.

We keep all the notation in Section 5. In particular, the underlying field of S has order $q = \ell^f$. By Section 2A, we may assume that $\ell \neq p \geq 11$. This assumption on p guarantees that Sylow p-subgroups of $\mathbb{G}$ are abelian. Recall also that e is the multiplicative order of q modulo p (when S is not of Suzuki or Ree type), $p^a = \Phi_e(q)_p$, and $\Phi_e^k = \Phi_e^{k_e}$ is the precise power of Φ_e dividing the generic order of $\mathbb{G}$. By Section 2A, we may assume that the Sylow p-subgroups of S are not cyclic, and thus $k_e \geq 2$. Also, $\boldsymbol{S}_e$ is a Sylow e-torus of a simple algebraic group $\boldsymbol{G}^*$ of simply connected type associated with a Steinberg endomorphism F such that $S = \mathbb{G}^*/\boldsymbol{Z}(\mathbb{G}^*)$ and $\mathbb{G}^* := \boldsymbol{G}^{*F}$, and $\boldsymbol{L}_e := \boldsymbol{C}_{\mathbb{G}^*}(\boldsymbol{S}_e)$ is a minimal e-split Levi subgroup of $\boldsymbol{G}^*$. Note that $\boldsymbol{L}_e$ is then a maximal torus of $\boldsymbol{G}^*$ (in other words, e is regular for $\boldsymbol{G}^*$), except the single case of type E_7 and $e = 4$. The relative Weyl groups $W(\boldsymbol{L}_e)$ are always finite complex reflection groups, and we will follow the notation for these groups in [Benard 1976]. Relative Weyl groups for various $\boldsymbol{L}_e$ are available in [Broué et al. 1993, Tables 1 and 3]. The structure of $\mathrm{Out}(S)$ is available in [Gorenstein et al. 1994, Theorem 2.5.12]. We will use these data freely without further notice.

It turns out that Theorem 5.4 is sufficient to prove Theorem 1.2 whenever $k_e \geq 3$. In fact, even when $k_e = 2$, Theorem 5.4 is also sufficient for Theorem 1.2(ii). We have to work harder, though, to achieve Theorem 1.2(i) in the case $k_e = 2$ for some types.

Proposition 8.1. *Theorem 1.2 holds for simple groups of exceptional types.*

Proof. (1) $\underline{S = G_2(q) \text{ and } S = F_4(q)}$: First we consider $S = G_2(q)$ (so $S = \mathbb{G}$) with $q > 2$. Then $e \in \{1, 2\}$ and $k_1 = k_2 = 2$. Also, the Sylow e-tori are maximal tori, and their relative Weyl groups are the dihedral group D_{12}. The bound (5.4) implies that $n(S) \geq 5 + (p+1)/12$ for $q = 3^f$ with odd f, and $n(S) \geq 6 + (p+1)/12$ otherwise. In any case it follows that $n(S) > 2(p-1)^{1/4}$, proving Theorem 1.2(ii) for $G_2(q)$.

Note that $\mathrm{Aut}(S)$ is a cyclic extension of S. First assume that $q \neq 3^f$ or G does not contain the graph automorphism of S. In particular, every unipotent character of S is extendible to G. Let $H := \langle S, \boldsymbol{C}_G(P)\rangle$, where P is a Sylow p-subgroup of G (and S as well by the assumption $p \nmid |G/S|$). Since $P\boldsymbol{C}_G(P)$

is contained in H, $B_0(H)$ is covered by a unique block of G, which is $B_0(G)$. It follows that, each unipotent character in $B_0(S)$ extends to an irreducible character in $B_0(H)$, which in turn lies under $|G/H|$ irreducible characters in $B_0(G)$. Therefore, the number of irreducible characters in $B_0(G)$ lying over unipotent characters of S is at least $k(D_{12})|G/H| = 6|G/H|$. When $q = 3^f$ and G does contain the nontrivial graph automorphism, similar arguments yield that the number of irreducible characters in $B_0(G)$ lying over unipotent characters of S is at least $5|G/H|$.

On the other hand, each G-orbit on semisimple characters (associated to p-elements) of S now has length at most $|G/H|$ by (5.1) and the fact that $H = \langle S, \boldsymbol{C}_G(P)\rangle$ fixes every conjugacy class of p-elements of S. Therefore, the bound (5.2) yields

$$n(G, \mathrm{Irr}_{ss}(B_0(S))) \geq \frac{p^2-1}{12|G/H|}.$$

This and the conclusion of the last paragraph imply that

$$k(B_0(G)) \geq 5|G/H| + \frac{p^2-1}{12|G/H|} \geq 2\sqrt{\frac{5(p^2-1)}{12}},$$

which in turn implies the desired bound $k(B_0(G)) > 2\sqrt{p-1}$ for all $p \geq 11$.

For $S = F_4(q)$, we have $e \in \{1, 2\}$ for which $k_e = 4$, or $e \in \{3, 4, 6\}$ for which $k_e = 2$. Therefore all the Sylow e-tori are maximal tori, and their relative Weyl groups are $G_{28} = \mathrm{GO}_4^+(3)$ for $e = 1, 2$; $G_5 = \mathrm{SL}_2(3) \times C_3$ for $e = 3, 6$; and $G_8 = C_4.\mathrm{S}_4$ for $e = 4$. Now we just follow along similar arguments as above to prove the theorem for this type.

(2) $\underline{S = {}^2F_4(q) \text{ with } q = 2^{2n+1} \geq 8 \text{ and } S = {}^3D_4(q)}$: These two types are treated in a fairly similar way as for G_2. Note that $\mathrm{Out}(S)$ here is always cyclic. First let $S = {}^2F_4(q)$. Then $e \in \{1, 2, 4^+, 4^-\}$ and $k_e = 2$ for all e. All the Sylow e-tori are maximal. The relative Weyl groups of these tori are D_{16}, $G_{12} = \mathrm{GL}_2(3)$, $G_8 = C_4.\mathrm{S}_4$ and G_8 for $e = 1, 2, 4^+$, and 4^-, respectively. One can now easily check the inequality $n(S) \geq 2(p-1)^{1/4}$, using (5.4). The bound $k(B_0(G)) > 2\sqrt{p-1}$ is proved similarly as in type G_2.

Now let $S = {}^3D_4(q)$. Then $e \in \{1, 2, 3, 6\}$ and $k_e = 2$ for all e. For $e \in \{3, 6\}$, a Sylow e-torus is maximal with the relative Weyl group $G_4 = \mathrm{SL}_2(3)$. For $e = 1$ or 2, Sylow e-tori of S are not maximal anymore but are contained in maximal tori of orders $\Phi_1^2(q)\Phi_3(q)$ and $\Phi_2^2(q)\Phi_6(q)$, respectively. The relative Weyl groups of these tori are both isomorphic to D_{12}. Now the routine estimates are applied to achieve the required bounds.

(3) $\underline{S = E_6(q) \text{ and } S = {}^2E_6(q)}$: These two types are approached similarly and we will provide details only for E_6. Then $e = 1$ for which $k_e = 6$, or $e = 2$ for which $k_e = 4$, or $e = 3$ for which $k_e = 3$, or $e \in \{4, 6\}$ for which $k_e = 2$.

Assume $e = 1$. Then $\boldsymbol{S}_1$ is a maximal torus and its Weyl group is $G_{35} = \mathrm{SO}_5(3)$. Theorem 5.4 then implies that

$$n(S) \geq k(\mathrm{SO}_5(3)) + \frac{p^6-1}{6(p-1)|\mathrm{SO}_5(3)|} = 25 + \frac{p^6-1}{311040(p-1)} > 2\sqrt{p-1},$$

proving both parts of Theorem 1.2 in this case. The case $e \in \{2, 3\}$ is similar. We note that $\boldsymbol{S}_3$ is a maximal torus with the relative Weyl group $G_{25} = 3^{1+2}.\mathrm{SL}_2(3)$, and a maximal torus containing a Sylow 2-torus has relative Weyl group G_{28}.

Assume $e = 4$. Then a maximal torus containing a Sylow 4-torus of $E_6(q)_{sc}$ has order $\Phi_4^2(q)\Phi_1^2(q)$ and its relative Weyl group is $G_8 = C_4.\mathrm{S}_4$, whose order is 96 and class number is 16. Now the bound (5.4) yields $n(S) > 2(p-1)^{1/4}$, proving part (ii) of the theorem.

We need to do more to obtain part (i) in this case. In fact, when $2\sqrt{p-1} \leq 16$, which means that $p \leq 65$, we have $n(S) > 16 \geq 2\sqrt{p-1}$, which proves part (i) as well. So let us assume that $p > 65$.

Note that $\mathrm{Out}(S)$ is a semidirect product $C_{(3,q-1)} \rtimes (C_f \times C_2)$, which may not be abelian but every unipotent character of S is still fully extendible to $\mathrm{Aut}(S)$ by [Malle 2008, Theorems 2.4 and 2.5]. As before, let $\mathbb{G}$ be the extension of S by diagonal automorphisms. Similar to the proof for type G_2, let $H := \langle G \cap \mathbb{G}, \boldsymbol{C}_G(P)\rangle$, where P is a Sylow p-subgroup of S. Each unipotent character in $B_0(S)$ then lies under at least $|\mathrm{Irr}(G/H)| = |G/H|$ irreducible characters in $B_0(G)$. (Here we note that G/H is abelian.) Thus, the number of irreducible characters in $B_0(G)$ lying over unipotent characters of S is at least $16|G/H|$.

As in Section 5D, here we have

$$|\mathrm{Irr}_{ss}(B_0(\mathbb{G}))| \geq \frac{p^2-1}{|W(\boldsymbol{L}_4)|} = \frac{p^2-1}{96}.$$

Let $\mathrm{Irr}_{ss}(B_0(S))$ be the set of restrictions of characters in $\mathrm{Irr}_{ss}(B_0(\mathbb{G}))$ to S. These restrictions are irreducible as the semisimple elements of $\mathbb{G}^*$ associated to these semisimple characters are p-elements whose orders are coprime to $|\boldsymbol{Z}(\mathbb{G}^*)| = \gcd(3, q-1)$. Moreover, if the restrictions of $\chi_{(t)}$ and $\chi_{(t_1)}$ to S are the same, then $(t) = (t_1 z)$ for some $z \in \boldsymbol{Z}(\mathbb{G}^*)$ (see [Tiep 2015, Proposition 5.1]), which happens only when z is trivial since t and t_1 are p-elements. It follows that

$$|\mathrm{Irr}_{ss}(B_0(S))| = |\mathrm{Irr}_{ss}(B_0(\mathbb{G}))| \geq \frac{p^2-1}{96}.$$

Note that $\mathbb{G} \cap G = \mathbb{G}$ or S and each G-orbit of relevant semisimple characters in $B_0(\mathbb{G} \cap G)$, and hence in $B_0(S)$, has length at most $|G/H|$. It follows that the number of irreducible characters in $B_0(G)$ lying over semisimple characters in $B_0(S)$ is at least $(p^2-1)/(96|G/H|)$. Together with the bound of $16|G/H|$ for the number of irreducible characters in $B_0(G)$ lying over unipotent characters of S, we deduce that

$$k(B_0(G)) \geq 16|G/H| + \frac{p^2-1}{96|G/H|} \geq 2\sqrt{\frac{16(p^2-1)}{96}},$$

and thus, when $p > 65$, the desired bound $k(B_0(G)) > 2\sqrt{p-1}$ follows.

The last case $e = 6$ can be argued in a similar way, with notice that a maximal torus containing a Sylow 6-torus of $E_6(q)_{sc}$ has order $\Phi_6^2(q)\Phi_3(q)$ and its relative Weyl group is $G_5 = \mathrm{SL}_2(3) \times C_3$, whose order is 72 and class number is 21.

(4) $\underline{S = E_7(q)}$: Then $e \in \{1, 2\}$ for which $k_e = 7$, or $e \in \{3, 6\}$ for which $k_e = 3$, or $e = 4$ for which $k_e = 2$. When $k_e > 2$, the bound (5.4) again is sufficient to achieve the desired bound $n(S) > 2\sqrt{p-1}$. In fact,

even for the case $k_e = 2$, we have $n(S) \geq 2(p-1)^{1/4}$. So it remains to prove Theorem 1.2(i) for $e = 4$, in which case e is not regular and the relative Weyl group of the minimal e-split Levi subgroup $\boldsymbol{L}_e = \boldsymbol{S}_e.A_1^3$ is G_8. The estimates are now similar to those in the case $e = 4$ of the type E_6.

(5) $\underline{S = E_8(q)}$: Then $e \in \{1, 2\}$ for which $k_e = 8$, or $e \in \{3, 4, 6\}$ for which $k_e = 4$, or $e \in \{5, 8, 10, 12\}$ for which $k_e = 2$. The standard approach as above works for all e with $k_e > 2$.

Assume that $e \in \{5, 10\}$. Then a Sylow e-torus of S is maximal and its relative Weyl group is $G_{16} \cong \mathrm{SL}_2(5) \times C_5$. A similar proof to the case of type G_2 yields $k(B_0(G)) \geq 2\sqrt{45(p^2-1)/600}$, which is certainly greater than $2\sqrt{p-1}$ for $p \geq 13$. On the other hand, we always have $k(B_0(G)) \geq 45 > 2\sqrt{p-1}$ for smaller p, and thus the desired bound holds for all p. Finally, the case $e \in \{8, 12\}$ is entirely similar, with notice that the relative Weyl groups of Sylow e-tori are $G_9 = C_8.\mathrm{S}_4$ and $G_{10} = C_{12}.\mathrm{S}_4$ for $e = 8$ and 12, respectively. □

Theorem 1.2 is now completely proved.

9. Proof of Theorems 1.1 and 1.3

We are now ready to prove the main results.

Proof of Theorems 1.1 and 1.3. First we remark that the "if" implication of Theorem 1.3 is clear, and moreover, we are done if the Sylow p-subgroups of G are cyclic, thanks to Section 2A.

Let (G, p) be a counterexample to either Theorem 1.1 or the "only if" implication of Theorem 1.3 with $|G|$ minimal. In particular, Sylow p-subgroups of G are not cyclic and $k(B_0(G)) \leq 2\sqrt{p-1}$. Let N be a minimal normal subgroup of G. Note that $N = G$ if G turns out to be simple.

Assume first that $p \mid |G/N|$. Then, since $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$ and by the minimality of $|G|$, we have

$$2\sqrt{p-1} \geq k(B_0(G)) \geq k(B_0(G/N)) \geq 2\sqrt{p-1},$$

and thus

$$k(B_0(G)) = k(B_0(G/N)) = 2\sqrt{p-1}.$$

The minimality of G again then implies that G/N is isomorphic to the Frobenius group $C_p \rtimes C_{\sqrt{p-1}}$. It follows that $p \mid |N|$, and thus there exists a nontrivial irreducible character $\theta \in \mathrm{Irr}(B_0(N))$. As $B_0(G)$ covers $B_0(N)$, there is some $\chi \in \mathrm{Irr}(B_0(G))$ lying over θ, implying that $k(B_0(G)) > k(B_0(G/N))$, a contradiction.

So we must have $p \nmid |G/N|$, and it follows that $p \mid |N|$. This in fact also yields that N is the unique minimal normal subgroup of G. Assume first that N is abelian. We then have that G is p-solvable, and hence Fong's theorem (see [Navarro 1998, Theorem 10.20]) implies that

$$k(B_0(G)) = k(B_0(G/\boldsymbol{O}_{p'}(G))) = k(G/\boldsymbol{O}_{p'}(G)),$$

which is greater than $2\sqrt{p-1}$ by the main result of [Maróti 2016].

We now may assume that $N \cong S_1 \times S_2 \times \cdots \times S_k$, a direct product of $k \in \mathbb{N}$ copies of a nonabelian simple group S. If S has cyclic Sylow p-subgroups, then G is not a counterexample for Theorem 1.1 by Lemma 2.1. Furthermore,

$$k(B_0(G)) \geq k(\boldsymbol{N}_G(P)/\boldsymbol{O}_{p'}(\boldsymbol{N}_G(P))) > 2\sqrt{p-1}$$

by the analysis in Section 2A, and thus G is not a counterexample for Theorem 1.3 either.

So the Sylow p-subgroups of S are not cyclic. Let n be the number of $\boldsymbol{N}_G(S_1)/N$-orbits on $\mathrm{Irr}(B_0(S_1))$. By Theorem 1.2(ii), we have $n \geq 2(p-1)^{1/4}$. Therefore, if $k \geq 2$, the number of G-orbits on $\mathrm{Irr}(B_0(N)) = \prod_{i=1}^k \mathrm{Irr}(B_0(S_i))$ is at least $n(n+1)/2 \geq 2(p-1)^{1/4}(2(p-1)^{1/4}+1)/2 > 2\sqrt{p-1}$, and it follows that $k(B_0(G)) > 2\sqrt{p-1}$, a contradiction. Hence, $N = S$ and G is then an almost simple group with socle S. Furthermore, $p \nmid |G/S|$. But such a group G cannot be a counterexample by Theorem 1.2(i). The proof is complete. □

In regard to Theorem 1.1, we remark that Kovács and Leedham-Green constructed, for any odd prime p, a family of p-groups P of order p^p with $k(P) = (p^3 - p^2 + p + 1)/2$; see [Pyber 1992]. Therefore the bound $k(B_0(G)) \geq 2\sqrt{p-1}$ cannot be replaced by $k(B_0(G)) \geq p^3$, even if one assumes $|G|$ to be divisible by a certain fixed power of p.

With Theorem 1.1 in mind, it follows that for any p-block B for a finite group such that $k(B) = k(B_0(H))$ for some finite group H of order divisible by p, we have $k(B) \geq 2\sqrt{p-1}$. In particular, we may record the following:

Corollary 9.1. *Let G be one of the classical groups $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$, or $\mathrm{GO}_{2n}^{\pm}(q)$. Let p be a prime dividing $|G|$ and not dividing q. Then for any p-block B of G with positive defect, we have $k(B) \geq 2\sqrt{p-1}$.*

Proof. If $p = 2$, then the statement is clear, so we assume p is odd. First, if $G = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$, the statement follows immediately from Theorem 1.1 and [Michler and Olsson 1983, Theorem (1.9)], which states that B has the same number of irreducible characters as the principal block of a product of lower-rank general linear and unitary groups of order also divisible by p.

Now suppose that G is $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$, or $\mathrm{GO}_{2n}^{\pm}(q)$. If B is a unipotent block, then by [Malle 2018, Proposition 5.4 and 5.5], B has the same number of irreducible characters as a block of an appropriate general linear group of order also divisible by p. (In the case $\mathrm{GO}_{2n}^{\pm}(q)$, we define a unipotent block to be one lying above a unipotent block of $\mathrm{SO}_{2n}^{\pm}(q)$.) Hence the statement holds if B is a unipotent block.

Now, the block B determines a class of semisimple p'-elements (s) of the dual group G^* (see [Cabanes and Enguehard 2004, Theorem 9.12]) such that B contains some member of $\mathcal{E}(G, (s))$. By [Enguehard 2008, Théorème 1.6], there exists a group $G(s)$ dual to $\boldsymbol{C}_{G^*}(s)$ such that $k(B) = k(b)$ for some unipotent block b of $G(s)$. Now, in the cases under consideration, $\boldsymbol{C}_{G^*}(s)$ and $G(s)$ are direct products of lower-rank classical groups of the types being considered here, completing the proof. □

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